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Propriétés qualitatives de solutions de problèmes dégénérés et/ou singuliers en calcul des variations

Benjamin Lledos

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THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*

Présentée et soutenue le *11/07/2023* par :

Benjamin LLEDOS

**Propriétés qualitatives de solutions de problèmes dégénérés et/ou
singuliers en calcul des variations**

JURY

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Le 11 juillet 2023

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et/ou singuliers en Calcul des Variations.**

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*Si j'ai pu voir plus loin, c'est que
je me tenais sur les épaules de
géants.*

Isaac Newton

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Résumé

Propriétés qualitatives pour des solutions de problèmes dégénérés et/ou singuliers en calcul des variations

Résumé : Cette thèse s'inscrit dans les domaines du calcul des variations, des équations aux dérivées partielles elliptiques et de la théorie géométrique de la mesure.

Dans le premier chapitre, nous prouvons un résultat d'unicité des solutions pour un problème de minimisation de la forme $\int_{\Omega} \varphi(\nabla u) - \lambda u$ avec λ une fonction constante. L'autonomie de cette fonctionnelle nous permet d'effectuer des translations de solutions comme dans [29] pour montrer qu'il existe au plus un minimiseur uniformément continu.

Dans le deuxième chapitre, nous considérons une fonction convexe φ comme celle présente dans [8] n'étant pas différentiable à l'origine et λ une fonction lipschitzienne. La preuve de l'unicité dans ce cas-là repose sur une étude de la régularité des ensembles de niveau.

Dans le troisième chapitre, on travaille avec un autre φ introduit dans [41] qui n'est pas strictement convexe sur plusieurs ensembles. Nous prouvons un résultat d'unicité en dimension deux reposant encore une fois sur une étude des lignes de niveau et sur un principe du maximum pour $\nabla\varphi(\nabla u)$ avec u un minimiseur.

Dans le quatrième chapitre, nous nous intéressons à la régularité des solutions faibles d'équations aux dérivées partielles elliptiques dégénérées de la forme $\operatorname{div} G(\nabla u) = f$. En dimension deux quand f est une constante nous présentons des résultats sur la continuité de $G(\nabla u)$ sous diverses hypothèses de croissance sur G . Nous établissons aussi une extension de [57] en élargissant la zone de dégénérescence de G .

Mots-Clés : Calcul des variations, équations aux dérivées partielles, théorie géométrique de la mesure, unicité, régularité, ensembles de niveau.

Qualitative properties for solutions of degenerate and/or singular problems in the calculus of variations

Abstract : This thesis belongs in the fields of calculus of variations, elliptic partial differential equations and geometric measure theory.

In the first chapter, we prove a uniqueness result for the solutions of a minimization problem of the form $\int_{\Omega} \varphi(\nabla u) - \lambda u$ with λ a constant function. The autonomy of this functional allows us to use translations of solutions as in [29] to prove that there exists at most one uniformly continuous minimizer.

In the second chapter, we consider a convex function φ as the one in [8] which is not differentiable at the origin and λ a Lipschitz continuous function. The proof of the uniqueness in this case relies on a study of the regularity of the level sets.

In the third chapter, we work with another φ introduced in [41] which is not strictly convex on several sets. We prove a uniqueness result in dimension two based again on a study of the level lines and on a maximum principle for $\nabla\varphi(\nabla u)$ with u a minimizer.

In the fourth chapter, we are interested in the regularity of weak solutions of degenerate elliptic partial differential equations of the form $\operatorname{div} G(\nabla u) = f$. In dimension two when f is a constant we present some results on the continuity of $G(\nabla u)$ under various growth assumptions on G . We also establish an extension of [57] by enlarging the degeneracy zone of G .

Keywords : Calculus of Variations, partial differential equations, geometric measure theory, uniqueness, regularity, level sets.

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Introduction

Dans cette thèse, nous nous intéressons aux questions d'unicité et de régularité pour les minimiseurs de problèmes dégénérés et/ou singuliers en calcul des variations. Nous étudions aussi les propriétés qualitatives de ces fonctions et généralisons l'étude de la régularité aux solutions des équations aux dérivées partielles elliptiques. Nous verrons les liens étroits qui unissent ces différentes notions. Cela est notamment le cas de la régularité des ensembles de niveau des minimiseurs qui est importante pour prouver l'unicité d'une telle solution.

0.1 Présentation du sujet

0.1.1 Présentation générale du calcul des variations

Dans cette section nous présentons le cadre général de cette thèse. Commençons par nous donner une structure classique avant d'étudier les résultats généraux pour certains problèmes de minimisation en calcul des variations. Soit $N \geq 1$ un entier naturel, on considère un ouvert Ω borné de \mathbb{R}^N et

$$\begin{aligned} F &: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\} \\ (x, u, z) &\mapsto F(x, u, z) \end{aligned}$$

une fonction mesurable. Cette fonction nous sert d'intégrande pour définir la fonctionnelle suivante :

$$I_F : u \rightarrow \int_{\Omega} F(x, u(x), \nabla u(x)) dx.$$

Soit \mathcal{E} un ensemble de fonctions tel que pour tout $u \in \mathcal{E}$ la quantité $I_F(u)$ soit bien définie, on peut introduire le problème de minimisation suivant :

$$\mathcal{P}_F : \inf_{u \in \mathcal{E}} I_F(u). \tag{0.1.1}$$

Nous appelons solution ou minimiseur du problème \mathcal{P}_F toute fonction $u \in \mathcal{E}$ telle que $I_F(u) \leq I_F(v)$ pour tout $v \in \mathcal{E}$. L'objectif de cette thèse est d'étudier les solutions d'un tel problème. Plusieurs questions naturelles peuvent être posées. Ce problème admet-il une solution ? Si une solution existe est-elle unique ? Quelle régularité peut-on espérer des minimiseurs ? Ont-ils certaines propriétés particulières ?

Les réponses à ces questions dépendent fortement des conditions dans lesquelles le problème \mathcal{P}_F est posé. En effet, la dimension N , la structure de F et le choix de \mathcal{E} sont autant de paramètres pouvant influencer sur les propriétés des solutions. De plus, ces choix peuvent en réalité être contraints si le problème a pour origine la

modélisation d'un phénomène particulier que cela soit en optimisation de forme, en physique des matériaux ou toute autre provenance. Par exemple, l'ensemble \mathcal{E} sur lequel on effectue la minimisation peut contenir des contraintes empêchant la preuve de l'existence d'un minimiseur alors que le choix d'un autre ensemble aurait permis d'obtenir une solution. Cette thèse s'articulant autour des questions d'unicité, de régularité et de l'étude de propriétés qualitatives des solutions nous allons nous placer dans un cadre garantissant l'existence d'au moins un minimiseur.

0.1.2 Existence en calcul des variations : la méthode directe

La méthode classique pour prouver l'existence d'une solution au problème 0.1.1 est appelée méthode directe. Pour ce faire, on introduit pour $p \geq 1$ les espaces de Sobolev suivant :

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega, \mathbb{R}) \text{ tel que } \nabla u \in L^p(\Omega, \mathbb{R}^N)\}.$$

Ici ∇u , le gradient de u au sens des distributions coïncide avec une fonction appartenant à l'espace $L^p(\Omega)$ des fonctions dont la puissance p est intégrable pour la mesure de Lebesgue. Si $u \in W^{1,p}(\Omega)$ on peut lui associer la norme de Sobolev suivante : $\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$. L'ensemble $W^{1,p}(\Omega)$ munit de cette norme est un espace de Banach. De plus, quand $p > 1$ cet espace est réflexif.

Dans cette thèse, l'ensemble \mathcal{E} sera en général un sous-ensemble de $W^{1,p}(\Omega)$ vérifiant une condition de Dirichlet sur le bord $\partial\Omega$ de Ω . Il existe plusieurs manières de définir cette notion de condition au bord pour des fonctions qui ne sont pas forcément continues. L'une d'entre elles est d'introduire l'espace $W_0^{1,p}(\Omega)$ comme étant l'adhérence dans $W^{1,p}(\Omega)$ pour la norme $\|\cdot\|_{W^{1,p}(\Omega)}$ des fonctions \mathcal{C}^∞ à support compact dans Ω . Si l'on se donne $\psi \in W^{1,p}(\Omega)$, on peut définir

$$W_\psi^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega) \text{ tel que } u - \psi \in W_0^{1,p}(\Omega)\}.$$

Une autre manière de procéder dans le cas d'un ouvert Ω ayant un bord $\partial\Omega$ lipschitzien est l'utilisation de l'opérateur trace, noté Tr . Cet opérateur est le prolongement à $W^{1,p}(\Omega)$ de la fonction restriction sur $\partial\Omega$ définie pour les fonctions continues sur $\bar{\Omega}$. Dans ce là, pour $\psi \in W^{1-\frac{1}{p},p}(\partial\Omega)$ on peut définir

$$W_\psi^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega) \text{ tel que } Tr(u) = \psi\}.$$

Dans cette thèse nous n'avons pas besoin d'une condition au bord aussi générale et ψ sera dans la majorité des cas une fonction lipschitzienne.

Dorénavant, on suppose donc que $\mathcal{E} = W_\psi^{1,p}(\Omega)$ avec $p > 1$. Nous avons alors le résultat suivant : si I_F est séquentiellement faiblement semi-continue inférieurement pour la topologie induite par la norme de Sobolev sur cet ensemble et si I_F est coercive sur $W_\psi^{1,p}(\Omega)$ dans le sens où :

$$\lim_{\|u\|_{W^{1,2}(\Omega)} \rightarrow +\infty} I_F(u) = +\infty$$

alors I_F admet au moins un minimiseur sur $\mathcal{E} = W_\psi^{1,p}(\Omega)$.

La preuve de ce résultat peut être trouvée dans les livres de Dacorogna [27, Théorème 3.30] et Giusti [37, Théorème 4.6]. La condition nécessaire et suffisante pour avoir la semi-continuité faible dans $W^{1,p}(\Omega)$ avec $p > 1$ est la convexité de $z \rightarrow F(x, u, z)$ pour tout $(x, u) \in \Omega \times \mathbb{R}$. Il existe aussi des conditions suffisantes garantissant la coercivité de I_F , la plus commune étant l'existence de $c_1 > 0$, $c_2 \in \mathbb{R}$, $c_3 \in L^1(\Omega)$ et $1 \leq q < p$ tels que :

$$F(x, u, z) \geq c_1|z|^p + c_2|u|^q + c_3(x)$$

pour presque tout $x \in \Omega$ et tout $(u, z) \in \mathbb{R} \times \mathbb{R}^N$ avec $|\cdot|$ la norme euclidienne usuelle sur \mathbb{R}^N . De plus, cette condition a aussi le mérite de garantir que I_F est bien définie sur $W_\psi^{1,p}(\Omega)$.

L'idée de la preuve de ce résultat d'existence est la suivante. Nous pouvons supposer que $\inf_{v \in W_\psi^{1,p}(\Omega)} I_F(v) < +\infty$ sinon il n'y a rien à faire. On peut donc consi-

dérer une suite minimisante $(u_n)_{n \in \mathbb{N}}$ de I_F sur $W_\psi^{1,p}(\Omega)$. Ensuite, nous utilisons la coercivité de I_F pour montrer que l'on peut borner les normes de Sobolev des éléments de cette suite. Grâce au théorème de Banach-Alaoglu et à la réflexivité de $W^{1,p}(\Omega)$ nous pouvons extraire une sous-suite convergente de $(u_n)_{n \in \mathbb{N}}$ qui tend vers $u \in W_\psi^{1,p}(\Omega)$. Finalement, la semi-continuité inférieure faible de I_F implique que u est un minimiseur de I_F sur $W_\psi^{1,p}(\Omega)$.

Maintenant que nous savons comment garantir l'existence d'une solution pour le problème \mathcal{P}_F (voir (0.1.1)) quand $\mathcal{E} = W_\psi^{1,p}(\Omega)$, on peut se demander si cette solution est unique.

0.1.3 Unicité en calcul des variations

Admettons que le problème \mathcal{P}_λ admette un minimiseur u dans $W_\psi^{1,p}(\Omega)$. Un moyen de garantir la non-existence d'une autre solution est de supposer que $(u, z) \rightarrow F(x, u, z)$ soit strictement convexe pour presque tout $x \in \Omega$. En effet, sous cette hypothèse la fonctionnelle I_F devient strictement convexe, elle admet donc au plus un minimiseur.

Nous pouvons donc nous demander ce qu'il en est dans le cas où F est convexe mais pas strictement convexe. Il n'y pas de réponse générale à cette question et il faut souvent étudier les problèmes au cas par cas pour essayer d'obtenir une preuve d'unicité ou de non-unicité. En effet, considérons la fonction convexe suivante définie sur \mathbb{R} :

$$\varphi(x) := \begin{cases} |x| & \text{si } |x| \leq 1, \\ \frac{1}{2}(|x|^2 + 1) & \text{si } |x| > 1. \end{cases} \quad (0.1.2)$$

Si $\Omega = (0, 1)$ alors il n'y a que la fonction $u(x) = x$ qui minimise $\int_0^1 \varphi(v'(x)) dx$ sur $W_\psi^{1,2}((0, 1))$ avec $\psi(0) = 0$ et $\psi(1) = 1$. En revanche, si $\psi(0) = 0$ et $\psi(1) = \frac{1}{2}$ toutes les fonctions 1-lipschitziennes croissantes satisfaisant la condition au bord sont solutions.

Une large partie de cette thèse est dédiée à l'étude de l'unicité des solutions pour des problèmes convexes qui ne sont pas strictement convexes. Plus précisément, comme il n'y a pas de théorie générale pour prouver l'unicité nous allons nous restreindre à une certaine famille de problèmes où la fonctionnelle à minimiser est de la forme :

$$I_\varphi : u \rightarrow \int_{\Omega} \varphi(\nabla u(x)) + \lambda(x)u(x)dx \quad (0.1.3)$$

avec $\varphi : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ une fonction convexe et $\lambda \in L^\infty(\Omega)$. L'objectif est d'explicitier des méthodes pour prouver ou infirmer l'unicité des solutions pour de tels problèmes.

Si l'existence d'une unique solution pour un problème donné est déjà une information importante en soi, il existe plusieurs applications de ce résultat. Nous pouvons par exemple prouver la non-existence de solution pour des problèmes non-convexes. En effet, si φ n'est pas une fonction convexe on ne peut pas garantir l'existence d'un minimiseur même sous les hypothèses de croissance de φ du type $\varphi(z) \geq c_1|z|^p + c_2$ avec $c_1 > 0$ et $c_2 \in \mathbb{R}$. En revanche, on peut introduire φ^{**} comme étant la plus grande minorante convexe de φ . Ainsi, la fonctionnelle $I_{\varphi^{**}}$ est convexe et le problème admet une solution u^{**} . D'après Ekeland et Teman [30], nous avons que

$$\inf_{u \in W_\psi^{1,p}(\Omega)} I_\varphi(u) = \min_{u \in W_\psi^{1,p}(\Omega)} I_{\varphi^{**}}(u).$$

Si l'on parvient à prouver que $I_{\varphi^{**}}(u^{**}) < I_\varphi(u^{**})$ alors u^{**} n'est pas solution du problème originel. De plus, si u^{**} est l'unique solution du problème convexifié alors pour tout $v \neq u$ on a

$$\inf_{u \in W_\psi^{1,p}(\Omega)} I_\varphi(u) = \min_{u \in W_\psi^{1,p}(\Omega)} I_{\varphi^{**}}(u) = I_{\varphi^{**}}(u^{**}) < I_{\varphi^{**}}(v) \leq I_\varphi(v).$$

Ainsi le problème originel n'admet pas de solution. En revanche si $I_{\varphi^{**}}(u^{**}) = I_\varphi(u^{**})$ alors u^{**} est solution de problème originel. Une autre application intervient dans le cas où notre problème admet certaines symétries. Il est alors possible de prouver que la solution u possède les mêmes symétries. Enfin, l'existence d'une unique solution ouvre la porte à la question de la régularité de celle-ci.

0.1.4 Régularité en calcul des variations

Si l'on se donne une solution u d'un certain problème de minimisation \mathcal{P}_F sur un espace de fonctions qui ne sont pas nécessairement continues on peut espérer que cette solution soit plus régulière que prévue.

La question de la continuité des solutions est importante si la minimisation est effectuée sur l'ensemble $W_\psi^{1,p}(\Omega)$ avec $1 < p \leq N$ qui ne s'injecte pas dans les fonctions uniformément continues sur Ω . Néanmoins, nous savons que si le bord de Ω est lipschitzien, si $\psi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$ et si il existe $d_1, d_2 > 0$ tels que :

$$|z|^p - d_1|u|^p - d_2 \leq F(x, u, z) \leq d_1(|z|^p + |u|^p) + d_2$$

alors les minimiseurs sont dans $\mathcal{C}^{0,\alpha}(\overline{\Omega})$ avec $0 < \alpha < 1$ par [37]. Il existe des résultats de régularité lipschitzienne locale comme dans les articles de Brasco [14], Celada, Cupini et Guidorzi [20] et Fonseca, Fusco et Marcellini [34] sous des hypothèses d'uniforme convexité à l'infini. Dans cette thèse, nous prouvons un résultat de régularité lipschitzienne globale sans majoration de la croissance de la fonctionnelle. Ce dernier résultat aura par ailleurs une importance capitale pour des preuves d'unicité.

Il n'est pas toujours vrai qu'une solution soit lipschitzienne. C'est notamment le cas quand $\inf_{u \in W_{\psi}^{1,p}(\Omega)} I_F(u) < \inf_{v \in Lip_{\psi}(\Omega)} I_F(v)$ où $Lip_{\psi}(\Omega)$ est l'ensemble des fonction lipschitziennes sur $\overline{\Omega}$ étant égales à ψ sur le bord de Ω . Cette situation est appelée phénomène de Lavrentiev. Un exemple d'occurrence d'un tel phénomène est dû à Mania quand $F(x, u, z) = (x - u^3)^2 z^6$, $\Omega = (0, 1)$, $\psi(0) = 0$ et $\psi(1) = 1$. On peut montrer que $\inf_{u \in Lip_{\psi}(0,1)} I_F(u) > \min_{u \in W_{\psi}^{1,1}(0,1)} I_F(u) = 0$ et que $u(x) = x^{\frac{1}{3}}$ est une solution.

La théorie de la régularité \mathcal{C}^1 des minimiseurs a connu des développements importants depuis un siècle. Nous pouvons notamment citer les travaux parallèles de De Giorgi dans [28] et Nash dans [59] qui ont montré le résultat suivant. On considère $F : \mathbb{R} \rightarrow \mathbb{R}$ une fonction convexe de classe \mathcal{C}^2 telle que

$$\frac{1}{C}|\xi|^2 \leq \langle D^2 F(z)\xi, \xi \rangle \leq C|\xi|^2 \quad (0.1.4)$$

pour un certain $C > 0$ et tous $z, \xi \in \mathbb{R}^N$. Si l'on souhaite étudier la régularité du minimiseur de

$$\int_{\Omega} F(\nabla u) \text{ sur } W_{\psi}^{1,2}(\Omega),$$

le résultat de De Giorgi et Nash affirme que la solution d'un tel problème est localement $\mathcal{C}^{1,\alpha}$ dans Ω . Dans le cas où l'on suppose de plus que F est lisse alors u est lisse aussi.

Cette condition (0.1.4) sur la matrice hessienne de F est appelée condition d'uniforme ellipticité. Nous pouvons nous demander ce qu'il se passe si cette hypothèse n'est pas vérifiée. Il est aussi intéressant de regarder les résultats que l'on peut obtenir selon la taille de la zone où la fonctionnelle n'est pas elliptique. C'est ce que l'on va étudier dans le dernier chapitre de cette thèse en généralisant cela à l'étude de la régularité pour des équations aux dérivées partielles elliptiques qui est par ailleurs aussi au coeur de la preuve du résultat de De Giorgi et Nash.

0.1.5 Équations d'Euler-Lagrange

Supposons que u soit un minimiseur lipschitzien sur $W_{\psi}^{1,p}(\Omega)$ de la fonctionnelle (0.1.3) avec φ une fonction convexe de classe \mathcal{C}^1 . Nous avons alors que u est une solution faible de l'équation

$$\operatorname{div}(\nabla \varphi(\nabla u)) = \lambda \text{ sur } \Omega. \quad (0.1.5)$$

Inversement si $u \in W_\psi^{1,2}(\Omega)$ est une solution faible de l'équation d'Euler-Lagrange (0.1.5) alors u est un minimiseur de I_φ . Cette équation a plusieurs intérêts. En premier lieu elle permet de créer un pont entre le calcul des variations et les équations aux dérivées partielles elliptiques. Cela est notamment utile pour des questions de régularité. En effet si $\lambda = 0$ et φ est une fonction uniformément convexe lisse vérifiant (0.1.4) alors les solutions de cette équation sont \mathcal{C}^∞ et cela est donc le cas pour les minimiseurs de I_φ .

Le quatrième chapitre de cette thèse est dédié à la régularité des solutions d'équations elliptique de la forme

$$\begin{cases} \operatorname{div} G(\nabla u) = f \text{ sur } \Omega, \\ u = \psi \text{ sur } \partial\Omega \end{cases}$$

avec G une fonction vérifiant $\langle G(z_1) - G(z_2), z_1 - z_2 \rangle \geq 0$ pour tous $z_1, z_2 \in \mathbb{R}^N$ n'étant pas nécessairement le gradient d'une fonction convexe. Nous obtenons notamment des résultats sur la continuité de $G(\nabla u)$ qui permettent d'avoir des informations sur la régularité locale de u . Comme nous étudierons des solutions localement lipschitziennes nous pouvons citer des articles établissant cette régularité : [14] énoncé dans le cadre des problèmes de minimisations, un article général de Esposito, Mingione et Trombetti [31] et [66] écrit par Tsubouchi dans le cadre de la somme entre le 1-Laplacien et le p -Laplacien.

Dans cette thèse, nous nous intéressons aussi à une autre application de cette équation. Il s'agit de travailler sur l'apport de cette équation à la géométrie des ensembles de niveau $[u = t]$ avec u un minimiseur de I_φ . Cela nous permet de montrer que les ensembles de niveau d'un minimiseur sont aussi des solutions d'un problème de minimisation. Grâce à cela nous pouvons prouver la régularité \mathcal{C}^1 dans certains cas pour de tels ensembles. Ces outils seront grandement utilisés pour prouver l'unicité des solutions dans certains problèmes présenté dans les deuxième et troisième chapitres.

0.2 Plan de la thèse et présentation des chapitres

Cette thèse est composée de quatre chapitres. Les trois premiers sont consacrés à la question de l'unicité pour des problèmes de la forme :

$$\min_{u \in W_\psi^{1,p}(\Omega)} \int_\Omega \varphi(\nabla u) - \lambda u. \quad (0.2.1)$$

Dans le premier chapitre, nous supposons que $\lambda \in \mathbb{R}$ est constante. Dans ce cas là, nous prouvons qu'il existe au plus une solution uniformément continue sur Ω quand il n'existe pas d'ouvert non vide de \mathbb{R}^N tel que φ soit affine dessus. Après avoir présenté des applications de ce résultat, nous utilisons l'invariance par translation de ce type de problème pour prouver le théorème.

Pour les deuxième et troisième chapitres, on se restreint à deux φ particuliers tels qu'il existe une large zone de \mathbb{R}^N sur laquelle ces fonctions ne sont pas strictement convexes. Dans ces deux cas on considère des λ non constants. L'existence de contre-exemples présentés dans ces deux chapitres nous pousse à restreindre notre étude à des λ ayant de faibles oscillations ou étant petits. La stratégie de la preuve repose sur un résultat de régularité lipschitzienne globale des solutions présenté dans le deuxième chapitre. Nous étudions ensuite la régularité des lignes de niveau des minimiseurs pour montrer que deux solutions sont égales presque partout. Le problème considéré dans le deuxième chapitre est singulier car φ n'est pas différentiable à l'origine et dégénéré car la matrice hessienne de φ a une valeur propre nulle sur la boule unité. Dans le cadre du troisième chapitre, le problème énoncé en dimension deux est seulement dégénéré mais la zone de dégénérescence n'est ni convexe ni même connexe car il s'agit d'une union d'anneaux concentriques. Cette alternance entre parties affines et strictement convexes crée des difficultés supplémentaires concernant la régularité des lignes de niveau et leur géométrie.

Le dernier chapitre est quant à lui dédié à l'étude de la régularité \mathcal{C}^1 des solutions d'équations aux dérivées partielles dégénérées de la forme

$$\begin{cases} \operatorname{div} G(\nabla u) = f & \text{sur } \Omega, \\ u = \psi & \text{sur } \partial\Omega. \end{cases}$$

Plus particulièrement, on prouve la continuité de $G(\nabla u)$ pour une large classe de fonctions monotones G en dimension deux. Cela a des conséquences sur la régularité \mathcal{C}^1 partielle des solutions et celle des lignes de niveau. Nous finirons avec un résultat valide en dimension $N \in \mathbb{N}$.

0.3 Unicité pour des problèmes dégénérés et/ou singuliers

Dans cette partie nous détaillons les résultats obtenus dans cette thèse sur la question de l'unicité des solutions dans le calcul des variations. Les fonctionnelles considérées ont la forme suivante :

$$I_\lambda : u \rightarrow \int_{\Omega} \varphi(\nabla u(x)) - \lambda(x)u(x)dx. \quad (0.3.1)$$

Ici, Ω est un ouvert borné de \mathbb{R}^N avec $N \geq 1$, $\varphi : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ est une fonction convexe et $\lambda \in L^\infty(\Omega)$. On se donne une condition de Dirichlet au bord ψ et un espace de Sobolev $W_\psi^{1,p}(\Omega)$ avec $p \geq 1$. L'objectif est d'identifier un cadre garantissant l'unicité d'une solution à

$$\mathcal{P}_\lambda : \inf_{u \in W_\psi^{1,p}(\Omega)} I_\lambda(u).$$

Comme dit précédemment si φ est strictement convexe alors ce problème admet au plus une solution. Nous nous demandons donc ce qu'il peut se passer si l'on

retire cette hypothèse. Pour ce faire on se restreint d'abord au cas où $\lambda \in \mathbb{R}$ et nous verrons plus tard un panorama de ce qu'il peut arriver quand nous autorisons λ à ne pas être constante.

0.3.1 Cas où λ est constante : invariance par translations

0.3.1.1 Un résultat général

La première contribution de cette thèse aux résultats d'unicité pour des problèmes de la forme (0.3.1) est présentée dans le premier chapitre. Dans cette partie on suppose qu'il n'existe pas d'ouvert non vide de \mathbb{R}^N tel que φ soit affine dessus. Cette condition peut être exprimée sous différentes formes. Par exemple, elle est équivalente à ce que les dimensions des projections des faces de l'épigraphe de φ soient inférieures ou égales à $N - 1$. Dans le cas où φ est de classe \mathcal{C}^1 cela revient à dire que le gradient de φ n'est jamais constant sur un ouvert non vide. Dans tous les cas nous avons le résultat suivant :

Théorème 0.3.1. *Pour un tel φ le problème \mathcal{P}_λ admet au plus un minimiseur uniformément continu sur Ω dans l'ensemble $W_\psi^{1,p}(\Omega)$.*

Nous pouvons remarquer que cette condition sur φ équivaut à être strictement convexe en dimension un. Cependant cette contrainte est plus faible à partir de la dimension deux. Cette condition sur φ peut être considérée comme optimale dans le sens où si l'on choisit $\varphi(z) := (|z| - 1)_+^2 := (\max\{|z| - 1, 0\})^2$ avec $\lambda = 0$ et $\psi = 0$ alors toutes les fonctions 1-lipschitziennes sont solutions de \mathcal{P}_λ . Ainsi pour des fonctionnelles du type (0.3.1) la contrainte annoncée est beaucoup plus générale que la stricte convexité et ne peut pas être affaiblie.

Avant de discuter des applications et de l'utilité de ce théorème nous pouvons remarquer que ce résultat ne donne pas d'informations sur l'existence ou la régularité des solutions. Il a donc un corollaire immédiat : si le problème \mathcal{P}_λ admet au moins une solution et que les solutions sont uniformément continues alors \mathcal{P}_λ admet une unique solution. Nous allons voir dans les applications et dans les idées de preuve du théorème que l'on peut prendre des ensembles plus généraux que $W_\psi^{1,p}(\Omega)$. Cela permet notamment d'appliquer ce résultat sur un ensemble de fonctions très irrégulières si l'on peut prouver par exemple que les minimiseurs sont eux uniformément continus.

0.3.1.2 Liens avec la littérature existante

Le cas où λ est une constante a été beaucoup étudié dans la littérature avec des φ différents. Nous pouvons appliquer le théorème 0.3.1 dans de nombreux cas ce qui a pour avantage d'unifier, de simplifier et parfois d'améliorer plusieurs preuves sur l'unicité des solutions en calcul des variations. Nous pouvons notamment citer les problèmes autour du gradient minimal quand on suppose que $\varphi(\cdot) := |\cdot|$. Par exemple, dans [63] Sternberg, Williams et Ziemer veulent minimiser (0.3.1) quand

$\lambda = 0$ parmi les fonctions dans

$$\{u \in BV(\Omega) \cap C^0(\overline{\Omega}), u = \psi \text{ sur } \partial\Omega\}. \quad (0.3.2)$$

Ici l'ensemble $BV(\Omega)$ est l'ensemble des fonctions à variations bornées sur Ω , c'est-à-dire l'ensemble des fonctions u dans $L^1(\Omega)$ telles que :

$$\sup_{\theta \in C_0^1(\Omega), |\theta| \leq 1} \left| \int_{\Omega} u \operatorname{div} \theta \right| < +\infty. \quad (0.3.3)$$

Nous pouvons remarquer que cet ensemble n'est pas dans un ensemble de Sobolev ce qui rend la question de l'existence plus ardue. Cependant, lorsque le bord de Ω est assez régulier, au moins $C^{1,1}$, les minimiseurs sont lipschitziens par [63, Théorème 5.9].

Il est possible d'appliquer le théorème 0.3.1 lorsque l'on travaille avec des problèmes pour lesquels l'existence et l'uniforme continuité des solutions découlent de la théorie classique. Nous pouvons citer [41] par exemple où Kawohl, Stará et Wittum considèrent une fonction φ de la forme :

$$\varphi(z) := \begin{cases} \frac{1}{2}|z|^2 & \text{si } |z| \leq 1, \\ |z| - \frac{1}{2} & \text{si } 1 < |z| < 2, \\ \frac{1}{4}|z|^2 + \frac{1}{2} & \text{si } 2 \leq |z|. \end{cases} \quad (0.3.4)$$

Ce problème énoncé en dimension deux a pour origine un problème d'élasticité obtenu en convexifiant le minimum des paraboles $t \rightarrow \frac{1}{2}t^2$ et $t \rightarrow \frac{1}{4}t^2 + \frac{1}{2}$. Dans cet article les auteurs prouvent un résultat partiel d'unicité en supposant que les lignes de niveau de $|\nabla u| = 1$ et $|\nabla u| = 2$ sont continues quand u est un minimiseur. Comme φ est une fonction radiale ayant un minimum strict en 0 nous pouvons appliquer le théorème 0.3.1. Ainsi nous pouvons garantir l'existence et l'unicité des solutions de \mathcal{P}_λ quand l'on minimise sur $W_\psi^{1,2}(\Omega)$ en dimension $N \geq 2$ sans avoir besoin de cette hypothèse sur la régularité des lignes de niveau de $|\nabla u|$.

Un autre exemple avec une fonctionnelle dépendant de la norme euclidienne provient d'articles de Kohn et Strang [42, 43, 44] quand φ est de la forme (0.1.2). Cette dernière fonction n'est donc pas différentiable en zéro. Ce problème ayant pour origine une question en optimisation de forme est étudié dans un article de Alibert, Bouchitté, Fragalà et Lucardesi [2]. Un premier résultat d'unicité est établi par Bouchitté et Bousquet dans [8]. Néanmoins nous pouvons remarquer que dans ce dernier article les auteurs supposent que $\partial\Omega$ est connexe. En appliquant le théorème 0.3.1 nous pouvons enlever cette hypothèse. D'autres applications sont présentées dans le premier chapitre, notamment lorsque l'on impose une contrainte ponctuelle sur la gradient comme $\{\nabla u(x) \in C \text{ pour presque tout } x \in \Omega\}$ avec C un ensemble convexe. C'est le cas de l'article de De Silva et Savin [29] ayant inspiré la preuve du théorème 0.3.1.

0.3.1.3 Invariance par translation

L'idée de la preuve du théorème 0.3.1 provient de [29]. Dans cet article les auteurs considèrent un problème en dimension deux avec $\lambda = 0$, $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$ une fonction convexe ayant pour domaine un polygone P et étant égale à $+\infty$ en dehors. La minimisation s'effectue sur les fonctions lipschitziennes sur Ω étant égales à ψ sur $\partial\Omega$. L'idée des auteurs est une preuve par l'absurde dans laquelle ils comparent une solution à la translation d'une autre solution. C'est cette idée là que nous allons généraliser dans le cadre de (0.3.1) quand $\lambda \in \mathbb{R}$.

Pour ce faire commençons par supposer que \mathcal{P}_λ admet deux minimiseurs différents et uniformément continus u et v parmi un ensemble de fonctions $\mathcal{E}(\Omega, \psi)$. On suppose que quelque soit $\alpha \in \mathbb{R}$ et quelque soit $\tau \in \mathbb{R}^N$ si $u \in \mathcal{E}(\Omega, \psi)$ alors $u(\cdot + \tau) + \alpha \in \mathcal{E}(\Omega - \tau, \psi + \alpha)$. Ainsi si u est une solution de \mathcal{P}_λ sur $\mathcal{E}(\Omega, \psi)$ alors l'autonomie en x de l'intégrande dans (0.1.3) quand $\lambda \in \mathbb{R}$ nous permet de prouver que $u(\cdot + \tau) + \alpha$ est un minimiseur sur $\mathcal{E}(\Omega - \tau, \psi + \alpha)$. Cette propriété est bien vérifiée quand $\mathcal{E}(\Omega, \psi) = W_\psi^{1,p}(\Omega)$ mais elle peut s'étendre à d'autres ensembles comme vu dans les exemples précédents.

Puisque u et v sont deux fonctions continues différentes il existe $x_0 \in \Omega$ tel que $u(x_0) \neq v(x_0)$. Nous pouvons supposer que sur un voisinage de x_0 nous avons que $u(x) > v(x) + \epsilon$ pour un certain $\epsilon > 0$.

A partir de là, nous utilisons l'autonomie de (0.3.1) en x pour prouver que

$$w_1 := \max\{u(x + \tau) - \frac{\epsilon}{2}, v(x)\} \text{ et } w_2 := \min\{u(x), u_2(x + \tau) + \frac{\epsilon}{2}\}$$

sont aussi des solutions du même problème quand τ est petit. Or nous savons que si u_1 et u_2 sont deux solutions du même problème alors pour presque tout $x \in \Omega$ la fonction φ est affine entre $\nabla u_1(x)$ et $\nabla u_2(x)$. Ainsi nous pouvons comparer les gradients de u , v , w_1 et w_2 pour presque tout $x \in \Omega$. Cela revient en réalité en faisant varier τ à comparer les gradients de u et v en un point aux gradients de u et v en d'autres points. Ainsi nous pouvons montrer qu'il existe un ensemble X sur lequel φ est affine tel que pour presque tout $x \in \Omega$, $\nabla u(x)$ et $\nabla v(x)$ appartiennent à X . Par hypothèse sur φ l'espace affine généré par l'ensemble X est de dimension strictement inférieure à N . L'espace vectoriel associé admet donc au moins une direction orthogonale. Cette direction étant orthogonale à $\nabla u(x) - \nabla v(x)$ pour presque tout $x \in \Omega$, on peut montrer que la fonction $u - v$ est constante dans cette direction. Comme Ω est borné, en suivant cette direction nous atteignons le bord de Ω où $u = v$. Ainsi $u(x_0) = v(x_0)$ ce qui contredit notre supposition $u(x_0) > v(x_0)$. En conclusion, nous avons une unique solution uniformément continue.

0.3.1.4 Limites du théorème

Malgré les nombreuses applications du théorème 0.3.1 nous pouvons évoquer certaines limites de ce résultat. Comme vu dans la sous-section précédente la preuve du théorème est assez rigide. En effet pour prouver que les translations de u et v sont des solutions nous devons imposer $\lambda \in \mathbb{R}$. Cette preuve ne peut donc pas

se généraliser au cas $\lambda \in L^\infty(\Omega)$. Le fait d'introduire des translations de u et v et d'en prendre le minimum ou le maximum rend la généralisation de ce résultat au cas vectoriel impossible. En effet, si l'on effectue une comparaison coordonnée par coordonnée on ne peut pas garantir que u et v soient différentes en un point et que chaque coordonnée de u soit plus grande que la coordonnée correspondante de v . Enfin, l'utilité de ce résultat dépend fortement de la connaissance a priori de l'existence d'un minimiseur et de la régularité de celui-ci.

0.4 Unicité quand $\lambda \in L^\infty(\Omega)$

Nous allons maintenant nous intéresser au cas $\lambda \in L^\infty(\Omega)$. Nous verrons dans le troisième chapitre qu'en général on ne peut pas espérer avoir une unique solution quand λ n'est pas une fonction constante. Nous avons prouvé que dans le cas où $\varphi(z) := (|z| - 1)_+^2$, construire un cadre où l'on a une infinité de solutions est très facile. Dans les deuxième et troisième chapitres nous étudierons des problèmes qui englobent le cas (0.1.2) où l'on a une singularité à l'origine et le cas (0.3.4) où la fonction φ est strictement convexe autour de 0. L'objectif est de trouver une méthode de preuve et des conditions sur λ pour garantir l'unicité. Ainsi nous aurons étudié les trois cas généraux typiques d'une fonction convexe à l'origine.

0.4.1 L'idée de Marcellini

Nous avons vu que la preuve du théorème 0.3.1 ne peut pas être généralisée au cas où λ n'est pas constante. Nous devons trouver un autre moyen pour prouver l'unicité. C'est en ce sens que l'idée utilisée par Marcellini en 1981 dans [54] est particulièrement intéressante. Dans cet article l'auteur considère le problème suivant :

$$\text{Minimiser } F_g(u) := \int_{\Omega} g(|\nabla u(x)|) dx \quad (0.4.1)$$

sur l'ensemble des fonctions lipschitziennes étant égales à ψ sur $\partial\Omega$. Ici on suppose que $g : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ est une fonction convexe ayant un minimum strict à l'origine et Ω est un ouvert convexe borné de \mathbb{R}^N avec $N \geq 2$ ayant un bord de classe \mathcal{C}^1 . Le résultat est le suivant : si $u \in \mathcal{C}^1(\overline{\Omega})$ est un minimiseur de F_g tel que ∇u ne s'annule jamais dans $\overline{\Omega}$ alors il s'agit de l'unique minimiseur.

La méthode utilisée pour obtenir ce résultat se décompose en deux étapes. Tout d'abord, les ensembles de niveau de cette solution u intersectent le bord de Ω . Ensuite, on montre que si v est une solution du même problème alors v est constante sur les ensembles de niveau de u . Comme $u = v$ sur le bord de Ω , la fonction $u - v$ est nulle sur tous les ensembles de niveau de u et par conséquent sur tout Ω .

Le cadre d'application du résultat de [54] est relativement restreint. En effet, il n'est pas vrai en général que le minimiseur u soit \mathcal{C}^1 sur $\overline{\Omega}$ sans hypothèse supplémentaire sur g , Ω et ψ . De plus, la condition $\nabla u \neq 0$ partout est compliquée à vérifier et n'est pas satisfaite dès que notre solution u admet un extremum local.

L'objectif d'une partie de la thèse est de trouver un cadre naturel pour pouvoir appliquer une méthode de ce type. Par méthode de ce type on entend pouvoir prouver que si u et v sont deux minimiseurs alors les ensembles de niveau de u intersectent une partie de Ω où $u = v$ et v est constant sur les ensembles de niveau de u . Cette idée de Marcellini a été reprise dans plusieurs articles. Dans [53], Lussardi et Mascolo considèrent le même cadre que (0.4.1) en dimension deux mais sans supposer Ω convexe, $u \in C^1(\bar{\Omega})$ et que le gradient de u ne s'annule jamais. Une première application d'une méthode de ce type en dimension quelconque avec un terme de la forme $\int_{\Omega} \lambda u$ où $\lambda \in \mathbb{R}$ est non nulle est l'article [8] avec φ comme dans (0.1.2). Dans ce papier le minimiseur n'est pas forcément globalement lipschitzien, un travail sur la régularité des ensembles de niveau supplémentaire est nécessaire. De plus, le fait que λ soit différent de 0 demande une adaptation supplémentaire de la preuve car les ensembles de niveau des minimiseurs n'intersectent plus nécessairement le bord de Ω .

Nous pouvons remarquer que tous les articles cités dans cette partie sont englobés par le cadre du théorème 0.3.1 ce qui rend plus intéressant l'étude du cas où λ est une fonction bornée non constante. Pour le cas λ variable nous allons chercher à appliquer ce type de méthode pour des minimiseurs globalement lipschitziens.

0.4.2 Régularité lipschitzienne globale des minimiseurs

Dans le deuxième chapitre nous présentons un résultat de régularité lipschitzienne globale des minimiseurs de \mathcal{P}_{λ} quand $\varphi(\cdot) := g(|\cdot|)$ est radiale, strictement convexe hors d'une boule et uniformément convexe à l'infini au sens suivant :

$$\liminf_{x \rightarrow +\infty} \frac{xg''(x)}{g'(x)} > 0. \quad (0.4.2)$$

Si l'on appelle α cette \liminf alors en posant $p = \frac{\alpha}{2} + 1$ on peut montrer que $\varphi(z)$ croît plus vite que $|z|^p$ quand $|z| \rightarrow +\infty$. Le bon cadre fonctionnel pour effectuer la minimisation est donc l'espace $W_{\psi}^{1,p}(\Omega)$. Nous avons le résultat suivant :

Théorème 0.4.1. *Supposons que Ω soit un ouvert borné simplement connexe de \mathbb{R}^N avec $N \geq 2$. Si $\partial\Omega$ est connexe de régularité $C^{1,1}$, $\psi \in C^{1,1}(\mathbb{R}^N)$ et si φ satisfait (0.4.2) alors les minimiseurs de \mathcal{P}_{λ} sont globalement lipschitziens sur $\bar{\Omega}$. De plus, la constante de Lipschitz est inférieure à une constante*

$$L := L(p, \|\lambda\|_{L^{\infty}(\Omega)}, N, |\Omega|, \text{diam}(\Omega), \|\psi\|_{C^{1,1}(\mathbb{R}^N)}, \kappa)$$

avec κ la courbure principale de Ω .

Ce résultat est nouveau pour plusieurs raisons. La première vient du fait que l'on ne suppose pas φ strictement convexe mais seulement strictement convexe hors d'une boule à l'inverse de [16]. Cela est naturel car on s'intéresse à la structure de φ quand le gradient des minimiseurs est grand. Un autre intérêt est que l'on ne majore pas la croissance de φ alors que c'est le cas pour les résultats classiques de régularité

Hölderienne globale. De plus c'est un résultat de régularité globale à la différence de [14], [20] et [34].

La preuve de ce résultat est obtenue en régularisant notre problème. On approche les fonctions φ et λ par une suite de fonctions lisses $(\varphi_\epsilon)_{\epsilon>0}$ et $(\lambda_\epsilon)_{\epsilon>0}$ avec $\epsilon > 0$ par convolution. Ainsi on peut travailler avec des minimiseurs lisses $(u_\epsilon)_{\epsilon>0}$. L'objectif est d'obtenir une estimation de la constante de Lipschitz de u_ϵ indépendante de $\epsilon > 0$. Pour ce faire on encadre les minimiseurs par des fonctions barrières. Nous utilisons la régularité de $\partial\Omega$ et ψ pour construire deux fonctions lipschitziennes l^- et l^+ telles que $l^-(x) \leq u(x) \leq l^+(x)$ pour tout $x \in \Omega$. Ensuite nous utilisons la même stratégie que Bousquet et Brasco dans [12] pour conclure.

Par ailleurs ce dernier article contient un autre de résultat de régularité lipschitzienne globale quand Ω est convexe. Dans ce cas-là les auteurs supposent moins de régularité sur Ω et ψ . En revanche ψ doit satisfaire la condition de pente bornée de rang $K > 0$ suivante : pour tout $y \in \partial\Omega$ il existe $\zeta_y^+, \zeta_y^- \in \mathbb{R}^N$ tels que $|\zeta_y^+|, |\zeta_y^-| \leq K$ et

$$\psi(y) + \langle \zeta_y^+, x - y \rangle \leq \psi(x) \leq \psi(y) + \langle \zeta_y^-, x - y \rangle \text{ pour tout } x \in \Omega.$$

Une autre différence entre ces deux résultats est la manière dont les fonctions barrières sont construites.

0.4.3 Unicité pour un problème singulier et dégénéré

Dans cette sous-section nous supposons que φ est une fonction dépendante de la norme euclidienne : $\varphi(\cdot) := g(|\cdot|)$. Ici g est une fonction convexe paire de classe \mathcal{C}^1 sur $\mathbb{R} \setminus \{0\}$, \mathcal{C}^2 et strictement convexe sur $(1, +\infty)$. De plus on suppose que la fonction convexe g vérifie $g(t) = t$ pour tout $0 \leq t \leq 1$. Comme la fonction g n'est pas dérivable en 0 le problème est dit singulier. Nous avons que le problème est aussi dégénéré car la matrice hessienne de $\varphi(z)$ possède une valeur propre nulle pour tout $z \in B_1(0)$. Un exemple de telle fonction φ est celui décrit dans (0.1.2).

Si l'on suppose que g , Ω et ψ vérifient aussi les hypothèses de la sous-section précédente nous donnant la régularité lipschitzienne globale des minimiseurs alors on a le théorème suivant qui est le résultat principal du deuxième chapitre :

Théorème 0.4.2. *On suppose que Ω est un ouvert connexe de \mathbb{R}^N avec $N \geq 2$ ayant un bord de classe $\mathcal{C}^{1,1}$ connexe et $\psi \in \mathcal{C}^{1,1}(\mathbb{R}^N)$. Si λ est une fonction lipschitzienne strictement positive sur Ω alors il existe*

$$C(p, N, |\Omega|, \text{diam}(\Omega), \max_{\Omega} \lambda, \min_{\Omega} \lambda, \|\psi\|_{\mathcal{C}^{1,1}(\mathbb{R}^N)}, \kappa) > 0$$

telle que si $\|\nabla \lambda\|_{L^\infty(\Omega)} \leq C$ alors le problème \mathcal{P}_λ admet une unique solution sur $W_\psi^{1,p}(\Omega)$.

Dans le deuxième chapitre, nous construisons des contre-exemples quand λ est lisse et strictement positive. Ainsi nous sommes obligé d'imposer une contrainte sur les variations de λ . Nous établissons aussi des principes de comparaisons permettant

notamment de montrer que quand $\lambda \in L^\infty(\Omega)$ est petit et $\psi \equiv 0$ alors la fonction constante égale à 0 est solution.

La stratégie de la preuve de ce théorème est la suivante. D'après Colombo et Figalli dans [26] il existe un ouvert $U \subset \Omega$ tel que si u est une solution de \mathcal{P}_λ alors $|\nabla u(x)| > 1$ pour tout $x \in U$ et $|\nabla u(x)| \leq 1$ pour presque tout $x \in \Omega \setminus U$. De plus, cet ensemble U ne dépend pas du choix du minimiseur. En utilisant la stricte convexité de φ hors de la boule unité on montre que si u et v sont deux solutions alors $\nabla u = \nabla v$ dans l'ouvert U . Comme λ est presque constante nous montrons que toutes les composantes connexes de U ont un bord qui intersecte le bord de Ω . Ainsi nous obtenons que $u = v$ sur $\partial\Omega \cup \bar{U}$. Une différence avec l'idée de Marcellini est la suivante, on ne travaille pas avec les ensembles de niveau de u mais avec le bord de ses ensembles de sur-niveau $E_t(u) := \{x \in \Omega, u(x) \geq t\}$ avec $t \in \mathbb{R}$. Notre objectif est donc de montrer que les composantes connexes du bord des ensembles $E_t(u)$ rencontrent $\partial\Omega \cup \bar{U}$ et que v est constante dessus.

Pour ce faire nous avons besoin de mieux comprendre la régularité et les propriétés de minimisation des ensembles de sur-niveau. C'est ce que l'on va décrire dans les prochaines sous-sections.

0.4.3.1 Ensembles de périmètre fini

La preuve de l'unicité repose sur une bonne compréhension de la régularité du bord des ensembles de sur-niveau $E_t(u)$. Nous avons pour presque tout $t \in \mathbb{R}$ que l'indicatrice de E_t :

$$\mathbb{1}_{E_t(u)} := \begin{cases} 1 & \text{si } x \in E_t(u), \\ 0 & \text{si } x \notin E_t(u) \end{cases}$$

est une fonction dans $BV(\Omega)$. Si l'indicatrice d'un ensemble E est dans $BV(\Omega)$ on dit alors que E est un ensemble ayant un périmètre fini dans Ω ou est de Caccioppoli. Dans ce cas là, son gradient faible est une mesure de Radon vectorielle que l'on note $D\mathbb{1}_E$ et sa variation totale est notée par $|D\mathbb{1}_E|$. Nous définissons le périmètre d'un tel ensemble de la manière suivante :

$$\text{Per}(E, \Omega) = \int_{\Omega} |D\mathbb{1}_E| = \sup \left\{ \int_{E \cap \Omega} \text{div} g \text{ avec } g \in \mathcal{C}_0^1(\Omega) \text{ et } \|g\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

Une première propriété classique du périmètre héritée des fonctions BV est sa semi-continuité inférieure qui peut être utilisée pour prouver l'existence d'ensembles ayant un périmètre minimal sous certaines conditions. Étudions maintenant le bord des ensembles de périmètre fini. Il existe plusieurs manières de définir la frontière d'un ensemble E selon nos besoins. Nous avons évidemment le bord topologique ∂E mais aussi la frontière essentielle $\partial^e E$ composée des $x \in \mathbb{R}^N$ tels que pour tout $r > 0$:

$$0 < |B_r(x) \cap E| < |B_r(x)|$$

avec $B_r(x) := \{y \in \mathbb{R}^N \text{ tel que } |x - y| < r\}$. Nous avons que $\partial^e E \subset \partial E$. Un autre type de bord défini dans le cadre des ensembles de périmètre fini est la frontière

réduite $\partial^*E \subset \partial^eE$ composée des $x \in \text{supp}D\mathbf{1}_E \cap \Omega$ tels que

$$\nu_E(x) := \lim_{r \rightarrow 0} \frac{\int_{B_r(x)} D\mathbf{1}_E}{\int_{B_r(x)} |D\mathbf{1}_E|}$$

existe et est de norme égale à un. Nous pouvons trouver dans les livres de Evans et Gariepy [32] et Giusti [38] une étude générale des propriétés des fonctions à variations bornées et des ensembles de périmètre fini. Nous énonçons quelques résultats importants utilisés de nombreuses fois dans cette thèse. Tout d'abord, si E est un ensemble de périmètre fini dans Ω alors pour tout ensemble $B \subset \Omega$ mesurable nous avons d'après [38, Théorème 4.4] que :

$$\int_B |D\mathbf{1}_E| = \mathcal{H}^{N-1}(B \cap \partial^*E)$$

où \mathcal{H}^s est la mesure de Hausdorff définie comme

$$\mathcal{H}^s(E) := \lim_{r \rightarrow 0} \inf_{\{\text{diam}(A_i) \leq r\}} \left\{ \sum_{i=1}^{+\infty} \text{diam}(A_i)^s \mid E \subset \bigcup_{i=1}^{+\infty} A_i \right\}.$$

De cette définition de mesure de Hausdorff découle la définition de la dimension de Hausdorff notée $\dim_{\mathcal{H}}(\cdot)$:

$$\dim_{\mathcal{H}}(E) := \inf\{s, \mathcal{H}^s(E) = 0\} = \sup\{s, \mathcal{H}^s(E) = +\infty\}.$$

Une formule très importante pour les fonctions BV est la formule de la co-aire qui donne pour tout $u \in BV(\Omega)$ et tout $B \subset \Omega$ mesurable l'égalité suivante :

$$\int_B |Du| = \int_{\mathbb{R}} \int_B |D\mathbf{1}_{E_t(u)}| dt.$$

Cette égalité peut se réécrire sous la forme :

$$\int_{\Omega} \mathbf{1}_B |Du| = \int_{\mathbb{R}} \mathcal{H}^{N-1}(B \cap \partial^*E_t(u)) dt.$$

Ainsi on obtient une formule de la co-aire pour des fonctions f mesurables positives :

$$\int_{\Omega} f |Du| = \int_{\mathbb{R}} \left(\int_{\partial^*E_t(u)} f d\mathcal{H}^{N-1} \right) dt.$$

Nous allons voir dans la prochaine sous-section un problème classique sur les ensembles de périmètre fini puis nous expliquerons comment l'adapter à la preuve de notre résultat.

0.4.3.2 Problème de Cheeger

Soit Ω un ouvert bornée de \mathbb{R}^N , on introduit la constante suivante :

$$h_{\Omega} := \inf_{E \subset \Omega} \frac{\text{Per}(E)}{|E|}$$

appelée constante de *Cheeger*. Un ensemble E qui minimise ce ratio est appelé ensemble de *Cheeger* de Ω . Dans le deuxième chapitre on montre que l'on a l'égalité suivante pour tout $p \geq 1$:

$$h_\Omega := \inf \left\{ \int_\Omega |\nabla v| \text{ avec } v \in W_0^{1,p}(\Omega), \int_\Omega v = 1 \right\}.$$

Comme pour les minimiseurs de fonctionnelles l'existence, l'unicité et la régularité des ensembles de *Cheeger* sont des questions importantes, on peut trouver un certain nombre d'informations sur ces sujets dans [60] écrit par Parini. Par exemple, si le bord de Ω est lipschitzien alors Ω admet au moins un ensemble de *Cheeger*. Dans ce là, la constante h_Ω est identique si la minimisation est effectuée sur les ensembles lisses compactement inclus dans Ω . Comme le ratio $E \rightarrow \frac{Per(E)}{|E|}$ est $\frac{1}{t}$ -homogène pour la dilatation de centre 0 et de rapport t nous avons automatiquement que le bord d'un ensemble de *Cheeger* rencontre le bord de Ω . De plus, nous pouvons montrer que la partie du bord de l'ensemble de *Cheeger* qui ne coïncide pas avec le bord de Ω est analytique hors d'un ensemble de dimension de Hausdoff plus petite ou égale à $N - 8$. Quand Ω est convexe il existe un unique ensemble de *Cheeger*. Cet ensemble est alors convexe et son bord est de classe $\mathcal{C}^{1,1}$. Lorsque Ω est une boule nous avons que Ω est l'unique ensemble de *Cheeger* de lui-même. En revanche, même si Ω est convexe l'ensemble de *Cheeger* n'est pas forcément Ω . En effet, en dimension deux par [60, Proposition 5.1] on a que l'ensemble de *Cheeger* d'un ensemble convexe Ω est

$$\{x \in \Omega, \text{dist}(x, \partial\Omega) \geq R\} \oplus B_R(0)$$

avec R tel que $|\{x \in \Omega, \text{dist}(x, \partial\Omega) \geq R\}| = \pi R^2$. Ici, si Ω_1 et Ω_2 sont deux ensembles alors $\Omega_1 \oplus \Omega_2 := \{x + y \text{ avec } x \in \Omega_1 \text{ et } y \in \Omega_2\}$. Si Ω est un carré ce dernier ensemble n'est pas le carré car les angles ne sont pas dans l'ensemble de *Cheeger*.

Dans la prochaine sous-section nous allons voir comment utiliser, adapter ces propriétés ou s'inspirer des preuves pour montrer notre résultat.

0.4.3.3 Pseudo-périmètre et problème de Pseudo-Cheeger

Dans le deuxième chapitre nous montrons que les ensembles de sur-niveau $E_t(u)$ d'une solution u de \mathcal{P}_λ sont des minimiseurs d'une certaine fonctionnelle définie sur les ensembles de périmètre fini. Nous avons déjà vu l'existence de l'ouvert U sur lequel la gradient de u a une norme strictement plus grande que 1. Cependant, d'après [26] on a un résultat plus fort : la fonction $x \mapsto \max(1, |\nabla\varphi(\nabla u(x))|)$ a un représentant continu sur Ω qui ne dépend pas du choix du minimiseur. Ainsi on peut définir un périmètre à poids que l'on appelle *pseudo-périmètre* :

$$\widetilde{Per}(F, V) = \int_{\partial^* F \cap V} \max(1, |\nabla\varphi(\nabla u)|) d\mathcal{H}^{N-1}$$

où F est un ensemble de périmètre fini dans Ω et $V \subset \Omega$ un ouvert. Comme le périmètre, le *pseudo-périmètre* est semi-continu inférieurement. L'intérêt d'introduire

cette nouvelle quantité est d'utiliser l'équation d'Euler-Lagrange associée à notre problème. En effet, si u est globalement lipschitzien il existe $\sigma \in L^\infty(\Omega)$ tel que

$$\sigma(x) := \begin{cases} \sigma(x) \text{ avec } |\sigma(x)| \leq 1 \text{ si } |\nabla u(x)| = 0, \\ \frac{\nabla u(x)}{|\nabla u(x)|} \text{ si } 0 < |\nabla u(x)| \leq 1, \\ \nabla \varphi(\nabla u(x)) \text{ si } 1 < |\nabla u(x)|. \end{cases}$$

et $\operatorname{div} \sigma = -\lambda$ faiblement. Ainsi si $E \Subset \Omega$ est un ensemble lisse on a la formule de Stokes suivante :

$$\int_E \lambda = \int_{\partial^* E} \langle \sigma, \nu_E \rangle d\mathcal{H}^{N-1}.$$

avec ν_E la normale unitaire extérieure à E . L'intégrande du membre de droite peut être bornée par $|\sigma|$ qui peut donc être majorée par $\max(1, |\nabla \varphi(\nabla u)|)$. Ainsi nous obtenons que $\int_E \lambda \leq \widetilde{Per}(E, \Omega)$. Le résultat principal autour du *pseudo-périmètre* est le suivant :

Théorème 0.4.3. *Supposons que u soit une solution lipschitzienne de \mathcal{P}_λ . Alors pour presque tout $t \in \mathbb{R}$, pour tout ensemble $F \subset \Omega$ de périmètre fini dans Ω tel que $F \Delta E_s \Subset \Omega$, on a :*

$$\widetilde{Per}(E_t, \Omega) - \int_{E_t} \lambda dx \leq \widetilde{Per}(F, \Omega) - \int_F \lambda dx$$

Ici, $F \Delta E \Subset \Omega := ((F \setminus E) \cup (E \setminus F))$ et $E_t(u) = [u \geq t]$.

Ainsi presque tous les ensembles de sur-niveau de notre solution u sont eux-mêmes solutions d'un problème de minimisation sur les ensembles de périmètre fini analogue à (0.3.1). La preuve de ce résultat repose sur une régularisation de σ pour pouvoir effectuer des intégrations par parties. Cette technique de preuve nous permet également de montrer que le membre de gauche dans le théorème vaut 0 si $E_t \Subset \Omega$, c'est le cas par exemple quand $\psi = 0$ et $t > 0$. Dans ce cas là on peut définir la notion d'ensemble de *pseudo-Cheeger* comme étant les ensembles F tels que :

$$\frac{\widetilde{Per}(F, \Omega)}{\int_F \lambda dx} = \inf_{D \Subset \Omega} \frac{\widetilde{Per}(D, \Omega)}{\int_D \lambda dx}.$$

Bien que cela n'ait pas été effectué dans cette thèse, l'étude des propriétés des ensembles de *pseudo-Cheeger* est un domaine intéressant en soi qui peut permettre de mieux comprendre la géométrie des ensembles de sur-niveau quand $\psi = 0$. Nous pouvons notamment citer l'article de Carlier et Comte [18] sur les problèmes de *Cheeger* à poids. Des résultats de régularité sur des ensembles minimisant une certaine quantité existent comme [15] où Bulanyi et Lemenant prouve que les ensembles solutions du problème de p -compliance en dimension deux sont $\mathcal{C}^{1,\alpha}$.

L'application principale du théorème 0.4.3 est la suivante. Pour un ensemble $F \subset \Omega$, dans $\partial^* F \cap \Omega \setminus \overline{U}$ le périmètre et le pseudo-périmètre coïncide. Dans ce cas là on peut utiliser un résultat de Tamanini ou Massari [55] qui nous donne pour

presque tout $t \in \mathbb{R}$ l'existence d'un ensemble ouvert $W_t \subset \Omega \setminus \bar{U}$ tel que $\partial^e E_t(u) \cap W_t$ soit une hypersurface \mathcal{C}^1 . De plus $\mathcal{H}^s(\Omega \setminus (W_t \cup \bar{U})) = 0$ pour tout $s > N - 8$ et $W_t \cap \partial^e E_t \subset \partial^* E_t$.

Cette régularité sur le bord des ensembles de sur-niveau nous permet de montrer que les composantes connexes de $\partial^e E_t$ intersectent \bar{U} et si v est une autre solution alors v est constante sur ces composantes connexes. C'est ainsi que l'on montre que $u = v$ sur les ensembles $\partial^* E_t$. Nous concluons grâce à la formule de la co-aire.

0.4.3.4 Questions ouvertes

Dans le deuxième chapitre nous présentons un principe de comparaison qui nous dit que $u_1 \leq u_2$ si $\psi_1 \leq \psi_2$ et $\lambda_1 \leq \lambda_2$. Comme λ_1 et λ_2 sont des fonctions il peut être intéressant de comprendre ce qu'il peut se passer si λ_2 est rarement plus petite que λ_1 .

Nous savons que si le maximum de u dans Ω est plus grand que le maximum de ψ sur $\partial\Omega$ alors l'ensemble où u atteint son maximum est de mesure strictement positive. Sur des exemples explicites comme

$$u(x) := C - \frac{N}{2\lambda} \left(\left| \frac{\lambda}{N} x \right|^2 - 1 \right)_+ \quad (0.4.3)$$

qui est solution quand $\Omega = B_1(0)$, $\psi = 0$, $\lambda \in \mathbb{R}_+^*$ et φ comme dans (0.1.2) nous pouvons voir l'existence d'un plateau où u est constante et vaut sa valeur maximale. L'étude de ce plateau commencée dans [2] fait intervenir un problème de frontière libre sur le gradient de u qui pourrait permettre de mieux comprendre la géométrie des solutions. L'exemple explicite (0.4.3) soulève les questions de la structure de l'ensemble U en général ainsi que de la convexité des ensembles de sur-niveau quand Ω est convexe.

Enfin, nous aimerions comprendre la régularité globale des ensembles de sur-niveau. Nous savons qu'ils sont \mathcal{C}^1 dans $\Omega \setminus \bar{U}$, dans U on peut utiliser la stricte convexité de φ pour utiliser des résultats classiques de régularité. Cependant, nous ne savons pas ce qu'il peut se passer sur ∂U . C'est en ce sens qu'une amélioration de [26] dans notre cadre pour obtenir la régularité $\mathcal{C}^{0,\alpha}$ de $\max(1, |\nabla\varphi(\nabla u)|)$ permettrait d'utiliser les résultats de [55] pour avoir des ensembles de sur-niveau \mathcal{C}^1 partout. Cela pourrait nous permettre de considérer une classe de fonctions λ plus large.

0.4.4 Unicité pour un problème dégénéré en dimension deux

Dans le troisième chapitre de cette thèse nous travaillons sur le problème suivant. Nous considérons φ une fonction convexe dépendante de la norme euclidienne étant strictement convexe autour de l'origine. Un exemple de telle fonction est donné dans (0.3.4). Le cadre général est donné dans le troisième chapitre mais pour plus de clarté nous allons considérer uniquement cet exemple dans l'introduction qui permet de présenter plus simplement les problématiques autour de ce type de fonctionnelle. Nous avons donc une fonction φ strictement convexe autour de l'origine, affine sur

une couronne et strictement convexe en dehors. Nous sommes donc dans le cadre d'un problème dégénéré mais pas singulier. Nous pourrions donc nous attendre à obtenir des résultats plus forts que dans le cadre précédent ou avec des hypothèses plus faibles. Cependant, le fait que la zone de dégénérescence ne soit pas convexe (on verra dans le troisième chapitre qu'elle peut ne pas être connexe) rend la preuve plus difficile.

Si l'on regarde les points communs avec la section précédente, l'un d'entre eux est le résultat de régularité lipschitzienne globale qui est valable dans ce cas là aussi. Grâce à Anzelotti et Giaquinta [5] nous pouvons aussi montrer l'existence d'un ensemble ouvert U tel que $U := \{|\nabla u| < 1\} \cup \{|\nabla u| > 2\}$ avec u une solution de \mathcal{P}_λ et φ comme dans (0.3.4). Cet ensemble U peut être défini indépendamment du choix de la solution. Nous avons toujours la continuité de $\max(2, |\nabla u|)$ par [26] et quand λ est presque constante on peut montrer que les composantes connexes de $U_2 := \{|\nabla u| > 2\}$ intersectent le bord de Ω . De plus la dépendance de φ en la norme euclidienne implique que les gradients de deux solutions sont colinéaires presque partout, cela permet donc de prouver que la différence entre deux solutions est constante sur les lignes de niveau si ces ensembles sont assez réguliers. Un dernier point commun est l'existence de contre-exemples quand les oscillations de λ sont trop importantes.

La principale différence dans la preuve provient du fait que même si l'on parvient à montrer que le bord des ensembles de sur-niveau intersectent l'ensemble $\partial\Omega \cup \bar{U}$ on ne sait pas a priori si toutes les solutions sont égales dessus. En effet, nous n'avons pas d'informations sur l'ensemble $U_1 := \{|\nabla u| < 1\}$. Ses composantes connexes n'ont aucune raison de rencontrer le bord de Ω et elles peuvent être en nombre dénombrables. On peut voir par exemple que même dans le cas $\lambda \in \mathbb{R}$, l'ensemble U_1 peut ne pas intersecter le bord de Ω . En effet la fonction

$$u(x) := \begin{cases} C - \frac{\lambda}{4}|x|^2 & \text{si } |x| \leq \frac{2}{\lambda}, \\ C + \frac{1}{\lambda} - \frac{\lambda}{2}|x|^2 & \text{si } \frac{2}{\lambda} < |x| \leq 1 \end{cases}$$

est solution de \mathcal{P}_λ quand $\lambda \in \mathbb{R}_+$, $\Omega = B_1(0) \subset \mathbb{R}^2$, $\psi = 0$, $C > 0$ telle que $u \equiv 0$ sur $\partial B_1(0)$ avec φ comme dans (0.3.4). Ici, l'ensemble U_1 est $B_{\frac{2}{\lambda}}(0)$.

Pour contourner ce problème nous avons besoin de plus de régularité sur les ensembles de sur-niveau a priori. C'est pourquoi dans ce chapitre nous travaillons en dimension deux. Ainsi ces ensembles ont un bord ayant une paramétrisation lipschitzienne en général. L'intérêt de la régularité des ensembles de sur-niveau de minimiseur en calcul des variations s'en retrouve donc renforcé car il permettrait d'entrevoir une preuve en toute dimension.

Le résultat principal du troisième chapitre est le suivant :

Théorème 0.4.4. *Supposons que Ω est un ouvert simplement connexe borné de \mathbb{R}^2 , que son bord $\partial\Omega$ est classe $\mathcal{C}^{1,1}$, $\psi \in \mathcal{C}^{1,1}(\mathbb{R}^2)$ et λ est lipschitzienne sur $\bar{\Omega}$ telle que $\min_{x \in \bar{\Omega}} \lambda(x) > 0$. Alors il existe une constante strictement positive*

$$C := C(p, N, |\Omega|, \max_{\Omega} \lambda, \min_{\Omega} \lambda, \|\psi\|_{\mathcal{C}^{1,1}(\Omega)}, \kappa)$$

telle que si $\|\nabla\lambda\|_{\mathcal{L}^\infty(\Omega)} \leq C$ alors \mathcal{P}_λ admet une unique solution sur $W_\psi^{1,p}(\Omega)$. Ici, κ est la courbure principale de Ω .

0.4.4.1 Continuité absolue sur les lignes

Nous présentons des résultats de régularité pour les lignes de niveau d'une solution u de \mathcal{P}_λ . Si f est une fonction lipschitzienne, on note $L_t(f)$ l'ensemble de niveau $[f = t]$ avec $t \in \mathbb{R}$. On introduit l'ensemble suivant $L_t^*(f)$ comme étant l'union de toutes les composantes connexes de $L_t(f)$ ayant une mesure de Hausdorff \mathcal{H}^1 strictement positive. Si f est lipschitzienne et à support compact dans \mathbb{R}^2 alors d'après [1, Théorème 2.5] nous avons que $\mathcal{H}^1(L_t(f) \setminus L_t^*(f)) = 0$ pour presque tout $t \in \mathbb{R}$ et les composantes connexes de $L_t^*(f)$ sont des courbes fermées simples ayant une paramétrisation lipschitzienne. Ainsi si u et v sont deux solutions lipschitziennes de \mathcal{P}_λ alors pour presque tout $t \in \mathbb{R}$ pour chaque composante connexe $l_t(u)$ de $L_t^*(u)$ nous avons que $u - v$ est constante sur la courbe $l_t(u)$. Donc pour presque tout $t \in \mathbb{R}$ et chaque composante connexe $l_t(u)$ de $L_t^*(u)$ si $l_t(u) \cap \partial\Omega \neq \emptyset$ on a $u = v$ dessus.

Si cela n'est pas le cas on utilise la technique suivante. On montre que $\sigma := \nabla\varphi(\nabla u)$ est une fonction dans $W_{loc}^{1,2}(\Omega)$. Pour ce faire on approche notre problème en convolant φ et λ . Les suites $(\varphi_\epsilon)_{\epsilon>0}$ et $(\lambda_\epsilon)_{\epsilon>0}$ nous permettent de définir des problèmes régularisés ayant une unique solution : u_ϵ . Nous obtenons ensuite une estimation de $\|\nabla\varphi_\epsilon(\nabla u_\epsilon)\|_{W_{loc}^{1,2}(\Omega)}$ uniforme en $\epsilon > 0$. Pour passer à la limite quand $\epsilon \rightarrow 0$ nous utilisons les mesures de Young : $(\nu_x)_{x \in \Omega}$. Ces mesures de probabilités vérifient la propriété suivante :

$$\int_{\Omega} H(x, \nabla u_\epsilon(x)) dx \rightarrow \int_{\Omega} \int_{\mathbb{R}^2} H(x, y) d\nu_x(y) dx$$

pour toute fonction de Carathéodory H .

Même si l'on sait que la fonction σ est dans $W_{loc}^{1,2}(\Omega)$ nous ne pouvons pas garantir qu'elle soit continue. Par contre on peut montrer qu'elle est absolument continue sur $l_t(u)$ pour presque tout $t \in \mathbb{R}$ avec $l_t(u)$ n'importe quelle composante connexe de $L_t^*(u)$ qui n'intersecte pas le bord de Ω . Cette absolue continuité nous permet de montrer que $|\sigma|$ est constante sur une telle courbe $l_t(u)$ si elle ne rencontre pas l'ensemble U . Dans le cadre général du troisième chapitre, hors de U la fonction $|\sigma|$ ne peut prendre qu'un nombre fini de valeurs. Par exemple, quand φ est comme dans (0.3.4) la seule valeur possible est 1 correspondant à la norme du gradient de φ sur l'anneau $\{1 \leq |z| \leq 2\}$. Cela nous permet donc d'avoir une contrainte sur les valeurs possibles de $|\sigma|$ sur $l_t(u)$ si $l_t(u)$ ne rencontre pas U . Nous allons nous servir de ce résultat pour prouver que $l_t(u)$ doit intersecter U ou $\partial\Omega$.

0.4.4.2 Principe du maximum pour σ

Dans cette sous-section nous discutons d'un principe du maximum sur $|\sigma|$ crucial dans la preuve du théorème 0.4.4. Pour presque tout $t \in \mathbb{R}$ supposons que $l_t(u)$ soit une composante connexe de $L_t^*(u)$ ne rencontrant $\partial\Omega \cup U$. Nous introduisons $F(l_t(u))$ la composante connexe bornée de $\mathbb{R}^2 \setminus l_t(u)$ obtenue grâce au théorème de Jordan.

Nous montrons que pour presque tout $s > t$ si $l_s(u)$ est une composante connexe de $L_s^*(u)$ telle que $l_s(u) \cap U$ est vide et $l_s(u) \subset F(l_t(u))$ alors $|\sigma|$ est constante sur $l_s(u)$ valant C_s et sur $l_t(u)$ valant C_t . De plus, nous avons que $C_s \leq C_t$. Ainsi la norme de σ diminue quand on progresse dans les lignes de niveau.

La preuve du théorème 0.4.4 se déroule de la manière suivante. On suppose qu'il existe $l_t(u)$ composante connexe de $L_t^*(u)$ telle que $l_t(u) \cap (U \cup \partial\Omega) = \emptyset$. Ainsi il existe $C_t > 0$ une constante telle que $|\sigma| = C_t$ sur $l_t(u)$. On obtient alors que

$$\int_{F(l_t(u))} \lambda = C_t \text{Per}(F(l_t(u)))$$

avec $\overline{F(l_t(u))}$ la composante connexe bornée de $\mathbb{R}^2 \setminus l_t(u)$. Nous avons aussi que si $F \subset \overline{F(l_t(u))}$ est un ensemble de périmètre fini alors

$$\int_F \lambda \leq C_t \text{Per}(F).$$

Comme $u > t$ sur $F(l_t(u))$ on se retrouve encore une fois avec des ensembles de sur-niveau solutions d'un problème de minimisation parmi des ensembles de périmètre finis.

Nous montrons par l'absurde qu'une telle courbe $l_t(u)$ ne peut pas exister en utilisant des idées similaires à celles utilisées pour les ensembles de *Cheeger*. Par exemple, un ensemble de *Cheeger* doit intersecter le bord de Ω . Ainsi toutes les composantes connexes de $L_t^*(u)$ intersectent $U \cup \partial\Omega$. Comme cette fois l'intersection s'effectue avec l'ensemble ouvert U et non \overline{U} on montre que $u - v$ ne peut prendre qu'un nombre dénombrable de valeurs quand u et v sont deux solutions de \mathcal{P}_λ . Comme ces deux fonctions sont lipschitziennes elles sont forcément égales.

0.4.4.3 Unicité en dimension supérieure

La principale différence entre le théorème 0.4.2 et le théorème 0.4.4 est que le second ne s'applique qu'à la dimension deux. Si l'on veut généraliser ce dernier résultat en dimension quelconque la compréhension de la régularité du bord des ensembles de sur-niveau est importante. On ne peut plus compter sur la paramétrisation lipschitzienne des courbes de $L_t^*(u)$, il nous faut donc un nouvel outil pour prouver que les composantes connexes de $\partial^* E_t(u)$ intersectent le bord de Ω ou l'ensemble U . Comme dans le problème précédent cette régularité pourrait provenir d'un problème de minimisation dont les ensembles de sur-niveau sont solutions. Cela soulève donc la question de la régularité de $\sigma = \nabla\varphi(\nabla u)$. Nous savons qu'elle est $W_{loc}^{1,2}(\Omega)$, nous aurions besoin de montrer qu'elle est $\mathcal{C}^{0,\alpha}$. Cependant nous ne savons presque rien sur sa continuité.

En revanche, nous pouvons montrer que en dimension deux quand $\lambda \in \mathbb{R}$ la fonction σ est continue. Nous allons détailler dans la section suivante les résultats du quatrième chapitre autour de la continuité de σ et de la régularité \mathcal{C}^1 des solutions de problèmes en calcul des variations mais aussi des solutions d'équations elliptiques dégénérées.

0.5 Régularité pour les solutions d'équations elliptiques dégénérées et/ou singulières.

Dans ce dernier chapitre nous étudions la régularité des solutions localement lipschitziennes de l'équation suivante :

$$\operatorname{div} G(\nabla u_0(x)) = f \text{ in } \Omega \quad (0.5.1)$$

avec G une fonction continue de \mathbb{R}^N dans \mathbb{R}^N et f une fonction continue sur $\bar{\Omega}$. Cette équation est à comprendre au sens faible, c'est-à-dire qu'une solution u_0 vérifie :

$$\int_{\Omega} \langle G(\nabla u_0), \nabla \theta \rangle = - \int_{\Omega} f \theta$$

pour toute fonction $\theta \in C_0^\infty(\Omega)$. On suppose que la fonction G est monotone dans le sens suivant :

$$\langle G(z_1) - G(z_2), z_1 - z_2 \rangle \geq 0 \quad (0.5.2)$$

pour tous $z_1, z_2 \in \mathbb{R}^N$. Il est important de noter que l'inégalité est large, ainsi nous sommes dans un cadre d'équations elliptiques pouvant être dégénérées. Généralement, il est donc vain d'espérer obtenir des solutions C^1 comme c'est le cas dans la littérature classique pour les équations *uniformément* elliptiques. En revanche on peut espérer prouver que $G(\nabla u_0)$ est continue. Nous allons aussi considérer le cas où G est singulière quand (0.5.2) ne peut être majorée par $C|z_1 - z_2|^2$ avec $C > 0$ pour tous $z_1, z_2 \in \mathbb{R}^N$. Cela arrive typiquement quand G a des dérivées partielles qui tendent vers $+\infty$ sur une droite de \mathbb{R}^2 . L'inégalité (0.5.2) est automatiquement satisfaite quand G est le gradient d'une fonction convexe. Dans ce cas là l'équation (0.5.1) peut être vue comme l'équation d'Euler-Lagrange de la fonctionnelle suivante :

$$\int_{\Omega} \varphi(\nabla v) + f v \quad (0.5.3)$$

avec $G = \nabla \varphi$.

Cependant une telle fonction G n'est pas nécessairement un gradient et nous allons énoncer des résultats qui n'ont pas besoin d'une telle restriction. Cela nous permet d'étudier des problèmes plus généraux que des problèmes de minimisation. Dans la plupart des cas que nous allons étudier dans ce chapitre la dimension N est égale à 2 et f est constante. Nous verrons un dernier résultat en dimension supérieure avec f variable. L'objectif principal du chapitre est d'étudier la continuité de $G(\nabla u_0)$ dans différents cas.

Les premiers résultats de régularité C^1 pour des équations dégénérées ou singulières proviennent du p -Laplacien où l'équation est soit singulière soit dégénérée à l'origine. Dans [65], Shuntaro Tsubouchi prouve que le gradient de u_0 est continu pour la somme du 1-Laplacien et du p -Laplacien avec f dans $L^q(\Omega)$ et $p > 1$, $q > N$.

Il existe des articles récents sur l'étude de la régularité des solutions quand la zone de dégénérescence est un ensemble convexe. On peut citer par exemple [26] qui nous

donne la continuité de $(|\nabla u_0| - 1)_+$ en toute dimension pour u_0 minimiseur de (0.5.3) quand φ est une fonction uniformément convexe hors de la boule unité. Une question naturelle est donc de savoir ce qu'il peut arriver quand la zone de dégénérescence n'est pas un ensemble convexe, pas connexe ou pas simplement connexe.

Les contributions de cette thèse à la régularité des minimiseurs se situent dans le cadre où f le membre de droite de 0.5.1 est au moins uniformément continu sur Ω . Parmi les cas où l'on impose moins de régularité sur f nous pouvons citer [4] dans lequel Ambrosio prouve que $G(\nabla u_0)$ est dans un espace de Bezoov quand f l'est aussi. Nous avons aussi [25] où Clop, Giova, Hatami et Passarelli di Napoli prouve un gain de régularité Sobolev de Du_0 quand f est dans une classe de Zygmund dans le cas vectoriel pour des problèmes dépendant de la norme euclidienne.

0.5.1 Résultats principaux

Nous allons présenter quatre résultats de régularité pour des solutions de (0.5.1). Dans le cadre du premier théorème nous supposons que le second membre f est nul.

Théorème 0.5.1. *On suppose que $G := G_1 + G_2 \in \mathcal{C}_{loc}^{0,1}(\mathbb{R}^2)$ et $f \equiv 0$. De plus, on demande que G_1 et G_2 vérifient chacune une condition supplémentaire :*

(A₁) *pour tout $L > 0$ il existe une constante $C_L > 0$ telle que pour tous $z_1, z_2 \in B_L(0)$:*

$$\langle G_1(z_1) - G_1(z_2), z_1 - z_2 \rangle \geq C_L |G_1(z_1) - G_1(z_2)|^2$$

et

(A₂) *la fonction G_2 est le gradient d'une fonction convexe $\varphi \in \mathcal{C}_{loc}^{1,1}(\mathbb{R}^2)$.*

Alors $G(\nabla u_0)$ est continue sur Ω pour toute fonction u_0 solution localement lipschitzienne de (0.5.1).

De plus pour tout $\Omega' \Subset \Omega$, nous avons un module de continuité explicite dépendant de la distance entre $\partial\Omega'$ et $\partial\Omega$, de la constante de Lipschitz L' de u_0 sur Ω' , de la constante de Lipschitz G sur $B_{L'}(0)$ et de $C_{L'}$. Une application directe de ce résultat est la suivante. Si $\varphi \in \mathcal{C}_{loc}^{1,1}(\mathbb{R}^2)$ est une fonction convexe alors tout minimiseur u_0 localement lipschitzien de

$$\int_{\Omega} \varphi(\nabla v)$$

vérifie $\nabla\varphi(\nabla u_0) \in \mathcal{C}^0(\Omega)$. De plus, ce théorème permet donc de généraliser cette étude de la régularité des minimiseurs au cadre des équations aux dérivées partielles elliptiques dégénérées. En effet, la fonction G_1 n'est pas nécessairement un gradient et n'est pas non plus strictement monotone. Des applications potentielles de ce théorème peuvent être les inclusions différentielles (voir [45]) où ce type d'équations peuvent apparaître.

Si l'on veut pouvoir considérer des $f \neq 0$ alors on doit imposer des contraintes supplémentaires sur G . C'est ce que nous faisons dans le cadre du deuxième théorème. Ici G sera le gradient d'une fonction convexe pouvant se décomposer en une

somme particulière. Plus précisément, pour le deuxième résultat on introduit la notion de *pseudo-norme* :

Définition 1. Une fonction $\mathcal{N} : \mathbb{R}^N \rightarrow \mathbb{R}_+$ est une pseudo-norme si $\mathcal{N}(0) = 0$, \mathcal{N} est positivement homogène et $\{z \in \mathbb{R}^N, \text{ tel que } \mathcal{N}(z) < 1\}$ est un ensemble ouvert borné strictement convexe ayant un bord de régularité $\mathcal{C}^{1,1}$.

On a alors le résultat suivant :

Théorème 0.5.2. On suppose que $f \equiv \lambda \in \mathbb{R}$ et $G = \sum_{i=1}^n G_i$ avec $n \in \mathbb{N}^*$. Ici, les fonctions $(G_i)_{1 \leq i \leq n}$ sont les gradients de fonctions convexes $(\varphi_i)_{1 \leq i \leq n}$ qui ont l'une des deux formes suivantes :

(A₃) $\varphi_i(z) := f_i(\mathcal{N}_i(z - \xi_i))$ avec $f_i \in \mathcal{C}_{loc}^{1,1}(\mathbb{R})$ une fonction convexe telle que $f_i'(z) = 0 \Leftrightarrow z = 0$, \mathcal{N}_i une pseudo-norme et $\xi_i \in \mathbb{R}^2$.

(A₄) $\varphi_i(z) := f_i(\langle z, \xi_i \rangle)$ avec $f_i \in \mathcal{C}_{loc}^{1,1}(\mathbb{R})$ une fonction convexe et $\xi_i \in \mathbb{R}^2 \setminus \{0\}$.

Alors $G(\nabla u_0)$ est continue sur Ω pour toute fonction u_0 solution localement lipschitzienne de (0.5.1).

Ici φ est la somme de fonctions convexes faisant partie de deux classes. D'un côté nous avons des fonctions se rapprochant des fonctions convexes dépendant d'une norme. De l'autre nous avons des fonctions ne dépendant que d'une seule coordonnée.

Si $g \in \mathcal{C}_{loc}^{1,1}(\mathbb{R})$ est une fonction convexe telle que $g'(t) = 0 \Leftrightarrow t = 0$ alors $G = \nabla[g(| \cdot |)]$ satisfait les hypothèses du théorème. Ainsi quand φ est comme dans (0.3.4) on obtient que $(|\nabla u_0| - 2)_+$ est continu quand u_0 est un minimiseur. Cela est donc une nouvelle preuve pour le cadre radial de [26] en dimension deux ou d'un résultat précédent de Santambrogio et Vespi [62]. Cependant dans ce cas ci, la zone de dégénérescence est un anneau. Ainsi, la nouveauté principale pour ce problème découlant de ce théorème est que nous avons aussi que $(|\nabla u_0| - 1)_-$ est continu quand $\lambda \in \mathbb{R}$ en dimension deux.

Toujours dans le cadre des fonctions radiales on peut utiliser le fait que $\nabla\varphi(\nabla u_0)$ soit continue pour étudier la régularité des lignes de niveau de u_0 . En effet, le représentant continu σ de $\nabla\varphi(\nabla u_0)$ est colinéaire presque partout avec ∇u_0 . On a alors que $\frac{\sigma}{|\sigma|} = \frac{\nabla u_0}{|\nabla u_0|}$ presque partout dans le sous-ensemble de Ω où σ est différente de 0. Ainsi on peut montrer que presque toutes les lignes de niveau de u_0 ont une normale continue et sont donc des courbes de classe \mathcal{C}^1 .

Un autre cas étudié dans la littérature est celui des problèmes orthotropiques. Par orthotropique on entend que $G = \nabla\varphi$ avec φ la somme de fonctions convexes $z \rightarrow \varphi_i(z)$ qui ne dépendent que d'une seule coordonnée de z . On peut citer notamment l'article de Ricciotti [61] où est étudiée la régularité \mathcal{C}^1 des minimiseurs quand $\varphi(z) := |z_1|^p + |z_2|^p$ avec $1 < p < 2$ et $z_i = \langle z, e_i \rangle$ où e_1, e_2 sont des vecteurs unitaires non colinéaires. Cette configuration ne rentre pas dans le cadre de notre théorème à cause des singularités de φ sur $\{z_1 z_2 = 0\}$.

En revanche, nous pouvons considérer $\varphi(z) := |z_1|^{p_1} + |z_2|^{p_2}$ avec $p_1, p_2 \geq 2$. Ainsi nous obtenons que $p_1 |\partial_1 u_0|^{p_1-1} \partial_1 u_0$ et $p_2 |\partial_2 u_0|^{p_2-1} \partial_2 u_0$ sont continues sur Ω .

L'inversibilité de $t \rightarrow p_i |t|^{p_i-1} t$ a pour conséquence la régularité \mathcal{C}^1 de u_0 . Ainsi nous avons une nouvelle preuve d'un résultat déjà connu grâce à Lindqvist et Ricciotti dans [48]. Nous pouvons aussi citer [11] dans lequel Bousquet et Brasco étudient le cas où les deux exposants sont égaux pouvant être plus petits que deux.

Attardons nous maintenant sur le cas où $1 < p_1 \leq 2 \leq p_2$. Dans ce cas nous avons une singularité à l'origine. Sous ces conditions là nous pouvons appliquer le résultat suivant :

Théorème 0.5.3. *On suppose que $f \equiv \lambda \in \mathbb{R}$ et $G = G_1 + G_2$ avec $G_i(z) := f'_i(\langle z, \xi_i \rangle) \xi_i$ où $f_1 \in \mathcal{C}_{loc}^{1,1}(\mathbb{R})$ et $f_2 \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}_{loc}^{1,1}(\mathbb{R} \setminus \{0\})$ sont deux fonctions convexes et $\xi_1, \xi_2 \in \mathbb{R}^2 \setminus \{0\}$ ne sont pas colinéaires. De plus, on suppose qu'il existe un module de convexité $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ continu satisfaisant $\omega(t) = 0 \Leftrightarrow t = 0$ et $r > 0$ tels que pour tous $x, y \in (-r, r)$ on a $(f'_2(x) - f'_2(y))(x - y) \geq \omega(|x - y|)$. Alors $G(\nabla u_0)$ est continue sur Ω pour toute fonction u_0 solution localement lipschitzienne de (0.5.1).*

Ce théorème couvre toujours le cas des fonctions orthotropiques mais cette fois ci avec une puissance strictement plus petite que deux et l'autre plus grande que deux. Avec la continuité de $G(\nabla u_0)$ nous pouvons encore montrer la régularité \mathcal{C}^1 de u_0 . En effet si f_1 est strictement convexe alors $\partial_1 u_0$ est continue et si f_2 est strictement convexe alors c'est $\partial_2 u_0$ qui l'est. Ainsi les deux derniers théorèmes présentés permettent de retrouver des résultats établis par Bousquet dans [9]. Cependant, on peut considérer des fonctions plus générales que des fonctions puissances. Par exemple, si l'on choisit $f_1(t) := (|t| - r)_+^2$ et $f_2(t) = t^{\frac{3}{2}}$ avec $r > 0$ alors nous avons la continuité de $G(\nabla u_0)$ alors que la zone où la fonction est à la fois singulière et dégénérée est le segment $[-r, r] \times \{0\}$.

Nous pouvons voir les résultats de régularité \mathcal{C}^1 dans le cadre de fonctions strictement convexe comme un analogue de ce qui est déjà connu en dimension un. Par exemple, dans le livre de Clarke [23, Théorème 15.5] nous pouvons voir que les minimiseurs lipschitziens sont \mathcal{C}^1 quand la fonctionnelle est strictement convexe par rapport au gradient.

Le dernier théorème de ce quatrième chapitre est une amélioration du résultat principal de [57, Théorème 2.1]. Dans cet article Mooney considère $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$ une fonction convexe \mathcal{C}^1 . Il introduit alors pour tout $k \in \mathbb{N}$ l'ouvert

$$\mathcal{O}_k := \left\{ \frac{1}{k} Id < D^2 \varphi(z) < k Id \right\}.$$

On suppose de plus que $D_\varphi := \mathbb{R}^N \setminus \bigcup_{k \in \mathbb{N}} \mathcal{O}_k$ est un ensemble fini de points coplanaires.

Dans ce cas là les minimiseurs localement lipschitziens de

$$\int_{\Omega} \varphi(\nabla u)$$

sont \mathcal{C}^1 . Dans cette thèse on généralise cette preuve au cas où la zone de dégénérescence est une union fini d'ensemble simplement connexes coplanaires. Nous ajoutons aussi un terme d'ordre inférieur de régularité Sobolev :

Théorème 0.5.4. *Supposons que $f \in W^{1,q}(\Omega)$ avec $q > N$ et que u_0 est une solution faible localement lipschitzienne de (0.5.1). Nous supposons que la zone de dégénérescence $D_G := \mathbb{R}^N \setminus \bigcup_{k \in \mathbb{N}} \{\frac{1}{k}Id < DG^s(z) < kId\}$ est incluse dans un plan et a un nombre fini de composantes connexes. De plus, nous demandons l'existence de $t_0 > 0$ tel que pour tout $0 \leq t \leq t_0$ les composantes connexes de $\bar{N}_t(D_G)$ soient simplement connexes. Alors $dist(\nabla u_0, D_G)$ et $\nabla u_0 \times dist(\nabla u_0, D_G)$ sont continues. De plus, si G est constante sur chaque composante connexe de D_G , la fonction $G(\nabla u_0)$ est continue.*

Ici $\bar{N}_t(D_G)$ désigne l'ensemble $\{z \in \mathbb{R}^N \text{ tel que } dist(z, D_G) \leq t\}$ et $DG^s := \frac{DG + DG^T}{2}$.

Les premiers résultats de régularité \mathcal{C}^1 pour des problèmes n'étant pas uniformément elliptiques ont été obtenus dans le cadre du p -Laplacien. Quand la zone de dégénérescence n'est plus seulement un point mais un convexe alors [26] garantit que la distance de ∇u_0 à ce convexe est continue et que u_0 est \mathcal{C}^1 autour des points ayant un gradient hors du convexe de dégénérescence. Ainsi le théorème 0.5.4 prouvant la continuité de ∇u_0 par rapport à D_G et sa régularité en dehors est une amélioration similaire du résultat de régularité \mathcal{C}^1 de [57, Théorème 2.1] quand D_G n'était composé que de points.

0.5.2 Techniques de preuve

Nous présentons les idées pour prouver les quatre résultats principaux du quatrième chapitre.

0.5.2.1 Régularisation du problème

Comme les preuves des théorèmes de ce chapitre utilisent les dérivées des solutions et de la fonction G nous devons régulariser notre problème. Nous approchons donc G par une suite de fonctions lisses $(G^m)_{m \in \mathbb{N}}$ qui convergent uniformément vers G sur tout ensemble compact de \mathbb{R}^N quand $m \rightarrow +\infty$. Une contrainte importante est que G^m doit avoir les mêmes propriétés que G . Dans le cas du théorème 0.5.2 où $G = \nabla[f(\mathcal{N}(\cdot - \xi))]$ on ne peut pas juste régulariser $f(\mathcal{N})$ car nous perdriions la structure de la pseudo-norme \mathcal{N} . Il faut donc régulariser f par convolution et approcher \mathcal{N} par une suite de pseudo-normes lisses hors de l'origine en utilisant les jauges d'ensembles convexes. Les autres théorèmes demandent aussi une régularisation particulière pour conserver la nature de G .

Après avoir choisi soigneusement la manière d'approcher notre problème, on obtient une suite de solutions lisses $(u_m)_{m \in \mathbb{N}}$ de l'équation :

$$\operatorname{div} G^m(\nabla u) = f_m \text{ sur } \Omega. \quad (0.5.4)$$

On montre que cette suite admet une limite faible $u \in W_{u_0}^{1,2}(\Omega)$. Grâce aux mesures de Young nous montrons que u est solution de la même équation que u_0 même si, a priori, ce sont deux fonctions différentes. De plus, $G^m(\nabla u_m) \rightarrow G(\nabla u)$

dans $L^1(\Omega)$. Une dernière proposition consiste à prouver que la fonction $G(\nabla u)$ ne dépend pas du choix de la solution u de (0.5.1). Ainsi il ne reste plus qu'à prouver la continuité uniforme en $m \in \mathbb{N}$ de $G^m(\nabla u_m)$ ou $\text{dist}(\nabla u_m, D_G)$ pour le dernier théorème.

0.5.2.2 Estimations de Sobolev

Les preuves des théorèmes 0.5.1, 0.5.2 et 0.5.3 reposent sur une estimation de la norme de Sobolev $W_{loc}^{1,2}(\Omega)$ de $G^m(\nabla u_m)$ indépendante du paramètre de régularisation $m \in \mathbb{N}$. Pour les deux premiers théorèmes nous avons que les fonctions G^m sont la somme de fonctions lipschitziennes $(G_i^m)_{1 \leq i \leq n}$. Dans ce cas là on montre que pour chaque i , $G_i^m(\nabla u_m) \in W_{loc}^{1,2}(\Omega)$ avec une norme indépendante de $m \in \mathbb{N}$. Ainsi, cela est aussi le cas pour leur somme $G^m(\nabla u_m)$. Les techniques de preuves utilisées ici peuvent être trouvées dans [29]. L'idée est de dériver l'équation (0.5.4) dans une direction et de choisir une bonne fonction test pour pouvoir utiliser l'hypothèse de régularité lipschitzienne de G .

Pour le théorème 0.5.3 nous procédons un peu différemment. Comme dans les cas précédents on obtient une estimation de la norme de $f'_1(\langle \nabla u_m, \xi_1 \rangle)$ indépendante de $m \in \mathbb{N}$ en combinant les idées du cadre précédent avec un résultat de Carstensen et Müller [19]. Pour $f'_2(\langle \nabla u_m, \xi_2 \rangle)$ on obtient avec les mêmes idées une estimation sur l'ensemble où $|\langle \nabla u_m, \xi_2 \rangle| \geq \alpha > 0$ évitant ainsi la singularité potentielle en l'origine.

0.5.2.3 Principe du maximum et théorème de Lebesgue

Un outil important dans les preuves des théorèmes 0.5.1, 0.5.2 et 0.5.3 est la proposition suivante valable uniquement en dimension deux :

Proposition 0.5.1. *Soit $H \in W_{loc}^{1,2}(\Omega, \mathbb{R}^2)$. Si pour tout $\epsilon > 0$ et tout $x_0 \in \Omega$ il existe $C(\epsilon, x_0) > 0$ tel que pour tout $0 < \delta < \text{dist}(x_0, \partial\Omega)$:*

$$\text{osc}_{B_\delta(x_0)} H \geq \epsilon \Rightarrow \text{osc}_{\partial B_\delta(x_0)} H \geq C(\epsilon, x_0),$$

alors H est continue en x_0 . Ici, $\text{osc}_{B_\delta(x_0)} H := \sup_{x,y \in B_\delta(x_0)} |H(x) - H(y)|$.

Ce résultat dû à Lebesgue [46] est utilisé dans les articles [29] et [61] pour prouver la régularité \mathcal{C}^1 des solutions en combinaison avec un principe du maximum classique (voir le livre de Gilbarg et Trudinger [36]) :

Proposition 0.5.2. *Soit u_m une solution lisse de (0.5.4). Alors pour tout vecteur $e \in \mathbb{S}^{N-1}$ tout ouvert $\Omega' \Subset \Omega$, on a que :*

$$\sup_{x \in \Omega'} \partial_e u_m(x) = \sup_{x \in \partial\Omega'} \partial_e u_m(x).$$

La stratégie générale des preuves des théorèmes 0.5.1, 0.5.2 et 0.5.3 est alors la suivante. Nous commençons par approcher notre problème par une suite de problèmes lisses. On obtient ainsi une suite de fonctions $(G^m)_{m \in \mathbb{N}}$ approchant G et une

suite de solutions régulières $(u_m)_{m \in \mathbb{N}}$. Nous pouvons montrer que la suite $G^m(\nabla u_m)$ est uniformément dans $W_{loc}^{1,2}(\Omega)$. Le principe du maximum nous sert en suite à montrer que l'on peut appliquer le théorème de Lebesgue à $G^m(\nabla u_m)$ pour tout $x_0 \in \Omega$ et tout $\epsilon > 0$ avec une constante $C(\epsilon, x_0)$ indépendante de $m \in \mathbb{N}$. Il est important de noter que c'est à ce moment là que l'on utilise que les fonctions $(G^m)_{m \in \mathbb{N}}$ est la même nature que G . En effet, l'application du principe du maximum classique pour pouvoir utiliser le résultat de Lebesgue dépend fortement de la forme de G . De plus, pour le théorème 0.5.1 nous utilisons un principe du maximum vectoriel provenant d'un article de Hartman et Nirenberg [40] valable seulement quand $\lambda = 0$ car le principe du maximum classique n'est pas suffisant. Ainsi les fonctions $G^m(\nabla u_m)$ sont uniformément continues en $m \in \mathbb{N}$ et en passant à la limite on obtient que $G(\nabla u_0)$ est continue sur Ω .

0.5.2.4 Idées de preuve pour le théorème 0.5.4

Comme dans les résultats précédents nous commençons par régulariser notre problème pour prouver le théorème 0.5.4. La preuve peut se décomposer en deux parties. La première est de montrer qu'il existe $\delta > 0$ tel que $\nabla u_m(B_\delta(x_0))$ est soit proche de $\nabla u_m(x_0)$ avec $\nabla u_m(x_0)$ loin de C_G l'enveloppe convexe de D_G soit proche de C_G . Pour ce faire nous utilisons [26, Théorème 1.1] nous donnant la continuité de $dist(\nabla u_0, C_G)$.

Ainsi dans le cas où $dist(\nabla u_0, C_G) \geq r > 0$, nous utilisons la théorie classique des équations uniformément elliptiques pour conclure. Dans le second cas, on utilise l'hypothèse que D_G est de dimension deux pour montrer qu'il existe $\delta' > 0$ tel que $\nabla u_m(B_{\delta'}(x_0))$ est soit proche de $\nabla u_m(x_0)$ soit proche de D_G . Pour ce faire, nous utilisons le fait que une certaine fonction η de ∇u_m convexe dans $N - 1$ directions et concave dans la dernière est une sous-solution de l'équation (0.5.1) linéarisée :

$$L_G(v) := \operatorname{div}(DG^m(\nabla u_m)\nabla v) = \tilde{f}$$

avec \tilde{f} une fonction dans $L^q(\Omega)$.

Ainsi pour tout $\epsilon > 0$ on peut trouver un $\delta_\epsilon > 0$ indépendant de $m \in \mathbb{N}$ tel que que soit $\nabla u_m(B_{\delta_\epsilon}(x_0))$ est dans $B_\epsilon(\nabla u_m(x_0))$ soit dans le ϵ -voisinage de D_G . En passant à la limite quand m tend vers $+\infty$ on obtient le résultat souhaité.

0.5.3 Ouvertures potentielles

Plusieurs questions restent en suspens autour des résultats du dernier chapitre. En dimension deux, nous pouvons nous demander ce qu'il se passe si G est une fonction monotone, lipschitzienne et $\lambda \in \mathbb{R}$. C'est-à-dire peut-on améliorer le théorème 0.5.1 au cas où λ est constante mais non nulle ? Un résultat intéressant serait de savoir si le théorème 0.5.2 reste vrai si λ est une fonction lipschitzienne non constante. La difficulté dans ce cas là est que l'on ne peut plus tirer les mêmes informations du principe du maximum. Cela pourrait permettre de mieux comprendre la régularité des lignes de niveau en dimension deux ce qui peut être utile pour des questions

d'unicité ou de propriétés qualitatives des solutions comme vu dans les chapitres précédents. La généralisation des théorèmes 0.5.1, 0.5.2 et 0.5.3 en toute dimension est aussi une piste intéressante. Pour cela il faudrait trouver un moyen de compenser l'absence du théorème de Lebesgue en dimension plus grande que deux. Enfin, l'extension du théorème 0.5.4 à des G ayant un D_G plus général permettrait de mieux comprendre les hypothèses sur D_G garantissant la continuité de $dist(\nabla u_0, D_G)$. On voudrait savoir si il est possible d'avoir D_G de dimension deux mais non simplement connexe par exemple ou simplement connexe mais de dimension supérieure. Il est important de noter que [57, Théorème 2.3] procure un exemple de solution pas C^1 quand la zone de dégénérescence n'est pas dans un plan en dimension supérieure ou égale à quatre.

A uniqueness result for a translation invariant problem in the Calculus of Variations

This chapter is based on the paper "*A uniqueness result for a translation invariant problem in the Calculus of Variations*" [52] accepted at the *Journal of Convex Analysis*.

Abstract

We present a uniqueness result of uniformly continuous solutions for a general minimization problem in the Calculus of Variations. We minimize the functional $\mathcal{I}_\lambda(u) := \int_\Omega \varphi(\nabla u) + \lambda u$ with φ a convex but not necessarily strictly convex function, Ω an open set of \mathbb{R}^N with $N \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. The proof is based on the two following main points : the functional \mathcal{I}_λ is invariant under translations and we assume that the function φ is not affine on any non-empty open set. This provides a shorter proof and/or an extension for some already known uniqueness results for functionals of the type \mathcal{I}_λ that are presented in the chapter.

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1.1 Introduction

1.1.1 Presentation of the problem

We study a uniqueness result for a non-strictly convex problem in the calculus of variations in dimension $N \geq 1$. We consider functionals of the following form:

$$\mathcal{I}_\lambda : u \rightarrow \int_{\Omega} \varphi(\nabla u(x)) + \lambda u(x) dx \quad (1.1.1)$$

with $\lambda \in \mathbb{R}$, Ω a bounded open set of \mathbb{R}^N and $\varphi : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ a measurable convex function.

We consider the problem:

$$\mathcal{P}_\lambda : \text{To minimize } \mathcal{I}_\lambda(v) \text{ on } \mathcal{E}(\Omega, \psi).$$

Here, $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous function in $W_{loc}^{1,1}(\mathbb{R}^N)$ and $\mathcal{E}(\Omega, \psi)$ is a subset of $W_\psi^{1,1}(\Omega)$, namely the set of functions in $W^{1,1}(\Omega)$ such that the extension outside Ω by $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$ is in $W_{loc}^{1,1}(\mathbb{R}^N)$. We work with the set $\mathcal{E}(\Omega, \psi)$ to cover many situations: in the following $\mathcal{E}(\Omega, \psi)$ can stand for $W_\psi^{1,1}(\Omega)$, $W_\psi^{1,p}(\Omega)$, $W_\psi^{1,p}(\Omega) \cap C^0(\bar{\Omega})$, $C_\psi^{0,1}(\bar{\Omega})$ or $\{v \in W_\psi^{1,1}(\Omega), \nabla v(x) \in K \text{ for a.e. } x \in \Omega\}$ with K a convex set.

We assume that \mathcal{I}_λ is well defined on $\mathcal{E}(\Omega, \psi)$ with values in $\mathbb{R} \cup \{+\infty\}$ and that there exists $v \in \mathcal{E}(\Omega, \psi)$ such that $\mathcal{I}_\lambda(v) < +\infty$.

We point out that φ is not necessarily strictly convex. Hence, no general result guarantees uniqueness. Another remarkable feature of the main result Theorem 1.1.2 is that $\varphi(z)$ can be equal to $+\infty$ for some $z \in \mathbb{R}^N$. Hence, this framework covers also the case when we put a pointwise constraint on the gradient. In the situations studied in [8] and [41], \mathcal{P}_λ is the convexification of a non-convex problem that arises in shape optimization. In these cases (see also [54]) uniqueness for \mathcal{P}_λ can be useful to prove existence or non-existence of a minimizer for related non-convex problems.

1.1.2 Main result

In order to present the main result of this paper we introduce the following sets:

Definition 1.1.1. We say that a convex set $F \in \mathbb{R}^N$ is an affine part of φ if $\forall z, \xi \in F$ we have:

$$\varphi(z) < +\infty, \varphi(\xi) < +\infty \text{ and } \varphi\left(\frac{z + \xi}{2}\right) = \frac{\varphi(z) + \varphi(\xi)}{2}.$$

The dimension of F is the dimension of the affine space generated by F .

We make the two following assumptions on the set $\mathcal{E}(\Omega, \psi)$:

(A1) The set $\mathcal{E}(\Omega, \psi)$ is invariant under translation. Namely, if $v \in \mathcal{E}(\Omega, \psi)$ then $\tilde{v}(\cdot) = v(\cdot + \tau) + \alpha \in \mathcal{E}(\Omega_\tau, \psi(\cdot + \tau) + \alpha)$ with $\Omega_\tau = \Omega - \tau$.

(A2) If $u \in \mathcal{E}(\Omega, \psi)$ and $v \in \mathcal{E}(V, u)$ with V a non empty open set in Ω , then the extension of v by u is in $\mathcal{E}(\Omega, \psi)$.

We make one assumption on φ :

(A3) The dimensions of the affine parts of φ are at most $N - 1$.

We now present the main theorem of this chapter:

Theorem 1.1.2. *Under those three assumptions, the problem \mathcal{P}_λ admits at most one uniformly continuous minimizer.*

The proof of Theorem 1.1.2 which is relatively short and simple, unifies results that are described in Section 2. For some of them it is even an extension to higher dimensions or to a more general set Ω .

The first two assumptions (A1) and (A2) are relatively standard. For instance, they are satisfied when $\mathcal{E}(\Omega, \psi)$ is $W_\psi^{1,1}(\Omega)$, $W_\psi^{1,p}(\Omega)$, $W_\psi^{1,p}(\Omega) \cap C^0(\overline{\Omega})$, $C^{0,1}(\overline{\Omega})$ or $\{v \in W_\psi^{1,1}(\Omega), \nabla v(x) \in K \text{ for a.e. } x \in K\}$ with K a convex set.

However, the last assumption (A3) on the dimensions of the affine parts of φ is necessary for the statement to be true. It is easy to find counterexamples when this condition is not satisfied. For instance if we consider the problem when $N = 1$, $\Omega = (0, 1)$, $\psi(0) = 0$, $\psi(1) = 1$, $\lambda = 0$ and $\varphi(\cdot) = |\cdot|$, every increasing function that satisfies the boundary condition is a solution. Another example of non uniqueness when the last assumption is not satisfied can be found in [21, Example 6] with $\varphi(\cdot) = (|\cdot| - 1)_+^2$.

In this chapter, we prove a uniqueness result but we do not provide any result about the existence or about the regularity of minimizers. In the cases where the existence is provided by the direct methods and the uniform continuity is standard, we obtain the following corollary of Theorem 1.1.2:

Corollary 1.1.3. *We assume that the problem \mathcal{P}_λ admits a minimizer and every minimizer is uniformly continuous. Under the three assumptions (A1), (A2) and (A3), the problem \mathcal{P}_λ has a unique minimizer.*

That is the case for instance when Ω is smooth, ψ smooth and φ satisfies a p -growth condition, some examples are given in Section 2. In some problems like in [10], if a uniformly continuous minimizer exists then all the minimizers are uniformly continuous. In this case we have:

Corollary 1.1.4. *We assume that if the problem \mathcal{P}_λ has a uniformly continuous minimizer then every minimizer is uniformly continuous. Under the three assumptions (A1), (A2) and (A3) if a minimizer of \mathcal{P}_λ is uniformly continuous then it is the only minimizer.*

The main idea of the proof comes from [29, Section 4]. In this paper, De Silva and Savin consider the minimization in dimension two of

$$I(u) = \int_{\Omega} \varphi(\nabla u) dx$$

on $\mathcal{E}(\Omega, \psi) = C_\psi^{0,1}(\overline{\Omega})$ with φ a convex function with domain $K := \{z \in \mathbb{R}^2, \varphi(z) < +\infty\}$ a closed polygon. In [29, Proposition 4.5], the authors use the fact the problem is invariant under translation to compare the gradient of a minimizer at a point with the gradient of another minimizer at another point.

1.1.3 Plan of the paper.

In the next section, we present some cases where we can apply Theorem 1.1.2. In Section 3, we present some classical results for functionals of the type (1.1.1). The last section is devoted to the proof of Theorem 1.1.2.

1.2 Applications

In this section we apply Theorem 1.1.2 to obtain uniqueness results for some functionals arising in the literature and we conclude by a subsection on a comparison principle.

1.2.1 Radial functions

In this subsection, we assume that $\varphi(z) = g(\|z\|)$ with $g : \mathbb{R} \rightarrow \mathbb{R}$ a convex function such that $g(0) < g(t)$ for all $t \neq 0$ and $\|\cdot\|$ a strictly convex norm.

Proposition 1.2.1. *For such a φ , the assumption (A3) is satisfied.*

Proof. Since g is strictly increasing and $\|\cdot\|$ is strictly convex, the lower level sets of φ are strictly convex. We assume by contradiction that there exists $z \in \mathbb{R}^N$ and $\epsilon > 0$ such that $B_\epsilon(z) \subset F$ where $B_\epsilon(z) := \{y \in \mathbb{R}^N \text{ such that } \|y - z\| < \epsilon\}$ and F is an affine part of φ . Without loss of generality, we can assume that $z \neq 0$. Hence, there exists $z' \in B_\epsilon(z) \setminus \{z\}$ such that $\|z'\| = \|z\|$. Since $z, z' \in F$, φ is affine on $[z, z']$. Since $\varphi(z) = \varphi(z')$, the segment $[z, z']$ is contained in the boundary of a lower level set of φ . That is a contradiction. Thus, the interior of F is empty. Hence, the dimension of F is at most $N - 1$. \square

In the rest of the subsection, we detail some examples where the norm considered is the Euclidean norm $|\cdot|$. If the set $\mathcal{E}(\Omega, \psi)$ is a subset of uniformly continuous functions that are in $W^{1,1}(\Omega)$ then every minimizer is uniformly continuous. In this case, Theorem 1.1.2 becomes: \mathcal{P}_λ admits at most one minimizer. For instance that is the case in [54, Theorem 3] where the set considered is $\mathcal{C}_\psi^{0,1}(\overline{\Omega})$. A minimization among Lipschitz continuous functions also appears in [53, Theorem 1.1] but only for the dimension two. Hence, Theorem 1.1.2 allows to generalize this result to higher dimensions.

This theorem can also be used to prove the uniqueness part for the least gradient problem considered in [63]:

$$\text{To minimize } \int_{\Omega} |\nabla u| dx \text{ on } \{u \in BV(\Omega) \cap \mathcal{C}^0(\overline{\Omega}), u = \psi \text{ on } \partial\Omega\}.$$

In fact, we can not directly apply Theorem 1.1.2 because $\mathcal{E}(\Omega, \psi) \not\subseteq W^{1,1}(\Omega)$ but [63, Theorem 5.9] provides that if Ω and ψ are smooth, then the minimizers are globally Lipschitz continuous on Ω .

In the articles [53], [54] and [63] the question of existence is much harder. Now, we look at problems where the direct method guarantees the existence of a minimizer. If φ satisfies a *p-growth* condition: there exist $C_1, \alpha > 0$ and $p > 1$ such that

$$C_1|z|^p - \alpha \leq \varphi(z) \quad (1.2.1)$$

for every $z \in \mathbb{R}^N$ then by [27, Theorem 3.30], \mathcal{P}_λ admits at least one minimizer on $W_\psi^{1,p}(\Omega)$.

In [8], we have:

$$\varphi(z) = \begin{cases} |z| & \text{if } |z| \leq 1, \\ \frac{1}{2}(|z|^2 + 1) & \text{if } |z| > 1, \end{cases} \quad (1.2.2)$$

and in [41], up to some rescaling we have :

$$\varphi(y) = \begin{cases} \frac{1}{2}|z|^2 & \text{if } |z| \leq 1, \\ |z| - \frac{1}{2} & \text{if } 1 < |z| < 2, \\ \frac{1}{4}(|z|^2 + 2) & \text{if } 2 \leq |z|. \end{cases}$$

In both cases, φ is a radial convex function with 0 as strict minimum that satisfies the assumption (A3) and a *p-growth* condition. Hence, these two problems, or any problem of this type, admit at least a minimizer. It remains to show that every minimizer is continuous.

By [37, Theorem 7.8], if Ω has a Lipschitz boundary, if ψ is Lipschitz continuous and if

$$\varphi(z) \leq C_2|z|^p + \alpha$$

with $C_2 > 0$ and p, α as in (1.2.1) then every minimizer of \mathcal{P}_λ on $W_\psi^{1,p}(\Omega)$ is globally Hölder continuous. In these cases by Corollary 1.1.3, \mathcal{P}_λ has a unique solution on $W_\psi^{1,2}(\Omega)$.

Remark 1.2.2. An interesting fact here is that we do not assume that the boundary of Ω is connected in contrast with [8] and [41]. In this way, Theorem 1.1.2 extends the main results of [8] and [41] with a shorter proof.

1.2.2 When Ω is convex and φ is superlinear

In this subsection we use some results from [10]. We assume that Ω is convex, $\lambda = 0$, ψ is continuous and φ is superlinear:

$$\lim_{|z| \rightarrow +\infty} \frac{\varphi(z)}{|z|} = +\infty.$$

Under those assumptions, [10, Theorem 1.4] ensures that \mathcal{P}_λ has at least one continuous solution on $\bar{\Omega}$ when $\mathcal{E}(\Omega, \psi) = W_\psi^{1,1}(\Omega)$. Hence if φ satisfies the assumption (A3) then Theorem 1.1.2 provides that \mathcal{P}_λ has a unique continuous minimizer.

Moreover, if we assume that the affine parts of φ have a uniformly bounded diameter and that there exists a uniformly continuous minimizer of \mathcal{P}_λ then by [10,

Proposition 1.6], see also Lemma 1.3.7, every minimizer is uniformly continuous on $\overline{\Omega}$. In this case if φ satisfies the assumption (A3) then by Corollary 1.1.4, \mathcal{P}_λ has a unique minimizer and this minimizer is uniformly continuous on $\overline{\Omega}$.

1.2.3 Functions defined on a bounded convex set

In this subsection we assume that K is a bounded convex set of \mathbb{R}^N , $\varphi(z) = \tilde{\varphi}(z) + \chi_K(z)$ with $\tilde{\varphi} : \mathbb{R}^N \rightarrow \mathbb{R}$ a convex function that satisfies the assumption (A3) on K and

$$\chi_K(z) = \begin{cases} 0 & \text{if } z \in K, \\ +\infty & \text{if } z \notin K. \end{cases}$$

In this case, for every minimizer u and for a.e. $x \in \Omega$, $\nabla u(x) \in K$. Hence, every minimizer is globally Lipschitz continuous on $\overline{\Omega}$. Thus, Theorem 1.1.2 becomes: \mathcal{P}_λ admits at most one minimizer on $\mathcal{E}(\Omega, \psi)$.

The main source of inspiration for this paper comes from [29, Section 4]. In this chapter, as mentioned in the introduction, φ is defined on a closed convex polygon in \mathbb{R}^2 and $\varphi \equiv +\infty$ outside. The function is strictly convex inside the polygon. Hence, φ satisfies the assumption (A3). Thus, if the minimizer exists, it is the only one. We can generalize it to higher dimensions with a polytope instead of a polygon.

This type of formulation $\varphi(z) = \tilde{\varphi}(z) + \chi_K(z)$ can appear when we introduce a constraint on the gradients of the minimizers. For instance in [64], the authors consider the least gradient problem with $\varphi(z) = |z|$ for all $z \in \mathbb{R}^N$ but the minimization is done on $\{u \in C^{0,1}(\overline{\Omega}), |\nabla u| \leq 1 \text{ a.e.}\}$. In this case, we can set $K = \overline{B_1(0)}$ and apply Theorem 1.1.2 to obtain uniqueness if a minimizer exists.

1.2.4 Comparison principle

If we assume that the minimizers of \mathcal{P}_λ are uniformly continuous then we have the following comparison principle:

Proposition 1.2.3. *Let ψ_1 and ψ_2 be two boundary conditions on $\partial\Omega$. If assumptions (A1), (A2), (A3) are satisfied and $\psi_1 \leq \psi_2$ on $\partial\Omega$ then $u_1 \leq u_2$ where u_1 is the uniformly continuous solution of \mathcal{P}_λ on $\mathcal{E}(\Omega, \psi_1)$ and u_2 is the uniformly continuous solution of \mathcal{P}_λ on $\mathcal{E}(\Omega, \psi_2)$.*

Proof. By assumptions on ψ_1 and ψ_2 , Lemma 1.3.3 and Lemma 1.3.4 below imply that $\min\{u_1, u_2\}$ is a solution of \mathcal{P}_λ on $\mathcal{E}(\Omega, \psi_1)$. Hence, by Theorem 1.1.2 we have $\min\{u_1, u_2\} = u_1$ and thus $u_1 \leq u_2$. \square

1.3 Preliminary results

In this section, we recall some classical and simple results that we use in the proof of Theorem 1.1.2. The first lemma shows that the problem is invariant under translations as in [37, Section 1.1].

Lemma 1.3.1. *If u is a minimizer of \mathcal{P}_λ on $\mathcal{E}(\Omega, \psi)$ then for every $\tau \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$, $u(\cdot + \tau) + \alpha$ is a minimizer on $\mathcal{E}(\Omega_\tau, \psi + \alpha)$ where $\Omega_\tau := \Omega - \tau$.*

Proof. We consider $v \in \mathcal{E}(\Omega_\tau, \psi + \alpha)$. By assumption (A1), $\tilde{v}(x) := v(x - \tau) - \alpha$ is in $\mathcal{E}(\Omega, \psi)$ and thus

$$\int_{\Omega} \varphi(\nabla \tilde{v}(x)) + \lambda \tilde{v}(x) dx \geq \int_{\Omega} \varphi(\nabla u(x)) + \lambda u(x) dx. \quad (1.3.1)$$

But

$$\begin{aligned} \int_{\Omega_\tau} \varphi(\nabla v(y)) + \lambda v(y) dy &= \int_{\Omega} \varphi(\nabla \tilde{v}(x)) + \lambda(\tilde{v}(x) + \alpha) dx \\ &= \int_{\Omega} \varphi(\nabla \tilde{v}(x)) + \lambda \tilde{v}(x) dx + \alpha \lambda |\Omega|. \end{aligned}$$

By the inverse change of variables applied to u and (1.3.1) we obtain

$$\int_{\Omega_\tau} \varphi(\nabla v(y)) + \lambda v(y) dy \geq \int_{\Omega_\tau} \varphi(\nabla(u(y + \tau) + \alpha)) + \lambda(u(y + \tau) + \alpha) dy.$$

Since v can be any function in $\mathcal{E}(\Omega_\tau, \psi + \alpha)$ we have the desired result. \square

Remark 1.3.2. This lemma uses the invariance under translations of this type of functionals with λ constant. Hence, it seems difficult to generalize Theorem 1.1.2 to more general problems. When λ is not constant, we can use different ideas to prove uniqueness for specific functionals. For instance, in an upcoming paper, we prove uniqueness for φ as in (1.2.2) with λ having small oscillations by comparing the level sets of two minimizers.

The following lemma establishes that a minimizer on Ω is still a minimizer on the subsets of Ω .

Lemma 1.3.3. *Let u be a minimizer of \mathcal{P}_λ on $\mathcal{E}(\Omega, \psi)$. For every non empty open set $V \subset \Omega$, $u|_V$ is a minimizer on V among all the functions in $\mathcal{E}(V, u)$.*

Proof. We assume by contradiction that this is not the case. There exists $v \in \mathcal{E}(V, u)$ such that

$$\int_V \varphi(\nabla v(x)) + \lambda v(x) dx < \int_V \varphi(\nabla u(x)) + \lambda u(x) dx.$$

If we define $w(x) = u(x)$ if $x \notin V$ and $w(x) = v(x)$ if $x \in V$ then by assumption (A2), $w \in \mathcal{E}(\Omega, \psi)$ and $\mathcal{I}_\lambda(w) < \mathcal{I}_\lambda(u)$. This contradicts the fact that u is a minimizer. \square

In the next lemma, we prove that the extension of a minimizer by a compatible minimizer is still a minimizer.

Lemma 1.3.4. *Let u be a minimizer of \mathcal{P}_λ on $\mathcal{E}(\Omega, \psi)$ and $V \subset \Omega$ a non empty open set. If v is a minimizer of \mathcal{P}_λ on $\mathcal{E}(V, u)$ then*

$$w = \begin{cases} v & \text{on } V, \\ u & \text{on } \Omega \setminus V, \end{cases}$$

is a minimizer of \mathcal{P}_λ on $\mathcal{E}(\Omega, \psi)$.

Proof. By assumption (A2) on \mathcal{E} , $w \in \mathcal{E}(\Omega, \psi)$. We compare $\mathcal{I}_\lambda(u)$ and $\mathcal{I}_\lambda(w)$:

$$\mathcal{I}_\lambda(u) = \int_{\Omega} \varphi(\nabla u) + \lambda u = \int_V \varphi(\nabla u) + \lambda u + \int_{\Omega \setminus V} \varphi(\nabla u) + \lambda u$$

and

$$\mathcal{I}_\lambda(w) = \int_{\Omega} \varphi(\nabla w) + \lambda w = \int_V \varphi(\nabla v) + \lambda v + \int_{\Omega \setminus V} \varphi(\nabla u) + \lambda u.$$

By Lemma 1.3.3, $\int_V \varphi(\nabla u) + \lambda u = \int_V \varphi(\nabla v) + \lambda v$. Hence, $\mathcal{I}_\lambda(u) = \mathcal{I}_\lambda(w)$. Thus, w is a minimizer of \mathcal{P}_λ on $\mathcal{E}(\Omega, \psi)$. \square

We introduce the notion of point of *approximate continuity* that we use in the proof of Theorem 1.1.2.

Definition 1.3.5. Let $f : \Omega \rightarrow \mathbb{R}^m$ be a measurable function, we say that x_1 is a point of *approximate continuity* if for every $\eta > 0$

$$\lim_{r \rightarrow 0} \frac{|\{x \in B_r(x_1), |f(x) - f(x_1)| \geq \eta\}|}{|B_r(x_1)|} = 0. \quad (1.3.2)$$

We recall a classical result that can be found in [32, Section 1.7, Theorem 3].

Proposition 1.3.6. *Let $f : \Omega \rightarrow \mathbb{R}^m$ be a measurable function. Hence, a.e. $x \in \Omega$ is a point of approximate continuity of f .*

In the last result of this section, we show that given three minimizers, φ is affine on the simplex generated by their gradients at almost every point:

Lemma 1.3.7. *Let u, v and w be three minimizers of \mathcal{P}_λ on $\mathcal{E}(\Omega, \psi)$. Then φ is affine on the simplex with vertices $[\nabla u(x), \nabla v(x), \nabla w(x)]$ for a.e. $x \in \Omega$.*

Proof. Since u is a solution of \mathcal{P}_λ ,

$$\mathcal{I}_\lambda(u) \leq \mathcal{I}_\lambda\left(\frac{u+v+w}{3}\right).$$

By the convexity of φ ,

$$\begin{aligned} \mathcal{I}_\lambda\left(\frac{u+v+w}{3}\right) &= \int_{\Omega} \varphi\left(\frac{\nabla u + \nabla v + \nabla w}{3}\right) + \lambda \frac{u+v+w}{3} dx \\ &\leq \frac{1}{3} \int_{\Omega} (\varphi(\nabla u) + \lambda u) dx + \frac{1}{3} \int_{\Omega} (\varphi(\nabla v) + \lambda v) dx + \frac{1}{3} \int_{\Omega} (\varphi(\nabla w) + \lambda w) dx \\ &= \frac{1}{3} \mathcal{I}_\lambda(u) + \frac{1}{3} \mathcal{I}_\lambda(v) + \frac{1}{3} \mathcal{I}_\lambda(w). \end{aligned}$$

Since v and w are other solutions,

$$I_\lambda(u) = \frac{1}{3}I_\lambda(u) + \frac{1}{3}I_\lambda(v) + \frac{1}{3}I_\lambda(w).$$

This implies that

$$\int_{\Omega} \varphi\left(\frac{\nabla u + \nabla v + \nabla w}{3}\right) dx = \int_{\Omega} \frac{1}{3}(\varphi(\nabla u) + \varphi(\nabla v) + \varphi(\nabla w)) dx.$$

Hence for a.e. $x \in \Omega$,

$$\varphi\left(\frac{\nabla u(x) + \nabla v(x) + \nabla w(x)}{3}\right) = \frac{1}{3}\left(\varphi(\nabla u(x)) + \varphi(\nabla v(x)) + \varphi(\nabla w(x))\right).$$

Hence, for a.e. $x \in \Omega$, the convex function φ is affine on the simplex with vertices the points $[\nabla u(x), \nabla v(x), \nabla w(x)]$. □

Remark 1.3.8. The fact that φ is affine on the simplex with vertices the points $[\nabla u(x), \nabla v(x), \nabla w(x)]$ is equivalent to $\nabla u(x), \nabla v(x), \nabla w(x) \in F$ where F is an affine part of φ .

Remark 1.3.9. The number of minimizers considered in Lemma 1.3.7 can be changed. For instance, we can prove it with $n \in \mathbb{N}$ minimizers. In this case we obtain that φ is affine on the convex hull of the gradients of the solutions and they are still in the same affine part F of φ .

1.4 Proof of Theorem 1.1.2

The proof of Theorem 1.1.2 is divided into three steps. The first one is very similar to the proof of [29, Proposition 4.5].

Step 1 Let u_1, u_2 be two uniformly continuous minimizers of \mathcal{P}_λ on $\mathcal{E}(\Omega, \psi)$. We call ω a common modulus of continuity of u_1 and u_2 namely: $\lim_{w \rightarrow 0^+} \omega(w) = 0$, ω is increasing and $\omega(a+b) \leq \omega(a) + \omega(b)$ for every $a, b \in \mathbb{R}_+$. We assume by contradiction that there exist $x_0 \in \Omega$ and $\epsilon > 0$ such that $u_1(x_0) - u_2(x_0) = \epsilon$.

We consider $\rho > 0$ such that $\omega(\rho) < \frac{\epsilon}{6}$. For every $y \in \Omega$, $u_1(y) - u_2(y) \leq 2\omega(\text{dist}(y, \partial\Omega))$. Hence, $\omega(2\rho) \leq \frac{\epsilon}{3} < \frac{\epsilon}{2} = \frac{u_1(x_0) - u_2(x_0)}{2} \leq \omega(\text{dist}(x_0, \partial\Omega))$. Thus, $B_{2\rho}(x_0) \Subset \Omega$. For every $\tau \in \mathbb{R}^N$ such that $|\tau| < \rho$ and for every $x \in B_\rho(x_0)$ we have $u_1(x) \geq u_1(x_0) - \omega(\rho)$ and $u_2(x + \tau) \leq u_2(x_0) + 2\omega(\rho)$. Hence, we have that

$$B_\rho(x_0) \subset \{x \in \Omega \cap \Omega_\tau, u_2(x + \tau) + \frac{\epsilon}{2} < u_1(x)\} \quad (1.4.1)$$

and for the same reasons:

$$B_\rho(x_0) \subset \{x \in \Omega \cap \Omega_\tau, u_1(x + \tau) - \frac{\epsilon}{2} > u_2(x)\} \quad (1.4.2)$$

with $\Omega_\tau = \Omega - \tau$.

For every $x \in \Omega \cap \Omega_\tau$, we have

$$\begin{aligned} |u_1(x) - u_2(x + \tau)| &\leq \omega(\rho) + \min\{|u_1(x) - u_2(x)|, |u_1(x + \tau) - u_2(x + \tau)|\} \\ &\leq \omega(\rho) + 2\omega(\text{dist}(x, \partial(\Omega \cap \Omega_\tau))). \end{aligned}$$

Since $\omega(\rho) < \frac{\epsilon}{6}$ we have:

$$\{x \in \Omega \cap \Omega_\tau, u_2(x + \tau) + \frac{\epsilon}{2} < u_1(x)\} \Subset \Omega \cap \Omega_\tau \quad (1.4.3)$$

and for the same reasons:

$$\{x \in \Omega \cap \Omega_\tau, u_1(x + \tau) - \frac{\epsilon}{2} > u_2(x)\} \Subset \Omega \cap \Omega_\tau. \quad (1.4.4)$$

We extend the function $v_\tau(x) := \min\{u_1(x), u_2(x + \tau) + \frac{\epsilon}{2}\}$ defined on $\Omega \cap \Omega_\tau$ by u_1 to Ω . By assumptions $(\mathcal{A}1)$, $(\mathcal{A}2)$ and the inclusion (1.4.3), this extension belongs to $\mathcal{E}(\Omega, \psi)$. Thanks to the inclusion (1.4.4), if we extend $w_\tau(x) := \max\{u_1(x + \tau) - \frac{\epsilon}{2}, u_2(x)\}$ by u_2 , we also have that $w_\tau \in \mathcal{E}(\Omega, \psi)$. By Lemma 1.3.1 and Lemma 1.3.3, on the open set $\{x \in \Omega \cap \Omega_\tau, u_2(x + \tau) + \frac{\epsilon}{2} < u_1(x)\}$, u_1 and $u_2(\cdot + \tau) + \frac{\epsilon}{2}$ are solutions of the same problem. By Lemma 1.3.4, v_τ is a minimizer of \mathcal{P}_λ on $\mathcal{E}(\Omega, \psi)$. With the same arguments we can prove that w_τ is also a minimizer of \mathcal{P}_λ on $\mathcal{E}(\Omega, \psi)$.

Step 2 In this crucial step of the proof we show that there exists an $(N-1)$ -dimensional set containing the essential ranges of the gradients of any minimizer. We consider $x_1 \in B_{\frac{\rho}{2}}(x_0)$ and $x'_1 \in B_{\frac{\rho}{2}}(x_0)$ two points of *approximate continuity* of $(\nabla u_1, \varphi(\nabla u_1))$ and $x_2 \in B_{\frac{\rho}{2}}(x_0)$ a point of *approximate continuity* of $(\nabla u_2, \varphi(\nabla u_2))$. By Proposition 1.3.6 and the fact that ∇u_1 , ∇u_2 and φ are measurable, that is the case for a.e. $(x_1, x'_1, x_2) \in B_{\frac{\rho}{2}}(x_0)^3$. We fix $\tau := x_2 - x_1$ and $\tau' := x'_1 - x_1$. Then by definition of v_τ and $w_{\tau'}$, x_1 is a point of *approximate continuity* of $(\nabla v_\tau, \varphi(\nabla v_\tau))$ and $(\nabla w_{\tau'}, \varphi(\nabla w_{\tau'}))$.

We define

$$f : x \rightarrow (\nabla u_1(x), \nabla w_{\tau'}(x), \nabla v_\tau(x), \varphi(\nabla u_1(x)), \varphi(\nabla w_{\tau'}(x)), \varphi(\nabla v_\tau(x))).$$

Since x_1 is a point of *approximate continuity* of f , we have for every $n \in \mathbb{N}$:

$$\lim_{r \rightarrow 0} \frac{|\{x \in B_r(x_1), |f(x) - f(x_1)| > \frac{1}{n}\}|}{|B_r(x_1)|} = 0.$$

Thus, for every $n \in \mathbb{N}$, there exist $r_n > 0$ and $x \in B_{r_n}(x_1)$ such that $|f(x) - f(x_1)| \leq \frac{1}{n}$. By Lemma 1.3.7 we can further assume that

$$\varphi\left(\frac{\nabla u_1(x) + \nabla w_{\tau'}(x) + \nabla v_\tau(x)}{3}\right) = \frac{\varphi(\nabla u_1(x)) + \varphi(\nabla w_{\tau'}(x)) + \varphi(\nabla v_\tau(x))}{3} < +\infty. \quad (1.4.5)$$

We introduce $K := \{z \in \mathbb{R}^N, \varphi(z) < +\infty\}$ the convex domain of φ and we distinguish two cases:

a) If the triplet (x_1, x'_1, x_2) is such that $\frac{\nabla u_1(x_1) + \nabla w_{\tau'}(x_1) + \nabla v_\tau(x_1)}{3} \in \text{Int}(K)$, then by continuity of φ on $\text{Int}(K)$ we have

$$\varphi\left(\frac{\nabla u_1(x_1) + \nabla w_{\tau'}(x_1) + \nabla v_\tau(x_1)}{3}\right) = \frac{\varphi(\nabla u_1(x_1)) + \varphi(\nabla w_{\tau'}(x_1)) + \varphi(\nabla v_\tau(x_1))}{3}.$$

By the definition of $w_{\tau'}$, v_{τ} and (1.4.1)-(1.4.2), this is equivalent to $\nabla u_1(x_1)$, $\nabla u_1(x'_1)$ and $\nabla u_2(x_2)$ being in the same affine part of φ .

Now, we prove that φ is affine on the convex hull of $F := \{\nabla u_1(x)$ with $x \in X_1$ and $\nabla u_2(x)$ with $x \in X_2\}$ with X_i the set of points of *approximate continuity* of $(\nabla u_i, \varphi(\nabla u_i))$. If we consider $y \in B_{\frac{\rho}{2}}(x_0)$ a point of *approximate continuity* of $(\nabla u_i, \varphi(\nabla u_i))$ and $y' \in B_{\frac{\rho}{2}}(x_0)$ a point of *approximate continuity* of $(\nabla u_j, \varphi(\nabla u_j))$ with $i, j \in \{1, 2\}$ we have that

$$\frac{\nabla u_1(x_1) + \nabla u_1(x'_1) + \nabla u_2(x_2) + \nabla u_i(y) + \nabla u_j(y')}{5} \in \text{Int}(K). \quad (1.4.6)$$

By using the functions v_{y-x_1} if $i = 2$ and w_{y-x_1} if $i = 1$, and similarly for j , we obtain the analogue of (1.4.5) for these five minimizers, see Remark 1.3.9. Hence, we can argue as above and rely on (1.4.6) to get that the five points $\nabla u_1(x_1)$, $\nabla u_1(x'_1)$, $\nabla u_2(x_2)$, $\nabla u_i(y)$ and $\nabla u_j(y')$ belong to the same affine part of φ . Thus, φ is affine on the segment $[\nabla u_i(y), \nabla u_j(y')]$. Hence, φ is affine on the convex hull of F . Thus by assumption (A3), there exists \mathcal{F} an $(N-1)$ -dimensional affine space such that for a.e. $x \in B_{\frac{\rho}{2}}(x_0)$, $\nabla u_1(x)$ and $\nabla u_2(x)$ belong to \mathcal{F} .

b) If there is no such triplet then for every couple (x_1, x'_1) of points of *approximate continuity*, we have that $\nabla u(x_1)$, $\nabla u(x'_1)$ belong to the same face of ∂K , namely the intersection of an affine hyperplane with ∂K . Otherwise the convex combination would be in $\text{Int}(K)$ contradicting the fact that we are not in case **a**). Hence, there exists an affine hyperplane \mathcal{F} such that for a.e. $x \in B_{\frac{\rho}{2}}(x_0)$, $\nabla u_1(x)$ is in \mathcal{F} . The fact that we are not in case **a**) gives that for a.e. $x \in B_{\frac{\rho}{2}}(x_0)$, $\nabla u_2(x)$ is in \mathcal{F} too.

Step 3 Since the dimension of \mathcal{F} is at most $N-1$ there exists a normal vector η to \mathcal{F} . The scalar product $\langle \nabla u_i(x), \eta \rangle = C$ is constant in $B_{\frac{\rho}{2}}(x_0)$ and is independent of $i = 1, 2$. Thus, $u_1 - u_2 = \epsilon > 0$ on $(x_0 + \mathbb{R}\eta) \cap B_{\frac{\rho}{2}}(x_0)$. Hence, the compact set $E_\epsilon := (u_1 - u_2)^{-1}(\epsilon) \Subset \Omega$ has no extreme point in Ω .

We consider $\tilde{x}_0 \in E_\epsilon$ such that $|\tilde{x}_0| = \max_{x \in E_\epsilon} |x|$. Since E_ϵ has no extreme point, there exist y_1 and y_2 in E_ϵ such that $\tilde{x}_0 \in (y_1, y_2)$. By the triangle inequality, either $|y_1| > |\tilde{x}_0|$ or $|y_2| > |\tilde{x}_0|$. That is a contradiction. Hence, we have $u_1 = u_2$ on Ω .

A uniqueness result for a non-strictly convex problem in the calculus of variations

This chapter is based on the paper "*A uniqueness result for a non-strictly convex problem in the calculus of variations*" [49] submitted. Some of the results of this chapter come from my master thesis.

Abstract

We prove the uniqueness of the solution for a non-strictly convex problem in the Calculus of Variations of the form $\int \varphi(\nabla v) - \lambda v$. Here, φ is a convex function not differentiable at the origin and λ is a Lipschitz function. To prove this result we show that under fairly general assumptions, the minimizers are globally Lipschitz continuous.

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2.1 Introduction

2.1.1 The Kohn and Strang's example

The motivation of this chapter is to study non strictly convex problems in the Calculus of Variations in dimension $N \geq 2$. A first example is given by the convexification of a nonconvex functional introduced by Kohn and Strang in [42, 43, 44]:

$$\text{Minimize } : u \rightarrow \int_{\Omega} F(\nabla u(x)) dx \quad (2.1.1)$$

on $W_{\psi}^{1,2}(\Omega)$ with

$$F(z) := \begin{cases} |z| & \text{if } |z| \leq 1, \\ \frac{1}{2}(|z|^2 + 1) & \text{if } 1 < |z|, \end{cases}$$

which is the convexification of

$$F_0(z) := \begin{cases} 0 & \text{if } z = 0, \\ \frac{1}{2}(|z|^2 + 1) & \text{if } 0 < |z|. \end{cases}$$

Here $|z|$ is the euclidean norm of z in \mathbb{R}^N . For this problem, the admissible functions u belong to the Sobolev space $W^{1,2}(\Omega)$ and have a prescribed trace $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ on the boundary $\partial\Omega$ of an open bounded set Ω of \mathbb{R}^N . It is important to notice that F is not strictly convex, hence the uniqueness is not guaranteed. Moreover, we observe that F is singular at the origin and F is strictly convex outside the unit ball. During the rest of the chapter we work with problems that share these three features.

2.1.2 The general Problem

More precisely, the aim of this paper is to study functionals of the following form:

$$\mathcal{I}_{\lambda} : u \mapsto \int_{\Omega} \varphi(\nabla u) - \lambda(x)u dx$$

where Ω is an open bounded set in \mathbb{R}^N and $\lambda \in L^{\infty}(\Omega)$. Throughout this paper, $\varphi := g(|\cdot|)$ with $g : \mathbb{R} \rightarrow \mathbb{R}_+$ a continuous even convex function \mathcal{C}^1 on $(0, +\infty)$ and \mathcal{C}^2 on $(1, +\infty)$. We assume that:

$$g(x) = x \text{ if } 0 \leq x \leq 1. \quad (\text{A1})$$

By convexity of g , we have $g(x) \geq x$ and $g'(x) \geq 1$ for every $x \in \mathbb{R}_+^*$. We suppose that g is strongly convex at $+\infty$ in the following sense:

$$\liminf_{x \rightarrow +\infty} \frac{xg''(x)}{g'(x)} > 0. \quad (\text{A2})$$

Moreover, we assume that

$$\limsup_{\{x>1, x\rightarrow 1\}} g''(x) < +\infty \text{ and } g'' > 0 \text{ on } (1, +\infty). \quad (\text{A3})$$

Remark 2.1.1. The assumption (A2) provides the existence of $x_0 > 1$, $p = \frac{\alpha}{2} + 1 > 1$ with $\alpha := \liminf_{x \rightarrow +\infty} \frac{xg''(x)}{g'(x)}$ and $C_p > 0$ such that $g'(x) \geq C_p x^{p-1}$ and $g(x) \geq C_p x^p$ for every $x \geq x_0$. Moreover, with the assumption (A1) and a possible smaller $C_p > 0$, we have that $g(x) \geq C_p |x|^p$ for every $x \in \mathbb{R}$.

In this setting, the admissible functions for \mathcal{I}_λ belong to the Sobolev space $W_\psi^{1,p}(\Omega)$ which is the subset of those functions in $W^{1,p}(\Omega)$ that have a prescribed trace $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$ on the boundary $\partial\Omega$. Our goal is to prove the uniqueness of solutions for the following minimization problem:

$$\mathcal{P}_\lambda : \min_{u \in W_\psi^{1,p}(\Omega)} \mathcal{I}_\lambda(u).$$

As in the preliminary example (2.1.1), φ is not strictly convex. Hence there is no obvious reason for \mathcal{P}_λ to have a unique solution. When λ is constant, we have by [52] that if \mathcal{P}_λ admits a uniformly continuous solution then it is the unique solution. We could ask ourselves if it is possible to generalize this result to the case when λ is non constant.

In **Section 5**, we prove that we can not hope to extend this result for any function λ :

Proposition 2.1.2. *When Ω is a ball and $\psi \equiv 0$, there exists $\lambda \in C^1(\overline{\Omega})$ such that the problem \mathcal{P}_λ with φ as in (2.1.1) has more than one minimizer.*

Thus, if we want to have uniqueness, λ can not be any function in $L^\infty(\Omega)$. The aim of this paper is to find some assumptions on λ to guarantee the uniqueness of minimizers and study its regularity properties.

2.1.3 The two main results

We assume that Ω is an open simply connected bounded set of \mathbb{R}^N with a Lipschitz continuous boundary $\partial\Omega$ and ψ is a Lipschitz continuous function on $\partial\Omega$.

Since we do not have uniqueness for any $\lambda \in L^\infty(\Omega)$, we restrict our study to a specific class of functions, namely the functions that have *small oscillations* in a sense to be specified subsequently. In general, every solution u of \mathcal{P}_λ is globally Lipschitz continuous and the $W^{1,\infty}$ norm of u does not depend on $\nabla\lambda$. By choosing $|\nabla\lambda|$ sufficiently small we can prove the two main theorems of the chapter.

The first one requires more regularity on $\partial\Omega$ and ψ . In **Section 4** we prove the following global Lipschitz estimate:

Theorem 2.1.3 (Global Lipschitz continuity for a general degenerate functional). *Let Ω be a connected bounded open set of \mathbb{R}^N with $N \geq 2$ and $\lambda \in L^\infty(\Omega)$. We assume that Ω has a $C^{1,1}$ connected boundary, $\psi \in C^{1,1}(\mathbb{R}^N)$. If φ satisfies the assumption (A2) then any minimizer u of \mathcal{P}_λ is globally Lipschitz-continuous on Ω . Moreover,*

$$\|\nabla u\|_{\mathcal{L}^\infty(\Omega)} \leq C(p, C_p, \|\lambda\|_{L^\infty(\Omega)}, N, |\Omega|, \text{diam}(\Omega), \|\psi\|_{C^{1,1}(\Omega)}, \kappa)$$

where κ is the maximum of the principal curvatures of $\partial\Omega$.

Observe that in this result we do not assume any upper bound on φ . Moreover, the above statement improves several global regularity results in the literature. For instance, in comparison with [16] we do not require that $\varphi'(0) = 0$, $\varphi \in \mathcal{C}^2(\mathbb{R}_+)$ and $\lambda \equiv 0$.

This theorem allows us to prove the first result about uniqueness of the chapter:

Theorem 2.1.4. *Let Ω be a connected bounded open set of \mathbb{R}^N with $N \geq 2$. We assume that Ω has a $C^{1,1}$ connected boundary, $\psi \in C^{1,1}(\mathbb{R}^N)$ and λ is Lipschitz continuous on $\overline{\Omega}$ with $\min_{\overline{\Omega}} \lambda > 0$. If φ satisfies the assumptions (A1), (A2) and (A3) then there exists a positive constant*

$$C(p, C_p, N, |\Omega|, \text{diam}(\Omega), \max_{\overline{\Omega}} \lambda, \min_{\overline{\Omega}} \lambda, \|\psi\|_{C^{1,1}(\Omega)}, \kappa)$$

such that if $\|\nabla \lambda\|_{\mathcal{L}^\infty(\Omega)} \leq C$ then \mathcal{P}_λ admits a unique solution on $W_\psi^{1,p}(\Omega)$. Here, κ is the maximum of the principal curvatures of $\partial\Omega$, p and C_p are the constants introduced in Remark 2.1.1.

The second main theorem requires less regularity on Ω and ψ but in that case Ω has to be convex. In order to state it we need to introduce the notion of bounded slope condition:

Definition 2.1.5. Let Ω be an open bounded set of \mathbb{R}^N . We say that a map $\psi : \partial\Omega \rightarrow \mathbb{R}$ satisfies the bounded slope condition of rank R if for every $y \in \partial\Omega$, there exist $\zeta_y^+, \zeta_y^- \in \mathbb{R}^N$ such that $|\zeta_y^+|, |\zeta_y^-| \leq R$ and

$$\psi(y) + \langle \zeta_y^-, x - y \rangle \leq \psi(x) \leq \psi(y) + \langle \zeta_y^+, x - y \rangle \quad (2.1.2)$$

for every $x \in \partial\Omega$.

If Ω is convex and ψ satisfies the bounded slope condition, [12, Main Theorem] asserts that every minimizer of \mathcal{P}_λ is globally Lipschitz-continuous. We can prove the following result:

Theorem 2.1.6. *Let Ω be a convex set, λ a Lipschitz continuous function on $\overline{\Omega}$, $\min_{\overline{\Omega}} \lambda > 0$ and $\psi : \partial\Omega \rightarrow \mathbb{R}$ satisfy the bounded slope condition of rank R . If φ satisfies the assumptions (A1), (A2) and (A3) then there exists a constant $C := C(p, C_p, N, \text{diam}(\Omega), \max_{\overline{\Omega}} \lambda, \min_{\overline{\Omega}} \lambda, R) > 0$ such that if $\|\nabla \lambda\|_{L^\infty(\Omega)} \leq C$ then \mathcal{P}_λ admits a unique solution on $W_\psi^{1,p}(\Omega)$. Here, p and C_p are the constants introduced in Remark 2.1.1.*

Links with the existent literature

The question of uniqueness in the calculus of variations has aroused much interest over the last decades. The first example (2.1.1) of this chapter comes from [42, 43, 44] and is also studied in an article of Alibert, Bouchitté, Fragalà and Lucardesi [2]. A first proof of uniqueness for this particular problem with a general Ω can be found in a paper of Bouchitté and Bousquet [8]. In [41], Kawohl, Stara and Wittum study the uniqueness of the solutions when the integrand is the convexification of two parabolas in dimension two. In this case, the proof requires different ideas because the region where the integrand is affine is no longer the unit ball but an annulus. A part of the structure of our proof has been inspired by a seminal paper by Marcellini [54] on the uniqueness when the integrand is convex and depends only on ∇u . In [53], Lussardi and Mascolo proved the result of Marcellini with fewer assumptions but only in dimension two. When λ is constant a shorter proof can be found in [52]. Many of these articles use the regularity of the level sets, here we rely particularly on an article by Massari [55]. The *pseudo-Cheeger problem* exploits ideas coming from papers by Buttazzo, Carlier and Comte [17], [18].

2.1.4 Structure of the proof

We want to prove the uniqueness of the solution for a variational problem. We rely on a method introduced by Marcellini in [54]. There, he considers the minimization of

$$v \rightarrow \int_{\Omega} g(|\nabla v(x)|) dx$$

with $g : [0, +\infty] \rightarrow [0, +\infty)$ an increasing convex function and Ω a bounded convex \mathcal{C}^1 set. The main result is the following: if there exists a solution $u \in \mathcal{C}^1(\bar{\Omega})$ such that ∇u does not vanish on $\bar{\Omega}$, then u is the unique solution in the class of Lipschitz continuous functions agreeing with u on $\partial\Omega$. The proof is based on two parts:

Part A Each level set of a solution u intersects the boundary $\partial\Omega$ of the domain.

Part B If v is another solution, then v is constant on the level sets of u .

Hence, if u and v are two solutions then $u = v$ on each level set of u . Since each level set of u intersects $\partial\Omega$ where $u = v$, we have $u = v$ on $\bar{\Omega}$.

In our case, we have a lower order term $\int_{\Omega} \lambda u dx$. This term changes the geometry of the level sets which do not necessarily intersect the boundary anymore. For instance, if Ω is the unit ball of \mathbb{R}^N , then for $\lambda \equiv \lambda_0 \in \mathbb{R}_+$, $\lambda_0 > h_{\Omega}$, $u \equiv 0$ is not a solution of \mathcal{P}_{λ} on $W_0^{1,p}(\Omega)$ (see Lemma 2.3.10). Hence, only one level set of the solution intersects $\partial\Omega$.

That is why we must adapt the above strategy like in [8]. Our proof is based on the five following parts:

Part 1 There exists an open set U independent of the choice of the minimizer u such that $|\nabla u| > 1$ on U . This is a generalization of [8, Proposition 1.2] which is a consequence of [26, Theorem 1.1].

Proposition 2.1.7. *Let u be a Lipschitz continuous solution. When $\lambda \in L^{\infty}(\Omega)$,*

there exists an open subset $U \subset \Omega$ such that $u \in \mathcal{C}^{1,\alpha}$ on U with $0 < \alpha < 1$, $|\nabla u(x)| > 1$ for every $x \in U$ and $|\nabla u(x)| \leq 1$ for a.e. $x \in \Omega \setminus U$. Moreover, the restriction of $|\nabla u|$ to U is uniformly continuous on $U \cap \Omega'$ for every $\Omega' \Subset \Omega$. Hence, the restriction of $|\nabla u|$ to U can be extended as a continuous function on $\overline{U} \cap \Omega$ which is equal to 1 on $\partial U \cap \Omega$.

Part 2 We study the minimizing properties of the super-level sets $E_t(u) = [u \geq t]$ of a solution u . In order to do it we introduce the notions of *Cheeger problem* and *weighted Cheeger problem*.

Definition 2.1.8. The Cheeger constant of Ω is defined as:

$$h_\Omega = \inf_{D \subset \Omega} \frac{\text{Per}(D, \mathbb{R}^N)}{|D|}.$$

A set $D \subset \overline{\Omega}$ of finite perimeter is said to be a *Cheeger set* if $\text{Per}(D, \mathbb{R}^N) = h_\Omega |D|$.

Here, $\text{Per}(D, \mathbb{R}^N)$ is the perimeter of D in the sense of Caccioppoli sets, see **Section 3** below for details. In this chapter we also use a *pseudo-perimeter* $\widetilde{\text{Per}}$ that is a weighted perimeter defined in **Section 6**. In this case, we replace $|D|$ by the quantity $\int_D \lambda dx$ in order to show that the super-level sets satisfy a variational problem:

Theorem 2.1.9. *Let u be a Lipschitz continuous solution of \mathcal{P}_λ and $\lambda \in L^\infty(\Omega)$. Then for a.e. $s \in \mathbb{R}$ and for every set $F \subset \Omega$ with finite perimeter in Ω such that $F \Delta E_s \Subset \Omega$, we have*

$$\widetilde{\text{Per}}(E_s, \Omega) - \int_{E_s} \lambda dx \leq \widetilde{\text{Per}}(F, \Omega) - \int_F \lambda dx.$$

In the above statement $F \Delta E_s := (E_s \setminus F) \cup (F \setminus E_s)$. This result is crucial in order to prove the regularity of the super-level sets.

Part 3 We establish the following result on the super-level sets inspired from [8, Proposition 1.3] where it is formulated in the setting of the Kohn and Strang's problem. It states that generically, the super-level sets $E_t(u) = [u \geq t]$ of a solution u have a regular boundary outside \overline{U} .

Proposition 2.1.10. *Let $\lambda \in L^\infty(\Omega)$, $\lambda \geq 0$ a.e. on Ω and let u be a locally Lipschitz continuous solution of \mathcal{P}_λ . For a.e. $t \in \mathbb{R}$, there exists an open set W_t in $\Omega \setminus \overline{U}$ such that $\partial^e E_t \cap W_t$ is a \mathcal{C}^1 hypersurface and $\mathcal{H}^s(\Omega \setminus (W_t \cup \overline{U})) = 0$ for every $s > N - 8$.*

Here, $\partial^e E_t$ is the essential boundary of E_t , namely, the set of those $x \in \mathbb{R}^N$ such that $\forall \rho > 0$,

$$0 < |E_t \cap B_\rho(x)| < |B_\rho(x)|. \quad (2.1.3)$$

With this proposition we can prove that almost every level set of a solution u intersects the boundary of the domain $\partial\Omega$ or the closure of the set U .

Part 4 We use the regularity of the super-level sets of u to prove that if v is another solution, then v is constant \mathcal{H}^{N-1} a.e. on the boundary of the super-level sets of u .

Part 5 Since $u-v$ is constant on each connected component of U we try to obtain information on the nature of U to obtain the uniqueness. When λ is a constant, the connected components of U intersect the boundary of Ω (see [8, Lemma 3.3]) and this allows us to conclude. In this paper, we prove an equivalent result when λ has *small oscillations*, see Lemma 2.7.1. The open set U can be very ugly, it is even possible that $|\partial U| > 0$ or that \bar{U} be not equal to the union of the closures of its connected components as in Proposition 2.5.13. This is a huge difficulty to prove the uniqueness for a larger class of functions λ .

2.1.5 Plan of the paper

In the next section, we present some elementary results in order to have some general information about the solutions. In the third section we introduce the *Cheeger problem* and some comparison principles. We prove global Lipschitz estimates in Section 4. The fifth section is devoted to the Euler equation and to find some cases where we do not have a unique solution. In Section 6, we study the *pseudo-Cheeger problem* in order to obtain information about the level sets of the solutions. The last section is devoted to the proof of the main theorems.

2.2 Preliminary results

In this section, we show some elementary results used in the proofs presented in the following parts.

Lemma 2.2.1. *If u is a solution of \mathcal{P}_λ on $W_\psi^{1,p}(\Omega)$, then $u + c$ is a solution of \mathcal{P}_λ on $W_{\psi+c}^{1,p}(\Omega)$ for each $c \in \mathbb{R}$.*

Proof. Consider $c \in \mathbb{R}$, u a solution of \mathcal{P}_λ on $W_\psi^{1,p}(\Omega)$ and $v \in W_{\psi+c}^{1,p}(\Omega)$. We have

$$\begin{aligned} \mathcal{I}_\lambda(v) &= \int_\Omega \varphi(\nabla v) - \lambda(v - c) - \lambda c = \int_\Omega \varphi(\nabla(v - c)) - \lambda(v - c) - \lambda c \\ &\geq \mathcal{I}_\lambda(u) - \int_\Omega \lambda c = \mathcal{I}_\lambda(u + c). \end{aligned}$$

Thus $u + c$ is a solution of \mathcal{P}_λ on $W_{\psi+c}^{1,p}(\Omega)$. \square

As in [8, Lemma 2.1] we have the following estimate on the L^∞ norm of u which can be established with a similar proof:

Lemma 2.2.2. *If $\lambda \in L^\infty(\Omega)$, $\lambda \geq 0$ a.e. on Ω , then there exists $C > 0$ which depends only on N such that for every $x \in \Omega$,*

$$\min_{\partial\Omega} \psi \leq u(x) \leq \max_{\partial\Omega} \psi + C \|\lambda\|_{\mathcal{L}^\infty(\Omega)}^N \|(u - \max_{\partial\Omega} \psi)_+\|_{L^1(\Omega)}.$$

Moreover, if $\sup_{\Omega} u > \max_{\partial\Omega} \psi$, then

$$|\{x \in \Omega : u(x) = \sup_{\Omega} u\}| \geq \frac{1}{C \|\lambda\|_{L^\infty(\Omega)}^N}.$$

A direct consequence of this lemma is the following remark:

Remark 2.2.3. If $\lambda \in L^\infty(\Omega)$, $\lambda \geq 0$ a.e. on Ω then the solutions on $W_0^{1,p}(\Omega)$ are non-negative.

The definition of φ allows us to compare the gradients of two solutions as in [8, lemma 3.1]:

Lemma 2.2.4. Let $\lambda \in L^\infty(\Omega)$, $\lambda \geq 0$ a.e. on Ω and let u, v be two solutions of \mathcal{P}_λ . Then for a.e. $x \in \Omega$, either

$$\max(|\nabla u(x)|, |\nabla v(x)|) \leq 1 \quad \text{and} \quad \nabla u(x), \nabla v(x) \text{ are positively colinear}$$

or

$$\nabla u(x) = \nabla v(x).$$

We have the following observation:

Remark 2.2.5. If \mathcal{P}_λ has a Lipschitz continuous minimizer then every minimizer is Lipschitz continuous.

We conclude this section with some uniform convexity results at infinity for φ .

Proposition 2.2.6. There exists $D(C_p, \alpha) > 0$ such that

$$D|z|^{p-2}|\xi|^2 \leq \langle \nabla^2 \varphi(z)\xi, \xi \rangle$$

for every $|z| \geq x_0$ and every $\xi \in \mathbb{R}^N$ with x_0 , α and C_p defined in Remark 2.1.1.

Proof. If $|z| > 1$ then

$$\langle \nabla^2 \varphi(z)\xi, \xi \rangle = \frac{g''(|z|)}{|z|^2} \langle z, \xi \rangle^2 + \frac{g'(|z|)}{|z|} |\xi|^2 - \frac{g'(|z|)}{|z|^3} \langle z, \xi \rangle^2$$

for every $\xi \in \mathbb{R}^N$. Hence, $\nabla^2 \varphi(z)$ has two eigenvalues: $g''(|z|)$ and $\frac{g'(|z|)}{|z|}$.

By assumption (A2) there exists $x_0 > 1$ such that for every $x \geq x_0$, $\frac{xg''(x)}{g'(x)} \geq \frac{\alpha}{2} > 0$. By Remark 2.1.1 and the fact that $g(0) = 0$, there exist $p > 1$ and $C_p > 0$ such that:

$$C_p x^p \leq g(x) \leq xg'(x)$$

for every $x \geq x_0$. Hence, $C_p x^{p-2} \leq \frac{g'(x)}{x} \leq \frac{2g''(x)}{\alpha}$ for every $x \geq x_0$. Thus, we have

$$D|z|^{p-2}|\xi|^2 \leq \langle \nabla^2 \varphi(z)\xi, \xi \rangle$$

for every $|z| \geq x_0$ and every $\xi \in \mathbb{R}^N$ with $D := \min\{\frac{\alpha}{2}, 1\}C_p$. \square

Definition 2.2.7. A convex function is said to be uniformly convex at infinity if there exist $M > 0$ and $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a function such that $\lim_{x \rightarrow +\infty} x\Phi(x) = +\infty$ and

$$\varphi(t\xi + (1-t)\zeta) \leq t\varphi(\xi) + (1-t)\varphi(\zeta) - t(1-t)\Phi(|\xi| + |\zeta|)|\xi - \zeta|^2$$

for every $\xi, \zeta \in \mathbb{R}^N$ such that $[\xi, \zeta] \cap B_M(0) = \emptyset$.

Remark 2.2.8. The lower bound in Proposition 2.2.6 is equivalent to the existence of $\tilde{D} > 0$ such that φ is uniformly convex outside $B_{x_0}(0)$ with $\Phi(x) = \tilde{D}x^{p-2}$:

$$\varphi(t\xi + (1-t)\zeta) \leq t\varphi(\xi) + (1-t)\varphi(\zeta) - \tilde{D}t(1-t)(|\xi| + |\zeta|)^{p-2}|\xi - \zeta|^2$$

for every $\xi, \zeta \in \mathbb{R}^N$ such that $[\xi, \zeta] \cap \overline{B_{x_0}(0)} = \emptyset$ and every $t \in [0, 1]$. Moreover, when $p = 2$, we can take $\tilde{D} = \frac{D}{2}$.

2.3 The Cheeger problem

2.3.1 BV functions and perimeter

In this part we sum up some results about the functions of bounded variations and the perimeters of Caccioppoli sets. The general results and their proofs can be found in [32, Chapter 5].

Definition 2.3.1. A function $f \in L^1(\Omega)$ has bounded variation in Ω if

$$\int_{\Omega} |Df| := \sup \left\{ \int_{\Omega} f \operatorname{div} g \mid g \in \mathcal{C}_0^1(\Omega; \mathbb{R}^N), |g(x)| \leq 1, \forall x \in \Omega \right\} < \infty.$$

We denote by $BV(\Omega)$ the set of functions of bounded variation in Ω .

If $f \in BV(\Omega)$, the distributional gradient of f is a vector valued Radon measure that we denote by Df and $|Df|$ is the total variation of Df .

Definition 2.3.2. Let E be a Borel set. We say that E has finite perimeter in Ω if the characteristic function χ_E of E belongs to $BV(\Omega)$ with

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

The perimeter $\operatorname{Per}(E, \Omega)$ is defined as:

$$\operatorname{Per}(E, \Omega) = \int_{\Omega} |D\chi_E| = \sup \left\{ \int_E \operatorname{div} g : g \in \mathcal{C}_0^1(\Omega; \mathbb{R}^N), |g(x)| \leq 1, \forall x \in \Omega \right\}.$$

Remark 2.3.3. When $\Omega = \mathbb{R}^N$, we just denote $\operatorname{Per}(E, \mathbb{R}^N)$ by $\operatorname{Per}(E)$.

Definition 2.3.4. A Borel set E is a Caccioppoli set if for every bounded subset Ω of \mathbb{R}^N , we have $\operatorname{Per}(E, \Omega) < \infty$.

Definition 2.3.5. The reduced boundary ∂^*E of E in Ω is the set of those $x \in (\text{supp}|D\chi_E|) \cap \Omega$ such that

$$\nu_E := \lim_{\rho \rightarrow 0} \frac{\int_{B_\rho(x)} D\chi_E}{\int_{B_\rho(x)} |D\chi_E|}$$

is well defined and $|\nu_E| = 1$.

By [32, Lemma 5.8.1] the reduced boundary is a subset of the essential boundary $\partial^e E$. By [38, Theorem 4.4], we have for every Borel set $F \subset \Omega$,

$$\int_F |D\chi_E| = \mathcal{H}^{N-1}(F \cap \partial^*E). \quad (2.3.1)$$

If $u \in BV(\Omega)$, for a.e. $s \in \mathbb{R}$, the super-level set $E_s = [u \geq s]$ has finite perimeter and by the co-area formula from [3, Theorem 3.40] we have for every Borel set $F \subset \Omega$ that

$$\int_F |Du| = \int_{\mathbb{R}} ds \int_F |D\chi_{E_s}|. \quad (2.3.2)$$

The two last equations give

$$\int_F |Du| = \int_{\mathbb{R}} \mathcal{H}^{N-1}(F \cap \partial^*E_s) ds. \quad (2.3.3)$$

If we write it with characteristic functions, we obtain

$$\int_{\Omega} \chi_F d|Du| = \int_{\mathbb{R}} ds \int_{\partial^*E_s} \chi_F d\mathcal{H}^{N-1}. \quad (2.3.4)$$

Thus by linearity and monotone convergence theorem we can generalize the last equality to every non-negative Borel function f ,

$$\int_{\Omega} f d|Du| = \int_{\mathbb{R}} ds \int_{\partial^*E_s} f d\mathcal{H}^{N-1}. \quad (2.3.5)$$

The following standard lemma asserts that the gradient of a Sobolev function u is orthogonal to the level sets of u , see e.g. [8, Lemma 2.6] for a proof.

Lemma 2.3.6. *Let $u \in W^{1,1}(\Omega)$. Then for a.e. $s \in \mathbb{R}$ and \mathcal{H}^{N-1} a.e. $x \in \partial^*E_s$, $\nabla u(x) \neq 0$. Moreover, we have*

$$\nu_{E_s} = \frac{D\chi_{E_s}}{|D\chi_{E_s}|} = \frac{\nabla u}{|\nabla u|} \quad |D\chi_{E_s}| \text{ a.e.} \quad (2.3.6)$$

2.3.2 Comparison principles

When $\psi \equiv 0$, the geometry of Ω and a comparison principle allow to find some information on the solutions.

Definition 2.3.7. The Cheeger constant of Ω is defined as:

$$h_\Omega = \inf_{D \subset \bar{\Omega}} \frac{\text{Per } D}{|D|}.$$

A set $D \subset \bar{\Omega}$ of finite perimeter is said to be a Cheeger set if $\text{Per } D = h_\Omega |D|$.

In this subsection we also use an equivalent definition of h_Ω :

Proposition 2.3.8. *We have the following inequality for every $p \geq 1$:*

$$h_\Omega = \tilde{h}_\Omega := \inf \left(\int_\Omega |\nabla u| : u \in W_0^{1,p}(\Omega), \int_\Omega u = 1 \right).$$

Proof. By [60, Proposition 3.1], there exists a Cheeger set E of Ω . By [60, Proposition 3.3] and [56, Theorem 2], there exists a sequence $(E_k)_{k \in \mathbb{N}}$ of smooth sets such that $\chi_{E_k} \rightarrow \chi_E$ in $\mathcal{L}_{loc}^1(\mathbb{R}^N)$, $\text{Per}(E_k) \rightarrow \text{Per}(E)$ and $E_k \Subset \Omega$.

By [32, Theorem 5.2.2], for each $k \in \mathbb{N}$ there exists a sequence $(f_k^n)_{n \in \mathbb{N}} \in BV(\Omega) \cap C^\infty(\Omega)$ such that $f_k^n \rightarrow \chi_{E_k}$ in $\mathcal{L}_{loc}^1(\mathbb{R}^N)$ and $\int_\Omega |Df_k^n| \rightarrow \int_\Omega |D\chi_{E_k}|$ when $n \rightarrow +\infty$. For every $k \in \mathbb{N}$, we introduce $\zeta_k \in C_0^\infty(\Omega)$, $\chi_{E_k} \leq \zeta_k \leq 1$. The function $g_k^n := f_k^n \zeta_k \in C_0^\infty(\Omega)$ is such that $g_k^n \rightarrow \chi_{E_k}$ in $\mathcal{L}_{loc}^1(\mathbb{R}^N)$ and $\int_\Omega |Dg_k^n| \leq \int_\Omega |Df_k^n| + \int_\Omega |\nabla \zeta_k| f_k^n \rightarrow \int_\Omega |D\chi_{E_k}|$ when n goes to $+\infty$.

By a diagonal process, we can extract a sequence $(g_k^{n_k})_{k \in \mathbb{N}}$ of smooth functions compactly supported in Ω such that

$$\liminf_{k \rightarrow +\infty} \frac{\int_\Omega |Dg_k^{n_k}|}{\int_\Omega g_k^{n_k}} \leq \frac{\text{Per } E}{|E|} = h_\Omega.$$

If we normalize the $(g_k^{n_k})_{k \in \mathbb{N}}$ we obtain that $\tilde{h}_\Omega \leq h_\Omega$. To prove the equality, we consider $(u_n)_{n \in \mathbb{N}}$ a minimizing sequence of $\int_\Omega |\nabla u|$ in $\{u \in W_0^{1,p}(\Omega), \int_\Omega u = 1\}$:

$$0 = \lim_{n \rightarrow +\infty} \int_\Omega |\nabla u_n| - \tilde{h}_\Omega.$$

We can assume that $u_n \geq 0$ a.e. for every $n \in \mathbb{N}$. By the co-area formula and Fubini's theorem, we have

$$\int_\Omega |\nabla u_n| - \tilde{h}_\Omega = \int_\Omega (|\nabla u_n| - \tilde{h}_\Omega u_n) = \int_{\mathbb{R}} \text{Per}(E_s^n) - \tilde{h}_\Omega |E_s^n|$$

where $E_s^n := [u_n \geq s]$. Since $\tilde{h}_\Omega \leq h_\Omega$ we get

$$\int_{\mathbb{R}} \text{Per}(E_s^n) - \tilde{h}_\Omega |E_s^n| \geq \int_{\mathbb{R}} \text{Per}(E_s^n) - h_\Omega |E_s^n| \geq 0.$$

Hence,

$$0 = \lim_{n \rightarrow +\infty} \int_\Omega |\nabla u_n| - \tilde{h}_\Omega \geq \lim_{n \rightarrow +\infty} \int_\Omega |\nabla u_n| - h_\Omega \geq 0.$$

Thus, $\tilde{h}_\Omega = h_\Omega$.

□

The first result of this part is the following:

Proposition 2.3.9. *Let $\lambda \in L^\infty(\Omega)$, $\lambda \geq 0$ a.e. on Ω .*

- *If $\inf_{x \in \Omega} \lambda(x) > h_\Omega$, then 0 is not a solution of \mathcal{P}_λ on $W_0^{1,p}(\Omega)$.*
- *If $\sup_{x \in \Omega} \lambda(x) \leq h_\Omega$, then 0 is the unique solution of \mathcal{P}_λ on $W_0^{1,p}(\Omega)$.*

The proof of this proposition is based on the two following lemmata, see [8, lemmata 1.5 and 4.1] for the specific case of the Kohn and Strang's problem:

Lemma 2.3.10. *Let $\lambda \equiv \lambda_0 \in \mathbb{R}_+$.*

- *If $\lambda > h_\Omega$, then 0 is not the solution of \mathcal{P}_λ on $W_0^{1,p}(\Omega)$.*
- *If $\lambda \leq h_\Omega$, then 0 is the unique solution of \mathcal{P}_λ on $W_0^{1,p}(\Omega)$.*

Proof of Lemma 2.3.10. If we suppose that 0 is the solution, then for every $v \in \mathcal{C}_0^\infty(\Omega)$,

$$\int_{\Omega} \varphi(\nabla v) - \lambda \int_{\Omega} v \geq 0.$$

Thus, by assumption (A1) on φ we have

$$\lambda \int_{\Omega} v \leq \int_{\Omega} \varphi(\nabla v) = \int_{\|\nabla v\| \leq 1} |\nabla v| + \int_{\|\nabla v\| > 1} \varphi(\nabla v).$$

By replacing v by sv for some $s > 0$ and dividing by s , one gets

$$\lambda \int_{\Omega} v \leq \int_{\|\nabla v\| \leq s^{-1}} |\nabla v| + \frac{1}{s} \int_{\|\nabla v\| > s^{-1}} \varphi(s\nabla v).$$

We now let $s \rightarrow 0^+$, by convexity of φ and the fact that $\varphi(0) = 0$ we have:

$$\lambda \int_{\Omega} v \leq \lim_{s \rightarrow 0} \int_{\|\nabla v\| \leq s^{-1}} |\nabla v| + \lim_{s \rightarrow 0} \int_{\|\nabla v\| > s^{-1}} \varphi(\nabla v).$$

Since the limit in the last term is 0 we get

$$\lambda \int_{\Omega} v \leq \int_{\Omega} |\nabla v|.$$

By density of smooth functions in $W_0^{1,p}(\Omega)$ and the second definition of h_Ω we deduce that $\lambda \leq h_\Omega$.

Assume now that $\lambda \leq h_\Omega$. Then for any $v \in W_0^{1,p}(\Omega)$ and u solution, we have

$$\int_{\Omega} \varphi(\nabla v) - \lambda \int_{\Omega} v \geq \int_{\Omega} \varphi(\nabla u) - \lambda \int_{\Omega} u.$$

By Lemma 2.2.2 we have that $u \geq 0$. Then by Proposition 2.3.8,

$$\int_{\Omega} \varphi(\nabla v) - \lambda \int_{\Omega} v \geq \int_{\Omega} |\nabla u| - h_\Omega \int_{\Omega} u \geq 0. \quad (2.3.7)$$

This proves that 0 is the solution of \mathcal{P}_λ . By [52], when $\lambda \equiv \lambda_0 \in \mathbb{R}_+$, if \mathcal{P}_λ has a uniformly continuous solution in $W_0^{1,p}(\Omega)$ then it is the unique solution. Thus, 0 is the unique solution of \mathcal{P}_λ on $W_0^{1,p}(\Omega)$. \square

Lemma 2.3.11. *Consider $\lambda_1, \lambda_2 \in L^\infty(\Omega)$, $\lambda_2 \geq \lambda_1 > 0$ a.e. on Ω , u_1 a solution of \mathcal{P}_{λ_1} on $u_1 + W_0^{1,p}(\Omega)$ and u_2 a solution of \mathcal{P}_{λ_2} on $u_2 + W_0^{1,p}(\Omega)$. We also assume that $u_1|_{\partial\Omega} \leq u_2|_{\partial\Omega}$. If $\lambda_2 > \lambda_1$ a.e. on Ω or if u_1 is the unique minimizer of \mathcal{P}_{λ_1} on $u_1 + W_0^{1,p}(\Omega)$ or if u_2 is the unique minimizer of \mathcal{P}_{λ_2} on $u_2 + W_0^{1,p}(\Omega)$, then $u_1 \leq u_2$ a.e. on Ω .*

Proof of Lemma 2.3.11. Since $I_{\lambda_1}(u_1) \leq I_{\lambda_1}(\min(u_1, u_2))$, we have

$$\int_{[u_1 > u_2]} \varphi(\nabla u_1) - \lambda_1 u_1 \leq \int_{[u_1 > u_2]} \varphi(\nabla u_2) - \lambda_1 u_2. \quad (2.3.8)$$

Since $I_{\lambda_2}(u_2) \leq I_{\lambda_2}(\max(u_1, u_2))$,

$$\int_{[u_1 > u_2]} \varphi(\nabla u_2) - \lambda_2 u_2 \leq \int_{[u_1 > u_2]} \varphi(\nabla u_1) - \lambda_2 u_1. \quad (2.3.9)$$

The sum of (2.3.8) and (2.3.9) gives

$$0 \leq \int_{[u_2 < u_1]} (\lambda_2 - \lambda_1)(u_2 - u_1).$$

We first assume that for a.e. $x \in \Omega$, $\lambda_2(x) > \lambda_1(x)$ and $|[u_2 < u_1]| > 0$. Since $u_2 - u_1$ is negative on the set $[u_2 < u_1]$, it follows that this last integral is also negative. This is absurd, thus if for a.e. $x \in \Omega$, $\lambda_2(x) > \lambda_1(x)$ then $u_1 \leq u_2$ a.e. on Ω , which completes the proof in that case. Otherwise, if u_1 is the unique minimum of \mathcal{P}_{λ_1} on $u_1 + W_0^{1,2}(\Omega)$ and $|[u_2 < u_1]| > 0$ the inequality (2.3.8) becomes strict and with the help of equality (2.3.9) we have

$$0 < \int_{[u_2 < u_1]} (\lambda_2 - \lambda_1)(u_2 - u_1).$$

Since the integral is non-positive this implies that $|[u_2 < u_1]| = 0$. Hence, $u_1 \leq u_2$ a.e. on Ω , which completes the proof in that case as well. If u_2 is the unique solution of \mathcal{P}_λ , then the proof is similar and we omit it. \square

Proof of Proposition 2.3.9. Let $\lambda \in L^\infty(\Omega)$.

If $\inf_\Omega \lambda > h_\Omega$, then for ϵ small enough $\inf_\Omega \lambda \geq h_\Omega + \epsilon$. By Lemma 2.3.11 for any solution u of \mathcal{P}_λ , $u \geq v$ where v is a solution of $\mathcal{P}_{h_\Omega + \frac{\epsilon}{2}}$. By Lemma 2.3.10, $v \neq 0$, thus $u \neq 0$.

If $\lambda \leq h_\Omega$, by Lemma 2.3.10, 0 is the only solution of \mathcal{P}_{h_Ω} . Then by Lemma 2.3.11 we have $u \leq 0$ and by Lemma 2.2.2, $u \equiv 0$. \square

When $\lambda \leq h_\Omega$, we have the following \mathcal{L}^∞ estimate on the solutions:

Proposition 2.3.12. *Let $\lambda \in L^\infty(\Omega)$, $0 \leq \lambda \leq h_\Omega$ a.e. on Ω and let u be a solution of \mathcal{P}_λ on $W_\psi^{1,p}(\Omega)$. We have for a.e. $x \in \bar{\Omega}$,*

$$\min_{\partial\Omega} \psi \leq u(x) \leq \max_{\partial\Omega} \psi.$$

Proof. We proved in Lemma 2.2.2 that $u \geq \min_{\partial\Omega} \psi$. We denote by $b := \max_{\partial\Omega} \psi$. Since $0 \leq \lambda \leq h_\Omega$, by Lemma 2.3.10 and Lemma 2.2.1, b is the unique solution of \mathcal{P}_λ on $W_b^{1,p}$. By lemma 2.3.11 we have $u \leq b$. \square

2.4 Global Lipschitz regularity

In this section, we study two situations where the minimizers of \mathcal{P}_λ are globally Lipschitz continuous.

2.4.1 Ω is convex.

In the following statement we assume that ψ satisfies the bounded slope condition of rank R introduced in Definition 2.1.5. This assumption on ψ and Remark 2.2.8 allow us to apply [12, Main Theorem] in our case:

Theorem 2.4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded convex set, φ an uniformly convex function at infinity, ψ a Lipschitz continuous function that satisfies the bounded slope condition of rank $R \geq 0$ and $\lambda \in L^\infty(\Omega)$. Then every minimizer u of the problem \mathcal{P}_λ is Lipschitz-continuous. More precisely, we have*

$$\|\nabla u\|_{\mathcal{L}^\infty(\Omega)} \leq \mathcal{L}(\Phi, N, R, \|\lambda\|_{\mathcal{L}^\infty(\Omega)}, \text{diam}(\Omega))$$

with Φ given in Definition 2.2.7.

2.4.2 Ω is smooth.

If Ω is not convex or if the bounded slope condition is not satisfied we assume more regularity on $\partial\Omega$ and on the boundary condition ψ .

We recall Theorem 2.1.3:

Theorem 2.4.2 (Global Lipschitz continuity for a general degenerate functional). *Let Ω be a connected bounded open set of \mathbb{R}^N with $N \geq 2$. We assume that Ω has a $\mathcal{C}^{1,1}$ connected boundary, $\psi \in \mathcal{C}^{1,1}(\mathbb{R}^N)$ and $\lambda \in L^\infty(\Omega)$. If φ satisfies the assumption (A2) then any minimizer u of \mathcal{P}_λ is globally Lipschitz-continuous on Ω . Moreover,*

$$\|\nabla u\|_{\mathcal{L}^\infty(\Omega)} \leq C(p, C_p, \|\lambda\|_{\mathcal{L}^\infty(\Omega)}, N, |\Omega|, \text{diam}(\Omega), \|\psi\|_{\mathcal{C}^{1,1}(\Omega)}, \kappa)$$

with κ being the maximum of the principal curvatures of Ω .

The proof of this result is divided into four parts. In **Step 1**, we start by approximating φ by smooth uniformly convex functions $(\varphi_\epsilon)_{\epsilon>0}$ in order to work with smooth minimizers $(u_\epsilon)_\epsilon$. Then in **Step 2**, we construct two Lipschitz continuous functions l^+ and l^- such that $l^- \leq u_\epsilon \leq l^+$ for every $\epsilon > 0$. In the subsequent **Step 3** we prove that u_ϵ is globally Lipschitz continuous uniformly in ϵ . We pass to the limit when ϵ goes to 0 to conclude in **Step 4**.

Step 1 By assumption (A2) on g , there exist $x_0 > 1$ and $0 < \alpha$ such that for every $x \geq x_0$,

$$xg''(x) \geq \frac{\alpha}{2}g'(x).$$

Moreover, we take $\alpha \leq 2$ for the rest of the proof which is possible if we only assume $\alpha \leq \liminf_{x \rightarrow +\infty} \frac{xg''(x)}{g'(x)}$.

We introduce the following notations from [12, Lemma A5]: $Q > x_0$, $J_Q(x) = (|x| - Q)_+^2$ and

$$\nu = \min\{1, \min_{x \in [2x_0, 4Q]} \tilde{D}|x|^{p-2}\} \quad (2.4.1)$$

with \tilde{D} introduced in Remark 2.2.8.

We define an approximation of g quadratic at $+\infty$:

$$\tilde{g}(x) := \begin{cases} g(x) & \text{if } |x| \leq 3Q, \\ \frac{1}{2}g''(3Q)(x - 3Q)^2 + g'(3Q)(x - 3Q) + g(3Q) & \text{if } |x| > 3Q. \end{cases} \quad (2.4.2)$$

By Remark 2.1.1 and since we can take $\alpha \leq 2$, we have that $x\tilde{g}''(x) \geq \frac{\alpha}{2}\tilde{g}'(x)$ for every $x \geq x_0$.

Let us consider $\eta \in \mathcal{C}_0^\infty((-1, 1))$ an even function such that $\eta \geq 0$, $\int_{\mathbb{R}} \eta = 1$. We define for every $0 < \epsilon < \frac{1}{x_0}$,

$$\eta_\epsilon := \frac{1}{\epsilon}\eta\left(\frac{\cdot}{\epsilon}\right), \quad g_\epsilon(x) := (\tilde{g} + \nu J_Q) * \eta_\epsilon(x) + \epsilon x^2, \quad \varphi_\epsilon(\cdot) := g_\epsilon(|\cdot|) \text{ and } \lambda_\epsilon(z) := \lambda * \tilde{\eta}_\epsilon(z)$$

where $(\tilde{\eta}_\epsilon)_{\epsilon > 0}$ is a mollifying sequence in \mathbb{R}^N . We prove the following regularity result on φ_ϵ :

Proposition 2.4.3. *The function φ_ϵ is smooth on \mathbb{R}^N .*

Proof. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth even function then $f(|\cdot|)$ is \mathcal{C}^1 . Hence, φ_ϵ is \mathcal{C}^1 on \mathbb{R}^N . For every integer $1 \leq i \leq N$, we have $\partial_i \varphi_\epsilon(z) = \frac{g'_\epsilon(|z|)z_i}{|z|}$ if $z \neq 0$ and $\partial_i \varphi_\epsilon(0) = 0$. By a Taylor expansion we can see that

$$f(x) := \begin{cases} \frac{g'_\epsilon(x)}{x} & \text{if } x \neq 0, \\ g''_\epsilon(0) & \text{if } x = 0 \end{cases}$$

is a smooth even function. Thus, $\partial_i \varphi_\epsilon(z) = f(|z|)z_i$ is \mathcal{C}^1 . Hence, φ_ϵ is \mathcal{C}^2 . With the same arguments we can prove that this is also the case for $f(|\cdot|)$. By induction we have that φ_ϵ is \mathcal{C}^∞ . \square

By the proof of [12, Lemma A5], we have the following uniform convexity result for every $0 < \epsilon < \frac{1}{2}x_0$:

$$g_\epsilon(tx + (1-t)y) \leq tg_\epsilon(x) + (1-t)g_\epsilon(y) - t(1-t)\frac{\nu}{2}|x - y|^2$$

for every $t \in [0, 1]$ and every $x, y \in \mathbb{R}$ such that $[x, y] \cap B_{\frac{3}{2}x_0}(0) = \emptyset$.

Proposition 2.4.4. *We consider $0 < \epsilon < \frac{1}{2}x_0$. If we set $\tilde{\nu} := \min\{\nu, \frac{1}{2}C_p, \frac{1}{2}C_p Q^{p-2}\} > 0$ then*

$$\varphi_\epsilon(t\xi + (1-t)\zeta) \leq t\varphi_\epsilon(\xi) + (1-t)\varphi_\epsilon(\zeta) - t(1-t)\frac{\tilde{\nu}}{2}|\xi - \zeta|^2$$

for every $t \in [0, 1]$ and every $\xi, \zeta \in \mathbb{R}^N$ such that $[\xi, \zeta] \cap B_{\frac{3}{2}x_0}(0) = \emptyset$.

Proof. By Remark 2.2.8, we have to find $\tilde{\nu} > 0$ such that $\langle \nabla^2 \varphi_\epsilon(z) \xi, \xi \rangle \geq \tilde{\nu} |\xi|^2$ for every $z \in \mathbb{R}^N \setminus B_{\frac{3}{2}x_0}(0)$ and every $\xi \in \mathbb{R}^N$. If we reproduce the beginning of the proof of Proposition 2.2.6, we obtain that $\langle \nabla^2 \varphi_\epsilon(z) \xi, \xi \rangle \geq \min\{g''(|z|), \frac{g'_\epsilon(|z|)}{|z|}\} |\xi|^2$. Since g_ϵ is uniformly convex outside $B_{\frac{3}{2}x_0}(0)$ we have that $g''(|z|) \geq \nu$. If $x > 2Q + \epsilon$, by definition of J_Q , we obtain that $\frac{g'_\epsilon(x)}{|x|} \geq \nu$. Hence,

$$\langle \nabla^2 \varphi_\epsilon(z) \xi, \xi \rangle \geq \min\left\{\nu, \min_{x \in [x_0, 3Q]} \frac{g'_\epsilon(x)}{x}\right\} |\xi|^2$$

for every $z \in \mathbb{R}^N$ with $|z| \geq \frac{3}{2}x_0$ and every $\xi \in \mathbb{R}^N$.

By Remark 2.1.1, $\frac{g'_\epsilon(x)}{x} \geq C_p x^{p-2}$ on $[x_0, 3Q]$. Thus, for every $x \in [x_0, 3Q]$, $\frac{g'_\epsilon(x)}{x} \geq C_p \min\{1, Q^{p-2}\}$. Then, we can set $\tilde{\nu} := \min\{\nu, \frac{1}{2}C_p, \frac{1}{2}C_p Q^{p-2}\} > 0$ such that

$$\langle \nabla^2 \varphi_\epsilon(z) \xi, \xi \rangle \geq \tilde{\nu} |\xi|^2$$

for every z with $|z| \geq \frac{3}{2}x_0$ and every $\xi \in \mathbb{R}^N$. \square

Step 2 We introduce the following problem:

$$\mathcal{P}_{\lambda_\epsilon}^\epsilon : \min_{u \in W_\psi^{1,2}(\Omega)} I_{\lambda_\epsilon}^\epsilon(u) \quad (2.4.3)$$

where the functional $I_{\lambda_\epsilon}^\epsilon$ is the following:

$$I_{\lambda_\epsilon}^\epsilon : u \in W_\psi^{1,2}(\Omega) \rightarrow \int_\Omega \varphi_\epsilon(\nabla u) - \lambda_\epsilon u dx.$$

For each $\epsilon > 0$, the problem $\mathcal{P}_{\lambda_\epsilon}^\epsilon$ has a unique solution that we call u_ϵ . We first prove that the solutions $(u_\epsilon)_{\epsilon > 0}$ are bounded uniformly in ϵ :

Proposition 2.4.5. *We have $\sup_{\epsilon > 0} \|u_\epsilon\|_{\mathcal{L}^\infty(\Omega)} < +\infty$. Moreover, the supremum can be bounded uniformly in $Q > x_0$.*

Proof. We introduce $\Psi = \max_{x \in \partial\Omega} |\psi(x)|$, $\Lambda = \sup_{x \in \Omega, \epsilon > 0} \lambda_\epsilon(x)$ and $B_\Omega = B_{\text{diam}(\Omega)}(x_\Omega)$ the smallest ball containing Ω . By Lemma 2.2.2, $u_\epsilon^* \geq \Psi$ on $\partial\Omega$ where u_ϵ^* is the solution of $\mathcal{P}_\Lambda^\epsilon$ on $W_\Psi^{1,2}(B_\Omega)$. Hence, by Lemma 2.3.11, we have that $\|u_\epsilon\|_{\mathcal{L}^\infty(\Omega)} \leq \|u_\epsilon^*\|_{\mathcal{L}^\infty(\Omega)} \leq \|u_\epsilon^*\|_{\mathcal{L}^\infty(B_\Omega)}$. By [22, Theorem 1], $\|u_\epsilon^*\|_{\mathcal{L}^\infty(B_\Omega)} \leq \Psi + \frac{N}{\Lambda} g_\epsilon^*\left(\frac{\Lambda}{N} \text{diam}(\Omega)\right)$ where $g_\epsilon^*(x) = \sup_{y \geq 0} xy - g_\epsilon(y)$. By strict convexity of g_ϵ , $g_\epsilon^*(x) = x g_\epsilon'^{-1}(x) - g_\epsilon(g_\epsilon'^{-1}(x))$.

By (2.4.2), we have $\lim_{x \rightarrow +\infty} \tilde{g}'(x) = +\infty$. Hence, there exists $x > 0$ such that $g'(x) = \frac{\Lambda}{N} \text{diam}(\Omega) + 1$. By (2.4.2), for every Q such that $3Q > x$, we have $\tilde{g}'(x) = g'(x) = \frac{\Lambda}{N} \text{diam}(\Omega) + 1$. Since \tilde{g} is convex, $\tilde{g}'(y) \geq \frac{\Lambda}{N} \text{diam}(\Omega) + 1$ for every $y \geq x$. Thus, for every $\epsilon < 1$, $g'_\epsilon(t) \geq \frac{\Lambda}{N} \text{diam}(\Omega) + 1$ for every $t \geq x + 1$. Hence, $g_\epsilon'^{-1}\left(\frac{\Lambda}{N} \text{diam}(\Omega)\right)$ is bounded uniformly in ϵ and Q . Thus, $\|u_\epsilon\|_{\mathcal{L}^\infty(\Omega)}$ is bounded uniformly in ϵ and Q . \square

We introduce

$$M := \sup_{x \in \Omega, \epsilon > 0} |u_\epsilon(x) - \psi(x)|.$$

The minimizer of (2.4.3) is a solution of the following partial differential equation:

$$\operatorname{div}(\nabla \varphi_\epsilon(\nabla u)) + \lambda_\epsilon = 0.$$

Hence,

$$L_\epsilon(u_\epsilon) := \sum_{i,j} \partial_{ij}^2 \varphi_\epsilon(\nabla u_\epsilon) \partial_{ij}^2 u_\epsilon + \lambda_\epsilon = 0. \quad (2.4.4)$$

This is a quasi-linear elliptic partial differential equation with smooth coefficients. Hence, $u_\epsilon \in C^\infty(\Omega)$, by [47, Theorem 1], $u_\epsilon \in C^1(\overline{\Omega})$ and by [36, Lemma 9.15], $u_\epsilon \in W_\psi^{2,2}(\Omega)$.

We establish the following estimate:

Proposition 2.4.6. *There exists $\mu := \mu(x_0, \alpha, N, \|\lambda\|_{\mathcal{L}^\infty(\Omega)}) > 0$ such that*

$$|z| \sum_{i,j} |\partial_{ij}^2 \varphi_\epsilon(z)| + \|\lambda_\epsilon\|_{\mathcal{L}^\infty(\Omega)} \leq \mu \sum_{i,j} \partial_{ij}^2 \varphi_\epsilon(z) z_i z_j$$

for every $|z| \geq \mu$ and every $0 < \epsilon < \frac{1}{2}x_0$.

Proof. By construction of g_ϵ , we have for every $x \geq \frac{3}{2}x_0$ that

$$xg_\epsilon''(x) = x \int_{\mathbb{R}} (\tilde{g}''(x-t) + 2\nu\chi_{\{y>Q\}}(x-t) + 2\epsilon)\eta_\epsilon(t)dt.$$

Since $x\tilde{g}''(x) \geq \frac{\alpha}{2}\tilde{g}'(x)$ for every $x \geq \frac{3}{2}x_0$, we get

$$\begin{aligned} xg_\epsilon''(x) &\geq \int_{\mathbb{R}} \frac{x}{x-t} \left(\frac{\alpha}{2}\tilde{g}'(x-t) + 2\nu(x-t)\chi_{\{y>Q\}}(x-t) + 2\epsilon(x-t) \right) \eta_\epsilon(t)dt \\ &\geq \frac{1}{2} \times \frac{\alpha}{2} \int_{\mathbb{R}} \left(\tilde{g}'(x-t) + 2\nu(x-t)\chi_{\{y>Q\}}(x-t) + 2\epsilon(x-t) \right) \eta_\epsilon(t)dt \\ &\geq \frac{1}{2} \times \frac{\alpha}{2} g_\epsilon'(x). \end{aligned}$$

Moreover,

$$\partial_{ij}^2 \varphi_\epsilon(z) = \frac{g_\epsilon''(|z|)}{|z|^2} z_i z_j + \frac{g_\epsilon'(|z|)}{|z|} \delta_{ij} - \frac{g_\epsilon'(|z|)}{|z|^3} z_i z_j$$

for every $z \in \mathbb{R}^N$. By the previous estimate for every $|z| \geq \frac{3}{2}x_0$ we have:

$$|z| \sum_{i,j} |\partial_{ij}^2 \varphi_\epsilon(z)| + \|\lambda\|_{\mathcal{L}^\infty(\Omega)} \leq C(\alpha, N) g_\epsilon''(|z|) |z| + \|\lambda\|_{\mathcal{L}^\infty(\Omega)}.$$

We also have $\sum_{i,j} \partial_{ij}^2 \varphi_\epsilon(z) z_i z_j = g_\epsilon''(|z|) |z|^2$ and $\inf_{x > \frac{3}{2}x_0, 0 < \epsilon < \frac{1}{2}x_0} g_\epsilon''(x) x > 0$.

Hence, there exists $\mu > \frac{3}{2}x_0$ independent of ϵ , $\tilde{\nu}$ and Q such that

$$|z| \sum_{i,j} |\partial_{ij}^2 \varphi(z)| + \|\lambda\|_{\mathcal{L}^\infty(\Omega)} \leq \mu \sum_{i,j} \partial_{ij}^2 \varphi(z) z_i z_j$$

for every $|z| \geq \mu$. □

By [36, Appendix 14.6] and the regularity assumption on Ω , for every $y \in \partial\Omega$ there exists y_0 such that $y \in \overline{B_R(y_0)} \cap \overline{\Omega} = \overline{B_R(y_0)} \cap \partial\Omega \neq \emptyset$ where $R := \frac{1}{\kappa} > 0$ is the inverse of the maximum of the principal curvatures of Ω . We define the distance function $d_y(x) := \text{dist}(x, \partial B_R(y_0))$ and $\mathcal{N}_y := \{x \in \Omega \mid d_y(x) < A\}$ with A a constant to be specified later.

We estimate the norm of ∇u_ϵ on the boundary of Ω :

Lemma 2.4.7. *For every $y \in \partial\Omega$, there exist two L_1 -Lipschitz continuous functions l_y^+ , l_y^- such that $l_y^-(y) = u_\epsilon(y) = l_y^+(y)$ and $l_y^- \leq u_\epsilon \leq l_y^+$ on Ω for every $0 < \epsilon < \frac{1}{2}x_0$ with*

$$L_1 := L_1(\|\lambda\|_{\mathcal{L}^\infty(\Omega)}, M, N, \kappa, \alpha, x_0, \|\nabla\psi\|_{\mathcal{L}^\infty(\mathbb{R}^N)}, \|\nabla^2\psi\|_{\mathcal{L}^\infty(\mathbb{R}^N)})$$

independent of ϵ , $\tilde{\nu}$ introduced in Proposition 2.4.4 and Q .

Proof. We follow the discussion of [36, Chapter 14, Section 1] to construct barriers for u_ϵ .

We introduce the following constants :

$$\tilde{\mu} = 4(\mu(1 + \|\nabla\psi\|_{\mathcal{L}^\infty(\mathbb{R}^N)}^2) + \|\nabla^2\psi\|_{\mathcal{L}^\infty(\mathbb{R}^N)}), \quad \omega = (1 + \frac{N-1}{R})\tilde{\mu}, \quad k = \tilde{\mu}\omega e^{\omega M} \text{ and} \\ A = \frac{1 - e^{-\omega M}}{\tilde{\mu}\omega}.$$

Thus, we can define the following smooth function:

$$F_y(x) = \frac{1}{\omega} \log(1 + kd_y(x)).$$

The constant A is chosen such that $F_y(x) = M$ on $\partial\mathcal{N}_y \cap \Omega$. Thus $\psi(x) \pm F_y(x)$ are two Lipschitz continuous functions, with $\psi(x) - F_y(x) \leq u_\epsilon(x) \leq \psi(x) + F_y(x)$ on $\partial\mathcal{N}_y$.

Moreover, using Proposition 2.4.6 we can prove as in [36, Chapter 14, Section 1] that $L_\epsilon(\psi(x) + F_y(x)) \leq L_\epsilon(u_\epsilon) \leq L_\epsilon(\psi(x) - F_y(x))$. By [36, Theorem 10.1], we have that $\psi(x) - F_y(x) \leq u_\epsilon(x) \leq \psi(x) + F_y(x)$ for every $x \in \Omega \cap \mathcal{N}_y$. Since, $F_y \geq M$ on $\Omega \setminus \mathcal{N}_y$, we have that $\psi - F_y \leq u_\epsilon \leq \psi + F_y$ on Ω .

If we set $L_1 := \frac{k}{\omega} + \|\nabla\psi\|_{\mathcal{L}^\infty(\mathbb{R}^N)}$ then $l_y^\pm(x) := \psi(x) \pm F_y(x)$ are L_1 -Lipschitz continuous on \mathcal{N}_y for every $y \in \partial\Omega$. □

Step 3 Now, we prove the main part of Theorem 2.1.3:

Lemma 2.4.8. *There exists a constant $\mathcal{L} := \mathcal{L}(\|\lambda\|_{\mathcal{L}^\infty(\Omega)}, M, N, \kappa, \alpha, x_0, \|\psi\|_{\mathcal{C}^{1,1}(\mathbb{R}^N)})$ independent of ϵ , $\tilde{\nu}$ and Q such that*

$$\|\nabla u_\epsilon\|_{\mathcal{L}^\infty(\bar{\Omega})} \leq \frac{\mathcal{L}}{\tilde{\nu}}.$$

Proof. We define the two following functions:

$$l^+(x) = \begin{cases} \inf_{y \in \partial\Omega} l_y^+(x) & \text{if } x \in \Omega, \\ \psi(x) & \text{if } x \notin \Omega \end{cases} \quad (2.4.5)$$

and

$$l^-(x) = \begin{cases} \sup_{y \in \partial\Omega} l_y^-(x) & \text{if } x \in \Omega, \\ \psi(x) & \text{if } x \notin \Omega \end{cases} \quad (2.4.6)$$

with $M := \sup_{x \in \Omega, \epsilon > 0} |u_\epsilon(x) - \psi(x)|$.

We prove that l^+ and l^- are continuous on \mathbb{R}^N . Since l^+ is the infimum on Ω of L_1 -Lipschitz continuous functions, we have that l^+ is L_1 -Lipschitz continuous in Ω . Let us consider $y \in \partial\Omega$ and $(x_n)_{n \in \mathbb{N}}$ a sequence in Ω converging to y . For n large enough, $x_n \in \mathcal{N}_y$. By definition of l^+ and the fact that $l_y^+(x) := \psi(x) + F_y(x)$ on \mathcal{N}_y , we have $\psi(x_n) \leq l^+(x_n) \leq \psi(x_n) + F_y(x_n)$. Since $F_y(z)$ decreases to 0 when $d_y(z)$ goes to 0 we have that $l^+(x_n) \rightarrow \psi(y)$ when $n \rightarrow +\infty$. Hence l^+ is continuous on \mathbb{R}^N .

The same reasoning leads to the same conclusion for l^- . Moreover, l^+ and l^- are L_1 -Lipschitz continuous on \mathbb{R}^N with L_1 independent of ϵ . Those two Lipschitz continuous functions are equal to ψ on $\partial\Omega$ and $l^-(x) \leq u_\epsilon(x) \leq l^+(x)$ on $\bar{\Omega}$, this is the definition of barriers in [12].

Now, using Proposition 2.4.4, can follow [12, Section 4.4] to prove that

$$\|\nabla u_\epsilon\|_{\mathcal{L}^\infty(\bar{\Omega})} \leq \frac{\mathcal{L}}{\tilde{\nu}}$$

with $\mathcal{L} := \mathcal{L}(\|\lambda\|_\infty, M, N, \kappa, \alpha, x_0, \|\psi\|_{\mathcal{C}^{1,1}(\mathbb{R}^N)})$ independent of ϵ , $\tilde{\nu}$ and Q . \square

Step 4 We conclude by the proof of Theorem 2.1.3.

Proof of Theorem 2.1.3. By definition of ν in (2.4.1), definition of $\tilde{\nu}$ in Proposition 2.4.4 and growing assumptions stated in Remark 2.1.1, there exists $C > 0$ such that $\frac{\mathcal{L}}{\tilde{\nu}} \leq CQ^{2-p}$ when $Q \rightarrow +\infty$. Thus, we can choose Q such that $\frac{\mathcal{L}}{\tilde{\nu}} \leq Q - 1$. Hence for $\epsilon < \frac{1}{2}x_0$, we have $\varphi_\epsilon = g * \eta_\epsilon(|\cdot|) + \epsilon|\cdot|^2$ on $B_{\frac{\epsilon}{\tilde{\nu}}}(0)$. Thus, if we consider the Euler-Lagrange equation (see Proposition 2.5.6) we can prove that u_ϵ is a minimizer on $W_\psi^{1,2}(\Omega)$ of

$$v \rightarrow \int_{\Omega} g * \eta_\epsilon(|\nabla v|) + \epsilon|\nabla v|^2 - \lambda_\epsilon v.$$

We proved that the family $(u_\epsilon)_{\epsilon > 0}$ is equi-bounded in $W^{1,\infty}(\Omega)$. Hence, by the Arzelà-Ascoli Theorem, $u_\epsilon \xrightarrow{*} \tilde{u}$ up to a sub-sequence in $W^{1,\infty}(\Omega)$. We have that $\tilde{u} \in W_\psi^{1,p}(\Omega)$ and we show that \tilde{u} is a minimizer of \mathcal{P}_λ .

Let $v \in W_\psi^{1,\infty}(\Omega)$, by minimality of u_ϵ we have

$$\int_{\Omega} g * \eta_\epsilon(|\nabla u_\epsilon|) + \epsilon|u_\epsilon|^2 - \lambda_\epsilon u_\epsilon \leq \int_{\Omega} g * \eta_\epsilon(|\nabla v|) + \epsilon|\nabla v|^2 - \lambda_\epsilon v. \quad (2.4.7)$$

By Jensen's inequality we have that $g * \eta_\epsilon \geq \varphi$ on $B_{\frac{\epsilon}{\bar{\nu}}}(0)$. Hence,

$$\int_{\Omega} \varphi(\nabla u_\epsilon) \leq \int_{\Omega} g * \eta_\epsilon(|\nabla u_\epsilon|).$$

By weak lower semi-continuity, we obtain that

$$\int_{\Omega} \varphi(\nabla \tilde{u}) \leq \liminf_{\epsilon \rightarrow 0} \int_{\Omega} g * \eta_\epsilon(|\nabla u_\epsilon|).$$

Thus,

$$\int_{\Omega} \varphi(\nabla \tilde{u}) - \lambda \tilde{u} \leq \liminf_{\epsilon \rightarrow 0} \int_{\Omega} g * \eta_\epsilon(|\nabla u_\epsilon|) + \epsilon|\nabla u_\epsilon|^2 - \lambda_\epsilon u_\epsilon. \quad (2.4.8)$$

By Equations (2.4.7) and (2.4.8) we have

$$\int_{\Omega} \varphi(\nabla \tilde{u}) - \lambda \tilde{u} \leq \liminf_{\epsilon \rightarrow 0} \int_{\Omega} g * \eta_\epsilon(|\nabla v|) + \epsilon|\nabla v|^2 - \lambda_\epsilon v.$$

We apply the dominated convergence theorem to this last term to obtain

$$\int_{\Omega} \varphi(\nabla \tilde{u}) - \lambda \tilde{u} \leq \int_{\Omega} \varphi(\nabla v) - \int_{\Omega} \lambda v.$$

By [13, Theorem 1.1] and the regularity of $\partial\Omega$ we have that \tilde{u} is a minimizer of \mathcal{P}_λ on $W_\psi^{1,p}(\Omega)$. By Lemma 2.2.4 we have $\|\nabla \tilde{u} - \nabla v\|_{\mathcal{L}^\infty(\Omega)} \leq 1$. Thus, $u \in W^{1,\infty}(\Omega)$ and

$$\|\nabla u\|_{\mathcal{L}^\infty(\Omega)} \leq C(p, C_p, \|\lambda\|_{\mathcal{L}^\infty(\Omega)}, N, |\Omega|, \text{diam}(\Omega), \|\psi\|_{\mathcal{L}^{1,1}(\Omega)}, \kappa).$$

□

2.5 Non uniqueness cases

2.5.1 The Euler-Lagrange equation

We establish the weak Euler-Lagrange equation for Lipschitz continuous minimizer to obtain some regularity information on $|\nabla u|$. A proof of the following lemma can be found in [8]:

Lemma 2.5.1 (Euler-Lagrange equation). *Let $\lambda \in L^\infty(\Omega)$ and u be a solution of \mathcal{P}_λ . If u is Lipschitz continuous there exists $\sigma \in L^\infty(\Omega; \mathbb{R}^N)$ such that $\text{div } \sigma = -\lambda$ and*

$$\sigma \in \partial\varphi(\nabla u) \quad \text{a.e.}$$

Here $\partial\varphi$ is the convex subdifferential of φ .

We first exploit the Euler-Lagrange equation to show that every solution is \mathcal{C}^1 on the set where the norm of its gradient is larger than 1. In fact we use [26, Theorem 1] and proceed as in [8]:

Lemma 2.5.2. *Let $\lambda \in L^\infty(\Omega)$ and u be a Lipschitz solution of \mathcal{P}_λ . Then there exists an open subset $U \subset \Omega$ such that $u \in \mathcal{C}^{1,\alpha}$ on U with $0 < \alpha < 1$, $|\nabla u(x)| > 1$ for every $x \in U$ and $|\nabla u(x)| \leq 1$ for a.e. $x \in \Omega \setminus U$. Moreover, the restriction of $|\nabla u|$ to U is uniformly continuous on $U \cap \Omega'$ for all $\Omega' \Subset \Omega$. Hence, the restriction of $|\nabla u|$ to U can be extended as a continuous function on $\bar{U} \cap \Omega$ which is equal to 1 on $\partial U \cap \Omega$.*

Proof. By [26, Theorem 1.1], for every continuous function $\mathcal{H} : \mathbb{R}^N \rightarrow \mathbb{R}$ such that $\mathcal{H} = 0$ on $B_1(0)$, the function $\mathcal{H}(\nabla u)$ has a continuous representative on Ω . By applying this result to $\mathcal{H}(y) = (|y| - 1)_+$, one obtains that $(|\nabla u| - 1)_+$ has a continuous representative on Ω and the set $U := [\mathcal{H}(\nabla u) > 0]$ is open.

On the open set U , $|\nabla u| > 1$ a.e. so that the function σ introduced in Lemma 2.5.1 satisfies $\sigma = \nabla\varphi(\nabla u)$ a.e. on U . Hence, u is a weak solution of the quasilinear elliptic equation:

$$\operatorname{div}(\nabla\varphi(\nabla u)) = -\lambda. \quad (2.5.1)$$

By [37, Theorem 8.8], $u \in \mathcal{C}_{loc}^{1,\alpha}$ on U for every $\alpha < 1$. Since $|\nabla u| = \mathcal{H}(\nabla u) + 1$ is uniformly continuous on $U \cap \Omega'$ for every $\Omega' \Subset \Omega$, it follows that $|\nabla u|$ extends as a continuous function on $\bar{U} \cap \Omega$ which is equal to 1 on $\partial U \cap \Omega$.

Finally, on $\Omega \setminus U$, $\mathcal{H}(\nabla u) = 0$ and thus $|\nabla u| \leq 1$ a.e. there. □

Proposition 2.5.3. *The open set U does not depend on the choice of a Lipschitz continuous solution u of \mathcal{P}_λ on $W_\psi^{1,p}(\Omega)$.*

Proof. Let u and v be two minimizers of the same problem. By Lemma 2.2.4 and the strict convexity of φ outside the unit ball, the functions $(|\nabla u(x)| - 1)_+$ and $(|\nabla v(x)| - 1)_+$ are equal almost everywhere and have a continuous representative. Hence, they have the same continuous representative. Thus U is uniquely defined. □

Remark 2.5.4. Since φ is differentiable on $\mathbb{R}^N \setminus \{0\}$, $\sigma(x) = \nabla\varphi(\nabla u(x))$ a.e. on $[\nabla u \neq 0]$. In particular, if $V := \Omega \setminus \bar{U}$,

$$\sigma(x) = \nabla\varphi(\nabla u(x)) \text{ on } U, \quad (2.5.2)$$

$$\sigma(x) = \frac{\nabla u(x)}{|\nabla u(x)|} \text{ a.e. } x \in [\nabla u \neq 0] \cap V, \quad (2.5.3)$$

$$|\sigma(x)| \leq 1 \text{ a.e. on } [\nabla u = 0]. \quad (2.5.4)$$

Since $\operatorname{div}\sigma = -\lambda$, one also has

$$\int_{\Omega} \lambda v \leq \int_{[\nabla u=0]} |\nabla v| + \int_{[\nabla u \neq 0]} \langle \nabla\varphi(\nabla u), \nabla v \rangle, \quad \forall v \in W_0^{1,p}(\Omega). \quad (2.5.5)$$

Proposition 2.5.5. *On $W_0^{1,p}(\Omega)$, 0 is a solution of \mathcal{P}_λ if and only if $U = \emptyset$.*

Proof. If 0 is a solution then by lemma 2.2.4 for any other solution u we have $|\nabla u| \leq 1$ thus $U = \emptyset$. If $U = \emptyset$ and u is a solution we have

$$\int_{\Omega} \varphi(\nabla u) - \lambda u dx = \int_{\Omega} \varphi(\nabla u) - \langle \sigma, \nabla u \rangle dx = 0.$$

The last equality relies on the equation (2.5.3). Then 0 is also a solution. \square

2.5.2 Counter-examples

In this subsection, we restrict our attention to the case $p = 2$ with φ as in (2.1.1) to simplify the presentation. When λ is no longer a constant it is possible to have more than one solution. We state the following reciprocal result of Lemma 2.5.1 in the case $p = 2$ to use the Euler equation to find some examples.

Proposition 2.5.6. *Let u be in $W^{1,2}(\Omega)$. If there exist $\sigma \in L^2(\Omega; \mathbb{R}^N)$ and $\lambda \in L^\infty(\Omega)$ such that $\operatorname{div} \sigma = -\lambda$ and*

$$\sigma \in \partial\varphi(\nabla u) \quad \text{a.e.}$$

then u is a minimizer of \mathcal{P}_λ on $W_u^{1,2}(\Omega)$.

Proof. Since $\sigma \in \partial\varphi(\nabla u)$, for every $v \in W_u^{1,2}(\Omega)$ we have:

$$\int_{\Omega} \varphi(\nabla v) \geq \int_{\Omega} \varphi(\nabla u) + \langle \sigma, \nabla v - \nabla u \rangle.$$

Since $\operatorname{div} \sigma = -\lambda$ this gives:

$$\int_{\Omega} \langle \sigma, \nabla v - \nabla u \rangle = \int_{\Omega} \lambda(v - u) + \int_{\partial^* \Omega} \langle \sigma, \nu_{\Omega} \rangle (v - u) \mathcal{H}^{N-1} = \int_{\Omega} \lambda(v - u).$$

Hence, $\mathcal{I}_\lambda(v) \geq \mathcal{I}_\lambda(u)$ for every $v \in W_u^{1,2}(\Omega)$. Thus, u is a minimizer of \mathcal{P}_λ on $W_u^{1,2}(\Omega)$. \square

A first example of non uniqueness arises when $\Omega = (-1, 1)$.

Proposition 2.5.7. *There exists λ a nonnegative Lipschitz continuous function on $[-1, 1]$ such that \mathcal{P}_λ has more than one solution on $W_0^{1,2}((-1, 1))$.*

Proof. Let us consider the following functions:

$$u(x) = \begin{cases} \frac{1}{2} & \text{if } |x| < \frac{1}{2}, \\ 1 - |x| & \text{if } |x| \geq \frac{1}{2}, \end{cases}$$

$$\lambda(x) = \begin{cases} 8(\frac{1}{2} - |x|) & \text{if } |x| < \frac{1}{2}, \\ 0 & \text{if } |x| \geq \frac{1}{2}. \end{cases}$$

$$\sigma(x) = \begin{cases} 1 - \int_{-\frac{1}{2}}^x \lambda & \text{if } |x| < \frac{1}{2}, \\ -\frac{x}{|x|} & \text{if } |x| \geq \frac{1}{2}, \end{cases}$$

Direct computations show that $\sigma \in \partial\varphi(u')$, $\lambda = -\sigma'$ and

$$\int_{-1}^1 \varphi(u') - \lambda u = 2 \int_0^1 \varphi(u') - \lambda u = 1 - \int_0^{\frac{1}{2}} \lambda = 0. \quad (2.5.6)$$

Then u and 0 are solutions of \mathcal{P}_λ on $W_0^{1,2}((-1,1))$ thanks to Proposition 2.5.6. \square

Remark 2.5.8. In dimension one, if $\lambda > 0$ then we have uniqueness. Indeed, since $\sigma' = -\lambda$ there is at most two points where $|\sigma| = 1$. If u and v are two solutions, by definition of σ , $u'(x) \in (0, 1]$ for at most two points $x \in \Omega$ and $v'(y) \in (0, 1]$ for at most two points $y \in \Omega$. Hence, for a.e. $x \in \Omega$ we either have $u'(x) = v'(x) = 0$ or $u'(x) = v'(x)$ with $x \in U$. Then $u \equiv v$.

In higher dimension, even with $\lambda > 0$ we can have two solutions, for example if we consider $\Omega = B_1(0)$ the unit ball in dimension two:

Proposition 2.5.9. *There exists $\lambda \in \mathcal{C}^1(\overline{B_1(0)})$, $\lambda > 0$ such that the solutions of \mathcal{P}_λ on $W_0^{1,2}(B_1(0))$ are not unique.*

Proof. Let us consider the following function:

$$u(x) = \begin{cases} \frac{1}{2} & \text{if } |x| < \frac{1}{2}, \\ 1 - |x| & \text{if } |x| \geq \frac{1}{2}. \end{cases}$$

By definition of u , $u(x) = f(|x|)$ with

$$f(r) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq r < \frac{1}{2}, \\ 1 - r & \text{if } r \geq \frac{1}{2}. \end{cases}$$

Our goal is to find $\sigma \in \mathcal{C}^0(\overline{B_1(0)})$ and $\lambda \in \mathcal{C}^1(\overline{B_1(0)})$ strictly positive and radial such that $\sigma \in \partial\varphi(\nabla u)$, $\lambda = -\operatorname{div} \sigma$ and

$$\int_{B_1(0)} \varphi(\nabla u) - \lambda u = 0. \quad (2.5.7)$$

In this case, u and 0 are solutions of \mathcal{P}_λ on $W_0^{1,2}(B_1(0))$. By the definition of u and the Euler-Lagrange equation, when $|x| \geq \frac{1}{2}$ we have necessarily that $\sigma = -\frac{x}{|x|}$ and $\lambda = \frac{1}{|x|}$. It remains to find σ and λ on $B_{\frac{1}{2}}(0)$.

Since we want 0 to be a solution, in polar coordinates we must have that

$$0 = \int_0^1 (|f'(r)| - \lambda(r)f(r))rdr = \int_0^{\frac{1}{2}} -\lambda(r)f(r)rdr - \int_{\frac{1}{2}}^1 rf'(r) + f(r)dr$$

$$= - \int_0^1 f'(r)rdr + \int_0^{\frac{1}{2}} -\lambda(r)f(r)rdr - \int_{\frac{1}{2}}^1 f(r)dr.$$

With an integration by parts we have

$$0 = \int_0^{\frac{1}{2}} (1 - r\lambda(r))f(r)dr = \frac{1}{2} \int_0^{\frac{1}{2}} (1 - r\lambda(r))dr.$$

Then

$$\int_0^{\frac{1}{2}} \lambda(r)rdr = \frac{1}{2}. \quad (2.5.8)$$

Conversely, if λ satisfies the equation (2.5.8) then (2.5.7) holds true. We are now looking for a σ such that $|\sigma| \leq 1$ on $B(0, \frac{1}{2})$ and $\lambda = -\operatorname{div} \sigma$ satisfies (2.5.8). In polar coordinates we have $-\lambda = \operatorname{div} \sigma = \frac{1}{r}\partial_r(r\sigma_r) + \frac{1}{r}\partial_\theta\sigma_\theta$. Since $\sigma_r = -1$ and $\sigma_\theta = 0$ when $|x| \geq \frac{1}{2}$, we set $\sigma_\theta \equiv 0$ on $B_1(0)$. Thus $-\lambda = \frac{1}{r}\partial_r(r\sigma_r)$ and (2.5.8) is verified for all smooth σ_r , $|\sigma_r| \leq 1$ such that $\sigma_r(\frac{1}{2}) = -1$ and $\sigma_r'(\frac{1}{2}) = 0$. Moreover λ is Lipschitz on $B_1(0)$. Since λ has to be bounded and positive, we require that σ_r is negative, strictly decreasing and growth at 0 in such a way that $\lim_{r \rightarrow 0} \frac{\sigma_r}{r}$ is finite. Since we want λ to be \mathcal{C}^1 , we have to take σ_r such that $\lim_{r \rightarrow \frac{1}{2}^+} \sigma_r''(r) = 0$ and $\sigma_r = -r$

around 0. In this case $\nabla\lambda(0) = 0$ and $\nabla\lambda(x) = -\frac{x}{|x|^3}$ for every x such that $|x| = \frac{1}{2}$.

Since $|\sigma| \leq 1$ when $|x| \leq \frac{1}{2}$ we have $\sigma \in \partial\varphi(\nabla u)$. Since σ is continuous we have $\operatorname{div} \sigma = -\lambda$. Thus 0 and u are two solutions of \mathcal{P}_λ on $W_0^{1,2}(B_1(0))$. \square

In the previous example $U = \emptyset$ but even when $U \neq \emptyset$ we can have two solutions.

Proposition 2.5.10. *There exists $\lambda \in \mathcal{C}^0(\overline{\Omega})$, $\lambda > 0$ such that the solutions of \mathcal{P}_λ on $W_0^{1,2}(\Omega)$ are not unique and $U \neq \emptyset$.*

Proof. Let us consider the following functions:

$$u(x) = \begin{cases} \frac{5}{4} + \frac{1}{2} \ln(2) & \text{if } |x| < \frac{1}{2}, \\ \frac{7}{4} + \frac{1}{2} \ln(2) - |x| & \text{if } \frac{1}{2} \leq |x| \leq 1, \\ 1 + \frac{1}{2} \ln(2) - \frac{1}{2} \ln(|x|) - \frac{|x|^2}{4} & \text{if } |x| > 1 \end{cases}$$

and

$$v(x) = \begin{cases} \frac{3}{4} + \frac{1}{2} \ln(2) & \text{if } |x| \leq 1, \\ 1 + \frac{1}{2} \ln(2) - \frac{1}{2} \ln(|x|) - \frac{|x|^2}{4} & \text{if } |x| > 1, \end{cases}$$

defined on $\Omega := B_2(0)$.

Here, the set U is $\{x \in B_2(0) / |x| > 1\}$ and we fix $\sigma = \nabla u = \nabla v$ on U . Hence, we define $\lambda(x) = -\operatorname{div} \sigma(x) = 1$ on this set. For $|x| \leq 1$, u and v are up to a constant the same functions as in the previous proof. Thus, since there is no continuity problem on $\partial B_1(0)$, we can take σ and λ as in the previous proof on $B_1(0)$.

Hence, u and v are solutions of the same problem \mathcal{P}_λ on $W_0^{1,2}(B_2(0))$ with $\lambda(x) = \frac{1}{|x|}$ on $B_1(0) \setminus B_{\frac{1}{2}}(0)$, $\lambda(x) = 1$ on $B_2(0) \setminus B_1(0)$ and λ defined as in the previous proposition on $B_{\frac{1}{2}}(0)$. \square

Definition 2.5.11. We say that a solution u of \mathcal{P}_λ is special when $|\nabla u| \in \{0\} \cup (1, +\infty)$ a.e. in Ω .

Remark 2.5.12. Before proving the uniqueness when λ is constant in [8], it was proved in [2] that if there exists a special solution then it is the unique solution. In the last proof v is a special solution, then when λ is not constant the existence of a special solution does not guarantee the uniqueness.

Proposition 2.5.13. *It is possible to have two solutions even if each level set intersects \bar{U} .*

Proof. For $N = 2$ and $\Omega := B_1(0)$, we introduce

$$A_n = \{x \in B_1(0), |\sin(n)| - \epsilon^{n+1} < |x| < |\sin(n)| + \epsilon^{n+1}\} \text{ for } n \in \mathbb{N} \text{ and} \\ A := \bigcup_{n \in \mathbb{N}} A_n.$$

The set A is dense in $B_1(0)$ but is not equal to $B_1(0)$ for ϵ small enough. We call $U_0 := B_{r_0}(0)$ the connected component of A that contains 0 and $U := A \setminus U_0$. By definition of A and its density, we have $|\partial U| > 0$. Since U is an open set we have $U = \bigcup_{i \in \mathbb{N}^*} U_i$ with $U_i := \{x \in \Omega, r_i < |x| < R_i\}$.

We construct two radial functions $u, v \in W_0^{1,2}(B_1(0))$, regular on U . We introduce $\tilde{u}, \tilde{v} : [0, 1] \rightarrow \mathbb{R}$ such that $u(x) = \tilde{u}(|x|)$ and $v(x) = \tilde{v}(|x|)$. We start by defining \tilde{u}' and \tilde{v}' :

- If $0 \leq r < r_0$ then $\tilde{u}'(r) = \tilde{v}'(r) = 0$.
- If there exists $i \in \mathbb{N}^*$ such that $r_i < r < R_i$ then $\tilde{u}'(r) = \tilde{v}'(r) = f_i(r) := \gamma(R_i - r_i)^2 f\left(\frac{r-r_i}{R_i-r_i}\right) - 1$ with $f(t) = -t^2(t-1)^2$ and $\gamma > 0$ independent of $i \in \mathbb{N}^*$ small enough such that $-\frac{f_i(r)}{r} - f_i'(r) > 0$.
- If $r \geq r_0$ and $r \notin (r_i, R_i)$ for every $i \in \mathbb{N}^*$ then $\tilde{u}'(r) = -1$ and $\tilde{v}'(r) = 0$.

Since \tilde{u}' and \tilde{v}' are bounded we can define \tilde{u} and \tilde{v} on $[0, 1]$ such that $\tilde{u}(1) = \tilde{v}(1) = 0$. It remains to find λ and σ such that u and v are minimizers of \mathcal{I}_λ on $W_0^{1,2}(B_1(0))$:

- u and v are constant on the disk U_0 . Hence, we define λ and σ on U_0 as in the previous counterexamples by changing $\frac{1}{2}$ by r_0 and adjusting the constants.
- On every U_i , u satisfies $\partial_r u < -1$. We set $\lambda = -\Delta u = -\frac{\partial_r u}{r} - \partial_r^2 u$. Hence, $\lambda(x) = \tilde{\lambda}(|x|)$ with $\tilde{\lambda}(r) = -\frac{\tilde{u}'(r)}{r} - \tilde{u}''(r) = -\frac{f_i(r)}{r} - f_i'(r) > 0$ for $r_i < r < R_i$. On U_i , we set $\sigma(x) = \nabla u(x) = f_i(|x|)\frac{x}{|x|}$. Hence $\operatorname{div} \sigma = -\lambda$ on U .
- As in the previous example, $\nabla v = 0$ and $\nabla u = -\frac{x}{|x|}$ on $\Omega \setminus (U_0 \cup U)$. We set $\lambda = \frac{1}{|x|}$ and $\sigma = -\frac{x}{|x|}$ on $\Omega \setminus (U_0 \cup U)$.

By definition of f , λ is continuous, strictly positive on Ω and σ is continuous on Ω . It remains to check that $\operatorname{div} \sigma = -\lambda$ on $\Omega \setminus (U_0 \cup U)$.

For every $i \in \mathbb{N}^*$, we have $|\tilde{\sigma}'(r)| = |f_i'(r)| \leq 4\gamma$ on (r_i, R_i) and $\int_{r_i}^{R_i} \tilde{\sigma}'(r) dr = 0$.

Moreover, $\tilde{\sigma} = -1$ on $[r_0, 1] \setminus \left\{ \bigcup_{i \in \mathbb{N}^*} (r_i, R_i) \right\}$. Hence, by considering the different cases we have that

$$\tilde{\sigma}(t) - \tilde{\sigma}(s) = \int_s^t \tilde{\sigma}'(r) \chi_{\bigcup_{i \in \mathbb{N}^*} (r_i, R_i)} dr$$

for every $t, s \in [r_0, 1]$. Hence, we have that $\tilde{\sigma}$ is Lipschitz continuous and $\tilde{\sigma}' = \tilde{\sigma}' \chi_{\cup_{i \in \mathbb{N}^*} (r_i, R_i)}$ a.e. on $[r_0, 1]$. Since σ is radial we have $\operatorname{div} \sigma(x) = \tilde{\sigma}'(|x|) + \frac{1}{|x|} \tilde{\sigma}(|x|)$. Thus, by definition of λ we obtain that $\operatorname{div} \sigma = -\lambda$.

Hence, u and v are two different solutions of the same problem \mathcal{P}_λ and $\forall s \in \mathbb{R}$ such that $\partial E_s \neq \emptyset$ we have $\partial E_s \cap \bar{U} \neq \emptyset$. \square

2.6 Level sets

In this section we consider only Lipschitz continuous minimizers. We adapt the definition of the perimeter to our problem in order to show that the level-sets of a Lipschitz continuous solution are generically \mathcal{C}^1 outside \bar{U} .

2.6.1 Pseudo-perimeter and the Pseudo-Cheeger problem

In this section, we show that super-level sets of a Lipschitz continuous solution u are solutions of a weighted Cheeger problem. To do so we use the function $\max(1, |\nabla \varphi(\nabla u)|)$ where u is a Lipschitz minimizer of \mathcal{P}_λ . By Proposition 2.5.2, the function $\max(1, |\nabla \varphi(\nabla u)|)$ is continuous on Ω . Moreover, it does not depend on the choice of the minimizer by Lemma 2.2.4. Hence, $\max(1, |\nabla \varphi(\nabla u)|)$ is intrinsic to the problem \mathcal{P}_λ and we can introduce the following quantity:

Definition 2.6.1. For every Caccioppoli set $F \subset \Omega$ and every measurable open set $V \subset \Omega$ we introduce:

$$\widetilde{Per}(F, V) = \int_{\partial^* F \cap V} \max(1, |\nabla \varphi(\nabla u)|) d\mathcal{H}^{N-1}$$

where u is a Lipschitz continuous minimizer of \mathcal{P}_λ . We call $\widetilde{Per}(F, V)$ the *pseudo-perimeter* of F in V .

In the following proposition, we use the notation ν_F which is defined in Definition 2.3.5.

Proposition 2.6.2. *Let $V \Subset \Omega$ be an open set and F a Caccioppoli set such that $\widetilde{Per}(F, V) < \infty$. Then we have*

$$\widetilde{Per}(F, V) = \sup_{g \in \mathcal{C}_0^1(V; \mathbb{R}^N), |g| \leq 1} \int_{\partial^* F \cap V} \max(1, |\nabla \varphi(\nabla u)|) \langle g, \nu_F \rangle d\mathcal{H}^{N-1}.$$

Proof. We have

$$\widetilde{Per}(F, V) \geq \sup_{g \in \mathcal{C}_0^1(V; \mathbb{R}^N), |g| \leq 1} \int_{\partial^* F \cap V} \max(1, |\nabla \varphi(\nabla u)|) \langle g, \nu_F \rangle d\mathcal{H}^{N-1}.$$

By [18, Section 2, Lemma 1] there exists a sequence $(g_k)_{k \in \mathbb{N}} \in \mathcal{C}_0^1(V; \mathbb{R}^N)$, with $|g_k| \leq 1$ such that

$$\langle g_k, \nu_F \rangle \rightarrow 1 \text{ in } \mathcal{L}^1_{\mathcal{H}^{N-1}}(\partial^* F \cap V).$$

Since $\max(1, |\nabla\varphi(\nabla u)|)$ is continuous by Lemma 2.5.2 and $\widetilde{Per}(F, V) < +\infty$, by the dominated convergence theorem we have

$$\int_{\partial^* F \cap V} \max(1, |\nabla\varphi(\nabla u)|) \langle g_k, \nu_F \rangle d\mathcal{H}^{N-1} \rightarrow \int_{\partial^* F \cap V} \max(1, |\nabla\varphi(\nabla u)|) d\mathcal{H}^{N-1} = \widetilde{Per}(F, V)$$

when $k \rightarrow +\infty$. Thus,

$$\widetilde{Per}(F, V) \leq \sup_{g \in \mathcal{C}_0^1(V; \mathbb{R}^N), |g| \leq 1} \int_{\partial^* F \cap V} \max(1, |\nabla\varphi(\nabla u)|) \langle g, \nu_F \rangle d\mathcal{H}^{N-1}.$$

□

In the following proposition we show that the *pseudo-perimeter* is lower semi-continuous. The proof is inspired by [18, Section 2, Lemma 2].

Proposition 2.6.3. *Let $V \Subset \Omega$ be an open set, $(F_k)_{k \in \mathbb{N}}$ a sequence of Caccioppoli sets and F a Caccioppoli set such that that $\chi_{F_k} \rightarrow \chi_F$ when $k \rightarrow \infty$ in $\mathcal{L}^1(\Omega)$. We have*

$$\widetilde{Per}(F, V) \leq \liminf_{k \rightarrow \infty} \widetilde{Per}(F_k, V).$$

Proof. Let us consider $f \in \mathcal{C}^1(V, \mathbb{R}_+)$, $\psi \in \mathcal{C}_0^1(V; \mathbb{R}^N)$, $|g| \leq 1$. Since

$$\int_{\partial^* F_k \cap V} f \langle \psi, \nu_{F_k} \rangle d\mathcal{H}^{N-1} = - \int_V \chi_{F_k} \operatorname{div}(f\psi) dx$$

and $\chi_{F_k} \rightarrow \chi_F$ when $k \rightarrow \infty$ in $\mathcal{L}^1(\Omega)$, the dominated convergence theorem gives that

$$\int_{\partial^* F \cap V} f \langle \psi, \nu_F \rangle d\mathcal{H}^{N-1} = \lim_{k \rightarrow \infty} \int_{\partial^* F_k \cap V} f \langle \psi, \nu_{F_k} \rangle d\mathcal{H}^{N-1}.$$

Thus

$$\int_{\partial^* F \cap V} f \langle \psi, \nu_F \rangle d\mathcal{H}^{N-1} \leq \liminf_{k \rightarrow \infty} \sup_{g \in \mathcal{C}_0^1(V; \mathbb{R}^N), |g| \leq 1} \int_{\partial^* F_k \cap V} f \langle g, \nu_{F_k} \rangle d\mathcal{H}^{N-1}.$$

This inequality is true for every $\psi \in \mathcal{C}_0^1(V; \mathbb{R}^N)$, $|\psi| \leq 1$, then

$$\sup_{\psi \in \mathcal{C}_0^1(V; \mathbb{R}^N), |\psi| \leq 1} \int_{\partial^* F \cap V} f \langle \psi, \nu_F \rangle d\mathcal{H}^{N-1} \leq \liminf_{k \rightarrow \infty} \sup_{g \in \mathcal{C}_0^1(V; \mathbb{R}^N), |g| \leq 1} \int_{\partial^* F_k \cap V} f \langle g, \nu_{F_k} \rangle d\mathcal{H}^{N-1}.$$

Let us introduce

$$H_f(F) := \sup_{g \in \mathcal{C}_0^1(V; \mathbb{R}^N), |g| \leq 1} \int_{\partial^* F \cap V} f \langle g, \nu_F \rangle d\mathcal{H}^{N-1}.$$

Then H_f is lower semi-continuous for every $f \in \mathcal{C}^1(V, \mathbb{R}_+)$. Since $\max(1, |\nabla\varphi(\nabla u)|)$ is continuous and nonnegative, there exists a sequence of nonnegative \mathcal{C}^1 functions $(f_n)_{n \in \mathbb{N}}$ converging uniformly to $\max(1, |\nabla\varphi(\nabla u)|)$ on \overline{V} such that for every $n \in \mathbb{N}$,

$f_n \leq \max(1, |\nabla\varphi(\nabla u)|)$. The proof of Proposition 2.6.2 applied to $(f_n)_{n \in \mathbb{N}}$ implies that

$$H_{\max(1, |\nabla\varphi(\nabla u)|)}(E) = \widetilde{Per}(E, V) \geq \sup_{n \in \mathbb{N}} \int_{\partial^* E \cap V} f_n d\mathcal{H}^{N-1} = \sup_{n \in \mathbb{N}} H_{f_n}(E)$$

for every Caccioppoli set E . Hence, $H_{\max(1, |\nabla\varphi(\nabla u)|)}(E) \geq \sup_{n \in \mathbb{N}} H_{f_n}(E)$ for every Caccioppoli set E .

Let us consider $(g_n)_{n \in \mathbb{N}}$ a maximizing sequence for $H_{\max(1, |\nabla\varphi(\nabla u)|)}(E)$. Since $(f_n)_{n \in \mathbb{N}}$ converges uniformly to $\max(1, |\nabla\varphi(\nabla u)|)$ we have

$$\lim_{n \rightarrow \infty} \int_{\partial^* E \cap V} f_n(g_n, \nu_E) d\mathcal{H}^{N-1} = H_{\max(1, |\nabla\varphi(\nabla u)|)}(E). \quad (2.6.1)$$

Hence, $H_{\max(1, |\nabla\varphi(\nabla u)|)}(E) \leq \sup_{n \in \mathbb{N}} H_{f_n}(E)$ for every Caccioppoli set E . We proved that

$$H_{\max(1, |\nabla\varphi(\nabla u)|)} = \sup_{n \in \mathbb{N}} H_{f_n}.$$

Hence, $H_{\max(1, |\nabla\varphi(\nabla u)|)}$ is lower semi-continuous as the supremum of lower semi-continuous functions. Thus, by Proposition 2.6.2 we have

$$\widetilde{Per}(F, V) \leq \liminf_{k \rightarrow \infty} \widetilde{Per}(F_k, V).$$

□

We generalize to Ω , [8, Proposition 2.7] that states that the super-level sets satisfy a minimization problem:

Theorem 2.6.4. *Let u be a Lipschitz continuous solution of \mathcal{P}_λ . Then for a.e. $s \in \mathbb{R}$, for every set $F \subset \Omega$ with finite perimeter in Ω such that $F \Delta E_s \Subset \Omega$,*

$$\widetilde{Per}(E_s, \Omega) - \int_{E_s} \lambda dx \leq \widetilde{Per}(F, \Omega) - \int_F \lambda dx.$$

Proof. We divide the proof into three steps.

Step 1 The first is identical to [8, Proposition 2.7, **Step 1**] with an approximation of σ by smooth functions.

We extend σ by 0 outside Ω and we define $\sigma_n := \sigma * \rho_n$, where $(\rho_n)_{n \geq 1} \subset \mathcal{C}_0^\infty(B_{1/n})$ is a sequence of mollifiers. Then, up to a sub-sequence, σ_n converges to σ a.e. on Ω and $\forall K \Subset \Omega$ compact, $\forall n \geq \frac{1}{\text{dist}(K, \partial\Omega)}$,

$$\text{div } \sigma_n = (\text{div } \sigma) * \rho_n = -\lambda * \rho_n =: -\lambda_n \quad \text{a.e. on } K.$$

Now we prove that there exists a sub-sequence that we do not relabel, $(\sigma_n)_{n \geq 1}$ such that for a.e. $s \in \mathbb{R}$, $\forall K \Subset \Omega$,

$$\lim_{n \rightarrow +\infty} \int_{K \cap \partial^* E_s} |\sigma_n - \sigma| = 0.$$

If $K \Subset \Omega$, then by the co-area formula (2.3.5), for every $n \geq 1$,

$$\int_{\mathbb{R}} \int_{K \cap \partial^* E_s} |\sigma_n - \sigma| d\mathcal{H}^{N-1} ds = \int_K |\nabla u| |\sigma_n - \sigma|. \quad (2.6.2)$$

The integrand in the right hand side is bounded from above by

$$\|\nabla u\|_{L^\infty(\Omega)} (\|\sigma_n\|_{L^\infty(\mathbb{R}^N)} + \|\sigma\|_{L^\infty(\mathbb{R}^N)}) \leq 2\|\nabla u\|_{L^\infty(\Omega)} \|\sigma\|_{L^\infty(\Omega)}.$$

Since $(\sigma_n)_{n \in \mathbb{N}}$ converges a.e. to σ on Ω , we can apply the dominated convergence theorem to obtain that

$$\lim_{n \rightarrow +\infty} \int_K |\nabla u| |\sigma_n - \sigma| = 0.$$

By equation (2.6.2), there exists a sub-sequence that we do not relabel, $(\sigma_n)_{n \in \mathbb{N}^*}$ such that for a.e. $s \in \mathbb{R}$:

$$\lim_{n \rightarrow +\infty} \int_{K \cap \partial^* E_s} |\sigma_n - \sigma| d\mathcal{H}^{N-1} = 0. \quad (2.6.3)$$

Let $(K_m)_{m \geq 1}$ be an increasing sequence of compact subsets of Ω such that $\bigcup_{m \geq 1} \text{int } K_m$ is equal to Ω . We apply the above reasoning to each K_m . Hence, with a diagonal process, we can extract a sub-sequence $(\sigma_n)_{n \geq 1}$ such that for a.e. $s \in \mathbb{R}$, (2.6.3) is valid for every K_m . Since every compact subset $K \Subset \Omega$ is inside K_m for m large enough, we have that (2.6.3) is valid for every compact subsets of Ω .

Step 2 For every $F \subset \Omega$ as in the statement of the proposition, for a.e. $s \in \mathbb{R}$ and for every $\theta \in \mathcal{C}_0^\infty(\Omega)$ such that $\theta \equiv 1$ on $F \Delta E_s$ and $0 \leq \theta \leq 1$, we claim that:

$$\int_{\mathbb{R}^N} \lambda(\chi_F - \chi_{E_s}) \theta \leq \int_{\partial^* F} \theta \max(1, |\nabla \varphi(\nabla u)|) d\mathcal{H}^{N-1} - \int_{\partial^* E_s} \theta \max(1, |\nabla \varphi(\nabla u)|) d\mathcal{H}^{N-1}. \quad (2.6.4)$$

We prove this inequality in the remaining part of **Step 2**. By the co-area formula and Remark 2.5.4, we have that $\nabla u(x) \neq 0$ and $\langle \sigma(x), \frac{\nabla u(x)}{|\nabla u(x)|} \rangle = \max(1, |\nabla \varphi(\nabla u)|)$ for a.e. $s \in \mathbb{R}$ and for \mathcal{H}^{N-1} a.e. $x \in \partial^* E_s$. We fix any s for which this property as well as (2.3.6) and (2.6.3) hold true.

Since $\nabla \theta = 0$ on $E_s \Delta F$, we get the following equality:

$$- \int_{\mathbb{R}^N} (\chi_F - \chi_{E_s}) \theta \text{div } \sigma_n = - \int_{\mathbb{R}^N} (\chi_F - \chi_{E_s}) \text{div } (\theta \sigma_n). \quad (2.6.5)$$

But $\theta \sigma_n \in \mathcal{C}_0^\infty(\Omega)$ and for every $n > \frac{1}{\text{dist}(\text{supp } \theta, \partial \Omega)}$,

$$|\theta \sigma_n| \leq \theta (|\sigma| * \rho_n) \leq \theta (\max(1, |\nabla \varphi(\nabla u)|) * \rho_n).$$

Hence,

$$\left| \int_{\mathbb{R}^N} \chi_F \text{div } (\theta \sigma_n) \right| = \left| \int_{\partial^* F} \theta \langle \sigma_n, \nu_F \rangle d\mathcal{H}^{N-1} \right| \leq \int_{\partial^* F \cap \Omega} \theta \max(1, |\nabla \varphi(\nabla u)|) * \rho_n d\mathcal{H}^{N-1}. \quad (2.6.6)$$

Since $\max(1, |\nabla\varphi(\nabla u)|)$ is continuous by Lemma 2.5.2, when $n \rightarrow \infty$, we have $\max(1, |\nabla\varphi(\nabla u)|) * \rho_n \rightarrow \max(1, |\nabla\varphi(\nabla u)|)$ uniformly on $\text{supp } \theta$. Thus, by the dominated convergence theorem we have:

$$\limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}^N} \chi_F \text{div}(\theta \sigma_n) \right| \leq \int_{\partial^* F} \theta \max(1, |\nabla\varphi(\nabla u)|) d\mathcal{H}^{N-1}. \quad (2.6.7)$$

Since we selected $s \in \mathbb{R}$ such that (2.3.6) is satisfied,

$$\int_{\mathbb{R}^N} \chi_{E_s} \text{div}(\theta \sigma_n) = - \int_{\mathbb{R}^N} \theta \left\langle \sigma_n, \frac{D\chi_{E_s}}{|D\chi_{E_s}|} \right\rangle d|D\chi_{E_s}| = - \int_{\mathbb{R}^N} \theta \left\langle \sigma_n, \frac{\nabla u}{|\nabla u|} \right\rangle d|D\chi_{E_s}|.$$

With (2.3.1) we obtain:

$$\int_{\mathbb{R}^N} \chi_{E_s} \text{div}(\theta \sigma_n) = - \int_{\partial^* E_s} \theta \left\langle \sigma_n, \frac{\nabla u}{|\nabla u|} \right\rangle d\mathcal{H}^{N-1}. \quad (2.6.8)$$

Now, since $0 \leq \theta \leq 1$ we get:

$$\left| \int_{\partial^* E_s} \theta \left\langle \sigma_n, \frac{\nabla u}{|\nabla u|} \right\rangle d\mathcal{H}^{N-1} - \int_{\partial^* E_s} \theta \left\langle \sigma, \frac{\nabla u}{|\nabla u|} \right\rangle d\mathcal{H}^{N-1} \right| \leq \int_{\text{supp } \theta \cap \partial^* E_s} |\sigma_n - \sigma| d\mathcal{H}^{N-1}.$$

Thanks to **Step 1**, the right hand side goes to 0 when $n \rightarrow +\infty$. Thus,

$$\lim_{n \rightarrow +\infty} \int_{\partial^* E_s} \theta \left\langle \sigma_n, \frac{\nabla u}{|\nabla u|} \right\rangle d\mathcal{H}^{N-1} = \int_{\partial^* E_s} \theta \left\langle \sigma, \frac{\nabla u}{|\nabla u|} \right\rangle d\mathcal{H}^{N-1}. \quad (2.6.9)$$

But for \mathcal{H}^{N-1} a.e. $x \in \partial^* E_s \cap \Omega$, $\langle \sigma(x), \nabla u(x) / |\nabla u(x)| \rangle = \max(1, |\nabla\varphi(\nabla u)|)$. Hence, it follows from (2.6.8) and (2.6.9) that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \chi_{E_s} \text{div}(\theta \sigma_n) = - \int_{\partial^* E_s} \theta \max(1, |\nabla\varphi(\nabla u)|) d\mathcal{H}^{N-1}. \quad (2.6.10)$$

Since $F \Delta E_s$ is compactly contained in Ω , we have for n large enough, $\text{div } \sigma_n = -\lambda_n \rightarrow -\lambda$ a.e. on $F \Delta E_s$ when $n \rightarrow \infty$. Thus

$$\int_{\mathbb{R}^N} (\chi_F - \chi_{E_s}) \theta \text{div } \sigma_n dx \rightarrow \int_{\mathbb{R}^N} (\chi_F - \chi_{E_s}) \theta \lambda dx. \quad (2.6.11)$$

The claim (2.6.4) is a consequence of (2.6.5), (2.6.6), (2.6.10) and (2.6.11).

Step 3 We conclude as in [8, Proposition 2.7]. We replace the function θ introduced in the previous step by a sequence $(\theta_k)_{k \geq 1}$ such that each θ_k satisfies the same assumptions as θ , and $\theta_k \rightarrow 1$ a.e. on Ω . By letting $k \rightarrow +\infty$ in (2.6.4), we obtain $\int_{\mathbb{R}^N} \lambda(\chi_F - \chi_{E_s})$ can be bounded from above by

$$\liminf_{k \rightarrow +\infty} \left(\int_{\partial^* F} \theta_k \max(1, |\nabla\varphi(\nabla u)|) d\mathcal{H}^{N-1} - \int_{\partial^* E_s} \theta_k \max(1, |\nabla\varphi(\nabla u)|) d\mathcal{H}^{N-1} \right).$$

By the dominated convergence theorem we have

$$\int_{\mathbb{R}^N} \lambda(\chi_F - \chi_{E_s}) \leq \widetilde{\text{Per}}(F, \Omega) - \widetilde{\text{Per}}(E_s, \Omega).$$

Hence, we get

$$\widetilde{\text{Per}}(E_s, \Omega) - \int_{E_s} \lambda dx \leq \widetilde{\text{Per}}(F, \Omega) - \int_F \lambda dx.$$

□

Remark 2.6.5. For $s \in \mathbb{R}$ and F as in the statement of the previous proposition. We consider $V \subset \Omega$ such that $F \Delta E_s \Subset V$. Since F and E_s are identical outside V , we have

$$\widetilde{\text{Per}}(E_s, V) - \int_{E_s \cap V} \lambda dx \leq \widetilde{\text{Per}}(F, V) - \int_{F \cap V} \lambda dx.$$

Proposition 2.6.6. *If $F \Subset \Omega$ is a set of finite perimeter, then*

$$\int_F \lambda dx \leq \widetilde{\text{Per}}(F, \Omega).$$

Moreover, for a.e. $s \in \mathbb{R}$, if $E_s \Subset \Omega$ we have that

$$\int_{E_s} \lambda dx = \widetilde{\text{Per}}(E_s, \Omega).$$

Proof. The first part of the proposition is a consequence of the equation (2.6.7) with $\theta \equiv 1$ on F . We can prove, with the same arguments as in the previous proof especially the equations (2.6.8) and (2.6.10) applied to E_s , the second part of the proposition. □

Remark 2.6.7. The minimization problem satisfied by the super-level sets is a kind of analogue of the problem \mathcal{P}_λ for sets. When $\psi \equiv 0$, we have for a.e. $s > 0$,

$$0 = \widetilde{\text{Per}}(E_s, \Omega) - \int_{E_s} \lambda dx = \inf_{F \Subset \Omega} \widetilde{\text{Per}}(F, \Omega) - \int_F \lambda dx.$$

If we call *pseudo-Cheeger set* a set $F \subset \Omega$ such that

$$\frac{\widetilde{\text{Per}}(F, \Omega)}{\int_F \lambda dx} = \inf_{D \Subset \Omega} \frac{\widetilde{\text{Per}}(D, \Omega)}{\int_D \lambda dx}$$

then for a.e. $s > 0$, E_s is a *pseudo-Cheeger set*.

2.6.2 Regularity of the level sets.

The minimizing property of the super-level sets of a solution allows us to show Proposition 2.1.10. This result states that the super-level sets of a solution are \mathcal{C}^1 , up to \bar{U} and a negligible set:

Proposition 2.6.8. *Given $\lambda \in L^\infty(\Omega)$, $\lambda \geq 0$ a.e. on Ω , let u be a locally Lipschitz continuous solution of \mathcal{P}_λ and U be the open set in Ω defined by $[|\nabla u| > 1]$. Then for a.e. $t \in \mathbb{R}$, there exists an open set W_t in $\Omega \setminus \bar{U}$ such that $W_t \cap \partial^e E_t$ is a \mathcal{C}^1 hypersurface and $\mathcal{H}^s(\Omega \setminus (W_t \cup \bar{U})) = 0$ for every $s > N - 8$ with $W_t \cap \partial^e E_t \subset \partial^* E_t$.*

Proof. By Proposition 2.6.4 and Remark 2.6.5, we have for a.e $t \in \mathbb{R}$, for every $x \in \Omega$, $\rho > 0$ in order that $B_\rho(x) \subset \Omega$ and every Caccioppoli set F such that $F \Delta E_t \Subset B_\rho(x)$:

$$\widetilde{Per}(E_t, B_\rho(x)) - \int_{E_t \cap B_\rho(x)} \lambda dx \leq \widetilde{Per}(F, B_\rho(x)) - \int_{F \cap B_\rho(x)} \lambda dx.$$

But if $B_\rho(x) \subset \Omega \setminus \overline{U}$ then

$$\widetilde{Per}(\cdot, B_\rho(x)) = Per(\cdot, B_\rho(x)).$$

Hence, for a.e $t \in \mathbb{R}$, for every $x \in \Omega \setminus \overline{U}$, $\rho > 0$ such that $B_\rho(x) \subset \Omega \setminus \overline{U}$ and every Caccioppoli set F that satisfies $F \Delta E_t \Subset B_\rho(x)$:

$$Per(E_t, B_\rho(x)) - \int_{E_t \cap B_\rho(x)} \lambda dx \leq Per(F, B_\rho(x)) - \int_{F \cap B_\rho(x)} \lambda dx.$$

Then by [55, Theorems 5.1 and 5.2], there exists an open set W_t in $\Omega \setminus \overline{U}$ such that $\partial^e E_t \cap W_t$ is an $N - 1$ dimensional manifold of class $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha < 1$ and $\mathcal{H}^s(\Omega \setminus (W_t \cup \overline{U})) = 0$ for every $s > N - 8$. Moreover, $W_t \cap \partial^e E_t \subset \partial^* E_t$. \square

2.6.3 Relation between U and super-level sets

The minimising property of the super-level sets allows us to show that the level sets intersect the set $\overline{U} = \{|\nabla u| > 1\}$ for some classes of functions λ .

We consider λ Lipschitz continuous on $\overline{\Omega}$, $\lambda > 0$. We assume that there exist $x_\lambda \in \mathbb{R}^N$ and $l > 0$ such that $\forall x \in \Omega$,

$$\lambda(x) - \|\nabla \lambda\|_{\mathcal{L}^\infty(\Omega)} |x - x_\lambda| \geq l. \quad (2.6.12)$$

Then we have:

Proposition 2.6.9. *If $E \Subset \Omega$ is a Caccioppoli set with $|E| > 0$ and*

$$\int_E \lambda = \widetilde{Per}(E, \Omega)$$

then $\partial E \cap \overline{U} \neq \emptyset$.

Proof. If $\partial E \cap \overline{U} = \emptyset$ then

$$\widetilde{Per}(E, \Omega) = Per(E, \Omega)$$

and for $t \geq 1$ close to 1, we have $\partial^e E^t \cap (\partial\Omega \cup \overline{U}) = \emptyset$ with $E^t = t(E - x_\lambda) + x_\lambda$. Thus we get

$$\int_{E^t} \lambda \leq \widetilde{Per}(E^t, \Omega) = Per(E^t, \Omega) = t^{N-1} Per(E, \Omega) = t^{N-1} \int_E \lambda.$$

Hence, we have

$$\int_E t\lambda(t(x - x_\lambda) + x_\lambda) - \lambda(x) dx \leq 0.$$

By (2.6.12) we get a contradiction when $t \rightarrow 1$. Thus, $\partial E \cap (\partial\Omega \cup \overline{U}) \neq \emptyset$. \square

Remark 2.6.10. If $\|\nabla\lambda\|_{\mathcal{L}^\infty(\Omega)} < K \inf_{\Omega} \lambda$ with $K := \frac{1}{\text{diam}(\Omega)}$ then λ satisfies (2.6.12) with any $x_\lambda \in \Omega$.

2.7 Proof of the main theorems

In this subsection we consider λ with small oscillations. We assume that there exists a constant $\mathcal{L} > 0$ such that

$$\|\nabla u\|_{\mathcal{L}^\infty(\Omega)} \leq \mathcal{L}$$

for every minimizer u of \mathcal{P}_λ . The following lemma shows us that when $\nabla\lambda$ is small then $U \cup \partial\Omega$ is connected.

Lemma 2.7.1. *If λ is Lipschitz continuous on $\bar{\Omega}$, $\min_{\Omega} \lambda > 0$ and $\|\nabla\lambda\|_{\mathcal{L}^\infty(\Omega)} < \frac{1}{G\mathcal{L}} \min_{\Omega} \lambda^2$ with $G = \sup_{x \in (1, \mathcal{L})} g''(x) + (N-1) \frac{g'(x)}{x}$. Then each connected component U_0 of the open set U satisfies*

$$\partial U_0 \cap \partial\Omega \neq \emptyset.$$

The proof of this lemma is inspired from the one of [2, Proposition 7.3] and [8, Lemma 3.3].

Proof. Assume by contradiction that $\bar{U}_0 \subset \Omega$. The function $|\nabla u|_{U_0}$ can be extended as a uniformly continuous function on \bar{U}_0 and $|\nabla u| \equiv 1$ on ∂U_0 , see Lemma 2.5.2. We introduce $\tilde{U}_0 \Subset U_0$ a smooth set. By continuity of ∇u on U , there exists $\delta > 0$ such that $|\nabla u(x)| \geq 1 + 2\delta$ for every $x \in \tilde{U}_0$. We regularize φ and λ by convolution to obtain a sequence $(u_\epsilon)_{\epsilon > 0}$ of smooth solutions on Ω to approximated problems of \mathcal{P}_λ . By equations (100) and (102) in [26], $(|\nabla u_\epsilon| - 1 - \delta)_+ \rightarrow (|\nabla u_0| - 1 - \delta)_+$ uniformly on \tilde{U}_0 with u_0 a minimizer of \mathcal{P}_λ on $W_\psi^{1,p}(\Omega)$. By Proposition 2.2.4 and the fact that $|\nabla u(x)| \geq 1 + 2\delta$ on \tilde{U}_0 we have that $|\nabla u_\epsilon| \rightarrow |\nabla u|$ uniformly on \tilde{U}_0 . If $\|\nabla\lambda\|_{\mathcal{L}^\infty(\Omega)} < \frac{\min_{\Omega} \lambda^2}{G\mathcal{L}}$ then (15.11) in [36] is satisfied for u_ϵ with ϵ small enough. Hence, we can apply [36, Theorem 15.1] to obtain $\sup_{\tilde{U}_0} |\nabla u_\epsilon| = \sup_{\partial\tilde{U}_0} |\nabla u_\epsilon|$. Hence, $\sup_{\tilde{U}_0} |\nabla u| = \sup_{\partial\tilde{U}_0} |\nabla u|$. Since \tilde{U}_0 can be any smooth subset of U_0 , we have $\sup_{U_0} |\nabla u| = \sup_{\partial U_0} |\nabla u| = 1$. Since $|\nabla u| > 1$ on U_0 that is a contradiction and $\partial U_0 \cap \partial\Omega \neq \emptyset$ as desired. \square

Remark 2.7.2. When λ is constant $\nabla\lambda = 0$, every λ strictly positive satisfies the assumptions of the previous lemma. We no longer need the assumption that the minimizers are globally Lipschitz continuous in this case.

The following lemma comes from [8, Lemma 3.8], it is useful to show that generically, other solutions are constant on the boundary of the connected components of E_s where $E_s := [u \geq s]$.

Lemma 2.7.3. *Let V be an open bounded subset of \mathbb{R}^N such that $\mathbb{R}^N \setminus V$ is connected. Let M be a \mathcal{C}^1 orientable hypersurface compactly contained in V . If $\mathcal{H}^{N-1}(M) < \infty$ and $\mathcal{H}^{N-2}(\overline{M} \setminus M) = 0$, then there exists a non-empty connected open set $E \subset V$ such that $\partial E \subset \overline{M}$.*

The following proposition shows that generically the level-sets of two solutions coincide.

Proposition 2.7.4. *Let u be a Lipschitz continuous solution of \mathcal{P}_λ on $W_\psi^{1,p}(\Omega)$ with λ satisfying (2.6.12) and the assumptions of Lemma 2.7.1. For a.e. $s \in \mathbb{R}$, for every connected component M of $\partial^e E_s \cap W_s$ we have $\overline{M} \cap (\partial\Omega \cup \overline{U}) \neq \emptyset$ where W_s is defined in Proposition 2.6.8.*

Proof. Let S be the set of those $x \in \Omega$ such that u is not differentiable at x or $\nabla u(x) = 0$.

By the co-area formula for Lipschitz continuous functions, see [32, Theorem 1, Section 3.4.2] we have

$$0 = \int_S |\nabla u| dx = \int_{\mathbb{R}} \mathcal{H}^{N-1}(u^{-1}(t) \cap S) dt.$$

Hence, for a.e. $t \in \mathbb{R}$, for \mathcal{H}^{N-1} a.e. $x \in u^{-1}(t)$ we have that $\nabla u \neq 0$. By proposition 2.6.8, for a.e. $t \in \mathbb{R}$, there exists an open set W_t such that $\partial^e E_t \cap W_t$ is an orientable \mathcal{C}^1 hypersurface with $\mathcal{H}^s(\Omega \setminus (W_t \cup \overline{U})) = 0$ for every $s > N - 8$. Thus a.e. $t \in \mathbb{R}$ satisfy all these conditions.

For such a t , we consider M a connected component of $\partial^e E_t \cap W_t$. Since M is a \mathcal{C}^1 hypersurface we have that $\nabla v(x)$ is orthogonal to M at x . Thus u and v are constant on M and on \overline{M} .

If we assume that $\overline{M} \cap (\partial\Omega \cup \overline{U}) = \emptyset$, we can show as in [8, Theorem 1.1, page 18] that $\mathcal{H}^{N-2}(\overline{M} \setminus M) = 0$. Then we can apply Lemma 2.7.3 with $V = \Omega$. Thus, there exists a non-empty connected open set $E \subset \Omega$ such that $\partial E \subset \overline{M}$. Since $u \equiv t$ on ∂E by Lemma 2.2.2, $u \geq t$ on E . For every $s > t$, we consider $F_s := [u \geq s] \cap E$. We have that $F_s \subset E$, hence, $F_s \cap \partial\Omega = \emptyset$. If $F_s \cap \overline{U} \neq \emptyset$ then there exists a connected component U_i of U such that $U_i \cap E \neq \emptyset$. By Lemma 2.7.1, $\partial E \cap U_i \neq \emptyset$ which is a contradiction with $\partial E \subset M$ and $\overline{M} \cap (\partial\Omega \cup \overline{U}) = \emptyset$. Thus, $F_s \cap \overline{U} = \emptyset$. By Proposition 2.6.6 applied to F_s with E instead of Ω , we obtain that $\int_{F_s} \lambda = \widetilde{Per}(F_s, E) = \widetilde{Per}(F_s, \Omega)$. By Proposition 2.6.9, $|F_s| = 0$ for a.e. $s > t$. Hence, $u = t$ on E and $\nabla u = 0$ a.e. on E . Then $E \subset u^{-1}(t) \cap S$. Thus, E is a non-empty open set with $|E| \leq |u^{-1}(t) \cap S| = 0$. That is a contradiction. Hence, $\overline{M} \cap (\partial\Omega \cup \overline{U}) \neq \emptyset$. \square

Now, we can prove the two main theorems:

Proof of Theorem 2.1.4 and Theorem 2.1.6. We consider two minimizers u and v . We assume that λ satisfies the assumptions of Lemma 2.7.1 and Remark 2.6.10. Since $\nabla u = \nabla v$ on U we have that $u = v$ on $\partial\Omega \cap \overline{U}$. By Proposition 2.7.4 and

the continuity of u and v that is also the case on $\partial^e E_s \cap W_s$ for a.e. $s \in \mathbb{R}$ with $E_s = [u \geq s]$. By (2.3.3) we have

$$\int_{[u \neq v]} |\nabla u| = \int_{\mathbb{R}} \mathcal{H}^{N-1}(\partial^* E_s \cap [u \neq v]).$$

We have for a.e. $s \in \mathbb{R}$, $\partial^* E_s \cap [u \neq v] \subset \partial^e E_s \setminus (\partial^e E_s \cap (W_s \cup \bar{U}))$. Since $\mathcal{H}^{N-1}(\Omega \setminus (\partial^e E_s \cap (W_s \cup \bar{U}))) = 0$, we obtain that $\mathcal{H}^{N-1}(\partial^* E_s \cap [u \neq v]) = 0$. Hence, $\nabla u = 0$ a.e. on $[u \neq v]$. With the same arguments we can prove that this is also the case for ∇v . Thus, $\nabla(u - v) = 0$ a.e. on Ω . Since $u - v$ is Lipschitz continuous on Ω and $u - v = 0$ on $\partial\Omega$, we have $u = v$ on $\bar{\Omega}$. \square

A uniqueness result for a two dimensional variational problem

This chapter is based on the paper "*A uniqueness result for a two dimensional variational problem*" [50] submitted.

Abstract

We investigate the uniqueness of the solutions for a non-strictly convex problem in the Calculus of Variations of the form $\int \varphi(\nabla v) - \lambda v$. Here, $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a convex function and λ is Lipschitz continuous. We prove the uniqueness when $\nabla \lambda$ is small and give some counterexamples when that is not the case. The proof is based on the global Lipschitz regularity of the minimizers and on the study of their level sets.

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3.1 Introduction

3.1.1 A model case

The motivation of this chapter is to study non strictly convex problems in the Calculus of Variations in dimension two as in the following model case:

$$\tilde{\mathcal{I}}_\lambda : u \rightarrow \int_\Omega F(\nabla u(x)) - \lambda(x)u(x)dx \quad (3.1.1)$$

where Ω is an open bounded set in \mathbb{R}^2 , $\lambda \in L^\infty(\Omega)$ and $F(y) = f(|y|)$ with

$$f(t) = \begin{cases} \frac{1}{2}|t|^2 & \text{if } |t| \leq 1, \\ |t| - \frac{1}{2} & \text{if } 1 < |t| < 2, \\ \frac{1}{4}|t|^2 + \frac{1}{2} & \text{if } 2 \leq |t|. \end{cases} \quad (3.1.2)$$

For this functional, the admissible functions u belong to the Sobolev space $W^{1,2}(\Omega)$ with a prescribed trace $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$ on the boundary $\partial\Omega$ of Ω . Our goal is to prove the uniqueness of solutions to the following minimization problem:

$$\tilde{\mathcal{P}}_\lambda : \min_{u \in W_\psi^{1,2}(\Omega)} \tilde{\mathcal{I}}_\lambda(u).$$

When $\lambda \equiv \lambda_0 \in \mathbb{R}_+$, this problem studied by Kawohl, Stara and Wittum in [41] arises as the convexification a non convex problem of shape optimization in the theory of elasticity. In this example, f is the convexification of the minimum of two parabolas: $t \mapsto \frac{1}{2}|t|^2$ and $t \mapsto \frac{1}{4}|t|^2 + \frac{1}{2}$. Observe in particular that f is affine on the interval $(1, 2)$. Since f is convex but not strictly convex, there is no obvious reason for $\tilde{\mathcal{P}}_\lambda$ to have a unique solution.

In fact, the authors of [41] rely on the assumption that the level sets of one minimizer u are star-shaped. Furthermore, they suppose that the boundary of the set in Ω where $f'(|\nabla u|) = 1$ is piecewise \mathcal{C}^1 . In this paper, we do not require such additional assumptions.

There is no general answer to the question of uniqueness for non strictly convex problems in the Calculus of Variations, especially when λ is not constant. Hence we restrict our attention to the framework (3.1.1), where f can be replaced by more general convex functions provided that they are strictly convex around the origin and at infinity.

3.1.2 Main results

More precisely, let $g : \mathbb{R} \mapsto \mathbb{R}$ be an even \mathcal{C}^1 convex function with $g(t) > g(0) = 0$ for all $t \neq 0$. Moreover, we assume that $g \in \mathcal{C}_{loc}^{1,1}(\mathbb{R} \setminus \{0\})$

We suppose that g has p -growth for $p > 1$, namely, there exist $C_1 > 0$ and $C_2 > 0$ such that:

$$C_1|t|^p \leq g(t) \leq C_2(1 + |t|^p) \text{ for all } t \in \mathbb{R}. \quad (3.1.3)$$

We introduce the following set of strict convexity of g :

$$\mathcal{SC} = \{x \in \mathbb{R}, \forall y \in \mathbb{R} \setminus \{x\}, \forall t \in]0, 1[, g(tx + (1-t)y) < tg(x) + (1-t)g(y)\}.$$

For instance, $\mathcal{SC} = (-\infty, -2) \cup (-1, 1) \cup (2, +\infty)$ when g is equal to the function f in (3.1.2).

We make some *structural assumptions* on \mathcal{SC} :

- The set \mathcal{SC} has finitely many connected components, in particular \mathcal{SC} is open and

$$\mathcal{SC} \cap \mathbb{R}_+ = \bigcup_{n=0}^N \mathcal{SC}_n$$

with $\mathcal{SC}_0 := [0, b_0)$, $\mathcal{SC}_n := (a_n, b_n)$ for every $n \in \mathbb{N}^*$, $n < N$ and $\mathcal{SC}_N := (a_N, +\infty)$. For every $n \in \mathbb{N}$, $n < N$ we introduce $d_n := g'(b_n) = g'(a_{n+1})$.

- We assume that g is \mathcal{C}^2 and $g'' > 0$ on $\mathcal{SC} \setminus \{0\}$ and that g is strongly convex at $+\infty$ in the following sense:

$$\liminf_{t \rightarrow +\infty} \frac{tg''(t)}{g'(t)} > 0.$$

We define $\varphi(\cdot) := g(|\cdot|)$ and for $\lambda \in L^\infty(\Omega)$ we introduce the following functional:

$$\mathcal{I}_\lambda : u \mapsto \int_{\Omega} \varphi(\nabla u(x)) - \lambda(x)u(x) dx$$

on $W_\psi^{1,p}(\Omega)$, where Ω is an open simply connected bounded set of \mathbb{R}^2 with a Lipschitz continuous boundary $\partial\Omega$ and ψ is a Lipschitz-continuous function on $\partial\Omega$. Here, $W_\psi^{1,p}(\Omega)$ is the subset of those functions in $W^{1,p}(\Omega)$ that are equal to ψ on $\partial\Omega$.

We introduce the minimization problem:

$$\mathcal{P}_\lambda : \min_{u \in W_\psi^{1,p}(\Omega)} \mathcal{I}_\lambda(u).$$

The main result of the paper is the following:

Theorem 3.1.1. *Let Ω be a simply connected bounded open set of \mathbb{R}^2 . We assume that Ω has a $\mathcal{C}^{1,1}$ boundary, $\psi \in \mathcal{C}^{1,1}(\mathbb{R}^2)$, λ is Lipschitz continuous on $\overline{\Omega}$, $\min_{x \in \overline{\Omega}} \lambda(x) > 0$.*

0. *There exists a positive constant*

$$C := C(N, |\Omega|, \max_{\overline{\Omega}} \lambda, \min_{\overline{\Omega}} \lambda, \|\psi\|_{\mathcal{C}^{1,1}(\Omega)}, \kappa)$$

where κ is the maximum curvature of Ω such that if $\|\nabla \lambda\|_{L^\infty(\Omega)} \leq C$ then \mathcal{P}_λ admits a unique solution on $W_\psi^{1,p}(\Omega)$.

Remark 3.1.2. When λ is constant, a more general result, true in any dimension, can be found in [52].

Moreover, the boundedness condition on $\nabla \lambda$ is optimal since:

Proposition 3.1.3. *There exists $\lambda \in \mathcal{C}^\infty(\overline{B_1(0)})$ with $\min_{B_1(0)} \lambda > 0$ such that \mathcal{P}_λ has more than one solution on $W_0^{1,p}(B_1(0))$.*

3.1.3 Ideas of the proof

We want to prove the uniqueness of the solution u for the variational problem \mathcal{P}_λ . We know by classical theory (see [27] and [37]) that the problem \mathcal{P}_λ admits at least one minimizer u , this function u is bounded, globally Hölder continuous by [37, Theorem 7.8] and locally Lipschitz continuous by [20, Theorem 1.1].

When $\lambda = 0$, the proof is substantially simplified. In this case, the strategy has been developed by Marcellini in [54] under additional assumptions and the proof itself in a general framework is due to Lussardi and Mascolo in [53]. In those two papers the proof is divided into two parts:

- **Part 1** If u and v are two solutions of the same problem, then v is constant on the level sets of u .
- **Part 2** The level sets of u intersect the boundary $\partial\Omega$ of Ω . Since u and v are equal on $\partial\Omega$ they are equal on Ω .

As observed in Remark 3.1.2, when $\lambda \equiv \lambda_0 \in \mathbb{R}_+$, a shorter proof can be found in [52] but when $\lambda \in W^{1,\infty}(\Omega)$ the proof requires new ideas and turns out to be fairly intricate. **Part 1** remains true but **Part 2** fails to be true. In fact, the term $u \mapsto \int_\Omega \lambda u$ changes the geometry of the level lines: they do not necessarily intersect the boundary $\partial\Omega$ of Ω . It is even possible that only one level set intersects the boundary, see Proposition 3.2.10.

A very important subset of Ω is the following:

Proposition 3.1.4. *There exists an open set U such that for every minimizer u , one has $u \in \mathcal{C}^1(U)$ and for every $x \in U$, $|\nabla u(x)| \in \mathcal{SC} \setminus \{0\}$ while for a.e. $x \notin U$, $|\nabla u(x)| \notin \mathcal{SC} \setminus \{0\}$.*

When $\lambda \neq 0$, the set $U \cup \partial\Omega$ plays the same role as the one played by $\partial\Omega$ in **Part 2** when there is no lower order term. However, the fact that $u = v$ on U is far from being obvious. Nevertheless, if u and v are two minimizers of the same problem, we can easily see that ∇u and ∇v are equal on U and we can even prove that this is also the case on the level sets that intersect U . The aim of the proof is to show that $u = v$ or $\nabla u = \nabla v$ on the level sets of u and v . Hence, for a.e. $x \in \Omega$, the Lipschitz map $w(x) := u(x) - v(x)$ is equal to 0 or $\nabla w(x) = 0$, thus $u - v = w = 0$.

This idea of using $U \cup \partial\Omega$ comes from a paper by Bouchitté and Bousquet [8] but in their case the fact that \mathcal{SC} is of the form $(1, +\infty)$ implies that the boundary of every connected component of U intersects $\partial\Omega$. Since $\nabla u = \nabla v$ on U and $u = v$ on $\partial\Omega$ we readily obtain that $u = v$ on U and this part of the proof is easier. We warn the reader that this paper is not a generalization of [8] because g has no singularity at the origin unlike in [8]. This singularity of g in [8] and in its generalization [49] creates some regularity issues that will not appear here.

For instance in our situation, we exploit the fact that thanks to [29], the function $\max(\alpha, g'(|\nabla u|))$ is in $W_{loc}^{1,2}(\Omega)$ for any $\alpha > 0$. Then we prove that $\max(d_0, g'(|\nabla u|))$ has a representative that is absolutely continuous on almost every level sets.

The other major difference between this paper and [8, 49] is that in these references, $\max(1, g'(|\nabla u|))$ is continuous on Ω which allows to prove their results for

any dimension larger or equal to two. Here, we heavily rely on two results which are only valid in dimension two: a general regularity result for Lipschitz continuous functions, see Theorem 3.2.8 below and the Jordan curve theorem. The latter is the reason why we assume that Ω simply connected: this prevents the existence of holes inside the connected components of the upper level set $E_s := [u > s]$ for $s \in \mathbb{R}$.

When λ is small we can prove that almost every level set intersects a connected component of U where $|\nabla u| < b_0$. This allows us to prove the following theorem:

Theorem 3.1.5. *When $0 \leq \lambda(x) \leq d_0 h_\Omega$ for a.e. $x \in \Omega$, the problem \mathcal{P}_λ admits a unique minimizer.*

Here, $d_0 = g'(b_0)$ and h_Ω is the Cheeger constant of Ω :

Definition 3.1.6. The Cheeger constant of Ω is defined as:

$$h_\Omega = \inf_{D \subset \bar{\Omega}} \frac{\text{Per}(D, \mathbb{R}^2)}{|D|}$$

where

$$\text{Per}(D, \mathbb{R}^2) = \sup \left\{ \int_D \text{div } \theta \mid \theta \in C_c^1(\mathbb{R}^2; \mathbb{R}^2), |\theta(x)| \leq 1, \forall x \in \mathbb{R}^2 \right\}$$

is called the *Perimeter* of the set D . A set $D \subset \bar{\Omega}$ of finite perimeter is said to be a *Cheeger set* if $\text{Per}(D, \mathbb{R}^2) = h_\Omega |D|$.

The proof of the main theorem will be based on an induction argument related to the family $\{d_n, n \in \mathbb{N}, 0 \leq n < N\}$ with the previous theorem as the initialization step.

We study the connected components $l_s(u)$ of $L_s(u) := u^{-1}(s) \subset \mathbb{R}^2$ such that $l_s(u)$ is a closed simple curve. The case $l_s(u) \cap \partial\Omega \neq \emptyset$ is easy because $u - v$ is constant on $l_s(u)$ and $u - v = 0$ on $\partial\Omega$ so that $u - v = 0$ on $l_s(u)$. Hence, we can assume that $l_s(u) \Subset \Omega$ and by the Jordan curve theorem, we can define F_s as the bounded connected component of $\mathbb{R}^2 \setminus l_s(u)$. If $l_s(u) \cap U = \emptyset$ then we use the following proposition:

Proposition 3.1.7. *There exists a representative f_0 of $\max(d_0, g'(|\nabla u|))$ such that for a.e. $s \in \mathbb{R}$, if $l_s(u) \cap U = \emptyset$ then f_0 is equal to a constant $C(l_s(u)) \in \{d_i, 0 \leq i < N\}$ on $l_s(u)$.*

Another important result is a maximum principle proved in section 5 for smooth approximations of our problem \mathcal{P}_λ . We first regularize the problem to obtain a sequence $(u_n)_{n \in \mathbb{N}}$ of smooth minimizers of smooth problems \mathcal{P}_{λ_n} , with $(g_n)_{n \in \mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}}$ smooth approximations of g and λ . In section 4, we use the fact that the sequence $(\nabla u_n)_{n \in \mathbb{N}}$ generates Young measures $(\nu_x)_{x \in \Omega}$ to prove that $g'_n(|\nabla u_n|) \rightarrow g'(|\nabla u|)$ a.e. in Ω when $n \rightarrow +\infty$. For such approximations we have:

Proposition 3.1.8. *For a.e. $s \in \mathbb{R}$, if $l_s(u)$ is a connected component of $L_s(u)$ which is a closed simple curve and such that $l_s(u) \Subset \Omega$ then*

$$\sup_{l_s(u)} \max(d_0, g'_n(|\nabla u_n|)) = \sup_{F_s} g'_n(|\nabla u_n|).$$

3.1.4 Plan of the paper.

In the next section, we recall some classical results and we introduce the notations and notions that are useful throughout the chapter. In Section 3, we study the regularity properties of the level sets of the minimizers. In the subsequent Section 4, we prove that $\max(\alpha, g'(|\nabla u|)) \in W_{loc}^{1,2}(\Omega)$. The maximum principle for $\max(d_0, |\sigma_n|)$ is proved in Section 5. Section 6 is dedicated to the proof of Theorem 3.1.1 and Theorem 3.1.5. In the last section, we state a possible extension to the main theorem.

3.2 Preliminary results

In this section we introduce some known results related to this problem.

3.2.1 Direct methods

We know by the direct method in the calculus of variations (see [27] and [37]) that the problem \mathcal{P}_λ admits at least one minimizer. We recall that every minimizer u is bounded, globally Hölder continuous by [37, Theorem 7.8] and locally Lipschitz continuous by [20, Theorem 1.1].

We begin this subsection by observing that the minimum of a minimizer is attained on the boundary of Ω .

Proposition 3.2.1. *Let u be a minimizer of \mathcal{P}_λ on $W_\psi^{1,p}(\Omega)$ with $\lambda \in L^\infty(\Omega)$ and $\lambda(x) \geq 0$ for a.e. $x \in \Omega$. Then $\min_{\bar{\Omega}} u = \min_{\partial\Omega} \psi$.*

Proof. Since $\min_{\bar{\Omega}} u \leq c := \min_{\partial\Omega} \psi$, we have to prove that $\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} \psi$. We introduce $w := \max(u, c)$. If there exists a point $x \in \Omega$ such that $u(x) < c$, then by continuity of u the set $[u < c]$ has a positive measure. We have $w = u$ and $\nabla w = \nabla u$ on $[u > c]$. Moreover, since $[u < c]$ has positive measure we have:

$$0 = \int_{[u \leq c]} g(|\nabla w|) < \int_{[u \leq c]} g(|\nabla u|) \quad \text{and} \quad - \int_{[u \leq c]} \lambda w = - \int_{[u \leq c]} \lambda c \leq - \int_{[u \leq c]} \lambda u.$$

Hence, $I_\lambda(w) < I_\lambda(u)$ on Ω , which contradicts the fact that u is a minimizer. Thus, $u \geq c$ on Ω . \square

We now prove that the gradients of two minimizers of the same problem are collinear. This property is used in many subsequent proofs.

Lemma 3.2.2. *Let u and v be two minimizers of \mathcal{P}_λ with $\lambda \in L^\infty(\Omega)$. Then $\nabla u(x)$ and $\nabla v(x)$ are collinear and g is affine on the interval $[|\nabla u(x)|, |\nabla v(x)|]$ for a.e. $x \in \Omega$.*

Proof. Since u is a solution of \mathcal{P}_λ ,

$$\mathcal{I}_\lambda(u) \leq \mathcal{I}_\lambda\left(\frac{u+v}{2}\right).$$

By the fact that g is non decreasing and the convexity of g and of the Euclidean norm,

$$\begin{aligned} \mathcal{I}_\lambda\left(\frac{u+v}{2}\right) &= \int_\Omega g\left(\left|\frac{\nabla u + \nabla v}{2}\right|\right) - \lambda \frac{u+v}{2} \\ &\leq \frac{1}{2} \int_\Omega (g(|\nabla u|) - \lambda u) + \frac{1}{2} \int_\Omega (g(|\nabla v|) - \lambda v) \\ &= \frac{1}{2} I_\lambda(u) + \frac{1}{2} I_\lambda(v). \end{aligned}$$

Since v is another solution,

$$I_\lambda(u) = \frac{1}{2} I_\lambda(u) + \frac{1}{2} I_\lambda(v).$$

This implies that

$$\int_\Omega g\left(\left|\frac{\nabla u + \nabla v}{2}\right|\right) = \int_\Omega \frac{1}{2} (g(|\nabla u|) + g(|\nabla v|)).$$

Hence for a.e. $x \in \Omega$,

$$g\left(\left|\frac{\nabla u(x) + \nabla v(x)}{2}\right|\right) = \frac{1}{2} (g(|\nabla u(x)|) + g(|\nabla v(x)|)). \quad (3.2.1)$$

Hence for a.e. $x \in \Omega$, $g \circ |\cdot|$ is affine on the segment $[\nabla u(x), \nabla v(x)]$. In view of the definition of g and the strict convexity of the lower level sets of $|\cdot|$, this means that $\nabla u(x)$ and $\nabla v(x)$ are collinear for a.e. $x \in \Omega$ but also that g is affine on $[|\nabla u(x)|, |\nabla v(x)|]$ for a.e. $x \in \Omega$. □

We use a result of [5] to introduce the following set where ∇u is continuous and $|\nabla u|$ takes its values in $\mathcal{SC} \setminus \{0\}$, which is defined in Section 3.1.2.

Proposition 3.2.3. *When $\lambda \in \mathcal{C}^0(\overline{\Omega})$, there exists an open set U such that $u \in \mathcal{C}^1(U)$ and for every $x \in U$, $|\nabla u(x)| \in \mathcal{SC} \setminus \{0\}$ while for a.e. $x \notin U$, $|\nabla u(x)| \notin \mathcal{SC} \setminus \{0\}$.*

Proof. By [5, Theorem 6.1], for a.e. $x \in \Omega$ such that $|\nabla u(x)| \in \mathcal{SC} \setminus \{0\}$ there exists a neighborhood \mathcal{V} of x such that $u \in \mathcal{C}^{1,\alpha}(\mathcal{V})$. Since $\mathcal{SC} \setminus \{0\}$ is open there exists $\epsilon > 0$ such that for every $x' \in B_\epsilon(x)$, $|\nabla u(x')| \in \mathcal{SC} \setminus \{0\}$. Let U be the set of such x , then U is open and for a.e. $x \notin U$, $|\nabla u(x)| \notin \mathcal{SC} \setminus \{0\}$. □

One of the interests of this set is the following:

Proposition 3.2.4. *The set U does not depend on the choice of a minimizer. Moreover, let u and v be two minimizers of \mathcal{P}_λ , then $\nabla u = \nabla v$ on U .*

Proof. Let us consider two minimizers u and v of the same problem. We define respectively, U_u and U_v as the open sets of the previous proposition for u and v . By Lemma 3.2.2 and strict convexity of g on \mathcal{SC} , we have that $\nabla u = \nabla v$ a.e. on U_u . Hence, $v \in \mathcal{C}^1(U_u)$ and for every $x \in U_u$, $|\nabla v(x)| \in \mathcal{SC} \setminus \{0\}$. Thus, by definition of U_v , we have that $U_u \subset U_v$. To prove the other inclusion we just have to exchange u and v . Hence, $U_u = U_v = U$ and $\nabla u = \nabla v$ on U . \square

A direct consequence of this result is that:

Remark 3.2.5. For every connected component U_i of U , $u - v$ is constant on U_i .

To conclude this section, we introduce the weak Euler-Lagrange equation associated to \mathcal{P}_λ :

$$\operatorname{div}(\nabla \varphi(\nabla u)) = -\lambda \text{ on } \Omega. \quad (3.2.2)$$

Remark 3.2.6. By Lemma 3.2.2 the function $\nabla \varphi(\nabla u) = g'(|\nabla u|) \frac{\nabla u}{|\nabla u|}$ is independent of the choice of the minimizer of \mathcal{P}_λ and will be denoted by σ .

3.2.2 Lipschitz regularity of a minimizer u and its level lines.

In this subsection, we recall some Lipschitz regularity results for u and its level lines.

We use the following result from [49, Theorem 1.6] to show that our minimizers are globally Lipschitz-continuous on Ω :

Proposition 3.2.7. *We assume that Ω has a $\mathcal{C}^{1,1}$ boundary and $\psi \in \mathcal{C}^{1,1}(\mathbb{R}^2)$. Then any minimizer u of \mathcal{P}_λ is globally Lipschitz-continuous on Ω . Moreover, there exists $L > 0$ such that*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq L(p, C_1, |\Omega|, \|\lambda\|_{L^\infty(\Omega)}, \|\psi\|_{\mathcal{C}^{1,1}(\mathbb{R}^2)}, \kappa)$$

where κ is the maximum of the curvatures of Ω and C_1 is introduced in (3.1.3).

For a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and for every $s \in \mathbb{R}$, we introduce the following notation : $L_s^*(f)$ is the union of all connected components $l_s(f)$ of $L_s(f) = f^{-1}(s) \subset \mathbb{R}^2$ such that $\mathcal{H}^1(l_s(f)) > 0$. Here, \mathcal{H}^1 is the one-dimensional Hausdorff measure.

We will apply the following theorem from [1, Theorem 2.5] to prove Proposition 3.2.9.

Theorem 3.2.8. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Lipschitz continuous function with compact support. For a.e. $s \in \mathbb{R}$, we have:*

- $\mathcal{H}^1(L_s(f) \setminus L_s^*(f)) = 0$.
- Every connected component $l_s(f)$ of $L_s(f)$ that is not a point is a closed simple curve with a Lipschitz parametrization γ_s .
- $L_s^*(f)$ has a countable number of connected components.

It follows that the level lines of a minimizer u have a Lipschitz-continuous parametrization:

Remark 3.2.9. Let u be a globally Lipschitz-continuous minimizer of \mathcal{P}_λ with $\lambda \in L^\infty(\Omega)$. We extend it outside Ω by ψ that can be assumed compactly supported. For a.e. $s \in \mathbb{R}$ every connected component of $L_s^*(u) \subset \mathbb{R}^2$ has a Lipschitz-continuous parametrization.

3.2.3 Explicit solution on the ball and counter-example.

The application of [22, Theorem 1] to our problem \mathcal{P}_λ gives an explicit form for the unique solution on $W_0^{1,p}(B_r(x_0))$ in dimension two when $\lambda \equiv \lambda_0 \in \mathbb{R}_+$:

Proposition 3.2.10. *When $\Omega = B_r(x_0)$ and λ is constant, the problem \mathcal{P}_λ admits a unique minimizer on $W_0^{1,p}(B_r(x_0))$. We can compute it explicitly:*

$$u(x) := C - \frac{2}{\lambda} g^* \left(\frac{\lambda}{2} |x - x_0| \right)$$

with $g^*(x) := \sup_{y \in \mathbb{R}} \langle x, y \rangle - g(y)$, and C the constant such that $u \in W_0^{1,p}(B_r(x_0))$.

The following proposition uses the Euler-Lagrange equation (3.2.2) to prove that a function is a minimizer.

Proposition 3.2.11. *Let u be in $W^{1,p}(\Omega)$ and φ a convex function. If there exist $\sigma \in \partial\varphi(\nabla u) \in L^{p'}(\Omega; \mathbb{R}^2)$ and $\lambda \in L^\infty(\Omega)$ such that $\operatorname{div} \sigma = -\lambda$ then u is a minimizer of \mathcal{P}_λ on $W_u^{1,p}(\Omega)$.*

Here $\partial\varphi$ is the convex subdifferential of φ .

Proof. Since $\sigma \in \partial\varphi(\nabla u)$, for every $w \in W_u^{1,p}(\Omega)$ we have:

$$\int_{\Omega} \varphi(\nabla w) \geq \int_{\Omega} \varphi(\nabla u) + \langle \sigma, \nabla w - \nabla u \rangle.$$

Since $\operatorname{div} \sigma = -\lambda$ we get:

$$\int_{\Omega} \langle \sigma, \nabla w - \nabla u \rangle = \int_{\Omega} \lambda(w - u).$$

Hence, $\mathcal{I}_\lambda(w) \geq \mathcal{I}_\lambda(u)$ for every $w \in W_u^{1,p}(\Omega)$. Thus, u is a minimizer of \mathcal{P}_λ on $W_u^{1,p}(\Omega)$. □

We apply this result to show that when λ is not constant, we can have more than one solution.

Proposition 3.2.12. *Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a non-strictly convex function such that $g(0) < g(t)$ for every $t > 0$. We assume that $g \in C^1(\mathbb{R}_+)$. Then, there exists $\lambda_\infty \in C^\infty(\overline{B_1(0)})$, $\lambda_\infty > 0$ such that $\mathcal{P}_{\lambda_\infty}$ has an infinite number of solutions on $W_0^{1,p}(B_1)$ with $\varphi(\cdot) = g(|\cdot|)$ and*

$$\mathcal{I}_{\lambda_\infty}(u) := \int_{B_1} \varphi(\nabla u(x)) - \lambda_\infty(x)u(x) dx.$$

Proof. We construct two different radial solutions u and v of the same problem. For every $x \in B_1(0)$, we set $u(x) := \tilde{u}(|x|)$ and $v(x) := \tilde{v}(|x|)$. Our goal is to define \tilde{u}' and \tilde{v}' on $(0, 1)$.

Since g is not strictly convex on \mathbb{R}_+ , there exist $a, b \in \mathbb{R}_+$ such that g' is constant on (a, b) and $g'(t) \neq g'(a)$ for every $t \notin [a, b]$.

a) We assume that the smallest possible a is strictly positive. Let us introduce a smooth increasing function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $f(t) = t$ for every $t \geq 0$ small and $f(t) = g'(a)$ for every $t \geq \frac{1}{2}$.

We use the fact that for every $x > 0$ if $x \in \partial g^*(y)$ then $g'(x) = y$. Hence for every $t > 0$, $g'(\partial g^*(f(t))) = \{f(t)\}$. For every $t \in (0, 1)$, we set $\tilde{u}'(t) = -x_t$ with $x_t \in \partial g^*(f(t))$ such that \tilde{u}' is measurable. Such a choice is possible by [24, Theorem 5.3, page 151]. In order to define \tilde{v}' , we set $\tilde{v}'(t) = \tilde{u}'(t)$ on $(0, \frac{1}{2})$ and $\tilde{v}'(t) = -b$ for every $t > \frac{1}{2}$.

Hence, $g'(|\tilde{v}'|) = g'(|\tilde{u}'|) = f$ is a smooth function.

b) We assume that $a = 0$. We set $\tilde{u}' = 0$ on $(0, 1)$, $\tilde{v}'(t) = 0$ on $(0, \frac{1}{2})$ and $\tilde{v}'(t) = -b$ on $(\frac{1}{2}, 1)$.

Now, we can set $u(x) := \int_{|x|}^1 -\tilde{u}'(t)dt$ and $v(x) := \int_{|x|}^1 -\tilde{v}'(t)dt$ that are Lipschitz-continuous on $B_1(0)$ and vanish at the boundary. It remains to find σ and λ_∞ .

In the case **a)**, we set for every $x \in B_1(0)$, $\sigma(x) = -f(|x|)\frac{x}{|x|}$ that is smooth by assumptions on f .

In the case **b)**, we set $\sigma(x) = -g'(0)\frac{x}{|x|}$ for every $x \in B_1(0)$ such that $|x| > \frac{1}{2}$. On $B_{\frac{1}{2}}(0)$, we set $\sigma(x) = -\tilde{f}(x)\frac{x}{|x|}$ with \tilde{f} smooth satisfying the same assumptions as f .

Finally, we set

$$\lambda_\infty(x) = -\operatorname{div} \sigma \in C^\infty(B_1(0)).$$

Hence, by Proposition 3.2.11, u and v are solutions of the same problem $\mathcal{P}_{\lambda_\infty}$. Moreover, a direct computation shows that $\lambda_\infty(x) > 0$ on $\overline{B_1(0)}$. \square

Now, we give an explicit counter-example where \mathcal{P}_λ has more than one solution with $\lambda > 0$ and $U \neq \emptyset$ with g is as in (3.1.2), namely:

$$g(t) = \begin{cases} \frac{1}{2}|t|^2 & \text{if } |t| \leq 1, \\ |t| - \frac{1}{2} & \text{if } 1 < |t| < 2, \\ \frac{1}{4}|t|^2 + \frac{1}{2} & \text{if } 2 \leq |t|. \end{cases}$$

Proposition 3.2.13. *There exists $\lambda \in C^\infty(\overline{B_1(0)})$, $\min_{\overline{B_1(0)}} \lambda > 0$ such that \mathcal{P}_λ has more than one solution on $W_0^{1,2}(B_1(0))$ and $U \neq \emptyset$.*

Proof. We take the same notations as in the previous proof. In this case, we have

$$g^*(t) = \begin{cases} \frac{1}{2}|t|^2 & \text{if } |t| \leq 1, \\ |t|^2 - \frac{1}{2} & \text{if } 1 < |t|. \end{cases}$$

For $t \in \mathbb{R}$, we define:

$$\theta(t) = \begin{cases} t & \text{if } t \leq \frac{1}{4}, \\ 3t - \frac{1}{2} & \text{if } \frac{1}{4} < t < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq t. \end{cases}$$

Hence, we can take f as the convolution of θ with a smooth standard mollifier. For instance,

$$f(t) := C \int_{t-\frac{1}{8}}^{t+\frac{1}{8}} \theta(s) \exp\left(-\frac{1}{\frac{1}{64} - |t-s|^2}\right) ds$$

with

$$C^{-1} = \int_{-\frac{1}{8}}^{\frac{1}{8}} \exp\left(-\frac{1}{\frac{1}{64} - |s|^2}\right) ds.$$

Then $\tilde{u}' = -f$ on $[0, 1]$, $\tilde{v}' = -f$ on $[0, \frac{5}{8}]$ and $\tilde{v}' = -2$ on $(\frac{5}{8}, 1]$. Thus, $u(x) := \int_{|x|}^1 -\tilde{u}'(t) dt$ and $v(x) := \int_{|x|}^1 -\tilde{v}'(t) dt$ are two solutions on $W_0^{1,2}(B_1(0))$ with $\lambda(x) := \frac{N-1}{|x|} f(|x|) + f'(|x|)$.

Moreover, $U \neq \emptyset$ since $|\nabla u| < 1$ on $B_{\frac{1}{2}}(0)$. \square

3.2.4 BV functions.

We start by giving the definitions of functions of bounded variations and sets of finite perimeter:

Definition 3.2.14. A function $f \in L^1(\Omega)$ has bounded variations in Ω if

$$\int_{\Omega} |Df| := \sup \left\{ \int_{\Omega} f \operatorname{div} \theta \, dx \mid \theta \in C_c^1(\Omega; \mathbb{R}^2), |\theta(x)| \leq 1, \forall x \in \Omega \right\} < \infty.$$

We denote by $BV(\Omega)$ the set of functions in $L^1(\Omega)$ having bounded variations in Ω .

If $f \in BV(\Omega)$, the distributional gradient of f is a vector valued Radon measure that we denote by Df and $|Df|$ is the total variation of Df .

Definition 3.2.15. Let E be a Borel set. We say that E has finite perimeter in Ω if the characteristic function χ_E of E belongs to $BV(\Omega)$. The perimeter $\operatorname{Per}(E, \Omega)$ is defined as:

$$\operatorname{Per}(E, \Omega) = \int_{\Omega} |D\chi_E| = \sup \left\{ \int_E \operatorname{div} \theta \mid \theta \in C_c^1(\Omega; \mathbb{R}^2), |\theta(x)| \leq 1, \forall x \in \Omega \right\}.$$

Definition 3.2.16. For a set E of finite perimeter in \mathbb{R}^2 , we define the reduced boundary $\partial^* E$ of E as the subset of $\operatorname{supp} |D\chi_E|$ such that for every $x \in \partial^* E$,

$$\nu_E(x) := \lim_{r \rightarrow 0} \frac{\int_{B_r(x)} D\chi_E}{\int_{B_r(x)} |D\chi_E|}$$

exists and $|\nu_E(x)| = 1$.

Remark 3.2.17. The reduced boundary ∂^*E is a subset of ∂E .

We recall the coarea formula for Lipschitz continuous functions from [32, Theorem 3.4.2.1, page 112] that will be useful throughout the chapter.

Proposition 3.2.18. (Coarea formula) *Let u be a Lipschitz continuous function with compact support and f be a nonnegative measurable function. Then*

$$\int_{\mathbb{R}^2} f|\nabla u|dx = \int_{\mathbb{R}} \int_{L_s(u)} f(x)d\mathcal{H}^1(x)ds$$

where $L_s(u) := u^{-1}(s) \subset \mathbb{R}^2$.

Remark 3.2.19. By replacing f by the indicator function

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

we observe that if $|A| = 0$ then for a.e. $s \in \mathbb{R}$, $\mathcal{H}^1(L_s(u) \cap A) = 0$.

Proposition 3.2.20. *Let v be a Lipschitz continuous function with compact support in \mathbb{R}^2 . For a.e. $s \in \mathbb{R}$ we have*

$$\frac{\nabla v(x)}{|\nabla v(x)|} = \frac{D\mathbf{1}_{[v>s]}(x)}{|D\mathbf{1}_{[v>s]}(x)|} \text{ for } \mathcal{H}^1 \text{ a.e. } x \in L_s(v).$$

Proof. By the vector valued coarea formula [3, Theorem 3.40] we have that

$$\int_A \nabla v = \int_{\mathbb{R}} \int_A D\mathbf{1}_{[v>s]} ds$$

for every Borel set A .

By linearity, for every linear combination of indicator functions χ , we have

$$\int_{\text{supp } v} \langle \chi, \nabla v \rangle = \int_{\mathbb{R}} \int_{\text{supp } v} \langle \chi, D\mathbf{1}_{[v>s]} \rangle ds.$$

By density, for every $\theta \in L^\infty(\text{supp } v; \mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^2} \langle \theta, \nabla v \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \langle \theta, D\mathbf{1}_{[v>s]} \rangle ds.$$

We fix $\theta := \frac{\nabla v}{|\nabla v|}$ when $\nabla v \neq 0$, $\theta := 0$ when $\nabla v = 0$ and obtain

$$\int_{\mathbb{R}^2} |\nabla v| = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left\langle \frac{\nabla v}{|\nabla v|}, D\mathbf{1}_{[v>s]} \right\rangle ds. \quad (3.2.3)$$

But, by [38, Theorem 4.4] we have for a.e. $s \in \mathbb{R}$ that $|D\mathbf{1}_{[v>s]}| = \mathcal{H}^1 \llcorner \partial^*[v > s]$. Hence,

$$\int_{\mathbb{R}^2} \langle \theta, D\mathbf{1}_{[v>s]} \rangle = \int_{\mathbb{R}^2} \left\langle \theta, \frac{D\mathbf{1}_{[v>s]}}{|D\mathbf{1}_{[v>s]|}} \right\rangle d|D\mathbf{1}_{[v>s]}| = \int_{\partial^*[v>s]} \left\langle \theta, \frac{D\mathbf{1}_{[v>s]}}{|D\mathbf{1}_{[v>s]|}} \right\rangle d\mathcal{H}^1. \quad (3.2.4)$$

Since $\langle \theta, \frac{D\mathbf{1}_{[v>s]}}{|D\mathbf{1}_{[v>s]}} \rangle \leq 1$ for a.e. $s \in \mathbb{R}$, with (3.2.3) and (3.2.4) we get

$$\int_{\mathbb{R}^2} |\nabla v| \leq \int_{\mathbb{R}} \mathcal{H}^1(\partial^*[v > s]) ds.$$

By remark 3.2.17, we have

$$\int_{\mathbb{R}^2} |\nabla v| \leq \int_{\mathbb{R}} \mathcal{H}^1(\partial^*[v > s]) ds \leq \int_{\mathbb{R}} \mathcal{H}^1(L_s(v)) ds.$$

By the coarea formula given in Proposition 3.2.18, the following equalities hold true:

$$\int_{\mathbb{R}^2} |\nabla v| = \int_{\mathbb{R}} \mathcal{H}^1(\partial^*[v > s]) ds = \int_{\mathbb{R}} \mathcal{H}^1(L_s(v)) ds.$$

Hence, we get $\langle \theta, \frac{D\mathbf{1}_{[v>s]}}{|D\mathbf{1}_{[v>s]}} \rangle = 1$ for \mathcal{H}^1 a.e. $x \in \partial^*[v > s]$ and $\mathcal{H}^1(L_s(v) \setminus \partial^*[v > s]) = 0$ for a.e. $s \in \mathbb{R}$. Thus, $\theta = \frac{D\mathbf{1}_{[v>s]}}{|D\mathbf{1}_{[v>s]}} \mathcal{H}^1$ a.e. on $L_s(v)$ for a.e. $s \in \mathbb{R}$, as desired. \square

3.3 Relation between the level lines and U.

In this section we use the Lipschitz regularity of the level lines of a minimizer u to prove that they are, in a generic sense, level sets for the other minimizers. We then study the case when a level line intersects the set U , which implies that the gradient of another solution is equal to ∇u on that particular level line.

3.3.1 Equality on level lines.

We first prove that the difference between two minimizers is constant on every connected component of almost every level sets.

Proposition 3.3.1. *Let u and v be two minimizers of the same problem \mathcal{P}_λ . There exists a negligible subset N_0 of \mathbb{R} such that for every $s \in S_0 := \mathbb{R} \setminus N_0$, for every connected component $l_s(u)$ of $L_s(u)$, the map $u - v$ is constant on $l_s(u)$.*

Proof. We consider that u and v are extended by ψ outside of Ω . By Proposition 3.2.9 there is a negligible set N_∞ such that for every $s \in \mathbb{R} \setminus N_\infty$, every connected component $l_s(u)$ of $L_s(u) \subset \mathbb{R}^2$ that is not a point has a parametrization that is Lipschitz continuous.

Since ∇u and ∇v are defined and collinear a.e. on \mathbb{R}^2 , by the coarea formula we obtain that there exists a negligible set N'_∞ such that for every $s \in \mathbb{R} \setminus N'_\infty$, ∇u and ∇v are defined and collinear \mathcal{H}^1 a.e. on $L_s(u)$. We set $N_0 := N_\infty \cup N'_\infty$.

Hence, for every $s \in \mathbb{R} \setminus N_0$ we have that ∇v is orthogonal to each Lipschitz connected curve $l_s(u)$. We introduce $\gamma_s : [0, \text{length}(l_s(u))] \mapsto l_s(u)$ a Lipschitz-continuous parametrization of $l_s(u)$. Then, by the chain rule we have $(v \circ \gamma_s)' = \langle \nabla v(\gamma_s), \gamma_s' \rangle$ a.e. on $[0, \text{length}(l_s(u))]$. By orthogonality of ∇v to $l_s(u)$, we have that v is constant on $l_s(u)$. \square

The following proposition is the first step to prove the uniqueness result.

Proposition 3.3.2. *For $s \in S_0$, if $l_s(u) \cap (\mathbb{R}^2 \setminus \Omega) \neq \emptyset$ then $u = v$ on $l_s(u)$.*

Proof. Thanks to the previous proposition we know that $u - v$ is constant on $l_s(u)$. Since u and v are extended by ψ outside Ω , we have $u \equiv v$ on $\mathbb{R}^2 \setminus \Omega$. By assumption, we have $l_s(u) \cap \mathbb{R}^2 \setminus \Omega \neq \emptyset$ then $u = v$ on $l_s(u)$. \square

3.3.2 Relation between U and the level lines

In this section we consider two minimizers u and v of I_λ with the same boundary condition. We know by Proposition 3.2.4 that $\nabla u = \nabla v$ on U . We will extend this result to the level lines that intersect U .

Notation. We denote by $S \subset \mathbb{R}$ the set of these s that satisfy the following conditions:

- $s \in S_0$ with S_0 defined in Proposition 3.3.1,
- $\nabla u \neq 0$ \mathcal{H}^1 a.e. on $L_s(u)$.

We introduce the following set:

$$\Gamma := \{l_s^i(u), s \in S \text{ and } i \in I_s\}$$

where the index set I_s corresponds to the non-constant curves $l_s^i(u)$ among the connected components of $L_s(u)$ inside Ω which do not intersect $\partial\Omega$.

Proposition 3.3.3. *Given $s \in S$ and $i \in I_s$, let F_s^i be the bounded connected component of $\mathbb{R}^2 \setminus l_s^i(u)$ given by the Jordan curve theorem. Then for every $i \in I_s$, $u > s$ on F_s^i .*

Proof. We fix $s \in S$ and we call E_s the set $[u > s] \subset \mathbb{R}^2$. By continuity of u , E_s is an open set. Let $l_s^i(u)$ a connected component of $L_s(u)$ such that $l_s^i(u) \Subset \Omega$. Since Ω is simply connected, $F_s^i \subset \Omega$ and by Proposition 3.2.1, we have that $u \geq s$ on F_s^i . If $l'_s(u)$ is a connected component of $L_s^*(u)$ that is inside F_s^i then for \mathcal{H}^1 a.e. $y \in l'_s(u)$, $\nabla u(y)$ is defined and $\nabla u(y) \neq 0$. Since $u \geq s$ on F_s^i , every point y in $l'_s(u)$ is a local minimum on F_s^i and hence either $\nabla u(y) = 0$ or $\nabla u(y)$ is not defined. Thus there is no such $l'_s(u)$ in F_s^i . By assumptions on s , we have $\mathcal{H}^1(L_s \setminus L_s^*) = 0$. Hence, $\mathcal{H}^1([u = s] \cap F_s^i) = 0$.

Let us assume that there exists $x \in F_s^i$ such that $u(x) = s$. Then by Proposition 3.2.1 for every $\epsilon < \text{dist}(x, l_s^i(u))$ there exists $y_\epsilon \in \partial B_\epsilon(x)$ such that $u(y_\epsilon) = s$ and we define the following set $Y := \{y_\epsilon, 0 < \epsilon < \text{dist}(x, l_s^i(u))\}$. We have

$$\mathcal{H}^1(Y) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^1(Y)$$

with

$$\mathcal{H}_\delta^1(Y) := \inf \left\{ \sum_{n \in \mathbb{N}} \text{diam}(V_n) \right\}$$

where the infimum is taken over the families of sets $(V_n)_{n \in \mathbb{N}}$ such that $Y \subset \bigcup_n V_n$ and $\text{diam}(V_n) < \delta$ for every $n \in \mathbb{N}$.

For every admissible $(V_n)_{n \in \mathbb{N}}$, we define E_n as the set of those ϵ such that $y_\epsilon \in V_n \cap Y$. We define $e_m := \inf\{\epsilon \in E_n\}$, $e^M := \sup\{\epsilon \in E_n\}$ and $\tilde{V}_n := [e_m, e^M]$. We have $\text{diam}(\tilde{V}_n) = e^M - e_m \leq \text{diam}(V_n) < \delta$ and $]0, \text{dist}(x, l_s^i(u))[\subset \bigcup \tilde{V}_n$.

Hence, $(\tilde{V}_n)_{n \in \mathbb{N}}$ is admissible for $\mathcal{H}_\delta^1((0, \text{dist}(x, l_s^i(u))))$ and

$$\mathcal{H}_\delta^1(Y) \geq \mathcal{H}_\delta^1((0, \text{dist}(x, l_s^i(u)))).$$

By taking the limit when δ goes to 0, we obtain:

$$\mathcal{H}^1(Y) \geq \mathcal{H}^1((0, \text{dist}(x, l_s^i(u))) = \text{dist}(x, l_s^i(u)).$$

Thus $\mathcal{H}^1([u = s] \cap F_s^i) \geq \text{dist}(x, l_s^i(u)) > 0$. That is a contradiction. Hence there is no such x , thus $u > s$ on F_s^i . \square

A direct consequence of that result is the following:

Proposition 3.3.4. *For every $s \in S$, I_s is countable. Moreover, for every $i \in I_s$, $l_s^i(u)$ is the boundary of a connected component F_s^i of $E_s = [u > s]$.*

Proof. By Theorem 3.2.8, I_s is countable. Let us consider $i \in I_s$. By the previous proposition, $l_s^i(u)$ is the boundary of a connected component of $E_s = [u > s]$. \square

We also have that:

Proposition 3.3.5. *For every $s \in S$, every connected component F_s of E_s , if $F_s \Subset \Omega$ then F_s is simply connected and its boundary is a closed simple curve $l_s(u)$ with Lipschitz parametrization.*

Proof. Since F_s is bounded, $\mathbb{R}^2 \setminus F_s$ has only one unbounded connected component. We call \tilde{F}_s the complement of this unbounded set. We claim that $\tilde{F}_s = F_s$. We have that $\partial \tilde{F}_s \subset \partial F_s$. Hence, $u \equiv s$ on $\partial \tilde{F}_s$. Since \tilde{F}_s is simply connected, $\partial \tilde{F}_s$ is a connected set in $L_s(u)$ with $\mathcal{H}^1(\partial \tilde{F}_s) > 0$. By Theorem 3.2.8, $\partial \tilde{F}_s$ is a closed subset of a closed simple curve with Lipschitz parametrization $l_s(u)$. Hence, \tilde{F}_s is a bounded set such that $\partial \tilde{F}_s \subset l_s(u)$. We have that \tilde{F}_s is an open set in $\mathbb{R}^2 \setminus l_s(u)$. Since, $\partial \tilde{F}_s \subset l_s(u)$, \tilde{F}_s is also closed in $\mathbb{R}^2 \setminus l_s(u)$. The fact that \tilde{F}_s is bounded and connected gives that \tilde{F}_s is the bounded connected component of $\mathbb{R}^2 \setminus l_s(u)$ and by the Jordan curve theorem we have $\partial \tilde{F}_s = l_s(u)$. By Proposition 3.3.3, $\tilde{F}_s \subset E_s$. Since \tilde{F}_s contains F_s , we get that $\tilde{F}_s = F_s$. Moreover, we proved that F_s is simply connected with $l_s(u)$ as boundary. \square

The main result of this subsection is the following:

Proposition 3.3.6. *For a.e. $s \in S$, for every $i \in I_s$, if $l_s^i(u) \cap U \neq \emptyset$ then $\nabla(u - v) = 0$ \mathcal{H}^1 a.e. on $l_s^i(u)$.*

In order to prove this result, we state two technical lemmata:

Lemma 3.3.7. *For a.e. $s \in S$, for every $i \in I_s$ there exists a decreasing sequence $(s_n)_{n \in \mathbb{N}}$ converging to s such that:*

- *There exists a simple connected curve $l_{s_n}(u)$ in L_{s_n} with Lipschitz parametrization that is inside F_s^i .*

- *$(F_{s_n})_{n \in \mathbb{N}}$ is an increasing sequence with $\bigcup_{n \in \mathbb{N}} F_{s_n} = F_s^i$.*

Here, F_t is the bounded connected component of $\mathbb{R}^2 \setminus l_t(u)$ given by the Jordan curve theorem and F_s^i is the bounded connected component of $\mathbb{R}^2 \setminus l_s^i(u)$.

Proof of Lemma 3.3.7. By Proposition 3.3.3 we have that $u > s$ on F_s^i . By the coarea formula 3.2.18 there exists $s_0 > s$, $s_0 \in S$ such that $\mathcal{H}^1(L_{s_0}(u) \cap F_s^i) > 0$. Moreover, by Theorem 3.2.8, $\mathcal{H}^1(L_{s_0}(u) \setminus L_{s_0}^*(u)) = 0$. Hence, there exists $l_{s_0}(u)$ in L_{s_0} satisfying the assumptions of Lemma 3.3.7. We next select $s < s_1 < s_0$ with $s_1 \in S$ such that $\mathcal{H}^1(L_{s_1}(u) \cap F_s^i) > 0$. We have that $F_{s_0} \subset E_{s_1}$. Hence, F_{s_0} is in one connected component of E_{s_1} , we call $l_{s_1}(u)$ the boundary of that connected component. By Proposition 3.3.5 we have that $l_{s_1}(u)$ is a simple connected curve with Lipschitz parametrization. We repeat this argument to find a sequence $(s_n)_{n \in \mathbb{N}}$ that satisfies the first part of the lemma. By construction $(F_{s_n})_{n \in \mathbb{N}}$ is an increasing sequence, it remains to prove that $\bigcup_{n \in \mathbb{N}} F_{s_n} = F_s^i$.

We introduce $F_\infty := \bigcup_{n \in \mathbb{N}} F_{s_n}$ a subset of F_s^i . If $y \in \partial F_\infty$ there exists a sequence $(y_n)_{n \in \mathbb{N}}$ such that $y_n \in \partial F_{s_n}$ and $y_n \rightarrow y$. By continuity of u and the fact that y_n converges to y we obtain that $u(y) = s$.

By Proposition 3.3.3 we have that $\partial F_\infty \subset \partial F_s^i$. We claim that $F_\infty = F_s^i$. Indeed, if those two sets are not equal there exists $x \in F_s^i \setminus F_\infty$. Since F_s^i is a connected open set, for every $y \in F_\infty$ there exists a continuous path from x to y included in F_s^i . By continuity this path must intersect $\partial F_\infty \subset \partial F_s^i$, contradicting the fact that the path is in F_s^i . Hence, $F_\infty = F_s^i$. □

Lemma 3.3.8. *Let u be a minimizer. For a.e. $s \in S$ and every $i \in I_s$, we consider a sequence $(l_{s_n})_{n \in \mathbb{N}}$ as in the previous lemma. Then, we have that $\lim_{n \rightarrow +\infty} D(l_{s_n}, l_s^i(u)) = 0$ where $D(E, F) := \sup_{e \in E} \inf_{f \in F} |e - f|$. Moreover, let v be another minimizer, if there exists a constant C such that $u - v \equiv C$ on every l_{s_n} then $\nabla(u - v) = 0$ \mathcal{H}^1 a.e on $l_s^i(u)$.*

Proof of Lemma 3.3.8. For every $\epsilon > 0$, we claim that there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $D(l_{s_n}, l_s^i(u)) < \epsilon$. Indeed, assume by contradiction that there exists $\epsilon > 0$ such that $\forall N$, there exist $n \geq N$ and $y_n \in l_{s_n}(u)$ such that $\forall x \in l_s^i(u)$, $d(y_n, x) \geq \epsilon$. Since all these y_n are in Ω , there exists a sequence $(y_{g(n)})_{n \in \mathbb{N}}$ converging towards some $y \in F_s^i$. We have that $d(y, x) \geq \epsilon$ for every $x \in l_s^i(u)$. By continuity of u , we have that $u(y) = s$. Thus, by Proposition 3.3.3, $y \in \partial F_s^i = l_s^i(u)$. That is a contradiction.

For \mathcal{H}^1 a.e. $x \in l_s^i(u)$ we have that $\nabla u(x) \neq 0$ and $\nabla v(x)$ exist. Moreover, for \mathcal{H}^1 a.e. $x \in l_s^i(u)$ we have that $\nabla u(x)$ and $\nabla v(x)$ are orthogonal to $l_s^i(u)$ at x in

the sense that $\langle \nabla u(x), \gamma'_s(\gamma_s^{-1}(x)) \rangle = 0$ where γ_s is a Lipschitz parametrization of $l_s^i(u)$. We consider $d_x := x + \mathbb{R}\nabla u(x)$. Let us call H_- and H_+ the two half-planes of $\mathbb{R}^2 \setminus d_x$. For every $r > 0$, $H_\pm \cap l_s^i(u) \cap B_r(x) \neq \emptyset$, otherwise it would contradict the fact that $\nabla u(x)$ is orthogonal to $l_s^i(u)$. A direct consequence is the fact that $H_- \cap F_s^i \neq \emptyset$ and $H_+ \cap F_s^i \neq \emptyset$.

We assume that for every $N \in \mathbb{N}$, there exists $\tilde{n} \geq N$ such that $l_{s_{\tilde{n}}}(u) \cap d_x = \emptyset$. Thus, we can assume that $l_{s_{\tilde{n}}}(u) \cap H_- = \emptyset$. Hence, there exist $y \in l_s^i(u)$ and $\epsilon_0 > 0$ such that $d(y, F_{s_{\tilde{n}}}) > \epsilon_0$. Since $l_{s_{\tilde{n}}}(u) \rightarrow l_s^i(u)$ in the sense of D that is absurd. Then, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $l_{s_n}(u) \cap d_x \neq \emptyset$. For every such n we take x_n as a point that minimizes $d(x, y)$ on $l_{s_n}(u) \cap d_x$. Since this sequence $(x_n)_{n \in \mathbb{N}}$ is bounded, it converges, up to a subsequence, to a point $x' \in \overline{F_s^i}$ such that $u(x') = s$. By Proposition 3.3.3, $x' \in l_s^i(u)$. If $x' \neq x$ then there exists $r > 0$ such that $d_x \cap B_r(x) \cap l_{s_n}(u) = \emptyset$ for every $n \in \mathbb{N}$ large enough. Hence, $d_x \cap B_r(x) \cap F_{s_n} = \emptyset$ for every $n \in \mathbb{N}$ large enough which contradicts the fact that $\bigcup_{n \in \mathbb{N}} F_{s_n} = F_s^i$. Thus, $x = x'$ and we can find a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in l_{s_n}(u) \cap d_x$ and $x_n \rightarrow x$. By assumption, $u - v \equiv C$ on $l_{s_n}(u)$ for n large enough. By continuity of $u - v$ we obtain that $u - v(x) = C$. Then, $\frac{(u-v)(x) - (u-v)(x_n)}{|x - x_n|} = \frac{C - C}{|x - x_n|} = 0$. Moreover, $\nabla(u - v)(x)$ is collinear to $\nabla u(x)$, hence, we obtain $\nabla(u - v)(x) = 0$. Since that is the case for \mathcal{H}^1 a.e. $x \in l_s^i(u)$, we have the desired conclusion. \square

Proof of Proposition 3.3.6. For every $s \in S$, $l_s^i(u)$ is a Lipschitz continuous closed curve such that ∇u and ∇v are defined and collinear \mathcal{H}^1 a.e. on $l_s^i(u)$ and $\nabla u \neq 0$ \mathcal{H}^1 a.e. on $l_s^i(u)$. If $l_s^i(u) \cap U \neq \emptyset$ then there exists U_i a connected component of U such that $l_s^i(u) \cap U_i \neq \emptyset$. By Proposition 3.2.5 and Proposition 3.3.1 we have $u - v \equiv C_i$ on $l_s^i(u)$. We consider the sequences $(s_n)_{n \in \mathbb{N}}$, $(l_{s_n}(u))_{n \in \mathbb{N}}$ and $(F_{s_n})_{n \in \mathbb{N}}$ from Lemma 3.3.7.

Hence, by the first part of Lemma 3.3.8, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $l_{s_n}(u) \cap U_i \neq \emptyset$. Thus, by Proposition 3.2.5 and Proposition 3.3.1, $u - v \equiv C_i$ on $l_{s_n}(u)$ for every $n \geq N$. By the second part of Lemma 3.3.8, we have that $\nabla(u - v) = 0$ \mathcal{H}^1 a.e. on $l_s^i(u)$. \square

3.4 $W^{1,2}$ regularity of $|\sigma|$.

In this section, we prove the following proposition:

Proposition 3.4.1. *For every $\alpha > 0$, the function $f := \max(\alpha, |\sigma|)$ is in $W^{1,2}(\Omega')$ for any $\Omega' \Subset \Omega$.*

We prove this result in four parts. In **Step 1**, we regularize our problem in order to work with smooth solutions $(u_n)_{n \in \mathbb{N}}$. Then in **Step 2**, we prove that $\|\max(\alpha, |\nabla \varphi_n(\nabla u_n)|)\|_{W^{1,2}(\Omega')}$ is uniformly bounded in $n \in \mathbb{N}$. In the subsequent **Step 3**, we show that $\max(\alpha, |\nabla \varphi_n(\nabla u_n)|) \rightarrow f$ a.e. on Ω . We conclude that f is in $W^{1,2}(\Omega')$ in **Step 4**.

Step 1 For every $n \in \mathbb{N}$, we introduce $(\rho_n)_{n \in \mathbb{N}}$ a standard mollifying sequence with $\text{supp } \rho_n \subset B_{\frac{1}{n}}(0)$. If we set $g_n := g * \rho_n$ and $\lambda_n := \lambda * \rho_n$ then $(g_n)_{n \in \mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}}$ are sequences of smooth approximations of g and λ . We consider $\varphi_n := g_n(|\cdot|) + \frac{1}{n}\theta(|\cdot|)$. The function θ is smooth quadratic around the origin such that $0 < \theta''(x)$ for every $x \in \mathbb{R}$ and

$$C_-|x|^p \leq \theta(z) \leq C_+(|x|^p + 1)$$

for all $|x| \geq 1$ with $0 < C_- < C_+$.

Let u_n be the minimizer of:

$$\mathcal{I}_n : v \rightarrow \int_{\Omega} \varphi_n(\nabla v(x)) - \lambda_n v(x) dx$$

on $W_{\psi}^{1,p}(\Omega)$.

Proposition 3.4.2. *The sequence $(u_n)_{n \in \mathbb{N}}$ is uniformly bounded in $W^{1,p}(\Omega)$. There exists a subsequence still denoted by $(u_n)_{n \in \mathbb{N}}$ that weakly converges in $W^{1,p}(\Omega)$ towards \tilde{u} . Moreover, \tilde{u} is a minimizer of \mathcal{P}_{λ} on $W_{\psi}^{1,p}(\Omega)$.*

Proof. By Proposition 3.2.7 we have that the sequence $(u_n)_{n \in \mathbb{N}}$ is uniformly bounded in $W^{1,\infty}(\Omega)$. Hence, we can extract a subsequence, still denoted by $(u_n)_{n \in \mathbb{N}}$, that converges strongly in $L^p(\Omega)$ and weakly in $W_{\psi}^{1,p}(\Omega)$ towards \tilde{u} . It remains to prove that \tilde{u} is a minimizer of \mathcal{P}_{λ} on $W_{\psi}^{1,p}(\Omega)$.

By Jensen's inequality, we have $\varphi_n \geq \varphi$. Hence

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} \varphi_n(\nabla u_n) \geq \liminf_{n \rightarrow +\infty} \int_{\Omega} \varphi(\nabla u_n). \quad (3.4.1)$$

By weak lower semi-continuity of I_{λ} , (3.4.1) and the fact that u_n is the minimizer for \mathcal{I}_n we have

$$\begin{aligned} \int_{\Omega} \varphi(\nabla \tilde{u}) - \lambda \tilde{u} &\leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \varphi(\nabla u_n) - \lambda u_n \\ &\leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \varphi_n(\nabla u_n) - \lambda_n u_n \leq \lim_{n \rightarrow +\infty} \int_{\Omega} \varphi_n(\nabla u) - \lambda_n u. \end{aligned} \quad (3.4.2)$$

By the dominated convergence theorem applied to the last quantity we obtain

$$\int_{\Omega} \varphi(\nabla \tilde{u}) - \lambda \tilde{u} \leq \int_{\Omega} \varphi(\nabla u) - \lambda u.$$

Hence, \tilde{u} is a minimizer on $W_{\psi}^{1,p}(\Omega)$. \square

Step 2 For every $n \in \mathbb{N}$, we introduce $\sigma_n := \nabla \varphi_n(\nabla u_n)$. In this part, we prove the following result on $f_n := \max(\alpha, |\sigma_n|)$:

Proposition 3.4.3. *For every $\alpha > 0$ and every $\Omega' \Subset \Omega$, the functions $\max(\alpha, |\sigma_n|)$ are uniformly bounded in $W^{1,2}(\Omega')$.*

Proof. By [29, Proposition 2.4], we have for every $b > 0$ and $k \in \{1, 2\}$ that:

$$\int_{\Omega' \cap \{\partial_k u_n \geq b\}} |\nabla \sigma_n|^2 \leq C_1 \left(b, \|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}, \sup_{b \leq t \leq \|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}} g_n''(t) + \frac{\theta''(t)}{n} \right).$$

Thus,

$$\int_{\Omega' \cap \{|\nabla u_n| \geq b\}} |\nabla \sigma_n|^2 \leq C_2 \left(b, \|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}, \sup_{b \leq t \leq \|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}} g_n''(t) + \frac{\theta''(t)}{n} \right).$$

Finally,

$$\int_{\Omega' \cap \{|\sigma_n| \geq g_n'(b)\}} |\nabla \sigma_n|^2 \leq C_2 \left(b, \|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}, \sup_{b \leq t \leq \|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}} g_n''(t) + \frac{\theta''(t)}{n} \right).$$

By Proposition 3.2.7 we have that $\|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}$ can be bounded by L uniformly in $n \in \mathbb{N}$. Moreover, $g \in \mathcal{C}_{loc}^{1,1}(\mathbb{R} \setminus \{0\})$ and g_n is a convolution of g . Hence, $\sup_{b \leq t \leq \|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}} g_n''(t) + \frac{\theta''(t)}{n}$ can be bounded by $\sup_{\frac{b}{2} \leq t \leq L} g''(t) + 1$ for every $n \in \mathbb{N}$ such that g_n'' is close enough to g'' on $(\frac{b}{2}, +\infty)$ and larger than $\sup_{b \leq t \leq L} g''(t)$. Namely, every $n \in \mathbb{N}$ larger than $\max\{\frac{1}{2b}, \sup_{b \leq t \leq L} \theta''(t)\}$. Thus, we get

$$\int_{\Omega' \cap \{|\sigma_n| \geq g_n'(b)\}} |\nabla \sigma_n|^2 \leq C_2(b, L, \sup_{\frac{b}{2} \leq t \leq L} g''(t) + 1)$$

for every $n \in \mathbb{N}$ larger than $\max\{\frac{1}{2b}, \sup_{b \leq t \leq L} g''(t)\}$.

By growing assumptions on g , for every $\alpha > 0$ we can find $b > 0$ such that $g_n'(b) \leq \alpha$ for $n \in \mathbb{N}$ large enough. Hence, for every $\alpha > 0$ and every $n \in \mathbb{N}$ large enough we have:

$$\int_{\Omega' \cap \{|\sigma_n| > \alpha\}} |\nabla \sigma_n|^2 \leq C(\alpha, g'', L).$$

Thus, the sequence $(f_n)_{n \in \mathbb{N}}$ is uniformly bounded in $W^{1,2}(\Omega')$. \square

Since the functions $f_n := \max(\alpha, |\sigma_n|)$ are uniformly bounded in $W^{1,2}(\Omega')$, we can extract a subsequence which converges weakly.

Step 3 We prove that $\sigma_n \rightarrow \sigma$ a.e. Ω up to a subsequence. To do so we use the Young measures associated to $(\nabla u_n)_{n \in \mathbb{N}}$.

Proposition 3.4.4. *We have the following equality*

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} \varphi(\nabla u_n) = \int_{\Omega} \varphi(\nabla \tilde{u}). \quad (3.4.3)$$

Proof. If we replace u by \tilde{u} in the last term of (3.4.2) we obtain that

$$\begin{aligned} \int_{\Omega} \varphi(\nabla \tilde{u}) - \lambda \tilde{u} &\leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \varphi(\nabla u_n) - \lambda u_n \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \varphi_n(\nabla u_n) - \lambda_n u_n \\ &\leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \varphi_n(\nabla \tilde{u}) - \lambda_n \tilde{u}. \end{aligned}$$

By Fatou's lemma with the lim sup the last term is equal to the first term. Hence all those inequalities are equalities, in particular:

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} \varphi(\nabla u_n) - \lambda u_n = \int_{\Omega} \varphi(\nabla \tilde{u}) - \lambda \tilde{u}.$$

Since $u_n \rightarrow \tilde{u}$ in $L^p(\Omega)$ we have that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \lambda u_n = \int_{\Omega} \lambda \tilde{u}.$$

Hence,

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} \varphi(\nabla u_n) = \int_{\Omega} \varphi(\nabla \tilde{u}).$$

□

We consider $(u_{\psi(n)})_{n \in \mathbb{N}}$ a subsequence such that

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} \varphi(\nabla u_n) = \lim_{n \rightarrow +\infty} \int_{\Omega} \varphi(\nabla u_{\psi(n)}). \quad (3.4.4)$$

In order to simplify the notations, we still denote $(u_{\psi(n)})_{n \in \mathbb{N}}$ by $(u_n)_{n \in \mathbb{N}}$.

Proposition 3.4.5. *For a.e. $x \in \Omega$ we have*

$$\varphi(\nabla \tilde{u}(x)) = \bar{\varphi}(x) := \int_{\mathbb{R}^2} \varphi(y) d\nu_x(y)$$

where ν_x is a probability measure that depends on x and on the weak convergence of $(\nabla u_n)_{n \in \mathbb{N}}$ towards $\nabla \tilde{u}$. Moreover, $\text{supp } \nu_x \subset \{y \in \mathbb{R}^2, \nabla \varphi(y) = \nabla \varphi(\nabla \tilde{u}(x))\}$ for a.e. $x \in \Omega$.

Proof. Let $(\nu_x)_{x \in \Omega}$ be the Young measures associated to a subsequence of $(\nabla u_n)_{n \in \mathbb{N}}$ given by [6, Theorem 2]. We have for every Carathéodory function F such that $\{F(\cdot, \nabla u_n(\cdot))\}_{n \in \mathbb{N}}$ is uniformly integrable:

$$\lim_{n \rightarrow +\infty} \int_{\Omega} F(x, \nabla u_n(x)) dx = \int_{\Omega} \bar{F}(x) dx \quad (3.4.5)$$

with $\bar{F}(x) = \int_{\mathbb{R}^2} F(x, y) d\nu_x(y)$. Moreover, for a.e. $x \in \Omega$,

$$\nabla \tilde{u}(x) = \int_{\mathbb{R}^2} y d\nu_x(y). \quad (3.4.6)$$

Since u_n is uniformly bounded in $W^{1,\infty}(\Omega)$,

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \varphi(\nabla u_n(x)) dx = \int_{\Omega} \bar{\varphi}(x) dx \text{ where } \bar{\varphi}(x) = \int_{\mathbb{R}^2} \varphi(y) d\nu_x(y). \quad (3.4.7)$$

If we combine this last equation with (3.4.4) we get

$$\int_{\Omega} \varphi(\nabla \tilde{u}) = \lim_{n \rightarrow +\infty} \int_{\Omega} \varphi(\nabla u_n(x)) dx = \int_{\Omega} \bar{\varphi}(x) dx.$$

If we apply the triangle inequality and Jensen's inequality to (3.4.6) we obtain for a.e. $x \in \Omega$,

$$\varphi(\nabla \tilde{u}(x)) \leq \int_{\mathbb{R}^2} \varphi(y) d\nu_x(y) dx = \bar{\varphi}(x). \quad (3.4.8)$$

If we combine the two last equations we obtain for a.e. $x \in \Omega$

$$\varphi(\nabla \tilde{u}(x)) = \bar{\varphi}(x). \quad (3.4.9)$$

By Jensen's inequality, φ is affine on $\text{supp } \nu_x$ and thus, for a.e. $x \in \Omega$ we have that $\text{supp } \nu_x \subset \{y \in \mathbb{R}^2, \nabla \varphi(y) = \nabla \varphi(\nabla \tilde{u}(x))\}$. □

Now, we can prove the following convergence result:

Proposition 3.4.6. *We have that $\sigma_n \rightarrow \sigma$ in $L^1(\Omega)$ when $n \rightarrow +\infty$. Here, $\sigma_n = \nabla \varphi_n(\nabla u_n)$.*

Proof. If we set $F(x, y) = |\nabla \varphi(y) - \sigma(x)|$ in (3.4.5), we obtain that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |\nabla \varphi(\nabla u_n(x)) - \sigma(x)| dx = \int_{\Omega} \int_{\mathbb{R}^2} |\nabla \varphi(y) - \sigma(x)| d\nu_x(y) dx = 0.$$

Since ∇u_n is uniformly bounded in $L^\infty(\Omega)$, we have that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |\nabla \varphi_n(\nabla u_n) - \nabla \varphi(\nabla u_n)| = 0.$$

Hence, by the triangle inequality,

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |\sigma_n(x) - \sigma(x)| dx = 0.$$

Hence, $\sigma_n \rightarrow \sigma$ in $L^1(\Omega)$ when $n \rightarrow +\infty$. □

Thanks to the previous Proposition, we can extract a subsequence, we do not relabel, such that $\sigma_n \rightarrow \sigma$ a.e. on Ω when $n \rightarrow +\infty$.

Step 4 Since $\sigma_n \rightarrow \sigma$ a.e. on Ω when $n \rightarrow +\infty$, we have that $f_n \rightarrow \max(\alpha, |\sigma|)$ a.e. on Ω . By Proposition 3.4.3, we have that $\max(\alpha, |\sigma|) \in W^{1,2}(\Omega')$.

3.5 Continuity of $|\sigma|$ on the level lines and a maximum principle

In this section, we prove that generically, $\max(d_0, |\sigma|)$ is continuous on the level lines of u and satisfies a maximum principle.

For $\Omega' \Subset \Omega$, we introduce

$$\Gamma' := \{l_s^i(u), s \in S \text{ and } i \in I'_s\}$$

where $S \subset \mathbb{R}$ is the set of those s that satisfy the conclusion of Theorem 3.2.8 and such that $\nabla u, \nabla v$ are defined, $\nabla u \neq 0$, ∇u and ∇v are collinear \mathcal{H}^1 a.e. on $L_s(u)$. The index set I'_s corresponds to the non-constant curves $l_s^i(u)$ among the connected components of $L_s(u)$ such that $l_s^i(u) \Subset \Omega'$.

Proposition 3.5.1. *There exists a representative f_0 of $\max(d_0, |\sigma|)$ that is absolutely continuous on $l_s^i(u)$ for a.e. $s \in S$ and every $i \in I_s$.*

Proof. We consider the sequence $(\sigma_n)_{n \in \mathbb{N}}$ introduced in the previous section. We have that $\sigma_n \rightarrow \sigma$ a.e. on Ω when $n \rightarrow +\infty$. By Proposition 3.4.1, we have $\|\nabla \max(d_0, |\sigma_n|)\|_{L^2(\Omega_1)} \leq C_1$ with $\Omega_1 \Subset \Omega$ and C_1 independent of $n \in \mathbb{N}$. Thus, there exists a constant C_2 independent of n such that

$$C_2 \geq \int_{\Omega_1} |\nabla \max(d_0, |\sigma_n|)|^2 |\nabla u| = \int_{\mathbb{R}} \int_{L_s(u) \cap \Omega_1} |\nabla \max(d_0, |\sigma_n|)|^2 d\mathcal{H}^1 ds$$

where the equality is given by Proposition 3.2.18. With Fatou's lemma we obtain that for a.e. $s \in S$,

$$\liminf_{n \rightarrow +\infty} \int_{L_s(u) \cap \Omega_1} |\nabla \max(d_0, |\sigma_n|)|^2 d\mathcal{H}^1 \leq C_3(s). \quad (3.5.1)$$

Here, the index I_s^1 corresponds to the non-constant curves $l_s^i(u)$ among the connected components of $L_s(u)$ such that $l_s^i(u) \Subset \Omega_1$. We define S_1 as the subset of S such that (3.5.1) holds. We have $|S \setminus S_1| = 0$.

Now, we fix $s \in S_1$ and $i \in I_s^1$. We can extract a subsequence such that for every $n \in \mathbb{N}$:

$$\int_{l_s^i(u)} |\nabla \max(d_0, |\sigma_n|)|^2 d\mathcal{H}^1 \leq 2C_3(s).$$

Let us call $\gamma_s^i : [0, \text{length}(l_s^i(u))] \mapsto l_s^i(u)$ a Lipschitz continuous parametrization of $l_s^i(u)$. We have that $\max(d_0, |\sigma_n|) \circ \gamma_s^i$ is bounded in $W^{1,2}([0, \text{length}(l_s^i(u))])$. By the Arzelà-Ascoli theorem there exists a subsequence of $\max(d_0, |\sigma_n|) \circ \gamma_s^i$ converging uniformly to $v \in C^0([0, \text{length}(l_s^i(u))])$.

Since $\max(d_0, |\sigma_n|) \rightarrow \max(d_0, |\sigma|)$ \mathcal{H}^1 a.e. on $l_s^i(u)$ for a.e. $t \in \mathbb{R}$ and every $j \in I_t$, we choose f_0 as a representative of $\max(d_0, |\sigma|)$ such that $f_0 = v \circ (\gamma_s^i)^{-1}$ on $l_s^i(u)$.

Now, we introduce an increasing sequence of open sets $(\Omega_k)_{k \in \mathbb{N}}$ such that $\chi_{\Omega_k} \rightarrow \chi_{\Omega}$ in $L^1(\mathbb{R}^2)$. For a.e. $s \in S_1$ and for every $i \in I_s^2 \setminus I_s^1$, we can define f_0 as we did

on Ω_1 . Hence, there exists $S_2 \subset S_1$ such that $|S \setminus S_2| = 0$ and for every $s \in S_2$ and every $i \in I_s^2$, we have f_0 absolutely continuous on $l_s^i(u)$.

Thus, we can select by induction a representative of $\max(d_0, |\sigma|)$ that is absolutely continuous on $l_s^i(u)$ for a.e. $s \in S$ and every $i \in I_s$. □

For a.e. $s \in S$, if $l_s^i(u) \cap U = \emptyset$ we have some additional information that will be useful in the final proof.

Proposition 3.5.2. *For a.e. $s \in S$, for every $i \in I_s$, if $l_s^i(u) \cap U = \emptyset$ then $f_0 = C_s^i$ is constant on $l_s^i(u)$ with $C_s^i \in \{d_n, n \in \mathbb{N}, 0 \leq n < N\}$.*

Proof. For a.e. $x \in \Omega \setminus U$ we have $|\sigma(x)| \in \{0\} \cup g'(\mathbb{R} \setminus \mathcal{SC})$. By the coarea formula for a.e. $s \in \mathbb{R}$, for \mathcal{H}^1 a.e. $x \in (\Omega \cap L_s(u)) \setminus U$ we have $|\sigma(x)| \in g'(\mathbb{R} \setminus \mathcal{SC})$. Hence, for a.e. $s \in \mathbb{R}$ if $l_s^i(u) \cap U = \emptyset$ then $f_0(l_s^i(u)) \subset g'(\mathbb{R} \setminus \mathcal{SC}) \cup f_0(X)$ for some $X \subset l_s^i(u)$ with $\mathcal{H}^1(X) = 0$.

Moreover for a.e. $s \in \mathbb{R}$ and every $i \in I_s$, $l_s^i(u)$ is a Lipschitz continuous curve such that f_0 is absolutely continuous on $l_s^i(u)$. Since $g'(\mathbb{R} \setminus \mathcal{SC})$ is finite and $f_0(X)$ is the image of a negligible set by an absolutely continuous function we have $|g'(\mathbb{R} \setminus \mathcal{SC}) \cup f_0(X)| = 0$. The continuity of f_0 on $l_s^i(u)$ implies that f_0 is constant on $l_s^i(u)$. Since $l_s^i(u) \cap U = \emptyset$ we obtain $f_0 = C_s^i \in \{d_n, n \in \mathbb{N}, 0 \leq n < N\}$. □

We use the notations of Section 4, where $(\sigma_n)_{n \in \mathbb{N}}$ is a smooth approximation that converges a.e. on Ω to σ . We prove the following maximum principle on $\max(d_0, |\sigma_n|)$:

Proposition 3.5.3. *We assume that Ω has a $\mathcal{C}^{1,1}$ boundary, $\psi \in \mathcal{C}^{1,1}(\mathbb{R}^2)$, λ is globally Lipschitz continuous on $\bar{\Omega}$ and $\lambda > 0$. There exists*

$$\Upsilon := \Upsilon(|\Omega|, \max_{\bar{\Omega}} \lambda, \min_{\bar{\Omega}} \lambda, \|\psi\|_{\mathcal{C}^{1,1}(\mathbb{R}^2)}, \kappa) > 0$$

with κ the maximum of the principal curvatures of $\partial\Omega$ such that if $\|\nabla\lambda\|_{L^\infty(\Omega)} \leq \Upsilon$ then for $n \in \mathbb{N}$ large enough, for a.e. every $s \in S$, for every $i \in I_s$, if $l_s^i(u) \cap U = \emptyset$ we have

$$\sup_{F_s^i} |\sigma_n| \leq \sup_{l_s^i(u)} \max(d_0, |\sigma_n|)$$

where F_s^i is the bounded connected component of $\mathbb{R}^2 \setminus l_s^i(u)$.

Remark 3.5.4. When λ is constant this result is true even if Ω and ψ are only Lipschitz continuous.

Proof. By the coarea formula in Proposition 3.2.18, $\sigma_n \rightarrow \sigma$ \mathcal{H}^1 a.e. on $l_s^i(u)$ for a.e. $s \in S$ and every $i \in I_s$. We apply the maximum principle from [36, Theorem

15.1] to $|\nabla u_n|$ on F_s^i . To do so we assume that

$$\|\nabla \lambda\|_{L^\infty(\Omega)} \leq \frac{\min_{\Omega} \lambda^2}{2 \times L \times \sup_{x \in [\frac{b_0}{2}, L]} g''(x) + \frac{g'(x)}{x}}.$$

Here L is the Lipschitz constant introduced in Proposition 3.2.7. For every $n \in \mathbb{N}$, there exists b_n such that $g'_n(b_n) = d_0$.

Hence, for $n \in \mathbb{N}$ large enough, $b_n \geq \frac{b_0}{2}$ and

$$\|\nabla \lambda_n\|_{L^\infty(\Omega)} \leq \frac{\min_{\Omega} \lambda_n^2}{\|\nabla u_n\|_{L^\infty(\Omega)} \times \sup_{x \in [\frac{b_n}{2}, L]} g''_n(x) + \frac{g'_n(x)}{x}}.$$

Thus, thanks to [36, Theorem 15.1, Equation (15.15)] for a.e. $s \in \mathbb{R}$ we have

$$\sup_{F_s} |\nabla u_n| \leq \sup_{l_s^i(u)} \max\left(\frac{b_0}{2}, |\nabla u_n|\right) \leq \sup_{l_s^i(u)} \max(b_n, |\nabla u_n|)$$

for $n \in \mathbb{N}$ large enough. Since g'_n is increasing we obtain $\sup_{F_s} |\sigma_n| \leq \sup_{l_s^i(u)} \max(d_0, |\sigma_n|)$. \square

Proposition 3.5.5. *Let us consider $s \in S$ and $i \in I_s$ such that $l_s^i(u) \cap U = \emptyset$ and $f_0 = C_s^i$ on $l_s^i(u)$. Then for a.e. $t > s$, for every $j \in I_t$ such that $l_t^j(u) \Subset F_s^i$ and $l_t^j(u) \cap U = \emptyset$, we have $\max(d_0, |\sigma|) = C_t^j \mathcal{H}^1$ a.e. on $l_t^j(u)$ with $C_t^j \in \{d_n, n \in \mathbb{N}, 0 \leq n < N\}$ not larger than C_s^i .*

Proof. By construction of f_0 in the proof of Proposition 3.5.1 and Proposition 3.5.2, we can construct a subsequence $\max(d_0, |\sigma_{\psi(n)}|)$ converging uniformly to C_s^i on $l_s^i(u)$ that also converges uniformly to C_t^j on $l_t^j(u)$. Hence, with the previous proposition we get:

$$\begin{aligned} C_t^j &\leq \limsup_{n \rightarrow +\infty} \sup_{l_t^j(u)} \max(d_0, |\sigma_{\psi(n)}|) \leq \limsup_{n \rightarrow +\infty} \sup_{F_s^i(u)} \max(d_0, |\sigma_{\psi(n)}|) \\ &\leq \lim_{n \rightarrow +\infty} \sup_{l_s^i(u)} \max(d_0, |\sigma_{\psi(n)}|) = C_s^i. \end{aligned}$$

Thus, we have that $C_t^j \leq C_s^i$. \square

3.6 Proof of the main theorem

3.6.1 Pseudo Cheeger problem

In this part we combine the maximum principle for $|\sigma|$ and the Euler-Lagrange equation to prove that the level sets are almost Cheeger sets.

We recall the definition of the Cheeger constant of a set:

Definition 3.6.1. The Cheeger constant of Ω is defined as:

$$h_\Omega = \inf_{D \subset \bar{\Omega}} \frac{\text{Per}(D, \mathbb{R}^2)}{|D|}$$

A set $D \subset \bar{\Omega}$ of finite perimeter is said to be a *Cheeger set* if $\text{Per}(D, \mathbb{R}^2) = h_\Omega |D|$.

Remark 3.6.2. There is no Cheeger set D of Ω such that $D \Subset \Omega$ because the function $t \mapsto \frac{\text{Per}(tD, \mathbb{R}^2)}{|tD|}$ is $\frac{1}{t}$ -homogeneous.

The following equality is a consequence of Proposition 3.5.2:

Proposition 3.6.3. For a.e. $s \in S$, for every $i \in I_s$ if $l_s^i(u) \cap U = \emptyset$ we have

$$\int_{F_s^i} \lambda = C_s^i \text{Per}(F_s^i)$$

where F_s^i is the bounded connected component of $\mathbb{R}^2 \setminus l_s^i(u)$ and C_s^i is the constant introduced Proposition in 3.5.2.

Proof. By [32, Section 5.11, Theorem 1], for a.e. $s \in S$ and for every $i \in I_s$, we have that $D\mathbf{1}_{F_s^i} \in BV(\Omega)$.

We consider the sequence $(\sigma_n)_{n \in \mathbb{N}}$ that converges a.e. on Ω towards σ introduced in Section 4. By [32, Section 5.8, Theorem 1] we obtain:

$$\begin{aligned} \int_{F_s^i} \text{div}(\sigma_n) dx &= \int_{\Omega} \mathbf{1}_{F_s^i}(x) \text{div}(\sigma_n) dx = - \int_{\Omega} \langle \sigma_n, \frac{D\mathbf{1}_{F_s^i}}{|D\mathbf{1}_{F_s^i}|} \rangle d|D\mathbf{1}_{F_s^i}| \\ &= - \int_{\partial^* F_s^i} \langle \sigma_n, \frac{D\mathbf{1}_{F_s^i}}{|D\mathbf{1}_{F_s^i}|} \rangle d\mathcal{H}^1. \end{aligned}$$

The set $\partial^* F_s^i$ is introduced in Definition 3.2.16. We can use Proposition 3.2.20 that gives:

$$- \int_{F_s^i} \text{div}(\sigma_n) dx = \int_{\partial^* F_s^i} \langle \sigma_n, \frac{\nabla u}{|\nabla u|} \rangle d\mathcal{H}^1.$$

But by the coarea formula $\sigma_n \rightarrow \sigma$ \mathcal{H}^1 a.e. on $\partial^* F_s^i \subset l_s^i(u)$ for a.e. $s \in \mathbb{R}$ and every $i \in I_s$. By Proposition 3.5.2 and since σ is collinear to $\frac{\nabla u}{|\nabla u|}$ \mathcal{H}^1 a.e. on $l_s^i(u)$, we get for such an s :

$$\lim_{n \rightarrow +\infty} - \int_{F_s^i} \text{div}(\sigma_n) dx = \int_{\partial^* F_s^i} |\sigma| d\mathcal{H}^1 = C_s^i \text{Per}(F_s^i). \quad (3.6.1)$$

Moreover,

$$- \int_{F_s^i} \text{div}(\sigma_n) = \int_{F_s^i} \lambda_n \rightarrow \int_{F_s^i} \lambda \quad (3.6.2)$$

when $n \rightarrow +\infty$, where $\lambda_n := \lambda * \rho_n$. Hence, with (3.6.1) and (3.6.2), we have the desired result:

$$\int_{F_s^i} \lambda = C_s^i \text{Per}(F_s^i)$$

for a.e. $s \in S$, for every $i \in I_s$ if $l_s^i(u) \cap U = \emptyset$.

□

We also have:

Proposition 3.6.4. *For every set $F \subset \overline{F_s^i}$ of finite perimeter we have*

$$\int_F \lambda \leq C_s^i \text{Per}(F).$$

Proof. We follow the same ideas developed in the previous proof. We have:

$$-\int_F \text{div}(\sigma_n) = \int_{\partial^* F} \langle \sigma_n, \nu_F \rangle d\mathcal{H}^1.$$

The term in the left hand side tends to $\int_F \lambda$ when $n \rightarrow +\infty$. For the term in the right hand side we get:

$$\int_{\partial^* F} \langle \sigma_n, \nu_F \rangle d\mathcal{H}^1 \leq \int_{\partial^* F} |\sigma_n| d\mathcal{H}^1.$$

By Proposition 3.5.3, $\sup_{F_s^i} |\sigma_n| \leq \sup_{l_s^i} \max(d_0, |\sigma_n|)$. By Proposition 3.5.1 and Proposition 3.5.2 we have $\max(d_0, |\sigma_n|) \rightarrow C_s^i \mathcal{H}^1$ a.e. on $l_s^i(u)$ when $n \rightarrow +\infty$. Hence,

$$\int_F \lambda \leq C_s^i \text{Per}(F).$$

□

3.6.2 Main proof.

We first prove Theorem 3.1.5:

Proof of Theorem 3.1.5. For a.e. $s \in \mathbb{R}$, for every $i \in I_s$ if $l_s^i(u) \cap (U \cup \partial\Omega) = \emptyset$ then by Proposition 3.6.3,

$$\int_{F_s^i} \lambda = C_s^i \text{Per}(F_s^i).$$

We assume that such a $l_s^i(u)$ exists. Since $F_s^i \Subset \Omega$ by Remark 3.6.2 and the previous equality, we have

$$h_\Omega < \frac{\text{Per}(F_s^i)}{|F_s^i|} = \frac{1}{C_s^i |F_s^i|} \int_{F_s^i} \lambda.$$

We have that $\|\lambda\|_{L^\infty(\Omega)} \leq d_0 h_\Omega$. Thus,

$$h_\Omega < \frac{d_0 h_\Omega}{C_s^i}.$$

Hence, $C_s^i < d_0$ which is a contradiction. Thus, for a.e. $s \in \mathbb{R}$ we have $l_s^i(u) \cap (U \cup \partial\Omega) \neq \emptyset$.

Let v be another minimizer. By Proposition 3.3.2 and Proposition 3.3.6, for a.e. $s \in \mathbb{R}$, on every connected component $l_s^i(u)$ of $L_s(u)$ that is not a point we have $u = v$ on $l_s^i(u)$ or $\nabla(u - v) = 0$ \mathcal{H}^1 a.e. on $l_s^i(u)$.

By the coarea formula:

$$\int_{\mathbb{R}^2 \cap [u \neq v]} |\nabla(u-v)| |\nabla u| = \int_{\mathbb{R}} \int_{L_s(u) \cap [u \neq v]} |\nabla(u-v)| d\mathcal{H}^1 ds.$$

By Theorem 3.2.8, for a.e. $s \in \mathbb{R}$, $\mathcal{H}^1(L_s \setminus L_s^*) = 0$ and L_s^* is composed by a countable number of curves $l_s^i(u)$. For every $i \in I_s$, we have:

$$\int_{l_s^i(u) \cap [u \neq v]} |\nabla(u-v)| d\mathcal{H}^1 = 0.$$

By Proposition 3.3.2, we get

$$\int_{L_s(u) \cap [u \neq v]} |\nabla(u-v)| d\mathcal{H}^1 = 0.$$

Hence,

$$\int_{\mathbb{R}^2 \cap [u \neq v]} |\nabla(u-v)| |\nabla u| dx = 0.$$

For the same reasons,

$$\int_{\mathbb{R}^2 \cap [u \neq v]} |\nabla(u-v)| |\nabla v| dx = 0.$$

Hence we have $\nabla(u-v) = 0$ a.e. on $[u \neq v]$. This implies that the map $u-v$ is constant on \mathbb{R}^2 . Since $u=v$ on $\partial\Omega$, we have that $u=v$ on Ω . \square

Now, we are ready to prove the main theorem:

Proof of Theorem 3.1.1. Let u and v be two minimizers of \mathcal{P}_λ . We assume that $\frac{\min \lambda}{\text{diam} \Omega} \leq \Upsilon < \frac{\min \lambda}{\text{diam} \Omega}$ up to decreasing the constant Υ from Proposition 3.5.3. For a.e. $s \in \mathbb{R}$, every $i \in I_s$, if $l_s^i(u) \cap (U \cup \partial\Omega) = \emptyset$ by Proposition 3.5.2, $|\sigma| = C_s^i$ \mathcal{H}^1 a.e. on $l_s^i(u)$ with $C_s^i \in \{d_n, 0 \leq n < N\}$.

We prove by induction on $0 \leq n < N$ that if $|\sigma| = d_n$ on $l_s^i(u)$ then $\nabla(u-v) = 0$ \mathcal{H}^1 a.e. on l_s^i .

Step 1 As an initialisation step, we assume that $C_s^i = d_0$. By Proposition 3.3.3, $u > s$ on F_s^i . By the coarea formula, for a.e. $t > s$, t belongs to \mathcal{S} . We assume that there exists $t > s$ and $j \in I_t$ such that $l_t^j(u) \cap U = \emptyset$ and $F_t \Subset F_s^i$. By Proposition 3.5.3, $|\sigma| = d_0$ a.e. on $l_t^j(u)$. Thus, by Proposition 3.6.3 we have that $\int_{F_t} \lambda = d_0 \text{Per}(F_t)$. For $r > 1$ close to 1 and $x_0 \in \Omega$, we introduce $F_t^r = r(F_t - x_0) + x_0 \Subset F_s$.

Hence, by Proposition 3.5.3 we have $|\sigma| \leq d_0$ on ∂F_t^r . Then, by Proposition 3.6.4,

$$r^2 \int_{F_t} \lambda(r(x-x_0) + x_0) dx = \int_{F_t^r} \lambda(y) dy \leq d_0 \text{Per}(F_t^r) = r d_0 \text{Per}(F_t) = r \int_{F_t} \lambda(x) dx.$$

Thus,

$$\int_{F_t} r\lambda(r(x - x_0) + x_0) - \lambda(x) \leq 0.$$

Since $\|\nabla\lambda\|_{L^\infty(\Omega)} < \frac{\min \lambda}{\text{diam}(\Omega)}$, we have that $r\lambda(r(x - x_0) + x_0) - \lambda(x) > 0$ for every $x \in F_s^i$. That is a contradiction. Hence, for a.e. $t > s$ and every $j \in I_t$ such that $l_t^j(u) \Subset F_s^i$ we have $l_t^j(u) \cap U \neq \emptyset$. By Proposition 3.3.6, $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_t^j(u)$. By the coarea formula, $\nabla(u - v) = 0$ a.e. in F_s^i . By Lemma 3.3.8, we have that $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_s^i(u)$.

Step 2 Now, we prove the induction part. We consider $1 \leq n < N$. Let us assume that for every $k < n$, for a.e. $t \in \mathbb{R}$ and every $j \in I_t$ if $l_t^j(u) \cap (U \cup \partial\Omega) = \emptyset$ and $C_t^j = d_k$ then $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_t^j(u)$.

If $l_s^i(u)$ is such that $l_s^i(u) \cap (U \cup \partial\Omega) = \emptyset$ and $C_s^i = d_n$, we consider $t > s$ such that $l_t^j(u) \cap U = \emptyset$ and $F_t \Subset F_s^i$. Hence, by Proposition 3.5.5, either $C_t^j = d_n$ or $C_t^j < d_n$. If $C_t^j = d_n$, then as in **Step 1** we construct $F_t^r \Subset F_s^i$ and we prove the $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_t^j(u)$. By induction we have $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_t^j(u)$ in the second case. Hence, $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_t^j(u)$. We can conclude as in **Step 1** that $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_s^i(u)$.

Step 3 For a.e. $s \in S$, we consider $l_s(u)$ a connected component of $L^*(u)$. If $l_s(u) \cap (\mathbb{R}^2 \setminus \Omega) \neq \emptyset$ then by Proposition 3.3.2, $u = v$ on $l_s(u)$. If $l_s(u) \subset \Omega$ and $l_s(u) \cap U \neq \emptyset$ then by Proposition 3.3.6, $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_s(u)$. Finally, thanks to **Step 2** if $l_s(u) \Subset \Omega$ and $l_s(u) \cap U = \emptyset$ then we have $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_s(u)$. Hence, we can prove with the coarea formula, as in the proof of Theorem 3.1.5, that $u = v$. □

3.7 Extensions

In this section we present an extension of the main theorem where \mathcal{SC} has a countable number of connected components. We assume that g is \mathcal{C}^2 and $g'' > 0$ on $\text{int}(\mathcal{SC}) \setminus \{0\}$ and:

$$\mathcal{SC} \cap \mathbb{R}_+ = \mathcal{SC}_\infty \cup \left(\bigcup_{n \in \mathbb{N}} \mathcal{SC}_n \right)$$

with $\mathcal{SC}_0 := [0, b_0)$, $\mathcal{SC}_n := (a_n, b_n)$ for every $n \in \mathbb{N}^*$ and \mathcal{SC}_∞ is defined below. We assume that $(a_n)_{n \in \mathbb{N}^*}$ and $(b_n)_{n \in \mathbb{N}}$ are strictly increasing sequences. Moreover, the sequence $(a_n)_{n \in \mathbb{N}^*}$ is bounded and $\lim_{n \rightarrow +\infty} a_n = \alpha$. For every $n \in \mathbb{N}$, $d_n := g'(b_n) = g'(a_{n+1})$ is an increasing sequence.

The connected component \mathcal{SC}_∞ is exceptional because $\mathcal{SC}_\infty := (a_\infty, +\infty)$ if $\alpha < a_\infty$ and $\mathcal{SC}_\infty := [\alpha, +\infty)$ if $\alpha = a_\infty$.

Proposition 3.7.1. *In that case Theorem 3.1.1 is still valid.*

Proof. With this new *structural assumptions*, the minimizers of \mathcal{P}_λ are still globally Lipschitz-continuous on Ω . We can define U as previously with $\text{int}(\mathcal{SC})$ instead of \mathcal{SC} . The function $\max(d_0, g'(\nabla u))$ is still in $H_{loc}^1(\Omega)$. Since $|g'(\mathbb{R} \setminus \mathcal{SC})| = 0$, Proposition 3.5.2 remains valid. Hence, the last crucial point is the end of the induction argument in **Step 2** of the proof of Theorem 3.1.1. We assume that there exists $l_s^i(u) \Subset \Omega$ such that $l_s^i(u) \cap U = \emptyset$ and $C_s^i = g'(a_\infty)$. Then for every $l_t \Subset F_s^i$, we either have that $C_t = g'(a_\infty)$ or $\nabla(u - v) = 0$ \mathcal{H}^1 a.e. on l_t . Hence, we have that $\nabla(u - v) = 0$ \mathcal{H}^1 a.e. on $l_s^i(u)$. Thus, $u = v$ on Ω . \square

Remark 3.7.2. The sets

$$[0, \frac{1}{2}) \cup \left(\bigcup_{n \in \mathbb{N}^*} \left(\frac{2^{2n} - 1}{2^{2n}}, \frac{2^{2n+1} - 1}{2^{2n+1}} \right) \right) \cup [1, +\infty)$$

and

$$[0, \frac{1}{2}) \cup \left(\bigcup_{n \in \mathbb{N}^*} \left(\frac{2^{2n} - 1}{2^{2n}}, \frac{2^{2n+1} - 1}{2^{2n+1}} \right) \right) \cup (2, +\infty)$$

satisfy the new *structural assumptions* made on \mathcal{SC} .

Regularity of the stress field for degenerate and/or singular elliptic problems

This chapter is based on the paper "*Regularity of the stress field for degenerate and/or singular elliptic problems*" [51] submitted. I would like to express particular thanks to Xavier Lamy for his discussions and remarks about the regularity of the solutions of partial differential equations which allowed me to improve this chapter.

Abstract

We investigate the regularity of the solutions of degenerate and/or singular elliptic equations. We prove the continuity of $G(\nabla u)$ where u is a locally Lipschitz solution of $\operatorname{div} G(\nabla u) = \lambda \in \mathbb{R}$ in dimension two under some growth assumptions on G . We also present a result true in any dimension stating that the distance between ∇u and the degeneracy set of G is continuous.

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4.1 Introduction

4.1.1 A first example

In this article, we establish the continuity of certain functions of the gradients of solutions for elliptic partial differential equations. For instance, let us consider a locally Lipschitz continuous function u defined on an open subset Ω of \mathbb{R}^2 that minimizes the following functional:

$$v \rightarrow \int_{\Omega} \varphi(\nabla v) - \lambda v \quad (4.1.1)$$

among the functions in $W_u^{1,2}(\Omega)$. The set $W_u^{1,2}(\Omega)$ is the set of functions $v \in L^2(\Omega)$ with a distributional gradient that is also in $L^2(\Omega)$ such that u and v share the same trace on the boundary $\partial\Omega$ of Ω . Here we assume that $\lambda \in \mathbb{R}_+$ and φ has the following form:

$$\varphi(z) := \begin{cases} \frac{1}{2}|z|^2 & \text{if } |z| \leq 1, \\ |z| - \frac{1}{2} & \text{if } 1 < |z| < 2, \\ \frac{1}{4}|z|^2 + \frac{1}{2} & \text{if } 2 \leq |z|. \end{cases} \quad (4.1.2)$$

This convex function is not strictly convex and is not \mathcal{C}^2 . Hence, we cannot apply the classical regularity theory for smooth strictly convex functions. The case where we do not have ellipticity at only one point has also been well studied. For instance in the case of the p -Laplacian when $\varphi(z) = |z|^p$ with $p > 1$ we know that the solutions are $\mathcal{C}^{1,\alpha}$. However, in our example $D^2\varphi(z)$ has an eigenvalue equal to 0 on the entire annulus $\{1 < |z| < 2\}$. Thus, we cannot use the results already known when the set of degeneracy is just a point.

For this kind of problems we know that we can not expect u to be \mathcal{C}^1 on Ω . In fact by [22, Theorem 1] the function

$$u(x) := \begin{cases} C - \frac{\lambda}{4}|x|^2 & \text{if } |x| \leq \frac{2}{\lambda}, \\ C + \frac{1}{\lambda} - \frac{\lambda}{2}|x|^2 & \text{if } \frac{2}{\lambda} < |x| \leq 1 \end{cases} \quad (4.1.3)$$

is a minimizer of (4.1.1) with φ as in (4.1.2) on the set $W_u^{1,2}(\Omega)$ with its own boundary condition.

The problem (4.1.1) with φ as in (4.1.2) was introduced by Kawohl, Stara and Wittum in [41] where the authors want to prove the uniqueness of the solutions. They assume that Ω has several symmetries in order to establish the Lipschitz continuity of the level sets of the minimizers. In our article, we prove that we do not need Ω to have any symmetry to obtain such a result. This shows that a good understanding of the regularity of the solutions can be useful to prove the uniqueness of the minimizers. Nevertheless, in the case of (4.1.1) with φ as in (4.1.2) and $\lambda \in \mathbb{R}_+$ we can use [52, Theorem 1.1] to obtain a direct proof of uniqueness.

Since in general, u is not \mathcal{C}^1 , one of our goals is to prove the continuity of $\nabla\varphi(\nabla u)$. This new result has important applications such as the local \mathcal{C}^∞ regularity of the

solution around points where the gradient has a norm either smaller than one or larger than two. This also gives that generically the level sets of a solution are C^1 curves.

More generally, the aim of the article is to prove this kind of continuity estimates for different types of convex functions defined on \mathbb{R}^2 that are not strictly convex. In fact, the results are stated in the larger framework of elliptic equations that also enclose the Euler-Lagrange equations associated to minimization problems. In Theorem 4.1.9, we partially generalize these results to any dimension $N \in \mathbb{N}$.

4.1.2 General problem

Let $G : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a continuous function with $N \in \mathbb{N}$. In this article we study the regularity of locally Lipschitz continuous weak solutions of the following equation:

$$\operatorname{div} G(\nabla u_0) = f \text{ in } \Omega \quad (4.1.4)$$

with Ω an open bounded set of \mathbb{R}^N and $f : \mathbb{R}^N \rightarrow \mathbb{R}$ in $W^{1,q}(\Omega)$ with $q > N$. We assume that G is non-decreasing in the following sense:

$$\langle G(z_1) - G(z_2), z_1 - z_2 \rangle \geq 0 \quad (4.1.5)$$

for every $z_1, z_2 \in \mathbb{R}^N$. By solution we mean every locally Lipschitz function u_0 such that

$$\int_{\Omega} \langle G(\nabla u_0), \nabla \theta \rangle = - \int_{\Omega} f \theta$$

for every function $\theta \in C_0^\infty(\Omega)$.

Many results state that the solutions are locally Lipschitz under suitable growth on G at infinity, see e.g. [14], [20], [31] and [34]. The main goal of the paper is to prove the continuity of the stress field $G(\nabla u_0)$ depending on the assumptions of G and f .

When G is the gradient of a convex function φ , we obtain a nonlinear elliptic equation that can be seen as the Euler-Lagrange equation associated to the minimization of

$$\int_{\Omega} \varphi(\nabla v) + f v. \quad (4.1.6)$$

If G is smooth and if there exists $C > 0$ such that

$$\frac{1}{C} |A - B|^2 \leq \langle G(A) - G(B), A - B \rangle \leq C |A - B|^2$$

for every $A, B \in \mathbb{R}^N$ then the solutions of (4.1.4) are C^1 when $f \in L^p(\Omega)$ with $p > N$, (see [33, Theorem 6.33]). This is the case for instance for the Poisson equation when $G = Id$. When there exist A and B two distinct vectors of \mathbb{R}^N such that $\langle G(A) - G(B), A - B \rangle = 0$ we say that the equation is degenerate. If we cannot bound from above $\langle G(A) - G(B), A - B \rangle$ by a constant times the quantity $|A - B|^2$

then we say that the equation is singular. In these two critical frameworks the C^1 regularity is not guaranteed.

However, the study of the regularity of the solutions for degenerate and/or singular equations with a large set of degeneracy and/or singularity is a recent and dynamic subject. In the seminal paper [5], the authors study the partial $C^{1,\alpha}$ regularity of the solutions. Namely, let u_0 be a minimizer of (4.1.6), if $x \in \Omega$ is a Lebesgue point of ∇u_0 such that φ is C^2 and $D^2\varphi$ is positive-definite on a neighborhood of $\nabla u_0(x)$ then there exists an open neighborhood U of x such that $u_0 \in C^{1,\alpha}(U)$. If we apply this result to φ as in (4.1.2) then there exist two open sets U_1 and U_2 such that $u_0 \in C^{1,\alpha}$ on these two sets. Moreover, $|\nabla u_0| < 1$ on U_1 , $|\nabla u_0| > 2$ on U_2 and for a.e. $x \in \Omega \setminus (U_1 \cup U_2)$ we have $1 \leq |\nabla u_0(x)| \leq 2$. The drawback of this result is that we do not know the behavior of ∇u_0 at the boundary of the set where $u \in C^{1,\alpha}$. For instance, does $|\nabla u_0(x)|$ converge to 2 when x converges to $\partial U_2 \cap \Omega$ from the inside of U_2 ?

Some results of our paper use ideas from [29]. In this article, De Silva and Savin prove in particular two theorems that state that the minimizers of (4.1.6) with $f \equiv 0$ are C^1 . They assume that φ is strictly convex and prove two results. In [29, Theorem 1.1], φ is not singular and degenerate on the same set. In [29, Theorem 1.2], φ is not singular except at a finite number of points.

There is a recent family of results when the set of degeneracy or singularity is convex. In this case $G = \nabla\varphi$ with φ a convex function that is strongly convex outside a convex set C containing the origin. Let us quote two results of continuity everywhere on Ω . The first one is an article of Santambrogio and Vespi [62, Theorem 11] in dimension two and the second one an article of Colombo and Figalli [26, Theorem 1.1]. In the latter case, the authors prove that $F(\nabla u)$ is continuous on Ω when F is a continuous function that vanishes on C . In the vectorial case, the article [7] extends [26] for a particular φ that is equal to $\frac{1}{p}(|\cdot| - 1)_+^p$ with $p > 1$. It would be interesting to extend the results of our paper to the vectorial case. When φ is as in (4.1.2), thanks to [26, Theorem 1.1] we get that $(|\nabla u_0| - 2)_+$ is continuous on Ω . In our paper we obtain a similar result for $(1 - |\nabla u_0|)_+$ even if the set of degeneracy is not convex.

4.1.3 Main results

We state the new results of this paper. Theorem 4.1.1, Theorem 4.1.4 and Theorem 4.1.6 are only valid in dimension two. Theorem 4.1.9 which is valid in any dimension extends [57, Theorem 2.1] with a larger class of degeneracy sets.

Theorem 4.1.1. *Let us assume that $G := G_1 + G_2 \in C_{loc}^{0,1}(\mathbb{R}^2)$ and that $f = 0$. We assume that:*

for every $L > 0$ there exists $C_L > 0$ such that for every $z_1, z_2 \in B_L(0)$: (A₁)

$$\langle G_1(z_1) - G_1(z_2), z_1 - z_2 \rangle \geq C_L |G_1(z_1) - G_1(z_2)|^2$$

and

the function G_2 is the gradient of a $\mathcal{C}_{loc}^{1,1}(\mathbb{R}^2)$ convex function φ . (A₂)

Then for every solution u_0 of (4.1.4) the function $G(\nabla u_0)$ is continuous.

We point out that (A₁) does not imply that G_1 is strictly increasing or a gradient of a convex function. Moreover, we have an explicit modulus of continuity:

Remark 4.1.2. Let u_0 be a solution of (4.1.4). Then for every $\epsilon > 0$ and $x_0 \in \Omega'' \Subset \Omega' \Subset \Omega$, $G(\nabla u_0(B_\delta(x_0))) \subset B_\epsilon(G(\nabla u(x_0)))$ with

$$\delta = \frac{\text{dist}(x_0, \partial\Omega'')}{2} \exp - \frac{2\pi \|G(\nabla u)\|_{W^{1,2}(\Omega'')}^2}{\epsilon^2}$$

and

$$\|G(\nabla u_0)\|_{W^{1,2}(\Omega'')}^2 \leq \frac{2L^2|\Omega'|}{K \text{dist}(\partial\Omega', \partial\Omega'')^2}$$

with $L = \|\nabla u_0\|_{L^\infty(\Omega')}$ and $K > 0$ depending only on $\|DG\|_{L^\infty(B_L(0))}$ and C_L the constant introduced in (A₁).

When $f \in \mathbb{R}$ is not equal to zero, we can extend the previous result under structural assumptions on G . In order to state the next result we need the following definition:

Definition 4.1.3. We say that a convex function $\mathcal{N} : \mathbb{R}^N \rightarrow \mathbb{R}_+$ is a *pseudo-norm* if $\mathcal{N}(0) = 0$, \mathcal{N} is positively homogeneous and $\{z \in \mathbb{R}^N \text{ such that } \mathcal{N}(z) < 1\}$ is an open strictly convex bounded set with a $\mathcal{C}^{1,1}$ continuous boundary.

It is important to notice that \mathcal{N} is not necessarily symmetric. Hence, the definition of a *pseudo-norm* is more general than the definition of a norm with $\mathcal{C}^{1,1}$ level sets.

The next theorem is stated when $f \equiv \lambda \in \mathbb{R}$ and G is the sum of gradients of convex functions:

Theorem 4.1.4. Let us assume that $f \equiv \lambda \in \mathbb{R}$ and $G = \sum_{i=1}^n G_i$ with $n \in \mathbb{N}$. Here the functions $(G_i)_{1 \leq i \leq n}$ are gradients of convex functions $(\varphi_i)_{1 \leq i \leq n}$ that have one of the two following forms:

$$\varphi_i(z) := f_i(\mathcal{N}_i(z - \xi_i)) \text{ and } f_i'(z) = 0 \Leftrightarrow z = 0 \quad (\text{A}_3)$$

with $f_i \in \mathcal{C}_{loc}^{1,1}(\mathbb{R})$ a convex function, \mathcal{N}_i a pseudo-norm and $\xi_i \in \mathbb{R}^2$.

$$\varphi_i(z) := f_i(\langle z, \xi_i \rangle) \quad (\text{A}_4)$$

where $f_i \in \mathcal{C}_{loc}^{1,1}(\mathbb{R})$ is a convex function and $\xi_i \in \mathbb{R}^2 \setminus \{0\}$.

Then for every solution u_0 of (4.1.4) the function $G(\nabla u_0)$ is continuous.

Remark 4.1.5. In this article we use the convention that $0 \notin \mathbb{N}$.

We can apply Theorem 4.1.4 to $G = \nabla\varphi$ with φ as in (4.1.2). Hence, $\nabla\varphi(\nabla u_0)$ is continuous. In this particular case, $[1 - |\nabla u_0|]_+$ is continuous which is a new feature that can not be obtained with [26], [57] or [62] since the set of degeneracy is an annulus. Thus, we know that $|\nabla u_0(x)|$ has to go to 1 when x converges to the boundary of the open set $\{x' \in \Omega \text{ such that } |\nabla u_0(x')| < 1\}$ from the inside. This new result is useful to study global regularity of the level lines.

Theorem 4.1.4 can also be used for orthotropic type functionals. By orthotropic we mean that $G = \nabla\varphi$ and φ is the sum of convex functions $z \mapsto \varphi_i(z)$ that depends only on one coordinate of z . Hence, if $\varphi(z) = |z_1|^{p_1} + |z_2|^{p_2}$ with $2 \leq p_1 \leq p_2$ and with $z_i := \langle z, e_i \rangle$, then $\nabla\varphi(\nabla u_0)$ is continuous. In the singular case where $1 < p_1 \leq 2 \leq p_2$ we have the following result:

Theorem 4.1.6. *Let us assume that $f \equiv \lambda \in \mathbb{R}$ and $G = G_1 + G_2$ with $G_i(z) := f'_i(\langle z, \xi_i \rangle)\xi_i$ where $f_1 \in \mathcal{C}_{loc}^{1,1}(\mathbb{R})$ and $f_2 \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}_{loc}^{1,1}(\mathbb{R} \setminus \{0\})$ are two convex functions and $\xi_1, \xi_2 \in \mathbb{R}^2 \setminus \{0\}$ are non colinear. Moreover, we assume that there exist $r > 0$ and a modulus of convexity $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ for f_2 , that is a continuous function satisfying $\omega(t) = 0 \Leftrightarrow t = 0$ such that for every $x, y \in (-r, r)$ we have that $(f'_1(x) - f'_2(y))(x - y) \geq \omega(|x - y|)$.*

Then for every solution u_0 of (4.1.4) the function $G(\nabla u_0)$ is continuous.

Theorem 4.1.6 can be used to prove some regularity results of the solutions in the case of orthotropic functionals with more general growth than power-type growth. In fact if $\varphi(z) = |z_1|^{p_1} + |z_2|^{p_2}$ with $1 < p_1 < 2 \leq p_2$ as in a case of [9] then the functional is singular on $\{z_1 = 0\}$ and degenerate on $\{z_2 = 0\}$. Hence this is only when $z = 0$ that we have both problems. In our case the set where φ is singular and degenerate at the same time can be a line. For instance, we can consider $f_1(t) := (|t| - r)_+^2$ and $f_2(t) = |t|^{\frac{3}{2}}$ for every $t \in \mathbb{R}$. But we do not have the \mathcal{C}^1 regularity in that case.

However, the two functions $(|\partial_1 u_0| - r)_+$ and $\frac{1}{|\partial_2 u_0|^{\frac{1}{2}}}\partial_2 u_0$ are continuous. We can see that the continuity of this last function implies the continuity of $\partial_2 u_0$. Hence, the regularity of $G(\nabla u_0)$ can be useful when we exploit the local properties of G . For instance, in the setting of Theorem 4.1.1, Theorem 4.1.4 or Theorem 4.1.6 we have a local \mathcal{C}^1 regularity result:

Proposition 4.1.7. *Under the assumptions of Theorem 4.1.1, Theorem 4.1.4 or Theorem 4.1.6, $G(\nabla u_0)$ has a continuous representative σ . If G is a homeomorphism between two open sets U and V then $u_0 \in \mathcal{C}^1(\sigma^{-1}(V))$.*

In the case where $\sigma^{-1}(V) = \Omega$ then u_0 is \mathcal{C}^1 . The above proposition can be seen as an extension of what is known in dimension one, see e.g. [23, Theorem 15.5] that states that a Lipschitz minimizer u_0 of $\int_a^b F(x, u(x), u'(x))dx$ is \mathcal{C}^1 when $y \rightarrow F(x, u_0(x), y)$ is strictly convex for a.e. $x \in (a, b)$.

This proposition is useful in the case of orthotropic functionals with $\varphi(z) = |z_1|^{p_1} + |z_2|^{p_2}$. The first result on this subject [29, Theorem 1.1] provides the \mathcal{C}^1 regularity of the minimizers when the problem is fully singular: $1 < p_1 \leq p_2 \leq 2$

or fully degenerate: $2 \leq p_1 \leq p_2$. We point out that we can use Theorem 4.1.4 and Proposition 4.1.7 to obtain a new proof of the C^1 regularity of u_0 in the degenerate case $2 \leq p_1 \leq p_2$.

The singular and degenerate case where $1 < p_1 < 2 < p_2$ is studied in [9] using ideas of [29] but Proposition 4.1.7 combined with Theorem 4.1.6 gives a new proof of the C^1 regularity in this case. The fully singular case $1 < p_1 \leq p_2 \leq 2$ is out of the scope of Theorem 4.1.4 and Theorem 4.1.6 unless $p_2 = 2$. Some other cases with different exponents can be found in the following papers [11], [48] and [61].

Under the assumptions of Theorem 4.1.4, when G is the gradient of a convex function φ that depends only on the Euclidean norm we have that $\frac{\nabla\varphi(\nabla u_0)}{|\nabla\varphi(\nabla u_0)|} = \frac{\nabla u_0}{|\nabla u_0|}$ is continuous when $\nabla u_0 \neq 0$. This allows to define the normal of the level sets as a continuous function. In that case we have the following result on the regularity of the level sets of a solution:

Proposition 4.1.8. *Let φ be a radial $C_{loc}^{1,1}(\mathbb{R}^2)$ convex function and u_0 a solution of (4.1.4) with $G = \nabla\varphi$. We denote by σ the continuous representative of $\nabla\varphi(\nabla u_0)$ obtained in Theorem 4.1.4. Then for a.e. $t \in \mathbb{R}$, if we set $L_t := [u_0 = t]$ then the connected components of $L_t \cap [\sigma \neq 0]$ are C^1 curves.*

Our last result is an improvement of [57, Theorem 2.1]. In this article Mooney considers a C^1 convex function φ and proves that the Lipschitz minimizers of

$$u \rightarrow \int_{\Omega} \varphi(\nabla u)$$

are C^1 under some assumptions on φ . He introduces the sets $\mathcal{O}_k := \{z \in \mathbb{R}^N, \frac{1}{k}|v|^2 < \langle D^2\varphi(z)v, v \rangle < k|v|^2 \text{ for every } v \in \mathbb{R}^N\}$ and $D_\varphi = \mathbb{R}^N \setminus \bigcup_{k \in \mathbb{N}} \mathcal{O}_k$. Then [57, Theorem 2.1] states that if φ is C^2 outside D_φ and if D_φ is a finite set of coplanar points then the solutions are C^1 .

In order to state the last theorem we assume that there exists a compact set D_G such that $G \in C^1(\mathbb{R}^N \setminus D_G)$ and $D_G = \mathbb{R}^N \setminus \bigcup_{k \in \mathbb{N}} \mathcal{O}_k$ with $\mathcal{O}_k := \{z \in \mathbb{R}^N, \frac{1}{k}|v|^2 < \langle DG(z)v, v \rangle < k|v|^2 \text{ for every } v \in \mathbb{R}^N\}$. For every $t \geq 0$, we introduce the closed t -neighborhood of a set U as $\overline{N}_t(U) := \{z \in \mathbb{R}^N \text{ such that } \text{dist}(z, U) \leq t\}$.

Theorem 4.1.9. *Let us assume that $f \in W^{1,q}(\Omega)$ with $q > N$, D_G is contained in a plane and has finitely many connected components. We assume that there exists $t_0 > 0$ such that for every $0 \leq t \leq t_0$ the connected components of $\overline{N}_t(D_G)$ are simply connected. Then for every solution u_0 of (4.1.4), $\text{dist}(\nabla u_0, D_G)$ and $\nabla u_0 \times \text{dist}(\nabla u_0, D_G)$ are continuous. Moreover, if G is constant on each connected components of D_G then $G(\nabla u_0)$ is continuous.*

This assumption on the simply connected neighborhoods is satisfied when the connected components of D_G are simply connected with a Lipschitz boundary. The main difference between this result and [57, Theorem 2.1] is that the degeneracy set D_G is not just points. However, even if the conclusion is weakened the solutions

are still \mathcal{C}^1 around points $x \in \Omega$ such that $\nabla u(x)$ is outside this set of degeneracy. Furthermore, we prove that the distance between ∇u and the degeneracy set is a continuous function.

This extension is natural in the sense that this is an improvement of [57] comparable to the improvement of [26] and [62] to the p -Laplacian case. In fact, we can look at [26] or [62] as an extension of what is known for the p -Laplacian case where the set of degeneracy is one point to the case where the set of degeneracy is larger. In this case, it is proven that the distance between ∇u_0 and the degeneracy set is continuous. This is exactly what we are doing in Theorem 4.1.9 with respect to [57, Theorem 2.1].

The study of the regularity of the solutions of (4.1.4) with a right-hand side $f \in L^q(\Omega)$ with $q > N$ is a widely studied subject in the classical framework of *uniform* elliptic equations and degenerate problems. It is the case in [7] and [26] for instance. In our case, we allow a right-hand side a the smaller set of Sobolev functions: $f \in W^{1,q}(\Omega)$ with $q > N$.

It is important to notice that the case where G is constant on each connected components of D_G does not cover the framework of Theorem 4.1.1, Theorem 4.1.4 and Theorem 4.1.6 since the connected components of D_G must be simply connected. That is not the case when $G = \nabla \varphi$ with φ as in (4.1.2) for instance.

4.1.4 Ideas of the proofs

The proof of the continuity of $G(\nabla u_0)$ in Theorem 4.1.1, Theorem 4.1.4 and Theorem 4.1.6 uses ideas from [29]. In this article the authors want to prove the \mathcal{C}^1 regularity for Lipschitz minimizers of the following functional:

$$v \rightarrow \int_{\Omega} F(\nabla v).$$

A major difference between our article and [29] is that we do not require that G is strictly increasing, which as expected weakens the conclusion. The solutions are not necessarily \mathcal{C}^1 as shown in (4.1.3) but Proposition 4.1.7 provides a partial answer to that.

We can divide the proofs of Theorem 4.1.1, Theorem 4.1.4 and Theorem 4.1.6 in four parts:

Part 1 We regularize G in order to work with smooth elliptic equations of the form

$$\operatorname{div} G^m(\nabla u_m) = f_m.$$

We have to be careful when we approximate our problem since the functions $(G^m)_{m \in \mathbb{N}}$ have to share some properties of G such as the *pseudo-norm* structure or the orthotropic form.

Part 2 As in [29], we want to prove that $\|G^m(\nabla u_m)\|_{W^{1,2}(\Omega)}$ can be bounded uniformly in $m \in \mathbb{N}$. Since Theorem 4.1.1 is stated for a function G that is not necessarily the gradient of a convex function we have to adapt some ideas of [29]

to the setting of partial differential equations. In the case of Theorem 4.1.1 and Theorem 4.1.4 we prove the following result:

Proposition 4.1.10. *We assume that $G \in \mathcal{C}^1(\mathbb{R}^N)$ satisfies the assumptions of Theorem 4.1.1 or Theorem 4.1.4. Then $G(\nabla u) \in W_{loc}^{1,2}(\Omega)$.*

We have an analogous result in the framework of Theorem 4.1.6. In our case we have to combine some results of [29] with an adaptation of [19, Theorem 2.1] to obtain Sobolev estimates in that framework. Hence, we can avoid the singularity at the origin with the following result:

Proposition 4.1.11. *We assume that $G \in \mathcal{C}^1(\mathbb{R}^N)$ satisfies the assumption of Theorem 4.1.6. Then for every $\Omega' \Subset \Omega$ and every $r > 0$:*

$$\int_{\Omega' \cap U_r} |\nabla[G(\nabla u)]|^2 \leq C(G, r, \Omega') \tag{4.1.7}$$

where $U_r := \{x \in \Omega \text{ such that } |\langle \nabla u(x), \xi_2 \rangle| \geq r\}$. Moreover, $G_1(\nabla u) \in W_{loc}^{1,2}(\Omega)$.

Part 3 We use this uniform estimate to obtain a uniform modulus of continuity. The original idea, specific to the dimension two, is due to Lebesgue and is used e.g. in [29, Lemma 2.1] and [48, Lemma 3.1]:

Proposition 4.1.12. *Let $H \in W_{loc}^{1,2}(\Omega)$. If for every $\epsilon > 0$ and every $x_0 \in \Omega$ there exists $C(\epsilon, x_0) > 0$ such that for every $0 < \delta < \text{dist}(x_0, \partial\Omega)$:*

$$\text{osc}_{B_\delta(x_0)} H \geq \epsilon \Rightarrow \text{osc}_{\partial B_\delta(x_0)} H \geq C(\epsilon, x_0),$$

then H is continuous at x_0 . Here, $\text{osc}_{B_\delta(x_0)} H := \sup_{x,y \in B_\delta(x_0)} |H(x) - H(y)|$.

The second tool is a classical maximum principle see e.g. [36, Theorem 3.1]:

Proposition 4.1.13. *Let u be a \mathcal{C}^3 solution of (4.1.4) with $G \in \mathcal{C}^2$ and $f \equiv \lambda \in \mathbb{R}$. Then for any $e \in \mathbb{S}^{N-1}$ and any open set $\Omega' \Subset \Omega$, we have that*

$$\sup_{x \in \Omega'} \partial_e u_m(x) = \sup_{x \in \partial\Omega'} \partial_e u_m(x).$$

This maximum principle is used as in [29] to prove that $G^m(\nabla u_m)$ satisfies the assumptions of the result from Lebesgue uniformly in $m \in \mathbb{N}$. Hence, the functions $G^m(\nabla u_m)$ are uniformly continuous in $m \in \mathbb{N}$.

Part 4 We pass to the limit when m goes to $+\infty$ and we prove that the sequence $G^m(\nabla u_m)$ converges uniformly to $G(\nabla u_0)$.

The strategy of the proof of Theorem 4.1.9 is different. Since the result is stated in any dimension we can not use the result from Lebesgue. The proof is an adaptation of the one from [57, Theorem 2.1]. In our case, the result is stated with partial differential equations and with a non-zero right hand side $f \in W^{1,q}(\Omega)$ with $q > N$ which create some technical difficulties.

The proof shows that one of the two following cases occurs:

- either $\nabla u(B_r(x_0))$ is outside the degeneracy set D_G for r small enough.
- or $\nabla u(B_r(x_0))$ is inside the convex hull of D_G when r is small enough.

In the first case, we are reduced to the framework of *uniform* elliptic partial differential equations and the conclusion follows from classical results. In the second case, we use the fact that the set of degeneracy D_G is in a plane to show that either $\nabla u(B_r(x_0))$ converges to a point outside D_G when $r \rightarrow 0$ or $\nabla u(B_r(x_0))$ is contained in a neighborhood of D_G when $r \rightarrow 0$.

4.1.5 Plan of the paper

In the following Section 2, we approximate our equation (4.1.4) by smooth equations in order to work with smooth functions. We also prove that if we pass to the limit we obtain a solution of (4.1.4). In Section 3, we prove a uniform continuity estimate for Theorem 4.1.1 and Theorem 4.1.4 thanks to a uniform Sobolev estimate. Section 4 is devoted to the proof of Theorem 4.1.6 for approximated solutions. In the subsequent Section 5, we prove an intermediate result for Theorem 4.1.9. Finally, we pass to the limit in Section 6 to obtain the final conclusions. Section 7 is an appendix about the convex gauge functionals used for the *pseudo-norms*.

4.2 Approximations of the solutions by smooth functions

In this article, we assume a priori that the solution u_0 of (4.1.4) with G as in (4.1.5) is locally Lipschitz continuous. This regularity can be obtained under a uniform convexity condition at infinity. For instance, we can apply [31, Theorem 4.1] or [14, Theorem 2.1] when there exist $C > 0$ and $R > 0$ such that $G \in C^1(\mathbb{R}^N \setminus \overline{B_R(0)})$ and

$$\frac{1}{C}|\xi|^2 \leq \langle DG(z)\xi, \xi \rangle \leq C|\xi|^2 \quad (4.2.1)$$

for every $z \in \mathbb{R}^N \setminus \overline{B_R(0)}$ and every $\xi \in \mathbb{R}^N$. Under these assumptions for every $\Omega' \Subset \Omega$ there exists a constant $L := L(\Omega', R, C)$ such that $\|\nabla u_0\|_{L^\infty(\Omega')} \leq L$.

Since we want to prove some local regularity results, we can assume that u_0 is globally Lipschitz continuous on Ω . Hence, we can change G outside a sufficiently large ball in order to assume that there exist $C > 0$ and $R > 0$ such that G satisfies (4.2.1).

In this section we describe an approximation argument for the proofs of the main theorems that has to be adapted for each theorem in order to have smooth approximations $(G^m)_{m \in \mathbb{N}}$ that share the same properties as G and (4.2.1) uniformly in $m \in \mathbb{N}$.

We begin with an infinitesimal version of the assumption (A_1) :

Lemma 4.2.1. *Let $L > 0$ and H be a C^1 function that satisfies (A_1) . Then there exists $C_L > 0$ such that for every $z \in B_L(0)$ and every $v \in \mathbb{R}^N$ we have:*

$$\langle DH(z)v, v \rangle \geq C_L |DH(z)v|^2.$$

Proof. This result is true when $v = 0$. By assumption (A_1) , we have that for every $z \in B_L(0)$, every $v \in \mathbb{R}^N \setminus \{0\}$ and every $0 < h < \frac{L-|z|}{|v|}$:

$$\langle H(z + hv) - H(z), hv \rangle \geq C_L |H(z + hv) - H(z)|^2.$$

By dividing this last equation by h^2 and letting h go to 0, we get:

$$\langle DH(z)v, v \rangle \geq C_L |DH(z)v|^2.$$

□

Remark 4.2.2. The result is also true without the condition (A_1) when DH is symmetric, nonnegative and bounded. In fact, for every $z, v \in \mathbb{R}^N$, we can work in an orthogonal basis where $DH(z)$ is diagonal. Hence, $\langle DH(z)v, DH(z)v \rangle \leq C \langle DH(z)v, v \rangle$ with C an upper bound of the largest eigenvalues of DH on $B_L(0)$.

We use this new version of (A_1) in order to approximate G in the framework of Theorem 4.1.1. In this section the constant $L > 0$ is such that $\|\nabla u_0\|_{L^\infty(\Omega)} \leq L$.

Proposition 4.2.3. *If G satisfies the assumptions of Theorem 4.1.1 then there exists a sequence of smooth functions $(G^m)_{m \in \mathbb{N}}$ converging uniformly to G on $B_L(0)$ that satisfy the same assumptions as G and (4.2.1) uniformly in $m \in \mathbb{N}$. Namely, for every $m \in \mathbb{N}$, G^m is the sum of two functions G_1^m and G_2^m such that G_2^m is a gradient of a convex function φ^m whose $C^{1,1}$ norm is independent of $m \in \mathbb{N}$ and there exists $C_1 > 0$ independent of $m \in \mathbb{N}$ such that for every $z, \xi \in \mathbb{R}^N$ we have*

$$\langle DG_1^m(z)\xi, \xi \rangle \geq C_1 |DG_1^m(z)\xi|^2. \tag{A'_1}$$

Moreover, DG^m is invertible everywhere for every $m \in \mathbb{N}$.

Proof. In the framework of Theorem 4.1.1, we have that $G = G_1 + G_2$ with $G_2 = \nabla \varphi$. Let $(\rho_m)_{m \in \mathbb{N}}$ be a standard radial mollifying sequence with support in $B_{\frac{1}{m}}(0)$. We introduce the convex function $\Phi(z) := (|z| - 2L)_+^2$ for every $z \in \mathbb{R}^2$ with L the Lipschitz constant of u_0 and $\theta \in C_0^\infty(B_{4L}(0))$ such that $0 \leq \theta \leq 1$ on \mathbb{R}^N , $\theta \equiv 1$ on $B_{3L}(0)$ and $\|\nabla \theta\|_{L^\infty(\mathbb{R}^N)} \leq \frac{2}{L}$.

Moreover, we introduce \tilde{G}_2 that is the gradient of a convex function $\tilde{\varphi}$ that is equal to φ on $B_{2L}(0)$ and quadratic at infinity. It can be done by post-composing φ with a smooth function Θ equals to the identity on $[-R, R]$ for a certain $R > 1$ such that $\varphi(B_{2L}(0)) \subset [-R + 1, R - 1]$ and equals to $2R$ outside $B_{2R}(0)$. Then we add the function Φ to it: $\tilde{\varphi} := \Theta(\varphi + \Phi) + K_2\Phi$ with $K_2 > 0$ such that $\tilde{\varphi}$ is convex. If we compute the Hessian matrix of $\tilde{\varphi}$ we obtain that:

$$D^2\tilde{\varphi} := \Theta'(\varphi + \Phi)D^2(\varphi + \Phi) + \Theta''(\varphi + \Phi)\nabla(\varphi + \Phi) \otimes \nabla(\varphi + \Phi) + K_2D^2\Phi.$$

Since Θ is equal to the identity on $[-R, R]$ and $\varphi(B_{2L}(0)) \subset [-R + 1, R - 1]$ there exists $\epsilon > 0$ such that for every $z \in B_{2L+\epsilon}(0)$, $D^2\tilde{\varphi}(z) = D^2(\varphi(z) + \Phi(z)) + K_2D^2\Phi(z) \geq 0$. Since $\varphi + \Phi$ is coercive there exists $L' > 2L + \epsilon$ such that for every

z with a norm larger than L' we have that $D^2\tilde{\varphi}(z) = K_2D^2\Phi(z)$. It remains to choose $K_2 > 0$ such that for every $z \in B_{L'}(0) \setminus B_{2L+\epsilon}(0)$ we have that:

$$K_2D^2\Phi(z) \geq |\Theta'(\varphi + \Phi)D^2(\varphi + \Phi) + \Theta''(\varphi + \Phi)\nabla(\varphi + \Phi) \otimes \nabla(\varphi + \Phi)|.$$

We set $G^m := G_1^m + \tilde{G}_2^m + \nabla\Phi^m + \frac{1}{m}Id$ where $G_1^m := \theta(G_1 * \rho_m) + K_1\nabla\Phi^m$, $\tilde{G}_2^m := \tilde{G}_2 * \rho_m$, $\Phi^m := \Phi * \rho_m$ and $K_1 > 0$ to be fixed later. Thanks to Φ and the regularity of G , G^m satisfies (4.2.1) uniformly in $m \in \mathbb{N}$ and $G_2^m := \tilde{G}_2^m + \nabla\Phi^m + \frac{1}{m}Id$ is the gradient of a convex function. Since we add the identity in G^m we have that DG^m is invertible everywhere. It remains to check that G_1^m satisfies the assumption (A'_1) uniformly in $m \in \mathbb{N}$.

For every $z_1, z_2 \in \mathbb{R}^2$ we have

$$\langle G_1 * \rho_m(z_1) - G_1 * \rho_m(z_2), z_1 - z_2 \rangle \geq \int_{\mathbb{R}^2} \langle G_1(z_1 - y) - G_1(z_2 - y), (z_1 - y) - (z_2 - y) \rangle \rho_m(y) dy.$$

Thus, by assumption (A_1) we obtain that

$$\langle G_1 * \rho_m(z_1) - G_1 * \rho_m(z_2), z_1 - z_2 \rangle \geq C_{3L+1} \int_{\mathbb{R}^2} |G_1(z_1 - y) - G_1(z_2 - y)|^2 \rho_m(y) dy$$

for every $z_1, z_2 \in B_{3L}(0)$. By Jensen's inequality we get that $\langle G_1 * \rho_m(z_1) - G_1 * \rho_m(z_2), z_1 - z_2 \rangle \geq C_{3L+1} |G_1 * \rho_m(z_1) - G_1 * \rho_m(z_2)|^2$. Hence, by Lemma 4.2.1 we obtain that

$$\langle DG_1 * \rho_m(z)\xi, \xi \rangle \geq C_{3L+1} |DG_1 * \rho_m(z)\xi|^2$$

for every $z \in B_{3L}(0)$, every $\xi \in \mathbb{R}^N$ and every $m \in \mathbb{N}$. In (A'_1) we can assume that $|\xi| = 1$. For every $z, \xi \in \mathbb{R}^N$ with $|\xi| = 1$ we have that:

$$\langle DG_1^m(z)\xi, \xi \rangle \geq \theta \langle D(G_1 * \rho_m)(z)\xi, \xi \rangle + K_1 \langle D^2\Phi^m(z)\xi, \xi \rangle - |\nabla\theta| |D(G_1 * \rho_m)(z)|$$

and

$$|DG_1^m(z)\xi|^2 \leq 4(|\nabla\theta|^2 |G_1 * \rho_m(z)|^2 + \theta^2 |D(G_1 * \rho_m)(z)|^2 + K_1^2 |D^2\Phi^m(z)\xi|^2).$$

If $z \in B_{3L}(0)$ then $\theta(z) = 1$ and $\nabla\theta(z) = 0$. Hence,

$$\langle DG_1^m(z)\xi, \xi \rangle \geq \frac{1}{4} \min\{C_{3L+1}, \frac{1}{2K_1}\} |DG_1^m(z)\xi|^2.$$

If $z \notin B_{4L}(0)$ then $DG_1^m = K_1D^2\Phi^m$. Thus,

$$\langle DG_1^m(z)\xi, \xi \rangle \geq \frac{1}{2K_1} |DG_1^m(z)\xi|^2.$$

Finally, if $z \in B_{4L}(0) \setminus B_{3L}(0)$ then we can bound $|\nabla\theta|$ from above by $\frac{2}{L}$. Hence, we want to find $C > 0$ such that:

$$\begin{aligned} K_1 \langle D^2\Phi^m(z)\xi, \xi \rangle &\geq 4C \left(\frac{4}{L^2} |G_1 * \rho_m(z)|^2 + |D(G_1 * \rho_m)(z)|^2 + K_1^2 |D^2\Phi^m(z)\xi|^2 \right) \\ &\quad + \frac{2}{L} |D(G_1 * \rho_m)(z)|. \end{aligned}$$

By definition of Φ , when m is large enough this is equivalent to

$$\frac{4K_1}{3} \geq \frac{2}{L} |D(G_1 * \rho_m)(z)| + 4C \left(\frac{4}{L^2} |G_1 * \rho_m(z)|^2 + |D(G_1 * \rho_m)(z)|^2 + 4K_1^2 \right).$$

By global Lipschitz regularity of G_1 on $B := B_{4L}(0)$ we can choose the two constants $K_1 := K_1(L, \|DG\|_{L^\infty(B)}) > 0$ and $C := C(L, \|DG\|_{L^\infty(B)}) > 0$ such that this last inequality is true. Hence, by taking C_1 as $\min\{C, \frac{1}{8K_1}, \frac{C_{3L+1}}{4}\}$ the assumption (A'_1) is satisfied uniformly in $m \in \mathbb{N}$. \square

In the case of Theorem 4.1.4 we proceed as follows:

Proposition 4.2.4. *If G satisfies the assumptions of Theorem 4.1.4 then there exists a sequence of C^4 functions $(G^m)_{m \in \mathbb{N}}$ converging to G uniformly on $B_L(0)$ that satisfy the same assumptions as G and (4.2.1) uniformly in $m \in \mathbb{N}$. Moreover, DG^m is invertible everywhere for every $m \in \mathbb{N}$.*

Proof. In the framework of Theorem 4.1.4, we have that $G = \sum_{i=1}^n \nabla \varphi_i$ with

$$\varphi_i(\cdot) = f_i(\mathcal{N}_i(\cdot - \xi_i)) \text{ or } \varphi_i(\cdot) = f_i(\langle \cdot, \xi_i \rangle).$$

For $L \geq \|\nabla u_0\|_{L^\infty(\Omega)}$ and every $1 \leq i \leq n$ we introduce:

$$\tilde{f}_i(t) := \begin{cases} f_i(-2L) + f'_i(-2L)(t+2L) + (t+2L)^2 & \text{if } t < -2L \\ f_i(t) & \text{if } -2L \leq t \leq 2L, \\ f_i(2L) + f'_i(2L)(t-2L) + (t-2L)^2 & \text{if } t > 2L, \end{cases}$$

and $\Phi(z) = (|\cdot| - 2L)_+^2$. We divide the rest of the proof in four steps.

Step 1 If $\varphi_i(\cdot) = f_i(\langle \cdot, \xi_i \rangle)$ then we set $G_i^m(\cdot) := \nabla[f_i^m(\langle \cdot, \xi_i \rangle)]$ for every $m \in \mathbb{N}$ with $f_i^m(\cdot) := \tilde{f}_i * \rho_m(\cdot) + \frac{1}{m} |\cdot|^2$.

Step 2 If $\varphi_i(\cdot) = f_i(\mathcal{N}_i(\cdot - \xi_i))$ then we proceed as follows. We introduce $C := (\mathcal{N}_i)^{-1}(\{[0, 1]\})$, then \mathcal{N}_i is the convex gauge γ_C of the convex set C . Then we regularize γ_C by convolution: $\gamma_C^m := \gamma_C * \rho_m$. For every $m \in \mathbb{N}$, the function γ_C^m is convex and has strictly convex lower level sets thanks to Proposition 4.7.2. By Sard's theorem, we can define C_m as $(\gamma_C^m)^{-1}(\{[0, r_m]\})$ with $r_m \rightarrow 1$ when $m \rightarrow +\infty$ selected such that C_m is smooth. Moreover, we can assume that there exists $r > 0$ independent of m such that $B_r(0)$ is in the interior of C_m . Then we define \mathcal{N}_i^m as the gauge of C_m . Hence, by Proposition 4.7.1 \mathcal{N}_i^m is a pseudo-norm smooth outside the origin. Moreover, for every $z \neq 0$ we have that

$$\nabla \mathcal{N}_i^m(z) = \frac{\nu_{C_m}(P_m(z))}{\langle \nu_{C_m}(P_m(z)), P_m(z) \rangle} \neq 0$$

where $P_m(z)$ is the intersection between $\mathbb{R}_+ z$ and ∂C_m and ν_{C_m} is the unit outward normal vector of C_m . In order to regularize f_i we set

$$f_i^m(t) := (\tilde{f}_i * \rho_m(\cdot) + \frac{1}{m} |\cdot|^2) \left((|t|^q + \frac{1}{m})^{\frac{1}{q}} - \left(\frac{1}{m}\right)^{\frac{1}{q}} + \alpha_i^m \right).$$

Here, $q \geq 6$ is chosen in order to have $f_i^m(\mathcal{N}_i^m)$ at least \mathcal{C}^5 for the upcoming computations, α_i^m is the only point where the strictly convex and coercive function $\tilde{f}_i * \rho_m(\cdot) + \frac{1}{m}|\cdot|^2$ attains its minimum. Finally we set $\varphi_i^m(\cdot) := f_i^m(\mathcal{N}_i^m(\cdot - \xi_i))$. Hence, φ_i^m is a strictly convex function such that $\nabla\varphi_i^m(z) = 0 \Leftrightarrow z = \xi_i$.

Step 3 We prove that φ_i^m and $\nabla\varphi_i^m$ converge uniformly to φ_i and $\nabla\varphi_i$ on every compact set when $m \rightarrow +\infty$. For every $z \in \mathbb{R}^2 \setminus \{0\}$ we have that

$$|\gamma_C(P_m(z)) - \gamma_C(P_C(z))| \leq |\gamma_C^m(P_m(z)) - \gamma_C(P_m(z))| + |\gamma_C(P_C(z)) - \gamma_C^m(P_m(z))|$$

with $P_C(z)$ the intersection of ∂C and \mathbb{R}_+z . By uniform convergence of γ_C^m to γ_C on compact sets the first term in the right-hand side converges to 0 when $m \rightarrow +\infty$ uniformly in $z \in \mathbb{R}^2 \setminus \{0\}$. The second term is equal to $|r_m - 1|$ and converges also to 0 uniformly in $z \in \mathbb{R}^2 \setminus \{0\}$. This means that $\gamma_C(P_m(z))$ converges uniformly to 1 on $\mathbb{R}^2 \setminus \{0\}$ when $m \rightarrow +\infty$. Hence, P_m converges to P_C uniformly on $\mathbb{R}^2 \setminus \{0\}$. By homogeneity of \mathcal{N}_i^m we get that $\mathcal{N}_i^m(z) = \frac{|z|}{|P_m(z)|}$ for every $z \neq 0$. The convergence of P_m combined with the fact that $\mathcal{N}_i^m(0) = 0 = \mathcal{N}_i(0)$ gives that \mathcal{N}_i^m converges uniformly to \mathcal{N}_i on every compact sets of \mathbb{R}^2 when $m \rightarrow +\infty$. Thus we obtain that φ_i^m converges uniformly to φ_i on every compact sets of \mathbb{R}^2 when $m \rightarrow +\infty$.

When $z = \xi_i$, $\nabla\varphi_i^m(\xi_i) = 0$ and for every $z \neq \xi_i$, we have that

$$\nabla\varphi_i^m(z) = (f_i^m)'(\mathcal{N}_i^m(z - \xi_i))\nabla\mathcal{N}_i^m(z - \xi_i).$$

Moreover, if we set $f_i^m := g_i^m(\Theta_q^m)$ with $\Theta_q^m(t) = (|t|^q + \frac{1}{m})^{\frac{1}{q}} - (\frac{1}{m})^{\frac{1}{q}}$ then

$$(f_i^m)''(t) = (g_i^m)'(\Theta_q^m(t))(\Theta_q^m)''(t) + (g_i^m)''(\Theta_q^m(t))((\Theta_q^m)'(t))^2.$$

The fact that $(g_i^m)'(0) = 0$ and $(\Theta_q^m)''(t) \leq \frac{C}{t}$ with C independent of $m \in \mathbb{N}$ gives that the functions $(f_i^m)_{m \in \mathbb{N}}$ are uniformly in $\mathcal{C}^{1,1}(\mathbb{R}^2)$. Hence, it only remains to check that $\nabla\mathcal{N}_i^m$ converge uniformly to $\nabla\mathcal{N}_i$ on $\mathbb{R}^2 \setminus \{0\}$. For every $z \in \mathbb{R}^2 \setminus \{0\}$ we have that $\nabla\mathcal{N}_i^m(z) = \frac{\nu_{C_m}(P_m(z))}{\langle \nu_{C_m}(P_m(z)), P_m(z) \rangle} \neq 0$. The function $\nu_{C_m}(P_m)$ is equal to $\frac{\nabla\gamma_C^m(P_m)}{|\nabla\gamma_C^m(P_m)|}$ that converges uniformly on $\mathbb{R}^2 \setminus \{0\}$ to $\frac{\nabla\gamma_C(P_C)}{|\nabla\gamma_C(P_C)|}$ that is equal to $\nu_C(P_C)$. Since there exists a small ball $B_r(0)$ with $r > 0$ independent of $m \in \mathbb{N}$ inside every C^m the scalar product $\langle \nu_{C_m}(P_m(z)), P_m(z) \rangle$ can be bounded from below by a positive constant independent of $m \in \mathbb{N}$. Hence, $\nabla\mathcal{N}_i^m$ converges uniformly on $\mathbb{R}^2 \setminus \{0\}$ to $\nabla\mathcal{N}_i$. Thus, $\nabla\varphi_i^m$ converges uniformly on every compact sets of \mathbb{R}^2 to $\nabla\varphi_i$.

Thanks to Proposition 4.7.3 the sets C_m have a Lipschitz continuous normal with a Lipschitz constant independent of $m \in \mathbb{N}$. Hence the functions $\nabla\varphi_i^m(z) := (f_i^m)'(\mathcal{N}_i^m(z))\nabla\mathcal{N}_i^m(z)$ if $z \neq 0$ and $\nabla\varphi_i^m(0) = 0$ are equi-Lipschitz continuous on each compact set of \mathbb{R}^2 .

Step 4 In order to have $(G^m)_{m \in \mathbb{N}}$ satisfying (4.2.1) uniformly in $m \in \mathbb{N}$ we add a term of the following form: $\nabla[(\Phi * \rho_m(|\cdot|) + \frac{1}{m}|\cdot|^2)]$. We define G^m as the following: $G^m := \sum_{i=1}^n G_i^m$. Since we add the identity in G^m we have that DG^m is invertible everywhere. Thus, G^m is a function that satisfies the assumptions of Theorem 4.1.4 and (4.2.1) uniformly in $m \in \mathbb{N}$. \square

We have the following result for Theorem 4.1.6:

Proposition 4.2.5. *If G satisfies the assumptions of Theorem 4.1.6 then there exists a sequence of smooth functions $(G^m)_{m \in \mathbb{N}}$ converging to G uniformly on $B_L(0)$ that satisfy the same assumptions as G and (4.2.1) uniformly in $m \in \mathbb{N}$. Moreover, DG^m is invertible everywhere for every $m \in \mathbb{N}$ and for every $0 < r < L$, $\sup_{r \leq t \leq L} (f_2^m)''(t)$ can be bounded uniformly in $m \in \mathbb{N}$.*

Proof. Let us consider $L > \|\nabla u_0\|_{L^\infty(\Omega)}$. In the framework of Theorem 4.1.6, we have that $G = G_1 + G_2$ with $G_i(\cdot) = f_i'(\langle \cdot, \xi_i \rangle) \xi_i$ for $i = 1, 2$. We introduce $\Phi_1(z) := (|\langle z, \xi_1 \rangle| - 2L)_+^2$, $\Phi_2(z) := (|\langle z, \xi_2 \rangle| - 2L)_+^2$ and $\Phi(z) := \Phi_1(z) + \Phi_2(z)$. We also introduce \tilde{f}_1 and \tilde{f}_2 that satisfy

$$\tilde{f}_i(t) := \begin{cases} f_i(-2L) + f_i'(-2L)(t + 2L) + (t + 2L)^2 & \text{if } t < -2L \\ f_i(t) & \text{if } -2L \leq t \leq 2L, \\ f_i(2L) + f_i'(2L)(t - 2L) + (t - 2L)^2 & \text{if } t > 2L. \end{cases}$$

Hence, we set for every $m \in \mathbb{N}$, $G_i^m(\cdot) := \nabla[(\tilde{f}_i * \rho_m)(\langle \cdot, \xi_i \rangle) + \frac{1}{m}|\langle \cdot, \xi_i \rangle|^2 + \Phi_i * \rho_m(\cdot)]$ for $i = 1, 2$. Then we can define G^m as the sum of these two functions: $G^m := G_1^m + G_2^m$. Since we add $\frac{1}{m}|\langle \cdot, \xi_1 \rangle|^2 + \frac{1}{m}|\langle \cdot, \xi_2 \rangle|^2$ in G^m we have that DG^m is invertible everywhere.

It remains to check that around the origin where $f_2^m(\cdot) := f_2 * \rho_m(\cdot) + \frac{1}{m}|\cdot|^2 + \Phi_i * \rho_m(\cdot)$ that this function has a uniform modulus of convexity ω without any dependence on $m \in \mathbb{N}$. For every $m > \frac{2}{r}$ and every $x, y \in (-\frac{r}{2}, \frac{r}{2})$ we have that

$$((f_2^m)'(x) - (f_2^m)'(y))(x - y) \geq \int_{B_{\frac{r}{2}}(0)} (f_2'(x-t) - f_2'(y-t))((x-t) - (y-t))\rho_m(t)dt.$$

By the uniform convexity assumption made on f_2 , we have that $(f_2'(x-t) - f_2'(y-t))((x-t) - (y-t)) \geq \omega(|x-y|)$ for every $x, y, t \in (-\frac{r}{2}, \frac{r}{2})$. Hence, $((f_2^m)'(x) - (f_2^m)'(y))(x - y) \geq \omega(|x-y|)$ for every $m \geq \frac{2}{r}$ and every $x, y \in (-\frac{r}{2}, \frac{r}{2})$. \square

In the case of Theorem 4.1.9 we just approximate G by $(G * \rho_m + \nabla \Phi^m + \frac{1}{m}Id)_{m \in \mathbb{N}}$ with $\Phi(\cdot) := (|\cdot| - 2L)_+^2$.

Proposition 4.2.6. *If G satisfies the assumptions of Theorem 4.1.9 then there exists a sequence of smooth functions $(G^m)_{m \in \mathbb{N}}$ converging to G uniformly on $B_L(0)$ that satisfy (4.2.1) uniformly in $m \in \mathbb{N}$. Moreover, for every $r > 0$ there exists $k_r \in \mathbb{N}$ such that for every $m \geq \frac{2}{r}$ and every $z \in \mathbb{R}^N$ such that $\text{dist}(z, D_G) \geq r$ we have $\frac{1}{k_r}Id < (DG^m(z))^s < k_r Id$ with $(DG^m)^s := \frac{DG^m + (DG^m)^T}{2}$.*

Proof. We regularize G as in the proof of Proposition 4.2.3. More precisely, we introduce $\Phi(\cdot) := (|\cdot| - 2L)_+^2$, $\theta \in C_0^\infty(B_{4L}(0))$ such that $0 \leq \theta \leq 1$ on \mathbb{R}^N , $\theta \equiv 1$ on $B_{3L}(0)$. For every $m \in \mathbb{N}$, we set $G^m := \theta(G * \rho_m) + \frac{1}{m}Id + K\nabla\Phi * \rho_m$ with

$K > 0$ such that G^m is increasing. Thus, for every $r > 0$ the support of ρ_m is inside $B_{\frac{r}{2}}(0)$ for every $m \geq \frac{2}{r}$. Since $\{z \in \mathbb{R}^N \text{ such that } \text{dist}(z, D_G) \geq \frac{r}{2}\}$ is inside $\mathcal{O}_{k'_r}$ for a certain k'_r , the conclusion follows. \square

In all the four cases, when $m \rightarrow +\infty$ we have that $G^m \rightarrow G$ uniformly on $B_L(0)$ with $L > 0$ selected such that $\|\nabla u_0\|_{L^\infty(\Omega)} \leq L$. Hence, up to a modification of G outside $B_L(0)$ we can assume that $G^m \rightarrow G$ uniformly on every compact sets of \mathbb{R}^N when $m \rightarrow +\infty$.

For every $m \in \mathbb{N}$, we can consider the following equation:

$$\begin{cases} \text{div } G_m(\nabla v(x)) = f_m & \text{in } \Omega, \\ v = u_0 & \text{on } \partial\Omega, \end{cases} \quad (4.2.2)$$

with $f_m := f * \rho_m$ and u_0 a globally Lipschitz continuous solution of (4.1.4).

By [33, Theorem 6.33], the solution u_m of (4.2.2) is C^3 inside Ω if $q \geq 6$ in the proof of Proposition 4.2.4. We have that (4.2.1) implies the existence of $C > 0$, $C' > 0$, $D \in \mathbb{R}$ and $D' > 0$ such that

$$C|z_1 - z_2|^2 + D \leq \langle G(z_1) - G(z_2), z_1 - z_2 \rangle \leq C'|z_1 - z_2|^2 + D' \quad (4.2.3)$$

for every $z_1, z_2 \in \mathbb{R}^N$. By the growth assumptions of G^m we have that the sequence $(u_m)_{m \in \mathbb{N}}$ is uniformly bounded in $W^{1,2}(\Omega)$:

Proposition 4.2.7. *The sequence $(u_m)_{m \in \mathbb{N}}$ is uniformly bounded in $W_{u_0}^{1,2}(\Omega)$.*

Proof. For every $m \in \mathbb{N}$, using the fact that u_m is a solution of (4.2.2) we obtain:

$$\int_{\Omega} \langle G_m(\nabla u_m), \nabla(u_m - u_0) \rangle = - \int_{\Omega} f_m(u_m - u_0)$$

Thanks to the first inequality in (4.2.3) we get:

$$\int_{\Omega} \langle G_m(\nabla u_0), \nabla(u_m - u_0) \rangle + C|\nabla u_m - \nabla u_0|^2 + D \leq - \int_{\Omega} f_m(u_m - u_0).$$

Hence, since $u_0 \in W^{1,\infty}(\Omega)$ and $\|f_m\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}$ we have:

$$\int_{\Omega} |\nabla u_m - \nabla u_0|^2 \leq C' \int_{\Omega} |\nabla u_m - \nabla u_0| + D'.$$

Applying Young's inequality on the first term of the right-hand side gives that $|\nabla u_m - \nabla u_0|$ is bounded in $L^2(\Omega)$. Thus, (u_m) is uniformly bounded in $W_{u_0}^{1,2}(\Omega)$. \square

By (4.2.1), we can assume that for every $\Omega' \Subset \Omega$ there exists $L_{\Omega'}$ such that $\|\nabla u_m\|_{L^\infty(\Omega')} \leq L_{\Omega'}$ for every $m \in \mathbb{N}$.

Since the sequence $(u_m)_{m \in \mathbb{N}}$ is uniformly bounded in $W^{1,2}(\Omega)$, we can extract a subsequence that converges weakly to a function $u \in W_{u_0}^{1,2}(\Omega)$. Moreover, for every subset $\Omega' \Subset \Omega$ we can use the Ascoli theorem to extract a subsequence of $(u_m)_{m \in \mathbb{N}}$

that converges uniformly to u on Ω' . Up to a diagonal process we can assume that the sequence $(u_m)_{m \in \mathbb{N}}$ converges locally uniformly to u on Ω .

We can prove that u is a solution of (4.1.4). To do so we use the following result on Young measures:

Lemma 4.2.8. *There exists a family of probability measures $(\nu_x)_{x \in \Omega}$ measurable with respect to x such that for a.e. $x \in \Omega$ and for ν_x -a.e. $y \in \mathbb{R}^N$ we have:*

$$\langle G(y) - G(\nabla u(x)), y - \nabla u(x) \rangle = 0. \tag{4.2.4}$$

Moreover, these probability measures satisfy the following property:

$$\int_{\Omega} H(x, \nabla u_m(x)) dx \rightarrow \int_{\Omega} \int_{\mathbb{R}^N} H(x, y) d\nu_x(y) dx \tag{4.2.5}$$

when $m \rightarrow +\infty$, for every bounded Carathéodory function $H : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$.

Proof. Since for every $\Omega' \Subset \Omega$ there exists a constant $L_{\Omega'}$ independent of $m \in \mathbb{N}$ such that $\|\nabla u_m\|_{L^\infty(\Omega')} \leq L_{\Omega'}$, we get:

$$\lim_{k \rightarrow +\infty} \sup_{m \in \mathbb{N}} |\{x \in \Omega, |\nabla u_m(x)| > k\}| = 0.$$

By [6, Theorem] and [6, Remark 3], the sequence $(\nabla u_m)_{m \in \mathbb{N}}$, up to an extraction, generates a family of Young measures denoted by $(\nu_x)_{x \in \Omega}$ satisfying (4.2.5). By weak convergence of $(\nabla u_m)_{m \in \mathbb{N}}$ to ∇u , we obtain that

$$\nabla u(x) = \int_{\mathbb{R}^N} y d\nu_x(y) \tag{4.2.6}$$

for a.e. $x \in \Omega$. Since u_m is a solution of (4.2.2), for every $\theta \in C_0^\infty(\Omega)$ we have that

$$\int_{\Omega} \langle G^m(\nabla u_m), \nabla \theta \rangle = - \int_{\Omega} f_m \theta.$$

When $m \rightarrow +\infty$, $G^m \rightarrow G$ on the compact set $B_L(0)$ where L is the uniform bound of $(\|\nabla u_m\|_{L^\infty(\text{supp } \theta)})$. Thus,

$$\lim_{m \rightarrow +\infty} \int_{\Omega} \langle G^m(\nabla u_m) - G(\nabla u_m), \nabla \theta \rangle = 0.$$

Hence, by (4.2.5) and the previous equation we have that

$$- \int_{\Omega} f \theta = \int_{\Omega} \langle \nabla \theta, \int_{\mathbb{R}^N} G(y) d\nu_x(y) \rangle dx.$$

We introduce X_θ :

$$X_\theta := \lim_{m \rightarrow +\infty} \int_{\Omega} H(x, \nabla u_m(x)) dx$$

with $H(x, y) := \theta(x)\langle G(y), y - \nabla u(x) \rangle$. Thus, by (4.2.5) we obtain:

$$X_\theta = \int_{\Omega} \theta(x) \int_{\mathbb{R}^N} \langle G(y), y - \nabla u(x) \rangle d\nu_x(y) dx.$$

Since u_m is solution of a (4.2.2), we get that X_θ is equal to:

$$\lim_{m \rightarrow +\infty} \int_{\Omega} \theta \langle G(\nabla u_m) - G^m(\nabla u_m), \nabla u_m - \nabla u \rangle - \langle G^m(\nabla u_m), \nabla \theta \rangle (u_m - u) - f_m \theta (u_m - u) dx.$$

Using the fact that $(u_m)_{m \in \mathbb{N}}$ converges uniformly to u on $\text{supp } \theta$ and that $(G^m)_{m \in \mathbb{N}}$ converges uniformly to G on the compact set $B_L(0)$ when $m \rightarrow +\infty$ imply that $X_\theta = 0$. But by (4.2.6) we get that

$$0 = X_\theta = \int_{\Omega} \theta \int_{\mathbb{R}^N} \langle G(y), y - \nabla u \rangle d\nu_x(y) dx = \int_{\Omega} \theta \int_{\mathbb{R}^N} \langle G(y) - G(\nabla u), y - \nabla u \rangle d\nu_x(y) dx \quad (4.2.7)$$

where the last equality comes from (4.2.6). Since (4.2.7) is true for every $\theta \in \mathcal{C}_0^\infty(\Omega)$, we obtain that for a.e. $x \in \Omega$,

$$\int_{\mathbb{R}^N} \langle G(y) - G(\nabla u(x)), y - \nabla u(x) \rangle d\nu_x(y) = 0$$

and the conclusion follows from the fact that $\langle G(y) - G(\nabla u(x)), y - \nabla u(x) \rangle \geq 0$. \square

We can make the following observation:

Remark 4.2.9. For a.e. $x \in \Omega$, if G is strictly increasing at $\nabla u(x)$, namely

$$\langle G(\nabla u(x)) - G(y), \nabla u(x) - y \rangle > 0$$

for every $y \in \mathbb{R}^N \setminus \{\nabla u(x)\}$, then $\nu_x = \delta_{\nabla u(x)}$ thanks to Lemma 4.2.8.

With this lemma we can show that:

Proposition 4.2.10. *Under the assumptions of Theorem 4.1.1, Theorem 4.1.4 or Theorem 4.1.6, for every bounded Carathéodory function $H : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ we have*

$$\int_{\Omega} H(x, G(\nabla u_m(x))) dx \rightarrow \int_{\Omega} H(x, G(\nabla u(x))) dx$$

when $m \rightarrow +\infty$.

Proof. Thanks to Lemma 4.2.8 and (4.2.5), it remains to prove the following result:

$$\text{supp } \nu_x \subset \{y \in \mathbb{R}^N \text{ such that } G(y) = G(\nabla u(x))\} \text{ for a.e. } x \in \Omega.$$

Since G is the sum of non decreasing functions $(G_i)_{1 \leq i \leq n}$, we have that $\langle G(y) - G(\nabla u(x)), y - \nabla u(x) \rangle$ is the sum of n non negative terms : $\langle G_i(y) - G_i(\nabla u(x)), y - \nabla u(x) \rangle$ for every $1 \leq i \leq n$. For a.e. $x \in \Omega$ and ν_x -a.e. $y \in \mathbb{R}^N$, since $\langle G(y) - G(\nabla u(x)), y - \nabla u(x) \rangle = 0$ for every $1 \leq i \leq n$ we get that $\langle G_i(y) - G_i(\nabla u(x)), y - \nabla u(x) \rangle = 0$. We distinguish two cases:

a) If the condition (A_1) is satisfied for G_i , for a.e. $x \in \Omega$ and ν_x -a.e. $y \in \mathbb{R}^N$, we have that $G_i(y) = G_i(\nabla u(x))$ at once.

b) If G_i is the gradient of a convex function φ_i , we obtain that for a.e. $x \in \Omega$ and ν_x -a.e. $y \in \mathbb{R}^N$, $G_i(y) = G_i(\nabla u(x))$ and the conclusion follows. \square

We have the following convergence result:

Remark 4.2.11. If we consider $H(x, y) := |G(\nabla u(x)) - G(y)|$ in (4.2.5), we get that $G(\nabla u_m) \rightarrow G(\nabla u)$ in $L^1(\Omega)$.

We can prove that u is a solution of (4.1.4):

Proposition 4.2.12. *Under the assumptions of Theorems 4.1.1, 4.1.4 or 4.1.6, the function u is a solution of*

$$\begin{cases} \operatorname{div} G(\nabla u(x)) = f & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

Proof. Since $(u_m)_m$ converges weakly to $u \in W_{u_0}^{1,2}(\Omega)$ we have that u satisfies the boundary condition. It remains to prove that

$$\int_{\Omega} \langle G(\nabla u), \nabla \theta \rangle = - \int_{\Omega} f \theta$$

for every $\theta \in C_0^\infty(\Omega)$. Since $\|\nabla u_m\|_{L^\infty(\operatorname{supp} \theta)}$ is uniformly bounded, $G^m(\nabla u_m) - G(\nabla u_m) \rightarrow 0$ in $L^1(\operatorname{supp} \theta)$ when $m \rightarrow +\infty$. By Remark 4.2.11, this implies that $G^m(\nabla u_m) \rightarrow G(\nabla u)$ in $L^1(\operatorname{supp} \theta)$. Since

$$\int_{\Omega} \langle G^m(\nabla u_m), \nabla \theta \rangle = - \int_{\Omega} f_m \theta$$

we have our desired result. \square

We conclude this section with a counterpart of Proposition 4.2.10 in the case of Theorem 4.1.9:

Proposition 4.2.13. *Under the assumptions of Theorem 4.1.9, we have that*

$$\int_{\Omega} |\operatorname{dist}(\nabla u_m, D_G) - \operatorname{dist}(\nabla u, D_G)| \rightarrow 0$$

and

$$\int_{\Omega} |\nabla u_m \times \operatorname{dist}(\nabla u_m, D_G) - \nabla u \times \operatorname{dist}(\nabla u, D_G)| \rightarrow 0$$

when $m \rightarrow +\infty$.

Proof. For every $x \in \Omega$, such that $\nabla u(x) \notin D_G$ we have that $\langle DG(\nabla u(x))A, A \rangle > 0$ for every $A \neq 0$. Hence, $\langle G(\nabla u(x)) - G(A), \nabla u(x) - A \rangle > 0$ for every $A \in \mathbb{R}^N \setminus \{\nabla u(x)\}$. Thanks to Remark 4.2.9, we get that $\nu_x = \delta_{\nabla u(x)}$. Since $(x, y) \rightarrow |\operatorname{dist}(y, D_G) - \operatorname{dist}(\nabla u(x), D_G)|$ is a Carathéodory function, this implies together with (4.2.5) that

$$\int_{\Omega} |\operatorname{dist}(\nabla u_m, D_G) - \operatorname{dist}(\nabla u, D_G)| \rightarrow \int_{\Omega} \int_{\mathbb{R}^N} |\operatorname{dist}(y, D_G) - \operatorname{dist}(\nabla u(x), D_G)| d\nu_x(y) dx = 0$$

and for the same reasons

$$\int_{\Omega} |\nabla u_m \times \operatorname{dist}(\nabla u_m, D_G) - \nabla u \times \operatorname{dist}(\nabla u, D_G)| \rightarrow 0$$

when $m \rightarrow +\infty$. \square

4.3 Uniform estimates for Theorem 4.1.1 and Theorem 4.1.4

In this section, we prove that $G^m(\nabla u_m)$ is continuous with a modulus of continuity independent of $m \in \mathbb{N}$ when we are under the assumptions of Theorem 4.1.1 or Theorem 4.1.4.

4.3.1 $W^{1,2}$ regularity of $G^m(\nabla u_m)$

In this subsection we show that $G^m(\nabla u_m) \in W_{loc}^{1,2}(\Omega)$ with a norm uniformly bounded in $m \in \mathbb{N}$. More precisely, our goal is to prove that in the framework of Theorem 4.1.1 and Theorem 4.1.4, for every $1 \leq i \leq n$ the function $G_i^m(\nabla u_m)$ is in $W_{loc}^{1,2}(\Omega)$ with a norm that does not depend on $m \in \mathbb{N}$. To do so we apply the same method as in [29, Proposition 2.4] to smooth functions, namely the regularized equations.

Proposition 4.3.1. *We assume that $G \in C^1(\mathbb{R}^N)$ satisfies the assumptions of Theorem 4.1.1 with (A_1) replaced by (A'_1) or the assumptions of Theorem 4.1.4 and that u is a C^2 solution of (4.1.4). If we write $G := \sum_{i=1}^n G_i$ as in those theorems then $G_i(\nabla u) \in W_{loc}^{1,2}(\Omega)$ for every $1 \leq i \leq n$ and $G(\nabla u) \in W_{loc}^{1,2}(\Omega)$. Moreover, the dependence on G of the norms of these quantities is only due to the Lipschitz constant of G and C_1 from assumption (A'_1) .*

Proof. Let us consider $\Omega'' \Subset \Omega' \Subset \Omega$. By differentiating (4.1.4), for every $e \in \mathbb{S}^1$, every $\theta \in C_0^1(\Omega')$ we have that

$$\int_{\Omega} \langle DG(\nabla u) \nabla \partial_e u, \nabla \theta \rangle = - \int_{\Omega} \partial_e f \theta.$$

In this last equality we choose the following test function: $\theta = \eta^2 \partial_e u$ with $\eta \in C_0^\infty(\Omega')$ and $\eta \equiv 1$ on Ω'' . Hence, we get:

$$\int_{\Omega} \langle DG(\nabla u) \nabla \partial_e u, \nabla \partial_e u \rangle \eta^2 = -2 \int_{\Omega} \eta \partial_e u \langle DG(\nabla u) \nabla \partial_e u, \nabla \eta \rangle - \int_{\Omega} \partial_e f \eta^2 \partial_e u.$$

Since $G = \sum_{i=1}^n G_i$ it can be rewritten as

$$\sum_{i=1}^n \int_{\Omega} \langle DG_i(\nabla u) \nabla \partial_e u, \nabla \partial_e u \rangle \eta^2 = -2 \sum_{i=1}^n \int_{\Omega} \eta \partial_e u \langle DG_i(\nabla u) \nabla \partial_e u, \nabla \eta \rangle - \int_{\Omega} \partial_e f \eta^2 \partial_e u. \quad (4.3.1)$$

If G_i satisfies the assumption (A'_1) then there exists $C_1 := C_1(L) > 0$ with $L := \|\nabla u\|_{L^\infty(\Omega')}$ such that

$$\langle DG_i(\nabla u) \nabla \partial_e u, \nabla \partial_e u \rangle \geq C_1 |DG_i(\nabla u) \nabla \partial_e u|^2.$$

If G_i is the gradient of a $C_{loc}^{1,1}$ convex function then by Remark 4.2.2 we obtain

$$\langle DG_i(\nabla u) \nabla \partial_e u, \nabla \partial_e u \rangle \geq \frac{1}{\|DG_i\|_{L^\infty(B_L(0))}} |DG_i(\nabla u) \nabla \partial_e u|^2.$$

We set $K_i(G_i, C_1) := \min\{1, C_1, \frac{1}{\|DG_i\|_{L^\infty(B_L(0))}}\}$.

Thanks to Young's inequality each term of the sum in the right-hand side can be bounded by

$$\alpha_i \int_{\Omega} |DG_i(\nabla u) \nabla \partial_e u|^2 \eta^2 + \frac{1}{\alpha_i} \int_{\Omega} |\nabla \eta|^2 |\nabla u|^2 + |\nabla f| \eta^2 |\nabla u|$$

with $0 < \alpha_i < K_i(G_i, C_1)$. We can take $\alpha_i = \frac{K_i(G_i, C_1)}{2}$ for instance. We introduce $K(G, C_1) := \min\{K_i, 1 \leq i \leq n\}$ and $\alpha := \min\{\alpha_i, 1 \leq i \leq n\}$.

Thus, since $\eta \equiv 1$ on Ω'' we have that

$$\sum_{i=1}^n \int_{\Omega''} |DG_i(\nabla u) \nabla \partial_e u|^2 \leq \frac{2L(L|\Omega| \text{dist}(\partial\Omega', \partial\Omega'')^{-2} + \|f\|_{W^{1,1}(\Omega)})}{\alpha(G, C_1) K(G, C_1)}. \quad (4.3.2)$$

Here the dependence on G in α and K is just the dependence on $\|DG_i\|_{L^\infty(B_L(0))}$. Hence, for every $e \in \mathbb{S}^1$ the function $\partial_e(G_i(\nabla u)) = DG_i(\nabla u) \nabla \partial_e u$ is in $L_{loc}^2(\Omega)$. Thus, $G_i(\nabla u) \in W_{loc}^{1,2}(\Omega)$ and we have an explicit estimate for the norm from (4.3.2). Moreover, $G(\nabla u)$ is also in $W_{loc}^{1,2}(\Omega)$ as the sum of $(G_i(\nabla u))_{1 \leq i \leq n}$. \square

We apply this result to G^m and u_m to prove a uniform estimate on the Sobolev norm of $G_i^m(\nabla u_m)$.

Proposition 4.3.2. *If $G \in C^{0,1}(\mathbb{R}^N)$ satisfies the assumption of Theorem 4.1.1 or Theorem 4.1.4, then, $G^m(\nabla u_m) \in W_{loc}^{1,2}(\Omega)$ with a norm independent from $m \in \mathbb{N}$. Moreover, $G_i^m(\nabla u_m) \in W_{loc}^{1,2}(\Omega)$ with a norm independent from $m \in \mathbb{N}$ for every $1 \leq i \leq n$.*

Proof. We introduce $\Omega'' \Subset \Omega' \Subset \Omega$. If the case of Theorem 4.1.1 the functions $(G_1^m)_{m \in \mathbb{N}}$ satisfy uniformly the assumption (A'_1) . Since the norm $\|\nabla u_m\|_{L^\infty(\Omega')}$ can be bounded uniformly in $m \in \mathbb{N}$, all the estimates of the previous proposition are independent of $m \in \mathbb{N}$ if we apply it to G^m and u_m . Hence, $\|G_i^m(\nabla u_m)\|_{W^{1,2}(\Omega'')}$ can be bounded uniformly in $m \in \mathbb{N}$. That is also the case for their sum: $G^m(\nabla u_m) \in W_{loc}^{1,2}(\Omega)$. \square

4.3.2 Continuity of $G^m(\nabla u_m)$

In this subsection we use the $W^{1,2}$ regularity that we obtained before in order to prove the continuity of $G^m(\nabla u_m)$.

The following proposition is crucial in the proofs of Theorem 4.1.1 and Theorem 4.1.4. As in [29, Lemma 2.1] and [48, Theorem 3.1] our strategy to prove it relies on a maximum principle that can be found in [36, Theorem 3.1] and a theorem due to Lebesgue stated in [46, page 388]:

Proposition 4.3.3. *Let $H \in W_{loc}^{1,2}(\Omega)$. If for every $\epsilon > 0$ and every $x_0 \in \Omega$ there exists $C(\epsilon, x_0) > 0$ such that for every $0 < \delta < \text{dist}(x_0, \partial\Omega)$:*

$$\text{osc}_{B_\delta(x_0)} H \geq \epsilon \Rightarrow \text{osc}_{\partial B_\delta(x_0)} H \geq C(\epsilon, x_0),$$

then H is continuous at x_0 . Here, $\text{osc}_{B_\delta(x_0)} H := \sup_{x,y \in B_\delta(x_0)} |H(x) - H(y)|$.

Proof. We argue by contradiction. Let us assume that there exist $\epsilon > 0$ and $x_0 \in \Omega$ such that for every $0 < \delta < \frac{\text{dist}(x_0, \partial\Omega)}{2}$ there are $x, y \in B_\delta(x_0)$ such that $|H(x) - H(y)| \geq \epsilon$. By assumption, there exist $x_1, x_2 \in \partial B_\delta(x_0)$ such that $|H(x_1) - H(x_2)| \geq C(\epsilon, x_0)$. Hence, there exists $e \in \mathbb{S}^1$ such that

$$C(\epsilon, x_0) \leq \langle H(x_1), e \rangle - \langle H(x_2), e \rangle.$$

The term in the right-hand side can be bounded from above by $\int_{\partial B_\delta(x_0)} |\nabla H| d\mathcal{H}^1$. By Cauchy-Schwarz inequality we obtain

$$\frac{C(\epsilon, x_0)^2}{2\pi\delta} \leq \int_{\partial B_\delta(x_0)} |\nabla H|^2 d\mathcal{H}^1.$$

By integrating over δ between a certain δ_ϵ and $\frac{\text{dist}(x_0, \partial\Omega)}{2}$ we have

$$\frac{C(\epsilon, x_0)^2}{2\pi} \ln \frac{\text{dist}(x_0, \partial\Omega)}{2\delta_\epsilon} \leq \|H\|_{W^{1,2}(B_{\frac{\text{dist}(x_0, \partial\Omega)}{2}}(x_0))}^2.$$

By taking $\delta_\epsilon > 0$ small enough we obtain a contradiction thanks to the fact that $H \in W_{loc}^{1,2}(\Omega)$. \square

Thanks to Proposition 4.3.1 we can make the following observation:

Remark 4.3.4. In Proposition 4.3.2, we have proved that $G^m(\nabla u_m)$ is bounded in $W_{loc}^{1,2}(\Omega)$ uniformly in $m \in \mathbb{N}$. Hence, the functions $(G^m(\nabla u_m))_{m \in \mathbb{N}}$ are uniformly continuous on any compact subset of Ω with a modulus of continuity independent of $m \in \mathbb{N}$ if they satisfy the assumptions of Proposition 4.3.3 with a constant $C(\epsilon, x_0)$ independent of $m \in \mathbb{N}$.

It remains to prove that under the assumptions of Theorem 4.1.1 and Theorem 4.1.4, the functions $G^m(\nabla u_m)$ or $G_i^m(\nabla u_m)$ satisfy the maximum principle stated in Proposition 4.3.3 with C that does not depend on $m \in \mathbb{N}$.

The following lemma, instrumental for the proof of Theorem 4.1.1 uses the fact that $f \equiv 0$:

Lemma 4.3.5. *Let u_m be a C^2 solution of (4.2.2) with $f \equiv 0$. We have that $\det(D^2 u_m) \leq 0$.*

Proof. Since u_m is a solution of (4.2.2) with $f \equiv 0$ we have that

$$\text{Tr}(DG^m(\nabla u_m)D^2u_m) = 0. \tag{4.3.3}$$

Thus,

$$\text{Tr}((DG^m)^s(\nabla u_m)D^2u_m) = 0 \tag{4.3.4}$$

where $(DG^m)^s = \frac{DG^m + (DG^m)^T}{2}$ is the symmetric part of DG^m . Thanks to Proposition 4.2.3, $(DG^m)^s$ is positive-definite. Hence, in a basis where $(DG^m)^s$ is diagonal the product of the two terms on the diagonal of D^2u_m is non-positive. Since D^2u_m is symmetric we obtain $\det(D^2u_m) \leq 0$. \square

As a consequence of Lemma 4.3.5, [40, Theorem 2] implies that:

Proposition 4.3.6. *Let u_m be a solution of (4.2.2) with $f \equiv 0$. Then for every $\Omega' \Subset \Omega$, we have that $\partial\nabla u_m(\Omega') \subset \nabla u_m(\partial\Omega')$.*

Remark 4.3.7. This last result is true in any dimension provided that the sign of $\det(D^2u_m)$ does not change. This is the case if u_m is convex everywhere or concave for instance. This proposition can be used as an improved version of [29, Lemma 3.2].

With this result we can prove the following lemma:

Lemma 4.3.8. *Let u_m be a solution of (4.2.2) with $f \equiv 0$. Then for every $\Omega' \Subset \Omega$ we have that $\partial\sigma(\Omega') \subset \sigma(\partial\Omega')$ where $\sigma := G^m(\nabla u_m)$.*

Proof. We consider $z \in \partial\sigma(\Omega')$, then there exists a sequence $(z_n)_{n \in \mathbb{N}}$ such that $z_n \in \sigma(\Omega')$ and $z_n \rightarrow z$ when $n \rightarrow +\infty$. Hence, there exists a sequence $(y_n)_{n \in \mathbb{N}}$ with $y_n \in \nabla u_m(\Omega')$ such that $G^m(y_n) = z_n$. Since the sequence $(y_n)_{n \in \mathbb{N}}$ is bounded by $L := \|\nabla u_m\|_{L^\infty(\Omega')}$, we can extract a subsequence converging to $y \in \overline{\nabla u_m(\Omega')}$. By continuity of G^m , $z_n \rightarrow z = G^m(y)$. Since, $DG^m(y)$ is invertible, by the inverse function theorem if $y \in \text{Int}(\nabla u_m(\Omega'))$ then $z \in \text{Int}(\sigma(\Omega'))$. Since this is not the case we have that $y \in \partial\nabla u_m(\Omega')$. Thanks to Proposition 4.3.6, we obtain that $y \in \nabla u_m(\partial\Omega')$, thus $z = G^m(y) \in \sigma(\partial\Omega')$. \square

This lemma leads to the proof of Theorem 4.1.1 in the regularized setting:

Proposition 4.3.9. *Under the assumptions of Theorem 4.1.1, $G^m(\nabla u_m)$ is continuous with a modulus of continuity independent of $m \in \mathbb{N}$.*

Proof. We just have to prove that $G^m(\nabla u_m)$ satisfies the assumption of Proposition 4.3.3 uniformly in $m \in \mathbb{N}$. Let us assume that there exist $x_1, x_2 \in B_\delta(x_0)$ such that $|G^m(\nabla u_m(x_1)) - G^m(\nabla u_m(x_2))| \geq r > 0$. By Lemma 4.3.8 with $\Omega' = B_\delta(x_0)$, the diameter of $G^m(\nabla u_m(B_\delta(x_0)))$ can be bounded from above by the diameter of $G^m(\nabla u_m(\partial\Omega'))$. Thus, there exist $x_3, x_4 \in \partial B_\delta(x_0)$ such that $|G^m(\nabla u_m(x_4)) - G^m(\nabla u_m(x_3))| \geq r$. Hence thanks to Proposition 4.3.3, $G^m(\nabla u_m)$ is continuous with a modulus of continuity independent of the parameter of regularization $m \in \mathbb{N}$. \square

In the remaining part of the section, we proceed to establish continuity estimates for $G^m(\nabla u_m)$ independent of $m \in \mathbb{N}$ under the assumptions of Theorem 4.1.4. We prove that for each $0 \leq i \leq n$, $G_i^m(\nabla u_m)$ is continuous.

To do so, we use the fact that ∇u_m satisfies the following classical maximum principle see e.g. [36, Theorem 3.1]:

Proposition 4.3.10. *Let u_m be a solution of (4.2.2). Then for any $e \in \mathbb{S}^{N-1}$ and any open set $\Omega' \Subset \Omega$, we have that*

$$\sup_{x \in \Omega'} \partial_e u_m(x) = \sup_{x \in \partial \Omega'} \partial_e u_m(x).$$

We start with the case when $G_i^m = \nabla \varphi_i^m$ where $\varphi_i^m(\cdot) = f_i^m(\mathcal{N}_i^m(\cdot - \xi_i)) \in \mathcal{C}_{loc}^{1,1}(\mathbb{R}^2)$ with f_i^m a convex function, \mathcal{N}_i^m a pseudo-norm and $\xi_i \in \mathbb{R}^2$. The pseudo-norm is introduced in Definition 4.1.3. We denote the non-oriented angle between two vectors z_1, z_2 by $\angle(z_1, z_2) \in [0, \pi]$ with the convention that $\angle(z, 0) = 0$. We can apply the following lemma to G_i^m :

Lemma 4.3.11. *Let us assume that $G_i^m = \nabla \varphi_i^m$ with $\varphi_i^m(\cdot) = f_i^m(\mathcal{N}_i^m(\cdot - \xi_i)) \in \mathcal{C}_{loc}^{1,1}(\mathbb{R}^2)$.*

- *For every $r > 0$ there exists $C(r) > 0$ independent of $m \in \mathbb{N}$ such that if $|\nabla \varphi_i^m(z)| \geq r$ then $|z - \xi_i| \geq C(r)$.*
 - *For every $0 < \theta \leq \pi$ there exists $0 < D(\theta) \leq \pi$ independent of $m \in \mathbb{N}$ such that if $\angle(\nabla \varphi_i^m(z), \nabla \varphi_i^m(z')) \geq \theta$ then $\angle(z - \xi_i, z' - \xi_i) \geq D(\theta)$ for every $z, z' \in \mathbb{R}^2 \setminus \{\xi_i\}$.*
- Moreover, $C(r) \rightarrow 0$ when $r \rightarrow 0$ and $D(\theta) \rightarrow 0$ when $\theta \rightarrow 0$.*

Proof. For every $r > 0$ we introduce

$$C(r) := \inf\{|z - \xi_i|, z \in \mathbb{R}^2 \text{ such that } |\nabla \varphi_i^m(z)| \geq r \text{ for some } m \in \mathbb{N}\}$$

and for every $0 < \theta \leq \pi$ we introduce

$$D(\theta) := \inf\{\angle(z - \xi_i, z' - \xi_i), z, z' \in \mathbb{R}^2 \setminus \{\xi_i\} \text{ such that } \angle(\nabla \varphi_i^m(z), \nabla \varphi_i^m(z')) \geq \theta \text{ for some } m \in \mathbb{N}\}.$$

In the definition of the constant $D(\theta)$ we can replace $\mathbb{R}^2 \setminus \{\xi_i\}$ by $B_R(\xi_i) \setminus B_\rho(\xi_i)$ with $R > \rho > 0$ since the direction of $\nabla \varphi_i^m$ is constant on the half-lines starting at ξ_i .

Since for every $m \in \mathbb{N}$, $\nabla \varphi_i^m(z) = 0$ only when $z = \xi_i$, $C(r) \rightarrow 0$ when $r \rightarrow 0$. The fact that for each $m \in \mathbb{N}$, the range of the gradient of \mathcal{N}_i^m is not in a half-line provides that $D(\theta) \rightarrow 0$ when $\theta \rightarrow 0$.

It remains to prove that $C(r) > 0$ and $D(\theta) > 0$. If $C(r) = 0$ then there exist $(z_n)_{n \in \mathbb{N}}$ and $(m_n)_{n \in \mathbb{N}}$ such that $z_n \rightarrow \xi_i$ when $n \rightarrow +\infty$ and $|\nabla \varphi_i^{m_n}(z_n)| \geq r$ for every $n \in \mathbb{N}$. We set $M := \limsup_{n \rightarrow +\infty} m_n$. If $M \in \mathbb{N}$ then up to an extraction we can assume that $m_n \equiv M$ for n large enough. Thus, $|\nabla \varphi_i^M(z_n)| \geq r$ and $z_n \rightarrow \xi_i$ when $n \rightarrow +\infty$ which is a contradiction with the fact that $\nabla \varphi_i^M(\xi_i) = 0$. If $M = +\infty$

then we combine the fact that $\nabla\varphi_i^m$ converges to $\nabla\varphi_i$ uniformly with the fact that $\nabla\varphi_i(\xi_i) = 0$ to obtain a contradiction. Hence, $C(r) > 0$.

If $D(\theta) = 0$ then there exist $(z_n)_{n \in \mathbb{N}}$, $(z'_n)_{n \in \mathbb{N}}$ and $(m_n)_{n \in \mathbb{N}}$ such that $\angle(z_n - \xi_i, z'_n - \xi_i) \rightarrow 0$ when $n \rightarrow +\infty$ and $\angle(\nabla\varphi_i^{m_n}(z_n), \nabla\varphi_i^{m_n}(z'_n)) \geq \theta$. If $M := \limsup_{n \rightarrow +\infty} m_n < +\infty$ we use the continuity of $\nabla\varphi_i^M$ to obtain a contradiction. If $M = +\infty$ we use the fact that the sequences $(z_n)_{n \in \mathbb{N}}$ and $(z'_n)_{n \in \mathbb{N}}$ are in $B_R(\xi_i) \setminus B_\rho(\xi_i)$ to extract two converging subsequences that tend to $z \neq \xi_i$ and $z' \neq \xi_i$. By uniform convergence of $\nabla\varphi_i^m$ to $\nabla\varphi_i$ we obtain that $\angle(z - \xi_i, z' - \xi_i) = 0$ and $\angle(\nabla\varphi_i(z), \nabla\varphi_i(z')) \geq \theta$. That contradicts the fact that $\nabla\varphi_i(z)$ and $\nabla\varphi_i(z')$ are colinear when z and z' are colinear. Hence, $D(\theta) > 0$. \square

The converse is also true:

Lemma 4.3.12. *Let us assume that $G_i^m = \nabla\varphi_i^m$ with $\varphi_i^m(\cdot) = f_i^m(\mathcal{N}_i^m(\cdot - \xi_i)) \in \mathcal{C}_{loc}^{1,1}(\mathbb{R}^2)$.*

• *For every $r > 0$ there exists $C'(r) > 0$ independent of $m \in \mathbb{N}$ such that if $|z - \xi_i| \geq r$ then $|\nabla\varphi_i^m(z)| \geq C'(r)$.*

• *For every $0 < \theta \leq \pi$ there exists $0 < D'(\theta) \leq \pi$ independent of $m \in \mathbb{N}$ such that if $\angle(z - \xi_i, z' - \xi_i) \geq \theta$ then $\angle(\nabla\varphi_i^m(z), \nabla\varphi_i^m(z')) \geq D'(\theta)$ for every $z, z' \in \mathbb{R}^2 \setminus \{\xi_i\}$.*

Moreover, $C'(r) \rightarrow 0$ when $r \rightarrow 0$ and $D'(\theta) \rightarrow 0$ when $\theta \rightarrow 0$.

Proof. We argue as in the proof of the previous lemma. For every $r > 0$ and every $0 < \theta \leq \pi$ we set

$$C'(r) := \inf\{|\nabla\varphi_i^m(z)| \text{ with } m \in \mathbb{N}, z \in \mathbb{R}^2 \text{ such that } |z - \xi_i| \geq r\}$$

and

$$D'(\theta) := \inf\{\angle(\nabla\varphi_i^m(z), \nabla\varphi_i^m(z')) \text{ with } m \in \mathbb{N}, z, z' \in \mathbb{R}^2 \setminus \{\xi_i\} \\ \text{such that } \angle(z - \xi_i, z' - \xi_i) \geq \theta\}.$$

The quantity $D'(\theta)$ is the same if we replace $\mathbb{R}^2 \setminus \{\xi_i\}$ by $\mathbb{R}^2 \setminus B_r(\xi_i)$ with $r > 0$ since the direction of $\nabla\varphi_i^m$ is constant on the half-lines starting at ξ_i . The continuity of $\nabla\varphi_i^m$ gives that $C'(r) \rightarrow 0$ when $r \rightarrow 0$ and $D'(\theta) \rightarrow 0$ when $\theta \rightarrow 0$.

If we assume that $C'(r) = 0$, then by uniform coercivity of $|\nabla\varphi_i^m|$ we can find $z \in \mathbb{R}^2$ and $M \in \mathbb{N} \cup \{+\infty\}$ such that $\nabla\varphi_i^M(z) = 0$ and $|z - \xi_i| \geq r$ with the convention $\varphi_i^{+\infty} = \varphi_i$. Since $\nabla\varphi_i^M(z) = 0 \Rightarrow z = \xi_i$ this is absurd. Hence, we have that $C'(r) > 0$.

If we assume that $D'(\theta) = 0$ then once again we can find $z, z' \in \mathbb{R}^2 \setminus B_r(\xi_i)$ and $M \in \mathbb{N} \cup \{+\infty\}$ such that $\angle(\nabla\varphi_i^M(z), \nabla\varphi_i^M(z')) = 0$ and $\angle(z - \xi_i, z' - \xi_i) \geq \theta$. Using the strict convexity of the level sets of φ_i^M we obtain a contradiction. Thus, we have $D'(\theta) > 0$. \square

With these two results we can prove that:

Proposition 4.3.13. *We set $\sigma^i := G_i^m(\nabla u_m)$ with $G_i^m = \nabla \varphi_i^m$ where the convex function $\varphi_i^m(\cdot) = f_i^m(\mathcal{N}_i^m(\cdot - \xi_i))$ satisfies (A₃). If $x_0 \in \Omega$ is such that $\sigma^i(x_0) = 0$ then σ^i is continuous at x_0 and the modulus of continuity is independent of $m \in \mathbb{N}$.*

Proof. Thanks to Proposition 4.3.1 and Proposition 4.3.3, it remains to prove that for every $\epsilon > 0$ there exists $\tilde{C}(\epsilon, x_0) > 0$ such that for every $\delta > 0$ if there exists $x \in B_\delta(x_0)$ such that $|\sigma^i(x)| \geq \epsilon$ then there exist $x_1, x_2 \in \partial B_\delta(x_0)$ such that $|\sigma^i(x_1) - \sigma^i(x_2)| \geq \tilde{C}(\epsilon, x_0)$.

By the first point of Lemma 4.3.11, $|\nabla u_m(x) - \xi_i| \geq C(\epsilon) > 0$ with $C(\epsilon)$ that does not depend on $m \in \mathbb{N}$. We set $e := \frac{\nabla u_m(x) - \xi_i}{|\nabla u_m(x) - \xi_i|}$. By the maximum principle from Proposition 4.3.10, there exists $x_1 \in \partial B_\delta(x_0)$ such that

$$\langle \nabla u_m(x_1) - \xi_i, e \rangle \geq \langle \nabla u_m(x) - \xi_i, e \rangle = |\nabla u_m(x) - \xi_i| \geq C(\epsilon).$$

In particular $|\nabla u_m(x_1) - \xi_i| \geq C(\epsilon)$ and by Lemma 4.3.12 we obtain that $|\sigma^i(x_1)| \geq C'(C(\epsilon))$. If we set $e_1 := \frac{\nabla u_m(x_1) - \xi_i}{|\nabla u_m(x_1) - \xi_i|}$ then once again by Proposition 4.3.10 there exists $x_2 \in \partial B_\delta(x_0)$ such that

$$\langle \nabla u_m(x_2) - \xi_i, e_1 \rangle \leq \langle \nabla u_m(x_0) - \xi_i, e_1 \rangle = \langle 0, e_1 \rangle = 0. \quad (4.3.5)$$

The last equality comes from the fact that $\sigma^i(x_0) = 0 \Rightarrow \nabla u_m(x_0) = \xi_i$. If $\nabla u_m(x_2) = \xi_i$ then $|\sigma^i(x_1) - \sigma^i(x_2)| = |\sigma^i(x_1)| \geq C'(C(\epsilon)) > 0$. Otherwise, by (4.3.5) we get that $\angle(\nabla u_m(x_1) - \xi_i, \nabla u_m(x_2) - \xi_i) \geq \frac{\pi}{2}$. In that case, by the second point of Lemma 4.3.12 we obtain that $\angle(\sigma^i(x_1), \sigma^i(x_2)) \geq D'(\frac{\pi}{2})$. Since $|\sigma^i(x_1)| \geq C'(C(\epsilon))$ there exists $\tilde{C}(\epsilon) > 0$ such that $|\sigma^i(x_1) - \sigma^i(x_2)| \geq \tilde{C}(\epsilon)$.

By taking $\tilde{C}(\epsilon) := \min\{\tilde{C}(\epsilon), C'(C(\epsilon))\}$ we can apply Proposition 4.3.3. The conclusion follows. \square

In the case where $\sigma^i(x_0) \neq 0$, we have the following lemma:

Lemma 4.3.14. *We set $\sigma^i := G_i^m(\nabla u_m)$ with $G_i^m = \nabla \varphi_i^m$ where the convex function $\varphi_i^m(\cdot) = f_i^m(\mathcal{N}_i^m(\cdot - \xi_i))$ satisfies (A₃). If $\sigma^i(x_0) \neq 0$ then for every $r > 0$, $\theta > 0$ there exists $0 < \delta(r, \theta) < \frac{\text{dist}(x_0, \partial\Omega)}{2}$ independent of $m \in \mathbb{N}$ such that for every $x \in B_\delta(x_0)$, either $|\sigma^i(x)| \leq r$ or the angle between $\sigma^i(x)$ and $\sigma^i(x_0)$ is smaller than θ .*

Proof. Given $0 < \delta < \frac{\text{dist}(x_0, \partial\Omega)}{2}$ let us assume that there exists $x \in B_\delta(x_0)$ such that $|\sigma^i(x)| > r$ and $\angle(\sigma^i(x), \sigma^i(x_0)) \geq \theta$. By Lemma 4.3.11 we have that

$$|\nabla u_m(x) - \xi_i| > C(r) \text{ and } \angle(\nabla u_m(x) - \xi_i, \nabla u_m(x_0) - \xi_i) \geq D(\theta). \quad (4.3.6)$$

If $\langle \nabla u_m(x) - \xi_i, \frac{\nabla u_m(x_0) - \xi_i}{|\nabla u_m(x_0) - \xi_i|} \rangle \leq -\frac{C(r)}{2}$ then by Proposition 4.3.10 there exist $x_1 \in \partial B_\delta(x_0)$ such that $\langle \nabla u_m(x_1) - \xi_i, \frac{\nabla u_m(x_0) - \xi_i}{|\nabla u_m(x_0) - \xi_i|} \rangle \leq -\frac{C(r)}{2}$ and $x_2 \in \partial B_\delta(x_0)$ such that

$$\langle \nabla u_m(x_2) - \xi_i, \frac{\nabla u_m(x_0) - \xi_i}{|\nabla u_m(x_0) - \xi_i|} \rangle \geq \langle \nabla u_m(x_0) - \xi_i, \frac{\nabla u_m(x_0) - \xi_i}{|\nabla u_m(x_0) - \xi_i|} \rangle > 0.$$

The last inequality comes from the assumption $\sigma^i(x_0) \neq 0$. In that case the angle between $\nabla u_m(x_1) - \xi_i$ and $\nabla u_m(x_2) - \xi_i$ is bounded from below by a constant $0 < \theta' \leq \pi$ depending only on $C(r)$ and a Lipschitz constant of u_m on $B_{\frac{\text{dist}(x_0, \partial\Omega)}{2}}(x_0)$ independent of $m \in \mathbb{N}$. By the second point of Lemma 4.3.12 the angle between $\sigma^i(x_1)$ and $\sigma^i(x_2)$ is bounded from below by $D'(\theta') > 0$. Since $|\nabla u_m(x_1) - \xi_i| \geq \frac{C(r)}{2}$ by the first point of Lemma 4.3.12 we obtain that $|\sigma^i(x_1)|$ is larger than $C'(\frac{C(r)}{2})$. Hence, there exists a constant $F(r) > 0$ such that $|\sigma^i(x_1) - \sigma^i(x_2)| \geq F(r)$.

If $\langle \nabla u_m(x) - \xi_i, \frac{\nabla u_m(x_0) - \xi_i}{|\nabla u_m(x_0) - \xi_i|} \rangle > -\frac{C(r)}{2}$ then by (4.3.6) there exists e a unit vector orthogonal to $\nabla u_m(x_0) - \xi_i$ and $F'(r, \theta) > 0$ such that $\langle \nabla u_m(x) - \xi_i, e \rangle > F'(r, \theta)$. Once again by Proposition 4.3.10 there exist $x_1, x_2 \in \partial B_\delta(x_0)$ such that

$$\langle \nabla u_m(x_1) - \xi_i, e \rangle \geq \langle \nabla u_m(x) - \xi_i, e \rangle > F'(r, \theta) > 0 = \langle \nabla u_m(x_0) - \xi_i, e \rangle \geq \langle \nabla u_m(x_2) - \xi_i, e \rangle.$$

If $\nabla u_m(x_2) = \xi_i$ then by Lemma 4.3.12, $|\sigma^i(x_1) - \sigma^i(x_2)| \geq C'(F'(r, \theta))$. Otherwise, the angle between $\nabla u_m(x_1) - \xi_i$ and $\nabla u_m(x_2) - \xi_i$ is bounded from below by a constant $0 < \theta' \leq \pi$ depending only on $r > 0$, $\theta > 0$ and the Lipschitz constant of u_m on $B_{\frac{\text{dist}(x_0, \partial\Omega)}{2}}(x_0)$ independent of $m \in \mathbb{N}$. We conclude as in the first case.

Hence, we have proved that for every $0 < \delta < \frac{\text{dist}(x_0, \partial\Omega)}{2}$ if there exists $x \in B_\delta(x_0)$ such that $|\sigma^i(x)| > r$ and the angle between $\sigma^i(x)$ and $\sigma^i(x_0)$ is larger than θ then there exist $x_1, x_2 \in \partial B_\delta(x_0)$ and $F(r, \theta) > 0$ such that $|\sigma^i(x_1) - \sigma^i(x_2)| \geq F(r, \theta)$. We can conclude as in the proof of Proposition 4.3.3:

$$F(r, \theta) \leq |\sigma^i(x_1) - \sigma^i(x_2)| \leq \int_{\partial B_\delta(x_0)} |\nabla \sigma^i| d\mathcal{H}^1.$$

Using the Cauchy-Schwarz inequality and integrating over δ between $\delta' > 0$ and $\frac{\text{dist}(x_0, \partial\Omega)}{2}$ we obtain

$$\frac{F(r, \theta)^2}{2\pi} \ln \frac{\text{dist}(x_0, \partial\Omega)}{2\delta'} \leq \|\sigma^i\|_{W^{1,2}(B_{\frac{\text{dist}(x_0, \partial\Omega)}{2}}(x_0))}^2.$$

The conclusion follows from the fact that $\sigma^i \in W_{loc}^{1,2}(\Omega)$ with a norm independent of $m \in \mathbb{N}$ by Proposition 4.3.1. □

The following lemma asserts that the component of σ^i is continuous in the direction of $\sigma^i(x_0)$.

Lemma 4.3.15. *We set $\sigma^i := G_i^m(\nabla u_m)$ with $G_i^m = \nabla \varphi_i^m$ where the convex function $\varphi_i^m(\cdot) = f_i^m(\mathcal{N}_i^m(\cdot - \xi_i))$ satisfies (A₃). If $\sigma^i(x_0) \neq 0$ then for every $\epsilon > 0$ there exists $\delta > 0$ independent of $m \in \mathbb{N}$ such that*

$$|\sigma^i(x_0)| - \epsilon \leq \left\langle \sigma^i(x), \frac{\sigma^i(x_0)}{|\sigma^i(x_0)|} \right\rangle \leq |\sigma^i(x_0)| + \epsilon$$

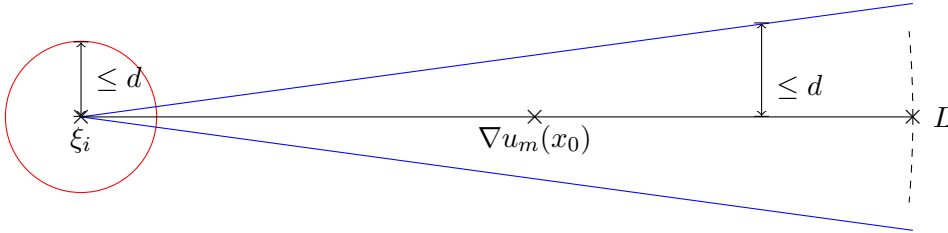
for every $x \in B_\delta(x_0)$.

Proof. Thanks to Lemma 4.3.14, for every $r > 0$ and $\theta > 0$ there exists $\delta(r, \theta) > 0$ such that for every $x \in B_\delta(x_0)$, $|\sigma^i(x)| \leq r$ or $\angle(\sigma^i(x), \sigma^i(x_0)) \leq \theta$. By the contrapositive statement of Lemma 4.3.12, there exist $\tilde{C}(r) > 0$ and $\tilde{D}(\theta) > 0$ such that $|\nabla u_m(x) - \xi_i| \leq \tilde{C}(r)$ or $\angle(\nabla u_m(x) - \xi_i, \nabla u_m(x_0) - \xi_i) \leq \tilde{D}(\theta)$ for every $x \in B_\delta(x_0)$. Moreover, we can choose them in such a way that $\tilde{C}(r) \rightarrow 0$ when $r \rightarrow 0$ and $\tilde{D}(\theta) \rightarrow 0$ when $\theta \rightarrow 0$.

For $\epsilon > 0$, we introduce $\eta > 0$ independent of $m \in \mathbb{N}$ such that the oscillations of G_i^m on the square of center ξ_i and sides of length 2η are smaller than $\frac{\epsilon}{2}$. Since $\tilde{C}(r)$ goes to 0 when r goes to 0 and $\tilde{D}(\theta)$ goes to 0 when θ goes to 0 we can choose $\delta > 0$ small enough such that for every $x \in B_\delta(x_0)$ we have

$$|\nabla u_m(x) - \xi_i| \leq \eta \text{ or } \angle(\nabla u_m(x) - \xi_i, \nabla u_m(x_0) - \xi_i) \text{ is as small as we want. (4.3.7)}$$

We introduce p_x the projection of $\nabla u_m(x)$ on $\mathbb{R}_+(\nabla u_m(x_0) - \xi_i) + \xi_i$. Hence, for every $d > 0$ we can choose δ_0 such that for every $x \in B_{\delta_0}(x_0)$ the distance between $\nabla u_m(x)$ and p_x is smaller than d . By uniform continuity of G_i^m in $m \in \mathbb{N}$, there exists ω a modulus of continuity independent of $m \in \mathbb{N}$ such that $|G_i^m(p_x) - G_i^m(\nabla u_m(x))| \leq \omega(d)$.



Let us argue by contradiction. We assume that for every $0 < \delta < \delta_0$ there exists $x \in B_\delta(x_0)$ such that $|\sigma^i(x_0)| + \epsilon < \langle \sigma^i(x), e \rangle$ or $\langle \sigma^i(x), e \rangle < |\sigma^i(x_0)| - \epsilon$ with $e := \frac{\sigma^i(x_0)}{|\sigma^i(x_0)|}$.

Case 1 We begin with the case where $\langle \sigma^i(x), e \rangle > |\sigma^i(x_0)| + \epsilon$. Since $|G_i^m(p_x) - \sigma^i(x)| \leq \omega(d)$ and $\langle \sigma^i(x), e \rangle \geq |\sigma^i(x_0)| + \epsilon$ we obtain that

$$\langle G_i^m(p_x), e \rangle \geq |\sigma^i(x_0)| = \langle \sigma^i(x_0), e \rangle$$

when d is small enough such that $\omega(d) \leq \epsilon$ without any dependence on $m \in \mathbb{N}$.

By definition of φ_i^m we have that $G_i^m(p_x) = (f_i^m)'(\mathcal{N}_i^m(p_x - \xi_i)) \nabla \mathcal{N}_i^m(p_x - \xi_i)$ and $\sigma^i(x_0) = (f_i^m)'(\mathcal{N}_i^m(\nabla u_m(x_0) - \xi_i)) \nabla \mathcal{N}_i^m(\nabla u_m(x_0) - \xi_i)$. Thus, these two vectors are positively colinear to e . This means that $(f_i^m)'(\mathcal{N}_i^m(p_x - \xi_i)) \geq (f_i^m)'(\mathcal{N}_i^m(\nabla u_m(x_0) - \xi_i))$. Thus, by strict convexity of f_i^m we have that

$$\langle \nabla u_m(x) - \nabla u_m(x_0), \nabla u_m(x_0) - \xi_i \rangle = \langle p_x - \nabla u_m(x_0), \nabla u_m(x_0) - \xi_i \rangle \geq 0.$$

Hence, by Proposition 4.3.10 there exists $x_1, x_2 \in \partial B_\delta(x_0)$ such that

$$\langle \nabla u_m(x_1) - \xi_i, e' \rangle \leq \langle \nabla u_m(x_0) - \xi_i, e' \rangle \leq \langle \nabla u_m(x) - \xi_i, e' \rangle \leq \langle \nabla u_m(x_2) - \xi_i, e' \rangle$$

with $e' := \frac{\nabla u_m(x_0) - \xi_i}{|\nabla u_m(x_0) - \xi_i|}$. We introduce p_1 and p_2 the projection of $\nabla u_m(x_1)$ and $\nabla u_m(x_2)$ on $\mathbb{R}_+(\nabla u_m(x_0) - \xi_i) + \xi_i$. Since $\langle p_1, e' \rangle \leq \langle \nabla u_m(x_0) - \xi_i, e' \rangle$ the convexity

of f_i^m gives that $\langle G_i^m(p_1), e \rangle \leq \langle \sigma^i(x_0), e \rangle$. For the same reasons, $\langle G_i^m(p_2), e \rangle \geq \langle G_i^m(p_x), e \rangle \geq \langle \sigma^i(x), e \rangle - \omega(d)$ where the last inequality comes from the fact that $|G_i^m(p_x) - \sigma^i(x)| \leq \omega(d)$.

We also have that $|\nabla u_m(x_1) - p_1|$ and $|\nabla u_m(x_2) - p_2|$ are smaller than d . Thus, $\langle \sigma^i(x_1), e \rangle \leq \langle G_i^m(p_1), e \rangle + \omega(d)$ and $\langle \sigma^i(x_2), e \rangle \geq \langle G_i^m(p_2), e \rangle - \omega(d)$. Hence,

$$\langle \sigma^i(x_1), e \rangle \leq \langle \sigma^i(x_0), e \rangle + \omega(d) \quad (4.3.8)$$

and $\langle \sigma^i(x_2), e \rangle \geq \langle \sigma^i(x), e \rangle - 2\omega(d)$. When d is sufficiently small with respect to ϵ we have that $|\sigma^i(x_2) - \sigma^i(x_1)| \geq \frac{\epsilon}{2}$.

Case 2 Let us assume that there exists $x \in B_\delta(x_0)$ such that $|\sigma^i(x_0)| - \epsilon > \langle \sigma^i(x), \frac{\sigma^i(x_0)}{|\sigma^i(x_0)|} \rangle$. Hence, if we apply Proposition 4.3.10 to $\langle \nabla u_m(\cdot) - \xi_i, \nabla u_m(x_0) - \xi_i \rangle$ we can show that there exists $x_1 \in \partial B_\delta(x_0)$ such that $\langle \nabla u_m(x_1) - \xi_i, \nabla u_m(x_0) - \xi_i \rangle \geq \langle \nabla u_m(x_0) - \xi_i, \nabla u_m(x_0) - \xi_i \rangle$. As in the previous case we can show that $\langle \sigma^i(x_1), e \rangle \geq |\sigma^i(x_0)| - \omega(d)$ that is an analogous result to (4.3.8) (up to interchanging \leq in \geq).

It remains to find a point $x_2 \in \partial B_\delta(x_0)$ such that $\langle \nabla u_m(x_2) - \xi_i, \nabla u_m(x_0) - \xi_i \rangle$ is sufficiently small. We distinguish two sub-cases. We start by assuming that $\nabla u_m(x) \in B_\eta(\xi_i)$. By Proposition 4.3.10 there exists $x_2 \in \partial B_\delta(x_0)$ such that

$$\langle \nabla u_m(x_2) - \xi_i, \nabla u_m(x_0) - \xi_i \rangle \leq \langle \nabla u_m(x) - \xi_i, \nabla u_m(x_0) - \xi_i \rangle.$$

By (4.3.7) we have that $\nabla u_m(x_2) \in B_\eta(\xi_i)$. Since the oscillations of G_i^m are smaller than $\frac{\epsilon}{2}$ on that set we have that $\langle \sigma^i(x_2), e \rangle \leq \langle \sigma^i(x), e \rangle + \frac{\epsilon}{2}$. In that case we have

$$\langle \sigma^i(x_2), e \rangle \leq \langle \sigma^i(x), e \rangle + \frac{\epsilon}{2} \leq \langle \sigma^i(x_0), e \rangle - \frac{\epsilon}{2} \leq \langle \sigma^i(x_1), e \rangle - \frac{\epsilon}{2} + \omega(d).$$

Thus, by taking d small enough $|\sigma^i(x_1) - \sigma^i(x_2)| \geq \frac{\epsilon}{4}$.

If $\nabla u_m(x) \notin B_\eta(\xi_i)$ then $\langle p_x - \xi_i, \nabla u_m(x_0) - \xi_i \rangle > 0$ with p_x the projection of $\nabla u_m(x)$ on $\mathbb{R}_+(\nabla u_m(x_0) - \xi_i) + \xi_i$. Hence, there exists $x_2 \in \partial B_\delta(x_0)$ such that $\langle \nabla u_m(x_2), e' \rangle \leq \langle \nabla u_m(x), e' \rangle$ which implies that $\langle p_x - p_2, \nabla u_m(x_0) - \xi_i \rangle \geq 0$. Hence, by convexity of f_i^m we have that $\langle G_i^m(p_2), e \rangle \leq \langle G_i^m(p_x), e \rangle$. Thus,

$$\langle \sigma^i(x_2), e \rangle \leq \langle \sigma^i(x), e \rangle + 2\omega(d) \leq \langle \sigma^i(x_0), e \rangle - \epsilon + 2\omega(d).$$

Once again, by taking d small enough $|\sigma^i(x_1) - \sigma^i(x_2)| \geq \frac{\epsilon}{4}$.

In any case, for every $0 < \delta < \delta_0$ such that there exists $x \in B_\delta(x_0)$ such that $|\sigma^i(x_0)| + \epsilon < \langle \sigma^i(x), e \rangle$ or $\langle \sigma^i(x), e \rangle < |\sigma^i(x_0)| - \epsilon$ we can find $x_1, x_2 \in \partial B_\delta(x_0)$ such that $|\sigma^i(x_1) - \sigma^i(x_2)| \geq \frac{\epsilon}{4}$. The conclusion follows Proposition 4.3.3. \square

The combination of the last three results gives the following proposition:

Proposition 4.3.16. *Under the assumptions of Theorem 4.1.4, in the case where $G_i^m = \nabla \varphi_i^m$ with $\varphi_i^m(\cdot) = f_i^m(\mathcal{N}_i^m(\cdot - \xi_i))$, we have that $G_i^m(\nabla u_m)$ is continuous on Ω with a modulus of continuity that does not depend on $m \in \mathbb{N}$.*

Now, let us focus on the case where G_i satisfies the assumption (A_4) .

Proposition 4.3.17. *Let us assume that $G_i^m = \nabla \varphi_i^m$ with $\varphi_i^m(\cdot) = f_i^m(\langle \cdot, \xi_i \rangle)$ and $\xi_i \neq 0$. Then if $G_i^m(\nabla u_m) \in W_{loc}^{1,2}(\Omega)$ we have that $\sigma^i := G_i^m(\nabla u_m)$ is continuous on Ω with a modulus of continuity that does not depend on $m \in \mathbb{N}$.*

Proof. For every $\epsilon > 0$ we have

$$|G_i^m(z_1) - G_i^m(z_2)| \geq \epsilon \Leftrightarrow |(f_i^m)'(\langle z_1, \xi_i \rangle) - (f_i^m)'(\langle z_2, \xi_i \rangle)| \geq \frac{\epsilon}{|\xi_i|} \quad (4.3.9)$$

for every $z_1, z_2 \in \mathbb{R}^2$.

Thus, for $\delta > 0$, if we assume that there exists $x \in B_\delta(x_0)$ such that

$$|G_i^m(\nabla u_m(x)) - G_i^m(\nabla u_m(x_0))| \geq \epsilon$$

then

$$|(f_i^m)'(\langle \nabla u_m(x), \xi_i \rangle) - (f_i^m)'(\langle \nabla u_m(x_0), \xi_i \rangle)| \geq \frac{\epsilon}{|\xi_i|}. \quad (4.3.10)$$

Up to a change of sign of ξ_i we can assume that $\langle \nabla u_m(x_0), \xi_i \rangle \leq \langle \nabla u_m(x), \xi_i \rangle$. By the maximum principle from Proposition 4.3.10 applied to $y \rightarrow \langle \nabla u_m(y), \xi_i \rangle$, there exist $x_1, x_2 \in \partial B_\delta(x_0)$ such that $\langle \nabla u_m(x_1), \xi_i \rangle \leq \langle \nabla u_m(x_0), \xi_i \rangle$ and $\langle \nabla u_m(x), \xi_i \rangle \leq \langle \nabla u_m(x_2), \xi_i \rangle$.

We use the fact that $(f_i^m)'$ is increasing with (4.3.10) to obtain that

$$|(f_i^m)'(\langle \nabla u_m(x_1), \xi_i \rangle) - (f_i^m)'(\langle \nabla u_m(x_2), \xi_i \rangle)| \geq \frac{\epsilon}{|\xi_i|}.$$

Thus, by (4.3.9), we get that $|G_i^m(\nabla u_m(x_1)) - G_i^m(\nabla u_m(x_2))| \geq \epsilon$. Once again, we can conclude thanks to Proposition 4.3.3. \square

Hence, we have proved the following result:

Proposition 4.3.18. *Under the assumptions of Theorem 4.1.4, the continuous functions $(G^m(\nabla u_m))_{m \in \mathbb{N}}$ have a uniform modulus of continuity on each compact subset of Ω .*

Proof. For every $m \in \mathbb{N}$ we have proved in Proposition 4.3.16 or in Proposition 4.3.17 that for every $1 \leq i \leq n$ the function $G_i^m(\nabla u_m)$ is continuous with a modulus of continuity independent of $m \in \mathbb{N}$. Hence that is the case for their sum, namely $G^m(\nabla u_m)$. \square

4.4 Uniform estimates for Theorem 4.1.6

In this section we study the case when $G = G_1 + G_2$ where $G_i(z) := f_i'(\langle z, \xi_i \rangle) \xi_i$ with $f_1 \in C_{loc}^{1,1}(\mathbb{R})$ and $f_2 \in C^1(\mathbb{R}) \cap C_{loc}^{1,1}(\mathbb{R} \setminus \{0\})$ two convex functions. Moreover, the right-hand side of (4.1.4) is a constant $\lambda \in \mathbb{R}$.

We begin by the following observation on ξ_1 and ξ_2 :

Proposition 4.4.1. *Under the assumptions of Theorem 4.1.6 we can assume that $\xi_1 = e_1$ and $\xi_2 = e_2$ are the two standard vectors of the canonical basis.*

Proof. Let us assume that A is an invertible linear matrix. We introduce the convex function $\varphi(z) := f_1(\langle z, \xi_1 \rangle) + f_2(\langle z, \xi_2 \rangle)$. Let us consider u a solution of (4.2.2), then u is a minimizer of

$$\min_{w \in W_u^{1,2}(\Omega)} \int_{\Omega} \varphi(\nabla w(x)) + \lambda w(x) dx \quad (4.4.1)$$

with $\varphi(z) = f_1(\langle z, \xi_1 \rangle) + f_2(\langle z, \xi_2 \rangle)$. For every $w \in W^{1,2}(\Omega)$, we have:

$$\int_{\Omega} \varphi(\nabla w(x)) + \lambda w(x) dx = |\det A| \int_{A^{-1}(\Omega)} \varphi(\nabla w(Ay)) + \lambda w(Ay) dy.$$

If we set $v(y) := w(Ay)$ then:

$$\int_{\Omega} \varphi(\nabla w(x)) + \lambda w(x) dx = |\det A| \int_{A^{-1}(\Omega)} \varphi((A^T)^{-1} \nabla v(y)) + \lambda v(y) dy.$$

Hence, since u is a minimizer of (4.4.1), we obtain that $u(A \cdot)$ is a minimizer of

$$\int_{A^{-1}(\Omega)} \varphi((A^T)^{-1} \nabla v(y)) + \lambda v(y) dy$$

on $W_{u(A \cdot)}^{1,2}(A^{-1}(\Omega))$. It remains to choose A such that $(A^{-1})\xi_1 = e_1$ and $(A^{-1})\xi_2 = e_2$.

Since proving that $\nabla \varphi(\nabla u) \in \mathcal{C}^0(\Omega)$ is equivalent to proving that $\nabla \varphi(\nabla u(A \cdot)) \in \mathcal{C}^0(A^{-1}(\Omega))$ we can assume that $\xi_1 = e_1$ and $\xi_2 = e_2$. \square

We want to establish continuity estimates for $G^m(\nabla u_m)$ independent of $m \in \mathbb{N}$ with u_m solution of (4.2.2). We start by proving the following lemma inspired by [29, Proposition 2.3]:

Lemma 4.4.2. *Let f_1 and f_2 be two smooth convex functions. Let u be a smooth solution of (4.2.2) with $G(z) := f_1'(\langle z, e_1 \rangle)e_1 + f_2'(\langle z, e_2 \rangle)e_2$. Then the function $f_1'(\partial_1 u)$ belongs to $W_{loc}^{1,2}(\Omega)$. Moreover, for every $\Omega'' \Subset \Omega' \Subset \Omega$ and every $L_{\Omega'} \geq \|\nabla u\|_{L^\infty(\Omega')}$ we have*

$$\|f_1'(\partial_1 u)\|_{W^{1,2}(\Omega'')} \leq C(L_{\Omega'}, \|f_1'\|_{L^\infty(B_{L_{\Omega'}})}, \|f_1''\|_{L^\infty(B_{L_{\Omega'}})}, \|f_2'\|_{L^\infty(B_{L_{\Omega'}})}, \text{dist}(\partial\Omega', \partial\Omega'')).$$

Proof. By local Lipschitz regularity of u , we already know that $f_1'(\partial_1 u) \in L_{loc}^2(\Omega)$. Since u is a solution of (4.2.2) we have that:

$$\int_{\Omega} \langle \nabla \varphi(\nabla u), \nabla \theta \rangle = - \int_{\Omega} \lambda \theta$$

for every $\theta \in \mathcal{C}_0^\infty(\Omega)$. If we differentiate the Euler-Lagrange equation in the first direction, the fact that $\lambda \in \mathbb{R}$ gives that

$$\int_{\Omega} \langle \nabla^2 \varphi(\nabla u) \nabla \partial_1 u, \nabla \theta \rangle = 0 \quad (4.4.2)$$

for every $\theta \in C_0^\infty(\Omega)$. If we replace θ by $\xi^2 f_1'(\partial_1 u)$ with $\xi \in C_0^\infty(\Omega')$ and $\Omega' \Subset \Omega$ we obtain

$$\sum_{i=1,2} \int_{\Omega} f_i''(\partial_i u) \partial_{1i} u \xi^2 f_1''(\partial_1 u) \partial_{1i} u = -2 \sum_{i=1,2} \int_{\Omega} f_i''(\partial_i u) \partial_{1i} u \xi \partial_i \xi f_1'(\partial_1 u).$$

Since the terms in the left hand side are non-negative we have that

$$\int_{\Omega} (f_1''(\partial_1 u) \partial_{11} u \xi)^2 \leq -2 \sum_{i=1,2} \int_{\Omega} f_i''(\partial_i u) \partial_{1i} u \xi \partial_i \xi f_1'(\partial_1 u).$$

With an integration by parts on the right hand side we get

$$\int_{\Omega} (\partial_1 [f_1'(\partial_1 u)])^2 \xi^2 \leq 2 \sum_{i=1,2} \int_{\Omega} f_i'(\partial_i u) \partial_1 (\xi \partial_i \xi) f_1'(\partial_1 u) + f_i'(\partial_i u) \xi \partial_i \xi \partial_1 [f_1'(\partial_1 u)].$$

With the Young and Hölder inequalities and the fact that $\|\nabla u\|_{L^\infty(\Omega')} \leq L_{\Omega'}$ we obtain that

$$\int_{\Omega} (\partial_1 [f_1'(\partial_1 u)])^2 \xi^2 \leq C(L_{\Omega'}, \|f_1'\|_{L^\infty(-L_{\Omega'}, L_{\Omega'})}, \|f_2'\|_{L^\infty(-L_{\Omega'}, L_{\Omega'})}, \|\xi\|_{W^{1,\infty}(\Omega)}).$$

Hence, if we take $\xi \in C_0^\infty(\Omega')$ such that $\xi \equiv 1$ on Ω'' with $\Omega'' \Subset \Omega'$ then we get that $\partial_1 [f_1'(\partial_1 u)]$ belongs to $L^2(\Omega'')$ and

$$\|\partial_1 [f_1'(\partial_1 u)]\|_{L^2(\Omega'')} \leq C(L_{\Omega'}, \|f_1'\|_{L^\infty(-L_{\Omega'}, L_{\Omega'})}, \|f_2'\|_{L^\infty(-L_{\Omega'}, L_{\Omega'})}, \text{dist}(\Omega', \Omega'')).$$

With the same strategy we can also prove that $\partial_2 [f_2'(\partial_2 u)] \in L^2(\Omega'')$ with the same estimate.

It remains to prove that $\partial_2 [f_1'(\partial_1 u)] \in L_{loc}^2(\Omega)$. We proceed as in the proof of [19, Theorem 2.1]. For $0 < h < \frac{1}{4}$ and $x \in \Omega''$, we introduce $\tau(x) = \frac{\sigma(x + h e_2) - \sigma(x)}{h}$ where $\sigma(x) = G(\nabla u(x))$. We set $\tau_1(x) := \frac{f_1'(\partial_1 u(x + h e_2)) - f_1'(\partial_1 u(x))}{h}$, we want to prove that $\|\tau_1\|_{L_{loc}^2(\Omega'')}$ is bounded uniformly in h .

Since f_1' is Lipschitz continuous and increasing there exists $K > 1$ such that

$$\tau_1(x)^2 \leq K \tau_1(x) \times \frac{\partial_1 u(x + h e_2) - \partial_1 u(x)}{h}.$$

Using the fact that f_2 is convex we have that:

$$\tau_1(x)^2 \leq K \langle \tau(x), \frac{\nabla u(x + h e_2) - \nabla u(x)}{h} \rangle.$$

With $\xi \in C_0^\infty(\Omega''')$ where $\Omega''' \Subset \Omega''$ and $0 < h < \text{dist}(\partial\Omega''', \partial\Omega'')$ we have

$$\|\xi\tau_1\|_{L^2(\Omega''')}^2 \leq K \int_{\Omega'''} \xi^2 \left\langle \tau(x), \frac{\nabla u(x + he_2) - \nabla u(x)}{h} \right\rangle.$$

Since $\text{div } \tau = 0$, an integration by parts gives that

$$\|\xi\tau_1\|_{L^2(\Omega''')}^2 \leq 2K \int_{\Omega'''} |\xi(x)| \frac{u(x + he_2) - u(x)}{h} \langle \nabla \xi(x), \tau(x) \rangle.$$

Thus,

$$\|\xi\tau_1\|_{L^2(\Omega''')}^2 \leq C(L_{\Omega'}, K, \|\xi\|_{W^{1,\infty}(\Omega'')}) \|\xi\tau\|_{L^2(\Omega''')}.$$

We already know that $\|\tau_2\|_{L^2(\Omega''')}$ is bounded uniformly in h thanks to the fact that $\partial_2[f_2'(\partial_2 u)] \in L^2(\Omega'')$. Hence, $\|\xi\tau_1\|_{L^2(\Omega''')}$ is bounded uniformly in h . Thus, if we take $\xi \equiv 1$ on a subset of Ω''' , we obtain that $\partial_2[f_1'(\partial_1 u)]$ belongs to $L_{loc}^2(\Omega''')$ with an explicit estimate. \square

If we apply this lemma with f_1^m and f_2^m we obtain an estimate on the Sobolev norm of $(f_1^m)'(\partial_1 u_m)$ independent of $m \in \mathbb{N}$. Hence, we can apply Proposition 4.3.17 to prove that $(f_1^m)'(\partial_1 u_m)$ is continuous with a modulus of continuity that does not depend on $m \in \mathbb{N}$. It remains to do the same for $(f_2^m)'(\partial_2 u_m)$. For every $r > 0$ we have the following result coming from [29, Proposition 2.4]:

Proposition 4.4.3. *Let f_1 and f_2 be two smooth convex functions. Let u be a smooth solution of (4.2.2) with $G(z) := f_1'(\langle z, e_1 \rangle)e_1 + f_2'(\langle z, e_2 \rangle)e_2$ and $f \equiv \lambda \in \mathbb{R}$. For every $r > 0$ and every $x_0 \in \Omega' \Subset \Omega$, we have*

$$\int_{\Omega' \cap U_r} |f_2''(\partial_2 u) \partial_{22} u|^2 \leq C(r, \alpha_r, \|f_1''\|_{L^\infty(B_{L_{\Omega'}})})$$

with $U_r := \{x \in \Omega, |\partial_2 u(x)| \geq r\}$, $L_{\Omega'} := \|\nabla u\|_{L^\infty(\Omega')}$ and $\alpha_r := \sup_{r \leq t \leq L_{\Omega'}} f_2''(t)$.

Proof. Since λ is a constant, the right-hand side of (4.2.2) vanishes when we differentiate the equation. By [29, Proposition 2.4], we have that

$$\int_{\Omega' \cap U_r} |f_1''(\partial_1 u) \partial_{11} u|^2 + |f_2''(\partial_2 u) \partial_{22} u|^2 \leq C(r, \alpha_r, \|f_1''\|_{L^\infty(B_{L_{\Omega'}})})$$

and the conclusion follows. \square

Since this proposition allows to avoid the values of f_2'' around the origin we can apply it with f_1^m and f_2^m . In that case the constant $C(r, \alpha_r, \|(f_1^m)''\|_{L^\infty(B_{L_{\Omega'}})})$ can be taken independent of $m \in \mathbb{N}$.

Let us use this estimate in order to prove the continuity of $(f_2^m)'(\partial_2 u_m)$ uniformly in $m \in \mathbb{N}$. Thanks to Proposition 4.2.5 there exist $r > 0$, $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a continuous function that satisfies $\omega(t) = 0 \Leftrightarrow t = 0$ and $M_r \in \mathbb{N}$ such that for every $m \geq M_r$, every $x, y \in (-\frac{r}{2}, \frac{r}{2})$ we have that

$$((f_2^m)'(x) - (f_2^m)'(y))(x - y) \geq \omega(|x - y|). \quad (4.4.3)$$

We prove the following alternative:

Lemma 4.4.4. *For every $t > 0$ there exist $0 < d_t < t$ and $\delta_0 > 0$ such that for every $m \geq M_r$, $\partial_2 u_m(x) > d_t$ for every $x \in B_{\delta_0}(x_0)$ or $\partial_2 u_m(x) < t$ for every $x \in B_{\delta_0}(x_0)$.*

Proof. By (4.4.3), for every $t > 0$ there exist $0 < d < t$ and C independent of $m \geq M_r$ such that $0 < C \leq (f_2^m)'(t) - (f_2^m)'(d)$. We introduce $F : \mathbb{R} \rightarrow \mathbb{R}$ a smooth increasing function such that $F(s) = 0$ for every $s < (f_2^m)'(d)$ and $F(s) = C$ for every $s > (f_2^m)'(t)$. We assume that for every $0 < \delta < \frac{\text{dist}(x_0, \partial\Omega)}{2}$ there exist x_1 and x_2 in $B_\delta(x_0)$ such that $\partial_2 u_m(x_1) < d < t < \partial_2 u_m(x_2)$. Once again, thanks to the maximum principle from Proposition 4.3.10, there exist x'_1 and x'_2 on $\partial B_\delta(x_0)$ such that $\partial_2 u_m(x'_1) < d < t < \partial_2 u_m(x'_2)$. Hence,

$$\begin{aligned} C &= F((f_2^m)'(\partial_2 u_m(x'_2))) - F((f_2^m)'(\partial_2 u_m(x'_1))) \leq \int_{\partial B_\delta(x_0)} |\nabla[F((f_2^m)'(\partial_2 u_m))]| d\mathcal{H}^1 \\ &\leq \|\nabla F\|_{L^\infty(\mathbb{R})} \int_{\partial B_\delta(x_0) \cap U_d} |(f_2^m)''(\partial_2 u_m) \nabla \partial_2 u_m| d\mathcal{H}^1 \end{aligned}$$

with $U_d = \{x \in \Omega, |\partial_2 u_m| \geq d\}$. Thus, as in the proof of Proposition 4.3.3, if we integrate between δ_0 and $\frac{\text{dist}(x_0, \partial\Omega)}{2}$ we obtain that:

$$\frac{C^2}{2\pi \|\nabla F\|_{L^\infty(\mathbb{R})}^2} \ln \frac{\text{dist}(x_0, \partial\Omega)}{2\delta_0} \leq \int_{B_{\frac{\text{dist}(x_0, \partial\Omega)}{2}}(x_0) \cap U_d} |(f_2^m)''(\partial_2 u_m) \nabla \partial_2 u_m|^2 d\mathcal{H}^1.$$

Hence, using Proposition 4.4.3 there exists $\delta_0 > 0$ such that $\partial_2 u_m(x) \geq d$ for every $x \in B_{\delta_0}(x_0)$ or $\partial_2 u_m(x) \leq t$ for every $x \in B_{\delta_0}(x_0)$. \square

We have the same result with $t < 0$:

Lemma 4.4.5. *For every $t < 0$ there exist $t < d'_t < 0$ and $\delta_0 > 0$ such that $\partial_2 u_m(x) < d'_t$ for every $x \in B_{\delta_0}(x_0)$ or $\partial_2 u_m(x) > t$ for every $x \in B_{\delta_0}(x_0)$.*

Proof. The proof is the same as the proof of Proposition 4.4.4 replacing $\partial_2 u$ by $-\partial_2 u$. \square

Hence we have proved:

Proposition 4.4.6. *For every $t > 0$ there exist $d_t > 0$ and $\delta_0 > 0$ such that for every $m \geq M_r$ we have that $-t < \partial_2 u(x) < t$ for every $x \in B_{\delta_0}(x_0)$ or $\partial_2 u_m(x) \in (-\infty, d'_{-t}) \cup (d_t, \infty)$ for every $x \in B_{\delta_0}(x_0)$.*

With this result we are ready to prove the continuity of $(f_2^m)'(\partial_2 u_m)$.

Proposition 4.4.7. *Under the assumptions of Theorem 4.1.6, $(f_2^m)'(\partial_2 u_m)$ is continuous with a modulus of continuity independent of $m \in \mathbb{N}$.*

Proof. For every $\epsilon > 0$, we want to find $\delta > 0$ such that

$$(f_2^m)'(\partial_2 u_m(B_\delta(x_0))) \subset B_\epsilon((f_2^m)'(\partial_2(u_m(x_0)))).$$

We introduce $C(\epsilon) > 0$ such that if $-C(\epsilon) \leq t \leq C(\epsilon)$ then

$$|(f_2^m)'(t) - (f_2^m)'(0)| \leq \frac{\epsilon}{2} \tag{4.4.4}$$

for every $m \geq M_r$. Then thanks to Proposition 4.4.6, we have two options.

If $\partial_2 u_m(x) \in (-\infty, d'_{-C(\epsilon)}) \cup (d_{C(\epsilon)}, \infty)$ for every $x \in B_{\delta_0}(x_0)$ we can assume that f_1 and f_2 are in $C_{loc}^{1,1}$ and apply Theorem 4.1.4.

Otherwise, $-C(\epsilon) \leq \partial_2 u_m(x) \leq C(\epsilon)$ for every $x \in B_{\delta_0}(x_0)$. In that case we conclude thanks to (4.4.4). \square

If we combine the results of this section, we have proved:

Proposition 4.4.8. *Under the assumptions of Theorem 4.1.6, the smooth functions $(G^m(\nabla u_m))_{m \in \mathbb{N}}$ are continuous with the same modulus of continuity on each compact subset of Ω .*

4.5 Uniform estimate for Theorem 4.1.9

This section is devoted to the proof of Theorem 4.1.9. We assume that there exists a compact set D_G such that $G \in C^1(\mathbb{R}^N \setminus D_G)$ and $D_G = \mathbb{R}^N \setminus \bigcup_{k \in \mathbb{N}} \mathcal{O}_k$ with $\mathcal{O}_k := \{z \in \mathbb{R}^N, \frac{1}{k}|v|^2 < \langle DG^s(z)v, v \rangle < k|v|^2 \text{ for every } v \in \mathbb{R}^N\}$ where $DG^s := \frac{DG+DG^T}{2}$. For every $r > 0$, we introduce the closed r -neighborhood of a set U : $\bar{N}_r(U) := \{y \in \mathbb{R}^N, \text{dist}(y, U) \leq r\}$. We assume that there exists $t_0 > 0$ such that for every $0 \leq t \leq t_0$ the connected components of $\bar{N}_t(D_G)$ are simply connected.

As in the previous sections, we want to obtain uniform estimates for smooth approximations of the original problem. Hence, we are working with the smooth function G^m from Proposition 4.2.6 and the smooth solution u_m of (4.2.2). Let r_0 be the smallest distance between two connected components of D_G . We introduce $\rho_0 < \min\{\frac{r_0}{8}, t_0\}$. In this section, we prove that:

Proposition 4.5.1. *For every $0 < t < \frac{\rho_0}{2}$ and for every subset $\Omega' \Subset \Omega$, the functions*

$$\text{dist}(\nabla u_m, \bar{N}_t(D_G)) \text{ and } \nabla u_m \times \text{dist}(\nabla u_m, \bar{N}_t(D_G))$$

are continuous with a uniform modulus of continuity in $m \geq \frac{2}{t}$. Moreover, for every $x_0 \in \Omega$, there exists $r > 0$ independent of $m \in \mathbb{N}$ such that $\nabla u_m(B_r(x_0))$ encounters at most one connected component D_G^0 of D_G . Furthermore if $\nabla u_m(B_r(x_0)) \cap D_G^0$ is not empty then $\nabla u_m(B_r(x_0)) \subset \bar{N}_{3\rho_0}(D_G^0)$.

We define

$$\tilde{\mathcal{O}}_k^t := \left(\bigcap_{m \geq \frac{2}{t}} \left\{ \frac{1}{k} Id < (DG^m)^s < k Id \right\} \right) \setminus \bar{N}_t(D_G).$$

Hence the sets $\tilde{\mathcal{O}}_k^t$ are independent of m when $m \geq \frac{2}{t}$.

Since every estimate of this section is independent of $m \geq \frac{2}{t}$ we can drop the subscript $m \in \mathbb{N}$ in order to simplify the notations. Moreover, since we want to prove continuity results for every $x_0 \in \Omega$ we can replace Ω by $B_{\text{dist}(x_0, \partial\Omega)}(x_0)$. By replacing $u(\cdot)$ by $u(\text{dist}(x_0, \partial\Omega) \cdot + x_0)$ we can assume that $\Omega = B_1(0)$. We introduce the constant $L > 0$ that is a Lipschitz constant of u_m on $B_1(0)$ uniform in $m \in \mathbb{N}$. In the remaining of the section $0 < t < \frac{\rho_0}{2}$ is fixed. Hence, we do not state the dependence on t in the constants of the following results.

4.5.1 Preliminary results

In this section, we introduce two results that are adaptations of [57, Proposition 3.1, Lemma 3.2] in the case where $f \neq 0$.

We introduce the following operator:

$$L_G^u : v \mapsto \text{div}(DG(\nabla u)\nabla v).$$

Remark 4.5.2. We have that $L_G^u(\partial_e u) = \partial_e f$ for every $e \in \mathbb{S}^{N-1}$.

We prove the following result:

Proposition 4.5.3. *Let $K \geq 0$ and $q > N$. We assume that v is in $W^{1,2}(B_1(0)) \cap L^\infty(B_1(0))$, $v \geq 0$ and solves $L_G^u(v) \geq g$ in the weak sense with $\{v > 0\} \subset \tilde{O}_k^t$ for some $k \in \mathbb{N}$ and $g \in L^q(B_1(0))$. Then for all $0 < \mu \leq 1$, there exists $\nu := \nu(\mu, N, k)$ such that if*

$$\frac{|\{v > 0 \cap B_{\frac{1}{2}}(0)\}|}{|B_{\frac{1}{2}}(0)|} \leq 1 - \mu$$

and

$$\|g\|_{L^q(B_1(0))} \leq \frac{\nu}{2} \max\{\sup_{B_1(0)} v, K\}$$

then

$$\sup_{B_{\frac{1}{4}}(0)} v \leq (1 - \nu) \max\{\sup_{B_1(0)} v, K\}.$$

Proof. Let us introduce $m := \sup_{B_1(0)} v$, then $m - v \geq 0$ and $L_G^u(m - v) \leq -g$. If we replace $DG(\nabla u)$ by the identity matrix on the set where $v = 0$ then by [36, Theorem 8.18], there exists $C_0 := C_0(N, k) > |B_1(0)|$ such that

$$2^N \int_{B_{\frac{1}{2}}(0)} m - v \leq C_0 \left(\inf\{m - v(x), x \in B_{\frac{1}{4}}(0)\} + k \|g\|_{L^q(B_1(0))} \right).$$

We estimate the left-hand side from below by integrating over the set $[v = 0]$. Thus,

$$2^N \mu m |B_{\frac{1}{2}}(0)| \leq C_0 \left(m - \sup_{B_{\frac{1}{4}}(0)} v + k \|g\|_{L^q(B_1(0))} \right).$$

Hence,

$$\sup_{B_{\frac{1}{4}}(0)} v \leq \left(1 - \frac{2^N \mu |B_{\frac{1}{2}}(0)|}{C_0}\right) m + k \|g\|_{L^q(B_1(0))}.$$

Since $\|g\|_{L^q(B_1(0))} \leq \frac{\nu}{2} \max\{\sup_{B_1(0)} v, K\}$, if we take $\nu = (1 + \frac{k}{2})^{-1} (\frac{2^N \mu |B_{\frac{1}{2}}(0)|}{C_0})$, we have:

$$\sup_{B_{\frac{1}{4}}(0)} v \leq (1 - \nu) \max\{\sup_{B_1(0)} v, K\}.$$

□

In the rest of the paper we are going to apply this proposition to functions of ∇u that are concave in one direction. To do so, we prove the following result:

Lemma 4.5.4. *Let $\tilde{f} \in C^1(\overline{B_1(0)})$ and v a smooth solution in $B_1(0)$ of*

$$\operatorname{div}(G(\nabla v)) = \tilde{f}. \tag{4.5.1}$$

Let η be a smooth function in a neighborhood of $\nabla v(B_1(0))$. We assume that

$$\nabla v(B_1(0)) \cap \{\eta > 0\} \subset \tilde{\mathcal{O}}_k^t.$$

Then there exists $\lambda := \lambda(k, N) > 0$ such that if in $\nabla v(B_1(0)) \cap \{\eta > 0\}$ the eigenvalues $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_N$ of $D^2\eta$ satisfy $\gamma_2 > 0$ and $0 \geq \gamma_1 \geq -\lambda\gamma_2$ then

$$L_G^v(\eta_+(\nabla v)) \geq \left(\langle \nabla \tilde{f}, \nabla \eta(\nabla v) \rangle - \lambda \gamma_2 k^3 \tilde{f}^2 \right) \mathbb{1}_{\{\eta(\nabla v) > 0\}}$$

in the weak sense.

Here, $\eta_+ := \max\{\eta, 0\}$.

Proof. Since v is a solution of (4.5.1) we have that

$$L_G^v(\eta(\nabla v)) = \operatorname{div}(DG(\nabla v)\nabla[\eta(\nabla v)]) = \sum_{i,j,s,l} D_i G_j(\nabla v) v_{js} \eta_{sl}(\nabla v) v_{li} + \langle \nabla \tilde{f}, \nabla \eta(\nabla v) \rangle.$$

As in [57, Lemma 3.2], for $x_0 \in \{\eta(\nabla v) > 0\}$ we can choose coordinates such that $\eta_{sl}(\nabla v(x_0)) = \gamma_s \delta_{sl}$. Hence,

$$L_G^v(\eta(\nabla v))(x_0) = \sum_s \gamma_s \sum_{i,j} D_i G_j(\nabla v) v_{js} v_{si} + \langle \nabla \tilde{f}, \nabla \eta(\nabla v) \rangle.$$

Since $\nabla v(B_1(0)) \cap \{\eta > 0\} \subset \tilde{\mathcal{O}}_k^t$ we obtain that

$$L_G^v(\eta(\nabla v))(x_0) \geq \gamma_1 k |\nabla v_1|^2 + \sum_{n=2}^N \gamma_n \frac{1}{k} |\nabla v_n|^2 + \langle \nabla \tilde{f}, \nabla \eta(\nabla v) \rangle.$$

This last inequality combined with the fact that $\gamma_1 \geq -\lambda\gamma_2$ on $\{\eta(\nabla v) > 0\}$ provides that

$$\gamma_2^{-1} L_G^v(\eta(\nabla v))(x_0) \geq k^{-1} \sum_{n=2}^N |\nabla v_n(x_0)|^2 - k\lambda |\nabla v_1(x_0)|^2 + \gamma_2^{-1} \langle \nabla \tilde{f}, \nabla \eta(\nabla v) \rangle.$$

If $L_G^v(\eta(\nabla v))(x_0) < \langle \nabla \tilde{f}, \nabla \eta(\nabla v) \rangle - \lambda \gamma_2 k^3 \tilde{f}^2$ then

$$k^{-1} \sum_{n=2}^N |\nabla v_n(x_0)|^2 - k\lambda |\nabla v_1(x_0)|^2 < -\lambda k^3 \tilde{f}^2.$$

Since $\sum_{n=2}^N |\nabla v_n(x_0)|^2 \geq \frac{1}{2} \sum_{(i,j) \neq (1,1)} v_{ij}^2(x_0)$ we have that

$$\left(\frac{\lambda^{-1} k^{-2}}{2} - 1\right) \sum_{(i,j) \neq (1,1)} v_{ij}^2(x_0) + k^2 \tilde{f}^2 < v_{11}^2(x_0). \quad (4.5.2)$$

We introduce $D^s G = \frac{DG + DG^T}{2}$. Since $Tr(DG^s(\nabla v)D^2v) = Tr(DG(\nabla v)D^2v) = \tilde{f}$, we obtain that

$$D_1 G_1^s(\nabla v)v_{11} = - \sum_{(i,j) \neq (1,1)} D_i G_j^s v_{ij} + \tilde{f}.$$

Thus, since $x_0 \in \{\eta(\nabla v) > 0\}$ and $\{\eta > 0\} \subset \tilde{\mathcal{O}}_k^t$ we have

$$v_{11}^2(x_0) \leq C(N, k) \sum_{(i,j) \neq (1,1)} v_{ij}^2(x_0) + k^2 \tilde{f}^2.$$

We get a contradiction with (4.5.2) when $\lambda \leq \frac{k^{-2}}{2(C(N,k)+1)}$. \square

4.5.2 When ∇u is close to the convex hull of D_G

Let C_G be the convex hull of D_G .

In this section we study the behavior of $\nabla u(B_\delta(0))$ when $\nabla u(B_\delta(0))$ is close to C_G . By [26, Proposition 4.3], we have the following result:

Proposition 4.5.5. *We assume that $\epsilon > 0$ and that*

$$\operatorname{div}(G(\nabla v)) = \tilde{f}$$

in $B_1(0)$ with $\tilde{f} \in L^q(B_1(0))$. We consider p such that $B_\epsilon(p) \cap \overline{N}_t(D_G) = \emptyset$. Let $k \in \mathbb{N}$ such that $B_\epsilon(p) \subset \tilde{\mathcal{O}}_k^t$. Then there exist $\delta_0 > 0$ and $\mu_0 > 0$ depending on the modulus of continuity of DG in $B_{\epsilon+\frac{t}{2}}(p)$, k and ϵ such that if $\|v - l_p\|_{L^\infty(B_1(0))} \leq \delta_0$ for some affine function l_p with $\nabla l_p = p$ and $\|\tilde{f}\|_{L^q(B_1(0))} \leq \delta_0 \mu_0$, then

$$\nabla v(B_{\frac{1}{2}}(0)) \subset B_\epsilon(p).$$

We can use this result to show that:

Lemma 4.5.6. *For every $\epsilon > 0$ there exist $\alpha := \alpha(\rho_0, DG, \|f\|_{L^q(\Omega)}, L, N, \epsilon)$, $\kappa := \kappa(DG, L, N, \epsilon) \leq \epsilon$ and $\mu_1 := \mu_1(DG, L, N, \epsilon)$ with L the Lipschitz constant of u on $B_1(0)$ such that if*

$$\frac{|\{\nabla u \in B_\kappa(p)\} \cap B_r(0)|}{|B_r(0)|} \geq 1 - \mu_1$$

for some $p \notin \overline{N_{t+\epsilon}(DG)}$ and $r \leq \alpha$ then

$$\nabla u(B_{\frac{r}{2}}(0)) \subset B_\epsilon(p).$$

Remark 4.5.7. When a dependence in DG appears in a constant it means that the constant depends only on the sets of ellipticity $\tilde{\mathcal{O}}_k^t$ and the modulus of continuity of DG^m outside $\overline{N}_t(DG)$ with $m \geq \frac{2}{t}$ the parameter of regularization. Since all those quantities are independent of $m \geq \frac{2}{t}$ we can just denote this dependence by DG .

Proof. For $r > 0$, we introduce $v(x) := \frac{1}{r}u(rx)$ in $B_1(0)$. Then

$$\operatorname{div}(G(\nabla v(x))) = rf(rx)$$

in $B_1(0)$. Since $q > N$ by taking r small enough we can assume that $\|rf(r\cdot)\|_{L^q(B_1(0))}$ is as small as we want. Hence, there exists $\alpha := \alpha(\rho_0, DG, \|f\|_{L^q(\Omega)}, L, N) > 0$ such that $\|rf(r\cdot)\|_{L^q(B_1(0))} \leq \delta_0\mu_0$ for every $r \leq \alpha$.

We show as in [26, Lemma 4.1] that there exists an affine function l_p such that $\|v - l_p\|_{L^\infty(B_1(0))} \leq \delta_0$. By Morrey inequalities, there exists a constant C_0 depending only N such that for every $x \in B_r(0)$ and every $w \in W^{1,2N}(B_1(0))$ we have

$$\left| w(x) - \frac{1}{|B_1(0)|} \int_{B_1(0)} w(y) dy \right| \leq C_0 \left(\frac{1}{|B_1(0)|} \int_{B_1(0)} |\nabla w(y)|^{2N} dy \right)^{\frac{1}{2N}}.$$

We set $l_p(x) := \langle p, x \rangle + \frac{1}{|B_1(0)|} \int_{B_1(0)} v(y) dy$. In that case, we obtain

$$|v(x) - l_p(x)| \leq C_0 \left(\frac{1}{|B_1(0)|} \int_{B_1(0)} |\nabla v(y) - p|^{2N} dy \right)^{\frac{1}{2N}}.$$

We estimate the right hand side by splitting the integral in two sets. The first one is $X := \{x \in B_1(0), \nabla v(x) \in B_\kappa(p)\}$. A direct computation gives that

$$\int_{B_1(0) \cap X} |\nabla v(y) - p|^{2N} dy \leq \kappa^{2N} |B_1(0)|.$$

Since the complement of X has a measure less than $\mu|B_1(0)|$, we have that

$$\int_{B_1(0) \setminus X} |\nabla v(y) - p|^{2N} dy \leq \mu |B_1(0)| (2L)^{2N}$$

with L the Lipschitz constant of u on $B_1(0)$. Thus,

$$|v(x) - l_p(x)| \leq C_0 (\kappa^{2N} + \mu(2L)^{2N})^{\frac{1}{2N}}.$$

Hence, it remains to take κ and μ small enough such that $C_0(\kappa^{2N} + \mu(2L)^{2N})^{\frac{1}{2N}} \leq \delta_0$ in order to apply Proposition 4.5.5 to v . The conclusion follows for u . \square

In the rest of the paper, we write the vectors $z \in \mathbb{R}^N$ in the following way: $z = (p, p')$ with $p \in \mathbb{R}^2$ and $p' \in \mathbb{R}^{N-2}$ and we assume that $D_G \subset \{p' = 0\}$. Now, we introduce the main result of this subsection that has the same conclusion as [57, Propositon 3.7].

Proposition 4.5.8. *For every $\epsilon > 0$, there exist $\beta := \beta(\|f\|_{W^{1,q}(\Omega)}, \kappa, \mu_1, N, DG, \epsilon)$, $s_0 := s_0(\kappa, \mu_1, L, DG, \epsilon)$ and $\sigma_0 := \sigma_0(\kappa, \mu_1, N, DG, \epsilon)$ with κ and μ_1 from Lemma 4.5.6 such that if $\nabla u(B_r(0)) \subset \{|p'| < \sigma_0\}$ with $r \leq \beta$ then either $\nabla u(B_{s_0 r}(0)) \subset B_\epsilon(p)$ for some $p \notin \overline{N}_{\epsilon+t}(D_G)$ or $\nabla u(B_{s_0 r}(0)) \subset \overline{N}_{\epsilon+t}(D_G)$.*

As in [57], we need the following preliminary lemma:

Lemma 4.5.9. *Let $(p_0, 0) \notin \overline{N}_{\epsilon+t}(D_G)$. There exist $\sigma_0 := \sigma_0(\kappa, \mu_1, N, DG, \epsilon)$, $\beta := \beta(\|f\|_{W^{1,q}(\Omega)}, \kappa, \mu_1, N, DG, \epsilon)$ and $C_0 := C_0(\kappa, \mu_1, N, DG, \epsilon)$ such that if*

$$\nabla u(B_r(0)) \subset \{|p'| < \sigma_0\} \cap \{|p - p_0| \geq \frac{\kappa}{4}\}$$

and

$$\frac{|\{\nabla u \in B_\kappa(p_0, 0)\} \cap B_{\frac{r}{2}}(0)|}{|B_{\frac{r}{2}}(0)|} < 1 - \mu_1,$$

for $r \leq \beta$ then

$$\nabla u(B_{\frac{r}{4}}(0)) \subset \{|p - p_0| \geq \frac{\kappa}{4} + C_0\}.$$

Proof. There exists $k \in \mathbb{N}$ such that $B_\epsilon(p_0, 0) \subset \tilde{\mathcal{O}}_k^t$. We follow the proof of [57, Lemma 3.8]. We set $v(x) = \frac{1}{r}u(rx)$ on $B_1(0)$ and we replace $f(x)$ by $rf(rx)$. We define

$$\eta_A(p, p') := \exp\left(\frac{A^2}{2}|p'|^2 - A|p|\right), \quad \eta_{A,p_0} := \eta_A(p - p_0, p') - \exp\left(-A\frac{\kappa}{2}\right).$$

In the basis $(\frac{p^\perp}{|p|}, \frac{p}{|p|}, e_3, \dots, e_N)$, on the set $\{|p'| < A^{-3}\}$, we have

$$(A^2\eta_A)^{-1}D^2\eta_A = \text{diag}(-(A|p|)^{-1}, 1, \dots, 1) + O(A^{-2})$$

and $\{\eta_A > \exp(-A\frac{\kappa}{2})\} \subset \{|p| < \frac{\kappa}{2} + A^{-5}\}$. Since $\nabla u(B_r(0)) \subset \{|p'| < \sigma_0\}$, for A large enough depending only on κ , if $\sigma_0 < A^{-3}$ then $\nabla v(B_1(0)) \cap \{\eta_{A,p_0} > 0\} \subset B_L \cap B_\kappa(p_0, 0)$. Since $(p_0, 0) \notin \overline{N}_{\epsilon+t}(D_G)$ we also have $B_L(0) \cap B_\kappa(p_0, 0) \subset B_L \setminus \overline{N}_t(D_G)$. Hence, we obtain that $\nabla v(B_1(0)) \cap \{\eta_{A,p_0} > 0\} \subset B_L \cap B_\kappa(p_0, 0) \subset B_L \setminus \overline{N}_t(D_G)$.

Once again if A is large enough depending only on $\lambda(k, N)$ from Lemma 4.5.4 and κ then the eigenvalues $\gamma_1 \leq \dots \leq \gamma_n$ of $D^2\eta_{A,p_0}$ satisfy $\gamma_2 > 0$ and $\gamma_1 > -\lambda\gamma_2$ in $\nabla v(B_1(0)) \cap \{\eta_{A,p_0} > 0\}$. Thanks to Lemma 4.5.4, the function $v_{p_0} := (\eta_{A,p_0})_+(\nabla v)$ satisfies

$$L_G(v_{p_0}) \geq \langle r^2 \nabla f(r \cdot), \nabla \eta_{A,p_0}(\nabla u(r \cdot)) \rangle - r^2 \lambda \gamma_2 k^3 f(r \cdot)^2 \mathbf{1}_{\{\eta_{A,p_0}(\nabla v) > 0\}}. \quad (4.5.3)$$

Let denote by ν_1 the constant $\nu(\mu_1, N, k)$ from Proposition 4.5.3. We select $\sigma_0(\kappa, \nu_1, A)$ is small in such a way that

$$\exp\left(-A\frac{\kappa}{4}\right) - \exp\left(-A\frac{\kappa}{2}\right) > (1 - \nu_1)\left(\exp\left(\frac{A^2}{2}\sigma_0^2 - A\frac{\kappa}{4}\right) - \exp\left(-A\frac{\kappa}{2}\right)\right).$$

Then if we look at the right-hand side of (4.5.3) since $q > N$ there exists a constant $\beta(\|f\|_{W^{1,q}(\Omega)}, \kappa, \mu_1, L, N, DG, \epsilon)$ such that if $r \leq \beta$, we can apply Proposition 4.5.3 to v_{p_0} with $K = \exp(\frac{A^2}{2}\sigma_0^2 - A\frac{\kappa}{4}) - \exp(-A\frac{\kappa}{2})$. Hence, we have that

$$\nabla v(B_{\frac{1}{4}}(0)) \subset \{\eta_{A,p_0} < (1 - \nu_1)(\exp(\frac{A^2}{2}\sigma_0^2 - A\frac{\kappa}{4}) - \exp(-A\frac{\kappa}{2}))\}.$$

Thus,

$$\nabla v(B_{\frac{1}{4}}(0)) \subset \{\exp(-A|p - p_0|) - \exp(-A\frac{\kappa}{2}) < (1 - \nu_1)\exp(-A\frac{\kappa}{4}) - \exp(-A\frac{\kappa}{2})\}.$$

Hence, $\nabla v(B_{\frac{1}{4}}(0)) \subset \{|p - p_0| \geq \frac{\kappa}{4} + C_0\}$ for some $C_0 := C_0(\kappa, \mu_1, N, DG, \epsilon) > 0$. The conclusion follows. \square

We prove Proposition 4.5.8:

Proof of Proposition 4.5.8. We follow the proof of [57, Proposition 3.7]. We introduce α, κ and μ_1 from Lemma 4.5.6 and β from Lemma 4.5.9. Taking $r \leq \min\{\alpha, \beta\}$, we use the following strategy: if there exist $(p_0, 0) \in B_{2L}(0) \setminus \overline{N}_{\epsilon+t}(D_G)$ and $n \in \mathbb{N}$ such that

$$\frac{|\{\nabla u \in B_\kappa(p_0, 0)\} \cap B_{2^{-2n}r}|}{|B_{2^{-2n}r}|} \geq 1 - \mu_1 \quad (4.5.4)$$

then we can apply Lemma 4.5.6 in order to obtain that $\nabla u(B_{2^{-2n-1}r}) \subset B_\epsilon(p_0, 0)$ and the conclusion follows.

If that is not the case then by Lemma 4.5.9 we obtain that for every $(p_0, 0) \in B_{2L}(0) \setminus \overline{N}_{\epsilon+t}(D_G)$ if

$$\nabla u(B_{2^{-2n+1}r}(0)) \subset \{|p'| < \sigma_0\} \cap \{|p - p_0| \geq \frac{\kappa}{4}\}$$

then

$$\nabla u(B_{2^{-2n-1}r}(0)) \subset \{|p - p_0| \geq \frac{\kappa}{4} + C_0\}. \quad (4.5.5)$$

The idea in [57, Proposition 3.7] is to use a covering argument with neighborhoods of lines. Unfortunately, we can not cover the set $B_{2L}(0) \setminus \overline{N}_{\epsilon+t}(D_G)$ with $\frac{\kappa}{4}$ neighborhoods of lines since $\overline{N}_{\epsilon+t}(D_G)$ is not a finite union of small balls.

Instead, since the connected components of $\overline{N}_{\epsilon+t}(D_G)$ are simply connected we can consider a finite family of points $(x_i)_{i \in I}$ in $(B_{2L}(0) \setminus \overline{N}_{\epsilon+t}(D_G)) \cap \{p' = 0\}$ with $I := [1, I_G] \subset \mathbb{N}$ such that $B_\kappa(x_1) \cap B_L(0) = \emptyset$, $|x_i - x_{i+1}| \leq C_0$ for every $i \in I \setminus \{I_G\}$ with C_0 from Lemma 4.5.9. Moreover, we assume that $(B_{2L}(0) \setminus \overline{N}_{\epsilon+t}(D_G)) \cap \{p' = 0\} \subset \bigcup_{i \in I} B_{\frac{\kappa}{4}}(x_i)$. By definition of L , $B_\kappa(x_1) \cap \nabla u(B_r(0)) = \emptyset$.

Thus, we can initiate the algorithm. By Lemma 4.5.9 we obtain that

$$\nabla u(B_{2^{-2}r}(0)) \subset \{|p - x_1| \geq \frac{\kappa}{4} + C_0\}.$$

Hence, $\nabla u(B_{2^{-2}r}(0)) \subset \{|q| < \sigma_0\} \cap \{|p - x_2| \geq \frac{\kappa}{4}\}$. If (4.5.4) is satisfied with $(p_0, 0) = x_2$ and $n = 3$ then we can conclude, otherwise $\nabla u(B_{2^{-4}r}(0)) \subset \{|p'| <$

$\sigma_0\} \cap \{|p - x_3| \geq \frac{\kappa}{4}\}$. The algorithm terminates after at most I_G steps with two potential conclusions. The first one is that $\nabla u(B_{2^{-2n_0-1}r}) \subset B_\epsilon(x_{n_0})$ for a certain $n_0 \in [1, I_G]$. The second one is that $\nabla u(B_{2^{-2I_G-1}r}) \subset B_{2L}(0) \setminus \bigcup_{i \in I} B_{\frac{\kappa}{4}}(x_i)$ which is a subset of $\bar{N}_{\epsilon+t}(D_G)$. \square

4.5.3 Reduction to the convex hull of D_G

In this subsection we present a result that states that either $\nabla u_m(B_r(0))$ is outside C_G the convex hull of D_G or is close to it. By [26, Theorem 1.1] we have the following proposition:

Proposition 4.5.10. *For every $s > 0$, there exists $\delta_s := \delta_s(L, DG, s) > 0$ such that either $\nabla u_m(B_{\delta_s}) \subset B_s(\nabla u_m(0))$ or $\nabla u_m(B_{\delta_s}) \subset \bar{N}_{4s}(C_G)$ for every $m \in \mathbb{N}$.*

Proof. By [26, Theorem 1.1], if H is a continuous function on \mathbb{R}^N such that $H \equiv 0$ on $\bar{N}_s(C_G)$ then $H(\nabla u_m)$ has a modulus of continuity depending on L and DG that does not depend on the parameter of regularity $m \in \mathbb{N}$. We can take $H(\cdot) := \text{dist}(\cdot, \bar{N}_s(C_G))$. In that case if $\nabla u_m(0) \in \bar{N}_{3s}(C_G)$ then there exists $\delta_s := \delta_s(M, G, r) > 0$ such that $\nabla u_m(B_{\delta_s}) \subset \bar{N}_{4s}(C_G)$. If we assume that $\nabla u_m(0) \notin \bar{N}_{3s}(C_G)$ then there exists $\delta'_s := \delta'_s(M, G, s) > 0$ such that $\nabla u_m(B_{\delta'_s}) \cap \bar{N}_{2s}(C_G) = \emptyset$. By classical results on uniform elliptic equations there exists $\delta_s := \delta_s(L, G, s) > 0$ such that $\nabla u_m(B_{\delta_s}) \subset B_s(\nabla u(0))$. \square

4.5.4 Proof of Proposition 4.5.1

To finish this section we reintroduce the subscript $m \in \mathbb{N}$. We can prove that for every $0 < t < \frac{\rho_0}{2}$, $\text{dist}(\nabla u_m, \bar{N}_t(D_G))$ and $\nabla u_m \times \text{dist}(\nabla u_m, \bar{N}_t(D_G))$ are uniformly continuous in $m \geq \frac{2}{t}$ with a similar strategy as in [57].

Proof of Proposition 4.5.1. We take $x_0 \in \Omega$ and $0 < \epsilon < r_0$, we consider σ_0 from Lemma 4.5.9. We apply Proposition 4.5.10 with $s = \frac{\sigma_0}{4}$. Hence, there exists $\delta_s > 0$ such that either $\nabla u_m(B_{\delta_s}(x_0)) \subset B_{\frac{\sigma_0}{4}}(\nabla u_m(x_0))$ or $\nabla u_m(B_{\delta_s}(x_0)) \subset \mathcal{N}_{\sigma_0}(C_G)$.

In the first case for every $0 < \epsilon_1 < \frac{\sigma_0}{4}$, we can find $\delta := \delta(L, DG, \epsilon_1)$ such that $\nabla u_m(B_\delta(x_0)) \subset B_{\epsilon_1}(\nabla u_m(x_0))$ thanks to Proposition 4.5.10.

In the second case, we apply Proposition 4.5.8 with $r = \min\{\beta, \delta_s\}$. Hence, in this case, we either have $\nabla u_m(B_{s_0r}(x_0)) \subset B_\epsilon(p)$ for some $p \notin \bar{N}_{\epsilon+t}(D_G)$ or $\nabla u_m(B_{s_0r}(x_0)) \subset \bar{N}_{\epsilon+t}(D_G)$. Since $\nabla u_m(B_{s_0r})$ is connected, by definition of ρ_0 the set $\nabla u_m(B_{s_0r})$ encounters at most one connected component D_G .

Hence, for every $\epsilon > 0$ and every $t > 0$ there exists

$$\delta_\epsilon(L, DG, N, \epsilon, \|f\|_{W^{1,q}(\Omega)}) > 0$$

such that either $\nabla u_m(B_{\delta_\epsilon}) \subset B_\epsilon(\nabla u_m(x_0))$ or $\nabla u_m(B_{\delta_\epsilon}) \subset \bar{N}_{\epsilon+t}(D_G)$ for every $m \geq \frac{2}{t}$.

Thus, we have that $\text{dist}(\nabla u_m, \bar{N}_t(D_G))$ and $\nabla u_m \times \text{dist}(\nabla u_m, \bar{N}_t(D_G))$ are continuous uniformly in $m \in \mathbb{N}$ with $m \geq \frac{2}{t}$. \square

4.6 Main proofs

We assume that G satisfies the assumptions of Theorem 4.1.1, Theorem 4.1.4 or Theorem 4.1.6. Before proving these three theorems, we show the following result:

Proposition 4.6.1. *If G satisfies the assumptions of Theorem 4.1.1, Theorem 4.1.4 or Theorem 4.1.6 then $G(\nabla u)$ does not depend on the choice of the solution u of (4.1.4) when the Dirichlet boundary condition is fixed.*

Proof. Let us assume that u and v are two solutions of the same equation (4.1.4) such that $u = v$ on $\partial\Omega$. Then we have:

$$\int_{\Omega} \langle G(\nabla u) - G(\nabla v), \nabla u - \nabla v \rangle = 0.$$

Since $\langle G(A) - G(B), A - B \rangle \geq 0$, we get that $\langle G(\nabla u(x)) - G(\nabla v(x)), \nabla u(x) - \nabla v(x) \rangle = 0$ for a.e. $x \in \Omega$. The condition (A_1) gives that $G_1(\nabla u(x)) = G_1(\nabla v(x))$ for a.e. $x \in \Omega$. In the other cases, we use the convexity of φ to get the same result. \square

With this proposition, we just have to show that $G(\nabla u)$ is continuous for u the solution of (4.1.4) obtained as the limit of $(u_m)_{m \in \mathbb{N}}$ when $m \rightarrow +\infty$:

Proof of Theorem 4.1.1, Theorem 4.1.4 and Theorem 4.1.6. Thanks to Proposition 4.3.18 and Proposition 4.4.8, we have that for every compact $\Omega' \Subset \Omega$, the family $(G^m(\nabla u_m))_{m \in \mathbb{N}}$ is equicontinuous. Hence, by the Arzelà-Ascoli Theorem, $G^m(\nabla u_m)$ converges to v uniformly on Ω' , up to a subsequence. Since $\|\nabla u_m\|_{L^\infty(\Omega')}$ is bounded uniformly in $m \in \mathbb{N}$, we have that $|G^m(\nabla u_m) - G(\nabla u_m)| \rightarrow 0$ in $L^1(\Omega')$. Thanks to Remark 4.2.11, we get that $G^m(\nabla u_m)$ converges to $G(\nabla u)$ in $L^1(\Omega')$. Thus, $v = G(\nabla u)$ is continuous for any solution of (4.1.4) thanks to Proposition 4.6.1. \square

We are ready to prove Proposition 4.1.7.

Proof of Proposition 4.1.7. On the set $\sigma^{-1}(V)$ we define $F(x)$ as $G^{-1}(\sigma(x))$. Thus, the function F is continuous and $F = \nabla u_0$ a.e. on $\sigma^{-1}(V)$. Hence, ∇u_0 has a continuous representative on $\sigma^{-1}(V)$. \square

Let us prove Proposition 4.1.8.

Proof of Proposition 4.1.8. We know that $G(\nabla u_0)$ has a continuous representative σ where $G = \nabla\varphi$ with φ a convex function that depends only on the Euclidean norm $|\cdot|$.

For every $t \in \mathbb{R}$ we introduce the super level set $E_t = [u_0 > t]$. By the co-area formula we obtain that for a.e. $t \in \mathbb{R}$, $D\mathbf{1}_{E_t} = \frac{\nabla u_0}{|\nabla u_0|} \mathcal{H}^{N-1} \llcorner \partial^e E_t$ where $\partial^e E_t$ is the support of the measure $D\mathbf{1}_{E_t}$. Since φ is radial $\frac{\sigma}{|\sigma|} = \frac{\nabla u_0}{|\nabla u_0|}$ a.e. on $\Omega \cap [\nabla u_0 \neq 0]$. Thus, by the co-area formula we get that $D\mathbf{1}_{E_t} = \frac{\sigma}{|\sigma|} \mathcal{H}^{N-1} \llcorner \partial^e E_t$ for a.e. $t \in \mathbb{R}$.

Hence, we get

$$\lim_{r \rightarrow 0} \frac{\int_{B_r(x)} D\mathbb{1}_{E_t}}{\int_{B_r(x)} |D\mathbb{1}_{E_t}|} = \frac{\sigma(x)}{|\sigma(x)|}$$

for every $x \in \partial^e E_t \cap [\sigma \neq 0]$. By [38, Theorem 4.11] we obtain that the connected components of $\partial^e E_t \cap [\sigma \neq 0]$ are \mathcal{C}^1 curves.

Let $x_0 \in [\sigma \neq 0]$. By continuity of σ this set is open. Hence, there exist $r > 0$ and $C > 0$ such that for every $x \in B_r(x_0)$ we have that $\langle \sigma(x), e \rangle \geq C$ with $e := \frac{\sigma(x_0)}{|\sigma(x_0)|}$. Since, $\sigma(x) = \nabla\varphi(\nabla u_0(x))$ for a.e. $x \in \Omega$ the continuity of $\nabla\varphi$ gives that $|\nabla u_0(x)| \geq \tilde{C} > 0$ for a.e. $x \in B_r(x_0)$. Since $\frac{\sigma}{|\sigma|} = \frac{\nabla u_0}{|\nabla u_0|}$ a.e. on $\Omega \cap [\nabla u_0 \neq 0]$, there exists $C' > 0$ such that for a.e. $x \in B_r(x_0)$, $\langle \nabla u_0(x), e \rangle \geq C' > 0$. By Lipschitz continuity of u_0 for a.e. $x \in B_{\frac{r}{2}}(x_0)$ and every $0 < t < \frac{r}{2}$ we have that

$$u_0(x + te) - u_0(x) = \int_0^t \langle \nabla u_0(x + se), e \rangle ds \geq C't > 0.$$

By continuity of u_0 , for every $x \in B_{\frac{r}{2}}(x_0)$ and every $0 < t < \frac{r}{2}$ we obtain that $u_0(x + te) - u_0(x) \geq C't > 0$. Hence, by continuity of u_0 , for every $\rho > 0$ we have that $0 < |B_\rho(x_0) \cap [u_0 > u_0(x_0)]| < B_\rho(x_0)$. This means that $x_0 \in \partial^e [u > u(x_0)]$.

Finally, let $t \in \mathbb{R}$ such that $\partial^e E_t \cap [\sigma \neq 0]$ is a \mathcal{C}^1 curve. By continuity of u_0 , for every $x \in \partial^e E_t \cap [\sigma \neq 0]$ we have that $u_0(x) = t$. Moreover, for every $x \in [\sigma \neq 0]$ such that $u_0(x) = t$ we have that $x \in \partial^e E_t$. Hence, for a.e. $t \in \mathbb{R}$, $[u_0 = t] \cap [\sigma \neq 0] = \partial^e E_t \cap [\sigma \neq 0]$. Hence, the connected components of $[u_0 = t] \cap [\sigma \neq 0]$ are \mathcal{C}^1 curves for a.e. $t \in \mathbb{R}$. □

In order to prove Theorem 4.1.9 we state the following proposition that we use instead of Proposition 4.6.1 in this case:

Proposition 4.6.2. *Let us assume that G satisfies the assumptions of Theorem 4.1.9. We consider two solutions u and v of (4.1.4) such that $u = v$ on $\partial\Omega$. Then $\nabla u(x) = \nabla v(x)$ for a.e. $x \in \Omega$ such that $\nabla u(x) \notin D_G$.*

Proof. As in the proof of Proposition 4.6.1, for a.e. $x \in \Omega$ we have that

$$\langle G(\nabla u(x)) - G(\nabla v(x)), \nabla u(x) - \nabla v(x) \rangle = 0.$$

If for some $x \in \Omega$, we have that $\nabla u(x) \notin D_G$ then there exists $k \in \mathbb{N}$ such that $\nabla u(x) \in \mathcal{O}_k := \{\frac{1}{k}Id < DG < kId\}$. Hence, for every $A \in \mathbb{R}^N \setminus \{\nabla u(x)\}$, we have that $\langle G(A) - G(\nabla v(x)), A - \nabla v(x) \rangle > 0$. Thus, $\nabla v(x) = \nabla u(x)$ for a.e. $x \in \Omega$ such that $\nabla u(x) \notin D_G$. □

Remark 4.6.3. Thanks to this proposition, $\text{dist}(\nabla u, D_G)$ and $\nabla u \times \text{dist}(\nabla u, D_G)$ do not depend on the choice of a solution of (4.1.4).

Finally, we prove Theorem 4.1.9:

Proof of Theorem 4.1.9. By Remark 4.2.11 and Remark 4.2.9, for every $\Omega' \Subset \Omega$ and every $t > 0$ we have:

$$(\text{dist}(\nabla u_m, \bar{N}_t(D_G)) - \text{dist}(\nabla u, \bar{N}_t(D_G))) \rightarrow 0 \text{ in } L^1(\Omega')$$

and

$$(\nabla u_m \times \text{dist}(\nabla u_m, \bar{N}_t(D_G)) - \nabla u \times \text{dist}(\nabla u, \bar{N}_t(D_G))) \rightarrow 0 \text{ in } L^1(\Omega')$$

when $m \rightarrow +\infty$. Thus, thanks to Proposition 4.5.1, the functions $\text{dist}(\nabla u, \bar{N}_t(D_G))$ and $\nabla u \times \text{dist}(\nabla u, \bar{N}_t(D_G))$ have a continuous representative σ_t and Σ_t respectively for every $t > 0$.

We introduce the following open subset of Ω , $\Omega_0 := \bigcup_{t>0} [\sigma_t > 0]$. Let us consider $x_0 \in \Omega_0$. Then there exist $t > 0$ and $\epsilon > 0$ such that $\sigma_t(x_0) > \epsilon$. By continuity of σ_t , there exists a neighborhood U of x_0 such that $\sigma_t \geq \frac{\epsilon}{2}$ on U . The continuity of Σ_t on U and the fact that $\sigma_t \geq \frac{\epsilon}{2}$ on U give that ∇u has a continuous representative on U . Hence, $\text{dist}(\nabla u, D_G)$ has a continuous representative on U . Thus, ∇u and $\text{dist}(\nabla u, D_G)$ have continuous representatives F_0 and σ_0 respectively on Ω_0 . Let us extend σ_0 by 0 on $\Omega \setminus \Omega_0$. We claim that this function σ is continuous and coincides a.e. with $\text{dist}(\nabla u, D_G)$. To prove the continuity we assume that there exists $x \in \Omega \setminus \Omega_0$ and $(x_n)_{n \in \mathbb{N}}$ in Ω_0 a sequence converging to x such that $(\sigma(x_n))_{n \in \mathbb{N}}$ does not converge to 0. This means that we can extract a subsequence from $(\sigma(x_n))_{n \in \mathbb{N}}$, still denoted $(\sigma(x_n))_{n \in \mathbb{N}}$ such that $\sigma(x_n) \geq l > 0$ when n is large enough. Thus, $\sigma_{\frac{l}{2}}(x_n) \geq \frac{l}{2}$ for every n large enough. By continuity of $\sigma_{\frac{l}{4}}$ we obtain that $x \in \Omega_0$ which is a contradiction. Hence, σ is continuous. Moreover, for a.e. $x \in \Omega_0$, $\sigma(x) := \text{dist}(\nabla u(x), D_G)$ and for a.e. $x \in \Omega$ such that $\sigma(x) = 0$ we have that $x \in D_G$. Thus, σ is a representative of $\text{dist}(\nabla u, D_G)$ and $F_0 \times \sigma$ is a continuous representative of $\nabla u \times \text{dist}(\nabla u, D_G)$.

To conclude, we assume that G is constant on each connected components of D_G . By Remark 4.2.9, there exists a subsequence of $(\nabla u_m)_{m \in \mathbb{N}}$ still denoted $(\nabla u_m)_{m \in \mathbb{N}}$ such that $\nabla u_m \rightarrow \nabla u$ a.e. on Ω_0 when $m \rightarrow +\infty$.

Let us assume that $(x_n)_{n \in \mathbb{N}}$ is a sequence in Ω_0 converging to $x_0 \in \Omega \setminus \Omega_0$ when $n \rightarrow +\infty$. By Proposition 4.5.1, there exists $r > 0$ such that for every $m \in \mathbb{N}$, $\nabla u_m(B_r(x_0)) \subset \bar{N}_{3\rho_0}(D_G^m)$ with D_G^m a connected component of D_G . By convergence a.e. on Ω_0 of ∇u_m we have that D_G^m is independent of m for m large enough. Let us call D_G^0 this connected component, by continuity of $\text{dist}(\nabla u, D_G)$ we obtain that $\text{dist}(\nabla u(x_n), D_G^0) \rightarrow 0$ when $n \rightarrow +\infty$. Since G is continuous on \mathbb{R}^N and constant on D_G^0 we have that $G(\nabla u(x_n)) \rightarrow G(\nabla u(x_0))$ when $n \rightarrow +\infty$. \square

4.7 Appendix: Regularity of the gauge function

Let C be a bounded convex set of \mathbb{R}^N such that its interior contains 0. We define the gauge associated to C as the following function:

$$\gamma_C(z) := \inf\{t > 0 \text{ such that } \frac{z}{t} \in C\}.$$

We have the following result about the regularity of γ_C :

Proposition 4.7.1. *Let $k \in \mathbb{N}$ and $0 \leq \alpha \leq 1$. If C is a strictly convex bounded set of \mathbb{R}^N of regularity $C^{k,\alpha}$ such that its interior contains the origin then the gauge γ_C associated to C is in $\mathcal{C}_{loc}^{k,\alpha}(\mathbb{R}^N \setminus \{0\})$.*

Proof. The function γ_C is convex on \mathbb{R}^N . For every $z \in \mathbb{R}^N$ we compute the convex subdifferential $\partial\gamma_C(z)$ of γ_C at the point z . By definition of the subdifferential we have

$$\partial\gamma_C(z) := \{y \in \mathbb{R}^N \text{ such that } \gamma_C(z') \geq \gamma_C(z) + \langle y, z' - z \rangle \text{ for every } z' \in \mathbb{R}^N\}.$$

By homogeneity of γ_C , for every $y \in \partial\gamma_C(z)$ we get that $\gamma_C(z') \geq \langle y, z' \rangle$ for every $z' \in \mathbb{R}^N$. By taking $z' = 0$ we get that $\gamma_C(z) \leq \langle y, z \rangle$. Hence, we have that $\gamma_C(z) = \langle y, z \rangle$ for every $y \in \partial\gamma_C(z)$. Thus, for every $z \in \mathbb{R}^N$, we obtain that

$$\partial\gamma_C(z) = \{y \in \mathbb{R}^N \text{ such that } \gamma_C(z) = \langle y, z \rangle \text{ and } \gamma_C(z') \geq \langle y, z' \rangle \text{ for every } z' \in \mathbb{R}^N\}. \quad (4.7.1)$$

This convex set is not empty since γ_C is a convex continuous function. We claim that when $z \neq 0$, $\partial\gamma_C(z)$ is reduced to a singleton. In fact, if there exist y_1 and y_2 two different points of $\partial\gamma_C(z)$ then $\langle y_1, z \rangle = \langle y_2, z \rangle = \gamma_C(z)$ and $\langle y_1, z' \rangle \leq 1$, $\langle y_2, z' \rangle \leq 1$ for every $z' \in \partial C$. Hence, C is on one side of the hyperplane $\langle \xi, y_1 \rangle = 1$, on one side of another hyperplane $\langle \xi, y_2 \rangle = 1$ and $\frac{z}{\gamma_C(z)} \in \partial C$ is in their intersection. This contradicts the fact that C is at least \mathcal{C}^1 . Thus for every $z \neq 0$, $\partial\gamma_C(z)$ contains only one vector. Hence, γ_C is differentiable at every $z \neq 0$. By homogeneity of γ_C we have that $\nabla\gamma_C(z)$ is positively colinear to $\nu_C(P_C(z))$ where ν_C is the unit outward normal vector to C and $P_C(z) := \frac{z}{\gamma_C(z)}$. By (4.7.1) we have that $\langle z, \nabla\gamma_C(z) \rangle = \gamma_C(z)$. Hence, $|\nabla\gamma_C(z)| \langle z, \nu_C(P_C(z)) \rangle = \gamma_C(z)$ for every $z \neq 0$. Again by homogeneity of γ_C , for every $z \neq 0$ we obtain that

$$\nabla\gamma_C(z) = \frac{\nu_C(P_C(z))}{\langle \nu_C(P_C(z)), P_C(z) \rangle}. \quad (4.7.2)$$

This scalar product in the denominator is not 0 because C contains a small ball centered at 0, thus for every $z' \in \partial C$ the normal vector $\nu_C(z')$ cannot be orthogonal to z' .

With this expression of the gradient of γ_C we can find the regularity of γ_C . In fact, we know that ν_C is $\mathcal{C}^{k-1,\alpha}$ continuous with $k \geq 1$. Since γ_C is Lipschitz continuous the map P_C is locally Lipschitz continuous on $\mathbb{R}^N \setminus \{0\}$. Hence, $\nabla\gamma_C$ is $\mathcal{C}_{loc}^{0,\alpha}$ continuous on $\mathbb{R}^N \setminus \{0\}$. Thus, P_C is $\mathcal{C}_{loc}^{1,\alpha}$ continuous on $\mathbb{R}^N \setminus \{0\}$. By a bootstrap argument we get that γ_C is $\mathcal{C}_{loc}^{k,\alpha}$ on $\mathbb{R}^N \setminus \{0\}$. □

We also prove a convexity result for the lower level sets of the convolution product of γ_C :

Proposition 4.7.2. *Let C be a strictly convex bounded set of \mathbb{R}^N such that its interior contains 0. If $(\rho_m)_{m \in \mathbb{N}}$ is a standard mollifying sequence then the lower level sets of $\gamma_C^m := \gamma_C * \rho_m$ are strictly convex for every $m \in \mathbb{N}$.*

Proof. Let us consider $z_1 \neq z_2$ on the boundary of a lower level set of γ_C^m . By continuity of γ_C^m , $\gamma_C^m(z_1) = \gamma_C^m(z_2) =: s$. Then for every $0 < t < 1$ we have:

$$\gamma_C^m(tz_1 + (1-t)z_2) = \int_{B_{\frac{1}{m}}(0)} \gamma_C(t(z_1 - y) + (1-t)(z_2 - y)) \rho_m(y) dy.$$

By convexity of γ_C we have

$$\gamma_C(t(z_1 - y) + (1-t)(z_2 - y)) \leq \gamma_C(z_1 - y) + (1-t)\gamma_C(z_2 - y). \quad (4.7.3)$$

If $\gamma_C(z_1 - y) = \gamma_C(z_2 - y)$ by strict convexity of C we get that

$$\gamma_C(t(z_1 - y) + (1-t)(z_2 - y)) < t\gamma_C(z_1 - y) + (1-t)\gamma_C(z_2 - y).$$

Hence, if we have equality in (4.7.3) this means that $z_1 - y$ and $z_2 - y$ are colinear. Since $z_1 \neq z_2$, for a.e. $y \in B_{\frac{1}{m}}(0)$ $z_1 - y$ and $z_2 - y$ are not colinear. Thus, $\gamma_C^m(t(z_1) + (1-t)(z_2)) < t\gamma_C^m(z_1) + (1-t)\gamma_C^m(z_2)$. This provides the strict convexity of the lower level sets of γ_C^m for every $m \in \mathbb{N}$. □

We prove that the approximations of a $\mathcal{C}^{1,1}$ strictly convex set through convolutions of γ_C are also $\mathcal{C}^{1,1}$ with a uniform norm.

Proposition 4.7.3. *Let C be a $\mathcal{C}^{1,1}$ strictly convex bounded set of \mathbb{R}^N such that its interior contains 0. We consider $\gamma_C^m := \gamma_C * \rho_m$ with $(\rho_m)_{m \in \mathbb{N}}$ a standard mollifying sequence and $r_m \rightarrow 1$ when $m \rightarrow +\infty$ such that $C_m := (\gamma_C^m)^{-1}(\{[0, r_m]\})$ is a smooth convex set containing 0. Then C_m is a $\mathcal{C}^{1,1}$ strictly convex set of \mathbb{R}^N and the Lipschitz constant of its outward normal vector can be bounded uniformly in $m \in \mathbb{N}$.*

Proof. By Proposition 4.7.2 the set C_m is strictly convex. Since $(r_m)_{m \in \mathbb{N}}$ converges to 1 and $(\gamma_C^m)_{m \in \mathbb{N}}$ converges uniformly to γ_C on \mathbb{R}^N when $m \rightarrow +\infty$ we have that

$$\lim_{m \rightarrow +\infty} \text{dist}(\partial C_m, \partial C) = 0.$$

Thanks to this last result and the fact that 0 is in the interior of C we can find $r > 0$ such that $B_r(0)$ is in C_m for every m large enough. Since C_m is a level set of γ_C^m for every $z \in \partial C_m$ we have that $\nu_{C_m}(z) = \frac{\nabla \gamma_C^m(z)}{|\nabla \gamma_C^m(z)|}$. By Proposition 4.7.1 and (4.7.2) the function $\nabla \gamma_C^m := \nabla \gamma_C * \rho_m$ is uniformly Lipschitz continuous on $\mathbb{R}^N \setminus B_r(0)$. Moreover, there exists $\kappa > 0$ such that $|\nabla \gamma_C^m(z)| \geq \kappa$ for every $m \in \mathbb{N}$ large enough according to $r > 0$ and C .

Hence, ν_{C_m} is Lipschitz continuous for every $m \in \mathbb{N}$ with a Lipschitz constant independent of $m \in \mathbb{N}$. □

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Propriétés qualitatives pour des solutions de problèmes dégénérés et/ou singuliers en calcul des variations

Résumé : Cette thèse s'inscrit dans les domaines du calcul des variations, des équations aux dérivées partielles elliptiques et de la théorie géométrique de la mesure.

Dans le premier chapitre, nous prouvons un résultat d'unicité des solutions pour un problème de minimisation de la forme $\int_{\Omega} \varphi(\nabla u) - \lambda u$ avec λ une fonction constante. L'autonomie de cette fonctionnelle nous permet d'effectuer des translations de solutions comme dans [29] pour montrer qu'il existe au plus un minimiseur uniformément continu.

Dans le deuxième chapitre, nous considérons une fonction convexe φ comme celle présente dans [8] n'étant pas différentiable à l'origine et λ une fonction lipschitzienne. La preuve de l'unicité dans ce cas-là repose sur une étude de la régularité des ensembles de niveau.

Dans le troisième chapitre, on travaille avec un autre φ introduit dans [41] qui n'est pas strictement convexe sur plusieurs ensembles. Nous prouvons un résultat d'unicité en dimension deux reposant encore une fois sur une étude des lignes de niveau et sur un principe du maximum pour $\nabla\varphi(\nabla u)$ avec u un minimiseur.

Dans le quatrième chapitre, nous nous intéressons à la régularité des solutions faibles d'équations aux dérivées partielles elliptiques dégénérées de la forme $\operatorname{div} G(\nabla u) = f$. En dimension deux quand f est une constante nous présentons des résultats sur la continuité de $G(\nabla u)$ sous diverses hypothèses de croissance sur G . Nous établissons aussi une extension de [57] en élargissant la zone de dégénérescence de G .

Mots-Clés : Calcul des variations, équations aux dérivées partielles, théorie géométrique de la mesure, unicité, régularité, ensembles de niveau.

Qualitative properties for solutions of degenerate and/or singular problems in the calculus of variations

Abstract : This thesis belongs in the fields of calculus of variations, elliptic partial differential equations and geometric measure theory.

In the first chapter, we prove a uniqueness result for the solutions of a minimization problem of the form $\int_{\Omega} \varphi(\nabla u) - \lambda u$ with λ a constant function. The autonomy of this functional allows us to use translations of solutions as in [29] to prove that there exists at most one uniformly continuous minimizer.

In the second chapter, we consider a convex function φ as the one in [8] which is not differentiable at the origin and λ a Lipschitz continuous function. The proof of the uniqueness in this case relies on a study of the regularity of the level sets.

In the third chapter, we work with another φ introduced in [41] which is not strictly convex on several sets. We prove a uniqueness result in dimension two based again on a study of the level lines and on a maximum principle for $\nabla\varphi(\nabla u)$ with u a minimizer.

In the fourth chapter, we are interested in the regularity of weak solutions of degenerate elliptic partial differential equations of the form $\operatorname{div} G(\nabla u) = f$. In dimension two when f is a constant we present some results on the continuity of $G(\nabla u)$ under various growth assumptions on G . We also establish an extension of [57] by enlarging the degeneracy zone of G .

Keywords : Calculus of Variations, partial differential equations, geometric measure theory, uniqueness, regularity, level sets.