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Some Homogenization problems in a periodic setting with a local defect

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Thèse de doctorat

École doctorale Sciences Mathématiques de Paris Centre (ED386)

Laboratoire Jacques-Louis Lions

Discipline **Mathématiques Appliquées**

Thèse présentée par **Sylvain Wolf**

Étude de problèmes d'homogénéisation dans un cadre
périodique avec défauts

Some Homogenization problems in a periodic setting with
defects

sous la direction de **Xavier Blanc**,
soutenue publiquement le **18/07/2022** devant le jury composé de

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La théorie de l'homogénéisation traite de l'étude des matériaux multi-échelles, c'est-à-dire de matériaux dont certaines propriétés physiques dépendent d'une micro-échelle, supposée très petite devant l'échelle d'observation. L'objectif principal est alors de remplacer ce matériau par un autre, plus simple, dont les propriétés physiques sont moyennées, ou homogénéisées. Bien que le cadre général de l'homogénéisation demande assez peu de propriétés sur la microstructure, une difficulté majeure est d'obtenir des quantités homogénéisées explicites et facilement calculables en pratique. Un cadre particulièrement agréable est le cadre périodique, pour lequel la description fine du matériau peut se limiter à son étude sur une cellule de périodicité. Une autre approche, qui est en fait apparue assez tôt, est l'approche stochastique. On ne connaît le matériau qu'en moyenne statistique et l'on impose des conditions sur la loi qu'il suit. Signalons qu'il existe d'autres cadres permettant d'obtenir des résultats précis d'homogénéisation, par exemple le cadre quasi-périodique. Une question intéressante, en ayant notamment en vue une modélisation plus proche de la réalité, est le traitement des défauts. Dans cette thèse, nous nous intéressons au cas de défauts locaux dans un fond périodique, que nous illustrons sur différents problèmes d'équations aux dérivées partielles elliptiques. Dans une première partie, on traite le cas de l'homogénéisation en milieu perforé. Puis, dans une seconde partie, on s'intéresse à l'homogénéisation d'équations non-linéaires dans lesquelles le coefficient oscillant est périodique et localement perturbé. Nous décrivons brièvement le contenu de chaque chapitre.

Dans le chapitre 1, nous présentons le contexte scientifique général et nous introduisons les résultats principaux exposés dans les chapitres suivants. Nous finissons par une brève section de perspectives. Ces perspectives s'appuient sur les résultats obtenus dans le corps de la thèse.

Dans le chapitre 2, nous étudions l'homogénéisation de l'équation de Poisson en milieu perforé. Les trous sont espacés d'une distance $\varepsilon \ll 1$ et sont de taille ε . Nous rappelons d'abord les résultats bien connus du cas périodique puis nous introduisons la classe de domaines non périodiques étudiée dans cette thèse. Cette géométrie s'appuie fortement sur le cadre périodique : lorsque la micro-échelle converge vers zéro, le domaine non-périodique coïncide avec le domaine périodique loin de l'origine. Nous construisons ensuite les outils classiques de l'homogénéisation : développement à deux échelles et correcteurs. Nous finissons enfin par des taux de convergence assez précis. Ces résultats redonnent, pour le cas sans défaut, ceux, bien connus, du cas périodique.

Dans le chapitre 3, nous travaillons sur le système de Stokes dans le même cadre que celui du chapitre précédent. Le comportement limite quand $\varepsilon \rightarrow 0$ est donné par la loi de Darcy. Comme pour l'équation de Poisson, nous construisons d'abord les correcteurs d'ordre 1 puis nous obtenons des taux de convergence dans un cadre cette fois-ci plus restreint.

Dans le chapitre 4, nous commençons la seconde partie de la thèse en étudiant des problèmes d'homogénéisation non linéaire à coefficients oscillants périodiques et localement perturbés. Ce cadre a été introduit dans [18, 19, 20] dans le cas linéaire. Nous traitons d'abord le cas d'une équation semi-linéaire. Nous présentons dans un premier temps des résultats d'homogénéisation dans le cas

périodique pour cette équation. Nous constatons ensuite que les outils principaux nécessaires à l’homogénéisation, tels que les correcteurs, sont couverts par le problème linéaire associé, déjà étudié dans des travaux précédents.

Dans le chapitre 5, nous nous concentrons sur une équation quasi linéaire. Contrairement au cas semi linéaire, l’équation du correcteur est non linéaire. Nous commençons par des rappels sur le cas périodique puis nous exhibons un cadre variationnel (qui correspond à des défauts suffisamment localisés, par exemple à support compact) permettant de construire les correcteurs. Nous étendons ensuite les résultats périodiques au cadre non périodique. Une hypothèse importante dans ce chapitre sera la non dégénérescence des correcteurs périodiques. Nous illustrons ces résultats par une description complète de la situation dans le cas unidimensionnel.

Articles écrits durant cette thèse

- [1] Blanc X., Wolf S., *Homogenization of the Poisson equation in a non periodically perforated domain*, Asymptotic Analysis vol. 126, no. 1-2, pp. 129–155, 2022.
- [2] Wolf S., *Homogenization of the Stokes system in a non periodically perforated domain*, Multiscale Model. Simul., 20(1), 72–106, 2022.
- [3] Wolf S., *Homogenization of the p -Laplace equation in a periodic setting with a local defect*, soumis, 2022.

Rapport écrit durant cette thèse

- [1] Wolf S., *Homogenization of the Poisson equation and Stokes system in some non-periodically perforated domains*, Oberwolfach report, 2021.

The homogenization theory deals with the study of multiscale materials. In these materials, some physical properties depend on a microscale that is assumed to be very small compared to the observation scale. The main purpose is to replace these media by simpler ones for which the physical properties are averaged (or homogenized). Though the general theory of homogenization requires few conditions on the microstructure, one major challenge is to obtain explicit homogenized quantities that are easily computable. The periodic case is such a fundamental example: the fine description of the material can be achieved simply by considering a single periodic cell. Another approach, that appeared early in the mathematical development of the homogenization theory, is to consider a stochastic setting. The material is only statistically known and we impose conditions on the law it follows. Let us mention that there exist other settings allowing precise homogenization results, for example the quasiperiodic setting. A relevant question, having particularly in mind a more realistic modeling, is the treatment of defects. In this thesis, we are interested in local defects that may appear in a periodic background. We examine different problems. In the first part, we study the case of perforated domains. In the second part, we are interested in nonlinear homogenization with oscillating coefficients that are local perturbations of a periodic background. We now briefly describe the content of each chapter.

In Chapter 1, we present the general scientific context and we introduce the main contributions of the thesis, that are detailed in subsequent chapters. We end by a brief perspective section which is based on the results obtained in this thesis.

In Chapter 2, we study the homogenization of Poisson equation in porous medium. The holes are separated by a distance $\varepsilon \ll 1$ and are of size ε . We first recall the well known results of the periodic case. We then introduce the class of nonperiodic domains that are considered in this part. The geometry strongly relies on the periodic framework: when the microscale tends to zero, the nonperiodic porous medium coincides with the periodic one far from the origin. We build the classical objects of the homogenization theory: two scale expansion and correctors. We end by proving precise convergence rates. We recover the results of the periodic case in this new setting.

In Chapter 3, we work on the Stokes system in the same framework as in the previous chapter. The behaviour of the system converges when $\varepsilon \rightarrow 0$ to Darcy's law. As for the Poisson equation, we build the first order correctors and we obtain convergence rates when imposing some conditions on the source term.

In Chapter 4, we begin the second part of the thesis by studying semilinear homogenization problems with oscillating coefficients that are periodic and locally perturbed. This setting has been introduced in [18, 19, 20] for the linear case. We first present homogenization results for periodic coefficients. We then notice that the main tools allowing for homogenization, such as the correctors, are obtained by studying the associated linear problem. Such problems have already been studied in previous works.

In Chapter 5, we focus on a quasilinear equation. Contrary to the semilinear case, the corrector equation is nonlinear. We start by recalling the main results of the periodic case. We then exhibit a variational framework (that corresponds to sufficiently fast decreasing defects, for example compactly supported) and we build the corresponding correctors. This allows to extend the convergence results from the periodic setting to the nonperiodic one. An important assumption in this chapter will be the non degeneracy of the periodic correctors. We also give a precise description of the situation in the one dimensional setting.

List of papers written during this PhD thesis

- [1] Blanc X., Wolf S., *Homogenization of the Poisson equation in a non periodically perforated domain*, Asymptotic Analysis vol. 126, no. 1-2, pp. 129–155, 2022.
- [2] Wolf S., *Homogenization of the Stokes system in a non periodically perforated domain*, Multiscale Model. Simul., 20(1), 72–106, 2022.
- [3] Wolf S., *Homogenization of the p -Laplace equation in a periodic setting with a local defect*, submitted, 2022.

Report written during this PhD thesis

- [1] Wolf S., *Homogenization of the Poisson equation and Stokes system in some non-periodically perforated domains*, Oberwolfach report, 2021.

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1.1 Contexte scientifique

Cette thèse traite de la théorie de l'homogénéisation. L'objectif principal de cette théorie est d'étudier des équations très oscillantes à une échelle microscopique $\varepsilon \ll 1$. Il peut s'agir d'une équation aux dérivées partielles (EDP) posée dans un matériau très hétérogène comme présenté sur la figure 1.1. En ingénierie, la résolution numérique d'une telle EDP (par exemple l'équation de la chaleur stationnaire) nécessite la discrétisation de l'espace en une maille de taille $H \ll \varepsilon$. En particulier, $\varepsilon \ll 1$ devient une contrainte très forte sur H et donc sur le temps de calcul. Pour se libérer de cette contrainte, l'idée est d'approcher théoriquement le milieu hétérogène par un milieu homogène. L'équation décrivant ce milieu homogène peut ensuite se discrétiser facilement. On adopte formellement la stratégie suivant $T_\varepsilon \underset{(1)}{\approx} T^* \underset{(2)}{\approx} (T^*)^H$ où

- T_ε est la température du milieu hétérogène et T^* la température dans le milieu homogénéisé ;
- $(T^*)^H$ est une discrétisation de T^* .

La seconde approximation (2) est numérique (et fournie par un théorème de convergence d'un schéma numérique quand $H \rightarrow 0$) tandis que la première (1) est théorique (et fournie par un théorème de convergence de la solution oscillante vers la solution homogénéisée quand $\varepsilon \rightarrow 0$). L'homogénéisation permet précisément d'effectuer l'approximation (1). D'autres motivations à l'homogénéisation sont

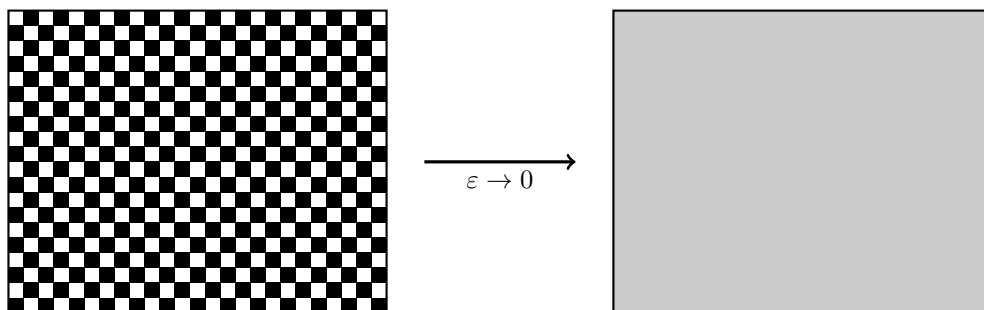


FIGURE 1.1 : Procédé d'homogénéisation : la figure de gauche représente une propriété physique d'un matériau (par exemple la conductivité thermique). Celle-ci est fortement hétérogène et l'on note ε sa taille caractéristique d'oscillation. On aimerait remplacer le matériau de gauche par le matériau de droite, homogénéisé.

issues de la physique et en particulier du passage d'une loi microscopique à une loi macroscopique. On peut par exemple citer la loi de Darcy, que nous étudierons dans le chapitre 3. Nous rappelons brièvement le contexte historique. Nous renvoyons à [92] pour une description plus détaillée et à [91, 6] pour une approche mathématique. En 1856, l'ingénieur Henry Darcy, en charge des conduites d'eau à Dijon, postule que la vitesse d'un fluide à travers un milieu poreux est linéaire en la différence de pression : $\vec{v} = -\kappa \nabla p$ où κ est une constante appelée perméabilité du milieu. Cette loi phénoménologique a été observée en faisant couler de l'eau dans un tube de sable vertical, ceci en vue de la filtrer [47, Note D]. Plus d'un siècle plus tard, une explication théorique, issue de l'homogénéisation, a été donnée à la loi de Darcy [99, 91]. En régime permanent, l'équation vérifiée par la vitesse du fluide est donnée par un système de Stokes. Le problème peut alors se reformuler en un problème d'homogénéisation : en notant ε la taille caractéristique d'un grain de sable, le système de Stokes vérifié par le fluide est posé dans un domaine Ω_ε qui comporte des "trous" (les grains de sable) de taille ε . On souhaite alors passer à la limite $\varepsilon \rightarrow 0$ dans l'EDP. Nous renvoyons à la sous-section 1.2.1.3 pour des résultats mathématiques précis.

Du point de vue mathématique, la théorie de l'homogénéisation déterministe s'est développée dans les années 1970 avec les travaux de De Giorgi-Spagnolo [96], Babuska [14], Sanchez-Palencia [91], Bensoussan-Lions-Papanicolaou [16], Murat-Tartar [83, 98, 101, 100] et, plus tard, N'Guetseng [85, 86], Allaire [5] sur la convergence à deux échelles, Braides-Defranceschi pour la minimisation de fonctionnelles [24], Cioranescu-Damlamian-Griso [33] pour la méthode d'éclatement périodique. Dans [16], une hypothèse importante est la périodicité de la micro-structure. Cela permet dans de nombreux cas d'obtenir des grandeurs homogénéisées explicites. Pour la simple équation de diffusion thermique stationnaire

$$-\operatorname{div} a \left(\frac{\cdot}{\varepsilon} \right) \nabla T_\varepsilon = f \quad \text{dans } \Omega, \quad T_\varepsilon = 0 \quad \text{sur } \partial\Omega \quad (1.1.1)$$

où a est un coefficient périodique, on a les résultats bien connus¹

$$T_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} T^*, \quad -\operatorname{div} a^* \nabla T^* = f \quad \text{dans } \Omega, \quad T^* = 0 \quad \text{sur } \partial\Omega \quad (1.1.2)$$

où a^* se calcule très simplement en résolvant d problèmes de cellule, appelés problèmes des correcteurs (ici, d est la dimension de l'espace ambiant). Ces résultats se généralisent au cas d'une suite de coefficients $(a_\varepsilon)_\varepsilon$ abstraite, uniformément bornée et coercive par rapport à ε (voir par exemple [101, Chapitre 6] ou encore [86] pour un autre cadre général abstrait dans un espace de fonctions admettant une moyenne). On a alors l'existence d'une sous-suite $(\varepsilon_k)_{k \in \mathbb{N}}$, $\varepsilon_k \xrightarrow{k \rightarrow +\infty} 0$, et d'un coefficient homogénéisé a^* (indépendant du second membre f mais dépendant a priori de la sous-suite $(\varepsilon_k)_{k \in \mathbb{N}}$) tel que

$$T_{\varepsilon_k} \xrightarrow{k \rightarrow +\infty} T^*, \quad -\operatorname{div} a^* \nabla T^* = f \quad \text{dans } \Omega, \quad T^* = 0 \quad \text{sur } \partial\Omega. \quad (1.1.3)$$

La question du calcul pratique de a^* se pose. Comme nous l'avons dit, ce calcul est remarquablement aisé dans le cas périodique. Pour autant, dans le cas général, il s'avère très compliqué à mener, notamment car a^* est défini par la limite (faible) d'une suite de fonctions, cf [101, Chapitre 13]. Un enjeu important de l'homogénéisation déterministe au delà du cas périodique est de trouver des classes de coefficients dont les coefficients homogénéisés se calculent facilement en pratique. Une telle classe (assez proche du cas périodique) est par exemple le cadre quasi-périodique [20] qui correspond, dans (1.1.1), à des coefficients a de la forme $a(y) = b(y_1, y_2)$ où b est périodique en chacune de ses variables (avec des périodes éventuellement non commensurables).

Parallèlement au développement de l'homogénéisation déterministe est apparue l'homogénéisation stochastique avec les travaux de Kozlov, Papanicolaou, Varadhan [71, 89]. Dans ce cas, le coefficient $a_\varepsilon = a(\cdot/\varepsilon, \omega)$ dépend d'un paramètre $\omega \in \Omega$ où $(\Omega, \mathcal{F}, \mathbb{P})$ est un espace de probabilité. On peut en fait voir Ω comme un ensemble de coefficients. Les hypothèses imposées sur \mathbb{P} sont la stationnarité (c'est-à-dire l'invariance de la loi par l'opération de translation d'un vecteur $x \in \mathbb{R}^d$) et l'ergodicité (qui peut être vue comme une décorrélation aux larges échelles). Dans ce cadre, on peut montrer, à

¹où la convergence a lieu en un certain sens que nous ne précisons pas.

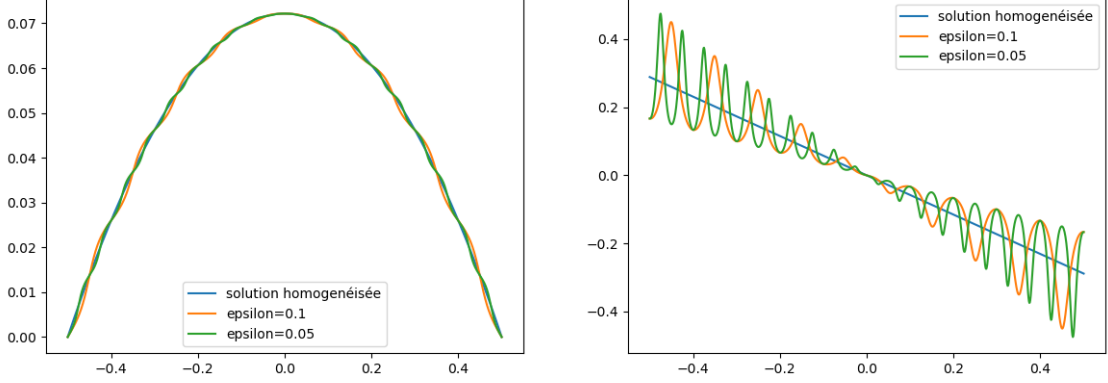


FIGURE 1.2 : Représentation graphique de la solution à l'équation $-(a(\cdot/\varepsilon)T'_\varepsilon)' = f$, $T_\varepsilon(-\frac{1}{2}) = T_\varepsilon(\frac{1}{2}) = 0$ où $f(x) = 1$, $a(y) = 2 + \cos(2\pi y)$, $\varepsilon = 0.1$ et $\varepsilon = 0.05$. La fonction u_ε est à gauche, sa dérivée est à droite.

nouveau dans le cas de la diffusion linéaire, que la suite $(T_\varepsilon(\cdot, \omega))_{\varepsilon>0}$ s'homogénéise avec un coefficient limite déterministe pour lequel il existe une formule explicite. Nous renvoyons à [41, Introduction] pour une synthèse à ce sujet.

Une fois la limite de la suite oscillante $(T_\varepsilon)_{\varepsilon>0}$ trouvée se pose la question du mode de convergence vers la solution homogénéisée T^* . Pour avoir une intuition des phénomènes en jeu, nous représentons sur la figure 1.2 un exemple en dimension 1. Nous constatons sur la figure de gauche que la solution exacte T_ε est proche de sa limite quand ε est petit. En revanche, ce n'est pas le cas de sa dérivée T'_ε , qui oscille autour de $(T^*)'$. Revenant au problème de référence (1.1.1), cela suggère que la convergence dans (1.1.2) peut être forte, mais que ce n'est pas le cas de la convergence de ∇T_ε , qui n'est en fait que faible. Pour obtenir la convergence forte des gradients, il est nécessaire d'introduire une famille de fonctions qui corrige les oscillations de T_ε à l'échelle ε . Ces fonctions sont appelées correcteurs. Dans (1.1.2)–(1.1.3), une approximation forte de ∇T_ε est typiquement de la forme (à, éventuellement, extraction d'une sous-suite près)

$$\nabla T_\varepsilon = \nabla T^* + \sum_{i=1}^d \nabla w_i^\varepsilon \partial_i T^* + \text{termes d'ordre plus élevé}, \tag{1.1.4}$$

où w_i^ε est une fonction appelée correcteur dans la direction² e_i à l'échelle ε . Sans rentrer dans les détails, signalons que les correcteurs peuvent être définis à l'aide d'une EDP ayant pour coefficient a_ε . Le calcul pratique des fonctions w_i^ε est un enjeu majeur de l'homogénéisation. Il est d'ailleurs instructif de commenter ce calcul dans les différents cadres introduits ci-dessus. Dans le cas périodique, il est remarquablement aisé. En effet, deux simplifications de taille sont possibles :

- (i) le coefficient oscillant a_ε étant de la forme $a(\cdot/\varepsilon)$, la fonction w_i^ε peut être écrite sous la forme $w_i(\cdot/\varepsilon)$ où, bien sûr, la fonction w_i est indépendante de ε ;
- (ii) le coefficient a étant périodique, la fonction w_i peut être cherchée périodique. L'EDP définissant w_i est alors posée sur le cube unité, voir (1.2.33).

Si l'on résume la situation, nous sommes passés, dans le cas périodique, d'un nombre infini de problèmes $(w_i^\varepsilon)_{\varepsilon \in (0,1)}$ à un nombre fini de problèmes $(w_i)_{\substack{1 \leq i \leq d \\ 1 \leq i \leq d}}$ posés sur le cube unité : le gain pratique est considérable. Dans le cadre quasi-périodique, on peut à nouveau chercher w_i^ε sous la forme $w_i(\cdot/\varepsilon)$.

²On note e_1, \dots, e_d la base canonique de \mathbb{R}^d .

La situation est alors similaire au cas périodique, à ceci près que l'équation donnant w_i est posée sur tout l'espace \mathbb{R}^d , ce qui rend son étude plus difficile. Un théorème d'existence est montré dans [20, théorème 5.8]. Dans le cadre déterministe général, la situation est plus délicate, voir [101, Théorème 13.1]. La faible structure imposée sur la suite de coefficients oscillants $(a_\varepsilon)_\varepsilon$ impose de résoudre, pour chaque indice k de la sous-suite $(\varepsilon_k)_{k \in \mathbb{N}}$, $\varepsilon_k \xrightarrow[k \rightarrow +\infty]{} 0$, une équation dépendant, entre autre, de a_{ε_k} . Cela est donc, d'un point de vue pratique, inutilisable. Finissons ce paragraphe par le cas stochastique (stationnaire et ergodique). La forme de l'approximation de la solution oscillante est analogue à (1.1.4) avec $w_i^\varepsilon = w_i(\cdot/\varepsilon, \omega)$, $\omega \in \Omega$. L'équation du correcteur $w_i(\cdot/\varepsilon, \omega)$ est alors posée sur tout l'espace. Il est montré dans [89, théorème 2], en utilisant la structure imposée sur l'ensemble de coefficients (voir aussi [65, lemme 1]), que cette équation est, pour presque tout coefficient $a(\cdot, \omega)$, bien posée.

Dans l'idée de mieux décrire le comportement de la solution oscillante, il est naturel d'essayer de démontrer des taux de convergence dans l'approximation (1.1.4). Cette question est parfois appelée homogénéisation quantitative, par opposition à l'homogénéisation qualitative, décrite ci-dessus, qui consiste à trouver la limite de la suite de fonctions oscillantes $(T_\varepsilon)_{\varepsilon > 0}$ quand $\varepsilon \rightarrow 0$. Cette question est un champ de recherche actif dans le cas stochastique. Elle ne sera pas abordée ici. Nous mentionnons cependant les travaux fondamentaux de A. Gloria, F. Otto, S. Neukamm d'une part et de S. Armstrong, T. Kuusi, J.-C. Mourrat d'autre part et nous donnons [10, 65] pour références. En ce qui concerne l'homogénéisation déterministe, les premiers résultats remontent aux années 1990 avec les contributions de M. Avellaneda et F. Lin [12, 11] (notons que l'on peut légèrement relaxer l'hypothèse de périodicité dans ce travail, voir la discussion 5.1 p. 846 dans [11]). On fait également référence à [93] pour un panorama sur l'homogénéisation quantitative périodique. En quelques mots, ces travaux montrent que, dans le cas périodique, et à nouveau pour l'équation (1.1.1), une approximation très simple de T_ε est disponible :

$$T_\varepsilon = T^* + \varepsilon \sum_{i=1}^d w_i \left(\frac{\cdot}{\varepsilon} \right) \partial_i T^* + \text{reste}, \quad (1.1.5)$$

où, grossièrement, le reste et son gradient se contrôlent par une puissance strictement positive de ε dans les normes de Lebesgue L^q , $q \in [1, +\infty]$. Pour autant, le cadre périodique, bien que très satisfaisant d'un point de vue des résultats mathématiques, semble bien restrictif du point de vue de la modélisation. À l'inverse, le cadre stochastique permet de traiter une classe de coefficients plus large mais semble plus coûteux à mettre en place d'un point de vue pratique. Dans cette thèse, nous souhaitons étudier un ensemble de problèmes non périodiques, tout en gardant un procédé d'homogénéisation simple à mettre en place et donc proche du cadre périodique. Nous aimerions, par exemple, prendre en compte la modélisation de défauts qui pourraient apparaître dans une micro-structure supposée périodique. Notre réflexion s'appuie sur les travaux antérieurs de X. Blanc, C. Le Bris, P.-L. Lions et, plus tard, M. Josien, dans le cas de l'équation modèle (1.1.1) (et, par la suite, pour des équations qui ne sont pas sous forme divergence [21]). Les résultats obtenus dans ces articles seront détaillés dans la section 1.2.2.1 de cette introduction. Ils pointent dans deux directions (complémentaires) :

- (A) construire une classe de coefficients non périodiques pour laquelle l'équation du correcteur admet une solution [19, 20, 18];
- (B) construire un cadre général (au-delà du cas périodique) déterministe permettant d'étendre les résultats quantitatifs de [11, 12], voir [17].

Du premier point a émergé l'idée d'une classe de coefficients périodiques avec défauts. S'appuyant sur le cas d'un coefficient périodique, pour lequel on dispose de nombreux résultats mathématiques, les auteurs de [19, 20, 18] traitent le cas d'une perturbation de ce coefficient. A l'échelle macroscopique, le défaut se concentre autour d'un point quand $\varepsilon \rightarrow 0$. Dans ce cadre, ils parviennent à résoudre l'équation du correcteur. En ce qui concerne le second point, des taux de convergence sont obtenus dans [17] sous certaines hypothèses dont, surtout, l'existence des correcteurs et une quantification de leur sous-linéarité à l'infini. Ce cadre contient bien sûr le cadre périodique avec défauts introduit

dans le premier point. Cette thèse vise à aborder d'autres problèmes d'homogénéisation au-delà du cas périodique, tout en gardant la même précision dans l'approximation de la solution oscillante. En s'appuyant sur les travaux de X. Blanc, C. Le Bris et P.-L. Lions, nous proposons des perturbations du cas périodique pour différents problèmes d'EDP elliptiques. Nous discuterons systématiquement, pour chacun des problèmes abordés dans ce manuscrit, les deux points (A) et (B) ci-dessus. Les problèmes que nous considérons ici sont de deux types :

- Équations oscillantes : il s'agit d'équations de la forme

$$-\operatorname{div} a\left(\frac{\cdot}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon\right) = f \quad \text{dans } \Omega \quad (1.1.6)$$

où Ω est un domaine borné et f est le terme source. Nous imposerons des données au bord de type Dirichlet homogène. Ce sont des équations de conservation où le flux a peut potentiellement être non linéaire. Le cas $a(y, u, \xi) = a(y)\xi$ correspond bien sûr à l'équation (1.1.1). On regardera des cas où la conductivité dépend de ξ , par exemple le cas du p -Laplacien $a(y, \xi) = a(y)|\xi|^{p-2}\xi$, $1 < p < +\infty$. Dans le cas scalaire, cela modélise, quand $|\nabla u_\varepsilon|$ est petit, un milieu très diffusif dans le cas $1 < p < 2$ et peu diffusif quand $p > 2$. Dans le cas vectoriel, de telles lois de puissance sont utilisées pour modéliser l'écoulement de fluides non-Newtoniens. Le cas $p > 2$ correspond à des fluides qui se rigidifient quand ils sont accélérés et le cas $p < 2$ à des fluides qui se liquéfient.

- Domaines perforés : il s'agit d'équations posées dans un domaine Ω_ε qui dépend de la micro-structure ε . Nous considérerons des domaines Ω_ε perforés par des trous de taille ε . Cela peut par exemple modéliser l'écoulement d'un fluide à travers un milieu poreux. Ce problème est alors relié à la loi de Darcy comme expliqué au début de l'introduction.

Beaucoup de références sont disponibles pour chacune des deux familles de problèmes ci-dessus. Nous en donnerons dans la section 1.2 ainsi qu'au cours des chapitres. Nous décrivons maintenant les résultats obtenus dans cette thèse.

1.2 Présentation de la thèse

Cette thèse se compose de deux parties distinctes. La première partie (chapitres 2 et 3) correspond aux articles [22, 107] (ainsi qu'au rapport [105]) et traite d'homogénéisation dans les milieux perforés pour un cadre périodique avec défauts. Nous y étudions l'équation de Poisson et le système de Stokes. La seconde partie concerne l'homogénéisation d'équations non linéaires à coefficients oscillants périodiques et localement perturbés. Nous étudions une équation semi linéaire dans le chapitre 4 ainsi qu'une classe d'équations quasi linéaires dans le chapitre 5. Le contenu de ce dernier chapitre fournira [106]. Nous détaillons dans la suite de cette section les résultats obtenus dans ces deux parties.

1.2.1 Homogénéisation en milieu perforé avec défaut

Dans les chapitres 2 et 3 (voir aussi [22, 107]), nous introduisons une classe de domaines perforés qui permettent l'homogénéisation de l'équation de Poisson et du système de Stokes en s'appuyant sur les résultats connus du cas périodique [71, 77, 40, 91, 99, 2]. Nous commençons par introduire cette classe de domaines.

1.2.1.1 Perturbation locale d'un domaine périodiquement perforé

Soit $d \geq 1$, on note $Q := (-\frac{1}{2}, \frac{1}{2})^d$ le cube unité de \mathbb{R}^d . Soit $k \in \mathbb{Z}^d$, on note

$$Q_k := Q + k = \prod_{i=1}^d \left(k_i - \frac{1}{2}, k_i + \frac{1}{2}\right)^d.$$

Soit $\Omega \subset \mathbb{R}^d$ un domaine (ouvert et connexe) borné et régulier (au moins de classe Lipschitz). On se fixe une perforation centrale, *i.e.* un ouvert $\mathcal{O}_0^{\text{per}}$ tel que $\mathcal{O}_0^{\text{per}} \subset\subset Q$, $Q \setminus \overline{\mathcal{O}_0^{\text{per}}}$ est connexe pour $d \geq 2$

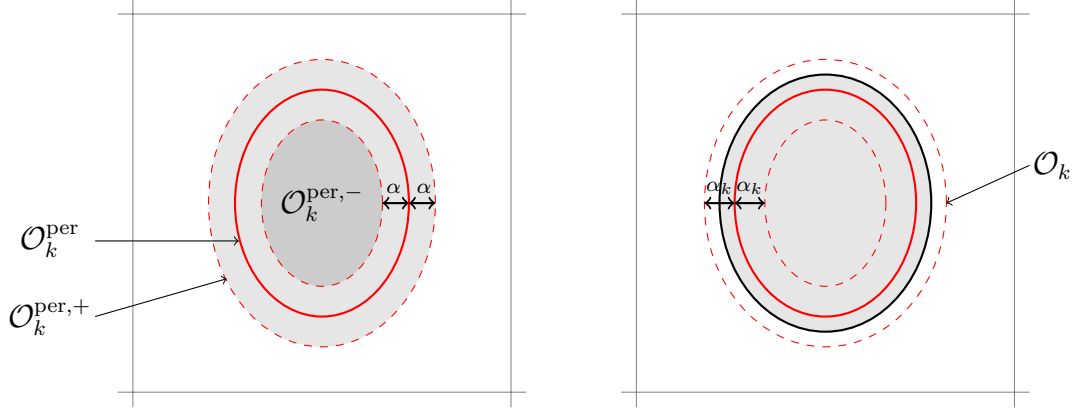


FIGURE 1.3 : Représentation graphique de l'hypothèse géométrique imposée sur les perforations non-périodiques.

et $\mathcal{O}_0^{\text{per}}$ est suffisamment régulier (de classe $\mathcal{C}^{2,\gamma}$ par exemple). Pour $k \in \mathbb{Z}^d$, on pose $\mathcal{O}_k^{\text{per}} := \mathcal{O}_0^{\text{per}} + k$. Le domaine périodiquement perforé est défini par :

$$\Omega_\varepsilon^{\text{per}} := \Omega \setminus \varepsilon \bigcup_{k \in Y_\varepsilon^{\text{per}}} \overline{\mathcal{O}_k^{\text{per}}}, \quad Y_\varepsilon^{\text{per}} := \{k \in \mathbb{Z}^d, \varepsilon Q_k \subset \Omega\}. \quad (1.2.1)$$

Dans (1.2.1), $0 < \varepsilon \ll 1$ représente la micro-structure du matériau étudié. Notons que le domaine $\Omega_\varepsilon^{\text{per}}$ est connexe et possède la même régularité que Ω et la perforation $\mathcal{O}_0^{\text{per}}$. Notons également que, dans (1.2.1), on a supprimé les perforations qui intersectent le bord macroscopique, ceci afin de garantir la connexité. Cette hypothèse sera systématique dans l'étude du système de Stokes (chapitre 3) mais n'est pas supposée dans l'étude de l'équation de Poisson (chapitre 2). On définit maintenant les perturbations de $\Omega_\varepsilon^{\text{per}}$ considérées dans les chapitres 2 et 3. On va pour cela définir une suite $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ de perforations non périodiques s'appuyant sur la distribution périodique $(\mathcal{O}_k^{\text{per}})_{k \in \mathbb{Z}^d}$ de trous. Pour $k \in \mathbb{Z}^d$, on fixe une perforation \mathcal{O}_k telle que $\mathcal{O}_k \subset\subset Q_k$ et $Q_k \setminus \overline{\mathcal{O}_k}$ est connexe. L'hypothèse géométrique principale (voir figure 1.3) que l'on impose est la suivante : il existe une suite $(\alpha_k)_{k \in \mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$ telle que

$$\forall k \in \mathbb{Z}^d, \quad \mathcal{O}_k^{\text{per},-}(\alpha_k) \subset \mathcal{O}_k \subset \mathcal{O}_k^{\text{per},+}(\alpha_k), \quad (\text{A})$$

où, pour $\alpha > 0$ et $k \in \mathbb{Z}^d$,

$$\mathcal{O}_k^{\text{per},-}(\alpha) := \{x \in \mathcal{O}_k^{\text{per}}, d(x, \partial \mathcal{O}_k^{\text{per}}) > \alpha\} \quad \text{et} \quad \mathcal{O}_k^{\text{per},+}(\alpha) := \{x \in Q_k, d(x, \mathcal{O}_k^{\text{per}}) < \alpha\},$$

et $d(x, A)$ désigne la distance euclidienne d'un point $x \in \mathbb{R}^d$ à l'ensemble A . Nous représentons sur la figure 1.4 un exemple schématique de perturbations considérées dans cette thèse. Dans la suite, on notera

$$\mathcal{O}^{\text{per}} := \bigcup_{k \in \mathbb{Z}^d} \mathcal{O}_k^{\text{per}} \quad \text{et} \quad \mathcal{O} := \bigcup_{k \in \mathbb{Z}^d} \mathcal{O}_k. \quad (1.2.2)$$

Comme pour le cas périodique, on définit le domaine perforé Ω_ε par

$$\Omega_\varepsilon := \Omega \setminus \varepsilon \bigcup_{k \in Y_\varepsilon} \overline{\mathcal{O}_k}, \quad Y_\varepsilon := \{k \in \mathbb{Z}^d, \varepsilon Q_k \subset \Omega\} \quad (1.2.3)$$

où, comme pour le cas périodique, on peut, dans certaines situations, inclure les perforations intersectant $\partial \Omega$, ou à l'inverse, imposer une certaine couche limite sans perforation autour du bord macroscopique. Dans la suite, nous aurons régulièrement besoin (pour des questions de régularité elliptique) de supposer que les perforations non périodiques sont régulières uniformément en $k \in \mathbb{Z}^d$. Une formalisation de cette hypothèse est donnée dans le chapitre 3 pour les régularités Lipschitz et \mathcal{C}^2 (mais, bien sûr, on peut formuler cela pour toute régularité). Nous présentons maintenant les résultats obtenus dans les chapitres 2 et 3.

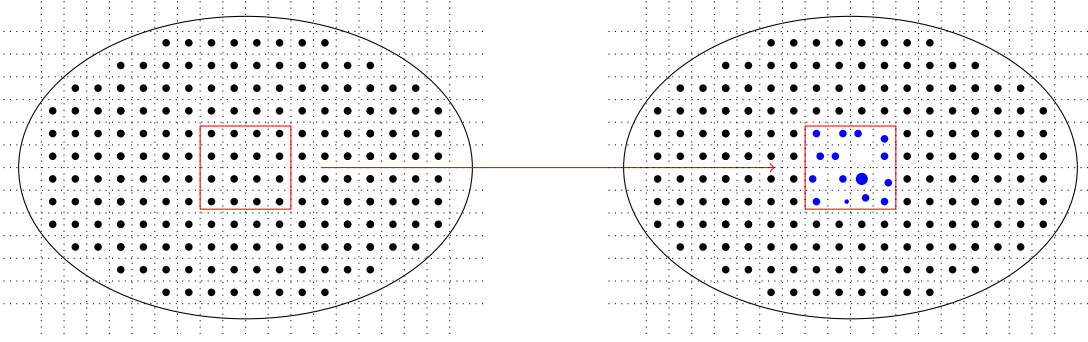


FIGURE 1.4 : Perturbations locales d'un domaine périodiquement perforé : la figure de gauche représente un domaine $\Omega_\varepsilon^{\text{per}}$ périodiquement perforé. La figure de droite montre un domaine localement perturbé Ω_ε . La zone rouge délimite le défaut (ici à support compact) introduit dans le domaine macroscopique. Lorsque $\varepsilon \rightarrow 0$, cette zone se concentre autour de l'origine.

1.2.1.2 Étude de l'équation de Poisson

Dans cette section, nous présentons les résultats de [22]. Nous y étudions l'équation suivante :

$$\begin{cases} -\Delta u_\varepsilon = f & \text{dans } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{sur } \partial\Omega_\varepsilon. \end{cases} \quad (1.2.4)$$

où Ω_ε est défini par $\Omega_\varepsilon := \Omega \setminus \varepsilon\bar{\mathcal{O}}$ et \mathcal{O} est un ensemble de perforations périodiques et localement perturbées, tel qu'introduit dans la sous-section 1.2.1.1. Dans (1.2.4), le terme source vérifie pour le moment $f \in L^2(\Omega)$. Le cas $\Omega_\varepsilon = \Omega_\varepsilon^{\text{per}}$ a été étudié dans [77]. Nous souhaitons généraliser ces résultats au cas d'un domaine périodique avec défauts. Nous commençons par étudier le comportement macroscopique de la suite $(u_\varepsilon)_{\varepsilon>0}$.

Une estimation *a priori* donne que, à extraction près, $u_\varepsilon/\varepsilon^2 \xrightarrow{\varepsilon \rightarrow 0} u^*$ dans $L^2(\Omega)$. Notons que ceci utilise la version suivante de l'inégalité de Poincaré-Friedrichs à l'échelle ε :

$$\exists C > 0, \quad \forall \varepsilon > 0, \quad \forall u \in H_0^1(\Omega_\varepsilon), \quad \|u\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla u\|_{L^2(\Omega_\varepsilon)}. \quad (1.2.5)$$

Cette inégalité est prouvée dans [99] pour le cas périodique. Nous la généralisons au cas périodique avec défaut dans [22, 107]³. L'étape suivante est d'effectuer un développement à deux échelles sous la forme

$$u_\varepsilon(x) = \varepsilon^2 u_2(x, y) + \varepsilon^3 u_3(x, y) + \varepsilon^4 u_4(x, y) + \dots, \quad y = \frac{x}{\varepsilon}, \quad (1.2.6)$$

où les fonctions u_i sont supposées régulières. Notons que, dans (1.2.6), les fonctions $u_i(x, \cdot)$ sont des fonctions variant à l'échelle microscopique et traduisent ainsi les propriétés microscopiques du matériau. Par conséquent, dans le cas périodique, elles seront supposées périodiques. Dans le cas périodique avec défaut, elles seront supposées périodiques et localement perturbées dans un sens précisé ci-dessous. Insérant l'expression (1.2.6) dans (1.2.4), on trouve, formellement,

$$\forall x \in \Omega, \quad \forall y \in \mathbb{R}^d, \quad u_i(x, y) = \sum_{|j|=i-2} w^{(j)}(y) D^j f(x), \quad (1.2.7)$$

où $D^j = \partial_{j_1} \cdots \partial_{j_d}$, $j_1 + \cdots + j_d = |j|$, désigne l'opération de dérivation de multi-indice $j \in \mathbb{N}^d$. Dans (1.2.7), les fonctions $w^{(j)}$ sont les correcteurs d'ordre j , qui se calculent récursivement. Pour $|j| = 0$ et 1, on a

$$\begin{cases} -\Delta_y w^{(0)} = 1 & \text{dans } \mathbb{R}^d \setminus \bar{\mathcal{O}} \\ w^{(0)} = 0 & \text{sur } \partial\mathcal{O}, \end{cases} \quad \begin{cases} -\Delta_y w^{(j)} = 2\partial_j w^{(0)} & \text{dans } \mathbb{R}^d \setminus \bar{\mathcal{O}} \\ w^{(j)} = 0 & \text{sur } \partial\mathcal{O}, \end{cases} \quad (1.2.8)$$

³Nous avons appris plus tard qu'elle avait déjà été démontrée dans un cadre différent (mais contenant le nôtre) dans [51].

où $j \in \llbracket 1, d \rrbracket$. Nous nous concentrons dans la suite sur le premier correcteur $w^{(0)}$, que l'on note w , et dont on admet momentanément l'existence dans l'espace

$$H_{\text{unif}}^1(\mathbb{R}^d) := \left\{ u \in H_{\text{loc}}^1(\mathbb{R}^d), \quad \sup_{k \in \mathbb{Z}^d} \|u\|_{H^1(Q+k)} < +\infty \right\}.$$

Dans [40], ce correcteur est utilisé dans le cas périodique pour trouver u^* par la méthode des fonctions test oscillantes dont on rappelle brièvement le principe.

On commence par prolonger u_ε par zéro dans les trous, cela fournit une fonction $u_\varepsilon \in H_0^1(\Omega)$. Soit $\phi \in \mathcal{D}(\Omega)$, on choisit ε assez petit pour que $\text{supp}(\phi) \subset \bigcup_{k \in Y_\varepsilon} \varepsilon \overline{Q_k}$ (voir (1.2.3) pour la notation).

On introduit $v_\varepsilon := w(\cdot/\varepsilon)\phi \in H_0^1(\Omega_\varepsilon)$ et on teste v_ε contre (1.2.4). On obtient

$$\varepsilon^{-1} \int_{\Omega} \nabla u_\varepsilon \cdot \nabla w \left(\frac{\cdot}{\varepsilon} \right) \phi = \int_{\Omega} f v_\varepsilon - \int_{\Omega} w \left(\frac{\cdot}{\varepsilon} \right) \nabla u_\varepsilon \cdot \nabla \phi. \quad (1.2.9)$$

De même, on teste l'équation du correcteur contre $u_\varepsilon \phi$:

$$\varepsilon^{-1} \int_{\Omega} \nabla w \left(\frac{\cdot}{\varepsilon} \right) \cdot \nabla u_\varepsilon \phi = -\varepsilon^{-1} \int_{\Omega} u_\varepsilon \nabla w \left(\frac{\cdot}{\varepsilon} \right) \cdot \nabla \phi + \varepsilon^{-2} \int_{\Omega} u_\varepsilon \phi. \quad (1.2.10)$$

En faisant la différence entre (1.2.9) et (1.2.10), on trouve

$$\int_{\Omega} f v_\varepsilon - \varepsilon^{-2} \int_{\Omega} u_\varepsilon \phi = \int_{\Omega} w \left(\frac{\cdot}{\varepsilon} \right) \nabla u_\varepsilon \cdot \nabla \phi - \varepsilon^{-1} \int_{\Omega} u_\varepsilon \nabla w \left(\frac{\cdot}{\varepsilon} \right) \cdot \nabla \phi. \quad (1.2.11)$$

On utilise alors l'estimation *a priori*

$$\|u_\varepsilon\|_{L^2(\Omega)} + \varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^2,$$

où C est une constante indépendante de ε , ainsi que la sous-linéarité du correcteur pour obtenir que

$$\int_{\Omega} (Af - u^*)\phi = 0, \quad A = L^2 - \lim_{\varepsilon \rightarrow 0} \text{faible} w \left(\frac{\cdot}{\varepsilon} \right),$$

pourvu que A existe. Remarquons que la preuve ci-dessus n'utilise en rien la périodicité, mais seulement que (1.2.8) admet une solution dans l'espace $H_{\text{unif}}^1(\mathbb{R}^d)$ tel que A existe. C'est en particulier le cas si la fonction w (ou plutôt son prolongement par zéro dans les trous) admet une moyenne sur \mathbb{R}^d . Dans le cas périodique, on a $A = \int_Q w^{\text{per}}(y) dy$, où w^{per} est la solution périodique de (1.2.8),

et donc $u^* = \left(\int_Q w^{\text{per}}(y) dy \right) f$. Dans le cas non périodique, nous prouvons d'abord l'existence du correcteur w . On pose

$$H^{1,\text{per}}(Q \setminus \overline{\mathcal{O}_0^{\text{per}}}) := \{ u \in H^{1,\text{loc}}(\mathbb{R}^d \setminus \overline{\mathcal{O}^{\text{per}}}), \quad u \text{ est périodique} \}$$

où l'exposant loc signifie localement dans \mathbb{R}^d .

Théorème 1. *Soit $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ une configuration non périodique de trous et Ω_ε le domaine perforé associé. Soit $\tilde{g} \in L^2(\mathbb{R}^d)$ et $g := 1_{\mathbb{R}^d \setminus \overline{\mathcal{O}^{\text{per}}}} + \tilde{g}$. Il existe une unique fonction $\tilde{w} \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ telle que $w := w^{\text{per}} + \tilde{w}$ est solution au sens des distributions de l'équation*

$$\begin{cases} -\Delta w = g & \text{dans } \mathbb{R}^d \setminus \overline{\mathcal{O}} \\ w = 0 & \text{sur } \partial \mathcal{O}, \end{cases} \quad (1.2.12)$$

où $w^{\text{per}} \in H^{1,\text{per}}(Q \setminus \overline{\mathcal{O}_0^{\text{per}}})$ est solution de l'EDP du correcteur périodique

$$\begin{cases} -\Delta w^{\text{per}} = 1 & \text{dans } Q \setminus \overline{\mathcal{O}_0^{\text{per}}} \\ w^{\text{per}} = 0 & \text{sur } \partial \mathcal{O}_0^{\text{per}}. \end{cases} \quad (1.2.13)$$

En particulier, w (prolongée par zéro dans les perforations), vérifie $w \in H_{\text{unif}}^1(\mathbb{R}^d)$.

Nous déduisons du Théorème 1 la limite $u_\varepsilon/\varepsilon^2 \xrightarrow{\varepsilon \rightarrow 0} u^*$ dans $L^2(\Omega)$ avec $u^* = Af$,

$$A := L^2 - \lim_{\varepsilon \rightarrow 0} \text{faible} w \left(\frac{\cdot}{\varepsilon} \right) = \int_Q w^{\text{per}}(y) dy,$$

puisque $\tilde{w} \in L^2(\mathbb{R}^d)$ (où \tilde{w} est prolongée par $-w^{\text{per}}$ dans les perforations non périodiques). En particulier, la limite faible de $u_\varepsilon/\varepsilon^2$ est la même que dans le cas périodique : ceci était à prévoir car Ω_ε et $\Omega_\varepsilon^{\text{per}}$ ne diffèrent qu'à l'échelle microscopique.

Avant d'exposer la suite des résultats obtenus dans le Chapitre 2, nous faisons deux commentaires sur le théorème 1. D'abord, on peut établir et prouver ce théorème pour un second membre plus général de la forme $g^{\text{per}} + \tilde{g}$ avec $g^{\text{per}} \in L^\infty, \text{per}(Q)$. Cela permet de construire les correcteurs d'ordre plus élevé. Ensuite, nous signalons que la preuve du théorème 1 repose sur des techniques variationnelles. Nous établissons d'abord une inégalité de Poincaré-Friedrichs puis nous résolvons (1.2.12) par un problème de minimisation.

Revenons maintenant aux résultats du chapitre 2. Après avoir montré l'existence de correcteurs, nous obtenons des taux de convergence. Les théorèmes suivants sont prouvés dans [22]. Ils sont énoncés avec la convention $\Omega_\varepsilon = \Omega \setminus \varepsilon\overline{\mathcal{O}}$. Nous commençons par un théorème de convergence H^1 . Pour $u \in W^{1,q}(\Omega_\varepsilon)$, on définit $|u|_{W^{1,q}(\Omega_\varepsilon)} := \|\nabla u\|_{L^q(\Omega_\varepsilon)}$.

Théorème 2. *Soit $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ une configuration non-périodique de trous et $\Omega_\varepsilon = \Omega \setminus \varepsilon\overline{\mathcal{O}}$ le domaine perforé associé. Soit $f \in \mathcal{D}(\Omega)$. Il existe une constante $C > 0$ indépendante de ε telle que*

$$\varepsilon^{-1} \|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{L^2(\Omega_\varepsilon)} + |u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^2.$$

Notons que les hypothèses du théorème 2 peuvent être affaiblies en $f \in H^2(\Omega)$ et $f|_{\partial\Omega} = 0$ si l'on suppose que les perforations sont uniformément de classe $\mathcal{C}^{1,\gamma}$. Si f ne s'annule pas sur $\partial\Omega$ alors une couche limite apparait et dégrade le taux de convergence, phénomène usuel en homogénéisation. Dans le théorème ci-dessous, nous obtenons un taux de convergence L^∞ pour des seconds membres f très réguliers.

Théorème 3. *Soit $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ une configuration non-périodique de trous. On suppose que les perforations sont uniformément de classe $\mathcal{C}^{1,\gamma}$ et on note $\Omega_\varepsilon = \Omega \setminus \varepsilon\overline{\mathcal{O}}$ le domaine perforé associé. Soit $f \in \mathcal{D}(\Omega)$. Il existe une constante $C > 0$ indépendante de ε telle que*

$$\|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon^3.$$

Dans l'annexe 2.5.2 du chapitre 2, nous améliorons ces résultats en atteignant toutes les normes de Sobolev $W^{1,q}(\Omega_\varepsilon)$, $q \in (1, +\infty)$. Nous nous appuyons pour cela sur [82, 71]. Pour des raisons de régularité, nous modifions légèrement la définition de Ω_ε . On définit

$$\Omega_\varepsilon := \Omega \setminus \varepsilon \bigcup_{k \in Y_\varepsilon} \overline{\mathcal{O}_k}, \quad Y_\varepsilon := \{k \in \mathbb{Z}^d, \varepsilon Q_k \subset \Omega\}. \quad (1.2.14)$$

Nous prouvons alors le résultat suivant, qui donne des taux de convergence optimaux.

Théorème 4 (Version raccourcie du théorème 2.5.6). *Supposons que Ω est de classe \mathcal{C}^2 . Soit $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ un ensemble de perforations non périodiques et Ω_ε défini par (1.2.14). On suppose que les perforations \mathcal{O}_k , $k \in \mathbb{Z}^d$ sont uniformément de classe \mathcal{C}^2 . Soit u_ε la solution de (1.2.4).*

1. *Si $f \in W^{2,q} \cap L^\infty(\Omega)$, alors il existe une constante $C > 0$ indépendante de ε telle que*

$$\varepsilon^{-1} \|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{L^q(\Omega_\varepsilon)} + |u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f|_{W^{1,q}(\Omega_\varepsilon)} \leq C\varepsilon^{1+1/q}. \quad (1.2.15)$$

2. Si $q \geq 2$, $f \in W^{2,q}(\Omega)$, $\Omega' \subset\subset \Omega$ et $\Omega'_\varepsilon := \Omega_\varepsilon \cap \Omega'$, il existe une constante $C > 0$ indépendante de ε telle que

$$\varepsilon^{-1} \|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{L^q(\Omega'_\varepsilon)} + |u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f|_{W^{1,q}(\Omega'_\varepsilon)} \leq C\varepsilon^2. \quad (1.2.16)$$

L'ensemble des résultats obtenus dans le chapitre 2 montre que le cadre introduit dans la section 1.2.1.1 est traitable par les outils issus de l'homogénéisation périodique. Par ailleurs, en utilisant les correcteurs *ad hoc*, nous pouvons obtenir une qualité d'approximation aussi fine que dans le cas périodique. Il est à ce stade important de souligner que le correcteur non périodique est plus compliqué à calculer, puisqu'il résout une EDP sur un domaine non borné. Il est alors naturel de se demander si, *en pratique*, il est réellement nécessaire de calculer la fonction w pour chaque situation. Nous avons vu que cela n'était pas utile si l'on souhaite seulement calculer u^* , qui ne dépend que de w^{per} . Cela n'est également pas avantageux si l'on souhaite une approximation assez grossière de u_ε , par exemple en norme $H^1(\Omega)$. Cette affirmation provient du théorème 2.5.6 (iii) (et du théorème 4, (2)), voir aussi la remarque 2.5.11. Nous nous contentons ici de résumer sur la figure 1.5 les cas où la construction *pratique* du correcteur non périodique se justifie.

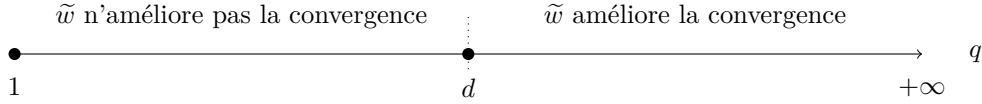


FIGURE 1.5 : Utilité du correcteur sur l'échelle des exposants de Lebesgue. En norme L^q (ou en norme $W^{1,q}$), le taux de convergence de $u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f$ vers zéro est meilleur que celui de $u_\varepsilon - \varepsilon^2 w^{\text{per}}(\cdot/\varepsilon) f$ vers zéro quand $q < d$.

1.2.1.3 Étude du système de Stokes

Dans cette section, nous introduisons les résultats du chapitre 3 en les replaçant dans leur contexte. Pour $f \in L^2(\Omega)^d$, nous étudions le système de Stokes ci-dessous :

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{dans } \Omega_\varepsilon \\ \operatorname{div} u_\varepsilon = 0 \\ u_\varepsilon = 0 & \text{sur } \partial\Omega_\varepsilon, \end{cases} \quad (1.2.17)$$

où Ω_ε est un domaine périodiquement ou non perforé et $f \in L^2(\Omega)^d$. Dans la suite, nous fixons un représentant quelconque de p_ε dans $L^2(\Omega_\varepsilon)/\mathbb{R}$. L'étude de (1.2.17) dans le cas $\Omega_\varepsilon = \Omega_\varepsilon^{\text{per}}$ a été effectuée dans [99, 91] puis dans [2] pour le cas d'une partie solide connexe (voir aussi [78]). Le premier résultat donne le comportement qualitatif de la suite $(u_\varepsilon, p_\varepsilon)$.

Théorème 5 ([91, 99]). *Soit $f \in L^2(\Omega)^d$ et $\Omega_\varepsilon^{\text{per}}$ défini par (1.2.1). Il existe un prolongement de u_ε et p_ε à Ω tel que l'on ait les convergences*

$$\begin{cases} \frac{u_\varepsilon}{\varepsilon^2} \xrightarrow{\varepsilon \rightarrow 0} u^* & \text{dans } L^2(\Omega) \\ p_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} p_0 & \text{dans } L^2(\Omega)/\mathbb{R}, \end{cases} \quad (1.2.18)$$

où le couple (u^*, p_0) est défini par $u^* = A(f - \nabla p_0)$ et ∇p_0 est donné par l'EDP

$$\begin{cases} -\operatorname{div} A(f - \nabla p_0) = 0 & \text{dans } \Omega \\ A(f - \nabla p_0) \cdot n = 0 & \text{sur } \partial\Omega. \end{cases} \quad (1.2.19)$$

La matrice A se calcule à l'aide des problèmes de correcteurs périodiques

$$\begin{cases} -\Delta w_j^{\text{per}} + \nabla p_j^{\text{per}} = e_j & \text{dans } Q \setminus \overline{\mathcal{O}_0^{\text{per}}} \\ \operatorname{div} w_j^{\text{per}} = 0 \\ w_j^{\text{per}} = 0 & \text{sur } \partial\mathcal{O}_0^{\text{per}} \end{cases} \quad (1.2.20)$$

et se définit par $A := \left(\int_Q \nabla w_i^{\text{per}} : \nabla w_j^{\text{per}} dy \right)_{1 \leq i, j \leq d}$. On a alors

$$\forall (i, j) \in \llbracket 1, d \rrbracket^2, \quad A_{i,j} = \int_Q w_j^{\text{per}} \cdot e_i = \int_Q w_i^{\text{per}} \cdot e_j.$$

Le théorème 5 montre que (u^*, p_0) suit la *Loi de Darcy*. On rappelle brièvement la stratégie utilisée pour prouver le Théorème 5, en remarquant qu'elle s'adapte au cas périodique avec défaut. En utilisant l'inégalité de Poincaré [99, Lemma 1] (et [107, Lemma 3.1] pour le cas non périodique), on a l'estimation *a priori*

$$\|u_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^2$$

(avec extension de u_ε par zéro dans les perforations). Ainsi, $u_\varepsilon/\varepsilon^2 \xrightarrow{\varepsilon \rightarrow 0} u^*$ dans $L^2(\Omega)^d$, à extraction près. Notons que, par [91, Chapter 7, Theorem 1.2], l'espace

$$H := \{u \in L^2(\Omega)^d, \operatorname{div} u = 0, u \cdot n|_{\partial\Omega} = 0\}$$

est fermé pour la norme $L^2(\Omega)^d$. Cela donne en particulier $u^* \cdot n|_{\partial\Omega} = 0$. La difficulté de la preuve est le prolongement de la pression p_ε dans les trous. En effet, contrairement à u_ε , il n'y a pas d'extension naturelle dans les perforations. Dans [99], une extension de p_ε à Ω est construite à l'aide d'un opérateur de restriction.

Lemme 6 (Lemma 4 dans [99]). *Il existe un opérateur $R_\varepsilon : H_0^1(\Omega)^d \rightarrow H_0^1(\Omega_\varepsilon)^d$ satisfaisant les propriétés suivantes : pour tout $u \in H_0^1(\Omega)^d$,*

$$\text{si } u = 0 \text{ dans } \Omega \setminus \overline{\Omega_\varepsilon} \text{ alors } R_\varepsilon u = u, \quad \text{si } \operatorname{div} u = 0 \text{ alors } \operatorname{div} R_\varepsilon u = 0, \quad (1.2.21)$$

$$\|\nabla R_\varepsilon u\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{-1} \|R_\varepsilon u\|_{L^2(\Omega_\varepsilon)} \leq C [\|\nabla u\|_{L^2(\Omega)} + \varepsilon^{-1} \|u\|_{L^2(\Omega)}], \quad (1.2.22)$$

où C est une constante indépendante de ε et u .

La preuve du lemme 6 repose sur la résolution d'un problème de Stokes dans $\varepsilon \left(Q_k^{\text{per}} \setminus \overline{\mathcal{O}_k^{\text{per}}} \right)$, $k \in Y_\varepsilon^{\text{per}}$. Sous une hypothèse d'uniforme régularité des perforations, la preuve s'étend au cas périodique avec défauts.

On définit, dans les cas périodique et non périodique, $F_\varepsilon \in H^{-1}(\Omega)^d$ par $\langle F_\varepsilon, u \rangle := \langle \nabla p_\varepsilon, R_\varepsilon u \rangle$ pour $u \in H_0^1(\Omega)^d$. Par (1.2.21), F_ε est en fait un gradient, que l'on notera ∇P_ε , $P_\varepsilon \in L^2(\Omega)/\mathbb{R}$ et, par ailleurs, $P_\varepsilon = p_\varepsilon$ (à une constante additive près) dans Ω_ε . Utilisant la propriété (1.2.22) et le système (1.2.17), on montre alors que

$$\forall \phi \in H_0^1(\Omega), \quad |\langle \nabla P_\varepsilon, \phi \rangle| \leq C [\|\phi\|_{L^2(\Omega)} + \varepsilon \|\nabla \phi\|_{L^2(\Omega)}].$$

En particulier, $\|P_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq C$ où C est une constante indépendante de ε . On a alors que, à extraction près, $P_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} p_0$ dans $L^2(\Omega)/\mathbb{R}$. On peut en fait montrer que cette convergence est, à nouveau à extraction près, forte. À ce stade, nous avons

$$\begin{cases} u_\varepsilon/\varepsilon^2 \xrightarrow{\varepsilon \rightarrow 0} u^* & \text{dans } L^2(\Omega)^d \\ P_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} p_0 & \text{dans } L^2(\Omega)/\mathbb{R}. \end{cases} \quad (1.2.23)$$

L'enjeu est maintenant d'identifier le couple (u^*, p_0) . Pour cela, comme pour l'équation de Poisson, nous introduisons les correcteurs associés au problème. Pour identifier ces correcteurs, on peut effectuer un développement à deux échelles :

$$\begin{cases} u_\varepsilon(x) = \varepsilon^2 u_2(x, y) + \varepsilon^3 u_3(x, y) + \varepsilon^4 u_4(x, y) + \cdots, & y = x/\varepsilon \\ p_\varepsilon(x) = p_0(x) + \varepsilon p_1(x, y) + \varepsilon^2 p_2(x, y) + \cdots, & y = x/\varepsilon, \end{cases} \quad (1.2.24)$$

où les fonctions u_i et p_i sont supposées régulières et les fonctions $u_i(x, \cdot)$ et $p_i(x, \cdot)$, $x \in \Omega$, traduisent les propriétés microscopiques du matériau (elles sont en particulier périodiques dans le cas périodique et périodique "avec défaut" dans le cadre plus général étudié ici). À cause de la contrainte sur la divergence, l'identification des fonctions u_i est plus délicate. Au premier ordre (voir [91, Chapter 7]), on trouve que, formellement,

$$u_\varepsilon(x) \approx \varepsilon^2 \sum_{j=1}^d w_j(y)(f_j - \partial_j p_0)(x) \quad \text{et} \quad p_\varepsilon \approx p_0(x) + \varepsilon \sum_{j=1}^d p_j(y)(f_j - \partial_j p_0)(x), \quad y = x/\varepsilon, \quad (1.2.25)$$

où $f = (f_1, \dots, f_d)$ et (w_j, p_j) sont les correcteurs, solution de

$$\begin{cases} -\Delta w_j + \nabla p_j = e_j & \text{dans } \mathbb{R}^d \setminus \overline{\mathcal{O}} \\ \operatorname{div} w_j = 0 \\ w_j = 0 & \text{sur } \partial\mathcal{O}, \end{cases} \quad (1.2.26)$$

et e_1, \dots, e_d sont les vecteurs de la base canonique de \mathbb{R}^d . Dans le cas périodique, l'équation (1.2.26) devient (1.2.20), qui est bien posée dans l'espace des fonctions localement H^1 et périodiques sur $Q \setminus \overline{\mathcal{O}}_0^{\text{per}}$. Dans le cas non périodique, on prouve le théorème d'existence suivant.

Théorème 7. *Soit $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ une suite de perforations non périodiques uniformément en $k \in \mathbb{Z}^d$ de classe Lipschitz. L'équation (1.2.26) admet, au sens des distributions, une solution de la forme $w_j = w_j^{\text{per}} + \tilde{w}_j$, $p_j = p_j^{\text{per}} + \tilde{p}_j$ où $(\tilde{w}_j, \tilde{p}_j) \in H^1 \times L^2_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})$ et $(w_j^{\text{per}}, p_j^{\text{per}})$ est le correcteur périodique i.e. solution de (1.2.20). Par ailleurs, il existe une constante $C > 0$ indépendante de ε telle que*

$$\|\tilde{p}_j - \langle \tilde{p}_j \rangle\|_{L^2(\frac{1}{\varepsilon}\Omega_\varepsilon)} \leq C\varepsilon^{-1},$$

où \tilde{p}_j est prolongée par sa valeur moyenne sur $\frac{1}{\varepsilon}\Omega \setminus \overline{\mathcal{O}}$ dans $\frac{1}{\varepsilon}\Omega_\varepsilon$ et $\langle \tilde{p}_j \rangle := \frac{1}{|\frac{1}{\varepsilon}\Omega_\varepsilon|} \int_{\frac{1}{\varepsilon}\Omega_\varepsilon} \tilde{p}_j$.

On peut ensuite passer à la limite faible $\varepsilon \rightarrow 0$ dans (1.2.25). On trouve que, toujours formellement, $u^* = A(f - \nabla p_0)$ avec $A = (\overline{w}_1, \dots, \overline{w}_d)_{1 \leq i \leq d} \in \mathcal{M}_d(\mathbb{R})$ et $\overline{w}_j := L^2$ -lim faible $w_j(\cdot/\varepsilon)$, si cette limite existe. Dans les cas périodique et non périodique, on a, en utilisant le théorème 7 et un lemme classique sur les fonctions périodiques, que $\overline{w}_j = \int_Q w_j^{\text{per}}(y) dy$. Cela montre formellement le théorème 5 dans le cadre (non) périodique. On peut alors suivre [99, Preuve du théorème 1], qui ne repose pas sur la périodicité, (et qui est essentiellement la même preuve que pour l'équation de Poisson) pour rendre cela rigoureux. Cela donne la convergence de $(u_\varepsilon, p_\varepsilon)$ vers la solution à la loi de Darcy dont la matrice de perméabilité A se calcule avec les correcteurs périodiques. Comme pour l'équation de Poisson, la limite macroscopique est inchangée par rapport au cas périodique.

Dans la suite, nous aimerions préciser la convergence de $(u_\varepsilon, p_\varepsilon)$. La première amélioration est fournie par [3] dans le cas périodique :

$$\frac{u_\varepsilon}{\varepsilon^2} - \sum_{j=1}^d w_j\left(\frac{\cdot}{\varepsilon}\right)(f_j - \partial_j p_0) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{dans } L^2(\Omega)^d, \quad (1.2.27)$$

où u_ε est étendue par zéro dans les perforations. Cette preuve s'étend à nouveau au cas périodique avec défauts, pourvu que les correcteurs $w_j, j = 1, \dots, d$ soient bornés. Ce résultat est prouvé dans l'annexe 3.4.3. Pour aller plus loin, nous aimerions obtenir des taux de convergence. Il s'avère que la situation est beaucoup plus délicate que pour l'équation de Poisson, premièrement à cause de la contrainte d'incompressibilité. A cela s'ajoute un phénomène de couche limite autour de $\partial\Omega$ qui n'est pas traitable de façon élémentaire. Dans [81], un taux de convergence en $O(\varepsilon^{1/6})$ dans (1.2.27) est obtenu pour le cas $d = 2$. Ce résultat a ensuite été amélioré, toujours dans le cas $d = 2$, dans [70] en $O(\varepsilon^{1/2})$ fournissant ainsi un taux optimal. Récemment, ce taux optimal a été montré pour toute

dimension dans [95]. Dans tous ces travaux, un élément central est la construction de fonctions permettant de corriger les données aux bords (boundary correctors). Une façon de s'affranchir des difficultés liées au bord macroscopique est, dans le cas périodique, de choisir $\varepsilon = 1/m$, $m \in \mathbb{N}^*$ et d'imposer des données au bord périodique sur un cube macroscopique, voir [80]. Signalons, à ce sujet, que des taux de convergences à tout ordre, et donc la construction de correcteurs d'ordre élevé, ont été proposés dans ce cadre [54]. Dans [107], nous nous plaçons dans un cadre permettant de s'affranchir du bord $\partial\Omega$ et nous supposons $d = 3$. On rappelle que, formellement,

$$u_\varepsilon = \varepsilon^2 \sum_{j=1}^d w_j \left(\frac{\cdot}{\varepsilon} \right) (f_j - \partial_j p_0) + \text{reste}.$$

La difficulté majeure provient de ce que $(f_j - \partial_j p_0)|_{\partial\Omega} \neq 0$. On impose alors les conditions suivantes :

$$\begin{cases} \operatorname{div} Af = 0 \\ f \in W_c^{3,\infty}(\Omega), \end{cases} \quad (1.2.28)$$

où l'indice c signifie à support compact, de sorte que $\nabla p_0 = 0$ dans Ω (voir (1.2.19)) et donc $(f_j - \partial_j p_0)$ est à support compact dans Ω . Les conditions (1.2.28) sont par exemple vérifiées si $f = A^{-1} \operatorname{rot}(\phi)$, $\phi \in \mathcal{D}(\Omega)^{3 \times 3}$. On prouve alors le taux de convergence suivant :

Théorème 8. *On suppose que $d = 3$ et que Ω est un domaine borné localement Lipschitzien. Soit $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ une suite de perforations non-périodiques uniformément en $k \in \mathbb{Z}^d$ de classe \mathcal{C}^2 . On note Ω_ε le domaine perforé défini par (1.2.3). On suppose que le terme source f satisfait (1.2.28). Soit $\Omega'' \subset\subset \Omega$. Il existe une constante $C > 0$ indépendante de ε tel que pour tout $\varepsilon < \varepsilon_0(f, \Omega'', \Omega)$,*

$$\begin{aligned} \left\| D^2 \left[u_\varepsilon - \varepsilon^2 w_j \left(\frac{\cdot}{\varepsilon} \right) f_j \right] \right\|_{[L^2(\Omega'' \cap \Omega_\varepsilon)]} + \varepsilon^{-1} \left\| \nabla \left[u_\varepsilon - \varepsilon^2 w_j \left(\frac{\cdot}{\varepsilon} \right) f_j \right] \right\|_{[L^2(\Omega_\varepsilon)]} \\ + \varepsilon^{-2} \left\| u_\varepsilon - \varepsilon^2 w_j \left(\frac{\cdot}{\varepsilon} \right) f_j \right\|_{[L^2(\Omega_\varepsilon)]} \leq C\varepsilon \end{aligned} \quad (1.2.29)$$

et

$$\left\| \nabla \left[p_\varepsilon - \varepsilon \left\{ p_j \left(\frac{\cdot}{\varepsilon} \right) - \lambda_\varepsilon^j \right\} f_j \right] \right\|_{L^2(\Omega'' \cap \Omega_\varepsilon)} + \left\| p_\varepsilon - \varepsilon \left\{ p_j \left(\frac{\cdot}{\varepsilon} \right) - \lambda_\varepsilon^j \right\} f_j \right\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C\varepsilon, \quad (1.2.30)$$

$$\text{où } \lambda_\varepsilon^j = \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} p_j \left(\frac{\cdot}{\varepsilon} \right).$$

Dans l'énoncé ci-dessus, nous avons utilisé la convention de sommation sur les indices répétés. Remarquons que la conclusion (1.2.30) peut se réduire à

$$\left\| \nabla \left[p_\varepsilon - \varepsilon \left\{ p_j \left(\frac{\cdot}{\varepsilon} \right) - \lambda_\varepsilon^j \right\} f_j \right] \right\|_{L^2(\Omega'' \cap \Omega_\varepsilon)} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C\varepsilon$$

sans perte d'information. Les correcteurs p_j permettent de corriger les oscillations de p_ε à l'échelle microscopique : ils améliorent donc la convergence du gradient. Le théorème 8, limité pour l'instant à des estimations Hilbertiennes, s'appuie sur [82, Theorem 4.1]. La majeure partie de la preuve du Théorème 8 consiste en la construction de fonctions auxiliaires permettant de corriger la divergence

$$\text{du reste } R_\varepsilon := u_\varepsilon - \varepsilon^2 \sum_{j=1}^3 w_j \left(\frac{\cdot}{\varepsilon} \right) f_j.$$

Nous notons que, dans le théorème 8, l'hypothèse d'uniforme régularité de classe \mathcal{C}^2 permet d'obtenir des constantes de régularité elliptique uniformes en k . Nous vérifions cela dans l'annexe 3.4.2. La technique de preuve pourrait s'adapter à l'équation de Poisson, ou à d'autres types de résultats de régularité. Nous ne détaillons pas ces passages, les preuves étant assez fastidieuses.

Nous finissons cette section en discutant l'*utilité pratique* des correcteurs non-périodiques. Remarquons d'abord l'estimation

$$\varepsilon^{-2} \left\| \varepsilon^2 \widetilde{w}_j \left(\frac{\cdot}{\varepsilon} \right) f_j \right\|_{L^2(\Omega_\varepsilon)} \leq C \varepsilon^{\frac{3}{2}} = O(\varepsilon).$$

Cela signifie que, utiliser w_j^{per} à la place de w_j dans (1.2.29), n'améliore par le taux de convergence en norme L^2 (et aussi en norme H^1). Pour exhiber des situations où le correcteur non périodique apporte, *en pratique*, une amélioration sur le taux de convergence, il faudrait obtenir des estimations L^q , $q > d$. Ce point sera discuté dans les perspectives, section 1.3 et pourrait donner lieu à un travail futur.

1.2.2 Homogénéisation d'équations non-linéaires à coefficients oscillants

Dans cette section, nous présentons les résultats obtenus dans les chapitres 4 et 5. L'objectif est de généraliser le cadre introduit dans [18, 19, 20] à des équations non-linéaires. Nous commençons par présenter ce cadre dans la section 1.2.2.1, que nous illustrons sur l'équation de diffusion linéaire.

1.2.2.1 Coefficients périodiques avec défauts

Nous présentons le cadre périodique avec défauts qui sera le coeur des chapitres 4 et 5. On fixe une application $A^{\text{per}} : \mathbb{R}^d \rightarrow \mathcal{M}_d(\mathbb{R})$ où $\mathcal{M}_d(\mathbb{R})$ désigne l'ensemble des matrices réelles de taille $d \times d$. On suppose que A^{per} est périodique et que

$$\exists \lambda > 0, \quad \forall y \in Q, \quad \forall \xi \in \mathbb{R}^d, \quad \lambda^{-1} |\xi|^2 \leq A^{\text{per}}(y) \xi \cdot \xi \leq \lambda |\xi|^2. \quad (1.2.31)$$

Pour simplifier, on suppose que A^{per} est symétrique *i.e.* $A^{\text{per}}(y) = A^{\text{per}}(y)^T$ pour tout $y \in Q$. On considère alors, pour $\varepsilon > 0$, le problème

$$\begin{cases} -\operatorname{div} A^{\text{per}}(\cdot/\varepsilon) \nabla u_\varepsilon = f & \text{dans } \Omega \\ u_\varepsilon \in H_0^1(\Omega), \end{cases} \quad (1.2.32)$$

où le second membre $f \in L^2(\Omega)$. L'homogénéisation périodique [16] montre que $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^*$ dans $H^1(\Omega)$ où u^* se calcule à l'aide de la matrice homogénéisée A^* . Cette matrice est définie par le problème de cellule

$$\begin{cases} -\operatorname{div} A^{\text{per}}(y) (p + \nabla w_p^{\text{per}}(y)) = 0 & \text{dans } Q \\ w_p \in H^{1,\text{per}}(Q), \end{cases} \quad (1.2.33)$$

où $p \in \mathbb{R}^d$. On a alors

$$A_{j,i}^* := \left(\int_Q A^{\text{per}}(y) (e_i + \nabla w_{e_i}^{\text{per}}(y)) dy \right) \cdot e_j \quad (1.2.34)$$

et u^* résout l'EDP

$$\begin{cases} -\operatorname{div} A^* \nabla u^* = f & \text{dans } \Omega \\ u^* \in H_0^1(\Omega). \end{cases} \quad (1.2.35)$$

Si l'on suppose la régularité de A^{per} (par exemple Hölderienne), les résultats fondamentaux [11] permettent d'obtenir des taux de convergence du reste $R_\varepsilon := u_\varepsilon - u^* - \varepsilon \sum_{j=1}^d w_{e_j}(\cdot/\varepsilon) \partial_j u^*$ vers zéro

dans différentes normes (H^1 , $W^{1,q}$, $\mathcal{C}^{0,\alpha}$), pourvu que le second membre de (1.2.32) soit dans le bon espace. L'objectif des travaux [19, 20, 18, 17] est de généraliser ces résultats au cas d'un coefficient périodique et localement perturbé au sens suivant : on se fixe un coefficient A^{per} satisfaisant (1.2.31) que l'on suppose de régularité Hölderienne. Soit $1 < r < +\infty$ et $\tilde{A} \in (L^r \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d))^{d \times d}$. On pose $A := A^{\text{per}} + \tilde{A}$ et l'on considère le problème d'homogénéisation

$$\begin{cases} -\operatorname{div} A(\cdot/\varepsilon) \nabla u_\varepsilon = f & \text{dans } \Omega \\ u_\varepsilon \in H_0^1(\Omega). \end{cases} \quad (1.2.36)$$

Dans (1.2.36), le coefficient périodique A^{per} est remplacé par le coefficient perturbé A . Notons que A tend vers A^{per} à l'infini et que l'exposant r mesure la décroissance du défaut \tilde{A} vers zéro à l'infini (voir la figure 1.6). Lorsque $\varepsilon \rightarrow 0$, $\tilde{A}(\cdot/\varepsilon)$ se concentre autour de l'origine, exprimant bien que la

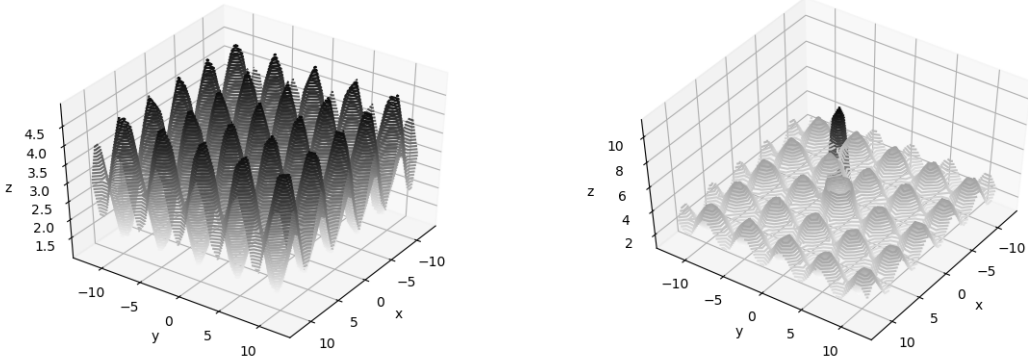


FIGURE 1.6 : Représentation d'un coefficient périodique avec défauts : la figure de gauche représente un coefficient périodique. La figure de droite représente un fond périodique qui est perturbé par un défaut ponctuel.

perturbation est locale. Le problème de cellule (1.2.33) est remplacé par l'équation :

$$-\operatorname{div} A(y)(p + \nabla w_p(y)) = 0 \quad \text{sur } \mathbb{R}^d, \quad (1.2.37)$$

où l'on impose la condition de sous-linéarité à l'infini $\frac{w_p(y)}{1 + |y|} \xrightarrow{|y| \rightarrow +\infty} 0$, essentielle pour obtenir des taux de convergence. La première difficulté est la résolution de (1.2.37) dans ce nouveau cadre [19, 20, 18]. Pour cela, on prend appui sur le problème périodique et l'on cherche w_p sous la forme $w_p = w_p^{\text{per}} + \tilde{w}_p$ où, formellement, \tilde{w}_p tend vers zéro à l'infini. L'équation vérifiée par \tilde{w}_p s'écrit sous la forme

$$-\operatorname{div} A \nabla \tilde{w}_p = \operatorname{div} g \quad \text{sur } \mathbb{R}^d, \quad g := \tilde{A}(p + \nabla w_p^{\text{per}}) \in L^r(\mathbb{R}^d)^d, \quad (1.2.38)$$

en utilisant le problème périodique. La résolution de (1.2.37) se fait en deux étapes :

- résoudre (1.2.38) pour des second membres agréables. Cela est fait dans [19] quand $g \in L^2(\mathbb{R}^d)^d$ (ce qui correspond à un défaut $\tilde{A} \in L^2(\mathbb{R}^d)$) par des techniques variationnelles ;
- établir un résultat de continuité $L^r - L^r$ pour l'application $g \mapsto \nabla v$, où v résout l'équation $-\operatorname{div} A \nabla v = g$, voir [18].

Cette analyse fournit un correcteur non périodique $w_p \in \left(W^{1,\infty} + \overset{\circ}{W}^{1,r} \right) (\mathbb{R}^d)$, où $\overset{\circ}{W}^{1,r}$ est l'espace de Sobolev homogène standard. On obtient alors (en dimension $d \geq 3$) la quantification suivante de la sous-linéarité à l'infini de w_p ($r \neq d$) :

$$\forall \rho > 1, \quad \sup_{|x-y| \leq \rho} |w_p(x) - w_p(y)| \leq C \rho^{1-\nu_r}, \quad \nu_r = \min(1, d/r).$$

Cela permet d'obtenir des taux de convergence : dans [17], les taux de convergence connus en périodique sont généralisés au cadre présenté ici (et, à vrai dire, à un cadre plus général pourvu que l'on sache construire des correcteurs ayant la bonne sous-linéarité à l'infini). Nous finissons cette section par deux remarques. D'abord, la perturbation est, comme pour le cas des domaines perforés, locale. En particulier, cela n'affecte pas le comportement macroscopique de la solution, ni même la

convergence forte du reste. Ensuite, la question de l'*utilité pratique* du correcteur se pose. À nouveau comme pour les domaines perforés, la construction de w_p , $p \in \mathbb{R}^d$ se justifie si l'on souhaite une approximation fine de u_ε (ou plutôt de son gradient).

1.2.2.2 Étude d'une équation semi-linéaire

Dans un premier temps, nous étudions une équation semi-linéaire dans le cadre périodique avec défauts présenté ci-dessus :

$$\begin{cases} -\operatorname{div} A(\cdot/\varepsilon)\nabla u_\varepsilon + u_\varepsilon|u_\varepsilon|^{p-2} = f & \text{dans } \Omega \\ u_\varepsilon = 0 & \text{sur } \partial\Omega, \end{cases} \quad (1.2.39)$$

où $f \in (H_0^1 \cap L^p(\Omega))'$, $1 < p < +\infty$ et A est de la forme $A = A^{\text{per}} + \tilde{A}$, avec les mêmes hypothèses que dans la section 1.2.2.1 sur A , à savoir

$$\begin{cases} \exists \lambda > 0, \forall \xi \in \mathbb{R}^d, \forall y \in \mathbb{R}^d, \lambda^{-1}|\xi|^2 \leq A(y)\xi \cdot \xi \leq \lambda|\xi|^2 & \text{et } \lambda^{-1}|\xi|^2 \leq A^{\text{per}}(y)\xi \cdot \xi \leq \lambda|\xi|^2 \\ A^{\text{per}}, A \in C^{0,\alpha}(\mathbb{R}^d)^{d \times d} & \text{et } \tilde{A} \in L^r(\mathbb{R}^d)^{d \times d}, \quad 1 < r < +\infty. \end{cases} \quad (1.2.40)$$

La première étape est d'étudier (1.2.39) quand $A = A^{\text{per}}$. Un développement à deux échelles standard suggère que $u_\varepsilon \approx u^* + \varepsilon \sum_{j=1}^d w_{e_j}(\cdot/\varepsilon)\partial_j u^*$, où w_{e_i} , $1 \leq i \leq d$ sont les correcteurs linéaires donnés par (1.2.33) et u^* est solution de

$$\begin{cases} -\operatorname{div} A^* \nabla u^* + u^*|u^*|^{p-2} = f & \text{dans } \Omega \\ u^* = 0 & \text{sur } \partial\Omega, \end{cases} \quad (1.2.41)$$

et A^* est donnée par (1.2.34). Nous formalisons ces observations dans le théorème suivant :

Théorème 9 (Theorem 4.1.1 issu du chapitre 4). *Soit $2 \leq p < +\infty$, $d \geq 1$, A^{per} un coefficient périodique, coercif, borné et de régularité Hölderienne et u^* donnée par (1.2.41). Pour $\varepsilon > 0$, on note $u_\varepsilon \in H_0^1 \cap L^p(\Omega)$ la solution de (1.2.39). On a la convergence faible $u_\varepsilon \rightharpoonup u^*$ dans $H_0^1 \cap L^p(\Omega)$ et la convergence forte $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^*$ dans $L^r(\Omega)$, $1 \leq r < \frac{2d}{d-2}$.*

Si l'on suppose de plus que $\nabla^2 u^ \in L^2(\Omega)$, on a la convergence $\nabla R_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$ dans $L^2(\Omega)$ où*

$$R_\varepsilon := u_\varepsilon - u^* - \varepsilon \sum_{i=1}^d w_{e_i}(\cdot/\varepsilon)\partial_i u^*.$$

Remarquons que, sous les hypothèses $f \in L^2(\Omega)$ et $p < 1 + \frac{d}{d-2}$ pour $d \geq 3$ et sans restriction sur p pour $d < 3$, on a $\nabla^2 u^* \in L^2(\Omega)$. La difficulté du théorème 9 est le passage à la limite dans le terme non-linéaire $u_\varepsilon|u_\varepsilon|^{p-2}$ puisque la suite $(u_\varepsilon)_{\varepsilon > 0}$ ne converge que faiblement dans $L^p(\Omega)$ quand $p > 2d/(d-2)$. Le théorème 9 montre que l'homogénéisation du problème (1.2.39) est essentiellement couverte par l'homogénéisation du problème (1.2.32). Le cas d'un coefficient périodique avec défauts se traite alors avec les mêmes outils que le cas linéaire [19, 20, 18], tout du moins pour ce qui concerne la construction des correcteurs. Dans la suite, nous souhaitons nous concentrer sur des problèmes non linéaires dont l'équation du correcteur diffère du cas linéaire.

1.2.2.3 Étude d'une équation quasi-linéaire

Le Chapitre 5 propose l'homogénéisation de l'équation de p -Laplace dans le cadre périodique avec défauts introduit dans la section 1.2.2.1. Nous y construisons les correcteurs non périodiques et nous obtenons un théorème de convergence forte de la solution oscillante et de son gradient. Pour fixer les idées, on considère, pour $p \geq 2$, l'équation

$$\begin{cases} -\operatorname{div} a\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon|\nabla u_\varepsilon|^{p-2} = f & \text{dans } \Omega \\ u_\varepsilon = 0 & \text{sur } \partial\Omega, \end{cases} \quad (1.2.42)$$

où $f \in W^{-1,p'}(\Omega)$ et $a : \mathbb{R}^d \rightarrow \mathbb{R}$ est un coefficient périodique avec défauts au sens de la section 1.2.2.1. On peut alors décomposer a sous la forme $a = a^{\text{per}} + \tilde{a}$. Le cas $a = a^{\text{per}}$ a été étudié dans [56, 57, 44], tout du moins pour ce qui concerne la limite de la suite $(u_\varepsilon)_{\varepsilon>0}$ et son gradient, et la construction des correcteurs. Dans cette section (et dans tout le chapitre 5), on suppose que $\tilde{a} \in L^{p'} \cap C^{0,\alpha}(\mathbb{R}^d)$. Nous rappelons d'abord les résultats du cas périodique :

Théorème 10 ([56, 57, 44] adapté à la forme de l'équation (1.2.42)). *Soit a^{per} un coefficient périodique tel que $\lambda^{-1} < a^{\text{per}} < \lambda$ où $\lambda > 0$. Soit $f \in W^{-1,p'}(\Omega)$ et u_ε satisfaisant (1.2.42). On a la convergence $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^*$ dans $W^{1,p}(\Omega)$ où u^* est l'unique solution de*

$$\begin{cases} -\operatorname{div} a^*(\nabla u^*) = f & \text{dans } \Omega \\ u^* \in W_0^{1,p}(\Omega). \end{cases} \quad (1.2.43)$$

L'application $a^* : \mathbb{R}^d \rightarrow \mathbb{R}^d$ est définie par :

$$\forall \xi \in \mathbb{R}^d, \quad a^*(\xi) = \int_Q a^{\text{per}}(y) (\xi + \nabla w_\xi^{\text{per}}(y)) |\xi + \nabla w_\xi^{\text{per}}(y)|^{p-2} dy, \quad (1.2.44)$$

où les fonctions $w_\xi^{\text{per}}, \xi \in \mathbb{R}^d$, sont les correcteurs périodiques et vérifient l'EDP

$$\begin{cases} -\operatorname{div} a^{\text{per}}(y) (\xi + \nabla w_\xi^{\text{per}}) |\xi + \nabla w_\xi^{\text{per}}|^{p-2} = 0 & \text{dans } Q \\ w_\xi^{\text{per}} \in W_{\text{per}}^{1,p}(Q). \end{cases} \quad (1.2.45)$$

Par ailleurs, on a la convergence forte

$$\nabla u_\varepsilon - \nabla u^* - \nabla w_{M_\varepsilon \nabla u^*}^{\text{per}} \left(\frac{\cdot}{\varepsilon} \right) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{dans } L^p(\Omega), \quad (1.2.46)$$

où l'on a noté $M_\varepsilon : L^p(\Omega) \rightarrow L^p(\Omega)$ l'opérateur de discrétisation défini par

$$M_\varepsilon \phi := \sum_{k, \varepsilon(Q+k) \subset \Omega} \begin{pmatrix} f \\ \int_{\varepsilon(Q+k)} \phi \end{pmatrix} \mathbf{1}_{\varepsilon(Q+k)}. \quad (1.2.47)$$

Une manière de deviner la forme de l'équation vérifiée par les correcteurs (1.2.45) est à nouveau d'effectuer un développement à deux échelles. Nous nous plaçons dans un cadre périodique, possiblement avec défaut. Pour alléger les notations, on pose

$$a^{\text{per}}(y, \xi) := a^{\text{per}}(y) \xi |\xi|^{p-2}, \quad a(y, \xi) := a(y) \xi |\xi|^{p-2} \quad \text{et} \quad \tilde{a}(y, \xi) := \tilde{a}(y) \xi |\xi|^{p-2}$$

pour $y \in \mathbb{R}^d, \xi \in \mathbb{R}^d$. On suppose que u_ε se développe sous la forme

$$u_\varepsilon = u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \dots, \quad y = x/\varepsilon, \quad (1.2.48)$$

où les fonctions u_i sont supposées régulières et $y \mapsto u_i(x, y)$ est périodique quand le coefficient a est périodique *i.e.* $a = a^{\text{per}}$. On note alors u_i sous la forme u_i^{per} . Dans le cas périodique avec défaut, on suppose que $u_i(x, \cdot)$ est une perturbation à l'échelle microscopique de u_i^{per} : pour tout $x \in \Omega$ et $y \in \mathbb{R}^d$, $u_i(x, \cdot) := u_i^{\text{per}}(x, \cdot) + \tilde{u}_i(x, \cdot)$ où $\nabla_y \tilde{u}_i(x, \cdot) \in L^p \cap C^{0,\alpha}(\mathbb{R}^d)$. On a

$$\nabla u_\varepsilon = \varepsilon^{-1} \nabla_y u_0 + \nabla_x u_0 + \nabla_y u_1 + \varepsilon (\nabla_x u_1 + \nabla_y u_2) + \dots, \quad y = x/\varepsilon, \quad x \in \Omega. \quad (1.2.49)$$

D'où, en utilisant l'homogénéité et en linéarisant autour du terme dominant,

$$a(\cdot/\varepsilon, \nabla u_\varepsilon) = \varepsilon^{1-p} a(y, \nabla_y u_0) + \varepsilon^{2-p} \partial_\xi a(y, \nabla_y u_0) \cdot [\nabla_x u_0 + \nabla_y u_1 + \varepsilon (\nabla_x u_1 + \nabla_y u_2) + \dots]. \quad (1.2.50)$$

Cela fournit

$$\begin{aligned} \operatorname{div} a(\cdot/\varepsilon, \nabla u_\varepsilon) &= \varepsilon^{-p} \operatorname{div}_y a(y, \nabla_y u_0) + \varepsilon^{1-p} \operatorname{div}_x a(y, \nabla_y u_0) \\ &\quad + \varepsilon^{1-p} \operatorname{div}_y \partial_\xi a(y, \nabla_y u_0) \cdot [\nabla_x u_0 + \nabla_y u_1 + \varepsilon (\nabla_x u_1 + \nabla_y u_2) + \dots] + \mathcal{O}(\varepsilon^{2-p}). \end{aligned}$$

En insérant ce développement dans (1.2.42) et en identifiant les puissances les plus élevées de $1/\varepsilon$, on trouve que $-\operatorname{div}_y a(y, \nabla_y u_0) = 0$ à x fixé. Dans le cas périodique, une estimation d'énergie et la monotonie de l'opérateur $a^{\text{per}}(y, \cdot)$ fournissent $\nabla_y u_0^{\text{per}} = 0$. Dans le cas non périodique, il faut montrer de plus que $\nabla_y \widetilde{u}_0 = 0$. On écrit que, formellement,

$$-\operatorname{div}_y a(y, \nabla_y \widetilde{u}_0) = 0. \quad (1.2.51)$$

En utilisant⁴ la monotonie de l'opérateur a , on obtient que $\nabla_y \widetilde{u}_0 = 0$ et donc $\nabla_y u_0 = 0$. D'où $u_0^{\text{per}}(x, y) = u_0^{\text{per}}(x)$ et $u_0(x, y) = u_0(x)$. Retournant à (1.2.49), on trouve que

$$\begin{aligned} \operatorname{div} a(\cdot/\varepsilon, \nabla u_\varepsilon) &= \varepsilon^{-1} \operatorname{div}_y a(y, \nabla_x u_0 + \nabla_y u_1) + \operatorname{div}_x a(y, \nabla_x u_0 + \nabla_y u_1) \\ &\quad + \operatorname{div}_y [\partial_\xi a(y, \nabla_x u_0 + \nabla_y u_1) \cdot (\nabla_x u_1 + \nabla_y u_2 + \dots)] + \mathcal{O}(\varepsilon). \end{aligned} \quad (1.2.52)$$

A nouveau en insérant dans (1.2.42) et en identifiant les puissances les plus élevées de $1/\varepsilon$, on obtient que $\operatorname{div}_y a(y, \nabla_x u_0 + \nabla_y u_1) = 0$. On en déduit que $u_1(x, y) = w_{\nabla u_0(x)}(y)$ où, pour $\xi \in \mathbb{R}^d$, w_ξ vérifie l'équation du correcteur

$$-\operatorname{div} a(y) (\xi + \nabla w_\xi) |\xi + \nabla w_\xi|^{p-2} = 0 \quad \text{dans } \mathbb{R}^d, \quad (1.2.53)$$

dans le cadre périodique (auquel cas $a = a^{\text{per}}$ et $w_\xi = w_\xi^{\text{per}}$ est périodique et bornée) comme dans le cadre non périodique (auquel cas on suppose que $w_\xi = w_\xi^{\text{per}} + \widetilde{w}_\xi$ est une perturbation de w_ξ^{per} , avec $\nabla \widetilde{w}_\xi \in L^p \cap L^\infty(\mathbb{R}^d)$). De même, $u_1^{\text{per}}(x, y) = w_{\nabla u_0^{\text{per}}(x)}^{\text{per}}(y)$. En exploitant à nouveau (1.2.52), on trouve que, nécessairement,

$$-\operatorname{div}_x a(y, \nabla_x u_0 + \nabla_y u_1) - \operatorname{div}_y [\partial_\xi a(y, \nabla_x u_0 + \nabla_y u_1) \cdot (\nabla_x u_1 + \nabla_y u_2)] = f.$$

On peut réécrire cette égalité sous la forme

$$-\operatorname{div}_x a(y, \nabla u_0(x) + \nabla w_{\nabla u_0(x)}(y)) - \operatorname{div}_y [\partial_\xi a(y, \nabla_x u_0 + \nabla_y u_1) \cdot (\nabla_x u_1 + \nabla_y u_2)] = f,$$

où $\nabla w_{\nabla u_0(x)}(y) = \nabla w_{\nabla u_0(x)}^{\text{per}}(y) + \nabla \widetilde{w}_{\nabla u_0(x)}(y)$ et $\widetilde{w}_{\nabla u_0(x)} \in L^p \cap L^\infty(\mathbb{R}^d)$, toujours à x fixé. En moyennant sur la variable rapide y (ce qui revient à moyenner sur un grand cube Q_R puis à faire $R \rightarrow +\infty$), on déduit⁵ que u_0 est solution de l'équation macroscopique (1.2.43) où a^* est donné par (1.2.44). En particulier, $\nabla u_0 = \nabla u_0^{\text{per}}$ et donc $u_0 = u_0^{\text{per}}$. Les calculs ci-dessus sont formels et ne constituent en rien une preuve. Notre objectif est maintenant de rendre ces observations rigoureuses.

La première étape est de résoudre (1.2.53). Pour cela, on s'appuie sur le cas linéaire et l'on exploite le problème périodique. On cherche w_ξ sous la forme $w_\xi = w_\xi^{\text{per}} + \widetilde{w}_\xi$ et l'on écrit l'équation que doit vérifier \widetilde{w}_ξ :

$$-\operatorname{div} a \left[\left(\xi + \nabla w_\xi^{\text{per}} + \nabla \widetilde{w}_\xi \right) \left| \xi + \nabla w_\xi^{\text{per}} + \nabla \widetilde{w}_\xi \right|^{p-2} - \left(\xi + \nabla w_\xi^{\text{per}} \right) \left| \xi + \nabla w_\xi^{\text{per}} \right|^{p-2} \right] = \operatorname{div} h, \quad (1.2.54)$$

où la fonction h est donnée par

$$h := \widetilde{a} \left(\xi + \nabla w_\xi^{\text{per}} \right) \left| \xi + \nabla w_\xi^{\text{per}} \right|^{p-2}. \quad (1.2.55)$$

Dans le cas linéaire (voir section 1.2.2.1) c'est-à-dire dans le cas $p = 2$, une première étape était de résoudre (1.2.54) dans un cadre variationnel. Ici, cela revient à imposer $h \in L^{p'}(\mathbb{R}^d)$, hypothèse

⁴On teste d'abord (1.2.51) contre une fonction de $\mathcal{D}(\mathbb{R}^d)$ puis on raisonne par densité pour approcher $\nabla_y \widetilde{u}_0$.

⁵Notons que, dans le cas périodique, on peut se limiter à intégrer sur Q .

vérifiée car $\tilde{a} \in L^{p'}(\mathbb{R}^d)$ (et $\nabla w_\xi^{\text{per}} \in L^\infty(Q)$, voir proposition 5.2.1). Multipliant formellement (1.2.54) par \tilde{w}_ξ et intégrant par parties, on obtient

$$\int a \left[\left(\xi + \nabla w_\xi^{\text{per}} + \nabla \tilde{w}_\xi \right) \left| \xi + \nabla w_\xi^{\text{per}} + \nabla \tilde{w}_\xi \right|^{p-2} - \left(\xi + \nabla w_\xi^{\text{per}} \right) \left| \xi + \nabla w_\xi^{\text{per}} \right|^{p-2} \right] \cdot \nabla \tilde{w}_\xi = - \int h \cdot \nabla \tilde{w}_\xi. \quad (1.2.56)$$

En utilisant des estimations sur le p -Laplacien (voir [68]), on déduit l'estimation

$$\|\nabla \tilde{w}_\xi\|_{L^p(\mathbb{R}^d)}^{p-1} \leq C \|h\|_{L^{p'}(\mathbb{R}^d)}.$$

Cela suggère de travailler dans l'espace $V := \{w \in W_{\text{loc}}^{1,1}(\mathbb{R}^d), \nabla w \in L^p(\mathbb{R}^d)\}$. Cependant, un calcul rapide (par exemple quand $p = 4$) montre que le membre de gauche de (1.2.56) n'est pas nécessairement défini sur V . Il faut donc restreindre cet espace. Pour cela, on fait appel à l'inégalité

$$(\xi|\xi|^{p-2} - \zeta|\zeta|^{p-2}) \cdot (\xi - \zeta) \geq c(|\xi|^{p-2} + |\zeta|^{p-2})|\xi - \zeta|^2,$$

et l'on constate que, nécessairement, $\nabla \tilde{w}_\xi \in L^2\left(\left|\xi + \nabla w_\xi^{\text{per}}\right|^{p-2} d\lambda\right)$ où $d\lambda$ désigne la mesure de Lebesgue. On introduit alors l'espace

$$W_{\xi + \nabla w_\xi^{\text{per}}} = \left\{ w \in W_{\text{loc}}^{1,1}(\mathbb{R}^d), \nabla w \in L^p(\mathbb{R}^d) \cap L^2\left(\left|\xi + \nabla w_\xi^{\text{per}}\right|^{p-2} d\lambda\right) \right\} / \mathbb{R}, \quad (1.2.57)$$

et l'on montre le théorème suivant.

Théorème 11 (Theorem 5.2.3 du chapitre 5). *Soit $a = a^{\text{per}} + \tilde{a}$ un coefficient coercif et borné tel que a^{per} soit également coercif et borné. On suppose que a et a^{per} sont lipschitziens. Soit $q \leq p'$ tel que $\tilde{a} \in L^q(\mathbb{R}^d)$. Il existe une unique fonction $\tilde{w}_\xi \in W_{\xi + \nabla w_\xi^{\text{per}}}$ solution au sens faible dans $W_{\xi + \nabla w_\xi^{\text{per}}}$ de (1.2.54)–(1.2.55), c'est-à-dire : pour tout $u \in W_{\xi + \nabla w_\xi^{\text{per}}}$,*

$$\int_{\mathbb{R}^d} a \left[\left(\xi + \nabla w_\xi^{\text{per}} + \nabla \tilde{w}_\xi \right) \left| \xi + \nabla w_\xi^{\text{per}} + \nabla \tilde{w}_\xi \right|^{p-2} - \left(\xi + \nabla w_\xi^{\text{per}} \right) \left| \xi + \nabla w_\xi^{\text{per}} \right|^{p-2} \right] \cdot \nabla u = - \int_{\mathbb{R}^d} h \cdot \nabla u.$$

En particulier, $w_\xi := w_\xi^{\text{per}} + \tilde{w}_\xi$ est solution de (1.2.53) au sens des distributions.

A ce stade, deux remarques s'imposent. D'abord, les résultats sont présentés pour l'équation de p -Laplace mais ils peuvent s'étendre à des opérateurs $a(y, \xi)$ plus généraux, voir la section 5.8.3. Ensuite, la méthode de preuve mise en place dans le chapitre 5 permet en fait de traiter le cas d'un coefficient $\tilde{a} \in L^q(\mathbb{R}^d)$, $q \in [p', 2]$. On obtient alors l'existence d'un correcteur dont le gradient est dans l'espace à poids $L^2\left(\left|\xi + \nabla w_\xi^{\text{per}}\right|^{p-2} d\lambda\right)$, mais on ne sait pas montrer que $\nabla \tilde{w}_\xi \in L^2(\mathbb{R}^d)$. En revanche, le cas $\tilde{a} \in L^{2+\delta}(\mathbb{R}^d)$ est ouvert, voir la section 1.3 ci-dessous.

Après avoir construit les correcteurs, nous aimerions obtenir l'équivalent du théorème 10 dans le cadre périodique avec défauts. Dans le cadre périodique, un outil important pour obtenir la convergence forte des gradients (1.2.46) est la continuité de l'application

$$\begin{cases} \mathbb{R}^d \longrightarrow L^p(Q) \\ \xi \longmapsto \nabla w_\xi^{\text{per}}. \end{cases} \quad (1.2.58)$$

Cette propriété est facile à montrer dans le cas périodique puisque le domaine d'étude est borné. Dans le cas non périodique, la situation est plus délicate. Nous faisons l'hypothèse suivante :

(H) Pour $\xi \in \mathbb{R}^d$, il existe une constante $C > 0$ et un rayon $r_{\min} > 0$ tels que pour tout $R > r_{\min}$ et tout $w \in H^1(Q \setminus Q_{1/2})$, on ait l'inégalité de Poincaré-Wirtinger à poids suivante :

$$\left\| \left| \xi + \nabla w_\xi^{\text{per}}(R \cdot) \right|^{\frac{p-2}{2}} \left(w - \fint_{Q \setminus Q_{1/2}} w \right) \right\|_{L^2(Q \setminus Q_{1/2})} \leq C \left\| \left| \xi + \nabla w_\xi^{\text{per}}(R \cdot) \right|^{\frac{p-2}{2}} \nabla w \right\|_{L^2(Q \setminus Q_{1/2})}. \quad (1.2.59)$$

Nous donnons dans la section 5.2.4 des conditions sur $\nabla w_\xi^{\text{per}}$ pour que **(H)** soit vérifiée. On peut alors montrer que, sous l'hypothèse **(H)**, l'application

$$\begin{cases} \mathbb{R}^d \longrightarrow L_{\text{unif}}^p(\mathbb{R}^d) \\ \xi \longmapsto \nabla w_\xi. \end{cases} \quad (1.2.60)$$

est continue. Cela permet d'obtenir le résultat suivant, qui s'appuie sur [57, 44].

Théorème 12 (Theorem 5.2.7 du chapitre 5). *Sous les hypothèses du Théorème 11, soit u_ε la solution de (1.2.53). Supposons que l'hypothèse **(H)** est vérifiée.*

(i) *On a $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^*$ dans $W^{1,p}(\Omega)$ et $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^*$ dans $L^p(\Omega)$ où u^* est solution de (1.2.43). Par ailleurs, on a la convergence faible*

$$a\left(\frac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon |\nabla u_\varepsilon|^{p-2} \xrightarrow{\varepsilon \rightarrow 0} a^*(\nabla u^*) \quad \text{dans } L^p(\Omega).$$

(ii) *On a la convergence forte*

$$\nabla u_\varepsilon - \nabla u^* - \nabla w_{M_\varepsilon \nabla u^*} \left(\frac{\cdot}{\varepsilon}\right) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{dans } L^p(\Omega), \quad (1.2.61)$$

où M_ε est défini dans (1.2.47).

(iii) *On a la convergence forte*

$$\nabla u_\varepsilon - \nabla u^* - \widetilde{\nabla w}_{M_\varepsilon \nabla u^*} \left(\frac{\cdot}{\varepsilon}\right) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{dans } L^p(\Omega), \quad (1.2.62)$$

où M_ε est défini dans (1.2.47).

Notons que les preuves des points (i) et (iii) du théorème 12 ne requièrent pas l'hypothèse **(H)**, comme on peut s'y attendre. Nous souhaitons dans la suite améliorer les résultats sur la continuité de l'application $\xi \mapsto \nabla w_\xi$, notamment en vue d'obtenir des taux de convergence. En particulier, nous aimerions, comme pour les correcteurs périodiques, obtenir la continuité hölderienne par rapport au vecteur ξ . Nous avons obtenu ce résultat, sous une condition de non dégénérescence du poids $\xi + \nabla w_\xi^{\text{per}}$:

Théorème 13 (Extrait du théorème 5.2.4). *Sous les hypothèses du théorème 11, supposons que pour tout $\xi \in \mathbb{R}^d$, on ait $|\xi + \nabla w_\xi^{\text{per}}| > c|\xi|$, où la constante $c > 0$ est indépendante de ξ . On a alors l'existence d'une constante $C = C(d, p, a, c)$ telle que pour tout $\xi, \eta \in \mathbb{R}^d$,*

$$\|\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta\|_{L^p(\mathbb{R}^d)} \leq C \left(|\xi|^{1-\widetilde{\beta}} + |\eta|^{1-\widetilde{\beta}} \right) |\xi - \eta|^{\widetilde{\beta}}, \quad \widetilde{\beta} := \frac{1}{p-1} \min(1, p-2).$$

La preuve du théorème 13 repose sur une linéarisation de l'équation du correcteur non périodique autour de $\xi + \nabla w_\xi^{\text{per}}$. On fait ensuite appel à [20]. L'hypothèse de non dégénérescence est très forte : elle est satisfaite en dimensions $d = 1, 2$ [30] et pour les "faux" cas multidimensionnels tels que les matériaux lamellés, voir Section 5.2.4. La difficulté dans l'homogénéisation du p -Laplacien provient essentiellement de ce que le coefficient linéarisé $\partial_\xi a(\cdot, \xi + \nabla w_\xi)$, qui se comporte comme $|\xi + \nabla w_\xi^{\text{per}}|^{p-2}$, peut être dégénéré. Des résultats récents étudient le cas non dégénéré dans le cadre de l'homogénéisation stochastique quantitative [42, 55]. Il peut s'agir d'opérateurs $a(y, \xi)$ à croissance quadratique et lipschitziens en ξ [55] ou d'opérateurs de la forme $a(y, \xi) = a(y)\xi(1 + |\xi|^{p-2})$. Dans ces travaux, un outil important est l'équation du correcteur linéarisé

$$-\operatorname{div} \partial_\xi a(\cdot, \xi + \nabla w_\xi) \cdot (h + \nabla w_{\xi, h}) = 0, \quad h \in \mathbb{R}^d \quad (1.2.63)$$

où, au moins formellement, $\nabla w_{\xi,h} = \partial_\xi \nabla w_\xi \cdot h$. Des bornes sur $\nabla w_{\xi,h}$ permettent de retrouver des résultats analogues au théorème 13 (sans hypothèse forte sur $\xi + \nabla w_\xi$). Nous pensons que, en s'appuyant sur les résultats du cadre linéaire, il est possible d'étudier (1.2.63), et donc les opérateurs considérés dans [55, 42], dans le cadre périodique avec défauts étudié dans cette thèse.

Nous terminons cette section par un résumé du chapitre 5. Nous y présentons l'homogénéisation de l'équation de p -Laplace dans un cadre périodique avec défauts. Nous prouvons que l'équation du correcteur admet, dans un cadre variationnel (qui correspond en fait à des défauts tendant suffisamment rapidement vers zéro à l'infini), une solution dans un espace de Sobolev à poids. Sous une hypothèse de non dégénérescence de ce poids, nous montrons, en utilisant des techniques issues du cas linéaire, la continuité de l'application $\xi \mapsto \nabla w_\xi$. Cela permet de reconstruire les oscillations de ∇u_ε à l'aide de la solution homogénéisée et des correcteurs périodiques et non périodiques. Remarquons que, comme dans le cas linéaire, le comportement macroscopique de u_ε n'est pas affecté par l'introduction du défaut local \tilde{a} . De même, l'utilisation du correcteur périodique dans la convergence forte du gradient n'affecte pas le mode de convergence, pourvu que l'on ne requiert pas une approximation de u_ε dans des normes très fines. Dans la section 5.3, nous illustrons cela dans le cas de la dimension 1. En effet, des calculs explicites permettent de montrer que

$$u'_\varepsilon - (u^*)' - w'_{(u^*)'} \left(\frac{\cdot}{\varepsilon} \right) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{et} \quad u'_\varepsilon - (u^*)' - \left(w_{(u^*)'}^{\text{per}} \right)' \left(\frac{\cdot}{\varepsilon} \right) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{dans} \quad L^p.$$

En revanche,

$$u'_\varepsilon - (u^*)' - w'_{(u^*)'} \left(\frac{\cdot}{\varepsilon} \right) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{dans} \quad L^\infty,$$

tandis que $u'_\varepsilon - (u^*)' - \left(w_{(u^*)'}^{\text{per}} \right)' \left(\frac{\cdot}{\varepsilon} \right)$ ne tend pas nécessairement vers zéro en norme L^∞ . Comme pour les cas linéaires, le correcteur non périodique améliore l'approximation *fine* de la solution oscillante. En dimension supérieure, ces problèmes sont hors de portée.

1.3 Perspectives

Dans cette section, nous présentons quelques questions ouvertes issues des travaux précédents.

Homogénéisation du système de Stokes. Une première piste possible concerne l'amélioration des taux de convergence obtenus dans le chapitre 3. Ils sont pour l'instant limités à l'exposant $q = 2$. Pourtant, les résultats obtenus pour l'équation de Poisson (mais aussi [82]) suggèrent que l'on pourrait, au moins sous l'hypothèse simplificatrice (1.2.28), généraliser ces taux au cas $q > 2$. Dans le cas $q > d$, on s'attend à ce que ces taux soient optimaux quand on utilise le correcteur non périodique. Au contraire, ils devraient se dégrader (quand q devient grand) lorsque l'on considère le reste "périodique"

$u_\varepsilon - \varepsilon^2 \sum_{j=1}^3 w_j^{\text{per}}(\cdot/\varepsilon) f_j$. Cela justifierait la construction *pratique* du correcteur non périodique lorsque

l'on souhaite une approximation *fine* de la solution. L'obtention de taux de convergence demanderait deux étapes :

- évaluer le second membre de l'EDP satisfaite par le reste $(R_\varepsilon, \pi_\varepsilon)$. Cette étape est bien sûr liée à une propriété de sous-linéarité des correcteurs. Nous montrons un tel résultat dans l'annexe 3.4.3 pour les correcteurs d'ordre (w_j, p_j) ;
- établir des estimées de régularité elliptique uniformes en ε pour un système de Stokes posé dans Ω_ε . De tels résultats ont été énoncés dans [82] pour le cas périodique. Une preuve de ces résultats a été soumise récemment [94] dans le cas d'un domaine perforé dans l'espace complet. Il serait intéressant de voir si cette preuve s'étend au cas périodique avec défauts étudié ici.

Une autre possibilité serait d'essayer de comprendre le comportement (dans l'intérieur du domaine) de $(u_\varepsilon, p_\varepsilon)$ dans le cas où f ne satisfait plus (1.2.28). On pourrait pour cela commencer par des simulations numériques.

Homogénéisation d'équations quasi-linéaires. La première piste que nous pourrions explorer est la résolution de l'équation du correcteur (5.1.7) dans le cas d'un défaut qui décroît moins rapidement à l'infini vers zéro : cela se traduirait ici par la condition $\tilde{a} \in L^q(\mathbb{R}^d)$, $q > 2$. Dans le cas linéaire, l'équation est résolue en étudiant la continuité $L^q - L^q$ de l'opérateur de Riesz. L'obtention d'un tel résultat dans le cas non linéaire semble plus délicate. Cela constituerait une amélioration ambitieuse des résultats précédents.

Une autre possibilité est d'étudier le cas $1 < p < 2$ dans le cadre du Chapitre 5. Dans le cas périodique, la restriction $p \geq 2$ n'est pas nécessaire pour prouver les théorèmes de convergence de u_ε et de son gradient. Cela devrait rester vrai pour le cadre non périodique. En revanche, la construction du correcteur non périodique, de même que les preuves des résultats de continuité sur l'application $\xi \mapsto \nabla w_\xi$, nécessitent, au moins sous leur forme actuelle, la condition $p \geq 2$. Il serait pertinent de reprendre les méthodes de preuve sous l'hypothèse $p < 2$.

Enfin, une dernière piste concernerait des taux de convergence dans le cas $p > 2$ pour l'équation de p -Laplace. Comme indiqué plus haut, des résultats sont déjà disponibles dans la littérature pour des équations non dégénérées [42, 55]. Ces résultats s'appuient, entre autres, sur la continuité Hölder de l'application $\xi \mapsto \nabla w_\xi$. Dans le cas périodique, cette continuité est simple à établir. Dans le cas non périodique, nous y sommes parvenus sous hypothèse de non dégénérescence du correcteur périodique. Il serait intéressant, sous cette hypothèse – certes assez forte –, de regarder dans quelle mesure la méthode présentée dans [42] peut être exploitée dans le cadre du chapitre 5.

Première partie

Homogénéisation en milieu perforé
avec défauts locaux

CHAPTER 2

HOMOGENIZATION OF THE POISSON EQUATION IN A NON-PERIODICALLY PERFORATED DOMAIN

This chapter is based on [22], written in collaboration with Xavier Blanc. We have added Subsection 2.5.2.

We study the Poisson equation in a perforated domain with homogeneous Dirichlet boundary conditions. The size of the perforations is denoted by $\varepsilon > 0$ and is proportional to the distance between neighbouring perforations. In the periodic case, the homogenized problem (obtained in the limit $\varepsilon \rightarrow 0$) is well understood (see [77]). We extend these results to a non-periodic case which is defined as a localized deformation of the periodic setting. We propose geometric assumptions that make precise this setting and we prove results which extend those of the periodic case: existence of a corrector, convergence to the homogenized problem and two-scale expansion.

2.1 Introduction

In this chapter, we study the following problem:

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \quad (2.1.1)$$

where f is a given smooth, compactly-supported function (this assumption may be relaxed, as we will see below in Remarks 2.1.4 and 2.2.4), and Ω_ε is a perforated domain that we make precise in the following. Our aim is to study the asymptotic behaviour of u_ε as $\varepsilon \rightarrow 0$, deriving a two-scale expansion and proving convergence estimates. In [77], these results were obtained in the periodic case (that is, if the perforations are a periodic array of period ε). Here, we adapt this work to a non-periodic setting. Using Assumptions **(A1)** and **(A2)** below, which are inspired from the setting developed in [19, 20, 18], we first prove the existence of a corrector (Theorem 2.2.1 below). While this result is trivial in the periodic case, it is not in the present setting. Then, we prove the convergence result stated in Theorem 2.2.2, which is a generalization of [77, Theorem 3.1] to the present setting. We also prove such a convergence in L^∞ norm (Theorem 2.2.3 below), a result which was not proved in [77]. As it is noticed in [99, 51] for the periodic case, the crucial point in order to prove such results is a Poincaré inequality with an explicit scaling in ε , for functions vanishing in the perforations (see Lemma 2.1.1 below in the periodic case and Lemma 2.3.2 in the non-periodic case).

To our knowledge, the first contributions on the homogenization of elliptic problems in perforated domains are [40, 97] in which qualitative results are obtained. The setting is periodic, the holes are "large" (i.e. they have the same size as the period), the equation is elliptic in divergence form and

various boundary conditions are considered, such as constant Dirichlet boundary conditions on the holes and Neumann conditions. This implies that the limit is not necessarily trivial, in contrast to [77], where, as we will see below, $u_\varepsilon(x) \approx \varepsilon^2 f(x)w(x/\varepsilon)$, for some periodic function w . The case of Robin boundary conditions along the holes is addressed in [34]. In this work, an important tool to study the problem is the so-called extension operator, which is introduced in [40]. In [36, 37], the case of operators in divergence form with periodic coefficients and with Robin boundary conditions is addressed, with the help of the periodic unfolding method (see [32, 31]). The case of eigenvalue problems is considered in [102]. In [38, 39], the geometry is periodic, the boundary conditions are of Dirichlet type but the holes are assumed to be asymptotically small compared to the period. The case of small holes with non-homogeneous Neumann boundary conditions is also developed in [43]. In [50], a perforated domain mixing large and small holes with a double periodicity structure is studied for the Poisson Equation with homogeneous Dirichlet boundary conditions. In [46], sufficient conditions on periodic holes are given which allow for homogenization. Sufficient conditions on the perforated domain are also provided in [51] for the homogenization of non-linear operators under divergence form with homogeneous Dirichlet boundary conditions, see also [49] for nonlinear problems.

In [28], a formalization in link with the H-convergence was proposed under general assumptions on the perforations. However, the computations are less explicit than in our setting. A general (non-periodic) perforated domain was also considered in [87]: this setting requires that, among other assumptions, the same perforation is reproduced in some cells of a periodic grid (but not necessarily all of them).

In the following subsection, we recall the results proved in the periodic setting in [77]. Then, in Subsection 2.1.2, we study the case of a locally perturbed periodic geometry. We give conditions on the perforations (inspired from [19, 20, 18]), which imply that, away from the defect, the perforations become periodic and which allow to prove convergence results similar to those of the periodic case. In Section 2.2, we give the main results of the article, together with some remarks and comments. Section 2.3 is devoted to a Poincaré-type inequality which is crucial in our proof. Finally, Section 2.4 is devoted to the proofs of the results stated in Section 2.2.

2.1.1 The periodic case

We start with some notations. We consider the d -dimensional unit cube $Q = (-\frac{1}{2}, \frac{1}{2})^d$ with $d \geq 2$. Let $\mathcal{O}_0^{\text{per}}$ be an open subset of Q such that $\mathcal{O}_0^{\text{per}} \subset\subset Q$ and $Q \setminus \overline{\mathcal{O}_0^{\text{per}}}$ is connected. For simplicity, we choose to impose that $\overline{\mathcal{O}_0^{\text{per}}}$ cannot intersect the boundary of Q . Assume, for elliptic regularity, that $\mathcal{O}_0^{\text{per}}$ is a $C^{1,\gamma}$ domain for some $0 < \gamma < 1$. We set, for all $k \in \mathbb{Z}^d$,

$$\mathcal{O}_k^{\text{per}} := \mathcal{O}_0^{\text{per}} + k \quad \text{and} \quad \mathcal{O}^{\text{per}} := \bigcup_{k \in \mathbb{Z}^d} \mathcal{O}_k^{\text{per}}. \quad (2.1.2)$$

If $k \in \mathbb{Z}^d$, we have $\mathcal{O}_k^{\text{per}} \subset\subset Q_k$ where $Q_k := Q + k$. Given $\varepsilon > 0$, denote by $\mathcal{O}_\varepsilon^{\text{per}}$ the set of ε -perforations:

$$\mathcal{O}_\varepsilon^{\text{per}} := \bigcup_{k \in \mathbb{Z}^d} \varepsilon \mathcal{O}_k^{\text{per}} = \bigcup_{k \in \mathbb{Z}^d} \varepsilon (\mathcal{O}_0^{\text{per}} + k) = \varepsilon \mathcal{O}^{\text{per}}. \quad (2.1.3)$$

We now define some useful functional spaces:

$$H^{1,\text{per}}(Q \setminus \overline{\mathcal{O}_0^{\text{per}}}) := \left\{ u \in H_{\text{loc}}^1(\mathbb{R}^d \setminus \overline{\mathcal{O}^{\text{per}}}) \text{ s.t. } u \text{ and } \partial_i u \text{ are } Q\text{-periodic for all } i \in \{1, \dots, d\} \right\}, \quad (2.1.4)$$

where the subscript *loc* means locally¹ in \mathbb{R}^d , and

$$H_0^{1,\text{per}}(Q \setminus \overline{\mathcal{O}_0^{\text{per}}}) := \left\{ u \in H^{1,\text{per}}(Q \setminus \overline{\mathcal{O}_0^{\text{per}}}) \text{ s.t. } u|_{\partial \mathcal{O}_0^{\text{per}}} = 0 \right\}. \quad (2.1.5)$$

¹*i.e.* a function u belongs to $H_{\text{loc}}^1(\mathbb{R}^d \setminus \overline{\mathcal{O}^{\text{per}}})$ if for all bounded open set U of \mathbb{R}^d , we have that $u \in H^1(U \cap (\mathbb{R}^d \setminus \overline{\mathcal{O}^{\text{per}}}))$.

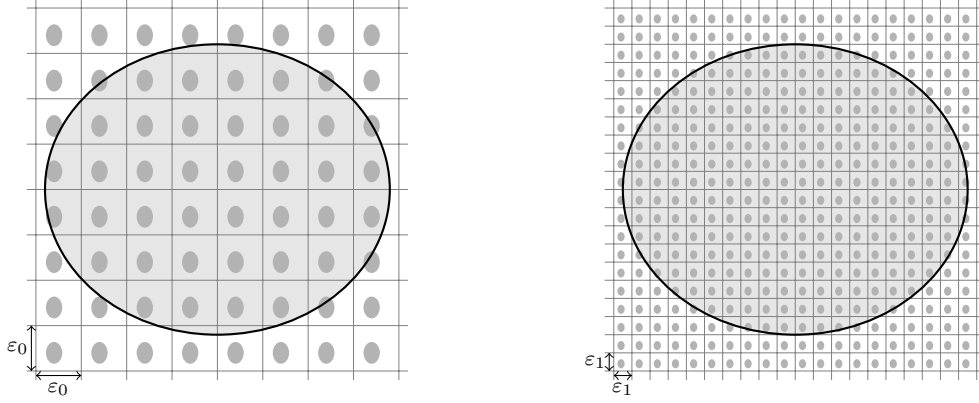


Figure 2.1: The periodic set for two choices of ε , ε_0 and $\varepsilon_1 = \varepsilon_0/2.5$.

The two spaces defined by (2.1.4) and (2.1.5) are Hilbert spaces for the norm

$$\|u\|_{H^{1,\text{per}}(Q \setminus \overline{\mathcal{O}_0^{\text{per}}})} := \left(\int_{Q \setminus \overline{\mathcal{O}_0^{\text{per}}}} |u|^2 + \int_{Q \setminus \overline{\mathcal{O}_0^{\text{per}}}} |\nabla u|^2 \right)^{1/2}.$$

In the sequel, a function of $H^{1,\text{per}}(Q \setminus \overline{\mathcal{O}_0^{\text{per}}})$ or of $H_0^{1,\text{per}}(Q \setminus \overline{\mathcal{O}_0^{\text{per}}})$ will naturally be extended to $\mathbb{R}^d \setminus \overline{\mathcal{O}^{\text{per}}}$ by periodicity. All along the paper, we will denote the H^1 -semi-norm on a set V by $|\cdot|_{H^1(V)}$:

$$|u|_{H^1(V)} := \left(\int_V |\nabla u|^2 \right)^{1/2}.$$

Let Ω be a bounded, open and connected domain of \mathbb{R}^d that is locally Lipschitz regular. For $\varepsilon > 0$, denote by $\Omega_\varepsilon := \Omega \setminus \overline{\mathcal{O}_\varepsilon^{\text{per}}}$. Note that Ω_ε is open and bounded but may not be connected. One has

$$\Omega_\varepsilon = \Omega \cap \left(\bigcup_{k \in \mathbb{Z}^d} \varepsilon(Q_k \setminus \overline{\mathcal{O}_k^{\text{per}}}) \right). \quad (2.1.6)$$

Figure 2.1 shows the set Ω_ε for two values ε_0 and ε_1 satisfying $\varepsilon_0 > \varepsilon_1$. The set Ω_ε is colored in light grey. We are interested in the Poisson problem (2.1.1). As we already mentioned, the source term f is supposed, as in [77], to be smooth and compactly supported in Ω . In fact (see Remark 3.3 of [77]), it is sufficient to assume that $f \in C^{m-2}(\overline{\Omega})$ and that $D^p f|_{\partial\Omega} = 0$ for all $|p| \leq m-2$, where m is the order of the two scale expansion of u_ε (that is, the order in ε that we seek in the approximation on u_ε). As pointed out in [75], the assumptions on f can be weakened further (see Remark 2.1.4 below).

By a simple application of the Lax-Milgram Lemma, we have the existence and the uniqueness of a solution u_ε to (2.1.1). In order to study the dependence of u_ε on ε , we will need the following Lemma which is a Poincaré-type inequality in $H_0^1(\Omega_\varepsilon)$. It is proved in [99, Lemma 1] (see also [23, Proposition 3.1]). A crucial point in the non-periodic case will be to have a similar result, with the same scaling in ε . This is done in Lemma 2.3.2 below. Note that a stronger result has in fact been independently proven in [51].

Lemma 2.1.1 (Lemma 1 of [99]). *There exists a constant $C_0 > 0$ independent of ε such that*

$$\forall u \in H_0^1(\Omega_\varepsilon), \quad \|u\|_{L^2(\Omega_\varepsilon)} \leq C_0 \varepsilon \|\nabla u\|_{L^2(\Omega_\varepsilon)}.$$

This allows to prove Lemma 2 of [99]:

Lemma 2.1.2 (Lemma 2 of [99]). *The solution u_ε of Problem (2.1.1) satisfies the estimates*

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \varepsilon^2 \quad \text{and} \quad \|u_\varepsilon\|_{H_0^1(\Omega_\varepsilon)} \leq C \varepsilon, \quad (2.1.7)$$

where C is a constant independent of ε .

Using a two-scale expansion of the form (see [77, Section 2])

$$u_\varepsilon(x) = u_0^{\text{per}}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1^{\text{per}}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2^{\text{per}}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 u_3^{\text{per}}\left(x, \frac{x}{\varepsilon}\right) + \cdots, \quad (2.1.8)$$

where the functions u_i^{per} are defined on $\Omega \times (Q \setminus \overline{\mathcal{O}_0^{\text{per}}})$, smooth and Q -periodic in the second variable, one proves that, at least formally,

$$u_\varepsilon(x) = \varepsilon^2 w^{\text{per}}\left(\frac{x}{\varepsilon}\right) f(x) + \cdots, \quad (2.1.9)$$

where w^{per} is the periodic solution of

$$\begin{cases} -\Delta w^{\text{per}} = 1, \\ w^{\text{per}} \in H_0^{1,\text{per}}(Q \setminus \overline{\mathcal{O}_0^{\text{per}}}). \end{cases} \quad (2.1.10)$$

We note that Problem (2.1.10) is well-posed. Indeed, it suffices to apply Lax-Milgram Lemma to the following variational form

$$\forall v \in H_0^{1,\text{per}}(Q \setminus \overline{\mathcal{O}_0^{\text{per}}}), \quad \int_{Q \setminus \overline{\mathcal{O}_0^{\text{per}}}} \nabla w^{\text{per}} \cdot \nabla v = \int_{Q \setminus \overline{\mathcal{O}_0^{\text{per}}}} v.$$

The following convergence result is proved in [77, Theorem 3.1] (take $m = 2$ there).

Theorem 2.1.3 (Consequence of Theorem 3.1 of [77]). *Assume that $\mathcal{O}_0^{\text{per}}$ is an open subset of Q such that $\mathcal{O}_0^{\text{per}} \subset\subset Q$. Let Ω_ε be defined by (2.1.6). Let $f \in \mathcal{D}(\Omega)$ and u_ε be the solution to (2.1.1). Then there exists a constant $C > 0$ independent of ε such that*

$$\varepsilon^{-1} \|u_\varepsilon - \varepsilon^2 w^{\text{per}}(\cdot/\varepsilon) f\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon - \varepsilon^2 w^{\text{per}}(\cdot/\varepsilon) f\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^2, \quad (2.1.11)$$

where w^{per} is the unique solution to (2.1.10).

Remark 2.1.4. *Since $\mathcal{O}_0^{\text{per}}$ is of class $C^{1,\gamma}$ for some $0 < \gamma < 1$, then Theorem 2.1.3 still holds true under the weaker hypotheses $f \in H^2 \cap L^\infty(\Omega)$ and $f|_{\partial\Omega} = 0$ (in the trace sense), see [75, Appendix A.2]. If we do not assume that f vanishes on $\partial\Omega$, $u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f$ does not vanish on $\partial\Omega$ either and we have the weaker estimate*

$$\|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^{3/2} \mathcal{N}(f),$$

where $\mathcal{N}(f) = \|f\|_{L^\infty} + \|\nabla f\|_{L^2} + \|\Delta f\|_{L^2}$.

2.1.2 The non-periodic case

We aim at extending the previous results to non-periodically perforated medium, in the special case of local perturbations of the periodic structure. More precisely, we define a reference periodic configuration by (2.1.2)–(2.1.3)–(2.1.6) and, for each $k \in \mathbb{Z}^d$, we denote by \mathcal{O}_k the (non-periodic) perforation in the cell Q_k . We recall that $\mathcal{O}_0^{\text{per}}$ is of class $C^{1,\gamma}$ for some $0 < \gamma < 1$. Our first assumption reads as follows:

(A1) For all $k \in \mathbb{Z}^d$ such that $\mathcal{O}_k \neq \emptyset$, \mathcal{O}_k is a locally Lipschitz open set such that $\mathcal{O}_k \subset\subset Q_k$ and $Q_k \setminus \overline{\mathcal{O}_k}$ is connected.

In [22], it is assumed for simplicity that each periodic cell contains a non-periodic perforation. In this chapter, we slightly adapt some proofs to cover the case $\mathcal{O}_k = \emptyset$ for a finite number of $k \in \mathbb{Z}^d$. We next introduce geometric tools. For $\alpha > 0$, define the Minkowski content of $\partial\mathcal{O}_0^{\text{per}}$ (i.e a widened boundary of $\mathcal{O}_0^{\text{per}}$) by the set

$$\mathcal{U}_0^{\text{per}}(\alpha) := \{x \in \mathbb{R}^d \text{ s.t. } \text{dist}(x, \partial\mathcal{O}_0^{\text{per}}) < \alpha\}.$$

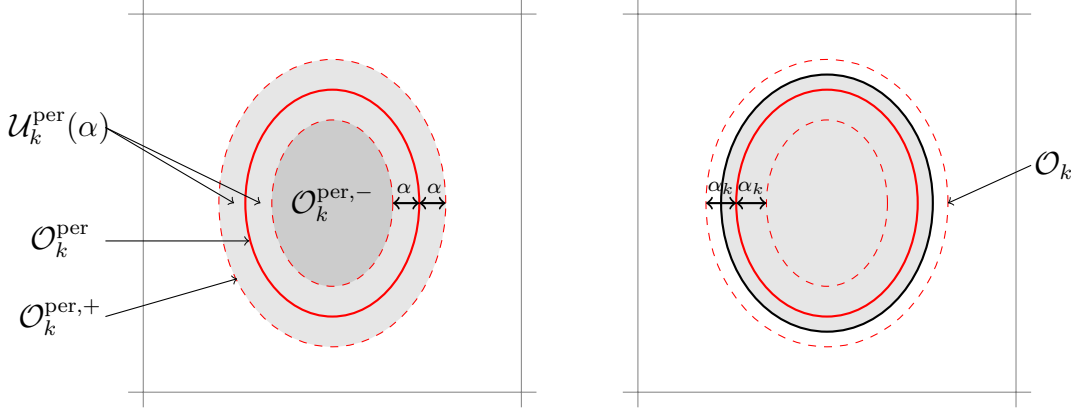


Figure 2.2: On the left, illustration of $\mathcal{O}_k^{\text{per}}$ (red), its widened boundary $\mathcal{U}_k^{\text{per}}$, its enlargement $\mathcal{O}_k^{\text{per},+}$ and its reduction $\mathcal{O}_k^{\text{per},-}$ (grey). On the right, \mathcal{O}_k .

Similarly, if $k \in \mathbb{Z}^d$ and $\alpha > 0$, denote the set

$$\mathcal{U}_k^{\text{per}}(\alpha) := \{x \in \mathbb{R}^d \text{ s.t. } \text{dist}(x, \partial\mathcal{O}_k^{\text{per}}) < \alpha\} = \mathcal{U}_0^{\text{per}}(\alpha) + k.$$

Now (see Figure 2.2 left), we define the reduction and the enlargement of $\mathcal{O}_k^{\text{per}}$ by

$$\mathcal{O}_k^{\text{per},-}(\alpha) := \mathcal{O}_k^{\text{per}} \setminus \overline{\mathcal{U}_k^{\text{per}}(\alpha)} \quad \text{and} \quad \mathcal{O}_k^{\text{per},+}(\alpha) := \mathcal{O}_k^{\text{per}} \cup \mathcal{U}_k^{\text{per}}(\alpha).$$

The following properties hold true:

$$\mathcal{O}_k^{\text{per},-}(\alpha) \subset \mathcal{O}_k^{\text{per},+}(\alpha) \quad \text{and} \quad \mathcal{U}_k^{\text{per}}(\alpha) = \mathcal{O}_k^{\text{per},+}(\alpha) \setminus \overline{\mathcal{O}_k^{\text{per},-}(\alpha)}.$$

Finally, we clearly have

$$\mathcal{O}_k^{\text{per},+}(\alpha) = \{x \in \mathbb{R}^d \text{ s.t. } \text{dist}(x, \mathcal{O}_k^{\text{per}}) < \alpha\}, \quad (2.1.12)$$

and

$$\mathcal{O}_k^{\text{per},-}(\alpha) = \{x \in \mathcal{O}_k^{\text{per}} \text{ s.t. } \text{dist}(x, \partial\mathcal{O}_k^{\text{per}}) > \alpha\}. \quad (2.1.13)$$

Assumption **(A2)** reads :

(A2) There exists a sequence $(\alpha_k)_{k \in \mathbb{Z}^d}$ such that $\alpha_k > 0$, $(\alpha_k)_{k \in \mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$ and

$$\forall k \in \mathbb{Z}^d, \mathcal{O}_k^{\text{per},-}(\alpha_k) \subset \mathcal{O}_k \subset \mathcal{O}_k^{\text{per},+}(\alpha_k) \quad (2.1.14)$$

i.e. \mathcal{O}_k is between the enlargement and the reduction of $\mathcal{O}_k^{\text{per}}$.

Note that if α_k is sufficiently large, $\mathcal{O}_k^{\text{per},-}(\alpha_k) = \emptyset$ and $\mathcal{O}_k \subset \mathcal{O}_k^{\text{per},+}(\alpha_k)$. Thus, there is potentially no restriction on a finite number of perforations \mathcal{O}_k . Figure 2.2 (right) explains Assumption **(A2)**. We define

$$\mathcal{O} := \bigcup_{k \in \mathbb{Z}^d} \mathcal{O}_k. \quad (2.1.15)$$

We split the domain $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ into two subdomains:

$$\mathbb{R}^d \setminus (\overline{\mathcal{O}} \cup \overline{\mathcal{O}^{\text{per}}}) \quad \text{and} \quad \mathcal{O}^{\text{per}} \setminus \overline{\mathcal{O}}.$$

Note that these domains are not necessarily connected. We split the boundary of the domain $\mathcal{O}^{\text{per}} \setminus \mathcal{O}$ into two parts (the one surrounding \mathcal{O}^{per} and the one surrounding \mathcal{O}). For $k \in \mathbb{Z}^d$, we define

$$\Gamma_1^k = \partial\mathcal{O}_k^{\text{per}} \setminus \overline{\mathcal{O}_k} \quad \text{and} \quad \Gamma_2^k = \partial\mathcal{O}_k \cap \mathcal{O}_k^{\text{per}} \quad \text{s.t.} \quad \partial(\mathcal{O}_k^{\text{per}} \setminus \overline{\mathcal{O}_k}) = \Gamma_1^k \cup \Gamma_2^k. \quad (2.1.16)$$

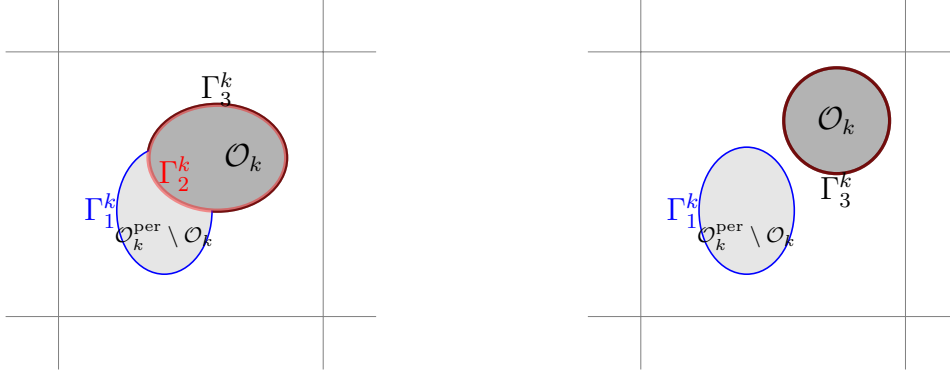


Figure 2.3: Pictures of perforated cells divided into two subdomains (white and light grey) with boundary $\Gamma_i, i = 1, 2, 3$. Left: $\mathcal{O}_k \cap \mathcal{O}_k^{\text{per}} \neq \emptyset$. Right: $\mathcal{O}_k \cap \mathcal{O}_k^{\text{per}} = \emptyset$

We denote by Γ_1 (resp. Γ_2) the union of the Γ_1^k (resp. Γ_2^k), $k \in \mathbb{Z}^d$:

$$\Gamma_1 = \bigcup_{k \in \mathbb{Z}^d} \Gamma_1^k \quad \text{and} \quad \Gamma_2 = \bigcup_{k \in \mathbb{Z}^d} \Gamma_2^k. \quad (2.1.17)$$

We also split the boundary of $\mathbb{R}^d \setminus \overline{\mathcal{O} \cup \mathcal{O}^{\text{per}}}$ into two parts. Write $\partial(\mathbb{R}^d \setminus \overline{\mathcal{O} \cup \mathcal{O}^{\text{per}}}) = \partial(\mathcal{O} \cup \mathcal{O}^{\text{per}})$, and define for $k \in \mathbb{Z}^d$

$$\Gamma_3^k = \partial\mathcal{O}_k \setminus \overline{\mathcal{O}_k^{\text{per}}} \quad \text{s.t.} \quad \partial(\mathcal{O} \cup \mathcal{O}^{\text{per}}) = \Gamma_1^k \cup \Gamma_3^k. \quad (2.1.18)$$

Note that $\partial\mathcal{O}_k = \Gamma_2^k \cup \Gamma_3^k$. Γ_3 denotes the union of the Γ_3^k over $k \in \mathbb{Z}^d$. Figure 2.3 explains the above definitions. From Assumption **(A2)**, we deduce Lemmas 2.5.1, 2.5.2 and 2.5.3, which are stated and proved in Appendix 2.5.1.

Remark 2.1.5. Assumption **(A2)** is a way to impose that the defect is localized. In [19, 20, 18], such an assumption is written as $a = a^{\text{per}} + \tilde{a}$, with $\tilde{a} \in L^q(\mathbb{R}^d)$, and a^{per} is periodic, where a is the coefficient of the considered elliptic equation. Here, writing a similar condition, we impose that the characteristic function of the perforations is a perturbation (i.e., a function in $L^q(\mathbb{R}^d)$) of the periodic case. For a characteristic function, being in $L^q(\mathbb{R}^d)$ is equivalent to being in $L^1(\mathbb{R}^d)$, hence the condition **(A2)**.

We will sometimes require the following uniform regularity assumption on the non-periodic perforations $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$:

(A3) There exists $\gamma \in (0, 1)$, $r > 0$ and $M > 0$ such that for all $k \in \mathbb{Z}^d$ and $x \in \partial\mathcal{O}_k$, there exists an open set $U_x \subset \mathbb{R}^{d-1}$, $0 \in U_x$ and a $\mathcal{C}^{1,\gamma}$ application $\Psi_x : U_x \rightarrow \mathbb{R}$ such that, after eventually translating and/or rotating the coordinate system, we have that

$$(Q_k \setminus \overline{\mathcal{O}_k}) \cap B(x, r) = \{x \in B(x, r), \quad x_d > \Psi_x(x_1, \dots, x_{d-1}), \quad (x_1, \dots, x_{d-1}) \in U_x\},$$

$$\partial\mathcal{O}_k \cap B(x, r) = \{x \in B(x, r), \quad x_d = \Psi_x(x_1, \dots, x_{d-1}), \quad (x_1, \dots, x_{d-1}) \in U_x\},$$

and $\|\Psi_x\|_{\mathcal{C}^{1,\gamma}(U_x)} \leq M$.

In the above Assumption **(A3)**, we will, if necessary, replace $\mathcal{C}^{1,\gamma}$ by any $\mathcal{C}^\ell, \mathcal{C}^{\ell,\gamma}$, $\ell \in \mathbb{N}$, $\gamma \in (0, 1)$. In this case, we say that the perforations \mathcal{O}_k are of class $\mathcal{C}^\ell, \mathcal{C}^{\ell,\gamma}$ uniformly in k .

2.2 Results

In order to state our main result, we first need to prove that a corrector exists:

Theorem 2.2.1 (Existence and uniqueness of the corrector). *Let $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ be a sequence of open sets satisfying Assumptions (A1)-(A2). Let \mathcal{O} be defined by (2.1.15), and*

$$g = \mathbf{1}_{\mathbb{R}^d \setminus \overline{\mathcal{O}^{\text{per}}}} + \tilde{g}, \quad \text{with } \tilde{g} \in L^2(\mathbb{R}^d).$$

There exists a unique $\tilde{w} \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ such that $w := w^{\text{per}} + \tilde{w} \in H^{1,\text{per}}(Q) + H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ is solution in the sense of distributions to

$$\begin{cases} -\Delta w = g \text{ in } \mathbb{R}^d \setminus \overline{\mathcal{O}} \\ w|_{\partial \mathcal{O}} = 0, \end{cases} \quad (2.2.1)$$

where $w^{\text{per}} \in H_0^{1,\text{per}}(Q \setminus \overline{\mathcal{O}_0^{\text{per}}})$ is the unique solution of the periodic corrector problem (2.1.10) extended by zero to \mathbb{R}^d .

Using Theorem 2.2.1 and a two-scale expansion, as it is done in the periodic case, we have the following result, which is the generalization of Theorem 2.1.3 to the present setting:

Theorem 2.2.2 (Convergence theorem in H^1 -norm). *Let $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ be a sequence of open sets satisfying Assumptions (A1)-(A2) and assume that \mathcal{O} is defined by (2.1.15). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and define for $\varepsilon > 0$ the perforated set $\Omega_\varepsilon := \Omega \setminus \varepsilon \overline{\mathcal{O}}$. Let $f \in \mathcal{D}(\Omega)$ and u_ε be the solution of Problem (2.1.1). Then there exists a constant $C > 0$ independent of ε such that*

$$\varepsilon^{-1} \|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^2, \quad (2.2.2)$$

where $w = w^{\text{per}} + \tilde{w} \in H^{1,\text{per}}(Q) + H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ is the unique solution of the corrector Problem (2.2.1) with $g = 1$.

We note that the constant C appearing in Theorem 2.2.2 is independent of ε but depends on f , on the non-periodic corrector constructed in Theorem 2.2.1 and on the Poincaré-Friedrichs constant of Ω_ε (denoted C in Lemma 2.3.2 below). Theorem 2.2.2 provides an error estimate of $u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f$ in $H^1(\Omega_\varepsilon)$ -norm. However, for this choice of norm, the use of a non-periodic corrector appears to be irrelevant, which means that we could also have used the periodic corrector w^{per} in (2.2.2) without changing the rate of convergence. Indeed, we have

$$\left\| \varepsilon^2 \tilde{w} \left(\frac{\cdot}{\varepsilon} \right) f \right\|_{H^1(\Omega_\varepsilon)} = O(\varepsilon^2). \quad (2.2.3)$$

In order to prove (2.2.3), we only deal with the leading order term of the above quantity, that is, the L^2 -norm of the gradient. One has

$$\begin{aligned} \int_{\Omega_\varepsilon} \left| \nabla \left[\varepsilon^2 \tilde{w} \left(\frac{\cdot}{\varepsilon} \right) f \right] (x) \right|^2 dx &\leq 2\varepsilon^2 \int_{\Omega_\varepsilon} \left| \nabla \tilde{w} \left(\frac{x}{\varepsilon} \right) \right|^2 |f(x)|^2 dx + 2\varepsilon^4 \int_{\Omega_\varepsilon} \tilde{w} \left(\frac{x}{\varepsilon} \right)^2 |\nabla f(x)|^2 dx \\ &\leq C\varepsilon^2 \int_{\Omega_\varepsilon} \left| \nabla \tilde{w} \left(\frac{x}{\varepsilon} \right) \right|^2 dx + C\varepsilon^4 \int_{\Omega_\varepsilon} \left| \tilde{w} \left(\frac{x}{\varepsilon} \right) \right|^2 dx. \end{aligned}$$

Thus, after the change of variable $y = x/\varepsilon$,

$$\int_{\Omega_\varepsilon} \left| \nabla \left[\varepsilon^2 \tilde{w} \left(\frac{\cdot}{\varepsilon} \right) f \right] (x) \right|^2 dx \leq C\varepsilon^{d+2} \int_{\mathbb{R}^d \setminus \mathcal{O}} |\nabla \tilde{w}(y)|^2 dy + C\varepsilon^{d+4} \int_{\mathbb{R}^d \setminus \mathcal{O}} |\tilde{w}(y)|^2 dy,$$

where C depends on the L^∞ -norms of f and ∇f . We thus have (2.2.3), which implies (since $d \geq 2$)

$$\left\| u_\varepsilon - \varepsilon^2 w^{\text{per}} \left(\frac{\cdot}{\varepsilon} \right) f \right\|_{H^1(\Omega_\varepsilon)} \leq \left\| u_\varepsilon - \varepsilon^2 w \left(\frac{\cdot}{\varepsilon} \right) f \right\|_{H^1(\Omega_\varepsilon)} + \left\| \varepsilon^2 \tilde{w} \left(\frac{\cdot}{\varepsilon} \right) f \right\|_{H^1(\Omega_\varepsilon)} = O(\varepsilon^2).$$

Thus, using w^{per} instead of w in convergence Theorem 2.2.2 does not change the order $O(\varepsilon^2)$ of the error.

The following Theorem states that the use of w instead of w^{per} improves the rate of convergence in L^∞ -norm for a non-periodic domain.

Theorem 2.2.3 (Convergence Theorem in L^∞ -norm). *Let $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ be a sequence of open sets satisfying Assumptions (A1)-(A2) and assume that \mathcal{O} is defined by (2.1.15). Assume that the perforations \mathcal{O}_k are uniformly in k of class $C^{1,\gamma}$ in the sense of Assumption (A3). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and define for $\varepsilon > 0$ the perforated set $\Omega_\varepsilon := \Omega \setminus \varepsilon \overline{\mathcal{O}}$. Let $f \in \mathcal{D}(\Omega)$ and u_ε be the solution of (2.1.1). Then there exists a constant $C > 0$ independent of ε such that*

$$\|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon^3,$$

where $w = w^{\text{per}} + \tilde{w} \in H^1_{\text{per}}(Q) + H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ is the unique solution of (2.2.1) with $g = 1_{\mathbb{R}^d \setminus \overline{\mathcal{O}}}$.

Note that $\|\varepsilon^2 \tilde{w}(\cdot/\varepsilon) f\|_{L^\infty(\Omega_\varepsilon)}$ is generally of order ε^2 exactly. We fix $K \subset \Omega$. One has

$$\left\| \left[\varepsilon^2 \tilde{w} \left(\frac{\cdot}{\varepsilon} \right) f \right] (\varepsilon \cdot) \right\|_{L^\infty(K)} \underset{\varepsilon \rightarrow 0}{\sim} \varepsilon^2 f(0) \|\tilde{w}\|_{L^\infty(K)}.$$

Besides, Theorem 2.2.3 implies

$$\left\| \left[u_\varepsilon - \varepsilon^2 w \left(\frac{\cdot}{\varepsilon} \right) f \right] (\varepsilon \cdot) \right\|_{L^\infty(K)} \leq C\varepsilon^3.$$

Thus,

$$\left\| \left[u_\varepsilon - \varepsilon^2 w^{\text{per}} \left(\frac{\cdot}{\varepsilon} \right) f \right] (\varepsilon \cdot) \right\|_{L^\infty(K)} \underset{\varepsilon \rightarrow 0}{\sim} C\varepsilon^2. \quad (2.2.4)$$

We have the same estimate as (2.2.4) when the $L^\infty(K)$ -norm is replaced by the $L^2(K)$ -norm. This proves that the convergence of $u_\varepsilon/\varepsilon^2 - w(\cdot/\varepsilon)f$ holds at the microscale in L^2 -norm when we use w . This is not the case when we use the periodic corrector w^{per} .

Remark 2.2.4. *This Remark is analogous to Remark 2.1.4 in the present non-periodic setting. The condition $f \in \mathcal{D}(\Omega)$ can be weakened in Theorem 2.2.2 provided that we use Lemma 2.4.11 proved below. Under Hölder regularity conditions on the perforations (in the sense of Assumption (A3)), one has thanks to Lemma 2.4.11 that $w \in L^\infty(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ and $\nabla w \in L^\infty(\mathbb{R}^d \setminus \overline{\mathcal{O}})$. Thus, if we assume that $f \in H^2(\Omega)$ and $f|_{\partial\Omega} = 0$ (in the trace sense), we obtain (see (2.4.26)),*

$$\|g_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq 2\|\nabla w\|_{L^\infty} \|\nabla f\|_{L^2} + \|w\|_{L^\infty} \|\Delta f\|_{L^2} \leq C\|f\|_{H^2(\Omega)}$$

for $\varepsilon < 1$. We deduce by integration by parts that $\|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^2$. If f does not vanish on $\partial\Omega$, we can prove that there exists a constant C independent of ε such that

$$\|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^{3/2} \mathcal{N}(f).$$

The proof is analogous to [75, Appendix A.2] provided we use Lemma 2.4.11 below. This requires $f \in H^2 \cap L^\infty(\Omega)$.

2.3 Poincaré-Friedrichs inequalities

The main ingredient of the proof of Theorem 2.2.1 is the following Poincaré-type inequality.

Theorem 2.3.1. *Let Q be the unit cube of \mathbb{R}^d and let U be an open subset of Q containing a box $\mathcal{R} = \prod_{i=1}^d [a_i, b_i]$. Then*

$$\forall v \in H^1(Q \setminus \overline{U}) \text{ s.t. } v|_{\partial U} = 0, \quad \int_{Q \setminus \overline{U}} |v|^2 \leq \frac{d}{|\mathcal{R}|} \int_{Q \setminus \overline{U}} |\nabla v|^2. \quad (2.3.1)$$

Similarly,

$$\forall v \in H^1(Q) \text{ s.t. } v|_U = 0, \quad \int_Q |v|^2 \leq \frac{d}{|\mathcal{R}|} \int_Q |\nabla v|^2.$$

An important point in (2.3.1) is that the constant is explicit and depends only on \mathcal{R} . This crucial point will allow us, with the help of Assumption **(A2)**, to prove Lemma 2.3.2 below, in which the fundamental point is that the constant does not depend on ε . We thus have an explicit scaling with respect to ε , similarly to the periodic case. This allows us to adapt the proofs of [77].

Proof. By density, it is enough to show the result for $v \in C^1(Q)$ satisfying $v = 0$ on U . Fix $x \in Q$ and write

$$v(x) - v(\hat{x}) = \int_0^1 \nabla v((1-t)\hat{x} + tx) \cdot (x - \hat{x}) dt,$$

where $\hat{x} = (\frac{a_i+b_i}{2} + x_i(b_i - a_i))_{1 \leq i \leq d} \in \mathcal{R}$. Note that $v(\hat{x}) = 0$ and $|x - \hat{x}|^2 \leq d$. Thus by the Cauchy-Schwarz inequality $|v(x)|^2 \leq d \int_0^1 |\nabla v((1-t)\hat{x} + tx)|^2 dt$. Integrating with respect to $x \in Q$ and exchanging the two integrals yields

$$\int_Q |v(x)|^2 dx \leq d \int_0^1 \left(\int_Q |\nabla v((1-t)\hat{x} + tx)|^2 dx \right) dt$$

Fix $t \in [0, 1]$ and define the diffeomorphism $\phi_t : Q \ni x \mapsto (1-t)\hat{x} + tx$. Note that $\phi_t(Q) \subset Q$ and that $|\det J(\phi_t)| = \prod_{i=1}^d [(1-t)(b_i - a_i) + t] \geq \prod_{i=1}^d (b_i - a_i)$. Thus by a change of variables,

$$\int_Q |\nabla v((1-t)\hat{x} + tx)|^2 dx \leq \frac{1}{\prod_{i=1}^d (b_i - a_i)} \int_Q |\nabla v|^2.$$

Integrating with respect to t concludes the proof. \square

Theorem 2.3.1 and Assumption **(A2)** allow to prove the following, which is a generalization to the present setting of Lemma 2.1.1. Note that the proof of the following Lemma slightly differs from the published version [22] since it is implicitly assumed there that each cell Q_k contains a perforation.

Lemma 2.3.2 (Poincaré-type inequality in $H_0^1(\Omega_\varepsilon)$). *Let $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ be a sequence of open sets such that $\mathcal{O}_k \subset\subset Q_k$. Suppose that the sequence $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ satisfies Assumption **(A2)**. Let Ω be an open subset of \mathbb{R}^d . Define for $\varepsilon > 0$,*

$$\Omega_\varepsilon = \Omega \setminus \varepsilon \overline{\mathcal{O}} = \Omega \cap \left(\bigcup_{k \in \mathbb{Z}^d} \varepsilon(Q_k \setminus \overline{\mathcal{O}_k}) \right).$$

There exists a constant $C > 0$ independent of ε such that

$$\forall u \in H_0^1(\Omega_\varepsilon), \quad \int_{\Omega_\varepsilon} u^2 \leq C \varepsilon^2 \int_{\Omega_\varepsilon} |\nabla u|^2.$$

Proof. We first recall (see Lemma 2.5.3 in the appendix) that the set $\mathcal{K} := \{k \in \mathbb{Z}^d, \mathcal{O}_k \cap \mathcal{O}_k^{\text{per}} = \emptyset\}$ is finite.

- *Step 1:* we define $A := \max_{k \in \mathcal{K}} |k|$. By definition of \mathcal{K} , the set $\mathcal{A} := \bigcup_{|k| \leq A+1} Q_k$ contains a

perforation that we denote by $\widehat{\mathcal{O}}_k$. Thus, by the standard Poincaré-Friederichs inequality, we get the existence of a constant $C_0 = C(\mathcal{A}, \widehat{\mathcal{O}}_k, d)$ such that

$$\forall w \in H_0^1(\mathbb{R}^d \setminus \overline{\mathcal{O}}), \quad \int_{\mathcal{A}} |w|^2 \leq C_0 \int_{\mathcal{A}} |\nabla w|^2. \quad (2.3.2)$$

In particular,

$$\forall k \in \mathcal{K}, \quad \forall w \in H_0^1(\mathbb{R}^d \setminus \overline{\mathcal{O}}), \quad \int_{Q_k \setminus \overline{\mathcal{O}_k}} |w|^2 \leq C_0 \int_{\mathcal{A}} |\nabla w|^2. \quad (2.3.3)$$

- *Step 2:* we show that there exists $\tilde{\rho} > 0$ such that for all $k \in \mathcal{K}^c$, there exists a box $\mathcal{R}_k \subset \mathcal{O}_k$ satisfying $|\mathcal{R}_k| \geq \tilde{\rho}$. Fix $k \in \mathbb{Z}^d$. By Lemma 2.5.2, there exists a ball $B_k \subset \mathcal{O}_k$ such that $|B_k| \geq \rho$ with ρ independent of k . Thus, there exists a box $\mathcal{R}_k \subset \mathcal{O}_k$ such that $|\mathcal{R}_k| \geq C(d)\rho$ where $C(d)$ is a constant depending only on d . We define $\tilde{\rho} := C(d)\rho > 0$. We next use Theorem 2.3.1 to obtain that

$$\forall k \in \mathcal{K}^c, \quad \forall w \in H_0^1(\mathbb{R}^d \setminus \overline{\mathcal{O}}), \quad \int_{Q_k \setminus \overline{\mathcal{O}_k}} w^2 \leq \frac{d}{\tilde{\rho}} \int_{Q_k \setminus \overline{\mathcal{O}_k}} |\nabla w|^2. \quad (2.3.4)$$

Summing (2.3.4) over k , then adding (2.3.3) to the result and defining $C := \max(|\mathcal{K}|C_0, d/\tilde{\rho})$, we obtain

$$\forall w \in H_0^1(\mathbb{R}^d \setminus \overline{\mathcal{O}}), \quad \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} w^2 \leq C \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} |\nabla w|^2. \quad (2.3.5)$$

Now, we fix $u \in H_0^1(\Omega_\varepsilon)$. We extend u by zero to $\mathbb{R}^d \setminus \varepsilon\overline{\mathcal{O}}$ and we define $v := u(\varepsilon \cdot)$. It is clear that $v \in H_0^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ and that

$$\forall y \in \mathbb{R}^d \setminus \overline{\mathcal{O}}, \quad \nabla v(y) = \varepsilon \nabla u(\varepsilon y). \quad (2.3.6)$$

Applying (2.3.5) with $w = v \in H_0^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ and using (2.3.6) yields

$$\int_{\frac{1}{\varepsilon}\Omega_\varepsilon} u^2(\varepsilon y) dy = \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} u^2(\varepsilon y) dy \leq C\varepsilon^2 \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} |\nabla u|^2(\varepsilon y) dy = C\varepsilon^2 \int_{\frac{1}{\varepsilon}\Omega_\varepsilon} |\nabla u|^2(\varepsilon y) dy.$$

Making the change of variables $x = \varepsilon y$ in each integral finally concludes the proof. \square

Remark 2.3.3. We deduce from the previous method of proof the following result that will be useful in Appendix 2.5.2. Let Ω' and Ω'' be two domains such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. We assume that $\Omega_\varepsilon := \Omega \setminus \varepsilon \bigcup_{k \in Y_\varepsilon} \overline{\mathcal{O}_k}$, where $Y_\varepsilon := \{k \in \mathbb{Z}^d, \varepsilon Q_k \subset \Omega\}$. Then there exists a constant $C > 0$ independent of ε such that for all $\varepsilon < \varepsilon_0(\Omega', \Omega'', \mathcal{O})$ small enough and $u \in H_0^1(\Omega_\varepsilon)$,

$$\|u\|_{L^2(\Omega_\varepsilon \cap \Omega')} \leq C\varepsilon \|\nabla u\|_{L^2(\Omega_\varepsilon \cap \Omega'')}. \quad (2.3.7)$$

We briefly prove (2.3.7). We choose $\varepsilon < \varepsilon_0(\Omega', \Omega'', A)$ so that

$$\bigcup_{k \in Y'_\varepsilon} \varepsilon \overline{Q_k} \subset \Omega'', \quad Y'_\varepsilon := \{k \in \mathbb{Z}^d, \varepsilon Q_k \cap \Omega' \neq \emptyset\} \quad \text{and} \quad \varepsilon \bigcup_{|k| \leq A+1} \overline{Q_k} (= \varepsilon \mathcal{A}) \subset \Omega''. \quad (2.3.8)$$

Let $u \in H_0^1(\Omega_\varepsilon)$. We extend u by zero in $\mathbb{R}^d \setminus \varepsilon\overline{\mathcal{O}}$. By (2.3.3) and a scaling argument, we have that

$$\forall k \in Y'_\varepsilon \cap \mathcal{K}, \quad \int_{\varepsilon(Q_k \setminus \overline{\mathcal{O}_k})} u^2 \leq C_0 \varepsilon^2 \int_{\varepsilon \mathcal{A}} |\nabla u|^2. \quad (2.3.9)$$

Similarly, by (2.3.4) and a scaling argument, we have that

$$\forall k \in Y'_\varepsilon \cap \mathcal{K}^c, \quad \int_{\varepsilon(Q_k \setminus \overline{\mathcal{O}_k})} u^2 \leq \frac{d}{\tilde{\rho}} \varepsilon^2 \int_{\varepsilon Q_k} |\nabla u|^2. \quad (2.3.10)$$

We sum (2.3.9) over $k \in Y'_\varepsilon \cap \mathcal{K}$ and (2.3.10) over $k \in Y'_\varepsilon \cap \mathcal{K}^c$. Adding the results and using (2.3.8) gives

$$\int_{\Omega_\varepsilon \cap \Omega'} u^2 \leq C\varepsilon^2 \int_{\Omega_\varepsilon \cap \Omega''} |\nabla u|^2. \quad (2.3.11)$$

This proves (2.3.7).

2.4 Proofs

2.4.1 Two-scale expansion

The aim of this section is to find an asymptotic equivalent of u_ε as ε goes to zero. We begin by the two scale expansion of u_ε . Write

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 u_3\left(x, \frac{x}{\varepsilon}\right) + \dots,$$

where the functions u_i are now defined on $\Omega \times (\mathbb{R}^d \setminus \overline{\mathcal{O}})$ and are of the form $u_i^{\text{per}} + \tilde{u}_i$. Suppose that $\tilde{u}_i(x, \cdot) \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ and use the u_i^{per} 's defined in Section 2.1.1 and extended by zero to \mathbb{R}^d . Because of the homogeneous Dirichlet boundary conditions on u_ε , we impose that $u_i(x, y) = 0$ for $y \in \partial\mathcal{O}$ and any $x \in \Omega$. The calculations leading to (2.1.9) (see [77, Section 2]) are still valid, so we have:

$$\begin{cases} -\Delta_y u_0 = 0 \\ -\Delta_y u_1 - 2(\nabla_x \cdot \nabla_y)u_0 = 0 \\ -\Delta_y u_2 - 2(\nabla_x \cdot \nabla_y)u_1 - \Delta_x u_0 = f \\ -\Delta_y u_3 - 2(\nabla_x \cdot \nabla_y)u_2 - \Delta_x u_1 = 0 \\ \dots \end{cases}, \quad (2.4.1)$$

where all these equations are posed on $\Omega \times (\mathbb{R}^d \setminus \overline{\mathcal{O}})$. These equations imply that u_0 and u_1 are constantly equal to zero. Indeed, fix $x \in \Omega$. Since $u_0^{\text{per}} \equiv 0$, we get that $\tilde{u}_0(x, \cdot)$ satisfies the PDE

$$\begin{cases} -\Delta_y \tilde{u}_0 = 0 \text{ in } \mathbb{R}^d \setminus \overline{\mathcal{O}}, \\ \tilde{u}_0|_{\partial\mathcal{O}} = 0. \end{cases}$$

Multiplying by $\tilde{u}_0(x, \cdot) \in H_0^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ and integrating by parts² yields $\tilde{u}_0(x, \cdot) \equiv 0$. Thus $u_0 \equiv 0$. Similarly, $u_1 \equiv 0$. We are now left with the following equation on u_2 :

$$\begin{cases} -\Delta_y u_2(x, y) = f(x) \text{ in } \mathbb{R}^d \setminus \overline{\mathcal{O}} \\ u_2(x, y) = 0, \quad x \in \Omega, y \in \partial\mathcal{O}. \end{cases} \quad (2.4.2)$$

According to (2.4.2), $u_2(x, y) = f(x)w(y)$, where w is a solution to the corrector equation (2.2.1) with $g \equiv 1$. This is why we introduced the corrector equation.

2.4.2 Proof of the existence of a corrector

The aim of this section is to prove Theorem 2.2.1. The difficulty of this theorem is that equation (2.2.1) is posed on an unbounded domain. We search for w in the form $w^{\text{per}} + \tilde{w}$, where we impose that $\tilde{w} \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$. We write the equation on \tilde{w} and prove by energy minimization that there is a solution. The equation we want to solve for \tilde{w} is

$$-\Delta \tilde{w} = \mathbf{1}_{\mathbb{R}^d \setminus \overline{\mathcal{O}^{\text{per}}}} + \tilde{g} + \Delta w^{\text{per}}, \quad (2.4.3)$$

where $\tilde{g} \in L^2(\mathbb{R}^d)$ and $w^{\text{per}} \in H^{1, \text{per}}(Q)$ is the solution to (2.1.10) defined in Section 2.1. We recall that w^{per} is extended by zero in \mathcal{O}^{per} . We impose that $\tilde{w} = -w^{\text{per}}$ on $\partial\mathcal{O}$. It is worth noticing that $w^{\text{per}} \notin H^2(Q)$ and thus the right-hand side of (2.4.3) cannot be in $L^2(\mathbb{R}^d \setminus \overline{\mathcal{O}})$. Thus the linear form of the weak formulation of (2.4.3) is not of the form $v \mapsto \int f v$. In fact, we will have to deal with boundary terms along $\partial\mathcal{O}^{\text{per}}$. These terms express the fact that Δw_{per} is a Dirac measure on $\partial\mathcal{O}^{\text{per}}$ (or that w^{per} has normal derivative jumps along $\partial\mathcal{O}^{\text{per}}$). In this section, we use that, since $\mathcal{O}_0^{\text{per}}$ is of class $\mathcal{C}^{1, \gamma}$, then [60, Theorem 8.34] gives that $w^{\text{per}} \in \mathcal{C}^{1, \gamma}(\mathbb{R}^d \setminus \overline{\mathcal{O}^{\text{per}}})$.

²Rigorously, we should first take $\phi \in \mathcal{D}(\mathbb{R}^d \setminus \overline{\mathcal{O}})$, multiply and integrate by parts, and then approximate $\tilde{u}_0(x, \cdot)$ in the H^1 -norm by a sequence $(\phi_n)_{n \in \mathbb{N}}$ of such functions.

Notation. We denote by $\frac{\partial u}{\partial n}\big|_{\text{ext}}$ (resp. $\frac{\partial u}{\partial n}\big|_{\text{int}}$) the exterior normal derivative of u on the outside (resp. inside) of a piecewise smooth closed surface Γ (when it is defined i.e u is H^2 on each side of the boundary).

Definition 2.4.1. We say that $\tilde{w} \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ is a weak solution of (2.4.3) if

$$\forall v \in C_c^1(\mathbb{R}^d \setminus \overline{\mathcal{O}}), \quad \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} \nabla \tilde{w} \cdot \nabla v + \int_{\Gamma_1} \frac{\partial w^{\text{per}}}{\partial n} \bigg|_{\text{ext}} v - \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} \tilde{g} v = 0 \quad (2.4.4)$$

and $\tilde{w}|_{\partial \mathcal{O}} = -w_{\text{per}}$ in the trace sense.

Note that, in the above Definition 2.4.1, we can replace Γ_1 by $\partial \mathcal{O}^{\text{per}}$ after extension of v by zero in \mathcal{O} . We check easily that \tilde{w} is solution to (2.4.3) in the sense of Definition 2.4.1 if and only if $w := w^{\text{per}} + \tilde{w}$ is solution in the distribution sense to (2.2.1).

Remark 2.4.2. We could also have written equation (2.4.3) as a system of PDEs coupled by transmission conditions:

$$\begin{cases} -\Delta \tilde{w} = \tilde{g} & \text{in } \mathbb{R}^d \setminus (\overline{\mathcal{O}} \cup \overline{\mathcal{O}^{\text{per}}}) \\ -\Delta \tilde{w} = \tilde{g} & \text{in } \mathcal{O}^{\text{per}} \setminus \overline{\mathcal{O}} \\ \tilde{w} = -w^{\text{per}} & \text{on } \Gamma_2 \cup \Gamma_3 \\ \frac{\partial \tilde{w}}{\partial n} \bigg|_{\text{ext}} + \frac{\partial \tilde{w}}{\partial n} \bigg|_{\text{int}} = \frac{\partial w^{\text{per}}}{\partial n} \bigg|_{\text{ext}} & \text{on } \Gamma_1. \end{cases} \quad (2.4.5)$$

The three first equations are obviously necessary. The last equation is necessary to guarantee that $w = w^{\text{per}} + \tilde{w} \in H_{\text{loc}}^2(\mathbb{R}^d \setminus \overline{\mathcal{O}})$.

Using standard tools of the calculus of variations, one easily proves the following:

Lemma 2.4.3. Assume that $\tilde{w} \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$. It is a weak solution of (2.4.3) in the sense of Definition 2.4.1, if and only if it is a solution to the following minimization problem:

$$\inf_{\tilde{w} \in V} \left\{ \frac{1}{2} \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} |\nabla \tilde{w}|^2 + \int_{\Gamma_1} \frac{\partial w^{\text{per}}}{\partial n} \bigg|_{\text{ext}} \tilde{w} - \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} \tilde{g} \tilde{w} \right\}, \quad (2.4.6)$$

where the minimization space V is defined by

$$V := \{ \tilde{w} \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}}) \text{ s.t. } \tilde{w}|_{\partial \mathcal{O}} = -w^{\text{per}} \}. \quad (2.4.7)$$

Definition 2.4.4. Let $\tilde{w} \in V$. We denote by \widetilde{W} its extension to \mathbb{R}^d defined by $\widetilde{W} = -w^{\text{per}}$ in \mathcal{O} .

The extension \widetilde{W} of \tilde{w} satisfies $\widetilde{W} \in H^1(\mathbb{R}^d)$ under Assumptions **(A1)**-**(A2)** on the sequence $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$. Figure 2.4 shows a function $\tilde{w} \in V$ (extended to \mathcal{O} by $-w^{\text{per}}$). In order to study the minimization problem (2.4.6), we will need the following Poincaré type inequality in V .

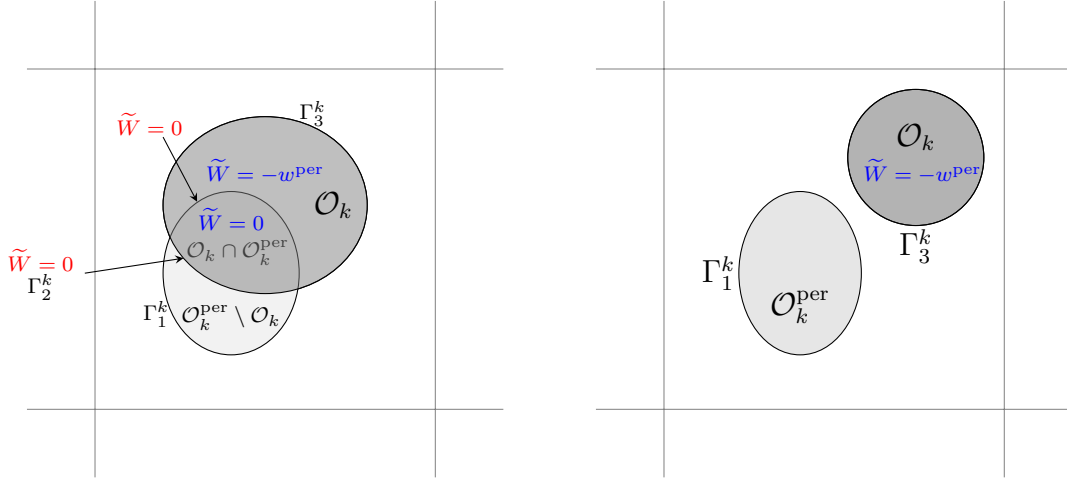
Lemma 2.4.5 (Poincaré-type inequality in V). Let $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ be a sequence of sets satisfying Assumptions **(A1)**-**(A2)**. Define $\mathcal{O} = \bigcup_{k \in \mathbb{Z}^d} \mathcal{O}_k$. Let w^{per} be the periodic corrector solution to (2.1.10).

There exist constants $C_0 > 0$ and $C_1 > 0$ such that for any $\tilde{w} \in V$,

$$\int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} \tilde{w}^2 \leq C_0 \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} |\nabla \tilde{w}|^2 + C_1. \quad (2.4.8)$$

Denoting by \widetilde{W} the extension of \tilde{w} (see Definition 2.4.4), we also have

$$\int_{\mathbb{R}^d} \widetilde{W}^2 \leq C_0 \int_{\mathbb{R}^d} |\nabla \widetilde{W}|^2 + 2C_1. \quad (2.4.9)$$

Figure 2.4: The function \tilde{w} (and its extension \tilde{W}) on a perforated cell with and without overlapping

Proof. The proof is very similar to the proof of Lemma 2.3.2. We fix $\tilde{w} \in V$ and we extend \tilde{w} by $-w^{\text{per}}$ in \mathcal{O} . This gives a function $\tilde{W} \in H^1(\mathbb{R}^d)$. Note that

$$\forall k \in \mathbb{Z}^d, \quad \tilde{W} = 0 \quad \text{in} \quad \mathcal{O}_k \cap \mathcal{O}_k^{\text{per}}. \quad (2.4.10)$$

- *Step 1.* We recall that $A := \max_{k \in \mathcal{K}} |k|$ and $\mathcal{K} := \{k \in \mathbb{Z}^d, \mathcal{O}_k \cap \mathcal{O}_k^{\text{per}} = \emptyset\}$. We define $\mathcal{A} := \bigcup_{|k| \leq A+1} \overline{Q}_k$. There exists $k \in \mathbb{Z}^d$ such that $|k| \leq A+1$ and $\mathcal{O}_k \cap \mathcal{O}_k^{\text{per}} \neq \emptyset$. Thus, the classical Poincaré inequality gives the existence of $C_2 = C(\mathcal{O}, \mathcal{O}^{\text{per}}, \mathcal{A}, d)$ such that

$$\int_{\mathcal{A}} |\tilde{W} + w^{\text{per}}|^2 \leq C_2 \int_{\mathcal{A}} |\nabla \tilde{W} + \nabla w^{\text{per}}|^2.$$

We recall that thanks to elliptic regularity theory, $w^{\text{per}} \in W^{1,\infty}(Q)$. This yields

$$\int_{\mathcal{A}} |\tilde{W}|^2 \leq 2C_2 \int_{\mathcal{A}} |\nabla \tilde{W}|^2 + 2C_2 \|w^{\text{per}}\|_{H^1(\mathcal{A})}^2 \leq 2C_2 \int_{\mathbb{R}^d} |\nabla \tilde{W}|^2 + 2C_2 |\mathcal{A}| \|w^{\text{per}}\|_{W^{1,\infty}(Q)}^2. \quad (2.4.11)$$

Now, the fact that $\tilde{W} = -w^{\text{per}}$ on \mathcal{O} implies

$$\begin{aligned} \int_{\mathcal{A} \setminus \overline{\mathcal{O}}} |\tilde{w}|^2 &\leq 2C_2 \int_{\mathcal{A} \setminus \overline{\mathcal{O}}} |\nabla \tilde{w}|^2 + 2C_2 \int_{\mathcal{A} \cap \overline{\mathcal{O}}} |\nabla w^{\text{per}}|^2 + 2C_2 |\mathcal{A}| \|w^{\text{per}}\|_{W^{1,\infty}(Q)}^2 \\ &\leq 2C_2 \int_{\mathcal{A} \setminus \overline{\mathcal{O}}} |\nabla \tilde{w}|^2 + 4C_2 |\mathcal{A}| \|w^{\text{per}}\|_{W^{1,\infty}(Q)}^2. \end{aligned} \quad (2.4.12)$$

- *Step 2.* We assume that $|k| \geq A$ so that $k \notin \mathcal{K}$. In particular, $\mathcal{O}_k \cap \mathcal{O}_k^{\text{per}} \neq \emptyset$. We use Lemma 2.5.2: there exists a ball $B_k \subset \mathcal{O}_k \cap \mathcal{O}_k^{\text{per}}$ such $|B_k| \geq \rho$ and thus a box $\mathcal{R}_k \subset \mathcal{O}_k \cap \mathcal{O}_k^{\text{per}}$ such that $|\mathcal{R}_k| \geq C(d)\rho$ where $C(d)$ depends only on the dimension. Theorem 2.3.1 gives the existence of a constant $C_3 = C(d)/\rho$ chosen greater than or equal to one such that

$$\int_{Q_k} \tilde{W}^2 \leq C_3 \int_{Q_k} |\nabla \tilde{W}|^2. \quad (2.4.13)$$

Recall that $\int_{\mathcal{O}_k} |\nabla \tilde{W}|^2 \leq \|\nabla w^{\text{per}}\|_{L^\infty(Q)}^2 |\mathcal{O}_k \setminus \mathcal{O}_k^{\text{per}}| \leq \|w^{\text{per}}\|_{W^{1,\infty}(Q)}^2 |\mathcal{O}_k \setminus \mathcal{O}_k^{\text{per}}|$. We thus have

$$\int_{Q_k \setminus \overline{\mathcal{O}_k}} \tilde{w}^2 \leq C_3 \int_{Q_k \setminus \overline{\mathcal{O}_k}} |\nabla \tilde{w}|^2 + C_3 \|w^{\text{per}}\|_{W^{1,\infty}(Q)}^2 |\mathcal{O}_k \setminus \mathcal{O}_k^{\text{per}}|. \quad (2.4.14)$$

Define

$$C_0 := \max(2C_2, C_3) \quad \text{and} \quad C_1 := C_0 \|w^{\text{per}}\|_{W^{1,\infty}(Q)}^2 \left[2|\mathcal{A}| + \sum_{k \in \mathbb{Z}^d} |\mathcal{O}_k \setminus \mathcal{O}_k^{\text{per}}| \right] < +\infty.$$

Summing (2.4.14) over k such that $|k| \geq A$ and adding (2.4.12) to the result, we get

$$\int_{\mathbb{R}^d \setminus \bar{\mathcal{O}}} \tilde{w}^2 \leq C_0 \int_{\mathbb{R}^d \setminus \bar{\mathcal{O}}} |\nabla \tilde{w}|^2 + C_1,$$

which is (2.4.8). The proof of the inequality (2.4.9) is straightforward using (2.4.8). \square

Using Lemma 2.4.5, we prove the following:

Lemma 2.4.6. *Suppose that the sequence $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ satisfies Assumption (A2). Let $\tilde{w} \in V$ and denote by $\tilde{W} \in H^1(\mathbb{R}^d)$ its extension (see Definition 2.4.4). Then, one has the following estimates:*

$$\left| \int_{\Gamma_1} \frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} \tilde{W} \right| \leq C + \frac{1}{4} \|\nabla \tilde{W}\|_{L^2(\mathbb{R}^d)}^2, \quad (2.4.15)$$

where C is a constant independent of \tilde{w} ,

$$\left| \int_{\mathbb{R}^d \setminus \bar{\mathcal{O}}} \tilde{g} \tilde{W} \right| \leq \|\tilde{g}\|_{L^2(\mathbb{R}^d)} \|\tilde{W}\|_{L^2(\mathbb{R}^d)} \quad (2.4.16)$$

and

$$\left| \int_{\mathcal{O}^{\text{per}} \setminus \bar{\mathcal{O}}} \tilde{W} \right| \leq |\mathcal{O}^{\text{per}} \setminus \bar{\mathcal{O}}|^{\frac{1}{2}} \|\tilde{W}\|_{L^2(\mathbb{R}^d)}. \quad (2.4.17)$$

Proof. Fix $\tilde{w} \in V$. Let us first show that $\tilde{W} \in H^1(\mathbb{R}^d)$. We clearly have that $\tilde{W} \in H_{\text{loc}}^1(\mathbb{R}^d)$. We then write

$$\int_{\mathcal{O}} \tilde{W}^2 + \int_{\mathcal{O}} |\nabla \tilde{W}|^2 = \int_{\mathcal{O} \setminus \bar{\mathcal{O}}^{\text{per}}} |w^{\text{per}}|^2 + \int_{\mathcal{O} \setminus \bar{\mathcal{O}}^{\text{per}}} |\nabla w^{\text{per}}|^2 \leq \|w^{\text{per}}\|_{W^{1,\infty}(Q)}^2 \sum_{k \in \mathbb{Z}^d} |\mathcal{O}_k \setminus \mathcal{O}_k^{\text{per}}|. \quad (2.4.18)$$

By Lemma 2.5.1, we conclude that $\tilde{W} \in H^1(\mathcal{O})$. This proves that $\tilde{W} \in H^1(\mathbb{R}^d)$. We next prove estimate (2.4.15). Standard elliptic regularity theory implies $\frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} \in L^\infty(\partial \mathcal{O}^{\text{per}})$. We apply the trace theorem [52, Theorem 1, p 258] for $p = 1$ to the open subset $\mathcal{O}_0^{\text{per}}$ (and thus to $\mathcal{O}_k^{\text{per}}$ by translation invariance with the same constant):

$$\begin{aligned} \left| \int_{\Gamma_1} \frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} \tilde{w} \right| &\leq \sum_{k \in \mathbb{Z}^d} \int_{\Gamma_1^k} \left| \frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} \tilde{w} \right| \leq \sum_{k \in \mathbb{Z}^d} \int_{\partial \mathcal{O}_k^{\text{per}}} \left| \frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} \tilde{W} \right| \\ &\leq \left\| \frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} \right\|_{L^\infty(\partial \mathcal{O}_0^{\text{per}})} \sum_{k \in \mathbb{Z}^d} \int_{\partial \mathcal{O}_k^{\text{per}}} |\tilde{W}| \leq C(w^{\text{per}}, \mathcal{O}^{\text{per}}) \sum_{k \in \mathbb{Z}^d} \left(\int_{\mathcal{O}_k^{\text{per}}} |\tilde{W}| + \int_{\mathcal{O}_k^{\text{per}}} |\nabla \tilde{W}| \right). \end{aligned} \quad (2.4.19)$$

Now, recall that $\tilde{W} = 0$ in $\mathcal{O}_k \cap \mathcal{O}_k^{\text{per}}$, so that using successively the Cauchy-Schwarz inequality and trace continuity (see [52, Theorem 1, p 272] with $p = 2$), we have

$$\left| \int_{\Gamma_1} \frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} \tilde{W} \right| \leq C \sum_{k \in \mathbb{Z}^d} |\mathcal{O}_k^{\text{per}} \setminus \mathcal{O}_k|^{1/2} \left(\|\tilde{W}\|_{L^2(\mathcal{O}_k^{\text{per}})} + \|\nabla \tilde{W}\|_{L^2(\mathcal{O}_k^{\text{per}})} \right).$$

We use the inequality $ab \leq D\frac{a^2}{2} + \frac{b^2}{2D}$ with D to be chosen later:

$$\left| \int_{\Gamma_1} \frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} \widetilde{W} \right| \leq \frac{CD}{2} \sum_{k \in \mathbb{Z}^d} |\mathcal{O}_k^{\text{per}} \setminus \mathcal{O}_k| + \frac{C}{D} \sum_{k \in \mathbb{Z}^d} \left(\|\widetilde{W}\|_{L^2(\mathcal{O}_k^{\text{per}})}^2 + \|\nabla \widetilde{W}\|_{L^2(\mathcal{O}_k^{\text{per}})}^2 \right). \quad (2.4.20)$$

Thus,

$$\left| \int_{\Gamma_1} \frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} \widetilde{W} \right| \leq \frac{CD}{2} \sum_{k \in \mathbb{Z}^d} |\mathcal{O}_k^{\text{per}} \setminus \mathcal{O}_k| + \frac{C}{D} \left(\|\widetilde{W}\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla \widetilde{W}\|_{L^2(\mathbb{R}^d)}^2 \right).$$

Lemma 2.4.5 implies

$$\frac{C}{D} \left(\|\widetilde{W}\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla \widetilde{W}\|_{L^2(\mathbb{R}^d)}^2 \right) \leq \frac{2CC_0}{D} \|\nabla \widetilde{W}\|_{L^2(\mathbb{R}^d)}^2 + \frac{CC_1}{D}.$$

Choosing $D = 8CC_0$ yields finally

$$\left| \int_{\partial \mathcal{O}^{\text{per}}} \frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} \widetilde{W} \right| \leq C \sum_{k \in \mathbb{Z}^d} |\mathcal{O}_k^{\text{per}} \setminus \mathcal{O}_k| + C + \frac{1}{4} \|\nabla \widetilde{W}\|_{L^2(\mathbb{R}^d)}^2, \quad (2.4.21)$$

with C being a constant independent of \widetilde{w} . We infer (2.4.15) thanks to Lemma 2.5.1. The two last estimates (2.4.16) and (2.4.17) are consequences of the Cauchy-Schwarz inequality. \square

Remark 2.4.7. Let $v \in H_0^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$. Computations (2.4.19)-(2.4.20) with \widetilde{w} replaced by v and $D = 1$ are valid and give

$$\left| \int_{\Gamma_1} \frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} v \right| \leq C |\mathcal{O}^{\text{per}} \setminus \overline{\mathcal{O}}| + C \|v\|_{H_0^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})}^2.$$

Thus, the linear form $v \mapsto \int_{\Gamma_1} \frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} v$ is continuous on $H_0^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$.

First, we prove below that the minimization space V is not empty:

Lemma 2.4.8. Let $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ satisfy Assumption (A1) and Assumption (A2). Then V defined by (2.4.7) is not empty.

Proof. We want to build a function $\phi \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ satisfying the boundary conditions $\phi = -w^{\text{per}}$ on $\partial \mathcal{O}$. We will first build ϕ on each cell Q_k . Let $k \in \mathbb{Z}^d$. If $\mathcal{O}_k = \emptyset$, we set $\phi_k(x) = 0$ in Q_k . We next treat the other cases.

Recall that $\delta_0^{\text{per}} = \text{dist}(\mathcal{O}_k^{\text{per}}, \partial Q_k)$ and that δ_0 is defined in Lemma 2.5.3 of the Appendix. Set

$$\varepsilon_k^{\text{per}} := \min(2\alpha_k, \delta_0^{\text{per}}/2) \quad \text{and} \quad \varepsilon_k := \min(\alpha_k, \delta_0/2)$$

and note that since $\alpha_k \xrightarrow{|k| \rightarrow +\infty} 0$, there exists k_0 such that

$$\forall |k| \geq k_0, \quad \varepsilon_k^{\text{per}} = 2\alpha_k \quad \text{and} \quad \varepsilon_k = \alpha_k.$$

Define $\mathcal{U}_k^{\text{per}}(\varepsilon_k^{\text{per}})$ (resp. $\mathcal{U}_k(\varepsilon_k)$) to be the $\varepsilon_k^{\text{per}}$ (resp. ε_k) Minkowski content of $\partial \mathcal{O}_k^{\text{per}}$ (resp. $\partial \mathcal{O}_k$), that is,

$$\mathcal{U}_k^{\text{per}}(\varepsilon_k^{\text{per}}) := \{x \in \mathbb{R}^d \text{ s.t. } \text{dist}(x, \partial \mathcal{O}_k^{\text{per}}) < \varepsilon_k^{\text{per}}\} \subset Q_k$$

and

$$\mathcal{U}_k(\varepsilon_k) := \{x \in \mathbb{R}^d \text{ s.t. } \text{dist}(x, \partial \mathcal{O}_k) < \varepsilon_k\} \subset Q_k.$$

Denote

$$\mathcal{O}_k^{\text{per},+}(\varepsilon_k^{\text{per}}) := \mathcal{O}_k^{\text{per}} \cup \mathcal{U}_k^{\text{per}}(\varepsilon_k^{\text{per}}) = \{x \in \mathbb{R}^d \text{ s.t. } \text{dist}(x, \mathcal{O}_k^{\text{per}}) < \varepsilon_k^{\text{per}}\} \subset Q_k$$

and

$$\mathcal{O}_k^+(\varepsilon_k) := \mathcal{O}_k \cup \mathcal{U}_k(\varepsilon_k) = \{x \in \mathbb{R}^d \text{ s.t. } \text{dist}(x, \mathcal{O}_k) < \varepsilon_k\} \subset Q_k.$$

Now, let $\chi_k \in C_c^\infty(Q_k)$ be a cut-off function satisfying

$$\begin{cases} 0 \leq \chi_k \leq 1 \text{ and } \chi_k \equiv 1 \text{ in } \mathcal{O}_k \\ \text{supp}(\chi_k) \subset \mathcal{O}_k^+, \text{ sup}(\nabla \chi_k) \subset \mathcal{U}_k(\varepsilon_k) \\ |\nabla \chi_k| \leq C/\varepsilon_k. \end{cases}$$

We define $\phi_k := -\chi_k w^{\text{per}}$. It is clear that $\phi_k \in H^1(\mathbb{R}^d)$ and that $\phi_k = -w^{\text{per}}$ on $\partial \mathcal{O}_k$. One defines

$$\phi(x) = \sum_{k \in \mathbb{Z}^d} \phi_k(x) = \sum_{k \in \mathbb{Z}^d} \phi_k(x) 1_{Q_k}(x).$$

Note that since $\text{supp}(\phi_k) \subset Q_k$, all terms but one (which depends on x) vanish in the above sum. Thus $\phi = -w^{\text{per}}$ on $\partial \mathcal{O}$. Our goal is to prove that $\phi \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ to conclude the proof. By Lemma 2.4.5, it is sufficient to show that $\nabla \phi \in L^2(\mathbb{R}^d \setminus \overline{\mathcal{O}})$. Showing this is equivalent to prove that

$$\sum_{k \in \mathbb{Z}^d} \|\nabla \phi_k\|_{L^2(\mathcal{U}_k(\varepsilon_k))}^2 < +\infty.$$

We are thus left to estimate each term $\|\nabla \phi_k\|_{L^2(\mathcal{U}_k(\varepsilon_k))}$ where $k \in \mathbb{Z}^d$. We study these terms only when $|k| \geq k_0$ and $k \notin \mathcal{K}$ where \mathcal{K} is defined in Lemma 2.5.1 of the Appendix (there is only a finite number of terms k such that $k \in \mathcal{K}$ and $|k| < k_0$). Let $k \in \mathbb{Z}^d$ such that $|k| \geq k_0$ and $k \notin \mathcal{K}$ that is $\mathcal{O}_k \cap \mathcal{O}_k^{\text{per}} \neq \emptyset$. One has, using Assumption **(A2)**, the inclusions

$$\mathcal{O}_k \subset \mathcal{O}_k^+(\alpha_k) \subset \mathcal{O}_k^{\text{per},+}(2\alpha_k) \quad \text{and} \quad \mathcal{U}_k(\alpha_k) \subset \mathcal{U}_k^{\text{per}}(2\alpha_k). \quad (2.4.22)$$

We write

$$\begin{aligned} \int_{\mathcal{U}_k(\alpha_k)} |\nabla(\chi_k w^{\text{per}})|^2 &\leq 2 \int_{\mathcal{U}_k(\alpha_k)} |\nabla w^{\text{per}}|^2 |\chi_k|^2 + 2 \int_{\mathcal{U}_k(\alpha_k)} |w^{\text{per}}|^2 |\nabla \chi_k|^2 \\ &\leq 2 \|\nabla w^{\text{per}}\|_{L^\infty(\mathcal{U}_k(\alpha_k))}^2 |\mathcal{U}_k(\alpha_k)| + 2 \|w^{\text{per}}\|_{L^\infty(\mathcal{U}_k(\alpha_k))}^2 \frac{C^2}{\alpha_k^2} |\mathcal{U}_k(\alpha_k)|. \end{aligned}$$

Using that $\nabla w^{\text{per}} \in L^\infty(\mathbb{R}^d)$, $d(\mathcal{U}_k(\alpha_k), \mathcal{O}_k^{\text{per}}) \leq \alpha_k$ and $w^{\text{per}} = 0$ in $\mathcal{O}_k^{\text{per}}$, we infer

$$\|w^{\text{per}}\|_{L^\infty(\mathcal{U}_k(\alpha_k))} \leq 2\alpha_k \|\nabla w^{\text{per}}\|_{L^\infty(Q)}.$$

We conclude that

$$\int_{\mathcal{U}_k(\alpha_k)} |\nabla \phi_k|^2 \leq C |\mathcal{U}_k(\alpha_k)| + C |\mathcal{U}_k(\alpha_k)| \alpha_k^2 / \alpha_k^2 \leq C |\mathcal{U}_k(\alpha_k)|.$$

Using (2.4.22), this yields

$$\int_{\mathcal{U}_k(\alpha_k)} |\nabla \phi_k|^2 \leq C |\mathcal{U}_k^{\text{per}}(2\alpha_k)|.$$

We deduce that for k large enough, $\int_{\mathcal{U}_k(\alpha_k)} |\nabla \phi_k|^2 \leq 2C\alpha_k$ (see (2.5.3)). Since $(\alpha_k)_{k \in \mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$, one concludes that $\phi \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$. \square

Proposition 2.4.9. *Under the assumptions **(A1)** and **(A2)**, the minimization Problem (2.4.6) has a solution.*

Proof. Let $(\widetilde{w}_n)_{n \in \mathbb{N}} \subset V$ be a minimizing sequence of Problem (2.4.6) which exists by Lemma 2.4.8, that is,

$$\frac{1}{2} \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} |\nabla \widetilde{w}_n|^2 + \int_{\Gamma_1} \frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} \widetilde{w}_n - \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} \widetilde{g} \widetilde{w}_n \xrightarrow{n \rightarrow +\infty} \inf_{u \in V} J(u).$$

We extend each \widetilde{w}_n by $-w^{\text{per}}$ in the perforations and denote by \widetilde{W}_n the extension (see Definition 2.4.4). The sequence

$$\frac{1}{2} \int_{\mathbb{R}^d} |\nabla \widetilde{W}_n|^2 + \int_{\Gamma_1} \left. \frac{\partial w^{\text{per}}}{\partial n} \right|_{\text{ext}} \widetilde{W}_n - \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} \widetilde{g} \widetilde{W}_n$$

admits an upper bound independent of n . We first prove that $\|\nabla \widetilde{W}_n\|_{L^2(\mathbb{R}^d)}$ is bounded independently of n . We use Lemma 2.4.5 and Lemma 2.4.6 to bound each term:

$$\begin{aligned} \left| \int_{\Gamma_1} \left. \frac{\partial w^{\text{per}}}{\partial n} \right|_{\text{ext}} \widetilde{W}_n \right| &\leq C + \frac{1}{4} \|\nabla \widetilde{W}_n\|_{L^2(\mathbb{R}^d)}^2, \\ \left| \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} \widetilde{g} \widetilde{W}_n \right| &\leq C \|\widetilde{W}_n\|_{L^2(\mathbb{R}^d)} \stackrel{\text{Lemma 2.4.5}}{\leq} C + C \|\nabla \widetilde{W}_n\|_{L^2(\mathbb{R}^d)}, \end{aligned}$$

where C denotes various constants independent of n . Hence, one gets

$$C \geq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \widetilde{W}_n|^2 + \int_{\partial \mathcal{O}^{\text{per}}} \left. \frac{\partial w^{\text{per}}}{\partial n} \right|_{\text{ext}} \widetilde{W}_n - \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} \widetilde{g} \widetilde{W}_n \geq \frac{1}{4} \|\nabla \widetilde{W}_n\|_{L^2(\mathbb{R}^d)}^2 - C \|\nabla \widetilde{W}_n\|_{L^2(\mathbb{R}^d)} - C,$$

and thus

$$\|\nabla \widetilde{W}_n\|_{L^2(\mathbb{R}^d)}^2 \leq C \|\nabla \widetilde{W}_n\|_{L^2(\mathbb{R}^d)} + C.$$

This proves that $\|\nabla \widetilde{W}_n\|_{L^2(\mathbb{R}^d)}$ is bounded independently of n . With Lemma 2.4.5, one deduces that $\|\widetilde{W}_n\|_{H^1(\mathbb{R}^d)}$ is also bounded independently of n . Thus, by weak compactness, there exists up to the extraction of a subsequence a weak limit $\widetilde{W} \in H^1(\mathbb{R}^d)$ such that

$$\widetilde{W}_n \rightharpoonup_{H^1} \widetilde{W} \quad \text{and} \quad \widetilde{W}_n \rightharpoonup_{L^2_{\text{loc}}} \widetilde{W}.$$

Denote $\widetilde{w} = \widetilde{W}|_{\mathbb{R}^d \setminus \overline{\mathcal{O}}}$. We first show that $\widetilde{w} \in V$. The strong convergence of $(\widetilde{W}_n)_{n \in \mathbb{N}}$ in L^2_{loc} and the equality $\widetilde{W}_n = -w^{\text{per}}$ in \mathcal{O} imply that $\widetilde{W} = -w^{\text{per}}$ in \mathcal{O} . For the boundary $\partial \mathcal{O}_k$, recall that the trace operator T_k (see [52, Theorem 1, p 272]) is weakly continuous from $H^1(\mathcal{O}_k)$ to $L^2(\partial \mathcal{O}_k)$. Thus

$$\widetilde{w}|_{\partial \mathcal{O}_k} = T_k \widetilde{W} = -T_k w^{\text{per}} = -w^{\text{per}}|_{\partial \mathcal{O}_k}.$$

Since this is true for all $k \in \mathbb{Z}^d$, we have proved that $\widetilde{w}|_{\partial \mathcal{O}} = -w^{\text{per}}$. Moreover, $\widetilde{w} \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$. Thus $\widetilde{w} \in V$. We can now pass to the limit $n \rightarrow +\infty$. Since $w \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}}) \mapsto \int |\nabla w|^2$ is convex and continuous (in the strong norm), it is weakly lower semi-continuous and thus

$$\int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} |\nabla \widetilde{w}|^2 \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} |\nabla \widetilde{w}_n|^2. \quad (2.4.23)$$

By weak H^1 -convergence, since $\widetilde{g} \in L^2(\mathbb{R}^d \setminus \overline{\mathcal{O}})$,

$$\int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} \widetilde{g} \widetilde{w}_n \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} \widetilde{g} \widetilde{w}. \quad (2.4.24)$$

Let us treat the remaining term. We first recall (see Remark 2.4.7) that the linear form $v \mapsto \int_{\Gamma_1} \left. \frac{\partial w^{\text{per}}}{\partial n} \right|_{\text{ext}} v$ is strongly and thus weakly continuous on $H^1_0(\mathbb{R}^d \setminus \overline{\mathcal{O}})$. We apply this continuity to $v_n := \widetilde{w}_n - \phi$, where ϕ was defined in the proof of Lemma 2.4.8. We deduce

$$\int_{\Gamma_1} \left. \frac{\partial w^{\text{per}}}{\partial n} \right|_{\text{ext}} \widetilde{w}_n \xrightarrow{n \rightarrow +\infty} \int_{\Gamma_1} \left. \frac{\partial w^{\text{per}}}{\partial n} \right|_{\text{ext}} \widetilde{w}. \quad (2.4.25)$$

Finally, collecting (2.4.23), (2.4.24) and (2.4.25) and letting $n \rightarrow +\infty$, we conclude that

$$\frac{1}{2} \int_{\mathbb{R}^d} |\nabla \tilde{w}|^2 + \int_{\Gamma_1} \frac{\partial w^{\text{per}}}{\partial n} \Big|_{\text{ext}} \tilde{w} - \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} \tilde{g} \tilde{w} \leq \inf_{u \in V} J(u).$$

This finishes the proof of the existence.

To conclude the proof, we prove uniqueness: let \tilde{w}_1 and \tilde{w}_2 be two functions of $H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ such that $w_i := w^{\text{per}} + \tilde{w}_i$, $i = 1, 2$ solve in the distribution sense (2.2.1). In particular, $\tilde{w}_1 - \tilde{w}_2 \in H_0^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ satisfies in the distribution sense $-\Delta(\tilde{w}_1 - \tilde{w}_2) = 0$. Thus,

$$\forall v \in \mathcal{D}(\mathbb{R}^d \setminus \overline{\mathcal{O}}), \quad \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} \nabla(\tilde{w}_1 - \tilde{w}_2) \cdot \nabla v = 0$$

By density of the test functions, we get that

$$\forall v \in H_0^1(\mathbb{R}^d \setminus \overline{\mathcal{O}}), \quad \int_{\mathbb{R}^d \setminus \overline{\mathcal{O}}} \nabla(\tilde{w}_1 - \tilde{w}_2) \cdot \nabla v = 0$$

We may choose $v = \tilde{w}_1 - \tilde{w}_2$ in the previous expression. This gives $\tilde{w}_1 - \tilde{w}_2 = \text{Cst. a.e.}$ and finally $\tilde{w}_1 - \tilde{w}_2 = 0$ by the boundary conditions. \square

Remark 2.4.10. *We could also have applied Lax-Milgram lemma to show that Problem (2.4.3) admits a weak solution. The ingredients are basically the same. Coercivity of the bilinear form is a direct consequence of Lemma 2.3.1 (see (2.3.5)). Continuity is proved using the same method as in the proof of Proposition 2.4.9, when passing to the limit in the minimizing sequence. This approach will be employed to prove the existence of a corrector for Stokes system in Chapter 3.*

2.4.3 Proof of the convergence results

2.4.3.1 H^1 convergence

Proof of Theorem 2.2.2. We first define the first order approximation of u_ε . Let $g = \mathbf{1}_{\mathbb{R}^d \setminus \mathcal{O}}$. With this choice of g , one has

$$\tilde{g} = \mathbf{1}_{\mathcal{O}^{\text{per}} \setminus \mathcal{O}} - \mathbf{1}_{\mathcal{O} \setminus \mathcal{O}^{\text{per}}} = \sum_{k \in \mathbb{Z}^d} \left(\mathbf{1}_{\mathcal{O}_k^{\text{per}} \setminus \mathcal{O}_k} - \mathbf{1}_{\mathcal{O}_k \setminus \mathcal{O}_k^{\text{per}}} \right).$$

Moreover, Lemma 2.5.1 implies that $\tilde{g} \in L^2(\mathbb{R}^d)$. Thus we can apply Theorem 2.2.1 and get the existence of a unique function $\tilde{w} \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ such that $w := w^{\text{per}} + \tilde{w}$ satisfies

$$\begin{cases} -\Delta w = 1 & \text{in } \mathbb{R}^d \setminus \overline{\mathcal{O}} \\ w|_{\partial \mathcal{O}} = 0. \end{cases}$$

in the sense of distribution. Note that $w \in H_{\text{loc}}^1(\mathbb{R}^d \setminus \mathcal{O})$. Now, set

$$\phi_\varepsilon := u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f.$$

Since $f \in \mathcal{D}(\Omega)$, $w = 0$ on $\partial \mathcal{O}$ and $w \in H_{\text{loc}}^1(\mathbb{R}^d \setminus \mathcal{O})$, one gets that $\phi_\varepsilon \in H_0^1(\Omega_\varepsilon)$. We have, in the distribution sense,

$$-\Delta \phi_\varepsilon = f + \Delta w \left(\frac{\cdot}{\varepsilon} \right) f + 2\varepsilon \nabla w \left(\frac{\cdot}{\varepsilon} \right) \cdot \nabla f + \varepsilon^2 w \left(\frac{\cdot}{\varepsilon} \right) \Delta f = f - f + \varepsilon g_\varepsilon = \varepsilon g_\varepsilon, \quad (2.4.26)$$

where

$$g_\varepsilon = 2\nabla w \left(\frac{\cdot}{\varepsilon} \right) \cdot \nabla f + \varepsilon w \left(\frac{\cdot}{\varepsilon} \right) \Delta f.$$

Note that $\|g_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ is bounded independently of ε . Next, we multiply (2.4.26) by ϕ_ε , integrate by parts and apply the Cauchy-Schwarz inequality:

$$\int_{\Omega_\varepsilon} |\nabla \phi_\varepsilon|^2 = \varepsilon \int_{\Omega_\varepsilon} g_\varepsilon \phi_\varepsilon \leq C\varepsilon \left(\int_{\Omega_\varepsilon} \phi_\varepsilon^2 \right)^{1/2}.$$

Thanks to Lemma 2.3.2, one concludes that

$$\left(\int_{\Omega_\varepsilon} |\nabla \phi_\varepsilon|^2 \right)^{1/2} \leq C\varepsilon^2 \quad \text{and} \quad \left(\int_{\Omega_\varepsilon} \phi_\varepsilon^2 \right)^{1/2} \leq C\varepsilon^3,$$

which finishes the proof. \square

2.4.3.2 L^∞ convergence

We first prove the following Lemma:

Lemma 2.4.11. *Let $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ be a sequence of open sets satisfying Assumptions **(A1)**-**(A2)**. Let w be the solution to (2.2.1) with $g = 1$. Then $w \in L^\infty(\mathbb{R}^d \setminus \overline{\mathcal{O}})$. Moreover, if the $C^{1,\gamma}$ norms of the charts that flatten $\partial\mathcal{O}_k$ are uniformly bounded in k , we have that $\nabla w \in L^\infty(\mathbb{R}^d \setminus \overline{\mathcal{O}})$.*

Proof. Let us first prove that $w \in L^\infty(\mathbb{R}^d \setminus \mathcal{O})$. Fix $k \in \mathbb{Z}^d$ and recall that

$$\begin{cases} -\Delta w = 1 & \text{in } Q_k \setminus \overline{\mathcal{O}_k} \\ w|_{\partial\mathcal{O}_k} = 0. \end{cases} \quad (2.4.27)$$

There exists a constant C independent of k such that

$$\|w\|_{L^\infty(\partial Q_k)} \leq C. \quad (2.4.28)$$

Proving (2.4.28) is equivalent to prove that $\|\tilde{w}\|_{L^\infty(\partial Q_k)} \leq C$. Lemma 2.5.3 implies that there exists $\delta > 0$ such that for all $k \in \mathbb{Z}^d$, $\overline{\mathcal{O}_k} \cup \overline{\mathcal{O}_k^{\text{per}}} \subset \prod_{i=1}^d [k_i - \frac{1}{2} + \delta, k_i + \frac{1}{2} - \delta]^d$. By translation invariance and since ∂Q is compact, there exists $x_1, x_2, \dots, x_\ell \in \partial Q$ such that

$$\forall k \in \mathbb{Z}^d, \quad \partial Q_k \subset \bigcup_{i=1}^{\ell} B(x_i + k, \delta/2). \quad (2.4.29)$$

On each ball $B(x_i + k, \delta)$, \tilde{w} satisfies $-\Delta \tilde{w} = 0$. De Giorgi-Nash-Moser Theory (see [110], Theorem 4.22, p. 155) implies that there exists a constant $C = C(d, \delta)$ independent of x_i and k such that

$$\sup_{B(x_i + k, \delta/2)} |\tilde{w}| \leq C(d, \delta) \left(\int_{B(x_i + k, \delta)} |\tilde{w}(x)|^2 dx \right)^{\frac{1}{2}} \leq C \|\tilde{w}\|_{L^2(\mathbb{R}^d \setminus \overline{\mathcal{O}})}. \quad (2.4.30)$$

The inclusion (2.4.29) together with (2.4.30) proves (2.4.28). We now apply the Maximum principle on w for each domain $Q_k \setminus \overline{\mathcal{O}_k}$. Let R be such that $Q_k \subset B(k, R)$. The functions

$$w^+(x) := w(x) + \frac{|x - k|^2}{2d} + \|w\|_{L^\infty(\partial Q_k)} \quad \text{and} \quad w^-(x) = w(x) + \frac{|x - k|^2 - R^2}{2d} - \|w\|_{L^\infty(\partial Q_k)}$$

are respectively supersolution and subsolution of (2.4.27). Thus, thanks to (2.4.28), $\|w\|_{L^\infty(Q_k \setminus \mathcal{O}_k)}$ is bounded independently of k . Hence $w \in L^\infty(\mathbb{R}^d \setminus \overline{\mathcal{O}})$.

For ∇w , we use Hölder Regularity results for the first derivatives. First recall that Assumption **(A1)** implies that $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ is connected. For all $x \in \mathbb{R}^d \setminus \overline{\mathcal{O}}$ such that $\text{dist}(x, \partial\mathcal{O}) > \delta/2$, there exists a ball B_x centered at x such that $\text{dist}(B_x, \partial\mathcal{O}) = \delta/2$. Interior estimates (see [60], Theorem 8.32, p. 210) give the existence of a constant $C = C(\delta, d)$ independent of x such that

$$\|w\|_{C^{1,\gamma}(B_x)} \leq C (\|w\|_{L^\infty(\mathbb{R}^d \setminus \mathcal{O})} + 1) \leq C.$$

We have proved that ∇w is bounded at a distance $\delta/2$ of $\partial\mathcal{O}$. For the proof up to the boundary $\partial\mathcal{O}$, we use Corollary 8.36 p. 212 of [60] with the sets $\Omega_k = \{x \text{ s.t. } \text{dist}(x, \partial\mathcal{O}_k) < \delta\} \setminus \overline{\mathcal{O}_k}$, $\Omega'_k = \{x \text{ s.t. } \text{dist}(x, \partial\mathcal{O}_k) < \delta/2\} \setminus \overline{\mathcal{O}_k}$ and $T_k = \partial\mathcal{O}_k$. We have $d' = \delta/2$ which is independent of k and thus

$$\|w\|_{C^{1,\gamma}(\Omega'_k)} \leq C(T_k, \delta, d) (\|w\|_{L^\infty(\mathbb{R}^d \setminus \mathcal{O})} + 1)$$

where the dependence on T_k appears through the $C^{1,\gamma}$ -norms of the charts that flatten T_k (see [60], p.210). By hypothesis, we get that $C(T_k) \leq C_0$. This concludes the proof. \square

Proof of Theorem 2.2.3. Fix $\varepsilon > 0$ and define $v_\varepsilon = u_\varepsilon(\varepsilon \cdot)$. Then $v_\varepsilon \in H_0^1(\frac{1}{\varepsilon}\Omega_\varepsilon)$ and satisfies

$$\begin{cases} -\Delta v_\varepsilon = \varepsilon^2 f(\varepsilon \cdot) \text{ in } \frac{1}{\varepsilon}\Omega_\varepsilon \\ v_\varepsilon = 0 \text{ on } \partial\left(\frac{1}{\varepsilon}\Omega_\varepsilon\right). \end{cases} \quad (2.4.31)$$

Define $\psi_\varepsilon := v_\varepsilon - \varepsilon^2 w f(\varepsilon \cdot) \in H_0^1(\frac{1}{\varepsilon}\Omega_\varepsilon)$ and note that

$$-\Delta \psi_\varepsilon = \varepsilon^3 [2\nabla w \cdot \nabla f(\varepsilon \cdot) + \varepsilon w \Delta f(\varepsilon \cdot)] =: \varepsilon^3 h_\varepsilon.$$

Lemma 2.4.11 and the fact that $f \in \mathcal{D}(\Omega)$ imply that $\|h_\varepsilon\|_{L^\infty(\frac{1}{\varepsilon}\Omega_\varepsilon)} \leq C$ for all $0 < \varepsilon < 1$. Define

$$\psi_\varepsilon^+ = \psi_\varepsilon + \varepsilon^3 \|h_\varepsilon\|_{L^\infty(\frac{1}{\varepsilon}\Omega_\varepsilon)} (w + \|w\|_{L^\infty}).$$

Then ψ_ε^+ is a supersolution of Problem (2.4.31). Thus, by the weak maximum principle (see [60] Theorem 8.1, p.179), one gets that $\psi_\varepsilon^+ \geq 0$ on $\frac{1}{\varepsilon}\Omega_\varepsilon$. Similarly,

$$\psi_\varepsilon^- = \psi_\varepsilon - \varepsilon^3 \|h_\varepsilon\|_{L^\infty(\frac{1}{\varepsilon}\Omega_\varepsilon)} (w + \|w\|_{L^\infty})$$

is a subsolution of (2.4.31) and thus $\psi_\varepsilon^- \leq 0$ on $\frac{1}{\varepsilon}\Omega_\varepsilon$. Finally,

$$-\varepsilon^3 \|h_\varepsilon\|_{L^\infty(\frac{1}{\varepsilon}\Omega_\varepsilon)} (w + \|w\|_{L^\infty}) \leq \psi_\varepsilon \leq \varepsilon^3 \|h_\varepsilon\|_{L^\infty(\frac{1}{\varepsilon}\Omega_\varepsilon)} (w + \|w\|_{L^\infty}).$$

The bound $\|h_\varepsilon\|_{L^\infty(\frac{1}{\varepsilon}\Omega_\varepsilon)} \leq C$ and Lemma 2.4.11 imply $\|\psi_\varepsilon\|_{L^\infty(\frac{1}{\varepsilon}\Omega_\varepsilon)} \leq C\varepsilon^3$. Rescaling back concludes the proof. \square

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2.5 Appendices

2.5.1 Proof of technical lemmas

Lemma 2.5.1. *Let $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ be a sequence of open sets satisfying Assumption (A2). Then,*

$$\sum_{k \in \mathbb{Z}^d} |\mathcal{O}_k \Delta \mathcal{O}_k^{\text{per}}| < +\infty, \quad (2.5.1)$$

where $A \Delta B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$ stands for the symmetric subset difference. Moreover, if

$$\mathcal{K} := \{k \in \mathbb{Z}^d \text{ s.t. } \mathcal{O}_k \cap \mathcal{O}_k^{\text{per}} = \emptyset\},$$

then $|\mathcal{K}| < +\infty$.

Proof. First note that, using (2.1.14),

$$\mathcal{O}_k \setminus \mathcal{O}_k^{\text{per}} \subset \mathcal{U}_k^{\text{per}}(\alpha_k) \quad \text{and} \quad \mathcal{O}_k^{\text{per}} \setminus \mathcal{O}_k \subset \mathcal{U}_k^{\text{per}}(\alpha_k). \quad (2.5.2)$$

We now use [53, Theorem 3.2.39] to control the measure of $\mathcal{U}_k^{\text{per}}(\alpha_k)$: there exists $\bar{\alpha} > 0$ such that

$$\forall \alpha < \bar{\alpha}, |\mathcal{U}_0^{\text{per}}(\alpha)| \leq 2C|\partial\mathcal{O}_0^{\text{per}}|\alpha. \quad (2.5.3)$$

By translation invariance, the above assertion is true for $\mathcal{U}_0^{\text{per}}(\alpha)$ replaced by $\mathcal{U}_k^{\text{per}}(\alpha)$:

$$\forall k \in \mathbb{Z}^d, \forall \alpha < \bar{\alpha}, |\mathcal{U}_k^{\text{per}}(\alpha)| \leq 2C|\partial\mathcal{O}_0^{\text{per}}|\alpha.$$

For k large enough such that $\alpha_k < \bar{\alpha}$, one thus has $|\mathcal{U}_k^{\text{per}}(\alpha_k)| \leq \tilde{C}\alpha_k$ where \tilde{C} is a constant. This, together with (2.5.2), proves (4.2.2). The fact that $|\mathcal{K}| < +\infty$ is a direct consequence of (4.2.2) and of the fact that for all $k \in \mathcal{K}$, $|\mathcal{O}_k \Delta \mathcal{O}_k^{\text{per}}| \geq |\mathcal{O}_k^{\text{per}} \setminus \mathcal{O}_k| = |\mathcal{O}_0^{\text{per}}|$. \square

Lemma 2.5.2. *Let $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ be a sequence of open sets satisfying Assumption (A2). There exists $\rho > 0$ such that*

$$\forall k \in \mathcal{K}^c, \exists B_k \text{ s.t. } |B_k| \geq \rho \text{ and } B_k \subset \mathcal{O}_k \cap \mathcal{O}_k^{\text{per}},$$

where B_k denotes an open ball and \mathcal{K} is defined in Lemma 2.5.1.

Proof. Since $\mathcal{O}_0^{\text{per}}$ is open, it contains a ball $B \subset \bar{B} \subset \mathcal{O}_0^{\text{per}}$. One has $\delta := \text{dist}(\bar{B}, \partial\mathcal{O}_0^{\text{per}}) > 0$. By translation invariance, for all $k \in \mathbb{Z}^d$, $B_k := B + k$ satisfies

$$B_k \subset \bar{B}_k \subset \mathcal{O}_k^{\text{per}} \quad \text{and} \quad \delta = \text{dist}(\bar{B}_k, \partial\mathcal{O}_k^{\text{per}}).$$

Since $(\alpha_k)_{k \in \mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$, there exists k_0 such that for all $|k| \geq k_0$, $\alpha_k \leq \delta/2$. Equation (2.1.13) implies $\bar{B}_k \subset \mathcal{O}_k^{\text{per},-}(\alpha_k)$ for $|k| \geq k_0$. This proves that

$$\forall |k| \geq k_0, \bar{B}_k \subset \mathcal{O}_k^{\text{per}} \cap \mathcal{O}_k.$$

If $|k| < k_0$ and $\mathcal{O}_k^{\text{per}} \cap \mathcal{O}_k \neq \emptyset$, there exists a ball B_k such that

$$B_k \subset \mathcal{O}_k^{\text{per}} \cap \mathcal{O}_k.$$

Defining $\rho = \min \left(\min_{|k| < k_0} |B_k|, |B| \right) > 0$ concludes the proof. \square

Lemma 2.5.3. *Let $(\mathcal{O}_k)_{k \in \mathbb{Z}^d}$ be a sequence of open sets satisfying Assumptions (A1)-(A2). There exists $\delta_0 > 0$ such that*

$$\forall k \in \mathbb{Z}^d, \text{dist}(\mathcal{O}_k, \partial Q_k) \geq \delta_0.$$

Proof. Recall that for all $k \in \mathbb{Z}^d$, $\mathcal{O}_k^{\text{per}} \subset \subset Q_k$. Thus, by translation invariance, there exists a constant $\delta_0^{\text{per}} > 0$ independent of k such that

$$\forall k \in \mathbb{Z}^d, \text{dist}(\mathcal{O}_k^{\text{per}}, \partial Q_k) = \delta_0^{\text{per}}.$$

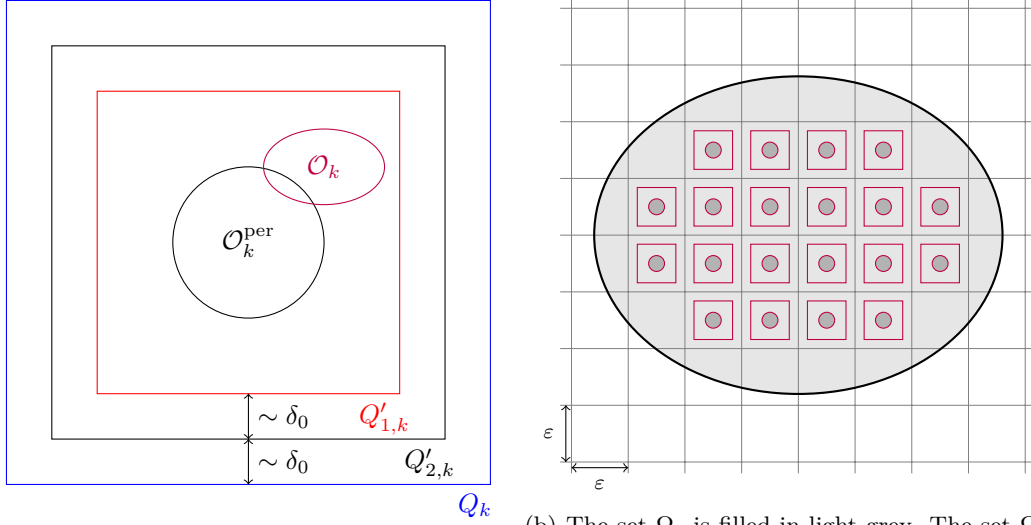
One has, using Assumption (A2) and in particular the inclusion $\mathcal{O}_k \subset \mathcal{O}_k^{\text{per},+}(\alpha_k)$,

$$\delta_k := \text{dist}(\mathcal{O}_k, \partial Q_k) \geq \delta_0^{\text{per}} - \alpha_k \geq \delta_0^{\text{per}}/2$$

for k large enough, say $|k| \geq k_0$. Since for all $|k| < k_0$, Assumption (A1) gives

$$\delta_k = \text{dist}(\mathcal{O}_k, \partial Q_k) > 0,$$

the Lemma is proved by defining $\delta_0 := \min \left(\frac{\delta_0^{\text{per}}}{2}, \min_{|k| < k_0} \delta_k \right)$. \square



(a) This drawing represents a periodic cell Q_k and the related notations.

(b) The set Ω_ε is filled in light grey. The set Ω'_ε (see (2.5.9)) is marked in purple. The setting is here periodic.

Figure 2.5: Drawing of the notations used in this section.

2.5.2 Improved rates of convergence

This section does not appear in the published version [22] of this chapter.

The goal of this section is to generalize the convergence Theorems 2.2.2 and 2.2.3 to Sobolev norms $W^{m,q}(\Omega_\varepsilon)$ for $1 < q < +\infty$ and $m \geq 0$ in the non-periodic setting introduced in Section 2.1.2. This will be made precise in Theorem 2.5.6 below, see also Remark 2.5.11.

We fix two smooth domains Q'_1 and Q'_2 such that

$$\left(-\frac{1}{2} + \frac{\delta_0}{2}, \frac{1}{2} - \frac{\delta_0}{2}\right)^d \subset\subset Q'_1 \subset\subset Q'_2 \subset\subset Q, \quad (2.5.4)$$

where δ_0 is given by Lemma 2.5.3. In particular, $Q'_i \subset Q$, $i = 1, 2$, $d(Q'_1, \partial Q'_2) > 0$ and $d(Q'_2, \partial Q) > 0$. We slightly change in this section the definition of the perforated domain. We set

$$\Omega_\varepsilon := \Omega \setminus \bigcup_{k \in Y_\varepsilon} \varepsilon \overline{O_k}, \quad (2.5.5)$$

where

$$Y_\varepsilon := \{k \in \mathbb{Z}^d, \quad \varepsilon Q_k \subset \Omega\}$$

and $(O_k)_{k \in \mathbb{Z}^d}$ is a set of non-periodic perforations satisfying **(A1)**-**(A2)**. With this definition, the set Ω_ε has the same regularity as Ω and the perforations O_k , $k \in \mathbb{Z}^d$. For $k \in \mathbb{Z}^d$, we define $Q'_{i,k} = Q'_i + k$, $k \in \mathbb{Z}^d$.

We first state and prove Theorem 2.5.4 which is a generalization of [82, Theorem 2.1] to the present non-periodic setting. Note that the proof of this Theorem is sketched in the periodic case in [82]. We detail it here, since the non-periodic case introduced in this chapter contains of course the periodic case.

Theorem 2.5.4. *We fix $1 < q < +\infty$. Let Ω be a bounded smooth domain and Ω_ε be defined by (2.5.5). Let us assume in addition that the perforations O_k , $k \in \mathbb{Z}^d$ are uniformly in k of class C^2 in the sense of Assumption **(A3)**. Let $f \in L^q(\Omega_\varepsilon)$. There exists a unique function $u_\varepsilon \in W^{2,q}(\Omega_\varepsilon)$*

solution to (2.1.1). Moreover, there exists a constant $C > 0$ independent of ε and f such that the following estimate holds true:

$$\|D^2 u_\varepsilon\|_{L^q(\Omega_\varepsilon)} + \varepsilon^{-1} \|\nabla u_\varepsilon\|_{L^q(\Omega_\varepsilon)} + \varepsilon^{-2} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \leq C \|f\|_{L^q(\Omega_\varepsilon)}. \quad (2.5.6)$$

If the perforations \mathcal{O}_k , $k \in \mathbb{Z}^d$ are in addition uniformly in k of class \mathcal{C}^{m+2} , $m \geq 0$ and $f \in W^{m,q}(\Omega_\varepsilon)$, then $u_\varepsilon \in W^{m+2,q}(\Omega_\varepsilon)$ and there exists a constant $C > 0$ independent of ε and f such that

$$\|D^{m+2} u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \leq C \sum_{i=0}^m \varepsilon^{i-m} \|D^i f\|_{L^q(\Omega_\varepsilon)}. \quad (2.5.7)$$

Proof of Theorem 2.5.4. Existence and uniqueness of a solution u_ε for fixed $\varepsilon > 0$ follow from standard elliptic regularity theory, see [60, Theorems 9.15 & 9.19]. We now prove the estimate (2.5.6). We introduce $v_\varepsilon := \varepsilon^{-2} u_\varepsilon(\varepsilon \cdot)$ and $f_\varepsilon := f(\varepsilon \cdot)$. We immediately check that

$$\begin{cases} -\Delta v_\varepsilon = f_\varepsilon \\ v_\varepsilon \in W_0^{1,q}(\frac{1}{\varepsilon} \Omega_\varepsilon). \end{cases}$$

For $k \in \mathbb{Z}^d$, we note that, thanks to (2.5.4) and Lemma 2.5.3 (see also Figure 2.5 left),

$$Q'_{i,k} \cap \left(\bigcup_{\ell \in \mathbb{Z}^d} \mathcal{O}_\ell \right) = \mathcal{O}_k, \quad i = 1, 2.$$

Since the perforations \mathcal{O}_k are uniformly in k regular, we have (see [60, Theorem 9.13] for a constant depending *a priori* on k and also [107, Appendix B] for a proof that this constant is in fact independent of k in the case of the Stokes system³) that there exists a constant $C > 0$ independent of k such that for all $k \in Y_\varepsilon$, the following estimate holds true:

$$\|v_\varepsilon\|_{W^{2,q}(Q'_{2,k} \setminus \overline{\mathcal{O}_k})}^q \leq C^q \{ \|f_\varepsilon\|_{L^q(Q_k \setminus \overline{\mathcal{O}_k})}^q + \|v_\varepsilon\|_{L^q(Q_k \setminus \overline{\mathcal{O}_k})}^q \}. \quad (2.5.8)$$

Summing (2.5.8) over k , we get that

$$\sum_{k \in Y_\varepsilon} \|v_\varepsilon\|_{W^{2,q}(Q'_{2,k} \setminus \overline{\mathcal{O}_k})}^q \leq C^q \{ \|f_\varepsilon\|_{L^q(\frac{1}{\varepsilon} \Omega_\varepsilon)}^q + \|v_\varepsilon\|_{L^q(\frac{1}{\varepsilon} \Omega_\varepsilon)}^q \}.$$

In particular, writing

$$\Omega'_\varepsilon := \bigcup_{k \in Y_\varepsilon} \varepsilon(Q'_{2,k} \setminus \overline{\mathcal{O}_k}) \subset \subset \Omega_\varepsilon, \quad (2.5.9)$$

this proves that

$$\|v_\varepsilon\|_{W^{2,q}(\frac{1}{\varepsilon} \Omega'_\varepsilon)}^q \leq C^q \{ \|f_\varepsilon\|_{L^q(\frac{1}{\varepsilon} \Omega_\varepsilon)}^q + \|v_\varepsilon\|_{L^q(\frac{1}{\varepsilon} \Omega_\varepsilon)}^q \}. \quad (2.5.10)$$

Scaling back (2.5.10) and taking the $1/q$ -th power provides

$$\|D^2 u_\varepsilon\|_{L^q(\Omega'_\varepsilon)} + \varepsilon^{-1} \|\nabla u_\varepsilon\|_{L^q(\Omega'_\varepsilon)} + \varepsilon^{-2} \|u_\varepsilon\|_{L^q(\Omega'_\varepsilon)} \leq C \{ \|f\|_{L^q(\Omega_\varepsilon)} + \varepsilon^{-2} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \}. \quad (2.5.11)$$

We now show that

$$\|u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \leq C \varepsilon^2 \|f\|_{L^q(\Omega_\varepsilon)}, \quad (2.5.12)$$

where C is independent of f and ε . We first assume that $q \geq 2$. Multiplying (2.1.1) by $u_\varepsilon |u_\varepsilon|^{q-2} \in W_0^{1,q'}(\Omega_\varepsilon)$ and integrating by parts, we obtain that

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot (q-1) |u_\varepsilon|^{q-2} \nabla u_\varepsilon = \int_{\Omega_\varepsilon} f \cdot u_\varepsilon |u_\varepsilon|^{q-2}.$$

³The adaptation to Poisson equation is similar.

Thus,

$$(q-1) \left(\frac{2}{q}\right)^2 \int_{\Omega_\varepsilon} |\nabla |u_\varepsilon|^{\frac{q}{2}}|^2 = \int_{\Omega_\varepsilon} f \cdot u_\varepsilon |u_\varepsilon|^{q-2}.$$

Applying Hölder inequality gives

$$\int_{\Omega_\varepsilon} |\nabla |u_\varepsilon|^{\frac{q}{2}}|^2 \leq C(q) \|f\|_{L^q(\Omega_\varepsilon)} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)}^{q/q'}.$$

We use the L^2 -Poincaré inequality (see Theorem⁴ 2.3.1) applied to $|u_\varepsilon|^{\frac{q}{2}} \in H_0^1(\Omega_\varepsilon)$ that

$$\int_{\Omega_\varepsilon} |u_\varepsilon|^q \leq C\varepsilon^2 \|f\|_{L^q(\Omega_\varepsilon)} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)}^{q/q'}.$$

Noticing that $q - q/q' = 1$, we conclude that (2.5.12) holds true for $q \geq 2$. The case $1 < q < 2$ is treated by duality: let $f \in L^q(\Omega_\varepsilon)$ and $u_\varepsilon \in W_0^{1,q}(\Omega_\varepsilon)$ solution to (2.1.1). For $g \in L^{q'}(\Omega_\varepsilon)$, we introduce $z_\varepsilon \in L^{q'}(\Omega_\varepsilon)$ the solution to (2.1.1) with source term g . We have, using the equations solved by u_ε and z_ε , and integration by parts, that

$$\int_{\Omega_\varepsilon} g u_\varepsilon = \int_{\Omega_\varepsilon} \nabla z_\varepsilon \cdot \nabla u_\varepsilon = \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla z_\varepsilon = \int_{\Omega_\varepsilon} f z_\varepsilon$$

By Hölder inequality together with (2.5.12) for $q' \geq 2$, we get that

$$\forall g \in L^{q'}(\Omega_\varepsilon), \quad \int_{\Omega_\varepsilon} g u_\varepsilon \leq \|z_\varepsilon\|_{L^{q'}(\Omega_\varepsilon)} \|f\|_{L^q(\Omega_\varepsilon)} \leq C\varepsilon^2 \|g\|_{L^{q'}(\Omega_\varepsilon)} \|f\|_{L^q(\Omega_\varepsilon)}.$$

Taking the supremum over g , we get that

$$\|u_\varepsilon\|_{L^q(\Omega)} = \sup \left\{ \int_{\Omega_\varepsilon} g u_\varepsilon, \quad \|g\|_{L^{q'}(\Omega_\varepsilon)} = 1 \right\} \leq C\varepsilon^2 \|f\|_{L^q(\Omega_\varepsilon)}.$$

Thus (2.5.12) is satisfied for all $1 < q < +\infty$ and (2.5.11) gives

$$\|D^2 u_\varepsilon\|_{L^q(\Omega'_\varepsilon)} + \varepsilon^{-1} \|\nabla u_\varepsilon\|_{L^q(\Omega'_\varepsilon)} + \varepsilon^{-2} \|u_\varepsilon\|_{L^q(\Omega'_\varepsilon)} \leq C \|f\|_{L^q(\Omega_\varepsilon)}. \quad (2.5.13)$$

It remains to show (2.5.13) in $\Omega_\varepsilon \setminus \overline{\Omega'_\varepsilon}$. We introduce a cut-off function $\chi \in \mathcal{D}(Q'_2)$ such that $\chi = 1$ in Q'_1 . We define $\chi_k^\varepsilon := \chi \left(\frac{\cdot - k}{\varepsilon} \right)$. By Lemma 2.5.3, the support of χ_k^ε contains strictly $\varepsilon \mathcal{O}_k$. Besides, there exists a constant $C = C(d, Q'_1, Q'_2) > 0$ such that

$$|\nabla \chi_k^\varepsilon| \leq C\varepsilon^{-1} \quad \text{and} \quad |D^2 \chi_k^\varepsilon| \leq C\varepsilon^{-2}$$

and the support of the χ_k^ε , $k \in \mathbb{Z}^d$ are disjoint. We define

$$\chi := 1 - \sum_{k \in Y_\varepsilon} \chi_k^\varepsilon \quad \text{and} \quad w_\varepsilon := u_\varepsilon \chi.$$

We have that⁵ $\chi = 1$ in $\Omega \setminus \bigcup_{k \in Y_\varepsilon} \varepsilon \overline{Q'_{2,k}}$ and that $\chi = 0$ around each perforation $\varepsilon \mathcal{O}_k$, $k \in Y_\varepsilon$. In particular, $w_\varepsilon \in W^{2,q} \cap W_0^{1,q}(\Omega)$. We have that

$$\begin{aligned} \|\Delta w_\varepsilon\|_{L^q(\Omega)} &= \left\| -f\chi + 2\nabla u_\varepsilon \cdot \nabla \chi + u_\varepsilon \Delta \chi \right\|_{L^q(\Omega)} \\ &\leq C \left\{ \|f\|_{L^q(\Omega)} + \varepsilon^{-1} \|\nabla u_\varepsilon\|_{L^q(\Omega_\varepsilon)} + \varepsilon^{-2} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \right\}, \end{aligned}$$

⁴The proof of this Theorem may be adapted to the new definition (2.5.5) of Ω_ε introduced in this section.

⁵See Figure 2.5 right. We have that $\chi = 1$ in the complement of the purple boxes.

Using elliptic regularity theory in Ω , see e.g. [60, Theorem 9.13], we get that

$$\|D^2 w_\varepsilon\|_{L^q(\Omega)} \leq C(\Omega) \{ \|f\|_{L^q(\Omega)} + \varepsilon^{-1} \|\nabla w_\varepsilon\|_{L^q(\Omega)} + \varepsilon^{-2} \|w_\varepsilon\|_{L^q(\Omega_\varepsilon)} \}. \quad (2.5.14)$$

Using the form of the function w_ε , we obtain that

$$\|D^2 w_\varepsilon\|_{L^q(\Omega)} \leq C(\Omega) \{ \|f\|_{L^q(\Omega)} + \varepsilon^{-1} \|\nabla u_\varepsilon\|_{L^q(\Omega_\varepsilon)} + \varepsilon^{-2} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \}. \quad (2.5.15)$$

Noticing that

$$w_\varepsilon = u_\varepsilon \quad \text{in} \quad \Omega_\varepsilon \setminus \overline{\Omega'_\varepsilon},$$

we get

$$\|D^2 u_\varepsilon\|_{L^q(\Omega_\varepsilon \setminus \overline{\Omega'_\varepsilon})} \leq C(\Omega) \{ \|f\|_{L^q(\Omega)} + \varepsilon^{-1} \|\nabla u_\varepsilon\|_{L^q(\Omega_\varepsilon)} + \varepsilon^{-2} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \}. \quad (2.5.16)$$

We next apply the following Gagliardo-Nirenberg inequality⁶

$$\|\nabla u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \leq C \left[\|D^2 u_\varepsilon\|_{L^q(\Omega_\varepsilon)}^{1/2} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)}^{1/2} + \varepsilon^{-1} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \right]$$

where the constant C is independent of ε . This yields

$$\|D^2 u_\varepsilon\|_{L^q(\Omega_\varepsilon \setminus \overline{\Omega'_\varepsilon})} \leq C \{ \|f\|_{L^q(\Omega)} + \varepsilon^{-1} \|D^2 u_\varepsilon\|_{L^q(\Omega_\varepsilon)}^{1/2} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)}^{1/2} + \varepsilon^{-2} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \}. \quad (2.5.17)$$

Summing (2.5.13) and (2.5.17) gives

$$\|D^2 u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \leq C \{ \|f\|_{L^q(\Omega)} + \varepsilon^{-1} \|D^2 u_\varepsilon\|_{L^q(\Omega_\varepsilon)}^{1/2} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)}^{1/2} + \varepsilon^{-2} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \}. \quad (2.5.18)$$

Thus

$$\|D^2 u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \leq C \left[\|f\|_{L^q(\Omega)} + \varepsilon^{-2} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \right] \stackrel{(2.5.12)}{\leq} C \|f\|_{L^q(\Omega_\varepsilon)},$$

where the constant C is independent of ε . This proves (2.5.6). The second part of the Theorem is proved similarly (by induction) using the scaling relations:

$$\varepsilon^{d/q} \|D^{m+2} v_\varepsilon\|_{L^q(\frac{1}{\varepsilon}\Omega_\varepsilon)} = \varepsilon^m \|D^{m+2} u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \quad \text{and} \quad \varepsilon^{d/q} \|f_\varepsilon\|_{W^{m,q}(\frac{1}{\varepsilon}\Omega_\varepsilon)} \sim \sum_{i=0}^m \varepsilon^i \|D^i f\|_{L^q(\Omega_\varepsilon)}$$

and higher elliptic regularity estimates, see [60, Theorem 9.15]. \square

Remark 2.5.5. *In Theorem 2.5.4, the proof of the inequality $\|u_\varepsilon\|_{L^q(\Omega_\varepsilon)} \leq C\varepsilon^2 \|f\|_{L^q(\Omega_\varepsilon)}$ relies on an integration by parts and Poincaré inequality and requires no more regularity assumptions on the perforations than Lipschitz continuity. We can also obtain this estimate when the perforated domain Ω_ε has the form $\Omega_\varepsilon = \Omega \setminus \varepsilon \bigcup_{k \in \mathbb{Z}^d} \overline{\mathcal{O}_k}$.*

Theorem 2.5.6. *We fix $\varepsilon > 0$, $q \in (1, +\infty)$ and we denote by w the solution given by Theorem 2.2.1. We assume that the perforations \mathcal{O}_k , $k \in \mathbb{Z}^d$ are of class \mathcal{C}^2 uniformly in k . Let Ω_ε be defined by (2.5.5).*

1. *Let $f \in W^{2,q}(\Omega)$ be compactly supported in Ω and u_ε be the solution to (2.1.1). Then there exists a constant $C > 0$ independent of ε such that for all $\varepsilon < \varepsilon_0(\Omega, f)$ small enough,*

$$\varepsilon^{-1} \|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{L^q(\Omega_\varepsilon)} + \|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{W^{1,q}(\Omega_\varepsilon)} \leq C\varepsilon^2. \quad (2.5.19)$$

⁶see [88, Remark 5. p. 126] for the inequality in Ω . We deduce this estimate in the perforated domain using the Stein extension operators (see [1, Chapter 5]) of each perforation. These operators are bounded uniformly in k since the perforations are uniformly in k of class \mathcal{C}^2 , see [60, Theorem 7.25].

2. Let $f \in W^{2,q} \cap L^\infty(\Omega)$ and u_ε be the solution to (2.1.1). Then there exists a constant $C > 0$ independent of ε such that for all $\varepsilon < \varepsilon(\Omega, \Omega')$

$$\varepsilon^{-1} \|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{L^q(\Omega_\varepsilon)} + |u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f|_{W^{1,q}(\Omega_\varepsilon)} \leq C\varepsilon^{1+1/q}. \quad (2.5.20)$$

3. Let $q \geq 2$, $f \in W^{2,q}(\Omega)$, u_ε be the solution to (2.1.1) and $\Omega' \subset\subset \Omega$. We write $\Omega'_\varepsilon := \Omega' \cap \Omega_\varepsilon$. Then there exists a constant $C > 0$ independent of ε such that for all $\varepsilon < \varepsilon_0(\Omega, \Omega', \mathcal{O}, d)$,

$$\varepsilon^{-1} \|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{L^q(\Omega'_\varepsilon)} + |u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f|_{W^{1,q}(\Omega'_\varepsilon)} \leq C\varepsilon^2. \quad (2.5.21)$$

4. Let $f \in W^{m+2,q}(\Omega)$, $m \geq 2$, be compactly supported in Ω and u_ε be the solution to (2.1.1). We assume that the perforations \mathcal{O}_k , $k \in \mathbb{Z}^d$ are uniformly of class \mathcal{C}^{m+2} . Then there exists a constant C independent of ε such that for all $\varepsilon < \varepsilon_0(\Omega, f)$,

$$\|D^{m+2}\{u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\}\| \leq C\varepsilon^{1-m}. \quad (2.5.22)$$

Proof. We prove the first point. We introduce, as in the proof of Theorem 2.2.2, the function

$$\phi_\varepsilon := u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f \in W_0^{1,q}(\Omega_\varepsilon).$$

We choose $\varepsilon > 0$ small enough such that

$$\text{supp}(f) \subset \bigcup_{k \in Y_\varepsilon} \varepsilon Q_k.$$

Since

$$\Omega_\varepsilon \cap \text{supp}(f) \subset \bigcup_{k \in Y_\varepsilon} \varepsilon(Q_k \setminus \overline{\mathcal{O}_k}),$$

we need to compute $\Delta \phi_\varepsilon$ in the set $\bigcup_{k \in Y_\varepsilon} \varepsilon(Q_k \setminus \overline{\mathcal{O}_k})$. We have, as in the proof of Theorem 2.2.2,

$$-\Delta \phi_\varepsilon = \varepsilon g_\varepsilon, \quad \text{where} \quad g_\varepsilon := 2\nabla w\left(\frac{\cdot}{\varepsilon}\right) \cdot \nabla f + \varepsilon w\left(\frac{\cdot}{\varepsilon}\right) \Delta f. \quad (2.5.23)$$

By Lemma 2.4.11, we have that $w(\cdot/\varepsilon) \in \mathcal{C}^{1,\gamma}(\mathbb{R}^d \setminus \varepsilon \overline{\mathcal{O}})$. Thus, since $f \in W^{2,q}(\Omega)$, we get that $g_\varepsilon \in L^q(\Omega)$ and

$$\|g_\varepsilon\|_{L^q(\Omega)} \leq \|w\|_{W^{1,\infty}(\mathbb{R}^d)} \|\nabla f\|_{W^{1,q}(\Omega)} \leq C.$$

Theorem 2.5.4 allows to conclude the first point. Note that this argument also gives point 4. Indeed, since the perforations are uniformly of class \mathcal{C}^{m+2} , we can prove as in Lemma 2.4.11 that $w \in W^{m+1,\infty}(\mathbb{R}^d \setminus \overline{\mathcal{O}})$. Thus, by Leibniz rule, we have that $g_\varepsilon \in W^{m,q}(\Omega_\varepsilon)$ and

$$\|D^i g_\varepsilon\|_{L^q(\Omega_\varepsilon)} \leq C\varepsilon^{-i}.$$

Theorem 2.5.4 and in particular (2.5.7) allow to deduce (2.5.22).

We now prove the second point. The proof is adapted from the case $(m, q) = (1, 2)$ of [75]. We fix a cut-off function $\chi \in \mathcal{D}(\Omega)$ such that the support of χ is included in $\{x \in \Omega, d(x, \partial\Omega) > \sqrt{d\varepsilon}\}$. Note that the support of χ is included in $\bigcup_{k \in Y_\varepsilon} \varepsilon \overline{Q_k}$. We assume that

$$\chi = 1 \quad \text{in} \quad \left\{x \in \Omega, \quad d(x, \partial\Omega) \geq 2\sqrt{d\varepsilon}\right\}.$$

We have the bounds

$$|\text{supp}(1 - \chi)| \leq C\varepsilon, \quad |\text{supp}(\nabla \chi)| \leq C\varepsilon, \quad |\nabla \chi| \leq C\varepsilon^{-1} \quad \text{and} \quad |D^2 \chi| \leq C\varepsilon^{-2},$$

where C is independent of ε . In particular,

$$\|1 - \chi\|_{L^q(\Omega_\varepsilon)} \leq C\varepsilon^{1/q}, \quad \|\nabla\chi\|_{L^q(\Omega_\varepsilon)} \leq C\varepsilon^{1/q-1} \quad \text{and} \quad \|D^2\chi\|_{L^q(\Omega_\varepsilon)} \leq C\varepsilon^{1/q-2}. \quad (2.5.24)$$

We have the inequality

$$\|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon)f\|_{W^{1,q}(\Omega_\varepsilon)} \leq \|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon)f\chi\|_{W^{1,q}(\Omega_\varepsilon)} + \|\varepsilon^2 w(\cdot/\varepsilon)(1 - \chi)f\|_{W^{1,q}(\Omega_\varepsilon)}. \quad (2.5.25)$$

The same inequality is obviously true with the $W^{1,q}$ -norm replaced by the L^q -norm. We note that, since $w \in W^{1,\infty}(\mathbb{R}^d)$ and $f \in W^{1,q} \cap L^\infty(\Omega)$,

$$\varepsilon^{-1} \|\varepsilon^2 w(\cdot/\varepsilon)(1 - \chi)f\|_{L^q(\Omega_\varepsilon)} + |\varepsilon^2 w(\cdot/\varepsilon)(1 - \chi)f|_{W^{1,q}(\Omega_\varepsilon)} \stackrel{(2.5.24)}{\leq} C\varepsilon^{1+1/q}, \quad (2.5.26)$$

In (2.5.25), it remains to estimate $\|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon)f\chi\|_{W^{1,q}(\Omega_\varepsilon)}$. We adopt the same strategy as in the proof of the first point. We introduce

$$R_\varepsilon := u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon)f\chi \in W_0^{1,q}(\Omega_\varepsilon).$$

The support of $f\chi$ is included in $\bigcup_{k \in Y_\varepsilon} \varepsilon\overline{Q}_k$. Since $\left(\bigcup_{k \in Y_\varepsilon} \varepsilon\overline{Q}_k\right) \cap \Omega_\varepsilon \subset \mathbb{R}^d \setminus \varepsilon\overline{O}$, we have that $R_\varepsilon \in W^{2,q} \cap W_0^{1,q}(\Omega_\varepsilon)$ and

$$\begin{aligned} \|\Delta R_\varepsilon\|_{L^q(\Omega_\varepsilon)} &= \|f(1 - \chi) + 2\varepsilon\nabla w(\cdot/\varepsilon) \cdot \nabla(f\chi) + \varepsilon^2 w(\cdot/\varepsilon)\Delta(f\chi)\|_{L^q(\Omega_\varepsilon)} \\ &\leq \|f\|_{L^\infty(\Omega)} \|1 - \chi\|_{L^q(\Omega)} + \varepsilon \|\nabla w\|_{L^\infty(\mathbb{R}^d)} \left\{ \|\nabla f\|_{L^q(\Omega)} + \|f\|_{L^\infty(\mathbb{R}^d)} \|\nabla\chi\|_{L^q(\Omega)} \right\} \\ &\quad + \varepsilon^2 \|w\|_{L^\infty(\mathbb{R}^d)} \left\{ \|\Delta f\|_{L^q(\Omega)} + 2\|\nabla f\|_{L^q(\Omega)} \|\nabla\chi\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\mathbb{R}^d)} \|D^2\chi\|_{L^q(\Omega)} \right\} \\ &\stackrel{(2.5.24)}{\leq} C\varepsilon^{1/q}. \end{aligned} \quad (2.5.27)$$

Applying Theorem 2.5.4 to R_ε , gathering (2.5.26) and (2.5.25) allows to deduce (2.5.20).

We now prove the third point. We fix Ω'' and Ω''' such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega''' \subset\subset \Omega$. Let χ be a cut-off function (independent of ε) such that $\chi = 1$ in Ω'' and $\text{supp}(\chi) \subset \Omega'''$. For $\varepsilon > 0$ small enough, we have that $\text{supp}(\chi) \subset \bigcup_{k \in Y_\varepsilon} \varepsilon Q_k$. Besides, we have the bounds $\|D^j\chi\|_{L^\infty(\Omega)} \leq C$, $j \geq 0$

where C is independent of ε . We introduce u_ε^χ and $u_\varepsilon^{1-\chi}$, the solutions to

$$\begin{cases} -\Delta u_\varepsilon^\chi = f\chi & \text{in } \Omega_\varepsilon \\ u_\varepsilon^\chi = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_\varepsilon^{1-\chi} = (1 - \chi)f & \text{in } \Omega_\varepsilon \\ u_\varepsilon^{1-\chi} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

We obviously have $u_\varepsilon = u_\varepsilon^\chi + u_\varepsilon^{1-\chi}$. By the first point, we have for $\varepsilon < \varepsilon_0(\Omega, f, \Omega')$,

$$\varepsilon^{-1} \|u_\varepsilon^\chi - \varepsilon^2 w(\cdot/\varepsilon)f\chi\|_{L^q(\Omega_\varepsilon)} + |u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon)f\chi|_{W^{1,q}(\Omega_\varepsilon)} \leq C\varepsilon^2. \quad (2.5.28)$$

We now use Lemma 2.5.7 below to bound for $\varepsilon > 0$ small enough

$$\varepsilon^{-1} \|u_\varepsilon^{1-\chi} - \varepsilon^2 w(\cdot/\varepsilon)f(1 - \chi)\|_{L^q(\Omega_\varepsilon \cap \Omega')} + |u_\varepsilon^{1-\chi} - \varepsilon^2 w(\cdot/\varepsilon)f(1 - \chi)|_{W^{1,q}(\Omega_\varepsilon \cap \Omega')} \leq C\varepsilon^2 \quad (2.5.29)$$

Gathering together (2.5.28) and (2.5.29) gives (2.5.21). \square

Lemma 2.5.7. *Let $q \geq 2$, $f \in L^q(\Omega)$ such that $f = 0$ in $\Omega'' \subset\subset \Omega$. Let u_ε be the solution to (2.1.1). Then for all $\Omega' \subset\subset \Omega''$, there exist constants $c, C, \delta > 0$ independent of ε, f and u_ε such that*

$$\varepsilon^{-1} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon \cap \Omega')} + |u_\varepsilon|_{W^{1,q}(\Omega_\varepsilon \cap \Omega')} \leq C\varepsilon \exp^{-\delta\varepsilon^{-1}}. \quad (2.5.30)$$

Note that Lemma 2.5.7 is adapted from [72, Lemma 3.25]. The proof is made for $q = 2$ there and we extend it to arbitrary $q \in [2, +\infty)$. The fact that the domain is not periodically perforated does not play a role in this proof.

Proof of Lemma 2.5.7. Let $0 \leq \tau \leq 1$ be a cut-off function such that $\tau = 1$ in Ω'_0 with $\Omega' \subset \subset \Omega'_0 \subset \subset \Omega''$ and $\text{supp}(\tau) \subset \Omega''$. We introduce $\phi := (e^{\mu\tau} - 1)u_\varepsilon|u_\varepsilon|^{q-2} \in W_0^{1,q'}(\Omega_\varepsilon)$ with $\mu > 0$ to be determined later. Multiplying (2.1.1) by ϕ and integrating by parts yields

$$(q-1) \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2} (e^{\mu\tau} - 1) + \mu \int_{\Omega_\varepsilon} (\nabla u_\varepsilon \cdot \nabla \tau) e^{\mu\tau} u_\varepsilon |u_\varepsilon|^{q-2} = \int_{\Omega_\varepsilon} f (e^{\mu\tau} - 1) u_\varepsilon |u_\varepsilon|^{q-2}.$$

Using that $f e^{\mu\tau} = f$ in Ω , we get that

$$(q-1) \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2} e^{\mu\tau} = -\mu \int_{\Omega_\varepsilon} (\nabla u_\varepsilon \cdot \nabla \tau) e^{\mu\tau} u_\varepsilon |u_\varepsilon|^{q-2} + (q-1) \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2}. \quad (2.5.31)$$

Applying the Cauchy-Schwarz inequality gives

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2} e^{\mu\tau} \leq c\mu \|\nabla \tau\|_{L^\infty} \left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2} e^{\mu\tau} \right)^{1/2} \left(\int_{\Omega_\varepsilon} |u_\varepsilon|^q e^{\mu\tau} \right)^{1/2} + \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2}. \quad (2.5.32)$$

Thanks to the Poincaré inequality, we have that

$$\begin{aligned} \int_{\Omega_\varepsilon} |u_\varepsilon|^q e^{\mu\tau} &\leq C\varepsilon^2 \int_{\Omega_\varepsilon} \left| \nabla \left(|u_\varepsilon|^{\frac{q}{2}} e^{\frac{\mu\tau}{2}} \right) \right|^2 \\ &\leq C_0\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2} e^{\mu\tau} + C_1\mu^2 \|\nabla \tau\|_{L^\infty}^2 \varepsilon^2 \int_{\Omega_\varepsilon} |u_\varepsilon|^q e^{\mu\tau}, \end{aligned} \quad (2.5.33)$$

where we have used that

$$\left| \nabla |u_\varepsilon|^{\frac{q}{2}} \right|^2 = \left(\frac{q}{2} \right)^2 \left| \nabla u_\varepsilon |u_\varepsilon|^{\frac{q-4}{2}} u_\varepsilon \right|^2 = \left(\frac{q}{2} \right)^2 |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2}. \quad (2.5.34)$$

We now choose

$$\mu := \min \left(\frac{1}{\sqrt{2C_1} \|\nabla \tau\|_{L^\infty} \varepsilon}, \frac{1}{2c \|\nabla \tau\|_{L^\infty} \sqrt{C_0} \varepsilon} \right). \quad (2.5.35)$$

From (2.5.33), we get that

$$\frac{1}{2} \int_{\Omega_\varepsilon} |u_\varepsilon|^q e^{\mu\tau} \leq C\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2} e^{\mu\tau}. \quad (2.5.36)$$

Inserting (2.5.36) in (2.5.32) and using (2.5.35), we deduce that

$$\frac{1}{2} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2} e^{\mu\tau} \leq \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2}.$$

Thus,

$$\int_{\Omega_\varepsilon \cap \Omega'_0} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2} \leq 2e^{-\mu} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2}.$$

Applying Theorem 2.5.4 and in particular (2.5.6), we obtain the bound

$$\int_{\Omega_\varepsilon \cap \Omega'_0} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2} \leq C e^{-\mu} \varepsilon^{2q-2} = C \varepsilon^{2q-2} e^{-\delta\varepsilon^{-1}}. \quad (2.5.37)$$

We next use Remark 2.3.3, (2.5.34) and (2.5.37) on the function $|u_\varepsilon|^{\frac{q}{2}}$ to deduce that

$$\int_{\Omega_\varepsilon \cap \Omega'_2} |u_\varepsilon|^q \leq C\varepsilon^2 \int_{\Omega_\varepsilon \cap \Omega'_0} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{q-2} \leq C\varepsilon^2 \varepsilon^{2q-2} e^{-\delta\varepsilon^{-1}} = C\varepsilon^{2q} e^{-\delta\varepsilon^{-1}},$$

where Ω'_2 is chosen such that $\Omega' \subset \subset \Omega'_2 \subset \subset \Omega'_0$. The Lemma is proved using Remark 2.5.8 (take $\Omega' = \Omega'_2$ there). \square

Remark 2.5.8. *The method of proof of Theorem 2.5.4 shows the following result: let $f \in L^q(\Omega_\varepsilon)$ and $u_\varepsilon \in W_0^{1,q}(\Omega_\varepsilon)$ be solution to (2.1.1), $1 < q < +\infty$. Then for all $\Omega' \subset\subset \Omega'_2 \subset\subset \Omega$, there exists a constant $C > 0$ independent of f , u_ε and ε such that for $\varepsilon > 0$ small enough,*

$$\varepsilon^{-1} \|\nabla u_\varepsilon\|_{L^q(\Omega_\varepsilon \cap \Omega')} \leq C [\|f\|_{L^q(\Omega_\varepsilon \cap \Omega'_2)} + \varepsilon^{-2} \|u_\varepsilon\|_{L^q(\Omega_\varepsilon \cap \Omega'_2)}]. \quad (2.5.38)$$

Remark 2.5.9. *In Theorem 2.5.4, the estimates (2.5.19), (2.5.20) and (2.5.21) can be proved for $\varepsilon > \varepsilon_0$ up to changing the constant C . In contrast, (2.5.22) requires ε to be small because of the perforations that are removed along the boundary.*

Remark 2.5.10. *Theorem 2.5.6 is obviously true for $q = 1$ by using Hölder inequality and the statement of Theorem 2.5.6 for $q = 1 + \delta$. It nevertheless requires to assume that $f \in W^{2,1+\delta}(\Omega)$.*

Remark 2.5.11. *This remark is to be compared to the discussion after Theorem 2.2.2. Theorem 2.5.4 shows that the non-periodic corrector w improves the convergence rates as long as we measure the remainder $u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon)f$ in $W^{1,q}$ -norm, $q > d$ and $f \in W^{2,q} \cap L^\infty(\Omega)$. Indeed, suppose that $|f| \geq \alpha > 0$ in $\Omega' \subset\subset \Omega$. Then*

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^{-(2+d/q)} \|\varepsilon^2 \tilde{w}(\cdot/\varepsilon)f\|_{L^q(\Omega_\varepsilon \cap \Omega')} \geq \alpha \liminf_{\varepsilon \rightarrow 0} \varepsilon^{-d/q} \|\tilde{w}(\cdot/\varepsilon)\|_{L^q(\Omega_\varepsilon \cap \Omega')} = \alpha \|\tilde{w}\|_{L^q(\mathbb{R}^d)} =: \beta.$$

We deduce that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \varepsilon^{-2-d/q} \|u_\varepsilon - \varepsilon^2 w^{per}(\cdot/\varepsilon)f\|_{L^q(\Omega_\varepsilon \cap \Omega')} \\ \geq \liminf_{\varepsilon \rightarrow 0} \varepsilon^{-2-d/q} \left\{ \|\varepsilon^2 \tilde{w}(\cdot/\varepsilon)f\|_{L^q(\Omega_\varepsilon \cap \Omega')} - \|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon)f\|_{L^q(\Omega_\varepsilon \cap \Omega')} \right\} \stackrel{(2.5.20)}{\geq} \beta > 0. \end{aligned} \quad (2.5.39)$$

This computation shows that $\|u_\varepsilon - \varepsilon^2 w^{per}(\cdot/\varepsilon)f\|_{L^q(\Omega_\varepsilon \cap \Omega')}$ is of order $O(\varepsilon^{2+d/q})$. Consequently, using the non-periodic corrector w in place of w^{per} allows to improve the convergence rates from $O(\varepsilon^{2+d/q})$ to $O(\varepsilon^3)$ when $f \in W^{2,q} \cap L^\infty(\Omega)$, $q > d$. This analysis was restricted to $f \in \mathcal{D}(\Omega)$ in Theorem 2.2.3.

CHAPTER 3

HOMOGENIZATION OF THE STOKES SYSTEM IN A NON-PERIODICALLY PERFORATED DOMAIN

This chapter is based on [107]. We have added Subsection 3.4.3.

In the previous chapter, we have studied the homogenization of the Poisson equation in a class of non periodically perforated domains. In this chapter, we examine the case of the Stokes system. We consider a porous medium in which the characteristic distance between two holes, denoted by ε , is proportional to the characteristic size of the holes. It is well known (see [2],[91] and [99]) that, when the holes are periodically distributed in space, the velocity converges to a limit given by the Darcy's law when the size of the holes tends to zero. We generalize these results to the setting of [22]. The non-periodic domains are defined as a local perturbation of a periodic distribution of holes. We obtain classical results of the homogenization theory in perforated domains (existence of correctors and regularity estimates uniform in ε) and we prove H^2 -convergence estimates for particular force fields. This chapter is based on [107].

3.1 Introduction

In this paper, we study the three dimensional Stokes system in a perforated domain for an incompressible fluid with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \quad (3.1.1)$$

In Equation (3.1.1), $\Omega_\varepsilon \subset \mathbb{R}^3$ denotes the perforated domain, the vector valued function f is the force field, the unknowns u_ε and p_ε refer respectively to the velocity and the pressure of the fluid. The distance between two neighbouring holes is denoted by ε . We assume that the characteristic size of the holes is ε . Our purpose is to understand the limit of $(u_\varepsilon, p_\varepsilon)$ when $\varepsilon \rightarrow 0$. We construct classical objects of the homogenization theory such as correctors (Theorem 3.2.1) and we give new rates of convergence of u_ε to its limit when f is smooth, compactly supported and $\operatorname{div}(Af) = 0$ where A is the so-called permeability tensor (see Theorem 3.2.3).

To our knowledge, the first paper on the homogenization of the Stokes system in perforated domains is [99]. In this work, Equation (3.1.1) is studied for a periodic distribution of perforations in the macroscopic domain Ω (that is, each cell of a periodic array of size ε contains a perforation). It is in particular proved that $(u_\varepsilon/\varepsilon^2, p_\varepsilon)$ converges in some sense to a couple (u_0, p_0) given by the Darcy's law. This result can be guessed by performing a standard two scale expansion of $(u_\varepsilon, p_\varepsilon)$, see [91].

Error estimates between u_ε and its first order term in ε are proved in [80, 81] for particular situations namely the two-dimensional case in [81] and the case of a periodic macroscopic domain in [80]. Sharp error estimates under general assumptions on f have been obtained in [95]. The case of boundary layers in an infinite two-dimensional rectangle has been addressed in [69]. The results of [99] have been extended in [2] to porous medium in which both solid and fluid parts are connected. The case of holes that scale differently as ε is examined in [4], see also the contributions [73, 79]. High order models for the homogenization of the Stokes system have been proposed in [54]. Homogenization in periodically perforated domains for non-linear problems has been developed in [8, 9] via variational techniques.

The homogenization in perforated domains beyond the periodic setting has been considered in a deterministic setting in [51], see also [84] for a problem of homogenization with defects in the scalar case. The case of non-zero boundary conditions for the Stokes system in a non-periodically perforated domain with small holes is presented in [67]. The homogenization in random porous medium has been considered in [62] for the Poisson equation and in [15, 64, 63], and the references therein, for the Stokes system. In this paper, we adapt the results of [99] to the setting of [22], that is to perforated domains that are defined as a local perturbation of the periodically perforated domain considered in [99]. This framework is inspired by the papers [19, 20, 18] (see [22, Remark 1.5]). The purpose of these works is to study the homogenization of elliptic PDEs with coefficients that are periodic and perturbed by a defect which belongs to L^r , $1 < r < +\infty$.

The paper is organized as follows. We recall in subsection 3.1.2 the main results of the homogenization of the Stokes system in the periodic case. We introduce in subsection 3.1.3 the non-periodic setting. We state in Section 3.2 the main results of this paper and we make some remarks. These results are proved in Section 3.3. Some technical Lemmas are given in Appendix 3.4.1. In Appendix 3.4.2, we give more specific geometric assumptions on the non-periodic perforations that allow to obtain the results of Section 3.2.

3.1.1 General notations

The canonical basis of \mathbb{R}^3 is denoted e_1, e_2, e_3 . We denote the euclidian scalar product between two vectors u and v by $u \cdot v$. The euclidian distance to a subset $A \subset \mathbb{R}^3$ will be written $d(\cdot, A)$. The diameter of A will be denoted by $\text{diam}(A)$. If A is a Lipschitz domain, we denote by n the outward normal vector. $|\cdot|$ will be the Lebesgue measure on \mathbb{R}^3 .

If A, B are two real matrices, we write $A : B := \sum_{i,j=1}^3 A_{i,j} B_{i,j}$. If X is a vector or a matrix, its transpose will be denoted by X^T . If $A \subset \mathbb{R}^3$, the complementary set of A will be written A^c . We define $Q :=]-\frac{1}{2}, \frac{1}{2}[^3$ and, for $k \in \mathbb{Z}^3$, $Q_k := \prod_{j=1}^3]-\frac{1}{2} + k_j, \frac{1}{2} + k_j[= Q + k$. If $x \in \mathbb{R}^3$ and $r > 0$, we denote by $B(x, r)$ the open ball centered in x of radius r .

The gradient operator of a real or vector valued function will be denoted $\nabla \cdot$ and the second order derivative of a real or vector valued function will be written $D^2 \cdot$. The divergence operator will be denoted $\text{div} \cdot$ and the scalar or vectorial Laplacian $\Delta \cdot$.

Functional spaces. If ω is an open subset of \mathbb{R}^3 , $m \in \mathbb{N}^*$ and $1 \leq q \leq +\infty$, we denote by $L^q(\omega)$ the standard Lebesgue spaces and $H^m(\omega), W^{m,q}(\omega)$ the standard Sobolev spaces. We denote by $[L^q(\omega)]^3$, $[H^m(\omega)]^3$ and $[W^{m,q}(\omega)]^3$ the spaces of vector valued functions whose components are respectively elements of $L^q(\omega)$, $H^m(\omega)$ and $W^{m,q}(\omega)$. The space $L^q(\omega)/\mathbb{R}$ corresponds to the equivalence classes for the relation \sim defined by: for all $f, g \in L^q(\omega)$, $f \sim g$ if and only if $f - g$ is a.e constant in ω . $\mathcal{D}(\omega)$ will be the set of smooth and compactly supported functions in ω . We denote by $\mathcal{C}^\infty(\omega)$ (resp. $\mathcal{C}^\infty(\bar{\omega})$) the set of smooth functions defined on ω (resp. $\bar{\omega}$).

3.1.2 Review of the periodic case

In this subsection, we recall the results of the homogenization of the Stokes system in periodically perforated domains with large holes. For more details, see [99, 91, 2].

Notations. We fix a locally Lipschitz bounded domain $\Omega \subset \mathbb{R}^3$ and a subset $\mathcal{O}_0^{\text{per}}$ such that $\mathcal{O}_0^{\text{per}} \subset\subset Q$, $\mathcal{O}_0^{\text{per}}$ is of class $\mathcal{C}^{2,\alpha}$ and $Q \setminus \overline{\mathcal{O}_0^{\text{per}}}$ is connected. We define for $k \in \mathbb{Z}^3$, $\mathcal{O}_k^{\text{per}} := \mathcal{O}_0^{\text{per}} + k$. \mathcal{O}^{per} will be the set of perforations, that is, $\mathcal{O}^{\text{per}} := \bigcup_{k \in \mathbb{Z}^3} \mathcal{O}_k^{\text{per}}$.

We define some periodic functional spaces that will be used in the sequel. Using the notations of our problem, we set for $1 \leq q \leq +\infty$,

$$L^{q,\text{per}} \left(Q \setminus \overline{\mathcal{O}_0^{\text{per}}} \right) := \left\{ u \in L^q_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\mathcal{O}^{\text{per}}}) \text{ s.t. } u \text{ is } Q\text{-periodic} \right\}$$

and

$$H^{1,\text{per}} \left(Q \setminus \overline{\mathcal{O}_0^{\text{per}}} \right) := \left\{ u \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\mathcal{O}^{\text{per}}}) \text{ s.t. } u \text{ is } Q\text{-periodic and } \partial_i u \text{ are } Q\text{-periodic, } i = 1, 2, 3 \right\},$$

where the subscript *loc* means locally in \mathbb{R}^3 . The space of H^1 -periodic vector valued functions will be $\left[H^{1,\text{per}} \left(Q \setminus \overline{\mathcal{O}_0^{\text{per}}} \right) \right]^3$. The space of H^1 -periodic functions that vanish on the perforations is

$$H_0^{1,\text{per}} \left(Q \setminus \overline{\mathcal{O}_0^{\text{per}}} \right) := \left\{ u \in H^{1,\text{per}} \left(Q \setminus \overline{\mathcal{O}_0^{\text{per}}} \right) \text{ s.t. } u = 0 \text{ on } \partial \mathcal{O}_0^{\text{per}} \right\}.$$

Similarly, we define $\left[H_0^{1,\text{per}} \left(Q \setminus \overline{\mathcal{O}_0^{\text{per}}} \right) \right]^3$. In the sequel, we use the summation convention on repeated indices.

For $\varepsilon > 0$, we denote $Y_\varepsilon^{\text{per}} := \{k \in \mathbb{Z}^3, \varepsilon Q_k \subset \Omega\}$. We define the periodically perforated domain $\Omega_\varepsilon^{\text{per}}$ by (see Figure 3.1)

$$\Omega_\varepsilon^{\text{per}} := \Omega \setminus \bigcup_{k \in Y_\varepsilon^{\text{per}}} \varepsilon \overline{\mathcal{O}_k^{\text{per}}}.$$

It is easily seen that $\Omega_\varepsilon^{\text{per}}$ is open and connected.

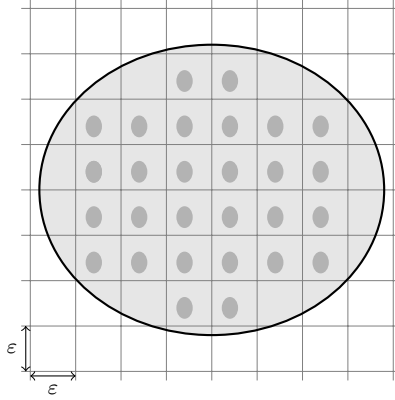


Figure 3.1: Periodic domain $\Omega_\varepsilon^{\text{per}}$

For $f \in [L^2(\Omega)]^3$, there exists a unique couple $(u_\varepsilon, p_\varepsilon) \in [H_0^1(\Omega_\varepsilon^{\text{per}})]^3 \times L^2(\Omega_\varepsilon^{\text{per}})/\mathbb{R}$ solution of System (3.1.1). The Poincaré inequality in perforated domains (see e.g. [99, Lemma 1]) and standard energy estimates yield the bound

$$\|u_\varepsilon\|_{[L^2(\Omega_\varepsilon^{\text{per}})]^3} \leq C\varepsilon^2$$

where C is a constant independent of ε . Thus, after extraction of a subsequence, $u_\varepsilon/\varepsilon^2$ converges L^2 -weakly to some limit velocity u^* . Besides, it can be proved (see [99, Theorem 1]) that the pressure p_ε converges $L^2(\Omega)/\mathbb{R}$ -strongly to the macroscopic pressure p_0 which is defined up to the addition of a constant. The couple (u^*, p_0) is determined by the Darcy's law which we recall here

$$\begin{cases} \operatorname{div}(u^*) = 0 & \text{in } \Omega, \\ u^* = A(f - \nabla p_0), \\ u^* \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.1.2)$$

In (3.1.2), the symmetric and positive definite matrix A is the so-called permeability tensor. Its coefficients are defined by

$$A_i^j = \int_{Q \setminus \overline{\mathcal{O}_0^{\text{per}}}} w_j^{\text{per}} \cdot e_i = \int_{Q \setminus \overline{\mathcal{O}_0^{\text{per}}}} \nabla w_i^{\text{per}} : \nabla w_j^{\text{per}}, \quad 1 \leq i, j \leq 3, \quad (3.1.3)$$

where the functions $w_j^{\text{per}}, j = 1, 2, 3$ are the cell periodic first correctors and solve the following Stokes problems:

$$\begin{cases} -\Delta w_j^{\text{per}} + \nabla p_j^{\text{per}} = e_j & \text{in } Q \setminus \overline{\mathcal{O}_0^{\text{per}}}, \\ \operatorname{div} w_j^{\text{per}} = 0, \\ w_j^{\text{per}} = 0 & \text{on } \partial \mathcal{O}_0^{\text{per}}. \end{cases} \quad (3.1.4)$$

We note that for fixed $j \in \{1, 2, 3\}$, Problem (3.1.4) is well-posed in the space $[H_0^{1,\text{per}}(Q \setminus \overline{\mathcal{O}_0^{\text{per}}})]^3 \times L^{2,\text{per}}(Q \setminus \overline{\mathcal{O}_0^{\text{per}}})/\mathbb{R}$ (see [91]). A central point in the proof of the convergence of p_ε to p_0 is the construction of an extension of the pressure p_ε in the periodic holes. This extension is constructed in [99] by a duality argument.

The corrector equations (3.1.4) can be guessed by a standard two-scale expansion of u_ε and p_ε of the form

$$\begin{aligned} u_\varepsilon &= u_0^{\text{exp}}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1^{\text{exp}}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2^{\text{exp}}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 u_3^{\text{exp}}\left(x, \frac{x}{\varepsilon}\right) + \dots, \\ p_\varepsilon &= p_0^{\text{exp}}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon p_1^{\text{exp}}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 p_2^{\text{exp}}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 p_3^{\text{exp}}\left(x, \frac{x}{\varepsilon}\right) + \dots \end{aligned}$$

where the functions $u_i^{\text{exp}}(x, \cdot)$ and $p_i^{\text{exp}}(x, \cdot)$ are Q -periodic for fixed $x \in \Omega$ (see [91, Section 7.2]). It can be proved that the function p_0^{exp} is independent of the microscopic variable and that $p_0^{\text{exp}}(x, \frac{x}{\varepsilon}) = p_0(x)$ for all $x \in \Omega$ (where p_0 is given by (3.1.2)). Besides, the functions u_0 and u_1 vanish and (we use, as indicated above, the summation convention over repeated indices)

$$u_2^{\text{exp}}\left(x, \frac{x}{\varepsilon}\right) = w_j^{\text{per}}\left(\frac{x}{\varepsilon}\right) (f_j - \partial_j p_0)(x) \quad \text{and} \quad p_1^{\text{exp}}\left(x, \frac{x}{\varepsilon}\right) = p_j^{\text{per}}\left(\frac{x}{\varepsilon}\right) (f_j - \partial_j p_0)(x).$$

We define the remainders

$$R_\varepsilon := u_\varepsilon - \varepsilon^2 w_j^{\text{per}}\left(\frac{\cdot}{\varepsilon}\right) (f_j - \partial_j p_0) \quad \text{and} \quad \pi_\varepsilon := p_\varepsilon - p_0 - \varepsilon p_j^{\text{per}}\left(\frac{\cdot}{\varepsilon}\right) (f_j - \partial_j p_0).$$

The strong convergence $R_\varepsilon/\varepsilon^2 \rightarrow 0$ in $L^2(\Omega_\varepsilon^{\text{per}})$ -norm is proved in [3, Theorem 1.3]. An H^1 -quantitative estimate of this convergence is given in [95], provided that Ω is of class $\mathcal{C}^{2,\alpha}$. We will provide a new H^2 -convergence estimate when $\operatorname{div}(Af) = 0$ and f is compactly supported in Ω (see Theorem 3.2.3 and Remark 3.2.5 below).

In what follows, we extend w_j^{per} by zero in the periodic perforations. The pressure p_j^{per} is extended by a constant λ_j (for example zero) in the perforations.

3.1.3 The non-periodic setting

We fix a periodic set of perforations as described in the previous subsection. We describe the non-periodic setting (see [22] for more details). For $k \in \mathbb{Z}^3$ and $\alpha > 0$, we define (see figure 3.3a)

$$\mathcal{O}_k^{\text{per},+}(\alpha) := \{x \in Q_k, \quad d(x, \mathcal{O}_k^{\text{per}}) < \alpha\},$$

and

$$\mathcal{O}_k^{\text{per},-}(\alpha) := \{x \in \mathcal{O}_k^{\text{per}}, \quad d(x, \partial \mathcal{O}_k^{\text{per}}) > \alpha\}.$$

For all $k \in \mathbb{Z}^3$, we fix an open subset \mathcal{O}_k of Q_k . We suppose that the sequence $(\mathcal{O}_k)_{k \in \mathbb{Z}^3}$ satisfies Assumptions **(A1)**-**(A5)** below. We define the non periodic set of perforations by

$$\mathcal{O} := \bigcup_{k \in \mathbb{Z}^3} \mathcal{O}_k.$$

(A1) For all $k \in \mathbb{Z}^3$, we have $\mathcal{O}_k \subset\subset Q_k$ and $Q_k \setminus \overline{\mathcal{O}_k}$ is connected.

(A2) For all $k \in \mathbb{Z}^3$, the perforation \mathcal{O}_k is Lipschitz continuous.

(A3) There exists a sequence $(\alpha_k)_{k \in \mathbb{Z}^3} \in \ell^1(\mathbb{Z}^3)$ such that for all $k \in \mathbb{Z}^3$, $\alpha_k > 0$ and we have the following chain inclusion:

$$\mathcal{O}_k^{\text{per},-}(\alpha_k) \subset \mathcal{O}_k \subset \mathcal{O}_k^{\text{per},+}(\alpha_k).$$

We refer to figure 3.3a for an illustration of **(A3)**.

The assumptions **(A1)**-**(A2)** are analogous to the one made on \mathcal{O}^{per} and guarantee connectedness and some regularity on the perforated domain. Assumption **(A3)** is the geometric assumption that makes precise that $(\mathcal{O}_k)_{k \in \mathbb{Z}^3}$ is a perturbation of $(\mathcal{O}_k^{\text{per}})_{k \in \mathbb{Z}^3}$. It is worth mentioning that, in the whole paper, we fix a periodic distribution of holes that is perturbed in the sense of Assumption **(A3)**. This models some local defects that could appear in this periodic setting. In particular, the translation of $(\mathcal{O}_k^{\text{per}})_{k \in \mathbb{Z}^3}$ by a fixed vector $\ell_0 \in \mathbb{R}^3 \setminus \{0\}$ does not satisfy Assumption **(A3)** but corresponds to another periodic set of perforations. We also point out that Assumption **(A3)** is linked to the method of proof which strongly uses that the non-periodic perforations are close to a periodic set of holes. We recall (see [22, Lemma A.1 and Lemma A.3]) that Assumptions **(A1)**-**(A3)** imply the following facts:

- There exists $\delta > 0$ such that for all $k \in \mathbb{Z}^3$, $d(\mathcal{O}_k, \partial Q_k) \geq \delta$. In other words, \mathcal{O}_k is strictly included in Q_k , uniformly with respect to k .
- We have

$$\sum_{k \in \mathbb{Z}^3} |\mathcal{O}_k \Delta \mathcal{O}_k^{\text{per}}| < +\infty \quad (3.1.5)$$

where Δ stands for the sets symmetric difference operator.

Using the first point above, we can introduce two smooth open sets Q' and Q'' such that (see Figure 3.2) $Q' \subset\subset Q \subset\subset Q''$ and for all $k \in \mathbb{Z}^3$, $(Q' + k) \cap \mathcal{O} = (Q'' + k) \cap \mathcal{O} = \mathcal{O}_k$. We define, for $k \in \mathbb{Z}^3$,

$$Q'_k := Q' + k \quad \text{and} \quad Q''_k := Q'' + k. \quad (3.1.6)$$

The sets Q'_k and Q''_k , $k \in \mathbb{Z}^3$ will be used several times in the sequel.

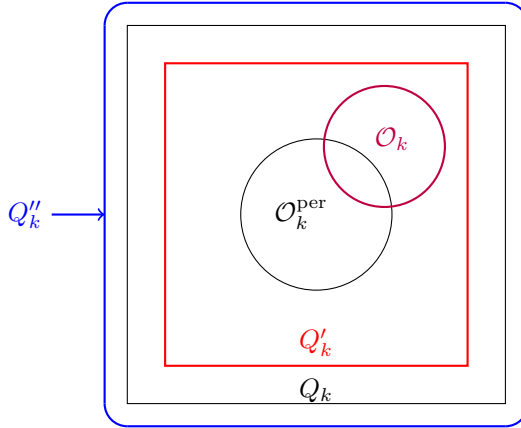


Figure 3.2: A cell Q_k , $k \in \mathbb{Z}^3$

(A4) This assumption is divided into two sub-assumptions **(A4)**₀ and **(A4)**₁.

(A4)₀ For all $1 < q < +\infty$, there exists a constant $C_q^0 > 0$ such that for all $k \in \mathbb{Z}^3$, the problem

$$\begin{cases} \operatorname{div} v = f & \text{in } Q_k \setminus \overline{\mathcal{O}_k}, \\ v = 0 & \text{on } \partial [Q_k \setminus \overline{\mathcal{O}_k}] \end{cases} \quad (3.1.7)$$

with $f \in L^q(Q_k \setminus \overline{\mathcal{O}_k})$ completed with the compatibility condition

$$\int_{Q_k \setminus \overline{\mathcal{O}_k}} f = 0 \quad (3.1.8)$$

admits a solution v such that $v \in [W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})]^3$ and

$$\|v\|_{[W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})]^3} \leq C_q^0 \|f\|_{L^q(Q_k \setminus \overline{\mathcal{O}_k})}. \quad (3.1.9)$$

(A4)₁ For all $1 < q < +\infty$, there exists a constant $C_q^1 > 0$ such that for all $k \in \mathbb{Z}^3$, Problem (3.1.7) with $f \in W_0^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})$ completed with the compatibility condition (3.1.8) admits a solution v such that $v \in [W_0^{2,q}(Q_k \setminus \overline{\mathcal{O}_k})]^3$ and

$$\|v\|_{[W_0^{2,q}(Q_k \setminus \overline{\mathcal{O}_k})]^3} \leq C_q^1 \|f\|_{W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})}. \quad (3.1.10)$$

(A5) For all $1 < q < +\infty$, there exists a constant $C_q > 0$ such that for all $k \in \mathbb{Z}^3$, if $(v, p) \in [W^{1,q}(Q_k'' \setminus \overline{\mathcal{O}_k})]^3 \times L^q(Q_k'' \setminus \overline{\mathcal{O}_k})$ is solution to the Stokes problem

$$\begin{cases} -\Delta v + \nabla p = f & \text{in } Q_k'' \setminus \overline{\mathcal{O}_k}, \\ \operatorname{div} v = 0, \\ v = 0 & \text{on } \partial \mathcal{O}_k \end{cases} \quad (3.1.11)$$

with $f \in L^q(Q_k'' \setminus \overline{\mathcal{O}_k})$, then $(v, p) \in [W^{2,q}(Q_k \setminus \overline{\mathcal{O}_k})]^3 \times W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})$ and

$$\|v\|_{[W^{2,q}(Q_k \setminus \overline{\mathcal{O}_k})]^3} + \|p\|_{W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})} \leq C_q [\|f\|_{L^q(Q_k'' \setminus \overline{\mathcal{O}_k})} + \|v\|_{[W^{1,q}(Q_k'' \setminus \overline{\mathcal{O}_k})]^3} + \|p\|_{L^q(Q_k'' \setminus \overline{\mathcal{O}_k})}]. \quad (3.1.12)$$

Remark 3.1.1. For each fixed $k \in \mathbb{Z}^3$, the estimates (3.1.9) and (3.1.10) are satisfied with constants $C_{q,k}^i$, $i = 0, 1$, depending on k , see [58, Theorem III.3.3]. Similarly, as long as \mathcal{O}_k is of class \mathcal{C}^2 , (3.1.12) is satisfied when k is fixed (see [58, Theorem IV.5.1]). Assumptions **(A4)**-**(A5)** require that the constants appearing in (3.1.9), (3.1.10) and (3.1.12) are uniform with respect to $k \in \mathbb{Z}^3$.

Assumptions **(A4)**-**(A5)** are the weakest possible given our method of proof. However, they are associated to PDEs and we would like a somewhat more geometric interpretation of these assumptions, in the spirit of **(A3)**. In fact, we may replace **(A4)**-**(A5)** by the likely stronger (but geometric) Assumptions **(A4)'**-**(A5)'** below.

We suppose that there exist $r > 0$ and $M > 0$ such that for all $k \in \mathbb{Z}^3$ and for all $x \in \partial \mathcal{O}_k$, there exists $\zeta_x : U_x \rightarrow \mathbb{R}$ where $U_x \subset \mathbb{R}^2$, $0 \in U_x$ and $r_x > r$ such that, after eventually rotating and/or translating the local coordinate system, we have that $\zeta_x(0) = 0$ and

$$(Q_k \setminus \overline{\mathcal{O}_k}) \cap B(x, r_x) = \{(y_1, y_2, y_3) \in B(x, r_x), \quad y_3 > \zeta_x(y_1, y_2) \text{ and } (y_1, y_2) \in U_x\}. \quad (3.1.13)$$

We assume the following uniform regularity properties:

(A4)' The functions ζ_x , $x \in \partial \mathcal{O}_k$ are Lipschitz functions with Lipschitz constant $\|\zeta_x\|_{\operatorname{Lip}(U_x)}$ satisfying $\|\zeta_x\|_{\operatorname{Lip}(U_x)} \leq M$.

(A5)' The functions ζ_x , $x \in \partial \mathcal{O}_k$ are of class \mathcal{C}^2 and satisfy $\nabla \zeta_x(0) = 0$ with the estimate $\|\zeta_x\|_{W^{2,\infty}(U_x)} \leq M$.

In Assumptions **(A4)'**-**(A5)'** above, we emphasize that M is independent of k and x .

We prove in Appendix 3.4.2 that Assumptions **(A3)** and **(A4)'** imply Assumption **(A4)** and that Assumptions **(A3)** and **(A5)'** imply Assumption **(A5)**. We also note that **(A5)'** implies **(A4)'**.

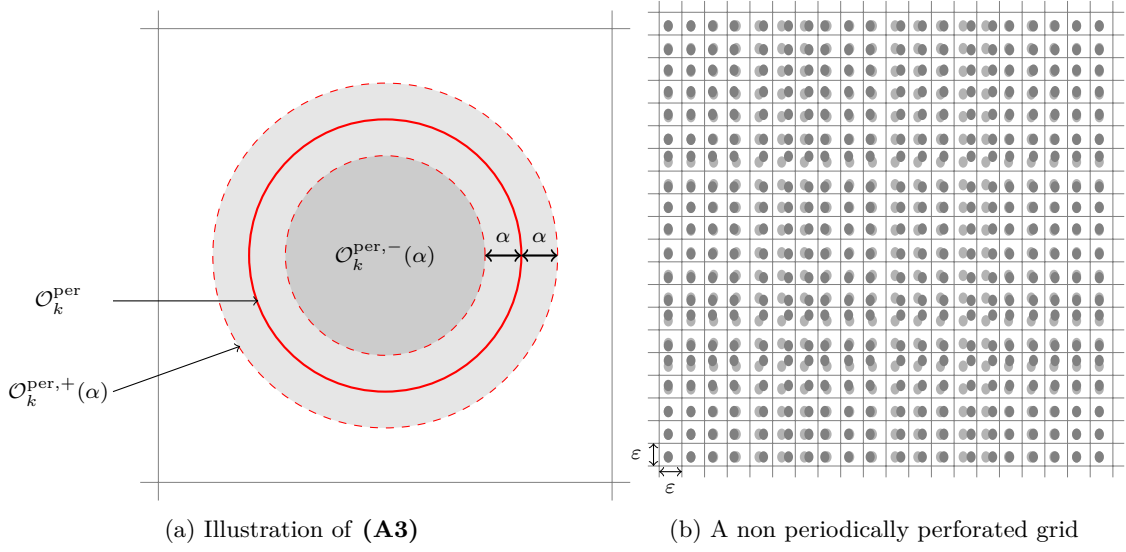


Figure 3.3: The non-periodic setting

Example 3.1.2. We give some examples of perforations satisfying **(A1)**-**(A5)**:

- Compactly supported perturbations, that is, we change $\mathcal{O}_k^{\text{per}}$ in a finite number of cells Q_k ;
- We remove a finite number of perforations;
- We make ℓ^1 -translations of the periodic perforations that is we choose a sequence $(\delta_k)_{k \in \mathbb{Z}^3}$ such that $\delta_k \in \mathbb{R}^3$, $\sum_{k \in \mathbb{Z}^3} |\delta_k| < +\infty$ and for all $k \in \mathbb{Z}^3$, $\mathcal{O}_k \subset\subset Q_k$ and $\mathcal{O}_k = \mathcal{O}_k^{\text{per}} + \delta_k$.

Remark 3.1.3. The assumption $\mathcal{O}_k \subset\subset Q_k$ is automatically implied by **(A3)** except for a finite number of cells. Dropping it would change some technical details but not the results of the paper.

The perforated domain. We assume that Ω is a smooth bounded domain of \mathbb{R}^3 . We denote

$$Y_\varepsilon := \{k \in \mathbb{Z}^3, \quad \varepsilon Q_k \subset \Omega\} \quad (3.1.14)$$

and define (see Figure 3.3b)

$$\Omega_\varepsilon := \Omega \setminus \bigcup_{k \in Y_\varepsilon} \varepsilon \overline{\mathcal{O}_k}. \quad (3.1.15)$$

We notice that Ω_ε is a bounded, locally Lipschitz and connected open subset of \mathbb{R}^3 .

For $f \in [L^2(\Omega)]^3$, there is a unique solution $(u_\varepsilon, p_\varepsilon) \in [H_0^1(\Omega_\varepsilon)]^3 \times L^2(\Omega_\varepsilon)/\mathbb{R}$ to the Stokes system

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \quad (3.1.16)$$

In the sequel, we study the homogenization of $(u_\varepsilon, p_\varepsilon)$.

3.2 Results

The first result concerns the existence of the first order correctors. We can perform a two scale expansion of the form

$$u_\varepsilon(x) = \varepsilon^2 \left[u_2^{\text{exp}} \left(x, \frac{x}{\varepsilon} \right) + \varepsilon u_3^{\text{exp}} \left(x, \frac{x}{\varepsilon} \right) + \dots \right], \quad p_\varepsilon(x) = p_0^{\text{exp}} \left(x, \frac{x}{\varepsilon} \right) + \varepsilon p_1^{\text{exp}} \left(x, \frac{x}{\varepsilon} \right) + \dots \quad (3.2.1)$$

to (3.1.16) and find that

$$p_0^{\text{exp}}\left(x, \frac{x}{\varepsilon}\right) = p_0(x),$$

$$u_2^{\text{exp}}(x, y) = \sum_{j=1}^3 w_j(y)(f_j - \partial_j p_0)(x) \quad \text{and} \quad p_1^{\text{exp}}(x, y) = \sum_{j=1}^3 p_j(y)(f_j - \partial_j p_0)(x), \quad (3.2.2)$$

where f_1, f_2, f_3 denote the components of the vector field f , (w_j, p_j) is solution to the following Stokes system for $j = 1, 2, 3$:

$$\begin{cases} -\Delta w_j + \nabla p_j = e_j & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\ \operatorname{div} w_j = 0, \\ w_j = 0 & \text{on } \partial\mathcal{O}. \end{cases} \quad (3.2.3)$$

and p_0 is given by the Darcy's law (3.1.2).

Theorem 3.2.1 (Existence of correctors). *Suppose that Assumptions (A1)-(A3) and (A4)₀ are satisfied. For all $j \in \{1, 2, 3\}$, System (3.2.3) admits in the distribution sense a solution (w_j, p_j) of the form*

$$w_j = w_j^{\text{per}} + \widetilde{w}_j \quad \text{and} \quad p_j = p_j^{\text{per}} + \widetilde{p}_j$$

where $(\widetilde{w}_j, \widetilde{p}_j) \in [H^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3 \times L^2_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})$. Moreover, we have the following estimate

$$\|\widetilde{p}_j - \langle \widetilde{p}_j \rangle\|_{L^2(\frac{1}{\varepsilon}\Omega_\varepsilon)} \leq C\varepsilon^{-1},$$

where C is a constant independent of ε and $\langle \widetilde{p}_j \rangle$ denotes the mean value of \widetilde{p}_j on $\frac{1}{\varepsilon}\Omega_\varepsilon$.

Note that the obvious solution $w_j = 0, p_j = e_j \cdot x$ does not satisfy the statement of Theorem 3.2.1. If it were the case, then there would exist $\widetilde{w}_j \in [H^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$ such that $w_j^{\text{per}} + \widetilde{w}_j = 0$. Necessarily, this implies that w_j^{per} is identically zero. However, recalling (3.1.4), this would imply that $\nabla p_j^{\text{per}} = e_j$ in $Q \setminus \overline{\mathcal{O}_0^{\text{per}}}$. Thus, by connectedness $p_j^{\text{per}} = e_j \cdot x$ in $Q \setminus \overline{\mathcal{O}^{\text{per}}}$ (up to a constant) which is impossible since p_j^{per} is periodic. We emphasize that the main point of Theorem 3.2.1 is to provide a solution (w_j, p_j) to (3.2.3) where w_j can be decomposed as a sum of the periodic corrector w_j^{per} (defined in (3.1.4)) and a perturbation $\widetilde{w}_j \in H^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})$ that, formally, vanish at infinity.

As pointed out in the introduction of the thesis (see Subsection 1.2.1.3), we can prove that there exists an extension of $(u_\varepsilon, p_\varepsilon)$ to Ω such that

$$\begin{cases} u_\varepsilon/\varepsilon^2 \xrightarrow{\varepsilon} u^* & \text{in } L^2(\Omega) \\ p_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} p_0 & \text{in } L^2(\Omega)/\mathbb{R}, \end{cases}$$

where (u^*, p_0) is given by the Darcy's law (3.1.2)–(3.1.3). In particular, the macroscopic behaviour of $(u_\varepsilon, p_\varepsilon)$ remains unchanged. We define

$$R_\varepsilon := u_\varepsilon - \varepsilon^2 \sum_{j=1}^3 w_j \left(\frac{\cdot}{\varepsilon}\right) (f_j - \partial_j p_0) \quad \text{and} \quad \pi_\varepsilon := p_\varepsilon - p_0 - \varepsilon \sum_{j=1}^3 p_j \left(\frac{\cdot}{\varepsilon}\right) (f_j - \partial_j p_0).$$

Following the ideas of the proof of [3, Theorem 1.3], we can prove under the assumption $f \in [W^{3,\infty}(\Omega)]^3$ that $R_\varepsilon/\varepsilon^2 \xrightarrow{\varepsilon \rightarrow 0} 0$ in the non-periodic setting for the $[L^2(\Omega)]^3$ -norm (where it is understood that u_ε and $w_j, j = 1, 2, 3$ are extended by zero in the perforations). This fact, though relevant because it makes (3.2.1) rigorous, is not strong enough to justify the construction of the non-periodic correctors $(w_j, p_j), j = 1, 2, 3$. Indeed, if we set

$$R_\varepsilon^{\text{per}} := u_\varepsilon - \varepsilon^2 \sum_{j=1}^3 w_j^{\text{per}} \left(\frac{\cdot}{\varepsilon}\right) (f_j - \partial_j p_0),$$

we notice that

$$R_\varepsilon = R_\varepsilon^{\text{per}} - \varepsilon^2 \sum_{j=1}^3 \widetilde{w}_j \left(\frac{\cdot}{\varepsilon} \right) (f_j - \partial_j p_0).$$

Since $\widetilde{w}_j \in [L^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$, one has for $j = 1, 2, 3$:

$$\begin{aligned} \left\| \widetilde{w}_j \left(\frac{\cdot}{\varepsilon} \right) (f_j - \partial_j p_0) \right\|_{[L^2(\Omega_\varepsilon)]^3} &= \varepsilon^{\frac{3}{2}} \|\widetilde{w}_j (f_j - \partial_j p_0)(\varepsilon \cdot)\|_{[L^2(\frac{1}{\varepsilon}\Omega_\varepsilon)]^3} \\ &\leq \varepsilon^{\frac{3}{2}} \|\widetilde{w}_j\|_{[L^2(\mathbb{R}^3)]^3} \|f_j - \partial_j p_0\|_{L^\infty(\Omega)} = C\varepsilon^{\frac{3}{2}}. \end{aligned}$$

Thus $R_\varepsilon^{\text{per}}/\varepsilon^2 = R_\varepsilon/\varepsilon^2 + O(\varepsilon^{3/2})$. This proves that $R_\varepsilon^{\text{per}}/\varepsilon^2 \xrightarrow{\varepsilon \rightarrow 0} 0$ for the $[L^2(\Omega)]^3$ -norm. So, using w_j^{per} instead of w_j does not change the convergence of u_ε to its first order asymptotic expansion.

Yet, since w_j and p_j , $j = 1, 2, 3$ are the *ad hoc* correctors for the non-periodic setting, there must be situations highlighting that the approximation of u_ε (resp. p_ε) by $\varepsilon^2 w_j(\cdot/\varepsilon)(f_j - \partial_j p_0)$ (resp. $p_0 + \varepsilon p_j(\cdot/\varepsilon)(f_j - \partial_j p_0)$) is improved in some sense when we use w_j instead of w_j^{per} . We exhibit in Theorem 3.2.3 such a situation (see Remark 3.2.6).

Before stating Theorem 3.2.3, we obtain in Theorem 3.2.2 H^2 -estimates for the solution of a Stokes system posed in Ω_ε (see [82, Theorem 4.1] for the periodic case).

Theorem 3.2.2 (Estimates for a Stokes problem). *Suppose that Assumptions (A4)₀ and (A5) are satisfied. Let $f \in [L^2(\Omega_\varepsilon)]^3$ and $(u, p) \in [H_0^1(\Omega_\varepsilon)]^3 \times L^2(\Omega_\varepsilon)/\mathbb{R}$ be solution of*

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \operatorname{div}(u_\varepsilon) = 0, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \quad (3.2.4)$$

Then $(u_\varepsilon, p_\varepsilon) \in [H^2(\Omega_\varepsilon)]^3 \times H^1(\Omega_\varepsilon)/\mathbb{R}$ and there exists a constant $C > 0$ such that for any domain $\Omega'' \subset\subset \Omega$ and all $\varepsilon < \varepsilon_0(\Omega'')$,

$$\begin{aligned} \|D^2 u_\varepsilon\|_{[L^2(\Omega'' \cap \Omega_\varepsilon)]^3} + \varepsilon^{-1} \|\nabla u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^3} + \varepsilon^{-2} \|u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^3} \\ + \|\nabla p_\varepsilon\|_{L^2(\Omega'' \cap \Omega_\varepsilon)} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C \|f\|_{[L^2(\Omega_\varepsilon)]^3}. \end{aligned}$$

Furthermore, the couple $(u_\varepsilon, p_\varepsilon)$ is unique in $[H^1(\Omega_\varepsilon)]^3 \times L^2(\Omega_\varepsilon)/\mathbb{R}$.

Theorem 3.2.3 (Convergence Theorem). *Suppose that assumptions (A1)-(A5) are satisfied. Let $f \in [W^{3,\infty}(\Omega)]^3$ be such that $\operatorname{div}(Af) = 0$ and f is compactly supported in Ω . There exists a constant $C > 0$ such that if $\Omega'' \subset\subset \Omega$ and $\varepsilon < \varepsilon_0(\Omega'', \Omega, f)$,*

$$\begin{aligned} \left\| D^2 \left[u_\varepsilon - \varepsilon^2 w_j \left(\frac{\cdot}{\varepsilon} \right) f_j \right] \right\|_{[L^2(\Omega'' \cap \Omega_\varepsilon)]^3} + \varepsilon^{-1} \left\| \nabla \left[u_\varepsilon - \varepsilon^2 w_j \left(\frac{\cdot}{\varepsilon} \right) f_j \right] \right\|_{[L^2(\Omega_\varepsilon)]^3} \\ + \varepsilon^{-2} \left\| u_\varepsilon - \varepsilon^2 w_j \left(\frac{\cdot}{\varepsilon} \right) f_j \right\|_{[L^2(\Omega_\varepsilon)]^3} \leq C\varepsilon \end{aligned} \quad (3.2.5)$$

and

$$\left\| \nabla \left[p_\varepsilon - \varepsilon \left\{ p_j \left(\frac{\cdot}{\varepsilon} \right) - \lambda_\varepsilon^j \right\} f_j \right] \right\|_{L^2(\Omega'' \cap \Omega_\varepsilon)} + \left\| p_\varepsilon - \varepsilon \left\{ p_j \left(\frac{\cdot}{\varepsilon} \right) - \lambda_\varepsilon^j \right\} f_j \right\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C\varepsilon, \quad (3.2.6)$$

where

$$\lambda_\varepsilon^j = \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} p_j \left(\frac{\cdot}{\varepsilon} \right).$$

Remark 3.2.4. *We note that Theorem 3.2.2 and Theorem 3.2.3 are valid in the periodic case (that is in the framework of subsection 3.1.2). This provides a new situation in which quantitative error estimates can be obtained, besides the ones of [80, 95].*

Remark 3.2.5. *The assumptions $\operatorname{div}(Af) = 0$ and f compactly supported in Ω make boundary effects disappear. Indeed, it is straightforward to see that in this case $\nabla p_0 = 0$ in Ω (see (3.1.2)). Since f is compactly supported, we have $\varepsilon^2 w_j(\cdot/\varepsilon) f_j = 0$ on $\partial\Omega$, so u_ε and its first order expansion coincide on $\partial\Omega$. This explains why the $O(\varepsilon^2)$ H^1 -convergence rate of R_ε obtained in Theorem 3.2.3 is sharper than the $O(\varepsilon^{3/2})$ H^1 -convergence rate obtained in [95, Theorem 1.1].*

Remark 3.2.6. *By applying Theorem 3.2.3, we get that $R_\varepsilon \in H^2(\Omega_\varepsilon)$. We now note that, in general, one has $R_\varepsilon^{\text{per}} \notin H^2(\Omega_\varepsilon)$. This follows from the fact that $w_j^{\text{per}}(\cdot/\varepsilon) \notin H^2(\Omega_\varepsilon)$ (unless of course $\Omega_\varepsilon = \Omega_\varepsilon^{\text{per}}$) for $j = 1, 2, 3$. This is due to the normal derivative jumps of $w_j^{\text{per}}(\cdot/\varepsilon)$ along the parts of $\varepsilon\partial\mathcal{O}^{\text{per}}$ that are included in Ω_ε . This shows that, in the non-periodic case, using the periodic corrector in (3.2.2) does not give the expected convergence rate, contrary to the non-periodic corrector.*

Remark 3.2.7. *Theorem 3.2.2 and Theorem 3.2.3 can be proved up to the boundary of Ω with the same convergence rates when Ω is of class \mathcal{C}^2 . The proof is rather technical and will be omitted here.*

Remark 3.2.8. *Theorem 3.2.2 can be proved for the H^m -norm, $m > 0$ in the periodic domain $\Omega_\varepsilon^{\text{per}}$ (see [82, Theorem 4.2]) and in the non-periodic domain Ω_ε , provided that we require higher regularity of \mathcal{O}_k in (A5)' (typically that \mathcal{O}_k is uniformly with respect to k of class \mathcal{C}^{m+2} , see [58, Theorem IV.5.1]): if $f \in [H^m(\Omega_\varepsilon)]^3$, then $(u_\varepsilon, p_\varepsilon) \in [H^{m+2}(\Omega_\varepsilon)]^3 \times H^{m+1}(\Omega_\varepsilon)/\mathbb{R}$ and there exists a constant C independent of ε such that*

$$\|D^{m+2}u_\varepsilon\|_{[L^2(\Omega'' \cap \Omega_\varepsilon)]^3} + \|D^{m+1}p_\varepsilon\|_{L^2(\Omega'' \cap \Omega_\varepsilon)} \leq C \sum_{i=0}^m \frac{1}{\varepsilon^i} \|D^{m-i}f\|_{[L^2(\Omega_\varepsilon)]^3}.$$

Remark 3.2.9. *This paper presents only the three dimensional case. All that follows is true in dimension greater than 3. As for the two dimensional case, Theorem 3.2.1 and Theorem 3.2.2 are valid.*

The rest of the paper is devoted to proofs. In Section 3.3.1, we give the proof of Theorem 3.2.2 in both periodic and non periodic perforated domains. We next prove in Section 3.3.2 the existence of the non-periodic correctors. Finally, Section 3.3.3 is devoted to the proof of the convergence Theorem 3.2.3. Some technical Lemmas, especially concerning divergence problems, are postponed to Appendix 3.4.1.

3.3 Proofs

3.3.1 Proof of Theorem 3.2.2

We first state the following Poincaré-Friedrichs inequality:

Lemma 3.3.1. *Suppose that Assumptions (A1) and (A3) are satisfied. There exists a constant $C > 0$ independent of ε such that for all $u \in [H_0^1(\Omega_\varepsilon)]^3$, one has*

$$\int_{\Omega_\varepsilon} |u|^2 \leq C\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla u|^2.$$

Proof. For simplicity, we assume that each cube Q_k contains a non-periodic perforation. We recall that Y_ε is defined by (3.1.14) and we define $Z_\varepsilon := \{k \in \mathbb{Z}^d, \varepsilon Q_k \cap \partial\Omega \neq \emptyset\}$. We have the decomposition

$$\Omega_\varepsilon = \left(\bigcup_{k \in Y_\varepsilon} \varepsilon(\overline{Q_k} \setminus \overline{\mathcal{O}_k}) \right) \cup \left(\bigcup_{k \in Z_\varepsilon} [(\varepsilon\overline{Q_k}) \cap \Omega] \right). \quad (3.3.1)$$

Thanks to Assumption (A3) and the proof of [22, Lemma 3.2], we know that there exists a constant $C > 0$ independent of k and ε such that for all $k \in Y_\varepsilon$, we have the inequality

$$\int_{\varepsilon(Q_k \setminus \overline{\mathcal{O}_k})} u^2 \leq C\varepsilon^2 \int_{\varepsilon(Q_k \setminus \overline{\mathcal{O}_k})} |\nabla u|^2. \quad (3.3.2)$$

We now fix $k \in Z_\varepsilon$. Thanks to the proof of [99, Lemma 1], there exists a constant $C > 0$ which is independent of k and ε such that

$$\int_{(\varepsilon Q_k) \cap \Omega} u^2 \leq C\varepsilon^2 \int_{(\varepsilon Q_k) \cap \Omega} |\nabla u|^2. \quad (3.3.3)$$

Summing the estimate (3.3.2) over $k \in Y_\varepsilon$, the estimate (3.3.3) over $k \in Z_\varepsilon$ and using (3.3.1) concludes the proof of Lemma 3.3.1. \square

Let $(u_\varepsilon, p_\varepsilon)$ be the solution of (3.1.16). We have by classical energy estimates the following inequalities:

$$\left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \right)^{\frac{1}{2}} \leq C\varepsilon \|f\|_{[L^2(\Omega_\varepsilon)]^3} \quad \text{and} \quad \left(\int_{\Omega_\varepsilon} |u_\varepsilon|^2 \right)^{\frac{1}{2}} \leq C\varepsilon^2 \|f\|_{[L^2(\Omega_\varepsilon)]^3} \quad (3.3.4)$$

which will be useful in the proof of Theorem 3.2.2.

Proof of Theorem 3.2.2. In this proof, C will denote various constants independent of ε that can change from one line to another. We fix $\Omega'' \subset \subset \Omega$. We first show the following estimate:

$$\|D^2 u_\varepsilon\|_{[L^2(\Omega'' \cap \Omega_\varepsilon)]^{3 \times 3}} + \|\nabla p_\varepsilon\|_{[L^2(\Omega'' \cap \Omega_\varepsilon)]^3} \leq C \left[\varepsilon^{-1} \|\nabla u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^{3 \times 3}} + \varepsilon^{-2} \|u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^3} + \|f\|_{[L^2(\Omega_\varepsilon)]^3} \right]. \quad (3.3.5)$$

Proof of (3.3.5): we study Problem (3.2.4) on each periodic cell $Q_k \setminus \overline{\mathcal{O}_k}$. Let $k \in Y_\varepsilon$ such that $\varepsilon Q_k'' \subset \Omega$. We recall that Q_k'' is introduced in (3.1.6) and we define in $Q_k'' \setminus \overline{\mathcal{O}_k}$ the functions

$$\begin{cases} U_\varepsilon^k := \varepsilon^{-2} u_\varepsilon(\varepsilon \cdot), \\ P_\varepsilon^k := \varepsilon^{-1} p_\varepsilon(\varepsilon \cdot) - \lambda_k, \\ F_\varepsilon^k := f(\varepsilon \cdot), \end{cases}$$

where $\lambda_k \in \mathbb{R}$ is chosen such that

$$\int_{Q_k'' \setminus \overline{\mathcal{O}_k}} P_\varepsilon^k = 0.$$

Then $(U_\varepsilon^k, P_\varepsilon^k) \in [H^1(Q_k'' \setminus \overline{\mathcal{O}_k})]^3 \times L^2(Q_k'' \setminus \overline{\mathcal{O}_k})$ and $(U_\varepsilon^k, P_\varepsilon^k)$ is solution to the following Stokes system

$$\begin{cases} -\Delta U_\varepsilon^k + \nabla P_\varepsilon^k = F_\varepsilon^k & \text{in } Q_k'' \setminus \overline{\mathcal{O}_k}, \\ \operatorname{div}(U_\varepsilon^k) = 0, \\ U_\varepsilon^k = 0 & \text{on } \partial \mathcal{O}_k. \end{cases} \quad (3.3.6)$$

By applying Assumption **(A5)** to System (3.3.6), we get the estimate

$$\|U_\varepsilon^k\|_{[H^2(Q_k \setminus \overline{\mathcal{O}_k})]^3} + \|P_\varepsilon^k\|_{H^1(Q_k \setminus \overline{\mathcal{O}_k})} \leq C \left[\|U_\varepsilon^k\|_{[H^1(Q_k'' \setminus \overline{\mathcal{O}_k})]^3} + \|P_\varepsilon^k\|_{L^2(Q_k'' \setminus \overline{\mathcal{O}_k})} + \|F_\varepsilon^k\|_{[L^2(Q_k'' \setminus \overline{\mathcal{O}_k})]^3} \right]. \quad (3.3.7)$$

Assumption **(A4)**₀ and [58, Lemma III.3.2] applied with $\Omega_1 := Q_k \setminus \overline{\mathcal{O}_k}$ and $\Omega_2 := Q_k'' \setminus \overline{\mathcal{O}_k}$ give a function $v \in [H_0^1(Q_k'' \setminus \overline{\mathcal{O}_k})]^3$ such that $\operatorname{div}(v) = P_\varepsilon^k$ and

$$\|v\|_{[H^1(Q_k'' \setminus \overline{\mathcal{O}_k})]^3} \leq C \|P_\varepsilon^k\|_{L^2(Q_k'' \setminus \overline{\mathcal{O}_k})}, \quad (3.3.8)$$

where C is independent of k . Thus,

$$\|P_\varepsilon^k\|_{L^2(Q_k'' \setminus \overline{\mathcal{O}_k})}^2 = \langle \nabla P_\varepsilon^k, v \rangle_{H^{-1} \times H_0^1(Q_k'' \setminus \overline{\mathcal{O}_k})} \leq \|\nabla P_\varepsilon^k\|_{[H^{-1}(Q_k'' \setminus \overline{\mathcal{O}_k})]^3} \|v\|_{[H^1(Q_k'' \setminus \overline{\mathcal{O}_k})]^3}. \quad (3.3.9)$$

Gathering together (3.3.8) and (3.3.9) yields

$$\|P_\varepsilon^k\|_{L^2(Q_k'' \setminus \overline{\mathcal{O}_k})} \leq C \|\nabla P_\varepsilon^k\|_{[H^{-1}(Q_k'' \setminus \overline{\mathcal{O}_k})]^3}. \quad (3.3.10)$$

The triangle inequality applied to the first equation of (3.3.6) then provides the inequality

$$\begin{aligned} \|\nabla P_\varepsilon^k\|_{[H^{-1}(Q_k'' \setminus \overline{\mathcal{O}_k})]^3} &\leq \|\Delta U_\varepsilon^k\|_{[H^{-1}(Q_k'' \setminus \overline{\mathcal{O}_k})]^3} + \|F_\varepsilon^k\|_{[H^{-1}(Q_k'' \setminus \overline{\mathcal{O}_k})]^3} \\ &\leq \|\nabla U_\varepsilon^k\|_{[L^2(Q_k'' \setminus \overline{\mathcal{O}_k})]^{3 \times 3}} + \|F_\varepsilon^k\|_{[L^2(Q_k'' \setminus \overline{\mathcal{O}_k})]^3}. \end{aligned} \quad (3.3.11)$$

Collecting (3.3.7), (3.3.10) and (3.3.11), we get

$$\|U_\varepsilon^k\|_{[H^2(Q_k \setminus \overline{\mathcal{O}_k})]^3} + \|P_\varepsilon^k\|_{H^1(Q_k \setminus \overline{\mathcal{O}_k})} \leq C \left[\|U_\varepsilon^k\|_{[H^1(Q_k'' \setminus \overline{\mathcal{O}_k})]^3} + \|F_\varepsilon^k\|_{[L^2(Q_k'' \setminus \overline{\mathcal{O}_k})]^3} \right].$$

In particular, we deduce

$$\|D^2 U_\varepsilon^k\|_{[L^2(Q_k \setminus \overline{\mathcal{O}_k})]^{3 \times 3}} + \|\nabla P_\varepsilon^k\|_{[L^2(Q_k \setminus \overline{\mathcal{O}_k})]^3} \leq C \left[\|U_\varepsilon^k\|_{[H^1(Q_k'' \setminus \overline{\mathcal{O}_k})]^3} + \|F_\varepsilon^k\|_{[L^2(Q_k'' \setminus \overline{\mathcal{O}_k})]^3} \right]. \quad (3.3.12)$$

Scaling back (3.3.12) gives

$$\begin{aligned} \|D^2 u_\varepsilon\|_{[L^2(\varepsilon Q_k \setminus \overline{\mathcal{O}_k})]^{3 \times 3}} + \|\nabla p_\varepsilon\|_{[L^2(\varepsilon Q_k \setminus \overline{\mathcal{O}_k})]^3} &\leq C [\varepsilon^{-1} \|\nabla u_\varepsilon\|_{[L^2(\varepsilon Q_k'' \setminus \overline{\mathcal{O}_k})]^{3 \times 3}} \\ &\quad + \varepsilon^{-2} \|u_\varepsilon\|_{[L^2(\varepsilon Q_k'' \setminus \overline{\mathcal{O}_k})]^3} + \|f\|_{[L^2(\varepsilon Q_k'' \setminus \overline{\mathcal{O}_k})]^3}]. \end{aligned} \quad (3.3.13)$$

Thus,

$$\begin{aligned} \|D^2 u_\varepsilon\|_{[L^2(\varepsilon Q_k \setminus \overline{\mathcal{O}_k})]^{3 \times 3}}^2 + \|\nabla p_\varepsilon\|_{[L^2(\varepsilon Q_k \setminus \overline{\mathcal{O}_k})]^3}^2 &\leq C [\varepsilon^{-2} \|\nabla u_\varepsilon\|_{[L^2(\varepsilon Q_k'' \setminus \overline{\mathcal{O}_k})]^{3 \times 3}}^2 \\ &\quad + \varepsilon^{-4} \|u_\varepsilon\|_{[L^2(\varepsilon Q_k'' \setminus \overline{\mathcal{O}_k})]^3}^2 + \|f\|_{[L^2(\varepsilon Q_k'' \setminus \overline{\mathcal{O}_k})]^3}^2]. \end{aligned} \quad (3.3.14)$$

We next sum (3.3.14) over $k \in \widetilde{Y}_\varepsilon$, where

$$\widetilde{Y}_\varepsilon := \{k \in \mathbb{Z}^3, \quad \varepsilon Q_k'' \subset \Omega\}.$$

We note that for $\varepsilon < \varepsilon_0(\Omega'')$, we have the chain of inclusion

$$\Omega'' \cap \Omega_\varepsilon \subset \bigcup_{k \in \widetilde{Y}_\varepsilon} \varepsilon (Q_k \setminus \overline{\mathcal{O}_k}) \subset \bigcup_{k \in \widetilde{Y}_\varepsilon} \varepsilon (Q_k'' \setminus \overline{\mathcal{O}_k}) \subset \Omega_\varepsilon.$$

We get

$$\begin{aligned} \|D^2 u_\varepsilon\|_{[L^2(\Omega'' \cap \Omega_\varepsilon)]^{3 \times 3}}^2 + \|\nabla p_\varepsilon\|_{[L^2(\Omega'' \cap \Omega_\varepsilon)]^3}^2 &\leq C [\varepsilon^{-2} \|\nabla u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^{3 \times 3}}^2 \\ &\quad + \varepsilon^{-4} \|u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^3}^2 + \|f\|_{[L^2(\Omega_\varepsilon)]^3}^2]. \end{aligned} \quad (3.3.15)$$

Estimate (3.3.5) is proved. We now conclude the proof of Theorem 3.2.2. We have, inserting (3.3.4) in the right hand side of (3.3.5),

$$\|D^2 u_\varepsilon\|_{[L^2(\Omega'' \cap \Omega_\varepsilon)]^{3 \times 3}} + \varepsilon^{-1} \|\nabla u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^{3 \times 3}} + \varepsilon^{-2} \|u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^3} + \|\nabla p_\varepsilon\|_{[L^2(\Omega'' \cap \Omega_\varepsilon)]^3} \leq C \|f\|_{[L^2(\Omega_\varepsilon)]^3}.$$

It remains to show that

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C \|f\|_{[L^2(\Omega_\varepsilon)]^3}. \quad (3.3.16)$$

By Lemma 3.4.3 stated in the appendix and the first line of (3.3.6), we get

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C \varepsilon^{-1} \left[\|\nabla u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^{3 \times 3}} + C \|f\|_{[H^{-1}(\Omega_\varepsilon)]^3} \right].$$

We now show that

$$\|f\|_{[H^{-1}(\Omega_\varepsilon)]^3} \leq C \varepsilon \|f\|_{[L^2(\Omega_\varepsilon)]^3}. \quad (3.3.17)$$

Indeed, for any $\phi \in [H_0^1(\Omega_\varepsilon)]^3$, we write that, using successively Cauchy-Schwarz inequality and Poincaré inequality (see Lemma 3.3.1),

$$\begin{aligned} \langle f, \phi \rangle &= \int_{\Omega_\varepsilon} f \cdot \phi \leq \|f\|_{[L^2(\Omega_\varepsilon)]^3} \|\phi\|_{[L^2(\Omega_\varepsilon)]^3} \leq C\varepsilon \|f\|_{[L^2(\Omega_\varepsilon)]^3} \|\nabla \phi\|_{[L^2(\Omega_\varepsilon)]^{3 \times 3}} \\ &\leq C\varepsilon \|f\|_{[L^2(\Omega_\varepsilon)]^3} \|\phi\|_{[H_0^1(\Omega_\varepsilon)]^3} \end{aligned}$$

Thus (3.3.17). Finally, we conclude with the use of (3.3.4) that

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C\varepsilon^{-1} \|\nabla u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^{3 \times 3}} + C\|f\|_{[L^2(\Omega_\varepsilon)]^3} \leq C\|f\|_{[L^2(\Omega_\varepsilon)]^3}.$$

This proves (3.3.16) and concludes the proof of Theorem 3.2.2. \square

3.3.2 Proof of Theorem 3.2.1

We use the periodic correctors $(w_j^{\text{per}}, p_j^{\text{per}})$ defined in (3.1.4) and we search w_j and p_j in the form $w_j = w_j^{\text{per}} + \widetilde{w}_j$ and $p_j = p_j^{\text{per}} + \widetilde{p}_j$. We recall (see the last paragraph of Subsection 3.1.3) that w_j^{per} is extended by zero in \mathcal{O}^{per} and that p_j^{per} is extended by a constant λ_j . The Stokes system defining $(\widetilde{w}_j, \widetilde{p}_j)$ is

$$\begin{cases} -\Delta \widetilde{w}_j + \nabla \widetilde{p}_j = e_j + \Delta w_j^{\text{per}} - \nabla p_j^{\text{per}} & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\ \operatorname{div} \widetilde{w}_j = 0, \\ \widetilde{w}_j = -w_j^{\text{per}} & \text{on } \partial \mathcal{O}. \end{cases} \quad (3.3.18)$$

The proof consists in applying Lax-Milgram's Lemma to (3.3.18). We first need to prove some preparatory Lemmas. In the sequel, we will use the notation

$$T_j := e_j + \Delta w_j^{\text{per}} - \nabla p_j^{\text{per}}$$

for $j \in \{1, 2, 3\}$.

Lemma 3.3.2. *Suppose that Assumption (A3) is satisfied. For all $1 < q < +\infty$, we have that $T_j \in [W^{-1, q'}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$, where $q' = q/(q-1)$.*

Proof. Let $\phi \in [\mathcal{D}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$. We extend ϕ by 0 in the perforations. We estimate $\langle T_j, \phi \rangle$ by an integration by parts:

$$\begin{aligned} \langle T_j, \phi \rangle &= \langle e_j + \Delta w_j^{\text{per}} - \nabla p_j^{\text{per}}, \phi \rangle \\ &= \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}}} e_j \cdot \phi - \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}}} \nabla w_j^{\text{per}} : \nabla \phi + \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}}} (p_j^{\text{per}} - \lambda_j) \operatorname{div}(\phi) \\ &= \int_{\mathbb{R}^3} e_j \cdot \phi - \int_{\mathbb{R}^3} \nabla w_j^{\text{per}} : \nabla \phi + \int_{\mathbb{R}^3} (p_j^{\text{per}} - \lambda_j) \operatorname{div}(\phi) \\ &= \int_{\mathbb{R}^3} e_j \cdot \phi - \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}^{\text{per}}}} \nabla w_j^{\text{per}} : \nabla \phi + \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}^{\text{per}}}} (p_j^{\text{per}} - \lambda_j) \operatorname{div}(\phi). \end{aligned}$$

Since w_j^{per} (resp. $p_j^{\text{per}} - \lambda_j$) is of class $\mathcal{C}^{2, \alpha}$ (resp. of class $\mathcal{C}^{1, \alpha}$) in $\mathbb{R}^3 \setminus \mathcal{O}^{\text{per}}$ (see [58, Theorem IV.7.1]), we may integrate by parts and find that

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}^{\text{per}}}} \nabla w_j^{\text{per}} : \nabla \phi &= \int_{\partial \mathcal{O}^{\text{per}}} \frac{\partial w_j^{\text{per}}}{\partial n} \cdot \phi - \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}^{\text{per}}}} \Delta w_j^{\text{per}} \cdot \phi, \\ \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}^{\text{per}}}} (p_j^{\text{per}} - \lambda_j) \operatorname{div}(\phi) &= \int_{\partial \mathcal{O}^{\text{per}}} (p_j^{\text{per}} - \lambda_j) \phi \cdot n - \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}^{\text{per}}}} \nabla p_j^{\text{per}} \cdot \phi, \end{aligned}$$

where we use the notations

$$\frac{\partial w_j^{\text{per}}}{\partial n} := \left(\frac{\partial w_j^{1, \text{per}}}{\partial n}, \frac{\partial w_j^{2, \text{per}}}{\partial n}, \frac{\partial w_j^{3, \text{per}}}{\partial n} \right)^T \quad \text{and} \quad w_j^{i, \text{per}} = w_j^{\text{per}} \cdot e_i,$$

for $i, j \in \{1, 2, 3\}$. Thus,

$$\begin{aligned}
\langle T_j, \phi \rangle &= \int_{\mathbb{R}^3} e_j \cdot \phi + \int_{\mathbb{R}^3 \setminus \mathcal{O}^{\text{per}}} [\Delta w_j^{\text{per}} - \nabla p_j^{\text{per}}] \cdot \phi + \int_{\partial \mathcal{O}^{\text{per}}} (p_j^{\text{per}} - \lambda_j) \phi \cdot n - \int_{\partial \mathcal{O}^{\text{per}}} \frac{\partial w_j^{\text{per}}}{\partial n} \cdot \phi \\
&= \int_{\mathbb{R}^3} e_j \cdot \phi - \int_{\mathbb{R}^3 \setminus \mathcal{O}^{\text{per}}} e_j \cdot \phi + \int_{\partial \mathcal{O}^{\text{per}}} (p_j^{\text{per}} - \lambda_j) \phi \cdot n - \int_{\partial \mathcal{O}^{\text{per}}} \frac{\partial w_j^{\text{per}}}{\partial n} \cdot \phi \\
&= \int_{\mathcal{O}^{\text{per}} \setminus \overline{\mathcal{O}}} e_j \cdot \phi + \int_{\partial \mathcal{O}^{\text{per}}} (p_j^{\text{per}} - \lambda_j) \phi \cdot n - \int_{\partial \mathcal{O}^{\text{per}}} \frac{\partial w_j^{\text{per}}}{\partial n} \cdot \phi. \\
&= (A) + (B) + (C)
\end{aligned}$$

We treat each term separately.

Term (A). By Hölder inequality and Assumption **(A3)** (more precisely (3.1.5)), we obtain that

$$\left| \int_{\mathcal{O}^{\text{per}} \setminus \overline{\mathcal{O}}} e_j \cdot \phi \right| \leq |\mathcal{O}^{\text{per}} \setminus \overline{\mathcal{O}}|^{\frac{1}{q'}} \|\phi\|_{[L^q(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3} \leq C \|\phi\|_{[W^{1,q}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3}.$$

Term (B). We have by standard regularity results (see [58, Theorem IV.7.1]) that $p_j^{\text{per}} \in L^\infty(\partial \mathcal{O}_0^{\text{per}})$. We apply a Trace Theorem $W^{1,1}(\mathcal{O}_0^{\text{per}}) \rightarrow L^1(\partial \mathcal{O}_0^{\text{per}})$ (see e.g. [52, Theorem 1, p. 258]) that yields a constant C , which is by translation invariance independent of k , such that for all $k \in \mathbb{Z}^3$,

$$\|\phi\|_{[L^1(\partial \mathcal{O}_k^{\text{per}})]^3} \leq C \|\phi\|_{[W^{1,1}(\mathcal{O}_k^{\text{per}})]^3}. \quad (3.3.19)$$

By applying (3.3.19) in the second inequality, we get

$$\begin{aligned}
\left| \int_{\partial \mathcal{O}^{\text{per}}} (p_j^{\text{per}} - \lambda_j) \phi \cdot n \right| &\leq \|p_j^{\text{per}} - \lambda_j\|_{L^\infty(\partial \mathcal{O}^{\text{per}})} \int_{\partial \mathcal{O}^{\text{per}}} |\phi| \\
&= C \sum_{k \in \mathbb{Z}^3} \int_{\partial \mathcal{O}_k^{\text{per}}} |\phi| \leq C \sum_{k \in \mathbb{Z}^3} \int_{\mathcal{O}_k^{\text{per}}} |\phi| + |\nabla \phi| = C \int_{\mathcal{O}^{\text{per}} \setminus \overline{\mathcal{O}}} |\phi| + |\nabla \phi|,
\end{aligned}$$

where we used in the last equality that $\phi = 0$ in \mathcal{O} . Using (3.1.5), we conclude thanks to Hölder inequality that

$$\left| \int_{\partial \mathcal{O}^{\text{per}}} (p_j^{\text{per}} - \lambda_j) \phi \cdot n \right| \leq C |\mathcal{O}^{\text{per}} \setminus \overline{\mathcal{O}}|^{\frac{1}{q'}} \left[\|\phi\|_{[L^q(\mathcal{O}^{\text{per}} \setminus \overline{\mathcal{O}})]^3} + \|\nabla \phi\|_{[L^q(\mathcal{O}^{\text{per}} \setminus \overline{\mathcal{O}})]^{3 \times 3}} \right] \leq C \|\phi\|_{[W^{1,q}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3}.$$

Term (C). The argument is similar to **Term (B)**. This gives the existence of a constant $C > 0$ such that:

$$\left| \int_{\partial \mathcal{O}^{\text{per}}} \frac{\partial w_j^{\text{per}}}{\partial n} \cdot \phi \right| \leq C \|\phi\|_{[W^{1,q}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3},$$

where C is independent of ϕ . We conclude that there exists a constant $C > 0$ such that

$$\forall \phi \in [\mathcal{D}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3, \quad |\langle T_j, \phi \rangle| \leq C \|\phi\|_{[W^{1,q}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3}.$$

This proves the Lemma. \square

Lemma 3.3.3. *Suppose that Assumptions **(A1)** and **(A3)** are satisfied. For all $1 < q < +\infty$, there exists a function $\phi_j \in [W^{1,q}(\mathbb{R}^3)]^3$ such that $\phi_j = w_j^{\text{per}}$ on $\partial \mathcal{O}$.*

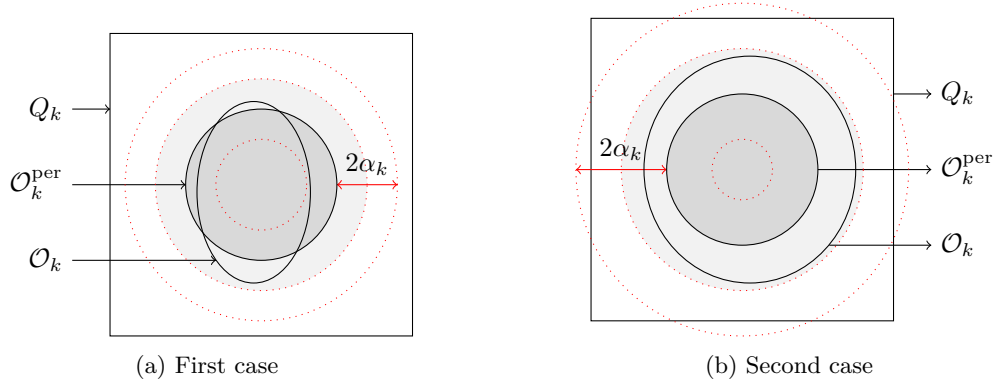


Figure 3.4: Illustration of the proof of Lemma 3.3.3

Proof. By Assumption **(A3)**, there exists a sequence $(\alpha_k)_{k \in \mathbb{Z}^3} \in \ell^1(\mathbb{Z}^3)$ such that for all $k \in \mathbb{Z}^3$, $\alpha_k > 0$ and

$$\{x \in \mathcal{O}_k^{\text{per}}, d(x, \partial \mathcal{O}_k^{\text{per}}) > \alpha_k\} \subset \mathcal{O}_k \subset \{x \in Q_k, d(x, \mathcal{O}_k^{\text{per}}) < \alpha_k\}.$$

Let $k \in \mathbb{Z}^3$.

If $\mathcal{O}_k \subset \mathcal{O}_k^{\text{per}}$, then we define the function χ_k by $\chi_k(x) = 0$ for all $x \in Q_k$.

If $\mathcal{O}_k \not\subset \mathcal{O}_k^{\text{per}}$, there are two cases (see Figure 3.4).

First case. We have $\{x \in \mathbb{R}^3, d(x, \mathcal{O}_k^{\text{per}}) < 2\alpha_k\} \subset Q_k$. We consider a function χ_k which is smooth and compactly supported such that

$$\begin{cases} \chi_k = 1 & \text{in } \{x \in Q_k, d(x, \mathcal{O}_k^{\text{per}}) < \alpha_k\}, \\ \chi_k = 0 & \text{in } \{x \in Q_k, d(x, \mathcal{O}_k^{\text{per}}) < 2\alpha_k\}^c. \end{cases}$$

We can choose χ_k such that the following estimates are satisfied:

$$|\chi_k| \leq 1 \ ; \ |\nabla \chi_k| \leq \frac{C}{\alpha_k} \quad \text{and} \quad \left| \text{supp}(\chi_k) \cap (Q_k \setminus \overline{\mathcal{O}_k^{\text{per}}}) \right| \leq C\alpha_k, \quad (3.3.20)$$

where the constants C are independent of k .

Second case. We have $\{x \in \mathbb{R}^3, d(x, \mathcal{O}_k^{\text{per}}) < 2\alpha_k\} \not\subset Q_k$. We consider a smooth and compactly supported function χ_k such that

$$\begin{cases} \chi_k = 1 & \text{in } \mathcal{O}_k, \\ \chi_k = 0 & \text{outside of } Q_k. \end{cases}$$

Because $\alpha_k \xrightarrow{|k| \rightarrow +\infty} 0$ and because there exists $\delta > 0$ such that

$$\forall k \in \mathbb{Z}^3, \quad d(\mathcal{O}_k, \partial Q_k) \geq \delta,$$

there are only a finite number of such configurations. After possible changes of the constant C , we can suppose that (3.3.20) is valid for all $k \in \mathbb{Z}^3$.

Conclusion. We define

$$\phi_j := \left(\sum_{k \in \mathbb{Z}^3} \chi_k \right) w_j^{\text{per}} \in \left[W_{\text{loc}}^{1,q}(\mathbb{R}^3) \right]^3.$$

We study the $W^{1,q}$ -local norm of ϕ_j . We fix $k \in \mathbb{Z}^3$; one has in Q_k :

$$|\nabla \phi_j| = |\nabla (\chi_k w_j^{\text{per}})| \leq |\nabla \chi_k| |w_j^{\text{per}}| + |\nabla w_j^{\text{per}}| |\chi_k|.$$

We now use that ∇w_j^{per} is bounded and the inequalities (3.3.20):

$$|\nabla \phi_j| \leq C \alpha_k^{-1} |w_j^{\text{per}}| + C.$$

To obtain that $|\nabla \phi_j|$ is bounded on its support, it suffices to show a bound of the type

$$|w_j^{\text{per}}| \leq C \alpha_k \quad \text{in } \{x \in Q_k, d(x, \mathcal{O}_k^{\text{per}}) < 2\alpha_k\}.$$

Since $w_j^{\text{per}} = 0$ on $\mathcal{O}_k^{\text{per}}$ and $\nabla w_j^{\text{per}} \in L^\infty(Q)$, this estimate follows from a classical Taylor inequality. We conclude that

$$\exists C > 0, \forall k \in \mathbb{Z}^3, \forall x \in Q_k, |\nabla \phi_j(x)| \leq C.$$

Because

$$\forall k \in \mathbb{Z}^3, |\text{supp}(\phi_j) \cap Q_k| = \left| \text{supp}(\chi_k) \cap \left(Q_k \setminus \overline{\mathcal{O}_k^{\text{per}}} \right) \right| = O(\alpha_k),$$

and because of Assumption **(A3)**, we conclude that $|\text{supp}(\phi_j)| < +\infty$ and so $\nabla \phi_j \in [L^q(\mathbb{R}^3)]^{3 \times 3}$. Similarly, $\phi_j \in [L^q(\mathbb{R}^3)]^3$. This concludes the Lemma. \square

We define, when $R > 0$,

$$\Omega^R := R\Omega \setminus \bigcup_{k \text{ s.t. } Q_k \subset R\Omega} \overline{\mathcal{O}_k}.$$

If $R = 1/\varepsilon$, one has $\Omega^R = \frac{1}{\varepsilon} \Omega_\varepsilon$.

Lemma 3.3.4. *Let $T \in [H^{-1}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$. The Stokes problem*

$$\begin{cases} -\Delta w + \nabla p = T & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\ \text{div}(w) = 0, \\ w = 0 & \text{on } \partial \mathcal{O} \end{cases} \quad (3.3.21)$$

admits a solution (w, p) such that $(w, p) \in [H_0^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3 \times L_{\text{loc}}^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})$ and $\nabla p \in [H^{-1}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$. Moreover, for all $R > 0$, we have the estimate

$$\|p - \lambda^R\|_{L^2(\Omega^R)} \leq CR \left[\|\nabla w\|_{[L^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^{3 \times 3}} + \|T\|_{[H^{-1}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3} \right], \quad \lambda^R = \frac{1}{|\Omega^R|} \int_{\Omega^R} p, \quad (3.3.22)$$

where C is a constant independent of T and R and p is extended by $\frac{1}{|R\Omega \setminus \overline{\mathcal{O}}|} \int_{R\Omega \setminus \overline{\mathcal{O}}} p$ in the perforations.

Proof. We consider the space $H := \{v \in [H_0^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3, \text{div}(v) = 0\}$. This is a Hilbert space as a closed subspace of $[H_0^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$. We formulate the following variational problem: find $w \in H$ such that

$$\forall v \in H, \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}}} \nabla w : \nabla v = \langle T, v \rangle. \quad (3.3.23)$$

We recall (see [22, Proof of Lemma 3.2]) that we dispose of a Poincaré inequality on $[H_0^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$ and thus of a Poincaré inequality on H . We can apply Lax Milgram's Lemma and find a solution $w \in H$ of (3.3.23). In particular, for each vector valued function $v \in [\mathcal{D}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$ such that $\text{div}(v) = 0$, we have

$$\langle \Delta w + T, v \rangle = 0.$$

Using [7, Theorem 2.1], this implies that there exists a distribution $p \in \mathcal{D}'(\mathbb{R}^3 \setminus \overline{\mathcal{O}})$ such that $\Delta w + T = \nabla p$. In particular, $\nabla p \in [H^{-1}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$.

Let¹ $n \in \mathbb{N}$ such that $\Omega \subset \subset (2n+1)Q$. We fix $R > 0$ and we define

$$Q^R := \left[\left(2 \left\lceil \frac{R}{2} \right\rceil + 1 \right) (2n+1) \right] Q \setminus \overline{\mathcal{O}} = [(2n_R+1)Q] \setminus \overline{\mathcal{O}}, \quad n_R := 2 \left\lceil \frac{R}{2} \right\rceil n + \left\lceil \frac{R}{2} \right\rceil + n,$$

where $\lceil \cdot \rceil$ denotes the ceiling function. We have that $R\Omega \setminus \overline{\mathcal{O}} \subset Q^R$ and

$$Q^R := \bigcup_{|k| \leq n_R} Q_k \setminus \overline{\mathcal{O}_k}.$$

Since $\nabla p \in [H^{-1}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$, we have $\nabla p \in [H^{-1}(Q^R)]^3$ and

$$\|\nabla p\|_{[H^{-1}(Q^R)]^3} \leq \|\nabla p\|_{[H^{-1}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3} \leq \|\nabla w\|_{[L^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^{3 \times 3}} + \|T\|_{[H^{-1}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3}$$

owing to the triangle inequality. Since $n_R = O(R)$, we have, arguing as in the proofs of Lemma 4.1.2 and Lemma 3.4.2, the existence of $C > 0$ independent of R such that

$$\left\| p - \fint_{Q^R} p \right\|_{L^2(Q^R)} \leq C(2n_R + 1) \|\nabla p\|_{H^{-1}(Q^R)} \leq CR \|\nabla p\|_{H^{-1}(Q^R)}.$$

This gives

$$\left\| p - \fint_{Q^R} p \right\|_{L^2(Q^R)} \leq CR \left[\|\nabla w\|_{[L^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^{3 \times 3}} + \|T\|_{[H^{-1}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3} \right], \quad (3.3.24)$$

where the constant C is independent of R . We now note that $\lambda^R = \frac{1}{|R\Omega \setminus \overline{\mathcal{O}}|} \int_{R\Omega \setminus \overline{\mathcal{O}}} p$ and that

$$\|p - \lambda^R\|_{L^2(\Omega^R)} = \left\| p - \fint_{R\Omega \setminus \overline{\mathcal{O}}} p \right\|_{L^2(R\Omega \setminus \overline{\mathcal{O}})} \leq 2 \left\| p - \fint_{Q^R} p \right\|_{L^2(R\Omega \setminus \overline{\mathcal{O}})}. \quad (3.3.25)$$

Gathering (3.3.24) and (3.3.25) concludes the proof of (3.3.22). \square

Proof of Theorem 3.2.1. We fix $j \in \{1, 2, 3\}$. Lemma 3.3.3 gives a function $\phi_j \in [H^1(\mathbb{R}^3)]^3$ such that $\phi_j = w_j^{\text{per}}$ on $\partial\mathcal{O}$. The problem

$$\begin{cases} \operatorname{div}(\tilde{v}_j) = \operatorname{div}(\phi_j) & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\ \tilde{v}_j = 0 & \text{on } \partial\mathcal{O} \end{cases}$$

admits a solution $\tilde{v}_j \in [H^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$ thanks to Lemma 3.4.4. Indeed, we just have to check that

$$\forall k \in \mathbb{Z}^3, \int_{\partial\mathcal{O}_k} \phi_j \cdot n = \int_{\partial\mathcal{O}_k} w_j^{\text{per}} \cdot n = \int_{\mathcal{O}_k} \operatorname{div}(w_j^{\text{per}}) = 0.$$

Defining $v_j := \tilde{v}_j - \phi_j$ yields a solution to the problem

$$\begin{cases} \operatorname{div}(v_j) = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\ v_j = -w_j^{\text{per}} & \text{on } \partial\mathcal{O}. \end{cases} \quad (3.3.26)$$

¹The following proof of (3.3.22) differs from [107] since in the proof there, we did not pay sufficiently attention to the perforations that intersect the boundary of Ω .

By Lemma 3.3.4, since $\Delta v_j \in [H^{-1}(\mathbb{R}^d \setminus \mathcal{O})]^3$, there exists a pair $(\widehat{v}_j, \widehat{p}_j) \in [H_0^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3 \times L_{\text{loc}}^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})$ solution of the Problem

$$\begin{cases} -\Delta \widehat{v}_j + \nabla \widehat{p}_j = T_j + \Delta v_j & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\ \operatorname{div}(\widehat{v}_j) = 0, \\ \widehat{v}_j = 0 & \text{on } \partial \mathcal{O}. \end{cases} \quad (3.3.27)$$

We set $\widetilde{w}_j := \widehat{v}_j + v_j$ and $\widetilde{p}_j = \widehat{p}_j$ and we finish the proof of Theorem 3.2.1. \square

Remark 3.3.5. *As for the homogenization of Poisson equation, see [22], we can prove the existence of \widetilde{w}_j using a minimization problem. This reads as finding a solution to*

$$\min_{v \in V} \left\{ \frac{1}{2} \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}}} \nabla v : \nabla v - \langle T_j, v \rangle \right\},$$

where the minimizing space is

$$V := \left\{ v \in [H^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3, \quad \operatorname{div}(v) = 0, \quad v|_{\partial \mathcal{O}} = -w_j^{\text{per}} \right\}.$$

3.3.3 Proof of Theorem 3.2.3

3.3.3.1 Strategy of the proof

We introduce

$$\mathcal{R}_\varepsilon := u_\varepsilon - \varepsilon^2 \sum_{j=1}^3 w_j \left(\frac{\cdot}{\varepsilon} \right) f_j \quad \text{and} \quad \mathcal{P}_\varepsilon := p_\varepsilon - \varepsilon \sum_{j=1}^3 \left(p_j \left(\frac{\cdot}{\varepsilon} \right) - \frac{1}{|\frac{1}{\varepsilon} \Omega_\varepsilon|} \int_{\frac{1}{\varepsilon} \Omega_\varepsilon} p_j \right) f_j.$$

The strategy of the proof is to find a Stokes system satisfied by $(\mathcal{R}_\varepsilon, \pi_\varepsilon)$ and then to apply Theorem 3.2.2. We need to compute the quantities

$$-\Delta \mathcal{R}_\varepsilon + \nabla \mathcal{P}_\varepsilon \quad \text{and} \quad \operatorname{div}(\mathcal{R}_\varepsilon). \quad (3.3.28)$$

The construction of auxiliary functions is necessary to correct the divergence equation satisfied by \mathcal{R}_ε , which doesn't have a suitable order in ε . This is done in subsection 3.3.3.2 below (Lemma 3.3.6). The proof of Theorem 3.2.3 is completed in subsection 3.3.3.3, in particular the computations (3.3.28).

3.3.3.2 Some auxiliary functions

We recall that the correctors w_j , $j \in \{1, 2, 3\}$ constructed in Theorem 3.2.1 are extended by zero in the non-periodic perforations. If $i \in \{1, 2, 3\}$, we denote $w_j^i := w_j \cdot e_i$ the i^{th} -component of w_j . Similarly, $w_j^{i, \text{per}}$ (resp. \widetilde{w}_j^i) will be the i^{th} -component of w_j^{per} (resp. \widetilde{w}_j^i). We recall that the definition of the matrix A is given in Equation (3.1.3).

Lemma 3.3.6. *Suppose that Assumption $(A4)_1$ is satisfied. Let $i, j \in \{1, 2, 3\}$ and χ be a function of class C^∞ with support in $Q \setminus \overline{Q'}$ such that $\int_Q \chi = 1$ where Q' is defined in (3.1.6) (see also Figure 3.2). We extend χ by periodicity to $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$. The problem*

$$\begin{cases} -\operatorname{div} z_j^i = w_j^i - \chi A_j^i & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\ z_j^i = 0 & \text{on } \partial \mathcal{O} \end{cases} \quad (3.3.29)$$

admits a solution $z_j^i \in [H_{0, \text{loc}}^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$. Besides, we have the estimate

$$\|z_j^i\|_{[H^2(\bigcup_{k \in Y_\varepsilon} (Q_k \setminus \overline{\mathcal{O}_k}))]^3} \leq C \varepsilon^{-\frac{3}{2}} \|w_j^{i, \text{per}}\|_{[H^1(Q)]^3} + C \varepsilon^{-1} \|\widetilde{w}_j^i\|_{[H^1(\mathbb{R}^3)]^3} \quad (3.3.30)$$

for all $\varepsilon > 0$ where C is a constant independent of ε .

Proof. We fix $i, j \in \{1, 2, 3\}$. We search z_j^i under the form $z_j^i = \nabla \Psi_j^i + g_j^i$.

Step 1. We build a function Ψ_j^i such that $\nabla \Psi_j^i \in [H^{2,\text{per}}(Q)]^3 + [H_{\text{loc}}^2(\mathbb{R}^3)]^3$ and

$$-\Delta \Psi_j^i = w_j^i - \chi A_j^i \quad \text{in } \mathbb{R}^3.$$

The periodic part of Ψ_j^i is defined by solving the problem

$$\begin{cases} -\Delta \Psi_j^{i,\text{per}} = w_j^{i,\text{per}} - \chi A_j^i & \text{in } Q, \\ \Psi_j^{i,\text{per}} \in H^{1,\text{per}}(Q). \end{cases} \quad (3.3.31)$$

Since $\int_Q (w_j^{i,\text{per}} - \chi A_j^i) = 0$, Problem (3.3.31) is well posed in $H^{1,\text{per}}(Q)/\mathbb{R}$. We choose $\Psi_j^{i,\text{per}}$ such that $\int_Q \Psi_j^{i,\text{per}} = 0$. Because $w_j^{i,\text{per}} - \chi A_j^i \in H^{1,\text{per}}(Q)$, standard elliptic regularity results state that $\nabla \Psi_j^{i,\text{per}} \in [H^{2,\text{per}}(Q)]^3$. Besides, there exists a constant C such that

$$\|\nabla \Psi_j^{i,\text{per}}\|_{[H^2(Q)]^3} \leq C \|w_j^{i,\text{per}} - \chi A_j^i\|_{[H^1(Q)]^3} \leq C \|w_j^{i,\text{per}}\|_{[H^1(Q)]^3}. \quad (3.3.32)$$

We now build the non-periodic part of Ψ_j^i . We extend \widetilde{w}_j^i by $-w_j^{i,\text{per}}$ in \mathcal{O} . We note that, with this extension, $\widetilde{w}_j^i \in [H^1(\mathbb{R}^3)]^3$. We consider the problem

$$-\Delta \widetilde{\Psi}_j^i = \widetilde{w}_j^i \quad \text{in } \mathbb{R}^3,$$

and its solution given formally by the Green function:

$$\widetilde{\Psi}_j^i = C_3 \frac{1}{|\cdot|} *_{\mathbb{R}^3} \widetilde{w}_j^i.$$

Thanks to the remarks after the proof of [61, Theorem 9.9] (see [61, p.235]), we have that $D^2 \widetilde{\Psi}_j^i \in [H^1(\mathbb{R}^3)]^{3 \times 3}$ and the Calderon-Zygmund estimates:

$$\|D^2 \widetilde{\Psi}_j^i\|_{[L^2(\mathbb{R}^3)]^{3 \times 3}} = \|\widetilde{w}_j^i\|_{L^2(\mathbb{R}^3)} \quad \text{and} \quad \|D^3 \widetilde{\Psi}_j^i\|_{[L^2(\mathbb{R}^3)]^{3 \times 3 \times 3}} = \|\nabla \widetilde{w}_j^i\|_{[L^2(\mathbb{R}^3)]^3}. \quad (3.3.33)$$

Using the Sobolev injection $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ for $\nabla \widetilde{\Psi}_j^i$, we deduce that $\nabla \widetilde{\Psi}_j^i \in [L^6(\mathbb{R}^3)]^3$ and, using (3.3.33), that the estimate

$$\|\nabla \widetilde{\Psi}_j^i\|_{[L^6(\mathbb{R}^3)]^3} \leq C \|\widetilde{w}_j^i\|_{L^2(\mathbb{R}^3)}$$

holds true. In particular, $\nabla \widetilde{\Psi}_j^i \in [L_{\text{loc}}^2(\mathbb{R}^3)]^3$ and, thanks to Hölder inequality, we have

$$\|\nabla \widetilde{\Psi}_j^i\|_{[L^2(\frac{1}{\varepsilon}\Omega_\varepsilon)]^3} \leq \frac{C}{\varepsilon} \|\nabla \widetilde{\Psi}_j^i\|_{[L^6(\mathbb{R}^3)]^3}.$$

We deduce that

$$\|\nabla \widetilde{\Psi}_j^i\|_{[L^2(\frac{1}{\varepsilon}\Omega_\varepsilon)]^3} \leq \frac{C}{\varepsilon} \|\nabla \widetilde{\Psi}_j^i\|_{[L^6(\mathbb{R}^3)]^3} \leq \frac{C}{\varepsilon} \|\widetilde{w}_j^i\|_{[L^2(\mathbb{R}^3)]^3}. \quad (3.3.34)$$

Finally, collecting (3.3.33) and (3.3.34), we get

$$\|\nabla \widetilde{\Psi}_j^i\|_{[H^2(\frac{1}{\varepsilon}\Omega_\varepsilon)]^3} \leq \frac{C}{\varepsilon} \|\widetilde{w}_j^i\|_{[H^1(\mathbb{R}^3)]^3}. \quad (3.3.35)$$

We define $\Psi_j^i := \Psi_j^{i,\text{per}} + \widetilde{\Psi}_j^i$ and verify that

$$-\Delta \Psi_j^i = w_j^{i,\text{per}} - \chi A_j^i + \widetilde{w}_j^i = w_j^i - \chi A_j^i \quad \text{on } \mathbb{R}^3.$$

We use the periodicity of $\nabla\Psi_j^{i,\text{per}}$ and write that

$$\begin{aligned} \|\nabla\Psi_j^i\|_{[H^2(\frac{1}{\varepsilon}\Omega_\varepsilon)]^3} &\leq \|\nabla\Psi_j^{i,\text{per}}\|_{[H^2(\frac{1}{\varepsilon}\Omega_\varepsilon)]^3} + \|\nabla\widetilde{\Psi}_j^i\|_{[H^2(\frac{1}{\varepsilon}\Omega_\varepsilon)]^3} \\ &\leq C\varepsilon^{-\frac{3}{2}}\|\nabla\Psi_j^{i,\text{per}}\|_{[H^2(Q)]^3} + \|\nabla\widetilde{\Psi}_j^i\|_{[H^2(\frac{1}{\varepsilon}\Omega_\varepsilon)]^3}, \end{aligned} \quad (3.3.36)$$

where the constant C is independent of ε . We make use of (3.3.35) and (3.3.32) and deduce that

$$\|\nabla\Psi_j^i\|_{[H^2(\frac{1}{\varepsilon}\Omega_\varepsilon)]^3} \leq C\varepsilon^{-\frac{3}{2}}\|w_j^{i,\text{per}}\|_{[H^1(Q)]^3} + C\varepsilon^{-1}\|\widetilde{w}_j^i\|_{[H^1(\mathbb{R}^3)]^3}. \quad (3.3.37)$$

Step 2. We introduce a cut-off function χ_1 such that $\chi_1 = 1$ in Q' and $\chi_1 = 0$ out of Q (see Figure 3.2). We fix $k \in \mathbb{Z}^3$ and define $\chi_1^k := \chi_1(\cdot - k)$. The goal of this step is to solve the following problem:

$$\begin{cases} \operatorname{div}(g_j^{i,k}) = 0 & \text{in } Q_k \setminus \overline{\mathcal{O}_k}, \\ g_j^{i,k} = -\nabla\Psi_j^i & \text{on } \partial\mathcal{O}_k, \\ g_j^{i,k} = 0 & \text{on } \partial Q_k. \end{cases} \quad (3.3.38)$$

We first solve

$$\begin{cases} \operatorname{div}(h_j^{i,k}) = \operatorname{div}(\chi_1^k \nabla\Psi_j^i) & \text{on } Q_k \setminus \overline{\mathcal{O}_k}, \\ h_j^{i,k} \in [H_0^2(Q_k \setminus \overline{\mathcal{O}_k})]^3. \end{cases} \quad (3.3.39)$$

The compatibility condition (3.1.8) is satisfied:

$$\int_{Q_k \setminus \overline{\mathcal{O}_k}} \operatorname{div}(\chi_1^k \nabla\Psi_j^i) = \int_{\partial\mathcal{O}_k} \chi_1^k \nabla\Psi_j^i \cdot n + \int_{\partial Q_k} \chi_1^k \nabla\Psi_j^i \cdot n = - \int_{\mathcal{O}_k} \Delta\Psi_j^i = 0.$$

Since

$$\operatorname{div}(\chi_1^k \nabla\Psi_j^i) \in H_0^1(Q_k \setminus \overline{\mathcal{O}_k}),$$

we obtain by Assumption **(A4)**₁ a solution $h_j^{i,k} \in [H_0^2(Q_k \setminus \overline{\mathcal{O}_k})]^3$ to (3.3.39) which satisfies the estimate

$$\|h_j^{i,k}\|_{[H^2(Q_k \setminus \overline{\mathcal{O}_k})]^3} \leq C\|\operatorname{div}(\chi_1^k \nabla\Psi_j^i)\|_{H^1(Q_k \setminus \overline{\mathcal{O}_k})} \leq C\|\nabla\Psi_j^i\|_{[H^2(Q_k \setminus \overline{\mathcal{O}_k})]^3}.$$

We extend $h_j^{i,k}$ by 0 to $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$. We then define $g_j^{i,k} := h_j^{i,k} - \chi_1^k \nabla\Psi_j^i$. We note that $g_j^{i,k} = 0$ out of Q_k and that $g_j^{i,k} \in [H^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$. Besides, $g_j^{i,k}$ solves Problem (3.3.38) and satisfies the estimate

$$\|g_j^{i,k}\|_{[H^2(Q_k \setminus \overline{\mathcal{O}_k})]^3} \leq C\|\nabla\Psi_j^i\|_{[H^2(Q_k)]^3}. \quad (3.3.40)$$

Step 3. We set

$$g_j^i(x) := g_j^{i,k}(x) \quad \text{if } x \in Q_k.$$

Then we have

$$\begin{cases} \operatorname{div}(g_j^i) = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\ g_j^i = -\nabla\Psi_j^i & \text{on } \partial\mathcal{O}. \end{cases}$$

Besides, $g_j^i \in [H_{\text{loc}}^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$ and summing (3.3.40) over $k \in Y_\varepsilon$ yields the estimate

$$\|g_j^i\|_{[H^2(\bigcup_{k \in Y_\varepsilon} Q_k \setminus \overline{\mathcal{O}_k})]^3} \leq C\|\nabla\Psi_j^i\|_{[H^2(\frac{1}{\varepsilon}\Omega_\varepsilon)]^3}. \quad (3.3.41)$$

We define $z_j^i := \nabla\Psi_j^i + g_j^i$. We have $z_j^i \in [H_{\text{loc}}^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$. Besides, z_j^i is a solution of (3.3.29) and, collecting (3.3.37) and (3.3.41), we prove the estimate (3.3.30):

$$\|z_j^i\|_{[H^2(\bigcup_{k \in Y_\varepsilon} Q_k \setminus \overline{\mathcal{O}_k})]^3} \leq C\|\nabla\Psi_j^i\|_{[H^2(\frac{1}{\varepsilon}\Omega_\varepsilon)]^3} \leq C\varepsilon^{-\frac{3}{2}}\|w_j^{i,\text{per}}\|_{[H^1(Q)]^3} + C\varepsilon^{-1}\|\widetilde{w}_j^i\|_{[H^1(\mathbb{R}^3)]^3}. \quad (3.3.42)$$

It remains to prove that $z_j^i \in [H_{0,\text{loc}}^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$. For that, we fix $k \in \mathbb{Z}^3$ and we notice that in a neighbourhood of the perforation $\partial\mathcal{O}_k$, the equality $z_j^i = h_j^{i,k} + (1 - \chi_1^k)\nabla\Psi_j^i = h_j^{i,k}$ is satisfied. Since $h_j^{i,k} \in [H_0^2(Q_k \setminus \overline{\mathcal{O}_k})]^3$, it proves that $z_j^i \in [H_{0,\text{loc}}^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})]^3$. This ends the proof. \square

3.3.3.3 Proof of convergence Theorem 3.2.3

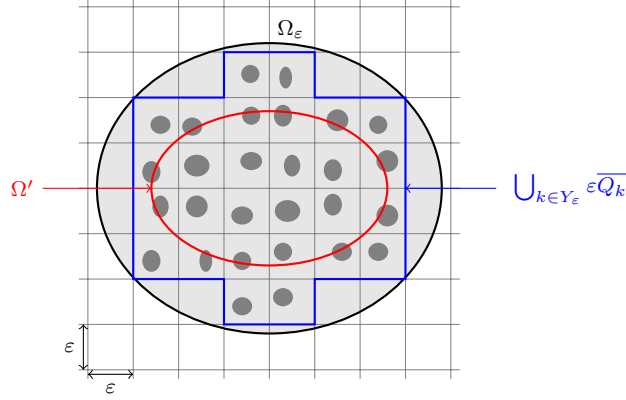


Figure 3.5: Proof of Theorem 3.2.3

Proof. We choose $\varepsilon > 0$ small enough such that

$$\text{supp}(f) \subset \bigcup_{k \in Y_\varepsilon} \varepsilon \overline{Q_k}.$$

We define (see Figure 3.5) $\Omega' := \{x \in \Omega \text{ s.t. } f(x) \neq 0\}$. We now set

$$u_\varepsilon^1 := \varepsilon^2 w_j \left(\frac{\cdot}{\varepsilon} \right) f_j + \varepsilon^3 z_j^i \left(\frac{\cdot}{\varepsilon} \right) \partial_i f_j$$

and

$$p_\varepsilon^1 := \varepsilon \left[p_j \left(\frac{\cdot}{\varepsilon} \right) - \lambda_\varepsilon^j \right] f_j, \quad \lambda_\varepsilon^j := \frac{1}{|\frac{1}{\varepsilon} \Omega_\varepsilon|} \int_{\frac{1}{\varepsilon} \Omega_\varepsilon} p_j.$$

We have $u_\varepsilon^1 \in [H_0^1(\Omega_\varepsilon)]^3$ and $p_\varepsilon^1 \in L^2(\Omega_\varepsilon)$ and thus

$$-\Delta u_\varepsilon^1 + \nabla p_\varepsilon^1 \in [H^{-1}(\Omega_\varepsilon)]^3.$$

Since (see Figure 3.5) $f = 0$ in $\Omega \setminus \Omega'$, we have that u_ε^1 and p_ε^1 are compactly supported in Ω . It is thus sufficient to compute $-\Delta u_\varepsilon^1 + \nabla p_\varepsilon^1$ in $\Omega_\varepsilon \cap \Omega'$. We notice that, for $\varepsilon > 0$ small enough, $\Omega_\varepsilon \cap \Omega' = \Omega' \setminus \varepsilon \overline{\mathcal{O}}$. Besides, thanks to Lemma 3.3.6, we have $z_j^i(\cdot/\varepsilon) \in [H^2(\Omega' \setminus \varepsilon \overline{\mathcal{O}})]^3$. We compute in $\Omega' \setminus \varepsilon \overline{\mathcal{O}}$:

$$\begin{aligned} \Delta u_\varepsilon^1 &= \Delta w_j \left(\frac{\cdot}{\varepsilon} \right) f_j + 2\varepsilon \partial_k w_j \left(\frac{\cdot}{\varepsilon} \right) \partial_k f_j + \varepsilon^2 w_j \left(\frac{\cdot}{\varepsilon} \right) \Delta f_j + \varepsilon \Delta z_j^i \left(\frac{\cdot}{\varepsilon} \right) \partial_i f_j \\ &\quad + 2\varepsilon^2 \partial_k z_j^i \left(\frac{\cdot}{\varepsilon} \right) \partial_k \partial_i f_j + \varepsilon^3 z_j^i \left(\frac{\cdot}{\varepsilon} \right) \Delta \partial_i f_j. \end{aligned}$$

and

$$\nabla p_\varepsilon^1 = \nabla p_j \left(\frac{\cdot}{\varepsilon} \right) f_j + \varepsilon \left\{ p_j \left(\frac{\cdot}{\varepsilon} \right) - \lambda_\varepsilon^j \right\} \nabla f_j.$$

Thus,

$$\begin{aligned} \Delta u_\varepsilon^1 - \nabla p_\varepsilon^1 &= \Delta w_j \left(\frac{\cdot}{\varepsilon} \right) f_j + 2\varepsilon \partial_k w_j \left(\frac{\cdot}{\varepsilon} \right) \partial_k f_j + \varepsilon^2 w_j \left(\frac{\cdot}{\varepsilon} \right) \Delta f_j + \varepsilon \Delta z_j^i \left(\frac{\cdot}{\varepsilon} \right) \partial_i f_j \\ &\quad + 2\varepsilon^2 \partial_k z_j^i \left(\frac{\cdot}{\varepsilon} \right) \partial_k \partial_i f_j + \varepsilon^3 z_j^i \left(\frac{\cdot}{\varepsilon} \right) \Delta \partial_i f_j - \nabla p_j \left(\frac{\cdot}{\varepsilon} \right) f_j - \varepsilon \left\{ p_j \left(\frac{\cdot}{\varepsilon} \right) - \lambda_\varepsilon^j \right\} \nabla f_j \quad (3.3.43) \\ &= -f_j e_j + \varepsilon f_\varepsilon = -f + \varepsilon f_\varepsilon, \end{aligned}$$

where

$$\begin{aligned} f_\varepsilon := & 2\partial_k w_j \left(\frac{\cdot}{\varepsilon}\right) \partial_k f_j + \varepsilon w_j \left(\frac{\cdot}{\varepsilon}\right) \Delta f_j + \Delta z_j^i \left(\frac{\cdot}{\varepsilon}\right) \partial_i f_j + 2\varepsilon \partial_k z_j^i \left(\frac{\cdot}{\varepsilon}\right) \partial_k \partial_i f_j \\ & + \varepsilon^2 z_j^i \left(\frac{\cdot}{\varepsilon}\right) \Delta \partial_i f_j - \left\{ p_j \left(\frac{\cdot}{\varepsilon}\right) - \lambda_\varepsilon^j \right\} \nabla f_j. \end{aligned}$$

Equation (3.3.43) is still valid in $\Omega_\varepsilon \setminus \Omega'$ (the LHS and RHS vanish). We define

$$R_\varepsilon := u_\varepsilon - u_\varepsilon^1 \quad \text{and} \quad \pi_\varepsilon := p_\varepsilon - p_\varepsilon^1.$$

Thus $(R_\varepsilon, \pi_\varepsilon) \in [H_0^1(\Omega_\varepsilon)]^3 \times L^2(\Omega_\varepsilon)$ and

$$-\Delta R_\varepsilon + \nabla \pi_\varepsilon = \varepsilon f_\varepsilon \quad \text{in} \quad \Omega_\varepsilon, \quad f_\varepsilon \in [L^2(\Omega_\varepsilon)]^3.$$

Using that $f \in [W^{3,\infty}(\Omega)]^3$, we infer

$$\begin{aligned} \|f_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^3} &\leq \left\| \nabla w_j \left(\frac{\cdot}{\varepsilon}\right) \right\|_{[L^2(\Omega' \setminus \varepsilon \bar{\mathcal{O}})]^{3 \times 3}} \|\nabla f_j\|_{[L^\infty(\Omega)]^3} + \varepsilon \left\| w_j \left(\frac{\cdot}{\varepsilon}\right) \right\|_{[L^2(\Omega' \setminus \varepsilon \bar{\mathcal{O}})]^3} \|\Delta f_j\|_{L^\infty(\Omega)} \\ &+ \left\| \Delta z_j^i \left(\frac{\cdot}{\varepsilon}\right) \right\|_{[L^2(\Omega' \setminus \varepsilon \bar{\mathcal{O}})]^3} \|\partial_i f_j\|_{L^\infty(\Omega)} + \varepsilon \left\| \nabla z_j^i \left(\frac{\cdot}{\varepsilon}\right) \right\|_{[L^2(\Omega' \setminus \varepsilon \bar{\mathcal{O}})]^{3 \times 3}} \|\nabla \partial_i f_j\|_{[L^\infty(\Omega)]^3} \\ &+ \varepsilon^2 \left\| z_j^i \left(\frac{\cdot}{\varepsilon}\right) \right\|_{[L^2(\Omega' \setminus \varepsilon \bar{\mathcal{O}})]^3} \|\Delta \partial_i f_j\|_{L^\infty(\Omega)} + \left\| p_j \left(\frac{\cdot}{\varepsilon}\right) - \lambda_\varepsilon^j \right\|_{L^2(\Omega' \setminus \varepsilon \bar{\mathcal{O}})} \|\nabla f_j\|_{[L^\infty(\Omega)]^3} \\ &\leq C\varepsilon^{\frac{3}{2}} \left[\|w_j\|_{[H^1(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon \bar{\mathcal{O}})]^3} + \|z_j^i\|_{[H^2(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon \bar{\mathcal{O}})]^3} + \|p_j - \lambda_\varepsilon^j\|_{L^2(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon \bar{\mathcal{O}})} \right] \\ &= C\varepsilon^{\frac{3}{2}} [(A) + (B) + (C)]. \end{aligned} \tag{3.3.44}$$

We treat each term separately. For (A), we have

$$\begin{aligned} \|w_j\|_{[H^1(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon \bar{\mathcal{O}})]^3} &\leq \|w_j^{\text{per}}\|_{[H^1(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon \bar{\mathcal{O}})]^3} + \|\widetilde{w}_j\|_{[H^1(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon \bar{\mathcal{O}})]^3} \\ &\leq C\varepsilon^{-\frac{3}{2}} \|\nabla w_j^{\text{per}}\|_{H^1(Q)} + \|\widetilde{w}_j\|_{[H^1(\mathbb{R}^3 \setminus \bar{\mathcal{O}})]^3}. \end{aligned} \tag{3.3.45}$$

For (B), we apply Lemma 3.3.6 (and especially (3.3.30)):

$$\|z_j^i\|_{[H^2(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon \bar{\mathcal{O}})]^3} \leq C\varepsilon^{-\frac{3}{2}} \|w_j^{i,\text{per}}\|_{[H^1(Q)]^3} + C\varepsilon^{-1} \|\widetilde{w}_j^i\|_{[H^1(\mathbb{R}^3)]^3} \tag{3.3.46}$$

For (C), Theorem 3.2.1 gives

$$\begin{aligned} \|p_j - \lambda_\varepsilon^j\|_{L^2(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon \bar{\mathcal{O}})} &\leq \|p_j^{\text{per}} - \lambda_\varepsilon^{j,\text{per}}\|_{L^2(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon \bar{\mathcal{O}})} + \|\widetilde{p}_j - \widetilde{\lambda}_\varepsilon^j\|_{L^2(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon \bar{\mathcal{O}})} \\ &\leq C\varepsilon^{-\frac{3}{2}} \|p_j^{\text{per}}\|_{L^2(Q)} + C\varepsilon^{-1}. \end{aligned} \tag{3.3.47}$$

Collecting (3.3.45), (3.3.46) and (3.3.47), we conclude that there exists a constant $C > 0$ independent of ε such that

$$\|f_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^3} \leq C.$$

We now study $\text{div}(R_\varepsilon)$. Using Lemma 3.3.6, we have in Ω_ε :

$$\text{div}(R_\varepsilon) = -\varepsilon^2 \chi \left(\frac{\cdot}{\varepsilon}\right) A_j^i \partial_i f_j - \varepsilon^3 z_j^i \left(\frac{\cdot}{\varepsilon}\right) \cdot \nabla \partial_i f_j.$$

We recall that $\text{div}(Af) = A_j^i \partial_i f_j = 0$. Thus,

$$-\text{div}(R_\varepsilon) = \varepsilon^3 z_j^i \left(\frac{\cdot}{\varepsilon}\right) \cdot \nabla \partial_i f_j.$$

We have that $\varepsilon^3 z_j^i \left(\frac{\cdot}{\varepsilon}\right) \cdot \nabla \partial_i f_j \in [H_0^1(\Omega_\varepsilon)]^3$ and $\int_{\Omega_\varepsilon} \varepsilon^3 z_j^i \left(\frac{\cdot}{\varepsilon}\right) \cdot \nabla \partial_i f_j = 0$. By Lemma 3.4.5 stated in the appendix, there exists $S_\varepsilon \in [H_0^2(\Omega_\varepsilon)]^3$ such that

$$\text{div}(S_\varepsilon) = \varepsilon^3 z_j^i \left(\frac{\cdot}{\varepsilon}\right) \cdot \nabla \partial_i f_j \quad \text{and} \quad \|S_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leq C\varepsilon^2 \left\| z_j^i \left(\frac{\cdot}{\varepsilon}\right) \cdot \nabla \partial_i f_j \right\|_{H_0^1(\Omega_\varepsilon)}.$$

Using that $f \in [W^{2,\infty}(\Omega)]^3$ and Lemma 3.3.6, we get

$$\left\| z_j^i \left(\frac{\cdot}{\varepsilon} \right) \cdot \nabla \partial_i f_j \right\|_{[H^1(\Omega_\varepsilon)]^3} \leq \frac{C}{\varepsilon}.$$

Thus

$$\|S_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leq C\varepsilon. \quad (3.3.48)$$

We now define $\widehat{R}_\varepsilon := R_\varepsilon + S_\varepsilon$. The pair $(\widehat{R}_\varepsilon, \pi_\varepsilon) \in [H_0^1(\Omega_\varepsilon)]^3 \times L^2(\Omega_\varepsilon)$ is solution to the following Stokes system:

$$\begin{cases} -\Delta \widehat{R}_\varepsilon + \nabla \pi_\varepsilon = \varepsilon f_\varepsilon - \Delta S_\varepsilon, \\ \operatorname{div}(\widehat{R}_\varepsilon) = 0, \\ \widehat{R}_\varepsilon|_{\partial\Omega_\varepsilon} = 0. \end{cases} \quad (3.3.49)$$

We notice that $\varepsilon f_\varepsilon - \Delta S_\varepsilon \in [L^2(\Omega_\varepsilon)]^3$ thus we may apply Theorem 3.2.2: for all $\Omega'' \subset \Omega$, we have for $\varepsilon < \varepsilon_0(\Omega'')$,

$$\|D^2 \widehat{R}_\varepsilon\|_{L^2(\Omega \cap \Omega'')} \leq C \|\varepsilon f_\varepsilon - \Delta S_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|S_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leq C\varepsilon,$$

and

$$\|\nabla \pi_\varepsilon\|_{L^2(\Omega'' \cap \Omega_\varepsilon)} \leq C\varepsilon.$$

By the triangle inequality and (3.3.48), we conclude that

$$\left\| D^2 \left[u_\varepsilon - \varepsilon^2 w_j \left(\frac{\cdot}{\varepsilon} \right) f_j \right] \right\|_{L^2(\Omega'' \cap \Omega_\varepsilon)} \leq C\varepsilon \quad \text{and} \quad \left\| \nabla \left[p_\varepsilon - \varepsilon \left\{ p_j \left(\frac{\cdot}{\varepsilon} \right) - \lambda_\varepsilon^j \right\} f_j \right] \right\|_{L^2(\Omega'' \cap \Omega_\varepsilon)} \leq C\varepsilon.$$

□

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3.4 Appendices

3.4.1 Technical Lemmas

We recall that if $R > 0$, we define

$$\Omega^R := R\Omega \setminus \bigcup_{k, Q_k \subset R\Omega} \overline{\mathcal{O}_k}. \quad (3.4.1)$$

Lemma 3.4.1 (Divergence Lemma on Ω^R). *Suppose that Assumption $(\mathbf{A4})_0$ is satisfied. Let $1 < q < +\infty$ and $R > 0$. Let $f \in L^q(\Omega^R)$ be such that*

$$\int_{\Omega^R} f = 0.$$

The problem

$$\begin{cases} -\operatorname{div}(v) = f & \text{in } \Omega_R, \\ v = 0 & \text{on } \partial\Omega_R \end{cases} \quad (3.4.2)$$

admits a solution $v \in [W^{1,q}(\Omega_R)]^3$ such that

$$\|v\|_{[W^{1,q}(\Omega_R)]^3} \leq CR \|f\|_{L^q(\Omega_R)}, \quad (3.4.3)$$

where $C > 0$ is a constant independent of f and R .

Proof. We first extend f by 0 in the perforations. We then solve the problem

$$\begin{cases} -\operatorname{div}(v_1) = f & \text{in } R\Omega, \\ v_1 \in [W_0^{1,q}(R\Omega)]^3. \end{cases} \quad (3.4.4)$$

By Lemma [58, Theorem III.3.1] and a simple scaling argument, Problem (3.4.4) admits a solution v_1 such that

$$\|v_1\|_{W^{1,q}(R\Omega)} \leq CR\|f\|_{L^q(R\Omega)}$$

with the constant C being independent of R . For $k \in \mathbb{Z}^3$ such that $Q_k \subset R\Omega$, we consider the problem

$$\begin{cases} \operatorname{div}(v_2^k) = 0 & \text{in } Q_k \setminus \overline{\mathcal{O}_k}, \\ v_2^k = 0 & \text{on } \partial Q_k, \\ v_2^k = -v_1 & \text{on } \partial \mathcal{O}_k. \end{cases} \quad (3.4.5)$$

The compatibility condition for (3.4.5) is satisfied:

$$-\int_{\partial \mathcal{O}_k} v_1 \cdot n = \int_{\mathcal{O}_k} \operatorname{div}(v_1) = -\int_{\mathcal{O}_k} f = 0.$$

Arguing as for Problem (3.3.38), we show that Problem (3.4.5) admits a solution $v_2^k \in [W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})]^3$ such that (the constant C is independent of k thanks to Assumption **(A4)**₀):

$$\|v_2^k\|_{[W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})]^3} \leq C\|v_1\|_{[W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})]^3}. \quad (3.4.6)$$

We extend v_2^k by zero to $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$. We define the function

$$v_2 := \sum_{k, Q_k \subset R\Omega} v_2^k 1_{Q_k \setminus \overline{\mathcal{O}_k}},$$

Summing (3.4.6) over k such that $Q_k \subset R\Omega$ yields

$$\|v_2\|_{W^{1,q}(\Omega_R)} \leq C\|v_1\|_{W^{1,q}(R\Omega)} \leq CR\|f\|_{L^q(Q_R)}.$$

We set $v = v_1 + v_2$ and notice that v satisfies the conclusion of Lemma 4.1.2. \square

Lemma 3.4.2. *Suppose that Assumption **(A4)**₀ is satisfied. Let $1 < q < +\infty$ and $R > 0$. Let $f \in \mathcal{D}'(\Omega^R)$ be such that $\nabla f \in [W^{-1,q}(\Omega^R)]^3$. Then $f \in L^q(\Omega^R)/\mathbb{R}$ and*

$$\|f\|_{L^q(\Omega^R)/\mathbb{R}} \leq CR\|\nabla f\|_{[W^{-1,q}(\Omega^R)]^3} \quad (3.4.7)$$

where C is a constant independent of f and R .

Proof. The fact that $f \in L^q(\Omega_R)/\mathbb{R}$ follows from [7, Lemma 2.7]. We now show the estimate (3.4.7). For $u \in L^1(\Omega^R)$, we denote $\lambda_u := \frac{1}{|\Omega^R|} \int_{\Omega^R} u$. We prove that there exists a constant C independent of R such that

$$\|f - \lambda_f\|_{L^q(\Omega_R)} \leq CR\|\nabla f\|_{[W^{-1,q}(\Omega^R)]^3}. \quad (3.4.8)$$

We argue by duality. We set $q' = q/(q-1)$. We fix a function $g \in L^{q'}(\Omega^R)$ and we define $\bar{g} := g - \lambda_g$.

We apply Lemma 4.1.2 to \bar{g} : there exists a function $v_g \in [W_0^{1,q'}(\Omega^R)]^3$ such that

$$\begin{cases} -\operatorname{div}(v_g) = \bar{g}, \\ \|v_g\|_{[W^{1,q'}(\Omega^R)]^3} \leq CR\|\bar{g}\|_{L^{q'}(\Omega^R)}. \end{cases}$$

Since $\|\bar{g}\|_{L^{q'}(\Omega_R)} \leq 2\|g\|_{L^{q'}(\Omega_R)}$, we have $\|v_g\|_{[W^{1,q'}(\Omega_R)]^3} \leq CR\|g\|_{L^{q'}(\Omega_R)}$. We now write :

$$\langle \nabla f, v_g \rangle_{[W^{-1,q} \times W_0^{1,q'}(\Omega^R)]^3} = - \int_{\Omega^R} (f - \lambda_f) \operatorname{div}(v_g) = - \int_{\Omega^R} (f - \lambda_f)(g - \lambda_g) = - \int_{\Omega^R} (f - \lambda_f)g.$$

Thus

$$\left| \int_{\Omega^R} (f - \lambda_f)g \right| \leq \|\nabla f\|_{[W^{-1,q}(\Omega^R)]^3} \|v_g\|_{[W_0^{1,q'}(\Omega^R)]^3} \leq CR\|\nabla f\|_{[W^{-1,q}(\Omega^R)]^3} \|g\|_{L^{q'}(\Omega^R)}.$$

Taking the supremum over g , we conclude the proof of the Lemma. \square

Lemma 3.4.3 (Scaling). *Suppose that Assumption $(\mathbf{A4})_0$ is satisfied. Let $1 < q < +\infty$. Let $\varepsilon > 0$ and Ω_ε be defined by (3.1.15). There exists a constant $C > 0$ independent of ε such that for all $f \in \mathcal{D}'(\Omega_\varepsilon)$ such that $\nabla f \in W^{-1,q}(\Omega_\varepsilon)$, we have $f \in L^q(\Omega_\varepsilon)/\mathbb{R}$ and the estimate*

$$\|f\|_{L^q(\Omega_\varepsilon)/\mathbb{R}} \leq C\varepsilon^{-1}\|\nabla f\|_{[W^{-1,q}(\Omega_\varepsilon)]^3}.$$

Proof. We apply Lemma 3.4.2 with $R = 1/\varepsilon$ and use a scaling argument. \square

Lemma 3.4.4. *Suppose that Assumption $(\mathbf{A4})_0$ is satisfied. Let $1 < q < +\infty$ and $F \in [W^{1,q}(\mathbb{R}^3)]^3$. Suppose that for all $k \in \mathbb{Z}^3$,*

$$\int_{\partial\mathcal{O}_k} F \cdot n = 0. \quad (3.4.9)$$

The problem

$$\begin{cases} -\operatorname{div}(v) = \operatorname{div}(F) & \text{in } \mathbb{R}^3 \setminus \bar{\mathcal{O}}, \\ v = 0 & \text{on } \partial\mathcal{O} \end{cases} \quad (3.4.10)$$

admits a solution $v \in [W^{1,q}(\mathbb{R}^3 \setminus \bar{\mathcal{O}})]^3$ such that

$$\|v\|_{[W^{1,q}(\mathbb{R}^3 \setminus \bar{\mathcal{O}})]^3} \leq C\|F\|_{[W^{1,q}(\mathbb{R}^3)]^3}$$

where C is a constant independent of F .

Proof. As in the proof of Lemma 3.3.6, we search the function v under the form $v = \nabla\Psi + v_1$ where

$$-\Delta\Psi = \operatorname{div}(F) \quad \text{in } \mathbb{R}^3,$$

that is, formally,

$$\Psi(x) = C \int_{\mathbb{R}^3} \frac{F(y) \cdot (x - y)}{|x - y|^3} dy$$

and

$$\begin{cases} \operatorname{div}(v_1) = 0 & \text{in } \mathbb{R}^3 \setminus \bar{\mathcal{O}}, \\ v_1 = -\nabla\Psi & \text{on } \partial\mathcal{O}. \end{cases}$$

Since $F \in [L^q(\mathbb{R}^3)]^3$, we know that $\nabla\Psi \in [L^q(\mathbb{R}^3)]^3$ and that there exists a constant $C > 0$ such that $\|\nabla\Psi\|_{[L^q(\mathbb{R}^3)]^3} \leq C\|F\|_{[L^q(\mathbb{R}^3)]^3}$ (see e.g. [58, Exercice II.11.9]). Besides, since $\operatorname{div}(F) \in L^q(\mathbb{R}^3)$, the estimate $\|D^2\Psi\|_{[L^q(\mathbb{R}^3)]^{3 \times 3}} \leq C\|\operatorname{div}(F)\|_{L^q(\mathbb{R}^3)}$ holds true (see e.g. [61, Theorem 9.9 & p. 235]). Thus,

$$\nabla\Psi \in [W^{1,q}(\mathbb{R}^3)]^3 \quad \text{and} \quad \|\nabla\Psi\|_{[W^{1,q}(\mathbb{R}^3)]^3} \leq C\|F\|_{[W^{1,q}(\mathbb{R}^3)]^3}.$$

We define the function v_1 on each cell $Q_k \setminus \bar{\mathcal{O}}_k$ as a solution to

$$\begin{cases} -\operatorname{div}(v_1^k) = 0 & \text{in } Q_k \setminus \bar{\mathcal{O}}_k, \\ v_1^k = 0 & \text{on } \partial Q_k, \\ v_1^k = -\nabla\Psi & \text{on } \partial\mathcal{O}_k. \end{cases} \quad (3.4.11)$$

Assumption $(\mathbf{A4})_0$ together with (3.4.9) guarantee that Problem (3.4.11) admits a solution that satisfies the estimate $\|v_1^k\|_{[W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})]^3} \leq C \|\nabla \Psi\|_{[W^{1,q}(Q_k)]^3}$. This proves the Lemma. \square

Lemma 3.4.5. *Suppose that Assumption $(\mathbf{A4})_1$ is satisfied. Let $g \in H_0^1(\Omega_\varepsilon)$ be such that*

$$\int_{\Omega_\varepsilon} g = 0.$$

The problem

$$\begin{cases} -\operatorname{div}(u) = g & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \quad (3.4.12)$$

admits a solution $u \in [H_0^2(\Omega_\varepsilon)]^3$ such that

$$\|u\|_{H^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon} \|g\|_{H^1(\Omega_\varepsilon)}, \quad (3.4.13)$$

where the constant C is independent of ε .

Proof. The proof is very similar to the proof of Lemma 3.3.6. We explain here only the main lines and refer to Subsection 3.3.3.2 for details. We first extend g by 0 in the perforations. We notice that

$$g \in H_0^1(\Omega) \quad \text{and} \quad \int_{\Omega} g = 0.$$

We consider the problem

$$\begin{cases} -\operatorname{div}(v) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.4.14)$$

Thanks to [58, Theorem III.3.3], Problem (3.4.14) admits a solution $v \in [H_0^2(\Omega)]^3$ such that

$$\|\nabla v\|_{H^1(\Omega)} \leq C(\Omega) \|g\|_{H^1(\Omega)} \quad \text{and} \quad \|v\|_{H^1(\Omega)} \leq C(\Omega) \|g\|_{L^2(\Omega)}. \quad (3.4.15)$$

We fix a cell Q_k such that $\varepsilon Q_k \subset \Omega$. We build a function $v_1^k \in [H^2(\varepsilon[Q_k \setminus \overline{\mathcal{O}_k})]^3$ such that

$$\begin{cases} \operatorname{div}(v_1^k) = 0 & \text{in } \varepsilon[Q_k \setminus \overline{\mathcal{O}_k}], \\ v_1^k = -v & \text{on } \varepsilon\partial\mathcal{O}_k, \\ \nabla v_1^k = -\nabla v & \text{on } \varepsilon\partial\mathcal{O}_k. \end{cases} \quad (3.4.16)$$

For that, we use a cut-off function $\chi_\varepsilon^k := \chi_1(\frac{\cdot}{\varepsilon} - k)$ as in **Step 2** of the proof of Lemma 3.3.6. We solve

$$\begin{cases} \operatorname{div}(w^k) = \operatorname{div}(\chi_\varepsilon^k v) & \text{in } \varepsilon[Q_k \setminus \overline{\mathcal{O}_k}], \\ w^k = 0 & \text{on } \varepsilon\partial[Q_k \setminus \overline{\mathcal{O}_k}] \end{cases} \quad (3.4.17)$$

and then set $v_1^k := w^k - \chi_\varepsilon^k v$. [58, Theorem III.3.3] together with Assumption $(\mathbf{A4})_1$ and a standard scaling argument show that Problem (3.4.16) admits a solution such that

$$\begin{aligned} & \varepsilon^2 \|D^2 v_1^k\|_{L^2(\varepsilon Q_k \setminus \overline{\mathcal{O}_k})} + \varepsilon \|\nabla v_1^k\|_{L^2(\varepsilon Q_k \setminus \overline{\mathcal{O}_k})} + \|v_1^k\|_{L^2(\varepsilon Q_k \setminus \overline{\mathcal{O}_k})} \\ & \leq C(\varepsilon^2 \|D^2 v\|_{L^2(\varepsilon Q_k)} + \varepsilon \|\nabla v\|_{L^2(\varepsilon Q_k)} + \|v\|_{L^2(\varepsilon Q_k)}), \end{aligned} \quad (3.4.18)$$

where the constant C is independent of k and ε . We extend v_1^k by zero to Ω_ε . We define

$$v_1 := \sum_{k \in Y_\varepsilon} v_1^k.$$

Then, after summation of (3.4.18) over k , the estimate

$$\begin{aligned} \varepsilon^2 \|D^2 v_1\|_{[L^2(\Omega_\varepsilon)]^{3 \times 3}} + \varepsilon \|\nabla v_1\|_{[L^2(\Omega_\varepsilon)]^{3 \times 3}} + \|v_1\|_{[L^2(\Omega_\varepsilon)]^3} \\ \leq C \left[\varepsilon^2 \|D^2 v\|_{[L^2(\Omega)]^{3 \times 3}} + \varepsilon \|\nabla v\|_{[L^2(\Omega)]^{3 \times 3}} + \|v\|_{[L^2(\Omega)]^3} \right] \end{aligned} \quad (3.4.19)$$

holds true. We note that the function $u := v + v_1$ satisfies the conclusion of Lemma 3.4.5. Furthermore, using (3.4.15) and (3.4.19), we get

$$\begin{aligned} \varepsilon^2 \|D^2 u\|_{[L^2(\Omega_\varepsilon)]^{3 \times 3}} + \varepsilon \|\nabla u\|_{[L^2(\Omega_\varepsilon)]^{3 \times 3}} + \|u\|_{[L^2(\Omega_\varepsilon)]^3} \\ \leq C \left[\varepsilon^2 \|D^2 v\|_{[L^2(\Omega)]^{3 \times 3}} + \varepsilon \|\nabla v\|_{[L^2(\Omega)]^{3 \times 3}} + \|v\|_{[L^2(\Omega)]^3} \right] \\ \leq C \left[\varepsilon^2 \|g\|_{H^1(\Omega)} + \|v\|_{[H^1(\Omega)]^3} \right] \leq C \left[\varepsilon^2 \|g\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)} \right] \\ \leq C \left[\varepsilon^2 \|g\|_{H^1(\Omega)} + \varepsilon \|g\|_{H^1(\Omega)} \right] \leq C \varepsilon \|g\|_{H^1(\Omega)}. \end{aligned} \quad (3.4.20)$$

where we used Lemma 3.3.1 on g in the last inequality. Thus (3.4.13) is proved. \square

3.4.2 Geometric assumptions

We prove in this section that Assumptions **(A3)** and **(A4)'** imply Assumption **(A4)** and that Assumptions **(A3)** and **(A5)'** imply Assumption **(A5)**. Appendix 3.4.2 follows the proofs of [58, Theorem III.3.1] and [58, Theorem IV.5.1] and makes precise the dependance of the constants appearing in these arguments. We begin by a covering Lemma.

Lemma 3.4.6. *Suppose that Assumption **(A3)** is satisfied. Let $0 < \rho < d(\partial\mathcal{O}_0^{\text{per}}, \partial Q)$. There exists $N \in \mathbb{N}^*$ such that for all $k \in \mathbb{Z}^3$, there exist $2N$ balls $B_i^k, i = 1, \dots, 2N$ such that*

- (i) *for all $i = 1, \dots, N$, we have that $B_i^k = B(\xi_i^k, \rho)$, $\xi_i^k \in \partial\mathcal{O}_k$ and $\{x \in Q_k \setminus \overline{\mathcal{O}_k}, d(x, \partial\mathcal{O}_k) < 3\rho/16\} \subset \bigcup_{i=1}^N B_i^k$;*
- (ii) *for all $i = N+1, \dots, 2N$, we have that $B_i^k = B(\xi_i^k, \rho/32)$, $\xi_i^k \in \{x \in Q_k \setminus \overline{\mathcal{O}_k}, d(x, \partial\mathcal{O}_k) > \rho/16\}$ and $\{x \in Q_k \setminus \overline{\mathcal{O}_k}, d(x, \partial\mathcal{O}_k) \geq 3\rho/16\} \subset \bigcup_{i=N+1}^{2N} B_i^k$.*

Moreover, there exist $2N$ balls $B_i^{0,\text{per}}, i = 1, \dots, 2N$ and $\eta = \eta(\rho) > 0$ such that

- (iii) *for all $i = 1, \dots, 2N$, $B_i^{0,\text{per}} \subset Q''$ and $\{x \text{ s.t. } d(x, Q \setminus \overline{\mathcal{O}_0^{\text{per}}}) < \eta\} \subset \bigcup_{i=1}^{2N} B_i^{0,\text{per}}$.*

- (iv) *there exists a bijection $\sigma : \{1, \dots, 2N\} \rightarrow \{1, \dots, 2N\}$ such that for all $i \in \{1, \dots, 2N-1\}$, we have that*

$$\Omega_{\sigma(i)}^{0,\text{per}} \cap \left(\bigcup_{s=i+1}^{2N} \Omega_{\sigma(s)}^{0,\text{per}} \right) \neq \emptyset \quad \text{and} \quad \Omega_j^{0,\text{per}} := B_j^{0,\text{per}} \cap \left(Q \setminus \overline{\mathcal{O}_0^{\text{per}}} \right).$$

- (v) *for all but a finite number of $k \in \mathbb{Z}^3$, we have that $B_i^{k,\text{per}} \subset B_i^k$ for all $i = 1, \dots, 2N$ and $\{x \text{ s.t. } d(x, Q_k \setminus \overline{\mathcal{O}_k}) < \eta/2\} \subset \bigcup_{i=1}^{2N} B_i^{k,\text{per}}$, where $B_i^{k,\text{per}} := B_i^{0,\text{per}} + k$.*

Remark 3.4.7. *Lemma 3.4.6.(iv) means that we can relabel the family $B_i^{0,\text{per}}, i = 1, \dots, 2N$ such that for all $i \in \{1, \dots, 2N-1\}$, we have that $\Omega_i^{0,\text{per}} \cap \left(\Omega_{i+1}^{0,\text{per}} \cup \dots \cup \Omega_{2N}^{0,\text{per}} \right) \neq \emptyset$.*

Proof. The proof of Lemma 3.4.6 relies on the periodic structure and on Assumption **(A3)**. We first fix by compactness N_0 balls $B_i^{0,\text{per}} = B(x_i, \rho/2), i = 1, \dots, N_0$ such that

$$\{x \in Q, \quad d(x, \partial\mathcal{O}_0^{\text{per}}) \leq \rho/4\} \subset \bigcup_{i=1}^{N_0} B_i^{0,\text{per}} \quad \text{and} \quad x_i \in \partial\mathcal{O}_0^{\text{per}}. \quad (3.4.21)$$

We note that there exists $\widehat{\rho} > 0$ such that for all $i \in \{1, \dots, N_0\}$, there exist two points $y_i \in B_i^{0,\text{per}} \cap \mathcal{O}_0^{\text{per}}$ and $z_i \in B_i^{0,\text{per}} \setminus \mathcal{O}_0^{\text{per}}$ satisfying $d(y_i, \partial\mathcal{O}_0^{\text{per}}) > \widehat{\rho}$ and $d(z_i, \partial\mathcal{O}_0^{\text{per}}) > \widehat{\rho}$. We define for each $k \in \mathbb{Z}^3$, $x_i^k := x_i + k$, $y_i^k := y_i + k$, $z_i^k := z_i + k$ and $B_i^{k,\text{per}} := B_i^{0,\text{per}} + k = B(x_i^k, \rho/2)$. By translation invariance, we obviously have (3.4.21) with 0 replaced by any $k \in \mathbb{Z}^3$.

We consider $k \in \mathbb{Z}^3$ such that $\alpha_k < \min(\rho/16, \widehat{\rho})$ (where we recall that α_k is introduced in **(A3)**). Then, by Assumption **(A3)** and (3.4.21), we have that

$$\{x \in Q_k, \quad d(x, \partial\mathcal{O}_k) < 3\rho/16\} \subset \{x \in Q_k, \quad d(x, \partial\mathcal{O}_k^{\text{per}}) < \rho/4\} \subset \bigcup_{i=1}^{N_0} B_i^{k,\text{per}}. \quad (3.4.22)$$

We next claim that each ball $B_i^{k,\text{per}}$, $i = 1, \dots, N_0$ intersects $\partial\mathcal{O}_k$. By definition, we have that $y_i^k \in B_i^{k,\text{per}} \cap \mathcal{O}_k^{\text{per}}$ and that $d(y_i^k, \partial\mathcal{O}_k^{\text{per}}) > \widehat{\rho} > \alpha_k$. Thus, by **(A3)**, we get that $y_i^k \in B_i^{k,\text{per}} \cap \mathcal{O}_k$. Similarly, we have that $z_i^k \in B_i^{k,\text{per}} \setminus \mathcal{O}_k$. Thus, there exists $\xi_i^k \in [y_i^k, z_i^k] \cap \partial\mathcal{O}_k$, proving that $\partial\mathcal{O}_k \cap B_i^{k,\text{per}} \neq \emptyset$. We fix an arbitrary point $\xi_i^k \in \partial\mathcal{O}_k \cap B_i^{k,\text{per}}$ and we notice that $B_i^{k,\text{per}} \subset B(\xi_i^k, \rho)$. By (3.4.22), we conclude that

$$\{x \in Q_k, \quad d(x, \partial\mathcal{O}_k) < 3\rho/16\} \subset \bigcup_{i=1}^{N_0} B_i^k. \quad (3.4.23)$$

It remains to cover $\{x \in Q_k \setminus \overline{\mathcal{O}_k}, \quad d(x, \partial\mathcal{O}_k) \geq 3\rho/16\}$. By **(A3)**, we have that

$$\{x \in Q_k \setminus \overline{\mathcal{O}_k}, \quad d(x, \partial\mathcal{O}_k) \geq 3\rho/16\} \subset \{x \in \overline{Q_k} \setminus \overline{\mathcal{O}_k^{\text{per}}}, \quad d(x, \partial\mathcal{O}_k^{\text{per}}) \geq \rho/8\}. \quad (3.4.24)$$

By compactness and translation invariance, we can cover the right hand side of (3.4.24) by N_1 balls $B_i^{k,\text{per}} = B(x_i^k, \rho/32)$, $i = N_0 + 1, \dots, N_0 + N_1$ where x_i^k is of the form $x_i^k = x_i + k$ and $x_i \in \{x \in Q \setminus \overline{\mathcal{O}_0^{\text{per}}}, \quad d(x, \partial\mathcal{O}_0^{\text{per}}) \geq \rho/8\}$. We set $B_i^k := B_i^{k,\text{per}}$ and $\xi_i^k := x_i^k$. By **(A3)**, we get that $\xi_i^k \in \{Q_k \setminus \overline{\mathcal{O}_k}, \quad d(x, \partial\mathcal{O}_k) > \rho/16\}$. With N to be fixed later, we have proved (i)-(ii) for $k \in \mathbb{Z}^3$ such that $\alpha_k < \min(\rho/16, \widehat{\rho})$.

We fix $k \in \mathbb{Z}^3$ such that $\alpha_k \geq \min(\rho/16, \widehat{\rho})$. We take any covering of $\{x \in Q_k, \quad d(x, \partial\mathcal{O}_k) < 3\rho/16\}$ with balls $B_i^k = B(\xi_i^k, \rho)$, $\xi_i^k \in \partial\mathcal{O}_k$ and $i \in \{1, \dots, N_0^k\}$. We then take any covering of $\{x \in Q_k \setminus \overline{\mathcal{O}_k}, \quad d(x, \partial\mathcal{O}_k) \geq 3\rho/16\}$ with balls $B_i^k = B(\xi_i^k, \rho/32)$, $\xi_i^k \in \{x \in Q_k \setminus \overline{\mathcal{O}_k}, \quad d(x, \partial\mathcal{O}_k) \geq 3\rho/16\}$ and $i \in \{N_0^k + 1, \dots, N_0^k + N_1^k\}$.

We set $N := \max_{k \in \mathbb{Z}^3} \{N_0^k, N_1^k\}$ where $N_0^k = N_0$ and $N_1^k = N_1$ if $\alpha_k < \min(\rho/16, \widehat{\rho})$. Note that because of **(A3)**, we have that $N < +\infty$. If $N_0^k < N$ or $N_1^k < N$, we duplicate one of the balls in order to define $2N$ balls $B_i^k, i = 1, \dots, 2N$. We proceed similarly for $B_i^{0,\text{per}}, i = 1, \dots, N_0 + N_1$. Assertions (i), (ii), (iii) and (v) are proved. We prove easily (iv) by connectedness of $Q \setminus \overline{\mathcal{O}_0^{\text{per}}}$. \square

Assumptions **(A3)** and **(A4)'** imply **(A4)**

Proof that **(A4)₀** is satisfied

Let $k \in \mathbb{Z}^3$. We formulate [58, Theorem III.3.1] in our particular setting: suppose that there exists $\Omega_i^k, i = 1, \dots, N_k$ such that

$$Q_k \setminus \overline{\mathcal{O}_k} = \bigcup_{i=1}^{N_k} \Omega_i^k, \quad (3.4.25)$$

where Ω_i^k is star-shaped with respect to a ball B_i^k of radius ρ_i^k such that $B_i^k \subset \subset \Omega_i^k$. We define for $i = 1, \dots, N_k - 1$

$$F_i^k := \Omega_i^k \cap \left(\bigcup_{s=i+1}^{N_k} \Omega_s^k \right)$$

and we assume that $F_i^k \neq \emptyset$ for all $i \in \{1, \dots, N_k - 1\}$. Then Problem (3.1.7) with $f \in L^q(Q_k \setminus \overline{\mathcal{O}_k})$ admits a solution v satisfying (3.1.9) with

$$C_q^0(k) \leq C(q) \left(\frac{\text{diam}(Q_k \setminus \overline{\mathcal{O}_k})}{\min_{i=1}^{N_k} \rho_i^k} \right)^3 \left(1 + \frac{\text{diam}(Q_k \setminus \overline{\mathcal{O}_k})}{\min_{i=1}^{N_k} \rho_i^k} \right) \left(1 + \frac{|Q_k \setminus \overline{\mathcal{O}_k}|^{1-1/q}}{\min_{i=1}^{N_k-1} |F_i^k|^{1-1/q}} \right)^{N_k}. \quad (3.4.26)$$

To bound $C_q^0(k)$ uniformly in k , it is sufficient to show that $Q_k \setminus \overline{\mathcal{O}_k}$ has a decomposition of the form (3.4.25) where N_k is independent of k , ρ_i^k and $|F_i^k|$ are uniformly bounded from below in k and i . We first explain how to find such a decomposition with N_k and ρ_i^k independent of k and i . By making precise the dependance on the geometry of $\partial\mathcal{O}_k$ at each step of the proof of [58, Lemma II.1.3], we can show that **(A4)'** implies that there exists $\rho > 0$ such that for all $k \in \mathbb{Z}^3$ and $\xi^k \in \partial\mathcal{O}_k$, there exists an open set G_{ξ^k} such that $\Omega_{\xi^k} := G_{\xi^k} \cap (Q_k \setminus \overline{\mathcal{O}_k})$ is star-shaped with respect to a ball of radius ρ strictly included in Ω_{ξ^k} and $B(\xi^k, \rho) \subset G_{\xi^k}$.

We next apply Lemma 3.4.6 with ρ given before and we denote by B_i^k , $i = 1, \dots, 2N$ the family of balls that we obtain. For $i = 1, \dots, N$, we define $G_i^k := G_{\xi_i^k}$ and $\Omega_i^k := \Omega_{\xi_i^k}$. For $i = N + 1, \dots, 2N$, we define $\Omega_i^k := B_i^k \cap (Q_k \setminus \overline{\mathcal{O}_k})$. Since $B_i^k \subset G_i^k$ for $i = 1, \dots, N$ and because B_i^k , $i = 1, \dots, 2N$ covers $Q_k \setminus \overline{\mathcal{O}_k}$, we have that (3.4.25) is satisfied with $\rho_i^k \geq \rho/32$ and $N_k = 2N$.

It remains to check that there exists a relabeling of the Ω_i^k 's such that we have that $\min_{i=1}^{2N-1} |F_i^k| \geq C$ where $C > 0$ is independent of k . We use Lemma 3.4.6.(iii)-(v). According to Remark 3.4.7, we relabel the $\Omega_i^{k,\text{per}}$ (note that this also implies a relabeling of the Ω_i^k 's) such that

$$\forall i \in \{1, \dots, 2N - 1\}, F_i^{k,\text{per}} := \Omega_i^{k,\text{per}} \cap \left(\Omega_{i+1}^{k,\text{per}} \cup \dots \cup \Omega_{2N}^{k,\text{per}} \right) \neq \emptyset.$$

We then fix $\rho' > 0$ such that for all $i \in \{1, \dots, 2N - 1\}$, we have that $F_i^{k,\text{per}}$ contains a ball $(B_i^{k,\text{per}})'$ of radius ρ' such that $(B_i^{k,\text{per}})' \subset\subset Q_k \setminus \overline{\mathcal{O}_k^{\text{per}}}$. We fix $k \in \mathbb{Z}^3$ such that Lemma 3.4.6.(v) is satisfied and such that

$$\alpha_k < \min_{i=1}^{2N-1} d \left((B_i^{k,\text{per}})', \partial\mathcal{O}_k^{\text{per}} \right). \quad (3.4.27)$$

Then, for all $i \in \{1, \dots, 2N - 1\}$, we have that

$$(B_i^{k,\text{per}})' \subset\subset Q_k \setminus \overline{\mathcal{O}_k}. \quad (3.4.28)$$

We then recall that

$$F_i^k = \Omega_i^k \cap \left(\bigcup_{s=i+1}^{2N} \Omega_s^k \right) = \left[G_i^k \cap \left(\bigcup_{s=i+1}^{2N} G_s^k \right) \right] \cap (Q_k \setminus \overline{\mathcal{O}_k}).$$

By Lemma 3.4.6.(v), we have that $B_j^{k,\text{per}} \subset G_j^k$ for all $j \in \{1, \dots, 2N\}$. Together with (3.4.28), this yields that $(B_i^{k,\text{per}})' \subset F_i^k$ for all $i \in \{1, \dots, 2N - 1\}$. Thus, $\min_{i=1}^{2N-1} |F_i^k| \geq \frac{4}{3}\pi\rho'^3$. Since by **(A3)** there are only a finite number of indices k such that (3.4.27) is not satisfied, we conclude that, after eventually relabeling these F_i^k 's, we have that $\min_{i=1}^{2N-1} |F_i^k| \geq C > 0$.

Proof that **(A4)₁** is satisfied

We briefly sketch the proof of **(A4)₁** and we refer to the proof of **(A4)₀** for some ingredients. Let $k \in \mathbb{Z}^3$ and $f \in W_0^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})$. To solve Problem (3.1.7), we use a decomposition of the form (3.4.25) with N_k uniform in k ($= N$) and Ω_i^k that is star-shaped with respect to a ball of radius ρ uniformly

bounded from below in k and i , as constructed in the proof of **(A4)**₀. We then write $f = f_1 + \dots + f_N$ where $f_i \in W_0^{1,q}(\Omega_i)$, $\int_{\Omega_i} f_i = 0$, $\|f_i\|_{W^{1,q}(\Omega_i^k)} \leq C_i^k \|f\|_{W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})}$ and we solve the Problem:

$$\begin{cases} \operatorname{div} v_i = f_i & \text{in } \Omega_i^k \\ v_i \in [W_0^{2,q}(\Omega_i^k)]^3. \end{cases}$$

Thanks to the estimate (III.3.23) of [58, p. 168], we have that

$$\|v_i\|_{[W^{2,q}(\Omega_i)]^3} \leq C(q, \rho) \|f_i\|_{W^{1,q}(\Omega_i^k)} \leq C(q, \rho) C_i^k \|f\|_{W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})}.$$

Extending v_i by zero to $Q_k \setminus \overline{\mathcal{O}_k}$ and setting $v := v_1 + \dots + v_N$, we have that v solves Problem (3.1.7) with the estimate

$$\|v\|_{[W^{2,q}(Q_k \setminus \overline{\mathcal{O}_k})]^3} \leq C(q, \rho) N C_i^k \|f\|_{W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})}.$$

We can conclude that **(A4)**₁ is satisfied if C_i^k is uniformly bounded in i and k . To prove that, we make precise the dependance of the constant controlling $\|f_i\|_{W^{1,q}(\Omega_i^k)}$ in the proof of [58, Lemma III.3.4.(vii)-(viii)]. This constant depends on N and on the maximum of the $W^{1,\infty}$ -norms of the functions Ψ_i^k , $i = 1, \dots, 2N$ and χ_i^k , $i = 1, \dots, 2N - 1$ where $\{\Psi_1^k, \dots, \Psi_{2N}^k\}$ is a partition of unity associated to $\{G_1^k, \dots, G_{2N}^k\}$ and $\chi_i^k \in \mathcal{D}(F_i^k)$ satisfies $\int_{F_i^k} \chi_i^k = 1$. Because of Lemma 3.4.6.(v), the family $\{\Psi_1^k, \dots, \Psi_{2N}^k\}$ may be chosen independently of k (by using the periodic balls), except for a finite number of indices k . Besides, still after the exclusion of a finite number of indices k , we have shown in the proof of **(A4)**₀ that F_i^k contains a ball of radius ρ' which is uniformly bounded in i and k . Thus, χ_i^k may be chosen as the translation of a reference function χ satisfying $\chi \in \mathcal{D}(B(0, \rho'))$ and $\int_{B(0, \rho')} \chi = 1$. This proves that $\max_{i=1}^{2N} C_i^k \leq C$ for all but a finite number of $k \in \mathbb{Z}^3$. Applying [58, Lemma III.3.4] for the remaining indices k , we conclude that $\max_{i=1}^{2N} C_i^k \leq C$ for all $k \in \mathbb{Z}^3$. This concludes the proof of **(A4)**₁.

Assumptions **(A3)** and **(A5)**' imply **(A5)**

We fix $f \in [L^q(Q_k'' \setminus \overline{\mathcal{O}_k})]^3$ and we consider the pair (v, p) solution to (3.1.11). We want to prove the regularity estimate (3.1.12). The interior regularity property is given by the following result (see [58, Theorem IV.4.1]):

$$\|D^2 v\|_{L^q(\Omega_k)^{3 \times 3 \times 3}} + \|\nabla p\|_{L^q(\Omega_k)^3} \leq C [\|v\|_{W^{1,q}(\Omega_k')^3} + \|p\|_{L^q(\Omega_k')} + \|f\|_{L^q(\Omega_k')^3}], \quad (3.4.29)$$

where $\Omega_k \subset\subset \Omega_k' \subset\subset Q_k'' \setminus \overline{\mathcal{O}_k}$ and C depends only on q and on the distance between Ω_k and $(\Omega_k')^c$. The regularity up to the boundary follows from the discussion [58, pp.271-274]. By tracing the dependance of the constants in these arguments, we can show that, under Assumption **(A5)**' , there exist a radius $\rho > 0$, a constant $d > 1$ and a constant $C > 0$ such that $d\rho < d(Q, \partial Q'')$ and for all $k \in \mathbb{Z}^3$ and $x \in \partial \mathcal{O}_k$, we have that

$$\begin{aligned} \|D^2 v\|_{L^q((Q_k \setminus \overline{\mathcal{O}_k}) \cap B(x, \rho))^{3 \times 3 \times 3}} + \|\nabla p\|_{L^q((Q_k \setminus \overline{\mathcal{O}_k}) \cap B(x, \rho))^3} &\leq C [\|v\|_{W^{1,q}((Q_k \setminus \overline{\mathcal{O}_k}) \cap B(x, d\rho))^3} \\ &+ \|p\|_{L^q((Q_k \setminus \overline{\mathcal{O}_k}) \cap B(x, d\rho))} + \|f\|_{L^q((Q_k \setminus \overline{\mathcal{O}_k}) \cap B(x, d\rho))^3}]. \end{aligned} \quad (3.4.30)$$

We combine estimates (3.4.29) and (3.4.30). We fix $k \in \mathbb{Z}^3$. Let $(B_i^k)_{i=1, \dots, 2N}$ be the family of balls given by Lemma 3.4.6 (applied with ρ defined by (3.4.30)). Thanks to (3.4.30) and the inequality

$$\forall a_1, \dots, a_p > 0, \quad a_1^q + \dots + a_p^q \leq (a_1 + \dots + a_p)^q \leq C_{p,q} (a_1^q + \dots + a_p^q), \quad (3.4.31)$$

we have for all $i \in \{1, \dots, N\}$,

$$\begin{aligned} \|D^2 v\|_{L^q((Q_k \setminus \overline{\mathcal{O}_k}) \cap B(\xi_i^k, \rho))^{3 \times 3 \times 3}} + \|\nabla p\|_{L^q((Q_k \setminus \overline{\mathcal{O}_k}) \cap B(\xi_i^k, \rho))^3} &\leq C [\|v\|_{W^{1,q}((Q_k \setminus \overline{\mathcal{O}_k}) \cap B(\xi_i^k, d\rho))^3} \\ &+ \|p\|_{L^q((Q_k \setminus \overline{\mathcal{O}_k}) \cap B(\xi_i^k, d\rho))} + \|f\|_{L^q((Q_k \setminus \overline{\mathcal{O}_k}) \cap B(\xi_i^k, d\rho))^3}]. \end{aligned} \quad (3.4.32)$$

Summing (3.4.32) over $i \in \{1, \dots, N\}$ and using that

$$U_k := \{x \in Q_k \setminus \overline{\mathcal{O}_k}, d(x, \partial\mathcal{O}_k) < 3\rho/16\} \subset \bigcup_{i=1}^N B(\xi_i^k, \rho) \quad \text{and} \quad (Q_k \setminus \overline{\mathcal{O}_k}) \cap B(\xi_i^k, d\rho) \subset Q_k'' \setminus \overline{\mathcal{O}_k}$$

yield

$$\|D^2 v\|_{L^q(U_k)^{3 \times 3 \times 3}}^q + \|\nabla p\|_{L^q(U_k)^3}^q \leq CN \left[\|v\|_{W^{1,q}(Q_k'' \setminus \overline{\mathcal{O}_k})^3}^q + \|p\|_{L^q(Q_k'' \setminus \overline{\mathcal{O}_k})}^q + \|f\|_{L^q(Q_k'' \setminus \overline{\mathcal{O}_k})^3}^q \right]. \quad (3.4.33)$$

We now apply (3.4.29) to $\Omega_k = \{x \in Q_k \setminus \overline{\mathcal{O}_k}, d(x, \partial\mathcal{O}_k) > \rho/8\}$ and $\Omega_k' = Q_k'' \setminus \overline{\mathcal{O}_k}$. We have that $d(\Omega_k, (\Omega_k')^c) = \min(d(Q, \partial Q''), \rho/8)$ is independent of k . Thus, using (3.4.29) and (3.4.31) yield

$$\|D^2 v\|_{L^q(\Omega_k)^{3 \times 3 \times 3}}^q + \|\nabla p\|_{L^q(\Omega_k)^3}^q \leq C \left[\|v\|_{W^{1,q}(Q_k'' \setminus \overline{\mathcal{O}_k})^3}^q + \|p\|_{L^q(Q_k'' \setminus \overline{\mathcal{O}_k})}^q + \|f\|_{L^q(Q_k'' \setminus \overline{\mathcal{O}_k})^3}^q \right], \quad (3.4.34)$$

where C is independent of k . Summing (3.4.32) and (3.4.34) and using that $U_k \cup \Omega_k = Q_k \setminus \overline{\mathcal{O}_k}$ together with (3.4.31) proves that **(A5)** is satisfied.

3.4.3 An additional result on the correctors

This section does not appear in the published version [107].

In this section, we prove Theorem 3.4.8 below. This is the first step in obtaining improved convergence rates *i.e.* in $W^{1,q}$ -norm for $q > 2$ for the homogenization of the Stokes system.

Theorem 3.4.8 (L^∞ -bounds on the correctors). *Assume that the perforations $(\mathcal{O}_k)_{k \in \mathbb{Z}^3}$ are of class $C^{2,\alpha}$ uniformly in k . Then $(w_j, p_j) \in W^{1,\infty} \times L^\infty(\mathbb{R}^3 \setminus \overline{\mathcal{O}})$ for $j = 1, 2, 3$.*

Note that the above Theorem gives the sublinearity of the correctors. In particular, we have that

$$\|\varepsilon p_j(\cdot/\varepsilon)\|_{L^q(\Omega_\varepsilon)/\mathbb{R}} \sim \varepsilon$$

for any $q \in (1, +\infty)$. This was restricted to $q = 2$ in Theorem 3.2.1.

We give the proof of Theorem 3.4.8 in the case of compactly supported perturbations. We indicate in the following where we use this assumption and why the argument is still valid in the case of general perturbations.

Proof of Theorem 3.4.8. We fix $j \in \{1, 2, 3\}$. We extend \widetilde{w}_j by $-w_j^{\text{per}}$ in \mathcal{O} . We have in particular that $\widetilde{w}_j \in H^1(\mathbb{R}^3)$. We show that there exists a constant $C > 0$ independent of $k \in \mathbb{Z}^3$ such that for all $k \in \mathbb{Z}^3$, we have

$$\|w_j\|_{W^{1,\infty}(Q_k)} \leq C \quad \text{and} \quad \|p_j\|_{L^\infty(Q_k)} \leq C.$$

We first prove that the velocity is bounded. We recall that

$$\begin{cases} -\Delta \widetilde{w}_j + \nabla \widetilde{p}_j = 0 \\ \operatorname{div} \widetilde{w}_j = 0 \end{cases}$$

in $Q_k'' \setminus \overline{Q_k'}$ for some domains $Q_k' = Q' + k$ and $Q_k'' = Q'' + k$ such that $Q' \subset\subset Q \subset\subset Q''$. We fix in what follows a smooth domain \widehat{Q} such that $Q \subset\subset \widehat{Q} \subset\subset Q''$. We also define $U := \widehat{Q} \setminus \overline{Q}$, where $Q' \subset\subset \widehat{Q} \subset\subset Q$. For $k \in \mathbb{Z}^3$, we define $\widehat{Q}_k := \widehat{Q} + k$ and $U_k := U + k$ and we note that $U_k \subset\subset Q_k'' \setminus \overline{Q_k'}$. By [58, Theorem IV.4.1 and Remark IV.4.1], we have that for all $m \geq 0$,

$$\|\widetilde{w}_j\|_{H^{m+2}(U_k)} \leq C \|\widetilde{w}_j\|_{H^1(Q_k'' \setminus \overline{Q_k'})}$$

where the constant C is independent of k by translation invariance. Choosing $m > 3/2$ and applying the injection $H^{m+2}(U_k) \hookrightarrow \mathcal{C}^{2,\alpha}(\overline{U_k})$ for some $\alpha > 0$ gives that

$$\|\widetilde{w}_j\|_{\mathcal{C}^{2,\alpha}(\overline{U_k})} \leq C \|\widetilde{w}_j\|_{H^1(Q'_k \setminus \overline{Q'_k})},$$

where the constant C is, here again, independent of k by translation invariance. In particular, since $\partial\widehat{Q}_k \subset \overline{U_k}$, we have that

$$\|\widetilde{w}_j\|_{\mathcal{C}^{2,\alpha}(\partial\widehat{Q}_k)} \leq C \|\widetilde{w}_j\|_{H^1(Q'_k \setminus \overline{Q'_k})} \leq C \|\widetilde{w}_j\|_{H^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})} \leq C.$$

This proves that

$$\|w_j\|_{\mathcal{C}^{2,\alpha}(\partial\widehat{Q}_k)} \leq \|w_j^{\text{per}}\|_{\mathcal{C}^{2,\alpha}(\partial\widehat{Q}_k)} + \|\widetilde{w}_j\|_{\mathcal{C}^{2,\alpha}(\partial\widehat{Q}_k)} \leq C. \quad (3.4.35)$$

Now, we recall that

$$\begin{cases} -\Delta w_j + \nabla p_j = e_j & \text{in } \widehat{Q}_k \setminus \overline{\mathcal{O}_k} \\ \operatorname{div} w_j = 0. \end{cases}$$

Schauder estimates (see [58, Theorem IV.7.1]) provide the existence of a constant λ_j^k such that

$$\begin{aligned} \|w_j\|_{\mathcal{C}^{2,\alpha}(\widehat{Q}_k \setminus \overline{\mathcal{O}_k})} + \|p_j - \lambda_j^k\|_{\mathcal{C}^{1,\alpha}(\widehat{Q}_k \setminus \overline{\mathcal{O}_k})} &\leq C_k \left[\|e_j\|_{\mathcal{C}^{0,\alpha}(\widehat{Q}_k \setminus \overline{\mathcal{O}_k})} + \|w_j\|_{\mathcal{C}^{2,\alpha}(\partial[\widehat{Q}_k \setminus \overline{\mathcal{O}_k}])} \right] \\ &\leq C_k \left[\|e_j\|_{\mathcal{C}^{0,\alpha}(\widehat{Q}_k \setminus \overline{\mathcal{O}_k})} + \|w_j\|_{\mathcal{C}^{2,\alpha}(\partial\widehat{Q}_k)} \right]. \end{aligned}$$

The constant C_k depends on the $\mathcal{C}^{2,\alpha}$ -norms of the charts flattening $\partial\mathcal{O}_k$ which are assumed to be bounded independently of k . Thus, C_k is independent of k and finally, using also (3.4.35), we derive that

$$\|w_j\|_{\mathcal{C}^{2,\alpha}(Q_k \setminus \overline{\mathcal{O}_k})} + \|p_j - \lambda_j^k\|_{\mathcal{C}^{1,\alpha}(Q_k \setminus \overline{\mathcal{O}_k})} \leq C. \quad (3.4.36)$$

This proves that the velocity is bounded. The rest of the proof is devoted to the pressure. Note that, thanks to (3.4.36), we have

$$\begin{aligned} \left\| p_j - \fint_{Q_k \setminus \overline{\mathcal{O}_k}} p_j \right\|_{L^\infty(Q_k \setminus \overline{\mathcal{O}_k})} &\leq \|p_j - \lambda_j^k\|_{L^\infty(Q_k \setminus \overline{\mathcal{O}_k})} + \left\| \lambda_j^k - \fint_{Q_k \setminus \overline{\mathcal{O}_k}} p_j \right\|_{L^\infty(Q_k \setminus \overline{\mathcal{O}_k})} \\ &\leq C + \left| \fint_{Q_k \setminus \overline{\mathcal{O}_k}} (\lambda_j^k - p_j) \right| \\ &\leq C + \|p_j - \lambda_j^k\|_{L^\infty(Q_k \setminus \overline{\mathcal{O}_k})} \leq C. \end{aligned}$$

This means that we may choose

$$\lambda_j^k = \fint_{Q_k \setminus \overline{\mathcal{O}_k}} p_j$$

in (3.4.36). We show in the following lines that the sequence $(\lambda_j^k)_{k \in \mathbb{Z}^3}$ is bounded. This will prove that

$$\|p_j\|_{L^\infty(Q_k \setminus \overline{\mathcal{O}_k})} \leq \|p_j - \lambda_j^k\|_{L^\infty(Q_k \setminus \overline{\mathcal{O}_k})} + |\lambda_j^k| \stackrel{(3.4.36)}{\leq} C + \max_{k \in \mathbb{Z}^3} |\lambda_j^k| \leq C(j),$$

concluding our proof. To show that $(\lambda_j^k)_{k \in \mathbb{Z}^3}$ is bounded, we are going to quantify its growth on neighbouring cells. Since λ_j^k is a mean-value on $Q_k \setminus \overline{\mathcal{O}_k}$, it would be convenient that for all but a finite number of $k \in \mathbb{Z}^3$, the volume $|Q_k \setminus \overline{\mathcal{O}_k}|$ is independent of k . Thus, we assume that there exists $K > 0$ such that for all $|k| \geq K$, $|Q_k \setminus \overline{\mathcal{O}_k}| = |Q_k \setminus \overline{\mathcal{O}_k^{\text{per}}}|$. Since

$$\lambda_j^k = \fint_{Q_k \setminus \overline{\mathcal{O}_k}} p_j^{\text{per}} + \fint_{Q_k \setminus \overline{\mathcal{O}_k}} \widetilde{p}_j,$$

it is sufficient to show that

$$\widetilde{\lambda}_j^k := \int_{Q_k \setminus \overline{\mathcal{O}_k}} \widetilde{p}_j$$

is bounded independently of k . We fix $i \in \{1, 2, 3\}$, we are going to show that

$$\left| \widetilde{\lambda}_j^k - \widetilde{\lambda}_j^{k+e_i} \right| \leq C \left[\|\nabla \widetilde{w}_j\|_{L^2(Q_k \cup Q_{k+e_i})} + \sqrt{\alpha_k} + \sqrt{\alpha_{k+e_i}} \right], \quad (3.4.37)$$

where C is independent of k and $|k| \geq K + 1$ and the sequence $(\alpha_k)_{k \in \mathbb{Z}^3}$ appears in Assumption **(A3)**. We fix $|k| \geq K$. We write the PDEs satisfied by $(\widetilde{w}_j, \widetilde{p}_j)$ on the (connected domains) $Q_k \setminus \overline{\mathcal{O}_k}$ and $\Omega_k^{e_i} := \text{Int} \left[(\overline{Q_k} \setminus \overline{\mathcal{O}_k}) \cup (\overline{Q_{k+e_i}} \setminus \overline{\mathcal{O}_{k+e_i}}) \right]$ ($\text{Int}(A)$ denotes the interior of A):

$$\begin{cases} -\Delta \widetilde{w}_j + \nabla \widetilde{p}_j = T_j & \text{in } Q_k \setminus \overline{\mathcal{O}_k} \\ \text{div } \widetilde{w}_j = 0 \\ \widetilde{w}_j = -w_j^{\text{per}} & \text{on } \partial \mathcal{O}_k \end{cases} \quad (3.4.38)$$

and

$$\begin{cases} -\Delta \widetilde{w}_j + \nabla \widetilde{p}_j = T_j & \text{in } \Omega_k^{e_i} \\ \text{div } \widetilde{w}_j = 0 \\ \widetilde{w}_j = -w_j^{\text{per}} & \text{on } \partial \mathcal{O}_k \cup \partial \mathcal{O}_{k+e_i}, \end{cases} \quad (3.4.39)$$

where

$$\|T_j\|_{H^{-1}(Q_k \setminus \overline{\mathcal{O}_k})} \leq C \sqrt{\alpha_k} \quad \text{and} \quad \|T_j\|_{H^{-1}(\Omega_k^{e_i})} \leq C (\sqrt{\alpha_k} + \sqrt{\alpha_{k+e_i}}).$$

By Assumption **(A4)**₀, there exists $v_1^k \in [H_0^1(Q_k \setminus \overline{\mathcal{O}_k})]^3$ such that

$$\text{div}(v_1^k) = \widetilde{p}_j - \int_{Q_k \setminus \overline{\mathcal{O}_k}} \widetilde{p}_j \quad \text{and} \quad \|v_1^k\|_{H_0^1(Q_k \setminus \overline{\mathcal{O}_k})} \leq C \left\| \widetilde{p}_j - \int_{Q_k \setminus \overline{\mathcal{O}_k}} \widetilde{p}_j \right\|_{L^2(Q_k \setminus \overline{\mathcal{O}_k})},$$

where the constant C is independent of k . Similarly, by Assumption **(A4)**₀ and [58] (for a bound on the constant appearing in (3.4.40)), there exists $v_2^k \in [H_0^1(\Omega_k^{e_i})]^3$ such that

$$\text{div}(v_2^k) = \widetilde{p}_j - \int_{\Omega_k^{e_i}} \widetilde{p}_j$$

and

$$\|v_2^k\|_{H_0^1(\Omega_k^{e_i})} \leq C \left\| \widetilde{p}_j - \int_{\Omega_k^{e_i}} \widetilde{p}_j \right\|_{L^2(\Omega_k^{e_i})}. \quad (3.4.40)$$

Testing (3.4.38) (resp. (3.4.39)) against v_1^k (resp. v_2^k) and noticing that

$$\int_{\Omega_k^{e_i}} \widetilde{p}_j = \frac{1}{2} \left(\widetilde{\lambda}_j^k + \widetilde{\lambda}_j^{k+e_i} \right)$$

yields

$$\left\| \widetilde{p}_j - \widetilde{\lambda}_j^k \right\|_{L^2(Q_k \setminus \overline{\mathcal{O}_k})} \leq C \left[\|\nabla \widetilde{w}_j\|_{L^2(Q_k)} + \sqrt{\alpha_k} \right] \quad (3.4.41)$$

and

$$\left\| \widetilde{p}_j - \frac{1}{2} \left(\widetilde{\lambda}_j^k + \widetilde{\lambda}_j^{k+e_i} \right) \right\|_{L^2(\Omega_k^{e_i})} \leq C \left[\|\nabla \widetilde{w}_j\|_{L^2(Q_k \cup Q_{k+e_i})} + \sqrt{\alpha_k} + \sqrt{\alpha_{k+e_i}} \right].$$

In particular,

$$\left\| \widetilde{p}_j - \frac{1}{2} \left(\widetilde{\lambda}_j^k + \widetilde{\lambda}_j^{k+e_i} \right) \right\|_{L^2(Q_k \setminus \overline{\mathcal{O}_k})} \leq C \left[\|\nabla \widetilde{w}_j\|_{L^2(Q_k \cup Q_{k+e_i})} + \sqrt{\alpha_k} + \sqrt{\alpha_{k+e_i}} \right] \quad (3.4.42)$$

Collecting (3.4.41) and (3.4.42) yields

$$\begin{aligned} \frac{|Q_k \setminus \overline{\mathcal{O}_k}|^{\frac{1}{2}}}{2} \left| \widetilde{\lambda}_j^k - \widetilde{\lambda}_j^{k+e_i} \right| &= \left\| \frac{1}{2} \left(\widetilde{\lambda}_j^k - \widetilde{\lambda}_j^{k+e_i} \right) \right\|_{L^2(Q_k \setminus \overline{\mathcal{O}_k})} \\ &\leq \left\| \widetilde{p}_j - \frac{1}{2} \left(\widetilde{\lambda}_j^k + \widetilde{\lambda}_j^{k+e_i} \right) \right\|_{L^2(Q_k \setminus \overline{\mathcal{O}_k})} + \left\| \widetilde{p}_j - \widetilde{\lambda}_j^k \right\|_{L^2(Q_k \setminus \overline{\mathcal{O}_k})} \\ &\leq C \left[\|\nabla \widetilde{w}_j\|_{L^2(Q_k \cup Q_{k+e_i})} + \sqrt{\alpha_k} + \sqrt{\alpha_{k+e_i}} \right]. \end{aligned} \quad (3.4.43)$$

Hence we have proved (3.4.37). Defining

$$\beta_j^k := \|\nabla \widetilde{w}_j\|_{L^2\left(\bigcup_{|k-\ell| \leq 1} Q_\ell\right)} + \sum_{|k-\ell| \leq 1} \sqrt{\alpha_\ell}$$

gives the existence of a constant C independent of k and i such that for all $|k| \geq K$ and $i = 1, 2, 3$,

$$\left| \widetilde{\lambda}_j^k - \widetilde{\lambda}_j^{k \pm e_i} \right| \leq C \beta_j^k. \quad (3.4.44)$$

If $|k| \leq K + 1$, we define

$$\beta_j^k = \max_{i=1,2,3} \left| \widetilde{\lambda}_j^k - \widetilde{\lambda}_j^{k \pm e_i} \right|,$$

so that (3.4.44) is valid for all $k \in \mathbb{Z}^3$. Since $\nabla \widetilde{w}_j \in [L^2(\mathbb{R}^3)]^{3 \times 3}$ and $(\alpha_k)_{k \in \mathbb{Z}^3} \in \ell^1(\mathbb{Z}^3)$, we have that $(\beta_j^k)_{k \in \mathbb{Z}^3} \in \ell^2(\mathbb{Z}^3)$. We set for all $k \in \mathbb{Z}^3$,

$$\gamma_j^k = \max_{|k-\ell| \leq 1} \beta_j^\ell,$$

then $(\gamma_j^k)_{k \in \mathbb{Z}^3} \in \ell^2(\mathbb{Z}^3)$ and if $\ell := (\ell_1, \ell_2, \ell_3)$ satisfies $|k - \ell|_\infty \leq 1$, then

$$\left| \widetilde{\lambda}_j^\ell - \widetilde{\lambda}_j^k \right| \leq \left| \widetilde{\lambda}_j^\ell - \widetilde{\lambda}_j^{(\ell_1, \ell_2, k_3)} \right| + \left| \widetilde{\lambda}_j^{(\ell_1, \ell_2, k_3)} - \widetilde{\lambda}_j^{(\ell_1, k_2, k_3)} \right| + \left| \widetilde{\lambda}_j^{(\ell_1, k_2, k_3)} - \widetilde{\lambda}_j^k \right| \leq 3\gamma_j^k.$$

Applying Lemma 3.4.9 below, we get that $(\widetilde{\lambda}_j^k)_{k \in \mathbb{Z}^3}$ is bounded. This concludes the proof of the Theorem. \square

In the following Lemma 3.4.9 and its proof, we denote by $|\cdot|$ the $|\cdot|_\infty$ norm on \mathbb{Z}^3 .

Lemma 3.4.9. *Let $(v_k)_{k \in \mathbb{Z}^3}$ be a real valued sequence such that there exists $(\gamma_k)_{k \in \mathbb{Z}^3} \in \ell^2(\mathbb{Z}^3)$ such that for all $\ell \in \mathbb{Z}^3$ such that $|k - j| \leq 1$,*

$$|v_j - v_k| \leq \gamma_k. \quad (3.4.45)$$

Then $(v_k)_{k \in \mathbb{Z}^3}$ is bounded.

Proof of Lemma 3.4.9. Let $\rho \in \mathcal{D}(B(0, \frac{1}{4}))$ such that $\int_Q \rho = 1$ and $\phi_0 := \rho * 1_Q$. We have that

$$\text{supp}(\phi_0) \subset Q + B(0, \frac{1}{4}) \subset \left] -\frac{3}{4}, \frac{3}{4} \right[\subset \bigcup_{|\ell| \leq 1} Q_\ell.$$

We define

$$\phi := \sum_{j \in \mathbb{Z}^3} v_j \phi_0(\cdot - j) = \sum_{j \in \mathbb{Z}^3} v_j \rho * 1_{Q_j}.$$

We obviously have $\phi \in L^2_{\text{loc}}(\mathbb{R}^3)$. Besides, $\phi = v_k$ in $\prod_{i=1}^3]k_i - \frac{1}{4}, k_i + \frac{1}{4}[$. We have

$$|\nabla\phi| = \sum_{k \in \mathbb{Z}^3} \left| \sum_{|k-j| \leq 1} v_j \nabla\phi_0(\cdot - j) \right| 1_{Q_k}.$$

On the other hand, for $x \in Q_k$,

$$\sum_{|k-j| \leq 1} v_j \nabla\phi_0(x - j) = v_k \sum_{|k-j| \leq 1} \nabla\phi_0(x - j) + \sum_{|k-j| \leq 1} (v_j - v_k) \nabla\phi_0(x - j).$$

We have

$$\sum_{|k-j| \leq 1} \nabla\phi_0(x - j) = \sum_{|k-j| \leq 1} \nabla\rho * 1_{Q_j}(x) = \int_{B(0,1/4)} \nabla\rho = 0.$$

Thus, for $x \in Q_k$,

$$|\nabla\phi(x)|^2 \leq 9 \sum_{|k-j| \leq 1} |v_j - v_k|^2 |\nabla\phi_0(x - j)|^2 \leq C\gamma_k^2$$

This proves that $\nabla\phi \in L^2(\mathbb{R}^3)$. Because of the inclusion $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)/\mathbb{R}$, there exists $M \in \mathbb{R}$ such that $\|\phi - M\|_{L^6(\mathbb{R}^3)} \leq C$. Thus,

$$\|\phi - M\|_{L^6(\bigcup_{k \in \mathbb{Z}^3} (\prod_{i=1}^3]k_i - \frac{1}{4}, k_i + \frac{1}{4}[))} \leq C$$

and finally

$$\sum_{k \in \mathbb{Z}^3} \frac{1}{8} |v_k - M|^6 \leq C^6.$$

This proves the Lemma. \square

Remark 3.4.10 (General configuration of holes). *Theorem 3.4.8 is proved for compactly supported perturbations. For general configurations, the method of proof still applies. We can for example build a partition $(R_k)_{k \in \mathbb{Z}^3}$ of \mathbb{R}^3 into rectangles such that for all $k \in \mathbb{Z}^3$, $Q'_k \subset\subset R_k \subset\subset Q''_k$ and for k large enough, $|R_k \setminus \overline{\mathcal{O}_k}| = |Q_k \setminus \overline{\mathcal{O}_k^{\text{per}}}|$ and $\overline{R_{k+e_i}} \cup \overline{R_k}$ is connected for all $i \in \{1, 2, 3\}$. One way to build this partition is to translate the upper face of each cube column by column starting from each cube $Q_{(k_1, k_2, 0)}$, $(k_1, k_2) \in \mathbb{Z}$.*

Deuxième partie

Homogénéisation d'équations non linéaires avec défauts locaux

CHAPTER 4

HOMOGENIZATION OF SEMILINEAR EQUATIONS IN A PERIODIC SETTING WITH A LOCAL DEFECT

In this chapter, we consider the homogenization of the semi-linear PDE

$$-\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) + u_\varepsilon |u_\varepsilon|^{p-2} = f, \quad 2 \leq p < +\infty \quad (4.0.1)$$

where $\varepsilon \ll 1$ and the coefficient A_ε is of the form $A_\varepsilon = A(\cdot/\varepsilon)$ where A is periodic (possibly perturbed by a local defect). We prove that, when ε tends to zero, u_ε converges to a homogenized limit u^* that is solution to an equation of the same type as (4.0.1). We also derive the coefficients of the homogenized equation and we compare it to the well-known corresponding linear problem:

$$\begin{cases} \operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega \\ u_\varepsilon \in H_0^1(\Omega). \end{cases} \quad (4.0.2)$$

In particular, we show that the correctors used for the homogenization of the semi-linear PDE (4.0.1) are the same as the ones used in the homogenization of (4.0.2). We then discuss the extension of the results beyond the periodic setting and, in particular, to the setting of [18, 19, 20].

4.1 Introduction

4.1.1 Setting of the Problem

In the sequel, we consider $d \geq 1$ and we denote by $Q := (-\frac{1}{2}, \frac{1}{2})^d$ the unit cube of \mathbb{R}^d . We fix $2 \leq p < +\infty$ and we set $p' = p/(p-1)$. Let Ω be a Lipschitz bounded domain of \mathbb{R}^d . We set $V := H_0^1 \cap L^p(\Omega)$ endowed with the norm

$$\|v\|_V := \|v\|_{L^p(\Omega)} + \|v\|_{H_0^1(\Omega)},$$

where

$$\|u\|_{H_0^1(\Omega)} := \|\nabla u\|_{L^2(\Omega)}.$$

It is clear that $(V, \|\cdot\|_V)$ is a Banach reflexive space. The topological dual space of V is denoted by V' and the pairing between V' and V will be written $\langle \cdot, \cdot \rangle$. We have $V' = H^{-1}(\Omega) + L^{p'}(\Omega)$. We fix $f \in V'$ and we consider the following semi-linear equation

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) + u_\varepsilon |u_\varepsilon|^{p-2} = f \\ u_\varepsilon \in V \end{cases} \quad (4.1.1)$$

where the matrix-valued coefficient A_ε is of the form $A_\varepsilon = A(\cdot/\varepsilon)$ and the matrix-valued function $A : Q \rightarrow \mathcal{M}_d(\mathbb{R})$ is Q -periodic, elliptic and bounded, that is,

$$\exists \alpha > 0, \quad \forall x \in Q, \quad \forall \xi \in \mathbb{R}^d, \quad \text{s.t.} \quad \alpha^{-1}|\xi|^2 \leq \xi^T A(x)\xi \leq \alpha|\xi|^2. \quad (4.1.2)$$

We discuss in Section 4.3 the case of a periodic coefficient that is locally perturbed by a defect (in the sense presented in Subsection 1.2.2.1). We also assume for the sake of simplicity that A is symmetric (see Remark 4.3.1 for the non-symmetric case) and that $A \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$, $0 < \alpha \leq 1$. The weak form of (4.1.1) is

$$\forall v \in V, \quad \int_{\Omega} A_\varepsilon \nabla u_\varepsilon \cdot \nabla v + \int_{\Omega} u_\varepsilon |u_\varepsilon|^{p-2} v = \langle f, v \rangle.$$

Problem (4.1.1) corresponds to the Euler-Lagrange equation of the minimization problem:

$$\min_{v \in V} \left\{ \frac{1}{2} \int_{\Omega} \nabla v \cdot A_\varepsilon \nabla v + \frac{1}{p} \int_{\Omega} |v|^p - \langle f, v \rangle \right\}. \quad (4.1.3)$$

We show that Problem (4.1.3), and thus Problem (4.1.1), admits a solution when ε is fixed. We define the functional

$$J_\varepsilon(v) := \frac{1}{2} \int_{\Omega} \nabla v \cdot A_\varepsilon \nabla v + \frac{1}{p} \int_{\Omega} |v|^p - \langle f, v \rangle, \quad \forall v \in V.$$

Using (4.2.7) and the Young inequality, we have that

$$\begin{aligned} J_\varepsilon(v) &\geq \alpha \|v\|_{H_0^1(\Omega)}^2 + \frac{1}{p} \|v\|_{L^p(\Omega)}^p - \|f\|_{V'} \left[\|v\|_{H_0^1(\Omega)} + \|v\|_{L^p(\Omega)} \right] \\ &\geq \frac{\alpha}{2} \|v\|_{H_0^1(\Omega)}^2 + \frac{1}{2p} \|v\|_{L^p(\Omega)}^p - \left[\frac{1}{2\alpha} \|f\|_{V'}^2 + \frac{2^{\frac{1}{p-1}}}{p'} \|f\|_{V'}^{p'} \right]. \end{aligned}$$

Thus, J_ε is coercive on V . Besides, J_ε is clearly strictly convex and continuous on V (which is a reflexive Banach space). This proves the existence and uniqueness of a solution $u_\varepsilon \in V$ to (4.1.3). Note that the uniqueness can also be proved by a monotonicity argument.

We immediately notice that, by integration by parts and thanks to (4.1.2), the following *a priori* estimate is satisfied

$$\|u_\varepsilon\|_V \leq C, \quad (4.1.4)$$

where the constant C is independent of ε . This implies that there exists $u^* \in V$ such that, after extraction of a subsequence:

- i $\nabla u_\varepsilon \rightharpoonup \nabla u^*$ in L^2 -weakly;
- ii $u_\varepsilon \rightharpoonup u^*$ in L^p -weakly;
- iii $u_\varepsilon \rightarrow u^*$ in L^r -strongly for all $r \in [1, 2^*)$ where $\begin{cases} 2^* = \frac{2d}{d-2} & \text{if } d > 2 \\ 2^* = +\infty & \text{if } d = 2. \end{cases}$

This point is a consequence of Rellich's Theorem.

The rest of this work is devoted to the identification of u^* (see Theorem 4.1.1).

4.1.2 Formal arguments and result

The formal asymptotic expansion of u_ε under the form

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2\left(x, \frac{x}{\varepsilon}\right) + \cdots, \quad x \in \Omega \quad (4.1.5)$$

where the functions $y \mapsto u_i(x, y)$ are periodic is done in [16, Section 16.1, pp. 200-202] in a more general setting. The cascade of equations implies in particular that $u_0(x, y) = u_0(x)$ and that

$$\begin{cases} -\operatorname{div}(A^* \nabla u_0) + u_0 |u_0|^{p-2} = f \\ u_0 \in H_0^1 \cap L^p(\Omega), \end{cases} \quad (4.1.6)$$

where $A^* = (A_1^*, \dots, A_n^*)$ is defined, as in the linear case, by

$$A_i^* = \int_Q A(y)(e_i + \nabla w_{e_i}) dy,$$

and $w_q, q \in \mathbb{R}^d$ are the standard linear correctors. We recall that they are defined as the solution to

$$\begin{cases} -\operatorname{div}(A(y)(q + \nabla w_q)) = 0 \\ w_q \in H^{1,\text{per}}(Q), \end{cases} \quad (4.1.7)$$

where

$$H^{1,\text{per}}(Q) := \{u \in H_{\text{loc}}^1(\mathbb{R}^d) \text{ s.t. } u \text{ is } Q\text{-periodic}\} / \mathbb{R}.$$

By standard elliptic regularity theory (see e.g. [59, Theorem 7.12]), we have that $w_q \in W^{1,\infty}(Q)$, since the coefficient A is Hölder continuous. The two scale expansion (4.1.5) is made rigorous for some particular cases in [16, Section 16.2 and 16.3]. Since these cases do not cover Equation (4.1.1) (in particular the whole range $p \in [2^*, +\infty)$), we prove in Theorem 4.1.1 that the two scale expansion (4.1.5) is indeed valid for the PDE (4.1.1).

Theorem 4.1.1. *Let $2 \leq p < +\infty$ and $f \in V'$. The sequence $(u_\varepsilon)_{\varepsilon>0}$ converges weakly in V (and strongly in L^r for all $r \in [1, 2^*)$) to the solution u_0 of (4.1.6). Besides, assume that $D^2 u_0 \in L^2(\Omega)$. Then we have the strong convergence*

$$\nabla R_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^2(\Omega), \quad (4.1.8)$$

where $R_\varepsilon := u_\varepsilon - u_0 - \varepsilon \sum_{i=1}^d w_i(\cdot/\varepsilon) \partial_i u_0$ and $w_i := w_{e_i}$.

Note that a sufficient condition to obtain $D^2 u_0 \in L^2(\Omega)$ is that $u_0 |u_0|^{p-2} \in L^2(\Omega)$ by elliptic regularity theory. It is the case if $u_0 \in L^{2(p-1)}(\Omega)$. The latter is true if $d = 2$. If $d \geq 3$, it is true if $2(p-1) \leq \frac{2d}{d-2}$ i.e. $p \leq 1 + \frac{d}{d-2}$. The proof of Theorem 4.1.1 relies on Murat and Tartar's compensated compactness method (the *Div-Curl* Lemma, see [83, Theorem 2, p. 478]) and follows the strategy of the linear case. The main difficulty is to pass to the limit in the non-linear term $u_\varepsilon |u_\varepsilon|^{p-2}$. This is done in the following Lemma:

Lemma 4.1.2. *Let $2 \leq p < +\infty$ and $f \in V'$. Then, after extraction of a subsequence, the sequence $u_\varepsilon |u_\varepsilon|^{p-2}$ converges L^p -strongly to $u^* |u^*|^{p-2}$ and u_ε converges L^p -strongly to u^* when $\varepsilon \rightarrow 0$.*

The proof of Lemma 4.1.2 is firstly achieved when $f \in L^\infty(\Omega)$ using maximum principle type arguments for semi-linear equations (see [26, Lemma 2, p. 3]). The general case $f \in V'$ is then deduced by density.

Remark 4.1.3. *We note that the proof of Lemma 4.1.2 is easy when $p < 2^*$. Indeed, we recall the standard inequality*

$$\forall x, y \in \mathbb{R}, \quad |x|x^{p-2} - y|y|^{p-2}| \leq C (|x|^{p-2} + |y|^{p-2}) |x - y|, \quad (4.1.9)$$

which is a consequence of the fact that $x \mapsto x|x|^{p-2} \in W_{\text{loc}}^{1,\infty}(\mathbb{R})$ and that $\frac{d}{dx}(x|x|^{p-2}) = (p-1)|x|^{p-2}$. Thanks to (4.1.9), we have that

$$|u_\varepsilon |u_\varepsilon|^{p-2} - u^* |u^*|^{p-2}|^{p'} \leq C \left(|u_\varepsilon|^{p'(p-2)} + |u^*|^{p'(p-2)} \right) |u_\varepsilon - u^*|^{p'}.$$

Using the Hölder inequality with the exponent p/p' and the identities $p'(p-2) = p-p'$ and $\frac{p/p'}{p/p' - 1} = \frac{p}{p-p'}$, we deduce that

$$\| |u_\varepsilon|^{p-2} - |u^*|^{p-2} \|_{L^{p'}(\Omega)}^{p'} \leq C \left(\|u_\varepsilon\|_{L^p(\Omega)}^{p-p'} + \|u^*\|_{L^p(\Omega)}^{p-p'} \right) \|u_\varepsilon - u^*\|_{L^p(\Omega)}^{p'}. \quad (4.1.10)$$

Using the a priori estimate (4.1.4) and the L^p -strong convergence $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^*$ after extraction of a subsequence, we conclude that Lemma 4.1.2 is satisfied when $p < 2^*$. Note that we have shown, thanks to (4.1.10), that proving $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^*$ in L^p -strong is sufficient to deduce Lemma 4.1.2.

4.2 Proof of the main result

4.2.1 Proof of Lemma 4.1.2

We first state the following Lemma.

Lemma 4.2.1. *Let $f \in L^\infty(\Omega)$ and u_ε be the solution to (4.0.1). After extraction of a subsequence, we have that $u_\varepsilon |u_\varepsilon|^{p-2}$ converges in the $L^{p'}(\Omega)$ strong topology to $u^* |u^*|^{p-2}$ when $\varepsilon \rightarrow 0$ and that u_ε converges in the $L^p(\Omega)$ strong topology to u^* when $\varepsilon \rightarrow 0$.*

Proof of Lemma 4.2.1. We note that it is enough to show that

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C \quad (4.2.1)$$

where C is a constant independent of ε . Indeed, we recall that the L^2 -strong convergence $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^*$ holds true after the extraction of a subsequence. We have, after another extraction of a subsequence, convergence almost everywhere to u^* . Thus,

$$u_\varepsilon |u_\varepsilon|^{p-2} \xrightarrow{\varepsilon \rightarrow 0} u^* |u^*|^{p-2} \quad \text{a.e. in } \Omega.$$

By the dominated convergence Theorem, we deduce the statement of Lemma 1. It is thus sufficient to prove (4.2.1). The proof requires two steps.

Step 1. We prove a Kato-type inequality which states as follows (see also [26, Lemma A.1, p. 13]): let $g \in L^{p'}(\Omega)$ and $u \in H_0^1(\Omega)$ such that

$$-\operatorname{div}(A_\varepsilon \nabla u) \leq g$$

in the weak sense *i.e.* for all $\phi \in V$ such that $\phi \geq 0$, we have

$$\int_\Omega A_\varepsilon \nabla u \cdot \nabla \phi \leq \int_\Omega g \phi. \quad (4.2.2)$$

Then, for all $\phi \in \mathcal{D}(\Omega)$ such that $\phi \geq 0$, we have the inequality

$$\int_\Omega A_\varepsilon \nabla u^+ \cdot \nabla \phi \leq \int_\Omega g \operatorname{sgn}^+(u) \cdot \phi, \quad (4.2.3)$$

where $\operatorname{sgn}^+(x) := \max(0, \frac{x}{|x|})$ for $x \neq 0$, $\operatorname{sgn}^+(0) = \frac{1}{2}$ and $x^+ := \frac{x+|x|}{2} = \max(x, 0)$ for $x \in \mathbb{R}$.

We prove (4.2.3). Let $\Psi \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ be a convex, nondecreasing and positive function such that Ψ' and Ψ'' are bounded. We first prove that

$$\forall \phi \in \mathcal{D}(\Omega), \quad \phi \geq 0, \quad \int_\Omega A_\varepsilon \nabla \Psi(u) \cdot \nabla \phi \leq \int_\Omega g \Psi'(u) \phi. \quad (4.2.4)$$

We will next approximate the function $u^+ := \max(0, u)$ by a sequence of functions $(\Psi_n(u))_{n \in \mathbb{N}}$ where $\Psi_n \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ is convex, nondecreasing and positive. For $\Psi \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ as above, it is easily proved that $\Psi(u) \in H^1(\Omega)$ and that

$$\nabla \Psi(u) = \Psi'(u) \nabla u.$$

Similarly, $\Psi'(u) \in H^1(\Omega)$ and

$$\nabla \Psi'(u) = \Psi''(u) \nabla u.$$

We fix $\phi \in \mathcal{D}(\Omega)$ such that $\phi \geq 0$. We have that

$$\int_{\Omega} A_{\varepsilon} \nabla \Psi(u) \cdot \nabla \phi = \int_{\Omega} A_{\varepsilon} \Psi'(u) \nabla u \cdot \nabla \phi = \int_{\Omega} A_{\varepsilon} \nabla u \cdot \Psi'(u) \nabla \phi.$$

On the other hand, $\Psi'(u)\phi \in H_0^1(\Omega)$ and $\nabla(\Psi'(u)\phi) = \phi \Psi''(u) \nabla u + \Psi'(u) \nabla \phi$. Thus,

$$\int_{\Omega} A_{\varepsilon} \nabla \Psi(u) \cdot \nabla \phi = - \int_{\Omega} A_{\varepsilon} \nabla u \cdot [\phi \Psi''(u) \nabla u] + \int_{\Omega} A_{\varepsilon} \nabla u \cdot \nabla(\Psi'(u)\phi).$$

The first term may be written in the form

$$\int_{\Omega} A_{\varepsilon} \nabla u \cdot [\phi \Psi''(u) \nabla u] = \int_{\Omega} \phi \Psi''(u) A_{\varepsilon} \nabla u \cdot \nabla u \geq \alpha \int_{\Omega} \phi \Psi''(u) |\nabla u|^2 \geq 0,$$

by the ellipticity condition (4.1.2) and since $\phi \geq 0$ and Ψ is convex. The second term satisfies

$$\int_{\Omega} A_{\varepsilon} \nabla u \cdot \nabla(\Psi'(u)\phi) \leq \int_{\Omega} g \Psi'(u) \phi,$$

since $\Psi'(u)\phi \in L^{\infty} \cap H_0^1(\Omega)$ and by (4.2.2). Thus, (4.2.4) is satisfied.

We define for $n \in \mathbb{N}^*$, $\Psi_n(x) := \frac{1}{2} \left[x + \sqrt{\frac{1}{n} + x^2} \right]$. Then it is easily checked that Ψ_n is positive, nondecreasing, convex, that Ψ_n' is bounded uniformly in n and that Ψ_n'' is bounded (but not uniformly in n). Besides, Ψ_n converges when $n \rightarrow +\infty$ uniformly to $x \mapsto \max(x, 0)$ and Ψ_n' converges pointwise to sgn^+ . Thus, $\Psi_n(u)$ converges when $n \rightarrow +\infty$ uniformly to u^+ and $\Psi_n'(u)$ converges to $\text{sgn}^+(u)$ in L^r for all $1 \leq r < +\infty$ by the dominated convergence Theorem. Since $\nabla u \in L^2(\Omega)$, we have that

$$\nabla \Psi_n(u) = \Psi_n'(u) \nabla u \xrightarrow{n \rightarrow +\infty} \text{sgn}^+(u) \nabla u = \nabla u^+ \quad \text{in } L^1(\Omega)$$

We fix $\phi \in \mathcal{D}(\Omega)$ such that $\phi \geq 0$. We have for all $n \in \mathbb{N}^*$, by (4.2.4)

$$\int_{\Omega} A_{\varepsilon} \nabla \Psi_n(u) \cdot \nabla \phi \leq \int_{\Omega} g \Psi_n'(u) \phi.$$

The above convergences allow to pass to the limit $n \rightarrow +\infty$ and find that

$$\int_{\Omega} A_{\varepsilon} \nabla u^+ \cdot \nabla \phi \leq \int_{\Omega} g \Psi'(u) \phi.$$

This proves the inequality (4.2.3).

Step 2. We set $U := \|f\|_{\infty}^{1/(p-1)}$ and we prove that $u_{\varepsilon} \leq U$ in Ω . We have that

$$-\text{div}(A_{\varepsilon} \nabla U) + U|U|^{p-2} = \|f\|_{\infty} \geq f.$$

We consequently deduce that, in the weak sense,

$$-\text{div} A_{\varepsilon} \nabla(u_{\varepsilon} - U) \leq U|U|^{p-2} - u_{\varepsilon}|u_{\varepsilon}|^{p-2} \quad \text{and} \quad U|U|^{p-2} - u_{\varepsilon}|u_{\varepsilon}|^{p-2} \in L^{p'}(\Omega)$$

We are in the context of **Step 1**. We may thus deduce that for all $\phi \in \mathcal{D}(\Omega)$ such that $\phi \geq 0$, the following inequality holds true:

$$\int_{\Omega} A_{\varepsilon} \nabla(u_{\varepsilon} - U)^+ \cdot \nabla \phi \leq \int_{\Omega} [U|U|^{p-2} - u_{\varepsilon}|u_{\varepsilon}|^{p-2}] \operatorname{sgn}^+(u_{\varepsilon} - U) \phi.$$

Since the function $x \mapsto x|x|^{p-2}$ is non-decreasing, we get that

$$\int_{\Omega} [U|U|^{p-2} - u_{\varepsilon}|u_{\varepsilon}|^{p-2}] \operatorname{sgn}^+(u_{\varepsilon} - U) \phi \leq 0.$$

Finally,

$$\forall \phi \in \mathcal{D}(\Omega), \quad \phi \geq 0, \quad \int_{\Omega} A_{\varepsilon} \nabla(u_{\varepsilon} - U)^+ \cdot \nabla \phi \leq 0. \quad (4.2.5)$$

By density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$, (4.2.5) is also true for all $\phi \in H_0^1(\Omega)$ such that $\phi \geq 0$ *a.e* in Ω . Thus, noticing that $(u_{\varepsilon} - U)^+ = 0$ on $\partial\Omega$ and taking $\phi = (u_{\varepsilon} - U)^+$ in (4.2.5) gives

$$\int_{\Omega} A_{\varepsilon} \nabla(u_{\varepsilon} - U)^+ \cdot \nabla(u_{\varepsilon} - U)^+ \leq 0.$$

Thus, by ellipticity and the boundary conditions, $(u_{\varepsilon} - U)^+ = 0$ and finally $u_{\varepsilon} \leq U$ in Ω . We have proved that u_{ε} is bounded uniformly in ε from above. Replacing f by $-f$ (and thus u_{ε} by $-u_{\varepsilon}$), we also show that u_{ε} is uniformly bounded from below. This concludes the proof of Lemma 4.2.1. \square

Proof of Lemma 4.1.2. In view of Remark 4.1.3, it is sufficient to show that u_{ε} converges L^p -strongly to u^* when $\varepsilon \rightarrow 0$, after extraction of a subsequence. We use Lemma 4.2.1 and we argue by density. We first prove that the (non-linear) application

$$\Psi : \begin{cases} V' \longrightarrow V \\ f \longmapsto u_{\varepsilon} \end{cases}$$

is continuous (and more precisely Hölder continuous). Note that the parameter ε is here fixed.

Let $g, h \in V'$ and $v_{\varepsilon}, w_{\varepsilon} \in V$ be the solutions to (4.1.1) associated to the source terms g, h respectively. We have that

$$-\operatorname{div} A_{\varepsilon} \nabla(v_{\varepsilon} - w_{\varepsilon}) + v_{\varepsilon}|v_{\varepsilon}|^{p-2} - w_{\varepsilon}|w_{\varepsilon}|^{p-2} = g - h. \quad (4.2.6)$$

We next multiply (4.2.6) by $v_{\varepsilon} - w_{\varepsilon}$ and we integrate by parts. The ellipticity condition (4.1.2) together with the following inequality (see [68, p. 298])

$$(x|x|^{p-2} - y|y|^{p-2})(x - y) \geq \beta|x - y|^p, \quad \beta > 0 \quad (4.2.7)$$

yield

$$\alpha \int_{\Omega} |\nabla(v_{\varepsilon} - w_{\varepsilon})|^2 + \beta \int_{\Omega} |v_{\varepsilon} - w_{\varepsilon}|^p \leq \langle g - h, v_{\varepsilon} - w_{\varepsilon} \rangle.$$

Thus,

$$\alpha \|\nabla(v_{\varepsilon} - w_{\varepsilon})\|_{L^2(\Omega)}^2 + \beta \|v_{\varepsilon} - w_{\varepsilon}\|_{L^p(\Omega)}^p \leq \|g - h\|_{V'} \|v_{\varepsilon} - w_{\varepsilon}\|_V.$$

We now apply the Young inequality:

$$\begin{aligned} \alpha \|\nabla(v_{\varepsilon} - w_{\varepsilon})\|_{L^2(\Omega)}^2 + \beta \|v_{\varepsilon} - w_{\varepsilon}\|_{L^p(\Omega)}^p &\leq \frac{1}{2\alpha} \|g - h\|_{V'}^2 \\ &+ \frac{\alpha}{2} \|\nabla(v_{\varepsilon} - w_{\varepsilon})\|_{L^2(\Omega)}^2 + \frac{\beta}{p} \|v_{\varepsilon} - w_{\varepsilon}\|_{L^p(\Omega)}^p + \frac{1}{p'\beta^{\frac{1}{p-1}}} \|g - h\|_{V'}^{p'}. \end{aligned}$$

Recalling that $p \geq 2$, this implies that there exists a constant $C = C(\alpha, \beta, p) > 0$ independent of ε such that

$$\|w_\varepsilon - v_\varepsilon\|_V \leq C \|g - h\|_{V'}^\gamma, \quad \gamma := \frac{1}{p-1}, \quad (4.2.8)$$

when $\|g - h\|_{V'} \leq 1$. This proves that Ψ is (locally) Hölder continuous.

We now use a density argument. We fix $k \in \mathbb{N}^*$, $f \in V'$ and $g^k \in L^\infty(\Omega)$ such that $\|f - g^k\|_{V'} \leq 1/k$. We denote by $(u_\varepsilon)_{\varepsilon>0} \subset V$ (resp. $(v_\varepsilon^k)_{\varepsilon>0} \subset V$) the sequence of solutions to (4.1.1) with source term f (resp. g^k). We know that, up to a subsequence, we have that u_ε and v_ε^k converge L^p -weakly to some limits u^* and $(v^*)^k$ as $\varepsilon \rightarrow 0$. By (4.2.8), we have that for all $\varepsilon > 0$,

$$\|u_\varepsilon - v_\varepsilon^k\|_{L^p(\Omega)} \leq Ck^{-\gamma} \quad (4.2.9)$$

and, since $u_\varepsilon - v_\varepsilon^k$ converges L^p -weakly to $u^* - (v^*)^k$, by weak semi-continuity of the norm, that

$$\|u^* - (v^*)^k\|_{L^p(\Omega)} \leq Ck^{-\gamma}. \quad (4.2.10)$$

Next, Lemma 4.1.2 applied to $g^k \in L^\infty(\Omega)$ and to the sequence $(v_\varepsilon^k)_{\varepsilon>0}$ gives, up to extracting a subsequence, that

$$\lim_{\varepsilon \rightarrow 0} \|v_\varepsilon^k - (v^*)^k\|_{L^p(\Omega)} = 0. \quad (4.2.11)$$

Thus, by the triangle inequality and gathering together (4.2.9), (4.2.10) and (4.2.11), we get that, along a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ (depending on k),

$$\limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon - u^*\|_{L^p(\Omega)} \leq 2Ck^{-\gamma}. \quad (4.2.12)$$

By a diagonal extraction procedure, we show that there exists a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that for all $k \in \mathbb{N}^*$, (4.2.12) is satisfied. In particular, u_ε converges along this subsequence to u^* strongly in $L^p(\Omega)$. \square

4.2.2 Proof of Theorem 4.1.1

It is sufficient to prove that for all converging subsequence $(u_{\varepsilon_k})_{k \in \mathbb{N}}$ of $(u_\varepsilon)_{\varepsilon>0}$, we have that

$$u_{\varepsilon_k} \xrightarrow[k \rightarrow +\infty]{} u_0 \quad \text{in } V, \quad \text{where } u_0 \text{ solves (4.1.6)}. \quad (4.2.13)$$

The whole sequence $(u_\varepsilon)_{\varepsilon>0}$ will then converge weakly to u_0 in V by uniqueness of the solution to the homogenized Problem (4.1.6).

Proof of (4.2.13). We fix an extraction of $(u_\varepsilon)_{\varepsilon>0}$ (that we still denote u_ε) and we write u^* its weak limit in V . We define, for $q \in \mathbb{R}^d$, $w_{q,\varepsilon} := w_q(\cdot/\varepsilon)$ and we consider the equality (on Ω),

$$A_\varepsilon \nabla u_\varepsilon \cdot (q + \nabla w_{q,\varepsilon}) = \nabla u_\varepsilon \cdot A_\varepsilon (q + \nabla w_{q,\varepsilon}), \quad (4.2.14)$$

which is valid since A is symmetric. We pass to the limit on each side of the above equality. For the LHS of (4.2.14), we recall that, since $p \geq 2$, we have that $V' \subset W^{-1,p'}(\Omega)$. This proves that $f \in W^{-1,p'}(\Omega)$ and, together with Lemma 4.1.2, that $f - u_\varepsilon |u_\varepsilon|^{p-2}$ is $W^{-1,p'}(\Omega)$ -strongly convergent after extraction of a subsequence. Thus, $\operatorname{div}[A_\varepsilon \nabla u_\varepsilon]$ is $W^{-1,p'}(\Omega)$ -strongly convergent. Besides, since A_ε is a.e. bounded, $p' \leq 2$ and $\|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq C$ (see (4.1.4)), we have that

$$\|A_\varepsilon \nabla u_\varepsilon\|_{L^{p'}(\Omega)} \leq C \|\nabla u_\varepsilon\|_{L^{p'}(\Omega)} \leq C \|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq C.$$

Thus, there exists by compactness $r^* \in L^{p'}(\Omega)^d$ such that $A_\varepsilon \nabla u_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} r^*$ in $L^{p'}(\Omega)^d$ after another extraction of a subsequence. Besides, because w_q is periodic, we have that $\int_Q \nabla w_q = 0$ and thus, because $\nabla w_q \in L^p(Q)$, that

$$q + \nabla w_q \left(\frac{\cdot}{\varepsilon} \right) \xrightarrow[\varepsilon \rightarrow 0]{} q \quad \text{in } L^p(\Omega).$$

Since $q + \nabla w_q \left(\frac{\cdot}{\varepsilon}\right) = \nabla [q \cdot x + \varepsilon w_q \left(\frac{\cdot}{\varepsilon}\right)]$, we deduce that $\mathbf{curl} [q + \nabla w_q \left(\frac{\cdot}{\varepsilon}\right)] = 0$. The *Div-Curl* Lemma (see e.g. [90, Theorem 2.1] – this is just an integration by parts in the present case) implies that

$$A_\varepsilon \nabla u_\varepsilon \cdot (q + \nabla w_{q,\varepsilon}) \xrightarrow{\varepsilon \rightarrow 0} r^* \cdot q \quad (4.2.15)$$

in the distribution sense. For the RHS of (4.2.14), we once again apply the *Div-Curl* Lemma, in the $L^2 - L^2$ version:

$$\nabla u_\varepsilon \cdot A_\varepsilon [q + \nabla w_{q,\varepsilon}] \xrightarrow{\varepsilon \rightarrow 0} \nabla u^* \cdot A^* q = A^* \nabla u^* \cdot q \quad (4.2.16)$$

in the distribution sense. Collecting (4.2.15) and (4.2.16) gives the identity $A^* \nabla u^* = r^*$. This shows that

$$A_\varepsilon \nabla u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} A^* \nabla u^* \quad \text{in } L^{p'}(\Omega),$$

after extraction of a subsequence. Since $\|A_\varepsilon \nabla u_\varepsilon\|_{L^2(\Omega)} \leq C$, the convergence also holds L^2 -weakly (after another extraction). Applying Lemma 4.1.2, we have proved that there exists a subsequence $(u_{\varepsilon_\ell})_{\ell \in \mathbb{N}}$ of $(u_\varepsilon)_{\varepsilon > 0}$ such that:

$$\forall v \in V, \quad \int_\Omega A_{\varepsilon_\ell} \nabla u_{\varepsilon_\ell} \cdot \nabla v + \int_\Omega u_{\varepsilon_\ell} |u_{\varepsilon_\ell}|^{p-2} v \xrightarrow{\ell \rightarrow +\infty} \int_\Omega A^* \nabla u^* \cdot \nabla v + \int_\Omega u^* |u^*|^{p-2} v.$$

Thus,

$$\forall v \in V, \quad \int_\Omega A^* \nabla u^* \cdot \nabla v + \int_\Omega u^* |u^*|^{p-2} v = \langle f, v \rangle,$$

which is the weak formulation of (4.1.6). This concludes the proof of (4.2.13) since, necessarily, $u^* = u_0$. It remains to prove (4.1.8). We have that

$$\nabla R_\varepsilon = \mathcal{R}_\varepsilon - \varepsilon w_i(\cdot/\varepsilon) \nabla \partial_i u^*, \quad \mathcal{R}_\varepsilon := \nabla u_\varepsilon - \sum_{i=1}^d (e_i + \nabla w_i(\cdot/\varepsilon)) \partial_i u^*.$$

Because $D^2 u^* \in L^2(\Omega)$ and the correctors w_i are bounded, it is sufficient to show that \mathcal{R}_ε tends strongly to zero in $L^2(\Omega)$. By coercivity, we have that

$$\|\mathcal{R}_\varepsilon\|_{L^2(\Omega)}^2 \leq \alpha \int_\Omega A_\varepsilon \mathcal{R}_\varepsilon \cdot \mathcal{R}_\varepsilon. \quad (4.2.17)$$

An immediate computation shows that

$$\begin{aligned} \frac{1}{\alpha} \|\mathcal{R}_\varepsilon\|_{L^2(\Omega)}^2 &\leq \int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon - \sum_{i=1}^d \int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot (e_i + \nabla w_i(\cdot/\varepsilon)) \partial_i u^* \\ &\quad - \sum_{i=1}^d \int_\Omega A_\varepsilon (e_i + \nabla w_i(\cdot/\varepsilon)) \cdot \nabla u_\varepsilon \partial_i u^* + \sum_{i,j=1}^d \int_\Omega A_\varepsilon (e_i + \nabla w_i(\cdot/\varepsilon)) \cdot (e_j + \nabla w_j(\cdot/\varepsilon)) \partial_i u^* \partial_j u^*. \end{aligned} \quad (4.2.18)$$

We treat each term of the RHS of (4.2.18) separately.

First term. Using (4.1.1) and testing against u_ε gives

$$\int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon = - \int_\Omega |u_\varepsilon|^p + \langle f, u_\varepsilon \rangle \xrightarrow{\varepsilon \rightarrow 0} - \int_\Omega |u^*|^p + \langle f, u^* \rangle = \int_\Omega A^* \nabla u^* \cdot \nabla u^* \quad (4.2.19)$$

where we used Lemma 4.1.2 and the convergence of $(u_\varepsilon)_{\varepsilon > 0}$.

Second term. We apply the *Div-Curl* Lemma. We first notice that $\operatorname{div} A_\varepsilon \nabla u_\varepsilon = u_\varepsilon |u_\varepsilon|^{p-2} - f$ converges $W^{-1,p'}(\Omega)$ strongly (we recall that $V' \subset W^{-1,p'}(\Omega)$). Besides, $A_\varepsilon \nabla u_\varepsilon$ converges $L^{p'}$ -weakly to $A^* \nabla u^*$ and $e_i + \nabla w_i(\cdot/\varepsilon)$ converges L^p -weakly to e_i . Thus

$$A_\varepsilon \nabla u_\varepsilon (e_i + \nabla w_i(\cdot/\varepsilon)) \xrightarrow{\varepsilon \rightarrow 0} A^* \nabla u^* \cdot e_i \quad \text{in } \mathcal{D}'(\Omega). \quad (4.2.20)$$

Since $\nabla w_i \in W^{1,\infty}(Q)$, we have the bound $\|A_\varepsilon \nabla u_\varepsilon(e_i + \nabla w_i(\cdot/\varepsilon))\|_{L^2(\Omega)} \leq C$. Consequently, we can upgrade (4.2.21), up to extracting a subsequence, into

$$A_\varepsilon \nabla u_\varepsilon(e_i + \nabla w_i(\cdot/\varepsilon)) \xrightarrow{\varepsilon \rightarrow 0} A^* \nabla u^* \cdot e_i \quad \text{in } L^2(\Omega). \quad (4.2.21)$$

Since $\partial_i u^* \in L^2(\Omega)$, we conclude that the second term tends to $\int_\Omega A^* \nabla u^* \cdot \nabla u^*$.

Third term. As for the previous term, we prove that, up to extracting a subsequence, it converges to $\int_\Omega A^* e_i \cdot \nabla u^* \partial_i u^* = \int_\Omega A^* \nabla u^* \cdot \nabla u^*$.

Fourth term. As a periodic function, we have that $A_\varepsilon(e_i + \nabla w_i(\cdot/\varepsilon)) \cdot (e_j + \nabla w_j(\cdot/\varepsilon))$ converges in the weak- \star in the L^∞ topology to $A^* e_i \cdot e_j$. Since $\partial_i u^* \partial_j u^* \in L^1(\Omega)$, we conclude that

$$\sum_{i,j=1}^d \int_\Omega A_\varepsilon(e_i + \nabla w_i(\cdot/\varepsilon)) \cdot (e_j + \nabla w_j(\cdot/\varepsilon)) \partial_i u^* \partial_j u^* \xrightarrow{\varepsilon \rightarrow 0} \sum_{i,j=1}^d \int_\Omega A^* e_i \cdot e_j \partial_i u^* \partial_j u^* = \int_\Omega A^* \nabla u^* \cdot \nabla u^*.$$

Collecting the above convergences, we conclude that, up to extracting a subsequence,

$$\lim_{\varepsilon \rightarrow 0} \|\mathcal{R}_\varepsilon\|_{L^2(\Omega)}^2 = 0.$$

Since the above arguments may be repeated for any subsequence of $(\mathcal{R}_\varepsilon)$, the whole sequence converges. This concludes the proof of the Theorem. \square

4.3 Remarks and extension to a periodic setting with defects

In this section, we discuss the homogenization of (4.1.1) in the periodic setting with defect introduced in [17, 18, 19, 20]. We first recall this setting: let $A : \mathbb{R}^d \rightarrow \mathcal{M}_d(\mathbb{R})$ be of the form $A = A^{\text{per}} + \tilde{A}$, where A^{per} is a periodic matrix valued tensor and $\tilde{A} \in L^r(\mathbb{R}^d)^{d \times d}$, $1 < r < +\infty$. We assume that A^{per} and A are elliptic and bounded in the sense (4.1.2) and that

$$A \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d), \quad A^{\text{per}} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d), \quad 0 < \alpha \leq 1.$$

We consider the following semi-linear PDE

$$-\text{div} \left[A \left(\frac{\cdot}{\varepsilon} \right) \nabla u_\varepsilon \right] + u_\varepsilon |u_\varepsilon|^{p-2} = f, \quad f \in V'. \quad (4.3.1)$$

In (4.3.1), the matrix \tilde{A} represents a local defect that is inserted, at the microscale, in a periodic material. The exponent r will be the integrability of the defect. Performing a two-scale expansion similar to the periodic case, we find that the corrector equation associated to (4.3.1) is

$$-\text{div} A(y)(q + \nabla w_q) = 0 \quad \text{in } \mathbb{R}^d, \quad q \in \mathbb{R}^d. \quad (4.3.2)$$

This PDE, similar to the corrector equation of the associated linear problem (4.0.2), has been studied in [20, 18]. It is proved that (4.3.2) admits a solution of the form $w_q^{\text{per}} + \tilde{w}_q$ where w_q^{per} is the periodic corrector (*i.e.* solution to (4.3.2) for $A = A^{\text{per}}$) and $\nabla \tilde{w}_q \in L^r \cap L^\infty(\mathbb{R}^d)^d$ where r is the integrability of the defect. In particular, we have the fundamental sublinearity property at infinity (see [17, Proposition 5.1]):

$$\forall \rho > 1, \quad \sup_{|x-y| \leq \rho} |w_q(x) - w_q(y)| \leq c \rho^{1-\nu_r}, \quad \nu_r = \min \left(1, \frac{d}{r} \right),$$

where c is a constant independent of ρ . Using the non-periodic correctors $w_q, q \in \mathbb{R}^d$, the proof of Theorem 4.1.1 extends *verbatim* to the periodic setting with defect introduced above. Let us note that, because of the integrability of the defect \tilde{A} and of the non-periodic corrector $\nabla \tilde{w}_q$, we get that

$$A \left(\frac{\cdot}{\varepsilon} \right) \left[q + \nabla w_q \left(\frac{\cdot}{\varepsilon} \right) \right] \xrightarrow{\varepsilon \rightarrow 0} \int_Q A^{\text{per}}(y)(q + \nabla w_q) dy \quad \text{in } L^2(\Omega)$$

and

$$q + \nabla w_q \left(\frac{\cdot}{\varepsilon} \right) \xrightarrow{\varepsilon \rightarrow 0} q \quad \text{in } L^p(\Omega).$$

This proves that $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^*$ in V with u^* being the solution to (4.1.6). In particular, the macroscopic behaviour of u_ε is the same as for the periodic case. Although we did not pursue in this direction, we believe that it is possible to obtain convergence rates in the homogenization of (4.1.1), as it is done in [17] for the linear case.

We close this section with two Remarks on the case of non-symmetric coefficients on the one hand, and the case $1 < p < 2$ on the other hand, both for the periodic and periodic with defect case:

Remark 4.3.1 (The non-symmetric case). *If A is non-symmetric, we use as in the linear case the adjoint cell corrector problem (see e.g. [35] or [16]). We briefly sketch the proof: we consider, instead of (4.1.7), the PDE*

$$\begin{cases} -\operatorname{div}(A^T(y)(q + \nabla w_q^T)) = 0 \\ w_q^T \in H^{1,\text{per}}(Q), \end{cases} \quad (4.3.3)$$

where A^T denotes the matrix transpose of A . We then write (4.2.14) under the form

$$A_\varepsilon \nabla u_\varepsilon \cdot (q + \nabla w_{q,\varepsilon}^T) = \nabla u_\varepsilon \cdot A_\varepsilon^T (q + \nabla w_{q,\varepsilon}^T).$$

As above, the Div-Curl Lemma allow to prove that, up to extracting a subsequence, $A_\varepsilon \nabla u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} (A^*)^T \nabla u^*$ in the $L^2(\Omega)$ topology, where

$$A^* := \left(\int_Q A^T(y)(e_1 + \nabla w_{e_1}^T(y)), \dots, \int_Q A^T(y)(e_d + \nabla w_{e_d}^T(y)) \right)_{1 \leq i \leq d}.$$

Since $A^* = (A^*)^T$, we conclude that $A_\varepsilon \nabla u_\varepsilon$ converges L^2 -weakly to $A^* \nabla u^*$. The rest of the proof of Theorem 4.1.1 is identical to the symmetric case.

Remark 4.3.2 (Case $1 < p < 2$). *This chapter only treats the case $p \geq 2$. The case $1 < p < 2$ can be easily addressed. In this case, $V = H_0^1(\Omega)$ and u_ε converges strongly in $L^p(\Omega)$ to u^* as $\varepsilon \rightarrow 0$ by Rellich Theorem. We can thus pass directly to the limit in the nonlinear term.*

CHAPTER 5

HOMOGENIZATION OF THE p -LAPLACE EQUATION IN A PERIODIC SETTING WITH A LOCAL DEFECT

This chapter is based on [106].

In this chapter, we consider the homogenization of the p -Laplace equation with a periodic coefficient that is perturbed by a local defect. This setting has been introduced in [19, 20] in the linear setting $p = 2$. We construct the correctors and we derive the limit of the oscillating function in the case $p > 2$ under the assumption that the periodic correctors are non degenerate. This chapter is based on [106].

5.1 Introduction

This chapter is concerned with the homogenization of non-linear degenerate elliptic equations in a periodic setting with defects. More precisely, we are interested in p -Laplacian type equations that are defined, for some $p \geq 2$, as

$$\begin{cases} -\operatorname{div} a\left(\frac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon |\nabla u_\varepsilon|^{p-2} = f & \text{in } \Omega \\ u_\varepsilon \in W_0^{1,p}(\Omega) \end{cases} \quad (5.1.1)$$

for a fixed bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$ and $f \in L^{p'}(\Omega)$. For $p = 2$, we recover the standard linear conductivity equation. In (5.1.1), the scalar-valued coefficient a is assumed to be of the form

$$a = a^{\text{per}} + \tilde{a}, \quad (5.1.2)$$

where a^{per} is a periodic coefficient with standard coercivity and boundedness condition and \tilde{a} is a perturbation of a^{per} such that $\tilde{a} \in L^q(\mathbb{R}^d)$ for some $1 \leq q \leq \frac{p}{p-1}$. We assume that the coefficient a itself is coercive and bounded and we choose $\lambda > 0$ such that

$$\forall y \in \mathbb{R}^d, \quad \lambda^{-1} < a^{\text{per}}(y) < \lambda \quad \text{and} \quad \lambda^{-1} < a(y) < \lambda. \quad (5.1.3)$$

For fixed $\varepsilon > 0$, Problem (5.1.1) is well-posed and corresponds to the Euler-Lagrange equation of the minimization Problem

$$\min_{v \in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p} \int_\Omega a\left(\frac{\cdot}{\varepsilon}\right) |\nabla v|^p - \int_\Omega f v \right\}. \quad (5.1.4)$$

The behaviour of (5.1.1) when $\varepsilon \rightarrow 0$ has been studied in the absence of perturbation, *i.e.* when $a = a^{\text{per}}$. It corresponds to a particular case of the homogenization of the equation

$$-\operatorname{div} A\left(\frac{\cdot}{\varepsilon}, \nabla u_\varepsilon\right) = f \quad (5.1.5)$$

under general growth and continuity conditions for the operator $A(y, \xi)$ (in our case, we have that $A(y) = a^{\text{per}}(y)\xi|\xi|^{p-2}$). The homogenized limit of (5.1.5) is derived in [56, 57]. It is proved that u_ε converges in the $W^{1,p}$ -weak topology, when $\varepsilon \rightarrow 0$, to u^* which is defined by the homogenized equation

$$\begin{cases} -\operatorname{div}A^*(\nabla u^*) = f \\ u^* \in W_0^{1,p}(\Omega), \end{cases} \quad (5.1.6)$$

where, for $\xi \in \mathbb{R}^d$, the homogenized operator is

$$A^*(\xi) := \int_Q A(y, \xi + \nabla w_\xi(y)) dy,$$

and the function $w_\xi \in W_{\text{per}}^{1,p}(Q)$ is the corrector in the direction ξ given as the periodic solution (up to an additive constant) to the equation

$$-\operatorname{div}A(\cdot, \xi + \nabla w_\xi) = 0. \quad (5.1.7)$$

The strong convergence of the gradient

$$\nabla u_\varepsilon - \nabla u^* - \nabla w_{\nabla u^*}(\cdot/\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^p(\Omega) \quad (5.1.8)$$

has been obtained in [44] with ∇u^* replaced by its discretization at small scale ε , for measurability reasons, see Section 5.2 below for the details. The periodic homogenization of the integral functionals corresponding to (5.1.1) is exposed in e.g. [25]. The stochastic case has been studied qualitatively in [45]. Recently, quantitative results for non-linear stochastic problems have been obtained in [55] with optimal convergence rates for non-degenerate non-linear operators with quadratic growth, see also [103] for the deterministic case. The case of stochastic non-degenerate operators with p -growth, $p > 2$, is addressed in [42].

In this chapter, we study the equation (5.1.7) when the perturbation \tilde{a} belongs to the space $L^q(\mathbb{R}^d)$ for $1 \leq q \leq \frac{p}{p-1}$ and to some Hölder space (see Theorem 5.2.3 below). We then derive the homogenized limit of the sequence $(u_\varepsilon)_{\varepsilon>0}$ and we study the convergence of the two-scale expansion (5.1.8) when we use, on the one hand, the periodic corrector and, on the other hand, the non-periodic corrector (corresponding respectively to the solutions of (5.1.7) when $A(y, \xi) = a^{\text{per}}(y)\xi|\xi|^{p-2}$ and $A(y, \xi) = a(y)\xi|\xi|^{p-2}$). We also illustrate the quantitative convergence of the two-scale expansion (5.1.8) in the one-dimensional setting and prove that, in this case, using the non-periodic corrector instead of the periodic corrector in fact improves the quality of convergence of (5.1.8). The main difficulty of this work is that Equation (5.1.7) is posed on the whole space \mathbb{R}^d . One major tool to obtain the strong convergence (5.1.8) in the non-periodic case is the continuity of the application $\xi \mapsto \nabla w_\xi$ (see Theorem 5.2.4 below). This will be proved under one of the two Assumptions **(A4)** or **(A4)'** below.

Before stating our main results, we would like to comment on the special case $p = 2$ for the homogenization of the Problem (5.1.1). This problem is very standard since the 70's for a periodic coefficient a , see e.g. [16] for qualitative results and [11] for quantitative results. It is worth mentioning that, in this case, the homogenization objects such as correctors, homogenized limits are explicit and very easy to compute. The setting (5.1.1)-(5.1.2) has first been introduced in [19] for $q = 2$. It models local defects that could appear, at the microscale, in a periodic background. The results obtained have been generalized to the case $1 \leq q < +\infty$ in [20, 18] and convergence rates have been proved in [17]. In [66], a new non-periodic setting has been introduced to model defects that are not local but rare at infinity. We stress that, in [19, 20, 18, 17, 66], as in the present work, the macroscopic behaviour of the oscillating solution remains the same as in the case of a periodic coefficient. This will be expressed, for the non-linear case, in Theorem 5.2.7 below.

The chapter is organized as follows. The main results of the paper are presented in Section 5.2. We develop in Section 5.3 explicit calculations in the one dimensional setting and obtain convergence

results. We then turn in Section 5.4 to the existence of the non-periodic correctors in any dimension. The properties of the non-periodic corrector are proved in Section 5.5. We then derive qualitative homogenization results in Section 5.6. We finally prove in Section 5.7 a weaker continuity result for the mapping $\xi \mapsto \nabla w_\xi$ that is enough to derive qualitative homogenization. We recall in Appendix 5.8.1 the proof of classical results in the periodic case. Technical inequalities are gathered in Appendix 5.8.2.

5.2 Main results

5.2.1 Notations

In the whole paper, $d \geq 1$ will be the dimension of the ambient space. The standard unit cube $(-\frac{1}{2}, \frac{1}{2})^d$ will be denoted Q . The euclidian norm will be written $|\cdot|$ as well as the Lebesgue measure of a measurable subset of \mathbb{R}^d . Let Ω be a bounded domain of \mathbb{R}^d . If $1 < q < +\infty$ is an exponent, we define its conjugate by $q' := q/(q-1)$. The euclidian open ball of \mathbb{R}^d centered in x and of radius $r > 0$ will be written $B(x, r)$. If $x = 0$, we write $B_r := B(0, r)$. We use similar notations for cubes, namely $Q(x, r)$ and Q_r . We define the mean-value operation for a measurable and integrable function u by

$$\mathop{\int}\limits_{B(x,r)} u := \frac{1}{|B(x,r)|} \int_{B(x,r)} u.$$

The indicator function of a measurable set A is denoted 1_A .

The standard Lebesgue and Sobolev spaces are denoted by $L^q(\Omega)$ and $W^{1,q}(\Omega)$. The associated norms are

$$\|u\|_{L^q(\Omega)} := \left(\int_{\Omega} |u|^q \right)^{1/q} \quad \text{and} \quad \|u\|_{W^{1,q}(\Omega)} := \left(\int_{\Omega} |u|^q \right)^{1/q} + \left(\int_{\Omega} |\nabla u|^q \right)^{1/q}.$$

The space L^q_{per} (resp. $W^{1,q}_{\text{per}}$) denotes the set of functions that are periodic and locally belong to L^q (resp. $W^{1,q}$). These two spaces are endowed with the norms

$$\|u\|_{L^q_{\text{per}}} := \left(\int_Q |u|^q \right)^{1/q} \quad \text{and} \quad \|u\|_{W^{1,q}_{\text{per}}} := \left(\int_Q |u|^q \right)^{1/q} + \left(\int_Q |\nabla u|^q \right)^{1/q}.$$

The space of uniformly L^q (resp. $W^{1,q}$) functions is denoted by L^q_{unif} (resp. $W^{1,q}_{\text{unif}}$). These spaces are endowed with the norms

$$\|u\|_{L^q_{\text{unif}}(\mathbb{R}^d)} := \sup_{x \in \mathbb{R}^d} \|u\|_{L^q(x+Q)} \quad \text{and} \quad \|u\|_{W^{1,q}_{\text{unif}}(\mathbb{R}^d)} := \sup_{x \in \mathbb{R}^d} \|u\|_{W^{1,q}(x+Q)}.$$

For $0 < \alpha < 1$, the space $\mathcal{C}^{0,\alpha}$ refers to the standard Hölder space endowed with the norm

$$\|u\|_{\mathcal{C}^{0,\alpha}} := \|u\|_{L^\infty} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

We define, for $\delta > 0$, the discretization operator $M_\delta : L^q(\Omega) \rightarrow L^q(\Omega)$ introduced in [44, 57]. If $\phi \in L^q(\Omega)$, we set

$$M_\delta \phi := \sum_{k \in \mathbb{Z}^d \text{ s.t. } \delta(Q+k) \subset \Omega} \left(\mathop{\int}\limits_{\delta(Q+k)} \phi \right) 1_{\delta(Q+k)}. \quad (5.2.1)$$

It is clear that M_δ is linear and bounded over $L^q(\Omega)$ and that $M_\delta \phi \xrightarrow{\delta \rightarrow 0} \phi$ in $L^q(\Omega)$.

5.2.2 The periodic case

We assume in this paragraph that $\tilde{a} = 0$ in (5.1.2). In this case, the corrector equation is, according to (5.1.7):

$$-\operatorname{div} a^{\text{per}}(y)(\xi + \nabla w_\xi^{\text{per}})|\xi + \nabla w_\xi^{\text{per}}|^{p-2} = 0. \quad (5.2.2)$$

The equation (5.2.2) admits a unique solution w_ξ^{per} in the space $W_{\text{per}}^{1,p}(Q)/\mathbb{R}$. Indeed, the weak formulation of (5.2.2) is

$$\forall \phi \in W_{\text{per}}^{1,p}(Q)/\mathbb{R}, \quad \int_Q a^{\text{per}}(y)(\xi + \nabla w_\xi^{\text{per}})|\xi + \nabla w_\xi^{\text{per}}|^{p-2} \cdot \nabla \phi = 0, \quad (5.2.3)$$

which is exactly the Euler-Lagrange equation of the minimization Problem

$$\min_{v \in W_{\text{per}}^{1,p}(Q)/\mathbb{R}} \left\{ \frac{1}{p} \int_Q a^{\text{per}}(y)|\xi + \nabla v|^p dy \right\}. \quad (5.2.4)$$

It is easy to see that the functional appearing in Problem (5.2.4) is strictly convex, coercive and continuous with respect to ∇v . Thus, (5.2.4) admits a minimizer w_ξ^{per} , the gradient of which is unique. We impose that $\int_Q w_\xi^{\text{per}} = 0$ so that w_ξ^{per} is itself unique. Besides, we have the following Proposition (see [57, 56, 44] or Appendix 5.8.1 below for a proof) gathering the main properties of the application $\xi \mapsto \nabla w_\xi^{\text{per}}$:

Proposition 5.2.1. *Let $a^{\text{per}} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a periodic and Lipschitz continuous coefficient satisfying (5.1.3).*

(i) *The map $\xi \mapsto \nabla w_\xi^{\text{per}}$ is homogeneous in the sense that for all $\xi \in \mathbb{R}^d$ and $t \in \mathbb{R}$,*

$$\nabla w_{t\xi}^{\text{per}} = t \nabla w_\xi^{\text{per}}. \quad (5.2.5)$$

(ii) *There exists a constant $C = C(d, p, a^{\text{per}}) > 0$ and an exponent $\alpha = \alpha(d, p, a^{\text{per}}) > 0$ such that for all $\xi \in \mathbb{R}^d$, $\nabla w_\xi^{\text{per}} \in C^{0,\alpha}(\mathbb{R}^d)$ and, moreover, we have the estimates*

$$\|\nabla w_\xi^{\text{per}}\|_{L_{\text{unif}}^p(\mathbb{R}^d)} \leq C|\xi| \quad \text{and} \quad \|\nabla w_\xi^{\text{per}}\|_{C^{0,\alpha}(\mathbb{R}^d)} \leq C|\xi|. \quad (5.2.6)$$

(iii) *There exists a constant $C = C(d, p, a^{\text{per}}) > 0$ such that for all $\xi, \eta \in \mathbb{R}^d$,*

$$\|\nabla w_\xi^{\text{per}} - \nabla w_\eta^{\text{per}}\|_{L_{\text{unif}}^p(\mathbb{R}^d)} \leq C [|\xi|^{1-\beta} + |\eta|^{1-\beta}] |\xi - \eta|^\beta, \quad \beta := \frac{1}{p-1}. \quad (5.2.7)$$

(iv) *There exists a constant $C = C(d, p, a^{\text{per}}) > 0$ such that for all $\xi, \eta \in \mathbb{R}^d$,*

$$\|\nabla w_\xi^{\text{per}} - \nabla w_\eta^{\text{per}}\|_{L^\infty(\mathbb{R}^d)} \leq C [|\xi|^{1-\gamma} + |\eta|^{1-\gamma}] |\xi - \eta|^\gamma, \quad \gamma := \frac{\beta p}{p + d/\alpha}, \quad (5.2.8)$$

where β is defined in (5.2.7) and α is given by (ii).

It is proved in [57] that u_ε converges weakly in $W^{1,p}(\Omega)$ to u^* which is defined by (5.1.6). Note that (5.1.6) is well posed due to the monotonicity of A^* (see [44] and [76, Corollary 8.1]). Convergence in the L^∞ -norm may be obtained in the one-dimensional setting, see Section 5.3 below.

5.2.3 Results in the non-periodic case

The first result of this contribution concerns the corrector equation (5.1.8) in the setting (5.1.1)-(5.1.2). For a fixed direction $\xi \in \mathbb{R}^d$, this equation, posed on the whole space \mathbb{R}^d , is

$$-\operatorname{div} a(y)(\xi + \nabla w_\xi)|\xi + \nabla w_\xi|^{p-2} = 0, \quad (5.2.9)$$

where the coefficient a is of the form $a := a^{\text{per}} + \tilde{a}$ and a^{per} is a periodic coefficient. We assume that a and a^{per} satisfy the following assumptions:

- (A1)** there exists $\lambda > 0$ such that (5.1.3) is satisfied;
- (A2)** the coefficients a and a^{per} are Lipschitz-continuous;
- (A3)** the perturbation \tilde{a} vanishes at infinity in the sense that $\tilde{a} \in L^{p'}(\mathbb{R}^d)$.

A few comments are in order. First, if \tilde{a} satisfies $\tilde{a} \in C^{0,1}(\mathbb{R}^d)$ and $\tilde{a} \in L^q(\mathbb{R}^d)$ for some $q \leq p'$ then \tilde{a} satisfies **(A3)** by interpolation. Second, Assumption **(A2)** allows to ensure local regularity (see Proposition 5.2.1 above) of the periodic and non-periodic correctors. Finally, the assumptions of [19] in the linear setting correspond to the case $p = 2$ in the assumptions **(A1)**-**(A2)**-**(A3)** above.

We now consider the equation (5.2.9) when the coefficient a has the non-periodic structure (5.1.2). For $u \in L^\infty(\mathbb{R}^d)$, we define the spaces

$$\mathcal{W}_u := \left\{ v \in W_{loc}^{1,1}(\mathbb{R}^d), \int_{\mathbb{R}^d} |\nabla v|^p + \int_{\mathbb{R}^d} |u|^{p-2} |\nabla v|^2 < +\infty \right\} \quad \text{and} \quad W_u := \mathcal{W}_u / \mathbb{R}. \quad (5.2.10)$$

The space W_u is endowed with the norm

$$\|v\|_{W_u} := \|\nabla v\|_{L^p(\mathbb{R}^d)} + \left\| |u|^{\frac{p-2}{2}} \nabla v \right\|_{L^2(\mathbb{R}^d)}. \quad (5.2.11)$$

In the sequel, we denote undifferently functions and equivalence classes for the relation: $f \sim g$ if and only if $f - g$ is almost everywhere constant. Lemma 5.4.1 below gathers some properties satisfied by spaces of the form (5.2.10). In order to solve (5.2.9), we seek for w_ξ of the form $w_\xi = w_\xi^{\text{per}} + \tilde{w}_\xi$ where w_ξ^{per} is the solution to (5.2.3) such that $\int_Q w_\xi^{\text{per}} = 0$. We transform the equation (5.2.9) into

$$-\operatorname{div} a \left[|\xi + \nabla w_\xi^{\text{per}} + \nabla \tilde{w}_\xi|^{p-2} (\xi + \nabla w_\xi^{\text{per}} + \nabla \tilde{w}_\xi) - |\xi + \nabla w_\xi^{\text{per}}|^{p-2} (\xi + \nabla w_\xi^{\text{per}}) \right] = \operatorname{div}(h), \quad (5.2.12)$$

where

$$h := \tilde{a}(\xi + \nabla w_\xi^{\text{per}})|\xi + \nabla w_\xi^{\text{per}}|^{p-2}. \quad (5.2.13)$$

Assumption **(A3)** and Proposition 5.2.1 (ii) ensure that $h \in L^{p'}(\mathbb{R}^d)^d$.

Definition 5.2.2. We say that $\tilde{w}_\xi \in W_{\xi + \nabla w_\xi^{\text{per}}}$ is a solution in the weak sense in $W_{\xi + \nabla w_\xi^{\text{per}}}$ to (5.2.12) if for all $w \in W_{\xi + \nabla w_\xi^{\text{per}}}$,

$$\int_{\mathbb{R}^d} a \left[|\xi + \nabla w_\xi^{\text{per}} + \nabla \tilde{w}_\xi|^{p-2} (\xi + \nabla w_\xi^{\text{per}} + \nabla \tilde{w}_\xi) - |\xi + \nabla w_\xi^{\text{per}}|^{p-2} (\xi + \nabla w_\xi^{\text{per}}) \right] \cdot \nabla w = - \int_{\mathbb{R}^d} h \cdot \nabla w.$$

We easily check using Appendix 5.8.2 that each integral appearing in Definition 5.2.2 is convergent. Note that if \tilde{w}_ξ is a solution to (5.2.12) in the sense of Definition 5.2.2, then it is a solution to (5.2.12) in the distribution sense but it is not clear that the converse holds true. This is true if the weight $\xi + \nabla w_\xi^{\text{per}}$ satisfies Assumption **(A4)'** below (see also Remark 5.2.13).

Theorem 5.2.3 (Existence of the non-periodic correctors). *Assume that the coefficient $a = a^{\text{per}} + \tilde{a}$ satisfies Assumptions **(A1)**-**(A2)**-**(A3)**. Then, for all $\xi \in \mathbb{R}^d$, there exists a unique solution w_ξ to equation (5.2.9) such that $w_\xi \in W_{loc}^{1,1}(\mathbb{R}^d)$, $w_\xi = w_\xi^{\text{per}} + \tilde{w}_\xi$, where $\tilde{w}_\xi \in W_{\xi + \nabla w_\xi^{\text{per}}}$ is solution in the weak sense in $W_{\xi + \nabla w_\xi^{\text{per}}}$ to (5.2.12)-(5.2.13).*

In view of Theorem 5.2.3, we denote in the sequel $\widetilde{w}_\xi \in \mathcal{W}_{\xi + \nabla w_\xi^{\text{per}}}$ the unique function such that $\int_Q \widetilde{w}_\xi = 0$. The function $w_\xi^{\text{per}} + \widetilde{w}_\xi$ is a solution to (5.2.9) and \widetilde{w}_ξ solves (5.2.12)-(5.2.13) in the sense of Definition 5.2.2. We also define

$$w_\xi := w_\xi^{\text{per}} + \widetilde{w}_\xi \in W_{\text{per}}^{1,p}(Q) + \mathcal{W}_{\xi + \nabla w_\xi^{\text{per}}}. \quad (5.2.14)$$

The analogous properties of those given in Proposition 5.2.1 are given in Theorem 5.2.4 below for the non-linear correctors w_ξ , $\xi \in \mathbb{R}^d$. In order to obtain continuity results for the application $\xi \mapsto \nabla w_\xi$, we need the following assumption:

(A4) There exists $c > 0$ independent of $\xi \in \mathbb{R}^d$ such that $|\xi + \nabla w_\xi^{\text{per}}| \geq c|\xi|$ on Q .

We comment in Subsection 5.2.4 on this assumption. We are able to prove the following Theorem.

Theorem 5.2.4. *Let $a := a^{\text{per}} + \widetilde{a}$ be a non-periodic coefficient satisfying Assumptions (A1)-(A2)-(A3). For $\xi \in \mathbb{R}^d$, let w_ξ be defined by (5.2.14).*

(i) *The map $\xi \mapsto \nabla w_\xi$ is homogeneous in the sense that for all $\xi \in \mathbb{R}^d$ and $t \in \mathbb{R}$,*

$$\nabla w_{t\xi} = t \nabla w_\xi. \quad (5.2.15)$$

(ii) *There exists a constant $C = C(d, p, a) > 0$ and an exponent $\alpha = \alpha(d, p, a) > 0$ such that for all $\xi \in \mathbb{R}^d$, $\nabla w_\xi \in L_{\text{unif}}^p(\mathbb{R}^d)$, $\nabla w_\xi \in C^{0,\alpha}(\mathbb{R}^d)$ and, moreover, we have the estimates*

$$\|\nabla w_\xi\|_{L_{\text{unif}}^p(\mathbb{R}^d)} \leq C|\xi| \quad \text{and} \quad \|\nabla w_\xi\|_{C^{0,\alpha}(\mathbb{R}^d)} \leq C|\xi|. \quad (5.2.16)$$

(iii) *Assume that Assumption (A4) is satisfied. Then there exists a constant $C = C(d, p, a, c) > 0$ independent of ξ and η such that, for all $\xi, \eta \in \mathbb{R}^d$,*

$$\|\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta\|_{L^p(\mathbb{R}^d)} \leq C \left(|\xi|^{1-\widetilde{\beta}} + |\eta|^{1-\widetilde{\beta}} \right) |\xi - \eta|^{\widetilde{\beta}}, \quad \widetilde{\beta} = \frac{\gamma}{p-1} \min(1, p-2). \quad (5.2.17)$$

where γ is given by (5.2.8).

(iv) *Assume that Assumption (A4) is satisfied. Then there exists a constant $C = C(d, p, a, c) > 0$ and an exponent $\widetilde{\gamma} > 0$ both independent of ξ and η such that, for all $\xi, \eta \in \mathbb{R}^d$,*

$$\|\nabla w_\xi - \nabla w_\eta\|_{L^\infty(\mathbb{R}^d)} \leq C \left(|\xi|^{1-\widetilde{\gamma}} + |\eta|^{1-\widetilde{\gamma}} \right) |\xi - \eta|^{\widetilde{\gamma}}. \quad (5.2.18)$$

An important tool to obtain Theorem 5.2.4(iii) is the following Theorem:

Theorem 5.2.5. *Let $a := a^{\text{per}} + \widetilde{a}$ be a non-periodic coefficient such that (A1)-(A2)-(A3)-(A4) are satisfied. For all $\xi \in \mathbb{R}^d$, we have that $\nabla \widetilde{w}_\xi \in L^{p'}(\mathbb{R}^d)$ and the estimate*

$$\|\nabla \widetilde{w}_\xi\|_{L^{p'}(\mathbb{R}^d)} \leq C|\xi| \quad (5.2.19)$$

holds true where $C = C(d, p, a, c) > 0$ is a constant independent of ξ .

Remark 5.2.6. *Note that, under Assumption (A4), the non-periodic part $\nabla \widetilde{w}_\xi$ of the corrector has the same integrability as the defect \widetilde{a} at infinity. This is reminiscent of the linear case $p = 2$, see [20].*

Using Theorem 5.2.4, we can prove qualitative results concerning the homogenization of (5.1.1) in the non-periodic setting.

Theorem 5.2.7. *Let Ω be a bounded smooth domain, $f \in L^{p'}(\Omega)$, $a := a^{\text{per}} + \widetilde{a}$ be a scalar-valued coefficient satisfying Assumptions (A1)-(A2)-(A3)-(A4). For $\varepsilon > 0$, let $u_\varepsilon \in W_0^{1,p}(\Omega)$ be the solution to (5.1.1).*

(i) We have that $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^*$ weakly in $W^{1,p}(\Omega)$ and $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^*$ strongly in $L^p(\Omega)$, where u^* solves Problem (5.1.6) and

$$\forall \xi \in \mathbb{R}^d, \quad a^*(\xi) = \int_Q a^{\text{per}}(y) |\xi + \nabla w_\xi^{\text{per}}| |\xi + \nabla w_\xi^{\text{per}}|^{p-2} dy. \quad (5.2.20)$$

Besides, we have the $L^p(\Omega)$ -weak convergence $a(\cdot/\varepsilon) \nabla u_\varepsilon |\nabla u_\varepsilon|^{p-2} \xrightarrow{\varepsilon \rightarrow 0} a^*(\nabla u^*)$.

(ii) We have the strong convergence

$$\nabla u_\varepsilon - \nabla u^* - \nabla w_{M_\varepsilon \nabla u^*} \left(\frac{\cdot}{\varepsilon} \right) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^p(\Omega), \quad (5.2.21)$$

where M_ε is defined by (5.2.1).

(iii) We have the strong convergence

$$\nabla u_\varepsilon - \nabla u^* - \nabla w_{M_\varepsilon \nabla u^*} \left(\frac{\cdot}{\varepsilon} \right) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^p(\Omega), \quad (5.2.22)$$

where M_ε is defined by (5.2.1).

We stress that, instead of assuming **(A4)**, Theorem 5.2.7 can be proved under the assumption that the mapping

$$\Phi_p : \begin{cases} \mathbb{R}^d \longrightarrow L^p_{\text{unif}}(\mathbb{R}^d) \\ \xi \longmapsto \nabla w_\xi \end{cases} \quad (5.2.23)$$

is continuous. This continuity can be obtained under the following Assumption **(A4)'** which is clearly weaker than Assumption **(A4)**:

(A4)' For $\xi \in \mathbb{R}^d$, there exists a constant $C > 0$ that may depend on ξ and $\nabla w_\xi^{\text{per}}$ such that the following weighted Poincaré-Wirtinger inequality holds true: there exists $r_{\min} > 0$ such that for all $R > r_{\min}$ and $w \in H^1(Q \setminus Q_{1/2})$,

$$\left\| |\xi + \nabla w_\xi^{\text{per}}(R \cdot)|^{\frac{p-2}{2}} \left(w - \int_{Q \setminus Q_{1/2}} w \right) \right\|_{L^2(Q \setminus Q_{1/2})} \leq C \left\| |\xi + \nabla w_\xi^{\text{per}}(R \cdot)|^{\frac{p-2}{2}} \nabla w \right\|_{L^2(Q \setminus Q_{1/2})}. \quad (5.2.24)$$

We comment in Subsection 5.2.4 on Assumption **(A4)'** and we will provide a sufficient condition on $\xi + \nabla w_\xi^{\text{per}}$ so that (5.2.24) is satisfied. We are able to prove the following Theorem:

Theorem 5.2.8. Assume that **(A1)**-**(A2)**-**(A3)**-**(A4)'** are satisfied. Then the mapping Φ_p defined by (5.2.23) is continuous.

We close this section by mentioning that the results of Theorem 5.2.7 can be improved in the one-dimensional setting. We devote Section 5.3 to convergence results in this particular case.

Remark 5.2.9. To derive Theorem 5.2.8, it is sufficient, instead of Assumption **(A4)'**, to assume that the set of smooth functions with compact support over \mathbb{R}^d , denoted by $C_0^\infty(\mathbb{R}^d)$, is dense in $W_{\xi + \nabla w_\xi^{\text{per}}}$. We show in Lemma 5.8.3 (see Appendix 5.8.2) that, as pointed out in [109], the density result is implied by Assumption **(A4)'**. In particular, the practical criterion given by Lemma 5.2.12 implies the density of $C_0^\infty(\mathbb{R}^d)$. Note that, under Assumption **(A4)'**, we can easily prove (by density) that (5.2.12)–(5.2.13) admits a unique solution in the distribution sense in $W_{\xi + \nabla w_\xi^{\text{per}}}$.

Remark 5.2.10. The method of proof of this paper allows to build the non-periodic correctors for a defect \tilde{a} that belongs to the dual space of $W_{\xi + \nabla w_\xi^{\text{per}}}$, see Lemma 5.4.1 (iii). This is in particular the case if $\tilde{a} \in L^2(\mathbb{R}^d)$. We are however not able to show that the non-periodic corrector satisfies $\nabla \tilde{w}_\xi \in L^2(\mathbb{R}^d)$ but only that $\nabla \tilde{w}_\xi \in L^2(|\xi + \nabla w_\xi^{\text{per}}|^{p-2} d\lambda)$, see Remark 5.2.11 below. More generally, building the non-periodic correctors for a defect $\tilde{a} \in L^{2+\delta} \cap C^{0,\alpha}(\mathbb{R}^d)$ is a challenging problem that we are unable to address for now. In the linear setting $p = 2$, this was achieved in [18] by studying the continuity from $L^q(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ for $q > 2$ of the Riesz operator associated to the coefficient a .

Remark 5.2.11. The space $W_{\xi + \nabla w_{\xi}^{\text{per}}}$ is in general different from the space $W^{\circ,1,p} \cap \dot{H}^1(\mathbb{R}^d)$, where $W^{\circ,1,p}$ and $\dot{H}^1(\mathbb{R}^d)$ are the standard homogeneous Sobolev spaces, unless $\xi + \nabla w_{\xi}^{\text{per}}$ does not vanish. Assume that there exists $x_0 \in Q$ such that $\xi + \nabla w_{\xi}^{\text{per}}(x_0) = 0$. We can assume by invariance translation that $x_0 = 0$. Owing to Proposition 5.2.1 (ii), we have that $|\xi + \nabla w_{\xi}^{\text{per}}(x)| \leq C|x|^{\alpha}$ in Q . Let $\phi \in \mathcal{D}(Q)$ be such that $\phi = 1$ on $B(0, 1/4)$. We define $\Psi := \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|k|^{\delta + \nu}} \phi(|k|^{\nu}(\cdot - k))$, where $\delta, \nu > 0$ will be chosen later. We have

$$\|\nabla \Psi\|_{L^p(\mathbb{R}^d)}^p = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|k|^{p\delta}} \int_{|k|^{-\nu}Q} |\nabla \phi(|k|^{\nu}x)|^p dx = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{\|\nabla \phi\|_{L^p(Q)}^p}{|k|^{p\delta + d\nu}}. \quad (5.2.25)$$

Besides, we have that

$$\begin{aligned} \|\nabla \Psi\|_{L^2(|\xi + \nabla w_{\xi}^{\text{per}}|^{p-2}d\lambda)}^2 &= \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|k|^{2\delta}} \int_{|k|^{-\nu}Q} |\nabla \phi(|k|^{\nu}x)|^2 |\xi + \nabla w_{\xi}^{\text{per}}(x)|^{p-2} dx \\ &\leq \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{C \|\nabla \phi\|_{L^2(Q)}^2}{|k|^{\alpha\nu(p-2) + d\nu + 2\delta}}. \end{aligned} \quad (5.2.26)$$

Finally,

$$\|\nabla \Psi\|_{L^2(\mathbb{R}^d)}^2 = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|k|^{2\delta}} \int_{|k|^{-\nu}Q} |\nabla \phi(|k|^{\nu}x)|^2 dx = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{\|\nabla \phi\|_{L^2(Q)}^2}{|k|^{d\nu + 2\delta}}. \quad (5.2.27)$$

We fix $\nu \in (\frac{d}{d+2}, 1)$ and $\delta \in (\max\{\frac{d(1-\nu)}{p}, \frac{d(1-\nu)-\alpha\nu(p-2)}{2}\}, \frac{d(1-\nu)}{2})$ so that $\nabla \Psi \in L^2(|\xi + \nabla w_{\xi}^{\text{per}}|^{p-2}d\lambda) \cap L^p(\mathbb{R}^d)$ and $\nabla \Psi \notin L^2(\mathbb{R}^d)$. Note that $\nabla \Psi \in \mathcal{C}^{0,\delta/\nu}(\mathbb{R}^d)$ so that this counter-example is consistent with the result of Theorem 5.2.4 (ii) since $\delta < \nu$.

5.2.4 Comments on the Assumptions

On Assumption (A4). Assumption (A4) is quite restrictive but is known to be true in dimension 1. Besides, it is proved in [30, Lemma 2, p. 404] that it is also satisfied in dimension $d = 2$.

We show here that Assumption (A4) is satisfied for laminate materials (in any dimension). Suppose that $a^{\text{per}}(x) = a_0(x_1)$ where $a_0 : \mathbb{R} \rightarrow \mathbb{R}$ is a periodic function. Let $\xi \neq 0$. In this case, the periodic corrector w_{ξ}^{per} is a function of the first variable *i.e.* $w_{\xi}^{\text{per}}(x) = w_{\xi}^0(x_1)$ and (5.2.2) becomes

$$-\frac{d}{dx_1} \left(a_0(x_1) \left(\xi_1 + \frac{dw_{\xi}^0}{dx_1} \right) \left| \left(\xi_1 + \frac{dw_{\xi}^0}{dx_1} \right)^2 + \xi_2^2 + \dots + \xi_d^2 \right|^{\frac{p-2}{2}} \right) = 0. \quad (5.2.28)$$

If there exists $i \geq 2$ such that $\xi_i \neq 0$, then $|\xi + \nabla w_{\xi}^{\text{per}}| \geq |\xi_i| > 0$. In the other cases, $\xi_i = 0$ for $i \geq 2$, thus $\xi_1 \neq 0$ and (5.2.28) reduces to:

$$-\frac{d}{dx_1} \left(a_0(x_1) \left(\xi_1 + \frac{dw_{\xi}^0}{dx_1} \right) \left| \xi_1 + \frac{dw_{\xi}^0}{dx_1} \right|^{p-2} \right) = 0. \quad (5.2.29)$$

There exists a constant $C(\xi)$ such that $(\xi_1 + \frac{dw_{\xi}^0}{dx_1})^{p-1} = C(\xi)/a_0(x_1)$, where $z^{p-1} := \text{sgn}(z)|z|^{p-1}$. If $C(\xi) = 0$, then $w_{\xi}^0(x_1) = -\xi_1 x_1$ which contradicts the periodicity of w_{ξ}^0 . Hence $C(\xi) \neq 0$. Thus

$$\left| \xi_1 + \frac{dw_{\xi}^0}{dx_1} \right| \geq (|C(\xi)|\lambda)^{\frac{1}{p-1}} > 0.$$

We have proved that $|\xi + \nabla w_\xi^{\text{per}}| \geq c(\xi) > 0$ on Q : this easily implies **(A4)**. Indeed, if by contradiction **(A4)** were not satisfied, then there would exist by homogeneity $(\xi_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ and $(x_n)_{n \in \mathbb{N}} \subset Q$ such that $|\xi_n| = 1$ and $|\xi_n + \nabla w_{\xi_n}^{\text{per}}(x_n)| \xrightarrow{n \rightarrow +\infty} 0$. By compactness, we have that up to a subsequence, $(x_n, \xi_n) \xrightarrow{n \rightarrow +\infty} (x, \xi)$. By Proposition 5.2.1 (ii) and Proposition 5.2.1 (iv), we obtain that $|\xi + \nabla w_\xi^{\text{per}}(x)| = 0$ which gives $c(\xi) = 0$: we thus reach a contradiction.

On Assumption (A4)'. The Assumption **(A4)'** is the weakest possible given our method of proof. It is satisfied in dimension $d = 1, 2$ because **(A4)** is satisfied. For higher dimensions, we provide here a sufficient condition implying **(A4)'**:

Lemma 5.2.12 (see [29] and [108]). *Assume that $d \geq 2$ and that $|\xi + \nabla w_\xi^{\text{per}}|^{2-p} \in L^{d/2}(Q)$, then **(A4)'** is satisfied.*

Proof. For the sake of completeness, we reproduce the proof here, see also [29, Lemma 8]. We fix $R > 1$. Then by periodicity, we immediately check that $|\xi + \nabla w_\xi^{\text{per}}(R \cdot)|^{2-p} \in L^{d/2}(Q \setminus Q_{1/2})$ and that

$$\| |\xi + \nabla w_\xi^{\text{per}}(R \cdot)|^{2-p} \|_{L^{d/2}(Q \setminus Q_{1/2})} \leq 4 \| |\xi + \nabla w_\xi^{\text{per}}|^{2-p} \|_{L^{d/2}(Q)}. \quad (5.2.30)$$

We then write, using Proposition 5.2.1 (ii), that

$$\begin{aligned} & \| |\xi + \nabla w_\xi^{\text{per}}(R \cdot)|^{\frac{p-2}{2}} (w - \fint_{Q \setminus Q_{1/2}} w) \|_{L^2(Q \setminus Q_{1/2})} \\ & \leq C |\xi|^{\frac{p-2}{2}} \| w - \fint_{Q \setminus Q_{1/2}} w \|_{L^2(Q \setminus Q_{1/2})} \leq C |\xi|^{\frac{p-2}{2}} \| \nabla w \|_{L^{\frac{2d}{d+2}}(Q \setminus Q_{1/2})} \\ & \leq C |\xi|^{\frac{p-2}{2}} \left(\int_{Q \setminus Q_{1/2}} |\xi + \nabla w_\xi^{\text{per}}(R \cdot)|^{p-2} |\nabla w|^2 \right)^{\frac{1}{2}} \left(\int_{Q \setminus Q_{1/2}} \{ |\xi + \nabla w_\xi^{\text{per}}(R \cdot)|^{2-p} \}^{d/2} \right)^{\frac{1}{d}}, \end{aligned} \quad (5.2.31)$$

where we used Poincaré-Sobolev embedding $W^{1, 2d/(d+2)}(Q \setminus Q_{1/2}) \hookrightarrow L^2(Q \setminus Q_{1/2})$ (see [58, Exercise II.5.9]) in the second inequality, Hölder inequality with exponents $r = (d+2)/d$, $r' = (d+2)/2$ and the fact that $dr'/(d+2) = d/2$ in the third inequality. Using (5.2.30) allows to conclude the proof. \square

If we assume that $\{\xi + \nabla w_\xi^{\text{per}} = 0\}$ is a finite number of points (in the case $d > 2$) and that all critical points have finite order, denoting by m the maximum order of the corresponding zero points, we have that $|\xi + \nabla w_\xi^{\text{per}}|^{2-p} \in L^{d/2}(Q)$ if and only if $m \frac{d}{2}(p-2) < d$ i.e. $p < 2 + 2/m$. Thus, in this case, Assumption **(A4)'** can be replaced by assuming that $p < 2 + 2/m$. Note also that if $\xi + \nabla w_\xi^{\text{per}}$ vanishes at order m along a line (or a curve) in dimension d , then $|\xi + \nabla w_\xi^{\text{per}}|^{2-p} \sim |x|^{m(2-p)}$ which is $L^{d/2}(Q)$ if and only if $\frac{d}{2}m(p-2) < d-1$ i.e. $p < 2 + \frac{2(d-1)}{dm}$.

Remark 5.2.13. *The Assumption **(A4)'** is used in the proof of Lemma 5.7.1 which allows to pass from solutions in the distribution sense to solutions in the sense of Definition 5.2.2 for PDEs of the form (5.2.12). We then take advantage of Lemma 5.7.1 in the proof of Theorem 5.2.8 by working locally in a concentration-compactness method.*

5.2.5 Extension to other non-linear operators

We have limited the presentation of the results to the simplest operator (5.1.1) in order to avoid some technicalities and the use of abstract existence Theorems for non-linear PDEs. However, the result of this paper extends to more general operators. We explain below the type of problems that we can address with the technique developed in this work.

The first direct extension concerns the equivalent of (5.1.1) when a is a matrix-valued coefficient. This corresponds to the following non-linear operator:

$$a(y, \xi) := \langle A(y)\xi, \xi \rangle^{\frac{p-2}{2}} A(y)\xi, \quad y \in \mathbb{R}^d, \quad \xi \in \mathbb{R}^d, \quad (5.2.32)$$

where A is of the form $A = A^{\text{per}} + \tilde{A}$. We assume that the matrix A^{per} is periodic and that A and A^{per} are symmetric and positive definite, that is,

$$\exists \lambda > 0, \quad \forall y \in \mathbb{R}^d, \quad \lambda^{-1}|\xi|^2 \leq \langle A(y)\xi, \xi \rangle \leq \lambda|\xi|^2 \quad \text{and} \quad \lambda^{-1}|\xi|^2 \leq \langle A^{\text{per}}(y)\xi, \xi \rangle \leq \lambda|\xi|^2.$$

The perturbation \tilde{A} satisfies $\tilde{A} \in L^{p'} \cap C^{0,1}(\mathbb{R}^d)^{d \times d}$. The periodic correctors can be defined thanks to variational techniques by considering the minimization problem

$$\min_{w_\xi^{\text{per}} \in H^{1,p^{\text{per}}}(Q)} \left\{ \frac{1}{p} \int_Q \left\langle A(y)(\xi + \nabla w_\xi^{\text{per}}), \xi + \nabla w_\xi^{\text{per}} \right\rangle^{p/2} \right\}.$$

The non-periodic equation corresponding to (5.2.12) is

$$-\text{div} \left[a(\cdot, \xi + \nabla w_\xi^{\text{per}} + \nabla \tilde{w}_\xi) - a(\cdot, \xi + \nabla w_\xi^{\text{per}}) \right] = \text{div}(h), \quad (5.2.33)$$

where

$$h := a^{\text{per}}(\cdot, \xi + \nabla w_\xi^{\text{per}}) - a(\cdot, \xi + \nabla w_\xi^{\text{per}}), \quad (5.2.34)$$

where $a^{\text{per}}(\cdot, \xi) := \langle A^{\text{per}}(\cdot)\xi, \xi \rangle^{\frac{p-2}{2}} A^{\text{per}}(\cdot)\xi$. It is easily proved that $f \in L^{p'}(\mathbb{R}^d)^d$ and that the method of proof of Section 5.4 extends to this case by studying the functional

$$\begin{aligned} F_\xi(v) := & \frac{1}{p} \int_{\mathbb{R}^d} \left\{ \left\langle A(y)(\xi + \nabla w_\xi^{\text{per}} + \nabla v), \xi + \nabla w_\xi^{\text{per}} + \nabla v \right\rangle^{p/2} - \left\langle A(y)(\xi + \nabla w_\xi^{\text{per}}), \xi + \nabla w_\xi^{\text{per}} \right\rangle^{p/2} \right. \\ & \left. - p \langle A(y)(\xi + \nabla w_\xi^{\text{per}}), \xi + \nabla w_\xi^{\text{per}} \rangle^{\frac{p-2}{2}} A(y)(\xi + \nabla w_\xi^{\text{per}}) \cdot \nabla v \right\} dy + \int_{\mathbb{R}^d} h \cdot \nabla v. \end{aligned}$$

Note that the inequalities given in Appendix 5.8.2 are valid for the matrix model (5.2.32). Concerning the continuity results for the application $\xi \mapsto \nabla \tilde{w}_\xi$, the results proved in sections 5.5, 5.6 and 5.7 still hold true.

The second less direct extension corresponds to non-variational operators, that is, PDEs that cannot be written as a minimization problem. We consider operators $a(y, \xi)$ that satisfy the following properties:

- (1) for all $\xi \in \mathbb{R}^d$, $a(\cdot, \xi)$ is a measurable function and $\xi \mapsto a(y, \cdot)$ for fixed $y \in \mathbb{R}^d$ is of class $C^1(\mathbb{R}^d)$ and of class $C^2(\mathbb{R}^d \setminus \{0\})$.
- (2) the application $\xi \mapsto a(y, \xi)$ is homogeneous *i.e.* $a(y, t\xi) = t^{p-1}a(y, \xi)$ for $t \in \mathbb{R}$ and $y, \xi \in \mathbb{R}^d$. We also assume that $a(\cdot, \xi)$ is a uniformly in ξ Lipschitz continuous function: there exists $\lambda > 0$ such that

$$\begin{aligned} \forall y, y' \in \mathbb{R}^d, \quad \forall \xi \in \mathbb{R}^d, \quad |a(y, \xi) - a(y', \xi)| &\leq \lambda |y - y'| |\xi|^{p-1}. \\ \forall y, y' \in \mathbb{R}^d, \quad \forall \xi \in \mathbb{R}^d, \quad |\partial_\xi a(y, \xi) - \partial_\xi a(y', \xi)| &\leq \lambda |y - y'| |\xi|^{p-2}. \end{aligned}$$

- (3) we have that $a(y, \xi) = a^{\text{per}}(y, \xi) + \tilde{a}(y, \xi)$ where $a^{\text{per}}(\cdot, \xi)$ is a periodic function satisfying the same homogeneity and regularity properties as a . We assume that the perturbation \tilde{a} satisfies:

$$\exists b \in L^{p'} \cap L^\infty(\mathbb{R}^d), \quad \forall \xi \in \mathbb{R}^d, \quad \forall y \in \mathbb{R}^d, \quad |\tilde{a}(y, \xi)| \leq b(y) |\xi|^{p-1} \quad \text{and} \quad |\partial_\xi \tilde{a}(y, \xi)| \leq b(y) |\xi|^{p-2}.$$
- (4) There exists $\lambda > 0$ such that

$$\begin{aligned} \{a(y, \xi) - a(y, \xi')\} \cdot \{\xi - \xi'\} &\geq \lambda^{-1} (|\xi|^{p-2} + |\xi'|^{p-2}) |\xi - \xi'|^2, \\ \{a^{\text{per}}(y, \xi) - a^{\text{per}}(y, \xi')\} \cdot \{\xi - \xi'\} &\geq \lambda^{-1} (|\xi|^{p-2} + |\xi'|^{p-2}) |\xi - \xi'|^2, \end{aligned}$$

and

$$\begin{aligned} |a(y, \xi) - a(y, \xi')| &\leq \lambda (|\xi|^{p-2} + |\xi'|^{p-2}) |\xi - \xi'| \\ |a^{\text{per}}(y, \xi) - a^{\text{per}}(y, \xi')| &\leq \lambda (|\xi|^{p-2} + |\xi'|^{p-2}) |\xi - \xi'|. \end{aligned}$$

We also assume that

$$\sup_{y \in \mathbb{R}^d} \sup_{|\xi|=1} |\partial_\xi^2 a(y, \xi)| \leq \lambda. \quad (5.2.35)$$

We define the operator

$$A : \begin{cases} W_{\xi + \nabla w_{\xi}^{\text{per}}} \longrightarrow (W_{\xi + \nabla w_{\xi}^{\text{per}}})' \\ \nabla v \longmapsto \begin{cases} W_{\xi + \nabla w_{\xi}^{\text{per}}} \longrightarrow \mathbb{R} \\ \nabla h \longmapsto \int_{\mathbb{R}^d} [a(\cdot, \xi + \nabla w_{\xi}^{\text{per}} + \nabla v) - a(\cdot, \xi + \nabla w_{\xi}^{\text{per}})] \cdot \nabla h. \end{cases} \end{cases} \quad (5.2.36)$$

We can show that A is hemicontinuous, bounded, coercive and strongly monotone. By [76, Corollary 8.1], the PDE $A(\nabla v) = \mathcal{F}$, where $\mathcal{F} := \text{div} \tilde{a}(\cdot, \xi + \nabla w_{\xi}^{\text{per}})$, admits a unique solution in $W_{\xi + \nabla w_{\xi}^{\text{per}}}$. The results of Section 5.7, which are sufficient to prove the qualitative homogenization of Section 5.6 (which is in fact the main result of this paper), only use the PDE and are thus directly generalized. The results of Section 5.5 can be proved using the PDE instead of the minimization problem (5.4.24). These extensions are detailed in Appendix 5.8.3 of this chapter.

Remark 5.2.14. *A simple example of a non-variational operator satisfying the above assumptions is $a(y, \xi) = A(y)\xi |\xi|^{p-2}$, where A is a positive definite and bounded symmetric matrix that can be written under the form $A = A^{\text{per}} + \tilde{A}$ where $\tilde{A} \in L^{p'} \cap C^{0,1}(\mathbb{R}^d)^{d \times d}$. We check that a is not variational: assume by contradiction that there exists a function $F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $a(y, \xi) = \partial_{\xi} F(y, \xi)$. In particular, thanks to Schwartz Theorem, we should have that for all $i, j \in \{1, \dots, d\}$,*

$$\partial_{\xi_j} [a(y, \xi)_i] = \partial_{\xi_i} [a(y, \xi)_j].$$

Expanding each term gives, for $\xi \neq 0$,

$$A(i, j) |\xi|^{p-2} + (p-2) [A(y)\xi]_i \xi_j |\xi|^{p-4} = A(j, i) |\xi|^{p-2} + (p-2) [A(y)\xi]_j \xi_i |\xi|^{p-4}$$

In particular, for all $\xi \neq 0$ and $(i, j) \in \{1, \dots, d\}^2$,

$$[A(y)\xi]_i \xi_j = [A(y)\xi]_j \xi_i.$$

This shows that A is a scalar matrix i.e. proportional to the identity.

Remark 5.2.15. *Assumption (5.2.35) is only needed in the proofs of Theorem 5.2.4 and Theorem 5.2.5. Note also that, together with homogeneity, this Assumption implies that for all $\delta > 0$,*

$$\sup_{y \in \mathbb{R}^d} \sup_{|\xi| = \delta} |\partial_{\xi}^2 a(y, \xi)| \leq \lambda \delta^{p-3}.$$

5.3 The one-dimensional setting

We consider the homogenization of (5.1.1) in the one-dimensional case. This equation reads as:

$$\begin{cases} - (a(\cdot/\varepsilon) u_{\varepsilon}' |u_{\varepsilon}'|^{p-2})' = f \\ u_{\varepsilon}(-\frac{1}{2}) = u_{\varepsilon}(\frac{1}{2}) = 0, \end{cases} \quad (5.3.1)$$

where a is of the form $a = a^{\text{per}} + \tilde{a}$ with $\tilde{a} \in L^q \cap C^{0,\alpha}(\mathbb{R})$, $1 < q < +\infty$ and a satisfies Assumption **(A1)**. In this section, we assume that $f \in L^{p'}(-\frac{1}{2}, \frac{1}{2})$. Direct computations show that

$$u_{\varepsilon}' = \left(\frac{-F + C_{\varepsilon}}{a(\cdot/\varepsilon)} \right)^{1/(p-1)}, \quad F(x) = \int_{-\frac{1}{2}}^x f, \quad (5.3.2)$$

where $x^{\frac{1}{p-1}} := \text{sgn}(x) |x|^{\frac{1}{p-1}}$ for $x \in \mathbb{R}$. The constant C_{ε} is such that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{-F + C_{\varepsilon}}{a(\cdot/\varepsilon)} \right)^{1/(p-1)} = 0. \quad (5.3.3)$$

We note that the function F is bounded and thus the sequence $(C_\varepsilon)_{\varepsilon>0}$ is bounded. Passing to the limit $\varepsilon \rightarrow 0$ in (5.3.2) and (5.3.3), we get that $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^*$ in $W^{1,p}(-\frac{1}{2}, \frac{1}{2})$ and $C_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} C^*$, where

$$(u^*)' = \left(\frac{-F + C^*}{a^*} \right)^{1/(p-1)}, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} (-F + C^*)^{1/(p-1)} = 0.$$

and the homogenized coefficient is defined by

$$a^* := \left(L^p - \text{weaklim}_{\varepsilon \rightarrow 0} a\left(\frac{\cdot}{\varepsilon}\right)^{-\frac{1}{p-1}} \right)^{-(p-1)}.$$

We easily show with the ingredients used in Remark 5.3.2 below that

$$a^* = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(a^{\text{per}})^{\frac{1}{p-1}}} \right)^{-(p-1)}.$$

The homogenized equation solved by u^* is

$$\begin{cases} -(a^*(u^*)'| (u^*)'|^{p-2})' = f \\ u^*(-\frac{1}{2}) = u^*(\frac{1}{2}) = 0. \end{cases}$$

The corrector equations (5.2.9) and (5.2.12)-(5.2.13) in the direction $\xi \in \mathbb{R}$ are easy to solve (see Remark 5.3.2 below):

$$\xi + w'_\xi = \xi \left(\frac{a^*}{a} \right)^{\frac{1}{p-1}} \quad \text{and} \quad \xi + (w_\xi^{\text{per}})' = \xi \left(\frac{a^*}{a^{\text{per}}} \right)^{\frac{1}{p-1}}. \quad (5.3.4)$$

Let $R_\varepsilon := u_\varepsilon - (u^*)' - w_{(u^*)'}(\cdot/\varepsilon)$. be the remainder between u_ε and its two scale expansion. When u^* is regular enough, we have that

$$\begin{aligned} R'_\varepsilon &= (u_\varepsilon^1)' - (u^*)'(1 + w'(\cdot/\varepsilon)) - \varepsilon w(\cdot/\varepsilon)(u^*)'' \\ &= \underbrace{\frac{(-F + C_\varepsilon)^{1/(p-1)} - (-F + C^*)^{1/(p-1)}}{a(\cdot/\varepsilon)^{1/(p-1)}}}_{=:(u_\varepsilon^1)'} - \varepsilon w(\cdot/\varepsilon)(u^*)'', \end{aligned} \quad (5.3.5)$$

where $w := w_1$. We concentrate in the sequel on the first term of (5.3.5), the second one being related to the regularity of u^* on the one hand (which is not related to homogenization) and to the sublinearity of w on the other hand. We prove briefly that w is sublinear: indeed, we can write $w' = (w^{\text{per}})' + \tilde{w}'$ where, due to Remark 5.3.1 below, $\tilde{w}' \in L^q(\mathbb{R}^d)$. By Hölder (or Morrey) inequality, we get immediately that \tilde{w} is sublinear. Since w^{per} is periodic and bounded, it is in particular also sublinear. This proves that w is sublinear. We use Lemma 5.8.2 stated in Appendix 5.8.2 to obtain the bound

$$|w'_\varepsilon - (u^*)'(1 + w'(\cdot/\varepsilon))| \leq \lambda |C_\varepsilon - C^*|^{1/(p-1)} \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{uniformly.} \quad (5.3.6)$$

We have obtained the L^∞ -strong convergence of $(u_\varepsilon^1)'$ to zero when we use the non-periodic corrector. Let us now introduce the "periodic" remainder $R_\varepsilon^{\text{per}}$ which is defined by $R_\varepsilon^{\text{per}} := u_\varepsilon - u^* - \varepsilon w_{(u^*)'}^{\text{per}}(\cdot/\varepsilon)$. We have that

$$\begin{aligned} (R_\varepsilon^{\text{per}})' &= \frac{(-F + C_\varepsilon)^{1/(p-1)}}{a(\cdot/\varepsilon)^{1/(p-1)}} - \frac{(-F + C^*)^{1/(p-1)}}{a^{\text{per}}(\cdot/\varepsilon)^{1/(p-1)}} + \varepsilon w_1^{\text{per}}(\cdot/\varepsilon)(u^*)'' \\ &= (u_\varepsilon^1)' + (-F + C^*)^{1/(p-1)} \left[\frac{1}{a(\cdot/\varepsilon)^{1/(p-1)}} - \frac{1}{a^{\text{per}}(\cdot/\varepsilon)^{1/(p-1)}} \right] + \varepsilon w_1^{\text{per}}(\cdot/\varepsilon)(u^*)''. \end{aligned} \quad (5.3.7)$$

The first term tends uniformly to zero while the second one does not tend to zero in L^∞ unless $\tilde{a} = 0$ or $C^* = 0$. Indeed, testing (5.3.7) at the microscale gives:

$$\left| (-F + C^*)^{1/(p-1)} \left[\frac{1}{a(\cdot/\varepsilon)^{1/(p-1)}} - \frac{1}{a^{\text{per}}(\cdot/\varepsilon)^{1/(p-1)}} \right] (\varepsilon x) \right| \geq \underbrace{c(p, \lambda) | -F(\varepsilon x) + C^* |^{1/(p-1)} |\tilde{a}(x)|}_{\xrightarrow[\varepsilon \rightarrow 0]{c(p, \lambda) | C^* |^{1/(p-1)} |\tilde{a}(x)| \neq 0}}.$$

This shows that the convergence of the remainder deteriorates when using w_ξ^{per} instead of w_ξ . We close this section by commenting on the integrability of the correctors in the particular 1D setting. We show in Remark 5.3.1 that, in this case, the exponent given by Theorem 5.2.5 is optimal for $q = p'$, see also Remark 5.2.6.

Remark 5.3.1. *Suppose that $\tilde{a} \in L^q(\mathbb{R}^d) \cap C^{0,\alpha}(\mathbb{R}^d)$, $1 < q < +\infty$. An explicit calculation shows that*

$$\tilde{w}_\xi' = - \left(\xi + (w_\xi^{\text{per}})' \right) + \left(\xi + (w_\xi^{\text{per}})' \right) \left(1 - \frac{\tilde{a}}{a} \right)^{\frac{1}{p-1}}, \quad (5.3.8)$$

and $|\xi + (w_\xi^{\text{per}})'| \geq c|\xi|$. Since $\tilde{a}(x) \xrightarrow[|x| \rightarrow +\infty]{} 0$, we have that

$$\tilde{w}_\xi' \underset{x \rightarrow \pm\infty}{\sim} - \frac{1}{p-1} \frac{\tilde{a}(\xi + (w_\xi^{\text{per}})')}{a}.$$

Thus $\tilde{w}_\xi' \in L^q(\mathbb{R}^d)$, that is \tilde{w}_ξ' has the same integrability as \tilde{a} and this exponent is optimal.

Remark 5.3.2. *We show below that there exists a unique solution w_ξ to (5.2.9) that is sublinear at infinity. This justifies, in dimension one, to search w_ξ under the form $w_\xi^{\text{per}} + \tilde{w}_\xi$ where $\tilde{w}_\xi' \in L^p(\mathbb{R})$.*

Assume that w_ξ is a sublinear solution to (5.3.4). Then, there exists a constant C such that $\xi + w_\xi' = (C/a)^{1/(p-1)}$. We have by sublinearity that

$$\xi = \lim_{x \rightarrow +\infty} \int_0^x (\xi + w_\xi') = \lim_{x \rightarrow +\infty} \int_0^x \left(\frac{C}{a} \right)^{\frac{1}{p-1}} = C^{\frac{1}{p-1}} \lim_{x \rightarrow +\infty} \int_0^x \left(\frac{1}{a} \right)^{\frac{1}{p-1}}.$$

However, by Lemma 5.8.2, we have that

$$\left| \int_0^x \left(\frac{1}{a} \right)^{\frac{1}{p-1}} - \int_0^x \left(\frac{1}{a^{\text{per}}} \right)^{\frac{1}{p-1}} \right| \leq \int_0^x \frac{|a^{\frac{1}{p-1}} - (a^{\text{per}})^{\frac{1}{p-1}}|}{a^{\frac{1}{p-1}} (a^{\text{per}})^{\frac{1}{p-1}}} \leq \text{Cst.} \int_0^x |\tilde{a}|^{\frac{1}{p-1}},$$

where Cst. denotes a constant depending only on p and λ . Since $\tilde{a} \in L^{p'}(\mathbb{R}^d)$, we get by Hölder inequality that

$$\int_0^x |\tilde{a}|^{\frac{1}{p-1}} \xrightarrow{x \rightarrow +\infty} 0.$$

This shows that

$$\lim_{x \rightarrow +\infty} \int_0^x \left(\frac{1}{a} \right)^{\frac{1}{p-1}} = \lim_{x \rightarrow +\infty} \int_0^x \left(\frac{1}{a^{\text{per}}} \right)^{\frac{1}{p-1}} = \left(\frac{1}{a^*} \right)^{\frac{1}{p-1}}$$

and gives that $C = \xi |\xi|^{p-2} a^*$. This shows that \tilde{w}_ξ is necessarily of the form (5.3.4).

Numerical experiments. We have implemented for $p = 3$ the solution to (5.1.1) in the 1D setting for $f(x) = 2x$ and

$$a(y) := a^{\text{per}}(y) + \tilde{a}(y) = 2 + \cos(2\pi y) + 10e^{-|y|}$$

on the domain $\Omega := (-\frac{1}{2}, \frac{1}{2})$. The boundary conditions are homogeneous Dirichlet conditions *i.e.* $u_\varepsilon(-\frac{1}{2}) = u_\varepsilon(\frac{1}{2}) = 0$. The coefficient a satisfies of course Assumptions **(A1)**-**(A3)**. The results are plotted on Figure 5.1. We comment on these results. We have plotted for different values of ε the

function u'_ε (which is labeled as 'exact solution'), the periodic two scale approximation $(u^*)' + (w^{\text{per}})'(\cdot/\varepsilon)(u^*)'$ (which is labeled as 'periodic two-scale approx.>') and the non-periodic two scale approximation $(u^*)' + w'(\cdot/\varepsilon)(u^*)'$ (which is labeled as 'non-periodic two-scale approx.>'). Tables 5.1 and 5.2 give numerical values for the periodic and non-periodic remainders in L^2 and L^∞ -norm for different values of ε . We see that on Figure 5.1, qualitatively, the non-periodic two-scale approximation fits efficiently the exact solution for each chosen value of ε . The periodic two-scale approximation corresponds to the exact solution far from the defect, which, as $\varepsilon \rightarrow 0$, concentrates around the origin. We notice that the non-periodic corrector is useful to reconstruct the oscillations of the exact solution locally around the defect. Tables 5.1 and 5.2 express the same idea: the L^∞ -norms of the periodic remainders remain unchanged as ε decreases whereas those of the non-periodic remainder decrease with ε . For the L^2 -norm, which is weaker than the L^∞ -norm, both norms decrease as ε gets closer to zero although the nonperiodic approximation is more accurate than the periodic approximation. This means that, depending on the precision we want (and also on the regularity on f and a), we may use the periodic corrector, which is much easier to compute, or the non-periodic corrector, if we seek for a fine approximation of the exact solution. This can also be seen theoretically since $R_\varepsilon^{\text{per}} - R_\varepsilon = \varepsilon \widetilde{w}'(\cdot/\varepsilon)(u^*)'$ and, for all $q \leq p$,

$$\|\varepsilon \widetilde{w}'(\cdot/\varepsilon)(u^*)'\|_{L^q(0,1)} \leq C \varepsilon^{d/p} \|(u^*)'\|_{L^\infty(0,1)} \|\widetilde{w}'\|_{L^p(\mathbb{R})}.$$

In any case, we get that $R_\varepsilon^{\text{per}} - R_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$ in L^q -norm, $q \leq p$ but not in L^∞ -norm. Another way to reformulate the preceding remark is the following: the non-periodic corrector provides a better approximation at the microscale.

ε	$\ R_\varepsilon^{\text{per}}\ _{L^\infty}$	$\ R_\varepsilon\ _{L^\infty}$
0.1	0.156	0.109
0.05	0.163	0.137
0.01	0.170	0.0657
0.005	0.170	0.0288
0.001	0.170	0.0245
0.0005	0.171	0.0136

Table 5.1: Numerical errors for different values of ε in L^∞ -norm.

ε	$\ R_\varepsilon^{\text{per}}\ _{L^2}$	$\ R_\varepsilon\ _{L^2}$
0.1	6.39	3.85
0.05	5.01	3.16
0.01	2.13	0.740
0.005	1.47	0.331
0.001	0.654	0.108
0.0005	0.46	0.0461

Table 5.2: Numerical errors for different values of ε in L^2 -norm.

5.4 Existence of the non-periodic correctors: proof of Theorem 5.2.3

We start this section with some preliminary results:

Lemma 5.4.1. *Let $\xi \in \mathbb{R}^d$ and $W_{\xi + \nabla w_\xi^{\text{per}}}$ be defined by (5.2.10).*

(i) *The space $W_{\xi + \nabla w_\xi^{\text{per}}}$ is a Banach space.*

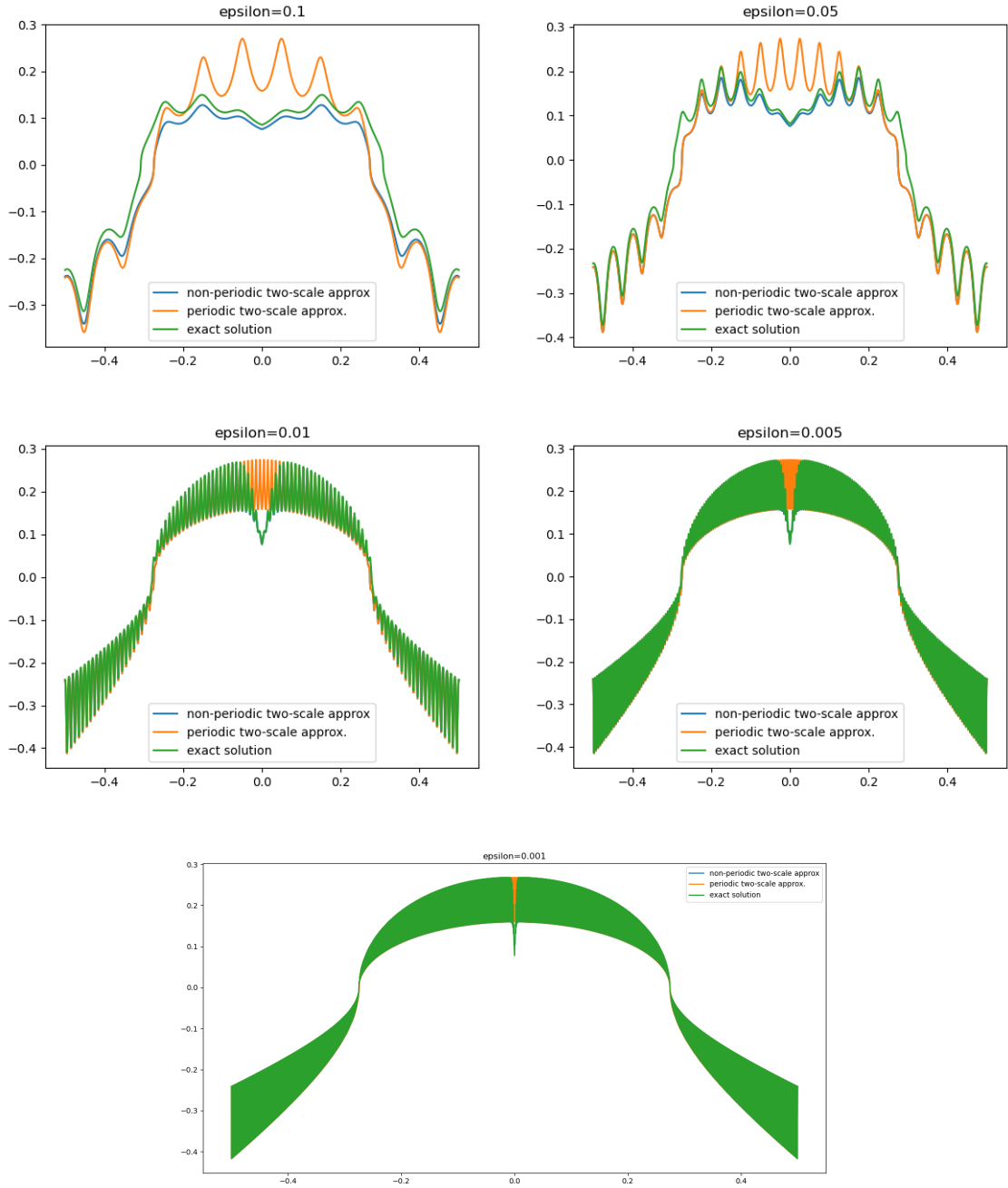


Figure 5.1: Numerical simulation in the particular 1D case.

(ii) Its topological dual space is

$$\left\{ -\operatorname{div}(g), \quad g = g_1 + g_2|\xi + \nabla w_\xi^{\text{per}}|^{p-2}, \quad g_1 \in L^{p'}(\mathbb{R}^d), \quad g_2|\xi + \nabla w_\xi^{\text{per}}|^{\frac{p-2}{2}} \in L^2(\mathbb{R}^d) \right\}.$$

(iii) Each bounded sequence in $W_{\xi + \nabla w_\xi^{\text{per}}}$ admits a weakly converging subsequence.

Proof of Lemma 5.4.1. We first check that $W_{\xi + \nabla w_\xi^{\text{per}}}$ is a Banach space. Let $(v_n)_{n \in \mathbb{N}}$ be a Cauchy

sequence in $W_{\xi+\nabla w_\xi^{\text{per}}}$. In particular we have that $(\nabla v_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R}^d)$ and that $(|\xi + \nabla w_\xi^{\text{per}}|^{\frac{p-2}{2}} \nabla v_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^d)$. Thus, there exist $\nabla v \in L^p(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d)$ such that

$$\nabla v_n \xrightarrow{n \rightarrow +\infty} \nabla v \text{ in } L^p(\mathbb{R}^d) \quad \text{and} \quad |\xi + \nabla w_\xi^{\text{per}}|^{\frac{p-2}{2}} \nabla v_n \xrightarrow{n \rightarrow +\infty} g \text{ in } L^2(\mathbb{R}^d). \quad (5.4.1)$$

Up to a subsequence, we have the a.e. convergences in (5.4.1). Thus $g = |\xi + \nabla w_\xi^{\text{per}}|^{p-2} \nabla v$ a.e. This gives that $v_n \xrightarrow{n \rightarrow +\infty} v$ in $W_{\xi+\nabla w_\xi^{\text{per}}}$. This proves (i).

We denote $d\lambda$ the Lebesgue measure over \mathbb{R}^d , $U := L^p(\mathbb{R}^d, d\lambda)^d$ and $V := L^2(\mathbb{R}^d, |\xi + \nabla w_\xi^{\text{per}}|^{p-2} d\lambda)^d$. It is clear that each function $-\text{div}(g)$ where $g = g_1 + g_2 |\xi + \nabla w_\xi^{\text{per}}|^{p-2}$, $(g_1, g_2) \in U' \times V'$ defines an element of $(W_{\xi+\nabla w_\xi^{\text{per}}})'$ by the following duality bracket:

$$\langle -\text{div}(g), v \rangle = \int_{\mathbb{R}^d} g_1 \cdot \nabla v + \int_{\mathbb{R}^d} g_2 \cdot \nabla v |\xi + \nabla w_\xi^{\text{per}}|^{p-2}, \quad v \in W_{\xi+\nabla w_\xi^{\text{per}}}.$$

Conversely, let $T \in (W_{\xi+\nabla w_\xi^{\text{per}}})'$. We consider the space $U \times V$ endowed with the norm $\|(u, v)\|_{U \times V} := \max(\|u\|_U, \|v\|_V)$. We introduce $\mathcal{V} := \{\nabla v, v \in W_{\xi+\nabla w_\xi^{\text{per}}}\}$ and the following linear form $T'(\nabla v, \nabla v) := T(v)$ defined over $D := \mathcal{V} \times \mathcal{V} \subset U \times V$. By the Hahn-Banach Theorem, there exists $\widetilde{T}' \in (U \times V)'$ such that $\widetilde{T}' = T'$ on D . We have the decomposition

$$\forall (u, v) \in U \times V, \quad \widetilde{T}'(u, v) = \widetilde{T}'(u, 0) + \widetilde{T}'(0, v) =: T_U(u) + T_V(v),$$

with $T_U \in U'$ and $T_V \in V'$. By the characterization of U' and V' , there exist $g_1 \in L^{p'}(\mathbb{R}^d)^d$ and $g_2 \in L^2(|\xi + \nabla w_\xi^{\text{per}}|^{p-2} d\lambda)^d$ such that

$$\forall (u, v) \in U \times V, \quad \widetilde{T}'(u, v) = \int_{\mathbb{R}^d} g_1 \cdot u + \int_{\mathbb{R}^d} g_2 \cdot v |\xi + \nabla w_\xi^{\text{per}}|^{p-2} d\lambda.$$

This proves (ii) since $T(v) = \widetilde{T}'(\nabla v, \nabla v)$ for $v \in W_{\xi+\nabla w_\xi^{\text{per}}}$.

Let $(v_n)_{n \in \mathbb{N}} \subset W_{\xi+\nabla w_\xi^{\text{per}}}$ be a bounded sequence. Then $(\nabla v_n)_{n \in \mathbb{N}}$ is bounded in $L^p(\mathbb{R}^d)$ -norm and $(|\xi + \nabla w_\xi^{\text{per}}|^{\frac{p-2}{2}} \nabla v_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\mathbb{R}^d)$ -norm. Thus, after extraction, we have that ∇v_n converges weakly in $L^p(\mathbb{R}^d)$ -norm to some $\nabla v \in L^p(\mathbb{R}^d)$ and that $|\xi + \nabla w_\xi^{\text{per}}|^{\frac{p-2}{2}} \nabla v_n$ converges weakly to some g in $L^2(\mathbb{R}^d)$ -norm. To identify g and $|\xi + \nabla w_\xi^{\text{per}}|^{\frac{p-2}{2}} \nabla v$ we note that for fixed $R > 0$, ∇v_n converges weakly to ∇v in $L^2(B_R)$ and thus $|\xi + \nabla w_\xi^{\text{per}}|^{\frac{p-2}{2}} \nabla v_n$ converges weakly to $|\xi + \nabla w_\xi^{\text{per}}|^{\frac{p-2}{2}} \nabla v$ in $L^2(B_R)$ since $|\xi + \nabla w_\xi^{\text{per}}| \in L^\infty(\mathbb{R}^d)$. By uniqueness of the weak limit, we conclude that $g = |\xi + \nabla w_\xi^{\text{per}}|^{\frac{p-2}{2}} \nabla v$ a.e. (in the Lebesgue measure sense) in B_R . Since R is arbitrary, we have that $g = |\xi + \nabla w_\xi^{\text{per}}|^{\frac{p-2}{2}} \nabla v$ a.e., in particular $v \in W_{\xi+\nabla w_\xi^{\text{per}}}$. It remains to show that v_n converges weakly to v in $W_{\xi+\nabla w_\xi^{\text{per}}}$ when $n \rightarrow +\infty$. But this is straightforward with the characterization (ii) and the weak convergences obtained above. \square

We now fix $\xi \in \mathbb{R}^d$, $h \in L^{p'}(\mathbb{R}^d)^d$, a coefficient a satisfying Assumptions **(A1)**-**(A2)**-**(A3)**. We introduce the functional F_ξ defined by

$$F_\xi(v) := \frac{1}{p} \int_{\mathbb{R}^d} a g_{\xi+\nabla w_\xi^{\text{per}}}(\nabla v) + \int_{\mathbb{R}^d} h \cdot \nabla v, \quad (5.4.2)$$

where the function g_ξ is defined by (5.8.8):

$$g_\xi(x) := |\xi + x|^p - |\xi|^p - p\xi|\xi|^{p-2} \cdot x.$$

Since $g_\xi(x) \geq 0$ over \mathbb{R}^d , we immediately have that F_ξ is defined over

$$V := \left\{ v \in W_{\text{loc}}^{1,1}(\mathbb{R}^d), \quad \nabla v \in L^p(\mathbb{R}^d) \right\} / \mathbb{R} \quad (5.4.3)$$

and takes its values in $\mathbb{R} \cup \{+\infty\}$. Note that since $F_\xi(v)$ only depends on ∇v , F_ξ is well-defined on the space of equivalence classes V . For $R > 0$, we define the mapping

$$F_\xi^R : \begin{cases} V \longrightarrow \mathbb{R} \\ v \longmapsto \frac{1}{p} \int_{B_R} a g_{\xi + \nabla w_\xi^{\text{per}}}(\nabla v) + \int_{\mathbb{R}^d} h \cdot \nabla v. \end{cases} \quad (5.4.4)$$

We gather in Lemmas 5.4.2 and 5.4.3 below the key properties satisfied by the functional F_ξ .

Lemma 5.4.2. *Let $\xi \in \mathbb{R}^d$, $h \in L^{p'}(\mathbb{R}^d)^d$, F_ξ be defined by (5.4.2) over V and the space $W_{\xi + \nabla w_\xi^{\text{per}}}$ be defined by (5.2.10).*

(i) *There exist two constants $c, C > 0$ such that for all $v \in W_{\xi + \nabla w_\xi^{\text{per}}}$,*

$$c \left[-1 + \|v\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}^2 \right] \leq F_\xi(v) \leq C \left[1 + \|v\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}^p \right]. \quad (5.4.5)$$

In particular, $F_\xi(v)$ is finite if and only if $v \in W_{\xi + \nabla w_\xi^{\text{per}}}$.

(ii) *The function F_ξ is convex over V and strictly convex over $W_{\xi + \nabla w_\xi^{\text{per}}}$.*

Proof of Lemma 5.4.2. The point (i) is a simple application of (5.8.9). Let $v \in W_{\xi + \nabla w_\xi^{\text{per}}}$, we have thanks to (5.8.9) together with Hölder inequality that

$$\begin{aligned} -\|h\|_{L^{p'}(\mathbb{R}^d)} \|\nabla v\|_{L^p(\mathbb{R}^d)} + c\lambda^{-1} \int_{\mathbb{R}^d} |\nabla v|^p + |\xi + \nabla w_\xi^{\text{per}}|^{p-2} |\nabla v|^2 &\leq F_\xi(v) \\ &\leq \|h\|_{L^{p'}(\mathbb{R}^d)} \|\nabla v\|_{L^p(\mathbb{R}^d)} + C\lambda \int_{\mathbb{R}^d} |\nabla v|^p + |\xi + \nabla w_\xi^{\text{per}}|^{p-2} |\nabla v|^2. \end{aligned} \quad (5.4.6)$$

We recall the following Young inequality

$$\|h\|_{L^{p'}(\mathbb{R}^d)} \|\nabla v\|_{L^p(\mathbb{R}^d)} \leq \frac{1}{p'} \|h\|_{L^{p'}(\mathbb{R}^d)}^{p'} + \frac{1}{p} \|\nabla v\|_{L^p(\mathbb{R}^d)}^p \quad (5.4.7)$$

and we notice that, since $x^2 \leq x^p + 1$ for $x \in \mathbb{R}^+$,

$$\|\nabla v\|_{L^p}^p \leq \|v\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}^p \quad \text{and} \quad \left\| |\xi + \nabla w_\xi^{\text{per}}|^{\frac{p-2}{2}} \nabla v \right\|_{L^2}^2 \leq \|v\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}^2 \leq 1 + \|v\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}^p. \quad (5.4.8)$$

Applying (5.4.7) and (5.4.8), we get that

$$F_\xi(v) \leq \frac{1}{p'} \|h\|_{L^{p'}(\mathbb{R}^d)}^{p'} + (2C\lambda + 1) \|v\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}^p.$$

This proves the right-most inequality of (5.4.5) after changing the constant C . For the left-most inequality, we write that, again by Young inequality

$$-\frac{\lambda}{2c} \|h\|_{L^{p'}(\mathbb{R}^d)}^{p'} + \frac{c\lambda^{-1}}{2} \int_{\mathbb{R}^d} |\nabla v|^p + c\lambda^{-1} \int_{\mathbb{R}^d} |\xi + \nabla w_\xi^{\text{per}}|^{p-2} |\nabla v|^2 \leq F_\xi(v).$$

Besides, noticing that

$$\left(\int_{\mathbb{R}^d} |\nabla v|^p \right)^{2/p} \leq \int_{\mathbb{R}^d} |\nabla v|^p + 1,$$

we get the lower bound

$$\frac{1}{2} \|v\|_{W_{\xi+\nabla w_{\xi}^{\text{per}}}}^2 - 1 \leq \int_{\mathbb{R}^d} |\nabla v|^p + \int_{\mathbb{R}^d} |\xi + \nabla w_{\xi}^{\text{per}}|^{p-2} |\nabla v|^2.$$

Thus,

$$- \left(\frac{\lambda}{2c} \|h\|_{L^{p'}(\mathbb{R}^d)}^{p'} + \frac{c\lambda^{-1}}{2} \right) + \frac{c\lambda^{-1}}{4} \|v\|_{W_{\xi+\nabla w_{\xi}^{\text{per}}}}^2 \leq F_{\xi}(v).$$

After changing the constant c , we get (5.4.5). This proves (i).

We prove (ii). Since the function $z \mapsto |z|^p$ is strictly convex, it is clear that the application $x \mapsto |\xi + x|^p$ is strictly convex. Thus, $g_{\xi+\nabla w_{\xi}^{\text{per}}}$ (see (5.8.8) for the definition) is strictly convex. This implies that for all $R > 0$,

$$F_{\xi}^R((1-t)v_1 + tv_2) \leq (1-t)F_{\xi}^R(v_1) + tF_{\xi}^R(v_2) \leq (1-t)F_{\xi}(v_1) + tF_{\xi}(v_2). \quad (5.4.9)$$

Letting $R \rightarrow +\infty$ in (5.4.9) gives by the monotone convergence Theorem that

$$F_{\xi}((1-t)v_1 + tv_2) \leq (1-t)F_{\xi}(v_1) + tF_{\xi}(v_2) \quad (5.4.10)$$

hence F_{ξ} is convex. As for the strict convexity, suppose that $F_{\xi}(v_i) < +\infty$, $i = 1, 2$ and that equality holds in (5.4.10) for some $t \in (0, 1)$. Then as all functions are integrable,

$$\int_{\mathbb{R}^d} \left\{ (1-t)g_{\xi+\nabla w_{\xi}^{\text{per}}}(\nabla v_1) + tg_{\xi+\nabla w_{\xi}^{\text{per}}}(\nabla v_2) - g_{\xi+\nabla w_{\xi}^{\text{per}}}((1-t)\nabla v_1 + tv_2) \right\} = 0.$$

By positivity, we get that $(1-t)g_{\xi+\nabla w_{\xi}^{\text{per}}}(\nabla v_1) + tg_{\xi+\nabla w_{\xi}^{\text{per}}}(\nabla v_2) - g_{\xi+\nabla w_{\xi}^{\text{per}}}((1-t)\nabla v_1 + tv_2) = 0$ a.e. The strict convexity of $g_{\xi+\nabla w_{\xi}^{\text{per}}}$ implies that $\nabla v_1 = \nabla v_2$ a.e. This concludes the proof. \square

Lemma 5.4.3. *Let $\xi \in \mathbb{R}^d$, $h \in L^{p'}(\mathbb{R}^d)^d$, F_{ξ} be defined by (5.4.2) over V and the space $W_{\xi+\nabla w_{\xi}^{\text{per}}}$ be defined by (5.2.10). Then the application F_{ξ} is Fréchet-differentiable over $W_{\xi+\nabla w_{\xi}^{\text{per}}}$. Its differential is given, for $v \in W_{\xi+\nabla w_{\xi}^{\text{per}}}$, by*

$$F'_{\xi}(v) \cdot u := \int_{\mathbb{R}^d} \left\{ a \left[(\xi + \nabla w_{\xi}^{\text{per}} + \nabla v) |\xi + \nabla w_{\xi}^{\text{per}} + \nabla v|^{p-2} - (\xi + \nabla w_{\xi}^{\text{per}}) |\xi + \nabla w_{\xi}^{\text{per}}|^{p-2} \right] + h \right\} \cdot \nabla u. \quad (5.4.11)$$

Proof of Lemma 5.4.3. We fix $v \in W_{\xi+\nabla w_{\xi}^{\text{per}}}$ and $u \in W_{\xi+\nabla w_{\xi}^{\text{per}}}$. We have that

$$F_{\xi}(v+u) - F_{\xi}(v) = \frac{1}{p} \int_{\mathbb{R}^d} a \left[g_{\xi+\nabla w_{\xi}^{\text{per}}}(\nabla v + \nabla u) - g_{\xi+\nabla w_{\xi}^{\text{per}}}(\nabla v) \right] + \int_{\mathbb{R}^d} h \cdot \nabla u. \quad (5.4.12)$$

We note that

$$\begin{aligned} & g_{\xi+\nabla w_{\xi}^{\text{per}}}(\nabla v + \nabla u) - g_{\xi+\nabla w_{\xi}^{\text{per}}}(\nabla v) \\ &= |\xi + \nabla w_{\xi}^{\text{per}} + \nabla v + \nabla u|^p - |\xi + \nabla w_{\xi}^{\text{per}} + \nabla v|^p - p(\xi + \nabla w_{\xi}^{\text{per}} + \nabla v) |\xi + \nabla w_{\xi}^{\text{per}} + \nabla v|^{p-2} \cdot \nabla u \\ & \quad + p \left((\xi + \nabla w_{\xi}^{\text{per}} + \nabla v) |\xi + \nabla w_{\xi}^{\text{per}} + \nabla v|^{p-2} - (\xi + \nabla w_{\xi}^{\text{per}}) |\xi + \nabla w_{\xi}^{\text{per}}|^{p-2} \right) \cdot \nabla u \\ &= A + B, \end{aligned} \quad (5.4.13)$$

where

$$A := |\xi + \nabla w_{\xi}^{\text{per}} + \nabla v + \nabla u|^p - |\xi + \nabla w_{\xi}^{\text{per}} + \nabla v|^p - p(\xi + \nabla w_{\xi}^{\text{per}} + \nabla v) |\xi + \nabla w_{\xi}^{\text{per}} + \nabla v|^{p-2} \cdot \nabla u \quad (5.4.14)$$

and

$$B := p \left((\xi + \nabla w_{\xi}^{\text{per}} + \nabla v) |\xi + \nabla w_{\xi}^{\text{per}} + \nabla v|^{p-2} - (\xi + \nabla w_{\xi}^{\text{per}}) |\xi + \nabla w_{\xi}^{\text{per}}|^{p-2} \right) \cdot \nabla u. \quad (5.4.15)$$

We note that, using the definition of g_ξ (5.8.8),

$$A = g_{\xi + \nabla w_\xi^{\text{per}} + \nabla v}(\nabla u) \leq C \left\{ |\nabla u|^p + |\xi + \nabla w_\xi^{\text{per}} + \nabla v|^{p-2} |\nabla u|^2 \right\},$$

where we have used the right-most part of inequality (5.8.9). Thus, applying the inequality $(b_1 + b_2)^{p-2} \leq C(p)(b_1^{p-2} + b_2^{p-2})$ for $b_1, b_2 \geq 0$, we get that

$$|A| \leq C \left\{ |\nabla u|^p + |\xi + \nabla w_\xi^{\text{per}}|^{p-2} |\nabla u|^2 + |\nabla v|^{p-2} |\nabla u|^2 \right\}. \quad (5.4.16)$$

We now note that, due to Hölder inequality and the fact that

$$\left(\frac{p}{2}\right)' \left(\frac{p}{2} - 1\right) = \frac{p}{2} \implies \left(\frac{p}{2}\right)' (p-2) = p,$$

we obtain

$$\int_{\mathbb{R}^d} |\nabla v|^{p-2} |\nabla u|^2 \leq \left(\int_{\mathbb{R}^d} |\nabla v|^p \right)^{1-2/p} \left(\int_{\mathbb{R}^d} |\nabla u|^p \right)^{2/p}. \quad (5.4.17)$$

Gathering (5.4.16), (5.4.17) and recalling the definition (5.2.11), we have proved that $A \in L^1(\mathbb{R}^d)$ and that

$$\int_{\mathbb{R}^d} |A| \leq C \left[\|u\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}^2 + \|u\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}^p \right], \quad (5.4.18)$$

where the constant C does not depend on u .

We now turn to estimating B , see (5.4.15). Using (5.8.7) and Cauchy-Schwarz inequality, we have that

$$\begin{aligned} |B| &\leq C \left[|\xi + \nabla w_\xi^{\text{per}} + \nabla v|^{p-2} + |\xi + \nabla w_\xi^{\text{per}}|^{p-2} \right] |\nabla v| |\nabla u| \\ &\leq C \left[|\xi + \nabla w_\xi^{\text{per}}|^{p-2} |\nabla v|^2 + |\xi + \nabla w_\xi^{\text{per}}|^{p-2} |\nabla u|^2 + |\nabla v|^{p-1} |\nabla u| \right] \\ &\leq C \left[|\xi + \nabla w_\xi^{\text{per}}|^{p-2} |\nabla v|^2 + |\xi + \nabla w_\xi^{\text{per}}|^{p-2} |\nabla u|^2 + |\nabla v|^p + |\nabla u|^p \right] \end{aligned} \quad (5.4.19)$$

where we used Young's inequality in form of $b_1 \leq b_1^p/p + b_2^{p'}/p'$ in the last inequality. This proves that $B \in L^1(\mathbb{R}^d)$ and that

$$\int_{\mathbb{R}^d} |B| \leq C \left[\|v\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}^2 + \|u\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}^2 + \|v\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}^p + \|u\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}^p \right], \quad (5.4.20)$$

where the constant C is independent of v and u . We can now conclude the proof of Lemma 5.4.3: using (5.4.12) and the notations (5.4.14) and (5.4.15), we have that

$$F_\xi(v+u) - F_\xi(v) - \left\{ \frac{1}{p} \int_{\mathbb{R}^d} B + \int_{\mathbb{R}^d} h \cdot \nabla u \right\} = \frac{1}{p} \int_{\mathbb{R}^d} A. \quad (5.4.21)$$

Defining

$$\begin{aligned} L_v(u) &:= \frac{1}{p} \int_{\mathbb{R}^d} B + \int_{\mathbb{R}^d} h \cdot \nabla u \\ &= \int_{\mathbb{R}^d} a \left((\xi + \nabla w_\xi^{\text{per}} + \nabla v) |\xi + \nabla w_\xi^{\text{per}} + \nabla v|^{p-2} - (\xi + \nabla w_\xi^{\text{per}}) |\xi + \nabla w_\xi^{\text{per}}|^{p-2} \right) \cdot \nabla u + \int_{\mathbb{R}^d} h \cdot \nabla u \end{aligned}$$

and noting that, thanks to (5.4.20), L_v is a bounded linear form on $W_{\xi + \nabla w_\xi^{\text{per}}}$, we have, gathering (5.4.21) and (5.4.18) together,

$$F_\xi(v+u) - F_\xi(v) - L_v(u) = O_{u \rightarrow 0} \left(\|u\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}^2 \right).$$

Lemma 5.4.3 is proved. \square

Proof of Theorem 5.2.3. We prove below that, for $h \in L^{p'}(\mathbb{R}^d)^d$, the PDE

$$-\operatorname{div} \left[|\xi + \nabla w_\xi^{\text{per}} + \nabla w_\xi|^{p-2} (\xi + \nabla w_\xi^{\text{per}} + \nabla \widetilde{w}_\xi) - |\xi + \nabla w_\xi^{\text{per}}|^{p-2} (\xi + \nabla w_\xi^{\text{per}}) \right] = \operatorname{div}(h), \quad (5.4.22)$$

admits a unique solution $\widetilde{w}_\xi \in W_{\xi + \nabla w_\xi^{\text{per}}}$ in the weak sense (see Definition 5.2.2). Theorem 5.2.3 is then proved by defining

$$h := \widetilde{a}(\xi + \nabla w_\xi^{\text{per}}) |\xi + \nabla w_\xi^{\text{per}}|^{p-2}. \quad (5.4.23)$$

Because of Proposition 5.2.1 (ii) and Assumptions **(A2)**-**(A3)**, it is clear that $h \in L^{p'}(\mathbb{R}^d)^d$. Since (5.4.22) is solvable for this choice of h , Theorem 5.2.3 is proved.

We are thus left to study the PDE (5.4.22) for an abstract right-hand side $h \in L^{p'}(\mathbb{R}^d)^d$. With Lemma 5.4.2, Lemma 5.4.3 and Lemma 5.4.1, we prove in a standard way that Problem (5.4.22) admits a unique solution. Indeed, let us consider the minimization Problem:

$$\min_{v \in W_{\xi + \nabla w_\xi^{\text{per}}}} F_\xi(v). \quad (5.4.24)$$

This Problem admits a unique solution. The existence is guaranteed by the following procedure: let $(v_n)_{n \in \mathbb{N}} \subset W_{\xi + \nabla w_\xi^{\text{per}}}$ be a minimizing sequence. Then, by the left-hand estimate of (5.4.5), we have

that the sequence $\left(\|v_n\|_{W_{\xi + \nabla w_\xi^{\text{per}}}} \right)_{n \in \mathbb{N}}$ is bounded (see (5.2.11) for the definition of $\|\cdot\|_{W_{\xi + \nabla w_\xi^{\text{per}}}}$).

By Lemma 5.4.1 (iii), we get that the sequence $(v_n)_{n \in \mathbb{N}}$ weakly converges, up to a subsequence, to some v in $W_{\xi + \nabla w_\xi^{\text{per}}}$ when $n \rightarrow +\infty$. Since by Lemma 5.4.2 (ii) and Lemma 5.4.3, F_ξ is convex and continuous over $W_{\xi + \nabla w_\xi^{\text{per}}}$, it is in particular weakly lower semi-continuous. Thus

$$F_\xi(v) \leq \liminf_{n \rightarrow +\infty} F_\xi(v_n) = \inf_{W_{\xi + \nabla w_\xi^{\text{per}}}} F_\xi.$$

This concludes the existence of a solution to (5.4.24). The uniqueness is given by the strict convexity of F_ξ , see Lemma 5.4.2 (ii). We finally note that the convexity of F_ξ together with its differentiability ensure that being a solution to Problem (5.4.24) is equivalent to solve the PDE (5.4.22), since (5.4.11) is exactly the weak form of (5.4.22) in the sense of Definition 5.2.2. Theorem 5.2.3 is proved. \square

Remark 5.4.4 (Regularity of F_ξ over V). *We have shown in Lemma 5.4.3 that F_ξ is regular over $W_{\xi + \nabla w_\xi^{\text{per}}}$. What happens if we enlarge this space, for example by studying F_ξ over V ?*

We can prove that F_ξ is weakly lower semi-continuous on V . Indeed, let us consider a sequence $(\nabla v_n)_{n \in \mathbb{N}}$ that weakly converges to some ∇v in $L^p(\mathbb{R}^d)$. We aim at proving that

$$F_\xi(v) \leq \liminf_{n \rightarrow +\infty} F_\xi(v_n). \quad (5.4.25)$$

We recall the notation (5.4.4). It is straightforward to check that F_R is convex and strongly continuous. Thus, F_R is weakly lower semi-continuous. However, for fixed $n \in \mathbb{N}$, we have $F_\xi^R(v_n) \leq F_\xi(v_n)$. Thus,

$$F_\xi^R(v) \leq \liminf_{n \rightarrow +\infty} F_\xi^R(v_n) \leq \liminf_{n \rightarrow +\infty} F_\xi(v_n). \quad (5.4.26)$$

We note that, since $g_z \geq 0$ for all $z \in \mathbb{R}^d$, we get, because of the monotone convergence Theorem, that

$$\int_{B_R} a g_{\xi + \nabla w_\xi^{\text{per}}}(\nabla v) \xrightarrow{R \rightarrow +\infty} \int_{\mathbb{R}^d} a g_{\xi + \nabla w_\xi^{\text{per}}}(\nabla v). \quad (5.4.27)$$

Letting $R \rightarrow +\infty$ in (5.4.26) and using (5.4.27) gives (5.4.25). However, the functional F_ξ is not continuous over V . To see that, let us denote by B a ball contained in the unit ball such that

$|\xi + \nabla w_\xi^{\text{per}}| > \eta > 0$ on B (such a ball exists otherwise, by Lemma 5.2.1 (ii), $\nabla w_\xi^{\text{per}} = -\xi$ on Q which contradicts the periodicity of $\nabla w_\xi^{\text{per}}$). Let $\phi \in \mathcal{C}_c^\infty(B)$, we define

$$\Psi := \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|k|^{d/2}} \phi(\cdot + k), \quad \|\nabla \Psi\|_{L^p(\mathbb{R}^d)}^p = \|\nabla \phi\|_{L^p(B)}^p \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|k|^{dp/2}} < +\infty.$$

However, because of (5.8.9), we have that

$$F_\xi(\Psi) \geq c \left\{ \|\nabla \Psi\|_{L^p(\mathbb{R}^d)}^p + \eta^{p-2} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|k|^d} \|\nabla \phi\|_{L^2(B)}^2 \right\} - \|h\|_{L^{p'}(\mathbb{R}^d)} \|\nabla \Psi\|_{L^p(\mathbb{R}^d)} = +\infty.$$

Thus F_ξ is not finite over V and, consequently, cannot be continuous.

5.5 Properties of the non-periodic correctors: proof of Theorem 5.2.4

5.5.1 A useful Lemma

We begin by introducing the following function: for all $\xi, \eta \in \mathbb{R}^d$, the function $G_{\xi, \eta}$ is defined over $\mathbb{R}^d \times \mathbb{R}^d$ by

$$G_{\xi, \eta}(X, Y) := |\xi + X|^p + |\eta + Y|^p - \left| \xi + \frac{X+Y}{2} \right|^p - \left| \eta + \frac{X+Y}{2} \right|^p - \frac{p}{2} (\xi|\xi|^{p-2} - \eta|\eta|^{p-2}) \cdot (X - Y). \quad (5.5.1)$$

The following Lemma gives a lower bound for $G_{\xi, \eta}$ that will allow to prove Theorem 5.2.5 (iii).

Lemma 5.5.1. *Suppose that $2 \leq p < 3$. For all $\delta > 0$, there exist constants $\gamma_p = \gamma(p) > 0$ and $c_p = c(p) > 0$ such that for all $X, Y \in \mathbb{R}^d$, for all $\xi \in \mathbb{R}^d \setminus B(0, \delta)$ and $\eta \in B(0, \delta/2)$, we have that*

$$G_{\xi, \eta}(X, Y) \geq \gamma_p |X - Y|^p - c_p \{ |\xi - \eta|^{p-2} |X - Y| + \delta^{p-3} |\xi - \eta| |X + Y| \} |X - Y|. \quad (5.5.2)$$

Suppose that $p \geq 3$. There exist constants $\gamma_p = \gamma(p) > 0$ and $c_p = c(p) > 0$ such that for all $X, Y \in \mathbb{R}^d$ and all $\xi, \eta \in \mathbb{R}^d$,

$$G_{\xi, \eta}(X, Y) \geq \gamma_p |X - Y|^p - c_p \{ |\xi - \eta|^{p-2} |X - Y| + |\xi - \eta| |X + Y|^{p-2} + (|\xi| + |\eta|)^{p-3} |\xi - \eta| |X + Y| \} |X - Y|. \quad (5.5.3)$$

Proof of Lemma 5.5.1. We first give the proof of Estimate (5.5.2). We have that $\xi \neq 0$ and $\eta \neq 0$. For all $X, Y \in \mathbb{R}^d$, we define $Z := \frac{X-Y}{2}$ and $T := \frac{X+Y}{2}$. Inequality (5.5.2) is equivalent to the following inequality: for any $Z, T \in \mathbb{R}^d$,

$$\begin{aligned} |\xi + T + Z|^p + |\eta + T - Z|^p - |\xi + T|^p - |\eta + T|^p - p(\xi|\xi|^{p-2} - \eta|\eta|^{p-2}) \cdot Z \\ \geq \gamma_p |Z|^p - c_p \{ |\xi - \eta|^{p-2} |Z| + \delta^{p-3} |\xi - \eta| |T| \} |Z|. \end{aligned} \quad (5.5.4)$$

We prove (5.5.4) for any $Z, T \in \mathbb{R}^d$. We fix $T \in \mathbb{R}^d$ and we introduce the function

$$\Phi_{\gamma_p}(Z) := |\xi + T + Z|^p + |\eta + T - Z|^p - \gamma_p |Z|^p,$$

where $\gamma_p > 0$ is to be chosen later. Since $p \geq 2$, the function Φ_{γ_p} is of class \mathcal{C}^2 . Besides, denoting by \mathbf{I} the identity matrix, we have that

$$\begin{aligned} \Phi_{\gamma_p}''(Z) &= p|\xi + T + Z|^{p-2} \mathbf{I} + p(p-2)|\xi + Z + T|^{p-4} (\xi + Z + T) \otimes (\xi + Z + T) + p|\eta + T - Z|^{p-2} \mathbf{I} \\ &\quad + p(p-2)|\eta + T - Z|^{p-4} (\eta + T - Z) \otimes (\eta + T - Z) - \gamma_p p |Z|^{p-2} \mathbf{I} - \gamma_p p (p-2) |Z|^{p-4} Z \otimes Z. \end{aligned}$$

Thus, for all $h \in \mathbb{R}^d$,

$$\begin{aligned} \Phi''_{\gamma}(Z)(h, h) &\geq p|\xi + T + Z|^{p-2}|h|^2 + p|\eta + T - Z|^{p-2}|h|^2 - \gamma_p p |Z|^{p-2}|h|^2 - \gamma_p p(p-2)|Z|^{p-4}(Z \cdot h)^2 \\ &\geq p \left[|\xi + T + Z|^{p-2} + |\eta + T - Z|^{p-2} - \gamma_p(p-1)|Z|^{p-2} \right] |h|^2. \end{aligned} \quad (5.5.5)$$

We next note that

$$\begin{aligned} |Z|^{p-2} &= \left| \frac{1}{2}(Z + \xi + T) + \frac{1}{2}(Z - \eta - T) + \frac{1}{2}(\eta - \xi) \right|^{p-2} \\ &\leq C(p) \left(|\xi + T + Z|^{p-2} + |\eta + T - Z|^{p-2} + |\xi - \eta|^{p-2} \right), \end{aligned} \quad (5.5.6)$$

where we have used the triangle inequality together with the fact that for all $m \geq 1$ and $p \geq 2$, there exists a constant $C(p, m)$ such that

$$\forall a_1, \dots, a_m \geq 0, \quad (a_1 + \dots + a_m)^{p-2} \leq C(p, m) \left(a_1^{p-2} + \dots + a_m^{p-2} \right).$$

Estimate (5.5.6) together with inequality (5.5.5) give that

$$\forall h \in \mathbb{R}^d, \quad \Phi''_{\gamma_p}(Z)(h, h) \geq -p|\xi - \eta|^{p-2}|h|^2$$

for

$$\gamma_p := \frac{1}{C(p)(p-1)}.$$

The function $\Phi_{\gamma_p} + \frac{p}{2}|\xi - \eta|^{p-2}|\cdot|^2$ is convex, hence

$$\forall Z \in \mathbb{R}^d, \quad \Phi_{\gamma_p}(Z) + \frac{p}{2}|\xi - \eta|^{p-2}|Z|^2 \geq \Phi_{\gamma_p}(0) + \nabla \Phi_{\gamma_p}(0) \cdot Z.$$

We have thus proved that

$$\Phi_{\gamma_p}(Z) \geq |\xi + T|^p + |\eta + T|^p + p \left[(\xi + T)|\xi + T|^{p-2} - (\eta + T)|\eta + T|^{p-2} \right] \cdot Z - \frac{p}{2}|\xi - \eta|^{p-2}|Z|^2.$$

This proves estimate (5.5.4) if $T = 0$. If $T \neq 0$, it remains to prove that

$$\left| (\xi + T)|\xi + T|^{p-2} - (\eta + T)|\eta + T|^{p-2} - \xi|\xi|^{p-2} + \eta|\eta|^{p-2} \right| \leq c_p \delta^{p-3} |\xi - \eta| |T|. \quad (5.5.7)$$

We want to apply the mean-value inequality to the function Ψ_T defined by

$$\Psi_T(x) := |x + T|^{p-2}(x + T) - x|x|^{p-2}, \quad x \in [\xi, \eta] \subset \mathbb{R}^d \setminus B(0, \delta/2),$$

which is differentiable over \mathbb{R}^d . We have that

$$\Psi'_T(x) = (|x + T|^{p-2} - |x|^{p-2}) \mathbf{I} + ((x + T) \otimes (x + T)|x + T|^{p-4} - x \otimes x|x|^{p-4}).$$

We now note that there exists a constant $C_p > 0$ such that for all $x \in \mathbb{R}^d \setminus B(0, \delta/2)$,

$$\left| |x + T|^{p-2} - |x|^{p-2} \right| \leq C_p \left\{ \delta^{p-3}|T| + |x|^{p-3}|T| \right\} \quad (5.5.8)$$

and

$$\left| (x + T) \otimes (x + T)|x + T|^{p-4} - x \otimes x|x|^{p-4} \right| \leq C_p \left\{ \delta^{p-3}|T| + |x|^{p-3}|T| \right\}. \quad (5.5.9)$$

Noting that $|x|^{p-3} \leq (\frac{1}{2})^{p-3}|\delta|^{p-3}$ since $p \leq 3$, we have proved (5.5.7). The proof of Lemma 5.5.1 is completed up to the justification of (5.5.8)-(5.5.9).

Proof of (5.5.8) and (5.5.9). We concentrate on the first inequality: assume first that $|T| \geq \frac{1}{2}|x| \geq \frac{1}{4}|\delta|$, then

$$\left| |x + T|^{p-2} - |x|^{p-2} \right| \leq C_p |T|^{p-2} \leq C_p \delta^{p-3} |T|. \quad (5.5.10)$$

We now treat the case $|T| \leq \frac{1}{2}|x|$. In particular $|\frac{T}{|x|}| \leq \frac{1}{2}$ and thus

$$||x + T|^{p-2} - |x|^{p-2}| = |x|^{p-2} \left| \left| \frac{x}{|x|} + \frac{T}{|x|} \right|^{p-2} - \left| \frac{x}{|x|} \right|^{p-2} \right| \leq C_p |x|^{p-2} \left| \frac{T}{|x|} \right| = C_p |T| |x|^{p-3}, \quad (5.5.11)$$

since the function $y \mapsto \left| \frac{x}{|x|} + y \right|^{p-2}$ is regular on $B(0, \frac{3}{4})$ with derivative uniformly bounded in x . Estimate (5.5.9) is proved the same way. We have concluded the proof.

Proof of (5.5.3). We assume that $p \geq 3$. With the above variables T and Z , (5.5.3) is equivalent to proving that for all Z, T, ξ and $\eta \in \mathbb{R}^d$, the following inequality holds true:

$$\begin{aligned} & |\xi + T + Z|^p + |\eta + T - Z|^p - |\xi + T|^p - |\eta + T|^p - p(\xi|\xi|^{p-2} - \eta|\eta|^{p-2}) \cdot Z \\ & \geq \gamma_p |Z|^p - c_p \{ |\xi - \eta|^{p-2} |Z| + |\xi - \eta| |T|^{p-2} + (|\xi| + |\eta|)^{p-3} |\xi - \eta| |T| \} |Z|. \end{aligned} \quad (5.5.12)$$

Applying the same method as for the proof of (5.5.2), we only have to prove that

$$\begin{aligned} & |(\xi + T)|\xi + T|^{p-2} - (\eta + T)|\eta + T|^{p-2} - \xi|\xi|^{p-2} + \eta|\eta|^{p-2}| \\ & \leq c_p \{ |\xi - \eta| |T|^{p-2} + (|\xi| + |\eta|)^{p-3} |\xi - \eta| |T| \}. \end{aligned} \quad (5.5.13)$$

We once again appeal to the mean-value inequality on Ψ_T , noticing that, in this case, see (5.5.10) and (5.5.11), we have for all $x \in \mathbb{R}^d$,

$$|\Psi'_T(x)| \leq C_p \{ |T|^{p-2} + |x|^{p-3} |T| \} \leq C_p \{ |T|^{p-2} + (|\xi| + |\eta|)^{p-3} |T| \}, \quad x \in [\xi, \eta]. \quad (5.5.14)$$

Note that, contrary to the case $p < 3$, estimate (5.5.14) does not depend on δ . This gives (5.5.13) and finally (5.5.12). \square

5.5.2 Proof of Theorem 5.2.5

We start this section with a Remark:

Remark 5.5.2. *The proofs of Theorem 5.2.4 (i) and 5.2.4 (ii) below do not use Theorem 5.2.5. Consequently, we may use freely the results of Theorem 5.2.4 (i) and 5.2.4 (ii) in the following proof.*

Proof of Theorem 5.2.5. By homogeneity, we can prove Theorem 5.2.5 for all $\xi \in \mathbb{R}^d$ such that $|\xi| = 1$. We fix such a $\xi \in \mathbb{R}^d$. By **(A4)**, there exists a constant $c > 0$ independent of ξ such that $|\xi + \nabla w_\xi^{\text{per}}| \geq c$. In the proof, we introduce the notations

$$C_\infty^{\text{per}} := \sup_{|\xi|=1} \|\xi + \nabla w_\xi^{\text{per}}\|_{L^\infty(Q)} \quad \text{and} \quad C_\infty := \sup_{|\xi|=1} \|\nabla \widetilde{w}_\xi\|_{L^\infty(\mathbb{R}^d)}, \quad (5.5.15)$$

where these quantities are well-defined owing to Proposition 5.2.1 (ii) and Theorem 5.2.4 (ii). We use the following Taylor inequality (5.5.16) for the function $y \mapsto (\xi + \nabla w_\xi^{\text{per}} + y)|\xi + \nabla w_\xi^{\text{per}} + y|^{p-2}$ which is of class C^2 over $B(0, 3c/4)$. For all $y \in \mathbb{R}^d$, we have, using also (5.8.7) when $|y| \geq c/2$,

$$\begin{aligned} & \left| (\xi + \nabla w_\xi^{\text{per}} + y)|\xi + \nabla w_\xi^{\text{per}} + y|^{p-2} - (\xi + \nabla w_\xi^{\text{per}})|\xi + \nabla w_\xi^{\text{per}}|^{p-2} \right. \\ & \quad \left. - \{ |\xi + \nabla w_\xi^{\text{per}}|^{p-2} \mathbf{I} + (p-2)|\xi + \nabla w_\xi^{\text{per}}|^{p-4} (\xi + \nabla w_\xi^{\text{per}}) \otimes (\xi + \nabla w_\xi^{\text{per}}) \} y \right| \\ & \leq C(p, c) |y|^2 \mathbf{1}_{\{|y| \leq c/2\}} + C(p) \left\{ |\xi + \nabla w_\xi^{\text{per}}|^{p-2} |y| + |y|^{p-1} \right\} \mathbf{1}_{\{|y| \geq c/2\}} \\ & \leq C(p, c) |y|^2 \mathbf{1}_{\{|y| \leq c/2\}} + C(p, c) \left\{ (C_\infty^{\text{per}})^{p-2} |y|^2 + |y|^{\max(2, p-1)} \right\} \mathbf{1}_{\{|y| \geq c/2\}} \\ & \leq C(p, c, C_\infty^{\text{per}}) (|y|^2 + |y|^{\max(2, p-1)}). \end{aligned} \quad (5.5.16)$$

By (5.5.16) applied with $y = \nabla \widetilde{w}_\xi$, we can write

$$\begin{aligned} & (\xi + \nabla w_\xi^{\text{per}} + \nabla \widetilde{w}_\xi) |\xi + \nabla w_\xi^{\text{per}} + \nabla \widetilde{w}_\xi|^{p-2} - (\xi + \nabla w_\xi^{\text{per}}) |\xi + \nabla w_\xi^{\text{per}}|^{p-2} \\ &= [|\xi + \nabla w_\xi^{\text{per}}|^{p-2} \mathbf{I} + (p-2) |\xi + \nabla w_\xi^{\text{per}}|^{p-4} (\xi + \nabla w_\xi^{\text{per}}) \otimes (\xi + \nabla w_\xi^{\text{per}})] \nabla \widetilde{w}_\xi + g_\xi(\nabla \widetilde{w}_\xi), \end{aligned} \quad (5.5.17)$$

where, using (5.5.15),

$$|g_\xi(\nabla \widetilde{w}_\xi)| \leq C(p, c, C_\infty^{\text{per}}, C_\infty) |\nabla \widetilde{w}_\xi|^2, \quad (5.5.18)$$

Thus, collecting (5.5.17) and (5.4.22), we get that $\nabla \widetilde{w}_\xi$ solves

$$\begin{aligned} -\operatorname{div} a [|\xi + \nabla w_\xi^{\text{per}}|^{p-2} \mathbf{I} + (p-2) |\xi + \nabla w_\xi^{\text{per}}|^{p-4} (\xi + \nabla w_\xi^{\text{per}}) \otimes (\xi + \nabla w_\xi^{\text{per}})] \nabla \widetilde{w}_\xi \\ = \operatorname{div}(h) + \operatorname{div}(ag_\xi(\nabla \widetilde{w}_\xi)) \end{aligned} \quad (5.5.19)$$

in the distribution sense. Equation (5.5.19) is of the form

$$-\operatorname{div}(A_\xi \nabla \widetilde{w}_\xi) = \operatorname{div}(h) + \operatorname{div}(ag_\xi(\nabla \widetilde{w}_\xi)), \quad (5.5.20)$$

where

$$A_\xi := a \left(|\xi + \nabla w_\xi^{\text{per}}|^{p-2} \mathbf{I} + (p-2) |\xi + \nabla w_\xi^{\text{per}}|^{p-4} (\xi + \nabla w_\xi^{\text{per}}) \otimes (\xi + \nabla w_\xi^{\text{per}}) \right). \quad (5.5.21)$$

We may write that $A_\xi = A_\xi^{\text{per}} + \widetilde{A}_\xi$, where

$$A_\xi^{\text{per}} := a^{\text{per}} \left(|\xi + \nabla w_\xi^{\text{per}}|^{p-2} \mathbf{I} + (p-2) |\xi + \nabla w_\xi^{\text{per}}|^{p-4} (\xi + \nabla w_\xi^{\text{per}}) \otimes (\xi + \nabla w_\xi^{\text{per}}) \right)$$

and

$$\widetilde{A}_\xi := \widetilde{a} \left(|\xi + \nabla w_\xi^{\text{per}}|^{p-2} \mathbf{I} + (p-2) |\xi + \nabla w_\xi^{\text{per}}|^{p-4} (\xi + \nabla w_\xi^{\text{per}}) \otimes (\xi + \nabla w_\xi^{\text{per}}) \right).$$

The matrix A_ξ^{per} is symmetric, periodic, Hölder continuous, bounded and coercive while the matrix $\widetilde{A}_\xi \in L^{p'} \cap L^\infty(\mathbb{R}^d)^{d \times d}$ by Assumption **(A3)**, in particular $\widetilde{A}_\xi \nabla \widetilde{w}_\xi \in L^{p'} \cap L^\infty(\mathbb{R}^d)^d$ due to Proposition 5.2.1 (ii) and Theorem 5.2.4 (ii). We write equation (5.5.20) as

$$-\operatorname{div} \left(A_\xi^{\text{per}} \nabla \widetilde{w}_\xi \right) = \operatorname{div}(h + ag_\xi(\nabla \widetilde{w}_\xi) + \widetilde{A}_\xi \nabla \widetilde{w}_\xi). \quad (5.5.22)$$

We have that $h \in L^{p'} \cap L^\infty(\mathbb{R}^d)$ and, thanks to the estimate (5.5.18) and the fact that $\nabla \widetilde{w}_\xi \in L^p(\mathbb{R}^d)$, that $ag_\xi(\nabla \widetilde{w}_\xi) \in L^{p/2} \cap L^\infty(\mathbb{R}^d)$. Thus

$$h + ag_\xi(\nabla \widetilde{w}_\xi) + \widetilde{A}_\xi \nabla \widetilde{w}_\xi \in \left(L^{\max(p', p/2)} \cap L^\infty(\mathbb{R}^d) \right)^d.$$

Applying [13, Theorem p. 247] and [12, Theorem A] to (5.5.22) gives $\nabla \widetilde{w}_\xi \in L^{\max(p', p/2)}(\mathbb{R}^d)$ with the estimate

$$\begin{aligned} \|\nabla \widetilde{w}_\xi\|_{L^{\max(p', p/2)}(\mathbb{R}^d)} &\leq C(d, p, c, C_\infty^{\text{per}}, \alpha) \|h + ag_\xi(\nabla \widetilde{w}_\xi) + \widetilde{A}_\xi \nabla \widetilde{w}_\xi\|_{L^{\max(p', p/2)}(\mathbb{R}^d)} \\ &\stackrel{(5.5.18)}{\leq} C(d, p, c, C_\infty^{\text{per}}, C_\infty, \lambda) (\|\widetilde{a}\|_{L^{\max(p', p/2)}(\mathbb{R}^d)} (C_\infty^{\text{per}})^{p-1} + \|\nabla \widetilde{w}_\xi\|_{L^{\max(p', p/2)}(\mathbb{R}^d)}^2) \\ &\quad + \|\widetilde{a}\|_{L^{\max(p', p/2)}(\mathbb{R}^d)} (C_\infty^{\text{per}})^{p-2} \|\nabla \widetilde{w}_\xi\|_{L^\infty(\mathbb{R}^d)} \\ &\leq C(\|\widetilde{a}\|_{L^{p'}(\mathbb{R}^d)}, \lambda, d, p, \alpha, c, C_\infty^{\text{per}}, C_\infty, C_p), \end{aligned} \quad (5.5.23)$$

where $C_p = \sup_{|\xi|=1} \|\nabla \widetilde{w}_\xi\|_{L^p(\mathbb{R}^d)}$. If $p' \geq p/2$, Theorem 5.2.5 is proved. Otherwise, $\nabla \widetilde{w}_\xi \in L^{p/2}(\mathbb{R}^d)^d$ and we iterate the argument. We have, thanks to (5.5.18), that

$$h + ag_\xi(\nabla \widetilde{w}_\xi) + \widetilde{A}_\xi \nabla \widetilde{w}_\xi \in \left(L^{\max(p', p/4)} \cap L^\infty(\mathbb{R}^d) \right)^d,$$

thus by [13], we get that $\nabla \widetilde{w}_\xi \in L^{\max(p', p/4)}(\mathbb{R}^d)^d$ and we can prove, similarly to (5.5.23) that

$$\|\nabla \widetilde{w}_\xi\|_{L^{\max(p', p/4)}(\mathbb{R}^d)} \leq C(\|\widetilde{a}\|_{L^{p'}(\mathbb{R}^d)}, \lambda, d, p, \alpha, c, C_\infty^{\text{per}}, C_\infty, C_p), \quad (5.5.24)$$

where the constant on the right-hand side of (5.5.24) is potentially greater than the one on the right-hand side of (5.5.23) but the dependance on the data remains the same. If $p' \geq p/4$, the Theorem is proved. Otherwise, we iterate similarly. The procedure ends at step k for which $p/2^k \leq p'$: we thus obtain that $\nabla \widetilde{w}_\xi \in L^{p'}(\mathbb{R}^d)^d$ and that there exists a constant $C_{\text{final}} := C(\widetilde{a}, \lambda, d, p, \alpha, c, C_\infty^{\text{per}}, C_\infty, C_p)$ such that

$$\|\nabla \widetilde{w}_\xi\|_{L^{p'}(\mathbb{R}^d)} \leq C_{\text{final}}.$$

Theorem 5.2.5 is proved. \square

5.5.3 Proof of Theorem 5.2.4

Proof of (i). This is due to Proposition 5.2.1 (i), to the form of the PDE (5.2.12)-(5.2.13) defining $\nabla \widetilde{w}_\xi$ and the fact that this PDE is uniquely solvable in the sense of Definition 5.2.2. Note that we use that for $t \neq 0$, $W_{\xi + \nabla w_\xi^{\text{per}}} = W_{t\xi + \nabla w_{t\xi}^{\text{per}}}$.

Proof of (ii). This result is proved in [104, Lemma 2.2] but we reproduce the proof here for the sake of completeness. Let $\xi \in \mathbb{R}^d$. By Definition 5.2.2 with $\nabla \phi = \nabla \widetilde{w}_\xi$, the inequality (5.8.5), Hölder inequality together with (5.4.23), we have

$$c \int_{\mathbb{R}^d} |\nabla \widetilde{w}_\xi|^p \leq \|f\|_{L^{p'}(\mathbb{R}^d)} \|\nabla \widetilde{w}_\xi\|_{L^p(\mathbb{R}^d)} \leq \|\widetilde{a}\|_{L^{p'}(\mathbb{R}^d)} \|\xi + \nabla w_\xi^{\text{per}}\|_{L^\infty(Q)}^{p-1} \|\nabla \widetilde{w}_\xi\|_{L^p(\mathbb{R}^d)}. \quad (5.5.25)$$

Thus, by Proposition 5.2.1 (ii) and (5.5.25), we obtain the first estimate of (5.2.16).

We show that there exists $\alpha > 0$ independent of ξ such that $\nabla w_\xi \in C^{0, \alpha}(\mathbb{R}^d)$. We introduce the function $\overline{w}_\xi := \xi \cdot x + w_\xi$, then $\nabla \overline{w}_\xi$ solves the standard homogeneous p -Laplace equation with varying coefficient a . Applying [74, Theorem 1], we get that $\nabla \overline{w}_\xi$ is continuous over \mathbb{R}^d . Besides, by [74, Theorem 4], there exists a constant $c \geq 1$ and a radius $r > 0$ depending only on d, p, λ and the Lipschitz constant of a , denoted a_{Lip} such that for all $x \in \mathbb{R}^d$,

$$|\nabla \overline{w}_\xi(x)| \leq c \left(\int_{B(x, r)} |\nabla \overline{w}_\xi|^{p'} \right)^{1/p'} \leq c \left(\int_{B(x, r)} |\nabla \overline{w}_\xi|^p \right)^{1/p}. \quad (5.5.26)$$

Due to the form of $\nabla \overline{w}_\xi$, see also (5.2.14) and the first estimate of (5.2.16), we have that

$$|\nabla \overline{w}_\xi(x)| \leq c|\xi| + cr^{-d/p} \|\nabla \widetilde{w}_\xi\|_{L^p(\mathbb{R}^d)} \leq C(d, p, \lambda, a_{\text{Lip}})|\xi|. \quad (5.5.27)$$

In particular, (5.5.27) proves that ∇w_ξ is bounded and that $\|\nabla w_\xi\|_{L^\infty(\mathbb{R}^d)} \leq C(d, p, \lambda, a_{\text{Lip}})|\xi|$. By Assumption **(A2)**, the non-linear operator $a(y, z) = a(y)z|z|^{p-2}$ falls into the scope of [48]. Let $x \in \mathbb{R}^d$, up to subtracting of $\overline{w}_\xi(x)$, we have by (5.5.27) that $|\overline{w}_\xi| \leq C(d, p, \lambda, a_{\text{Lip}})|\xi|$ on $B(x, 2)$. Thus, applying [48, Theorem 2], there exist $\alpha > 0$ and $C_0 > 0$ depending only on $\lambda, a_{\text{Lip}}, p, d, p$, and $C(d, p, \lambda, a_{\text{Lip}})|\xi|$ such that $\nabla \overline{w}_\xi \in C^{0, \alpha}(B(x, 1))$ and

$$[\nabla \overline{w}_\xi]_{C^{0, \alpha}(B(x, 1))} \leq C_0. \quad (5.5.28)$$

To specify the dependence of C_0 in ξ , we first take $|\xi| = 1$ and we then apply the homogeneity, Theorem 5.2.4 (i). This gives that $C_0 = C_0(p, d, \lambda, a_{\text{Lip}})|\xi|$ and concludes the proof of (ii), gathering (5.5.27) and (5.5.28) and the fact that

$$\|\nabla \widetilde{w}_\xi\|_{C^{0, \alpha}(\mathbb{R}^d)} \leq |\xi| + \|\nabla \overline{w}_\xi\|_{C^{0, \alpha}(\mathbb{R}^d)}.$$

Proof of (iii). We assume that $2 \leq p < 3$. Let us fix $\xi \in \mathbb{R}^d$ such that $|\xi| = 1$. In the proof, $c > 0$ will denote a universal constant given by **(A4)**. We consider $\eta \in \mathbb{R}^d$ such that $\xi \neq \eta$. In the sequel, we fix $\delta_0 \in (0, 1)$ such that $C(1 + 2^{1-\gamma})\delta_0^\gamma + \delta_0 \leq c/2$, where C and γ are given by (5.2.8).

Case 1. We assume that $|\xi - \eta| \geq \delta_0$. Then, thanks to Theorem 5.2.4 (ii), we have that

$$\|\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta\|_{L^p(\mathbb{R}^d)} \leq C_p + C_p |\eta|. \quad (5.5.29)$$

We now note that for all $0 < \tilde{\beta} \leq 1$,

$$C_p + C_p |\eta| \leq \begin{cases} \left(\frac{C_p}{\delta_0^{\tilde{\beta}}}\right) |\xi - \eta|^{\tilde{\beta}} (1 + |\eta|) \leq 2^{\tilde{\beta}} \left(\frac{C_p}{\delta_0^{\tilde{\beta}}}\right) |\xi - \eta|^{\tilde{\beta}} (1 + |\eta|^{1-\tilde{\beta}}) & \text{if } |\eta| \leq 2. \\ C_p C(\tilde{\beta}) |\eta| - 1 \left|\tilde{\beta}\right| (1 + |\eta|^{1-\tilde{\beta}}) \leq C_p C(\tilde{\beta}) |\xi - \eta|^{\tilde{\beta}} (1 + |\eta|^{1-\tilde{\beta}}) & \text{if } |\eta| > 2, \end{cases} \quad (5.5.30)$$

where we used that the function $x \mapsto \frac{1+x}{|x-1|^{\tilde{\beta}}(1+x^{1-\tilde{\beta}})}$ is bounded on $[2, +\infty[$. Thus

$$\|\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta\|_{L^p(\mathbb{R}^d)} \leq C(\delta_0, \tilde{\beta}, C_p) |\xi - \eta|^{\tilde{\beta}} (1 + |\eta|^{1-\tilde{\beta}}). \quad (5.5.31)$$

This gives (5.2.17).

Case 2. We assume that $|\xi - \eta| < \delta_0$. Then, by the choice of δ_0 and Proposition 5.2.1 (iv), we have that

$$\|\xi + \nabla w_\xi^{\text{per}} - \{\eta + \nabla w_\eta^{\text{per}}\}\|_{L^\infty(Q)} \leq \frac{c}{2} \quad \text{and} \quad |\xi + \nabla w_\xi^{\text{per}}| \geq c. \quad (5.5.32)$$

Recalling the notation (5.4.2), we have that

$$F_\xi(\nabla \widetilde{w}_\xi) + F_\eta(\nabla \widetilde{w}_\eta) < F_\xi\left(\frac{\nabla \widetilde{w}_\xi + \nabla \widetilde{w}_\eta}{2}\right) + F_\eta\left(\frac{\nabla \widetilde{w}_\xi + \nabla \widetilde{w}_\eta}{2}\right) < +\infty, \quad (5.5.33)$$

where we have used that $\xi \neq \eta$, F_z admits a unique minimizer for $z \in \mathbb{R}^d$ and $\nabla \widetilde{w}_\xi \in L^2(\mathbb{R}^d)$, $\nabla \widetilde{w}_\eta \in L^2(\mathbb{R}^d)$. We recall that

$$F_z^R(\nabla v) := \int_{B_R} a_{g_z + \nabla w_z^{\text{per}}}(\nabla v) + \int_{\mathbb{R}^d} f_z \cdot \nabla v, \quad z \in \mathbb{R}^d, \quad \nabla v \in L^p(\mathbb{R}^d) \quad (5.5.34)$$

and that $R \mapsto F_z^R(\nabla v)$ is a non-decreasing function. Thus, for R large enough, we have the inequality

$$F_\xi^R(\nabla \widetilde{w}_\xi) + F_\eta^R(\nabla \widetilde{w}_\eta) - F_\xi^R\left(\frac{\nabla \widetilde{w}_\xi + \nabla \widetilde{w}_\eta}{2}\right) - F_\eta^R\left(\frac{\nabla \widetilde{w}_\xi + \nabla \widetilde{w}_\eta}{2}\right) \leq 0. \quad (5.5.35)$$

We now use Lemma 5.5.1 applied with $\delta = c$. Taking into account (5.5.32), this gives

$$\begin{aligned} G_{\xi + \nabla w_\xi^{\text{per}}, \eta + \nabla w_\eta^{\text{per}}}(\nabla \widetilde{w}_\xi, \nabla \widetilde{w}_\eta) &\geq \gamma_p |\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta|^p - c_p \left\{ |\xi + \nabla w_\xi^{\text{per}} - (\eta + \nabla w_\eta^{\text{per}})|^{p-2} \|\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta\| \right. \\ &\quad \left. + c^{p-3} |\xi + \nabla w_\xi^{\text{per}} - (\eta + \nabla w_\eta^{\text{per}})| \|\nabla \widetilde{w}_\xi + \nabla \widetilde{w}_\eta\| \right\} |\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta|. \end{aligned} \quad (5.5.36)$$

For all $R > 0$, we can integrate (5.5.36) over the ball B_R . Using the notation (5.5.34) and the form of the map $G_{\xi, \eta}(X, Y)$, see (5.5.1), this yields

$$\begin{aligned} F_\xi^R(\nabla \widetilde{w}_\xi) + F_\eta^R(\nabla \widetilde{w}_\eta) - F_\xi^R\left(\frac{\nabla \widetilde{w}_\xi + \nabla \widetilde{w}_\eta}{2}\right) - F_\eta^R\left(\frac{\nabla \widetilde{w}_\xi + \nabla \widetilde{w}_\eta}{2}\right) - \frac{1}{2} \int_{\mathbb{R}^d} (h_\xi - h_\eta) \cdot (\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta) \\ \geq \gamma_p \int_{B_R} a |\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta|^p - c_p \int_{B_R} a \left\{ |\xi + \nabla w_\xi^{\text{per}} - (\eta + \nabla w_\eta^{\text{per}})|^{p-2} \|\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta\| \right. \\ \quad \left. + c^{p-3} |\xi + \nabla w_\xi^{\text{per}} - (\eta + \nabla w_\eta^{\text{per}})| \|\nabla \widetilde{w}_\xi + \nabla \widetilde{w}_\eta\| \right\} |\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta|, \end{aligned} \quad (5.5.37)$$

where $h_z = \tilde{a}(z + \nabla w_z^{\text{per}})|z + \nabla w_z^{\text{per}}|^{p-2}$ for $z \in \mathbb{R}^d$. For R large enough, we get because of (5.5.35) that

$$-\frac{1}{2} \int_{\mathbb{R}^d} (h_\xi - h_\eta) \cdot (\nabla \tilde{w}_\xi - \nabla \tilde{w}_\eta) \geq \gamma_p \int_{B_R} a |\nabla \tilde{w}_\xi - \nabla \tilde{w}_\eta|^p - c_p \int_{B_R} a \left\{ |\xi + \nabla w_\xi^{\text{per}} - (\eta + \nabla w_\eta^{\text{per}})|^{p-2} \cdot |\nabla \tilde{w}_\xi - \nabla \tilde{w}_\eta| + c^{p-3} |\xi + \nabla w_\xi^{\text{per}} - (\eta + \nabla w_\eta^{\text{per}})| |\nabla \tilde{w}_\xi + \nabla \tilde{w}_\eta| \right\} |\nabla \tilde{w}_\xi - \nabla \tilde{w}_\eta|, \quad (5.5.38)$$

Letting $R \rightarrow +\infty$ in (5.5.38) and using Theorem 5.2.5, we get by the monotone convergence Theorem that

$$-\frac{1}{2} \int_{\mathbb{R}^d} (f_\xi - f_\eta) \cdot (\nabla \tilde{w}_\xi - \nabla \tilde{w}_\eta) \geq \gamma_p \int_{\mathbb{R}^d} |\nabla \tilde{w}_\xi - \nabla \tilde{w}_\eta|^p - c_p \int_{\mathbb{R}^d} \left\{ |\xi + \nabla w_\xi^{\text{per}} - (\eta + \nabla w_\eta^{\text{per}})|^{p-2} \cdot |\nabla \tilde{w}_\xi - \nabla \tilde{w}_\eta| + c^{p-3} |\xi + \nabla w_\xi^{\text{per}} - (\eta + \nabla w_\eta^{\text{per}})| |\nabla \tilde{w}_\xi + \nabla \tilde{w}_\eta| \right\} |\nabla \tilde{w}_\xi - \nabla \tilde{w}_\eta|.$$

Thus, applying the Hölder inequality, Proposition 5.2.1 (iv) and Theorem 5.2.5 under the form

$$\|\nabla \tilde{w}_z\|_{L^{p'}(\mathbb{R}^d)} \leq C|z|, \quad z \in \mathbb{R}^d,$$

we get

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla \tilde{w}_\xi - \nabla \tilde{w}_\eta|^p &\leq C \left(\|\tilde{a}\|_{L^{p'}} |\xi - \eta|^\gamma \|\nabla \tilde{w}_\xi - \nabla \tilde{w}_\eta\|_{L^p(\mathbb{R}^d)} + \right. \\ &\quad \left. c_p \{ C |\xi - \eta|^{\gamma(p-2)} + c^{p-3} C |\xi - \eta|^\gamma \} \|\nabla \tilde{w}_\xi - \nabla \tilde{w}_\eta\|_{L^p} \right). \end{aligned}$$

Thus

$$\|\nabla \tilde{w}_\xi - \nabla \tilde{w}_\eta\|_{L^p(\mathbb{R}^d)}^{p-1} \leq C |\xi - \eta|^{\gamma(p-2)}.$$

This gives (5.2.17) when $|\xi| = 1$. The case $|\xi| \neq 1$ is treated by homogeneity.

Gathering Case 1 and Case 2, we have proved Theorem 5.2.4 (iii) for $p \in [2, 3)$. The proof of the case $p \geq 3$ is performed using the same method and (5.5.3).

Remark 5.5.3. *As suggested by (5.5.3), the assumptions of Theorem 5.2.4(iii) may be weakened when $p \geq 3$. In this case, it is sufficient to assume, instead of (A4), that $\nabla \tilde{w}_\xi \in L^{p'}(\mathbb{R}^d)$, that $\nabla w_\eta \in L^{p'}(\mathbb{R}^d)$ and that we have an estimate of the form (5.2.19).*

Proof of (iv). It is analogous to the proof of Proposition 5.2.1 (iv).

5.6 Qualitative Homogenization: proof of Theorem 5.2.7

The proof of Theorem 5.2.7 is an adaptation of [57] and [44, Theorem 2.1] to the present setting. We start with the following central Lemma:

Lemma 5.6.1. *For $\xi \in \mathbb{R}^d$, let us write $\nabla \tilde{w}_\xi$ the solution to (5.2.12)-(5.2.13) given by Theorem 5.2.3. Assume that the application*

$$\begin{cases} \mathbb{R}^d \longrightarrow L_{\text{unif}}^p(\mathbb{R}^d) \\ \xi \longmapsto \nabla \tilde{w}_\xi \end{cases} \quad (5.6.1)$$

is continuous. Then for all $\Psi \in L^p(\Omega)^d$,

$$\limsup_{\varepsilon \rightarrow 0} \int_{\Omega} \left| \nabla \widetilde{w_{M_\varepsilon \Psi}} \left(\frac{\cdot}{\varepsilon} \right) \right|^p = 0. \quad (5.6.2)$$

Proof of Lemma 5.6.1. We first show the following assertion:

$$\forall \delta > 0, \quad \exists A > 0, \quad \forall |x| \geq A, \quad \forall \xi \in \mathbb{R}^d, \quad \|\nabla \widetilde{w}_\xi\|_{L^p(x+Q)} \leq \delta |\xi|. \quad (5.6.3)$$

By contradiction, if (5.6.3) does not hold, there exists $\delta > 0$ and two sequences $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ and $(\xi_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ such that $|x_n| \xrightarrow{n \rightarrow +\infty} +\infty$ and $\|\nabla \widetilde{w}_{\xi_n}\|_{L^p(x_n+Q)} \geq \delta |\xi_n|$. By Theorem 5.2.4 (i), we can assume that $|\xi_n| = 1$. Thus, up to a subsequence, $\xi_n \xrightarrow{n \rightarrow +\infty} \xi$. However, by (5.6.1), for all n large enough, we have that $\|\nabla \widetilde{w}_{\xi_n} - \nabla \widetilde{w}_\xi\|_{L^p(x_n+Q)} \leq \delta/2$. Thus, for n large enough, we have that $\|\nabla \widetilde{w}_\xi\|_{L^p(x_n+Q)} \geq \delta/2$. Since $|x_n| \xrightarrow{n \rightarrow +\infty} +\infty$, this contradicts that $\nabla \widetilde{w}_\xi \in L^p(\mathbb{R}^d)$. Thus (5.6.3) is satisfied.

We now turn to the proof of (5.6.2). By an immediate application of the Jensen inequality, we have that

$$\forall B \in \mathbb{N} \cup \{+\infty\}, \quad \sum_{|k| < B, \varepsilon(Q+k) \subset \Omega} \varepsilon^d |\Psi_\varepsilon^k|^p \leq \int_{\Omega \cap B_\infty(0, \varepsilon B)} |\Psi|^p, \quad \Psi_\varepsilon^k := \int_{\varepsilon(Q+k)} \Psi, \quad (5.6.4)$$

where $B_\infty(x, r)$ denotes the ball centered in x and of radius $r > 0$ for the $|\cdot|_\infty$ -norm on \mathbb{R}^d . Let $\delta > 0$ and A be given by (5.6.3). We have that

$$\begin{aligned} \int_\Omega \left| \nabla \widetilde{w}_{M_\varepsilon \Psi} \left(\frac{\cdot}{\varepsilon} \right) \right|^p &= \sum_{k \in \mathbb{Z}^d, \varepsilon(Q+k) \subset \Omega} \varepsilon^d \int_{Q+k} |\nabla \widetilde{w}_{\Psi_\varepsilon^k}|^p \\ &\leq \sum_{|k| < A, \varepsilon(Q+k) \subset \Omega} \varepsilon^d \int_{Q+k} |\nabla \widetilde{w}_{\Psi_\varepsilon^k}|^p + \sum_{|k| \geq A, \varepsilon(Q+k) \subset \Omega} \varepsilon^d \int_{Q+k} |\nabla \widetilde{w}_{\Psi_\varepsilon^k}|^p \\ &\stackrel{(5.6.3), (5.2.16)}{\leq} C \sum_{|k| < A, \varepsilon(Q+k) \subset \Omega} \varepsilon^d |\Psi_\varepsilon^k|^p + \delta^p \sum_{|k| \geq A, \varepsilon(Q+k) \subset \Omega} \varepsilon^d |\Psi_\varepsilon^k|^p \\ &\stackrel{(5.6.4)}{\leq} C \int_{B_\infty(0, \varepsilon A) \cap \Omega} |\Psi|^p + \delta^p \int_\Omega |\Psi|^p. \end{aligned} \quad (5.6.5)$$

By the dominated convergence Theorem, we have that

$$\limsup_{\varepsilon \rightarrow 0} \int_\Omega \left| \nabla \widetilde{w}_{M_\varepsilon \Psi} \left(\frac{\cdot}{\varepsilon} \right) \right|^p \leq \delta^p \int_\Omega |\Psi|^p. \quad (5.6.6)$$

Since (5.6.6) is true for all $\delta > 0$, we have proved (5.6.2). \square

We now state the analogous of [44, Lemma 3.5] to the present non-periodic setting. Before that, we introduce for $\xi, y \in \mathbb{R}^d$ the notations

$$p^{\text{per}}(y, \xi) := \xi + \nabla w_\xi^{\text{per}}(y) \quad \text{and} \quad p(y, \xi) := \xi + \nabla w_\xi(y) = p^{\text{per}}(y, \xi) + \nabla \widetilde{w}_\xi(y). \quad (5.6.7)$$

Lemma 5.6.2. *Assume that the Assumptions of Lemma 5.6.1 are satisfied. Let $\Psi \in L^p(\Omega)$ and $\Phi \in L^p(\Omega)$ such that $\Phi = \sum_{j=1}^m \eta_j 1_{\Omega_j}$ where $\bigcup_{j=1}^m \Omega_j \subset \subset \Omega$, $\Omega_k \cap \Omega_\ell = \emptyset$ for $k \neq \ell$ and $|\partial \Omega_j| = 0$ for $j \in \{1, m\}$. Then there exists a constant $C > 0$ independent of ε , Ψ and Φ such that*

$$\limsup_{\varepsilon \rightarrow 0} \|p(\cdot/\varepsilon, M_\varepsilon \Psi) - p(\cdot/\varepsilon, \Phi)\|_{L^p(\Omega)} \leq C \left\{ \|\Psi\|_{L^p(\Omega)}^{1-\beta} + \|\Phi\|_{L^p(\Omega)}^{1-\beta} \right\} \|\Psi - \Phi\|_{L^p(\Omega)}^\beta, \quad (5.6.8)$$

where β is given by Proposition 5.2.1 (iii).

Proof of Lemma 5.6.2. We first notice that

$$\int_\Omega \left| \nabla \widetilde{w}_\Phi \left(\frac{\cdot}{\varepsilon} \right) \right|^p = \sum_{j=1}^m \int_{\Omega_j} \left| \nabla \widetilde{w}_{\eta_j} \left(\frac{\cdot}{\varepsilon} \right) \right|^p \leq \varepsilon^d \sum_{j=1}^m \int_{\mathbb{R}^d} |\nabla \widetilde{w}_{\eta_j}|^p \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (5.6.9)$$

With the notations (5.6.7), we have, applying [44, Lemma 3.5], that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \|p(\cdot/\varepsilon, M_\varepsilon \Psi) - p(\cdot/\varepsilon, \Phi)\|_{L^p(\Omega)} &\leq \underbrace{\limsup_{\varepsilon \rightarrow 0} \|p^{\text{per}}(\cdot/\varepsilon, M_\varepsilon \Psi) - p^{\text{per}}(\cdot/\varepsilon, \Phi)\|_{L^p(\Omega)}}_{\leq \text{RHS of (5.6.8)}} \\ &+ \underbrace{\limsup_{\varepsilon \rightarrow 0} \left\{ \|\nabla \widetilde{w}_{M_\varepsilon \Psi}(\cdot/\varepsilon)\|_{L^p(\Omega)} + \|\nabla \widetilde{w}_\Phi(\cdot/\varepsilon)\|_{L^p(\Omega)} \right\}}_{=0 \text{ by Lemma 5.6.1 and (5.6.9)}}. \end{aligned}$$

□

With these tools, we can prove Theorem 5.2.7. The first point (i) is detailed in Appendix 5.8.4 since it is mainly a rewriting of [57]. Note that for this point, the continuity of $\xi \mapsto \nabla \widetilde{w}_\xi$ is not needed. The only result on the non-periodic correctors ∇w_ξ , $\xi \in \mathbb{R}^d$ that is used is Theorem 5.2.4 (ii). The proof of Theorem 5.2.7 (ii) follows the proof of [44, Theorem 2.1]. In the following, we sketch the proof of Theorem 5.2.7 (ii) by insisting on the points that differ from [44]. The proof of Theorem 5.2.7 (iii) follows from Theorem 5.2.7 (ii) together with Lemma 5.6.1.

Sketch of proof of Theorem 5.2.7 (ii). Since $M_\varepsilon \nabla u^*$ converges to ∇u^* when $\varepsilon \rightarrow 0$ in $L^p(\Omega)$, it is sufficient to show, using the notation (5.6.7) that

$$R_\varepsilon := \nabla u_\varepsilon - p(\cdot/\varepsilon, M_\varepsilon \nabla u^*) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^p(\Omega). \quad (5.6.10)$$

During the proof, we introduce a step function Φ as in Lemma 5.6.2 satisfying $\|\nabla u^* - \Phi\|_{L^p(\Omega)} \leq \delta$. By monotonicity of the p -Laplace operator, see (5.8.5), and Assumption **(A1)**, we have that

$$\begin{aligned} \lambda^{-1} c \|R_\varepsilon\|_{L^p(\Omega)}^p &\leq \int_{\Omega} \langle a(\cdot/\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - a(\cdot/\varepsilon) |p(\cdot/\varepsilon, M_\varepsilon \nabla u^*)|^{p-2} p(\cdot/\varepsilon, M_\varepsilon \nabla u^*), \nabla u_\varepsilon - p(\cdot/\varepsilon, M_\varepsilon \nabla u^*) \rangle \\ &= A_\varepsilon - B_\varepsilon - C_\varepsilon + D_\varepsilon, \end{aligned} \quad (5.6.11)$$

where

$$\begin{aligned} A_\varepsilon &:= \int_{\Omega} a(\cdot/\varepsilon) |\nabla u_\varepsilon|^p, \quad B_\varepsilon := \int_{\Omega} a(\cdot/\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot p(\cdot/\varepsilon, M_\varepsilon \nabla u^*) \\ C_\varepsilon &:= \int_{\Omega} a(\cdot/\varepsilon) |p(\cdot/\varepsilon, M_\varepsilon \nabla u^*)|^{p-2} p(\cdot/\varepsilon, M_\varepsilon \nabla u^*) \cdot \nabla u_\varepsilon \quad \text{and} \quad D_\varepsilon := \int_{\Omega} a(\cdot/\varepsilon) |p(\cdot/\varepsilon, M_\varepsilon \nabla u^*)|^p. \end{aligned}$$

The term A_ε is obviously treated by the L^p -weak convergence $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u^*$:

$$A_\varepsilon = \int_{\Omega} f u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} f u^* = \int_{\Omega} a^*(\nabla u^*) \cdot \nabla u^*. \quad (5.6.12)$$

We study the term B_ε when $M_\varepsilon \nabla u^*$ is replaced by Φ . This gives:

$$\int_{\Omega} a(\cdot/\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot p(\cdot/\varepsilon, \Phi) = \sum_{j=1}^m \int_{\Omega_j} a(\cdot/\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot p(\cdot/\varepsilon, \eta_j).$$

We then apply the standard div-curl Lemma, keeping in mind that $a(\cdot/\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon$ converges $L^p(\Omega)$ -weakly to $a^*(\nabla u^*)$ when $\varepsilon \rightarrow 0$, that $p(\cdot/\varepsilon, \eta_j)$ converges L^p -weakly to η_j and that, thanks to Theorem 5.2.4 (ii), $a(\cdot/\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot p(\cdot/\varepsilon, \eta_j)$ is bounded in $L^p(\Omega)$, uniformly with respect to ε . Thus,

$$\sum_{j=1}^m \int_{\Omega_j} a(\cdot/\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot p(\cdot/\varepsilon, \eta_j) \xrightarrow{\varepsilon \rightarrow 0} \sum_{j=1}^m \int_{\Omega_j} a^*(\nabla u^*) \cdot \eta_j = \int_{\Omega} a^*(\nabla u^*) \cdot \Phi.$$

In view of Lemma 5.6.2, we obtain that

$$\limsup_{\varepsilon \rightarrow 0} \left| B_\varepsilon - \int_{\Omega} a^*(\nabla u^*) \cdot \Phi \right| = O(\delta^\beta), \quad (5.6.13)$$

where the O is independent of δ . The term C_ε is also treated by replacing $M_\varepsilon \nabla u^*$ by Φ and using the div-curl Lemma. Noticing that $a(\cdot/\varepsilon) p(\cdot/\varepsilon, \eta_j) |p(\cdot/\varepsilon, \eta_j)|^{p-2} \xrightarrow{\varepsilon \rightarrow 0} a^*(\eta_j)$ in $L^{p'}(\Omega)$, we obtain that

$$\limsup_{\varepsilon \rightarrow 0} \left| C_\varepsilon - \int_{\Omega} a^*(\Phi) \cdot \nabla u^* \right| = O(\delta^\beta). \quad (5.6.14)$$

We introduce $D_\varepsilon^{\text{per}} := \int_{\Omega} a^{\text{per}}(\cdot/\varepsilon) |p^{\text{per}}(\cdot/\varepsilon, M_\varepsilon \nabla u^*)|^p$. By [44, Step 1, pp.1161-1162], we have that

$$D_\varepsilon^{\text{per}} \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} a^*(\nabla u^*) \cdot \nabla u^*. \quad (5.6.15)$$

Besides, since $|x|^p - |y|^p| \leq C(|x|^{p-1} + |y|^{p-1})|x - y|$ for all $x, y \in \mathbb{R}^d$, we get

$$\begin{aligned} |D_\varepsilon - D_\varepsilon^{\text{per}}| &\leq \int_{\Omega} \left| \tilde{a}\left(\frac{\cdot}{\varepsilon}\right) \left| p\left(\frac{\cdot}{\varepsilon}, M_\varepsilon \nabla u^*\right) \right|^p \right. \\ &\quad \left. + C \int_{\Omega} \left| a^{\text{per}}\left(\frac{\cdot}{\varepsilon}\right) \right|^p \left(\left| p\left(\frac{\cdot}{\varepsilon}, M_\varepsilon \nabla u^*\right) \right|^{p-1} + \left| p^{\text{per}}\left(\frac{\cdot}{\varepsilon}, M_\varepsilon \nabla u^*\right) \right|^{p-1} \right) \left| \nabla \widetilde{w}_{M_\varepsilon \nabla u^*}\left(\frac{\cdot}{\varepsilon}\right) \right| \right|. \end{aligned} \quad (5.6.16)$$

We show that each term of the RHS of (5.6.16) vanishes as $\varepsilon \rightarrow 0$. We use Theorem 5.2.4 (ii) and (5.6.4) with $B = +\infty$, which imply that there exists a constant $C > 0$ independent of ε such that

$$\int_{\Omega} \left| p\left(\frac{\cdot}{\varepsilon}, M_\varepsilon \nabla u^*\right) \right|^p + \left| p^{\text{per}}\left(\frac{\cdot}{\varepsilon}, M_\varepsilon \nabla u^*\right) \right|^p \leq C \|\nabla u^*\|_{L^p(\Omega)}^p. \quad (5.6.17)$$

With the Hölder inequality and Lemma 5.6.1, we prove that the second term of the RHS of (5.6.16) tends to zero as $\varepsilon \rightarrow 0$. As for the first term, we write that

$$\begin{aligned} \int_{\Omega} \left| \tilde{a}\left(\frac{\cdot}{\varepsilon}\right) \left| p\left(\frac{\cdot}{\varepsilon}, M_\varepsilon \nabla u^*\right) \right|^p \right| &\leq C \int_{\Omega} \left| \tilde{a}\left(\frac{\cdot}{\varepsilon}\right) \left| p\left(\frac{\cdot}{\varepsilon}, \Phi\right) \right|^p \right| + C \int_{\Omega} \left| \tilde{a}\left(\frac{\cdot}{\varepsilon}\right) \left| p\left(\frac{\cdot}{\varepsilon}, M_\varepsilon \nabla u^*\right) - p\left(\frac{\cdot}{\varepsilon}, \Phi\right) \right|^p \right| \\ &\stackrel{(5.2.16), (5.6.17)}{\leq} C \underbrace{\left\| \tilde{a}\left(\frac{\cdot}{\varepsilon}\right) \right\|_{L^1(\Omega)}}_{=O(\varepsilon^{d/p'})} \|\Phi\|_{L^\infty(\mathbb{R}^d)}^p + C \|\tilde{a}\|_{L^\infty(\mathbb{R}^d)} \left\| \left| p\left(\frac{\cdot}{\varepsilon}, M_\varepsilon \nabla u^*\right) - p\left(\frac{\cdot}{\varepsilon}, \Phi\right) \right|^p \right\|_{L^p(\mathbb{R}^d)}, \end{aligned} \quad (5.6.18)$$

where we used that $\tilde{a} \in L^\infty(\mathbb{R}^d)$ and the bound $|p(y, \xi)| \leq C|\xi|$ where $C > 0$ is independent of y and ξ . Collecting (5.6.15), (5.6.16), (5.6.18) and Lemma 5.6.2, we have proved that

$$\limsup_{\varepsilon \rightarrow 0} \left| D_\varepsilon - \int_{\Omega} a^*(\nabla u^*) \cdot \nabla u^* \right| = O(\delta^{p\beta}). \quad (5.6.19)$$

Finally, collecting (5.6.12), (5.6.13), (5.6.14), (5.6.19) and (5.6.11), we obtain that

$$\limsup_{\varepsilon \rightarrow 0} \|R_\varepsilon\|_{L^p(\Omega)}^p \leq \int_{\Omega} |a^*(\nabla u^*) \cdot (\nabla u^* - \Phi)| + \int_{\Omega} |(a^*(\nabla u^*) - a^*(\Phi)) \cdot \nabla u^*| + O(\delta^\beta). \quad (5.6.20)$$

Using the following property of a^* , see [44, Remark 1.3],

$$\|a^*(\nabla u^*) - a^*(\Phi)\|_{L^{p'}(\Omega)} \leq C \left[\|\nabla u^*\|_{L^p(\Omega)}^{p-2} + \|\Phi\|_{L^p(\Omega)}^{p-2} \right] \|\nabla u^* - \Phi\|_{L^p(\Omega)},$$

we conclude that $\limsup_{\varepsilon \rightarrow 0} \|R_\varepsilon\|_{L^p(\Omega)}^p = O(\delta^\beta)$ where the O is independent of δ . Since this is true for all $\delta > 0$, we conclude that $\|R_\varepsilon\|_{L^p(\Omega)} \xrightarrow{\varepsilon \rightarrow 0} 0$.

Remark 5.6.3. *Using the same strategy as above, it is straightforward to show that Theorem 5.2.7 holds with the operator M_ε replaced by M_{ε^ν} , $0 < \nu < 1$. In this case, the continuity of the application $\xi \mapsto \nabla \widetilde{w}_\xi$ is not needed and we only use that $\nabla \widetilde{w}_\xi \in L^p(\mathbb{R}^d)$.*

5.7 Continuity of $\xi \mapsto \nabla w_\xi$: proof of Theorem 5.2.8

5.7.1 Preliminary Lemmas

We begin this section with the following lemma that defines weak solution of PDEs of the form (5.7.1):

Lemma 5.7.1. *Let $\xi \in \mathbb{R}^d$, a coefficient a satisfying Assumption (A1) and $h \in L^{p'}(\mathbb{R}^d)^d$. Assume that Assumption (A4)' is satisfied. Let $v \in W_{\xi + \nabla w_\xi^{\text{per}}}$ be solution in the distribution sense to the following PDE:*

$$-\operatorname{div} a [(\xi + \nabla w_\xi^{\text{per}} + \nabla v)|\xi + \nabla w_\xi^{\text{per}} + \nabla v|^{p-2} - (\xi + \nabla w_\xi^{\text{per}})|\xi + \nabla w_\xi^{\text{per}}|^{p-2}] = \operatorname{div}(h). \quad (5.7.1)$$

Then ∇v solves (5.7.1) in the weak sense of Definition 5.2.2: for all $w \in W_{\xi + \nabla w_\xi^{\text{per}}}$,

$$\int_{\mathbb{R}^d} a [(\xi + \nabla w_\xi^{\text{per}} + \nabla v)|\xi + \nabla w_\xi^{\text{per}} + \nabla v|^{p-2} - (\xi + \nabla w_\xi^{\text{per}})|\xi + \nabla w_\xi^{\text{per}}|^{p-2}] \cdot \nabla w = - \int_{\mathbb{R}^d} h \cdot \nabla w. \quad (5.7.2)$$

Proof of Lemma 5.7.1. We define $u := \xi + \nabla w_\xi^{\text{per}}$ in the proof. We fix $w \in W_u$. In the following, χ will denote a smooth and compactly supported function with support in $Q(0, 1)$ such that $\chi = 1$ in $Q(0, \frac{1}{2})$. We fix $R > 0$ and we introduce the function

$$\Phi_R := \left(w - \int_{Q_R \setminus Q_{R/2}} w \right) \chi \left(\frac{\cdot}{R} \right).$$

By the Poincaré-Wirtinger inequality, we have that $\Phi_R \in W_0^{1,p}(Q_R)$. By (5.8.7), we have the bound

$$\begin{aligned} a |(u + \nabla v)|u + \nabla v|^{p-2} - u|u|^{p-2}| &\leq \lambda C [|u + \nabla v|^{p-2} + |u|^{p-2}] |\nabla v| \\ &\leq \lambda C(p) [|u|^{p-2} |\nabla v| + |\nabla v|^{p-1}], \end{aligned} \quad (5.7.3)$$

where we have used the inequality $(b_1 + b_2)^{p-2} \leq C(p)(b_1^{p-2} + b_2^{p-2})$ for $b_1, b_2 \geq 0$. Thus, since $\nabla v \in L^p(Q_R)$ and $p = p'(p-1)$,

$$\operatorname{div} a [(u + \nabla v)|u + \nabla v|^{p-2} - u|u|^{p-2}] \in W^{-1,p'}(Q_R) \quad \text{and} \quad \operatorname{div}(h) \in W^{-1,p'}(Q_R).$$

Consequently, we can test (5.7.1) against Φ_R and obtain, after expanding $\nabla \Phi_R$,

$$\begin{aligned} &\int_{Q_R} a [(u + \nabla v)|u + \nabla v|^{p-2} - u|u|^{p-2}] \cdot \chi \left(\frac{\cdot}{R} \right) \nabla w \\ &+ \frac{1}{R} \int_{Q_R} a [(u + \nabla v)|u + \nabla v|^{p-2} - u|u|^{p-2}] \cdot \left(w - \int_{Q_R \setminus Q_{R/2}} w \right) \nabla \chi \left(\frac{\cdot}{R} \right) = \\ &- \int_{Q_R} h \cdot \chi \left(\frac{\cdot}{R} \right) \nabla w - \frac{1}{R} \int_{Q_R} h \cdot \left(w - \int_{Q_R \setminus Q_{R/2}} w \right) \nabla \chi \left(\frac{\cdot}{R} \right). \end{aligned} \quad (5.7.4)$$

We now recall the following Poincaré-Wirtinger inequality:

$$\left\| w - \int_{Q_R \setminus Q_{R/2}} w \right\|_{L^p(Q_R \setminus Q_{R/2})} \leq CR \|\nabla w\|_{L^p(Q_R \setminus Q_{R/2})} \quad (5.7.5)$$

which is simply a rescaled version of the L^p inequality on $Q \setminus Q_{1/2}$. Besides, thanks to Assumption (A4)' (and its rescaled version), we have that

$$\left\| |u|^{\frac{p-2}{2}} \left(w - \int_{Q_R \setminus Q_{R/2}} w \right) \right\|_{L^2(Q_R \setminus Q_{R/2})} \leq CR \| |u|^{\frac{p-2}{2}} \nabla w \|_{L^2(Q_R \setminus Q_{R/2})}.$$

This yields, together with (5.7.3), Hölder inequality and the inclusion $\text{supp}(\nabla\chi) \subset Q_R \setminus Q_{R/2}$, that

$$\begin{aligned} & \left| \int_{Q_R \setminus Q_{R/2}} a [(u + \nabla v)|u + \nabla v|^{p-2} - u|u|^{p-2}] \cdot \left(w - \int_{Q_R \setminus Q_{R/2}} w \right) \nabla\chi \left(\frac{\cdot}{R} \right) \right| \\ & \leq \lambda CR \|\nabla\chi\|_{L^\infty} \left\{ \|\nabla v\|_{L^p(Q_R \setminus Q_{R/2})}^{p-1} \|\nabla w\|_{L^p(Q_R \setminus Q_{R/2})} \right. \\ & \quad \left. + \left\| |u|^{\frac{p-2}{2}} \nabla v \right\|_{L^2(Q_R \setminus Q_{R/2})} \left\| |u|^{\frac{p-2}{2}} \nabla w \right\|_{L^2(Q_R \setminus Q_{R/2})} \right\} \end{aligned} \quad (5.7.6)$$

and

$$\left| \int_{Q_R} h \cdot \left(w - \int_{Q_R \setminus Q_{R/2}} w \right) \nabla\chi \left(\frac{\cdot}{R} \right) \right| \leq CR \|\nabla\chi\|_{L^\infty} \|h\|_{L^{p'}(Q_R \setminus Q_{R/2})} \|\nabla w\|_{L^p(Q_R \setminus Q_{R/2})}. \quad (5.7.7)$$

Collecting (5.7.4), (5.7.6) and (5.7.7) and recalling that $v, w \in W_u$, we have that

$$\left| \int_{\mathbb{R}^d} \chi \left(\frac{\cdot}{R} \right) \{ a [(u + \nabla v)|u + \nabla v|^{p-2} - u|u|^{p-2}] + h \} \cdot \nabla w \right| \xrightarrow{R \rightarrow +\infty} 0. \quad (5.7.8)$$

On the other hand, by the dominated convergence Theorem, again since $v, w \in W_u$, we have that

$$\begin{aligned} & \int_{\mathbb{R}^d} \chi \left(\frac{\cdot}{R} \right) \{ a [(u + \nabla v)|u + \nabla v|^{p-2} - u|u|^{p-2}] + h \} \cdot \nabla w \\ & \xrightarrow{R \rightarrow +\infty} \int_{\mathbb{R}^d} \{ a [(u + \nabla v)|u + \nabla v|^{p-2} - u|u|^{p-2}] + h \} \cdot \nabla w. \end{aligned} \quad (5.7.9)$$

Thus (5.7.2) is satisfied. \square

The next lemma allows to pass to the limit in PDEs of the form (5.7.1).

Lemma 5.7.2. *Let $(\nabla\phi_n)_{n \in \mathbb{N}} \subset L^\infty(\mathbb{R}^d)^d$, $(a_n)_{n \in \mathbb{N}} \subset L^\infty(\mathbb{R}^d)$, $(h_n)_{n \in \mathbb{N}} \subset L^{p'}(\mathbb{R}^d)^d$ and $(v_n)_{n \in \mathbb{N}} \subset V$ (see (5.4.3)), such that $v_n \in W_{\nabla\phi_n}$ for all $n \in \mathbb{N}$. We assume that, for all $n \in \mathbb{N}$:*

1. *The coefficient a_n satisfies Assumption (A1) with λ uniformly bounded in n .*
2. *The function ∇v_n is solution, in the distribution sense, to*

$$-\text{div } a_n [(\nabla\phi_n + \nabla v_n)|\nabla\phi_n + \nabla v_n|^{p-2} - \nabla\phi_n|\nabla\phi_n|^{p-2}] = \text{div}(h_n). \quad (5.7.10)$$

We also assume the following convergences:

- (i) $\nabla\phi_n \xrightarrow{n \rightarrow +\infty} \nabla\phi$ in $L^\infty(\mathbb{R}^d)$;
- (ii) $a_n \xrightarrow{n \rightarrow +\infty} a$ in $L^\infty_{loc}(\mathbb{R}^d)$;
- (iii) $\nabla v_n \xrightarrow{n \rightarrow +\infty} \nabla v$ in $L^p(\mathbb{R}^d)$ and $|\nabla\phi_n|^{\frac{p-2}{2}} \nabla v_n \xrightarrow{n \rightarrow +\infty} |\nabla\phi|^{\frac{p-2}{2}} \nabla v$ in $L^2(\mathbb{R}^d)$ which $v \in W_{\nabla\phi}$;
- (iv) $h_n \xrightarrow{n \rightarrow +\infty} h$ in $L^{p'}_{loc}(\mathbb{R}^d)$ with $h \in L^{p'}(\mathbb{R}^d)$.

Then $\nabla v_n \xrightarrow{n \rightarrow +\infty} \nabla v$ in $L^p_{loc}(\mathbb{R}^d)$ and ∇v is solution in the distribution sense to

$$-\text{div } a [(\nabla\phi + \nabla v)|\nabla\phi + \nabla v|^{p-2} - \nabla\phi|\nabla\phi|^{p-2}] = \text{div}(h). \quad (5.7.11)$$

Remark 5.7.3. *If all other assumptions are satisfied, the assumption $|\nabla\phi_n|^{\frac{p-2}{2}}\nabla v_n \rightharpoonup |\nabla\phi|^{\frac{p-2}{2}}\nabla v$ in $L^2(\mathbb{R}^d)$ can be weakened in $|\nabla\phi_n|^{\frac{p-2}{2}}\nabla v_n$ is L^2 -weakly convergent. Indeed, following the proof of Lemma 5.4.2(iii), we can prove that if $\nabla v_n \rightharpoonup \nabla v$ in $L^p(\mathbb{R}^d)$, $\nabla\phi_n \rightharpoonup \nabla\phi$ in $L^\infty(\mathbb{R}^d)$ and $|\nabla\phi_n|^{\frac{p-2}{2}}\nabla v_n$ is L^2 -weakly convergent, then $|\nabla\phi_n|^{\frac{p-2}{2}}\nabla v_n \rightharpoonup |\nabla\phi|^{\frac{p-2}{2}}\nabla v$ in $L^2(\mathbb{R}^d)$. In particular, we have that $v \in W_{\nabla\phi}$.*

Proof of Lemma 5.7.2. We fix two bounded smooth domains B, B' such that $B \subset\subset B'$. Let $\chi \in \mathcal{D}(B')$ such that $\chi = 1$ on B and $0 \leq \chi \leq 1$ in B' . We introduce the function

$$\Psi_n := \underbrace{\left\{ \phi_n + v_n - (v + \phi) - \int_{B'} [\phi_n + v_n - (v + \phi)] \right\}}_{:=\Psi_n^1} \chi \in W_0^{1,p}(B').$$

We immediately check that, up to extracting a subsequence, we have by the Rellich compactness Theorem and (i), (iii) that

$$\Psi_n^1 \xrightarrow{n \rightarrow +\infty} 0 \text{ in } L^p(B') \quad \text{and} \quad \nabla \Psi_n^1 \xrightarrow{n \rightarrow +\infty} 0 \text{ in } L^p(B'). \quad (5.7.12)$$

Thus,

$$\Psi_n \xrightarrow{n \rightarrow +\infty} 0 \text{ in } L^p(B') \quad \text{and} \quad \nabla \Psi_n \xrightarrow{n \rightarrow +\infty} 0 \text{ in } L^p(B'). \quad (5.7.13)$$

Since $\Psi_n \in W_0^{1,p}(B')$, we can test Ψ_n against (5.7.10). We re-organize the terms and get that

$$\begin{aligned} & \int_{B'} a(\nabla\phi_n + \nabla v_n) |\nabla\phi_n + \nabla v_n|^{p-2} \cdot (\nabla\phi_n + \nabla v_n - \nabla\phi - \nabla v) \chi \\ &= - \int_{B'} a(\nabla\phi_n + \nabla v_n) |\nabla\phi_n + \nabla v_n|^{p-2} \cdot \Psi_n^1 \nabla \chi + \int_{B'} (a - a_n) (\nabla\phi_n + \nabla v_n) |\nabla\phi_n + \nabla v_n|^{p-2} \cdot \nabla \Psi_n \\ & \quad - \int_{B'} h_n \cdot \nabla \Psi_n + \int_{B'} a_n |\nabla\phi_n|^{p-2} \nabla\phi_n \cdot \nabla \Psi_n \\ &= -A_n + B_n - C_n + D_n. \end{aligned} \quad (5.7.14)$$

We study each term separately. The term A_n vanishes when $n \rightarrow +\infty$ since

$$|(\nabla\phi_n + \nabla v_n) |\nabla\phi_n + \nabla v_n|^{p-2}| \leq C_p [|\nabla\phi_n|^{p-1} + |\nabla v_n|^{p-1}],$$

which is bounded in $L^{p'}(B')$, uniformly with respect to n by (i) and (iii) and (5.7.12). The term B_n vanishes as $n \rightarrow +\infty$ by (5.7.13) and since, by (i), (ii) and (iii):

$$(a - a_n) (\nabla\phi_n + \nabla v_n) |\nabla\phi_n + \nabla v_n|^{p-2} \xrightarrow{n \rightarrow +\infty} 0 \text{ in } L^{p'}(B').$$

The term C_n vanishes by (5.7.13) and the L_{loc}^p -strong convergence of the sequence $(h_n)_{n \in \mathbb{N}}$. The term D_n vanishes by (5.7.13) and the convergence of $a_n |\nabla\phi_n|^{p-2} \nabla\phi_n$ to $a |\nabla\phi|^{p-2} \nabla\phi$ in $L^\infty(B')$. We have proved that

$$\int_{B'} a(\nabla\phi_n + \nabla v_n) |\nabla\phi_n + \nabla v_n|^{p-2} \cdot (\nabla\phi_n + \nabla v_n - \nabla\phi - \nabla v) \chi \xrightarrow{n \rightarrow +\infty} 0. \quad (5.7.15)$$

However, since $(\nabla\phi + \nabla v) |\nabla\phi + \nabla v|^{p-2} \in L^{p'}(B')$ and because of (5.7.12) and (i), we also have that

$$\int_{B'} a(\nabla\phi + \nabla v) |\nabla\phi + \nabla v|^{p-2} \cdot (\nabla\phi_n + \nabla v_n - \nabla\phi - \nabla v) \chi \xrightarrow{n \rightarrow +\infty} 0. \quad (5.7.16)$$

The difference between (5.7.15) and (5.7.16) gives that

$$\int_{B'} a \left[(\nabla\phi_n + \nabla v_n) |\nabla\phi_n + \nabla v_n|^{p-2} - (\nabla\phi + \nabla v) |\nabla\phi + \nabla v|^{p-2} \right] \cdot (\nabla\phi_n + \nabla v_n - \nabla\phi - \nabla v) \chi \xrightarrow{n \rightarrow +\infty} 0. \quad (5.7.17)$$

Using (5.7.17), (5.8.5), that $\chi \geq 0$ and $\chi = 1$ in B , we get

$$\int_B |\nabla\phi_n + \nabla v_n - \nabla\phi - \nabla v|^p \xrightarrow{n \rightarrow +\infty} 0. \quad (5.7.18)$$

By (i), we obtain that $\nabla v_n \xrightarrow{n \rightarrow +\infty} \nabla v$ in $L^p(B)$ up to a subsequence. We easily show that the convergence in fact holds for the whole sequence. We consequently get the $L^p_{\text{loc}}(\mathbb{R}^d)$ -convergence since B is arbitrary.

We now pass to the limit $n \rightarrow +\infty$ in (5.7.10). Let $\Psi \in \mathcal{D}(\mathbb{R}^d)$, we test (5.7.10) against Ψ . By (iv), it is clear that

$$\int_{\mathbb{R}^d} h_n \cdot \nabla \Psi \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^d} h \cdot \nabla \Psi. \quad (5.7.19)$$

Besides, by (5.8.7), (i), (ii) and the $L^p_{\text{loc}}(\mathbb{R}^d)$ -convergence of ∇v_n , we have that

$$\begin{aligned} & a_n \{ (\nabla\phi_n + \nabla v_n) |\nabla\phi_n + \nabla v_n|^{p-2} - \nabla\phi_n |\nabla\phi_n|^{p-2} \} \\ & \xrightarrow{n \rightarrow +\infty} a \{ (\nabla\phi + \nabla v) |\nabla\phi + \nabla v|^{p-2} - \nabla\phi |\nabla\phi|^{p-2} \} \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^d). \end{aligned}$$

This shows that

$$\int_{\mathbb{R}^d} a \{ (\nabla\phi + \nabla v) |\nabla\phi + \nabla v|^{p-2} - \nabla\phi |\nabla\phi|^{p-2} \} \cdot \nabla \Psi = - \int_{\mathbb{R}^d} h \cdot \nabla \Psi,$$

and concludes the proof of the Lemma 5.7.2. \square

5.7.2 Proof of Theorem 5.2.8

Let $(\xi_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ such that $\xi_n \xrightarrow{n \rightarrow +\infty} \xi$ for $\xi \in \mathbb{R}^d$. We aim at showing that $\nabla w_{\xi_n} \xrightarrow{n \rightarrow +\infty} \nabla w_\xi$ in $L^p_{\text{unif}}(\mathbb{R}^d)$. By Proposition 5.2.1 (iii), it is sufficient to show that $\nabla \widetilde{w}_{\xi_n} \xrightarrow{n \rightarrow +\infty} \nabla \widetilde{w}_\xi$ in $L^p_{\text{unif}}(\mathbb{R}^d)$.

Step 1. We have that $\nabla \widetilde{w}_{\xi_n} \xrightarrow{n \rightarrow +\infty} \nabla \widetilde{w}_\xi$ in $L^p(\mathbb{R}^d)$. Indeed, by (5.8.5), (5.8.6) and the form of h , see (5.2.13), we have the following a priori estimate: there exists a constant $C = C(d, p, a, a^{\text{per}}) > 0$ such that for all $n \in \mathbb{N}$,

$$\|\widetilde{w}_{\xi_n}\|_{W_{\xi_n + \nabla w_{\xi_n}^{\text{per}}}} \leq C(|\xi_n| + |\xi_n|^{p/2}). \quad (5.7.20)$$

In particular, there exists $v \in V$ (see (5.4.3) for the definition of V) and $w \in L^2(\mathbb{R}^d)$ such that

$$\nabla \widetilde{w}_{\xi_n} \xrightarrow{n \rightarrow +\infty} \nabla v \text{ in } L^p(\mathbb{R}^d) \quad \text{and} \quad |\xi_n + \nabla w_{\xi_n}^{\text{per}}|^{\frac{p-2}{2}} \nabla \widetilde{w}_{\xi_n} \xrightarrow{n \rightarrow +\infty} w \text{ in } L^2(\mathbb{R}^d). \quad (5.7.21)$$

Taking into account Remark 5.7.3, we have that $v \in W_{\xi + \nabla w_\xi^{\text{per}}}$ and

$$|\xi_n + \nabla w_{\xi_n}^{\text{per}}|^{\frac{p-2}{2}} \nabla \widetilde{w}_{\xi_n} \xrightarrow{n \rightarrow +\infty} |\xi + \nabla w_\xi^{\text{per}}|^{\frac{p-2}{2}} \nabla v \quad \text{in } L^2(\mathbb{R}^d).$$

We apply Lemma 5.7.2 with

$$\begin{cases} a_n = a, & v_n = \widetilde{w}_{\xi_n} \\ \phi_n = \xi_n \cdot x + w_{\xi_n}^{\text{per}}, & \phi = \xi \cdot x + w_\xi^{\text{per}} \\ h_n = \widetilde{a} |\xi_n + \nabla w_{\xi_n}^{\text{per}}|^{p-2} (\xi_n + \nabla w_{\xi_n}^{\text{per}}), & h = \widetilde{a} |\xi + \nabla w_\xi^{\text{per}}|^{p-2} (\xi + \nabla w_\xi^{\text{per}}). \end{cases}$$

The required convergences follow from (5.7.21) and Proposition 5.2.1 (ii) and (iv). We get that $\nabla \widetilde{w}_{\xi_n}$ converges when $n \rightarrow +\infty$ to ∇v in $L^p_{\text{loc}}(\mathbb{R}^d)$ and that ∇v solves (5.2.12) in the distribution sense. Thus, ∇v solves (5.2.12) in the weak sense in $W_{\xi + \nabla w_\xi^{\text{per}}}$ (see Definition 5.2.2) by Lemma 5.7.1, with

h given by (5.2.13). In addition, by Theorem 5.2.3, the solution of this PDE is unique in $W_{\xi + \nabla w_{\xi}^{\text{per}}}$. Thus $\nabla v = \nabla \widehat{w}_{\xi}$ and this concludes Step 1 since the sequence $(\nabla \widehat{w}_{\xi_n})_{n \in \mathbb{N}}$ has one possible limit.

Step 2. Suppose by contradiction that $\nabla \widehat{w}_{\xi_n}$ does not converge to $\nabla \widehat{w}_{\xi}$ in $L^p_{\text{unif}}(\mathbb{R}^d)$ when $n \rightarrow +\infty$. Then there exists $\delta > 0$, a subsequence of $(\xi_n)_{n \in \mathbb{N}}$ that we do not relabel and a sequence $(x_n)_{n \in \mathbb{N}}$ such that

$$\forall n \in \mathbb{N}, \quad \|\nabla \widehat{w}_{\xi_n} - \nabla \widehat{w}_{\xi}\|_{L^p(B(x_n, 1))} \geq \delta. \quad (5.7.22)$$

Up to another extraction, we can suppose that $x_n \xrightarrow{n \rightarrow +\infty} x$ in \mathbb{T}^d , where \mathbb{T}^d denotes the d -dimensional torus. Since by Step 1, we know that $\nabla \widehat{w}_{\xi_n} \xrightarrow{n \rightarrow +\infty} \nabla \widehat{w}_{\xi}$ in $L^p_{\text{loc}}(\mathbb{R}^d)$, we necessarily have that, up to extracting a subsequence, $|x_n| \xrightarrow{n \rightarrow +\infty} +\infty$ in \mathbb{R}^d . We introduce the shifted functions

$$\widehat{w}_n^1 := \widehat{w}_{\xi_n}(\cdot + x_n) \quad \text{and} \quad \widehat{w}_n^2 := \widehat{w}_{\xi}(\cdot + x_n), \quad (5.7.23)$$

$$\widehat{\phi}_n^1 := \xi_n \cdot x + w_{\xi_n}^{\text{per}}(\cdot + x_n) \quad \text{and} \quad \widehat{\phi}_n^2 := \xi \cdot x + w_{\xi}^{\text{per}}(\cdot + x_n). \quad (5.7.24)$$

In particular, (5.7.22) gives

$$\forall n \in \mathbb{N}, \quad \|\nabla \widehat{w}_n^1 - \nabla \widehat{w}_n^2\|_{L^p(B(0, 1))} \geq \delta. \quad (5.7.25)$$

We show in the sequel that, up to a subsequence, for $i = 1, 2$,

$$\nabla \widehat{w}_n^i \xrightarrow{n \rightarrow +\infty} \nabla v^i \text{ in } L^p_{\text{loc}}(\mathbb{R}^d) \quad \text{and} \quad \nabla v^i = 0 \text{ a.e.} \quad (5.7.26)$$

In particular, passing to the limit $n \rightarrow +\infty$ in (5.7.25) will lead to a contradiction.

Step 3. Proof of (5.7.26). We prove (5.7.26) for $i = 1$, the proof is standard for $i = 2$. We have that $\nabla \widehat{w}_n^1$ solves in the distribution sense the PDE

$$-\text{div}(\cdot + x_n) [(\nabla \widehat{\phi}_n^1 + \nabla \widehat{w}_n^1) |\nabla \widehat{\phi}_n^1 + \nabla \widehat{w}_n^1|^{p-2} - \nabla \widehat{\phi}_n^1 |\nabla \widehat{\phi}_n^1|^{p-2}] = \text{div} \left(\tilde{a}(\cdot + x_n) \nabla \widehat{\phi}_n^1 |\nabla \widehat{\phi}_n^1|^{p-2} \right) \quad (5.7.27)$$

and that $\widehat{w}_n^1 \in W_{\nabla \widehat{\phi}_n^1}$. Since

$$\|\widehat{w}_n^1\|_{W_{\nabla \widehat{\phi}_n^1}} = \|\widehat{w}_{\xi_n}\|_{W_{\xi_n + \nabla w_{\xi_n}^{\text{per}}}},$$

we get because of (5.7.20) that the sequences $(\|\nabla \widehat{w}_n^1\|_{L^p(\mathbb{R}^d)})_{n \in \mathbb{N}}$ and $(\|\nabla \widehat{\phi}_n^1\|^{\frac{p-2}{2}} \nabla \widehat{w}_n^1\|_{L^2(\mathbb{R}^d)})_{n \in \mathbb{N}}$ are uniformly bounded in n . Thus, up to extracting a subsequence,

$$\nabla \widehat{w}_n^1 \xrightarrow{n \rightarrow +\infty} \nabla v^1 \text{ in } L^p(\mathbb{R}^d) \quad \text{and} \quad \nabla \widehat{w}_n^1 |\nabla \widehat{\phi}_n^1|^{\frac{p-2}{2}} \xrightarrow{n \rightarrow +\infty} \widehat{w}^1 \text{ in } L^2(\mathbb{R}^d). \quad (5.7.28)$$

We may apply Lemma 5.7.2 with

$$\begin{cases} a_n = a(\cdot + x_n), & \nabla \phi_n = \nabla \widehat{\phi}_n^1, \\ \nabla v_n = \nabla \widehat{w}_n^1 & \text{and} \quad h_n = \tilde{a}(\cdot + x_n) \nabla \widehat{\phi}_n^1 |\nabla \widehat{\phi}_n^1|^{p-2}. \end{cases}$$

We check the required convergences.

- (i) We have that $\nabla \widehat{\phi}_n^1$ converges in $L^\infty(\mathbb{R}^d)$ to $\nabla \widehat{\phi}^1 : x \mapsto \xi \cdot x + \nabla w_{\xi}^{\text{per}}(\cdot + x)$. Indeed, by periodicity, it is enough to check that $\nabla \widehat{\phi}_n^1 \xrightarrow{n \rightarrow +\infty} \nabla \widehat{\phi}^1$ in Q . For all $y \in Q$,

$$\begin{aligned} |\nabla \widehat{\phi}_n^1(y) - \nabla \widehat{\phi}^1(y)| &\leq |\xi_n - \xi| + |\nabla w_{\xi_n}^{\text{per}}(x_n + y) - \nabla w_{\xi_n}^{\text{per}}(x + y)| + |\nabla w_{\xi_n}^{\text{per}}(x + y) - \nabla w_{\xi}^{\text{per}}(x + y)| \\ &\leq |\xi_n - \xi| + C|\xi_n| |x_n - x|_{\mathbb{T}}^\alpha + \{|\xi_n|^{1-\gamma} + |\xi|^{1-\gamma}\} |\xi_n - \xi|^\gamma \end{aligned}$$

where we used Proposition 5.2.1 (ii) together with Proposition 5.2.1 (iv) in the last inequality and $|\cdot|_{\mathbb{T}}$ denotes the euclidian norm on \mathbb{T}^d . This proves the result by convergence of the sequences $(\xi_n)_{n \in \mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$.

- (ii) We have $a_n = a^{\text{per}}(\cdot + x_n) + \tilde{a}(\cdot + x_n)$. Since $x_n \xrightarrow[n \rightarrow +\infty]{} x$ in \mathbb{T}^d , we have by Assumption **(A2)** that $a^{\text{per}}(\cdot + x_n) \xrightarrow[n \rightarrow +\infty]{} a^{\text{per}}(\cdot + x)$ in $L^\infty(Q)$. Let B be a bounded domain, then since $\tilde{a} \in \mathcal{C}^{0,\alpha} \cap L^{p'}(\mathbb{R}^d)$, we have $\tilde{a} \xrightarrow[|x| \rightarrow +\infty]{} 0$. Thus $\tilde{a}(\cdot + x_n) \xrightarrow[n \rightarrow +\infty]{} 0$ in $L^\infty(B)$ and finally a_n converges locally uniformly to $\hat{a} := a^{\text{per}}(\cdot + x)$ when $n \rightarrow +\infty$.
- (iii) This is (5.7.28).
- (iv) By the same argument as in (ii), we have that $h_n \xrightarrow[n \rightarrow +\infty]{} 0$ in $L_{\text{loc}}^{p'}(\mathbb{R}^d)$.

We have proved that, up to extracting a subsequence, $\nabla \widehat{w}_n^1 \xrightarrow[n \rightarrow +\infty]{} \nabla \widehat{v}^1$ in $L_{\text{loc}}^p(\mathbb{R}^d)$ where $\widehat{v}^1 \in W_{\xi + \nabla w_\xi^{\text{per}}(\cdot + x)}$ solves in the distribution sense the PDE

$$\begin{aligned} -\operatorname{div} a^{\text{per}}(\cdot + x) \left[(\xi + \nabla w_\xi^{\text{per}}(\cdot + x) + \nabla \widehat{v}^1) |\xi + \nabla w_\xi^{\text{per}}(\cdot + x) + \nabla \widehat{v}^1|^{p-2} \right. \\ \left. - (\xi + \nabla w_\xi^{\text{per}}(\cdot + x)) |\xi + \nabla w_\xi^{\text{per}}(\cdot + x)|^{p-2} \right] = 0. \end{aligned}$$

Introducing $\widehat{V}^1 := \widehat{v}^1(\cdot - x)$, we get that $\widehat{V}^1 \in W_{\xi + \nabla w_\xi^{\text{per}}}$ and that $\nabla \widehat{V}^1$ solves in the distribution sense the PDE

$$-\operatorname{div} a^{\text{per}} \left[(\xi + \nabla w_\xi^{\text{per}} + \nabla \widehat{V}^1) |\xi + \nabla w_\xi^{\text{per}} + \nabla \widehat{V}^1|^{p-2} - (\xi + \nabla w_\xi^{\text{per}}) |\xi + \nabla w_\xi^{\text{per}}|^{p-2} \right] = 0. \quad (5.7.29)$$

Applying Lemma 5.7.1 to (5.7.29) with $w := \widehat{V}^1$ gives now

$$\int_{\mathbb{R}^d} a^{\text{per}}(\cdot + x) \left[(\xi + \nabla w_\xi^{\text{per}} + \nabla \widehat{V}^1) |\xi + \nabla w_\xi^{\text{per}} + \nabla \widehat{V}^1|^{p-2} - (\xi + \nabla w_\xi^{\text{per}}) |\xi + \nabla w_\xi^{\text{per}}|^{p-2} \right] \cdot \nabla \widehat{V}^1 = 0.$$

Applying (5.8.5) allows to conclude that $\nabla \widehat{V}^1 = 0$ in \mathbb{R}^d and thus $\nabla \widehat{v}^1 = 0$. This proves (5.7.26) and concludes the proof of Theorem 5.2.8.

Remark 5.7.4. *From the above theorem, we can deduce that*

$$\begin{cases} \mathbb{R}^d \longrightarrow L^\infty(\mathbb{R}^d) \\ \xi \longmapsto \nabla \widetilde{w}_\xi \end{cases}$$

is continuous.

The continuity of $\xi \mapsto \nabla w_\xi^{\text{per}}$ is due to Proposition 5.2.1 (iv). We prove that $\xi \mapsto \nabla \widetilde{w}_\xi$ is continuous for the $L^\infty(\mathbb{R}^d)$ topology. By contradiction, suppose that there exists $\xi \in \mathbb{R}^d$, two sequences $(\xi_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ and $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ and a $\delta > 0$ such that $\xi_n \xrightarrow[n \rightarrow +\infty]{} \xi$ and

$$\forall n \in \mathbb{N}, \quad |\nabla \widetilde{w}_{\xi_n}(x_n) - \nabla \widetilde{w}_\xi(x_n)| \geq \delta.$$

By Theorem 5.2.4 (ii), there exists η independent of n such that

$$\forall n \in \mathbb{N}, \quad \forall y \in B(x_n, \eta), \quad |\nabla \widetilde{w}_{\xi_n}(y) - \nabla \widetilde{w}_\xi(y)| \geq \frac{\delta}{2}.$$

Thus, for all $n \in \mathbb{N}$,

$$\|\nabla \widetilde{w}_{\xi_n} - \nabla \widetilde{w}_\xi\|_{L_{\text{unif}}^p(\mathbb{R}^d)} \geq |B(0, 1)|^{1/p} \delta \eta^{d/p} > 0.$$

which is a contradiction with Theorem 5.2.8.

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5.8 Appendices

5.8.1 Proof of Proposition 5.2.1

Proof of (i). This point is obvious by the form of (5.2.9) and the uniqueness of w_ξ^{per} in $W_{\text{per}}^{1,p}(Q)/\mathbb{R}$.

Proof of (ii). The first estimate follows for example from (5.2.4) and in particular:

$$\frac{1}{p} \int_Q a^{\text{per}}(y) |\xi + \nabla w_\xi^{\text{per}}|^p \leq \frac{1}{p} \int_Q a^{\text{per}} |\xi|^p,$$

together with **(A1)**. The proof of the second estimate is exactly the same as the one of Theorem 5.2.4 (ii) (see Subsection 5.5.3) with a replaced by a^{per} and w_ξ replaced by w_ξ^{per} .

Proof of (iii). Let $\xi_1, \xi_2 \in \mathbb{R}^d$. Applying (5.2.3) with $\phi = w_{\xi_1}^{\text{per}} - w_{\xi_2}^{\text{per}}$ with $\xi = \xi_i, i = 1, 2$ and making the difference between the two expressions gives:

$$\int_Q a^{\text{per}} \{ (\xi_1 + \nabla w_{\xi_1}^{\text{per}}) |\xi_1 + \nabla w_{\xi_1}^{\text{per}}|^{p-2} - (\xi_2 + \nabla w_{\xi_2}^{\text{per}}) |\xi_2 + \nabla w_{\xi_2}^{\text{per}}|^{p-2} \} \cdot \{ \nabla w_{\xi_1}^{\text{per}} - \nabla w_{\xi_2}^{\text{per}} \} = 0. \quad (5.8.1)$$

Thus, adding the term

$$\int_Q a^{\text{per}} \{ (\xi_1 + \nabla w_{\xi_1}^{\text{per}}) |\xi_1 + \nabla w_{\xi_1}^{\text{per}}|^{p-2} - (\xi_2 + \nabla w_{\xi_2}^{\text{per}}) |\xi_2 + \nabla w_{\xi_2}^{\text{per}}|^{p-2} \} \cdot \{ \xi_1 - \xi_2 \} = 0 \quad (5.8.2)$$

in the left and right-hand side of (5.8.1), applying (5.8.5) on the left-hand side and (5.8.7) on the right-hand side provides

$$\begin{aligned} c \int_Q |\xi_1 + \nabla w_{\xi_1}^{\text{per}} - (\xi_2 + \nabla w_{\xi_2}^{\text{per}})|^p \\ \leq C \int_Q (|\xi_1 + \nabla w_{\xi_1}^{\text{per}}|^{p-2} + |\xi_2 + \nabla w_{\xi_2}^{\text{per}}|^{p-2}) |\xi_1 + \nabla w_{\xi_1}^{\text{per}} - (\xi_2 + \nabla w_{\xi_2}^{\text{per}})| |\xi_1 - \xi_2|. \end{aligned}$$

We apply the Hölder inequality on the right-hand side with exponents $p/(p-2)$, p and p find, using (5.2.16):

$$c \left(\int_Q |\xi_1 + \nabla w_{\xi_1}^{\text{per}} - (\xi_2 + \nabla w_{\xi_2}^{\text{per}})|^p \right)^{1-1/p} \leq C [|\xi_1|^{p-2} + |\xi_2|^{p-2}] |\xi_1 - \xi_2|.$$

This yields (5.2.16) by taking the $1/(p-1)$ -th power of the above inequality.

Proof of (iv). We argue by contradiction. Suppose that there exist three sequences $(x_n)_{n \in \mathbb{N}} \subset Q$, $(\xi_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ and $(\eta_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ such that for all $n \in \mathbb{N}$, $|\xi_n| = 1$, $0 < |\xi_n - \eta_n| \leq \frac{1}{2}$ and

$$\left| \nabla w_{\xi_n}^{\text{per}}(x_n) - \nabla w_{\eta_n}^{\text{per}}(x_n) \right| \geq n |\xi_n - \eta_n|^\gamma.$$

By Proposition 5.2.1 (ii), we have for n large enough that

$$\forall y \in B(x_n, \delta_n), \quad \left| \nabla w_{\xi_n}^{\text{per}}(y) - \nabla w_{\eta_n}^{\text{per}}(y) \right| \geq \frac{n}{2} |\xi_n - \eta_n|^\gamma, \quad (5.8.3)$$

where $\delta_n := |\xi_n - \eta_n|^{\gamma/\alpha}$. Integrating (5.8.3) over $B(x_n, \delta_n)$, we get that

$$\|\nabla w_{\xi_n}^{\text{per}} - \nabla w_{\eta_n}^{\text{per}}\|_{L^p(Q)}^p \geq |B_1| \left(\frac{n}{2}\right)^p |\xi_n - \eta_n|^{p\gamma} \delta_n^d = Cn^p |\xi_n - \eta_n|^{\gamma(p+d/\alpha)} = Cn^p |\xi_n - \eta_n|^{\beta p}. \quad (5.8.4)$$

However, by (5.2.16), $\|\nabla w_{\xi_n}^{\text{per}} - \nabla w_{\eta_n}^{\text{per}}\|_{L^p(Q)}^p \leq C|\xi_n - \eta_n|^{\beta p}$. This is a contradiction with (5.8.4) when taking $n \rightarrow +\infty$. We have proved (5.2.8) for $|\xi| = 1$ and $|\xi - \eta| \leq \frac{1}{2}$. The other cases are treated by homogeneity and with the help of (5.2.16) as in (5.5.30).

5.8.2 Some technical inequalities

We gather in this subsection some useful inequalities. We first have

$$(x|x|^{p-2} - y|y|^{p-2}) \cdot (x - y) \geq c|x - y|^p, \quad (5.8.5)$$

$$(x|x|^{p-2} - y|y|^{p-2}) \cdot (x - y) \geq c[|x|^{p-2} + |y|^{p-2}] |x - y|^2, \quad (5.8.6)$$

$$|x|x|^{p-2} - y|y|^{p-2}| \leq C[|x|^{p-2} + |y|^{p-2}] |x - y|. \quad (5.8.7)$$

In the above inequalities (5.8.5)-(5.8.7), c and C refer to universal constants that only depend on p . For a proof of these inequalities, we refer to [68]. For $\xi, x \in \mathbb{R}^d$, we introduce the function

$$g_\xi(x) := |\xi + x|^p - |\xi|^p - p\xi|\xi|^{p-2} \cdot x. \quad (5.8.8)$$

We have the following lemma:

Lemma 5.8.1. *There exist two constants $c, C > 0$ depending only on p such that*

$$\forall \xi, x \in \mathbb{R}^d, \quad c[|x|^2|\xi|^{p-2} + |x|^p] \leq g_\xi(x) \leq C[|x|^2|\xi|^{p-2} + |x|^p]. \quad (5.8.9)$$

Proof of Lemma 5.8.1. We introduce $h_\xi(x) := |\xi + x|^p$. The function h_ξ is differentiable, strictly convex and

$$g_\xi(x) = h_\xi(x) - h_\xi(0) - h'_\xi(0) \cdot x$$

for all $x \in \mathbb{R}^d$. This proves that $g_\xi > 0$ on $\mathbb{R}^d \setminus \{0\}$. We fix $\xi \in \mathbb{R}^d$ such that $|\xi| = 1$ and we notice that for all $x \neq 0$,

$$\frac{g_\xi(x)}{|x|^p} \geq \frac{(|x| - 1)^p - 1 - p|x|}{|x|^p} \xrightarrow{|x| \rightarrow +\infty} 1,$$

We can choose $A > 1$ independently of ξ such that $|x| \geq A$ implies

$$\frac{1}{4}[|x|^p + |x|^2] \leq g_\xi(x) \leq \frac{3}{2}[|x|^p + |x|^2].$$

It remains to study g_ξ on $B(0, A)$. We first notice that for $y \neq -\xi$ and $h \in \mathbb{R}^d$,

$$\begin{aligned} h''_\xi(y)(h, h) &= p(p-2) (|\xi + y|^{p-2} \mathbf{I}_d + (\xi + y)^T (\xi + y) |\xi + y|^{p-4}) \cdot (h, h) \\ &= p(p-2) |\xi + y|^{p-2} (|h|^2 + |h(y + \xi)|^2) \geq p(p-2) |\xi + y|^{p-2} |h|^2. \end{aligned}$$

We fix $0 < \varepsilon < 1$ independent of ξ such that for all $y \in B(0, \varepsilon)$, we have that $|\xi + y|^{p-2} \geq \frac{1}{p(p-2)}$. Let $x \in B(0, \varepsilon)$, the Taylor Formula for h_ξ (or α -convexity) yields on the one hand

$$g_\xi(x) = \frac{1}{2} \int_0^1 (1-t)^2 h''_\xi(tx) \cdot (x, x) dt \geq \frac{|x|^2}{2} \geq \frac{1}{4}|x|^p + \frac{1}{4}|x|^2, \quad (5.8.10)$$

and, on the other hand,

$$\forall x \in B(0, A), \quad g_\xi(x) \leq C(A)|x|^2 \leq C(A)[|x|^2 + |x|^p].$$

It remains to prove that the left-side inequality of (5.8.9) is true on $C_{\varepsilon,A} := \overline{B(0,A)} \setminus B(0,\varepsilon)$. Because $x \mapsto g_\xi(x)/(|x|^p + |x|^2)$ is (strictly) positive on $C_{\varepsilon,A}$, we have

$$c_\xi := \min_{x \in C_{\varepsilon,A}} \frac{g_\xi(x)}{|x|^p + |x|^2} > 0.$$

Suppose by contradiction that $\inf_{|\xi|=1} c_\xi = 0$, then there exist two sequences $(\xi_n)_{n \in \mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$ such that

$$|\xi_n| = 1, \quad x_n \in C_{\varepsilon,A} \quad \text{and} \quad \frac{g_{\xi_n}(x_n)}{|x_n|^p + |x_n|^2} \xrightarrow{n \rightarrow +\infty} 0.$$

By compactness, we can assume that the two sequences $(\xi_n)_{n \in \mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$ converge respectively to ξ and x . Since $|x_n| \leq A$, we have that $g_{\xi_n}(x_n) \xrightarrow{n \rightarrow +\infty} 0$. Thus $g_\xi(x) = 0$ which is a contradiction.

This proves that $\inf_{|\xi|=1} c_\xi = c > 0$ and concludes the proof of (5.8.9) when $|\xi| = 1$.

For $\xi \neq 0$, we note that

$$g_\xi = |\xi|^p g_{\frac{\xi}{|\xi|}} \left(\frac{\cdot}{|\xi|} \right), \quad (5.8.11)$$

and the conclusion still holds true with C independent of ξ . \square

Lemma 5.8.2. *There exists a constant $C > 0$ such that for all $x, h \in \mathbb{R}$,*

$$\left| (x+h)^{1/(p-1)} - x^{1/(p-1)} \right| \leq C|h|^{1/(p-1)}. \quad (5.8.12)$$

Lemma 5.8.3. *Assume that Hypothesis (A4)' is satisfied. Then $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in $W_{\xi+\nabla w_\xi^{\text{per}}}$.*

Proof of Lemma 5.8.3. Let $v \in W_{\xi+\nabla w_\xi^{\text{per}}}$ and $\varepsilon > 0$. There exists $R = R(\varepsilon) > 1$ such that

$$\left\| |\xi + \nabla w_\xi^{\text{per}}|^{\frac{p-2}{2}} \nabla v \right\|_{L^2(Q_{R/2}^c)} + \|\nabla v\|_{L^p(Q_{R/2}^c)} < \varepsilon. \quad (5.8.13)$$

Let χ_R be a cut-off function such that $\chi_R = 1$ in $Q_{R/2}$ and $\chi = 0$ in $Q_{3R/4}^c$. We have that $|\chi_R| + R|\nabla \chi_R| \leq C$ where C depends only on the dimension d (and in particular not on R). We introduce

$$w_R := \left(v - \int_{Q_R \setminus Q_{R/2}} v \right) \chi_R,$$

We have immediately that w_R is compactly supported in Q_R and that $w_R \in W_0^{1,p}(Q_R)$. Thus there exists a function $\Phi \in \mathcal{C}_0^\infty(Q_R)$ such that

$$\|w_R - \Phi\|_{W^{1,p}(Q_R)} \leq \varepsilon R^{-\frac{dp}{p-2}} \quad (\leq \varepsilon). \quad (5.8.14)$$

We extend Φ by zero outside Q_R . By Hölder inequality, we have that

$$\|w_R - \Phi\|_{H_0^1(Q_R)} \leq \varepsilon. \quad (5.8.15)$$

We next show that

$$\|v - \Phi\|_{W_{\xi+\nabla w_\xi^{\text{per}}}^{\text{per}}} \leq C(\xi, d, p, a^{\text{per}}, C_{\text{poinc}})\varepsilon, \quad (5.8.16)$$

where C_{poinc} denotes the maximum between the L^p Poincaré-Wirtinger constant on $Q \setminus Q_{1/2}$ and the weighted L^2 Poincaré-Wirtinger constant, given by Assumption (A4)', on $Q \setminus Q_{1/2}$. By the triangle inequality, we have that

$$\|v - \Phi\|_{W_{\xi+\nabla w_\xi^{\text{per}}}^{\text{per}}} \leq \|v - w_R\|_{W_{\xi+\nabla w_\xi^{\text{per}}}^{\text{per}}} + \|w_R - \Phi\|_{W_{\xi+\nabla w_\xi^{\text{per}}}^{\text{per}}}. \quad (5.8.17)$$

We study separately each term of (5.8.17). By Proposition 5.2.1 (iv), (5.8.14) and (5.8.15), we have that

$$\begin{aligned} \|w_R - \Phi\|_{W_{\xi+\nabla w_{\xi}^{\text{per}}}} &= \|\nabla w_R - \nabla \Phi\|_{L^p(Q_R)} + \left\| |\xi + \nabla w_{\xi}^{\text{per}}|^{\frac{p-2}{2}} (\nabla w_R - \nabla \Phi) \right\|_{L^2(Q_R)} \\ &\leq \|w_R - \Phi\|_{W^{1,p}(Q_R)} + C|\xi|^{\frac{p-2}{2}} \|w_R - \Phi\|_{H^1(Q_R)} \leq \left(C(d, p, a^{\text{per}}) |\xi|^{\frac{p-2}{2}} + 1 \right) \varepsilon. \end{aligned}$$

As for the first term of (5.8.17), we write that

$$\nabla v - \nabla w_R = \nabla v(1 - \chi_R) + \frac{1}{R} \left(v - \int_{Q_R \setminus Q_{R/2}} v \right) \nabla \chi(\cdot/R)$$

Thus, applying the L^p Poincaré-Wirtinger inequality, we have that

$$\|\nabla v - \nabla w_R\|_{L^p(\mathbb{R}^d)} \leq \|\nabla v\|_{L^p(Q_{R/2}^c)} + C_{\text{poinc}} \|\nabla v\|_{L^p(Q_R \setminus Q_{R/2})} \stackrel{(5.8.13)}{\leq} (1 + C_{\text{poinc}}) \varepsilon. \quad (5.8.18)$$

As for the $L^2 \left(|\xi + \nabla w_{\xi}^{\text{per}}|^{\frac{p-2}{2}} d\lambda \right)$ norm, we use Assumption **(A4)'** to obtain that

$$\begin{aligned} \|\nabla v - \nabla w_R\|_{L^2 \left(|\xi + \nabla w_{\xi}^{\text{per}}|^{\frac{p-2}{2}} d\lambda \right)} &\leq \left\| |\xi + \nabla w_{\xi}^{\text{per}}|^{\frac{p-2}{2}} \nabla v \right\|_{L^2(Q_{R/2}^c)} \\ &\quad + C_{\text{poinc}} \left\| |\xi + \nabla w_{\xi}^{\text{per}}|^{\frac{p-2}{2}} \nabla v \right\|_{L^2(Q_R \setminus Q_{R/2})} \leq (1 + C_{\text{poinc}}) \varepsilon. \end{aligned} \quad (5.8.19)$$

Gathering together (5.8.17), (5.8.18) and (5.8.19), we get (5.8.16) and conclude the proof of the Lemma. \square

5.8.3 Extension to other non-linear operators

This section does not appear in the paper [106]. This is a complement to Subsection 5.2.5 above.

We fix an operator $a : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying the properties (1)-(2)-(3)-(4) introduced in Subsection 5.2.5. We also define the operator $A : W_{\xi+\nabla w_{\xi}^{\text{per}}} \rightarrow \left(W_{\xi+\nabla w_{\xi}^{\text{per}}} \right)'$ by (5.2.36). We have the two following preliminary Lemmas:

Lemma 5.8.4. *The space $W_{\xi+\nabla w_{\xi}^{\text{per}}}$ is reflexive and separable*

Proof of Lemma 5.8.4. Owing to Lemma 5.4.1 and [27, Theorem 3.19], $W_{\xi+\nabla w_{\xi}^{\text{per}}}$ is a reflexive Banach space. To prove that it is separable, we adapt [27, Proposition 8.1] to the present situation. Consider the mapping

$$\mathcal{G} : \begin{cases} W_{\xi+\nabla w_{\xi}^{\text{per}}} \mapsto L^p(\mathbb{R}^d) \times L^2 \left(|\xi + \nabla w_{\xi}^{\text{per}}|^{p-2} d\lambda \right) \\ v \mapsto (\nabla v, \nabla v), \end{cases}$$

where the space $X := L^p(\mathbb{R}^d) \times L^2 \left(|\xi + \nabla w_{\xi}^{\text{per}}|^{p-2} d\lambda \right)$ is endowed with the norm

$$\|(u_1, u_2)\|_X = \|u_1\|_{L^p(\mathbb{R}^d)} + \|u_2\|_{L^2 \left(|\xi + \nabla w_{\xi}^{\text{per}}|^{p-2} d\lambda \right)} = \|u_1\|_{L^p(\mathbb{R}^d)} + \left\| |\xi + \nabla w_{\xi}^{\text{per}}|^{\frac{p-2}{2}} u_2 \right\|_{L^2(\mathbb{R}^d)}.$$

As a product of two separables spaces, X is separable. By [27, Proposition 3.25], we get that $\mathcal{G} \left(W_{\xi+\nabla w_{\xi}^{\text{per}}} \right)$ is also separable. Let us denote by S a countable and dense subset of $\mathcal{G} \left(W_{\xi+\nabla w_{\xi}^{\text{per}}} \right)$. Then $\mathcal{G}^{-1}(S) \subset W_{\xi+\nabla w_{\xi}^{\text{per}}}$ is dense in $W_{\xi+\nabla w_{\xi}^{\text{per}}}$ since \mathcal{G} is an isometry. Thus $W_{\xi+\nabla w_{\xi}^{\text{per}}}$ is separable. \square

Lemma 5.8.5. *Assume that the non-linear operator a satisfies the Assumptions of Subsection 5.2.5. Then there exists a constant $c > 0$ such that*

$$\forall \xi, \xi' \in \mathbb{R}^d, \quad \forall y \in \mathbb{R}^d, \quad \{a(y, \xi) - a(y, \xi')\} \cdot \{\xi - \xi'\} \geq c|\xi - \xi'|^p. \quad (5.8.20)$$

The main lemma concerning the existence of the non-periodic correctors is the following:

Lemma 5.8.6. *Assume that a satisfies (1) and (4) in Subsection 5.2.5. Let A be defined by (5.4.17). Then, the operator A is hemicontinuous, bounded, coercive and strongly monotone. In particular, the equation $A(\nabla v) = \mathcal{F}$ for $\mathcal{F} \in \left(W_{\xi+\nabla w_\xi^{\text{per}}}\right)'$ admits a unique solution.*

Proof of Lemma 5.8.6. We first show the following properties:

- *Hemicontinuity:* let $u, v, h \in W_{\xi+\nabla w_\xi^{\text{per}}}$. We want to show that for $s, t \in [0, 1]$,

$$|\langle A(u+tv) - A(u+sv), h \rangle| \xrightarrow{|t-s| \rightarrow 0} 0, \quad (5.8.21)$$

where $\langle \cdot, \cdot \rangle$ stands for the duality product in $W_{\xi+\nabla w_\xi^{\text{per}}}$. We first notice that due to Assumption (4) in Subsection 5.2.5,

$$\begin{aligned} & \left| a\left(\cdot, \xi + \nabla w_\xi^{\text{per}} + \nabla u + t\nabla v\right) - a\left(\cdot, \xi + \nabla w_\xi^{\text{per}} + \nabla u + s\nabla v\right) \right| \lesssim \lambda|t-s| \\ & \left(|\xi + \nabla w_\xi^{\text{per}}|^{p-2} + |\nabla u|^{p-2} + |\nabla v|^{p-2} \right) |\nabla v| \lesssim \lambda|t-s| \left(|\xi + \nabla w_\xi^{\text{per}}|^{p-2} |\nabla v| + |\nabla v|^{p-1} + |\nabla u|^{p-1} \right), \end{aligned}$$

where we used Young inequality in the last inequality. Using the above inequality (we first integrate on the ball B_R and let finally $R \rightarrow +\infty$), we get that

$$\begin{aligned} & |\langle A(u+tv) - A(u+sv), h \rangle| \\ & \lesssim \lambda|t-s| \left(\|\nabla v\|_{W_{\xi+\nabla w_\xi^{\text{per}}}} \|\nabla h\|_{W_{\xi+\nabla w_\xi^{\text{per}}}} + \left\{ \|\nabla u\|_{W_{\xi+\nabla w_\xi^{\text{per}}}}^{p-1} + \|\nabla v\|_{W_{\xi+\nabla w_\xi^{\text{per}}}}^{p-1} \right\} \|\nabla h\|_{W_{\xi+\nabla w_\xi^{\text{per}}}} \right). \end{aligned}$$

This proves (5.8.21).

- *A is bounded:* let $v, h \in W_{\xi+\nabla w_\xi^{\text{per}}}$. We show immediately due to Assumption (4) in Subsection 5.2.5 that

$$|\langle A(v), h \rangle| \lesssim \|\nabla v\|_{W_{\xi+\nabla w_\xi^{\text{per}}}} \|\nabla h\|_{W_{\xi+\nabla w_\xi^{\text{per}}}} + \|\nabla v\|_{W_{\xi+\nabla w_\xi^{\text{per}}}}^{p-1} \|\nabla h\|_{W_{\xi+\nabla w_\xi^{\text{per}}}}.$$

- *A is strongly monotone:* let $u, v \in W_{\xi+\nabla w_\xi^{\text{per}}}$. We fix $R > 0$. For all $y \in B_R$, we have thanks to Lemma 5.8.5 that

$$\begin{aligned} & \{a(y, \xi + \nabla w_\xi^{\text{per}} + \nabla u) - a(y, \xi + \nabla w_\xi^{\text{per}}) - [a(y, \xi + \nabla w_\xi^{\text{per}} + \nabla v) - a(y, \xi + \nabla w_\xi^{\text{per}})]\} \cdot \nabla(u-v) \\ & = \{a(y, \xi + \nabla w_\xi^{\text{per}} + \nabla u) - a(y, \xi + \nabla w_\xi^{\text{per}} + \nabla v)\} \cdot \nabla(u-v) \geq c|\nabla u - \nabla v|^p. \end{aligned} \quad (5.8.22)$$

We integrate (5.8.22) over B_R and we let $R \rightarrow +\infty$. By dominated convergence Theorem, this gives

$$\langle A(\nabla u) - A(\nabla v), \nabla u - \nabla v \rangle \geq c\|\nabla u - \nabla v\|_{L^p(\mathbb{R}^d)}^p. \quad (5.8.23)$$

This consequently implies the strong monotonicity of A .

- *A is coercive:* let $v \in W_{\xi+\nabla w_\xi^{\text{per}}}$. Applying Assumption (4) in Subsection 5.2.5 and Lemma 5.8.5, we get that

$$\begin{aligned} \langle A(\nabla v), \nabla v \rangle & = \frac{1}{2} \langle A(\nabla v), \nabla v \rangle + \frac{1}{2} \langle A(\nabla v), \nabla v \rangle \\ & \geq \frac{\lambda^{-1}}{2} \int_{\mathbb{R}^d} |\xi + \nabla w_\xi^{\text{per}}|^{p-2} |\nabla v|^2 + \frac{c}{2} \int_{\mathbb{R}^d} |\nabla v|^p \geq \frac{1}{2} \min(\lambda^{-1}, c) \left(\frac{1}{2} \|\nabla v\|_{W_{\xi+\nabla w_\xi^{\text{per}}}}^2 - 1 \right). \end{aligned}$$

Thus,

$$\frac{\langle A(\nabla v), \nabla v \rangle}{\|\nabla v\|_{W_{\xi+\nabla w_\xi^{\text{per}}}}} \longrightarrow +\infty \quad \text{as} \quad \|\nabla v\|_{W_{\xi+\nabla w_\xi^{\text{per}}}} \longrightarrow +\infty,$$

which gives the coercivity of A . This shows the first part of the Lemma. The surjectivity of A is deduced from [76, Corollary 8.1]. \square

The application of Lemma 5.8.6 with $\mathcal{F} = \text{div}(h)$ where h is defined by (5.2.34) gives the existence and uniqueness of the non-periodic corrector $w_\xi \in W_{\xi+\nabla w_\xi^{\text{per}}}$ satisfying the conclusion of Theorem 5.2.3. Indeed, it is sufficient to check that $\mathcal{F} \in \left(W_{\xi+\nabla w_\xi^{\text{per}}}\right)'$. This follows from Assumption (3) introduced in Subsection 5.2.5:

$$\int_{\mathbb{R}^d} |h|^{p'} = \int_{\mathbb{R}^d} \left| \tilde{a}(\cdot, \xi + \nabla w_\xi^{\text{per}}) \right|^{p'} \stackrel{(3)}{\leq} \int_{\mathbb{R}^d} b(y)^{p'} |\xi + \nabla w_\xi^{\text{per}}|^p < +\infty,$$

since $b \in L^{p'}(\mathbb{R}^d)$ and $\nabla w_\xi^{\text{per}} \in L^\infty(\mathbb{R}^d)$. Thus, $\mathcal{F} \in W^{-1,p'}(\mathbb{R}^d) \subset \left(W_{\xi+\nabla w_\xi^{\text{per}}}\right)'$. Theorem 5.2.3 is proved in the extended setting of Subsection 5.2.5.

We now turn to the proof of Theorem 5.2.4. We only detail the proof of point (iii) since the other assertions are adaptations of the p -Laplacian case to the present setting. Let us first note that the proof of Theorem 5.2.5 still works for the non-variational operators considered in this section.

Proof of Theorem 5.2.5. The proof consists in a linearization. Assume that $|\xi| = 1$. Using Assumption **(A4)** and (5.2.35), we first have

$$a\left(y, \xi + \nabla w_\xi^{\text{per}} + \nabla \tilde{w}_\xi\right) - a\left(y, \xi + \nabla w_\xi^{\text{per}}\right) = \partial_\xi a\left(y, \xi + \nabla w_\xi^{\text{per}}\right) \cdot \nabla \tilde{w}_\xi + O_c\left(|\nabla \tilde{w}_\xi|^2\right), \quad (5.8.24)$$

where the symbol O_c is uniform in $y \in \mathbb{R}^d$ but depends, among other, on the constant c appearing in Assumption **(A4)**. We note that, due to (4) in Subsection 5.2.5, we have for $h \in \mathbb{R}^d$,

$$h^T \cdot \partial_\xi a\left(y, \xi + \nabla w_\xi^{\text{per}}\right) \cdot h \geq \lambda^{-1} |\xi + \nabla w_\xi^{\text{per}}|^{p-2} |h|^2.$$

Thus, Assumption **(A4)** gives that the matrix $\partial_\xi a\left(y, \xi + \nabla w_\xi^{\text{per}}\right)$ is strongly elliptic. It is also bounded and Hölder regular: indeed, we fix $y \in \mathbb{R}^d$ and $y' \in B(y, \eta)$, $\eta = \eta(d, c, p, a^{\text{per}})$, such that

$$|\nabla w_\xi^{\text{per}}(y) - \nabla w_\xi^{\text{per}}(y')| \leq \frac{c}{2}.$$

By the triangle inequality, the inequality (2) and Proposition 5.2.1, we have that

$$\begin{aligned} \left| \partial_\xi a\left(y, \xi + \nabla w_\xi^{\text{per}}(y)\right) - \partial_\xi a\left(y', \xi + \nabla w_\xi^{\text{per}}(y')\right) \right| &\leq \left| \partial_\xi a\left(y, \xi + \nabla w_\xi^{\text{per}}(y)\right) - \partial_\xi a\left(y', \xi + \nabla w_\xi^{\text{per}}(y)\right) \right| \\ &+ \left| \partial_\xi a\left(y', \xi + \nabla w_\xi^{\text{per}}(y)\right) - \partial_\xi a\left(y', \xi + \nabla w_\xi^{\text{per}}(y')\right) \right| \leq \lambda C |y - y'| |\xi|^{p-2} + \lambda C |\nabla w_\xi^{\text{per}}(y) - \nabla w_\xi^{\text{per}}(y')|, \end{aligned}$$

where we used (5.2.35) in the last inequality. Since $\nabla w_\xi^{\text{per}} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$, we get that $\partial_\xi a\left(y, \xi + \nabla w_\xi^{\text{per}}\right)$ is Hölder regular, uniformly in $y \in \mathbb{R}^d$. Besides, we have the decomposition

$$\partial_\xi a^{\text{per}}\left(y, \xi + \nabla w_\xi^{\text{per}}\right) + \partial_\xi \tilde{a}\left(y, \xi + \nabla w_\xi^{\text{per}}\right), \quad \partial_\xi \tilde{a}\left(y, \xi + \nabla w_\xi^{\text{per}}\right) \in L^{p'}(\mathbb{R}^d)^{d \times d}.$$

Applying the results of [12, 13], we deduce that $\nabla \tilde{w}_\xi \in L^{\max(p', p/2)}(\mathbb{R}^d)$. Iterating the argument, we conclude that $\nabla \tilde{w}_\xi \in L^{p'}(\mathbb{R}^d)$. \square

We now detail the proof of Theorem 5.2.4 (iii) in the general setting. We define the homogeneous Sobolev spaces

$$W^{1,q}(\mathbb{R}^d) := \left\{ v \in W_{\text{loc}}^{1,1}(\mathbb{R}^d), \quad \nabla v \in L^q(\mathbb{R}^d) \right\}, \quad 1 < q < +\infty$$

and

$$\mathring{H}^1(\mathbb{R}^d) := \left\{ v \in W_{\text{loc}}^{1,1}(\mathbb{R}^d), \quad \nabla v \in L^2(\mathbb{R}^d) \right\}.$$

Proof of Theorem 5.2.4 (iii). Let $\xi, \eta \in \mathbb{R}^d$ such that $|\xi| = 1$ and $|\xi - \eta| \leq \frac{1}{2}$. We note that, thanks to Assumption **(A4)**, we have that $W_{\xi + \nabla w_\xi^{\text{per}}} = W_{\eta + \nabla w_\eta^{\text{per}}} = W^{1,p} \cap \mathring{H}^1(\mathbb{R}^d)$. Besides, $\nabla \widetilde{w}_\xi \in W^{1,p} \cap W^{1,p'}(\mathbb{R}^d)$ thanks to Theorem 5.2.5 and the same holds true for $\nabla \widetilde{w}_\eta$. Writing the difference between the PDEs satisfied by $\nabla \widetilde{w}_\xi$ and $\nabla \widetilde{w}_\eta$, we have that

$$\begin{aligned} & -\operatorname{div} \left[a \left(y, \xi + \nabla w_\xi^{\text{per}} + \nabla \widetilde{w}_\xi \right) - a \left(y, \xi + \nabla w_\xi^{\text{per}} \right) - a \left(y, \eta + \nabla w_\eta^{\text{per}} + \nabla \widetilde{w}_\eta \right) + a \left(y, \eta + \nabla w_\eta^{\text{per}} \right) \right] \\ &= \operatorname{div} \left[\underbrace{\widetilde{a} \left(y, \xi + \nabla w_\xi^{\text{per}} \right) - \widetilde{a} \left(y, \eta + \nabla w_\eta^{\text{per}} \right)}_{=: h_{\xi, \eta}} \right] \end{aligned} \quad (5.8.25)$$

in the weak sense in $W^{1,p} \cap \mathring{H}^1(\mathbb{R}^d)$. We note that, due to Assumption (3) of Subsection 5.2.5 and the L^∞ -bounds on the periodic correctors, we have that

$$\begin{aligned} |h_{\xi, \eta}| &\leq b(y) \left[|\xi + \nabla w_\xi^{\text{per}}|^{p-2} + |\eta + \nabla w_\eta^{\text{per}}|^{p-2} \right] \left| \xi + \nabla w_\xi^{\text{per}} - (\eta + \nabla w_\eta^{\text{per}}) \right| \\ &\leq Cb(y) \left\{ |\xi|^{p-1-\gamma} + |\eta|^{p-1-\gamma} \right\} |\xi - \eta|^\gamma, \end{aligned} \quad (5.8.26)$$

where γ is given by Proposition 5.2.1 (iv). Thus, $h_{\xi, \eta} \in L^{p'}(\mathbb{R}^d)$ and

$$\|h_{\xi, \eta}\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|b\|_{L^{p'}(\mathbb{R}^d)} \left\{ |\xi|^{p-1-\gamma} + |\eta|^{p-1-\gamma} \right\} |\xi - \eta|^\gamma \leq C|\xi - \eta|^\gamma. \quad (5.8.27)$$

We now test (5.8.25) against the function $\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta$. This yields

$$\begin{aligned} & \int_{\mathbb{R}^d} \left[a \left(y, \xi + \nabla w_\xi^{\text{per}} + \nabla \widetilde{w}_\xi \right) - a \left(y, \xi + \nabla w_\xi^{\text{per}} \right) - a \left(y, \eta + \nabla w_\eta^{\text{per}} + \nabla \widetilde{w}_\eta \right) \right. \\ & \quad \left. + a \left(y, \eta + \nabla w_\eta^{\text{per}} \right) \right] \cdot \left[\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta \right] dy = - \int_{\mathbb{R}^d} h_{\xi, \eta} \cdot \left[\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta \right]. \end{aligned} \quad (5.8.28)$$

As in the proof of Lemma 5.5.1, we introduce

$$T := \frac{\nabla \widetilde{w}_\xi + \nabla \widetilde{w}_\eta}{2} \quad \text{and} \quad Z := \frac{\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta}{2}. \quad (5.8.29)$$

We also define the function

$$\Psi_{\bar{\gamma}}(s) = [\Phi_{\bar{\gamma}}(sZ) - \Phi_{\bar{\gamma}}(0)] \cdot Z, \quad s \in [0, 1],$$

where, for $\widehat{Z} \in \mathbb{R}^d$,

$$\Phi_{\bar{\gamma}}(\widehat{Z}) := a(y, \xi + \nabla w_\xi^{\text{per}} + T + \widehat{Z}) - a(y, \eta + \nabla w_\eta^{\text{per}} + T - \widehat{Z}) - \bar{\gamma}a(y, \widehat{Z}),$$

with $\bar{\gamma} > 0$ to be chosen later. The function $\Psi_{\bar{\gamma}}$ is of class \mathcal{C}^1 over $[0, 1]$ and

$$\begin{aligned} \Psi_{\bar{\gamma}}'(s) &= Z^T \Phi_{\bar{\gamma}}'(sZ) \cdot Z \\ &= Z^T \left(\partial_\xi a(y, \xi + \nabla w_\xi^{\text{per}} + T + sZ) + \partial_\xi a(y, \eta + \nabla w_\eta^{\text{per}} + T - sZ) - \bar{\gamma} \partial_\xi a(y, sZ) \right) \cdot Z \\ &\stackrel{(4)}{\geq} \left\{ \lambda^{-1} \left| \xi + \nabla w_\xi^{\text{per}} + T + sZ \right|^{p-2} + \lambda^{-1} \left| \eta + \nabla w_\eta^{\text{per}} + T - sZ \right|^{p-2} - 2\bar{\gamma} \lambda s^{p-2} |Z|^{p-2} \right\} |Z|^2. \end{aligned}$$

We note, as in inequality (5.5.6), that

$$\begin{aligned} |sZ|^{p-2} &\leq C_p \left\{ \left| \xi + \nabla w_\xi^{\text{per}} + T + sZ \right|^{p-2} + \left| \eta + \nabla w_\eta^{\text{per}} + T - sZ \right|^{p-2} + \left| \xi + \nabla w_\xi^{\text{per}} - (\eta + \nabla w_\eta^{\text{per}}) \right|^{p-2} \right\} \\ &\stackrel{\text{Prop.5.2.1(iv)}}{\leq} C \left\{ \left| \xi + \nabla w_\xi^{\text{per}} + T + sZ \right|^{p-2} + \left| \eta + \nabla w_\eta^{\text{per}} + T - sZ \right|^{p-2} + |\xi - \eta|^{\gamma(p-2)} \right\}. \end{aligned}$$

Thus, there exists $\bar{\gamma} > 0$ independent of ξ, η, Z and T such that

$$\forall s \in [0, 1], \quad \Psi'_{\bar{\gamma}}(s) \geq -\lambda^{-1} |\xi - \eta|^{\gamma(p-2)} |Z|^2.$$

In particular, we get by integration that

$$\Psi_{\bar{\gamma}}(1) - \Psi_{\bar{\gamma}}(0) \geq -\lambda^{-1} |\xi - \eta|^{\gamma(p-2)} |Z|^2.$$

All in all, we have proved that there exists $\bar{\gamma} > 0$ such that

$$\begin{aligned} &\left\{ a(y, \xi + \nabla w_\xi^{\text{per}} + \nabla \widetilde{w}_\xi) - a(y, \eta + \nabla w_\eta^{\text{per}} + \nabla \widetilde{w}_\eta) - a(y, \xi + \nabla w_\xi^{\text{per}} + T) + a(y, \eta + \nabla w_\eta^{\text{per}} + T) \right\} \cdot Z \\ &\geq \bar{\gamma} a(y, Z) \cdot Z - \lambda^{-1} |\xi - \eta|^{\gamma(p-2)} |Z|^2 \stackrel{(4)}{\geq} \lambda^{-1} \bar{\gamma} |Z|^p - \lambda^{-1} |\xi - \eta|^{\gamma(p-2)} |Z|^2. \end{aligned} \quad (5.8.30)$$

It remains to bound

$$\left| a(y, \xi + \nabla w_\xi^{\text{per}} + T) - a(y, \eta + \nabla w_\eta^{\text{per}} + T) - \left\{ a(y, \xi + \nabla w_\xi^{\text{per}}) - a(y, \eta + \nabla w_\eta^{\text{per}}) \right\} \right|. \quad (5.8.31)$$

We apply as in Lemma 5.5.1 the mean-value inequality to the function

$$\chi_T(x) := a(y, x + T) - a(y, x), \quad x \in \left[\xi + \nabla w_\xi^{\text{per}}, \eta + \nabla w_\eta^{\text{per}} \right],$$

which derivative is given by

$$\chi'_T(x) := \partial_\xi a(y, x + T) - \partial_\xi a(y, x), \quad x \in \left[\xi + \nabla w_\xi^{\text{per}}, \eta + \nabla w_\eta^{\text{per}} \right].$$

We choose η sufficiently close to ξ so that for all $y \in Q$,

$$\left[\xi + \nabla w_\xi^{\text{per}}(y), \eta + \nabla w_\eta^{\text{per}}(y) \right] \subset B\left(\xi + \nabla w_\xi^{\text{per}}(y), \frac{c}{2}\right) \subset \mathbb{R}^d \setminus B\left(0, \frac{c}{2}\right).$$

By separating the cases between $|T| \geq \frac{1}{2}|x|$ and $|T| \leq \frac{1}{2}|x|$ and using (5.2.35) together with homogeneity, we easily show, as for the p -Laplacian case, that, for $p < 3$,

$$|\chi'_T(x)| \lesssim c^{p-3} |T|,$$

proving that

$$\begin{aligned} &\left| a(y, \xi + \nabla w_\xi^{\text{per}} + T) - a(y, \eta + \nabla w_\eta^{\text{per}} + T) - a(y, \xi + \nabla w_\xi^{\text{per}}) + a(y, \eta + \nabla w_\eta^{\text{per}}) \right| \\ &\lesssim c^{p-3} |T| \left| \xi + \nabla w_\xi^{\text{per}} - (\eta + \nabla w_\eta^{\text{per}}) \right| \lesssim c^{p-3} |\xi - \eta|^\gamma |T|, \end{aligned} \quad (5.8.32)$$

where we used Proposition 5.2.1 (iv) in the last inequality. The case $p \geq 3$ is treated analogously (we obtain a bound analogous to (5.5.14)).

Let us go back to (5.8.28). Gathering together (5.8.30) and (5.8.32), we get that

$$\begin{aligned} &\left\{ a(y, \xi + \nabla w_\xi^{\text{per}} + \nabla \widetilde{w}_\xi) - a(y, \eta + \nabla w_\eta^{\text{per}} + \nabla \widetilde{w}_\eta) - a(y, \xi + \nabla w_\xi^{\text{per}}) + a(y, \eta + \nabla w_\eta^{\text{per}}) \right\} \cdot Z \\ &\geq \bar{\gamma} a(y, Z) \cdot Z - C \lambda^{-1} |\xi - \eta|^{\gamma(p-2)} |Z|^2 - C c^{p-3} |T| |\xi - \eta|^\gamma |Z|. \end{aligned}$$

Noting that, thanks to Theorem 5.2.5, $T \in L^{p'}(\mathbb{R}^d)^d$, we get, for $p < 3$, :

$$\begin{aligned} \bar{\gamma} \int_{\mathbb{R}^d} |\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta|^p &\leq C \lambda^{-1} |\xi - \eta|^{\gamma(p-2)} \int_{\mathbb{R}^d} |\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta|^2 \\ &\quad + C c^{p-3} |\xi - \eta|^\gamma \int_{\mathbb{R}^d} |\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta| |\nabla \widetilde{w}_\xi + \nabla \widetilde{w}_\eta| + \int_{\mathbb{R}^d} |h_{\xi, \eta}| |\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta| \\ &\leq C(d, p, a, \lambda, c) |\xi - \eta|^{\gamma(p-2)} \int_{\mathbb{R}^d} |\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta| |\nabla \widetilde{w}_\xi + \nabla \widetilde{w}_\eta| + \int_{\mathbb{R}^d} |h_{\xi, \eta}| |\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta|. \end{aligned}$$

Using (5.8.27) and the bound given by Theorem 5.2.5 together with Hölder inequality, we conclude finally that,

$$\bar{\gamma} \left(\int_{\mathbb{R}^d} |\nabla \widetilde{w}_\xi - \nabla \widetilde{w}_\eta|^p \right)^{1/p} \leq C |\xi - \eta|^{\frac{\gamma(p-2)}{p-1}},$$

where the constant C does not depend on ξ and η . The estimate (5.2.18) is deduced for any ξ and η by homogeneity. The proof for the case $p \geq 3$ follows the same lines. \square

We close this section by commenting on Theorem 5.2.7 and Theorem 5.2.8 in the two remarks below.

Remark 5.8.7 (About Theorem 5.2.7 for non-variational operators). *The main result of Section 5.6 is Lemma 5.6.1 which only uses the continuity of the application $\xi \mapsto \nabla \widetilde{w}_\xi$ and its homogeneity. It thus holds true in the present setting, as well as Lemma 5.6.2. The proof of Theorem 5.2.7 relies on the approximation of ∇u^* by a piecewise constant function Φ . The analysis remains similar since (5.8.20) still holds true.*

Remark 5.8.8 (About Theorem 5.2.8 for non-variational operators). *The arguments exposed in Section 5.6 use only the PDE and not the minimization problem. They consequently almost directly extend to the present setting with the suitable adaptation of notations and using the regularity conditions imposed in Subsection 5.2.5 on the non-periodic operator a .*

5.8.4 Derivation of the weak limit

In this section, we prove Theorem 5.2.7 (i). This proof follows the lines of [57]. We first note that we have the following *a priori* estimates

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C \quad \text{and} \quad \left\| a\left(\frac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon |\nabla u_\varepsilon|^{p-2} \right\|_{L^{p'}(\Omega)} \leq C.$$

Up to extracting a subsequence, we have by Rellich's compactness Theorem, that

$$u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u_0 \quad \text{in } L^p(\Omega), \quad \nabla u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \nabla u_0 \quad \text{in } L^p(\Omega)$$

and

$$a\left(\frac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon |\nabla u_\varepsilon|^{p-2} \xrightarrow{\varepsilon \rightarrow 0} a_0 \quad \text{in } L^{p'}(\Omega).$$

The main goal is to show that

$$a_0 = a^*(\nabla u_0). \tag{5.8.33}$$

This will imply in particular that

$$-\operatorname{div} a^*(\nabla u_0) = f$$

in the weak sense in $W_0^{1,p}(\Omega)$. Consequently, this in turn implies that $u_0 = u^*$ by strict monotonicity of the operator a^* . Finally, by uniqueness of the limit, the whole sequence $(u_\varepsilon)_{\varepsilon > 0}$ converges in $W^{1,p}(\Omega)$.

Proof of (5.8.33). We introduce a parameter $\nu > 0$ and we define $Q_k^\nu := 2^{-\nu}(Q + k)$ for $k \in \mathbb{Z}^d$. We also set $I_\nu := \{k \in \mathbb{Z}^d, Q_k^\nu \subset \Omega\}$ and the following discretization of ∇u_0 :

$$\langle \nabla u_0 \rangle_k^\nu := \int_{Q_k^\nu} \nabla u_0, \quad \forall k \in I_\nu.$$

We fix $k \in I_\nu$ and we introduce the functions

$$v_k^{\nu, \varepsilon} := \varepsilon w_{\langle \nabla u_0 \rangle_k^\nu} \left(\frac{\cdot}{\varepsilon} \right) + \langle \nabla u_0 \rangle_k^\nu \cdot x, \quad \nabla v_k^{\nu, \varepsilon} = \nabla w_{\langle \nabla u_0 \rangle_k^\nu} \left(\frac{\cdot}{\varepsilon} \right) + \langle \nabla u_0 \rangle_k^\nu.$$

We check that:

- $v_k^{\nu, \varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \langle \nabla u_0 \rangle_k^\nu \cdot x$ in $L^p(\Omega)$;
- $\nabla v_k^{\nu, \varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \langle \nabla u_0 \rangle_k^\nu$ in $L^p(\Omega)$;
- $\operatorname{div} a_\varepsilon \nabla v_k^{\nu, \varepsilon} |\nabla v_k^{\nu, \varepsilon}|^{p-2} = 0$ in Ω ;
- $a_\varepsilon \nabla v_k^{\nu, \varepsilon} |\nabla v_k^{\nu, \varepsilon}|^{p-2} \xrightarrow{\varepsilon \rightarrow 0} a^*(\langle \nabla u_0 \rangle_k^\nu)$ in $L^{p'}(\Omega)$.

The first point follows from [17, Proposition 5.1] which shows that if $\nabla w_\xi \in (L^{\infty, \text{per}} + L^p)(\mathbb{R}^d)$ then w_ξ is sublinear and we have the bound

$$|\nabla w_\xi(x) - \nabla w_\xi(y)| \leq C|x - y|^{1-\nu_p}, \quad \nu_p := \min\left(1, \frac{d}{p}\right).$$

This shows in particular that

$$\varepsilon \left\| w_{\langle \nabla u_0 \rangle_k^\nu} \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^p(\Omega)} \leq C\varepsilon^{\nu_p} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

This gives the first point. The second point follows from the weak convergence of the periodic part $\nabla w_{\langle \nabla u_0 \rangle_k^\nu}^{\text{per}} \left(\frac{\cdot}{\varepsilon} \right)$ to $\int_Q \nabla w_{\langle \nabla u_0 \rangle_k^\nu}^{\text{per}} = 0$ and the fact that $\nabla w_{\langle \nabla u_0 \rangle_k^\nu} \in L^p(\mathbb{R}^d)$. The third point follows from the corrector equation (5.2.9). The last point follows on the one hand from the convergence (of the periodic function)

$$a_\varepsilon^{\text{per}} \left(\nabla w_{\langle \nabla u_0 \rangle_k^\nu}^{\text{per}} \left(\frac{\cdot}{\varepsilon} \right) + \langle \nabla u_0 \rangle_k^\nu \right) \left| \nabla w_{\langle \nabla u_0 \rangle_k^\nu}^{\text{per}} \left(\frac{\cdot}{\varepsilon} \right) + \langle \nabla u_0 \rangle_k^\nu \right|^{p-2} \xrightarrow{\varepsilon \rightarrow 0} a^*(\langle \nabla u_0 \rangle_k^\nu) \quad \text{in } L^{p'}(\Omega)$$

and on the other hand from the estimate (with $\nabla v_k^{\nu, \varepsilon, \text{per}} := \nabla w_{\langle \nabla u_0 \rangle_k^\nu}^{\text{per}} \left(\frac{\cdot}{\varepsilon} \right) + \langle \nabla u_0 \rangle_k^\nu$)

$$\begin{aligned} & \left| a_\varepsilon \nabla v_k^{\nu, \varepsilon} |\nabla v_k^{\nu, \varepsilon}|^{p-2} - a_\varepsilon^{\text{per}} \nabla v_k^{\nu, \varepsilon, \text{per}} |\nabla v_k^{\nu, \varepsilon, \text{per}}|^{p-2} \right| \\ & \leq |\tilde{a}_\varepsilon| |\nabla v_k^{\nu, \varepsilon}|^{p-1} + C(d, p, \lambda) \left(|\nabla v_k^{\nu, \varepsilon}|^{p-2} + |\nabla v_k^{\nu, \varepsilon, \text{per}}|^{p-2} \right) |\nabla w_{\langle \nabla u_0 \rangle_{k, \nu}} \left(\frac{\cdot}{\varepsilon} \right)| \\ & \leq C(d, p, a) |\tilde{a}_\varepsilon| |\langle \nabla u_0 \rangle_k^\nu|^{p-1} + C(d, p, a) |\langle \nabla u_0 \rangle_k^\nu|^{p-2} |\nabla w_{\langle \nabla u_0 \rangle_{k, \nu}} \left(\frac{\cdot}{\varepsilon} \right)|, \end{aligned}$$

where we used Proposition 5.2.1 (ii) and Theorem 5.2.4 (ii). Since $\tilde{a} \in L^{p'}(\mathbb{R}^d)$ and $\nabla w_{\langle \nabla u_0 \rangle_{k, \nu}} \in L^p(\mathbb{R}^d)$, we get that

$$a_\varepsilon \nabla v_k^{\nu, \varepsilon} |\nabla v_k^{\nu, \varepsilon}|^{p-2} - a_\varepsilon^{\text{per}} \nabla v_k^{\nu, \varepsilon, \text{per}} |\nabla v_k^{\nu, \varepsilon, \text{per}}|^{p-2} \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^{p'}(\Omega),$$

proving the last point above.

Before going further, we estimate in the following lines the quantity $\sum_{k \in I_\nu} \int_{Q_k^\nu} |\nabla v_k^{\nu, \varepsilon}|^p$. Using L^∞ -bound given by Theorem 5.2.4 (ii), we have that

$$\sum_{k \in I_\nu} \int_{Q_k^\nu} |\nabla v_k^{\nu, \varepsilon}|^p \leq \sum_{k \in I_\nu} 2^{-d\nu} |\langle \nabla u_0 \rangle_k^\nu|^p \leq \sum_{k \in I_\nu} 2^{-d\nu} \int_{Q_k^\nu} |\nabla u_0|^p \leq \int_\Omega |\nabla u_0|^p. \quad (5.8.34)$$

We now proceed with the proof of (5.8.33). We fix $\phi \in \mathcal{D}(\Omega)$ and $\eta_\nu \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that for all $k \in \mathbb{Z}^d$, $\eta_\nu \in \mathcal{C}_0^\infty(Q_k^\nu)$ and $0 \leq \eta_\nu \leq 1$. The function η can simply be chosen under the form $\sum_{k \in \mathbb{Z}^d} \eta(2^\nu(\cdot - 2^{-\nu}k))$, where $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ with compact support included in Q and $0 \leq \eta \leq 1$. The idea is to estimate

$$\left| \underbrace{\int_\Omega a_\varepsilon \nabla u_\varepsilon |\nabla u_\varepsilon|^{p-2} \cdot \phi \eta_\nu - \sum_{k \in I_\nu} \int_{Q_k^\nu} a_\varepsilon \nabla v_k^{\nu, \varepsilon} |\nabla v_k^{\nu, \varepsilon}|^{p-2} \cdot \phi \eta_\nu}_{\mathcal{I}} \right| \quad (5.8.35)$$

as $\varepsilon \rightarrow 0$, then $\eta \rightarrow 1_Q$ in $L^p(Q)$ and finally $\nu \rightarrow +\infty$. We choose ν large enough so that $\text{supp}(\phi) \subset \bigcup_{k \in I_\nu} Q_k^\nu$. We have, using the Hölder inequality together with (5.8.7) that

$$\begin{aligned} |\mathcal{I}| &\leq \sum_{k \in I_\nu} \int_{Q_k^\nu} a_\varepsilon \left| |\nabla u_\varepsilon|^{p-2} - |\nabla v_k^{\nu, \varepsilon}|^{p-2} \right| |\phi \eta_\nu| \\ &\leq \lambda \left(\sum_{k \in I_\nu} \int_{Q_k^\nu} |\nabla u_\varepsilon|^p + |\nabla v_k^{\nu, \varepsilon}|^p \right)^{\frac{p-2}{p}} \left(\sum_{k \in I_\nu} \int_{Q_k^\nu} |\nabla u_\varepsilon - \nabla v_k^{\nu, \varepsilon}|^p \eta_\nu^p \right)^{\frac{1}{p}} \left(\sum_{k \in I_\nu} \int_{Q_k^\nu} |\phi|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Using (5.8.34) and the *a priori* estimates on ∇u_ε , we find that

$$|\mathcal{I}| \leq C \|\phi\|_{L^\infty} \left(\sum_{k \in I_\nu} \int_{Q_k^\nu} |\nabla u_\varepsilon - \nabla v_k^{\nu, \varepsilon}|^p \eta_\nu^p \right)^{\frac{1}{p}} \leq C \|\phi\|_{L^\infty} \left(\sum_{k \in I_\nu} \int_{Q_k^\nu} |\nabla u_\varepsilon - \nabla v_k^{\nu, \varepsilon}|^p \eta_\nu \right)^{\frac{1}{p}}, \quad (5.8.36)$$

where the constant C depends, among other quantities, on $\|\nabla u_0\|_{L^p(\Omega)}$. We now estimate the quantity

$\sum_{k \in I_\nu} \int_{Q_k^\nu} |\nabla u_\varepsilon - \nabla v_k^{\nu, \varepsilon}|^p \eta_\nu$. By monotonicity, we have that

$$\int_{Q_k^\nu} |\nabla u_\varepsilon - \nabla v_k^{\nu, \varepsilon}|^p \eta_\nu \leq c\lambda \int_{Q_k^\nu} \left(a_\varepsilon |\nabla u_\varepsilon|^{p-2} - a_\varepsilon |\nabla v_k^{\nu, \varepsilon}|^{p-2} \right) \cdot (\nabla u_\varepsilon - \nabla v_k^{\nu, \varepsilon}) \eta_\nu \quad (5.8.37)$$

Let us recall that $\eta_\nu \in \mathcal{C}_0^\infty(Q_k^\nu)$. We can pass to the limit $\varepsilon \rightarrow 0$ in the RHS of (5.8.37). This is done by the div-curl Lemma (which is in fact an integration by parts in this case) using the convergences stated at the beginning of the section:

$$\begin{aligned} \int_{Q_k^\nu} \left(a_\varepsilon |\nabla u_\varepsilon|^{p-2} - a_\varepsilon |\nabla v_k^{\nu, \varepsilon}|^{p-2} \right) \cdot (\nabla u_\varepsilon - \nabla v_k^{\nu, \varepsilon}) \eta_\nu \\ \xrightarrow{\varepsilon \rightarrow 0} \int_{Q_k^\nu} (a_0 - a^*(\langle \nabla u_0 \rangle_k^\nu)) \cdot (\nabla u_0 - \langle \nabla u_0 \rangle_k^\nu) \eta_\nu. \end{aligned} \quad (5.8.38)$$

Passing to the limit $\varepsilon \rightarrow 0$ in (5.8.35) and using (5.8.38), we obtain that

$$\left| \int_\Omega a_0 \cdot \phi \eta_\nu - \sum_{k \in I_\nu} \int_{Q_k^\nu} a^*(\langle \nabla u_0 \rangle_k^\nu) \phi \eta_\nu \right| \leq \sum_{k \in I_\nu} \int_{Q_k^\nu} (a_0 - a^*(\langle \nabla u_0 \rangle_k^\nu)) \cdot (\nabla u_0 - \langle \nabla u_0 \rangle_k^\nu) \eta_\nu. \quad (5.8.39)$$

This yields since $0 \leq \eta_\nu \leq 1$,

$$\left| \int_{\Omega} a_0 \cdot \phi \eta_\nu - \sum_{k \in I_\nu} \int_{Q_k^\nu} a^* (\langle \nabla u_0 \rangle_k^\nu) \phi \eta_\nu \right| \leq \sum_{k \in I_\nu} \int_{Q_k^\nu} |a_0 - a^* (\langle \nabla u_0 \rangle_k^\nu)| |\nabla u_0 - \langle \nabla u_0 \rangle_k^\nu|. \quad (5.8.40)$$

We want to let $\eta_\nu \rightarrow 1$ in $L^p(\Omega)$ in the LHS. This is possible for the first term since $a_0 \phi \in L^{p'}(\Omega)$. For the second term, we recall the bound $|a^*(\xi)| \leq C|\xi|^{p-1}$ (which is a consequence of Proposition 5.2.1 (ii)) and we deduce that

$$\left\| \sum_{k \in I_\nu} a^* (\langle \nabla u_0 \rangle_k^\nu) 1_{Q_k^\nu} \right\|_{L^{p'}(\Omega)}^{p'} \leq C \sum_{k \in I_\nu} |\langle \nabla u_0 \rangle_k^\nu|^p |Q_k^\nu| \leq C \int_{\Omega} |\nabla u_0|^p. \quad (5.8.41)$$

This allows to pass to the limit in η_ν in (5.8.40). We obtain

$$\left| \int_{\Omega} a_0 \cdot \phi - \sum_{k \in I_\nu} \int_{Q_k^\nu} a^* (\langle \nabla u_0 \rangle_k^\nu) \phi \right| \leq \sum_{k \in I_\nu} \int_{Q_k^\nu} |a_0 - a^* (\langle \nabla u_0 \rangle_k^\nu)| |\nabla u_0 - \langle \nabla u_0 \rangle_k^\nu|. \quad (5.8.42)$$

We finally pass to the limit $\nu \rightarrow +\infty$ in (5.8.42). Using the Hölder inequality, we have that

$$\begin{aligned} & \sum_{k \in I_\nu} \int_{Q_k^\nu} |a_0 - a^* (\langle \nabla u_0 \rangle_k^\nu)| |\nabla u_0 - \langle \nabla u_0 \rangle_k^\nu| \\ & \leq \left(\sum_{k \in I_\nu} \int_{Q_k^\nu} |a_0 - a^* (\langle \nabla u_0 \rangle_k^\nu)|^{p'} \right)^{\frac{1}{p'}} \left(\sum_{k \in I_\nu} \int_{Q_k^\nu} |\nabla u_0 - \langle \nabla u_0 \rangle_k^\nu|^p \right)^{\frac{1}{p}}. \end{aligned} \quad (5.8.43)$$

Using (5.8.41) and that $a_0 \in L^{p'}(\Omega)$, we get that

$$\begin{aligned} & \sum_{k \in I_\nu} \int_{Q_k^\nu} |a_0 - a^* (\langle \nabla u_0 \rangle_k^\nu)| |\nabla u_0 - \langle \nabla u_0 \rangle_k^\nu| \\ & \leq C \left(\|\nabla u_0\|_{L^p(\Omega)}, \|a_0\|_{L^{p'}(\Omega)} \right) \left(\sum_{k \in I_\nu} \int_{Q_k^\nu} |\nabla u_0 - \langle \nabla u_0 \rangle_k^\nu|^p \right)^{\frac{1}{p}}. \end{aligned} \quad (5.8.44)$$

The RHS of (5.8.44) tends to zero as $\nu \rightarrow +\infty$. Inserting this into (5.8.43), we have that

$$\limsup_{\nu \rightarrow +\infty} \left| \int_{\Omega} a_0 \cdot \phi - \sum_{k \in I_\nu} \int_{Q_k^\nu} a^* (\langle \nabla u_0 \rangle_k^\nu) \phi \right| = 0.$$

We now recall that (see [57, Proposition 2.7])

$$\forall \xi, \xi' \in \mathbb{R}^d, \quad |a^*(\xi) - a^*(\xi')| \leq C (|\xi|^{p-2} + |\xi'|^{p-2}) |\xi - \xi'|.$$

This proves that $\sum_{k \in I_\nu} a^* (\langle \nabla u_0 \rangle_k^\nu) \phi 1_{Q_k^\nu}$ converges as $\nu \rightarrow +\infty$ to $a^*(\nabla u_0) \phi$ in $L^{p'}(\Omega)$. We consequently obtain that

$$\int_{\Omega} a_0 \cdot \phi - a^*(\nabla u_0) \phi = 0.$$

Since this is true for all $\phi \in \mathcal{D}(\Omega)$, we conclude that $a_0 = a^*(\nabla u_0)$ a.e. This concludes the proof. \square

BIBLIOGRAPHY

- [1] Robert A Adams and John JF Fournier. *Sobolev spaces*. Elsevier, 2003.
- [2] Grégoire Allaire. Homogenization of the Stokes flow in a connected porous medium. *Asymptotic Analysis*, 2(3):203–222, 1989.
- [3] Grégoire Allaire. Continuity of the Darcy’s law in the low-volume fraction limit. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 18(4):475–499, 1991.
- [4] Grégoire Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes I. Abstract framework, a volume distribution of holes. *Archive for Rational Mechanics and Analysis*, 113(3):209–259, 1991.
- [5] Grégoire Allaire. Homogenization and two-scale convergence. *SIAM Journal on Mathematical Analysis*, 23(6):1482–1518, 1992.
- [6] Grégoire Allaire. One-phase newtonian flow. In *Homogenization and Porous Media*, pages 45–76. Springer, 1997.
- [7] Chérif Amrouche and Vivette Girault. Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension. *Czechoslovak Mathematical Journal*, 44(1):109–140, 1994.
- [8] Nadia Ansini and Andrea Braides. Separation of scales and almost-periodic effects in the asymptotic behaviour of perforated periodic media. *Acta Applicandae Mathematica*, 65(1):59–81, 2001.
- [9] Nadia Ansini and Andrea Braides. Asymptotic analysis of periodically-perforated nonlinear media. *Journal de mathématiques pures et appliquées*, 81(5):439–451, 2002.
- [10] Scott Armstrong, Tuomo Kuusi, and Jean-Christophe Mourrat. *Quantitative stochastic homogenization and large-scale regularity*, volume 352. Springer, 2019.
- [11] Marco Avellaneda and Fang-Hua Lin. Compactness methods in the theory of homogenization. *Communications on Pure and Applied Mathematics*, 40(6):803–847, 1987.
- [12] Marco Avellaneda and Fang Hua Lin. Lp bounds on singular integrals in homogenization. *Communications on pure and applied mathematics*, 44(8-9):897–910, 1991.
- [13] Marco Avellaneda, Fang-Hua Lin, and J-L Lions. Un théorème de liouville pour des équations elliptiques à coefficients périodiques. *Comptes rendus de l’Académie des sciences. Série 1, Mathématique*, 309(5):245–250, 1989.

- [14] Ivo Babuška. Homogenization approach in engineering. In *Computing methods in applied sciences and engineering*, pages 137–153. Springer, 1976.
- [15] A Yu Beliaev and Serguei M Kozlov. Darcy equation for random porous media. *Communications on pure and applied mathematics*, 49(1):1–34, 1996.
- [16] Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou. *Asymptotic Analysis for Periodic Structures*. North-Holland, Amsterdam, 1978.
- [17] Xavier Blanc, Marc Josien, and Claude Le Bris. Precised approximations in elliptic homogenization beyond the periodic setting. *Asymptotic Analysis*, 116(2):93–137, 2020.
- [18] Xavier Blanc, Claude Le Bris, and P-L Lions. On correctors for linear elliptic homogenization in the presence of local defects. *Communications in Partial Differential Equations*, 43(6):965–997, 2018.
- [19] Xavier Blanc, Claude Le Bris, and Pierre-Louis Lions. A possible homogenization approach for the numerical simulation of periodic microstructures with defects. *Milan J. Math.*, 80(2):351–367, 2012.
- [20] Xavier Blanc, Claude Le Bris, and Pierre-Louis Lions. Local profiles for elliptic problems at different scales: defects in, and interfaces between periodic structures. *Comm. Partial Differential Equations*, 40(12):2173–2236, 2015.
- [21] Xavier Blanc, Claude Le Bris, and Pierre-Louis Lions. On correctors for linear elliptic homogenization in the presence of local defects: the case of advection-diffusion. *J. Math. Pures Appl.*, 2018. To appear.
- [22] Xavier Blanc and Sylvain Wolf. Homogenization of the Poisson equation in a non periodically perforated domain. *Asymptotic Analysis*, vol 126:129–155, 2022.
- [23] Alain Bourgeat and Roland Tapiéro. Homogenization in a perforated domain including a thin full interlayer. In *Flow in porous media (Oberwolfach, 1992)*, volume 114 of *Internat. Ser. Numer. Math.*, pages 25–36. Birkhäuser, Basel, 1993.
- [24] Andrea Braides and Anneliese Defranceschi. *Homogenization of multiple integrals*, volume 12. Oxford University Press, 1998.
- [25] Andrea Braides et al. *Gamma-convergence for Beginners*, volume 22. Clarendon Press, 2002.
- [26] Haim Brezis. Semilinear equations in \searrow^N without condition at infinity. *Applied Mathematics and Optimization*, 12(1):271–282, 1984.
- [27] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science & Business Media, 2010.
- [28] Marc Briane, Alain Damlamian, and Patrizia Donato. H -convergence for perforated domains. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XIII(Paris, 1994/1996)*, volume 391 of *Pitman Res. Notes Math. Ser.*, pages 62–100. Longman, Harlow, 1998.
- [29] Giuseppe Cardone, Svetlana E Pastukhova, and Carmen Perugia. Estimates in homogenization of degenerate elliptic equations by spectral method. *Asymptotic Analysis*, 81(3-4):189–209, 2013.
- [30] Kirill D Cherednichenko and Valery P Smyshlyaev. On full two-scale expansion of the solutions of nonlinear periodic rapidly oscillating problems and higher-order homogenised variational problems. *Archive for rational mechanics and analysis*, 174(3):385–442, 2004.

- [31] Doina Cioranescu, Alain Damlamian, Patrizia Donato, Georges Griso, and Rachad Zaki. The periodic unfolding method in domains with holes. *SIAM Journal on Mathematical Analysis*, 44(2):718–760, 2012.
- [32] Doïna Cioranescu, Alain Damlamian, and Georges Griso. Periodic unfolding and homogenization. *C. R. Math. Acad. Sci. Paris*, 335(1):99–104, 2002.
- [33] Doina Cioranescu, Alain Damlamian, and Georges Griso. The periodic unfolding method in homogenization. *SIAM Journal on Mathematical Analysis*, 40(4):1585–1620, 2008.
- [34] Doïna Cioranescu and Patrizia Donato. Homogénéisation du problème de Neumann non homogène dans des ouverts perforés. *Asymptotic Anal.*, 1(2):115–138, 1988.
- [35] Doina Cioranescu and Patrizia Donato. *An introduction to homogenization*, volume 17. Oxford University Press Oxford, 1999.
- [36] Doïna Cioranescu, Patrizia Donato, and Rachad Zaki. Periodic unfolding and Robin problems in perforated domains. *C. R. Math. Acad. Sci. Paris*, 342(7):469–474, 2006.
- [37] Doïna Cioranescu, Patrizia Donato, and Rachad Zaki. The periodic unfolding method in perforated domains. *Port. Math. (N.S.)*, 63(4):467–496, 2006.
- [38] Doïna Cioranescu and François Murat. Un terme étrange venu d’ailleurs. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. II (Paris, 1979/1980)*, volume 60 of *Res. Notes in Math.*, pages 98–138, 389–390. Pitman, Boston, Mass.-London, 1982.
- [39] Doïna Cioranescu and François Murat. Un terme étrange venu d’ailleurs. II. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. III (Paris, 1980/1981)*, volume 70 of *Res. Notes in Math.*, pages 154–178, 425–426. Pitman, Boston, Mass.-London, 1982.
- [40] Doïna Cioranescu and Jeannine Saint Jean Paulin. Homogenization in open sets with holes. *J. Math. Anal. Appl.*, 71(2):590–607, 1979.
- [41] Nicolas Clozeau. *Quantitative estimates in stochastic homogenization of elliptic equations and systems*. PhD thesis, Sorbonne université, 2021.
- [42] Nicolas Clozeau and Antoine Gloria. Quantitative nonlinear homogenization: control of oscillations. *arXiv preprint arXiv:2104.04263*, 2021.
- [43] Carlos Conca and Patrizia Donato. Non-homogeneous Neumann problems in domains with small holes. *ESAIM: Mathematical Modelling and Numerical Analysis*, 22(4):561–607, 1988.
- [44] Gianni Dal Maso and Anneliese Defranceschi. Correctors for the homogenization of monotone operators. *Differential and Integral Equations*, 3(6):1151–1166, 1990.
- [45] Gianni Dal Maso and Luciano Modica. Nonlinear stochastic homogenization. *Annali di matematica pura ed applicata*, 144(1):347–389, 1986.
- [46] Alain Damlamian and Patrizia Donato. Which sequences of holes are admissible for periodic homogenization with Neumann boundary condition? *ESAIM Control Optim. Calc. Var.*, 8:555–585, 2002. A tribute to J. L. Lions.
- [47] Henry Darcy. *Les fontaines publiques de la ville de Dijon: exposition et application...* Victor Dalmont, 1856.
- [48] Emmanuele DiBenedetto. $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. Technical report, Wisconsin Univ-Madison mathematics research center, 1982.

- [49] Patrizia Donato and Gioconda Moscariello. On the homogenization of some nonlinear problems in perforated domains. *Rendiconti del Seminario Matematico della Università di Padova*, 84:91–108, 1990.
- [50] Patrizia Donato and Jeannine Saint Jean Paulin. Homogenization of the Poisson equation in a porous medium with double periodicity. *Japan journal of industrial and applied mathematics*, 10(2):333, 1993.
- [51] Patrizia Donato and Colette Picard. Convergence of Dirichlet problems for monotone operators in a class of porous media. *Ricerche di Matematica*, 49:245–268, 2000.
- [52] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.
- [53] Herbert Federer. *Geometric measure theory. Repr. of the 1969 ed.* Berlin: Springer-Verlag, repr. of the 1969 ed. edition, 1996.
- [54] Florian Feppon. High order homogenization of the stokes system in a periodic porous medium. 2020.
- [55] Julian Fischer and Stefan Neukamm. Optimal homogenization rates in stochastic homogenization of nonlinear uniformly elliptic equations and systems. *arXiv preprint arXiv:1908.02273*, 2019.
- [56] Nicola Fusco and Gioconda Moscariello. Further results on the homogenization of quasilinear operators. 1985.
- [57] Nicola Fusco and Gioconda Moscariello. On the homogenization of quasilinear divergence structure operators. *Annali di matematica pura ed applicata*, 146(1):1–13, 1986.
- [58] Giovanni Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems*. Springer Science & Business Media, 2011.
- [59] Mariano Giaquinta. *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. (AM-105), Volume 105*. Princeton University Press, 2016.
- [60] David Gilbarg and Neil Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [61] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Springer, 2015.
- [62] Arianna Giunti. Convergence rates for the homogenization of the Poisson problem in randomly perforated domains. *arXiv preprint arXiv:2007.13386*, 2020.
- [63] Arianna Giunti. Derivation of Darcy’s law in randomly punctured domains. *arXiv preprint arXiv:2101.01046*, 2021.
- [64] Arianna Giunti and Richard M Höfer. Homogenisation for the Stokes equations in randomly perforated domains under almost minimal assumptions on the size of the holes. In *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*, volume 36, pages 1829–1868. Elsevier, 2019.
- [65] Antoine Gloria, Stefan Neukamm, and Felix Otto. A regularity theory for random elliptic operators. *arXiv preprint arXiv:1409.2678*, 2014.
- [66] Rémi Goudey. A periodic homogenization problem with defects rare at infinity. *arXiv preprint arXiv:2109.05506*, 2021.
- [67] Matthieu Hillairet. On the homogenization of the Stokes problem in a perforated domain. *Archive for Rational Mechanics and Analysis*, 230(3):1179–1228, 2018.

- [68] Tadeusz Iwaniec. Projections onto gradient fields and L^p -estimates for degenerated elliptic operators. *Studia Mathematica*, 75(3):293–312, 1983.
- [69] Willi Jäger and Andro Mikelić. On the flow conditions at the boundary between a porous medium and an impervious solid. 1994.
- [70] Gaspard Jankowiak and Alexei Lozinski. Non-conforming multiscale finite element method for Stokes flows in heterogeneous media. Part II: Error estimates for periodic microstructure. *arXiv preprint arXiv:1802.04389*, 2018.
- [71] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin, 1994.
- [72] Vasili Vasilievitch Jikov, Sergei M Kozlov, and Olga Arsenievna Oleinik. *Homogenization of differential operators and integral functionals*. Springer Science & Business Media, 2012.
- [73] Wenjia Jing. A unified homogenization approach for the Dirichlet problem in perforated domains. *SIAM Journal on Mathematical Analysis*, 52(2):1192–1220, 2020.
- [74] Tuomo Kuusi and Giuseppe Mingione. A nonlinear stein theorem. *Calculus of Variations and Partial Differential Equations*, 51(1):45–86, 2014.
- [75] Claude Le Bris, Frédéric Legoll, and Alexei Lozinski. An MsFEM type approach for perforated domains. *Multiscale Model. Simul.*, 12(3):1046–1077, 2014.
- [76] Hervé Le Dret. *Nonlinear Elliptic Partial Differential Equations*. Springer, 2018.
- [77] Jacques-Louis Lions. Asymptotic expansions in perforated media with a periodic structure. *Rocky Mountain J. Math.*, 10(1):125–140, 1980.
- [78] Robert Lipton and Marco Avellaneda. Darcy’s law for slow viscous flow past a stationary array of bubbles. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 114(1-2):71–79, 1990.
- [79] Yong Lu. Homogenization of Stokes equations in perforated domains: a unified approach. *Journal of Mathematical Fluid Mechanics*, 22(3):1–13, 2020.
- [80] Eduard Marušić-Paloka. Asymptotic expansion for a flow in a periodic porous medium. *Comptes Rendus de l’Academie des Sciences-Series IIB-Mechanics-Physics-Chemistry-Astronomy*, 325(7):369–374, 1997.
- [81] Eduard Marusic-Paloka and Andro Mikelić. An error estimate for correctors in the homogenization of the Stokes and Navier-Stokes equations in a porous medium. *Boll. Unione Mat. Ital*, 10(3):661–671, 1996.
- [82] Nader Masmoudi. Some uniform elliptic estimates in a porous medium. *Comptes Rendus Mathematique*, 339(12):849–854, 2004.
- [83] François Murat. Compacité par compensation. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 5(3):489–507, 1978.
- [84] Sergueĭ A Nazarov and Gudrun Thäter. Asymptotics at infinity of solutions to the Neumann problem in a sieve-type layer. *Comptes Rendus Mecanique*, 331(1):85–90, 2003.
- [85] Gabriel Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM Journal on Mathematical Analysis*, 20(3):608–623, 1989.
- [86] Gabriel Nguetseng. Homogenization structures and applications i. *Zeitschrift für Analysis und ihre Anwendungen*, 22(1):73–108, 2003.

- [87] Gabriel Nguetseng. Homogenization in perforated domains beyond the periodic setting. *J. Math. Anal. Appl.*, 289(2):608–628, 2004.
- [88] L Nirenberg. On elliptic partial differential equations, estratto dagli annali della scuola normale superiore di pisa serie iii xiii. *Fasc. II*, 1959.
- [89] George C Papanicolaou. Boundary value problems with rapidly oscillating random coefficients. In *Colloquia Math. Soc., Janos Bolyai*, volume 27, pages 853–873, 1979.
- [90] Dan Polisevski. The Div-Curl Lemma revisited. *arXiv preprint arXiv:0712.2133*, 2007.
- [91] Enrique Sanchez-Palencia. Fluid flow in porous media. *Non-homogeneous media and vibration theory*, pages 129–157, 1980.
- [92] Ben Schweizer. Darcy’s law and groundwater flow modelling. 2015.
- [93] Zhongwei Shen. *Periodic homogenization of elliptic systems*. Springer, 2018.
- [94] Zhongwei Shen. Compactness and large-scale regularity for Darcy’s law. *arXiv preprint arXiv:2104.05074*, 2021.
- [95] Zhongwei Shen. Sharp convergence rates for Darcy’s law. *Communications in Partial Differential Equations*, 2022.
- [96] E De Giorgi-S Spagnolo. Sulla convergenza degli integrali dell’energia per operatori ellittici del secondo ordine. *Boll. Un. Mat. Ital.*, 4(8):391–411, 1973.
- [97] Luc Tartar. Cours Peccot au collège de france. *Manuscript (unpublished)*, 1977.
- [98] Luc Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt symposium*, volume 4, pages 136–212, 1979.
- [99] Luc Tartar. Incompressible fluid flow in a porous medium. Convergence of the homogenization process. Appendix in E. Sanchez-Palencia, Nonhomogeneous media and vibration theory. *Lecture Notes in Phys*, 127, 1980.
- [100] Luc Tartar. H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 115(3-4):193–230, 1990.
- [101] Luc Tartar. *The general theory of homogenization: a personalized introduction*, volume 7. Springer Science & Business Media, 2009.
- [102] Muthusamy Vanninathan. Homogenization of eigenvalue problems in perforated domains. *Proc. Indian Acad. Sci. Math. Sci.*, 90(3):239–271, 1981.
- [103] Li Wang, Qiang Xu, and Peihao Zhao. Quantitative estimates on periodic homogenization of nonlinear elliptic operators. *arXiv preprint arXiv:1807.10865*, 2018.
- [104] Li Wang, Qiang Xu, and Peihao Zhao. Convergence rates on periodic homogenization of p-laplace type equations. *Nonlinear Analysis: Real World Applications*, 49:418–459, 2019.
- [105] Sylvain Wolf. Homogenization of the poisson equation and the stokes system in some non periodically perforated domains. *Oberwolfach report*, 14, 2021.
- [106] Sylvain Wolf. Homogenization of p-laplace type equations in a periodic setting with defects. *Submitted*, 2022.
- [107] Sylvain Wolf. Homogenization of the Stokes system in a non periodically perforated domain. *Multiscale, Model. and Simul.*, vol 20(1):72–106, 2022.

-
- [108] Vasilii Vasil'evich Zhikov and Svetlana Evgenievna Pastukhova. Homogenization of degenerate elliptic equations. *Siberian Mathematical Journal*, 49(1):80–101, 2008.
- [109] Vasilii Vasil'evich Zhikov. Weighted sobolev spaces. *Sbornik: Mathematics*, 189(8):1139, 1998.
- [110] Xiao Zhong. De Giorgi–Nash–Moser theory. In *Harmonic and geometric analysis*, Adv. Courses Math. CRM Barcelona, pages 145–170. Birkhäuser/Springer Basel AG, Basel, 2015.

Étude de problèmes d'homogénéisation dans un cadre périodique avec défauts

Résumé

Cette thèse s'intéresse à des problèmes d'homogénéisation dans un cadre périodique avec défauts. Elle se divise en deux parties. Dans une première partie, nous étudions le cas de l'homogénéisation en milieu perforé. En s'appuyant sur l'homogénéisation périodique des problèmes de Poisson et de Stokes, nous construisons une notion de perturbation locale du domaine perforé périodique. Cela permet d'obtenir les mêmes types de résultats, à la fois pour l'équation de Poisson et pour le système de Stokes, que dans le cas périodique. Nous construisons en particulier les correcteurs associés aux problèmes et nous obtenons des taux de convergence vers la solution homogénéisée. Dans une seconde partie, nous regardons des équations dont les coefficients sont oscillants à l'échelle $\varepsilon \ll 1$. Partant à nouveau du cas d'un coefficient périodique, nous étudions des perturbations locales de ce coefficient. Ce cadre a été introduit par X. Blanc, C. Le Bris et P.-L. Lions dans le cas linéaire. Nous étudions ici des cas non-linéaires. Nous construisons les correcteurs associés et nous obtenons des théorèmes de convergence sous certaines hypothèses.

Mots-clés: homogénéisation périodique, homogénéisation non-linéaire, milieu perforé, équations elliptiques.

Some Homogenization problems in a periodic setting with defects

Abstract

This PhD thesis focuses on some homogenization problems in a periodic setting with defects. Our study is divided in two parts. In the first part, we study the homogenization in porous medium. Relying on the case of periodically perforated domains, we build a framework that allows to address local perturbation of the periodic setting for the Poisson problem and the Stokes system. We obtain the same type of results as in the periodic case for both equations. We build in particular the correctors and we obtain convergence rates to the homogenized solution. In a second part, we consider equations with oscillating coefficients at scale $\varepsilon \ll 1$. Given a periodic background, we introduce some local perturbation of the coefficient. This setting has been introduced by X. Blanc, C. Le Bris and P. L. Lions for the linear case. We study here nonlinear cases. We build the associated correctors and we obtain convergence theorems under some assumptions.

Keywords: periodic homogenization, nonlinear homogenization, porous medium, elliptic PDEs.

