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**Étude de deux problèmes de propagation d'ondes en milieu électromagnétique dispersif: 1) Stabilité en temps long dans un milieu de Drude-Lorentz; 2) Transmission entre une couche de metamateriau et un diélectrique.**

Luis Alejandro Rosas Martinez

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INSTITUT  
POLYTECHNIQUE  
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Thèse de doctorat



# Study of two wave propagation problems in electromagnetic dispersive media: 1) Long-time stability analysis in Drude-Lorentz media; 2) Transmission between a slab of metamaterial and a dielectric

Thèse de doctorat de l'Institut Polytechnique de Paris  
préparée à l'École nationale supérieure de techniques avancées

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Thèse présentée et soutenue à Palaiseau, le 16 novembre 2023, par

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*To my parents*

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## Introduction (English)

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Over the last twenty years, the study of wave propagation in dispersive media, and more notoriously negative index materials in electromagnetism, has recovered significantly more interest since the rapid development of the area of **metamaterials**. In the classical theory of propagation, electromagnetic waves propagating through a determined material is well modelled by Maxwell's equations subject to two physical quantities which determine the properties of material: the dielectric permittivity  $\varepsilon$  and magnetic permeability  $\mu$  of the material. Usually, materials in nature have both quantities positive ( $\varepsilon, \mu > 0$ ), moreover, most of materials tend to have a permeability close to  $\mu_0$  and a permittivity larger than  $\varepsilon_0$ , where  $\varepsilon_0$  and  $\mu_0$  are, respectively, the permittivity and permeability of the vacuum [23, 24]. In this sense, as the name can suggest, metamaterials offer an alternative to realize all possible configurations trespassing the concept of "classical material" and consequently opening a wide source of new properties non-found in nature.

### Introduction to metamaterials

Metamaterials, or also well-known as LHM (left-handed materials, first proposed by Veselago in 1968 [86]) or NIM (negative-refractive index materials, media discovered much earlier [57, 58, 83]), are formally defined in [23, 24] as **"macroscopic composites of periodic or non-periodic structure, whose function is due to both the cellular architecture and the chemical composition"**. More recently, Muamer Kadic, Graeme W. Milton, Martin van Hecke, and Martin Wegener gave the following definition (see [49]): **"Metamaterials are rationally designed composites made of tailored building blocks, which are composed of one or more constituent bulk materials, leading to effective medium properties beyond those of their ingredients."**

In much simpler words, metamaterials are artificial materials with physical properties difficult to find in nature. Metamaterials consist of a periodic assembly of a large number of resonant micro-structures (see Figure 1). Indeed, those micro-structures at the level of the base cell confer the original and interesting properties. For this specific kind of metamaterials, the witty idea for building them consists in cleverly choosing the structure as well as the resonators in order to obtain the desired effectiveness of the medium, hence by passing an homogenization process, the resulting effective homogeneous media present the so-claimed properties.

### Negative materials

As previously stated, in the study of classic dispersive electromagnetic media, we encounter "classical materials" whose electrical permeability and magnetic permittivity are

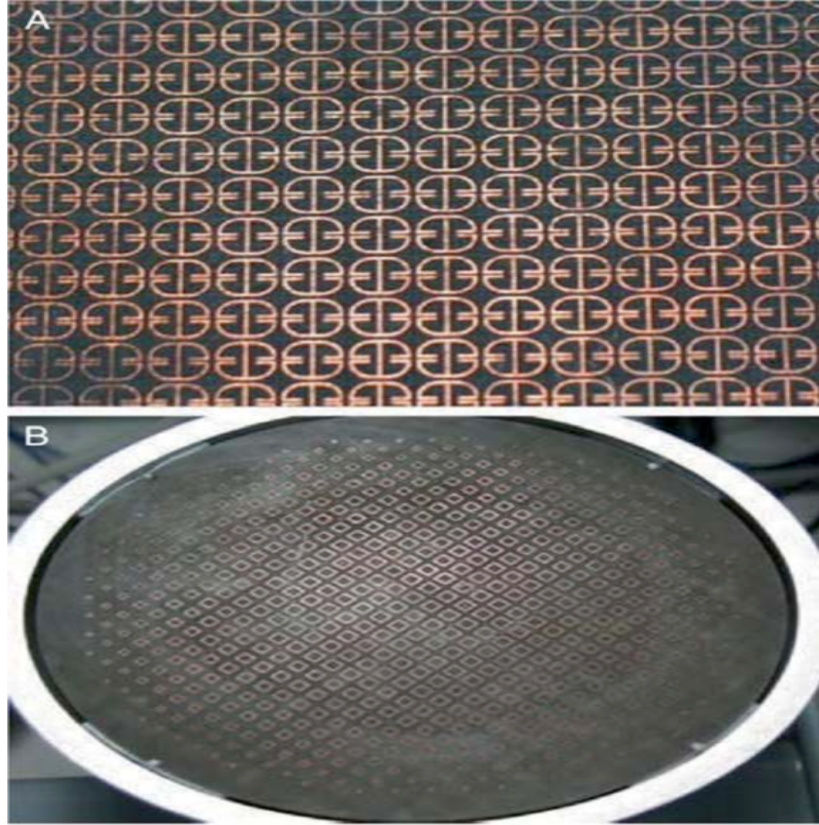


Figure 1: Two typical metamaterial structures in the microwave regime. A) A periodic structure (homogeneous medium) above, B) A non-periodic structure (inhomogeneous medium) below.

strictly positive quantities. As "classical materials", one firstly mentions the *dielectrics* or *dielectric media*, which are electrical insulators (materials with low conductivity) with the capacity of being polarized in the presence of an electric field. Examples of dielectrics are: glass, wood, petroleum, and gases as air and nitrogen. It is also very useful the concept of *high vacuum*, nearly lossless dielectric even though its relative dielectric constant is only unity [55], thus the vacuum can be considered a dielectric for our theoretical purposes. A good approximation of negative materials among natural materials are metals (e.g. gold, silver and copper) in the optical frequency range since their permittivity exhibits a negative part compared to their imaginary part which is relatively small (see [68]). However, as natural media, their permeability remains positive. Metamaterials are a second example of negative materials among manufactured materials. Furthermore, they exhibit a permittivity or /and permeability negative in some frequency range. Among the most important phenomena issued by the negative properties of a material, we find:

- the spatial localization of electromagnetic waves over materials surfaces, this phenomenon receives the name of *surface plasmons*;
- the phenomenon of negative refraction, which translates as the change of direction of wave propagation of a beam transmitted over the interface between a negative media and a dielectric, in this case one refers to materials of type LHM and NIM.

---

## Plasmon surfaces

An electron gas in a solid, as for example the free electrons of metals, has the capacity to undergo collective motions that have been named plasma oscillations. These collective plasma oscillations are due to the long range nature of the Coulomb interaction between conduction electrons in metals, as was pointed out for the first time by Pines and Bohm [75] and discussed in detail by Pines [74]. The presence of surfaces introduces new modes of plasma oscillations in addition to the bulk one. These new modes can be excited by incident electrons or photons and can be detected experimentally. They are also strongly dependent on the properties of the surfaces so that they can be used to make sensors. Moreover these surface oscillations allow to shrink optical signals in dimensions much smaller than their vacuum wavelength offering the way to have optical circuits of nanometric dimensions. The quanta of these oscillations are called plasmons (first introduced by R. Ritchie in 1957 [78] and later in 1960 introduced the term surface plasmon by Stern and Ferrel [84]). Plasmons are electromagnetic excitations coupled to electrons in a metal or doped semiconductor.

Recently these waves have been used in new applications concerning the design of biosensors, cancer therapies, the production of efficient photovoltaic cells and many others (see [56]). Optical sensors based on surface plasmons, often referred to as surface plasmon resonance (SPR) sensors, exploit the sensitivity of the propagation constant of a surface plasmon to refractive index changes occurring in its evanescent field. A change in the refractive index produces a change in the propagation constant of the surface plasmon, which results in a change in the characteristics of the light wave interacting with it. The first sensor of this kind was demonstrated by Nylander and Liedberg [67] for gas detection.

From a mathematical point of view, the existence of these waves is mainly due to the fact that at optical frequencies, some metals like silver or gold have a dielectric permittivity  $\epsilon$  with a small imaginary part and a negative real part (see [7, 13, 14, 21, 32, 46, 70, 77, 90, 93] for some more rigorous examples and explanations). Neglecting the imaginary part, for these ranges of frequencies, we are led to consider a real-valued  $\epsilon$  which is negative in the metal and positive in the air around the metal.

For more detail on plasmonics, see for instance [5].

## Applications of metamaterials

The huge regain of interest to the area of metamaterials has arose a considerable amount of notable applications and potential applications. We present two of the most famous of these.

For LHM, the most remarkable implementation is in the domain of superlens [72] (see Figure 2), which can be widely used in the super-resolution medical imaging, optical imaging, and nondestructive detection. The first superlens in the microwave regime was realized in 2004 [40]. The importance of superlens resides in their non-submitting ability to the diffraction limit phenomenon and are able to produce super resolution images [72].

For more general metamaterials, the cloaking devices have attracted more and more attention [41, 51, 66, 73, 91] (see Figure 3). Like out of Harry Potter's saga, by choosing the good parameters, the successful demonstrations of invisible cloaks experimentally in the microwave regime [54, 82] make it possible to realize cloaking devices in the future, that is to say, we are not private in the out coming decades of acquiring invisibility cloaks being



Figure 2: Perfect lens with metamaterials, experiment led in Imperial college of London.

capable of deviating light beams conferring us invisible! Experiments on this subjects are still not conclusive, indeed, for building this type of metamaterials we should be able to realize pretty small periodic structures landing on the size of visible light wavelength (400 to 800 nanometers), which is still not technologically achieved. However, nowadays it is possible to invisibilize objects for the microwaves whose wavelength oscillates between 1 millimeter and 1 meter [82].

## Dispersive electromagnetic waves: mathematical models

The dispersion phenomenon is ubiquitous in electromagnetism: the light velocity in a medium has generally a dependency on the frequency of the wave, which can be seen as an effect of "inertia" at the microscopic level. Despite its omnipresence, this frequency dispersion has raised numerous theoretical issues and generated intense discussions especially since the introduction of metamaterials and negative index materials. This follows in particular the extensive interest for spectacular devices like the perfect lens [72] and the invisibility cloak or carpet [59], and more recently plasmonic structures like corners with negative permittivity [6,46]. In this effervescence, apparent contradictions have been pointed in physics literature about the causality principle and the notion of electromagnetic energy. These contradictions are generally associated with an improper modeling of frequency dispersion, precisely in the range of frequency where the optical properties offered by metamaterials occur. However, most of mathematical studies in this context concerning the frequency domain and not the time-dependent equations. Therefore, they are insufficient to describe the whole physics of these materials since the frequencies are correlated by dispersion that must be taken into account rigorously.

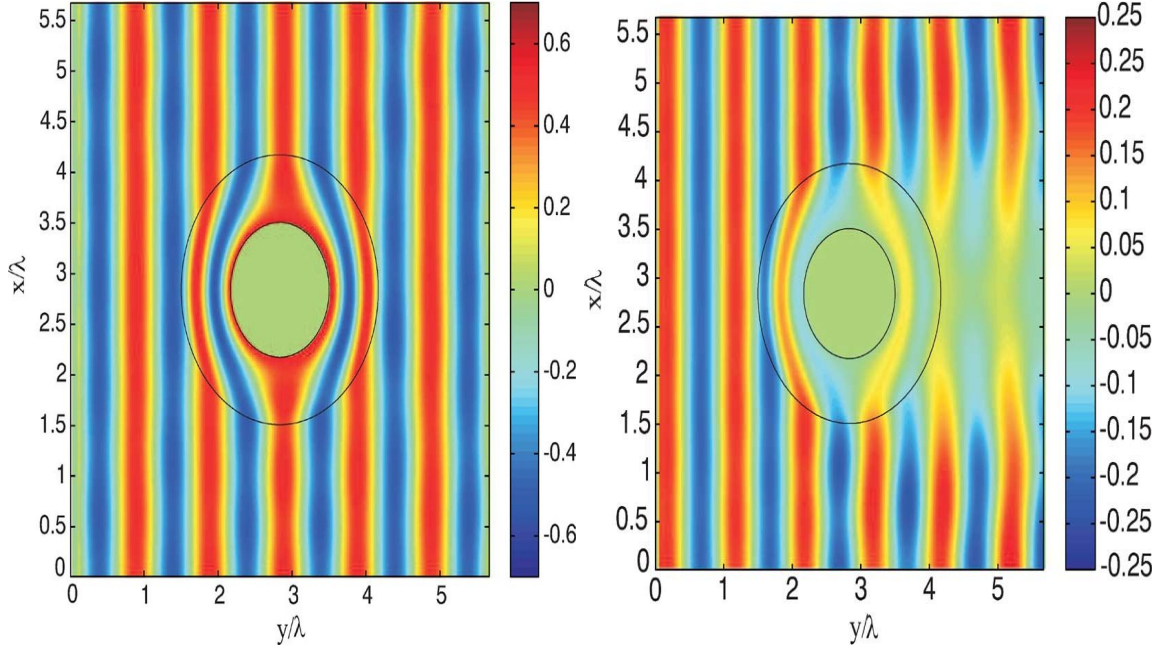


Figure 3: Normalized magnetic field distribution of the lossless (left) and lossy (right) cloaking device with plane wave excitation [1].

Under this understanding, we aim in this PhD thesis to go further in the analysis of dispersive systems to predict in particular their time-dependent behaviour. The approach we will intend in this thesis is based on a first mathematical tool that comes from complex analysis and more precisely the theory of Herglotz functions, i.e. analytic functions of the upper-half plane with non-negative imaginary part, which precisely define the permittivity and permeability of causal and passive linear media as functions of the frequency [17, 19]. Indeed, for dispersive electromagnetic systems, an important progress was made by A. Figotin, J.Schenker, B.Gralak, and A.Tip [37–39, 85] and developed by M.Cassier, C. Hazard, P.Joly and M.Kachanovska in [15, 17]. Using Herglotz functions properties in the frequency domain (in particular their representation theorem), they rewrite the dissipative and dispersive time-dependent Maxwell’s equations as a conservative system. In the time-domain, it corresponds to the introduction of additional variables (to the electrical and magnetic fields).

As explained e.g. in details in [14], in the domain of electromagnetism, Maxwell’s equations relate the space variations of the electric and magnetic fields  $\mathbf{E}(\mathbf{x}, t)$  and  $\mathbf{H}(\mathbf{x}, t)$  (where  $\mathbf{x} \in \mathbb{R}^3$  denotes the space variable and  $t > 0$  the time parameter) to the time variations of the corresponding electric and magnetic inductions  $\mathbf{D}(\mathbf{x}, t)$  and  $\mathbf{B}(\mathbf{x}, t)$ :

$$\partial_t \mathbf{D} - \nabla \times \mathbf{H} = 0, \quad \partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \mathbf{x} \in \mathbb{R}^3, t > 0. \quad (0.0.1)$$

The former equations are completed by so-called *constitutive laws* that characterize the material in which electromagnetic waves propagate, the constitutive laws make the link between the electric (or magnetic) field and the corresponding induction. In this work, we shall restrict ourselves to materials which are local in space (i.e. the induction at a given point only depends on the corresponding field in this point). In standard dielectric media, it is common to assume that the relationship is also local in time (typically the electric induction  $\mathbf{D}$  at a given point in time depends only on the electric field  $\mathbf{E}$ ). If,

moreover, one assumes that the medium is isotropic (roughly speaking, the relationship between  $\mathbf{D}$  and  $\mathbf{E}$  does not see the orientation of the fields), it is natural to suggest that the fields are proportional

$$\mathbf{D}(\mathbf{x}, t) = \varepsilon(\mathbf{x}) \mathbf{E}(\mathbf{x}, t), \quad \mathbf{B}(\mathbf{x}, t) = \mu(\mathbf{x}) \mathbf{H}(\mathbf{x}, t), \quad (0.0.2)$$

where at any point  $\mathbf{x}$ ,  $\varepsilon(\mathbf{x})$  and  $\mu(\mathbf{x})$  are positive real numbers called respectively the electric permittivity and the magnetic permeability of the material at a space point  $\mathbf{x} \in \mathbb{R}^3$ . The fact that they may depend of  $\mathbf{x}$  characterizes the possible heterogeneity of the material. In the vacuum, these coefficients are of course independent of  $\mathbf{x}$  (we speak of a homogeneous media):  $\varepsilon(\mathbf{x}) = \varepsilon_0 \approx (36 \pi)^{-1} 10^{-9} \text{ Fm}^{-1}$ ,  $\mu(\mathbf{x}) = \mu_0 = 4 \pi 10^{-7} \text{ Hm}^{-1}$ . However, (0.0.9) cannot hold and it must be seen only as an approximation. It appears that simple proportionality laws can be valid only in the vacuum, otherwise this would violate some physical principles [50]. In order to be consistent with such physical principles, one needs to abandon the idea that the constitutive laws are local in time and to accept e.g. that  $\mathbf{D}(\mathbf{x}, t)$  depends on the history of the values of  $\mathbf{E}$  between 0 and  $t$ , i.e.  $\mathbf{D}(\mathbf{x}, t)$  is a function depending on the values in  $\{E(\mathbf{x}, s) \mid 0 \leq s \leq t\}$ , in other words

$$\mathbf{D}(\mathbf{x}, t) = F(\mathbf{x}, t; \{E(\mathbf{x}, s) \mid 0 \leq s \leq t\}).$$

The above obeys a fundamental physical principle: *the causality principle*. Adding *the time invariance principle*, i.e. that the material behaves the same way whatever the time one observes it, one infers that the function  $F$  is also independent of time:  $F(\mathbf{x}, t; \cdot) = F(\mathbf{x}; \cdot)$ . To translate the above in more mathematical terms, it is useful to go to the *frequency domain* expressed by means of the Fourier-Laplace transform. The Fourier-Laplace transform  $\hat{u}$  of a function  $u$  is the function defined in the upper complex half-space  $\mathbb{C}^+ := \{\omega \in \mathbb{C} \mid \text{Im}(\omega) > 0\}$  and defined by the following integral formula

$$\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} u(t) e^{i\omega t} dt. \quad (0.0.3)$$

Dispersive (isotropic) electromagnetic materials are most often defined as materials in which the proportionality laws of the form (0.0.9) hold true in the frequency domain, that is to say, they are satisfied by the Fourier-Laplace transforms of the fields, rather than by the fields themselves. In this case there is no reason to require that  $\varepsilon$  and  $\mu$  are real and independent of the frequency. That is why a dispersive isotropic medium will be defined as obeying constitutive laws of the form

$$\widehat{\mathbf{D}}(\mathbf{x}, \omega) = \varepsilon(\mathbf{x}, \omega) \widehat{\mathbf{E}}(\mathbf{x}, \omega), \quad \widehat{\mathbf{B}}(\mathbf{x}, \omega) = \mu(\mathbf{x}, \omega) \widehat{\mathbf{H}}(\mathbf{x}, \omega), \quad (0.0.4)$$

where for each  $\mathbf{x} \in \mathbb{R}^3$ ,  $\omega \in \mathbb{C}^+$ ,  $\omega \in \mathbb{C}^+ \mapsto \varepsilon(\mathbf{x}, \omega)$  (the permittivity) and  $\omega \in \mathbb{C}^+ \mapsto \mu(\mathbf{x}, \omega)$  (the permeability) are non-trivial functions of the frequency that describe the dispersivity of the medium. For non-dispersive materials, these functions are real positive and constant, i.e. (0.0.9) holds. Of course, these functions satisfy some particular properties imposed by physical or mathematical reasons:

- **Causality principle.** To ensure the causality of  $\mathbf{D}(\mathbf{x}, t)$  (or  $\mathbf{B}(\mathbf{x}, t)$ ) provided that  $\mathbf{E}(\mathbf{x}, t)$  (or  $\mathbf{H}(\mathbf{x}, t)$ ) is causal, it is natural to impose

$$\forall \mathbf{x} \in \mathbb{R}^3, \quad \omega \mapsto \varepsilon(\mathbf{x}, \omega) \quad \text{and} \quad \omega \mapsto \mu(\mathbf{x}, \omega) \quad \text{are analytic in } \mathbb{C}^+. \quad (0.0.5)$$

- 
- **Reality principle.** A second requirement is that if  $\mathbf{D}(\mathbf{x}, t)$  (or  $\mathbf{B}(\mathbf{x}, t)$ ) is real then  $\mathbf{E}(\mathbf{x}, t)$  (or  $\mathbf{E}(\mathbf{x}, t)$ ) is real too. According to (0.0.11), the invariance of  $\mathbb{C}^+$  under the map  $\omega \mapsto -\bar{\omega}$  and Fourier-Laplace properties of real-valued functions, the reality principle traduces as the condition

$$\forall \mathbf{x} \in \mathbb{R}^3 \quad \forall \omega \in \mathbb{C}^+, \quad \varepsilon(\mathbf{x}, -\bar{\omega}) = \overline{\varepsilon(\mathbf{x}, \omega)} \quad \text{and} \quad \mu(\mathbf{x}, -\bar{\omega}) = \overline{\mu(\mathbf{x}, \omega)}. \quad (0.0.6)$$

- **High frequency principle.** A fundamental property from the physical point of view is that, at high frequency, any material "behaves as the vacuum". Mathematically, this amounts to requiring that

$$\forall \mathbf{x} \in \mathbb{R}^3 \quad \forall \eta > 0, \quad \lim_{\substack{\text{Im}(\omega) \geq \eta \\ |\omega| \rightarrow +\infty}} \varepsilon(\mathbf{x}, \omega) = \varepsilon_0, \quad \lim_{\substack{\text{Im}(\omega) \geq \eta \\ |\omega| \rightarrow +\infty}} \mu(\mathbf{x}, \omega) = \mu_0. \quad (0.0.7)$$

This means that the material is "less and less dispersive" at high frequencies.

The following PhD work is composed of two independent parts related to wave propagation in dispersive materials with a start point in the temporal regime, that is to say, starting from an approach said stationary where we reconstruct a transitory wave as a superposition of periodic waves. The first part of the PhD work consist in the analysis of long time behaviour of the solution of Maxwell's equations in dissipative generalized Lorentz materials, this represents the content of the [Chapter 1](#) and [Chapter 2](#), these ones develop two distinct approaches for the treatment of the subject. On the other hand, the second part is devoted to the study of Maxwell's equations in time regime, delimited in a slab of metamaterial immersed in a dielectric material (for example the vacuum). This second part comprehends [Chapter 3](#), [Chapter 4](#) and [Chapter 5](#).

## Part I Long time behaviour of the solution of Maxwell's equations in dissipative generalized Lorentz materials

### Context

The analysis of decreasing and decreasing rates of the energy of solutions for dispersive and dissipative models for linear wave propagation has been over long periods of time studied in the literature. This research has primarily been motivated by applications in visco-elasticity [[22](#), [26](#), [35](#)] and electromagnetism [[36](#), [61](#)], but it has recently seen a resurgence of interest due to its relevance to metamaterials, take for instance [[17](#), [37](#), [63](#), [65](#), [65](#)], where it is given a suitable construction of mathematical models congruent with physically motivated principles such as causality and passivity principles (see also [[19](#), [39](#), [92](#)]).

### Content of the Part I

In [Chapter 1](#) we develop for Generalized dissipative Drude-Lorentz media a Lyapunov technique for proving the polynomial stability of the electromagnetic energy, in a more pedestrian way to say, the idea is to derive a differential inequality via a energy balance derived from Maxwell's equations, hence by Gronwall-like inequalities in the spatial Fourier domain and Plancherel's isometry we reach upto the desired result. It is worthwhile mentioning that other stability results for dispersive/dissipative results have also been obtained via the Lyapunov technique (see [[35](#), [36](#), [61](#)]), nevertheless, neither of those

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techniques apply in our context given the lack of sign properties assumed for the convolution kernels implied in the memory effects of our model.

Another ph

In [Chapter 2](#), we treat the same problem as in [Chapter 1](#) using in this occasion the approach of modal decomposition. This approach is technically more complex than the one in the first chapter, but it has significantly several advantages. First, it is more physical in many ways, as it explicitly refers to the dispersion relation of the medium via a wave plane decomposition, this feature makes it easier for physicists to understand. Second, it leads to optimal results, as it allows us to obtain both upper and lower bounds for the stability of the system, this feature could not be reached via the first approach presented by means of energy balance. Third, it allows to recover the same polynomial stability results under the same dissipation hypothesis as [Chapter 1](#), moreover, we achieve to obtain polynomial stability results under weaker conditions on the dissipation coefficients, in particular, one obtains in certain configurations weaker polynomial decay rates than the ones observed in [Chapter 1](#).

## Part II Wave guides of a slab of metamaterial

### Context

Most of materials found in nature have positive physical parameters for most frequencies. However, it is well-know since last century that metals (such as gold) have a negative electrical permittivity for a wide range of frequencies. The Drude model is a simple model that can be used to describe the behavior of metals in this range. In the Drude model, electrons in a metal are assumed to be free to move. This leads to a negative permittivity for frequencies below the plasma frequency of the metal. Indeed, if by applying fundamental dynamic principles to an electron and by linking its displacement to the polarization and the electric field, one shows the very good approximation

$$\varepsilon(\omega) = \varepsilon_0 \left( 1 - \frac{\Omega_e^2}{\omega^2} \right),$$

where  $\varepsilon_0$  stands for the permittivity in the vacuum and  $\Omega_e$  for the plasma pulsation of the metal.

The negative permittivity of metals has several interesting properties. For example, it allows surface plasmons to propagate at the interface between a dielectric and a metal. Surface plasmons are waves that are confined to the interface. They have been used in a variety of applications in micro and nano technologies, such as optical sensing and imaging.

This second part of the thesis work places itself and tries to solve one of the perspectives of the previous work made by Maxence Cassier, Christophe Hazard, and Patrick Joly [[14–16](#)] in the study of wave propagation in a settle of two semi-spaces, one fulfilled of a dielectric as the vacuum and the other of a dispersive material, Drude material. In this work we intend to study the wave propagation considering a new separable geometry, namely, the flat lens: a slab of dispersive material within a dielectric.

### Content of the Part II

The second part is composed by [Chapter 3](#) and [Chapter 4](#) and enrolls itself as continuation of the works of Bonnet-Joly, Dermenjian-Guillot, Ricardo Weder, and Calvin Hayden



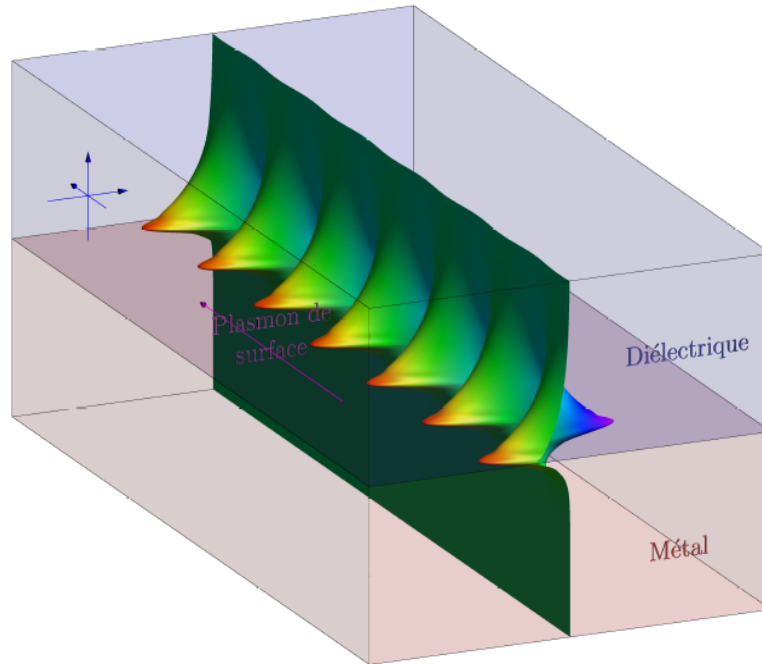


Figure 4: Scheme representation of a surface plasmon.

Wilcox for the study of wave propagation in stratified media [7, 32, 90, 93], Maxence Cassier, Christophe Hazard, and Patrick Joly for their researches of a bi-layered medium that involves a dispersive material modeled by Drude constitutive laws [14–16]. This study is also developed in the original framework of Drude model.

Chapter 3 treats the general features of the second part of the thesis. In Section 3.1, the physical and mathematical framework of the problem related to this study is presented. For this purpose, we begin with Maxwell’s equations in time domain presented as modeling the problem in the case of the particular but enough rich Drude material. Subsequently, a decomposition of solutions of our problem in Transverse Electric modes (TE) and Transverse Magnetic modes (TM) is given. Once chosen to work arbitrarily on mode TM, the original problem is simplified and reformulated under the form of an abstract Schrödinger equation implying an unbounded operator  $\mathbb{A}$  defined in a suitable Hilbert space of finite energy functions. In Section 3.2, the symmetry of our problem leads us to consider the Fourier transform ending up with the reduced Hamiltonians  $\mathbb{A}_k$ , where  $k \in \mathbb{R}$  is seen as the wave-length of the involved solutions. Later in Section 3.3, the spectral theory of the reduced Hamiltonian is evoked, the concept of modal decomposition is presented and the *dispersion relation* (relation between the frequency  $\omega$  and the wave-length  $k$ ) is derived.

Chapter 4 constitutes the soul of this second part of the thesis, where we find the set solution for the *dispersion relation* and we confirm the presence of plasmonic waves. In Section 4.1, we transform the *dispersion relation* in an equivalent *dimensionless dispersion relation* the which we decompose as a countable collection of *dimensionless dispersion systems*  $(\mathcal{DS}_n)$ . Next in Section 4.2, we develop an abstract general technique for the resolution of the systems  $(\mathcal{DS}_n)$ . Finally, we show that each system  $(\mathcal{DS}_n)$  defines a curve solution (*dispersion curves*) named  $\mathcal{C}_n$ , in Section 4.3, Section 4.4 and Section 4.5, we

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investigate the analytic properties of  $\mathcal{C}_n$  divided, respectively, for the cases  $n \geq 2$ ,  $n = 1$  and  $n = 0$ .

Chapter 5 presents the comparisons of our study in this second part of the thesis with respect the previous studies: 1) the case of a non-dispersive slab material, treated in what we refer ourselves as *the classical case* standing for the study of wave propagation in stratified media (see [29–31, 44, 88–90, 93, 94] for the case of acoustic waves, [42, 90] for electromagnetic waves and [32, 43, 81] in the domain of elasticity); 2) the case of two-layer media containing a dispersive Drude material [14–16].

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## Introduction (Français)

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Au cours des vingt dernières années, l'étude de la propagation des ondes dans les milieux dispersifs, et plus particulièrement des matériaux à indice négatif en électromagnétisme, a suscité beaucoup plus d'intérêt depuis le développement rapide du domaine des **métamatériaux**. Dans la théorie classique de la propagation, les ondes électromagnétiques se propageant à travers un matériau déterminé sont bien modélisées par les équations de Maxwell, soumises à deux grandeurs physiques qui déterminent les propriétés du matériau : la permittivité diélectrique  $\varepsilon$  et la perméabilité magnétique  $\mu$  du matériau. Habituellement, les matériaux dans la nature ont des valeurs positives pour ces grandeurs ( $\varepsilon, \mu > 0$ ). De plus, la plupart des matériaux tendent à avoir une perméabilité proche de  $\mu_0$  et une permittivité supérieure à  $\varepsilon_0$ , où  $\varepsilon_0$  et  $\mu_0$  représentent respectivement la permittivité et la perméabilité du vide [23, 24]. En ce sens, comme son nom peut le suggérer, les métamatériaux offrent une alternative pour réaliser toutes les configurations possibles dépassant le concept de "matériau classique" et ouvrant ainsi une large gamme de nouvelles propriétés non trouvées dans la nature.

### Introduction aux métamatériaux

Les métamatériaux, également connus sous le nom de MGL (matériaux gauchers, proposés pour la première fois par Veselago en 1968 [86]) ou MIR (matériaux à indice de réfraction négatif, découverts bien plus tôt [57, 58, 83]), sont formellement définis dans [23, 24] comme des **"composites macroscopiques de structure périodique ou non périodique, dont la fonctionnalité est due à l'architecture cellulaire et à la composition chimique"**. Plus récemment, Muamer Kadic, Graeme W. Milton, Martin van Hecke et Martin Wegener ont donné la définition suivante (voir [49]) : **"Les métamatériaux sont des composites conçus de blocs de construction adaptés, composés d'un ou plusieurs matériaux de base constitutifs, donnant lieu à des propriétés de milieu effectif au-delà de celles de leurs composants"**.

En termes plus simples, les métamatériaux sont des matériaux artificiels aux propriétés physiques difficiles à trouver dans la nature. Ils consistent en un assemblage périodique d'un grand nombre de microstructures résonantes (voir Figure 5). En effet, ces microstructures au niveau de la cellule de base confèrent les propriétés originales et intéressantes. Pour ce type spécifique de métamatériaux, l'idée ingénieuse pour les construire consiste à choisir habilement la structure ainsi que les résonateurs afin d'obtenir l'efficacité souhaitée du milieu. Ainsi, en passant par un processus d'homogénéisation, les milieux homogènes effectifs résultants présentent les propriétés revendiquées.

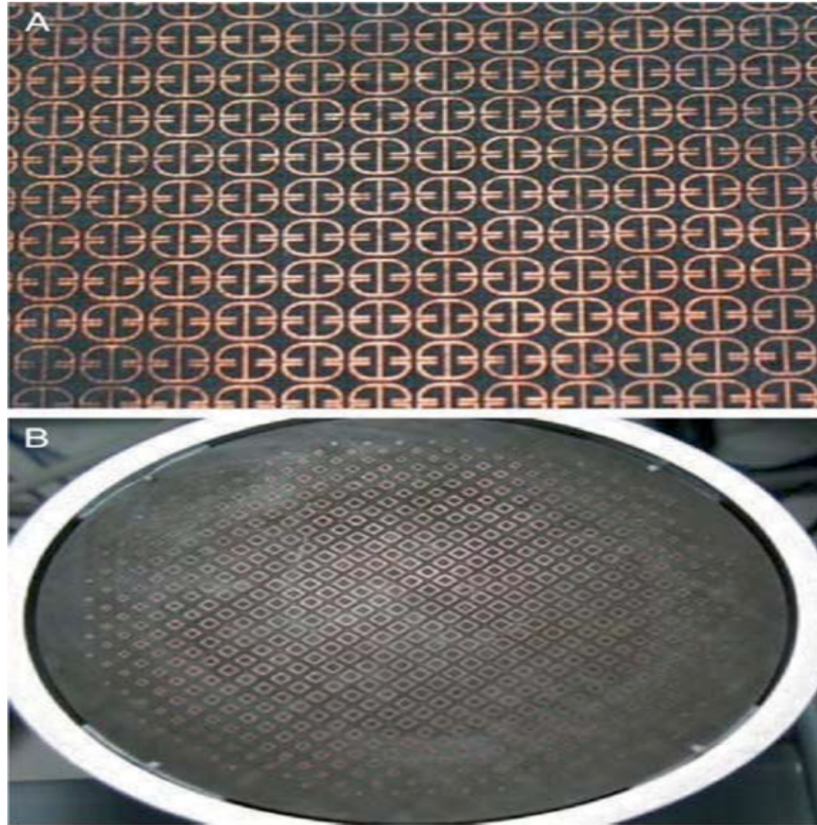


Figure 5: Deux structures métamatériaux typiques dans le régime micro-ondes. A) Une structure périodique (milieu homogène) ci-dessus, B) Une structure non périodique (milieu non homogène) ci-dessous.

### Matériaux négatifs

Comme mentionné précédemment, dans l'étude des milieux électromagnétiques dispersifs classiques, on rencontre des "matériaux classiques" dont la perméabilité électrique et la permittivité magnétique sont strictement des grandeurs positives. Parmi les "matériaux classiques", on mentionne tout d'abord les *diélectriques* ou les *milieux diélectriques*, qui sont des isolants électriques (matériaux à faible conductivité) capables de se polariser en présence d'un champ électrique. Des exemples de diélectriques sont : le verre, le bois, le pétrole et les gaz tels que l'air et l'azote. Il est également très utile de considérer le concept de *vide haut*, un diélectrique presque sans perte même si sa constante diélectrique relative est seulement l'unité [55]. Ainsi, le vide peut être considéré comme un diélectrique à des fins théoriques. Une bonne approximation de matériaux négatifs parmi les matériaux naturels sont les métaux (par exemple, l'or, l'argent et le cuivre) dans la gamme de fréquences optiques car leur permittivité présente une partie négative par rapport à leur partie imaginaire qui est relativement petite (voir [68]). Cependant, en tant que médias naturels, leur perméabilité reste positive. Les métamatériaux constituent un deuxième exemple de matériaux négatifs parmi les matériaux fabriqués. De plus, ils présentent une permittivité et/ou une perméabilité négative(s) sur une plage de fréquences spécifique(s) en fonction de leur conception, ce qui les rend intéressants pour les applications dans la manipulation des ondes électromagnétiques.

Parmi les phénomènes les plus importants issus des propriétés négatives d'un matériau, nous trouvons :

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- la localisation spatiale des ondes électromagnétiques sur les surfaces des matériaux, ce phénomène est appelé les *plasmons de surface*;
  - le phénomène de réfraction négative, qui se traduit par le changement de direction de la propagation d'une onde transmise sur l'interface entre un milieu négatif et un diélectrique, dans ce cas, il s'agit de matériaux de type LHM et NIM.

### Plasmons de surface

Un gaz d'électrons dans un solide, comme par exemple les électrons libres des métaux, a la capacité de subir des mouvements collectifs qui ont été nommés oscillations de plasma. Ces oscillations collectives de plasma sont dues à la nature à longue portée de l'interaction de Coulomb entre les électrons de conduction dans les métaux, comme l'ont souligné pour la première fois Pines et Bohm [75] et discuté en détail par Pines [74]. La présence de surfaces introduit de nouveaux modes d'oscillations de plasma en plus de celui du volume. Ces nouveaux modes peuvent être excités par des électrons ou des photons incidents et peuvent être détectés expérimentalement. Ils dépendent également fortement des propriétés des surfaces de sorte qu'ils peuvent être utilisés pour fabriquer des capteurs. De plus, ces oscillations de surface permettent de réduire les signaux optiques dans des dimensions beaucoup plus petites que leur longueur d'onde dans le vide, offrant ainsi la possibilité d'avoir des circuits optiques de dimensions nanométriques. Les quanta de ces oscillations sont appelés plasmons (introduits pour la première fois par R. Ritchie en 1957 [78] et plus tard en 1960, le terme plasmon de surface a été introduit par Stern et Ferrel [84]). Les plasmons sont des excitations électromagnétiques couplées aux électrons dans un métal ou un semi-conducteur dopé.

Récemment, ces ondes ont été utilisées dans de nouvelles applications concernant la conception de biocapteurs, les thérapies contre le cancer, la production de cellules photovoltaïques efficaces et bien d'autres (voir [56]). Les capteurs optiques basés sur les plasmons de surface, souvent appelés capteurs de résonance des plasmons de surface (SPR), exploitent la sensibilité de la constante de propagation d'un plasmon de surface aux changements d'indice de réfraction se produisant dans son champ évanescent. Un changement dans l'indice de réfraction produit un changement de la constante de propagation du plasmon de surface, ce qui se traduit par un changement dans les caractéristiques de l'onde lumineuse interagissant avec lui. Le premier capteur de ce type a été démontré par Nylander et Liedberg [67] pour la détection de gaz.

D'un point de vue mathématique, l'existence de ces ondes est principalement due au fait qu'aux fréquences optiques, certains métaux comme l'argent ou l'or ont une permittivité diélectrique avec une petite partie imaginaire et une partie réelle négative (voir [7, 13, 14, 21, 32, 46, 70, 77, 90, 93] pour des exemples et des explications plus rigoureux). En négligeant la partie imaginaire, pour ces plages de fréquences, nous sommes amenés à considérer une  $\epsilon$  réelle qui est négative dans le métal et positive dans l'air autour du métal.

Pour plus de détails sur la plasmonique, voir par exemple [5].

### Applications des métamatériaux

Le regain d'intérêt considérable pour le domaine des métamatériaux a suscité un nombre notable d'applications et d'applications potentielles remarquables. Nous présentons deux des plus célèbres.

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Pour les matériaux à indice de réfraction négatif (LHM), la mise en œuvre la plus remarquable se situe dans le domaine de la super lentille [72] (voir Figure 6), qui peut être largement utilisée dans l'imagerie médicale à super-résolution, l'imagerie optique et la détection non destructive. La première super lentille dans le régime des micro-ondes a été réalisée en 2004 [40]. L'importance des super lentilles réside dans leur capacité à ne pas être soumises au phénomène de limite de diffraction et à produire des images à super-résolution [72].



Figure 6: Lentille parfaite avec des métamatériaux, expérience menée à l'Imperial College de Londres.

Pour les métamatériaux plus généraux, les dispositifs de camouflage ont attiré de plus en plus d'attention [41, 51, 66, 73, 91] (voir Figure 7). Comme dans la saga Harry Potter, en choisissant les bons paramètres, les démonstrations réussies de capes d'invisibilité expérimentales dans le régime des micro-ondes [54, 82] rendent possible la réalisation de dispositifs de camouflage dans le futur, c'est-à-dire que nous ne sommes pas loin, dans les décennies à venir, d'acquérir des capes d'invisibilité capables de dévier les faisceaux lumineux nous rendant invisibles ! Les expériences sur ce sujet ne sont toujours pas concluantes. En effet, pour construire ce type de métamatériaux, nous devrions être en mesure de réaliser des structures périodiques assez petites atteignant la taille de la longueur d'onde de la lumière visible (400 à 800 nanomètres), ce qui n'est toujours pas technologiquement réalisé. Cependant, de nos jours, il est possible de rendre invisibles des objets pour les micro-ondes dont la longueur d'onde oscille entre 1 millimètre et 1 mètre [82].

## Ondes électromagnétiques dispersives : modèles mathématiques

Le phénomène de dispersion est omniprésent en électromagnétisme : la vitesse de la lumière dans un milieu dépend généralement de la fréquence de l'onde, ce qui peut

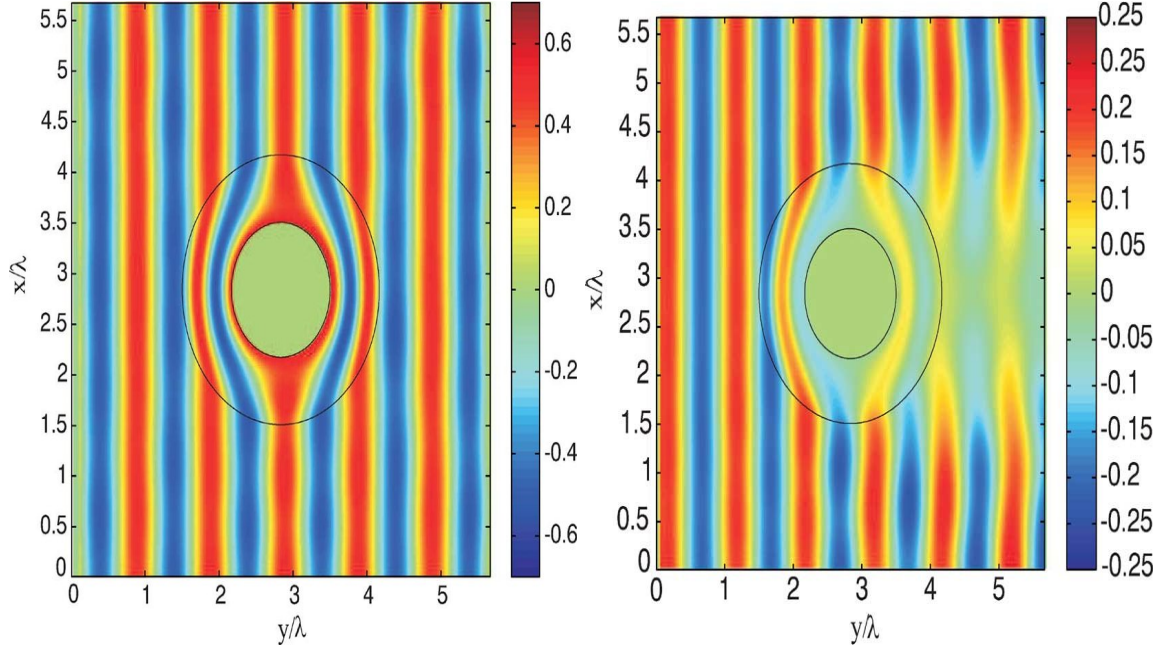


Figure 7: Distribution normalisée du champ magnétique du dispositif de camouflage sans perte (gauche) et avec perte (droite) avec une excitation d'onde plane [1].

être interprété comme un effet d'"inertie" au niveau microscopique. Malgré sa présence généralisée, cette dispersion en fréquence a soulevé de nombreuses questions théoriques et engendré des discussions intenses, en particulier depuis l'introduction des métamatériaux et des matériaux à indice négatif. Cela fait suite notamment à l'intérêt considérable pour des dispositifs spectaculaires tels que la lentille parfaite [72] et la cape ou le tapis d'invisibilité [59], et plus récemment, pour des structures plasmoniques comme les coins avec une permittivité négative [6, 46]. Dans cette effervescence, des contradictions apparentes ont été relevées dans la littérature physique concernant le principe de causalité et la notion d'énergie électromagnétique. Ces contradictions sont généralement associées à une modélisation incorrecte de la dispersion en fréquence, précisément dans la plage de fréquences où les propriétés optiques offertes par les métamatériaux se produisent. Cependant, la plupart des études mathématiques dans ce contexte concernent le domaine fréquentiel et non les équations dépendant du temps. Par conséquent, elles sont insuffisantes pour décrire toute la physique de ces matériaux puisque les fréquences sont corrélées par une dispersion qui doit être prise en compte rigoureusement.

Dans cette optique, notre objectif dans cette thèse de doctorat est d'approfondir l'analyse des systèmes dispersifs pour prédire en particulier leur comportement dépendant du temps. L'approche que nous envisageons dans cette thèse repose sur un premier outil mathématique issu de l'analyse complexe et plus précisément de la théorie des fonctions de Herglotz, c'est-à-dire des fonctions analytiques du demi-plan supérieur avec une partie imaginaire non négative, qui définissent précisément la permittivité et la perméabilité de milieux linéaires causaux et passifs en fonction de la fréquence [17, 19].

En effet, pour les systèmes électromagnétiques dispersifs, des progrès importants ont été réalisés par A. Figotin, J. Schenker, B. Gralak et A. Tip [37–39, 85] et développés par M. Cassier, C. Hazard, P. Joly et M. Kachanovska dans [15, 17]. En utilisant les propriétés des fonctions de Herglotz dans le domaine fréquentiel (en particulier leur théorème de représentation), ils réécrivent les équations de Maxwell dissipatives et disper-

sives dépendantes du temps comme un système conservatif. Dans le domaine temporel, cela correspond à l'introduction de variables supplémentaires (aux champs électriques et magnétiques).

Comme expliqué en détail, par exemple, dans [14], dans le domaine de l'électromagnétisme, les équations de Maxwell relient les variations spatiales des champs électrique et magnétique  $\mathbf{E}(\mathbf{x}, t)$  et  $\mathbf{H}(\mathbf{x}, t)$  (où  $\mathbf{x} \in \mathbb{R}^3$  désigne la variable spatiale et  $t > 0$  le paramètre temporel) aux variations temporelles des inductions électriques et magnétiques correspondantes  $\mathbf{D}(\mathbf{x}, t)$  et  $\mathbf{B}(\mathbf{x}, t)$  :

$$\partial_t \mathbf{D} - \nabla \times \mathbf{H} = 0, \quad \partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \mathbf{x} \in \mathbb{R}^3, t > 0. \quad (0.0.8)$$

Les équations précédentes sont complétées par des lois constitutives qui caractérisent le matériau dans lequel les ondes électromagnétiques se propagent, les lois constitutives établissent le lien entre le champ électrique (ou magnétique) et l'induction correspondante. Dans ce travail, nous nous limiterons aux matériaux qui sont locaux dans l'espace (c'est-à-dire que l'induction en un point donné dépend uniquement du champ correspondant en ce point). Dans les milieux diélectriques standard, il est courant de supposer que la relation est également locale dans le temps (typiquement, l'induction électrique  $\mathbf{D}$  à un instant donné dépend uniquement du champ électrique  $\mathbf{E}$ ). De plus, si l'on suppose que le milieu est isotrope (en gros, la relation entre  $\mathbf{D}$  et  $\mathbf{E}$  ne tient pas compte de l'orientation des champs), il est naturel de suggérer que les champs sont proportionnels :

$$\mathbf{D}(\mathbf{x}, t) = \varepsilon(\mathbf{x}), \mathbf{E}(\mathbf{x}, t), \quad \mathbf{B}(\mathbf{x}, t) = \mu(\mathbf{x}), \mathbf{H}(\mathbf{x}, t), \quad (0.0.9)$$

où en tout point  $\mathbf{x}$ ,  $\varepsilon(\mathbf{x})$  et  $\mu(\mathbf{x})$  sont des nombres réels positifs appelés respectivement la permittivité électrique et la perméabilité magnétique du matériau en un point spatial  $\mathbf{x} \in \mathbb{R}^3$ . Le fait qu'ils puissent dépendre de  $\mathbf{x}$  caractérise l'hétérogénéité possible du matériau. Dans le vide, ces coefficients sont bien sûr indépendants de  $\mathbf{x}$  (on parle d'un milieu homogène) :  $\varepsilon(\mathbf{x}) = \varepsilon_0 \approx (36, \pi)^{-1}, 10^{-9}, Fm^{-1}$ ,  $\mu(\mathbf{x}) = \mu_0 = 4, \pi, 10^{-7}, Hm^{-1}$ . Cependant, (0.0.9) ne peut pas être valide et doit être vue uniquement comme une approximation. Il apparaît que des lois de proportionnalité simples ne peuvent être valables que dans le vide, sinon cela violerait certains principes physiques [50].

Pour être cohérent avec de tels principes physiques, il est nécessaire d'abandonner l'idée que les lois constitutives sont locales dans le temps et d'accepter, par exemple, que  $\mathbf{D}(\mathbf{x}, t)$  dépend de l'historique des valeurs de  $\mathbf{E}$  entre 0 et  $t$ , c'est-à-dire que  $\mathbf{D}(\mathbf{x}, t)$  est une fonction dépendant des valeurs dans  $E(\mathbf{x}, s)$ ,  $|, 0 \leq s \leq t$ , en d'autres termes :

$$\mathbf{D}(\mathbf{x}, t) = F(\mathbf{x}, t; \{E(\mathbf{x}, s) \mid 0 \leq s \leq t\}).$$

Ce qui obéit à un principe physique fondamental : le principe de causalité. En ajoutant le principe d'invariance temporelle, c'est-à-dire que le matériau se comporte de la même manière quel que soit le temps où on l'observe, on en déduit que la fonction  $F$  est également indépendante du temps :  $F(\mathbf{x}, t; \cdot) = F(\mathbf{x}; \cdot)$ .

Pour traduire cela de manière plus mathématique, il est utile de passer au domaine fréquentiel exprimé au moyen de la transformée de Fourier-Laplace. La transformée de Fourier-Laplace  $\hat{u}$  d'une fonction  $u$  est la fonction définie dans la moitié supérieure du plan complexe  $\mathbb{C}^+ := \omega \in \mathbb{C}, \text{Im}(\omega) > 0$  et définie par la formule intégrale suivante :



$$\widehat{u}(\omega) = \frac{1}{\sqrt{2, \pi}}, \int_0^{+\infty} u(t), e^{i, \omega, t} dt. \quad (0.0.10)$$

Les matériaux électromagnétiques dispersifs (isotropes) sont le plus souvent définis comme des matériaux dans lesquels les lois de proportionnalité de la forme (0.0.9) sont vérifiées dans le domaine fréquentiel, c'est-à-dire qu'elles sont satisfaites par les transformées de Fourier-Laplace des champs, plutôt que par les champs eux-mêmes. Dans ce cas, il n'y a aucune raison d'exiger que  $\varepsilon$  et  $\mu$  soient réels et indépendants de la fréquence. C'est pourquoi un milieu isotrope dispersif sera défini comme obéissant à des lois constitutives de la forme :

$$\widehat{\mathbf{D}}(\mathbf{x}, \omega) = \varepsilon(\mathbf{x}, \omega), \widehat{\mathbf{E}}(\mathbf{x}, \omega), \quad \widehat{\mathbf{B}}(\mathbf{x}, \omega) = \mu(\mathbf{x}, \omega), \widehat{\mathbf{H}}(\mathbf{x}, \omega), \quad (0.0.11)$$

où pour chaque  $\mathbf{x} \in \mathbb{R}^3$ ,  $\omega \in \mathbb{C}^+$ ,  $\omega \in \mathbb{C}^+ \mapsto \varepsilon(\mathbf{x}, \omega)$  (la permittivité) et  $\omega \in \mathbb{C}^+ \mapsto \mu(\mathbf{x}, \omega)$  (la perméabilité) sont des fonctions non triviales de la fréquence qui décrivent la dispersivité du milieu. Pour les matériaux non dispersifs, ces fonctions sont réelles, positives et constantes, c'est-à-dire que (0.0.9) est vérifié. Bien sûr, ces fonctions satisfont certaines propriétés particulières imposées par des raisons physiques ou mathématiques :

- **Principe de causalité.** Pour assurer la causalité de  $\mathbf{D}(\mathbf{x}, t)$  (ou  $\mathbf{B}(\mathbf{x}, t)$ ) à condition que  $\mathbf{E}(\mathbf{x}, t)$  (ou  $\mathbf{H}(\mathbf{x}, t)$ ) soit causale, il est naturel d'imposer

$$\forall \mathbf{x} \in \mathbb{R}^3, \quad \omega \mapsto \varepsilon(\mathbf{x}, \omega) \quad \text{et} \quad \omega \mapsto \mu(\mathbf{x}, \omega) \quad \text{sont analytiques dans } \mathbb{C}^+. \quad (0.0.12)$$

- **Principe de réalité.** Une deuxième exigence est que si  $\mathbf{D}(\mathbf{x}, t)$  (ou  $\mathbf{B}(\mathbf{x}, t)$ ) est réel, alors  $\mathbf{E}(\mathbf{x}, t)$  (ou  $\mathbf{H}(\mathbf{x}, t)$ ) est également réel. Selon (0.0.11), l'invariance de  $\mathbb{C}^+$  sous l'application  $\omega \mapsto -\bar{\omega}$  et les propriétés de Fourier-Laplace des fonctions réelles, le principe de réalité se traduit par la condition

$$\forall \mathbf{x} \in \mathbb{R}^3 \quad \forall \omega \in \mathbb{C}^+, \quad \varepsilon(\mathbf{x}, -\bar{\omega}) = \overline{\varepsilon(\mathbf{x}, \omega)} \quad \text{et} \quad \mu(\mathbf{x}, -\bar{\omega}) = \overline{\mu(\mathbf{x}, \omega)}. \quad (0.0.13)$$

- **Principe de haute fréquence.** Une propriété fondamentale du point de vue physique est que, à haute fréquence, tout matériau "se comporte comme le vide". Mathématiquement, cela revient à exiger que

$$\forall \mathbf{x} \in \mathbb{R}^3 \quad \forall \eta > 0, \quad \lim_{\text{Im}(\omega) \geq \eta \quad |\omega| \rightarrow +\infty} \varepsilon(\mathbf{x}, \omega) = \varepsilon_0, \quad \lim_{\text{Im}(\omega) \geq \eta \quad |\omega| \rightarrow +\infty} \mu(\mathbf{x}, \omega) = \mu_0. \quad (0.0.14)$$

Cela signifie que le matériau est "de moins en moins dispersif" à haute fréquence.

Le travail de thèse suivant se compose de deux parties indépendantes liées à la propagation des ondes dans des matériaux dispersifs, avec un point de départ dans le régime temporel, c'est-à-dire, en partant d'une approche dite stationnaire où nous reconstruisons une onde transitoire comme une superposition d'ondes périodiques. La première

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partie du travail de thèse consiste en l'analyse du comportement à long terme de la solution des équations de Maxwell dans des matériaux de Lorentz généralisés dissipatifs, cela représente le contenu du [Chapter 1](#) et du [Chapter 2](#), qui développent deux approches distinctes pour le traitement du sujet. En revanche, la deuxième partie est consacrée à l'étude des équations de Maxwell dans le régime temporel, délimitée dans une lame de métamatériau immergée dans un matériau diélectrique (par exemple le vide). Cette seconde partie comprend [Chapter 3](#), [Chapter 4](#) et [Chapter 5](#).

## Partie I Comportement à long terme de la solution des équations de Maxwell dans les matériaux de Lorentz généralisés dissipatifs

### Contexte

L'analyse des taux de décroissance de l'énergie des solutions pour les modèles dispersifs et dissipatifs de la propagation des ondes linéaires a été étudiée pendant de longues périodes dans la littérature. Cette recherche a été principalement motivée par des applications en visco-élasticité [[22](#), [26](#), [35](#)] et en électromagnétisme [[36](#), [61](#)], mais elle a récemment suscité un regain d'intérêt en raison de sa pertinence pour les métamatériaux, comme le montrent par exemple [[17](#), [37](#), [63](#), [65](#), [65](#)], où une construction appropriée de modèles mathématiques congruents avec des principes motivés par la physique tels que les principes de causalité et de passivité est présentée (voir également [[19](#), [39](#), [92](#)]).

### Contenu de la Partie I

Dans le [Chapter 1](#), nous développons pour les milieux Drude-Lorentz dissipatifs généralisés une technique de Lyapunov pour prouver la stabilité polynomiale de l'énergie électromagnétique. En termes plus simples, l'idée est de dériver une inégalité différentielle via un bilan énergétique issu des équations de Maxwell, ainsi, par des inégalités de type Gronwall dans le domaine spatial de Fourier et l'isométrie de Plancherel, nous parvenons au résultat souhaité. Il convient de mentionner que d'autres résultats de stabilité pour des modèles dispersifs/dissipatifs ont également été obtenus via la technique de Lyapunov (voir [[35](#), [36](#), [61](#)]), cependant, aucune de ces techniques ne s'applique dans notre contexte étant donné le manque de propriétés de signe supposées pour les noyaux de convolution impliqués dans les effets de mémoire de notre modèle.

Dans le [Chapter 2](#), nous traitons le même problème que dans le [Chapter 1](#) en utilisant cette fois-ci l'approche de la décomposition modale. Cette approche est techniquement plus complexe que celle du premier chapitre, mais elle présente plusieurs avantages significatifs. Premièrement, elle est plus physique à bien des égards, car elle se réfère explicitement à la relation de dispersion du milieu via une décomposition en plans d'onde, ce qui facilite la compréhension pour les physiciens. Deuxièmement, elle conduit à des résultats optimaux, car elle nous permet d'obtenir à la fois des bornes supérieures et inférieures pour la stabilité du système, ce qui n'a pas pu être atteint par la première approche présentée au moyen d'un bilan énergétique. Troisièmement, elle permet de retrouver les mêmes résultats de stabilité polynomiale sous les mêmes hypothèses de dissipation que le [Chapter 1](#); de plus, nous parvenons à obtenir des résultats de stabilité polynomiale sous des conditions plus faibles sur les coefficients de dissipation, en particulier, dans certaines configurations, on obtient des taux de décroissance polynomiale plus faibles que ceux observés dans le [Chapter 1](#).

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## Partie II Guides d'ondes d'une plaque de métamatériau

### Contexte

La plupart des matériaux trouvés dans la nature ont des paramètres physiques positifs pour la plupart des fréquences. Cependant, il est bien connu depuis le siècle dernier que les métaux (comme l'or) ont une permittivité électrique négative pour une large gamme de fréquences. Le modèle de Drude est un modèle simple qui peut être utilisé pour décrire le comportement des métaux dans cette plage. Dans le modèle de Drude, on suppose que les électrons dans un métal sont libres de se déplacer. Cela conduit à une permittivité négative pour les fréquences en dessous de la fréquence de plasma du métal. En effet, en appliquant des principes dynamiques fondamentaux à un électron et en liant son déplacement à la polarisation et au champ électrique, on montre la très bonne approximation

$$\varepsilon(\omega) = \varepsilon_0, \left(1 - \frac{\Omega_e^2}{\omega^2}\right),$$

où  $\varepsilon_0$  représente la permittivité dans le vide et  $\Omega_e$  la pulsation de plasma du métal.

La permittivité négative des métaux a plusieurs propriétés intéressantes. Par exemple, elle permet aux plasmons de surface de se propager à l'interface entre un diélectrique et un métal. Les plasmons de surface sont des ondes qui sont confinées à l'interface. Ils ont été utilisés dans une variété d'applications en micro et nanotechnologies, telles que la détection optique et l'imagerie.

Cette deuxième partie du travail de thèse se place et tente de résoudre l'une des perspectives du travail précédent réalisé par Maxence Cassier, Christophe Hazard et Patrick Joly [14–16] dans l'étude de la propagation des ondes dans un ensemble de deux demi-espaces, l'un rempli d'un diélectrique comme le vide et l'autre d'un matériau dispersif, le matériau de Drude. Dans ce travail, nous avons l'intention d'étudier la propagation des ondes en considérant une nouvelle géométrie séparable, à savoir, la lentille plate : une plaque de matériau dispersif dans un diélectrique.

### Contenu de la Partie II

La deuxième partie est composée des [Chapter 3](#) et [Chapter 4](#) et s'inscrit dans la continuité des travaux de Bonnet-Joly, Dermenjian-Guillot, Ricardo Weder et Calvin Hayden Wilcox pour l'étude de la propagation des ondes dans les milieux stratifiés [7, 32, 90, 93], de Maxence Cassier, Christophe Hazard et Patrick Joly pour leurs recherches sur un milieu bicouche impliquant un matériau dispersif modélisé par les lois constitutives de Drude [14–16]. Cette étude est également développée dans le cadre original du modèle de Drude.

Le [Chapter 3](#) traite des caractéristiques générales de la deuxième partie de la thèse. Dans la [Section 3.1](#), le cadre physique et mathématique du problème lié à cette étude est présenté. À cette fin, nous commençons par les équations de Maxwell dans le domaine temporel présentant la modélisation du problème dans le cas du matériau de Drude, particulièrement riche mais suffisamment précis. Ensuite, une décomposition des solutions de notre problème en modes transverses électriques (TE) et en modes transverses magnétiques (TM) est donnée. Une fois choisi de travailler de manière arbitraire sur le mode TM, le problème original est simplifié et reformulé sous la forme d'une équation de Schrödinger abstraite impliquant un opérateur non borné  $\mathbb{A}$  défini dans un espace de Hilbert approprié des fonctions d'énergie finie. Dans la [Section 3.2](#), la symétrie de notre problème nous amène à considérer la transformée de Fourier pour obtenir les Hamiltoniens

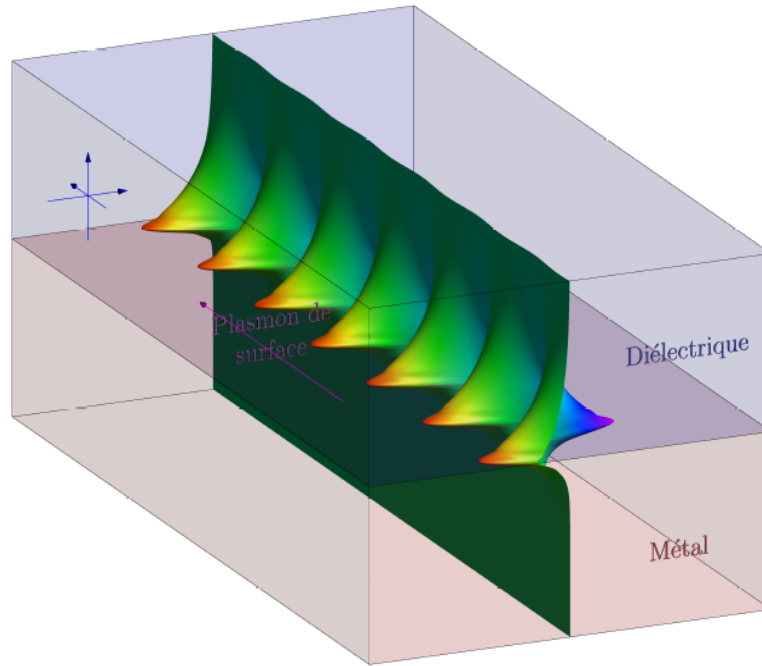


Figure 8: Représentation schématique d'un plasmon de surface.

réduits  $\mathbb{A}_k$ , où  $k \in \mathbb{R}$  est considéré comme la longueur d'onde des solutions impliquées. Ensuite, dans la [Section 3.3](#), la théorie spectrale de l'Hamiltonien réduit est évoquée, le concept de décomposition modale est présenté et la *relation de dispersion* (relation entre la fréquence  $\omega$  et la longueur d'onde  $k$ ) est dérivée.

Le [Chapter 4](#) constitue l'essence de cette deuxième partie de la thèse, où nous trouvons l'ensemble des solutions pour la *relation de dispersion* et confirmons la présence d'ondes plasmoniques. Dans la [Section 4.1](#), nous transformons la *relation de dispersion* en une *relation de dispersion sans dimension* équivalente que nous décomposons en une collection dénombrable de *systèmes de dispersion sans dimension* ( $\mathcal{DS}_n$ ). Ensuite, dans la [Section 4.2](#), nous développons une technique générale abstraite pour la résolution des systèmes ( $\mathcal{DS}_n$ ). Enfin, nous montrons que chaque système ( $\mathcal{DS}_n$ ) définit une courbe de solution (*courbes de dispersion*) nommée  $\mathcal{C}_n$ , dans les [Section 4.3](#), [Section 4.4](#) et [Section 4.5](#), nous étudions les propriétés analytiques de  $\mathcal{C}_n$  divisées, respectivement, pour les cas  $n \geq 2$ ,  $n = 1$  et  $n = 0$ .

Le [Chapter 5](#) présente les comparaisons de notre étude dans cette deuxième partie de la thèse par rapport aux études précédentes : 1) le cas d'un matériau en plaque non dispersif, traité dans ce que nous appelons nous-mêmes le *cas classique* représentant l'étude de la propagation des ondes dans les milieux stratifiés (voir [[29–31](#), [44](#), [88–90](#), [93](#), [94](#)] pour le cas des ondes acoustiques, [[42](#), [90](#)] pour les ondes électromagnétiques et [[32](#), [43](#), [81](#)] dans le domaine de l'élasticité); 2) le cas de deux milieux en couches contenant un matériau dispersif de Drude [[14–16](#)].

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# PART I

MAXWELL'S EQUATIONS IN METAMATERIALS:  
LONG TIME BEHAVIOR OF SOLUTIONS

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## CHAPTER 1

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# Long time behaviour of the solution of Maxwell's equations in dissipative generalized Lorentz materials (I) A frequency dependent Lyapunov function approach

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It is well-known that electromagnetic dispersive structures such as metamaterials can be modelled by generalized Drude-Lorentz models. The present paper is the first of two articles dedicated to dissipative generalized Drude-Lorentz open structures. We wish to quantify the loss in such media in terms of the long time decay rate of the electromagnetic energy for the corresponding Cauchy problem. By using an approach based on frequency dependent Lyapunov estimates, we show that this decay is polynomial in time. These results extend to an unbounded structure the ones obtained for bounded media in [65] via a quite different method based on the notion of cumulated past history and semi-group theory. A great advantage of the approach developed here is to be less abstract and directly connected to the physics of the system via energy balances.

## 1.1 Introduction and motivation

The study of the long time behaviour of solutions of dispersive and dissipative models for linear wave propagation has already been extensively studied in the literature, primarily for applications in visco-elasticity and more recently in electromagnetism. The subject has recently known a regain of interest related to metamaterials. We can refer for instance, in electromagnetism, to the article [17] in which we presented a systematic construction of mathematical models compatible with physically motivated principle such as causality and passivity (see also [19, 39, 92]). The common point to all these models is that the constitutive laws include memory effects corresponding to time convolution nonlocal operators that induce dispersion (the velocity of waves is frequency dependent) and dissipation (the energy decay of the solution) that are in often intimately related.

For such models one of the most natural question is the study of the long time behaviour of the corresponding Cauchy problem: prove that the energy of the solution tends to 0 when  $t$  tends to  $+\infty$  and study the rate of decay. This is of course closely related to the theory of control and stabilization of dynamical systems where one commonly distinguishes the notion of exponential stability (which corresponds to an exponential decay of the energy) and polynomial stability (the energy decays as the inverse of a positive power of  $t$ ).

### 1.1.1 Maxwell's equations in dispersive media

#### 1.1.1.1 General features

As said previously in the introduction, Maxwell's equations relate the electric and magnetic inductions  $\mathbf{D}(\mathbf{x}, t)$  and  $\mathbf{B}(\mathbf{x}, t)$  ( $\mathbf{x} \in \mathbb{R}^3$  and  $t > 0$  are respectively the space and time variables) to the electric and magnetic fields  $\mathbf{E}(\mathbf{x}, t)$  and  $\mathbf{H}(\mathbf{x}, t)$ :

$$\begin{cases} \partial_t \mathbf{D} - \nabla \times \mathbf{H} = 0, \\ \partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0. \end{cases} \quad (1.1.1)$$

On the other hand, one defines the electric polarization and magnetization by

$$\begin{cases} \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}_{\text{tot}}, & \mathbf{P}_{\text{tot}} : \text{electric polarization}, \\ \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}_{\text{tot}}, & \mathbf{M}_{\text{tot}} : \text{magnetization}. \end{cases} \quad (1.1.2)$$

where  $\varepsilon_0 > 0$  and  $\mu_0 > 0$  are the vacuum permittivity and permeability. The above equations are completed by the following non local constitutive laws (we consider the case of a homogeneous medium)

$$\begin{cases} \mathbf{P}_{\text{tot}}(\cdot, t) = \varepsilon_0 \int_0^t \chi_e(t-s) \mathbf{E}(\cdot, s) ds, \\ \mathbf{M}_{\text{tot}}(\cdot, t) = \mu_0 \int_0^t \chi_m(t-s) \mathbf{H}(\cdot, s) ds, \end{cases} \quad (1.1.3)$$

where  $\chi_e$  and  $\chi_m$  are the electrical and magnetic susceptibilities of the material (convolutions products being understood in the distributional sense, see for e.g. [20, 95], for  $(\chi_e, \chi_m)$  not in  $L^1_{loc}(\mathbb{R}^+)$ ).

In the Fourier-Laplace domain

$$\mathbf{E}(\cdot, t) \longrightarrow \hat{\mathbf{E}}(\cdot, \omega) = \int_0^{+\infty} \mathbf{E}(\cdot, t) e^{i\omega t} dt, \quad \text{Im } \omega > 0,$$

(1.1.2) and (1.1.3) reduce to

$$\begin{cases} \hat{\mathbf{D}}(\cdot, \omega) = \varepsilon(\omega) \hat{\mathbf{E}}(\cdot, \omega), \\ \hat{\mathbf{B}}(\cdot, \omega) = \mu(\omega) \hat{\mathbf{H}}(\cdot, \omega), \end{cases} \quad (1.1.4)$$

where the complex permittivity  $\varepsilon(\omega)$  and the complex permeability  $\mu(\omega)$  are given in terms of the Fourier-Laplace transform of the susceptibility functions:

$$\varepsilon(\omega) = \varepsilon_0 (1 + \hat{\chi}_e(\omega)) \quad \text{and} \quad \mu(\omega) = \mu_0 (1 + \hat{\chi}_m(\omega)). \quad (1.1.5)$$

where  $\varepsilon(\omega) \rightarrow \varepsilon_0$  and  $\mu(\omega) \rightarrow \mu_0$  when  $\omega \rightarrow \infty$  in  $\mathbb{C}^+ := \{\omega \in \mathbb{C} / \text{Im } \omega > 0\}$ . In other words, the material behaves as the vacuum at high frequencies. In the frequency domain, passivity, causality and the high frequency behaviour are traduced by the fact that (see [4, 19, 92, 95] for more details)

$$\omega \mapsto \omega \varepsilon(\omega) \quad \text{and} \quad \omega \mapsto \omega \mu(\omega) \quad \text{are } \textit{Herglotz functions}, \quad (1.1.6)$$

that is to say analytic functions from  $\mathbb{C}^+$  into its closure  $\overline{\mathbb{C}^+}$ . Furthermore as the susceptibilities  $\chi_e$  and  $\chi_m$  are real-valued functions in the time domain, the permittivity and permeability satisfy  $\varepsilon(\omega) = \varepsilon(-\bar{\omega})$  and  $\mu(\omega) = \mu(-\bar{\omega})$ ,  $\forall \omega \in \mathbb{C}^+$ .

**Remark 1.1.1.** [About the notion of passivity] The condition (1.1.6) is the condition which is most often used to define passive materials: we called it *mathematical passivity* in [17]. In the same article, we define the related notion *physical passivity* which is associated to the Cauchy problem associated to (1.1.1, 1.1.2, 1.1.3), seen as an evolution problem with respect to the electromagnetic field  $(\mathbf{E}, \mathbf{H})$ . In other words, we look at the free evolution of the system i.e in the absence of external sources. More precisely a material is *physical passive* if and only if the electromagnetic energy

$$\mathcal{E}(t) \equiv \mathcal{E}(\mathbf{E}, \mathbf{H}, t) := \frac{1}{2} \left( \varepsilon_0 \int_{\mathbb{R}^3} |\mathbf{E}(\mathbf{x}, t)|^2 d\mathbf{x} + \mu_0 \int_{\mathbb{R}^3} |\mathbf{H}(\mathbf{x}, t)|^2 d\mathbf{x} \right). \quad (1.1.7)$$

can never exceeds its value at  $t = 0$ , that is to say

$$\forall t \geq 0, \quad \mathcal{E}(t) \leq \mathcal{E}(0). \quad (1.1.8)$$

It is emphasized in [17] that the above property does not mean that the electromagnetic energy is a decreasing function of time.

### 1.1.1.2 The generalized Lorentz media

In this paper, we shall focus on the most well-known subclass of models: the (dissipative) generalized Lorentz media (see e.g. [2, 17, 48, 52, 53]). Such model will be called *local* because of the relationship between  $\mathbf{D}$  and  $\mathbf{E}$  or  $\mathbf{B}$  and  $\mathbf{H}$  can be written with ordinary differential equations. More precisely, these correspond to

$$\mathbf{P}_{\text{tot}} = \varepsilon_0 \sum_{j=1}^{N_e} \Omega_{e,j}^2 \mathbf{P}_j, \quad \mathbf{M}_{\text{tot}} = \mu_0 \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 \mathbf{M}_\ell, \quad (1.1.9)$$



where each  $\mathbf{P}_j$  (resp. each  $\mathbf{M}_\ell$ ) is related to  $\mathbf{E}$  (resp.  $\mathbf{H}$ ) by an ordinary differential equation

$$\begin{cases} \partial_t^2 \mathbf{P}_j + \alpha_{e,j} \partial_t \mathbf{P}_j + \omega_{e,j}^2 \mathbf{P}_j = \mathbf{E}, & 1 \leq j \leq N_e, \\ \partial_t^2 \mathbf{M}_\ell + \alpha_{m,\ell} \partial_t \mathbf{M}_\ell + \omega_{m,\ell}^2 \mathbf{M}_\ell = \mathbf{H}, & 1 \leq \ell \leq N_m, \end{cases} \quad (1.1.10)$$

completed by initial conditions centered at 0

$$\begin{cases} \mathbf{P}_j(\cdot, 0) = \partial_t \mathbf{P}_j(\cdot, 0) = 0, & 1 \leq j \leq N_e, \\ \mathbf{M}_\ell(\cdot, 0) = \partial_t \mathbf{M}_\ell(\cdot, 0) = 0, & 1 \leq \ell \leq N_m. \end{cases} \quad (1.1.11)$$

In the above equations, the (real) coefficients  $\Omega_{e,j}, \Omega_{m,\ell}, \omega_{e,j}, \omega_{m,\ell}$  are supposed to satisfy

$$\Omega_{e,j} > 0, \quad \omega_{e,j} \geq 0, \quad 1 \leq j \leq N_e, \quad \Omega_{m,\ell} > 0, \quad \omega_{m,\ell} \geq 0, \quad 1 \leq \ell \leq N_m, \quad (1.1.12)$$

while for stability/dissipation issues the coefficients  $(\alpha_{m,\ell}, \alpha_{m,\ell})$  must non-negative

$$\alpha_{e,j} \geq 0, \quad 1 \leq j \leq N_e, \quad \alpha_{m,\ell} \geq 0 \quad 1 \leq \ell \leq N_m. \quad (1.1.13)$$

Moreover, the reader will easily check that one can assume without any loss of generality that the couples  $(\alpha_{e,j}, \omega_{e,j})$  (resp.  $(\alpha_{m,\ell}, \omega_{m,\ell})$ ) are all distinct the ones from the others.

Note that (1.1.9, 1.1.10) corresponds to

$$(a) \quad \varepsilon(\omega) = \varepsilon_0 \left( 1 - \sum_{j=1}^{N_e} \frac{\Omega_{e,j}^2}{\omega^2 + i \alpha_{e,j} \omega - \omega_{e,j}^2} \right), \quad (b) \quad \mu(\omega) = \mu_0 \left( 1 - \sum_{\ell=1}^{N_m} \frac{\Omega_{m,\ell}^2}{\omega^2 + i \alpha_{m,\ell} \omega - \omega_{m,\ell}^2} \right). \quad (1.1.14)$$

Straightforward calculations show that (1.1.9, 1.1.10) are equivalent to (1.1.3) with

$$\chi_e = \sum_{j=1}^{N_e} \Omega_{e,j}^2 \chi_{e,j}, \quad \chi_m = \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 \chi_{m,\ell} \quad (1.1.15)$$

where the expression of each  $\chi_{\nu,j}$  for  $\nu = e, m$  and  $j \in \{1, \dots, N_\nu\}$  is given by

$$\begin{aligned} (i) \quad & \chi_{\nu,j}(t) = 2 \delta_{\nu,j}^{-1} \sinh(\delta_{\nu,j} t/2) e^{-(\alpha_{\nu,j} t/2)}, \quad \text{if } \alpha_{\nu,j} > 2\omega_{\nu,j}, \\ (ii) \quad & \chi_{\nu,j}(t) = 2 \delta_{\nu,j}^{-1} \sin(\delta_{\nu,j} t/2) e^{-(\alpha_{\nu,j} t/2)}, \quad \text{if } \alpha_{\nu,j} < 2\omega_{\nu,j}, \\ (iii) \quad & \chi_{\nu,j}(t) = t e^{-(\alpha_{\nu,j} t/2)}, \quad \text{if } \alpha_{\nu,j} = 2\omega_{\nu,j}, \end{aligned} \quad (1.1.16)$$

where we have set  $\delta_{\nu,j} = \sqrt{\alpha_{\nu,j}^2 - 4\omega_{\nu,j}^2}$  if  $\alpha_{\nu,j} \geq 2\omega_{\nu,j}$  and  $\delta_{\nu,j} = \sqrt{4\omega_{\nu,j}^2 - \alpha_{\nu,j}^2}$  if  $\alpha_{\nu,j} < 2\omega_{\nu,j}$ .

Note that each kernel  $\chi_{\nu,j}$  is not monotonous with respect to time as soon as  $\omega_{\nu,j} > 0$  or  $\alpha_{\nu,j} > 0$  and tends to 0 when  $t \rightarrow +\infty$  if (and only if)  $\alpha_{\nu,j} > 0$  (see figure [Figure 1.1](#), first two pictures). As a consequence  $\chi_\nu$  does not tend to 0 at infinity as soon as one of the  $\alpha_{\nu,j}$  vanishes (see figure [Figure 1.1](#), third picture).

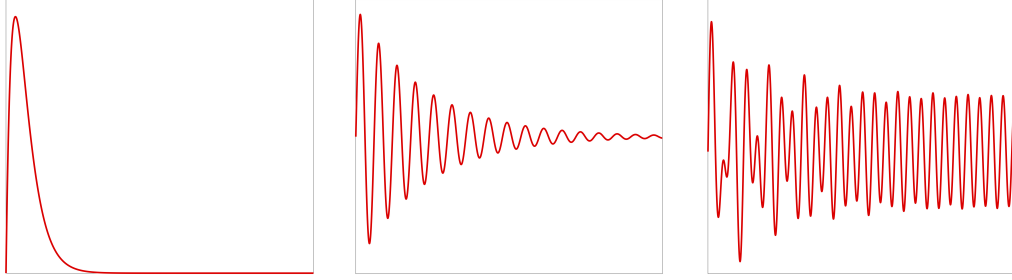


Figure 1.1: Kernels as functions of time. Left:  $\chi_{\nu,j}$  for  $\alpha_{\nu,j} > 2\omega_{\nu,j} > 0$ . Center:  $\chi_{\nu,j}$  for  $0 < \alpha_{\nu,j} < 2\omega_{\nu,j}$ . Right:  $\chi_\nu$  for  $N_\nu = 2, \alpha_{\nu,1} = 0, \alpha_{\nu,2} > 0$ .

### 1.1.2 A brief review of the literature

As said in introduction, there are already many existing results on the long time behaviour of the solution of dissipative dispersive systems. In this paragraph, we discuss in some detail some of the most significant contributions that are in close connection with the present work.

In the article [37], the authors considered a very abstract evolution model that in particular includes (1.1.3) with  $\chi_m = 0$  and a function  $\chi_e$  for which  $\omega \hat{\chi}_e(\omega)$  is a Herglotz function. If one assumes that this function satisfies the additional assumption that

$$\text{for a. e. } \omega \in \mathbb{R}, \quad \gamma_e(\omega) := \lim_{\zeta \in \mathbb{C}^+ \rightarrow \omega} \text{Im}(\zeta \hat{\chi}_e(\zeta)) \text{ exists,} \quad (1.1.17)$$

(which is satisfied by most of dispersive materials in physics and in particular by generalized Lorentz materials) then the sufficient dissipation condition (6.4) of [37] is equivalent to

$$\text{for a. e. } \omega \in \mathbb{R}, \quad \gamma_e(\omega) > 0, \quad \gamma_e^{-1} \in L^1_{loc}(\mathbb{R}). \quad (1.1.18)$$

When applied to the generalized Lorentz model (see Appendix A.1), namely when  $\varepsilon(\omega)$  is given by (1.1.14) the above condition corresponds to

$$\exists 1 \leq j \leq N_e \text{ such that } \omega_{e,j} = 0 \text{ and } \alpha_{e,j} > 0. \quad (1.1.19)$$

Under this condition, it is proven that the electromagnetic energy (see the formula (1.1.7)) tends to 0 for any initial data  $(\mathbf{E}_0, \mathbf{H}_0)$  in  $L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3$ :

$$\forall (\mathbf{E}_0, \mathbf{H}_0) \in L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3, \quad \lim_{t \rightarrow +\infty} \mathcal{E}(t) = 0. \quad (1.1.20)$$

This result is proven in [17] (section 4.4) in a much more pedestrian way on a toy problem corresponding the Drude model with  $N_e = N_m = 1, \omega_{e,1} = \omega_{m,1} = 0$  and  $\alpha_{e,1}, \alpha_{e,m} > 0$ .

In the above references, the question of the rate of convergence to 0 of the electromagnetic energy is not discussed. This question is addressed in a series of work by S. Nicaise and her collaborator C. Pignotti [63], [65] (which generalizes [63]) (see also [64] for local dissipation models). These works consider the initial value problem in a bounded domain  $\Omega \subset \mathbb{R}^3$  with perfectly conducting boundary conditions for which they prove *polynomial stability* in the sense mentioned below (see estimate (1.1.25)). The conditions for polynomial stability in [65] are two fold:

- (i) The first condition is expressed in the time domain, more precisely in terms of regularity

and decay properties of the kernels  $\chi_e$  and  $\chi_m$ :

$$\chi = \chi_e \text{ or } \chi_m \text{ satisfies } \chi \in C^2(\mathbb{R}^+), \quad \lim_{t \rightarrow +\infty} \chi'(t) = 0, \quad |\chi''(t)| \leq C e^{-\delta t} \text{ (with } C, \delta > 0\text{).} \quad (1.1.21)$$

Note that (1.1.21) implies that it exists  $C_1 > 0$  such that  $|\chi'(t)| \leq C_1 \delta^{-1} e^{-\delta t}$  and  $|\chi(t)| \leq C_1$  for  $t \geq 0$  (see e.g. the appendix of [64] for the details). Hence, it follows (by integrations by parts of the Fourier-Laplace integral) that

$$\omega \widehat{\chi}(\omega) = i \widehat{\chi}'(\omega) + i \chi(0) \quad \text{and} \quad \omega \widehat{\chi}(\omega) = -(\widehat{\chi}''(\omega) + \chi'(0)) \omega^{-1} + i \chi(0), \quad \forall \omega \in \mathbb{C}^+ \quad (1.1.22)$$

where, as the Laplace-Fourier transform of a  $L^1$  causal function,  $\omega \mapsto \widehat{\chi}'(\omega)$  and  $\omega \mapsto \widehat{\chi}''(\omega)$  are analytic on  $\mathbb{C}^+$ , continuous on  $\overline{\mathbb{C}^+}$  and decay to 0 when  $\omega \rightarrow \infty$  in  $\overline{\mathbb{C}^+}$ . Thus, using (1.1.22), one observes that (1.1.21) implies that the functions  $\omega \mapsto \omega \widehat{\chi}_e(\omega)$  and  $\omega \mapsto \omega \widehat{\chi}_m(\omega)$  are analytic on  $\mathbb{C}^+$  and can be extended as continuous and bounded functions in the closed upper half-plane  $\overline{\mathbb{C}^+}$ . Furthermore,  $\widehat{\chi} = \widehat{\chi}_e, \widehat{\chi}_m$ , one has  $\omega \widehat{\chi}(\omega) = i\chi(0) - \chi'(0)\omega^{-1} + o(\omega^{-1})$  when  $\omega \rightarrow \infty$  in  $\overline{\mathbb{C}^+}$ .

(ii) The second condition is expressed in the frequency domain for real frequencies. It also has two parts. The first one is a strict positivity condition

$$\forall \omega \in \mathbb{R}^*, \quad \gamma_e(\omega) := \text{Im } \omega \widehat{\chi}_e(\omega) > 0, \quad \gamma_m(\omega) := \text{Im } \omega \widehat{\chi}_m(\omega) > 0, \quad (1.1.23)$$

which is completed by the additional assumption

$$\exists \omega_0, p, C > 0 \text{ such that: } |\omega| \geq \omega_0 \implies \gamma_e(\omega) \geq C |\omega|^{-p}, \quad \gamma_m(\omega) \geq C \omega^{-p}, \quad (1.1.24)$$

that means the the (strictly positive) functions  $\gamma_e(\omega)$  and  $\gamma_m(\omega)$  do not decay too fast at infinity.

Under assumptions (1.1.21), (1.1.23) and (1.1.24), the authors of [65] prove, for  $H^1$  initial data, decay estimates of the form

$$\mathcal{E}(t) \leq C_p t^{-\frac{2}{p}} (\|\mathbf{E}_0\|_{H^1(\Omega)}^2 + \|\mathbf{H}_0\|_{H^1(\Omega)}^2). \quad (1.1.25)$$

When specified to the case of the Lorentz kernels, it is easy to see (with (1.1.15) and (1.1.16)) that, in addition to the non-negativity of the coefficients  $(\alpha_{e,j}, \alpha_{m,\ell})$ , the conditions (1.1.21), (1.1.23) and (1.1.24) (which is then satisfied for  $p = 2$ ) correspond to the following strong dissipation condition.

**Definition 1.1.2** (Strong Dissipation for Lorentz models).

$$\forall 1 \leq j \leq N_e, \quad \alpha_{e,j} > 0, \quad \forall 1 \leq \ell \leq N_m, \quad \alpha_{m,\ell} > 0. \quad (1.1.26)$$

It is worthwhile mentioning briefly the techniques of proof for (1.1.20) in [37] and (1.1.25) in [65].

The technique used in [37] is difficult to describe in a few lines but we can give some of the main ideas. The authors use an augmented formulation of the evolution problem where, typically, the convolutions (1.1.3) are hidden behind the introduction of new unknowns. In the very abstract framework of [37], these new unknowns can be seen as elements of an adequately constructed Hilbert space. In the case of the Maxwell's equations, these new unknowns are functions of  $(\mathbf{x}, t)$  but also of additional variable  $\xi$  that varies in  $\mathbb{R}$  (or a subset of  $\mathbb{R}$ ), see [37] and also [17,39]. The fundamental property of the obtained "augmented" system is that it is conservative: in other

words, it is an evolution problem associated with a self-adjoint operator  $\mathcal{A}_c$  in the augmented Hilbert space:

$$\frac{d\mathcal{U}_c}{dt} + i\mathcal{A}_c\mathcal{U}_c = 0.$$

This allows to use tools from spectral theory of self-adjoint operators and to obtain an adequate (Fourier-like) integral representation of the solution of the original problem. The convergence result (1.1.20) then appears as a consequence of the spectral theorem and the use of Riemann-Lebesgue theorem on compact sets, that is justified by the assumption (1.1.18).

Although quite different, the approach of [65] also starts from another augmented formulation of the original system in which the convolutions (1.1.3) are again replaced by additional purely differential equations. The construction of this augmented model relies of the very nice trick of Dafermos [25] for treating viscoelasticity. This implies to introduce an additional time variable  $s$  (that plays a similar role than  $\xi$  in [17]) and additional unknowns: the so called cumulated past histories of the fields  $\mathbf{E}$  and  $\mathbf{H}$ . The convolution operators are then replaced by non homogeneous transport equations in the  $(s, t)$  plane. Contrary to [37], the augmented system is not conservative and is written as an autonomous evolution problem involving an operator  $\mathcal{A}_a$  which is not self-adjoint but maximal dissipative:

$$\frac{d\mathcal{U}_a}{dt} + i\mathcal{A}_a\mathcal{U}_a = 0.$$

This problem arises from the application of the theory of semi-groups. In particular, estimates (1.1.25) are obtained by the application of theorem 2.4 of [8]. Applying this theorem essentially requires to establish localization results for the spectrum of  $\mathcal{A}_a$  (inside  $\mathbb{C}^- = \{\omega \in \mathbb{C} \mid \text{Im}(\omega) < 0\}$ ) and appropriate estimates for its resolvent, using the conditions (1.1.22, 1.1.23, 1.1.24).

### 1.1.3 About Lyapunov techniques

To conclude this short bibliographical review, it is worthwhile mentioning that other stability results for dispersive/dissipative results have also been obtained via the Lyapunov technique: roughly speaking, the idea is to derive some differential inequality (in time) for a certain functional of the solution, namely the Lyapunov function  $\mathcal{L}$ , which dominates the energy (or some function of the energy). The stability estimates are then obtained from the time integration of the differential inequality.

In the context of dissipative systems with memory, this type of technique was introduced in particular to show exponential stability in the theory of linear viscoelasticity [22, 26, 35] (see also [36, 61] in the context of electromagnetism) and used in [61] where the authors establish polynomial stability estimates associated to various damping phenomena in electromagnetism to model, for instance, a rigid electric conductor or the ionized atmosphere (see also Remark 1.1.3). However their technique cannot be applied to our case because it requires sign properties of the derivatives of convolution kernels, which clearly prevents from time oscillations as in (1.1.15). To give an idea about why this kind of assumption appears, let us come back the system (1.1.1, 1.1.2, 1.1.3) in the whole space  $\mathbb{R}^3$ , provided that the kernels  $\chi_e$  and  $\chi_m$  are of class  $C^3$  on  $\mathbb{R}^+$ . Adapting the techniques of [35] to dispersive Maxwell's equations, one can show (formally) the following identity (see Appendix A.2)

$$\frac{d}{dt}\mathcal{L}(\mathbf{E}, \mathbf{H})(t) + \mathcal{D}(\mathbf{E}, \mathbf{H})(t) = 0 \tag{1.1.27}$$

where the Lyapunov function  $\mathcal{L}(\mathbf{E}, \mathbf{H})$  is a kind of augmented energy, namely

$$\mathcal{L}(\mathbf{E}, \mathbf{H})(t) = \mathcal{E}(\mathbf{E}, \mathbf{H})(t) + \mathcal{E}_{ad}(\mathbf{E}, \mathbf{H})(t) \quad (1.1.28)$$

with the additional energy

$$\left| \begin{aligned} \mathcal{E}_{ad}(\mathbf{E}, \mathbf{H})(t) = & \frac{\varepsilon_0}{2} \chi_e'(t) \int_{\mathbb{R}^3} |\mathbf{E}_p(\mathbf{x}, t)|^2 d\mathbf{x} + \frac{\mu_0}{2} \chi_m'(t) \int_{\mathbb{R}^3} |\mathbf{H}_p(\mathbf{x}, t)|^2 d\mathbf{x} \\ & - \frac{\varepsilon_0}{2} \int_0^t \chi_e''(t-s) \left( \int_{\mathbb{R}^3} |\mathbf{E}_p(\mathbf{x}, t) - \mathbf{E}_p(\mathbf{x}, s)|^2 d\mathbf{x} \right) ds \\ & - \frac{\mu_0}{2} \int_0^t \chi_m''(t-s) \left( \int_{\mathbb{R}^3} |\mathbf{H}_p(\mathbf{x}, t) - \mathbf{H}_p(\mathbf{x}, s)|^2 d\mathbf{x} \right) ds. \end{aligned} \right. \quad (1.1.29)$$

and the functional  $\mathcal{D}(\mathbf{E}, \mathbf{H})$  is given by

$$\left| \begin{aligned} \mathcal{D}(\mathbf{E}, \mathbf{H})(t) = & \varepsilon_0 \chi_e(0) \int_{\mathbb{R}^3} |\mathbf{E}(\mathbf{x}, t)|^2 d\mathbf{x} + \mu_0 \chi_m(0) \int_{\mathbb{R}^3} |\mathbf{H}(\mathbf{x}, t)|^2 d\mathbf{x} \\ & - \frac{\varepsilon_0}{2} \chi_e''(t) \int_{\mathbb{R}^3} |\mathbf{E}_p(\mathbf{x}, t)|^2 d\mathbf{x} - \frac{\mu_0}{2} \chi_m''(t) \int_{\mathbb{R}^3} |\mathbf{H}_p(\mathbf{x}, t)|^2 d\mathbf{x} \\ & + \frac{\varepsilon_0}{2} \int_0^t \chi_e'''(t-s) \left( \int_{\mathbb{R}^3} |\mathbf{E}_p(\mathbf{x}, t) - \mathbf{E}_p(\mathbf{x}, s)|^2 d\mathbf{x} \right) ds \\ & + \frac{\mu_0}{2} \int_0^t \chi_m'''(t-s) \left( \int_{\mathbb{R}^3} |\mathbf{H}_p(\mathbf{x}, t) - \mathbf{H}_p(\mathbf{x}, s)|^2 d\mathbf{x} \right) ds. \end{aligned} \right. \quad (1.1.30)$$

In the above formula (1.1.29) and (1.1.30), we have introduced the primitives (in time) of the fields  $\mathbf{F} = \mathbf{E}, \mathbf{H}$ :

$$\mathbf{F}_p(\mathbf{x}, t) = \int_0^t f(\mathbf{x}, s) ds, \quad \forall t \geq 0 \text{ and a.e. } \mathbf{x} \in \mathbb{R}^3.$$

Sufficient conditions to ensure stability estimates and time decay results simply amount to check that

$$\mathcal{L}(\mathbf{E}, \mathbf{H}) \geq \mathcal{E}(\mathbf{E}, \mathbf{H}) \quad (\iff \mathcal{E}_{ad}(\mathbf{E}, \mathbf{H}) \geq 0) \quad \text{and} \quad \mathcal{D}(\mathbf{E}, \mathbf{H}) \geq 0. \quad (1.1.31)$$

It is clear on (1.1.29) and (1.1.30) that (2.5.3) lead to the (sufficient) conditions

$$\chi_\nu(0) \geq 0, \quad \chi_\nu'(t) \geq 0, \quad \chi_\nu''(t) \leq 0 \quad \text{and} \quad \chi_\nu^{(3)}(t) \geq 0 \quad \forall t \in \mathbb{R}^+, \quad \nu = e, m. \quad (1.1.32)$$

Such conditions are fulfilled, for instance, if there exists  $\beta_\nu > 0$  such that

$$\chi_\nu(0) > 0, \quad \chi_\nu'(t) \geq 0, \quad -\chi_\nu''(t) \geq \beta_\nu \chi_\nu'(t) \quad \text{and} \quad \chi_\nu'''(t) \geq -\beta_\nu \chi_\nu''(t) \quad (1.1.33)$$

which yields the existence of a constant  $\delta > 0$  such that  $\mathcal{D}(\mathbf{E}, \mathbf{H}) \geq \delta \mathcal{L}(\mathbf{E}, \mathbf{H})$  so that immediately implies the *exponential stability* of Maxwell's equations in the sense that

$$\mathcal{E}(t) \leq C e^{-\delta t}. \quad (1.1.34)$$

An elementary example of susceptibility kernels satisfying (1.1.33) is given by  $\chi_\nu(t) = 2 - e^{-t}$ .

Unfortunately, the conditions (2.5.3) are useless for analysing the stability of Maxwell's generalized Lorentz materials given by (1.1.10) and (1.1.12), the conditions (1.1.32) are only satisfied by non-dissipative Drude materials for which  $\chi_e(t) = \Omega_e^2 t$  and  $\chi_m(t) = \Omega_m^2 t$ .

**Remark 1.1.3.** It is worthwhile to come back here to what we said in the remark 1.1.1 and more precisely on the possible equivalence (generally conjectured) between the two notions of *mathematical passivity* and *physical passivity*. What we show above is that the conditions (1.1.32) are sufficient conditions for physical passivity. As a consequence, finding functions  $\chi_e$  and  $\chi_m$  satisfying (1.1.32) but such that the Herglotz property (1.1.6) would not hold, would provide a counter example to the equivalence.

The conditions (1.1.32) are clearly reminiscent of the notion of Bernstein functions [3, 80], i.e. positive continuous function  $f : [0, \infty[ \rightarrow (0, \infty)$ ,  $C^\infty$  on  $(0, +\infty)$  and whose derivative  $f'$  is a completely monotonous functions, which means that the sign of the successive derivatives of  $f$  alternate with the order of derivation:

$$\forall j \in \mathbb{N} \setminus \{0\} \text{ and } \forall t > 0, \quad (-1)^{j+1} f^{(j)}(t) \geq 0. \quad (1.1.35)$$

Completely monotone functions are also characterized as the Laplace transforms of positive Borel measures on  $[0, +\infty)$  (see, e.g. [3, 80]). It is known (see [45], Theorem 3.2 and corollary 3.14) that Bernstein functions satisfy the Herglotz property

$$\omega \hat{f}(\omega) \text{ is an Herglotz function, where } \hat{f}(\omega) \text{ is the Laplace-Fourier transform of } f. \quad (1.1.36)$$

However, when the alternating sign property (1.1.35) is only true for  $j \leq 3$ , which corresponds to (1.1.32), (1.1.36) could a priori fail.

#### 1.1.4 Objectives and outline of our work

We revisit in this first part of the thesis (composed by the Chapter 1 and Chapter 2) the stability theory of (1.1.1, 1.1.2, 1.1.9, 1.1.10), that is to say Maxwell's equations in generalized dissipative Lorentz media. For the simplicity of exposition, we shall consider the problem posed in the whole space  $\mathbb{R}^3$  that authorizes the use of the Fourier transform in space. We have a double objective

- propose new constructive proofs of stability estimates based on elementary tools that avoids any use of "black box" results of abstract mathematical theory,
- extend the existing results with less restrictive assumptions than those appearing in [65] or [64], namely to the case of the weak dissipativity assumption defined as follows (see (1.1.37)).

**Definition 1.1.4** (Weak Dissipation for generalized Lorentz models).

$$\sum_{j=1}^{N_e} \alpha_{e,j} + \sum_{\ell=1}^{N_m} \alpha_{m,\ell} > 0. \quad (1.1.37)$$

In this first paper, we shall restrict ourselves to the strong dissipation assumption (1.1.26) and wish to recover in a quite explicit form the results from [65] with a technique inspired by the Lyapunov approach. Compared to more standard Lyapunov methods, we introduce frequency dependent Lyapunov functions (understand spatial frequency or wave numbers) that in particular allows to distinguish the respective roles of low and high frequencies (the role of low frequencies, that does not appear in a bounded domain as in [65] is due to the fact that we work in an unbounded domain).

In the second paper, we shall use, in addition to the Fourier transform, a spectral representation

of the solution that will permit us to derive sharp asymptotic long time estimates. This approach is less tricky than the frequency dependent Lyapunov approach but is technically more involved because non self-adjoint operators have to be handled. It also has the interest to only assume the weak dissipativity condition (2.1.10) and to provide optimal results. We also think that, in both papers, the arguments we shall use are quite close to physical notions (plane waves, dispersion analysis, energy balance) which should make these papers more accessible to physicists.

The outline of the present paper is as follows. For pedagogical purpose and to emphasize the main ideas that guided our computations, we first consider in section 1.2 the case of the (single) dissipative Drude model that corresponds to the particular case of (1.1.10) when  $N_e = N_m = 1$  and  $\omega_{e,1} = \omega_{e,m} = 0$  (this corresponds to the toy problem considered in the section 4.4 of [17]). In section 1.3, we shall extend the technical developments of the previous section to the generalized Lorentz models (1.1.10), emphasizing the changes to be done in order to treat this more general model. Our stability results are compared with the ones of [65]. In section 1.4, we present how to apply our method to bounded domains (section 1.4.1) and extend our results to generalized Drude-Lorentz models (section 1.4.2). Finally, the Appendix 1.4.2 gives the proofs of technical results used through the paper.

## 1.2 The case of the Drude model

In this section we are interested in studying the behaviour for long times of the solutions of the electric and magnetic fields, respectively,  $\mathbf{E}$  and  $\mathbf{H}$ , of the Drude model whose dissipative formulation is obtained by introducing the time-derivatives of the polarization term  $\mathbf{P}$  and the magnetization term  $\mathbf{M}$  (where for the Drude model  $\mathbf{P}_{\text{tot}} = \varepsilon_0 \Omega_e^2 \mathbf{P}$  and  $\mathbf{M}_{\text{tot}} = \mu_0 \Omega_m^2 \mathbf{M}$ ). The unknowns of the problem are

$$\begin{cases} \mathbf{E}(\mathbf{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3, & \mathbf{H}(\mathbf{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3, \\ \partial_t \mathbf{P}(\mathbf{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3, & \partial_t \mathbf{M}(\mathbf{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3, \end{cases}$$

and satisfy the governing equations

$$\begin{cases} \varepsilon_0 \partial_t \mathbf{E} - \nabla \times \mathbf{H} + \varepsilon_0 \Omega_e^2 \partial_t \mathbf{P} = 0, & (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_*^+, & (1.2.1a) \\ \mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} + \mu_0 \Omega_m^2 \partial_t \mathbf{M} = 0, & (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_*^+, & (1.2.1b) \\ \partial_t^2 \mathbf{P} + \alpha_e \partial_t \mathbf{P} = \mathbf{E}, & (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_*^+, & (1.2.1c) \\ \partial_t^2 \mathbf{M} + \alpha_m \partial_t \mathbf{M} = \mathbf{H}, & (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_*^+, & (1.2.1d) \end{cases}$$

completed by initial conditions

$$\begin{cases} \mathbf{E}(\cdot, 0) = \mathbf{E}_0, & \mathbf{H}(\cdot, 0) = \mathbf{H}_0, \\ \partial_t \mathbf{P}(\cdot, 0) = \partial_t \mathbf{M}(\cdot, 0) = 0. \end{cases} \quad (1.2.2)$$

In (1.2.1), the coefficients  $(\varepsilon_0, \mu_0, \Omega_e, \Omega_m)$  are strictly positive. The coefficients  $(\alpha_e, \alpha_m)$  are strictly positive damping coefficients.

Setting  $\mathcal{H} = L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3$ , Proposition A.3 in the Appendix 1.4.2 (see also remark 1.4.4) insures that for  $(\mathbf{E}_0, \mathbf{H}_0) \in L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3$ , the system admits a unique mild solution  $\mathbf{U} = (\mathbf{E}, \mathbf{H}, \partial_t \mathbf{P}, \partial_t \mathbf{M})$  in  $C^0(\mathbb{R}^+, \mathcal{H})$  which is a strong solution in  $C^1(\mathbb{R}^+, \mathcal{H})$  as soon as  $(\mathbf{E}_0, \mathbf{H}_0)$  belongs to  $H^1(\mathbb{R}^3)^3 \times H^1(\mathbb{R}^3)^3$ . The goal of what follows is to analyze their influence

on the long time behaviour (decay) of the solution. More precisely, our goal is in particular to obtain decay rates for the standard electromagnetic energy defined as following

$$\mathcal{E}(t) \equiv \mathcal{E}(\mathbf{E}, \mathbf{H}, t) := \frac{1}{2} \left( \varepsilon_0 \|\mathbf{E}(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + \mu_0 \|\mathbf{H}(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 \right). \quad (1.2.3)$$

This will be done through the following (augmented) energy

$$\mathcal{L}(t) \equiv \mathcal{L}(\mathbf{E}, \mathbf{H}, \mathbf{P}, \mathbf{M}, t) := \mathcal{E}(\mathbf{E}, \mathbf{H}, t) + \frac{1}{2} \left( \varepsilon_0 \Omega_e^2 \|\partial_t \mathbf{P}(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + \mu_0 \Omega_m^2 \|\partial_t \mathbf{M}(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 \right), \quad (1.2.4)$$

that is a decreasing function of time, according to the energy identity

$$\frac{d}{dt} \mathcal{L}(t) + \alpha_e \varepsilon_0 \Omega_e^2 \int_{\mathbb{R}^3} |\partial_t \mathbf{P}(\mathbf{x}, t)|^2 d\mathbf{x} + \mu_0 \alpha_m \Omega_m^2 \int_{\mathbb{R}^3} |\partial_t \mathbf{M}(\mathbf{x}, t)|^2 d\mathbf{x} = 0, \quad (1.2.5)$$

that is easily demonstrated by standard arguments (see [17], section 4.4.1 and remark 1.2.1).

For this purpose we shall use a Lyapunov function approach, based on the use the 3D spatial Fourier transform  $\mathcal{F}$  defined by:

$$\mathbb{G}(\mathbf{k}) = \mathcal{F}(\mathbf{G})(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \mathbf{G}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \quad \forall \mathbf{G} \in L^1(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3,$$

where we denote  $\mathbf{k} \in \mathbb{R}^3$  the dual variable of  $\mathbf{x}$  (or wave vector). We recall that  $\mathcal{F}$  extends by density as a unitary transformation from  $L^2(\mathbb{R}_x^3)^3$  to  $L^2(\mathbb{R}_k^3)^3$ . We set  $\mathbb{E}(\mathbf{k}, t) := \mathcal{F}\{\mathbf{E}(\cdot, t)\}(\mathbf{k})$  and analogously  $\mathbb{H}, \mathbb{P}$  and  $\mathbb{M}$  the Fourier transforms of  $\mathbf{H}, \mathbf{P}$  and  $\mathbf{M}$ , respectively. Accordingly we denote  $\mathbb{E}_0$  the Fourier transform of  $\mathbf{E}_0$  and  $\mathbb{H}_0$  the Fourier transform of  $\mathbf{H}_0$ . Then  $(\mathbb{E}, \mathbb{H}, \mathbb{P}, \mathbb{M})$  satisfy

$$\begin{cases} \varepsilon_0 \partial_t \mathbb{E} - i(\mathbf{k} \times \mathbb{H}) + \varepsilon_0 \Omega_e^2 \partial_t \mathbb{P} = 0, & (1.2.6a) \end{cases}$$

$$\begin{cases} \mu_0 \partial_t \mathbb{H} + i(\mathbf{k} \times \mathbb{E}) + \mu_0 \Omega_m^2 \partial_t \mathbb{M} = 0, & (1.2.6b) \end{cases}$$

$$\begin{cases} \partial_t^2 \mathbb{P} + \alpha_e \partial_t \mathbb{P} = \mathbb{E}, & (1.2.6c) \end{cases}$$

$$\begin{cases} \partial_t^2 \mathbb{M} + \alpha_m \partial_t \mathbb{M} = \mathbb{H}. & (1.2.6d) \end{cases}$$

In the above formula, the usual cross product on  $\mathbb{R}^3 \times \mathbb{R}^3$  has been here extended to  $\mathbb{C}^3 \times \mathbb{C}^3$  via

$$\mathbf{a} \times \mathbf{b} = (\mathbf{a}_2 \mathbf{b}_3 - \mathbf{a}_3 \mathbf{b}_2, \mathbf{a}_3 \mathbf{b}_1 - \mathbf{a}_1 \mathbf{b}_3, \mathbf{a}_1 \mathbf{b}_2 - \mathbf{a}_2 \mathbf{b}_1).$$

According to (1.2.2), the above system is complemented by

$$\mathbb{E}(\mathbf{k}, 0) = \mathbb{E}_0(\mathbf{k}), \quad \mathbb{H}(\mathbf{k}, 0) = \mathbb{H}_0(\mathbf{k}), \quad \partial_t \mathbb{P}(\mathbf{k}, 0) = \partial_t \mathbb{M}(\mathbf{k}, 0) = 0. \quad (1.2.7)$$

Let us note that by multiplying (1.2.6a) by  $\overline{\mathbb{E}}$ , using (1.2.6c) and taking the real part, we obtain that

$$\frac{1}{2} \frac{d}{dt} (\varepsilon_0 |\mathbb{E}(\mathbf{k}, t)|^2 + \varepsilon_0 \Omega_e^2 |\partial_t \mathbb{P}(\mathbf{k}, t)|^2) + \alpha_e \varepsilon_0 \Omega_e^2 |\partial_t \mathbb{P}(\mathbf{k}, t)|^2 - \operatorname{Re}(i(\mathbf{k} \times \mathbb{H}(\mathbf{k}, t)) \cdot \overline{\mathbb{E}(\mathbf{k}, t)}) = 0. \quad (1.2.8)$$

Analogously we have from (1.2.6b) and (1.2.6d) that

$$\frac{1}{2} \frac{d}{dt} (\mu_0 |\mathbb{H}(\mathbf{k}, t)|^2 + \mu_0 \Omega_m^2 |\partial_t \mathbb{M}(\mathbf{k}, t)|^2) + \alpha_m \mu_0 \Omega_m^2 |\partial_t \mathbb{M}(\mathbf{k}, t)|^2 + \operatorname{Re}(i(\mathbf{k} \times \mathbb{E}(\mathbf{k}, t)) \cdot \overline{\mathbb{H}(\mathbf{k}, t)}) = 0. \quad (1.2.9)$$



Finally by adding up (1.2.8) and (1.2.9) and using  $\operatorname{Re}(i(\mathbf{k} \times \mathbb{E}(\mathbf{k}, t)) \cdot \overline{\mathbb{H}(\mathbf{k}, t)}) - \operatorname{Re}(i(\mathbf{k} \times \mathbb{H}(\mathbf{k}, t)) \cdot \mathbb{E}(\mathbf{k}, t)) = 0$ , we show that

$$\frac{d}{dt} \mathcal{L}_{\mathbf{k}} + \mathcal{D}_{\alpha, \mathbf{k}} = 0, \quad (1.2.10)$$

where we have set

$$\left\{ \begin{array}{ll} \mathcal{L}_{\mathbf{k}}(t) = \mathcal{E}_{\mathbf{k}}(t) + \mathcal{E}_{\Omega, \mathbf{k}}(t), & \text{the Lyapunov density,} \\ \mathcal{E}_{\mathbf{k}}(t) := \frac{1}{2}(\varepsilon_0 |\mathbb{E}(\mathbf{k}, t)|^2 + \mu_0 |\mathbb{H}(\mathbf{k}, t)|^2), & \text{the energy density,} \\ \mathcal{E}_{\Omega, \mathbf{k}}(t) := \frac{1}{2}(\varepsilon_0 \Omega_e^2 |\partial_t \mathbb{P}(\mathbf{k}, t)|^2 + \mu_0 \Omega_m^2 |\partial_t \mathbb{M}(\mathbf{k}, t)|^2), & \text{the additional energy density,} \end{array} \right. \quad (1.2.11)$$

and the decay density (the index  $\alpha$  is here to emphasize the fact that this is the term in (1.2.10) which involves the damping coefficients  $\alpha_e$  and  $\alpha_m$ )

$$\mathcal{D}_{\alpha, \mathbf{k}}(t) := \alpha_e \varepsilon_0 \Omega_e^2 |\partial_t \mathbb{P}(\mathbf{k}, t)|^2 + \alpha_m \mu_0 \Omega_m^2 |\partial_t \mathbb{M}(\mathbf{k}, t)|^2. \quad (1.2.12)$$

We employ the term "density" to refer to the fact that one works at fixed  $\mathbf{k}$ : the electromagnetic energy  $\mathcal{E}$ , for instance, is obtained, via Plancherel's theorem, by integration over  $\mathbf{k}$  of the energy density  $\mathcal{E}_{\mathbf{k}}$

$$\mathcal{E}(t) = \int \mathcal{E}_{\mathbf{k}}(t) d\mathbf{k}. \quad (1.2.13)$$

**Remark 1.2.1.** The reader will note that (1.2.5) can be recovered by integrating the above identity over  $\mathbf{k} \in \mathbb{R}^3$  and applying Plancherel's theorem.

We can not exploit only (1.2.10) for studying the long time behaviour of  $\mathcal{L}_{\mathbf{k}}(t)$  because we can not estimate  $\mathcal{L}_{\mathbf{k}}(t)$  with the help of  $\mathcal{D}_{\alpha, \mathbf{k}}(t)$ , which does not involve  $\mathbb{E}$  and  $\mathbb{H}$ . On the other hand, we see on (1.2.6) that, in order to control  $\mathbb{E}$  and  $\mathbb{H}$ , we need the second order derivatives  $\partial_t^2 \mathbb{P}, \partial_t^2 \mathbb{M}$ . This suggests to look at the system obtained after time differentiation of (1.2.6). Then in the same way that we obtained (1.2.10), we have (with obvious notation)

$$\frac{d}{dt} \mathcal{L}_{\mathbf{k}}^1 + \mathcal{D}_{\alpha, \mathbf{k}}^1 = 0, \quad (1.2.14)$$

where we have introduced the first order densities (where "first order" refers to that fact that first order derivatives of the electromagnetic field are involved)

$$\left\{ \begin{array}{ll} \mathcal{L}_{\mathbf{k}}^1(t) = \mathcal{E}_{\mathbf{k}}^1(t) + \mathcal{E}_{\Omega, \mathbf{k}}^1(t), & \text{first order Lyapunov density,} \\ \mathcal{E}_{\mathbf{k}}^1(t) := \frac{1}{2}(\varepsilon_0 |\partial_t \mathbb{E}(\mathbf{k}, t)|^2 + \mu_0 |\partial_t \mathbb{H}(\mathbf{k}, t)|^2) & \text{first order energy density,} \\ \mathcal{E}_{\Omega, \mathbf{k}}^1(t) := \frac{1}{2}(\varepsilon_0 \Omega_e^2 |\partial_t^2 \mathbb{P}(\mathbf{k}, t)|^2 + \mu_0 \Omega_m^2 |\partial_t^2 \mathbb{M}(\mathbf{k}, t)|^2), & \text{first order add. energy density,} \end{array} \right. \quad (1.2.15)$$

and the first order decay density

$$\mathcal{D}_{\alpha, \mathbf{k}}^1(t) := \alpha_e \varepsilon_0 \Omega_e^2 |\partial_t^2 \mathbb{P}(\mathbf{k}, t)|^2 + \alpha_m \mu_0 \Omega_m^2 |\partial_t^2 \mathbb{M}(\mathbf{k}, t)|^2, \quad (1.2.16)$$

which precisely involves  $\partial_t^2 \mathbb{P}, \partial_t^2 \mathbb{M}$ . The next step is to combine (1.2.10) and (1.2.14). In what follows, we shall use some standard notation, to begin with

$$\langle \mathbf{k} \rangle = (1 + |\mathbf{k}|^2)^{\frac{1}{2}} \quad (1.2.17)$$

and to compare two positive functions  $f(\mathbf{k}, t)$  and  $g(\mathbf{k}, t)$ , the notation

$$f(\mathbf{k}, t) \lesssim g(\mathbf{k}, t) \iff \exists C > 0 \text{ such that } \forall \mathbf{k} \in \mathbb{R}^3, \forall t \geq 0, \quad f(\mathbf{k}, t) \leq C g(\mathbf{k}, t). \quad (1.2.18)$$

Performing the linear combination (1.2.10) +  $\langle \mathbf{k} \rangle^{-2}$  (1.2.14) we obtain (cf. remark 1.2.2)

$$\frac{d}{dt} \mathcal{L}_{\mathbf{k}}^{(1)} + \mathcal{D}_{\alpha, \mathbf{k}}^{(1)} = 0, \quad (1.2.19)$$

where we have introduced the first order cumulated densities for the Lyapunov function, the energy and the additional energy, (note the difference of notation between  $\mathcal{L}_{\mathbf{k}}^{(1)}$  and  $\mathcal{L}_{\mathbf{k}}^1$ , etc ...)

$$\mathcal{L}_{\mathbf{k}}^{(1)} := \mathcal{L}_{\mathbf{k}} + \langle \mathbf{k} \rangle^{-2} \mathcal{L}_{\mathbf{k}}^1, \quad \mathcal{E}_{\mathbf{k}}^{(1)} := \mathcal{E}_{\mathbf{k}} + \langle \mathbf{k} \rangle^{-2} \mathcal{E}_{\mathbf{k}}^1, \quad \mathcal{E}_{\Omega, \mathbf{k}}^{(1)} := \mathcal{E}_{\Omega, \mathbf{k}}^1 + \langle \mathbf{k} \rangle^{-2} \mathcal{E}_{\Omega, \mathbf{k}}^1, \quad (1.2.20)$$

and the first order cumulated decay density

$$\mathcal{D}_{\alpha, \mathbf{k}}^{(1)} := \mathcal{D}_{\alpha, \mathbf{k}} + \langle \mathbf{k} \rangle^{-2} \mathcal{D}_{\alpha, \mathbf{k}}^1. \quad (1.2.21)$$

**Remark 1.2.2.** The weight  $\langle \mathbf{k} \rangle^{-2}$  is here to compensate the time derivation in the expressions of  $\mathcal{L}_{\mathbf{k}}^1$  and  $\mathcal{D}_{\alpha, \mathbf{k}}^1$  so that the quantities summed in the definitions (1.2.20) and (1.2.21) have the same ‘‘homogeneity’’.

The key point is that (1.2.19) can be exploited thanks to the following lemma.

| <b>Lemma 1.2.3</b>  |
|---|
| <p>Assume that <math>\alpha_e &gt; 0</math> and <math>\alpha_m &gt; 0</math>. Then the following estimate holds</p> $\mathcal{L}_{\mathbf{k}}^{(1)}(t) \lesssim \langle \mathbf{k} \rangle^2 \mathcal{D}_{\alpha, \mathbf{k}}^{(1)}(t). \quad (1.2.22)$ |

*Proof.* In what follows, for conciseness, we omit to mention the (implicit) dependence of various quantities with respect to  $t$  and/or  $\mathbf{k}$ . We recall that  $\mathcal{L}_{\mathbf{k}}^{(1)} = \mathcal{E}_{\mathbf{k}}^{(1)} + \mathcal{E}_{\Omega, \mathbf{k}}^{(1)}$ .

Since both  $\alpha_e$  and  $\alpha_m$  are strictly positive, we immediately observe that we can control the cumulated energy additional  $\mathcal{E}_{\Omega, \mathbf{k}}^{(1)}$  with  $\mathcal{D}_{\alpha, \mathbf{k}}^{(1)}$  :

$$\mathcal{E}_{\Omega, \mathbf{k}} \lesssim \mathcal{D}_{\alpha, \mathbf{k}} \text{ and } \mathcal{E}_{\Omega, \mathbf{k}}^1 \lesssim \mathcal{D}_{\alpha, \mathbf{k}}^1 \text{ which implies, by (1.2.20, 1.2.21), } \mathcal{E}_{\Omega, \mathbf{k}}^{(1)} \lesssim \mathcal{D}_{\alpha, \mathbf{k}}^{(1)}. \quad (1.2.23)$$

For estimating the energy density  $\mathcal{E}_{\mathbf{k}}$ , it is natural to use the constitutive equations (1.2.6c) and (1.2.6d) to deduce, again because  $\alpha_e$  and  $\alpha_m$  are strictly positive,

$$|\mathbb{E}|^2 \lesssim |\partial_t \mathbb{P}|^2 + |\partial_t^2 \mathbb{P}|^2 \lesssim \mathcal{D}_{\alpha, \mathbf{k}} + \mathcal{D}_{\alpha, \mathbf{k}}^1, \quad |\mathbb{H}|^2 \lesssim |\partial_t \mathbb{M}|^2 + |\partial_t^2 \mathbb{M}|^2 \lesssim \mathcal{D}_{\alpha, \mathbf{k}} + \mathcal{D}_{\alpha, \mathbf{k}}^1. \quad (1.2.24)$$

Thus, by definition of  $\mathcal{E}_{\mathbf{k}}$ ,

$$\mathcal{E}_{\mathbf{k}} \lesssim \mathcal{D}_{\alpha, \mathbf{k}} + \mathcal{D}_{\alpha, \mathbf{k}}^1. \quad (1.2.25)$$

For estimating the first order energy density  $\mathcal{E}_{\mathbf{k}}^1$ , we need to bound  $\partial_t \mathbb{E}$  and  $\partial_t \mathbb{H}$  for which it is natural to use Maxwell equations (1.2.6a) and (1.2.6b). This is where we need to introduce  $\mathbf{k}$ -dependent coefficients in our estimates. Indeed, from (1.2.6a) and (1.2.6b),

$$\begin{aligned} |\partial_t \mathbb{E}|^2 &\lesssim |\mathbf{k} \times \mathbb{H}|^2 + |\partial_t \mathbb{P}|^2 \lesssim |\mathbf{k}|^2 |\mathbb{H}|^2 + |\partial_t \mathbb{P}|^2, \\ |\partial_t \mathbb{H}|^2 &\lesssim |\mathbf{k} \times \mathbb{E}|^2 + |\partial_t \mathbb{M}|^2 \lesssim |\mathbf{k}|^2 |\mathbb{E}|^2 + |\partial_t \mathbb{M}|^2. \end{aligned} \quad (1.2.26)$$

Adding the two inequalities yields, by definition of  $\mathcal{E}_{\mathbf{k}}^1$  (1.2.15) and  $\mathcal{D}_{\alpha,\mathbf{k}}$  (1.2.16) (we use again  $\alpha_e, \alpha_m > 0$ )

$$\mathcal{E}_{\mathbf{k}}^1 \lesssim |\mathbf{k}|^2 \mathcal{E}_{\mathbf{k}} + \mathcal{D}_{\alpha,\mathbf{k}} \lesssim \langle \mathbf{k} \rangle^2 \mathcal{D}_{\alpha,\mathbf{k}} + |\mathbf{k}|^2 \mathcal{D}_{\alpha,\mathbf{k}}^1 \quad (1.2.27)$$

where, for the second inequality, we have used (1.2.25). Finally, performing (1.2.25) +  $\langle \mathbf{k} \rangle^{-2}$  (1.2.27), gives

$$\mathcal{E}_{\mathbf{k}}^{(1)} \lesssim \mathcal{D}_{\alpha,\mathbf{k}} + \mathcal{D}_{\alpha,\mathbf{k}}^1 \leq \langle \mathbf{k} \rangle^2 \mathcal{D}_{\alpha,\mathbf{k}}^{(1)} \quad (\text{since } 1 \leq \langle \mathbf{k} \rangle^2). \quad (1.2.28)$$

Finally, (1.2.22) results from  $\mathcal{L}_{\mathbf{k}}^{(1)} = \mathcal{E}_{\mathbf{k}}^{(1)} + \mathcal{E}_{\Omega,\mathbf{k}}^{(1)}$ , (1.2.23) and (1.2.28).  $\square$

The inequality eqrefestimacion chida in Lemma 1.2.3 means the existence of a constant  $\sigma = \sigma(\varepsilon_0, \mu_0, \alpha_e, \alpha_m, \Omega_e, \Omega_m) > 0$  such that

$$\mathcal{D}_{\alpha,\mathbf{k}}^{(1)} \geq \sigma \langle \mathbf{k} \rangle^{-2} \mathcal{L}_{\mathbf{k}}^{(1)}.$$

Combined with (1.2.14), this gives the differential inequality

$$\frac{d}{dt} \mathcal{L}_{\mathbf{k}}^{(1)} + \sigma \langle \mathbf{k} \rangle^{-2} \mathcal{L}_{\mathbf{k}}^{(1)} \leq 0$$

which gives by integration in time:

$$\mathcal{L}_{\mathbf{k}}^{(1)}(t) \leq \mathcal{L}_{\mathbf{k}}^{(1)}(0) e^{-\frac{\sigma t}{\langle \mathbf{k} \rangle^2}}. \quad (1.2.29)$$

Next, we show how to control  $\mathcal{L}_{\mathbf{k}}^{(1)}(0)$  in terms on the initial data as

$$\mathcal{L}_{\mathbf{k}}^{(1)}(0) \lesssim |\mathbb{E}(\mathbf{k}, 0)|^2 + |\mathbb{H}(\mathbf{k}, 0)|^2. \quad (1.2.30)$$

Indeed, by definition and initial conditions (1.2.7),

$$\mathcal{L}_{\mathbf{k}}(0) \lesssim |\mathbb{E}(\mathbf{k}, 0)|^2 + |\mathbb{H}(\mathbf{k}, 0)|^2. \quad (1.2.31)$$

On the other hand, from (1.2.15), one has

$$\mathcal{L}_{\mathbf{k}}^1(0) \lesssim |\partial_t \mathbb{E}(\mathbf{k}, 0)|^2 + |\partial_t \mathbb{H}(\mathbf{k}, 0)|^2 + |\partial_t^2 \mathbb{P}(\mathbf{k}, 0)|^2 + |\partial_t^2 \mathbb{M}(\mathbf{k}, 0)|^2.$$

By (1.2.6a, 1.2.6b) at  $t = 0$ ,  $|\partial_t \mathbb{E}(\mathbf{k}, 0)|^2 + |\partial_t \mathbb{H}(\mathbf{k}, 0)|^2 \lesssim |\mathbf{k}|^2 (|\mathbb{E}(\mathbf{k}, 0)|^2 + |\mathbb{H}(\mathbf{k}, 0)|^2)$  since  $\partial_t \mathbb{P}(\mathbf{k}, 0)$  and  $\partial_t \mathbb{M}(\mathbf{k}, 0)$  vanish, while (1.2.6c, 1.2.6d) give  $|\partial_t^2 \mathbb{P}(\mathbf{k}, 0)|^2 + |\partial_t^2 \mathbb{M}(\mathbf{k}, 0)|^2 \lesssim |\mathbb{E}(\mathbf{k}, 0)|^2 + |\mathbb{H}(\mathbf{k}, 0)|^2$ . Thus

$$\mathcal{L}_{\mathbf{k}}^1(0) \lesssim \langle \mathbf{k} \rangle^2 (|\mathbb{E}(\mathbf{k}, 0)|^2 + |\mathbb{H}(\mathbf{k}, 0)|^2). \quad (1.2.32)$$

Finally (1.2.30) results from (1.2.31) +  $\langle \mathbf{k} \rangle^{-2}$  (1.2.32). At last, substituting (1.2.30) into (1.2.29), as  $\mathcal{L}_{\mathbf{k}} \leq \mathcal{L}_{\mathbf{k}}^{(1)}$

$$\mathcal{L}_{\mathbf{k}}(t) \lesssim (|\mathbb{E}(\mathbf{k}, 0)|^2 + |\mathbb{H}(\mathbf{k}, 0)|^2) e^{-\frac{\sigma t}{\langle \mathbf{k} \rangle^2}}. \quad (1.2.33)$$

This inequality says that the Lyapunov density decays exponentially in time for each  $|\mathbf{k}|$ . It also says the decay rate depends on  $|\mathbf{k}|$  and tends to 0 when  $|\mathbf{k}|$  tends to  $+\infty$ . This is the reason why, when coming back to the augmented energy  $\mathcal{L}(t)$  (cf. (1.2.4)) we shall obtain only polynomial decay. This is summarized in the main theorem of this section:

**Theorem 1.2.4**

For any  $(\mathbf{E}_0, \mathbf{H}_0) \in L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3$ , the total energy tends to 0 when  $t$  tends to  $+\infty$  :

$$\lim_{t \rightarrow +\infty} \mathcal{L}(t) = 0. \quad (1.2.34)$$

If  $(\mathbf{E}_0, \mathbf{H}_0) \in H^m(\mathbb{R}^3)^3 \times H^m(\mathbb{R}^3)^3$  for some integer  $m > 0$ , one has a polynomial decay rate

$$\mathcal{L}(t) \lesssim \left( \|\mathbf{E}_0\|_{H^m(\mathbb{R}^3)}^2 + \|\mathbf{H}_0\|_{H^m(\mathbb{R}^3)}^2 \right) t^{-m}. \quad (1.2.35)$$

*Proof.* From the respective definitions of  $\mathcal{L}$  and  $\mathcal{L}_{\mathbf{k}}$ , by Plancherel's identity and (1.2.33), we have

$$\mathcal{L}(t) = \int_{\mathbb{R}^3} \mathcal{L}_{\mathbf{k}}(t) \, d\mathbf{k} \lesssim \int_{\mathbb{R}^3} (|\mathbb{E}_0(\mathbf{k})|^2 + |\mathbb{H}_0(\mathbf{k})|^2) e^{-\frac{\sigma t}{\langle \mathbf{k} \rangle^2}} \, d\mathbf{k}. \quad (1.2.36)$$

From Lebesgue's dominated convergence theorem, we first conclude that (1.2.34) holds for any initial data  $(\mathbf{E}_0, \mathbf{H}_0) \in L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3$ .

Next, in order to exploit the Sobolev regularity of the initial for obtaining (1.2.35), we rewrite (1.2.36) as follows (we simply make appear artificially the factor  $\langle \mathbf{k} \rangle^{2m}/t^m$ )

$$\begin{aligned} \mathcal{L}(t) &\lesssim t^{-m} \int_{\mathbb{R}^3} \langle \mathbf{k} \rangle^{2m} (|\mathbb{E}_0(\mathbf{k})|^2 + |\mathbb{H}_0(\mathbf{k})|^2) (t/\langle \mathbf{k} \rangle^2)^m e^{-\frac{\sigma t}{\langle \mathbf{k} \rangle^2}} \, d\mathbf{k} \\ &\lesssim t^{-m} \int_{\mathbb{R}^3} \langle \mathbf{k} \rangle^{2m} (|\mathbb{E}_0(\mathbf{k})|^2 + |\mathbb{H}_0(\mathbf{k})|^2) F(t/\langle \mathbf{k} \rangle^2) \, d\mathbf{k}. \end{aligned}$$

where we have set  $F_m(r) := r^m e^{-\sigma r}$ ,  $r \geq 0$  which is clearly bounded on  $\mathbb{R}^+$ .

Setting  $C_m := \sup_{r \geq 0} F_m(r) = (m/(\sigma e))^m$ , by Fourier characterization of Sobolev norms, we get

$$\mathcal{L}(t) \lesssim C_m t^{-m} \left( \|\mathbf{E}_0\|_{H^m(\mathbb{R}^3)}^2 + \|\mathbf{H}_0\|_{H^m(\mathbb{R}^3)}^2 \right).$$

□

### 1.3 The case of the generalized Lorentz model

In this section, our goal is to extend the results in the latest section to the case of the (generalized) Lorentz model. The evolution (Cauchy) problem reads as follows

$$\text{Find } \begin{cases} \mathbf{E}(\mathbf{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3 & \mathbf{H}(\mathbf{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3 \\ \mathbf{P}_j(\mathbf{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3, 1 \leq j \leq N_e, \mathbf{M}_\ell(\mathbf{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3, 1 \leq \ell \leq N_m, \end{cases}$$

such that (for all  $1 \leq j \leq N_e, 1 \leq \ell \leq N_m$ )

$$\left\{ \begin{array}{l} \varepsilon_0 \partial_t \mathbf{E} - \nabla \times \mathbf{H} + \varepsilon_0 \sum_{j=1}^{N_e} \Omega_{e,j}^2 \partial_t \mathbf{P}_j = 0, \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_*^+, \quad (1.3.1a) \\ \mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} + \mu_0 \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 \partial_t \mathbf{M}_\ell = 0, \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_*^+, \quad (1.3.1b) \\ \partial_t^2 \mathbf{P}_j + \alpha_{e,j} \partial_t \mathbf{P}_j + \omega_{e,j}^2 \mathbf{P}_j = \mathbf{E}, \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_*^+, j \in \{1, \dots, N_e\} \quad (1.3.1c) \\ \partial_t^2 \mathbf{M}_\ell + \alpha_{m,\ell} \partial_t \mathbf{M}_\ell + \omega_{m,\ell}^2 \mathbf{M}_\ell = \mathbf{H}, \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_*^+, \ell \in \{1, \dots, N_m\} \quad (1.3.1d) \end{array} \right.$$

completed by the following divergence free initial conditions

$$\left\{ \begin{array}{l} \mathbf{E}(\cdot, 0) = \mathbf{E}_0, \quad \mathbf{H}(\cdot, 0) = \mathbf{H}_0 \quad \text{with} \quad \nabla \cdot \mathbf{E}_0 = \nabla \cdot \mathbf{H}_0 = 0, \\ \mathbf{P}(\cdot, 0) = \mathbf{M}(\cdot, 0) = \partial_t \mathbf{P}(\cdot, 0) = \partial_t \mathbf{M}(\cdot, 0) = 0, \end{array} \right. \quad (1.3.2)$$

where for simplifying some formulas, we treat the  $\mathbf{P}_j$  and the  $\mathbf{M}_\ell$  collectively setting

$$\mathbf{P} = (\mathbf{P}_j)_{j=1}^{N_e} \quad \text{and} \quad \mathbf{M} = (\mathbf{M}_\ell)_{\ell=1}^{N_m}. \quad (1.3.3)$$

In the above equations, the coefficients  $(\Omega_{e,j}, \Omega_{m,\ell}, \alpha_{e,j}, \omega_{e,j})$  are supposed to satisfy (see remark 1.3.1)

$$\begin{aligned} 0 < \Omega_{e,1} \leq \dots \leq \Omega_{e,N_e}, \quad 0 < \Omega_{m,1} \leq \dots \leq \Omega_{m,N_m}, \\ \omega_{e,j} > 0, \quad 1 \leq j \leq N_e, \quad \omega_{m,\ell} > 0, \quad 1 \leq \ell \leq N_m, \end{aligned} \quad (1.3.4)$$

and one can assume without any loss of generality that the couples  $(\alpha_{e,j}, \omega_{e,j})$  (resp.  $(\alpha_{m,\ell}, \omega_{m,\ell})$ ) are all distinct the ones from the others.

**Remark 1.3.1.** Note that one recovers the Drude model of section with  $N_e = N_m = 1$  if  $\omega_{e,1} = \omega_{m,1} = 0$ .

Setting  $\mathcal{H} = L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^{3N_e} \times L^2(\mathbb{R}^3)^{3N_e} \times L^2(\mathbb{R}^3)^{3N_m} \times L^2(\mathbb{R}^3)^{3N_m}$ , [Proposition A. 3](#) insures that for

$$(\mathbf{E}_0, \mathbf{H}_0) \in L^2(\mathbb{R}^3)^3,$$

(1.3.1) admits a unique mild solution  $\mathbf{U} = (\mathbf{E}, \mathbf{H}, \mathbf{P}, \partial_t \mathbf{P}, \mathbf{M}, \partial_t \mathbf{M})$  in  $C^0(\mathbb{R}^+, \mathcal{H})$ , which is a strong solution in  $C^1(\mathbb{R}^+, \mathcal{H})$  as soon as  $(\mathbf{E}_0, \mathbf{H}_0) \in H^1(\mathbb{R}^3)^3 \times H^1(\mathbb{R}^3)^3$ .

The equations (1.3.1) are completed by the initial conditions (1.3.2). If the initial electric and magnetic fields are divergence free, all vector fields appearing in (1.3.1) are divergence free at any time, see Lemma 3.13 of [53], or [Proposition A. 4](#) of [Appendix A.4](#).

$$\nabla \cdot \mathbf{E}(\cdot, t) = \nabla \cdot \mathbf{H}(\cdot, t) = \nabla \cdot \mathbf{P}_j(\cdot, t) = \nabla \cdot \mathbf{M}_\ell(\cdot, t) = 0, \quad \forall t > 0, \quad \forall j, \ell. \quad (1.3.5)$$

The equivalent for Lorentz of the identity (1.2.5) (for Drude) is (see [17])

$$\frac{d}{dt} \mathcal{L}(t) + \varepsilon_0 \sum_{j=1}^{N_e} \alpha_{j,e} \Omega_{j,e}^2 \int_{\mathbb{R}^3} |\mathbf{P}_j(\mathbf{x}, t)|^2 d\mathbf{x} + \mu_0 \sum_{\ell=1}^{N_m} \alpha_{m,\ell} \Omega_{m,\ell}^2 \int_{\mathbb{R}^3} |\mathbf{M}_\ell(\mathbf{x}, t)|^2 d\mathbf{x} = 0. \quad (1.3.6)$$

We assume the strong dissipation assumption (1.1.26), namely all the damping coefficients are

positive:  $\alpha_{e,j}, \alpha_{m,\ell} > 0$  for all  $1 \leq j \leq N_e, 1 \leq \ell \leq N_m$ .

Next, we observe that the space Fourier transforms  $(\mathbb{E}, \mathbb{H}, \mathbb{P}_j, \mathbb{M}_\ell)$  of  $(\mathbf{E}, \mathbf{H}, \mathbf{P}_j, \mathbf{M}_\ell)$  satisfy

$$\left\{ \begin{array}{l} \varepsilon_0 \partial_t \mathbb{E} - \mathbf{i} \mathbf{k} \times \mathbb{H} + \varepsilon_0 \sum_{j=1}^{N_e} \Omega_{e,j}^2 \partial_t \mathbb{P}_j = 0, \end{array} \right. \quad (1.3.7a)$$

$$\left\{ \begin{array}{l} \mu_0 \partial_t \mathbb{H} + \mathbf{i} \mathbf{k} \times \mathbb{E} + \mu_0 \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 \partial_t \mathbb{M}_\ell = 0, \end{array} \right. \quad (1.3.7b)$$

$$\left\{ \begin{array}{l} \partial_t^2 \mathbb{P}_j + \alpha_{e,j} \partial_t \mathbb{P}_j + \omega_{e,j}^2 \mathbb{P}_j = \mathbb{E}, \end{array} \right. \quad (1.3.7c)$$

$$\left\{ \begin{array}{l} \partial_t^2 \mathbb{M}_\ell + \alpha_{m,\ell} \partial_t \mathbb{M}_\ell + \omega_{m,\ell}^2 \mathbb{M}_\ell = \mathbb{H}. \end{array} \right. \quad (1.3.7d)$$

According to (2.1.1), we shall set

$$\mathbb{P} := (\mathbb{P}_j)_{j=1}^{N_e}, \quad \mathbb{M} := (\mathbb{M}_\ell)_{\ell=1}^{N_m}, \quad (1.3.8)$$

and will also use the condensed notation

$$|\mathbb{P}|^2 := \sum_{j=1}^{N_e} |\mathbb{P}_j|^2, \quad \text{and} \quad |\mathbb{M}|^2 := \sum_{\ell=1}^{N_m} |\mathbb{M}_\ell|^2. \quad (1.3.9)$$

To study the long time behaviour of the solution of (1.3.7), it is natural to try to use the same approach than for the Drude model in section 1.2. As a matter of fact proceeding as for obtaining (1.2.10), we can derive from (1.3.7), and the identity  $\text{Re}(\mathbf{i}(\mathbf{k} \times \mathbb{E}(\mathbf{k}, t)) \cdot \mathbb{H}(\mathbf{k}, t)) - \text{Re}(\mathbf{i}(\mathbf{k} \times \mathbb{H}(\mathbf{k}, t)) \cdot \overline{\mathbb{E}(\mathbf{k}, t)}) = 0$ , the relation

$$\frac{d}{dt} \mathcal{L}_{\mathbf{k}} + \mathcal{D}_{\alpha, \mathbf{k}} = 0, \quad (1.3.10)$$

where we have introduced the energy densities

$$\left\{ \begin{array}{l} \mathcal{L}_{\mathbf{k}}(t) = \mathcal{E}_{\mathbf{k}}(t) + \mathcal{E}_{\Omega, \mathbf{k}}(t), \\ \mathcal{E}_{\mathbf{k}}(t) := \frac{1}{2} (\varepsilon_0 |\mathbb{E}(\mathbf{k}, t)|^2 + \mu_0 |\mathbb{H}(\mathbf{k}, t)|^2) \\ \mathcal{E}_{\Omega, \mathbf{k}}(t) := \frac{1}{2} \left( \varepsilon_0 \sum_{j=1}^{N_e} \Omega_{e,j}^2 |\partial_t \mathbb{P}_j(\mathbf{k}, t)|^2 + \mu_0 \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 |\partial_t \mathbb{M}_\ell(\mathbf{k}, t)|^2 \right) \\ \quad + \frac{1}{2} \left( \varepsilon_0 \sum_{j=1}^{N_e} \omega_{e,j}^2 \Omega_{e,j}^2 |\mathbb{P}_j(\mathbf{k}, t)|^2 + \mu_0 \sum_{\ell=1}^{N_m} \omega_{m,\ell}^2 \Omega_{m,\ell}^2 |\mathbb{M}_\ell(\mathbf{k}, t)|^2 \right), \end{array} \right. \quad (1.3.11)$$

and the decay density

$$\mathcal{D}_{\alpha, \mathbf{k}}(t) := \varepsilon_0 \sum_{j=1}^{N_e} \alpha_{e,j} \Omega_{e,j}^2 |\partial_t \mathbb{P}_j(\mathbf{k}, t)|^2 + \mu_0 \sum_{\ell=1}^{N_m} \alpha_{m,\ell} \Omega_{m,\ell}^2 |\partial_t \mathbb{M}_\ell(\mathbf{k}, t)|^2. \quad (1.3.12)$$

The main novelty, with respect to the Drude case, is the apparition of the second line in the definition of  $\mathcal{E}_{\Omega, \mathbf{k}}$ , that involves the fields  $\mathbb{P}$  and  $\mathbb{M}$  and not only their time derivative.

Reasoning with the time derivatives of the fields as for the Drude case, we also have

$$\frac{d}{dt} \mathcal{L}_{\mathbf{k}}^1 + \mathcal{D}_{\alpha, \mathbf{k}}^1 = 0, \quad (1.3.13)$$

having defined the first order energy densities

$$\left\{ \begin{array}{l} \mathcal{L}_{\mathbf{k}}^1(t) = \mathcal{E}_{\mathbf{k}}^1(t) + \mathcal{E}_{\Omega, \mathbf{k}}^1(t), \\ \mathcal{E}_{\mathbf{k}}^1(t) := \frac{1}{2} (\varepsilon_0 |\partial_t \mathbb{E}(\mathbf{k}, t)|^2 + \mu_0 |\partial_t \mathbb{H}(\mathbf{k}, t)|^2) \\ \mathcal{E}_{\Omega, \mathbf{k}}^1(t) := \frac{1}{2} \left( \varepsilon_0 \sum_{j=1}^{N_e} \Omega_{e,j}^2 |\partial_t^2 \mathbb{P}_j(\mathbf{k}, t)|^2 + \mu_0 \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 |\partial_t^2 \mathbb{M}_\ell(\mathbf{k}, t)|^2 \right) \\ \quad + \frac{1}{2} \left( \varepsilon_0 \sum_{j=1}^{N_e} \omega_{e,j}^2 \Omega_{e,j}^2 |\partial_t \mathbb{P}_j(\mathbf{k}, t)|^2 + \mu_0 \sum_{\ell=1}^{N_m} \omega_{m,\ell}^2 \Omega_{m,\ell}^2 |\partial_t \mathbb{M}_\ell(\mathbf{k}, t)|^2 \right), \end{array} \right. \quad (1.3.14)$$

and the first order decay density

$$\mathcal{D}_{\alpha, \mathbf{k}}^1(t) := \varepsilon_0 \sum_{j=1}^{N_e} \alpha_{e,j} \Omega_{e,j}^2 |\partial_t^2 \mathbb{P}_j(\mathbf{k}, t)|^2 + \mu_0 \sum_{\ell=1}^{N_m} \alpha_{e,\ell} \Omega_{m,\ell}^2 |\partial_t^2 \mathbb{M}_\ell(\mathbf{k}, t)|^2. \quad (1.3.15)$$

However, this time, (1.3.10) and (1.3.13) will not be sufficient to proceed as in the Drude case because we need to control the term in the second line of the definition (1.3.11) of  $\mathcal{E}_{\Omega, \mathbf{k}}$ , in other words  $\mathbb{P}$  and  $\mathbb{M}$ . Because these fields only appear the constitutive laws (1.3.7c) and (1.3.7d), we need to adopt a different strategy with respect to section 1.2:

- (i) This time, the constitutive laws (1.3.7c) and (1.3.7d) are used to control  $\mathbb{P}$  and  $\mathbb{M}$  (and no longer  $\mathbb{E}$  and  $\mathbb{H}$ ) in function of  $\mathbb{E}$ ,  $\partial_t \mathbb{P}$ ,  $\partial_t^2 \mathbb{P}$ ,  $\mathbb{H}$ ,  $\partial_t \mathbb{M}$  and  $\partial_t^2 \mathbb{M}$ .
- (ii) We then need to control  $\mathbb{E}$  and  $\mathbb{H}$  in another manner: this will be done by using the Maxwell's equations (1.3.7a, 1.3.7b), via  $\mathbf{k} \times \mathbb{E}$  (resp.  $\mathbf{k} \times \mathbb{H}$ ) (this control will thus degenerate when  $|\mathbf{k}|$  tends to 0) in function of  $\partial_t \mathbb{H}$  and  $\partial_t \mathbb{M}$  (resp.  $\partial_t \mathbb{E}$  and  $\partial_t \mathbb{P}$ ). This will use  $|\mathbf{k} \times \mathbb{E}| = |\mathbf{k}| |\mathbb{E}|$  (resp.  $|\mathbf{k} \times \mathbb{H}| = |\mathbf{k}| |\mathbb{H}|$ ), which is the counterpart in Fourier space of the free divergence property (1.3.5).
- (iii) Finally, to control  $\partial_t \mathbb{E}$  and  $\partial_t \mathbb{H}$ , the idea is to use again (1.3.7c) and (1.3.7d), but this time after time differentiation. Doing so, we control  $\partial_t \mathbb{E}$  and  $\partial_t \mathbb{H}$  with  $\partial_t^2 \mathbb{P}$  and  $\partial_t^2 \mathbb{M}$ , which do appear in the definitions of  $\mathcal{D}_{\alpha, \mathbf{k}}$  and  $\mathcal{D}_{\alpha, \mathbf{k}}^1$ , but also the third order derivatives  $\partial_t^3 \mathbb{P}$  and  $\partial_t^3 \mathbb{M}$ .

That is why, in order to make appear a damping function containing the third order derivatives, we differentiate the equations of the problem once more in time, which leads to the identity

$$\frac{d}{dt} \mathcal{L}_{\mathbf{k}}^2 + \mathcal{D}_{\alpha, \mathbf{k}}^2 = 0, \quad (1.3.16)$$

having defined the second order energy densities

$$\left\{ \begin{array}{l} \mathcal{L}_{\mathbf{k}}^2(t) = \mathcal{E}_{\mathbf{k}}^2(t) + \mathcal{E}_{\Omega, \mathbf{k}}^2(t), \\ \mathcal{E}_{\mathbf{k}}^2(t) := \frac{1}{2} (\varepsilon_0 |\partial_t^2 \mathbb{E}(\mathbf{k}, t)|^2 + \mu_0 |\partial_t^2 \mathbb{H}(\mathbf{k}, t)|^2) \\ \mathcal{E}_{\Omega, \mathbf{k}}^2(t) := \frac{1}{2} \left( \varepsilon_0 \sum_{j=1}^{N_e} \Omega_{e,j}^2 |\partial_t^3 \mathbb{P}_j(\mathbf{k}, t)|^2 + \mu_0 \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 |\partial_t^3 \mathbb{M}_\ell(\mathbf{k}, t)|^2 \right) \\ \quad + \frac{1}{2} \left( \varepsilon_0 \sum_{j=1}^{N_e} \omega_{e,j}^2 \Omega_{e,j}^2 |\partial_t^2 \mathbb{P}_j(\mathbf{k}, t)|^2 + \mu_0 \sum_{\ell=1}^{N_m} \omega_{m,\ell}^2 \Omega_{m,\ell}^2 |\partial_t^2 \mathbb{M}_\ell(\mathbf{k}, t)|^2 \right), \end{array} \right. \quad (1.3.17)$$

and the second order decay density

$$\mathcal{D}_{\alpha, \mathbf{k}}^2(t) := \varepsilon_0 \sum_{j=1}^{N_e} \alpha_{e,j} \Omega_{e,j}^2 |\partial_t^3 \mathbb{P}_j(\mathbf{k}, t)|^2 + \mu_0 \sum_{\ell=1}^{N_m} \alpha_{e,\ell} \Omega_{m,\ell}^2 |\partial_t^3 \mathbb{M}_\ell(\mathbf{k}, t)|^2. \quad (1.3.18)$$

Finally, proceeding as for obtaining (1.2.19), we deduce from equations (1.3.10) to (1.3.18) that

$$\frac{d}{dt} \mathcal{L}_{\mathbf{k}}^{(2)} + \mathcal{D}_{\alpha, \mathbf{k}}^{(2)} = 0, \quad (1.3.19)$$

where we have introduced the second order cumulated energy densities

$$\mathcal{L}_{\mathbf{k}}^{(2)} := \sum_{j=0}^2 \langle \mathbf{k} \rangle^{-2j} \mathcal{L}_{\mathbf{k}}^j \equiv \mathcal{E}_{\mathbf{k}}^{(2)} + \mathcal{E}_{\Omega, \mathbf{k}}^{(2)}, \quad \mathcal{E}_{\mathbf{k}}^{(2)} := \sum_{j=0}^2 \langle \mathbf{k} \rangle^{-2j} \mathcal{E}_{\mathbf{k}}^j, \quad \mathcal{E}_{\Omega, \mathbf{k}}^{(2)} := \sum_{j=0}^2 \langle \mathbf{k} \rangle^{-2j} \mathcal{E}_{\Omega, \mathbf{k}}^j, \quad (1.3.20)$$

and the second order cumulated decay density

$$\mathcal{D}_{\alpha, \mathbf{k}}^{(2)} := \sum_{j=0}^2 \langle \mathbf{k} \rangle^{-2j} \mathcal{D}_{\alpha, \mathbf{k}}^j. \quad (1.3.21)$$

In (1.3.20, 1.3.21), by convention,  $\mathcal{L}_{\mathbf{k}}^0 = \mathcal{L}_{\mathbf{k}}$ , etc ... The key point is that, according to the process (i)(ii)(iii) described above, we can bound  $\mathcal{L}_{\mathbf{k}}^{(2)}$  in terms of  $\mathcal{D}_{\alpha, \mathbf{k}}^{(2)}$  :

**Lemma 1.3.2**

Assume that the strong dissipation assumption holds. Then, one has the following estimate

$$\mathcal{L}_{\mathbf{k}}^{(2)}(t) \lesssim (\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2}) \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}(t). \quad (1.3.22)$$

*Proof.* Before entering the technical details, let us first give the main ideas and steps of the proof. The goal is to control  $\mathcal{L}_{\mathbf{k}}^{(2)}$  with the help of  $\mathcal{D}_{\alpha, \mathbf{k}}^{(2)}$ . Towards this goal, we observe that

$$\mathcal{L}_{\mathbf{k}}^{(2)} \text{ is a squared norm in } \mathbb{U} := (\mathbb{E}, \partial_t \mathbb{E}, \partial_t^2 \mathbb{E}, \mathbb{H}, \partial_t \mathbb{H}, \partial_t^2 \mathbb{H}, \mathbb{P}, \partial_t \mathbb{P}, \partial_t^2 \mathbb{P}, \partial_t^3 \mathbb{P}, \mathbb{M}, \partial_t \mathbb{M}, \partial_t^2 \mathbb{M}, \partial_t^3 \mathbb{M}),$$

$$\mathcal{D}_{\alpha, \mathbf{k}}^{(2)} \text{ is a squared norm in } \mathbb{V} := (\partial_t \mathbb{P}, \partial_t^2 \mathbb{P}, \partial_t^3 \mathbb{P}, \partial_t \mathbb{M}, \partial_t^2 \mathbb{M}, \partial_t^3 \mathbb{M}),$$

The idea that we shall develop is that, roughly speaking,  $\mathcal{D}_{\alpha, \mathbf{k}}^{(2)}$  is also a norm with respect to  $\mathbb{V}$  in  $\mathbb{U}$  along the linear manifold (in the  $\mathbb{U}$ -space) generated by the equations (1.3.7a, 1.3.7b, 1.3.7c, 1.3.7d)



and their time-derivatives. More precisely, we shall be able to control the terms which are missing in  $\mathbb{V}$  namely  $\mathbb{E}, \partial_t \mathbb{E}, \partial_t^2 \mathbb{E}, \mathbb{H}, \partial_t \mathbb{H}, \partial_t^2 \mathbb{H}, \mathbb{P}, \mathbb{M}$  with the terms appearing in  $\mathcal{D}_{\alpha, \mathbf{k}}^{(2)}$  namely  $\partial_t \mathbb{P}, \partial_t^2 \mathbb{P}, \partial_t^3 \mathbb{P}, \partial_t \mathbb{M}, \partial_t^2 \mathbb{M}, \partial_t^3 \mathbb{M}$ . This will be done in the following order

$$\begin{aligned}
 (a) \quad & \left\{ \begin{array}{l} \text{Control } \partial_t \mathbb{E} \text{ with } \partial_t \mathbb{P}, \partial_t^2 \mathbb{P}, \partial_t^3 \mathbb{P} \text{ using } \partial_t(1.3.7c), \\ \text{Control } \partial_t \mathbb{H} \text{ with } \partial_t \mathbb{M}, \partial_t^2 \mathbb{M}, \partial_t^3 \mathbb{M} \text{ using } \partial_t(1.3.7d), \end{array} \right. \\
 (b) \quad & \left\{ \begin{array}{l} \text{Control } \mathbb{E} \text{ with } \partial_t \mathbb{H}, \partial_t \mathbb{M} \text{ using } (1.3.7b), \\ \text{Control } \mathbb{H} \text{ with } \partial_t \mathbb{E}, \partial_t \mathbb{P} \text{ using } (1.3.7a), \end{array} \right. \\
 (c) \quad & \left\{ \begin{array}{l} \text{Control } \partial_t^2 \mathbb{E} \text{ with } \partial_t \mathbb{H}, \partial_t \mathbb{P} \text{ using } \partial_t(1.3.7a), \\ \text{Control } \partial_t^2 \mathbb{H} \text{ with } \partial_t \mathbb{E}, \partial_t \mathbb{M} \text{ using } \partial_t(1.3.7b), \end{array} \right. \\
 (d) \quad & \left\{ \begin{array}{l} \text{Control } \mathbb{P} \text{ with } \partial_t \mathbb{P}, \partial_t^2 \mathbb{P} \text{ and } \mathbb{E} \text{ using } (1.3.7c), \\ \text{Control } \mathbb{M} \text{ with } \partial_t \mathbb{M}, \partial_t^2 \mathbb{M} \text{ and } \mathbb{H} \text{ using } (1.3.7d). \end{array} \right.
 \end{aligned}$$

Of course the constants in the estimates issued from using Maxwell's equations (1.3.7a, 1.3.7b), that is to say the ones of steps (b) and (c), will be  $\mathbf{k}$ -dependent. It is worth while mentioning that these equations are used in a different manner in (b) and (c). In step (b), they are used to estimate lower order time derivatives of the field  $(\mathbb{E}, \mathbb{H})$  with higher order time derivatives: as a consequence, the constants in the estimates will blow up when  $\mathbf{k}$  tend to 0. At the contrary, in step (c), they are used to estimate higher order time derivatives of the field  $(\mathbb{E}, \mathbb{H})$  with lower order time derivatives: as a consequence, the constants in the estimates will blow up when  $|\mathbf{k}|$  tend to  $+\infty$ .

Let us now enter in the details of the proof.

*Step 1 : Control of the energy density  $\mathcal{E}_{\mathbf{k}}^{(2)}$ .* We control below each of the terms  $\langle \mathbf{k} \rangle^{-2j} \mathcal{E}_{\mathbf{k}}^j$  appearing in the sum (1.3.20) defining  $\mathcal{E}_{\mathbf{k}}^{(2)}$ .

(a) Case  $j = 1$  : control of  $\langle \mathbf{k} \rangle^{-2} \mathcal{E}_{\mathbf{k}}^1$ .

Using the strong dissipation assumption (2.1.8), we obtain from differentiating (1.3.7c) with respect to time and a summation over  $j$  according to (1.3.8, 1.3.9)

$$|\partial_t \mathbb{E}|^2 \lesssim |\partial_t \mathbb{P}|^2 + |\partial_t^2 \mathbb{P}|^2 + |\partial_t^3 \mathbb{P}|^2. \quad (1.3.23)$$

On the other hand, by definition of each  $\mathcal{D}_{\alpha, \mathbf{k}}^j$ , and making appear at the right hand sides the terms in the sum that defines  $\mathcal{D}_{\alpha, \mathbf{k}}^{(2)}$ , we have

$$\begin{aligned}
 |\partial_t^3 \mathbb{P}|^2 \lesssim \mathcal{D}_{\alpha, \mathbf{k}}^2 & \implies \langle \mathbf{k} \rangle^{-2} |\partial_t^3 \mathbb{P}|^2 \lesssim \langle \mathbf{k} \rangle^{-2} \mathcal{D}_{\alpha, \mathbf{k}}^2 = \langle \mathbf{k} \rangle^2 (\langle \mathbf{k} \rangle^{-4} \mathcal{D}_{\alpha, \mathbf{k}}^2), \\
 |\partial_t^2 \mathbb{P}|^2 \lesssim \mathcal{D}_{\alpha, \mathbf{k}}^1 & \implies \langle \mathbf{k} \rangle^{-2} |\partial_t^2 \mathbb{P}|^2 \lesssim \langle \mathbf{k} \rangle^{-2} \mathcal{D}_{\alpha, \mathbf{k}}^1 \lesssim \langle \mathbf{k} \rangle^2 (\langle \mathbf{k} \rangle^{-2} \mathcal{D}_{\alpha, \mathbf{k}}^1) \quad (\langle \mathbf{k} \rangle \geq 1), \\
 |\partial_t \mathbb{P}|^2 \lesssim \mathcal{D}_{\alpha, \mathbf{k}} & \implies \langle \mathbf{k} \rangle^{-2} |\partial_t \mathbb{P}|^2 \lesssim \langle \mathbf{k} \rangle^{-2} \mathcal{D}_{\alpha, \mathbf{k}} \leq \langle \mathbf{k} \rangle^2 (\mathcal{D}_{\alpha, \mathbf{k}}) \quad (\langle \mathbf{k} \rangle \geq 1).
 \end{aligned}$$

After summation of the above three inequalities and by definition of  $\mathcal{D}_{\alpha, \mathbf{k}}^{(2)}$ , we deduce from

(1.3.23) that

$$\langle \mathbf{k} \rangle^{-2} |\partial_t \mathbb{E}|^2 \lesssim \langle \mathbf{k} \rangle^2 \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}. \quad (1.3.24)$$

Analogously from differentiating (1.3.7d) with respect to time, we get  $\langle \mathbf{k} \rangle^{-2} |\partial_t \mathbb{H}|^2 \lesssim \langle \mathbf{k} \rangle^2 \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}$  which, combined with (1.3.24) and the definition (1.3.14) of  $\mathcal{E}_{\mathbf{k}}^1$ , leads to

$$\langle \mathbf{k} \rangle^{-2} \mathcal{E}_{\mathbf{k}}^1 \lesssim \langle \mathbf{k} \rangle^2 \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}. \quad (1.3.25)$$

(b) Case  $j = 0$  : control of  $\mathcal{E}_{\mathbf{k}}$ .

Using the divergence free property (1.3.5) (i.e.  $\mathbf{k} \cdot \mathbb{H} = 0$ ), (1.3.7a) and (1.3.24), we get

$$|\mathbf{k}|^2 |\mathbb{H}|^2 = |\mathbf{k} \times \mathbb{H}|^2 \lesssim |\partial_t \mathbb{E}|^2 + |\partial_t \mathbb{P}|^2 \lesssim \langle \mathbf{k} \rangle^4 \mathcal{D}_{\alpha, \mathbf{k}}^{(2)} + \mathcal{D}_{\alpha, \mathbf{k}} \lesssim \langle \mathbf{k} \rangle^4 \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}$$

since  $\mathcal{D}_{\alpha, \mathbf{k}} \leq \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}$  and  $1 \leq \langle \mathbf{k} \rangle^4$ . Thus  $|\mathbb{H}|^2 \lesssim |\mathbf{k}|^{-2} \langle \mathbf{k} \rangle^4 \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}$ , or equivalently

$$|\mathbb{H}|^2 \lesssim (\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2}) \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}. \quad (1.3.26)$$

Similarly, from (1.3.7b) we obtain  $|\mathbb{E}|^2 \lesssim (\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2}) \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}$  which, combined with (1.3.26) and the definition (1.3.11) of  $\mathcal{E}_{\mathbf{k}}$ , leads to

$$\mathcal{E}_{\mathbf{k}} \lesssim (\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2}) \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}. \quad (1.3.27)$$

(c) Case  $j = 2$  : control of  $\langle \mathbf{k} \rangle^{-4} \mathcal{E}_{\mathbf{k}}^2$ .

Differentiating (1.3.7a) in time and then using (1.3.25), we get

$$|\partial_t^2 \mathbb{E}|^2 \lesssim |\mathbf{k}|^2 |\partial_t \mathbb{H}|^2 + |\partial_t^2 \mathbb{P}|^2 \lesssim \langle \mathbf{k} \rangle^6 \mathcal{D}_{\alpha, \mathbf{k}}^{(2)} + \mathcal{D}_{\alpha, \mathbf{k}}^1.$$

Thus, as  $\mathcal{D}_{\alpha, \mathbf{k}}^1 \leq \langle \mathbf{k} \rangle^2 \mathcal{D}_{\alpha, \mathbf{k}}^{(2)} \leq \langle \mathbf{k} \rangle^6 \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}$ , it follows that:

$$\langle \mathbf{k} \rangle^{-4} |\partial_t^2 \mathbb{E}|^2 \lesssim \langle \mathbf{k} \rangle^2 \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}. \quad (1.3.28)$$

Analogously, from (1.3.7b), we get  $\langle \mathbf{k} \rangle^{-4} |\partial_t^2 \mathbb{H}|^2 \lesssim \langle \mathbf{k} \rangle^2 \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}$  which, combined with (1.3.28) gives

$$\langle \mathbf{k} \rangle^{-4} \mathcal{E}_{\mathbf{k}}^2 \lesssim \langle \mathbf{k} \rangle^2 \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}. \quad (1.3.29)$$

Using (1.3.27, 1.3.25, 1.3.29) in the definition (1.3.20) of the cumulated energy density  $\mathcal{E}_{\mathbf{k}}^{(2)}$ , we get

$$\mathcal{E}_{\mathbf{k}}^{(2)} \lesssim (\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2}) \mathcal{D}_{\alpha, \mathbf{k}}^{(2)}. \quad (1.3.30)$$

*Step 2: Control of the energy density  $\mathcal{E}_{\Omega, \mathbf{k}}^{(2)}$ .* We immediately observe from (1.3.11), (1.3.12) (def. of  $\mathcal{L}_{\mathbf{k}}$  and  $\mathcal{D}_{\mathbf{k}}$ ), (1.3.14), (1.3.15) (def. of  $\mathcal{L}_{\mathbf{k}}^1$  and  $\mathcal{D}_{\mathbf{k}}^1$ ), (1.3.17) and (1.3.18) (def. of  $\mathcal{L}_{\mathbf{k}}^2$  and  $\mathcal{D}_{\mathbf{k}}^2$ ) that

$$\mathcal{E}_{\Omega, \mathbf{k}} \lesssim |\mathbb{P}|^2 + |\mathbb{M}|^2 + \mathcal{D}_{\alpha, \mathbf{k}}, \quad \mathcal{E}_{\Omega, \mathbf{k}}^1 \lesssim \mathcal{D}_{\alpha, \mathbf{k}} + \mathcal{D}_{\alpha, \mathbf{k}}^1, \quad \mathcal{E}_{\Omega, \mathbf{k}}^2 \lesssim \mathcal{D}_{\alpha, \mathbf{k}}^1 + \mathcal{D}_{\alpha, \mathbf{k}}^2$$

thus, with the adequate linear combination,  $\mathcal{E}_{\Omega, \mathbf{k}}^{(2)} \lesssim \mathcal{D}_{\alpha, \mathbf{k}}^{(2)} + |\mathbb{P}|^2 + |\mathbb{M}|^2 + \langle \mathbf{k} \rangle^{-2} \mathcal{D}_{\alpha, \mathbf{k}} + \langle \mathbf{k} \rangle^{-4} \mathcal{D}_{\alpha, \mathbf{k}}^1$ , i.e.

$$\mathcal{E}_{\Omega, \mathbf{k}}^{(2)} \lesssim \mathcal{D}_{\alpha, \mathbf{k}}^{(2)} + |\mathbb{P}|^2 + |\mathbb{M}|^2. \quad (1.3.31)$$

It remains to control  $\mathbb{P}$  and  $\mathbb{M}$  which corresponds to the point (d) above. Using the constitutive

equation (1.3.7c) and the definitions (1.3.12, 1.3.15) of  $(\mathcal{D}_{\alpha,\mathbf{k}}, \mathcal{D}_{\alpha,\mathbf{k}}^1)$ , together with (1.3.27), yields to

$$|\mathbb{P}|^2 \lesssim |\partial_t \mathbb{P}|^2 + |\partial_t^2 \mathbb{P}|^2 + |\mathbb{E}|^2 \lesssim \mathcal{D}_{\alpha,\mathbf{k}} + \mathcal{D}_{\alpha,\mathbf{k}}^1 + (\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2}) \mathcal{D}_{\alpha,\mathbf{k}}^{(2)},$$

thus, as  $\mathcal{D}_{\alpha,\mathbf{k}} \leq \mathcal{D}_{\alpha,\mathbf{k}}^{(2)}$  and  $\mathcal{D}_{\alpha,\mathbf{k}}^1 \leq \langle \mathbf{k} \rangle^2 \mathcal{D}_{\alpha,\mathbf{k}}^{(2)}$ , (see (1.3.21)),

$$|\mathbb{P}|^2 \lesssim (\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2}) \mathcal{D}_{\alpha,\mathbf{k}}^{(2)}. \quad (1.3.32)$$

Likewise from (1.3.7d), we have

$$|\mathbb{M}|^2 \lesssim (\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2}) \mathcal{D}_{\alpha,\mathbf{k}}^{(2)}. \quad (1.3.33)$$

Using (1.3.32) and (1.3.33) in (1.3.31), we finally obtained

$$\mathcal{E}_{\Omega,\mathbf{k}}^{(2)} \lesssim (\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2}) \mathcal{D}_{\alpha,\mathbf{k}}^{(2)}. \quad (1.3.34)$$

The announced estimate (1.3.22) follows from (1.3.30), (1.3.34) and the definition of  $\mathcal{L}_{\mathbf{k}}^{(2)}$ .  $\square$

Proceeding as in section 1.2 for obtaining (1.2.29), we deduce that, for some constant  $\sigma > 0$ ,

$$\forall \mathbf{k} \neq 0, \quad \mathcal{L}_{\mathbf{k}}^{(2)}(t) \leq \mathcal{L}_{\mathbf{k}}^{(2)}(0) e^{-\frac{\sigma t}{\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2}}}. \quad (1.3.35)$$

We shall use an estimate for the initial value  $\mathcal{L}_{\mathbf{k}}^{(2)}(0)$  which is the equivalent of the estimate for the Drude case. As getting this estimate is slightly more lengthy and tedious than for the Drude case, we give it in a lemma whose proof is delayed in the Appendix 1.4.2, section Appendix A.1.

| <b>Lemma 1.3.3</b>   |
|--|
| $\mathcal{L}_{\mathbf{k}}^{(2)}(0) \lesssim  \mathbb{E}_0(\mathbf{k}) ^2 +  \mathbb{H}_0(\mathbf{k}) ^2. \quad (1.3.36)$ |

Therefore, using (1.3.36) in (1.3.35), and as  $\mathcal{L}_{\mathbf{k}} \leq \mathcal{L}_{\mathbf{k}}^{(2)}$ , we have

$$\mathcal{L}_{\mathbf{k}}(t) \leq (|\mathbb{E}(\mathbf{k}, 0)|^2 + |\mathbb{H}(\mathbf{k}, 0)|^2) e^{-\frac{\sigma t}{\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2}}}. \quad (1.3.37)$$

The exponential decay rate in (1.3.35) degenerates when  $|\mathbf{k}| \rightarrow +\infty$ , as in the Drude case, but also when  $|\mathbf{k}| \rightarrow \mathbf{0}$  reason why the low (spatial) frequencies will need a special treatment leading to new assumptions involving the moments of the initial data. For this reason, we introduce the spaces, for  $p \in \mathbb{N}$ ,

$$\left\{ \begin{array}{l} L_p^1(\mathbb{R}^3) := \{u \in L^1(\mathbb{R}^3) / (1 + |\mathbf{x}|)^p u \in L^1(\mathbb{R}^3)\} \\ L_{p,0}^1(\mathbb{R}^3) := \left\{ u \in L_p^1(\mathbb{R}^3) / \forall \alpha \text{ such that } |\alpha| \leq p-1, \int_{\mathbb{R}^d} x^\alpha u(\mathbf{x}) d\mathbf{x} = 0 \right\} \end{array} \right. \quad (1.3.38)$$

where  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$  denotes a multi-index with "length"  $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$  and

$$\partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} \quad \text{and} \quad \mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \quad \text{for any } \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3. \quad (1.3.39)$$

We point out that, from the definition (1.3.38), one has  $L_0^1(\mathbb{R}^3) = L_{0,0}^1(\mathbb{R}^3) = L^1(\mathbb{R}^3)$ .

We are now in position to state the main theorem which is expressed in terms of the augmented

energy  $\mathcal{L}(t)$  that naturally replaces the one defined by (1.2.4) for the Drude case, namely

$$\begin{aligned} \mathcal{L}(t) := & \mathcal{E}(\mathbf{E}, \mathbf{H}, t) + \frac{1}{2} \left( \varepsilon_0 \sum_{j=1}^{N_e} \Omega_{e,j}^2 \|\partial_t \mathbf{P}_j(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + \mu_0 \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 \|\partial_t \mathbf{M}_\ell(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 \right), \\ & + \frac{1}{2} \left( \varepsilon_0 \sum_{j=1}^{N_e} \omega_{e,j}^2 \Omega_{e,j}^2 \|\mathbf{P}_j(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + \mu_0 \sum_{\ell=1}^{N_m} \omega_{m,\ell}^2 \Omega_{m,\ell}^2 \|\mathbf{M}_\ell(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 \right). \end{aligned} \quad (1.3.40)$$

#### Theorem 1.3.4

For any  $(\mathbf{E}_0, \mathbf{H}_0) \in L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3$  satisfying the free divergence condition, the total energy tends to 0 when  $t$  tends to  $+\infty$ :

$$\lim_{t \rightarrow +\infty} \mathcal{L}(t) = 0. \quad (1.3.41)$$

Moreover if for some integers  $m \geq 1$  and  $p \geq 0$ ,

$$(\mathbf{E}_0, \mathbf{H}_0) \in H^m(\mathbb{R}^3)^3 \times H^m(\mathbb{R}^3)^3, \quad (\mathbf{E}_0, \mathbf{H}_0) \in L_{p,0}^1(\mathbb{R}^3)^3 \times L_{p,0}^1(\mathbb{R}^3)^3, \quad (1.3.42)$$

one has a polynomial decay rate

$$\mathcal{L}(t) \leq \frac{C_{\text{HF}}^m(\mathbf{E}_0, \mathbf{H}_0)}{t^m} + \frac{C_{\text{LF}}^p(\mathbf{E}_0, \mathbf{H}_0)}{t^{p+\frac{3}{2}}} \quad (1.3.43)$$

where the above constants satisfy

$$\begin{cases} C_{\text{HF}}^m(\mathbf{E}_0, \mathbf{H}_0) \lesssim \|\mathbf{E}_0\|_{H^m(\mathbb{R}^3)}^2 + \|\mathbf{H}_0\|_{H^m(\mathbb{R}^3)}^2, \\ C_{\text{LF}}^p(\mathbf{E}_0, \mathbf{H}_0) \lesssim \sup_{|\alpha|=p} \|x^\alpha \mathbf{E}_0\|_{L^1}^2 + \sup_{|\alpha|=p} \|x^\alpha \mathbf{H}_0\|_{L^1}^2. \end{cases}$$

*Proof.* From the respective definitions of  $\mathcal{L}$  and  $\mathcal{L}_{\mathbf{k}}$ , see (1.3.11) and (1.3.40), we can use Plancherel's identity in (1.3.37) to obtain

$$\mathcal{L}(t) = \int_{\mathbb{R}^3} \mathcal{L}_{\mathbf{k}}(t) \, d\mathbf{k} \lesssim \int_{\mathbb{R}^3} (|\mathbb{E}_0(\mathbf{k})|^2 + |\mathbb{H}_0(\mathbf{k})|^2) e^{-\frac{\sigma t}{\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2}}} \, d\mathbf{k}. \quad (1.3.44)$$

Thus, from Lebesgue's dominated convergence theorem, we deduce (2.1.33).

Next, as announced before, we treat low and high (space) frequencies separately. We begin with high frequencies who can be treated as for the Drude model.

(i) If  $|\mathbf{k}| \geq 1$ , then  $\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2} \leq \langle \mathbf{k} \rangle^2 + 1 \leq 2 \langle \mathbf{k} \rangle^2$ . Thus

$$\int_{|\mathbf{k}| \geq 1} \mathcal{L}_{\mathbf{k}}(t) \, d\mathbf{k} \lesssim \int_{|\mathbf{k}| \geq 1} (|\mathbb{E}_0(\mathbf{k})|^2 + |\mathbb{H}_0(\mathbf{k})|^2) e^{-\frac{\sigma t}{2 \langle \mathbf{k} \rangle^2}} \, d\mathbf{k}.$$

Bounding, in the right hand side, the integral over  $|\mathbf{k}| \geq 1$  by the one over  $\mathbb{R}^3$ , we can proceed

as in the proof of Theorem 1.2.4 (with  $\sigma/2$  instead of  $\sigma$ , cf. (1.2.36)) to obtain

$$\int_{|\mathbf{k}| \geq 1} \mathcal{L}_{\mathbf{k}}(t) \, d\mathbf{k} \lesssim \left( \frac{2m}{\sigma e t} \right)^m \left( \|\mathbf{E}_0\|_{H^m(\mathbb{R}^3)}^2 + \|\mathbf{H}_0\|_{H^m(\mathbb{R}^3)}^2 \right). \quad (1.3.45)$$

(ii) If  $|\mathbf{k}| \leq 1$ , then  $\langle \mathbf{k} \rangle^2 + |\mathbf{k}|^{-2} \leq 3 |\mathbf{k}|^{-2}$ , (both 1 and  $|\mathbf{k}|^2$  are smaller than  $|\mathbf{k}|^{-2}$ ) thus

$$\int_{|\mathbf{k}| \leq 1} \mathcal{L}_{\mathbf{k}}(t) \, d\mathbf{k} \lesssim \int_{\mathbb{R}^3} \left( |\mathbb{E}_0(\mathbf{k})|^2 + |\mathbb{H}_0(\mathbf{k})|^2 \right) e^{-\frac{|\mathbf{k}|^2 \sigma t}{3}} \, d\mathbf{k}. \quad (1.3.46)$$

The behaviour of the right hand side is obviously dominated by what happens when  $|\mathbf{k}|$  tends to 0.

The condition  $(\mathbf{E}_0, \mathbf{H}_0) \in L_{p,0}^1(\mathbb{R}^3)^3 \times L_{p,0}^1(\mathbb{R}^3)^3$  implies in particular that

$$\forall \alpha / |\alpha| \leq p-1, \quad \partial^\alpha \mathbb{E}_0(0) = \partial^\alpha \mathbb{H}_0(0) = 0.$$

Furthermore, as  $(\mathbf{E}_0, \mathbf{H}_0) \in L_{p,0}^1(\mathbb{R}^3)^3$ , their Fourier transform  $\mathbb{E}_0$  and  $\mathbb{H}_0$  are bounded functions of class  $\mathcal{C}^p$  whose partial derivatives are bounded up to the order  $p$ . Consequently, using a Taylor expansion at 0 truncated at order  $p$ , we have

$$|\mathbb{E}_0(\mathbf{k})| \lesssim |\mathbf{k}|^p \sup_{|\alpha|=p} \|\partial_\alpha \mathbb{E}_0\|_{L^\infty}, \quad |\mathbb{H}_0(\mathbf{k})| \lesssim |\mathbf{k}|^p \sup_{|\alpha|=p} \|\partial_\alpha \mathbb{H}_0\|_{L^\infty}$$

which implies, using well known properties of the Fourier transform,

$$|\mathbb{E}_0(\mathbf{k})| \lesssim |\mathbf{k}|^p \sup_{|\alpha|=p} \| |x|^\alpha \mathbf{E}_0 \|_{L^1}, \quad |\mathbb{H}_0(\mathbf{k})| \lesssim |\mathbf{k}|^p \sup_{|\alpha|=p} \| |x|^\alpha \mathbf{H}_0 \|_{L^1}$$

Substituting the above in (1.3.46) yields

$$\int_{|\mathbf{k}| \leq 1} \mathcal{L}_{\mathbf{k}}(t) \, d\mathbf{k} \lesssim \left( \int_{\mathbb{R}^3} |\mathbf{k}|^{2p} e^{-\frac{2|\mathbf{k}|^2 \sigma t}{3}} \, d\mathbf{k} \right) \left[ \sup_{|\alpha|=p} \| |x|^\alpha \mathbf{E}_0 \|_{L^1}^2 + \sup_{|\alpha|=p} \| |x|^\alpha \mathbf{H}_0 \|_{L^1}^2 \right].$$

With the change of variable  $\xi = \sqrt{\sigma t} \mathbf{k}$ , we compute that, for some constant  $C(p) > 0$ ,

$$\int_{\mathbb{R}^3} |\mathbf{k}|^{2p} e^{-\frac{2|\mathbf{k}|^2 \sigma t}{3}} \, d\mathbf{k} = \frac{1}{(\sigma t)^{p+\frac{3}{2}}} \int_{\mathbb{R}^d} |\xi|^{2p} e^{-\frac{2|\xi|^2}{3}} \, d\xi \equiv \frac{C(p)}{(\sigma t)^{p+\frac{3}{2}}}.$$

Finally, with another constant  $C$  that only depends on  $p$  and the parameters of the model, we get

$$\int_{|\mathbf{k}| \leq 1} \mathcal{L}_{\mathbf{k}}(t) \, d\mathbf{k} \lesssim \frac{C}{t^{p+\frac{3}{2}}} \left[ \sup_{|\alpha|=p} \| |x|^\alpha \mathbf{E}_0 \|_{L^1}^2 + \sup_{|\alpha|=p} \| |x|^\alpha \mathbf{H}_0 \|_{L^1}^2 \right]. \quad (1.3.47)$$

At the end, the final estimate (1.3.43) is obtained by joining (1.3.45) and (1.3.47).  $\square$

**Remark 1.3.5** (Comparison with the estimates of [65]). The reader will check that, if we drop the second term in the estimate (1.3.43), which is specific to the problem in the whole space, our results for  $m = 1$  coincide qualitatively with the ones of [65], cf. (1.1.25) with  $p = 2$ .

## 1.4 Extensions

### 1.4.1 The problem in a bounded domain

One can consider the evolution problem associated to equations (2.1.1) (or (1.2.1)) but posed in a bounded Lipschitz domain  $\Omega$  of  $\mathbb{R}^3$  and completed, for instance, with perfectly conducting conditions (in some sense the "Dirichlet" problem for Maxwell's equations)

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega, \quad (\text{with } \mathbf{n} \text{ the unit normal vector to } \partial\Omega), \quad (1.4.1)$$

where  $\partial\Omega$  denotes the boundary of  $\Omega$  and  $\mathbf{n}$  the unit outward normal vector of  $\partial\Omega$ . In such a case, the analysis of sections 1.2 and 1.3 can be extended. The main difference is the use of the Fourier transform in space has to be replaced by an adequate modal expansion. More precisely, we introduce the cavity eigenvalue problem:

$$\left\{ \begin{array}{l} \text{Find } k \in \mathbb{R} \text{ and } (\mathbf{u}, \mathbf{v}) \neq 0 \in L^2(\Omega)^3 \times L^2(\Omega)^3 \text{ such that} \\ i \begin{pmatrix} 0 & \nabla \times \\ -\nabla \times & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = k \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega, \quad \text{and } \mathbf{u} \times \mathbf{n} = \mathbf{0}, \quad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \end{array} \right. \quad (1.4.2)$$

which corresponds to find the eigenvalues of the self-adjoint Maxwell operator  $\mathcal{A}$  in the closed subspace of  $L^2(\Omega)^3$ ,  $\mathcal{H} := \{(\mathbf{u}, \mathbf{v}) \in L^2(\Omega)^3 / \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{v} = 0 \text{ and } \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$ , namely

$$\mathcal{A} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = i \begin{pmatrix} \nabla \times \mathbf{v} \\ -\nabla \times \mathbf{u} \end{pmatrix}, \quad \forall (\mathbf{u}, \mathbf{v}) \in D(\mathcal{A}) := \mathcal{H} \cap (H_0(\text{rot}; \Omega) \times H(\text{rot}; \Omega)), \quad (1.4.3)$$

where  $H(\text{rot}; \Omega) := \{\mathbf{v} \in L^2(\Omega)^3 \mid \nabla \times \mathbf{v} \in L^2(\Omega)^3\}$  and  $H_0(\text{rot}; \Omega) := \{\mathbf{u} \in H(\text{rot}; \Omega) \mid \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega\}$ . One shows, see e.g. [27] chapter IX, that the operator  $\mathcal{A}$  has a compact resolvent with a finite dimensional kernel (see Remark 1.4.1). From the theory of selfadjoint operators with compact resolvent, and using the symmetries of Maxwell's equations, one knows that there is a countable infinity of cavity modes

$$(\pm k_n, \mathbf{u}_n^\pm, \mathbf{v}_n^\pm) \in \mathbb{R} \times L^2(\Omega)^3 \times L^2(\Omega)^3, \quad n \in \mathbb{N}, \quad \text{with } k_n > 0, \quad k_n \rightarrow +\infty \quad (1.4.4)$$

with  $\mathbf{u}_n^+ = \mathbf{u}_n^-$  and  $\mathbf{v}_n^+ = -\mathbf{v}_n^-$  in such a way that  $\{(\mathbf{u}_n^\pm, \mathbf{v}_n^\pm), n \in \mathbb{N}\}$  form an orthonormal basis of the Hilbert space  $(\text{Ker } \mathcal{A})^\perp$ . Then, one can decompose the electromagnetic field as

$$\mathbf{E}(\cdot, t) = \sum_{\pm} \sum_{n=0}^{+\infty} \mathbb{E}_n^\pm(t) \mathbf{u}_n^\pm, \quad \mathbf{H}(\cdot, t) = \sum_{\pm} \sum_{n=0}^{+\infty} \mathbb{H}_n^\pm(t) \mathbf{v}_n^\pm,$$

and the auxiliary field  $\mathbf{P}_j$  and  $\mathbf{M}_\ell$  (understood as in (1.3.1)) accordingly

$$\mathbf{P}_j(\cdot, t) = \sum_{\pm} \sum_{n=0}^{+\infty} \mathbb{P}_{j,n}^\pm(t) \mathbf{u}_n^\pm, \quad \mathbf{M}_\ell(\cdot, t) = \sum_{\pm} \sum_{n=0}^{+\infty} \mathbb{M}_{\ell,n}^\pm(t) \mathbf{v}_n^\pm.$$

The rest of the analysis follows exactly the same lines as for  $\Omega = \mathbb{R}^3$ , modulo the following substitutions

$$\mathbf{k} \in \mathbb{R}^3 \rightarrow \{(\pm, n), n \in \mathbb{N}\}, \quad \int_{\mathbb{R}^3} d\mathbf{k} \rightarrow \sum_{\pm} \sum_{n=0}^{+\infty}, \quad \dots \quad \text{etc.}$$

The results are then similar to the one of [Theorem 1.3.4](#) without the second term in the right hand side of the estimate [\(1.3.43\)](#) and the hypothesis of the second line of [\(1.3.42\)](#), provided some modifications on the assumptions for the initial data which must now satisfy

$$(\mathbf{E}_0, \mathbf{H}_0) \in \mathcal{H}_0 := (\text{Ker } \mathcal{A})^\perp, \quad (\text{orthogonality in } \mathcal{H}) \quad (\text{see also remark } \a href="#">1.4.1) \quad (1.4.5)$$

and the Sobolev regularity [\(1.3.42\)](#)(first line) must be replaced by

$$(\mathbf{E}_0, \mathbf{H}_0) \in D(\mathcal{A}^m) \cap \mathcal{H}_0, \quad (\text{see also remark } \a href="#">1.4.2). \quad (1.4.6)$$

**Remark 1.4.1.** The (finitely dimensional) space  $\text{Ker } \mathcal{A}$  is the space of electromagnetic static fields

$$\text{Ker } \mathcal{A} = \{(\mathbf{u}, \mathbf{v}) \in L^2(\Omega)^3 / \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{v} = 0, \nabla \times \mathbf{u} = \nabla \times \mathbf{v} = 0 \text{ on } \Omega \text{ and } \mathbf{u} \times \mathbf{n} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

When  $\Omega$  in *simply connected*, it is known that  $\text{Ker } \mathcal{A} = \{0\}$  and that its dimension increases with the complexity of the topology of  $\Omega$ . For the proof of these assertions, we refer to chapter IX of [\[27\]](#). In some sense, the condition [\(1.4.5\)](#) can be seen as a substitute to the second condition in [\(1.3.42\)](#).

**Remark 1.4.2.** The condition  $(\mathbf{E}_0, \mathbf{H}_0) \in D(\mathcal{A}^m)$  implies  $(\mathbf{E}_0, \mathbf{H}_0) \in H_{loc}^m(\Omega)^3 \times H_{loc}^m(\Omega)^3$ . Furthermore, for  $C^\infty$  domains  $\Omega$ , the condition  $(\mathbf{E}_0, \mathbf{H}_0) \in D(\mathcal{A}^m)$  (see e.g. [\[27\]](#), chapter IX) is equivalent to  $(\mathbf{E}_0, \mathbf{H}_0) \in H^m(\Omega)^3 \times H^m(\Omega)^3$  for any integer  $m > 0$ .

## 1.4.2 The case of mixed Drude-Lorentz models

In the sums [\(1.1.14\)](#) defining  $\varepsilon(\omega)$  and  $\mu(\omega)$  the resonance frequency  $\omega_{e,j}$  or  $\omega_{m,\ell}$  are either strictly positive or zero. For instance, for  $\varepsilon(\omega)$ , we shall say that

$$\frac{\Omega_{e,j}^2}{\omega^2 + i\alpha_{e,j}\omega - \omega_{e,j}^2}, \quad \omega_{e,j} > 0 \text{ is a Lorentz term,} \quad \frac{\Omega_{e,j}^2}{\omega^2 + i\alpha_{e,j}\omega} \text{ is a Drude term.} \quad (1.4.7)$$

In [Section 1.2](#) (standard Drude model), we consider the case where  $\varepsilon(\omega)$  and  $\mu(\omega)$  contained a single Drude term while in [Section 1.3](#), we consider the case where  $\varepsilon(\omega)$  and  $\mu(\omega)$  only contained Lorentz terms, because of assumption (see [\(1.3.4\)](#), second line).

It is natural to look at the cases where  $\varepsilon(\omega)$  (resp.  $\mu(\omega)$ ) contains Lorentz terms but also Drude terms whose number is  $N_{d,e}$  (respectively  $N_{d,m}$ ). It appears that our Lyapunov approach can easily handle these cases (modulo minor additional manipulations) with the following results:

- If  $N_{d,e} > 0$  and  $N_{d,m} > 0$ , the result is the same as for the Drude model (see [Theorem 1.2.4](#)).
- If  $N_{d,e} > 0$  and  $N_{d,m} = 0$  (or the contrary), the result is the one for (generalized) Lorentz (see [Theorem 1.3.4](#)).

## Appendix A

### A.1. On the dissipation condition of [\[37\]](#) for Lorentz models

Let us recall that, when the limit [\(1.1.17\)](#) exists almost everywhere on the real axis (which is the case for (generalized) Lorentz models), the sufficient dissipation condition [\(6.4\)](#) given in [\[37\]](#) reads

$$\text{for a. e. } \omega \in \mathbb{R}, \quad \text{Im } \omega \hat{\chi}_e(\omega) \geq \gamma(\omega)^{-1} > 0, \quad \gamma \in L_{loc}^1(\mathbb{R}). \quad (1.4.8)$$

By virtue of the expression (1.1.14)(a) of the complex permittivity  $\varepsilon(\omega)$  and the formula (1.1.5), we have, as soon as  $\omega \in \mathbb{R}$  and  $\omega^2 + i\alpha_{e,j}\omega - \omega_{e,j}^2 \neq 0$  for all  $j \in \{1, \dots, N_e\}$ :

$$\operatorname{Im} \omega \hat{\chi}_e(\omega) = \sum_{j=1}^{N_e} \frac{\alpha_{e,j} \Omega_{e,j}^2 \omega^2}{|\omega^2 + i\alpha_{e,j}\omega - \omega_{e,j}^2|^2}.$$

Let  $J_0 := \{j \in \{1, \dots, N_e\} / \omega_{e,j} = 0\}$  and  $J_+ := \{j \in \{1, \dots, N_e\} / \alpha_{e,j} > 0\}$ .

(i)  $J_+ = \emptyset$ , i. e. for all  $j \in \{1, \dots, N_e\}$ ,  $\alpha_{e,j} = 0$  then  $\operatorname{Im} \omega \hat{\chi}_e(\omega) = 0$  and (1.1.18) can not hold.

(ii)  $J_+ \neq \emptyset$ . We distinguish two subcases.

(a)  $J_+ \cap J_0 \neq \emptyset$ . This means that one  $\omega_{e,j}$  vanishes, for instance  $\omega_{e,1} = 0$ , and  $\alpha_{e,1} > 0$ . Then

$$\operatorname{Im} \omega \hat{\chi}_e(\omega) = \frac{\alpha_{e,1} \Omega_{e,1}^2 \omega^2}{|\omega^2 + i\alpha_{e,1}\omega|^2} + \sum_{j=2}^{N_e} \frac{\alpha_{e,j} \Omega_{e,j}^2 \omega^2}{|\omega^2 + i\alpha_{e,j}\omega - \omega_{e,j}^2|^2} \geq \frac{\alpha_{e,1} \Omega_{e,1}^2}{|\omega + i\alpha_{e,1}|}$$

in which case (1.1.18)(ii) holds true with  $\gamma : \omega \mapsto (\alpha_{e,1} \Omega_{e,1}^2)^{-1} |\omega + i\alpha_{e,1}| \in L_{loc}^1(\mathbb{R})$ .

(b)  $J_+ \cap J_0 = \emptyset$ . In this case  $\operatorname{Im} \omega \hat{\chi}_e(\omega) \underset{\omega \rightarrow 0}{\sim} \left( \sum_{j \in J_+} \frac{\alpha_{e,j} \Omega_{e,j}^2}{\omega_{e,j}^4} \right) \omega^2$ , and (1.1.18) can not hold.

## A.2. On the energy identity (1.1.27)

Let  $\mathbf{E}, \mathbf{H} \in C^0(\mathbb{R}^+, H(\operatorname{rot}; \mathbb{R}^3)) \cap C^1(\mathbb{R}^+, L^2(\mathbb{R}^3)^3)$  and  $\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B} \in C^1(\mathbb{R}^+, L^2(\mathbb{R}^3)^3)$  be solutions of the equations (1.1.1), (1.1.2) and (1.1.3) where  $\chi_\nu \in C^3(\mathbb{R}^+)$  for  $\nu = e, m$ . We explain in this appendix how to obtain the energy identity (1.1.27) with (1.1.29) and (1.1.30).

From equations (1.1.1), (1.1.2) and (1.1.3) it is straightforward to deduce the identity (with  $\mathcal{E}(t)$  the electromagnetic energy, see (2.1.7))

$$\frac{d\mathcal{E}(t)}{dt} + \mathcal{I}(t) = 0, \quad \mathcal{I}(t) := \int_{\mathbb{R}^3} \partial_t \mathbf{P}_{\text{tot}}(\mathbf{x}, t) \mathbf{E}(\mathbf{x}, t) d\mathbf{x} + \int_{\mathbb{R}^3} \partial_t \mathbf{M}_{\text{tot}}(\mathbf{x}, t) \mathbf{H}(\mathbf{x}, t) d\mathbf{x}. \quad (1.4.9)$$

Differentiating the constitutive laws (1.1.3) in time, we have

$$\begin{aligned} \partial_t \mathbf{P}_{\text{tot}}(\mathbf{x}, t) &= \varepsilon_0 \int_0^t \chi_e'(t-s) \mathbf{E}(\mathbf{x}, s) ds + \varepsilon_0 \chi_e(0) \mathbf{E}(\mathbf{x}, t) \\ \partial_t \mathbf{M}_{\text{tot}}(\mathbf{x}, t) &= \mu_0 \int_0^t \chi_m'(t-s) \mathbf{H}(\mathbf{x}, t-s) ds + \mu_0 \chi_m(0) \mathbf{H}(\mathbf{x}, t) \end{aligned}$$



from which we deduce that

$$\begin{cases} \mathcal{I}(t) = \mathcal{I}_e(t) + \mathcal{I}_m(t) & \text{with} \\ \mathcal{I}_e(t) := \varepsilon_0 \chi_\varepsilon(0) \int_{\mathbb{R}^3} |\mathbf{E}(\mathbf{x}, t)|^2 d\mathbf{x} + \varepsilon_0 \int_{\mathbb{R}^3} \left( \mathbf{E}(\mathbf{x}, t) \cdot \int_0^t \chi'_\varepsilon(t-s) \mathbf{E}(\mathbf{x}, s) ds \right) d\mathbf{x}, \\ \mathcal{I}_m(t) := \mu_0 \chi_m(0) \int_{\mathbb{R}^3} |\mathbf{H}(\mathbf{x}, t)|^2 d\mathbf{x} + \mu_0 \int_{\mathbb{R}^3} \left( \mathbf{H}(\mathbf{x}, t) \cdot \int_0^t \chi'_m(t-s) \mathbf{H}(\mathbf{x}, s) ds \right) d\mathbf{x}. \end{cases} \quad (1.4.10)$$

It remains to transform  $\mathcal{I}_e(t)$  and  $\mathcal{I}_m(t)$ .

The basic technical ingredient concerns convolution type quadratic forms:

$$Qu(t) := \int_0^t k'(t-s) u(s) u'(t) ds, \quad (1.4.11)$$

where  $k(t)$  is a given convolution kernel. When  $k(t) = C \delta(t)$ , formally  $Qu(t) = C \frac{d}{dt} |u(t)|^2$ . The next lemma (lemma 3.2 in [61]) generalizes this observation to a smooth kernel. For completeness, we provide here a constructive proof (which is not given in [61]).

**Lemma A. 1**

Given  $k \in C^2(\mathbb{R}^+)$  and  $u \in C^1(\mathbb{R}^+)$ ,  $Qu(t)$  (defined in (1.4.11)) satisfies

$$\begin{cases} Qu(t) = \frac{1}{2} \frac{d}{dt} \left[ (k(t) - k(0)) |u(t)|^2 - \int_0^t k'(t-s) (u(s) - u(t))^2 ds \right] \\ \quad - \frac{1}{2} k'(t) |u(t)|^2 + \frac{1}{2} \int_0^t k''(t-s) (u(s) - u(t))^2 ds. \end{cases} \quad (1.4.12)$$

*Proof.* The guiding idea is to make appear time derivatives of square quantities in the expression of  $Qu(t)$ .

The main trick is to write  $u(s) = u(t) + (u(s) - u(t))$  in (1.4.11), in such a way that

$$Qu(t) = \frac{1}{2} (k(t) - k(0)) \frac{d}{dt} |u(t)|^2 + \int_0^t k'(t-s) (u(s) - u(t)) u'(t) ds. \quad (1.4.13)$$

On the one hand, one has

$$\frac{1}{2} (k(t) - k(0)) \frac{d}{dt} |u(t)|^2 = \frac{1}{2} \frac{d}{dt} \left( (k(t) - k(0)) |u(t)|^2 \right) - \frac{1}{2} k'(t) |u(t)|^2. \quad (1.4.14)$$

On the other hand, observing that  $(u(t) - u(s)) u'(t) = \frac{1}{2} \frac{d}{dt} [(u(s) - u(t))^2]$  we have

$$\int_0^t k'(t-s) (u(s) - u(t)) u'(t) ds = -\frac{1}{2} \int_0^t k'(t-s) \frac{d}{dt} [(u(s) - u(t))^2] ds,$$

i.e., since  $k'(t-s) \frac{d}{dt} [(u(s) - u(t))^2] = \frac{d}{dt} [k'(t-s) (u(s) - u(t))^2] - k''(t-s) (u(s) - u(t))^2$ ,

$$\left| \int_0^t k'(t-s) (u(s) - u(t)) u'(t) ds = -\frac{1}{2} \int_0^t \frac{d}{dt} [k'(t-s) (u(s) - u(t))^2] ds \right. \\ \left. + \frac{1}{2} \int_0^t k''(t-s) (u(s) - u(t))^2 ds. \right.$$

Finally  $\int_0^t \frac{d}{dt} [k'(t-s) (u(s) - u(t))^2] ds = \frac{d}{dt} \int_0^t k'(t-s) (u(s) - u(t))^2 ds$ , thus

$$\left| \int_0^t k'(t-s) (u(s) - u(t)) u'(t) ds = -\frac{1}{2} \frac{d}{dt} \int_0^t k'(t-s) (u(s) - u(t))^2 ds \right. \\ \left. + \frac{1}{2} \int_0^t k''(t-s) (u(s) - u(t))^2 ds. \right. \quad (1.4.15)$$

Finally, substituting (1.4.14) and (1.4.15) in (1.4.13) leads to (1.4.12).  $\square$

Now, we wish to transform the integrand (in space) in the second term of the expression (1.4.10) of  $\mathcal{I}_e(t)$  by making appear a quantity of the form (1.4.11). For this, it is useful to introduce primitives (in time) of the fields. More precisely, for  $\mathbf{F} = \mathbf{E}, \mathbf{H}$ , we define

$$\mathbf{F}_p(\mathbf{x}, t) = \int_0^t f(\mathbf{x}, s) ds, \quad \forall t \geq 0 \text{ and a.e. } \mathbf{x} \in \mathbb{R}^3.$$

In order to transform  $\mathcal{I}_e(t)$ , we first perform an integration by parts in time to get, since  $\mathbf{E}_p(\mathbf{x}, 0) = 0$ ,

$$\int_0^t \chi_e'(t-s) \mathbf{E}(\mathbf{x}, s) ds = \int_0^t \chi_e'(t-s) \partial_t \mathbf{E}_p(\mathbf{x}, s) ds = \chi_e'(0) \mathbf{E}_p(\mathbf{x}, t) + \int_0^t \chi_e''(t-s) \mathbf{E}_p(\mathbf{x}, s) ds.$$

As a consequence, since  $\mathbf{E} = \partial_t \mathbf{E}_p$ , we have

$$\mathbf{E}(\mathbf{x}, t) \cdot \int_0^t \chi_e'(t-s) \mathbf{E}(\mathbf{x}, s) ds = \frac{\chi_e'(0)}{2} \frac{d}{dt} |\mathbf{E}_p(\mathbf{x}, t)|^2 + \int_0^t \chi_e''(t-s) \mathbf{E}_p(\mathbf{x}, s) \cdot \partial_t \mathbf{E}_p(\mathbf{x}, t) ds.$$

The second term in the right hand side of the above expression is a sum of terms of the form (1.4.11) with  $k = \chi_e'$ . Thus, integrating in space and then using the lemma 1, and substituting the resulting equality in the expression (1.4.10) of  $\mathcal{I}_e(t)$  gives (note that the terms involving  $\chi_e'(0)$  cancel each other)

$$\left| \mathcal{I}_e(t) = \frac{\varepsilon_0}{2} \frac{d}{dt} \left( \chi_e'(t) \int_{\mathbb{R}^3} |\mathbf{E}_p(\mathbf{x}, t)|^2 d\mathbf{x} \right) - \frac{\varepsilon_0}{2} \frac{d}{dt} \left( \int_0^t \chi_e''(t-s) \left( \int_{\mathbb{R}^3} |\mathbf{E}_p(\mathbf{x}, t) - \mathbf{E}_p(\mathbf{x}, s)|^2 d\mathbf{x} \right) ds \right) \right. \\ \left. + \varepsilon_0 \chi_e(0) \int_{\mathbb{R}^3} |\mathbf{E}(\mathbf{x}, t)|^2 d\mathbf{x} - \frac{\varepsilon_0}{2} \chi_e''(t) \int_{\mathbb{R}^3} |\mathbf{E}_p(\mathbf{x}, t)|^2 d\mathbf{x} \right. \\ \left. + \frac{\varepsilon_0}{2} \int_0^t \chi_e'''(t-s) \left( \int_{\mathbb{R}^3} |\mathbf{E}_p(\mathbf{x}, t) - \mathbf{E}_p(\mathbf{x}, s)|^2 d\mathbf{x} \right) ds. \right. \quad (1.4.16)$$

Analogously, we have

$$\begin{aligned} \mathcal{I}_m(t) &= \frac{\mu_0}{2} \frac{d}{dt} \left( \chi'_m(t) \int_{\mathbb{R}^3} |\mathbf{H}_p(\mathbf{x}, t)|^2 d\mathbf{x} \right) - \frac{\mu_0}{2} \frac{d}{dt} \left( \int_0^t \chi''_m(t-s) \left( \int_{\mathbb{R}^3} |\mathbf{H}_p(\mathbf{x}, t) - \mathbf{H}_p(\mathbf{x}, s)|^2 d\mathbf{x} \right) ds \right) \\ &\quad + \mu_0 \chi_m(0) \int_{\mathbb{R}^3} |\mathbf{H}(\mathbf{x}, t)|^2 d\mathbf{x} - \frac{\mu_0}{2} \chi''_m(t) \int_{\mathbb{R}^3} |\mathbf{H}_p(\mathbf{x}, t)|^2 d\mathbf{x} \\ &\quad + \frac{\mu_0}{2} \int_0^t \chi'''_m(t-s) \left( \int_{\mathbb{R}^3} |\mathbf{H}_p(\mathbf{x}, t) - \mathbf{H}_p(\mathbf{x}, s)|^2 d\mathbf{x} \right) ds. \end{aligned} \tag{1.4.17}$$

Finally, (1.1.27) is simply obtained by gathering (1.4.9), (1.4.10), (1.4.16) and (1.4.17).

### A.3. Estimating $\mathcal{L}_{\mathbf{k}}^{(2)}(0)$ in the Lorentz case

Below, we use the notation of Section 1.3 and our goal is to prove the estimate of Lemma 1.3.3, namely

$$\mathcal{L}_{\mathbf{k}}^{(2)}(0) \lesssim |\mathbb{E}_0(\mathbf{k})|^2 + |\mathbb{H}_0(\mathbf{k})|^2. \tag{1.4.18}$$

First, by definition of  $\mathcal{L}_{\mathbf{k}}(0)$  and since  $\mathbb{P}(\mathbf{k}, 0) = \mathbb{M}(\mathbf{k}, 0) = \partial_t \mathbb{P}(\mathbf{k}, 0) = \partial_t \mathbb{M}(\mathbf{k}, 0) = 0$ ,

$$\mathcal{L}_{\mathbf{k}}(0) = \frac{1}{2} (\varepsilon_0 |\mathbb{E}_0(\mathbf{k})|^2 + \mu_0 |\mathbb{H}_0(\mathbf{k})|^2). \tag{1.4.19}$$

Next we estimate  $\langle \mathbf{k} \rangle^{-2} \mathcal{L}_{\mathbf{k}}^1(0) = \langle \mathbf{k} \rangle^{-2} \mathcal{E}_{\mathbf{k}}^1(0) + \langle \mathbf{k} \rangle^{-2} \mathcal{E}_{\Omega, \mathbf{k}}^1(0)$ .

From equations (1.3.7a) and (1.3.7b) at  $t = 0$ , since  $\partial_t \mathbb{P}(\mathbf{k}, 0) = \partial_t \mathbb{M}(\mathbf{k}, 0) = 0$ ,

$$\langle \mathbf{k} \rangle^{-2} \mathcal{E}_{\mathbf{k}}^1(0) \lesssim \langle \mathbf{k} \rangle^{-2} (|\mathbf{k} \times \mathbb{E}_0(\mathbf{k})|^2 + |\mathbf{k} \times \mathbb{H}_0(\mathbf{k})|^2) \lesssim |\mathbb{E}_0(\mathbf{k})|^2 + |\mathbb{H}_0(\mathbf{k})|^2.$$

Since  $\mathbb{P}(\mathbf{k}, 0) = \mathbb{M}(\mathbf{k}, 0) = \partial_t \mathbb{P}(\mathbf{k}, 0) = \partial_t \mathbb{M}(\mathbf{k}, 0) = 0$ , from (1.3.7c) and (1.3.7d) at  $t = 0$ , we have

$$\langle \mathbf{k} \rangle^{-2} \mathcal{E}_{\Omega, \mathbf{k}}^1(0) \lesssim \langle \mathbf{k} \rangle^{-2} (|\partial_t^2 \mathbb{P}(\mathbf{k}, 0)|^2 + |\partial_t^2 \mathbb{M}(\mathbf{k}, 0)|^2) \lesssim |\mathbb{E}_0(\mathbf{k})|^2 + |\mathbb{H}_0(\mathbf{k})|^2$$

and as a consequence

$$\langle \mathbf{k} \rangle^{-2} \mathcal{L}_{\mathbf{k}}^1(0) \lesssim |\mathbb{E}_0(\mathbf{k})|^2 + |\mathbb{H}_0(\mathbf{k})|^2. \tag{1.4.20}$$

Finally we estimate  $\langle \mathbf{k} \rangle^{-4} \mathcal{L}_{\mathbf{k}}^2(0) = \langle \mathbf{k} \rangle^{-4} \mathcal{E}_{\mathbf{k}}^2(0) + \langle \mathbf{k} \rangle^{-4} \mathcal{E}_{\Omega, \mathbf{k}}^2(0)$ .

For bounding  $\langle \mathbf{k} \rangle^{-4} \mathcal{E}_{\mathbf{k}}^2(0)$ , we differentiate (1.3.7a) (resp. (1.3.7b)) and evaluate the resulting equations at  $t = 0$  to express  $\partial_t^2 \mathbb{E}(\mathbf{k}, 0)$  (resp.  $\partial_t^2 \mathbb{H}(\mathbf{k}, 0)$ ) in terms of  $\mathbf{k} \times \partial_t \mathbb{H}(\mathbf{k}, 0)$  and  $\partial_t^2 \mathbb{P}(\mathbf{k}, 0)$  (resp.  $\mathbf{k} \times \partial_t \mathbb{E}(\mathbf{k}, 0)$  and  $\partial_t^2 \mathbb{M}(\mathbf{k}, 0)$ ). From this, we deduce

$$\langle \mathbf{k} \rangle^{-4} \mathcal{E}_{\mathbf{k}}^2(0) \lesssim \langle \mathbf{k} \rangle^{-4} (|\mathbf{k}|^2 \mathcal{E}_{\mathbf{k}}^1(0) + |\partial_t^2 \mathbb{P}(\mathbf{k}, 0)|^2 + |\partial_t^2 \mathbb{M}(\mathbf{k}, 0)|^2) \lesssim \langle \mathbf{k} \rangle^{-2} \mathcal{L}_{\mathbf{k}}^1(0) \lesssim |\mathbb{E}_0(\mathbf{k})|^2 + |\mathbb{H}_0(\mathbf{k})|^2, \tag{1.4.21}$$

where for the last inequality we have used (1.4.20). For the last term we first observe that

$$\langle \mathbf{k} \rangle^{-4} \mathcal{E}_{\Omega, \mathbf{k}}^2(0) \lesssim \langle \mathbf{k} \rangle^{-4} (|\partial_t^3 \mathbb{P}(\mathbf{k}, 0)|^2 + |\partial_t^3 \mathbb{M}(\mathbf{k}, 0)|^2 + |\partial_t^2 \mathbb{P}(\mathbf{k}, 0)|^2 + |\partial_t^2 \mathbb{M}(\mathbf{k}, 0)|^2)$$

and we obtain  $\partial_t^3 \mathbb{P}(\mathbf{k}, 0)$  (resp.  $\partial_t^3 \mathbb{M}(\mathbf{k}, 0)$ ) in function of  $\partial_t \mathbb{E}(\mathbf{k}, 0)$  and  $\partial_t^2 \mathbb{P}(\mathbf{k}, 0)$  (resp.  $\partial_t \mathbb{H}(\mathbf{k}, 0)$  and  $\partial_t^2 \mathbb{M}(\mathbf{k}, 0)$ ) from equations (1.3.7c) and (1.3.7d) after time differentiation. This leads to

$$\langle \mathbf{k} \rangle^{-4} \mathcal{E}_{\Omega, \mathbf{k}}^2(0) \lesssim \langle \mathbf{k} \rangle^{-4} \langle \mathbf{k} \rangle^2 \mathcal{L}_{\mathbf{k}}^1(0) \lesssim \langle \mathbf{k} \rangle^{-2} \mathcal{L}_{\mathbf{k}}^1(0) \lesssim |\mathbb{E}_0(\mathbf{k})|^2 + |\mathbb{H}_0(\mathbf{k})|^2 \tag{1.4.22}$$

where uses (1.4.20) for the last inequality. Adding the two inequalities (1.4.21) and (1.4.22) we obtain

$$\langle \mathbf{k} \rangle^{-4} \mathcal{L}_{\mathbf{k}}^2(0) \lesssim |\mathbb{E}_0(\mathbf{k})|^2 + |\mathbb{H}_0(\mathbf{k})|^2. \quad (1.4.23)$$

Finally, (1.4.18) is deduced from (1.4.19), (1.4.20) and (1.4.23).

#### A.4. Well-posedness and regularity of the solutions of the Cauchy problem in generalized Lorentz media

The (Cauchy) problem (1.3.1) can be rewritten as a generalized Schrödinger evolution problem:

$$\frac{d\mathbf{U}}{dt} + i\mathbb{A}\mathbf{U} = 0 \text{ with } \mathbf{U}(0) = \mathbf{U}_0, \quad (1.4.24)$$

where the Hamiltonian  $\mathbb{A}$  is an unbounded operator on the Hilbert-space:

$$\mathcal{H} := L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^{3N_e} \times L^2(\mathbb{R}^3)^{3N_e} \times L^2(\mathbb{R}^3)^{3N_m} \times L^2(\mathbb{R}^3)^{3N_m}, \quad (1.4.25)$$

endowed by the following inner product :

for any  $\mathbf{U} = (\mathbf{E}, \mathbf{H}, \mathbf{P}, \dot{\mathbf{P}}, \mathbf{M}, \dot{\mathbf{M}})$  and  $\mathbf{U}' = (\mathbf{E}', \mathbf{H}', \mathbf{P}', \mathbf{M}', \dot{\mathbf{P}}', \dot{\mathbf{M}}')$ , where  $(\mathbf{P}, \mathbf{M})$  is defined as in (1.3.3), (and the same for  $(\mathbf{P}', \mathbf{M}')$ ,  $(\dot{\mathbf{P}}, \dot{\mathbf{M}})$  and  $(\dot{\mathbf{P}}', \dot{\mathbf{M}}')$ )

$$\left| \begin{aligned} (\mathbf{U}, \mathbf{U}')_{\mathcal{H}} = & \frac{\varepsilon_0}{2} (\mathbf{E}, \mathbf{E}')_{L^2} + \frac{\mu_0}{2} (\mathbf{H}, \mathbf{H}')_{L^2} + \frac{\varepsilon_0}{2} \sum_{j=1}^{N_e} \omega_{e,j}^2 \Omega_{e,j}^2 (\mathbf{P}_j, \mathbf{P}'_j)_{L^2} + \frac{\varepsilon_0}{2} \sum_{j=1}^{N_e} \Omega_{e,j}^2 (\dot{\mathbf{P}}_j, \dot{\mathbf{P}}'_j)_{L^2} \\ & + \frac{\mu_0}{2} \sum_{\ell=1}^{N_m} \Omega_{e,\ell}^2 (\mathbf{M}_\ell, \mathbf{M}'_\ell)_{L^2} + \frac{\mu_0}{2} \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 (\dot{\mathbf{M}}_\ell, \dot{\mathbf{M}}'_\ell)_{L^2}. \end{aligned} \right.$$

More precisely, if we introduce  $\mathbb{A} : D(\mathbb{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  defined by

$$\forall \mathbf{U} = (\mathbf{E}, \mathbf{H}, \mathbf{P}, \dot{\mathbf{P}}, \mathbf{M}, \dot{\mathbf{M}}) \in D(\mathbb{A}), \quad \mathbb{A}\mathbf{U} := -i \begin{pmatrix} -\varepsilon_0^{-1} \nabla \times \mathbf{H} + \sum \Omega_{e,j}^2 \dot{\mathbf{P}}_j \\ \mu_0^{-1} \nabla \times \mathbf{E} + \sum \Omega_{m,\ell}^2 \dot{\mathbf{M}}_\ell \\ -\dot{\mathbf{P}} \\ (\alpha_{e,j} \dot{\mathbf{P}}_j + \omega_{e,j}^2 \mathbf{P}_j - \mathbf{E})_j \\ -\dot{\mathbf{M}} \\ (\alpha_{m,\ell} \dot{\mathbf{M}}_\ell + \omega_{m,\ell}^2 \mathbf{M}_\ell - \mathbf{H})_\ell \end{pmatrix} \quad (1.4.26)$$

the domain  $D(\mathbb{A})$  (dense in  $\mathcal{H}$ ) being given by

$$D(\mathbb{A}) := H(\text{rot}; \mathbb{R}^3) \times H(\text{rot}; \mathbb{R}^3) \times L^2(\mathbb{R}^3)^{3N_e} \times L^2(\mathbb{R}^3)^{3N_e} \times L^2(\mathbb{R}^3)^{3N_m} \times L^2(\mathbb{R}^3)^{3N_m}, \quad (1.4.27)$$

we observe that one can rewrite (1.3.1) as (1.4.24) with the initial condition  $\mathbf{U}_0 = (\mathbf{E}_0, \mathbf{H}_0, 0, 0, 0, 0) \in \mathcal{H}$ . We point that in the formula (1.4.26), we omit in the summation bounds 1 and  $N_e$  (resp. 1 and  $N_m$ ) for sum on  $j$  (resp.  $\ell$ ) indices.

The well-posedness of (1.4.24) is ensured by the following lemma [Lemma A. 2](#). This lemma holds under the conditions (1.1.13) and (1.3.4) on the coefficients of the system (1.3.1). In particular, no dissipation condition is required.

**Lemma A. 2**

The operator  $-i\mathbb{A}$  is a maximal dissipative operator.

*Proof.* To show that  $-i\mathbb{A}$  is a maximal dissipative operator (see [28], theorem 8 p 340) is equivalent to show one hand that  $-i\mathbb{A}$  is dissipative, i.e.  $\text{Im}(\mathbb{A}\mathbf{U}, \mathbf{U}) \leq 0$  for all  $\mathbf{U} \in D(\mathbb{A})$  and that there exists  $\omega \in \mathbb{C}^+$  such that  $\mathbb{A} - \omega\mathbf{I}$  is surjective. Thus the proof will consist in two steps.

**Step 1:  $-i\mathbb{A}$  is dissipative.** Performing  $(\mathbb{A}\mathbf{U}, \mathbf{U})_{\mathcal{H}}$  thanks to (1.4.26), after integration by parts, i. e.  $(\nabla \times \mathbf{H}, \mathbf{E})_{L^2} = (\mathbf{H}, \nabla \times \mathbf{E})_{L^2} = (\nabla \times \mathbf{E}, \mathbf{H})_{L^2}$ , one finds that

$$\text{Im}(\mathbb{A}\mathbf{U}, \mathbf{U}) = -\frac{\varepsilon_0}{2} \sum_{j=1}^{N_e} \alpha_{e,j} \Omega_{e,j}^2 \|\dot{\mathbf{P}}_j\|_{L^2}^2 - \frac{\mu_0}{2} \sum_{\ell=1}^{N_m} \alpha_{m,\ell} \Omega_{m,\ell}^2 \|\dot{\mathbf{M}}_\ell\|_{L^2}^2 \leq 0.$$

**Step 2: for any  $\omega \in \mathbb{C}^+$ ,  $(\mathbb{A} - \omega\mathbf{I})D(\mathbb{A}) = \mathcal{H}$ .** We prove also the injectivity of the operator  $\mathbb{A} - \omega\mathbf{I}$  by showing that for any  $\mathbf{F} = (\mathbf{e}, \mathbf{h}, \mathbf{p}, \dot{\mathbf{p}}, \mathbf{m}, \dot{\mathbf{m}}) \in \mathcal{H}$  the system:

$$(\mathbb{A} - \omega\mathbf{I})\mathbf{U} = \mathbf{F} \quad (1.4.28)$$

admits a unique solution  $\mathbf{U} = (\mathbf{E}, \mathbf{H}, \mathbf{P}, \dot{\mathbf{P}}, \mathbf{M}, \dot{\mathbf{M}}) \in D(\mathbb{A})$ . To prove this, one first eliminates  $\dot{\mathbf{P}}$  and  $\dot{\mathbf{M}}$  in the system (1.4.28) by using that  $\dot{\mathbf{P}} = -i\omega\mathbf{P} - i\dot{\mathbf{p}}$  and  $\dot{\mathbf{M}} = -i\omega\mathbf{M} - i\dot{\mathbf{m}}$  and obtain the following expression for  $\mathbf{P}$  and  $\mathbf{M}$  in term of  $\mathbf{E}$  and  $\mathbf{H}$ :

$$\begin{cases} \mathbf{P}_j = -\frac{\mathbf{E}}{q_{e,j}(\omega)} + \mathbf{F}_{p,j}(\omega) & \text{where } \mathbf{F}_{p,j}(\omega) = \frac{(-i\alpha_{e,j} - \omega)\mathbf{p}_j - i\dot{\mathbf{p}}_j}{q_{e,j}(\omega)}, \end{cases} \quad (1.4.29a)$$

$$\begin{cases} \mathbf{M}_\ell = -\frac{\mathbf{H}}{q_{m,\ell}(\omega)} + \mathbf{F}_{m,\ell}(\omega) & \text{where } \mathbf{F}_{m,\ell}(\omega) = \frac{(-i\alpha_{m,\ell} - \omega)\mathbf{m}_\ell - i\dot{\mathbf{m}}_\ell}{q_{m,\ell}(\omega)}, \end{cases} \quad (1.4.29b)$$

where  $q_{e,j}(\omega) = \omega^2 + i\alpha_{e,j}\omega - \omega_{e,j}^2 \neq 0$  and  $q_{m,\ell}(\omega) = \omega^2 + i\alpha_{m,\ell}\omega - \omega_{m,\ell}^2 \neq 0$  for  $\omega \in \mathbb{C}^+$ . We point out that if  $\omega_*$  is zero of  $q_e$ ,  $-\bar{\omega}_*$  is also zero of  $q_e$ . Thus for  $\omega_* \notin i\mathbb{R}^-$ ,  $\text{Im}(\omega_*) = -\alpha_{e,j}/2 \leq 0$ . Therefore  $\omega \in \mathbb{C}^+$  is not a zero of  $q_{e,j}$ . The same holds for the polynomials  $q_{m,\ell}$ . Thus, it follows the expressions of  $\dot{\mathbf{P}}$  and  $\dot{\mathbf{M}}$  in terms of  $\mathbf{E}$  and  $\mathbf{H}$ :

$$\begin{cases} \dot{\mathbf{P}}_j = \frac{i\omega\mathbf{E}}{q_{e,j}(\omega)} + \dot{\mathbf{F}}_{p,j}(\omega) & \text{where } \dot{\mathbf{F}}_{p,j}(\omega) = \frac{i\omega_{e,j}^2\mathbf{p}_j - \omega\dot{\mathbf{p}}_j}{q_{e,j}(\omega)}, \end{cases} \quad (1.4.30a)$$

$$\begin{cases} \dot{\mathbf{M}}_\ell = \frac{i\omega\mathbf{H}}{q_{m,\ell}(\omega)} + \dot{\mathbf{F}}_{m,\ell}(\omega) & \text{where } \dot{\mathbf{F}}_{m,\ell}(\omega) = \frac{i\omega_{m,\ell}^2\mathbf{m}_\ell - \omega\dot{\mathbf{m}}_\ell}{q_{m,\ell}(\omega)}. \end{cases} \quad (1.4.30b)$$

Eliminating  $\dot{\mathbf{P}}_j$  and  $\dot{\mathbf{M}}_\ell$  in the two first equations of (1.4.28) yields to the following system in  $\mathbf{E}$  and  $\mathbf{H}$ :

$$\begin{cases} -\nabla \times \mathbf{H} - i\omega\varepsilon(\omega)\mathbf{E} = \mathbf{F}_e(\omega) & \text{with } \mathbf{F}_e(\omega) = \varepsilon_0[\mathbf{i}\mathbf{e} - \sum_{j=1}^{N_e} \Omega_{e,j}^2 \dot{\mathbf{F}}_{p,j}(\omega)], \end{cases} \quad (1.4.31a)$$

$$\begin{cases} \nabla \times \mathbf{E} - i\omega\mu(\omega)\mathbf{H} = \mathbf{F}_h(\omega) & \text{with } \mathbf{F}_h(\omega) = \mu_0[\mathbf{i}\mathbf{h} - \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 \dot{\mathbf{F}}_{m,\ell}(\omega)], \end{cases} \quad (1.4.31b)$$

where  $\varepsilon(\omega)$  and  $\mu(\omega)$  are defined by (1.1.14). Substituting

$$\mathbf{H} = \frac{i}{\omega\mu(\omega)} \left( -\nabla \times \mathbf{E} + \mathbf{F}_h(\omega) \right) \quad (1.4.32)$$

in (1.4.31a) yields the following equation (satisfied in the sense of distributions)

$$-\frac{1}{\omega\mu(\omega)} \nabla \times (\nabla \times \mathbf{E}) + \omega\varepsilon(\omega) \mathbf{E} = -\frac{1}{\omega\mu(\omega)} \nabla \times \mathbf{F}_h(\omega) + i \mathbf{F}_e(\omega). \quad (1.4.33)$$

One shows using standard arguments that  $\mathbf{E} \in H(\text{rot}; \mathbb{R}^3)$  is solution of (1.4.33) if and only if  $\mathbf{E}$  is solution to the following variational problem in  $H(\text{rot}; \mathbb{R}^3)$ :

$$a(\mathbf{E}, \psi) = l(\psi), \quad \forall \psi \in H(\text{rot}; \mathbb{R}^3),$$

where the sesquilinear form  $a$  and the antilinear form  $l$  are defined for all  $\phi, \psi \in H(\text{rot}; \mathbb{R}^3)$ :

$$a(\phi, \psi) = \int_{\mathbb{R}^3} \left( -\frac{(\nabla \times \phi) \cdot \overline{(\nabla \times \psi)}}{\omega\mu(\omega)} + \omega\varepsilon(\omega) \phi \cdot \overline{\psi} \right) dx, \quad l(\psi) = \int_{\mathbb{R}^3} \left( -\frac{\mathbf{F}_h(\omega)}{\omega\mu(\omega)} \cdot \overline{(\nabla \times \psi)} + i \mathbf{F}_e(\omega) \cdot \overline{\psi} \right) dx.$$

By using the Cauchy-Schwarz inequality, it is clear that  $a$  (resp.  $l$ ) is continuous on  $H(\text{rot}; \mathbb{R}^3)^2$  (resp. on  $H(\text{rot}; \mathbb{R}^3)$ ). Furthermore,  $\omega \mapsto \omega\varepsilon(\omega)$  and  $\omega \mapsto \omega\mu(\omega)$  are non-constant Herglotz functions. Thus,  $\omega \mapsto -1/(\omega\mu(\omega))$  is also a non-constant Herglotz function. Hence, one shows that for all  $\omega \in \mathbb{C}^+$ :

$$|a(\phi, \phi)| \geq |\text{Im } a(\phi, \phi)| \geq \gamma(\omega) \|\phi\|_{H(\text{rot}; \mathbb{R}^3)}^2,$$

$$\text{where } \gamma(\omega) = \min \left( \text{Im} \left( -(\omega\mu(\omega))^{-1} \right), \text{Im} (\omega\varepsilon(\omega)) \right) > 0.$$

Thus, by the Lax-Milgram theorem, (1.4.33) admits a unique solution in  $H(\text{rot}; \mathbb{R}^3)$ . Then, from (1.4.32), (1.4.29a), (1.4.30a), (1.4.29b), (1.4.30b),  $\mathbf{H}, \mathbf{P}, \dot{\mathbf{P}}, \mathbf{M}, \dot{\mathbf{M}}$  are defined uniquely in term of  $\mathbf{E}$  as elements of  $L^2(\mathbb{R}^3)^3$ . Moreover, with (1.4.31a),  $\nabla \times \mathbf{H} \in L^2(\mathbb{R}^3)^3$ , thus (1.4.28) admits a unique solution  $\mathbf{U}$  in  $D(\mathbb{A})$ .  $\square$

The operator  $-i\mathbb{A}$  is maximal dissipative. Hence,  $\mathbb{A}$  is a closed operator (see e.g. [28], theorem 8 page 340) and its spectrum  $\sigma(\mathbb{A})$  is contained in the closure of the lower half plane  $\overline{\mathbb{C}^-}$ . Furthermore, it follows from the Lumer-Phillips theorem (see e.g. [28], theorem 7 pages 336-337) that  $-i\mathbb{A}$  is the generator of a contraction semi-group  $\{\mathcal{S}(t)\}_{t \geq 0}$  of class  $\mathcal{C}^0$ . This implies the following standard results on the well-posedness of (1.4.24) and the regularity of its solutions (where the Hilbert space  $D(\mathbb{A})$  is endowed with the graph norm defined by:  $\|\mathbf{U}\|_{D(\mathbb{A})}^2 = \|\mathbf{U}\|_{\mathcal{H}}^2 + \|\mathbb{A}\mathbf{U}\|_{\mathcal{H}}^2, \forall \mathbf{U} \in D(\mathbb{A})$ ).

### Proposition A. 3

Let  $m$  be an integer satisfying  $m \geq 1$ . If the initial condition  $\mathbf{U}_0 \in D(\mathbb{A}^m)$  then the Cauchy problem (1.4.24) admits a unique strong solution  $\mathbf{U} \in C^m(\mathbb{R}^+, \mathcal{H}) \cap C^{m-1}(\mathbb{R}^+, D(\mathbb{A}))$  given by  $\mathbf{U}(t) = \mathcal{S}(t)\mathbf{U}_0$  for  $t \geq 0$ . If  $\mathbf{U}_0 \in \mathcal{H}$ , (1.4.24) admits a unique mild solution  $\mathbf{U} \in C^0(\mathbb{R}^+, \mathcal{H})$ .

For the proof of this classical result, we refer to [28] (theorem 1 page 399) for the case  $m = 1$  and to [10] (Theorem 7.5 page 191) for the case  $m > 1$ . For the definition of mild solutions when the initial data  $\mathbf{U}_0$  is in  $\mathcal{H}$ , we refer to [28] pages 404-405.

**Remark 1.4.3.** As the operator  $-i\mathbb{A}$  is dissipative, the identity  $d\mathcal{L}(t)/dt = 2\text{Im}(\mathbb{A}\mathbf{U}(t), \mathbf{U}(t))_{\mathcal{H}} \leq$

0 with  $\mathcal{L}(t) = (\mathbf{U}(t), \mathbf{U}(t))_{\mathcal{H}}$  (the abstract version of (1.2.5) or (1.3.6)) implies the decay in time of  $\mathcal{L}(t)$ .

We now define  $\mathcal{H}_{\text{div},0}$  the closed subspace of  $\mathcal{H}$  given by

$$\mathcal{H}_{\text{div},0} := \{\mathbf{U} = (\mathbf{E}, \mathbf{H}, \mathbf{P}, \dot{\mathbf{P}}, \mathbf{M}, \dot{\mathbf{M}}) \in \mathcal{H} \mid \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{H} = \nabla \cdot \mathbf{P}_j = \nabla \cdot \dot{\mathbf{P}}_j = \nabla \cdot \mathbf{M}_\ell = \nabla \cdot \dot{\mathbf{M}}_\ell = 0\}, \quad (1.4.34)$$

(where in the previous definition the indices  $j$  and  $\ell$  belong to  $\{1, \dots, N_e\}$  and  $\{1, \dots, N_m\}$ ).

Finally, the following proposition explains that if the initial data are divergence free, the components of the solution to (1.4.24) remain divergence free for any  $t \geq 0$ .

**Proposition A. 4**

Let  $\mathbf{U}_0 \in D(\mathbb{A}) \cap \mathcal{H}_{\text{div},0}$  and  $\mathbf{U} : \mathbb{R}^+ \rightarrow \mathcal{H}$  be the unique strong solution of (1.4.24) with  $\mathbf{U}_0$  as initial condition then  $\mathbf{U}(t) \in D(\mathbb{A}) \cap \mathcal{H}_{\text{div},0}$  for any  $t \geq 0$ . If  $\mathbf{U}_0 \in \mathcal{H}_{\text{div},0}$  then the mild solution  $\mathbf{U}$  of (1.4.24) satisfies also  $\mathbf{U}(t) \in \mathcal{H}_{\text{div},0}$  for any  $t \geq 0$ .

*Proof.* We assume first that the initial condition is regular enough, namely  $\mathbf{U}_0 \in D(\mathbb{A}^2)$ . Hence by Proposition A. 3, the strong solution  $\mathbf{U}$  of (1.4.24) belongs to  $C^2(\mathbb{R}^+, \mathcal{H}) \cap C^1(\mathbb{R}^+, D(\mathbb{A}))$  and thus the evolution equation (1.4.24) (or equivalently the system of equations (1.3.1)) holds also at  $t = 0$  and one can differentiate it for  $t \geq 0$ . Taking the divergence in the distributional sense of the equation (1.3.1a) for  $t \geq 0$  and taking the divergence of the equations  $\partial_t((1.3.1c))$  for  $t \geq 0$  and  $j \in \{1, \dots, N_e\}$  leads to:

$$\partial_t \nabla \cdot \mathbf{E} + \sum_{j=1}^{N_e} \Omega_{e,j}^2 \partial_t \nabla \cdot \mathbf{P}_j = 0 \quad \text{and} \quad \partial_t^3 \nabla \cdot \mathbf{P}_j + \alpha_{e,j} \partial_t^2 \nabla \cdot \mathbf{P}_j + \omega_{e,j}^2 \partial_t \nabla \cdot \mathbf{P}_j = \partial_t \nabla \cdot \mathbf{E}. \quad (1.4.35)$$

Thus, substituting  $\partial_t \nabla \cdot \mathbf{E}$  in the second equation of (1.4.35) gives

$$\partial_t^3 \nabla \cdot \mathbf{P}_j + \alpha_{e,j} \partial_t^2 \nabla \cdot \mathbf{P}_j + \omega_{e,j}^2 \partial_t \nabla \cdot \mathbf{P}_j + \sum_{j'=1}^{N_e} \Omega_{e,j'}^2 \partial_t \nabla \cdot \mathbf{P}_{j'} = 0 \quad \text{for } j \in \{1, \dots, N_e\} \text{ and } t \geq 0, \quad (1.4.36)$$

with initial conditions:  $\nabla \cdot \mathbf{P}_j(\cdot, 0) = \partial_t \nabla \cdot \mathbf{P}_j(\cdot, 0) = 0$  (since  $\mathbf{U}_0 \in \mathcal{H}_{\text{div},0}$ ). Then taking the divergence of equation (1.3.1c) evaluated at  $t = 0$  yields

$$\partial_t^2 \nabla \cdot \mathbf{P}_j(\cdot, 0) = -\alpha_{e,j} \partial_t \nabla \cdot \mathbf{P}_j(\cdot, 0) - \omega_{e,j}^2 \partial_t \nabla \cdot \mathbf{P}_j(\cdot, 0) + \nabla \cdot \mathbf{E}(\cdot, 0) = 0.$$

Thus, from (1.4.36), one deduces  $\nabla \cdot \mathbf{P}_j(\cdot, t) = \nabla \cdot \partial_t \mathbf{P}_j(\cdot, t) = 0$  for any  $t \geq 0$  and  $j \in \{1, \dots, N_e\}$ . It follows with (1.4.35) that  $\partial_t \nabla \cdot \mathbf{E}(\cdot, t) = 0$  with  $\nabla \cdot \mathbf{E}(\cdot, 0) = 0$ . Therefore, one gets also  $\nabla \cdot \mathbf{E}(\cdot, t) = 0$ . Similarly using (1.3.1b) and (1.3.1d) and their derivatives, one shows that:  $\nabla \cdot \mathbf{M}_\ell(\cdot, t) = \nabla \cdot \partial_t \mathbf{M}_\ell(\cdot, t) = \nabla \cdot \mathbf{H}(\cdot, t) = 0$   $t \geq 0$  and  $\ell \in \{1, \dots, N_e\}$  and concludes that  $\mathbf{U}(t) \in D(\mathbb{A}) \cap \mathcal{H}_{\text{div},0}$  for any  $t \geq 0$ .

If  $\mathbf{U}_0$  is less regular, i.e.  $\mathbf{U}_0$  in  $\mathcal{H}_{\text{div},0}$ , one obtains that  $\mathbf{U}(t) = S(t)\mathbf{U}_0 \in \mathcal{H}_{\text{div},0}$  from the previous reasoning by using a density argument on the initial condition and the fact that the elements  $S(t)$  of the  $C^0$  semi-group corresponding to (1.4.24) are contractions. Moreover, if  $\mathbf{U}_0 \in D(\mathbb{A})$ , one has, by Proposition A. 3,  $\mathbf{U}(t) \in D(\mathbb{A})$  and thus  $\mathbf{U}(t) \in D(\mathbb{A}) \cap \mathcal{H}_{\text{div},0}$ .  $\square$

**Remark 1.4.4.** The (Cauchy) problem (1.3.1) assumes that all the resonance frequencies satisfy  $\omega_{e,j}, \omega_{m,\ell} > 0$ , in other words that  $\varepsilon(\omega)$  and  $\mu(\omega)$  only contained Lorentz terms (in the sense of

Section 1.4.2). However, all the results in this appendix still hold with a similar proof if they contain Drude terms. In that case, if  $N_{d,e}$  and  $N_{d,m}$  are the number of electric and magnetic Drude terms, one has to redefine  $\mathcal{H}$  in (1.4.25) as

$$\mathcal{H} := L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^{3(N_e - N_{d,e})} \times L^2(\mathbb{R}^3)^{3N_e} \times L^2(\mathbb{R}^3)^{3(N_m - N_{d,m})} \times L^2(\mathbb{R}^3)^{3N_m}$$

( $\mathbf{P}_j$  (resp.  $\mathbf{M}_\ell$ ) are no longer unknowns of the evolution problem for Drude terms).



## CHAPTER 2

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# Long time behaviour of the solution of Maxwell's equations in dissipative generalized Lorentz materials (II) A modal approach

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## 2.1 Introduction

### 2.1.1 Motivation

This paper is devoted to the study of the long time behaviour, of solutions of the Cauchy problem for Maxwell's equations in a large class dissipative and dispersive materials: the generalized Lorentz media. We refer for instance to [16, 17, 19, 69, 96] for the interest of considering such media with respect to applications, in particular in the context of metamaterials. We are more precisely interested in the decay of the electromagnetic energy. The evolution problem we consider is the following

$$\text{Find} \quad \begin{cases} \mathbf{E}(\mathbf{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3 & \mathbf{H}(\mathbf{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3 \\ \mathbf{P}_j(\mathbf{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3, \quad 1 \leq j \leq N_e, \quad \mathbf{M}_\ell(\mathbf{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3, \quad 1 \leq \ell \leq N_m, \end{cases}$$

such that (for all  $1 \leq j \leq N_e, 1 \leq \ell \leq N_m$ )

$$\left\{ \begin{array}{l} \varepsilon_0 \partial_t \mathbf{E} - \nabla \times \mathbf{H} + \varepsilon_0 \sum_{j=1}^{N_e} \Omega_{e,j}^2 \partial_t \mathbf{P}_j = 0, \\ \mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} + \mu_0 \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 \partial_t \mathbf{M}_\ell = 0, \end{array} \right. \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}^{+,*}, \quad (2.1.1a)$$

$$\left\{ \begin{array}{l} \mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} + \mu_0 \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 \partial_t \mathbf{M}_\ell = 0, \\ \partial_t^2 \mathbf{P}_j + \alpha_{e,j} \partial_t \mathbf{P}_j + \omega_{e,j}^2 \mathbf{P}_j = \mathbf{E}, \end{array} \right. \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}^{+,*}, \quad (2.1.1b)$$

$$\left\{ \begin{array}{l} \partial_t^2 \mathbf{P}_j + \alpha_{e,j} \partial_t \mathbf{P}_j + \omega_{e,j}^2 \mathbf{P}_j = \mathbf{E}, \\ \partial_t^2 \mathbf{M}_\ell + \alpha_{m,\ell} \partial_t \mathbf{M}_\ell + \omega_{m,\ell}^2 \mathbf{M}_\ell = \mathbf{H}, \end{array} \right. \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}^{+,*}, \quad (2.1.1c)$$

$$\left\{ \begin{array}{l} \partial_t^2 \mathbf{M}_\ell + \alpha_{m,\ell} \partial_t \mathbf{M}_\ell + \omega_{m,\ell}^2 \mathbf{M}_\ell = \mathbf{H}, \\ \partial_t^2 \mathbf{P}_j + \alpha_{e,j} \partial_t \mathbf{P}_j + \omega_{e,j}^2 \mathbf{P}_j = \mathbf{E}, \end{array} \right. \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}^{+,*}, \quad (2.1.1d)$$

completed by the divergence free initial conditions

$$\left\{ \begin{array}{l} \mathbf{E}(\cdot, 0) = \mathbf{E}_0 \in \mathbf{L}^2(\mathbb{R}^3), \quad \mathbf{H}(\cdot, 0) = \mathbf{H}_0 \in \mathbf{L}^2(\mathbb{R}^3) \quad \text{with} \quad \nabla \cdot \mathbf{E}_0 = \nabla \cdot \mathbf{H}_0 = 0, \\ \mathbf{P}(\cdot, 0) = \mathbf{M}(\cdot, 0) = \partial_t \mathbf{P}(\cdot, 0) = \partial_t \mathbf{M}(\cdot, 0) = 0, \end{array} \right. \quad (2.1.2)$$

where  $\mathbf{L}^2(\mathbb{R}^3) = L^2(\mathbb{R}^3)^3$  and, for simplicity, we have used the notations

$$\mathbf{P} = (\mathbf{P}_j) \quad \text{and} \quad \mathbf{M} = (\mathbf{M}_\ell) \quad \text{with} \quad (\mathbf{P}_j) := (\mathbf{P}_j)_{j=1}^{N_e} \quad \text{and} \quad (\mathbf{M}_\ell) := (\mathbf{M}_\ell)_{\ell=1}^{N_m}.$$

In (2.1.1), the fields  $\mathbf{E}$  and  $\mathbf{H}$  are respectively the electric and magnetic fields while the  $\mathbf{P}_j$  and the  $\mathbf{M}_\ell$  are auxiliary unknowns (polarization and magnetization fields respectively). The coefficients  $(\omega_{e,j}, \omega_{m,\ell})$  and  $(\Omega_{e,j}, \Omega_{m,\ell})$  are supposed to satisfy

$$0 < \Omega_{e,1} \leq \dots \leq \Omega_{e,N_e}, \quad 0 < \Omega_{m,1} \leq \dots \leq \Omega_{m,N_m}, \quad (2.1.3)$$

$$\omega_{e,j} > 0, \quad 1 \leq j \leq N_e, \quad \omega_{m,\ell} > 0, \quad 1 \leq \ell \leq N_m.$$

These coefficients are responsible for dispersion effects. Finally, the coefficients  $(\alpha_{e,\ell}, \alpha_{m,\ell})$  must be non negative

$$\alpha_{e,j} \geq 0, \quad 1 \leq j \leq N_e, \quad \alpha_{m,\ell} \geq 0 \quad 1 \leq \ell \leq N_m. \quad (2.1.4)$$

The presence of positive coefficients  $(\alpha_{e,\ell}, \alpha_{m,\ell})$  is responsible for dissipation effects. The reader will notice that one can assume without any loss of generality that the couples  $(\alpha_{e,j}, \omega_{e,j})$  (resp.  $(\alpha_{m,\ell}, \omega_{m,\ell})$ ) are all distinct the ones from the others.

Let us introduce the two rational functions

$$\varepsilon(\omega) = \varepsilon_0 \left( 1 - \sum_{j=1}^{N_e} \frac{\Omega_{e,j}^2}{q_{e,j}(\omega)} \right) \quad \text{and} \quad \mu(\omega) = \mu_0 \left( 1 - \sum_{\ell=1}^{N_m} \frac{\Omega_{m,\ell}^2}{q_{m,\ell}(\omega)} \right), \quad (2.1.5)$$

where the second order polynomials  $q_{e,j}$  and  $q_{m,\ell}$  are defined by

$$q_{e,j}(\omega) = \omega^2 + i\alpha_{e,j}\omega - \omega_{e,j}^2 \quad \text{and} \quad q_{m,\ell}(\omega) = \omega^2 + i\alpha_{m,\ell}\omega - \omega_{m,\ell}^2. \quad (2.1.6)$$

As functions of the frequency  $\omega$ ,  $\varepsilon$  and  $\mu$  are respectively the permittivity and permeability of the generalized Lorentz media associated to the Maxwell's system (2.1.1) (see Section 1.1 of Chapter 1 for more details about their physical interpretation). We denote by  $\mathcal{P}_e$  (resp.  $\mathcal{P}_m$ ) the set of poles of the function  $\varepsilon$  (resp.  $\mu$ ). We also introduce  $\mathcal{Z}_e$  (resp.  $\mathcal{Z}_m$ ) the set of zeros of  $\varepsilon$  (resp.  $\mu$ ). Generalized Lorentz media are causal and passive electromagnetic materials, as defined in [18] for instance. The counterpart of causality and passivity in the frequency domain is that  $\omega \mapsto \omega \varepsilon(\omega)$  and  $\omega \mapsto \omega \mu(\omega)$  are Herglotz functions of the frequency. Herglotz functions are analytic functions  $f$  in the upper half-plane  $\mathbb{C}^+ := \{\omega \in \mathbb{C} \mid \text{Im} \omega > 0\}$  whose imaginary part  $\text{Im} f$  is non-negative on  $\mathbb{C}^+$ . For more details about passive electromagnetic media and Herglotz functions, we refer to [17–19].

We make the two following assumptions which are linked to the irreducibility of the dispersion relation associated to the system (2.1.1) that involves the product  $\varepsilon(\omega)\mu(\omega)$  (see Section 2.3.2.1):

- (H<sub>1</sub>): the electric polynomials  $q_{e,j}$  (see (2.1.6)) with distinct indices  $j$  do not have common roots. The same holds for the magnetic polynomials  $q_{m,\ell}$  with distinct indices  $\ell$ .
- (H<sub>2</sub>): The zeros of the permittivity  $\varepsilon$  are not poles of the permeability  $\mu$  and vice versa. Namely, one assumes that  $\mathcal{P}_e \cap \mathcal{Z}_m = \emptyset$  and  $\mathcal{P}_m \cap \mathcal{Z}_e = \emptyset$ .

**Remark 2.1.1.** It is easy to see that, as  $\alpha_{e,j} \geq 0$ , the roots of  $q_{e,j}$  have a non positive imaginary part. More precisely, as the discriminant of  $q_{e,j}$  is  $\delta_{e,j} = 4\omega_{e,j}^2 - \alpha_{e,j}^2$

- If  $\alpha_{e,j} < 2\omega_{e,j}$ , the two roots of  $q_{e,j}$  are  $-i\alpha_{e,j}/2 \pm \delta_{e,j}^{1/2}/2 \notin i\mathbb{R}^-$ ,
- If  $\alpha_{e,j} \geq 2\omega_{e,j}$ , the two roots of  $q_{e,j}$  are  $-i(\alpha_{e,j} \pm |\delta_{e,j}|^{1/2})/2 \in i\mathbb{R}^-$ .

Thus, when  $\alpha_{e,j} < 2\omega_{e,j}$ , the two roots of  $q_{e,j}$ ,  $\omega_*$  and  $-\bar{\omega}_* \notin i\mathbb{R}^-$ , are distinct. Moreover, as  $\alpha_{e,j} = -2\text{Im}(\omega_*)$  and  $\omega_{e,j} = |\omega_*|$ ,  $q_{e,j}$  can not share a common root with an other electric polynomial  $q_{e,j'}$  since by assumption  $(\alpha_{e,j}, \omega_{e,j}) \neq (\alpha_{e,j'}, \omega_{e,j'})$  for  $j \neq j'$ . Therefore, one only needs to assume (H<sub>1</sub>) for electric polynomials  $q_{e,j'}$  for which  $\alpha_{e,j'} \geq 2\omega_{e,j'}$ . The same properties hold for the magnetic polynomials  $q_{m,\ell}$  with obvious changes.

The electromagnetic energy is defined as

$$\mathcal{E}(t) \equiv \mathcal{E}(\mathbf{E}, \mathbf{H}, t) := \frac{1}{2} \left( \varepsilon_0 \int_{\mathbb{R}^3} |\mathbf{E}(\mathbf{x}, \mathbf{t})|^2 d\mathbf{x} + \mu_0 \int_{\mathbb{R}^3} |\mathbf{H}(\mathbf{x}, \mathbf{t})|^2 d\mathbf{x} \right). \quad (2.1.7)$$

The present chapter is the "sequel" of Chapter 1 in which we addressed the question of long time decay of  $\mathcal{E}(t)$  with a (frequency dependent) Lyapunov function approach. We were able to prove some *polynomial stability* results under the so-called *strong dissipation assumption*,

**Definition 2.1.2** (Strong Dissipation for Lorentz models).

$$\forall 1 \leq j \leq N_e, \quad \alpha_{e,j} > 0, \quad \forall 1 \leq \ell \leq N_m, \quad \alpha_{m,\ell} > 0. \quad (2.1.8)$$

By polynomial stability, we mean the energy decays more rapidly than a negative power of  $t$ :

$$\mathcal{E}(t) \leq C t^{-s} \text{ for some } s > 0. \quad (2.1.9)$$

In this paper, we address the same question as in [18] via a *modal approach*. Admittedly, this approach is technically more involved than the one in [18], but it presents many advantages:

- it is by many aspects more *physical* (in particular it refers explicitly to the dispersion relation of the medium) thus more easily understandable by physicists,
- it leads to *optimal* results in the sense that upper bounds of the type (2.1.9) can be completed by corresponding lower bounds,
- it allows us to obtain polynomial stability results under a condition on the dissipation coefficients which is strictly weaker than (2.1.8), namely the *weak dissipation* assumption:

**Definition 2.1.3** (Weak Dissipation for Lorentz models).

$$\sum_{j=1}^{N_e} \alpha_{e,j} + \sum_{\ell=1}^{N_m} \alpha_{m,\ell} > 0. \quad (2.1.10)$$

The reader will notice that the *strong dissipation* assumption imposes that all the coefficients  $(\alpha_{e,j}, \alpha_{m,\ell})$  are strictly positive while the *weak dissipation* assumption means only that at least one of them is strictly positive.

In [Chapter 1](#), we made in [Section 1.2](#) a rather extensive analysis of the literature addressing problems similar to the one we consider here, with application in viscoelasticity [25, 26, 35] or electromagnetism [36, 61]. We discussed in particular various existing techniques for obtaining *polynomial* or *exponential stability* results (Sections 1.2 and 1.3). The closest works to ours are the paper by Figotin-Schechter [37] and the papers [63–65] by S. Nicaise and his collaborator C. Pignotti. In [37], the authors prove that the electromagnetic energy of the solutions of a large class of linear dissipative models tends to zero but without addressing the question of the decay rate. In [63–65], the question of the decay rate of the solution is studied in detail in bounded domains for perfectly conducting materials. However, when specified for Maxwell’s equations in generalized Lorentz media, the results of [37] and [63–65] only applied when the strong dissipation assumption (2.1.8) is satisfied. Furthermore, under the weak dissipation assumption (2.1.10), we enlighten for some cases (when (2.1.8) does not hold) new decay rates which are still polynomial but with a smaller exponent in  $1/t$  than the ones observed in the literature [18, 63–65].

The outline of the rest of this chapter is as follows. First in [Section 2.1.2](#), we rewrite the Maxwell’s time-dependent system as a Schrödinger evolution problem which involves a maximal dissipative Hamiltonian  $\mathbb{A}$ . Then in [Section 2.1.3](#), we state the main results of the article. In [Section 2.2](#), we show how to reduce the analysis of the Schrödinger evolution problem to the study of an infinity of Ordinary Differential Equations (ODE) systems in time parametrized by one real parameter. This is done by exploiting the fact that we work with constant coefficients which allows us to use the Fourier transform in space and the isotropy of the model. The equations are then parametrized by  $|\mathbf{k}|$ , the modulus of the wave vector  $\mathbf{k}$ , namely the dual variable of the space variable. Each of this ODE system involves a finite dimensional dissipative operator  $\mathbb{A}_{|\mathbf{k}|,\perp}$  and the rest of the analysis is devoted to estimates, for large  $t$  of the exponentials

$$e^{-i\mathbb{A}_{|\mathbf{k}|,\perp} t} \text{ for } |\mathbf{k}| \in \mathbb{R}^+, \quad (2.1.11)$$

which obviously depends on the eigenvalues of  $\mathbb{A}_{|\mathbf{k}|,\perp}$  and more precisely on their imaginary parts. [Section 2.3](#) is devoted to general spectral properties of  $\mathbb{A}_{|\mathbf{k}|,\perp}$  at fixed  $|\mathbf{k}|$ : we characterise its eigenvalues as the solutions of a rational equation parametrised by  $|\mathbf{k}|$ , namely the dispersion relation of the medium, whose analysis allows us to provide a diagonalizability criterion for  $\mathbb{A}_{|\mathbf{k}|,\perp}$ . Then, we need to distinguish in [Section 2.4](#), [Section 2.5](#) and [Section 2.6](#), respectively, the large, small and intermediate values of  $|\mathbf{k}|$ . In [Section 2.4](#), devoted to large  $|\mathbf{k}|$ , we provide asymptotic expansions (when  $|\mathbf{k}| \rightarrow +\infty$ ) of the solutions of dispersion relation which show the diagonalisability of  $\mathbb{A}_{|\mathbf{k}|,\perp}$  for  $|\mathbf{k}|$  large enough. After diagonalization of the exponential [\(2.1.11\)](#), we obtain optimal decay estimates in time of the high spatial frequency components (i.e. large values of  $|\mathbf{k}|$ ) of the solution of the evolution problem, based on the behaviour of the imaginary part of the eigenvalues of  $\mathbb{A}_{|\mathbf{k}|,\perp}$  and uniform bounds (in  $|\mathbf{k}|$ ) of the associated (non-orthogonal) spectral projectors. A similar strategy is adopted in [Section 2.5](#) to derive optimal estimates for the low frequency components (i.e. small values of  $|\mathbf{k}|$ ) of the solution. [Section 2.6](#) deals with estimates for intermediate values of  $|\mathbf{k}|$ . The method is quite different and somewhat less technical and precise than the method of [Section 2.4](#) and [Section 2.5](#) but it is sufficient to show (in [Section 2.7](#)) that, the corresponding part of the solution decays exponentially fast in time. [Section 2.7](#) is devoted to the proofs of our main results, after regrouping the estimates of the three previous sections and deriving estimates in space variable norms via Plancherel's theorem: this is where the polynomial decay due to large and small values of  $|\mathbf{k}|$  is put in evidence. Finally, [Appendix B](#) contains technical results used for the spectral analysis of  $\mathbb{A}_{|\mathbf{k}|,\perp}$ . In particular, in [Appendix B.2](#), [Lemma B. 1](#) provides a useful implicit function-like result for some (scalar) nonlinear equations in the complex plane. This lemma is used for the analysis of the dispersion relation.

### 2.1.2 Maxwell's equations in dissipative generalized Lorentz media

The (Cauchy) problem [\(2.1.1, 2.1.2\)](#) can be rewritten as a Schrödinger evolution problem

$$\frac{d\mathbf{U}}{dt} + i\mathbb{A}\mathbf{U} = 0 \quad \text{with} \quad \mathbf{U}(0) = \mathbf{U}_0, \quad (2.1.12)$$

where the Hamiltonian  $\mathbb{A}$  is an unbounded operator on the Hilbert-space:

$$\mathcal{H} := \mathbf{L}^2(\mathbb{R}^3)^N = \mathbf{L}^2(\mathbb{R}^3) \times \mathbf{L}^2(\mathbb{R}^3) \times \mathbf{L}^2(\mathbb{R}^3)^{N_e} \times \mathbf{L}^2(\mathbb{R}^3)^{N_e} \times \mathbf{L}^2(\mathbb{R}^3)^{N_m} \times \mathbf{L}^2(\mathbb{R}^3)^{N_m} \quad (2.1.13)$$

where  $N = 2 + 2(N_e + N_m)$ .  $\mathcal{H}$  is endowed by the following inner product:

for any  $\mathbf{U} = (\mathbf{E}, \mathbf{H}, \mathbf{P}, \dot{\mathbf{P}}, \mathbf{M}, \dot{\mathbf{M}}) \in \mathcal{H}$  and  $\mathbf{U}' = (\mathbf{E}', \mathbf{H}', \mathbf{P}', \dot{\mathbf{P}}', \mathbf{M}', \dot{\mathbf{M}}') \in \mathcal{H}$ , where  $(\mathbf{P}, \mathbf{M})$  is defined as in [\(2.1.1\)](#), (and the same for  $(\mathbf{P}', \mathbf{M}')$ ,  $(\dot{\mathbf{P}}, \dot{\mathbf{M}})$  and  $(\dot{\mathbf{P}}', \dot{\mathbf{M}}')$ )

$$\left| \begin{aligned} (\mathbf{U}, \mathbf{U}')_{\mathcal{H}} &= \frac{\varepsilon_0}{2} (\mathbf{E}, \mathbf{E}')_{\mathbf{L}^2} + \frac{\mu_0}{2} (\mathbf{H}, \mathbf{H}')_{\mathbf{L}^2} \\ &+ \frac{\varepsilon_0}{2} \sum_{j=1}^{N_e} \omega_{e,j}^2 \Omega_{e,j}^2 (\mathbf{P}_j, \mathbf{P}'_j)_{\mathbf{L}^2} + \frac{\varepsilon_0}{2} \sum_{j=1}^{N_e} \Omega_{e,j}^2 (\dot{\mathbf{P}}_j, \dot{\mathbf{P}}'_j)_{\mathbf{L}^2} \\ &+ \frac{\mu_0}{2} \sum_{\ell=1}^{N_m} \omega_{m,\ell}^2 \Omega_{m,\ell}^2 (\mathbf{M}_\ell, \mathbf{M}'_\ell)_{\mathbf{L}^2} + \frac{\mu_0}{2} \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 (\dot{\mathbf{M}}_\ell, \dot{\mathbf{M}}'_\ell)_{\mathbf{L}^2}, \end{aligned} \right. \quad (2.1.14)$$

where  $(\mathbf{u}, \mathbf{v})_{\mathbf{L}^2} = \int_{\mathbb{R}^3} \mathbf{u} \cdot \bar{\mathbf{v}} \, d\mathbf{x}$  denotes the standard inner product of  $\mathbf{L}^2(\mathbb{R}^3)$ .

(the symbol  $\cdot$  stands here for the operation defined on  $\mathbb{C}^3 \times \mathbb{C}^3$  by  $\mathbf{a} \cdot \mathbf{b} := \mathbf{a}_1 \mathbf{b}_1 + \mathbf{a}_2 \mathbf{b}_2 + \mathbf{a}_3 \mathbf{b}_3$ ).

The Hamiltonian  $\mathbb{A} : D(\mathbb{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  is defined by

$$\forall \mathbf{U} = (\mathbf{E}, \mathbf{H}, \mathbf{P}, \dot{\mathbf{P}}, \mathbf{M}, \dot{\mathbf{M}}) \in D(\mathbb{A}), \quad \mathbb{A}\mathbf{U} := -i \begin{pmatrix} -\varepsilon_0^{-1} \nabla \times \mathbf{H} + \sum \Omega_{e,j}^2 \dot{\mathbf{P}}_j \\ \mu_0^{-1} \nabla \times \mathbf{E} + \sum \Omega_{m,\ell}^2 \dot{\mathbf{M}}_\ell \\ -\dot{\mathbf{P}} \\ \alpha_{e,j} \dot{\mathbf{P}}_j + \omega_{e,j}^2 \mathbf{P}_j - \mathbf{E} \\ -\dot{\mathbf{M}} \\ \alpha_{m,\ell} \dot{\mathbf{M}}_\ell + \omega_{m,\ell}^2 \mathbf{M}_\ell - \mathbf{H} \end{pmatrix} \quad (2.1.15)$$

with  $D(\mathbb{A})$  (dense in  $\mathcal{H}$ ) being given by

$$D(\mathbb{A}) := H(\text{rot}; \mathbb{R}^3) \times H(\text{rot}; \mathbb{R}^3) \times L^2(\mathbb{R}^3)^{3N_e} \times L^2(\mathbb{R}^3)^{3N_e} \times L^2(\mathbb{R}^3)^{3N_m} \times L^2(\mathbb{R}^3)^{3N_m}, \quad (2.1.16)$$

where  $H(\text{rot}; \mathbb{R}^3) := \{\mathbf{u} \in \mathbf{L}^2(\mathbb{R}^3) \mid \nabla \times \mathbf{u} \in \mathbf{L}^2(\mathbb{R}^3)\}$ .

**Remark 2.1.4.** Let us point out that, in the sequel, as in the formula (2.1.15), we shall often omit the summation bounds 1 and  $N_e$  (resp. 1 and  $N_m$ ) for sum on  $j$  (resp.  $\ell$ ) indices.

We notice that one can rewrite the evolution problem (2.1.1, 2.1.2) as (2.1.12) with the initial condition

$$\mathbf{U}_0 = (\mathbf{E}_0, \mathbf{H}_0, 0, 0, 0, 0) \in \mathcal{H} \quad \text{satisfying} \quad \nabla \cdot \mathbf{E}_0 = \nabla \cdot \mathbf{H}_0 = 0. \quad (2.1.17)$$

Moreover, its associated electromagnetic energy  $\mathcal{E}(t)$ , defined in (2.1.7), is dominated by  $\|\mathbf{U}(t)\|^2$ .

We prove in Lemma A.2 of [18] that the operator  $-i\mathbb{A}$  is maximal dissipative. Thus, it generates a contraction semi-group  $\{\mathcal{S}(t)\}_{t \geq 0}$  of class  $\mathcal{C}^0$  (see e.g. Theorem 4.3 page 14 of [71]). Hence, the Cauchy problem (2.1.12) is well-posed and stable. More precisely (see propositions A.3 of [18]), it admits a unique mild solution

$$\mathbf{U} = \mathcal{S}(t)\mathbf{U}_0 \in C^0(\mathbb{R}^+, \mathcal{H}),$$

which is a strong solution,

$$U \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathbb{A})),$$

as soon as  $\mathbf{U}_0 \in D(\mathbb{A})$ , with  $D(\mathbb{A})$  is endowed with the usual graph norm  $\|\cdot\|_{D(\mathbb{A})}$  defined by

$$\|\mathbf{U}\|_{D(\mathbb{A})}^2 = \|\mathbf{U}\|_{\mathcal{H}}^2 + \|\mathbb{A}\mathbf{U}\|_{\mathcal{H}}^2, \quad \forall \mathbf{U} \in D(\mathbb{A}). \quad (2.1.18)$$

We introduce now the well-known Hodge decomposition in electromagnetism which consists to write any  $\mathbf{L}^2(\mathbb{R}^3)$ -fields as the orthogonal sum of a gradient (i.e. a curl free field) and a divergence free field (see e.g. [27]). Namely, one has

$$\mathbf{L}^2(\mathbb{R}^3) = \nabla W^1(\mathbb{R}^3) \oplus \mathbf{L}_{\text{div}0}^2, \quad (2.1.19)$$

where  $\nabla W^1(\mathbb{R}^3) = \{\nabla u \mid u \in W^1(\mathbb{R}^3)\}$  with

$$W^1(\mathbb{R}^3) = \{u \in L_{loc}^2(\mathbb{R}^3) \mid (1 + |x|^2)^{-1/2} u \in L^2(\mathbb{R}^3) \text{ and } \nabla u \in \mathbf{L}^2(\mathbb{R}^3)\}$$

the standard Beppo-Levy space on  $\mathbb{R}^3$ , and  $\mathbf{L}_{\text{div}0}^2 = \{\mathbf{u} \in \mathbf{L}^2(\mathbb{R}^3) \mid \nabla \cdot \mathbf{u} = 0\}$  the space of 3D free divergence fields in  $\mathbb{R}^3$ .

Following the decomposition (2.1.19) for each copy  $\mathbf{L}^2(\mathbb{R}^3)$  of  $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^3)^N$ , one deduces immediately that the Hilbert  $\mathcal{H}$  admits the following orthogonal decomposition:

$$\mathcal{H} = \mathcal{H}_{\parallel} \oplus^{\perp} \mathcal{H}_{\perp} \text{ where } \mathcal{H}_{\parallel} = \nabla W^1(\mathbb{R}^3)^N \text{ and } \mathcal{H}_{\perp} = (\mathbf{L}_{\text{div}0}^2)^N. \quad (2.1.20)$$

Let us now introduce  $D(\mathbb{A}_{\parallel}) = D(\mathbb{A}) \cap \mathcal{H}_{\parallel}$  and  $D(\mathbb{A}_{\perp}) = D(\mathbb{A}) \cap \mathcal{H}_{\perp}$ . As an immediate consequence of (2.1.15), (2.1.16), (2.1.18) and (2.1.20), one has

$$\mathbb{A}(D(\mathbb{A}_{\parallel})) \subset \mathcal{H}_{\parallel}, \quad \mathbb{A}(D(\mathbb{A}_{\perp})) \subset \mathcal{H}_{\perp} \text{ and } D(\mathbb{A}) = D(\mathbb{A}_{\parallel}) \oplus^{\perp} D(\mathbb{A}_{\perp}).$$

Thus one can reduce the operator  $\mathbb{A}$  as a sum of two closed and densely defined operators  $\mathbb{A}_{\parallel} : D(\mathbb{A}_{\parallel}) \subset \mathcal{H}_{\parallel} \mapsto \mathcal{H}_{\parallel}$  and  $\mathbb{A}_{\perp} : D(\mathbb{A}_{\perp}) \subset \mathcal{H}_{\perp} \mapsto \mathcal{H}_{\perp}$  in the sense that

$$\mathbb{A} = \mathbb{A}_{\parallel} \oplus \mathbb{A}_{\perp} \text{ with } \mathbb{A}_{\parallel} \mathbf{U} = \mathbb{A} \mathbf{U}, \quad \forall \mathbf{U} \in D(\mathbb{A}_{\parallel}) \text{ and } \mathbb{A}_{\perp} \mathbf{U} = \mathbb{A} \mathbf{U}, \quad \forall \mathbf{U} \in D(\mathbb{A}_{\perp}). \quad (2.1.21)$$

Moreover, since any element of  $\mathcal{H}_{\parallel}$  is made of curl free vector fields, it is readily seen on (2.1.15) that  $D(\mathbb{A}_{\parallel}) = \mathcal{H}_{\parallel}$  and that  $\mathbb{A}_{\parallel}$  is bounded.

From (2.1.21), we deduce using the Hille-Yosida approximation of  $\mathbb{A}$  (see Corollary 3.5 page 11 of [71]) that a similar reduction holds for the semi group  $\{\mathcal{S}(t)\}_{t \geq 0}$ . Namely,  $\mathbb{A}_{\parallel}$  and  $\mathbb{A}_{\perp}$  generate two contraction semi-groups  $\{\mathcal{S}_{\parallel}(t)\}_{t \geq 0}$  and  $\{\mathcal{S}_{\perp}(t)\}_{t \geq 0}$  on the Hilbert spaces  $\mathcal{H}_{\parallel}$  and  $\mathcal{H}_{\perp}$  such that for all  $t \geq 0$ :

$$\mathcal{S}(t) = \mathcal{S}_{\parallel}(t) \oplus \mathcal{S}_{\perp}(t), \quad \mathcal{S}_{\parallel}(t) \mathbf{U}_0 = \mathcal{S}(t) \mathbf{U}_0, \quad \forall \mathbf{U}_0 \in \mathcal{H}_{\parallel}, \quad \mathcal{S}_{\perp}(t) \mathbf{U}_0 = \mathcal{S}(t) \mathbf{U}_0, \quad \forall \mathbf{U}_0 \in \mathcal{H}_{\perp}.$$

In more physical terms, it means that if the components of the initial conditions  $\mathbf{U}_0 \in \mathcal{H}$  are curl free, then the solution  $\mathbf{U}(t)$  of (2.1.12) remains curl free and the corresponding dynamics is trivial since it corresponds to a damped harmonic oscillator in finite dimension (these are non propagative solutions of the Maxwell's equations (2.1.1)).

Oppositely, the interesting case is when  $\mathbf{U}_0 \in \mathcal{H}_{\perp}$  (e.g. for the divergence free initial conditions (2.1.2)). Then the solution  $\mathbf{U}(t)$  of (2.1.12) is divergence free, propagative and given by

$$\mathbf{U}(t) = \mathcal{S}_{\perp}(t) \mathbf{U}_0 \in \mathcal{H}_{\perp}, \quad \forall t \geq 0.$$

### 2.1.3 Statement of the main results

#### 2.1.3.1 The long-time energy decay rate

To expose our main result, Theorem 2.1.7, we need first to define an adapted functional framework:

a. For each  $p \geq 0$ , we introduce the Banach space

$$\mathcal{L}_p(\mathbb{R}^3) = \{v \in \mathcal{S}'(\mathbb{R}^3) / |\mathbf{k}|^{-p}(1 + |\mathbf{k}|^p) \widehat{v} \in L^{\infty}(\mathbb{R}^3)\} \quad (2.1.22)$$

(where  $\mathbf{k} \mapsto \widehat{v}(\mathbf{k})$  denotes here the Fourier transform of a scalar tempered distributions  $v \in \mathcal{S}'(\mathbb{R}^3)$ ) equipped with the norm

$$\|v\|_{\mathcal{L}_p} := \left\| |\mathbf{k}|^{-p}(1 + |\mathbf{k}|^p) \widehat{v} \right\|_{L^{\infty}(\mathbb{R}^3)}. \quad (2.1.23)$$

We point out that if  $v \in \mathcal{L}_p(\mathbb{R}^3)$ , then one has  $|\widehat{v}(\mathbf{k})| \leq \|v\|_{\mathcal{L}_p} |\mathbf{k}|^p$  for a.e.  $\mathbf{k} \in \mathbb{R}^3$ . The reader will also note that for  $q > p$ ,  $\mathcal{L}_q(\mathbb{R}^3) \subset \mathcal{L}_p(\mathbb{R}^3)$ .

Identifying a function in  $\mathcal{L}_p(\mathbb{R}^3)$  is not trivial since it requires to look at the same time at the function and its Fourier transform. See however [Remark 2.1.5](#) for some more concrete examples of spaces of functions included in  $\mathcal{L}_p(\mathbb{R}^3)$ .

We shall introduce  $\mathcal{L}_p(\mathbb{R}^3) = \mathcal{L}_p(\mathbb{R}^3)^3$  the vector valued version of  $\mathcal{L}_p(\mathbb{R}^3)$  endowed with the norm  $\|\cdot\|_{\mathcal{L}_p}$  defined by

$$\forall \mathbf{u} = (u_1, u_2, u_3) \in \mathcal{L}_p(\mathbb{R}^3), \quad \|\mathbf{u}\|_{\mathcal{L}_p}^2 = \|u_1\|_{\mathcal{L}_p}^2 + \|u_2\|_{\mathcal{L}_p}^2 + \|u_3\|_{\mathcal{L}_p}^2.$$

Then, we define on  $\mathcal{L}_p(\mathbb{R}^3)^N$  the norm  $\|\cdot\|_{\mathcal{L}_p^N}$  by:  $\forall \mathbf{U} = (\mathbf{E}, \mathbf{H}, \mathbf{P}, \dot{\mathbf{P}}, \mathbf{M}, \dot{\mathbf{M}}) \in \mathcal{L}_p(\mathbb{R}^3)^N$ ,

$$\|\mathbf{U}\|_{\mathcal{L}_p^N}^2 = \|\mathbf{E}\|_{\mathcal{L}_p}^2 + \|\mathbf{H}\|_{\mathcal{L}_p}^2 + \sum_j (\|\mathbf{P}_j\|_{\mathcal{L}_p}^2 + \|\dot{\mathbf{P}}_j\|_{\mathcal{L}_p}^2) + \sum_\ell (\|\mathbf{M}_\ell\|_{\mathcal{L}_p}^2 + \|\dot{\mathbf{M}}_\ell\|_{\mathcal{L}_p}^2).$$

b. For each  $m \geq 0$ , the space  $\mathbf{H}^m(\mathbb{R}^3) = H^m(\mathbb{R}^3)^3$  where  $H^m(\mathbb{R}^3)$  denotes the standard Sobolev space defined on scalar functions, endowed with the norm  $\|\cdot\|_{\mathbf{H}^m}$  defined by

$$\forall \mathbf{u} = (u_1, u_2, u_3) \in \mathbf{H}^m(\mathbb{R}^3), \quad \|\mathbf{u}\|_{\mathbf{H}^m(\mathbb{R}^3)}^2 = \|u_1\|_{H^m(\mathbb{R}^3)}^2 + \|u_2\|_{H^m(\mathbb{R}^3)}^2 + \|u_3\|_{H^m(\mathbb{R}^3)}^2,$$

where  $\|\cdot\|_{H^m(\mathbb{R}^3)}$  is the usual Sobolev norm on  $H^m(\mathbb{R}^3)$  (see e.g. [27] page 500). Then, one defines the norm  $\|\cdot\|_{\mathbf{H}^m(\mathbb{R}^3)^N}$  for any  $\mathbf{U} = (\mathbf{E}, \mathbf{H}, \mathbf{P}, \dot{\mathbf{P}}, \mathbf{M}, \dot{\mathbf{M}}) \in \mathbf{H}^m(\mathbb{R}^3)^N$  by

$$\left| \begin{aligned} \|\mathbf{U}\|_{\mathbf{H}^m(\mathbb{R}^3)^N}^2 &= \|\mathbf{E}\|_{\mathbf{H}^m(\mathbb{R}^3)}^2 + \|\mathbf{H}\|_{\mathbf{H}^m(\mathbb{R}^3)}^2 \\ &+ \sum_j (\|\mathbf{P}_j\|_{\mathbf{H}^m(\mathbb{R}^3)}^2 + \|\dot{\mathbf{P}}_j\|_{\mathbf{H}^m(\mathbb{R}^3)}^2) \\ &+ \sum_\ell (\|\mathbf{M}_\ell\|_{\mathbf{H}^m(\mathbb{R}^3)}^2 + \|\dot{\mathbf{M}}_\ell\|_{\mathbf{H}^m(\mathbb{R}^3)}^2). \end{aligned} \right. \quad (2.1.24)$$

**Remark 2.1.5.** [ On the space  $\mathcal{L}_p(\mathbb{R}^3)$  ] A first trivial observation is that for any  $p \geq 0$ ,  $\mathcal{L}_p(\mathbb{R}^3)$  contains all integrable functions whose Fourier transform is supported outside a ball centered at the origin.

Moreover, for  $p \in \mathbb{N}$ , let us introduce the weighted  $L^1$  space (already used in [18]):

$$L_p^1(\mathbb{R}^3) := \{v \in L^1(\mathbb{R}^3) / (1 + |\mathbf{x}|)^p v \in L^1(\mathbb{R}^3)\} \quad (L_0^1(\mathbb{R}^3) = L^1(\mathbb{R}^3)), \quad (2.1.25)$$

endowed with the norm

$$\|v\|_{L_p^1(\mathbb{R}^3)} := \|(1 + |\mathbf{x}|)^p v\|_{L^1(\mathbb{R}^3)}. \quad (2.1.26)$$

In particular, functions of  $L_p^1(\mathbb{R}^3)$  have existing moments up to order  $p$ . Let us introduce, for  $p \geq 1$ , the closed subspace of  $L_p^1(\mathbb{R}^3)$  of functions whose moments up to order  $p - 1$  vanish:

$$L_{p,0}^1(\mathbb{R}^3) := \{u \in L_p^1(\mathbb{R}^3) / \forall |\alpha| \leq p - 1, \int \mathbf{x}^\alpha u d\mathbf{x} = 0\}, \quad (2.1.27)$$

where  $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$  denotes a multi-index with "length"  $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$  and where  $\mathbf{x}^\alpha := \mathbf{x}_1^{\alpha_1} \mathbf{x}_2^{\alpha_2} \mathbf{x}_3^{\alpha_3}$  for all  $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ . Moreover, by convention,  $L_{0,0}^1(\mathbb{R}^3) := L^1(\mathbb{R}^3)$ .

Note that since any function in  $L_p^1(\mathbb{R}^3)$  belongs in particular to  $L^1(\mathbb{R}^3)$ , its Fourier transform  $\widehat{v}(\mathbf{k})$  is well defined and belongs to  $C_0^0(\mathbb{R}^3)$ , namely the closed subspace of  $L^\infty(\mathbb{R}^3)$  made of



continuous functions that tend to 0 at infinity. We claim that

$$\forall p \in \mathbb{N}, \quad L_{p,0}^1(\mathbb{R}^3) \subset \mathcal{L}_p. \quad (2.1.28)$$

Indeed, from well-known properties of the Fourier transform, for all  $v \in L_{p,0}^1(\mathbb{R}^3)$ , one has

$$\partial^\alpha \widehat{v}(\mathbf{k}) \in C_0^0(\mathbb{R}^3) \text{ for } |\alpha| \leq p \quad \text{and, if } p > 0, \quad \partial^\alpha \widehat{v}(0) = 0, \text{ for } |\alpha| \leq p - 1. \quad (2.1.29)$$

Moreover, from the Taylor expansion of  $\widehat{v}(\mathbf{k})$  at 0, one gets (by the Taylor-Lagrange formula):

$$\exists C(p) > 0 \quad | \widehat{v}(\mathbf{k}) | \leq C(p) \|v\|_{L_p^1(\mathbb{R}^3)} |\mathbf{k}|^p, \quad \forall \mathbf{k} \in \mathbb{R}^3 \quad \text{and} \quad \forall v \in L_{p,0}^1(\mathbb{R}^3). \quad (2.1.30)$$

Thus, using the fact that  $\|\widehat{v}\|_{L^\infty(\mathbb{R}^3)} \leq \|v\|_{L^1(\mathbb{R}^3)} \leq \|v\|_{L_p^1(\mathbb{R}^3)}$ , one deduces with (2.1.30) that

$$|\widehat{v}(\mathbf{k})| \lesssim \min(1, |\mathbf{k}|^p) \|v\|_{L_p^1(\mathbb{R}^3)}, \quad \forall \mathbf{k} \in \mathbb{R}^3 \quad \text{and} \quad \forall v \in L_{p,0}^1(\mathbb{R}^3). \quad (2.1.31)$$

Then, it follows from (2.1.23) and (2.1.31) that the embedding (2.1.28) is continuous, more precisely:

$$\exists C(p) > 0, \quad \forall v \in L_{p,0}^1(\mathbb{R}^3), \quad \|v\|_{\mathcal{L}_p} \leq C(p) \|v\|_{L_p^1(\mathbb{R}^3)}. \quad (2.1.32)$$

It also follows from (2.1.28) that, in particular,  $\mathcal{L}_p$  contains all compactly supported integrable functions (in the  $\mathbf{x}$  variable) whose moments up to order  $p - 1$  vanish.

Next, we introduce the notion of critical configuration associated to the Maxwell's system (2.1.1), which will influence the long-time decay rate of the electromagnetic energy. It will be enlightened in our proof via the analysis of the dispersion curves associated to this system in the high frequency regime (see Section 2.4.1).

**Definition 2.1.6.** We say that the Maxwell's system (2.1.1) is in a critical configuration if the weak dissipation condition (2.1.10) holds and if one of the following conditions is satisfied:

- a.  $\forall \ell \in \{1, \dots, N_m\}, \alpha_{m,\ell} = 0$  and  $\exists j \in \{1, \dots, N_e\} \mid \alpha_{e,j} = 0$  and  $\omega_{e,j} \notin \{\omega_{m,\ell}\}$ .
- b.  $\forall j \in \{1, \dots, N_e\}, \alpha_{e,j} = 0$  and  $\exists \ell \in \{1, \dots, N_m\} \mid \alpha_{m,\ell} = 0$  and  $\omega_{m,\ell} \notin \{\omega_{e,j}\}$ .

When none of these condition holds, this system (2.1.1) is in a non-critical configuration.

We point out that, as the reader will easoly check, under the weak dissipation assumption (2.1.10), the critical configuration can occur only if  $N_e \geq 2$  or  $N_m \geq 2$ .

We are now able to state our main result under the weak dissipation assumption concerning the decay of the total  $\|\mathbf{U}(t)\|_{\mathcal{H}}^2$  associated to the Schrödinger evolution problem (2.1.12) with divergence-free initial conditions  $\mathbf{U}_0 \in \mathcal{H}_\perp$  which contain in particular initial conditions of the form (2.1.17). Thus, this result applies to the Maxwell's system (2.1.1) with initial conditions (2.1.2).

We shall use below, as well as in the rest of this paper, the following

**Notation 1.** To compare two positive functions  $f(y)$  and  $g(y)$ , where  $y \in Y$  and  $y = \mathbf{x}, \mathbf{k}, t$  or any combination of the variables, we introduce the notation:

$$f \lesssim g \iff \exists C > 0 \quad | \quad f(y) \leq C g(y), \quad \forall y \in Y,$$

where the constant  $C$  depends only on  $N_e, N_m$  and the coefficients of the system (2.1.1).

**Theorem 2.1.7**

Let assume that the Maxwell's system (2.1.1) satisfy the weak dissipation assumption (2.1.10) and the irreducibility assumptions (H<sub>1</sub>) and (H<sub>2</sub>). Then, for any initial condition  $\mathbf{U}_0 \in \mathcal{H}_\perp$ , the total energy  $\|\mathbf{U}(t)\|_{\mathcal{H}}^2$  of the solution  $\mathbf{U}$  of (2.1.12) tends to 0 when  $t$  tends to  $+\infty$ . Namely, one has

$$\lim_{t \rightarrow +\infty} \|\mathbf{U}(t)\|_{\mathcal{H}}^2 = 0. \quad (2.1.33)$$

Moreover if for some real numbers  $m > 0$  and  $p \geq 0$ ,  $\mathbf{U}_0 \in \mathbf{H}^m(\mathbb{R}^3)^N \cap \mathcal{L}_p^N \cap \mathcal{H}_\perp$  then the decay rate of  $\|\mathbf{U}(t)\|^2$  is polynomial. However, the associated exponents in  $1/t$  depend on the configuration (see Definition 2.1.6) of the corresponding Maxwell's system (2.1.1). More precisely,

a. If the Maxwell system is in a non-critical configuration, then one has

$$\|\mathbf{U}(t)\|_{\mathcal{H}}^2 \leq \frac{C_{\text{HF}}^m(\mathbf{U}_0)}{t^m} + \frac{C_{\text{LF}}^p(\mathbf{U}_0)}{t^{p+\frac{3}{2}}}, \quad \forall t > 0, \quad (2.1.34)$$

b. If the Maxwell system is in a critical configuration, then one has

$$\|\mathbf{U}(t)\|_{\mathcal{H}}^2 \leq \frac{C_{\text{HF}}^m(\mathbf{U}_0)}{t^{\frac{m}{2}}} + \frac{C_{\text{LF}}^p(\mathbf{U}_0)}{t^{p+\frac{3}{2}}}, \quad \forall t > 0, \quad (2.1.35)$$

where in (2.1.34) and (2.1.35) the constants satisfy

$$C_{\text{HF}}^m(\mathbf{U}_0) \lesssim \|\mathbf{U}_0\|_{\mathbf{H}^m(\mathbb{R}^3)^N}^2 \quad \text{and} \quad C_{\text{LF}}^p(\mathbf{U}_0) \lesssim \|\mathbf{U}_0\|_{\mathcal{L}_p^N}^2. \quad (2.1.36)$$

**Remark 2.1.8.** The energy  $\|\mathbf{U}(t)\|_{\mathcal{H}}^2$  is dominated in (2.1.34) and (2.1.35) by a sum of two terms.

- The first one involves the Sobolev regularity of the initial condition and is linked, as it will appear in the proof, to the decay of the high spatial frequency components of the energy.
- The second is related to the condition  $\mathbf{U}_0 \in \mathcal{L}_p^N$  and thus to the decay of the low spatial frequency components of the energy.

The presence of the second term is directly related to the fact that the domain of propagation is unbounded and does not appear in the control of the energy in bounded domains for perfectly conducting materials (see, [63–65]).

**Remark 2.1.9.** For generalized Lorentz media, this Theorem generalizes under the weak dissipation assumption (2.1.10) the results obtained under the more restrictive strong dissipation assumption (2.1.8) in [63–65] for bounded domains and in [18] for the whole space  $\mathbb{R}^3$ .

As (2.1.8) excludes the critical configuration, the weaker decay (2.1.35) could not be observed.

### 2.1.3.2 Optimality of the bounds

In Theorem 2.1.7, we prove a long-time polynomial decay rate for  $\|\mathbf{U}(t)\|$  (cf. (2.1.34) and (2.1.35)) for solutions  $\mathbf{U}(t)$  of the Cauchy problem (2.1.12) which satisfies divergence free initial conditions  $\mathbf{U}_0 \in \mathcal{H}_\perp$ . In this section, we analyse the question of the optimality of this decay rate.

The estimates (2.1.34) and (2.1.35) involved a sum of two terms which are related to the decay of the low and high spatial frequency components of the solution (see Remark 2.1.8). Thus, to speak about optimal polynomial exponents, we need to decouple these two terms by separating the high and low frequency behaviors of the solution  $\mathbf{U}$ . Therefore, we introduce the 3D spatial Fourier transform defined by

$$\mathbb{G}(\mathbf{k}) = \mathcal{F}(\mathbf{G})(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \mathbf{G}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \quad \forall \mathbf{G} \in \mathbf{L}^1(\mathbb{R}^3) \cap \mathbf{L}^2(\mathbb{R}^3),$$

which extends to a unitary transformation from  $\mathbf{L}^2(\mathbb{R}_{\mathbf{x}}^3)$  to  $\mathbf{L}^2(\mathbb{R}_{\mathbf{k}}^3)$ . Applying  $\mathcal{F}$  to each copy  $\mathbf{L}^2(\mathbb{R}^3)$  of  $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^3)^N$  defines a unitary transform, still denoted  $\mathcal{F}$ , in  $\mathcal{H}$ .

We now introduce for  $(m, p) \in \mathbb{R}^{+,*} \times \mathbb{R}^+$  the spaces

$$\mathcal{H}_{\perp, \text{HF}}^m = \{\mathbf{V} \in \mathcal{H}_{\perp} \cap \mathbf{H}^m(\mathbb{R}^3)^N \mid \exists k_+ > 0 \mid \text{supp}(\mathcal{F}(\mathbf{V})) \subset \mathbb{R}^3 \setminus B(0, k_+)\}, \quad (2.1.37)$$

$$\mathcal{H}_{\perp, \text{LF}}^p = \{\mathbf{V} \in \mathcal{H}_{\perp} \cap \mathcal{L}_p(\mathbb{R}^3)^N \mid \exists k_- > 0 \mid \text{supp}(\mathcal{F}(\mathbf{V})) \subset \overline{B(0, k_-)}\}. \quad (2.1.38)$$

In the spirit of [63], we define for any  $m > 0$  the high frequency optimal polynomial exponent of solutions  $\mathbf{U}(t)$  of the Cauchy problem (2.1.12) with initial conditions  $\mathbf{U}_0 \in \mathcal{H}_{\perp, \text{HF}}^m$  as follows:

$$\gamma_m^{\text{HF}} = \sup\{\gamma \in \mathbb{R}^+ \mid \forall \mathbf{U}_0 \in \mathcal{H}_{\perp, \text{HF}}^m, \exists C(\mathbf{U}_0) > 0 \mid \|\mathbf{U}(t)\|_{\mathcal{H}}^2 \lesssim \frac{C(\mathbf{U}_0)}{t^\gamma}, \quad \forall t \geq 1\}. \quad (2.1.39)$$

Similarly, we define for any  $p \geq 0$  the low frequency optimal polynomial exponent:

$$\gamma_p^{\text{LF}} = \sup\{\gamma \in \mathbb{R}^+ \mid \forall \mathbf{U}_0 \in \mathcal{H}_{\perp, \text{LF}}^p, \exists C(\mathbf{U}_0) > 0 \mid \|\mathbf{U}(t)\|_{\mathcal{H}}^2 \lesssim \frac{C(\mathbf{U}_0)}{t^\gamma}, \quad \forall t \geq 1\}. \quad (2.1.40)$$

We are now able to state our result concerning the low frequency and high frequency optimal polynomial decay rates.

### Theorem 2.1.10

Let  $(m, p) \in \mathbb{R}^{+,*} \times \mathbb{R}^+$ . The exponents  $\gamma_m^{\text{HF}}$  and  $\gamma_p^{\text{LF}}$  are given as follows:

- a. If the Maxwell system is in a non-critical configuration,  $\gamma_m^{\text{HF}} = m$  and  $\gamma_p^{\text{LF}} = p + 3/2$ .
- b. If the Maxwell system is in a critical configuration,  $\gamma_m^{\text{HF}} = m/2$  and  $\gamma_p^{\text{LF}} = p + 3/2$ .

We point out that in Section 2.7.1, we show that the upperbounds (2.1.34) and (2.1.35) give the right lower bounds on the exponents  $\gamma_m^{\text{HF}}$  and  $\gamma_p^{\text{LF}}$ . In Section 2.7.2, we show that these lower-bounds are also upper-bounds and thus deduce the value of these exponents.

## 2.2 Fourier reduction

### 2.2.1 The reduced Hamiltonian $\mathbb{A}_{\mathbf{k}}$

The homogeneity of the propagation medium allows us to reduce the spectral analysis of the operator  $\mathbb{A}$  defined in (2.1.15) to the spectral analysis of a family of (non-self-adjoint) operators  $(\mathbb{A}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{R}^3}$  on a finite dimensional space.

We introduce the space  $\mathbf{C}^N$  with  $\mathbf{C} = \mathbb{C}^3$  endowed with the inner product  $(\cdot, \cdot)_{\mathbf{C}^N}$  defined via

the expression (2.1.14) by replacing  $\mathbf{U} \in \mathcal{H}$  by  $\mathbb{U} = (\mathbb{E}, \mathbb{H}, \mathbb{P}, \dot{\mathbb{P}}, \mathbb{M}, \dot{\mathbb{M}}) \in \mathbf{C}^N$  with

$$\mathbb{P} = (\mathbb{P}_j), \dot{\mathbb{P}} = (\dot{\mathbb{P}}_j), \mathbb{M} = (\mathbb{M}_\ell), \dot{\mathbb{M}} = (\dot{\mathbb{M}}_\ell),$$

(the same for  $\mathbf{U}'$ ) and the  $\mathbf{L}^2$  inner product  $(\cdot, \cdot)_{\mathbf{L}^2}$  by the usual one in  $\mathbf{C}$ :  $(\mathbb{E}, \mathbb{E}') = \mathbb{E} \cdot \overline{\mathbb{E}'}$ .

The corresponding norms in  $\mathbf{C}$  or  $\mathbf{C}^N$  are both denoted  $|\cdot|$  for simplicity.

In this functional framework,  $\mathbb{A}$  is unitarily equivalent via  $\mathcal{F}$  to a direct integral of operators  $\mathbb{A}_{\mathbf{k}}$  defined on the finite dimension space  $\mathbf{C}^N$ . Namely, one has:

$$\mathbb{A} = \mathcal{F}^* \left( \int_{\mathbb{R}^3}^{\oplus} \mathbb{A}_{\mathbf{k}} d\mathbf{k} \right) \mathcal{F} \quad \text{i.e.} \quad \mathcal{F}(\mathbb{A}\mathbf{U})(\mathbf{k}) = \mathbb{A}_{\mathbf{k}} \mathcal{F}\mathbf{U}(\mathbf{k}), \quad \forall \mathbf{U} \in D(\mathbb{A}) \text{ and a.e. } \mathbf{k} \in \mathbb{R}^3, \quad (2.2.1)$$

where the (bounded) linear operator  $\mathbb{A}_{\mathbf{k}} : \mathbf{C}^N \mapsto \mathbf{C}^N$  is given by

$$\forall \mathbb{U} = (\mathbb{E}, \mathbb{H}, \mathbb{P}, \dot{\mathbb{P}}, \mathbb{M}, \dot{\mathbb{M}}) \in \mathbf{C}^N, \quad \mathbb{A}_{\mathbf{k}}\mathbb{U} = \begin{pmatrix} -\frac{\mathbf{k} \times \mathbb{H}}{\varepsilon_0} - i \sum \Omega_{e,j}^2 \dot{\mathbb{P}}_j \\ \frac{\mathbf{k} \times \mathbb{E}}{\mu_0} - i \sum \Omega_{m,\ell}^2 \dot{\mathbb{M}}_\ell \\ i \dot{\mathbb{P}} \\ -i \alpha_{e,j} \dot{\mathbb{P}}_j - i \omega_{e,j}^2 \mathbb{P}_j + i \mathbb{E} \\ i \dot{\mathbb{M}} \\ -i \alpha_{m,\ell} \dot{\mathbb{M}}_\ell - i \omega_{m,\ell}^2 \mathbb{M}_\ell + i \mathbb{H} \end{pmatrix}. \quad (2.2.2)$$

We point out that  $\mathbb{A}_{\mathbf{k}}$  is deduced from the definition (2.1.15) of  $\mathbb{A}$  by replacing the curl operator  $\nabla \times$  operator by its spatial Fourier counterpart  $i\mathbf{k} \times$ . The usual cross product on  $\mathbb{R}^3 \times \mathbb{R}^3$  has been here extended to  $\mathbf{C} \times \mathbf{C}$  via the formula  $\mathbf{a} \times \mathbf{b} = (\mathbf{a}_2 \mathbf{b}_3 - \mathbf{a}_3 \mathbf{b}_2, \mathbf{a}_3 \mathbf{b}_1 - \mathbf{a}_1 \mathbf{b}_3, \mathbf{a}_1 \mathbf{b}_2 - \mathbf{a}_2 \mathbf{b}_1)$ .

**Remark 2.2.1.** A simple computation shows that for any  $\mathbb{U} = (\mathbb{E}, \mathbb{H}, \mathbb{P}, \dot{\mathbb{P}}, \mathbb{M}, \dot{\mathbb{M}}) \in \mathbf{C}^N$ :

$$\text{Im}(\mathbb{A}_{\mathbf{k}}\mathbb{U}, \mathbb{U})_{\mathbf{C}^N} = - \sum_{j=1}^{N_e} \alpha_{e,j} \Omega_{e,j}^2 |\dot{\mathbb{P}}_j|^2 - \sum_{\ell=1}^{N_m} \alpha_{m,\ell} \Omega_{m,\ell}^2 |\dot{\mathbb{M}}_\ell|^2 \leq 0.$$

Thus, for all  $\mathbf{k} \in \mathbb{R}^3$ , the spectrum of  $\mathbb{A}_{\mathbf{k}}$ ,  $\sigma(\mathbb{A}_{\mathbf{k}})$  is included in  $\overline{\mathbb{C}^-} = \{\omega \in \mathbb{C} \mid \text{Im}(\omega) \leq 0\}$ .

Hence,  $-i\mathbb{A}(\mathbf{k})$  is maximal dissipative and the resolvent of  $\mathbb{A}(\mathbf{k})$ ,  $R_{\mathbb{A}_{\mathbf{k}}}(\omega) = (\mathbb{A}_{\mathbf{k}} - \omega \text{Id})^{-1}$  is well-defined in  $\mathbb{C}^+$  and satisfies:

$$\|R_{\mathbb{A}_{\mathbf{k}}}(\omega)\| \leq \text{Im}(\omega)^{-1}, \quad \forall \omega \in \mathbb{C}^+ \text{ and } \forall \mathbf{k} \in \mathbb{R}^3.$$

This estimate justifies the existence of the direct integral  $\int_{\mathbb{R}^3}^{\oplus} \mathbb{A}_{\mathbf{k}} d\mathbf{k}$  (see [62] page 5).

From an operator point of view, one deduces from (2.2.1) (see e.g. Theorem 4.2 of [62]) that the contractions  $S(t)$  of the semigroup  $\{\mathcal{S}(t)\}_{t \geq 0}$  are also unitarily equivalent via  $\mathcal{F}$  to a direct integral of contractions  $e^{-i\mathbb{A}_{\mathbf{k}}t}$  on  $\mathbf{C}^N$ . More precisely, one has for any  $t \geq 0$ :

$$S(t) = \mathcal{F}^* \left( \int_{\mathbb{R}^3}^{\oplus} e^{-i\mathbb{A}_{\mathbf{k}}t} d\mathbf{k} \right) \mathcal{F}, \quad \text{i.e.} \quad \mathcal{F}(S(t)\mathbf{U})(\mathbf{k}) = e^{-i\mathbb{A}_{\mathbf{k}}t} \mathcal{F}\mathbf{U}(\mathbf{k}), \quad \forall \mathbf{U} \in \mathcal{H} \text{ a.e. } \mathbf{k} \in \mathbb{R}^3. \quad (2.2.3)$$

From a more practical point of view, it means that the Fourier transform  $\mathbb{U} = \mathcal{F}\mathbf{U}$  of the solution

$\mathbf{U}$  of the Schrödinger evolution problem (2.1.12) with initial condition  $\mathbf{U}_0 \in \mathcal{H}$  satisfies a family of Ordinary Differential Equations (ODE) in  $\mathbf{C}^N$  parametrized by the wave vector  $\mathbf{k}$ . Namely, one has for a.e.  $\mathbf{k} \in \mathbb{R}^3$ :

$$\frac{d\mathbb{U}(\mathbf{k}, t)}{dt} + i\mathbb{A}_{\mathbf{k}}\mathbb{U}(\mathbf{k}, t) = 0 \quad \text{for } t \geq 0 \quad \text{with } \mathbb{U}(\mathbf{k}, 0) = \mathbb{U}_0(\mathbf{k}) = (\mathcal{F}\mathbf{U}_0)(\mathbf{k}). \quad (2.2.4)$$

The solution of each this ODE is given by  $\mathbb{U}(\mathbf{k}, t) = e^{-i\mathbb{A}_{\mathbf{k}}t}\mathbb{U}_0(\mathbf{k})$ ,  $\forall t \geq 0$ . We can now introduce the counterpart of the Hodge decomposition (2.1.20) in the (spatial) frequency domain. In this perspective, one decomposes orthogonally the space  $\mathbf{C}^N$ , for all wave vector  $\mathbf{k} \neq 0$ , as

$$\mathbf{C}^N = \mathbf{C}_{\mathbf{k},\parallel}^N \oplus \mathbf{C}_{\mathbf{k},\perp}^N \quad \text{where } \mathbf{C}_{\mathbf{k},\parallel} := \text{span } \mathbf{k} \quad \text{and } \mathbf{C}_{\mathbf{k},\perp} := \{\mathbf{k}\}^\perp. \quad (2.2.5)$$

One shows easily that  $\mathbf{C}_{\mathbf{k},\parallel}^N$  and  $\mathbf{C}_{\perp,\mathbf{k}}^N$  are stable by  $\mathbb{A}_{\mathbf{k}}$ . Thus, one can reduce  $\mathbb{A}_{\mathbf{k}}$  as a sum of two operators

$$\mathbb{A}_{\mathbf{k},\parallel} : \mathbf{C}_{\mathbf{k},\parallel}^N \rightarrow \mathbf{C}_{\mathbf{k},\parallel}^N \quad \text{and } \mathbb{A}_{\mathbf{k},\perp} : \mathbf{C}_{\mathbf{k},\perp}^N \rightarrow \mathbf{C}_{\mathbf{k},\perp}^N$$

in such way that

$$\mathbb{A}_{\mathbf{k}} = \mathbb{A}_{\mathbf{k},\parallel} \oplus \mathbb{A}_{\mathbf{k},\perp} \quad \text{with } \mathbb{A}_{\mathbf{k},\parallel}\mathbf{U} = \mathbb{A}\mathbf{U}, \forall \mathbf{U} \in \mathbf{C}_{\mathbf{k},\parallel}^N \quad \text{and } \mathbb{A}_{\mathbf{k},\perp}\mathbf{U} = \mathbb{A}_{\mathbf{k}}\mathbf{U}, \forall \mathbf{U} \in \mathbf{C}_{\mathbf{k},\perp}^N. \quad (2.2.6)$$

The above relation is nothing but the counterpart of (2.1.21) for Fourier components. Indeed, following the decompositions (2.1.21) and (2.2.6), one proves easily that the operators  $\mathbb{A}_{\perp}$  and the contractions  $S_{\perp}(t)$  of its associated semigroup  $\{S_{\perp}(t)\}_{t \geq 0}$  are also unitarily equivalent via  $\mathcal{F}$  to direct integral of operators  $\mathbb{A}_{\mathbf{k},\perp}$  and  $e^{-i\mathbb{A}_{\mathbf{k},\perp}t}$ :

$$\begin{aligned} \mathbb{A}_{\perp} &= \mathcal{F}^* \left( \int_{\mathbb{R}^3}^{\oplus} \mathbb{A}_{\mathbf{k},\perp} d\mathbf{k} \right) \mathcal{F} \quad \text{i.e. } \mathcal{F}(\mathbb{A}_{\perp}\mathbf{U})(\mathbf{k}) = \mathbb{A}_{\mathbf{k},\perp} \mathcal{F}\mathbf{U}(\mathbf{k}), \forall \mathbf{U} \in D(\mathbb{A}_{\perp}), \text{ a.e. } \mathbf{k} \in \mathbb{R}^3, \\ S_{\perp}(t) &= \mathcal{F}^* \left( \int_{\mathbb{R}^3}^{\oplus} e^{-i\mathbb{A}_{\mathbf{k},\perp}t} d\mathbf{k} \right) \mathcal{F} \quad \text{i.e. } \mathcal{F}(S_{\perp}(t)\mathbf{U})(\mathbf{k}) = e^{-i\mathbb{A}_{\mathbf{k},\perp}t} \mathcal{F}\mathbf{U}(\mathbf{k}), \forall \mathbf{U} \in \mathcal{H}_{\perp}, \text{ a.e. } \mathbf{k} \in \mathbb{R}^3. \end{aligned}$$

Similarly, the operators  $\mathbb{A}_{\parallel}$  and  $S_{\parallel}(t)$  for  $t \geq 0$  are also unitarily equivalent via  $\mathcal{F}$  to direct integral of operators and the latter relations hold also if one replaces all  $\perp$ -symbols by  $\parallel$ -symbols.

The direct integral decomposition of the operators  $S_{\perp}(t)$  implies that the Fourier transform  $\mathbb{U} = \mathcal{F}\mathbf{U}$  of the solution  $\mathbf{U}$  of the Schrödinger evolution problem (2.1.12) with initial divergence free conditions  $\mathbf{U}_0 \in \mathcal{H}_{\perp}$  (as e.g. in (2.1.2)) satisfies for a.e.  $\mathbf{k} \in \mathbb{R}^3$ :

$$\mathbb{U}(\mathbf{k}, t) = e^{-i\mathbb{A}_{\mathbf{k},\perp}t} \mathbb{U}_0(\mathbf{k}), \quad \forall t \geq 0 \quad \text{where } \mathbb{U}_0(\mathbf{k}) = (\mathcal{F}\mathbf{U}_0)(\mathbf{k}) \in \mathbf{C}_{\mathbf{k},\perp}^N. \quad (2.2.7)$$

### 2.2.2 From $\mathbb{A}_{\mathbf{k}}$ to $\mathbb{A}_{|\mathbf{k}|}$

In the expression of the solution (2.2.7), the space  $\mathbf{C}_{\perp,\mathbf{k}}$  depends on  $\mathbf{k}$  which complicates slightly the analysis of the decay of the Fourier components  $\mathbb{U}(\mathbf{k}, t)$ . To remediate to this point, one can use the isotropic character of the medium.

We introduce  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  the canonical orthonormal basis of  $\mathbf{C}$  and for  $\mathbf{k} \neq 0$  the unit vector  $\widehat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ . We now construct a rotation  $\mathbf{R}_{\mathbf{k}} : \mathbf{C} \rightarrow \mathbf{C}$  that maps the vector  $\widehat{\mathbf{k}}$  into  $\mathbf{e}_3$  as follows

- If  $\widehat{\mathbf{k}} \neq \pm \mathbf{e}_3$ , we set  $\mathbf{w}_{\mathbf{k}} := (\widehat{\mathbf{k}} \times \mathbf{e}_3)/|\widehat{\mathbf{k}} \times \mathbf{e}_3| \in \mathbf{C}_{\mathbf{k},\perp}$  so that  $(\widehat{\mathbf{k}}, \mathbf{w}_{\mathbf{k}}, \widehat{\mathbf{k}} \times \mathbf{w}_{\mathbf{k}})$  is an orthonormal basis of  $\mathbf{C}$ . We then define  $\mathbf{R}_{\mathbf{k}}$  in this basis by

$$\mathbf{R}_{\mathbf{k}}(\widehat{\mathbf{k}}) = \mathbf{e}_3, \quad \mathbf{R}_{\mathbf{k}}(\mathbf{w}_{\mathbf{k}}) = \mathbf{e}_1, \quad \mathbf{R}_{\mathbf{k}}(\widehat{\mathbf{k}} \times \mathbf{w}_{\mathbf{k}}) = \mathbf{e}_2.$$

- If  $\widehat{\mathbf{k}} = \pm \mathbf{e}_3$ , we simply set  $\mathbf{R}_{\mathbf{k}} = \text{Id}_{\mathbf{C}}$ .

Finally, we define the operator  $\mathcal{R}_{\mathbf{k}} : \mathbf{C}^N \rightarrow \mathbf{C}^N$  defined for all  $\mathbb{U} = (\mathbb{E}, \mathbb{H}, \mathbb{P}, \dot{\mathbb{P}}, \mathbb{M}, \dot{\mathbb{M}}) \in \mathbf{C}^N$  by

$$\mathcal{R}_{\mathbf{k}} \mathbb{U} = (\mathbf{R}_{\mathbf{k}} \mathbb{E}, \mathbf{R}_{\mathbf{k}} \mathbb{H}, (\mathbf{R}_{\mathbf{k}} \mathbb{P}_j), (\mathbf{R}_{\mathbf{k}} \dot{\mathbb{P}}_j), (\mathbf{R}_{\mathbf{k}} \mathbb{M}_\ell), (\mathbf{R}_{\mathbf{k}} \dot{\mathbb{M}}_\ell)). \quad (2.2.8)$$

Using the identity (we let its proof to the reader):

$$\mathbf{R}_{\mathbf{k}}^* (\mathbf{R}_{\mathbf{k}}(\mathbf{u}) \times \mathbf{R}_{\mathbf{k}}(\mathbf{v})) = \mathbf{u} \times \mathbf{v}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{C}$$

applied to  $\mathbf{u} = \mathbf{k}$  (and thus with  $\mathbf{R}_{\mathbf{k}}(\mathbf{u}) = |\mathbf{k}| \mathbf{e}_3$ ), one observes with (2.2.2) that

$$\mathbb{A}_{\mathbf{k}} = \mathcal{R}_{\mathbf{k}}^* \mathbb{A}_{|\mathbf{k}|} \mathcal{R}_{\mathbf{k}} \quad \text{where} \quad \mathbb{A}_{|\mathbf{k}|} := \mathbb{A}_{|\mathbf{k}| \mathbf{e}_3}. \quad (2.2.9)$$

Thus,  $\mathbb{A}_{\mathbf{k}}$  and  $\mathbb{A}_{|\mathbf{k}|}$  are unitarily equivalent. Furthermore, as

$$\mathcal{R}_{\mathbf{k}}(\mathbf{C}_{\mathbf{k},\parallel}^N) = \mathbf{C}_{\parallel}^N \quad \text{and} \quad \mathcal{R}_{\mathbf{k}}(\mathbf{C}_{\perp,\mathbf{k}}^N) = \mathbf{C}_{\perp}^N \quad \text{where} \quad \mathbf{C}_{\parallel} := \mathbf{C}_{\parallel, \mathbf{e}_3} \quad \text{and} \quad \mathbf{C}_{\perp} := \mathbf{C}_{\perp, \mathbf{e}_3}, \quad (2.2.10)$$

one has by (2.2.6):

$$\mathbb{A}_{\mathbf{k},\parallel} = \mathcal{R}_{\mathbf{k}}^* \mathbb{A}_{|\mathbf{k}|,\parallel} \mathcal{R}_{\mathbf{k}} \quad \text{and} \quad \mathbb{A}_{\mathbf{k},\perp} = \mathcal{R}_{\mathbf{k}}^* \mathbb{A}_{|\mathbf{k}|,\perp} \mathcal{R}_{\mathbf{k}}. \quad (2.2.11)$$

In particular, it shows with (2.2.7) that the Fourier transform  $\mathbb{U} = \mathcal{F}\mathbf{U}$  of the solution  $\mathbf{U}$  of (2.1.12) with initial divergence free conditions  $\mathbf{U}_0 \in \mathcal{H}_{\perp}$  satisfies for a.e.  $\mathbf{k} \in \mathbb{R}^3$ :

$$\mathbb{U}(\mathbf{k}, t) = \mathcal{R}_{\mathbf{k}}^* e^{-i\mathbb{A}_{|\mathbf{k}|,\perp} t} \mathcal{R}_{\mathbf{k}} \mathbb{U}_0(\mathbf{k}), \quad \forall t \geq 0 \quad \text{where} \quad \mathbb{U}_0(\mathbf{k}) = (\mathcal{F}\mathbf{U}_0)(\mathbf{k}) \in \mathbf{C}_{\mathbf{k},\perp}^N. \quad (2.2.12)$$

Thus, to estimate the norm of  $|\mathbb{U}(\mathbf{k}, t)|$ , one only needs to analyse the spectral properties of  $\mathbb{A}_{|\mathbf{k}|,\perp}$  on the space  $\mathbf{C}_{\perp}^N$  which is now independent of  $\mathbf{k}$ .

## 2.3 Modal analysis

The proofs of [Theorem 2.1.7](#) and [Theorem 2.1.10](#) are based on the spectral properties of  $\mathbb{A}_{|\mathbf{k}|}$ . These theorems are proved in [Section 2.7](#) based on results established from [Section 2.3](#) to [Section 2.6](#).

### 2.3.1 Spectrum and resolvent of the finite dimensional operators $\mathbb{A}_{|\mathbf{k}|}$

Using the expression (2.1.5), the rational functions  $\varepsilon$  and  $\mu$  can be rewritten as

$$\varepsilon(\omega) = \varepsilon_0 \frac{P_e(\omega)}{Q_e(\omega)} \quad \text{and} \quad \mu(\omega) = \mu_0 \frac{P_m(\omega)}{Q_m(\omega)}, \quad (2.3.1)$$

where the unitary polynomials  $P_e$  and  $Q_e$  (resp.  $P_m$  and  $Q_m$ ) are of degree  $2N_e$  (resp.  $2N_m$ ). More precisely,  $Q_e$  and  $P_e$  are explicitly given in term of the  $q_{e,j}$  by

$$Q_e(\omega) = \prod_{j=1}^{N_e} q_{e,j}(\omega) \quad \text{and} \quad P_e(\omega) = Q_e(\omega) - \sum_{j=1}^{N_e} \Omega_{e,j}^2 \prod_{k=1, k \neq j}^{N_e} q_{e,k}(\omega). \quad (2.3.2)$$

The reader will verify that, with (2.3.2), the assumption  $(\mathbf{H}_1)$  implies that  $P_e$  and  $Q_e$  do not share common zero. Therefore the representation (2.3.1) of the rational function  $\varepsilon$  is irreducible.

Thus,  $\mathcal{P}_e$ , the set of poles of  $\varepsilon$ , is exactly the set of roots of  $Q_e$  which is the union over  $j$  of the

roots of  $q_{e,j}$  and is therefore included in  $\overline{\mathbb{C}^-}$  (see Remark 2.1.1).

Moreover, by (H<sub>1</sub>) and Remark 2.1.1, the multiplicity of a root  $\omega_*$  of  $Q_e$  is either 1 or 2.

Obviously, the equivalent of (2.3.2) and the above properties hold for the polynomials  $P_m$  and  $Q_m$ .

We introduce the sets  $\mathcal{P} := \mathcal{P}_e \cup \mathcal{P}_m$  and  $\mathcal{Z} := \mathcal{Z}_e \cup \mathcal{Z}_m$ , where we recall that  $\mathcal{P}_e$  (resp.  $\mathcal{P}_m$ ) is the set of poles of the function  $\varepsilon$  (resp.  $\mu$ ) and  $\mathcal{Z}_e$  (resp.  $\mathcal{Z}_m$ ) the set of zeros of  $\varepsilon$  (resp.  $\mu$ ) and the rational function  $\mathcal{D} : \mathbb{C} \setminus \mathcal{P} \rightarrow \mathbb{C}$  defined by

$$\mathcal{D}(\omega) = \omega^2 \varepsilon(\omega) \mu(\omega), \quad \forall \omega \in \mathbb{C} \setminus \mathcal{P}. \quad (2.3.3)$$

Using (H<sub>2</sub>), it is easy to see that  $\mathcal{D}(\omega)$  is irreducible. From (2.3.1) and (2.3.2), one deduces immediately that its numerator is of degree  $N = 2 + 2N_e + 2N_m$  and its denominator of degree  $2N_e + 2N_m$ . Moreover, its poles  $p$  and zeros  $z$  are respectively the elements  $\mathcal{P}$  and  $\mathcal{Z} \cup \{0\}$ .

We introduce for any  $\mathbf{k} \in \mathbb{R}^3$ , the set

$$S(|\mathbf{k}|) := \{\omega \in \mathbb{C} \setminus \mathcal{P} \mid \mathcal{D}(\omega) = |\mathbf{k}|^2\}.$$

The following proposition establishes the link between  $S(|\mathbf{k}|)$  and the spectrum of the operator  $\mathbb{A}_{|\mathbf{k}|,\perp}$  for  $\mathbf{k} \neq 0$ .

**Proposition 2.3.1**

Let assume (H<sub>1</sub>) and (H<sub>2</sub>). Then for any  $\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}$ , one has

$$\sigma(\mathbb{A}_{|\mathbf{k}|,\perp}) = S(|\mathbf{k}|). \quad (2.3.4)$$

Moreover, each eigenvalue  $\omega \in \sigma(\mathbb{A}_{|\mathbf{k}|,\perp})$  has a geometric multiplicity of 2.

Finally, we conclude this section with the resolvent

$$R_{|\mathbf{k}|}(\omega) := (\mathbb{A}_{|\mathbf{k}|,\perp} - \omega \mathbb{I})^{-1}. \quad (2.3.5)$$

To obtain a readable expression, it appears useful to introduce some intermediate operators.

We first define four linear operators in  $\mathcal{L}(\mathbf{C}_\perp^N, \mathbf{C}_\perp)$ : given  $\mathbb{F} := (\mathbf{e}, \mathbf{h}, \mathbf{p}_j, \dot{\mathbf{p}}_j, \mathbf{m}_\ell, \dot{\mathbf{m}}_\ell) \in \mathbf{C}_\perp^N$

$$\begin{aligned} \mathbb{A}_{e,j}(\omega) \mathbb{F} &:= \frac{(-i\alpha_{e,j} - \omega) \mathbf{p}_j - i \dot{\mathbf{p}}_j}{q_{e,j}(\omega)}, & \mathbb{A}_{m,\ell}(\omega) \mathbb{F} &:= \frac{(-i\alpha_{m,\ell} - \omega) \mathbf{m}_\ell - i \dot{\mathbf{m}}_\ell}{q_{m,\ell}(\omega)}, \\ \dot{\mathbb{A}}_{e,j}(\omega) \mathbb{F} &:= \frac{i\omega_{e,j}^2 \mathbf{p}_j - \omega \dot{\mathbf{p}}_j}{q_{e,j}(\omega)}, & \dot{\mathbb{A}}_{m,\ell}(\omega) \mathbb{F} &:= \frac{i\omega_{m,\ell}^2 \mathbf{m}_\ell - \omega \dot{\mathbf{m}}_\ell}{q_{m,\ell}(\omega)}, \end{aligned} \quad (2.3.6)$$

from which we define two more operators in  $\mathcal{L}(\mathbf{C}_\perp^N, \mathbf{C}_\perp)$ :

$$\mathbb{A}_e(\omega) \mathbb{F} = -\varepsilon_0 \left( \mathbf{e} + i \sum \Omega_{e,j}^2 \dot{\mathbb{A}}_{e,j}(\omega) \mathbb{F} \right), \quad \mathbb{A}_m(\omega) \mathbb{F} = -\mu_0 \left( \mathbf{h} + i \sum \Omega_{m,\ell}^2 \dot{\mathbb{A}}_{m,\ell}(\omega) \mathbb{F} \right). \quad (2.3.7)$$

Finally, we shall also define the finite subset of  $\mathbb{C}$  :

$$\mathcal{S}_{\mathcal{T}} := \mathcal{P} \cup \mathcal{Z}_m \cup \{0\}. \quad (2.3.8)$$

**Proposition 2.3.2**

Let  $\mathbf{k} \in \mathbb{R}^3$ . For any  $\omega \in \mathbb{C} \setminus (S(|\mathbf{k}|) \cup \mathcal{S}_{\mathcal{T}})$ , the resolvent is given by

$$R_{|\mathbf{k}|}(\omega) = \mathcal{V}_{|\mathbf{k}|}(\omega) \mathcal{S}_{|\mathbf{k}|}(\omega) + \mathcal{T}(\omega) \quad (2.3.9)$$

with  $\mathcal{S}_{|\mathbf{k}|}(\omega) \in \mathcal{L}(\mathbf{C}_{\perp}^N, \mathbf{C}_{\perp})$  defined by: given  $\mathbb{F} := (\mathbf{e}, \mathbf{h}, \mathbf{p}_j, \dot{\mathbf{p}}_j, \mathbf{m}_{\ell}, \dot{\mathbf{m}}_{\ell}) \in \mathbf{C}_{\perp}^N$

$$\mathcal{S}_{|\mathbf{k}|}(\omega) \mathbb{F} := \frac{\omega \mu(\omega) \mathbb{A}_e(\omega) \mathbb{F} - |\mathbf{k}| \mathbf{e}_3 \times \mathbb{A}_m(\omega) \mathbb{F}}{\mathcal{D}(\omega) - |\mathbf{k}|^2}, \quad (2.3.10)$$

$\mathcal{V}_{|\mathbf{k}|}(\omega) \in \mathcal{L}(\mathbf{C}_{\perp}, \mathbf{C}_{\perp}^N)$  defined by: given  $\mathbb{X} \in \mathbf{C}_{\perp}$

$$\left| \begin{aligned} \mathcal{V}_{|\mathbf{k}|}(\omega) \mathbb{X} &:= \left( \mathbb{X}, 0, -\left(\frac{\mathbb{X}}{q_{e,j}(\omega)}\right), \left(\frac{i \omega \mathbb{X}}{q_{e,j}(\omega)}\right), 0, 0 \right) \\ &+ \frac{|\mathbf{k}|}{\omega \mu(\omega)} \left( 0, \mathbf{e}_3 \times \mathbb{X}, 0, 0, -\left(\frac{\mathbf{e}_3 \times \mathbb{X}}{q_{m,\ell}(\omega)}\right), \left(\frac{i \omega \mathbf{e}_3 \times \mathbb{X}}{q_{m,\ell}(\omega)}\right) \right) \end{aligned} \right. \quad (2.3.11)$$

and finally  $\mathcal{T}(\omega) \in \mathcal{L}(\mathbf{C}_{\perp}^N)$  defined by : given  $\mathbb{F} := (\mathbf{e}, \mathbf{h}, \mathbf{p}_j, \dot{\mathbf{p}}_j, \mathbf{m}_{\ell}, \dot{\mathbf{m}}_{\ell}) \in \mathbf{C}_{\perp}^N$

$$\left| \begin{aligned} \mathcal{T}(\omega) \mathbb{F} &:= \left( 0, \frac{\mathbb{A}_m(\omega) \mathbb{F}}{\omega \mu(\omega)}, 0, 0, -\left(\frac{\mathbb{A}_m(\omega) \mathbb{F}}{\omega \mu(\omega) q_{m,\ell}(\omega)}\right), \left(\frac{i \mathbb{A}_m(\omega) \mathbb{F}}{\mu(\omega) q_{m,\ell}(\omega)}\right) \right) \\ &+ \left( 0, 0, (\mathbb{A}_{p,j}(\omega) \mathbb{F}), (\dot{\mathbb{A}}_{\dot{p},j}(\omega) \mathbb{F}), (\mathbb{A}_{m,\ell}(\omega) \mathbb{F}), (\dot{\mathbb{A}}_{\dot{m},\ell}(\omega) \mathbb{F}) \right). \end{aligned} \right. \quad (2.3.12)$$

The (purely computational) proofs of [Proposition 2.3.1](#) and [Proposition 2.3.2](#) are delayed to the Appendix.

**Remark 2.3.3.** The function  $\omega \mapsto R_{|\mathbf{k}|}(\omega)$  is well-defined and analytic for  $\omega \in \mathbb{C} \setminus \sigma(\mathbb{A}_{|\mathbf{k}|,\perp})$ . The set  $\mathcal{S}_{\mathcal{T}}$  is the set of singularities for  $\omega \mapsto \mathcal{T}(\omega)$  but they are removable singularities in the expression [\(2.3.9\)](#).

## 2.3.2 The dispersion relation

### 2.3.2.1 General properties of the dispersion relation

By [Proposition 2.3.1](#), for a fixed wave number  $\mathbf{k} \neq 0$ , the eigenvalues of  $\mathbb{A}_{|\mathbf{k}|,\perp}$  are the solutions  $\omega \in \mathbb{C} \setminus \mathcal{P}$  of the equation

$$\mathcal{D}(\omega) = |\mathbf{k}|^2, \quad (2.3.13)$$

(with  $\mathcal{D}(\omega)$  defined by [\(2.3.3\)](#)) referred in physics as the dispersion relation. We make in this section several important remarks and results on this equation.

As a consequence of [Remark 2.2.1](#), one has  $\sigma(\mathbb{A}_{|\mathbf{k}|}) \subset \overline{\mathbb{C}^-}$ , thus  $\sigma(\mathbb{A}_{|\mathbf{k}|,\perp}) \subset \sigma(\mathbb{A}_{|\mathbf{k}|}) \subset \overline{\mathbb{C}^-}$ . We give here an elementary proof that for  $\mathbf{k} \neq 0$ , the spectrum  $\sigma(\mathbb{A}_{|\mathbf{k}|,\perp})$  is included in the lower open complex half plane

$$\sigma(\mathbb{A}_{|\mathbf{k}|,\perp}) \subset \mathbb{C}^-, \quad (2.3.14)$$

based on the dispersion relation [\(2.3.13\)](#). Assume by contradiction that there exists  $\omega \in \overline{\mathbb{C}^+} \cap \sigma(\mathbb{A}_{|\mathbf{k}|,\perp})$ . From [Proposition 2.3.1](#),  $\omega$  satisfies [\(2.3.13\)](#) and therefore  $\omega \neq 0$  and  $\omega \notin \mathcal{P}$ . Taking



the real part and the imaginary part of (2.3.13) leads to

$$\operatorname{Re}(\omega \varepsilon(\omega)) \operatorname{Re}(\omega \mu(\omega)) = |\mathbf{k}|^2 + \operatorname{Im}(\omega \varepsilon(\omega)) \operatorname{Im}(\omega \mu(\omega)) > 0 \quad (2.3.15)$$

$$\operatorname{Re}(\omega \varepsilon(\omega)) \operatorname{Im}(\omega \mu(\omega)) + \operatorname{Re}(\omega \mu(\omega)) \operatorname{Im}(\omega \varepsilon(\omega)) = 0 \quad (2.3.16)$$

where the rational Herglotz functions  $\omega \mapsto \omega \varepsilon(\omega)$  and  $\omega \mapsto \omega \mu(\omega)$  are analytic and have a positive imaginary part on  $\mathbb{C}^+$ . Furthermore, one has for  $\omega \in \mathbb{R} \setminus \mathcal{P}$

$$\operatorname{Im}(\omega \varepsilon(\omega)) = \varepsilon_0 |\omega|^2 \sum_{j=1}^{N_e} \frac{\Omega_{e,j}^2 \alpha_{e,j}}{|q_{e,j}(\omega)|^2} \geq 0 \quad \text{and} \quad \operatorname{Im}(\omega \mu(\omega)) = \mu_0 |\omega|^2 \sum_{\ell=1}^{N_m} \frac{\Omega_{m,\ell}^2 \alpha_{m,\ell}}{|q_{m,\ell}(\omega)|^2} \geq 0. \quad (2.3.17)$$

Thus, by the weak dissipation assumption (2.1.10), at least one of the coefficients  $\alpha_{e,j}$  or  $\alpha_{m,\ell}$  is positive and it follows from (2.3.17) that

$$\operatorname{Im}(\omega \varepsilon(\omega)) > 0 \quad \text{or} \quad \operatorname{Im}(\omega \mu(\omega)) > 0 \quad \text{on} \quad \mathbb{R}^* \setminus \mathcal{P}. \quad (2.3.18)$$

Hence, (2.3.15) implies that  $\operatorname{Re}(\omega \varepsilon(\omega))$  and  $\operatorname{Re}(\omega \mu(\omega))$  have the same sign (and do not vanish), whereas (2.3.16) implies that they have opposite sign, which leads to a contradiction. Thus, for  $\mathbf{k} \neq 0$ , the solutions of the dispersion relation  $\mathcal{D}(\omega) = |\mathbf{k}|^2$  all lie in  $\mathbb{C}^-$ .

Using the fact that  $\varepsilon(-\bar{\omega}) = \overline{\varepsilon(\omega)}$  and  $\mu(-\bar{\omega}) = \overline{\mu(\omega)}$ , one observes that if  $\omega$  is a solution of the dispersion relation then  $-\bar{\omega}$  is also a solution of this equation. Thus for a fixed  $\mathbf{k} \neq 0$ , the set  $\sigma(\mathbb{A}_{|\mathbf{k}|,\perp})$  is invariant by the transformation  $\omega \rightarrow -\bar{\omega}$ .

From the irreducible form (2.3.3) of  $\mathcal{D}$ , one deduces that the dispersion relation is equivalent to a polynomial equation of degree  $N$ . Namely, for a fixed  $\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}$ , one has:

$$\mathcal{D}(\omega) = |\mathbf{k}|^2 \iff D_{|\mathbf{k}|}(\omega) = 0, \quad D_{|\mathbf{k}|}(\omega) := \varepsilon_0 \mu_0 \omega^2 P_e(\omega) P_m(\omega) - |\mathbf{k}|^2 Q_e(\omega) Q_m(\omega) \quad (2.3.19)$$

is a polynomial of degree  $N = 2 + 2N_e + 2N_m$  with dominant coefficient  $\varepsilon_0 \mu_0$ . Thus, one has

$$D_{|\mathbf{k}|}(\omega) \sim \varepsilon_0 \mu_0 \omega^{2+2N_e+2N_m}, \quad (|\omega| \rightarrow +\infty). \quad (2.3.20)$$

This leads to the following corollary on the diagonalizability of  $\mathbb{A}_{|\mathbf{k}|,\perp}$ .

#### Corollary 2.3.4

For  $\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}$ ,  $\mathbb{A}_{|\mathbf{k}|,\perp}$  is diagonalizable on  $\mathbf{C}_\perp^N$  if and only if the roots of the polynomials  $D_{|\mathbf{k}|}$  defined in (2.3.19) are simple.

*Proof.* By Proposition 2.3.1 and relation (2.3.19), the eigenvalues of  $\mathbb{A}_{|\mathbf{k}|,\perp}$  are the solutions of the polynomial equation  $D_{|\mathbf{k}|}(\omega) = 0$  of degree  $N$ . Furthermore, each distinct solution of this equation is an eigenvalue of geometric multiplicity 2. As the space  $\mathbf{C}_\perp^N$  is of dimension  $2N$ , see (2.2.10) the result follows immediately from a simple argument of dimension.  $\square$

We end this section with two paragraphs: one on the poles  $\mathcal{P}$  and one on the zeros  $\mathcal{Z} \cup \{0\}$  of the rational function  $\mathcal{D}$  associated to the dispersion relation. Indeed (as we will see in the Section 2.3.3 and in the parts Section 2.4 and Section 2.5) poles and zeros play a key roles for the asymptotics of eigenvalues of  $\mathbb{A}_{|\mathbf{k}|}$  respectively for  $|\mathbf{k}| \gg 1$  and  $|\mathbf{k}| \ll 1$ .

### 2.3.2.2 Properties of the poles of the rational function $\mathcal{D}$

As we saw in [Section 2.3.1](#), the elements of  $\mathcal{P}$  are exactly the zeros of the denominator of the rational function  $\mathcal{D}$ , that is the polynomial of  $Q_e Q_m$  of degree  $2(N_e + N_m)$ . Thus,  $\mathcal{P}$  contains  $2(N_e + N_m)$  elements (counted with multiplicity) and this set is invariant by the transformation  $\omega \mapsto -\bar{\omega}$  (see [Remark 2.1.1](#)).

The multiplicity  $\mathbf{m}_p$  of a pole  $p \in \mathcal{P}$ , it can not take any arbitrary value. Indeed, the elements of  $\mathcal{P}$  are precisely (see [Section 2.3.1](#)) the roots of the 2-nd order polynomials  $q_{e,j}$  or  $q_{m,\ell}$  for  $j \in \{1, \dots, N_e\}$  and  $\ell \in \{1, \dots, N_m\}$ . Thus,  $\mathcal{P} \subset \overline{\mathbb{C}^-}$  (see [Remark 2.1.1](#)) and one has:

- If  $p \notin i\mathbb{R}^{-,*}$  then  $\mathbf{m}_p = 1$  if  $p \notin \mathcal{P}_e \cap \mathcal{P}_m$  or  $\mathbf{m}_p = 2$  if  $\omega \in \mathcal{P}_e \cap \mathcal{P}_m$ ,
- Else if  $p \in i\mathbb{R}^{-,*}$ , then by  $(H_1)$ ,  $\mathbf{m}_p \in \{1, 2\}$  if  $p \notin \mathcal{P}_e \cap \mathcal{P}_m$  or  $\mathbf{m}_p \in \{2, 3, 4\}$  if  $p \in \mathcal{P}_e \cap \mathcal{P}_m$ .

We will see in [Section 2.3.3](#) and [Section 2.4.3](#) that the important poles for our analysis for  $|\mathbf{k}| \gg 1$  are the ones that lie on the real axis. These poles are associated to polynomials  $q_{e,j}$  or  $q_{m,\ell}$  for which  $\alpha_{e,j} = 0$  or  $\alpha_{m,\ell} = 0$ . Thus, they are of the form  $p = \pm \omega_{e,\ell} \in \mathcal{P}_e$  or  $p = \pm \omega_{m,\ell} \in \mathcal{P}_m$  with  $\omega_{e,j}, \omega_{m,\ell} > 0$  and have multiplicity  $\mathbf{m}_p = 1$  if  $p \notin \mathcal{P}_e \cap \mathcal{P}_m$  or  $\mathbf{m}_p = 2$  if  $p \in \mathcal{P}_e \cap \mathcal{P}_m$ .

### 2.3.2.3 Properties of the zeros of the rational function $\mathcal{D}$

As we saw in [Section 2.3.1](#), the elements of  $\mathcal{Z} \cup \{0\}$  are the zeros of the rational function  $\mathcal{D}$ , that is the zeros of the polynomial  $\omega^2 P_e P_m$  of degree  $N = 2(N_e + N_m) + 2$ . Hence  $\mathcal{Z} \cup \{0\}$  contains  $N$  elements (counted with multiplicity).  $\mathcal{D}$  is defined as the product of the two (non-constant) rational Herglotz functions  $\omega \mapsto \omega \varepsilon(\omega)$  and  $\omega \mapsto \omega \mu(\omega)$  which satisfy (using [\(2.1.5\)](#))  $\text{Im}(\omega \varepsilon(\omega)) > 0$  and  $\text{Im}(\omega \mu(\omega)) > 0$  on  $\mathbb{C}^+$ . Thus, these functions could not vanish in the upper-half plane  $\mathbb{C}^+$  (indeed this property holds more generally for any non-constant Herglotz function as a consequence of the open mapping theorem for analytic functions, see for e.g. [\[19\]](#)). Therefore,  $\mathcal{Z} \cup \{0\}$  is included in  $\overline{\mathbb{C}^-}$ . Furthermore, as  $\varepsilon(-\bar{\omega}) = \varepsilon(\omega)$  and  $\mu(-\bar{\omega}) = \mu(\omega)$ , this set is also invariant by the transformation  $\omega \mapsto -\bar{\omega}$ .

Concerning the multiplicity  $\mathbf{m}_z$  of a zero  $z \in \mathcal{Z} \cup \{0\}$ , one observes that:

- 0 has a multiplicity  $\mathbf{m}_0 = 2$  of  $\mathcal{D}$  since (by [\(2.1.3\)](#) and [\(2.1.5\)](#)) at the vicinity of 0:

$$\mathcal{D}(\omega) \sim \omega^2 \varepsilon(0) \mu(0), \quad \varepsilon(0) = \varepsilon_0 \left( 1 + \sum_{j=1}^{N_e} \frac{\Omega_{e,j}^2}{\omega_{e,j}^2} \right) > 0, \quad \mu(0) = \mu_0 \left( 1 + \sum_{\ell=1}^{N_m} \frac{\Omega_{m,\ell}^2}{\omega_{e,\ell}^2} \right) > 0. \quad (2.3.21)$$

- If  $z \notin i\mathbb{R}^-$ , then  $-\bar{z}$  is a distinct zero with the same multiplicity  $\mathbf{m}_z$ .
- From the property [\(2.3.18\)](#) of  $\varepsilon(\omega)$  and  $\mu(\omega)$ , one immediately sees that
  - a. If there exists two indices  $j_0$  and  $\ell_0$  such that  $\alpha_{e,j_0} > 0$  and  $\alpha_{m,\ell_0} > 0$  and  $\mathcal{D}$  has no non-zero real zeros, i. e.  $\mathbb{R}^* \cap \mathcal{Z} = \emptyset$ .
  - b. If not, either all  $\alpha_{e,j}$  vanish in which case  $\mathbb{R}^* \cap \mathcal{Z} = \mathcal{Z}_e$ , either all  $\alpha_{m,\ell}$  vanish in which case  $\mathbb{R}^* \cap \mathcal{Z} = \mathcal{Z}_m$  and these zeros  $z \in \mathcal{Z}_e$  (resp.  $\mathcal{Z}_m$ ) have multiplicity  $\mathbf{m}_z = 1$  (this is easily deduced from the graph of the function  $\omega \in \mathbb{R} \setminus \mathcal{P}_e \mapsto \varepsilon(\omega)$  or  $\omega \in \mathbb{R} \setminus \mathcal{P}_m \mapsto \mu(\omega)$  given by [\(2.1.5\)](#)).

We point out that the second scenario occurs in particular in the critical configurations described by [Definition 2.1.6](#).

We will see in [Section 2.3.3](#) and [Section 2.5](#) that only the real zeros  $\mathbb{R} \cap (\mathcal{Z} \cup \{0\})$  can contribute at the main order to the asymptotic of the Fourier-components of  $|\mathbf{U}(k, t)|$  for  $|\mathbf{k}| \ll 1$ .

### 2.3.3 Main lines of the analysis

The strategy for proving [Theorem 2.1.7](#) is quite clear and simple:

- (i) For each  $\mathbf{k} \in \mathbb{R}^3$ , one estimates individually  $\mathbb{U}(\mathbf{k}, t)$  using formula [\(2.2.7\)](#), or more precisely [\(2.2.12\)](#), via the estimation of the exponential  $e^{-i\mathbb{A}_{|\mathbf{k}|,\perp} t}$ .
- (ii) One gathers the above estimates to estimate the  $L^2$ -norms of the various fields  $\mathbf{E}, \mathbf{H}, \dots$  via the norm  $\|\mathbb{U}(\cdot, t)\|_{\mathcal{H}}$  thanks to Plancherel's theorem.

For the first step, the key property is the fact that, as emphasized in [Section 2.3.2.1](#) (see [\(2.3.14\)](#)), the spectrum of  $\mathbb{A}_{|\mathbf{k}|,\perp}$  is included in the complex half-plane  $\mathbb{C}^-$ . As a consequence, each  $|\mathbb{U}(\mathbf{k}, t)|$  will decay exponentially to 0 for large  $t$ .

The reason why the exponential decay is lost for  $\|\mathbb{U}(\cdot, t)\|_{\mathcal{H}}$  and degenerates into a polynomial decay, is linked to the fact that the rate of decay of  $|\mathbb{U}(\mathbf{k}, t)|$  depends on  $\mathbf{k}$  and degenerates when  $|\mathbf{k}|$  tends to 0 or  $+\infty$ . This decay rate is of course linked to the distance of  $\sigma(\mathbb{A}_{|\mathbf{k}|,\perp})$  to the real axis that can be deduced from the analysis of the dispersion relation  $\mathcal{D}(\omega) = |\mathbf{k}|^2$ , where  $\mathcal{D}(\omega) = \omega^2 \varepsilon(\omega) \mu(\omega)$ , that characterize the eigenvalues of  $\mathbb{A}_{|\mathbf{k}|,\perp}$  (see [Proposition 2.3.1](#)).

- When  $|\mathbf{k}|$  is bounded from below and above, this distance is uniformly bounded from below by a strictly positive number which results into a uniform exponential decay of the corresponding  $|\mathbb{U}(\mathbf{k}, t)|$ 's.
- When  $|\mathbf{k}|$  tends to 0 or  $+\infty$  this distance tends to 0 and obtaining sharp estimates of  $|\mathbb{U}(\mathbf{k}, t)|$  requires to analyse the asymptotic behaviour of the imaginary parts  $\text{Im } \omega(|\mathbf{k}|)$  of those eigenvalues  $\omega(|\mathbf{k}|) \in \sigma(\mathbb{A}_{|\mathbf{k}|,\perp})$  whose distance to the real axis tends to 0.

From this observation, it is natural to split the analysis into three steps depending on the values of the "space frequency"  $|\mathbf{k}|$  :

- (a) For "mid frequencies", typically  $k_- \leq |\mathbf{k}| \leq k_+$  (with  $k_- > 0$ ), one will essentially abandon the spectral approach to the profit of standard techniques for ODE's combined with compactness arguments (the region  $k_- \leq |\mathbf{k}| \leq k_+$  is compact). This will be detailed in [Section 2.6](#).
- (b) For "high frequencies",  $|\mathbf{k}| \geq k_+$ , eigenvalues can be arbitrarily close to the real axis when  $|\mathbf{k}| \rightarrow +\infty$ : considering the limit of the equation  $\mathcal{D}(\omega) = |\mathbf{k}|^2$  when  $|\mathbf{k}| \rightarrow +\infty$  it is natural to look at where, for real  $\omega$ , the function  $\mathcal{D}(\omega)$  tends to  $+\infty$ . This only occurs in one of the following situation when

- $\omega \rightarrow \pm\infty$  in which case  $\mathcal{D}(\omega) \sim \omega^2/c^2$ , so that one expects the existence of two branches of eigenvalues  $\omega_\infty(|\mathbf{k}|)$  and  $-\bar{\omega}_\infty(|\mathbf{k}|)$  where

$$\omega_\infty(|\mathbf{k}|) = c|\mathbf{k}| + o(|\mathbf{k}|), \quad |\mathbf{k}| \rightarrow +\infty,$$

where  $c := \sqrt{\varepsilon_0 \mu_0}^{-1}$  is the speed of light in the vacuum.

- $\omega \rightarrow \pm\omega_{\nu,q}$  with  $\alpha_{\nu,q} = 0$ ,  $\nu = e$  or  $m$  where  $\omega_{\nu,q}$  is a real pole of  $\mathcal{D}$  of multiplicity 1 if  $\omega_{\nu,q} \notin \mathcal{P}_e \cap \mathcal{P}_m$  or 2 if not (see [Section 2.3.2.2](#)).

If  $\omega_{\nu,q} \notin \mathcal{P}_e \cap \mathcal{P}_m$ , one expects the existence of two branches of eigenvalues  $\omega_{\nu,q}(|\mathbf{k}|)$  and  $-\bar{\omega}_{\nu,q}(|\mathbf{k}|)$  such that

$$\omega_{\nu,q}(|\mathbf{k}|) = \omega_{\nu,q} + o(1), \quad |\mathbf{k}| \rightarrow +\infty.$$

Oppositely, if  $\omega_{e,q_1} = \omega_{m,q_2} \in \mathcal{P}_e \cap \mathcal{P}_m$ , one expects the existence of four branches of eigenvalues  $\omega_{e,q_1}(|\mathbf{k}|)$ ,  $-\bar{\omega}_{e,q_1}(|\mathbf{k}|)$ ,  $\omega_{m,q_2}(|\mathbf{k}|)$ ,  $-\bar{\omega}_{m,q_2}(|\mathbf{k}|)$  such that  $\omega_{e,q_1}(|\mathbf{k}|)$  and  $\omega_{m,q_2}(|\mathbf{k}|)$  tends to  $\omega_{e,q_1}$  as  $|\mathbf{k}| \rightarrow +\infty$ .

In the above cases, a more precise analysis of the asymptotic behaviour  $\text{Im } \omega_\infty(|\mathbf{k}|)$  and  $\text{Im } \omega_{\nu,q}(|\mathbf{k}|)$  when  $|\mathbf{k}| \rightarrow +\infty$  plays a crucial role.

- (c) For “low frequencies”,  $0 < |\mathbf{k}| \leq k_-$ , again some eigenvalues can be arbitrarily close to the real axis when  $|\mathbf{k}| \rightarrow 0$ : considering the limit of the equation  $\mathcal{D}(\omega) = |\mathbf{k}|^2$  when  $|\mathbf{k}| \rightarrow 0$  it is natural to look at where, for real  $\omega$ , the function  $\mathcal{D}(\omega)$  vanishes. This only occurs in one of the following situation when

- $\omega \rightarrow 0$  in which case by (2.3.21):  $\mathcal{D}(\omega) \sim \omega^2/c_0^2$  with  $c_0 := (\varepsilon(0)\mu(0))^{-1/2}$ . Thus, one expects the existence of branches of eigenvalues  $\omega_0(|\mathbf{k}|)$  and  $-\bar{\omega}_0(|\mathbf{k}|)$  such that

$$\omega_0(|\mathbf{k}|) = c_0 |\mathbf{k}| + o(|\mathbf{k}|), \quad |\mathbf{k}| \rightarrow 0.$$

- If we are in the second scenario described in Section 2.3.2.3, non-zero real zeros exist and are of multiplicity  $\mathbf{m}_z = 1$ . Thus, in this case, one has  $\omega \rightarrow \pm z_\nu$ , for  $\nu = e$  or  $m$ , where  $z_\nu$  is a zero of  $\mathcal{D}(\omega)$ . One then expects the existence of two branches of eigenvalues  $\omega_{z_\nu}(|\mathbf{k}|)$  and  $-\bar{\omega}_{z_\nu}(|\mathbf{k}|)$  such that

$$\omega_{z_\nu}(|\mathbf{k}|) = z_\nu + o(1), \quad |\mathbf{k}| \rightarrow 0.$$

In the above cases, a more precise analysis of the asymptotic behaviour of  $\text{Im } \omega_0(|\mathbf{k}|)$  and  $\text{Im } \omega_{z_\nu}(|\mathbf{k}|)$  when  $|\mathbf{k}| \rightarrow 0$  plays a crucial role.

The analysis of point (b) will explain why the polynomial stability is limited by the Sobolev regularity of the initial data with the first term in the right hand sides of the estimates (2.1.34) and (2.1.35), while the analysis of point (c) will explain why this polynomial stability in the second term in the right hand sides of (2.1.34) and (2.1.35) is related to the low frequency behavior of the Fourier components of the solution (and thus involved naturally the spaces  $\mathcal{L}_p(\mathbb{R}^3)^N$ ).

Let us mention that, since the matrices  $\mathbb{A}_{|\mathbf{k}|,\pm}$  are not normal, the estimate of  $e^{-i\mathbb{A}_{|\mathbf{k}|,\pm} t}$  in the regions  $|\mathbf{k}| \geq k_+$  and  $|\mathbf{k}| \leq k_-$  cannot be reduced to the study of their eigenvalues. That is why we shall complete the analysis by proving that

- for  $k_-$  and  $k_+$  well chosen, the matrices  $\mathbb{A}_{|\mathbf{k}|,\pm}$  are diagonalizable in the above regions,
- the associated spectral projectors can be bounded uniformly in  $|\mathbf{k}|$  in each region.

## 2.4 Asymptotic analysis for large spatial frequencies $|\mathbf{k}| \gg 1$

As this section is the longest of the article, it seems useful to describe its structure. It is made of three main subsections:

- Section 2.4.1: Asymptotics of dispersion curves for  $|\mathbf{k}| \gg 1$ .
- Section 2.4.2: Spectral decomposition of the solution for  $|\mathbf{k}| \gg 1$ .
- Section 2.4.3: Large time estimate of  $\mathbb{U}(\mathbf{k}, t)$  for  $|\mathbf{k}| \gg 1$ .

Section 2.4.3 puts together the results of Section 2.4.1 and Section 2.4.2. It is decomposed into subsections dedicated to the estimate each of the components of the solution issued from Section 2.4.2.

From the technical point of view, the proof of the asymptotic expansions will be based of a lemma proved in the appendix Appendix B.2., namely the Lemma B. 1, that can be seen as a kind of implicit function theorem for functions in the complex plane. This will also be the case in Section 2.5.3.

### 2.4.1 Asymptotics of dispersion curves for $|\mathbf{k}| \gg 1$

In this section, we focus on long time estimates of the high (spatial) frequency components  $\mathbb{U}(\mathbf{k}, t)$  (see (2.2.7)) of the solution. As explained in the above paragraph, the decay of  $\mathbb{U}(\mathbf{k}, t)$  is related to the analysis of the solutions of the dispersion relation (2.3.13) for  $|\mathbf{k}| \gg 1$ . Roughly speaking, as  $|\mathbf{k}|^2 \rightarrow +\infty$  when  $|\mathbf{k}| \rightarrow +\infty$ , the solutions of (2.3.13) must satisfy  $|\mathcal{D}(\omega)| \rightarrow +\infty$  as  $|\mathbf{k}| \rightarrow +\infty$ . Thus, we observe two scenarios: either they diverge to  $\infty$  or they converge to a pole  $p \in \mathcal{P}_e \cup \mathcal{P}_m$  around which, If  $p \in \mathcal{P}$  is a pole of multiplicity  $\mathfrak{m}_p$ ,  $\mathcal{D}$  can be rewritten as

$$\mathcal{D}(\omega) = (\omega - p)^{-\mathfrak{m}_p} f(\omega) \text{ with } f \text{ analytic on a vicinity of } p \text{ and } f(p) = A_p \neq 0. \quad (2.4.1)$$

This leads to the following proposition.

#### Proposition 2.4.1

It exists  $k_+ > 0$  such that for  $|\mathbf{k}| \geq k_+$ , the solutions of the rational dispersion relation (2.3.13) (or of its equivalent polynomial form (2.3.19)) are all simple. These solutions form  $N$  distinct branches which are  $C^\infty$ -smooth functions (with respect to  $|\mathbf{k}|$ ) characterized by their asymptotic expansion for large  $|\mathbf{k}|$ . More precisely

- For any  $p \in \mathcal{P}$  with multiplicity  $\mathfrak{m}_p$ , for large enough  $|\mathbf{k}|$ , there exists  $\mathfrak{m}_p$  distinct branches of solutions  $\omega_{p,n}$ ,  $n = 1, \dots, \mathfrak{m}_p$  of (2.3.13) satisfying

$$\omega_{p,n}(|\mathbf{k}|) = p + a_{p,n} |\mathbf{k}|^{-\frac{2}{\mathfrak{m}_p}} (1 + o(1)), \quad a_{p,n} = |A_p|^{1/\mathfrak{m}_p} e^{i\frac{\theta_p}{\mathfrak{m}_p}} e^{\frac{2in\pi}{\mathfrak{m}_p}} \quad (|\mathbf{k}| \rightarrow +\infty) \quad (2.4.2)$$

where  $A_p$  is defined in (2.4.1) and  $\theta_p \in (-\pi, \pi]$  is the principal argument of  $A_p$ .

- There are 2 distinct branches of solutions  $\omega_{\pm\infty}$  of (2.3.13) that diverge to  $\infty$  as

$$\omega_{\pm\infty}(|\mathbf{k}|) = \pm c |\mathbf{k}| (1 + o(1)) \quad \text{with } c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}} \quad (|\mathbf{k}| \rightarrow +\infty). \quad (2.4.3)$$

Moreover, there are no other solutions of (2.3.13) (for  $|\mathbf{k}| \geq k_+$ ) than those described above.

*Proof. Step 1: construction of the branches of solutions  $\omega_{p,n}(|\mathbf{k}|)$ .*

Let  $p \in \mathcal{P}$  a pole of multiplicity  $\mathfrak{m}_p$ . Then, the rational function  $\mathcal{D}$  can be factorized as in (2.4.1). Then  $p$  is a zero of multiplicity  $\mathfrak{m}_p$  of the function  $\omega \mapsto \mathcal{D}(\omega)^{-1} = (\omega - p)^{\mathfrak{m}_p} f(\omega)^{-1}$  which is analytic on an open neighbourhood of  $p$  and satisfies  $f(p)^{-1} = A_p^{-1} \neq 0$ . Moreover, for  $|\mathbf{k}| \neq 0$ , solving the dispersion relation:

$$\mathcal{D}(\omega) = |\mathbf{k}|^2 \quad \text{is equivalent to solve the equation} \quad \mathcal{D}(\omega)^{-1} = |\mathbf{k}|^{-2} = (|\mathbf{k}|^{-\frac{2}{\mathfrak{m}_p}})^{\mathfrak{m}_p}.$$

Then, by applying the [Lemma B. 1](#) of the [Appendix B.2](#). with

$$\mathcal{D}(\omega) = \mathcal{D}(\omega)^{-1}, \quad z = p, \quad g(\omega) = f(\omega)^{-1}, \quad \mathbf{m} = \mathbf{m}_p, \quad A = A_p^{-1} \text{ and } \zeta = |\mathbf{k}|^{-2/\mathbf{m}_p},$$

we deduce for  $|\mathbf{k}|$  large enough the existence of  $\mathbf{m}_p$  distinct branches of solutions:  $|\mathbf{k}| \mapsto \omega_{p,n}(|\mathbf{k}|)$  of the equation  $\mathcal{D}(\omega) = |\mathbf{k}|^2$  which are  $C^\infty$  functions with respect to  $|\mathbf{k}|$  and satisfy [\(2.4.2\)](#).

**Step 2: construction of the 2 distinct branches of solutions  $\omega_{\pm\infty}(|\mathbf{k}|)$ .**

To reduce ourselves to the application of [Lemma B. 1](#) as in step 1, the trick consists in saying that  $\mathcal{D}(\omega) = \omega^2 \varepsilon(\omega) \mu(\omega) = |\mathbf{k}|^2$  is equivalent to  $\mathcal{D}(\omega)^{-1} = \omega^{-2} \varepsilon(\omega)^{-1} \mu(\omega)^{-1} = |\mathbf{k}|^{-2}$ . Then we introduce the new unknown  $\xi = 1/\omega$ , so that  $|\omega| \rightarrow +\infty \Leftrightarrow \xi \rightarrow 0$  and

$$\mathcal{D}(\omega) = \omega^2 \varepsilon(\omega) \mu(\omega) = |\mathbf{k}|^2 \iff \xi^2 \varepsilon(1/\xi)^{-1} \mu(1/\xi)^{-1} = |\mathbf{k}|^{-2}, \text{ modulo } \xi = 1/\omega. \quad (2.4.4)$$

Then we introduce the rational function  $\mathcal{G}(\xi)$ :

$$\mathcal{G}(\xi) := \xi^2 g(\xi), \quad g(\xi) := \varepsilon(1/\xi)^{-1} \mu(1/\xi)^{-1} \quad \text{well defined for } \xi \in \mathbb{C} \setminus \{1/z, z \in \mathcal{Z}\}$$

after having remark that  $g(\xi)$  could be extended analytically at  $\xi = 0$  via

$$g(0) = \lim_{\xi \rightarrow 0} \varepsilon(\xi^{-1})^{-1} \mu(\xi^{-1})^{-1} = (\varepsilon_0 \mu_0)^{-1} = c^2 \neq 0.$$

Thus, we can now apply the [Lemma B. 1](#), replacing  $\omega$  par  $\xi$ , with

$$z = 0, \quad g(\xi) := \varepsilon(1/\xi)^{-1} \mu(1/\xi)^{-1}, \quad \mathbf{m} = 2, \quad A = g(0) = c^2 \text{ and } \zeta = |\mathbf{k}|^{-1}.$$

From [Lemma B. 1](#), we deduce the existence of two distinct analytic functions  $\zeta \mapsto \xi_{\pm}(\zeta)$ , defined at the vicinity of 0, such that  $\mathcal{G}(\xi_{\pm}(\zeta)) = \zeta^2$  and such that

$$\xi_{\pm}(\zeta) = \pm c^{-1} \zeta (1 + o(1)), \quad \text{as } \zeta \rightarrow 0. \quad (2.4.5)$$

Thus, setting  $\omega_{\pm\infty}(|\mathbf{k}|) = \xi_{\pm}(|\mathbf{k}|^{-1})^{-1}$ , we construct two branches of solutions of  $\mathcal{D}(\omega) = |\mathbf{k}|^2$  which admit by [\(2.4.5\)](#) the asymptotic expansion [\(2.4.3\)](#).

**Step 3: conclusion.** In step 1, since the sum of the  $\mathbf{m}_p$ 's over  $p \in \mathcal{P}$  is equal to  $2N_e + 2N_m$ , we have constructed for  $|\mathbf{k}|$  large enough,  $2N_e + 2N_m$  solutions, namely  $\{\omega_{p,n}(|\mathbf{k}|), p \in \mathcal{P}, n \leq \mathbf{m}_p\}$  which are all distinct due to the asymptotics [\(2.4.2\)](#).

In step 2, we have constructed for  $|\mathbf{k}|$  large enough, 2 additional solutions  $\omega_{\pm\infty}(|\mathbf{k}|)$  that are distinct thanks to [\(2.4.3\)](#).

From both asymptotics [\(2.4.2\)](#) and [\(2.4.3\)](#), none of this two solutions can coincide with any of the ones of step 1. Therefore, with  $\{\omega_{p,n}(|\mathbf{k}|), p \in \mathcal{P}, n \leq \mathbf{m}_p\} \cup \{\omega_{\pm\infty}(|\mathbf{k}|)\}$ , we have constructed  $2N_e + 2N_m + 2$  distinct solutions of [\(2.3.13\)](#). Since [\(2.3.13\)](#) is equivalent to a polynomial equation of degree  $2N_e + 2N_m + 2$ , cf. [\(2.3.19\)](#), there are no other solutions.  $\square$

## 2.4.2 Spectral decomposition of the solution for $|\mathbf{k}| \gg 1$

In the physics literature, the solutions of the dispersion relation [\(2.3.13\)](#)  $|\mathbf{k}| \rightarrow \omega_{p,n}(|\mathbf{k}|)$  and  $|\mathbf{k}| \rightarrow \omega_{\pm\infty}(|\mathbf{k}|)$  (given here for large  $|\mathbf{k}|$  by [Proposition 2.4.1](#)) are refereed as the dispersion curves. The asymptotics [\(2.4.2\)](#) and [\(2.4.3\)](#) of these curves show that they do not cross each other for  $|\mathbf{k}| \gg 1$ . Thus, combining [Proposition 2.4.1](#) and [Corollary 2.3.4](#) yields immediately to

the following property on the operator  $\mathbb{A}_{|\mathbf{k}|,\perp}$ .

**Corollary 2.4.2**

It exists  $k_+ > 0$  such that for  $|\mathbf{k}| \geq k_+$ ,  $\mathbb{A}_{|\mathbf{k}|,\perp}$  is diagonalizable on  $\mathbf{C}_\perp^N$ .

To express the diagonal decomposition of the solution  $\mathbb{U}(\mathbf{k}, t)$ , we split (see Section 2.3.2.2) the sets of poles  $\mathcal{P} \subset \mathbb{C}^-$  in three disjoint subsets :  $\mathcal{P} = \mathcal{P}_- \cup \mathcal{P}_s \cup \mathcal{P}_d$  with

$$\mathcal{P}_- := \mathcal{P} \cap \mathbb{C}^-, \quad \mathcal{P}_s := \{p \in \mathcal{P} \cap \mathbb{R} \mid \mathbf{m}_p = 1\}, \quad \text{and} \quad \mathcal{P}_d := \{p \in \mathcal{P} \cap \mathbb{R} \mid \mathbf{m}_p = 2\}.$$

We point out that the weak dissipation condition (2.1.10) implies that at least one pole of  $\mathcal{P}$  lies in  $\mathbb{C}^-$ , thus  $\mathcal{P}_- \neq \emptyset$  whereas the strong dissipation condition (2.1.8) implies that all poles lie in  $\mathbb{C}^-$ , that is  $\mathcal{P}_- = \mathcal{P}$ . From Section 2.3.2.2, one has also that  $\mathcal{P}_d = \mathcal{P}_e \cap \mathcal{P}_m \cap \mathbb{R}$ .

Using Proposition 2.3.1 and Proposition 2.4.1, we introduce the following partition of the spectrum of  $\mathbb{A}_{|\mathbf{k}|,\perp}$  for  $|\mathbf{k}| \geq k_+$ :

$$\left| \begin{aligned} \sigma(\mathbb{A}_{|\mathbf{k}|,\perp}) &= \{\omega_{\pm\infty}(|\mathbf{k}|)\} \cup \{\omega_p(|\mathbf{k}|), p \in \mathcal{P}_s\} \cup \{\omega_{p,r}(|\mathbf{k}|), p \in \mathcal{P}_d, r \in \{1, 2\}\} \\ &\cup \{\omega_{p,n}(|\mathbf{k}|), p \in \mathcal{P}_-, n = 1, \dots, \mathbf{m}_p\}. \end{aligned} \right.$$

(where we set  $\omega_p(|\mathbf{k}|) := \omega_{p,1}(|\mathbf{k}|)$  for real pole  $p \in \mathcal{P}_s$  of multiplicity  $\mathbf{m}_p = 1$ .) We refer to the Figure 2.1 for an illustration of the behavior of the dispersion curves for  $|\mathbf{k}| \gg 1$ . For  $|\mathbf{k}| \geq k_+$ ,

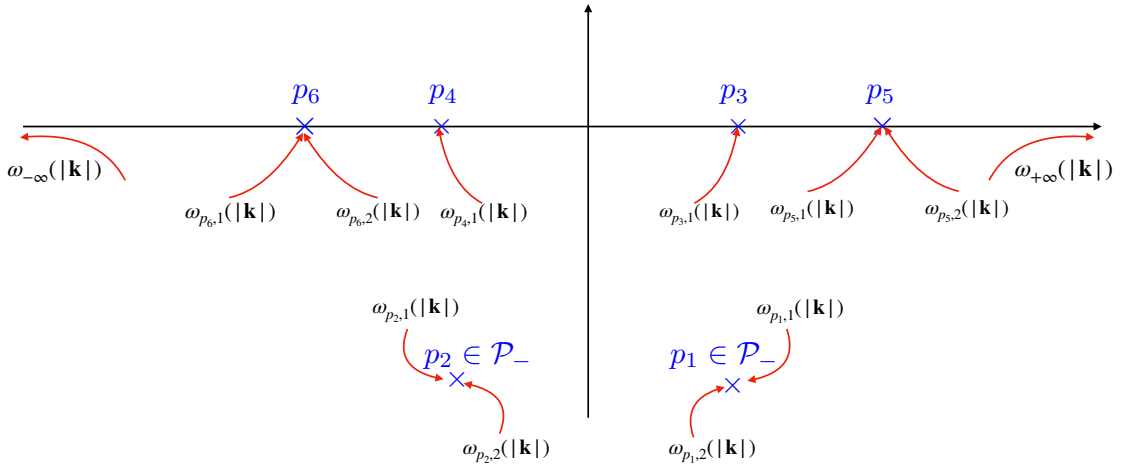


Figure 2.1: Sketch of a configuration of the dispersion curves for large  $|\mathbf{k}|$  large in the case where  $\mathcal{P} = \mathcal{P}_- \cup \mathcal{P}_s \cup \mathcal{P}_d$  with  $\mathcal{P}_- = \{p_1, p_2\}$ ,  $\mathcal{P}_s = \{p_3, p_4\}$  and  $\mathcal{P}_d = \{p_5, p_6\}$ .

as  $\mathbb{A}_{|\mathbf{k}|,\perp}$  is diagonalizable by Corollary 2.3.4,  $\mathbf{C}_\perp^N$  can be decomposed as

$$\mathbf{C}_\perp^N = \bigoplus_{\pm} V_{|\mathbf{k}|,\pm\infty} \oplus \bigoplus_{p \in \mathcal{P}_s} V_{|\mathbf{k}|,p} \oplus \bigoplus_{p \in \mathcal{P}_d} \bigoplus_{r=1}^2 V_{|\mathbf{k}|,p,r} \oplus \bigoplus_{p \in \mathcal{P}_-} \bigoplus_{n=1}^{\mathbf{m}_p} V_{|\mathbf{k}|,p,n} \quad (2.4.6)$$

$$\left\{ \begin{aligned} V_{|\mathbf{k}|,\pm\infty} &= \ker(\mathbb{A}_{|\mathbf{k}|,\perp} - \omega_{\pm\infty}(|\mathbf{k}|) \text{Id}), \quad V_{|\mathbf{k}|,p,n} = \ker(\mathbb{A}_{|\mathbf{k}|,\perp} - \omega_{p,n}(|\mathbf{k}|) \text{Id}), \quad n \leq \mathbf{m}_p, \\ V_{|\mathbf{k}|,p} &= \ker(\mathbb{A}_{|\mathbf{k}|,\perp} - \omega_p(|\mathbf{k}|) \text{Id}), \quad V_{|\mathbf{k}|,p,r} = \ker(\mathbb{A}_{|\mathbf{k}|,\perp} - \omega_{p,r}(|\mathbf{k}|) \text{Id}), \quad r = 1, 2, \end{aligned} \right. \quad (2.4.7)$$

where the above direct sums are (in general) non-orthogonal. Following these direct sums, one decomposes uniquely any vector  $x \in \mathbf{C}_{\perp}^N$  as

$$\left\{ \begin{array}{l} x = \sum_{\pm\infty} x_{|\mathbf{k}|,\pm\infty} + \sum_{p \in \mathcal{P}_s} x_{|\mathbf{k}|,p} + \sum_{p \in \mathcal{P}_d} \sum_{r=1}^2 x_{|\mathbf{k}|,p,r} + \sum_{p \in \mathcal{P}_-} \sum_{n=1}^{m_p} x_{|\mathbf{k}|,p,n} \\ x_{|\mathbf{k}|,\pm\infty} \in V_{|\mathbf{k}|,\pm\infty}, \quad x_{|\mathbf{k}|,p} \in V_{|\mathbf{k}|,p} \quad \text{and} \quad x_{|\mathbf{k}|,p,r} \in V_{|\mathbf{k}|,p,r}, \quad x_{|\mathbf{k}|,p,n} \in V_{|\mathbf{k}|,p,n}. \end{array} \right.$$

Then, we define the spectral projectors  $\Pi_{p,n}(|\mathbf{k}|)$ ,  $p \in \mathcal{P}_-$  and  $n \in \{1, \dots, m_p\}$ ,  $\Pi_{\pm\infty}(|\mathbf{k}|)$ ,  $\Pi_p(|\mathbf{k}|)$  for  $p \in \mathcal{P}_s$  and  $\Pi_{p,r}(|\mathbf{k}|)$  for  $p \in \mathcal{P}_d$  and  $r \in \{1, 2\}$  associated receptively to the eigenvalues  $\omega_{p,n}(|\mathbf{k}|)$ ,  $\omega_{\pm\infty}(|\mathbf{k}|)$ ,  $\omega_p(|\mathbf{k}|)$  and  $\omega_{p,r}(|\mathbf{k}|)$  by:

$$\left\{ \begin{array}{l} \Pi_{\pm\infty}(|\mathbf{k}|)(x) = x_{|\mathbf{k}|,\pm\infty}, \quad \Pi_{p,n}(|\mathbf{k}|)(x) = x_{|\mathbf{k}|,p,n}, \\ \Pi_p(|\mathbf{k}|)(x) = x_{|\mathbf{k}|,p}, \quad \Pi_{p,r}(|\mathbf{k}|)(x) = x_{|\mathbf{k}|,p,r}. \end{array} \right. \quad (2.4.8)$$

From [Proposition 2.3.1](#), the geometric multiplicity of each eigenvalues in  $\sigma(\mathbb{A}_{|\mathbf{k}|,\perp})$  is two. Thus, all the  $\Pi_{p,n}(|\mathbf{k}|)$ ,  $\Pi_{\pm\infty}(|\mathbf{k}|)$ ,  $\Pi_p(|\mathbf{k}|)$  and  $\Pi_{p,r}(|\mathbf{k}|)$  are rank two projectors. We emphasize that the dissipative operator  $\mathbb{A}_{|\mathbf{k}|,\perp}$  is not normal, thus its spectral projectors are not orthogonal. For  $|\mathbf{k}| \geq k_+$ , as  $\mathbb{A}_{|\mathbf{k}|,\perp}$  is diagonalizable (by [Corollary 2.3.4](#)), one has

$$\begin{aligned} \mathbb{A}_{|\mathbf{k}|,\perp} &= \sum_{\pm} \omega_{\pm\infty}(|\mathbf{k}|) \Pi_{\pm\infty}(|\mathbf{k}|) + \sum_{p \in \mathcal{P}_s} \omega_p(|\mathbf{k}|) \Pi_p(|\mathbf{k}|) + \sum_{p \in \mathcal{P}_d} \sum_{r=1}^2 \omega_{p,r}(|\mathbf{k}|) \Pi_{p,r}(|\mathbf{k}|) \\ &+ \sum_{p \in \mathcal{P}_-} \sum_{n=1}^{m_p} \omega_{p,n}(|\mathbf{k}|) \Pi_{p,n}(|\mathbf{k}|). \end{aligned}$$

Thus, for  $|\mathbf{k}| > k_+$ , the solution  $\mathbb{U}(\mathbf{k}, t)$  given by [\(2.2.12\)](#) can be expressed for all  $t \geq 0$  as

$$\mathbb{U}(\mathbf{k}, t) = \mathbb{U}_{\infty}(\mathbf{k}, t) + \mathbb{U}_s(\mathbf{k}, t) + \mathbb{U}_d(\mathbf{k}, t) + \mathbb{U}_-(\mathbf{k}, t) \quad (2.4.9)$$

where

$$\left\{ \begin{array}{l} \mathbb{U}_{\infty}(\mathbf{k}, t) = \sum_{\pm} e^{-i\omega_{\pm\infty}(|\mathbf{k}|)t} \mathcal{R}_{\mathbf{k}}^* \Pi_{\pm\infty}(|\mathbf{k}|) \mathcal{R}_{\mathbf{k}} \mathbb{U}_0(\mathbf{k}), \quad (i) \\ \mathbb{U}_s(\mathbf{k}, t) = \sum_{p \in \mathcal{P}_s} e^{-i\omega_p(|\mathbf{k}|)t} \mathcal{R}_{\mathbf{k}}^* \Pi_p(|\mathbf{k}|) \mathcal{R}_{\mathbf{k}} \mathbb{U}_0(\mathbf{k}), \quad (ii) \\ \mathbb{U}_d(\mathbf{k}, t) = \sum_{p \in \mathcal{P}_d} \sum_{r=1}^2 e^{-i\omega_{p,r}(|\mathbf{k}|)t} \mathcal{R}_{\mathbf{k}}^* \Pi_{p,r}(|\mathbf{k}|) \mathcal{R}_{\mathbf{k}} \mathbb{U}_0(\mathbf{k}), \quad (iii) \\ \mathbb{U}_-(\mathbf{k}, t) = \sum_{p \in \mathcal{P}_-} \sum_{n=1}^{m_p} e^{-i\omega_{p,n}(|\mathbf{k}|)t} \mathcal{R}_{\mathbf{k}}^* \Pi_{p,n}(|\mathbf{k}|) \mathcal{R}_{\mathbf{k}} \mathbb{U}_0(\mathbf{k}). \quad (iv) \end{array} \right. \quad (2.4.10)$$

### 2.4.3 Estimates of $\mathbb{U}(\mathbf{k}, t)$ for $|\mathbf{k}| \gg 1$

In each subsection of this section, we shall make appear a lower bound  $k_+ > 0$  for giving a sense to  $|\mathbf{k}| \gg 1$  via  $|\mathbf{k}| \geq k_+$ . A priori, the value of  $k_+$  will change from one section to the other but we can always choose a value for  $k_+$  that is larger than its previous values. This convention will be adopted systematically without being explicitly mentioned.



### 2.4.3.1 Orientation

In what follows we are going to estimate successively, in [Lemma 2.4.5](#), [Lemma 2.4.9](#), [Lemma 2.4.12](#) and [Lemma 2.4.13](#), each of the terms appearing in the decomposition [\(2.4.9\)](#).

For  $\mathbb{U}_\infty(\mathbf{k}, t)$ ,  $\mathbb{U}_s(\mathbf{k}, t)$  and  $\mathbb{U}_d(\mathbf{k}, t)$ , which will be treated in [Section 2.4.3.2](#) to [Section 2.4.3.4](#), we shall bound separately each of the terms of the sums in [\(2.4.10\)](#)(i) to [\(2.4.10\)](#)(iii). For each eigenvalue  $\omega(|\mathbf{k}|) \in \{\omega_{\pm\infty}(|\mathbf{k}|), \omega_p(|\mathbf{k}|), \omega_{p,r}(|\mathbf{k}|)\}$ , we shall first estimate in [Lemma 2.4.3](#), [Lemma 2.4.6](#) and [Lemma 2.4.10](#) the corresponding spectral projector  $\Pi(|\mathbf{k}|)$ . Many approaches are possible. It appeared useful to use here the expression of  $\Pi(|\mathbf{k}|)$  provided by the Riesz-Dunford functional calculus (see e.g. [\[34\]](#), sections VII.1 and VII.3), in terms of a contour integral in the complex plane whose integrand involves the resolvent  $R_{|\mathbf{k}|}(\omega)$  studied in [Section 2.3.1](#). This contour will be taken as a (positively oriented) circle  $\mathcal{C}_{|\mathbf{k}|}$  centered at  $\omega(|\mathbf{k}|)$

$$\mathcal{C}_{|\mathbf{k}|} := \{\omega \in \mathbb{C} / |\omega - \omega(|\mathbf{k}|)| = \rho_{|\mathbf{k}|}\}, \quad (2.4.11)$$

whose radius  $\rho_{|\mathbf{k}|}$  must be chosen in such a way that

$$\begin{cases} \text{(i) } \mathcal{C}_{|\mathbf{k}|} \text{ does not enclose or intersect any other eigenvalue,} \\ \text{(ii) } \mathcal{C}_{|\mathbf{k}|} \text{ does not enclose or intersect any point of the set } \mathcal{S}_{\mathcal{T}}. \end{cases} \quad (2.4.12)$$

This will be automatically achieved if we take  $\rho_{|\mathbf{k}|}$  equal to half of the distance of  $\omega(|\mathbf{k}|)$  to all the points that one wants to avoid, namely

$$\rho_{|\mathbf{k}|} = 1/2 \operatorname{dist}(\omega(|\mathbf{k}|), \Omega(|\mathbf{k}|)), \quad \Omega(|\mathbf{k}|) = \left( \sigma(\mathbb{A}_{|\mathbf{k}|, \perp}) \cup \mathcal{S}_{\mathcal{T}} \right) \setminus \{\omega(|\mathbf{k}|)\}. \quad (2.4.13)$$

We then have the formula

$$\Pi(|\mathbf{k}|) = -\frac{1}{2i\pi} \int_{\mathcal{C}_{|\mathbf{k}|}} R_{|\mathbf{k}|}(\omega) d\omega = -\frac{1}{2i\pi} \int_{\mathcal{C}_{|\mathbf{k}|}} \mathcal{V}_{|\mathbf{k}|}(\omega) \mathcal{S}_{|\mathbf{k}|}(\omega) d\omega, \quad (2.4.14)$$

where the first equality is justified by [\(2.4.12\)](#)(i) and the second by the expression of the resolvent:  $R_{|\mathbf{k}|}(\omega) = \mathcal{V}_{|\mathbf{k}|}(\omega) \mathcal{S}_{|\mathbf{k}|}(\omega) + \mathcal{T}(\omega)$  (given in [Proposition 2.3.2](#)) and by [\(2.4.12\)](#)(ii) since as  $\mathcal{S}_{\mathcal{T}}$  is defined as the set of singularities of  $\omega \mapsto \mathcal{T}(\omega)$ , this function is analytic on  $\mathbb{C} \setminus \mathcal{S}_{\mathcal{T}}$ .

From [\(2.4.14\)](#), we deduce the inequality that we shall use systematically in the following, namely

$$\|\Pi(|\mathbf{k}|)\| \leq \rho_{|\mathbf{k}|} \sup_{\omega \in \mathcal{C}_{|\mathbf{k}|}} (\|\mathcal{V}_{|\mathbf{k}|}(\omega)\| \|\mathcal{S}_{|\mathbf{k}|}(\omega)\|). \quad (2.4.15)$$

In a second step, we shall concentrate on the exponentials appearing in each factor whose estimate for large  $\mathbf{k}$  will rely on the asymptotic expansion of the eigenvalues (and more particularly their imaginary parts), see [Lemma 2.4.4](#), [Lemma 2.4.11](#) and [Lemma 2.4.7](#). These results will be transformed into sharp exponential decay estimates for  $|\mathbb{U}_\infty(\mathbf{k}, t)|$ ,  $|\mathbb{U}_s(\mathbf{k}, t)|$  and  $|\mathbb{U}_d(\mathbf{k}, t)|$ , in which the rate of decay degenerates for  $|\mathbf{k}| \rightarrow +\infty$ , see [Lemma 2.4.5](#), [Lemma 2.4.9](#) and [Lemma 2.4.12](#).

Finally, the last term [\(2.4.10\)](#)  $\mathbb{U}_-(\mathbf{k}, t)$  will not be treated by using [\(2.4.10\)](#)(iv) but an alternative expression directly issued from the Riesz-Dunford functional calculus (see [Section 2.4.3.5](#)). In fact, the exponential decay of  $|\mathbb{U}_-(\mathbf{k}, t)|$  will be, contrary to the previous terms, uniform with respect to  $|\mathbf{k}|$ , so that it will not contribute at the end to the large time equivalent of  $\mathbf{U}(\cdot, t)$ . This is the reason why we can be satisfied with rough estimates.

2.4.3.2 Estimates of  $\mathbb{U}_{\pm\infty}(\mathbf{k}, t)$  for  $|\mathbf{k}| \gg 1$ 

In the following Lemma, we estimate the spectral projectors  $\Pi_{\pm\infty}(|\mathbf{k}|)$ .

**Lemma 2.4.3**

The spectral projectors  $\Pi_{\pm\infty}(|\mathbf{k}|)$  are uniformly bounded for large  $|\mathbf{k}|$ .

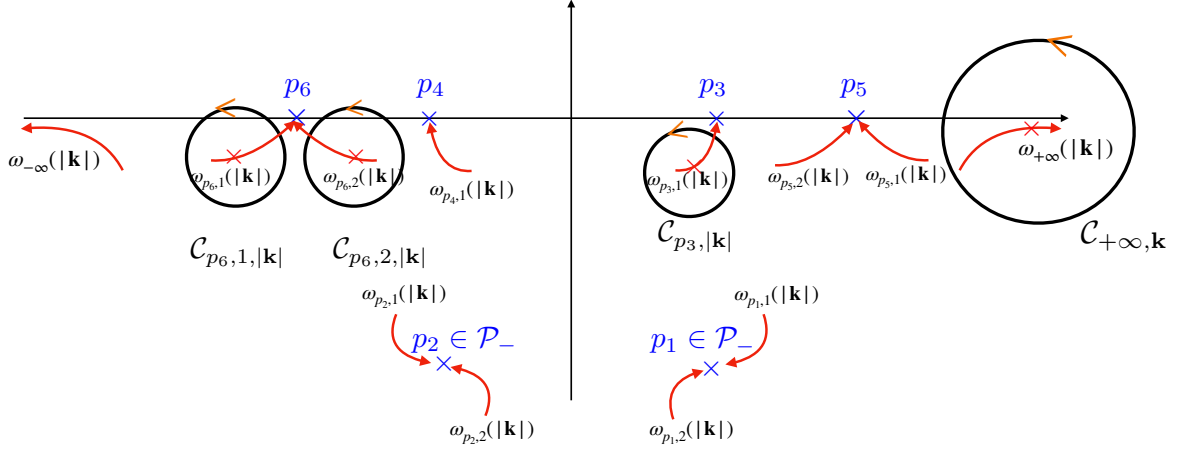


Figure 2.2: Contours of integration  $\mathcal{C}_{|\mathbf{k}|,+\infty}$ , and  $\mathcal{C}_{|\mathbf{k}|,p_3}$  used for the estimate of  $\Pi_{+\infty}(|\mathbf{k}|)$ ,  $\Pi_{p_3}(|\mathbf{k}|)$ ,  $p_3 \in \mathcal{P}_s$ ,  $\Pi_{p_{6,1}}(|\mathbf{k}|)$  and  $\Pi_{p_{6,2}}(|\mathbf{k}|)$  for  $p_6 \in \mathcal{P}_d$  (corresponding to the Figure 2.1).

*Proof.* We follow the approach described in Section 2.4.3.1 for  $\omega(|\mathbf{k}|) = \omega_{\pm\infty}(|\mathbf{k}|)$  and denote  $\mathcal{C}_{\pm\infty,|\mathbf{k}|}$  the corresponding contour, see (2.4.11) and Figure 2.2.

**Step 1 : Estimate of  $\rho_{|\mathbf{k}|}$ .** Defining  $\rho_{|\mathbf{k}|}$  by (2.4.13) for  $\omega(|\mathbf{k}|) = \omega_{\pm\infty}(|\mathbf{k}|)$  and using the asymptotic behaviour (2.4.3), it is clear that

$$\rho_{|\mathbf{k}|} \sim \frac{1}{2} c|\mathbf{k}|, \quad (|\mathbf{k}| \rightarrow +\infty), \quad \text{thus} \quad \rho_{|\mathbf{k}|} \lesssim |\mathbf{k}| \quad \text{for } |\mathbf{k}| \text{ large enough.} \quad (2.4.16)$$

**Step 2 : Estimate of  $(\mathcal{D}(\omega) - |\mathbf{k}|^2)^{-1}$ .** This term is of interest because it appears in the expression (2.3.10) of  $\mathcal{S}_{|\mathbf{k}|}(\omega)$ . Using (2.3.1), (2.3.3), we compute that

$$(\mathcal{D}(\omega) - |\mathbf{k}|^2)^{-1} = \frac{Q_e(\omega) Q_m(\omega)}{D_{|\mathbf{k}|}(\omega)}, \quad \text{with } D_{|\mathbf{k}|}(\omega) \text{ given in (2.3.19).} \quad (2.4.17)$$

According to Proposition 2.4.1, we know that  $D_{|\mathbf{k}|}(\omega)$  is a polynomial of degree  $N$  with simple roots so that, according to (2.3.20), it can be factorized as (note that we distinguish below  $\omega_{\pm\infty}(|\mathbf{k}|)$  that go to  $\infty$  with  $|\mathbf{k}|$ , from the other roots which remain bounded)

$$D_{|\mathbf{k}|}(\omega) = \varepsilon_0 \mu_0 D_{b,|\mathbf{k}|}(\omega) (\omega - \omega_{+\infty}(|\mathbf{k}|)) (\omega - \omega_{-\infty}(|\mathbf{k}|)) \quad (2.4.18)$$

with  $D_{b,|\mathbf{k}|}(\omega)$  given by

$$D_{b,|\mathbf{k}|}(\omega) = \prod_{p \in \mathcal{P}_-} \prod_{n=1}^{m_p} (\omega - \omega_{p,n}(|\mathbf{k}|)) \prod_{p \in \mathcal{P}_s} (\omega - \omega_p(|\mathbf{k}|)) \prod_{p \in \mathcal{P}_d} \prod_{r=1}^2 (\omega - \omega_{p,r}(|\mathbf{k}|)). \quad (2.4.19)$$

In the same way, thanks to (2.4.2), for any  $\omega(|\mathbf{k}|) \in \{\omega_{p,m}(|\mathbf{k}|), \omega_p(|\mathbf{k}|), \omega_{p,r}(|\mathbf{k}|)\}$ ,

$$|\omega - \omega(|\mathbf{k}|)| \geq |\omega_{\pm\infty}(|\mathbf{k}|) - \omega(|\mathbf{k}|)| - |\omega - \omega_{\pm\infty}(|\mathbf{k}|)| \sim c|\mathbf{k}|/2, \quad (|\mathbf{k}| \rightarrow +\infty). \quad (2.4.20)$$

Thus, by definition (2.4.19) of  $D_{b,|\mathbf{k}|}(\omega)$ , we deduce from (2.4.22) and (2.4.20), that

$$|D_{b,|\mathbf{k}|}(\omega)| \geq C |\mathbf{k}|^{2(N_e+N_m)}, \quad \forall \omega \in \mathcal{C}_{\pm\infty,|\mathbf{k}|}, \quad \text{for some } C > 0. \quad (2.4.21)$$

Next, if  $\omega \in \mathcal{C}_{\pm\infty,|\mathbf{k}|}$ , by the reverse triangular inequality

$$|\omega - \omega_{\mp\infty}(|\mathbf{k}|)| \geq |\omega_{\pm\infty}(|\mathbf{k}|) - \omega_{\mp\infty}(|\mathbf{k}|)| - |\omega - \omega_{\pm\infty}(|\mathbf{k}|)| \sim 3c|\mathbf{k}|/2, \quad (|\mathbf{k}| \rightarrow +\infty) \quad (2.4.22)$$

since, by (2.4.3),  $|\omega_{\pm\infty}(|\mathbf{k}|) - \omega_{\mp\infty}(|\mathbf{k}|)| \sim 2c|\mathbf{k}|$  and  $|\omega - \omega_{\pm\infty}(|\mathbf{k}|)| = \rho_{|\mathbf{k}|} \sim c|\mathbf{k}|/2$ . Thus, using (2.4.22) and (2.4.21) in (2.4.18) yields, as  $|\omega - \omega_{\pm\infty}(|\mathbf{k}|)| = \rho_{|\mathbf{k}|}$  on  $\mathcal{C}_{\pm\infty,|\mathbf{k}|}$ ,

$$|D_{|\mathbf{k}|}(\omega)|^{-1} \lesssim \rho_{|\mathbf{k}|}^{-1} |\mathbf{k}|^{-(2(N_e+N_m)+1)}, \quad \forall \omega \in \mathcal{C}_{\pm\infty,|\mathbf{k}|}. \quad (2.4.23)$$

For bounding  $Q_e(\omega)Q_m(\omega)$ , that appears in (2.4.17), we use an upper bound for  $\omega \in \mathcal{C}_{\pm\infty,|\mathbf{k}|}$

$$|\omega| \leq |\omega_{\pm\infty}(|\mathbf{k}|)| + \rho_{|\mathbf{k}|} \sim 3c|\mathbf{k}|/2, \quad |\mathbf{k}| \rightarrow +\infty, \quad \forall \omega \in \mathcal{C}_{\pm\infty,|\mathbf{k}|}, \quad (2.4.24)$$

to deduce that, as  $Q_e Q_m$  is a polynomial of degree  $2(N_e + N_m)$ ,

$$|Q_e(\omega) Q_m(\omega)| \lesssim |\omega|^{2(N_e+N_m)} \lesssim |\mathbf{k}|^{2(N_e+N_m)}, \quad \forall \omega \in \mathcal{C}_{\pm\infty,|\mathbf{k}|}. \quad (2.4.25)$$

Thus, using (2.4.25) and (2.4.23) in (2.4.17), we get, for  $|\mathbf{k}|$  large enough,

$$|(\mathcal{D}(\omega) - |\mathbf{k}|^2)^{-1}| \lesssim \rho_{|\mathbf{k}|}^{-1} |\mathbf{k}|^{-1}, \quad \forall \omega \in \mathcal{C}_{\pm\infty,|\mathbf{k}|}. \quad (2.4.26)$$

**Step 3: Estimates of  $\mathcal{S}_{|\mathbf{k}|}(\omega)$  and  $\mathcal{V}_{|\mathbf{k}|}(\omega)$ .** Owing to (2.3.10), to estimate  $\mathcal{S}_{|\mathbf{k}|}(\omega)$ , it remains to estimate  $\omega\mu(\omega)\mathbb{A}_e(\omega) - |\mathbf{k}|\mathbf{e}_3 \times \mathbb{A}_m(\omega)$ . We claim that,

$$\text{for } |\mathbf{k}| \text{ large enough, } \quad \|\omega\mu(\omega)\mathbb{A}_e(\omega) - |\mathbf{k}|\mathbf{e}_3 \times \mathbb{A}_m(\omega)\| \lesssim |\mathbf{k}|, \quad \forall \omega \in \mathcal{C}_{\pm\infty,|\mathbf{k}|}, \quad (2.4.27)$$

which, combined with (2.4.23) provides, via (2.3.10),

$$\|\mathcal{S}_{|\mathbf{k}|}(\omega)\| \lesssim \rho_{|\mathbf{k}|}^{-1}, \quad \forall \omega \in \mathcal{C}_{\pm\infty,|\mathbf{k}|}. \quad (2.4.28)$$

To prove (2.4.27), notice first that along  $\mathcal{C}_{\pm\infty,|\mathbf{k}|}$ ,

$$|\omega| \geq |\omega_{\infty}(|\mathbf{k}|)| - \rho_{|\mathbf{k}|} \sim c|\mathbf{k}|/2 \quad \text{when } |\mathbf{k}| \rightarrow +\infty. \quad (2.4.29)$$

Thus, as  $\mu(\omega) = \mu_0 + o(1)$  as  $|\omega| \rightarrow \infty$ , with the upper bound (2.4.24), one gets

$$\|\omega\mu(\omega)\mathbb{A}_e(\omega) - |\mathbf{k}|\mathbf{e}_3 \times \mathbb{A}_m(\omega)\| \lesssim |\mathbf{k}| (\|\mathbb{A}_e(\omega)\| + \|\mathbb{A}_m(\omega)\|), \quad \forall \omega \in \mathcal{C}_{\pm\infty,|\mathbf{k}|}.$$

Finally, we prove  $\mathbb{A}_e(\omega)$  and  $\mathbb{A}_m(\omega)$  are uniformly bounded (in  $\mathbf{k}$ ) on  $\mathcal{C}_{\pm\infty,|\mathbf{k}|}$ . Towards this goal, we first estimate the operators  $(\mathbb{A}_{e,j}(\omega), \dot{\mathbb{A}}_{e,j}(\omega), \mathbb{A}_{m,\ell}(\omega), \dot{\mathbb{A}}_{m,\ell}(\omega))$ . From (2.3.6),

$$\|\mathbb{A}_{e,j}(\omega)\|, \|\dot{\mathbb{A}}_{e,j}(\omega)\| \lesssim \frac{1 + |\omega|}{|q_{e,j}(\omega)|}, \quad \|\mathbb{A}_{m,\ell}(\omega)\|, \|\dot{\mathbb{A}}_{m,\ell}(\omega)\| \lesssim \frac{1 + |\omega|}{|q_{m,\ell}(\omega)|}, \quad (2.4.30)$$

Thus, using (2.4.29), as  $q_{e,j}(\omega), q_{m,\ell}(\omega) \sim \omega^2$  when  $|\omega| \rightarrow +\infty$  (see (2.1.6)), one has

$$|q_{e,j}(\omega)|^{-1} \lesssim |\mathbf{k}|^{-2}, \quad |q_{m,\ell}(\omega)|^{-1} \lesssim |\mathbf{k}|^{-2} \quad \text{for } \omega \in \mathcal{C}_{\pm\infty,|\mathbf{k}|}. \quad (2.4.31)$$

Thus, using the above inequalities and the upper bound (2.4.24) in (2.4.30), we get

$$\|\mathbb{A}_{e,j}(\omega)\|, \|\dot{\mathbb{A}}_{e,j}(\omega)\| \lesssim |\mathbf{k}|^{-1}, \quad \|\mathbb{A}_{m,\ell}(\omega)\|, \|\dot{\mathbb{A}}_{m,\ell}(\omega)\| \lesssim |\mathbf{k}|^{-1}.$$

The formulas (2.3.7) show that  $\mathbb{A}_e(\omega)$  and  $\mathbb{A}_m(\omega)$  are the sum on a fixed (independent of  $\omega$ ) operator with a (fixed) linear combination of  $(\mathbf{e}, \mathbf{h}, \mathbb{A}_{e,j}(\omega), \dot{\mathbb{A}}_{e,j}(\omega), \mathbb{A}_{m,\ell}(\omega), \dot{\mathbb{A}}_{m,\ell}(\omega))$ . They are thus uniformly bounded (in  $\mathbf{k}$ ) on  $\mathcal{C}_{\pm\infty,|\mathbf{k}|}$ , proving (2.4.27).

Finally, using (2.4.24), (2.4.31) and that, on  $\mathcal{C}_{\pm\infty,|\mathbf{k}|}$ ,  $|\omega\mu(\omega)|^{-1} \lesssim \mu_0^{-1} |\mathbf{k}|^{-1}$ , when  $|\mathbf{k}| \rightarrow +\infty$ , one sees on the definition (2.3.11) of the operator  $\mathcal{V}_{|\mathbf{k}|}(\omega)$  that, for  $|\mathbf{k}|$  large enough,

$$\|\mathcal{V}_{|\mathbf{k}|}(\omega)\| \lesssim 1, \quad \forall \omega \in \mathcal{C}_{\pm\infty,|\mathbf{k}|}. \quad (2.4.32)$$

**Conclusion.** It suffices to use (2.4.28) and (2.4.32) in (2.4.15) for  $\Pi(|\mathbf{k}|) = \Pi_{\pm\infty}(|\mathbf{k}|)$ , to conclude that  $\|\Pi_{\pm\infty}(|\mathbf{k}|)\| \lesssim 1$  for  $|\mathbf{k}|$  large enough.  $\square$

The next lemma is about the asymptotic expansion (in powers of  $|\mathbf{k}|$ ) of  $\omega_{\pm\infty}(|\mathbf{k}|)$  when  $|\mathbf{k}| \rightarrow +\infty$ . It is important to push the expansion up to the first apparition of a negative imaginary part, since it will govern the decay of  $\mathcal{U}_{\infty}(|\mathbf{k}|, t)$  for large  $|\mathbf{k}|$ .

#### Lemma 2.4.4

The eigenvalues  $\omega_{\pm\infty}(|\mathbf{k}|)$  satisfy the following asymptotic expansion

$$\omega_{\pm\infty}(|\mathbf{k}|) = \pm c |\mathbf{k}| \pm \frac{A_{1,\infty}}{2c} |\mathbf{k}|^{-1} - i \frac{A_{2,\infty}}{2c^2} |\mathbf{k}|^{-2} + o(|\mathbf{k}|^{-2}), \quad \text{as } |\mathbf{k}| \rightarrow +\infty, \quad (2.4.33)$$

where  $A_{1,\infty}$  and  $A_{2,\infty}$  are two real constants given by

$$A_{1,\infty} = \sum \Omega_{e,j}^2 + \sum \Omega_{m,\ell}^2 \quad \text{and} \quad A_{2,\infty} = \sum \alpha_{e,j} \Omega_{e,j}^2 + \sum \alpha_{m,\ell} \Omega_{m,\ell}^2, \quad (2.4.34)$$

with  $A_{1,\infty} > 0$  and  $A_{2,\infty} > 0$  thanks to the weak dissipation condition (2.1.10). In particular

$$\text{Im}(\omega_{\pm\infty}(|\mathbf{k}|)) = -\frac{A_{2,\infty}}{2c^2} |\mathbf{k}|^{-2} + o(|\mathbf{k}|^{-2}), \quad \text{as } |\mathbf{k}| \rightarrow +\infty. \quad (2.4.35)$$

*Proof.* Since  $\omega_{-\infty}(|\mathbf{k}|) = \overline{-\omega_{+\infty}(|\mathbf{k}|)}$  (see Section 2.3.2.1), we only prove (2.4.33) for

$$\omega_{+\infty}(|\mathbf{k}|) = \xi_+(|\mathbf{k}|^{-1})^{-1}, \quad (\text{cf. the proof of Proposition 2.4.1, step 2})$$

**Step 1: Proof of  $\omega_{+\infty}(|\mathbf{k}|) = c|\mathbf{k}| + O(|\mathbf{k}|^{-1})$**

The proof of Proposition 2.4.1 defines the functions  $\zeta \mapsto \xi_{\pm}(\zeta)$  as the branches of solutions of the equation  $G(\xi) = \zeta^2$  near  $\xi = 0$ , where  $G(\xi) = \xi^2 g(\xi)$  and  $g(\xi) = \varepsilon(1/\xi)^{-1} \mu(1/\xi)^{-1}$  is analytic near 0 and even. In particular  $g'(0) = 0$  and, by Lemma B.1 (with  $z = 0, g(0) = c^2$  and  $\mathbf{m} = 2$ ), and more precidely (2.7.39), we deduce that

$$\xi_+(\zeta) = c^{-1} \zeta (1 + O(\zeta^2)) \text{ near } 0, \quad \text{thus } \omega_{+\infty}(|\mathbf{k}|) = c|\mathbf{k}| + O(|\mathbf{k}|^{-1}) \text{ at } \infty. \quad (2.4.36)$$

**Step 2: Computations of the terms of order  $|\mathbf{k}|^{-1}$  and  $|\mathbf{k}|^{-2}$**

By analyticity of  $\xi_+(\zeta)$ , and using the step 1, we know that, for some coefficients  $(A_1, A_2)$ ,

$$\xi_+(\zeta) = c^{-1}\zeta (1 + A_1 \zeta^2 + A_2 \zeta^3 + O(\zeta^4)). \quad (2.4.37)$$

In order to compute  $A_1$  and  $A_2$ , we are going to proceed by identification in the equation

$$\xi_+(\zeta)^2 g(\xi_+(\zeta)) = \zeta^2. \quad (2.4.38)$$

Let us compute the Taylor expansion in the left hand side of (2.4.38). By (2.1.5,2.1.6), one has

$$\begin{aligned} \varepsilon(1/\xi) &= \varepsilon_0 \left[ 1 - \left( \sum \Omega_{e,j}^2 \right) \xi^2 + i \left( \sum \alpha_{e,j} \Omega_{e,j}^2 \right) \xi^3 + O(\xi^4) \right], \quad \text{as } \xi \rightarrow 0, \\ \mu(1/\xi) &= \mu_0 \left[ 1 - \left( \sum \Omega_{m,\ell}^2 \right) \xi^2 + i \left( \sum \alpha_{m,\ell} \Omega_{m,\ell}^2 \right) \xi^3 + O(\xi^4) \right], \quad \text{as } \xi \rightarrow 0. \end{aligned}$$

It follows that, with  $(A_{1,\infty}, A_{2,\infty})$  the coefficients defined by (2.4.34),

$$g(\xi) = c^2 (1 + A_{1,\infty} \xi^2 - i A_{2,\infty} \xi^3 + O(\xi^4)), \quad \text{as } \xi \rightarrow 0, \quad (2.4.39)$$

that is to say, as  $\xi(\zeta) = O(\zeta)$ ,

$$\xi_+(\zeta)^2 g(\xi_+(\zeta)) = c^2 (\xi_+(\zeta)^2 + A_{1,\infty} \xi_+(\zeta)^4 - i A_{2,\infty} \xi_+(\zeta)^5 + O(\zeta^6)), \quad \text{as } \zeta \rightarrow 0. \quad (2.4.40)$$

On the other hand, using (2.4.37), we have, truncating the expansions at the order 5 in  $\zeta$ ,

$$\begin{aligned} \xi_+(\zeta)^2 &= c^{-2} \zeta^2 (1 + 2 A_1 \zeta^2 + 2 A_2 \zeta^3 + O(\zeta^4)), \\ \xi_+(\zeta)^4 &= c^{-4} \zeta^4 (1 + O(\zeta^2)), \quad \xi_+(\zeta)^5 = c^{-5} \zeta^5 (1 + O(\zeta^2)), \end{aligned}$$

which we can substitute into (2.4.40) to obtain

$$\xi_+(\zeta)^2 g(\xi_+(\zeta)) = \zeta^2 (1 + 2 A_1 \zeta^2 + 2 A_2 \zeta^3) + A_{1,\infty} c^{-2} \zeta^4 - i A_{2,\infty} c^{-3} \zeta^5 + O(\zeta^6). \quad (2.4.41)$$

Using the above in (2.4.38) leads to the equations

$$2 A_1 + A_{1,\infty} c^{-2} = 0, \quad 2 A_2 - i A_{2,\infty} c^{-3} = 0.$$

from which we conclude easily. The remaining details are left to the reader.  $\square$

We conclude this section by estimating the term  $\mathcal{U}_\infty(\mathbf{k}, t)$  in the decomposition (2.4.9).

**Lemma 2.4.5**

For any constant  $C \in (0, A_{2,\infty}/(2c^2))$  with  $A_{2,\infty}$  given by (2.4.34), there exists  $k_+ > 0$  such that for  $|\mathbf{k}| \geq k_+$ , the functions  $\mathcal{U}_\infty(\mathbf{k}, t)$  defined by (2.4.9) and (2.4.10) satisfy

$$|\mathcal{U}_\infty(\mathbf{k}, t)| \lesssim e^{-\frac{C}{|\mathbf{k}|^2} t} |\mathcal{U}_0(\mathbf{k})|, \quad \forall t \geq 0. \quad (2.4.42)$$

*Proof.* The functions  $\mathcal{U}_\infty(\mathbf{k}, t)$  are defined by (2.4.9) and (2.4.10) for  $|\mathbf{k}|$  large enough. As the operator  $\mathcal{R}_\mathbf{k}$  and its inverse  $\mathcal{R}_\mathbf{k}^*$  are unitary, one has

$$|\mathcal{U}_\infty(\mathbf{k}, t)| \leq \sum_{\pm} e^{\text{Im}(\omega_{\pm\infty}(|\mathbf{k}|)) t} \|\Pi_{\pm\infty}(|\mathbf{k}|)\| |\mathcal{U}_0(\mathbf{k})|, \quad \forall t \geq 0.$$

Thus, the upper bound (2.4.42) follows immediately from (2.4.35) (see Lemma 2.4.4) and the

inequality  $\|\Pi_{\pm\infty}(|\mathbf{k}|)\| \lesssim 1$  (see Lemma 2.4.3).  $\square$

### 2.4.3.3 Estimates of $\mathcal{U}_s(\mathbf{k}, t)$ for $|\mathbf{k}| \gg 1$

We estimate now the term  $\mathcal{U}_s(\mathbf{k}, t)$  in (2.4.9) which involves in particular the projectors  $\Pi_p(|\mathbf{k}|)$ .

#### Lemma 2.4.6

The spectral projectors  $\Pi_p(|\mathbf{k}|)$ ,  $p \in \mathcal{P}_s$  are uniformly bounded for large  $|\mathbf{k}|$ .

*Proof.* We follow the approach described in Section 2.4.3.1 for  $\omega(|\mathbf{k}|) = \omega_p(|\mathbf{k}|)$ ,  $p \in \mathcal{P}_s$  and denote  $\mathcal{C}_{p,|\mathbf{k}|}$  the corresponding contour, see (2.4.11) and Figure 2.2.

**Step 1 : estimate of  $\rho_{|\mathbf{k}|}$ .** Obviously, for  $|\mathbf{k}|$  large enough, since  $\omega_p(|\mathbf{k}|)$  tends to  $p$  when  $|\mathbf{k}|$ , the infimum defining  $\rho_{|\mathbf{k}|}$  in (2.4.13) is attained at the point  $p$ , i.e. :

$$\rho_{|\mathbf{k}|} = \frac{1}{2} |\omega_p(|\mathbf{k}|) - p| \sim \frac{1}{2} |A_p| |\mathbf{k}|^{-2} \quad (|\mathbf{k}| \rightarrow +\infty), \quad \text{with } |A_p| > 0, \quad (2.4.43)$$

according to (2.4.2) for  $\mathbf{m}_p = 1$  since  $p \in \mathcal{P}_s$ .

For the rest of the proof, we essentially follow the proof of Lemma 2.4.3.

#### Step 2 : Estimate of $(\mathcal{D}(\omega) - |\mathbf{k}|^2)^{-1}$ .

First, due to (2.4.2) and (2.4.43), it is clear that there exists  $C > 0$  such that , for  $|\mathbf{k}|$  large enough,

$$\forall \tilde{p} \in \mathcal{P} \setminus \{p\}, \quad \forall \omega \in \mathcal{C}_{p,|\mathbf{k}|}, \quad |\omega - \omega_{\tilde{p},n}(|\mathbf{k}|)|, |\omega - \omega_{\tilde{p}}(|\mathbf{k}|)|, |\omega - \omega_{\tilde{p},r}(|\mathbf{k}|)| \geq C > 0.$$

Thus, for  $|\mathbf{k}|$  large enough, the function  $D_{b,|\mathbf{k}|}(\omega)$  given by the product (2.4.19) satisfies

$$\forall \omega \in \mathcal{C}_{p,|\mathbf{k}|}, \quad |D_{b,|\mathbf{k}|}(\omega)| \geq C \rho_{|\mathbf{k}|} \quad \text{thus} \quad |D_{b,|\mathbf{k}|}(\omega)|^{-1} \lesssim \rho_{|\mathbf{k}|}^{-1}. \quad (2.4.44)$$

On the other hand, due to (2.4.2) and (2.4.43),

$$|\omega - \omega_{\pm\infty}(|\mathbf{k}|)| = c|\mathbf{k}| + o(|\mathbf{k}|) \quad \text{uniformly for } \omega \in \mathcal{C}_{p,|\mathbf{k}|}.$$

It is then immediate to infer from the definition (2.4.18) of  $D_{|\mathbf{k}|}(\omega)$  that, for  $|\mathbf{k}|$  large enough,

$$|D_{|\mathbf{k}|}(\omega)|^{-1} \lesssim \rho_{|\mathbf{k}|}^{-1} |\mathbf{k}|^{-2}, \quad \forall \omega \in \mathcal{C}_{p,|\mathbf{k}|}. \quad (2.4.45)$$

Moreover,  $p \in \mathcal{P}_s$  is a simple zero of the product  $Q_e Q_m$ . Hence, for any  $\omega \in \mathcal{C}_{p,|\mathbf{k}|}$

$$|Q_e(\omega) Q_m(\omega)| \lesssim |\omega - p| \lesssim \rho_{|\mathbf{k}|} + |\omega_p(|\mathbf{k}|) - p| \sim \frac{3}{2} |A_p| |\mathbf{k}|^{-2} \quad (2.4.46)$$

As a consequence, from the expression (2.4.17) of  $(\mathcal{D}(\omega) - |\mathbf{k}|^2)^{-1}$ , it is clear that

$$|(\mathcal{D}(\omega) - |\mathbf{k}|^2)^{-1}| \lesssim \rho_{|\mathbf{k}|}^{-1} |\mathbf{k}|^{-4}, \quad \forall \omega \in \mathcal{C}_{p,|\mathbf{k}|}. \quad (2.4.47)$$

#### Step 3: Estimates of $\mathcal{S}_{|\mathbf{k}|}(\omega)$ and $\mathcal{V}_{|\mathbf{k}|}(\omega)$ .

To estimate  $\mathcal{S}_{|\mathbf{k}|}(\omega)$ , it remains to estimate  $\omega\mu(\omega)\mathbb{A}_e(\omega) - |\mathbf{k}| \mathbf{e}_3 \times \mathbb{A}_m(\omega)$ . Due to this, we shall

be led to distinguish two cases,  $p \in \mathcal{P}_m$  and  $p \in \mathcal{P}_e$ , and show that, for  $|\mathbf{k}|$  large enough,

$$\left\{ \begin{array}{l} \text{If } p \in \mathcal{P}_m, \quad \forall \omega \in \mathcal{C}_{p,|\mathbf{k}|}, \quad \|\mathcal{S}_{|\mathbf{k}|}(\omega)\| \lesssim \rho_{|\mathbf{k}|}^{-1} |\mathbf{k}|^{-1}, \quad \|\mathcal{V}_{|\mathbf{k}|}(\omega)\| \lesssim |\mathbf{k}|, \quad (i) \\ \text{If } p \in \mathcal{P}_e, \quad \forall \omega \in \mathcal{C}_{p,|\mathbf{k}|}, \quad \|\mathcal{S}_{|\mathbf{k}|}(\omega)\| \lesssim \rho_{|\mathbf{k}|}^{-1} |\mathbf{k}|^{-2}, \quad \|\mathcal{V}_{|\mathbf{k}|}(\omega)\| \lesssim |\mathbf{k}|^2. \quad (ii) \end{array} \right. \quad (2.4.48)$$

*Proof of (2.4.48)(i).* As  $p \in \mathcal{P}_m \cap \mathcal{P}_s$ ,  $p \notin \mathcal{P}_e$  and by  $\mathbf{H}_1$ , there is a unique index  $\ell_0$  such that  $p = \pm\omega_{m,\ell_0}$ . Thanks to (2.4.30), all the operators  $(\mathbb{A}_{p,j}(\omega), \dot{\mathbb{A}}_{p,j}(\omega), \mathbb{A}_{m,\ell}(\omega), \dot{\mathbb{A}}_{m,\ell}(\omega))$  are uniformly bounded along  $\mathcal{C}_{p,|\mathbf{k}|}$  except  $(\mathbb{A}_{m,\ell_0}(\omega), \dot{\mathbb{A}}_{m,\ell_0}(\omega))$ , as  $p$  is a simple pole of  $\omega \mapsto \mathbb{A}_{m,\ell_0}(\omega)$  and  $\omega \mapsto \dot{\mathbb{A}}_{m,\ell_0}(\omega)$ . More precisely, as  $|q_{m,\ell_0}(\omega)|^{-1} \sim |\omega - p|^{-1}$  when  $\omega \rightarrow p$ , for  $|\mathbf{k}|$  large enough,

$$\forall \omega \in \mathcal{C}_{p,|\mathbf{k}|}, \quad |q_{m,\ell_0}(\omega)|^{-1} \lesssim |\omega - p|^{-1} = \rho_{|\mathbf{k}|}^{-1} \lesssim |\mathbf{k}|^2, \quad \text{according to (2.4.43)} \quad (2.4.49)$$

Thus, using the bounds (2.4.30) and the fact that  $\mathcal{C}_{p,|\mathbf{k}|}$  lies in a bounded set,

$$\forall \omega \in \mathcal{C}_{p,|\mathbf{k}|}, \quad \|\mathbb{A}_{m,\ell_0}(\omega)\|, \|\dot{\mathbb{A}}_{m,\ell_0}(\omega)\| \lesssim |\mathbf{k}|^2. \quad (2.4.50)$$

Thanks to formulas (2.3.7) for  $(\mathbb{A}_e(\omega), \mathbb{A}_m(\omega))$ , we deduce that

$$\forall \omega \in \mathcal{C}_{p,|\mathbf{k}|}, \quad \|\mathbb{A}_e(\omega)\| \lesssim 1, \quad \|\mathbb{A}_m(\omega)\| \lesssim |\mathbf{k}|^2. \quad (2.4.51)$$

Moreover, as  $p \in \mathcal{P}_m$  is a simple pole of  $\mu(\omega)$  and  $\mathcal{C}_{p,|\mathbf{k}|}$  uniformly bounded :

$$|\mu(\omega)| \sim \mu_p |\omega - p|^{-1}, \quad \text{with } \mu_p > 0, \quad \text{thus } |\mu(\omega)| \lesssim |\omega - p|^{-1} \lesssim |\mathbf{k}|^2 \text{ along } \mathcal{C}_{p,|\mathbf{k}|}, \quad (2.4.52)$$

which, joined to (2.4.51), gives (the dominant term is the second one)

$$\forall \omega \in \mathcal{C}_{p,|\mathbf{k}|}, \quad \|\omega \mu(\omega) \mathbb{A}_e(\omega) - |\mathbf{k}| \mathbf{e}_3 \times \mathbb{A}_m(\omega)\| \lesssim |\mathbf{k}|^3. \quad (2.4.53)$$

The estimate for  $\mathcal{S}_{|\mathbf{k}|}(\omega)$  in (2.4.48)(i) then follows from (2.4.47), (2.4.53) and (2.3.10).

For  $\mathcal{V}_{|\mathbf{k}|}(\omega)$ , we first observe that  $|\mu(\omega)| \sim \mu_p |\omega - p|^{-1}$  when  $\omega \rightarrow p$  also implies that

$$|\mu(\omega)|^{-1} \lesssim |\mathbf{k}|^{-2} \text{ along } \mathcal{C}_{p,|\mathbf{k}|}. \quad (2.4.54)$$

On the other hand one sees on formula (2.3.11) that the dominant term in  $\mathcal{V}_{|\mathbf{k}|}(\omega)$  on  $\mathcal{C}_{p,|\mathbf{k}|}$  is asymptotically proportional to the function

$$|\mathbf{k}| \mu(\omega)^{-1} q_{m,\ell_0}(\omega)^{-1}.$$

This explains the estimate for  $\mathcal{V}_{|\mathbf{k}|}(\omega)$  in (2.4.48)(i) since, according to the inequalities (2.4.49) and (2.4.54),  $|\mu(\omega)|^{-1} |q_{m,\ell_0}(\omega)^{-1}| \lesssim 1$  on  $\mathcal{C}_{p,|\mathbf{k}|}$ .

*Proof of (2.4.48)(ii).* The proof is very similar to the one of (i) and we shall simply point out the difference with (i). As  $p \in \mathcal{P}_e \cap \mathcal{P}_s$ ,  $p \notin \mathcal{P}_m$  and there is a unique index  $j_0$  such that  $p = \pm\omega_{e,j_0}$ .

This time, all the operators  $(\mathbb{A}_{p,j}(\omega), \dot{\mathbb{A}}_{p,j}(\omega), \mathbb{A}_{m,\ell}(\omega), \dot{\mathbb{A}}_{m,\ell}(\omega))$  are uniformly bounded along  $\mathcal{C}_{p,|\mathbf{k}|}$  except  $(\mathbb{A}_{e,j_0}(\omega), \dot{\mathbb{A}}_{e,j_0}(\omega))$  and, similarly to (2.4.51), one shows that

$$\|\mathbb{A}_e(\omega)\| \lesssim |\mathbf{k}|^2, \quad \|\mathbb{A}_m(\omega)\| \lesssim 1. \quad (2.4.55)$$

The difference with (i) is that, this time,  $\mu(\omega)$  remains bounded on  $\mathcal{C}_{p,|\mathbf{k}|}$ , reason why we have

$$\|\omega\mu(\omega)\mathbb{A}_e(\omega) - |\mathbf{k}| \mathbf{e}_3 \times \mathbb{A}_m(\omega)\| \lesssim |\mathbf{k}|^2, \quad (2.4.56)$$

which leads to the estimate for  $\mathcal{S}_{|\mathbf{k}|}(\omega)$  in (2.4.48)(ii).

For  $\mathcal{V}_{|\mathbf{k}|}(\omega)$ , one sees on formula (2.3.11) that, this time, the dominant terms in  $\mathcal{V}_{|\mathbf{k}|}(\omega)$  on  $\mathcal{C}_{p,|\mathbf{k}|}$  is now proportional to

$$|q_{e,j_0}(\omega)^{-1}| \lesssim |\omega - p|^{-1} \lesssim |\rho_{\mathbf{k}}|^{-1} \lesssim |\mathbf{k}|^2.$$

This explains the estimate for  $\mathcal{V}_{|\mathbf{k}|}(\omega)$  in (2.4.48)(ii).

**Conclusion.** The important observation is that, from (2.4.48), in all cases

$$\|\mathcal{S}_{|\mathbf{k}|}(\omega)\| \|\mathcal{V}_{|\mathbf{k}|}(\omega)\| \lesssim \rho_{|\mathbf{k}|}^{-1}.$$

One concludes by applying (2.4.15) to  $\Pi(|\mathbf{k}|) = \Pi_p(|\mathbf{k}|)$ . □

We now give the asymptotic expansion of the eigenvalues  $\omega_p(|\mathbf{k}|)$  for large  $|\mathbf{k}|$ . As the study of their imaginary part requires to distinguish two cases, for the sake of readability, we delay it to the [Corollary 2.4.8](#).

#### Lemma 2.4.7

Let  $p \in \mathcal{P}_s$ . The eigenvalue  $\omega_p(|\mathbf{k}|)$  satisfies the following asymptotic expansion

$$\omega_p(|\mathbf{k}|) = p + A_{p,2} |\mathbf{k}|^{-2} + A_{p,4} |\mathbf{k}|^{-4} + o(|\mathbf{k}|^{-4}), \quad \text{as } |\mathbf{k}| \rightarrow +\infty, \quad (2.4.57)$$

where the complex numbers  $A_{p,2}$  and  $A_{p,4}$  are given by

$$\begin{aligned} \text{if } p = \pm \omega_{e,j_p} \in \mathcal{P}_e, \quad A_{p,2} = -\frac{1}{2} \varepsilon_0 p \mu(p) \Omega_{e,j_p}^2 \neq 0 \quad \text{and} \quad A_{p,4} = \frac{1}{2} \left( (\omega^2 \mu h_{e,p})^2 \right)'(p). \\ \text{if } p = \pm \omega_{m,\ell_p} \in \mathcal{P}_m, \quad A_{p,2} = -\frac{1}{2} \mu_0 p \varepsilon(p) \Omega_{m,\ell_p}^2 \neq 0 \quad \text{and} \quad A_{p,4} = \frac{1}{2} \left( (\omega^2 \varepsilon h_{m,p})^2 \right)'(p). \end{aligned} \quad (2.4.58)$$

where for any  $\omega \in (\mathbb{C} \setminus \mathcal{P}) \cup \{p\}$ ,

$$\text{if } p = \pm \omega_{e,j_p}, \quad h_{e,p}(\omega) = (\omega - p) \varepsilon(\omega), \quad \text{if } p = \pm \omega_{e,j_p}, \quad h_{m,p}(\omega) = (\omega - p) \mu(\omega). \quad (2.4.59)$$

*Proof.* Let  $p \in \mathcal{P}_s$ . Thus, either  $p \in \mathcal{P}_e$  or  $p \in \mathcal{P}_m$ . We assume without a loss of generality that  $p \in \mathcal{P}_e$ . The proof is done by symmetric arguments if  $p \in \mathcal{P}_m$ .

We are in the situation covered by the step 1 of the proof of [Proposition 2.4.1](#), in the case  $\mathbf{m}_p = 1$ , see also (2.4.1). As  $p$  is a simple pole of  $\omega \varepsilon(\omega)$ , there exists (by  $\mathbf{H}_1$ ) a unique index  $1 \leq j_p \leq N_e$  such that  $p = \pm \omega_{e,j_p}$ . In that case the set  $\{\omega_{p,n}(|\mathbf{k}|), 1 \leq n \leq \mathbf{m}_p\}$  is reduced to one single function denoted  $|\mathbf{k}| \mapsto \omega_p(|\mathbf{k}|)$ . It is constructed as the unique branch of solutions of the equation  $\mathcal{D}(\omega)^{-1} = \zeta$  with  $\zeta = |\mathbf{k}|^{-2}$  that converges to  $p$  when  $\zeta \rightarrow 0$ .

Next, we make more precise the factorization of  $\mathcal{D}(\omega)^{-1} \equiv (\omega^2 \varepsilon \mu)^{-1}$  corresponding to (2.4.1) for  $\mathbf{m}_p = 1$ . By definition (2.4.59) of  $h_{e,p}$ , we can write

$$\mathcal{D}(\omega)^{-1} = (\omega - p) g_p(\omega) \quad \text{with } g_p(\omega) := (\omega^2 \mu h_{e,p})(\omega)^{-1}.$$



From the definition of  $\varepsilon(\omega)$ , we also have

$$h_{e,p}(\omega) = \varepsilon_0 (\omega - p) \left( 1 - \sum_{j \neq j_p} \frac{\Omega_{e,j}^2}{q_{e,j}(\omega)} \right) - \varepsilon_0 \frac{\Omega_{e,j_p}^2}{\omega + p}, \quad \left( \text{thus } h_{e,p}(p) = -\varepsilon_0 \frac{\Omega_{e,j_p}^2}{2p} \right) \quad (2.4.60)$$

which shows that  $g_p$  is analytic at the neighbourhood of  $p$  (by  $(H_2)$ ,  $\mu(p) \neq 0$ ) and that

$$g_p(p) = -2 (\varepsilon_0 p \mu(p) \Omega_{e,j_p}^2)^{-1} \neq 0.$$

By formula (2.7.39), with  $\zeta = |\mathbf{k}|^{-2}$ ,  $z = p$ ,  $g = g_p$ ,  $\mathbf{m} = 1$ ,  $a_n^{-1} = g(p)^{-1}$  we get

$$\omega_p(|\mathbf{k}|) = p - \varepsilon_0 p \mu(p) \frac{\Omega_{e,j_p}^2}{2} |\mathbf{k}|^{-2} - \frac{g'_p(p) |\mathbf{k}|^{-4}}{g_p(p)^3} + o(|\mathbf{k}|^{-4}), \quad \text{as } |\mathbf{k}| \rightarrow +\infty. \quad (2.4.61)$$

This is nothing but (2.4.57) since  $-g'_p/g_p^2 = (1/g_p)' = (\omega^2 \mu h_{e,p})'$  by definition of  $g_p$ . Thus, one gets

$$-\frac{g'_p(p)}{g_p(p)^3} = (\omega^2 \mu h_{e,p})'(p) (\omega^2 \mu h_{e,p})(p) = \frac{1}{2} \left( (\omega^2 \mu h_{e,p})^2 \right)'(p).$$

Finally, note that, since  $\mathcal{P}_e \cap \mathcal{Z}_m = \emptyset$ , (see  $(H_2)$ ),  $p \notin \mathcal{Z}_m$  so that  $\mu(p) \neq 0$  thus  $A_{p,2} \neq 0$ .  $\square$

The behaviour on the imaginary part of  $A_{2,p}$  leads us to make the distinction between the critical and non critical configurations. For the ease of the reader, we recall below that the critical case corresponds to one of the following two situations (see Definition 2.1.6):

- a.  $\forall \ell \in \{1, \dots, N_m\}$ ,  $\alpha_{m,\ell} = 0$  and  $\exists j \in \{1, \dots, N_e\} \mid \alpha_{e,j} = 0$  and  $\omega_{e,j} \notin \{\omega_{m,\ell}\}$ .
- b.  $\forall j \in \{1, \dots, N_e\}$ ,  $\alpha_{e,j} = 0$  and  $\exists \ell \in \{1, \dots, N_m\} \mid \alpha_{m,\ell} = 0$  and  $\omega_{m,\ell} \notin \{\omega_{e,j}\}$ .

#### Corollary 2.4.8

Following the notations introduced in Lemma 2.4.7. In the non critical case,

$$\forall p \in \mathcal{P}_s, \quad \text{Im } A_{2,p} < 0. \quad (2.4.62)$$

As a consequence, there exists  $C > 0$  such that, for  $|\mathbf{k}|$  large enough,

$$\forall p \in \mathcal{P}_s, \quad \text{Im } \omega_p(|\mathbf{k}|) < -C |\mathbf{k}|^{-2}. \quad (2.4.63)$$

In the critical case, in situation 1,  $\mathcal{P}_e \cap \mathcal{P}_s \neq \emptyset$  and

$$\forall p \in \mathcal{P}_e \cap \mathcal{P}_s, \quad \text{Im } A_{2,p} = 0 \text{ and } \text{Im } A_{4,p} < 0, \quad \forall p \in \mathcal{P}_m \cap \mathcal{P}_s, \quad \text{Im } A_{2,p} < 0, \quad (2.4.64)$$

while, in situation 2,  $\mathcal{P}_m \cap \mathcal{P}_s \neq \emptyset$  and

$$\forall p \in \mathcal{P}_m \cap \mathcal{P}_s, \quad \text{Im } A_{2,p} = 0 \text{ and } \text{Im } A_{4,p} < 0, \quad \forall p \in \mathcal{P}_e \cap \mathcal{P}_s, \quad \text{Im } A_{2,p} < 0. \quad (2.4.65)$$

As a consequence, there exists  $C > 0$  such that, for  $|\mathbf{k}|$  large enough

$$\forall p \in \mathcal{P}_s, \quad \text{Im } \omega_p(|\mathbf{k}|) < -C |\mathbf{k}|^{-4}. \quad (2.4.66)$$

*Proof. Non critical case.* We prove (2.4.62) for  $p \in \mathcal{P}_s \cap \mathcal{P}_e$ , the case  $p \in \mathcal{P}_s \cap \mathcal{P}_m$  is similar.

According to formula (2.4.58), we simply have to check that  $\text{Im}(p\mu(p)) > 0$ . Thanks to (2.3.17), this will be true as soon as, at least one index  $\ell$ ,  $\alpha_{m,\ell} > 0$ . However,  $p \in \mathcal{P}_s \cap \mathcal{P}_e$  means that  $p = \pm \omega_{e,j}$  for some  $j$  with  $\alpha_{e,j} = 0$  and also that  $\omega_{e,j} \notin \{\omega_{m,\ell}\}$  (since  $p$  is a simple pole). Thus, if all  $\alpha_{m,\ell}$  vanished, we would be in the situation 1 of the critical case, which is excluded.

Of course, (2.4.63) is a direct consequence of (2.4.62).

**Critical case.** We prove (2.4.64) in situation 1, the proof of (2.4.65) in situation 2 being similar.

First  $\mathcal{P}_e \cap \mathcal{P}_s$  is non empty since any  $\omega_{e,j}$  such that  $\alpha_{e,j} = 0$  (a set which is itself non empty by definition of the situation 1) belongs to  $\mathcal{P}_e \cap \mathcal{P}_s$ .

Let  $p \in \mathcal{P}_e \cap \mathcal{P}_s$ . In situation 1, all  $\alpha_{m,\ell}$  vanish thus, by (2.3.17),  $\mu(\omega)$  and  $\mu'(\omega)$  are real-valued for  $\omega \in \mathbb{R} \setminus \mathcal{P}_m$ . In particular,  $\text{Im}(p\mu(p)) = 0$  which implies by (2.4.58) that  $\text{Im} A_{2,p} = 0$ . Next, as  $h_{e,p}(p) \in \mathbb{R}$  by (2.4.60), using (2.4.58) again yields

$$A_{p,4} = (\omega^2 \mu h_{e,p})'(p) (\omega^2 \mu h_{e,p})(p) \Rightarrow \text{Im} A_{p,4} = p^4 \mu^2(p) h_{e,p}(p) \text{Im} h'_{e,p}(p). \quad (2.4.67)$$

Moreover,  $p \in \mathcal{P}_e \cap \mathcal{P}_s$  implies  $p = \pm \omega_{e,k}$  for some  $k$  such that  $\alpha_{e,k} = 0$ . However, from the formulas for  $\mu(\omega)$  (2.1.5), the polynomials  $q_{e,j}$  (2.1.6) and  $h_{e,p}$  (2.4.60), one computes that

$$h_{e,p}(p) = -\varepsilon_0 \frac{\Omega_{e,k}^2}{2p} < 0 \quad \text{and} \quad \text{Im} h'_{e,p}(p) = \varepsilon_0 p \sum_{j \neq k} \frac{\alpha_{e,j} \Omega_{e,j}^2}{|q_{e,j}(p)|^2}, \quad (2.4.68)$$

The weak dissipation condition (2.1.10) implies that at least one  $\alpha_{e,j}$  for  $j \neq k$  is strictly positive so that  $\text{Im} h'_{e,p}(p) > 0$ , implying by (2.4.67) that  $\text{Im} A_{p,4} < 0$ .

Finally, for  $p \in \mathcal{P}_s \cap \mathcal{P}_m$ , the proof of  $\text{Im} A_{p,2} < 0$  uses (2.4.58), (2.3.17) and the fact that at least one  $\alpha_{e,j}$  for  $j \neq k$  is strictly positive. The proof of (2.4.64) is thus complete.

Finally (2.4.66) is easily deduced from (2.4.57), (2.4.64) and (2.4.65) after remarking that, for  $|\mathbf{k}|$  large enough,  $|\mathbf{k}|^{-4} \lesssim |\mathbf{k}|^{-2}$  which implies  $-|\mathbf{k}|^{-2} \lesssim -|\mathbf{k}|^{-4}$  (details are again left to the reader).  $\square$

Thus, proceeding as in the proof of Lemma 2.4.5, one shows using Lemma 2.4.6 and Corollary 2.4.8 the following result.

**Lemma 2.4.9**

There exists  $k_+ > 0$  and  $C > 0$  such that the function  $\mathbb{U}_s(\mathbf{k}, t)$  defined by (2.4.9) and (2.4.34) satisfy the following estimates:

a. If the system (2.1.1) is in a non-critical configuration, then

$$|\mathbb{U}_s(\mathbf{k}, t)| \lesssim e^{-\frac{Ct}{|\mathbf{k}|^2}} |\mathbb{U}_0(\mathbf{k})|, \quad \forall t \geq 0 \quad \text{and} \quad \forall |\mathbf{k}| \geq k_+. \quad (2.4.69)$$

b. If the system (2.1.1) is in a critical configuration, then

$$|\mathbb{U}_s(\mathbf{k}, t)| \lesssim e^{-\frac{Ct}{|\mathbf{k}|^4}} |\mathbb{U}_0(\mathbf{k})|, \quad \forall t \geq 0 \quad \text{and} \quad \forall |\mathbf{k}| \geq k_+. \quad (2.4.70)$$

#### 2.4.3.4 Estimates of $\cup_d(\mathbf{k}, t)$ for $|\mathbf{k}| \gg 1$

This time we estimate  $\cup_d(\mathbf{k}, t)$  in (2.4.9) which involves in particular the projectors  $\Pi_{p,r}(|\mathbf{k}|)$ .

|  |
|--|
| <b>Lemma 2.4.10</b>  |
| The projectors $\Pi_{p,r}( \mathbf{k} )$ , $p \in \mathcal{P}_d$ , $r = 1, 2$ are uniformly bounded for large $ \mathbf{k} $ . |

The projectors  $\Pi_{p,r}(|\mathbf{k}|)$ ,  $p \in \mathcal{P}_d$ ,  $r = 1, 2$  are uniformly bounded for large  $|\mathbf{k}|$ .

*Proof.* We follow again the approach of Section 2.4.3.1 for  $\omega_{p,r}(|\mathbf{k}|)$ ,  $p \in \mathcal{P}_d$ ,  $r = 1, 2$  and denote  $\mathcal{C}_{p,r,|\mathbf{k}|}$  the corresponding contour (see (2.4.11) and Figure 2.2). Without any loss of generality, we can restrict ourselves to  $r = 1$ .

**Step 1 : estimate of  $\rho_{|\mathbf{k}|}$ .** Thanks to the asymptotic (2.4.2), it is clear that the distance from  $\omega_{p,1}(|\mathbf{k}|)$  to any other eigenvalue that is different from  $\omega_{p,2}(|\mathbf{k}|)$  remains bounded from below, for large  $|\mathbf{k}|$ , by a strictly positive constant. The same observation holds true for distance from  $\omega_{p,1}(|\mathbf{k}|)$  to any other point of the set  $\mathcal{S}_{\mathcal{T}}$  than  $p$ .

Oppositely, the distances  $|\omega_{p,1}(|\mathbf{k}|) - p|$  and  $|\omega_{p,1}(|\mathbf{k}|) - \omega_{p,2}(|\mathbf{k}|)|$  tend to 0 when  $|\mathbf{k}| \rightarrow +\infty$ . More precisely, from (2.4.2) applied with  $\mathbf{m}_p = 2$ , since  $p \in \mathcal{P}_d$ , one deduces that

$$|\omega_{p,1}(|\mathbf{k}|) - p| \sim |A_p|^{\frac{1}{2}} |\mathbf{k}|^{-1}, \quad |\omega_{p,1}(|\mathbf{k}|) - \omega_{p,2}(|\mathbf{k}|)| \sim 2 |A_p|^{\frac{1}{2}} |\mathbf{k}|^{-1}, \quad (|\mathbf{k}| \rightarrow +\infty) \quad (2.4.71)$$

with  $|A_p| > 0$ . As a consequence, for  $|\mathbf{k}|$  large enough, we have

$$\rho_{|\mathbf{k}|} = \frac{1}{2} |\omega_p(|\mathbf{k}|) - p| \sim \frac{1}{2} |A_p|^{\frac{1}{2}} |\mathbf{k}|^{-1} \quad (|\mathbf{k}| \rightarrow +\infty), \quad \text{with } |A_p| > 0. \quad (2.4.72)$$

**Step 2 : Estimate of  $(\mathcal{D}(\omega) - |\mathbf{k}|^2)^{-1}$ .** It is very similar to the one in Lemma 2.4.6. One observes thanks to (2.4.2) that, when  $|\mathbf{k}| \rightarrow +\infty$ , in the product (2.4.18) defining  $D_{b,|\mathbf{k}|}(\omega)$ , only two terms, namely  $\omega - \omega_{p,1}(|\mathbf{k}|)$  and  $\omega - \omega_{p,2}(|\mathbf{k}|)$ , are not bounded from below when  $\omega$  describes  $\mathcal{C}_{p,1,|\mathbf{k}|}$  and  $|\mathbf{k}| \rightarrow +\infty$ . Thus for some  $C > 0$ ,  $\omega \in \mathcal{C}_{p,1,|\mathbf{k}|}$  and  $|\mathbf{k}|$  large enough

$$D_{b,|\mathbf{k}|}(\omega) \geq C |\omega - \omega_{p,1}(|\mathbf{k}|)| |\omega - \omega_{p,2}(|\mathbf{k}|)| = C \rho_{|\mathbf{k}|} |\omega - \omega_{p,2}(|\mathbf{k}|)|. \quad (2.4.73)$$

By the reverse triangular inequality, along  $\mathcal{C}_{p,1,|\mathbf{k}|}$ ,

$$|\omega - \omega_{p,2}(|\mathbf{k}|)| \geq |\omega_{p,1}(|\mathbf{k}|) - \omega_{p,2}(|\mathbf{k}|)| - |\omega - \omega_{p,1}(|\mathbf{k}|)| = |\omega_{p,1}(|\mathbf{k}|) - \omega_{p,2}(|\mathbf{k}|)| - \rho_{|\mathbf{k}|},$$

thus, using the equivalent (2.4.72) for  $\rho_{|\mathbf{k}|}$  and  $|\omega_{p,1}(|\mathbf{k}|) - \omega_{p,2}(|\mathbf{k}|)| \sim 2 |A_p|^{\frac{1}{2}} |\mathbf{k}|^{-1}$ ,

$$|\omega - \omega_{p,2}(|\mathbf{k}|)| \geq |\omega_{p,1}(|\mathbf{k}|) - \omega_{p,2}(|\mathbf{k}|)| - \rho_{|\mathbf{k}|} \sim \frac{3}{2} |A_p|^{\frac{1}{2}} |\mathbf{k}|^{-1}.$$

Therefore, from (2.4.73), we deduce that

$$|D_{b,|\mathbf{k}|}(\omega)|^{-1} \lesssim \rho_{|\mathbf{k}|}^{-1} |\mathbf{k}|, \quad \forall \omega \in \mathcal{C}_{p,1,|\mathbf{k}|}.$$

Thus, proceeding as in Lemma 2.4.6 (observe that by passing from  $|D_{b,|\mathbf{k}|}(\omega)|^{-1}$  to  $|D_{|\mathbf{k}|}(\omega)|^{-1}$ , see (2.4.44) and (2.4.45), one loses two powers of  $|\mathbf{k}|$ ),

$$|D_{|\mathbf{k}|}(\omega)|^{-1} \lesssim \rho_{|\mathbf{k}|}^{-1} |\mathbf{k}|^{-1}, \quad \forall \omega \in \mathcal{C}_{p,1,|\mathbf{k}|}. \quad (2.4.74)$$

Moreover,  $p \in \mathcal{P}_d$  is a double zero of the product  $Q_e Q_m$ . Hence, for any  $\omega \in \mathcal{C}_{p,1,|\mathbf{k}|}$

$$|Q_e(\omega) Q_m(\omega)| \lesssim |\omega - p|^2 \leq (\rho_{|\mathbf{k}|} + |\omega_p(|\mathbf{k}|) - p|)^2 \sim \frac{9}{4} |A_p| |\mathbf{k}|^{-2} \quad (\text{by (2.4.71, 2.4.72)}). \quad (2.4.75)$$

As a consequence, from (2.4.17), we finally get by product of (2.4.74) and (2.4.75),

$$|(\mathcal{D}(\omega) - |\mathbf{k}|^2)^{-1}| \lesssim \rho_{|\mathbf{k}|}^{-1} |\mathbf{k}|^{-3}, \quad \forall \omega \in \mathcal{C}_{p,1,|\mathbf{k}|}. \quad (2.4.76)$$

### Step 3 : Estimates of $\mathcal{S}_{|\mathbf{k}|}(\omega)$ and $\mathcal{V}_{|\mathbf{k}|}(\omega)$

As  $p \in \mathcal{P}_m \cap \mathcal{P}_e$ , there is a unique pair of indices of indices  $\{\ell_0, j_0\}$  such that  $p = \pm \omega_{e,j_0} = \pm \omega_{m,\ell_0}$ . The situation is a kind of mix between the two situations (i) and (ii) met in the step 3 of the proof of Lemma 2.4.6.

Using (2.4.30) one sees that all the operators  $(\mathbb{A}_{p,j}(\omega), \dot{\mathbb{A}}_{p,j}(\omega), \mathbb{A}_{m,\ell}(\omega), \dot{\mathbb{A}}_{m,\ell}(\omega))$  are uniformly bounded along  $\mathcal{C}_{p,1,|\mathbf{k}|}$  except  $(\mathbb{A}_{e,j_0}(\omega), \dot{\mathbb{A}}_{e,j_0}(\omega), \mathbb{A}_{m,\ell_0}(\omega), \dot{\mathbb{A}}_{m,\ell_0}(\omega))$ .

Proceeding as for (2.4.51) in the proof of Lemma (2.4.6), one easily gets (we omit the details)

$$\|\mathbb{A}_e(\omega)\| \lesssim |\mathbf{k}|, \quad \|\mathbb{A}_m(\omega)\| \lesssim |\mathbf{k}|. \quad (2.4.77)$$

As  $p$  is a simple pole of  $\mu(\omega)$ , we obtain, similarly to (2.4.52) in the proof of Lemma 2.4.6,

$$\forall \omega \in \mathcal{C}_{p,1,|\mathbf{k}|}, \quad |\mu(\omega)| \lesssim |\omega - p|^{-1} \lesssim |\mathbf{k}|. \quad (2.4.78)$$

which, joined to (2.4.77), gives

$$\|\omega \mu(\omega) \mathbb{A}_e(\omega) - |\mathbf{k}| \mathbf{e}_3 \times \mathbb{A}_m(\omega)\| \leq |\mathbf{k}|^2. \quad (2.4.79)$$

Finally using (2.4.76) and (2.4.79) in the definition of (2.3.10), we get

$$\|\mathcal{S}_{|\mathbf{k}|}(\omega)\| \lesssim \rho_{|\mathbf{k}|}^{-1} |\mathbf{k}|^{-1}, \quad \forall \omega \in \mathcal{C}_{p,1,|\mathbf{k}|}. \quad (2.4.80)$$

For  $\mathcal{V}_{|\mathbf{k}|}(\omega)$ , we first observe that  $|\mu(\omega)| \sim \mu_p |\omega - p|^{-1}$  ( $\omega \rightarrow p$ ) also implies that along  $\mathcal{C}_{p,1,|\mathbf{k}|}$ ,

$$|\mu(\omega)|^{-1} \lesssim \mu_p^{-1} \rho_{|\mathbf{k}|}^{-1} \lesssim |\mathbf{k}|^{-1} \text{ as } |\mathbf{k}| \rightarrow +\infty.$$

Thus, the function  $|\mathbf{k}| \mu(\omega)^{-1}$  is bounded for  $|\mathbf{k}|$  large enough. As a consequence, one sees on formula (2.3.11) that  $\mathcal{V}_{|\mathbf{k}|}(\omega)$  blows with the same rate than  $q_{e,j_0}(\omega)^{-1}$  and  $q_{m,\ell_0}(\omega)^{-1}$ , that is to say proportionally to  $\rho_{|\mathbf{k}|}^{-1}$ , that is to say proportionally to  $|\mathbf{k}|$ . Therefore

$$\|\mathcal{V}_{|\mathbf{k}|}(\omega)\| \lesssim |\mathbf{k}|, \quad \forall \omega \in \mathcal{C}_{p,1,|\mathbf{k}|}. \quad (2.4.81)$$

**Conclusion.** Again, by (2.4.80) and (2.4.81),  $\|\mathcal{S}_{|\mathbf{k}|}(\omega)\| \|\mathcal{V}_{|\mathbf{k}|}(\omega)\| \leq \rho_{|\mathbf{k}|}^{-1}$  and one concludes with (2.4.15) for  $\Pi(|\mathbf{k}|) = \Pi_p(|\mathbf{k}|)$ .  $\square$

We now give the asymptotic expansion of the eigenvalues  $\omega_{p,r}(|\mathbf{k}|)$  for large  $|\mathbf{k}|$ .

**Lemma 2.4.11**

For  $p \in \mathcal{P}_d$ , then  $p = \pm \omega_{e,j_p} = \pm \omega_{m,\ell_p}$  for some  $(j_p, \ell_p)$  and

$$\omega_{p,r}(|\mathbf{k}|) = p + (-1)^r \frac{\Omega_{e,j_p} \Omega_{m,\ell_p}}{2c} |\mathbf{k}|^{-1} + A_{p,2} |\mathbf{k}|^{-2} + o(|\mathbf{k}|^{-2}), \quad \text{as } |\mathbf{k}| \rightarrow +\infty, \quad (2.4.82)$$

where the complex number  $A_{p,2}$  is given by

$$A_{p,2} = \frac{1}{2} (\omega^2 h_{e,j_p} h_{m,\ell_p})'(p), \quad (2.4.83)$$

where the functions  $h_{e,p}$  and  $h_{m,p}$  are defined in (2.4.59). Moreover, one has

$$\text{Im } A_{p,2} < 0. \quad (2.4.84)$$

*Proof.* Let  $p \in \mathcal{P}_d$ . Then (by  $\text{H}_1$ ), there exist two unique indices  $j_p \in \{1, \dots, N_e\}$  and  $\ell_p \in \{1, \dots, N_m\}$  such that  $p = \pm \omega_{e,j_p} = \pm \omega_{m,\ell_p}$ . Then the two functions  $|\mathbf{k}| \mapsto \omega_{p,r}(|\mathbf{k}|)$  are defined in the proof of Proposition 2.4.1 for  $|\mathbf{k}|$  large enough via the Lemma B. 1 as the two branches of solutions of the equation  $\mathcal{D}(\omega)^{-1} = \zeta^2$  with  $\zeta = |\mathbf{k}|^{-1}$  on a vicinity of  $p$ . Here,  $p \in \mathcal{P}_e \cap \mathcal{P}_m$  is a common simple pole of the rational functions  $\omega \varepsilon$  and  $\omega \mu$ . More precisely, from the definition (2.4.58) of  $(h_{e,p}, h_{m,p})$  (see Lemma 2.4.7) we have

$$\mathcal{D}(\omega) = \omega^2 h_{e,p}(\omega) h_{m,p}(\omega) (\omega - p)^{-2}.$$

which emphasizes the fact that  $p$  is a double pole of the  $\mathcal{D}(\omega)$ , thus a double zero of  $\mathcal{D}^{-1}$ :  $\mathcal{D}(\omega)^{-1} = (\omega - p)^2 g(\omega)$  where  $g(\omega) := (\omega^2 h_{e,p}(\omega) h_{m,p}(\omega))^{-1}$  is analytic at the vicinity of  $p$  and satisfies (using the expression (2.4.60) for  $h_{e,p}$  and its equivalent for  $h_{m,p}$ ):

$$g(p) = 4c^2 \Omega_{e,j_p}^{-2} \Omega_{m,\ell_p}^{-2} > 0.$$

Thus, using the asymptotic formula (2.7.39) of Lemma B. 1 (with  $\zeta = |\mathbf{k}|^{-1}$ ,  $z = p$ ,  $\mathbf{m} = 2$  and  $a_1 = -\sqrt{g(p)} = -2c/(\Omega_{e,j_p} \Omega_{m,\ell_p})$ ,  $a_2 = \sqrt{g(p)}$  the two roots of  $X^2 = g(p)$ , see (2.7.39)) yields

$$\omega_{p,r}(|\mathbf{k}|) = p + a_r^{-1} |\mathbf{k}|^{-1} - \frac{a_r^{-2} g'(p)}{2g(p)} |\mathbf{k}|^{-2} + o(|\mathbf{k}|^{-2}), \quad \text{as } |\mathbf{k}| \rightarrow +\infty. \quad (2.4.85)$$

To conclude, it remains to remark that  $a_r^{-2} = g(p)^{-1}$  and  $g^{-1} = \omega^2 h_{e,p} h_{m,p}$  which gives

$$-\frac{a_r^{-2} g'(p)}{g(p)} = -\frac{g'(p)}{g^2(p)} = (g^{-1})'(p) = (\omega^2 h_{e,p} h_{m,p})'(p). \quad (2.4.86)$$

Finally, it remains to show that  $\text{Im } A_{p,2} < 0$ . From (2.4.84) and the expressions (2.4.59) of  $h_{e,p}$  and  $h_{m,p}$ , it follows (as  $p^2$ ,  $h_{e,p}(p)$ ,  $h_{m,p}(p)$  are real) that

$$\text{Im } A_{p,2} = \frac{p^2}{2} [h_{m,p}(p) \text{Im } h'_{e,p}(p) + h_{e,p}(p) \text{Im } h'_{m,p}(p)].$$

Finally using that  $h_{e,p}(p) = -\varepsilon_0 \Omega_{e,j_p}^2 / (2p)$  and  $h_{m,p}(p) = -\mu_0 \Omega_{m,\ell_p}^2 / (2p)$ , the expression (2.4.68) for  $\text{Im}(h'_{e,p}(p))$  and its equivalent form for  $\text{Im}(h'_{m,p}(p))$  gives that

$$\text{Im } A_{p,2} = -c^{-2} p^2 \left( \frac{\Omega_{m,\ell_p}^2}{4} \sum_{j=1, j \neq j_p}^{N_e} \frac{\Omega_{e,j}^2 \alpha_{e,j}}{|q_{e,j}(p)|^2} + \frac{\Omega_{e,j_p}^2}{4} \sum_{\ell=1, \ell \neq \ell_p}^{N_m} \frac{\Omega_{m,\ell}^2 \alpha_{m,\ell}}{|q_{m,\ell}(p)|^2} \right)$$

(where  $c^{-2} = \varepsilon_0 \mu_0$ ). Thus, this term is negative by the weak dissipation condition (2.1.10) since at least one coefficient  $\alpha_{e,j}$  or  $\alpha_{m,\ell}$  is positive. Finally, combining (2.4.85) and (2.4.86) gives (2.4.82).  $\square$

We now estimate the term  $\mathbb{U}_d(\mathbf{k}, t)$  for large  $|\mathbf{k}|$ . Proceeding as in the proof of Lemma 2.4.5, one shows using Lemma 2.4.10 and Lemma 2.4.11 the following result.

**Lemma 2.4.12**

If  $\mathcal{P}_d \neq \emptyset$ , then it exists  $k_+ > 0$  and  $C > 0$  such that the function  $\mathbb{U}_d(\mathbf{k}, t)$  defined by (2.4.9) and (2.4.34) satisfy the following estimate

$$|\mathbb{U}_d(\mathbf{k}, t)| \lesssim e^{-\frac{Ct}{|\mathbf{k}|^2}} |\mathbb{U}_0(\mathbf{k})|, \quad \forall t \geq 0 \text{ and } \forall |\mathbf{k}| \geq k_+. \quad (2.4.87)$$

**2.4.3.5 Estimates of  $\mathbb{U}_-(\mathbf{k}, t)$  for  $|\mathbf{k}| \gg 1$**

As announced in Section 2.4.3.1, since we simply want to obtain a ‘‘rough’’ exponential decay estimate for  $\mathbb{U}_-(\mathbf{k}, t)$ , we do not need to separate the analysis in three different lemmas as in Section 2.4.3.2, Section 2.4.3.3 and Section 2.4.3.4 but give a direct proof using Riesz-Dunford functional calculus.

**Lemma 2.4.13**

There exists  $\delta > 0$  and  $k_+ > 0$  such that  $\mathbb{U}_-(\mathbf{k}, t)$ , defined by (2.4.9) and (2.4.10), satisfies

$$|\mathbb{U}_-(\mathbf{k}, t)| \lesssim e^{-\delta t} |\mathbb{U}_0(\mathbf{k})|, \quad \forall t \geq 0, \quad \forall |\mathbf{k}| \geq k_+. \quad (2.4.88)$$

*Proof.* We introduce a simple closed contour  $\Gamma$ , included in  $\mathbb{C}^-$  such that all the poles of  $\mathcal{P}_-$  lies inside  $\Gamma$  (see Figure 2.3). We denote  $\delta$  the distance from  $\Gamma$  to the real axis:

$$\delta = \min\{-\text{Im}(\omega), \omega \in \Gamma\} > 0.$$

For  $|\mathbf{k}|$  large enough, by (2.4.2) and (2.4.3),  $\Gamma$  encloses all eigenvalues  $\omega_{p,n}(|\mathbf{k}|)$  for  $p \in \mathcal{P}_-$  and  $n \in \{1, \dots, \mathbf{m}_p\}$  but no other elements of the spectrum of  $A_{|\mathbf{k}|}$ . Then, by the Riesz-Dunford functional calculus, it exists  $k_+ > 0$  such that we have the formula:

$$\mathbb{U}_-(\mathbf{k}, t) = -\frac{\mathcal{R}_{\mathbf{k}}^*}{2i\pi} \int_{\Gamma} e^{i\omega t} R_{|\mathbf{k}|}(\omega) \mathcal{R}_{\mathbf{k}} \mathbb{U}_0(\mathbf{k}) d\omega, \quad \text{for } |\mathbf{k}| \geq k_+. \quad (2.4.89)$$

As  $\mathcal{R}_{\mathbf{k}}$  is unitary, it leads to

$$|\mathbb{U}_-(\mathbf{k}, t)| \lesssim e^{-\delta t} \left( \max_{\omega \in \Gamma} \|R_{|\mathbf{k}|}(\omega)\| \right) |\mathbb{U}_0(\mathbf{k})|, \quad (2.4.90)$$

where  $\|\cdot\|$  denotes the operator norm of  $\mathcal{L}(\mathbb{C}_{\perp}^N)$ . It remains now to estimate  $R_{|\mathbf{k}|}(\omega)$  from its expression  $R_{|\mathbf{k}|}(\omega) = \mathcal{V}_{|\mathbf{k}|}(\omega) \mathcal{S}_{|\mathbf{k}|}(\omega) + \mathcal{T}(\omega)$  in Proposition 2.3.2 (choosing  $\Gamma$  such that  $\Gamma \cap \mathcal{Z}_m = \emptyset$ ). The more involved step concerns the estimate of  $\mathcal{S}_{|\mathbf{k}|}(\omega)$ .

**Step 1 : Estimate of  $\mathcal{S}_{|\mathbf{k}|}(\omega)$ .** The expression of  $\mathcal{S}_{|\mathbf{k}|}(\omega)$ , given in (2.3.10), involved the term  $(\mathcal{D}(\omega) - |\mathbf{k}|^2)^{-1}$ . We first bound this term by using (2.4.17), (2.4.18) and (2.4.19). From the asymptotic behaviour (2.4.2), it follows that  $D_{b,|\mathbf{k}|}(\omega)$  given by (2.4.19) satisfies:

$$|D_{b,|\mathbf{k}|}(\omega)^{-1}| \lesssim 1, \quad \forall \omega \in \Gamma. \quad (2.4.91)$$

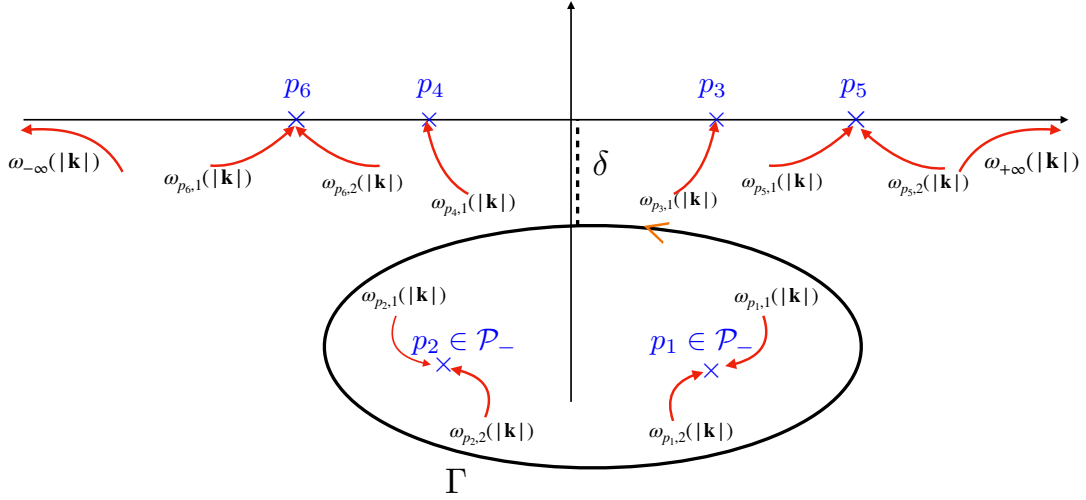


Figure 2.3: Contour integration for the estimate of  $\mathbf{U}_-(\mathbf{k}, t)$  in the case where  $\mathcal{P}_- = \{p_1, p_2\}$  (corresponding to the Figure 2.1).

From the asymptotic expansion (2.4.3),  $|\omega - \omega_{\pm\infty}(|\mathbf{k}|)|^{-1} = c^{-1}|\mathbf{k}|^{-1} + o(|\mathbf{k}|^{-1})$ . Hence, it yields with (2.4.18) and (2.4.91):

$$|D_{|\mathbf{k}|}(\omega)^{-1}| \lesssim |\mathbf{k}|^{-2}, \quad \text{and thus with (2.4.17)} \quad |(D(\omega) - |\mathbf{k}|^2)^{-1}| \lesssim |\mathbf{k}|^{-2} \quad \forall \omega \in \Gamma. \quad (2.4.92)$$

On the other hand, from the definitions (2.3.6) and (2.3.7), it is immediate that

$$\|\omega \mu(\omega) \mathbb{A}_e(\omega) - |\mathbf{k}| \mathbf{e}_3 \times \mathbb{A}_h(\omega)\| \lesssim |\mathbf{k}|, \quad \forall \omega \in \Gamma. \quad (2.4.93)$$

Combining (2.4.92) and (2.4.93) with (2.3.10) yields

$$\|\mathcal{S}_{|\mathbf{k}|}(\omega)\| \lesssim |\mathbf{k}|^{-1}, \quad \forall \omega \in \Gamma \text{ and } |\mathbf{k}| \geq k_+. \quad (2.4.94)$$

**Step 2 : Final estimate.** From the expression (2.3.11) and (2.3.12) of  $\mathcal{V}_{|\mathbf{k}|}(\omega)$  and  $\mathcal{T}(\omega)$  we deduce that

$$\|\mathcal{V}_{|\mathbf{k}|}(\omega)\| \lesssim |\mathbf{k}|, \quad \|\mathcal{T}(\omega)\| \lesssim 1, \quad \forall \omega \in \Gamma, \quad (2.4.95)$$

thus by (2.4.94)  $\|\mathcal{R}_{|\mathbf{k}|}(\omega)\| \lesssim 1, \forall \omega \in \Gamma$  which we substitute into (2.4.90) to obtain (2.4.88).  $\square$

### 2.4.3.6 The global estimates

**Theorem 2.4.14**

It exists  $k_+ > 0$  such that for  $|\mathbf{k}| \geq k_+$ , the spatial Fourier components  $|\mathbb{U}(\mathbf{k}, t)|$  of the solution of (2.1.12) with initial condition  $\mathbf{U}_0 \in \mathcal{H}_\perp$  satisfy the following estimates:

- a. If the Maxwell system is in a non-critical configuration, then there  $\exists C, \tilde{C} > 0$  such that

$$|\mathbb{U}(\mathbf{k}, t)| \leq \tilde{C} e^{-\frac{Ct}{|\mathbf{k}|^2}} |\mathbb{U}_0(\mathbf{k})|, \quad \forall t \geq 0. \quad (2.4.96)$$

- b. If the Maxwell system is in a critical configuration, then there  $\exists C, \tilde{C} > 0$  such that

$$|\mathbb{U}(\mathbf{k}, t)| \leq \tilde{C} e^{-\frac{Ct}{|\mathbf{k}|^4}} |\mathbb{U}_0(\mathbf{k})|, \quad \forall t \geq 0. \quad (2.4.97)$$

*Proof.* The inequalities (2.4.96) and (2.4.97) follow immediately from the expression (2.4.9) of  $|\mathbb{U}(\mathbf{k}, t)|$  for  $|\mathbf{k}|$  large enough and the estimates of the four terms  $|\mathbb{U}_\infty(\mathbf{k}, t)|$ ,  $|\mathbb{U}_s(\mathbf{k}, t)|$ ,  $|\mathbb{U}_d(\mathbf{k}, t)|$  and  $|\mathbb{U}_-(\mathbf{k}, t)|$  given respectively by Lemma 2.4.5, Lemma 2.4.9, Lemma 2.4.12 and Lemma 2.4.13 (since for  $|\mathbf{k}|$  large enough,  $t \geq 0$  and  $C_1, C_2 > 0$ :  $e^{-\delta t} \leq e^{-C_1 |\mathbf{k}|^{-2} t} \leq e^{-C_2 |\mathbf{k}|^{-4} t}$ ).  $\square$

We prove in the following result that the estimates of Theorem 2.4.14 are optimal for an infinite family of well chosen initial conditions  $\mathbf{U}_0 \in \mathcal{H}_{\perp, \text{HF}}^m$  for any fixed  $m > 0$ .



**Theorem 2.4.15**

Let  $m, k_+ > 0$  and  $\phi : \mathbb{R}^+ \mapsto \mathbb{R}$  be any a measurable and bounded function satisfying:

$$\text{supp } \phi \subset [k_+, +\infty[ \text{ and } 0 < |\phi(|\mathbf{k}|)| \lesssim (1 + |\mathbf{k}|^2)^{-s/2} \text{ for } |\mathbf{k}| > k_+ + 1 \text{ and } s > \frac{3}{2} + m. \quad (2.4.98)$$

a. If the Maxwell system is in a non-critical configuration, one defines (for  $k_+$  sufficiently large) an initial condition  $\mathbf{U}_0$  of (2.1.12) via its Fourier transform:

$$\mathbb{U}_0(\mathbf{k}) = \mathcal{F}(\mathbf{U}_0)(\mathbf{k}) = \phi(|\mathbf{k}|) \mathcal{R}_{\mathbf{k}}^* \frac{\mathcal{V}_{|\mathbf{k}|}(\omega_{+\infty}(|\mathbf{k}|)) \mathbf{e}_1}{|\mathcal{V}_{|\mathbf{k}|}(\omega_{+\infty}(|\mathbf{k}|)) \mathbf{e}_1|}, \quad \forall \mathbf{k} \in \mathbb{R}^{3,*}. \quad (2.4.99)$$

Then,  $\mathbf{U}_0 \in \mathcal{H}_{\perp, \text{HF}}^m$  and  $\exists C, \tilde{C} > 0$  such that the associated solution  $\mathbf{U}$  of (2.1.12) satisfy

$$\tilde{C} e^{-\frac{Ct}{|\mathbf{k}|^2}} |\mathbb{U}_0(\mathbf{k})| \leq |\mathbb{U}(\mathbf{k}, t)|, \quad \forall t \geq 0 \text{ and } \forall \mathbf{k} \in \mathbb{R}^{3,*}. \quad (2.4.100)$$

b. If the Maxwell system is in a critical configuration, by Corollary 2.4.8, it exists (at least one)  $p \in \mathcal{P}_s$  such that

$$\text{Im } \omega_p(|\mathbf{k}|) = \text{Im } A_{p,4} |\mathbf{k}|^{-4} + o(|\mathbf{k}|^{-4}), \text{ as } |\mathbf{k}| \mapsto +\infty, \text{ with } \text{Im } A_{p,4} < 0. \quad (2.4.101)$$

Defining (for  $k_+$  sufficiently large) as an initial condition  $\mathbf{U}_0$  for (2.1.12) by

$$\mathbb{U}_0(\mathbf{k}) = \mathcal{F}(\mathbf{U}_0)(\mathbf{k}) = \phi(|\mathbf{k}|) \mathcal{R}_{\mathbf{k}}^* \frac{\mathcal{V}_{|\mathbf{k}|}(\omega_p(|\mathbf{k}|)) \mathbf{e}_1}{|\mathcal{V}_{|\mathbf{k}|}(\omega_p(|\mathbf{k}|)) \mathbf{e}_1|}, \quad \forall \mathbf{k} \in \mathbb{R}^{3,*}, \quad (2.4.102)$$

then,  $\mathbf{U}_0 \in \mathcal{H}_{\perp, \text{HF}}^m$  and  $\exists C, \tilde{C} > 0$  such that the associated solution  $\mathbf{U}$  of (2.1.12) satisfy

$$\tilde{C} e^{-\frac{Ct}{|\mathbf{k}|^4}} |\mathbb{U}_0(\mathbf{k})| \leq |\mathbb{U}(\mathbf{k}, t)|, \quad \forall t \geq 0 \text{ and } \mathbf{k} \in \mathbb{R}^{3,*}. \quad (2.4.103)$$

In other words, the estimates (2.4.96) and (2.4.97) are optimal for an infinite family of solutions.

*Proof.* We separate it in two steps.

**Step 1. Proof of the lower bound (2.4.100):**

From Proposition 2.4.1, we know for  $|\mathbf{k}| > k_+$ ,  $|\mathbf{k}| \mapsto \omega_{+\infty}(|\mathbf{k}|)$  is well defined and satisfy (by Lemma 2.4.4) the asymptotic expansion (2.4.32).

Furthermore, the proof of Proposition 2.3.1 in appendix Section 2.7.2.2, shows that any  $\mathbb{U}$  in the two-dimensional eigenspace  $\ker(\mathbb{A}_{|\mathbf{k}|} - \omega_{+\infty}(|\mathbf{k}|) \text{Id})$  is of the form  $\mathbb{U} = (\mathbb{E}, \mathbb{H}, \mathbb{P}, \dot{\mathbb{P}}, \mathbb{M}, \dot{\mathbb{M}})$  for some  $\mathbb{E} \in \mathbf{C}_{\perp}$  and  $(\mathbb{H}, \mathbb{P}, \dot{\mathbb{P}}, \mathbb{M}, \dot{\mathbb{M}})$  deduced from  $\mathbb{E}$  by formula (2.7.33, 2.7.34), with  $\omega = \omega_{+\infty}(|\mathbf{k}|)$ .

This can be expressed with the help of the operator  $\mathcal{V}_{|\mathbf{k}|}(\omega) \in \mathcal{L}(\mathbf{C}_{\perp}, \mathbf{C}_{\perp}^N)$ , see (2.3.11), as follows

$$\ker(\mathbb{A}_{|\mathbf{k}|} - \omega_{+\infty}(|\mathbf{k}|) \text{Id}) = \mathcal{V}_{|\mathbf{k}|}(\omega_{+\infty}(|\mathbf{k}|))(\mathbf{C}_{\perp}). \quad (2.4.104)$$

Let  $m \in \mathbb{N}^*$ . Let us define  $\mathbf{U}_0$  via its Fourier transform  $\mathbb{U}_0(\mathbf{k})$  and formula (2.4.99) where  $\mathcal{R}_{\mathbf{k}}$ , see (2.2.8), is unitary. For  $s > 3/2 + m$ ,  $\mathbf{k} \mapsto (1 + |\mathbf{k}|^2)^{m/2} \mathbb{U}_0(\mathbf{k}) \in \mathbf{L}^2(\mathbb{R}^3)^N$ , thus  $\mathbf{U}_0 \in \mathbf{H}^m(\mathbb{R}^3)^N$  and thus belongs by construction to  $\mathcal{H}_{\perp, \text{HF}}^m$ .

Next, thanks to formula (2.2.12) for  $\mathbb{U}(\mathbf{k}, t)$ , (2.4.99) and (2.4.104), one has

$$\mathbb{U}(\mathbf{k}, t) = e^{-iA_{\mathbf{k}}t} \mathbb{U}_0(\mathbf{k}) = e^{-i\omega_{+\infty}(|\mathbf{k}|)t} \mathbb{U}_0(\mathbf{k}) \quad \text{and thus} \quad |\mathbb{U}(\mathbf{k}, t)| = e^{-\text{Im}\omega_{+\infty}(|\mathbf{k}|)t} |\mathbb{U}_0(\mathbf{k})|.$$

Moreover, using Lemma 2.4.4 and more precisely the expansion (2.4.33) for  $\omega_{+\infty}(|\mathbf{k}|)$ , one knows that for large enough  $|\mathbf{k}|$ , with  $A_{2,\infty} > 0$  defined by (2.4.34),

$$-\frac{A_{2,\infty}}{3c^2|\mathbf{k}|^2} \leq \text{Im}\omega_{+\infty}(|\mathbf{k}|) < 0 \quad (\text{simply because } 1/3 < 1/2).$$

We thus get  $e^{-\frac{A_{2,\infty}t}{3c^2|\mathbf{k}|^2}} |\mathbb{U}_0(\mathbf{k})| \lesssim |\mathbb{U}(\mathbf{k}, t)|$ , which achieves the proof of (2.4.100).

**Step 2 Proof of the lower bound (2.4.103):**

If the Maxwell system is in the critical configuration, to show that the estimate (2.4.97), one only needs to define  $\mathbf{U}_0$  by (2.4.102) (instead of (2.4.99)) replacing  $\omega_{\infty}(|\mathbf{k}|)$  by  $\omega_p(|\mathbf{k}|)$  satisfying (2.4.101) (instead of (2.4.33)). □

## 2.5 Asymptotic analysis for small spatial frequencies $|\mathbf{k}| \ll 1$

As the reader can expect, the structure of this section is similar to the one of Section 2.4 with three subsections:

- Section 2.5.1: Asymptotics of dispersion curves for  $0 < |\mathbf{k}| \ll 1$ .
- Section 2.5.2: Spectral decomposition of the solution for  $0 < |\mathbf{k}| \ll 1$ .
- Section 2.5.3: Large time estimate of  $\mathbb{U}(\mathbf{k}, t)$  for  $0 < |\mathbf{k}| \ll 1$ .

This section is however shorter than Section 2.4 because much less particular cases appear.

### 2.5.1 Asymptotics of the dispersion curves for $|\mathbf{k}| \ll 1$

This section is the counterpart of Section 2.4.1 for low spatial frequencies. In other words, we focus here on long time estimates of the low (spatial) frequency components  $\mathbb{U}(\mathbf{k}, t)$  (see (2.2.7)) of the solution. As explained in Section 2.3.3, the decay of  $\mathbb{U}(\mathbf{k}, t)$  for  $|\mathbf{k}| \ll 1$  is related to the analysis of the solutions of the dispersion relation (2.3.13) at low frequencies. Roughly speaking, as  $|\mathbf{k}|^2 \rightarrow 0$  when  $|\mathbf{k}| \rightarrow 0$ , the solutions of (2.3.13) must satisfy  $|\mathcal{D}(\omega)| \rightarrow 0$  as  $|\mathbf{k}| \rightarrow 0$ . Thus, they converge to a zero  $z \in \mathcal{Z} \cup \{0\}$  of the rational function  $\mathcal{D}$ . Thus, as one can rewrite  $\mathcal{D}$  at the vicinity of a zero  $z$  of multiplicity  $\mathbf{m}_z$  as

$$\mathcal{D}(\omega) = (\omega - z)^{\mathbf{m}_z} g(\omega) \quad \text{with } g \text{ analytic on a vicinity of } z \text{ and } g(z) = A_z \neq 0. \quad (2.5.1)$$

This leads to the following proposition (the equivalent of Proposition 2.4.1 in Section 2.4.1).

**Proposition 2.5.1**

It exists  $k_- > 0$  such that for  $0 < |\mathbf{k}| \leq k_-$ , the solutions of the rational dispersion relation (2.3.13) (or of its equivalent polynomial form (2.3.19)) are all simple. These solutions form  $N$  distinct branches which are  $C^\infty$ -smooth functions of  $|\mathbf{k}| \in (0, k_-]$ . These branches are characterized by their asymptotic expansion for small  $|\mathbf{k}|$ . More precisely, if  $z \in \mathcal{Z} \cup \{0\}$  is a zero of multiplicity  $\mathbf{m}_z$  of  $\mathcal{D}$ , there exists  $\mathbf{m}_z$  distinct branches of solutions  $\omega_{z,n}$ ,  $n = 1, \dots, \mathbf{m}_z$  of (2.3.13) satisfying

$$\omega_{z,n}(|\mathbf{k}|) = z + a_{z,n}^{-1} |\mathbf{k}|^{\frac{2}{\mathbf{m}_z}} (1 + o(1)), \quad a_{z,n} = |A_z|^{1/\mathbf{m}_z} e^{i\frac{\theta_z}{\mathbf{m}_z}} e^{\frac{2in\pi}{\mathbf{m}_z}} \quad (|\mathbf{k}| \rightarrow +\infty) \quad (2.5.2)$$

where  $A_z$  is defined in (2.5.1) and  $\theta_z \in (-\pi, \pi]$  is the principal argument of  $A_z$ .

*Proof. Step 1: Construction of the branches of solutions  $\omega_{z,n}(|\mathbf{k}|)$ .*

Let  $z \in \mathcal{Z} \cup \{0\}$  a zero of multiplicity  $\mathbf{m}_z$ . Then, the rational function  $\mathcal{D}$  can be factorized as in (2.5.1). Moreover, the dispersion relation can be rewritten as

$$\mathcal{D}(\omega) = |\mathbf{k}|^2 = (|\mathbf{k}|^{\frac{2}{\mathbf{m}_z}})^{\mathbf{m}_z}.$$

Thus, by applying Lemma B.1 of the appendix Appendix B.2. with

$$\mathcal{G}(\omega) = \mathcal{D}(\omega), \quad \mathbf{m} = \mathbf{m}_z, \quad A = A_z \text{ and } \zeta = |\mathbf{k}|^{2/\mathbf{m}_z},$$

we deduce the existence of  $k_- > 0$  and  $\mathbf{m}_z$  distinct  $C^\infty$  branches of solutions  $|\mathbf{k}| \in (0, k_-] \mapsto \omega_{z,n}(|\mathbf{k}|)$  of the dispersion  $\mathcal{D}(\omega) = |\mathbf{k}|^2$ . Moreover, for the solutions  $\omega_{z,n}(|\mathbf{k}|)$ , we can use the asymptotic expansion (2.7.39) of Lemma B.1 with  $\zeta = |\mathbf{k}|^{2/\mathbf{m}_z}$  which gives (2.5.2).

**Step 2: Conclusion.** From the asymptotic (2.5.2),  $\omega_{z,n}(|\mathbf{k}|)$  for  $z \in \mathcal{Z} \cup \{0\}$  and  $n \in \{1, \dots, \mathbf{m}_z\}$  are all distinct for  $|\mathbf{k}|$  small enough and positive. Since the sum of the  $\mathbf{m}_z$ 's, for  $z \in \mathcal{Z} \cup \{0\}$  is equal to  $2N_e + 2N_m + 2$ , we have constructed  $2N_e + 2N_m + 2$  distinct solutions of (2.3.13). As (2.3.13) is equivalent to a polynomial equation of degree  $2N_e + 2N_m + 2$ , cf. (2.3.19), these solutions are simple roots and there are no other solutions for small positive  $|\mathbf{k}|$ .  $\square$

### 2.5.2 Spectral decomposition of the solution for $|\mathbf{k}| \ll 1$

Our goal here is to obtain for small  $|\mathbf{k}|$  a decomposition of  $\mathbb{U}(\mathbf{k}, t)$  similar to the one obtained for large  $|\mathbf{k}|$  (see (2.4.9, 2.4.10)).

For this, we first notice that, combining Proposition 2.5.1 and Corollary 2.3.4 yields immediately to the following property on the operator  $\mathbb{A}_{|\mathbf{k}|, \perp}$ .

**Corollary 2.5.2**

It exists  $k_- > 0$  such that for  $0 < |\mathbf{k}| \leq k_-$ ,  $\mathbb{A}_{|\mathbf{k}|, \perp}$  is diagonalizable on  $\mathbf{C}_\perp^N$ .

To obtain the decomposition of  $\mathbb{U}(\mathbf{k}, t)$ , we split (see Section 2.3.2.3) the sets of zeros  $\{0\} \cup \mathcal{Z} \subset \overline{\mathbb{C}^-}$  of  $\mathcal{D}$  in three disjoint subsets :  $\{0\} \cup \mathcal{Z} = \{0\} \cup \mathcal{Z}_s \cup \mathcal{Z}_-$  where the multiplicity of  $z = 0$  is  $\mathbf{m}_0 = 2$  and

$$\mathcal{Z}_- := \mathcal{Z} \cap \mathbb{C}^- \text{ and } \mathcal{Z}_s := \{z \in \mathcal{Z} \cap \mathbb{R} \mid \mathbf{m}_z = 1\}.$$

**Remark 2.5.3.** [On the set  $\mathcal{Z}_-$ ] We point out that the weak dissipation condition (2.1.10) implies (see (2.3.18) and (2.3.17)) that at least one element of  $\mathcal{Z}$  lies in  $\mathbb{C}^-$ , thus  $\mathcal{Z}_- \neq \emptyset$ .

The strong dissipation condition (2.1.8) implies that all elements of  $\mathcal{Z}$  lie in  $\mathbb{C}^-$ , that is  $\mathcal{Z}_- = \mathcal{Z}$ .

**Remark 2.5.4.** [On the set  $\mathcal{Z}_s$ ] According to the analysis of the zeros of made in Section 2.3.2.3, the structure of the set  $\mathcal{Z}_s$  strongly relies on the condition

$$\exists (j_0, \ell_0) \text{ such that } \alpha_{e,j_0} > 0 \text{ and } \alpha_{m,\ell_0} > 0. \quad (2.5.3)$$

More precisely, if (2.5.3) holds,  $\mathcal{Z}_s = \emptyset$ . Oppositely, with the weak dissipation condition (2.1.10), if (2.5.3) does not hold, either all  $\alpha_{e,j}$  vanish in which case  $\mathcal{Z}_s = \mathcal{Z}_e$  either all  $\alpha_{\ell,m}$  vanish and  $\mathcal{Z}_s = \mathcal{Z}_m$ .

According to Proposition 2.3.1 and Proposition 2.5.1, we have the corresponding partition of the spectrum of  $\mathbb{A}_{|\mathbf{k}|,\perp}$  for  $0 < |\mathbf{k}| \leq k_-$ :

$$\sigma(\mathbb{A}_{|\mathbf{k}|,\perp}) = \{\omega_{0,r}(|\mathbf{k}|), r = 1, 2\} \cup \{\omega_z(|\mathbf{k}|), z \in \mathcal{Z}_s\} \cup \{\omega_{z,n}(|\mathbf{k}|), z \in \mathcal{Z}_-, n = 1, \dots, m_z\}, \quad (2.5.4)$$

where, for simplicity, we write  $\omega_z(|\mathbf{k}|)$  instead of  $\omega_{z,1}(|\mathbf{k}|)$  for the simple zeros zero  $z \in \mathcal{Z}_s$ . We point out that “by symmetry” of the dispersion relation, it holds  $\omega_{0,2}(|\mathbf{k}|) = -\bar{\omega}_{0,1}(|\mathbf{k}|)$ .

For a sketch of the small  $|\mathbf{k}|$  behaviour of the dispersion curves, see Figure 2.4.

For  $0 < |\mathbf{k}| \leq k_-$ , as  $\mathbb{A}_{|\mathbf{k}|,\perp}$  is diagonalizable by Corollary 2.5.2,  $\mathbf{C}_\perp^N$  can be decomposed as

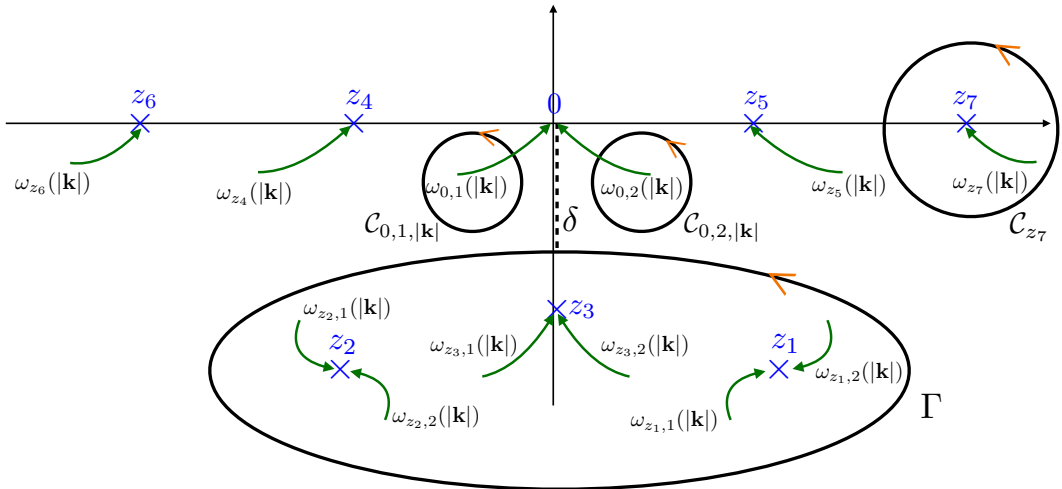


Figure 2.4: Sketch of a configuration of dispersion curves for small values of  $|\mathbf{k}|$  in the case where  $\{0\} \cup \mathcal{Z} = \{0\} \cup \mathcal{Z}_- \cup \mathcal{Z}_s$  with  $\mathcal{Z}_- = \{z_1, z_2, z_3\}$  and  $\mathcal{Z}_s = \{z_4, z_5, z_6, z_7\}$ .

$$\mathbf{C}_\perp^N = \bigoplus_{r=1}^2 V_{|\mathbf{k}|,0,r} \bigoplus_{z \in \mathcal{Z}_s} V_{|\mathbf{k}|,z} \bigoplus_{z \in \mathcal{Z}_-} \bigoplus_{n=1}^{m_z} V_{|\mathbf{k}|,z,n} \quad (2.5.5)$$

where we have defined the 2D (see below) subspaces

$$\begin{cases} V_{|\mathbf{k}|,0,r} = \ker (\mathbb{A}_{|\mathbf{k}|,\perp} - \omega_{0,r}(|\mathbf{k}|) \text{Id}), & r = 1, 2 \\ V_{|\mathbf{k}|,z} = \ker (\mathbb{A}_{|\mathbf{k}|,\perp} - \omega_z(|\mathbf{k}|) \text{Id}), \\ V_{|\mathbf{k}|,z,n} = \ker (\mathbb{A}_{|\mathbf{k}|,\perp} - \omega_{z,n}(|\mathbf{k}|) \text{Id}), & 1 \leq n \leq \mathbf{m}_z, \end{cases} \quad (2.5.6)$$

where the above direct sums are (in general) non-orthogonal. Thus, one decomposes uniquely any vector  $x \in \mathbf{C}_{\perp}^N$  as

$$\left\{ \begin{array}{l} x = \sum_{r=1}^2 x_{|\mathbf{k}|,0,r} + \sum_{z \in \mathcal{Z}_s} x_{|\mathbf{k}|,z} + \sum_{z \in \mathcal{Z}_-} \sum_{n=1}^{\mathbf{m}_z} x_{|\mathbf{k}|,z,n} \\ x_{|\mathbf{k}|,0,r} \in V_{|\mathbf{k}|,0,r}, \quad x_{|\mathbf{k}|,z} \in V_{|\mathbf{k}|,z} \quad \text{and} \quad x_{|\mathbf{k}|,z,n} \in V_{|\mathbf{k}|,z,n}. \end{array} \right.$$

Then, we define the spectral projectors  $\Pi_{0,r}(|\mathbf{k}|)$  for  $r \in \{1, 2\}$ ,  $\Pi_z(|\mathbf{k}|)$  for  $z \in \mathcal{Z}_s$  and  $\Pi_{z,n}(|\mathbf{k}|)$ ,  $z \in \mathcal{Z}_-$  and  $n \in \{1, \dots, \mathbf{m}_z\}$  associated to the eigenvalues  $\omega_{0,r}(|\mathbf{k}|)$ ,  $\omega_z(|\mathbf{k}|)$  and  $\omega_{z,n}(|\mathbf{k}|)$  by:

$$\Pi_{0,r}(|\mathbf{k}|) = x_{|\mathbf{k}|,0,r}, \quad \Pi_z(|\mathbf{k}|)(x) = x_{|\mathbf{k}|,z} \quad \text{and} \quad \Pi_{z,n}(|\mathbf{k}|)(x) = x_{|\mathbf{k}|,z,n}. \quad (2.5.7)$$

From [Proposition 2.3.1](#), the geometric multiplicity of each eigenvalues in  $\sigma(\mathbb{A}_{|\mathbf{k}|,\perp})$  is two. Thus, all these projectors are rank two projectors. We again emphasize that the dissipative operator  $\mathbb{A}_{|\mathbf{k}|,\perp}$  is not normal, thus its spectral projectors are not orthogonal.

For  $0 < |\mathbf{k}| \leq k_-$ , as  $\mathbb{A}_{|\mathbf{k}|,\perp}$  is diagonalizable (by [Corollary 2.3.4](#)), one has

$$\mathbb{A}_{|\mathbf{k}|,\perp} = \sum_{r=1}^2 \omega_{0,r}(|\mathbf{k}|) \Pi_{0,r}(|\mathbf{k}|) + \sum_{z \in \mathcal{Z}_s} \omega_z(|\mathbf{k}|) \Pi_z(|\mathbf{k}|) + \sum_{z \in \mathcal{Z}_-} \sum_{n=1}^{\mathbf{m}_z} \omega_{z,n}(|\mathbf{k}|) \Pi_{z,n}(|\mathbf{k}|). \quad (2.5.8)$$

Thus, for  $0 < |\mathbf{k}| \leq k_-$ , the solution  $\mathbb{U}(\mathbf{k}, t)$  given by [\(2.2.12\)](#) can be expressed for all  $t \geq 0$  as

$$\mathbb{U}(\mathbf{k}, t) = \mathbb{U}_{z,0}(\mathbf{k}, t) + \mathbb{U}_{z,s}(\mathbf{k}, t) + \mathbb{U}_{z,-}(\mathbf{k}, t) \quad (2.5.9)$$

where

$$\left\{ \begin{array}{l} \mathbb{U}_{z,0}(\mathbf{k}, t) = \sum_{r=1}^2 e^{-i\omega_{0,r}(|\mathbf{k}|)t} \mathcal{R}_{\mathbf{k}}^* \Pi_{0,r}(|\mathbf{k}|) \mathcal{R}_{\mathbf{k}} \mathbb{U}_0(\mathbf{k}), \quad (i) \\ \mathbb{U}_{z,s}(\mathbf{k}, t) = \sum_{z \in \mathcal{Z}_s} e^{-i\omega_z(|\mathbf{k}|)t} \mathcal{R}_{\mathbf{k}}^* \Pi_z(|\mathbf{k}|) \mathcal{R}_{\mathbf{k}} \mathbb{U}_0(\mathbf{k}), \quad (ii) \\ \mathbb{U}_{z,-}(\mathbf{k}, t) = \sum_{z \in \mathcal{Z}_-} \sum_{n=1}^{\mathbf{m}_z} e^{-i\omega_{z,n}(|\mathbf{k}|)t} \mathcal{R}_{\mathbf{k}}^* \Pi_{z,n}(|\mathbf{k}|) \mathcal{R}_{\mathbf{k}} \mathbb{U}_0(\mathbf{k}). \quad (iii) \end{array} \right. \quad (2.5.10)$$

## 2.5.3 Estimates of the low frequency components of the solution

### 2.5.3.1 Orientation

Proceeding as for large  $|\mathbf{k}|$  in [Section 2.4.3](#), we estimate successively, in [Lemma 2.5.7](#), [Lemma 2.5.10](#) and [Lemma 2.5.11](#), the three terms appearing in the decomposition [\(2.5.9\)](#). An important difference with [Section 2.4.3](#) is that we consider bounded values of  $|\mathbf{k}|$ , namely  $0 < |\mathbf{k}| < k^-$ , for some  $k^- > 0$ , which will simplify some estimates via compactness arguments.

Two cases will have to be distinguished.

**Case 1:** estimate of  $\mathbb{U}_{z,0}(\mathbf{k}, t)$  and  $\mathbb{U}_{z,s}(\mathbf{k}, t)$ , This will be done in [Section 2.5.3.2](#) and [Section 2.5.3.3](#), by bounding separately each of the terms of the sums in (2.5.10)(i) and (2.5.10)(ii).

For each eigenvalue  $\omega(|\mathbf{k}|) \in \{\omega_{0,r}(|\mathbf{k}|), \omega_z(|\mathbf{k}|)\}$ , we shall first estimate in [Lemma 2.5.5](#) and [Lemma 2.5.8](#) the corresponding spectral projector  $\Pi(|\mathbf{k}|)$  uniformly in  $|\mathbf{k}|$  for  $|\mathbf{k}|$  small enough and positive. To this aim as in [Section 2.4.3](#), we use the Dunford-Riesz functional calculus that says that, if  $\Gamma_{|\mathbf{k}|}$  is a close contour in  $\mathbb{C}$  that encloses  $\omega(|\mathbf{k}|)$  but no other eigenvalue of  $\mathbb{A}_{|\mathbf{k}|,\perp}$ , so that

$$\Pi(|\mathbf{k}|) = -\frac{1}{2i\pi} \int_{\Gamma_{|\mathbf{k}|}} R_{|\mathbf{k}|}(\omega) d\omega. \quad (2.5.11)$$

For  $\Pi_z(|\mathbf{k}|)$  in [Lemma 2.5.8](#), since  $\omega_z(|\mathbf{k}|)$  is a simple eigenvalues that converges to  $z$  while remaining well separated from the other eigenvalues. Hence, using [Proposition 2.3.2](#), we can choose for  $\Gamma_{|\mathbf{k}|}$  a fixed circle  $\Gamma$  (i.e. independent of  $|\mathbf{k}|$ ) centered at  $z$ , along which the operator  $\mathcal{R}_{|\mathbf{k}|}(\omega)$ , by continuity in  $(|\mathbf{k}|, \omega)$  and compactness of  $[0, k^-] \times \Gamma$ , remain bounded. Thus, the uniform estimate of  $\Pi_z(|\mathbf{k}|)$  becomes obvious.

For the projector  $\Pi_{0,1}(|\mathbf{k}|)$  (this is the same for  $\Pi_{0,2}(|\mathbf{k}|)$ ), this is more complicated because the two distinct eigenvalues  $\omega_{0,r}(|\mathbf{k}|)$  ( $r = 1, 2$ ) converge both to 0 while remaining far way from the other eigenvalues any other point of the set  $\mathcal{S}_{\mathcal{T}}$  than 0. For this reason, we shall take as  $\Gamma_{|\mathbf{k}|}$  the circle  $\mathcal{C}_{0,1,|\mathbf{k}|}$  centered at  $\omega_{0,1}(|\mathbf{k}|)$  with radius  $\rho_{|\mathbf{k}|} = |\omega_{0,2}(|\mathbf{k}|) - \omega_{0,1}(|\mathbf{k}|)|/4$  in order to prevent  $\omega_{0,2}(|\mathbf{k}|)$  from being inside  $\mathcal{C}_{0,1,|\mathbf{k}|}$ . This circle will also not contain 0 or any other point of  $\mathcal{S}_{\mathcal{T}}$  (by [Proposition 2.5.1](#)). As a consequence, in (2.5.11),  $R_{|\mathbf{k}|}(\omega)$  can be replaced by the product  $\mathcal{V}_{|\mathbf{k}|}(\omega) \mathcal{S}_{|\mathbf{k}|}(\omega)$ , (see formula (2.3.9) for  $R_{|\mathbf{k}|}(\omega)$  and the definition (2.3.8) of  $\mathcal{S}_{\mathcal{T}}$ ) leading, similarly to (2.4.15), to

$$\|\Pi(|\mathbf{k}|)\| \leq \rho_{|\mathbf{k}|} \sup_{\omega \in \mathcal{C}_{|\mathbf{k}|}} (\|\mathcal{V}_{|\mathbf{k}|}(\omega)\| \|\mathcal{S}_{|\mathbf{k}|}(\omega)\|), \quad (2.5.12)$$

Moreover the operators  $\mathcal{V}_{|\mathbf{k}|}(\omega)$  and  $\mathcal{S}_{|\mathbf{k}|}(\omega)$  will blow up when  $|\mathbf{k}|$  tend to 0 but this will be compensated by the fact that the radius of  $\rho_{|\mathbf{k}|}$  tends to 0.

In a second step, we concentrate on the exponentials appearing in each factor of the terms involved in  $\mathbb{U}_{z,0}(\mathbf{k}, t)$  and  $\mathbb{U}_{z,s}(\mathbf{k}, t)$  whose estimate for small  $|\mathbf{k}|$  rely on the asymptotic expansion of the eigenvalues (and more particularly their imaginary parts), see [Lemma 2.5.6](#) and [Lemma 2.5.9](#).

**Case 2:** estimate of  $\mathbb{U}_{z,-}(\mathbf{k}, t)$ . This term will be treated as the term  $\mathbb{U}_-(\mathbf{k}, t)$  (in [Section 2.4.3.5](#)), because the set  $\{\omega_{z,n}(|\mathbf{k}|)\}$  will remain far from the real axis for small enough  $|\mathbf{k}|$ . Thanks to this property, the exponential decay of  $|\mathbb{U}_{z,-}(\mathbf{k}, t)|$  will be, contrary to the previous terms  $\mathbb{U}_{z,0}(\mathbf{k}, t)$  and  $\mathbb{U}_{z,s}(\mathbf{k}, t)$ , uniform with respect to  $|\mathbf{k}|$ , so that it will not contribute at the end to the large time equivalent of  $\mathbf{U}(\cdot, t)$ . For proving this, we shall not use (2.5.10)(ii) but an alternative formula directly issued from the Riesz-Dunford functional calculus. Compared to [Section 2.4.3.5](#), its estimate in [Lemma 2.5.11](#) will be simplified by a compacity argument.

### 2.5.3.2 Estimates of $\mathbb{U}_{z,0}(\mathbf{k}, t)$ for $0 < |\mathbf{k}| \leq k_-$

In the following Lemma, we estimate the spectral projectors  $\Pi_{0,r}(|\mathbf{k}|)$  (for  $r = 1, 2$ ) using (2.5.12) with a simple closed contour  $\Gamma_{|\mathbf{k}|}$  satisfying the two following properties:

$$\begin{cases} (i) & \Gamma_{|\mathbf{k}|} \text{ encloses } \omega(|\mathbf{k}|) \text{ but no other eigenvalue of } \mathbb{A}_{|\mathbf{k}|,\perp}, \\ (ii) & \Gamma_{|\mathbf{k}|} \text{ does not enclose any point of the set } \mathcal{S}_{\mathcal{T}}. \end{cases} \quad (2.5.13)$$

#### Lemma 2.5.5

It exists  $k_- > 0$  such that the spectral projectors  $\Pi_{0,r}(|\mathbf{k}|)$ ,  $r \in \{1, 2\}$  are uniformly bounded for  $0 < |\mathbf{k}| \leq k_-$ .

*Proof.* Let  $r \in \{1, 2\}$ . We follow the approach of the Section 2.4.3 and introduce as  $\Gamma_{|\mathbf{k}|}$  the circle  $\mathcal{C}_{0,r,|\mathbf{k}|}$  (positively oriented) of center  $\omega_{0,r}(|\mathbf{k}|)$  and radius

$$\rho_{|\mathbf{k}|} = \frac{1}{4} |\omega_{0,1}(|\mathbf{k}|) - \omega_{0,2}(|\mathbf{k}|)|, \quad (2.5.14)$$

(see Figure 2.4 for an example of contour  $\mathcal{C}_{0,r,|\mathbf{k}|}$  with  $r = 1, 2$ .)

From the asymptotics (2.5.2) applied with  $\mathbf{m}_0 = 2$  ( $z = 0$  is a double zero), one deduces that

$$|\omega_{0,1}(|\mathbf{k}|)| \sim |A_0|^{\frac{1}{2}} |\mathbf{k}|, \quad |\omega_{0,1}(|\mathbf{k}|) - \omega_{0,2}(|\mathbf{k}|)| \sim 2 |A_0|^{\frac{1}{2}} |\mathbf{k}|, \quad (|\mathbf{k}| \rightarrow 0) \quad (2.5.15)$$

so that, for  $|\mathbf{k}|$  small enough,  $\Gamma_{|\mathbf{k}|}$  does not enclose  $\omega_{0,2}(|\mathbf{k}|)$  nor 0 (and of course any other point of  $\mathcal{S}_{\mathcal{T}}$ ). As explained in Section 2.5.3.1, we can apply the inequality (2.5.12). As  $\rho_{|\mathbf{k}|} \lesssim |\mathbf{k}|$ , according to (2.5.14) and (2.5.15), we thus obtain

$$\|\Pi_{0,r}(|\mathbf{k}|)\| \lesssim |\mathbf{k}| \sup_{\omega \in \mathcal{C}_{0,r,|\mathbf{k}|}} (\|\mathcal{V}_{|\mathbf{k}|}(\omega)\| \|\mathcal{S}_{|\mathbf{k}|}(\omega)\|). \quad (2.5.16)$$

It remains to estimate the operators  $\mathcal{V}_{|\mathbf{k}|}(\omega)$  and  $\mathcal{S}_{|\mathbf{k}|}(\omega)$  given by the formulas (2.3.10) and (2.3.11). Without any loss of generality, we can restrict ourselves to  $r = 1$  for the rest of the proof.

**Step 1: Estimate of  $(\mathcal{D}(\omega) - |\mathbf{k}|^2)^{-1}$ .**

This term is involved in the expression (2.3.10) of  $\mathcal{S}_{|\mathbf{k}|}(\omega)$ . Let us recall that (see (2.4.17))

$$(\mathcal{D}(\omega) - |\mathbf{k}|^2)^{-1} = Q_e(\omega) Q_m(\omega) D_{|\mathbf{k}|}(\omega)^{-1}.$$

where  $D_{|\mathbf{k}|}(\omega)$  is the polynomial defined by (2.3.19). Using the Proposition 2.5.1, we know that  $D_{|\mathbf{k}|}(\omega)$  can be factorized as the following for  $|\mathbf{k}|$  sufficiently small and positive:

$$D_{|\mathbf{k}|}(\omega) = \varepsilon_0 \mu_0 (\omega - \omega_{0,1}(|\mathbf{k}|)) (\omega - \omega_{0,2}(|\mathbf{k}|)) \prod_{z \in \mathcal{Z}_s} (\omega - \omega_z(|\mathbf{k}|)) \prod_{z \in \mathcal{Z}_-} \prod_{n=1}^{m_z} (\omega - \omega_{z,n}(|\mathbf{k}|)). \quad (2.5.17)$$

Since  $\omega_z(|\mathbf{k}|)$ ,  $z \in \mathcal{Z}_s$  or  $\omega_{z,n}(|\mathbf{k}|)$ ,  $z \in \mathcal{Z}_-$  converge to  $z \neq 0$ , cf. (2.5.2), the last two products in (2.5.17) remain bounded from below in modulus when  $\omega$  describes  $\mathcal{C}_{p,r,|\mathbf{k}|}$  and  $|\mathbf{k}|$  is small enough.

On the other hand, it is easy to see that for  $|\mathbf{k}|$  small enough ( $\mathcal{C}_{0,1,|\mathbf{k}|}$  has been chosen for that),

$$\forall \omega \in \mathcal{C}_{0,1,|\mathbf{k}|}, \quad |\omega - \omega_{0,1}(|\mathbf{k}|)| \geq C |\mathbf{k}|, \quad |\omega - \omega_{0,2}(|\mathbf{k}|)| \geq C |\mathbf{k}|,$$

Thus from (2.5.17), we infer that

$$\forall \omega \in \mathcal{C}_{0,1,|\mathbf{k}|}, \quad |D_{|\mathbf{k}|}(\omega)|^{-1} \lesssim |\mathbf{k}|^{-2}. \quad (2.5.18)$$

Thus, as the polynomial  $Q_e(\omega)Q_m(\omega)$  remains obviously bounded for bounded  $\omega$ , we deduce that there exists  $k_- > 0$  such that

$$\forall \omega \in \mathcal{C}_{0,1,|\mathbf{k}|}, \quad |\mathcal{D}(\omega) - |\mathbf{k}|^2|^{-1} \lesssim |\mathbf{k}|^{-2}, \quad \text{for } 0 < |\mathbf{k}| \leq k_-. \quad (2.5.19)$$

### Step 2: Estimates of $\mathcal{S}_{|\mathbf{k}|}(\omega)$ and $\mathcal{V}_{|\mathbf{k}|}(\omega)$

First, it follows from their definitions (2.3.6) that  $\mathbb{A}_{p,j}(\omega)$ ,  $\dot{\mathbb{A}}_{p,j}(\omega)$ ,  $\mathbb{A}_{m,\ell}(\omega)$  and  $\dot{\mathbb{A}}_{m,\ell}(\omega)$  are uniformly bounded in operator norm when  $\omega$  does not approach 0 (indeed 0 is not a zero of the polynomials  $q_{e,j}$  and  $q_{m,\ell}$ ). Thus  $\|\mathbb{A}_e(\omega)\| \lesssim 1$  and  $\|\mathbb{A}_m(\omega)\| \lesssim 1$  according to (2.3.7).

Moreover,  $\mu(\omega) = \mu(0) + o(1)$  as  $\omega \rightarrow 0$  and  $|\omega| \lesssim |\mathbf{k}|$  on  $\mathcal{C}_{0,1,|\mathbf{k}|}$ , thus, for  $|\mathbf{k}|$  small enough,

$$\forall \omega \in \mathcal{C}_{0,1,|\mathbf{k}|}, \quad \|\omega\mu(\omega)\mathbb{A}_e(\omega) - |\mathbf{k}| \mathbf{e}_3 \times \mathbb{A}_m(\omega)\| \lesssim |\omega| + |\mathbf{k}| \lesssim |\mathbf{k}|. \quad (2.5.20)$$

Combining (2.5.19) and (2.5.20) in (2.3.10) yields that it exists  $k_- > 0$ :

$$\forall \omega \in \mathcal{C}_{0,1,|\mathbf{k}|}, \quad \|\mathcal{S}_{|\mathbf{k}|}(\omega)\| \lesssim |\mathbf{k}|^{-1}, \quad \text{for } 0 < |\mathbf{k}| \leq k_-. \quad (2.5.21)$$

Finally, as, along  $\mathcal{C}_{0,1,|\mathbf{k}|}$ ,  $|\omega| \geq |\rho_{|\mathbf{k}|,1} - |\omega_{0,1}(|\mathbf{k}|)| \sim \frac{1}{2} |A_0|^{\frac{1}{2}} |\mathbf{k}|$  as  $|\mathbf{k}| \rightarrow 0$ , we deduce that

$$\forall \omega \in \mathcal{C}_{0,1,|\mathbf{k}|}, \quad \frac{|\mathbf{k}|}{|\omega| |\mu(\omega)|} \lesssim \frac{|\mathbf{k}|}{|\omega|} \lesssim 1, \quad \text{for } 0 < |\mathbf{k}| \leq k_-.$$

Since  $|q_{e,j}(\omega)|^{-1}$  and  $|q_{m,\ell}(\omega)|^{-1}$  remain bounded along  $\mathcal{C}_{0,1,|\mathbf{k}|}$ , by the formula (2.3.11) for  $\mathcal{V}_{|\mathbf{k}|}(\omega)$ ,

$$\|\mathcal{V}_{|\mathbf{k}|}(\omega)\| \lesssim 1, \quad \forall \omega \in \mathcal{C}_{0,1,|\mathbf{k}|} \text{ and } 0 < |\mathbf{k}| \leq k_-. \quad (2.5.22)$$

**Conclusion:** Combining (2.5.20) and (2.5.22) in eq. (2.5.16) finally implies that it exists  $k_- > 0$  such that  $\Pi_{0,1}(|\mathbf{k}|)$  is uniformly bounded for  $0 < |\mathbf{k}| \leq k_-$ . □

The next lemma is about the asymptotic expansion (in powers of  $|\mathbf{k}|$ ) of  $\omega_{0,r}(|\mathbf{k}|)$  when  $|\mathbf{k}| \rightarrow 0$ . It is important to push the expansion up to the first apparition of a negative imaginary part, since it will govern the decay of  $\mathbb{U}_{z,0}(\mathbf{k}, t)$  for small values of  $|\mathbf{k}|$ .

#### Lemma 2.5.6

For  $r \in \{1, 2\}$ , the eigenvalue  $\omega_{0,r}(|\mathbf{k}|)$  satisfies the following asymptotic expansions, with  $c_0 = (\varepsilon(0)\mu(0))^{-1/2} > 0$ ,

$$\omega_{z,r}(|\mathbf{k}|) = (-1)^r c_0 |\mathbf{k}| - \frac{1}{2} (\varepsilon\mu)'(0) c_0^4 |\mathbf{k}|^2 + o(|\mathbf{k}|^2), \quad \text{as } |\mathbf{k}| \rightarrow 0, \quad (2.5.23)$$

with moreover

$$\text{Im} \left( -(\varepsilon\mu)'(0) \right) < 0. \quad (2.5.24)$$



*Proof.* Using (2.3.21), one has  $\mathcal{D}(\omega) = \omega^2 g(\omega)$  where the rational function  $g = \varepsilon(\cdot)\mu(\cdot)$  is analytic at the vicinity of 0 and satisfies  $g(0) = \varepsilon(0)\mu(0) > 0$ . Thus applying Lemma B. 1 with  $z = 0$ ,  $\mathbf{m}_0 = 2$ ,  $\zeta = |\mathbf{k}|$ ,  $a_1 = -g(0)^{1/2} = -c_0^{-1}$  and  $a_2 = c_0^{-1}$ , the asymptotic formula (2.7.39) yields:

$$\omega_{z,r}(|\mathbf{k}|) = a_r^{-1}|\mathbf{k}|^1 - \frac{g'(0)}{2g(0)^2}|\mathbf{k}|^2 + o(|\mathbf{k}|^2), \quad \text{as } |\mathbf{k}| \rightarrow 0.$$

This is precisely the asymptotic (2.5.23). It remains to prove (2.5.24).

$\mu(0) > 0$  and  $\varepsilon(0) > 0$ , thus it is equivalent to show that  $\mu(0) \operatorname{Im} \varepsilon'(0) + \varepsilon(0) \operatorname{Im} \mu'(0) > 0$ . The Taylor expansion of the rational functions  $\varepsilon$  and  $\mu$  (see (2.1.5) and (2.1.6)) gives

$$\operatorname{Im} \varepsilon'(0) = \varepsilon_0 \sum_{j=1}^{N_e} \frac{\alpha_{e,j} \Omega_{e,j}^2}{\omega_{e,j}^4} \geq 0 \quad \text{and} \quad \operatorname{Im} \mu'(0) = \mu_0 \sum_{\ell=1}^{N_m} \frac{\alpha_{m,\ell} \Omega_{m,\ell}^2}{\omega_{m,\ell}^4} \geq 0.$$

Furthermore, by the weak dissipation condition (2.1.10) at least one coefficient  $\alpha_{e,j}$  or  $\alpha_{e,m}$  is positive, thus one has  $\operatorname{Im} \varepsilon'(0) > 0$  or  $\operatorname{Im} \mu'(0) > 0$  and we can conclude.  $\square$

Combining the Lemma 2.5.5 and Lemma 2.5.6 (formula (2.5.23) and (2.5.24)) and proceeding as in the proof of Lemma 2.4.5 gives the following estimate for  $\mathbb{U}_{z,0}(\mathbf{k}, t)$ .

**Lemma 2.5.7**

It exists  $k_- > 0$  and  $C > 0$  such that the function  $\mathbb{U}_{z,0}(\mathbf{k}, t)$  defined by (2.5.9) and (2.5.10), satisfies the following estimate

$$|\mathbb{U}_{z,0}(\mathbf{k}, t)| \lesssim e^{-C|\mathbf{k}|^2 t} |\mathbb{U}_0(\mathbf{k})|, \quad \forall t \geq 0 \quad \text{and} \quad 0 < |\mathbf{k}| \leq k_-. \quad (2.5.25)$$

**2.5.3.3 Estimates of  $\mathbb{U}_{z,s}(\mathbf{k}, t)$  for  $0 < |\mathbf{k}| \leq k_-$**

This time we estimate  $\mathbb{U}_{z,s}(\mathbf{k}, t)$  in (2.5.9) which involves in particular the projectors  $\Pi_z(|\mathbf{k}|)$ .

**Lemma 2.5.8**

There exists  $k_- > 0$  such that the spectral projectors  $\Pi_z(|\mathbf{k}|)$ ,  $z \in \mathcal{Z}_s$  are uniformly bounded for  $0 < |\mathbf{k}| \leq k_-$ .

*Proof.* The proof of this lemma has already been sketched in Section 2.5.3.1 (paragraph about  $\Pi_z(|\mathbf{k}|)$ ). The details, similar to some already treated cases, are left to the reader.  $\square$

We now give the asymptotic expansion of the eigenvalues  $\omega_z(|\mathbf{k}|)$  for small  $|\mathbf{k}|$ .

**Lemma 2.5.9**

Let  $z \in \mathcal{Z}_s$ . The eigenvalue  $\omega_z(|\mathbf{k}|)$  satisfies the following asymptotic expansion

$$\omega_z(|\mathbf{k}|) = z + A_z |\mathbf{k}|^2 + o(|\mathbf{k}|^2), \quad \text{as } |\mathbf{k}| \rightarrow 0, \quad (2.5.26)$$

where the complex number  $A_z$  is defined by (two disjoint cases have to be distinguished) :

$$\text{if } z \in \mathcal{Z}_e, \quad A_z = \frac{(\omega\varepsilon)'(z)^{-1}}{z\mu(z)}, \quad \text{if } z \in \mathcal{Z}_m, \quad A_z = \frac{(\omega\mu)'(z)^{-1}}{z\varepsilon(z)}, \quad (2.5.27)$$

and satisfies in all cases  $\text{Im } A_z < 0$ .

*Proof.* The elements of  $\mathcal{Z}_s$  are simple zero ( $m_z = 1$ ) of the rational function  $\mathcal{D}$ , it means that either  $z \in \mathcal{Z}_e$  or  $z \in \mathcal{Z}_m$ . We will do the proof by assuming that  $z \in \mathcal{Z}_e$ , the proof for  $z \in \mathcal{Z}_m$  can be done with obvious ‘‘symmetric arguments’’.

Using [Proposition 2.5.1](#) and more precisely the asymptotic expansion (2.5.2) (with here  $m_z = 1$ ) yields to (2.5.26) with  $A_z = g(z)^{-1}$  where  $g$  is defined by (2.5.1), that is to say

$$g(\omega) = \frac{\mathcal{D}(\omega)}{\omega - z} = \frac{\omega\varepsilon(\omega)}{\omega - z} \omega\mu(\omega) \quad \text{for } \omega \in \mathbb{C} \setminus (\mathcal{P} \cup \{z\}).$$

Since  $z \in \mathcal{Z}_e$  is a real simple zero of  $\omega\varepsilon(\cdot)$  thanks to assumption  $(H_2)$ , i. e.  $\mathcal{Z}_e \cap \mathcal{P}_m = \emptyset$ , it is not a zero nor a pole  $\omega\mu(\cdot)$ . Thus,  $g$  can indeed be extended analytically for  $\omega = z$  with

$$g(z) = (\omega\varepsilon)'(z) z\mu(z).$$

Thus, one has  $A_z = \frac{1}{g(z)} = \frac{1}{z\mu(z)(\omega\varepsilon)'(z)}$  that is (2.5.27).

We now show now that  $\text{Im } A_z < 0$ . As  $z \in \mathcal{Z}_e$ , one observes on one hand (see [Remark 2.5.4](#)) that  $\mathcal{Z}_s = \mathcal{Z}_e$  and that all the  $\alpha_{e,j}$  vanish. Thus,  $\omega\varepsilon(\omega)$  is real-valued on the real axis (outside the poles  $\mathcal{P}_e$  of  $\varepsilon(\omega)$ ) and from expression (2.1.5), one deduces easily that  $(\omega\varepsilon)'(\omega) > 0$  on  $\mathbb{R} \setminus \mathcal{P}_e$ . In particular, as  $z \in \mathbb{R} \setminus \mathcal{P}_e$ , one has

$$(\omega\varepsilon)'(z) > 0.$$

On the other hand, as all the  $\alpha_{e,j}$  vanish, one knows by the weak dissipation condition (2.1.10) that at least one coefficient  $\alpha_{m,\ell} > 0$ . Therefore by the formula (2.3.17),  $\text{Im}(z\mu(z)) > 0$ . Thus, one deduces that

$$\text{Im}((z\mu(z))^{-1}) = -\text{Im}(z\mu(z))/|z\mu(z)|^2 < 0.$$

Together with  $(\omega\varepsilon)'(z) > 0$ , this implies that  $\text{Im}(A_z)$  is negative.  $\square$

Combining [Lemma 2.5.8](#) and [Lemma 2.5.9](#) (formula (2.5.26) and (2.5.27)) and proceeding as in the proof of [Lemma 2.4.5](#) gives the following estimate for  $\mathcal{U}_{z,s}(\mathbf{k}, t)$ .

**Lemma 2.5.10**

If  $\mathcal{Z}_s \neq \emptyset$ , then it exists  $k_- > 0$  and  $C > 0$  such that the function  $\mathcal{U}_{z,s}(\mathbf{k}, t)$  defined by defined by (2.5.9) and (2.5.10), satisfies the following estimate

$$|\mathcal{U}_{z,s}(\mathbf{k}, t)| \lesssim e^{-C|\mathbf{k}|^2 t} |\mathcal{U}_0(\mathbf{k})|, \quad \forall t \geq 0 \quad \text{and} \quad 0 < |\mathbf{k}| \leq k_-. \quad (2.5.28)$$

### 2.5.3.4 Estimates of $\mathbb{U}_{z,-}(\mathbf{k}, t)$ for $0 < |\mathbf{k}| \leq k_-$

As announced in [Section 2.5.3.1](#), since we simply want to obtain a “rough” exponential decay estimate for  $\mathbb{U}_{z,-}(\mathbf{k}, t)$ . We give a direct proof of it using Riesz-Dunford functional calculus.

#### Lemma 2.5.11

There exists  $\delta > 0$  and  $k_- > 0$  such that  $\mathbb{U}_{z,-}(\mathbf{k}, t)$ , defined by [\(2.5.9\)](#) and [\(2.5.10\)](#), satisfies

$$|\mathbb{U}_{z,-}(\mathbf{k}, t)| \lesssim e^{-\delta t} |\mathbb{U}_0(\mathbf{k})|, \quad \forall t \geq 0, \quad \forall 0 < |\mathbf{k}| \leq k_-. \quad (2.5.29)$$

*Proof.* The proof is similar to the one of [Lemma 2.4.13](#) and even shorter since  $|\mathbf{k}|$  is bounded here. We introduce a fixed simple closed contour  $\Gamma$ , included in  $\mathbb{C}^-$  such that all the zeros of  $\mathcal{Z}_-$  lie inside  $\Gamma$  (see [Figure 2.4](#)). We choose also  $\Gamma$  such that  $\Gamma \cap \mathcal{P} = \emptyset$ . We denote by

$$\delta = \min\{-\operatorname{Im}(\omega), \omega \in \Gamma\} > 0 \quad (2.5.30)$$

the distance from  $\Gamma$  to the real axis.

For  $|\mathbf{k}|$  positive and small enough,  $\mathbb{U}_{z,-}(\mathbf{k}, t)$  is on one hand well-defined by [\(2.5.9\)](#) and [\(2.5.10\)](#) and on the other hand, by [Proposition 2.5.1](#),  $\Gamma$  encloses all eigenvalues  $\omega_{p,n}(|\mathbf{k}|)$  for  $z \in \mathcal{Z}_-$  and  $n \in \{1, \dots, \mathbf{m}_z\}$  but no other elements of the spectrum of  $\mathbb{A}_{|\mathbf{k}|}$ . Thus by the Riesz-Dunford functional calculus, one gets that it exists  $k_- > 0$  such that

$$\mathbb{U}_{z,-}(\mathbf{k}, t) = -\frac{\mathcal{R}_{\mathbf{k}}^*}{2i\pi} \int_{\Gamma} e^{i\omega t} R_{|\mathbf{k}|}(\omega) \mathcal{R}_{\mathbf{k}} \mathbb{U}_0(\mathbf{k}) d\omega, \quad \text{for } 0 < |\mathbf{k}| \leq k_-, \quad (2.5.31)$$

where the resolvent  $R_{\mathbf{k}}(\omega)$  is well-defined by [Proposition 2.3.2](#) for  $(|\mathbf{k}|, \omega) \in K = [0, k_-] \times \Gamma$ . Moreover, the function  $(|\mathbf{k}|, \omega) \mapsto R_{\mathbf{k}}(\omega)$ , valued in  $\mathcal{L}(\mathbb{C}_{\perp})$ , is continuous on the compact  $K$ . Then, using the fact that  $\mathcal{R}_{\mathbf{k}}$  is unitary and the definition [\(2.5.30\)](#) of  $\delta$  (which implies that  $|e^{i\omega t}| \leq e^{-\delta t}$  on  $\Gamma$ ), it follows from [\(2.5.31\)](#) that

$$|\mathbb{U}_{z,-}(\mathbf{k}, t)| \leq C e^{-\delta t} |\mathbb{U}_0(\mathbf{k})|, \quad \forall |\mathbf{k}| \in (0, k_-] \quad \text{where } C = \max_{(|\mathbf{k}|, \omega) \in K} \|R_{|\mathbf{k}|}(\omega)\|.$$

□

### 2.5.3.5 The global estimate

#### Theorem 2.5.12

It exists  $k_- > 0$ ,  $C, \tilde{C} > 0$  such that for  $0 < |\mathbf{k}| \leq k_-$ , the spatial Fourier components  $|\mathbb{U}(\mathbf{k}, t)|$  of the solution of [\(2.1.12\)](#) with initial condition  $\mathbf{U}_0 \in \mathcal{H}_{\perp}$  satisfy:

$$|\mathbb{U}(\mathbf{k}, t)| \leq \tilde{C} e^{-C|\mathbf{k}|^2 t} |\mathbb{U}_0(\mathbf{k})|, \quad \forall t \geq 0. \quad (2.5.32)$$

*Proof.* The proof is similar to the one of [Theorem 2.4.14](#) using [Lemma 2.5.7](#), [Lemma 2.5.10](#) and [Lemma 2.5.11](#). □

We prove in the following result that the estimates of [Theorem 2.5.12](#) is optimal for an infinite family of well chosen initial conditions  $\mathbf{U}_0 \in \mathcal{H}_{\perp, \text{LF}}^p$  for any fixed  $p \geq 0$ .

**Theorem 2.5.13**

Let  $p \geq 0$  and  $k_- > 0$  and  $\phi : \mathbb{R}^+ \mapsto \mathbb{R}$  be any measurable and bounded function satisfying

$$\text{supp } \phi \subset [0, k_-] \text{ and } 0 < |\phi(|\mathbf{k}|)| \lesssim |\mathbf{k}|^p \text{ for } |\mathbf{k}| \leq k_-. \quad (2.5.33)$$

If the initial condition  $\mathbf{U}_0$  of (2.1.12) is defined (for  $k_-$  small enough) via its Fourier transform:

$$\mathbb{U}_0(\mathbf{k}) = \mathcal{F}(\mathbf{U}_0)(\mathbf{k}) = \phi(|\mathbf{k}|) \mathcal{R}_{\mathbf{k}}^* \frac{\mathcal{V}_{|\mathbf{k}|}(\omega_{0,1}(|\mathbf{k}|)) \mathbf{e}_1}{|\mathcal{V}_{|\mathbf{k}|}(\omega_{0,1}(|\mathbf{k}|)) \mathbf{e}_1|}, \quad \forall \mathbf{k} \in \mathbb{R}^{3,*}, \quad (2.5.34)$$

then  $\mathbf{U}_0 \in \mathcal{H}_{\perp, \text{LF}}^p$  and  $\exists C, \tilde{C} > 0$  such that the associated solution  $\mathbf{U}$  of (2.1.12) satisfy

$$\tilde{C} e^{Ct|\mathbf{k}|^2} |\mathbb{U}_0(\mathbf{k})| \leq |\mathbb{U}(\mathbf{k}, t)|, \quad \forall t \geq 0 \text{ and } \forall \mathbf{k} \in \mathbb{R}^{3,*}. \quad (2.5.35)$$

In other words, the estimate (2.5.32) is optimal for an infinite family of solutions.

*Proof.* Using (2.5.33) and (2.5.34), it is clear that  $\mathbf{U}_0 \in \mathcal{H}_{\perp, \text{LF}}^p$ . The rest of the proof uses the following asymptotic expansion of  $\text{Im } \omega_{0,1}(|\mathbf{k}|)$  (given by Lemma 2.5.6):

$$\text{Im } \omega_{0,1}(|\mathbf{k}|) = \text{Im} \left( -(\varepsilon\mu)'(0) |\mathbf{k}|^2 + o(|\mathbf{k}|^2) \right), \text{ as } |\mathbf{k}| \rightarrow 0 \text{ with } \text{Im} \left( -(\varepsilon\mu)'(0) \right) < 0$$

and is similar to the proof of Theorem 2.4.15. Therefore, the details are left to the reader.  $\square$

## 2.6 Estimates of “mid frequencies” components of the solution

The goal is to prove (see Theorem 2.6.2) that for intermediate frequency components, i. e., for  $0 < k_- \leq |\mathbf{k}| \leq k_+$ , the Fourier components of the solution  $\mathbb{U}(\mathbf{k}, t)$  decay exponentially with a uniform exponent depending only on the compact set  $K = [k_-, k_+]$ . The proof is based on three key ingredients: the Dunford decomposition in linear algebra, the following perturbation Lemma and finally a compacity argument.

**Lemma 2.6.1**

Let  $\mathbb{M} \in \mathcal{L}(\mathbb{C}_{\perp}^N)$ . We assume that there exists two constants  $C, \alpha > 0$  such that

$$\|e^{t\mathbb{M}}\| \leq C e^{-\alpha t}, \quad \forall t \geq 0.$$

Then, for any perturbation  $\Delta \in \mathcal{L}(\mathbb{C}_{\perp}^N)$ , one has the following estimate:

$$\|e^{t(\mathbb{M}+\Delta)}\| \leq C e^{(-\alpha+C\|\Delta\|)t}, \quad \forall t \geq 0.$$

*Proof.* The proof, based on the Duhamel formula and the Grönwall’s lemma, is done e.g. in lemma 1.6 page 97 of [12] or in the Proposition 4.2.18 page 406 of [47].  $\square$

**Theorem 2.6.2**

Let  $K = [k_-, k_+]$  be a compact interval of  $\mathbb{R}^{+,*}$ , then there exist two constants  $C > 0$  and  $\beta > 0$  (depending only on  $K$ ) such that the spatial Fourier components  $|\mathcal{U}(\mathbf{k}, t)|$  of the solution of (2.1.12) with initial condition  $\mathcal{U}_0 \in \mathcal{H}_\perp$  satisfy

$$|\mathcal{U}(\mathbf{k}, t)| \leq C e^{-\beta t} |\mathcal{U}_0(\mathbf{k})|, \quad \forall t \geq 0 \text{ and } \forall \mathbf{k} \in \mathbb{R}^3 \mid |\mathbf{k}| \in K. \quad (2.6.1)$$

*Proof.* Let  $k \in K$  be fixed. By virtue of (2.3.14),  $\sigma(\mathbb{A}_{k,\perp})$  is included in the lower half-plane  $\mathbb{C}^-$ , thus it follows from the Dunford Decomposition of  $\mathbb{A}_{k,\perp}$  (see e.g. Corollary 2.26 page 106 of [47]) that it exists  $\alpha_0 > 0$  and  $C_k > 0$  such that

$$\|e^{-i\mathbb{A}_{k,\perp} t}\| \leq C_k e^{-\alpha_k t}, \quad \forall t \geq 0. \quad (2.6.2)$$

As the function  $|\mathbf{k}| \mapsto -i\mathbb{A}_{|\mathbf{k}|,\perp}$  is clearly continuous from  $\mathbb{R}^{+,*}$  to  $\mathcal{L}(\mathbf{C}_\perp^N)$ , it exists an open interval  $(k - \eta_k, k + \eta_k)$  such that

$$\| -i(\mathbb{A}_{|\mathbf{k}|,\perp} - \mathbb{A}_{k,\perp}) \| < \frac{\alpha_k}{2C_k}, \quad \forall |\mathbf{k}| \in (k - \eta_k, k + \eta_k) \cap K. \quad (2.6.3)$$

Hence, applying Lemma 2.6.1 with

$$\mathbb{M} = -i\mathbb{A}_{k,\perp} \quad \text{and} \quad \Delta = -i(\mathbb{A}_{|\mathbf{k}|,\perp} - \mathbb{A}_{k,\perp})$$

yields the following inequality for all  $t \geq 0$  and  $\forall |\mathbf{k}| \in (k - \eta_k, k + \eta_k) \cap K$ :

$$\|e^{-i\mathbb{A}_{|\mathbf{k}|,\perp} t}\| \leq C_k e^{-\beta_k t} \quad \text{with} \quad \beta_k = \alpha_k - C_k \|\Delta\|. \quad (2.6.4)$$

Thus, one deduces immediately from (2.6.3) and (2.6.4) that

$$\|e^{-i\mathbb{A}_{|\mathbf{k}|,\perp} t}\| \leq C_k e^{-\frac{\alpha_k}{2} t}, \quad \forall t \geq 0 \text{ and } \forall |\mathbf{k}| \in (k - \eta_k, k + \eta_k) \cap K. \quad (2.6.5)$$

As the compact  $K \subset \bigcup_{k \in K} (k - \eta_k, k + \eta_k)$ , it exists  $N > 0$  and  $k_1, k_2, \dots, k_N \in K$  such that

$$K \subset \bigcup_{n \leq N} (k_n - \eta_{k_n}, k_n + \eta_{k_n}). \quad (2.6.6)$$

With  $\alpha_{k_i} > 0$  and  $C_{k_i} > 0$  given by (2.6.2) for  $k = k_i$ , we define

$$C = \max_{i=1,\dots,N} C_{k_i} > 0 \quad \text{and} \quad \alpha = \min_{i=1,\dots,N} \alpha_{k_i} > 0.$$

Then, combining (2.6.5) and (2.6.6) yields

$$\|e^{-i\mathbb{A}_{|\mathbf{k}|,\perp} t}\| \leq C e^{-\frac{\alpha}{2} t}, \quad \forall t \geq 0 \text{ and } \forall |\mathbf{k}| \in K.$$

As the operator  $\mathcal{R}_\mathbf{k}$  is unitary, one finally deduces, with (2.2.12), the estimate (2.6.1).  $\square$

## 2.7 Proof of the main Theorems of the Chapter 2

### 2.7.1 Proof of Theorem 2.1.7 (decay rate estimates)

This section constitutes the last step of the proof of Theorem 2.1.7. For this final stage, we follow the approach developed in the proofs of Theorem 1.2.4 and Theorem 1.3.4 of Chapter 1. For the sake of readability, we recall here the argument, which rely on Plancherel's identity and the three decay estimates proved for high, low and intermediate frequencies respectively in Theorem 2.4.14, Theorem 2.5.12 and Theorem 2.6.2.

Let  $k_-$  and  $k_+$  be two fixed positive real numbers satisfying  $k_- < k_+$  such that the estimates (2.4.96) and (2.5.32) of Theorem 2.4.14 and Theorem 2.5.12 holds.

Using the Plancherel identity, any solution of (2.1.12) with initial condition  $\mathbf{U}_0 \in \mathcal{H}_\perp$  satisfies

$$\|\mathbf{U}(t)\|_{\mathcal{H}}^2 = \int_{|\mathbf{k}| < k_-} |\mathbb{U}(\mathbf{k}, t)|^2 d\mathbf{k} + \int_{k_- \leq |\mathbf{k}| \leq k_+} |\mathbb{U}(\mathbf{k}, t)|^2 d\mathbf{k} + \int_{k_+ < |\mathbf{k}|} |\mathbb{U}(\mathbf{k}, t)|^2 d\mathbf{k}. \quad (2.7.1)$$

#### Step 1: proof of the convergence result (2.1.33)

In a non critical configuration, using the low frequency estimate (2.5.32), the intermediate frequency estimate (2.6.2) and the high frequency estimate (2.4.96) of  $\mathbb{U}(\mathbf{k}, t)$ , one gets there exist three constants  $C_1 > 0$ ,  $C_2 > 0$  and  $C_3 > 0$  (independent of  $\mathbf{U}$ ) such that

$$\|\mathbf{U}(t)\|_{\mathcal{H}}^2 \lesssim \int_{|\mathbf{k}| < k_-} e^{-C_1 |\mathbf{k}|^2 t} |\mathbb{U}_0(\mathbf{k})|^2 d\mathbf{k} + e^{-C_2 t} \|\mathbb{U}_0\|_{\mathcal{H}}^2 + \int_{k_+ < |\mathbf{k}|} e^{-C_3 |\mathbf{k}|^{-2} t} |\mathbb{U}_0(\mathbf{k})|^2 d\mathbf{k}. \quad (2.7.2)$$

When  $t \rightarrow +\infty$ , the second term in (2.7.2) converges exponentially to 0, whereas the first and third terms tends to 0 by the Lebesgue's dominated convergence Theorem.

In a non critical configuration, one obtains (2.1.33) by the same reasoning only after replacing the high frequency estimate (2.4.96) by (2.4.97) (thus it consists to substitute the factor  $e^{-C_3 |\mathbf{k}|^{-2} t}$  by  $e^{-C_3 |\mathbf{k}|^{-4} t}$  in the second integral of (2.7.2)).

#### Step 2: Estimate of the low frequency term in (2.7.2)

We assume now that the initial condition  $\mathbf{U}_0 \in \mathcal{H}_\perp \cap \mathcal{L}_p^N$  for some integer  $p \geq 0$ .

Thus, by definition on the  $\mathcal{L}_p^N$  norm (see (2.1.23)),  $|\mathbb{U}_0(\mathbf{k})| \leq |\mathbf{k}|^p \|\mathbf{U}_0\|_{\mathcal{L}_p^N}$  for a.e.  $\mathbf{k} \in \mathbb{R}^3$  and consequently

$$\int_{|\mathbf{k}| < k_-} e^{-C_1 |\mathbf{k}|^2 t} |\mathbb{U}_0(\mathbf{k})|^2 d\mathbf{k} \leq \left( \int_{\mathbb{R}^3} e^{-C_1 |\mathbf{k}|^2 t} |\mathbf{k}|^{2p} d\mathbf{k} \right) \|\mathbf{U}_0\|_{\mathcal{L}_p^N}^2. \quad (2.7.3)$$

Since, with the change of variable  $\xi = \sqrt{C_1 t} \mathbf{k}$ ,

$$\int_{\mathbb{R}^3} |\mathbf{k}|^{2p} e^{-C_1 |\mathbf{k}|^2 t} d\mathbf{k} = \frac{1}{(C_1 t)^{p+\frac{3}{2}}} \int_{\mathbb{R}^3} |\xi|^{2p} e^{-|\xi|^2} d\xi \equiv \frac{C(p)}{t^{p+\frac{3}{2}}},$$

one concludes from (2.7.3) that

$$\int_{|\mathbf{k}| < k_-} e^{-C_1 |\mathbf{k}|^2 t} |\mathbb{U}_0(\mathbf{k})|^2 d\mathbf{k} \leq \frac{C(p)}{t^{p+\frac{3}{2}}} \|\mathbf{U}_0\|_{\mathcal{L}_p^N}^2. \quad (2.7.4)$$

**Step 3: Estimate of the high frequency term in (2.7.2)**
**In a non critical configuration:**

We assume here that the Maxwell's system is in a non critical configuration and that the initial condition  $\mathbf{U}_0 \in \mathcal{H}_\perp \cap \mathbf{H}^m(\mathbb{R}^3)^N$  for some  $m > 0$ . We estimate here the third term of (2.7.2) corresponding to the high frequency contribution. To this aim, we denote by  $\langle \mathbf{k} \rangle$  the quantity:  $\langle \mathbf{k} \rangle := (1 + |\mathbf{k}|^2)^{1/2}$ . Then, using the fact that  $e^{-C_3|\mathbf{k}|^{-2}t} \leq e^{-C_3\langle \mathbf{k} \rangle^{-2}t}$ , we rewrite this third term as follows (we simply make appear artificially the factor  $\langle \mathbf{k} \rangle^m/t^m$ )

$$\begin{aligned} \int_{k_+ < |\mathbf{k}|} e^{-C_3|\mathbf{k}|^{-2}t} |\mathcal{U}_0(\mathbf{k})|^2 d\mathbf{k} &\leq t^{-m} \int_{k_+ < |\mathbf{k}|} \langle \mathbf{k} \rangle^{2m} |\mathcal{U}_0(\mathbf{k})|^2 \left( \frac{t}{\langle \mathbf{k} \rangle^2} \right)^m e^{-C_3\langle \mathbf{k} \rangle^{-2}t} d\mathbf{k} \\ &\leq t^{-m} \int_{\mathbb{R}^3} \langle \mathbf{k} \rangle^{2m} |\mathcal{U}_0(\mathbf{k})|^2 F_m(t/\langle \mathbf{k} \rangle^2) d\mathbf{k}, \end{aligned}$$

where we have set  $F_m(r) := r^m e^{-C_3 r}$ ,  $r \geq 0$  which satisfies:  $\sup_{r \geq 0} F_m(r) = \tilde{C}_m := (m/(C_3 e))^m$ .

Thus, by Fourier characterization of Sobolev norms:

$$\int_{k_+ < |\mathbf{k}|} e^{-C_3|\mathbf{k}|^{-2}t} |\mathcal{U}_0(\mathbf{k})|^2 d\mathbf{k} \lesssim \tilde{C}_m t^{-m} \|\mathbf{U}_0\|_{\mathbf{H}^m(\mathbb{R}^3)^N}^2, \quad \forall t > 0. \quad (2.7.5)$$

**In a critical configuration:**

We assume here that the Maxwell's system is in a critical configuration with again an initial condition  $\mathbf{U}_0 \in \mathcal{H}_\perp \cap \mathbf{H}^m(\mathbb{R}^3)^N$ . The proof is similar to the one of the estimate (2.7.5). We simply enlightens the differences. In a critical configuration, one has to replace the estimate (2.4.96) by (2.4.97). Thus, in (2.7.2), one has to substitute the factor  $e^{-C_3|\mathbf{k}|^{-2}t}$  by  $e^{-C_3|\mathbf{k}|^{-4}t}$  in the second integral of (2.7.2). Then, one has

$$\begin{aligned} \int_{k_+ < |\mathbf{k}|} e^{-C_3|\mathbf{k}|^{-4}t} |\mathcal{U}_0(\mathbf{k})|^2 d\mathbf{k} &\leq t^{-m/2} \int_{k_+ < |\mathbf{k}|} \langle \mathbf{k} \rangle^{2m} |\mathcal{U}_0(\mathbf{k})|^2 \left( \frac{t}{\langle \mathbf{k} \rangle^4} \right)^{m/2} e^{-C_3\langle \mathbf{k} \rangle^{-4}t} d\mathbf{k} \\ &\leq t^{-m/2} \int_{\mathbb{R}^3} \langle \mathbf{k} \rangle^{2m} |\mathcal{U}_0(\mathbf{k})|^2 F_{m/2}(t/\langle \mathbf{k} \rangle^4) d\mathbf{k}. \end{aligned}$$

Setting  $\tilde{C}_{m/2} := \sup_{r \geq 0} F_{m/2}(r)$ , this yields, by definition (2.1.24) of the  $\mathbf{H}^m(\mathbb{R}^3)^N$  norm,

$$\int_{k_+ < |\mathbf{k}|} e^{-C_3|\mathbf{k}|^{-2}t} |\mathcal{U}_0(\mathbf{k})|^2 d\mathbf{k} \leq \tilde{C}_{m/2} t^{-m/2} \|\mathbf{U}_0\|_{\mathbf{H}^m(\mathbb{R}^3)^N}^2, \quad \forall t > 0. \quad (2.7.6)$$

**Step 4: Proof of the estimate (2.1.34) and (2.1.35).**

In the non critical case (resp. the critical case), it suffices to substitute (2.7.4) and (2.7.5) (resp. (2.7.4) and (2.7.6)) into (2.7.2) to obtain (2.1.34) (resp. (2.1.35)).

**2.7.2 Proof of Theorem 2.1.10 (optimality decay rate estimates)**
**2.7.2.1 Optimality of the high frequency polynomial decay rate**
**Determination of  $\gamma_m^{\text{HF}}$  in a non-critical configuration**

We assume first that the Maxwell system is in a non-critical configuration.

First, we emphasize that for any initial conditions  $\mathbf{U}_0 \in \mathcal{H}_{\perp, \text{HF}}^m$  (see (2.1.37) for the definition of  $\mathcal{H}_{\perp, \text{HF}}^m$ ) which does not contain any low frequency Fourier components (since as  $\mathbf{U}_0 \in \mathcal{H}_{\perp, \text{HF}}^m$ ,  $\text{supp}(\mathbf{U}_0) \subset \mathbb{R}^3 \setminus B(0, k_+)$ ), we have the upper-bound given by (2.7.2) and (2.7.5), namely

$$\|\mathbf{U}(t)\|_{\mathcal{H}}^2 \lesssim \int_{k_+ < |\mathbf{k}|} e^{-C_3 |\mathbf{k}|^{-4} t} |\mathbf{U}_0(\mathbf{k})|^2 d\mathbf{k} \lesssim \frac{\|\mathbf{U}_0\|_{\mathbf{H}^m(\mathbb{R}^3)^N}^2}{t^m}, \quad \forall t > 0. \quad (2.7.7)$$

Clearly, (2.7.7) implies that  $\gamma_m^{\text{HF}}$  (defined by (2.1.39)) exists and satisfies  $\gamma_m^{\text{HF}} \geq m$ . To show that  $\gamma_m^{\text{HF}} = m$ , we construct for any  $\varepsilon > 0$  an initial condition  $\mathbf{U}_{0, \varepsilon} \in \mathcal{H}_{\perp, \text{HF}}^m$  such that the associated solution of (2.1.12) satisfies

$$\|\mathbf{U}_{\varepsilon}(t)\|_{\mathcal{H}}^2 \geq \frac{C}{t^{m+\varepsilon}}, \quad \text{for some } C > 0, \quad \forall t \geq 1.$$

To this aim, according to Theorem 2.4.15, we choose  $\mathbf{U}_{0, \varepsilon}$  in the form (2.4.99), more precisely

$$\begin{cases} \mathbf{U}_{0, \varepsilon}(\mathbf{k}) = \phi_{\varepsilon}(|\mathbf{k}|) \mathcal{R}_{\mathbf{k}}^* \frac{\mathcal{V}_{|\mathbf{k}|}(\omega_{+\infty}(|\mathbf{k}|)) \mathbf{e}_1}{|\mathcal{V}_{|\mathbf{k}|}(\omega_{+\infty}(|\mathbf{k}|)) \mathbf{e}_1}, \\ \phi_{\varepsilon}(|\mathbf{k}|) = (1 + |\mathbf{k}|^2)^{-(\frac{3}{4} + \frac{m}{2} + \frac{\varepsilon}{2})} \text{ if } |\mathbf{k}| \geq k_+, = 0 \text{ otherwise,} \end{cases} \quad (2.7.8)$$

where the exponent  $\frac{3}{4} + \frac{m}{2} + \frac{\varepsilon}{2}$  has been chosen just above in order to ensure the  $H^m$  regularity of  $\mathbf{U}_{0, \varepsilon}$  (see (2.5.33)). As moreover  $\mathbf{U}_{0, \varepsilon}(\mathbf{k})$  is supported in  $\{|\mathbf{k}| \geq k_+\}$ ,  $\mathbf{U}_{0, \varepsilon} \in \mathcal{H}_{\perp, \text{HF}}^m$ .

Therefore, applying the inequality (2.4.100) of Theorem 2.4.15, we get, as  $|\mathbf{U}_{0, \varepsilon}(\mathbf{k})| = \phi_{\varepsilon}(|\mathbf{k}|)$ ,

$$e^{-\frac{2Ct}{|\mathbf{k}|^2}} (1 + |\mathbf{k}|^2)^{-(\frac{3}{2} + m + \varepsilon)} \lesssim |\mathbf{U}_{\varepsilon}(\mathbf{k}, t)|^2, \quad \text{for a.e. } \mathbf{k} \in \mathbb{R}^3 \mid |\mathbf{k}| \geq k_+ \text{ and } \forall t \geq 0. \quad (2.7.9)$$

By Plancherel identity,  $\|\mathbf{U}_{\varepsilon}(t)\|_{\mathcal{H}}^2 = \int_{k_+}^{\infty} \|\mathbf{U}_{\varepsilon}(\mathbf{k}, t)\|^2 d\mathbf{k}$ . Therefore, using (2.7.9), one gets

$$I_{\varepsilon}(t) = \int_{|\mathbf{k}| \geq k_+} e^{-\frac{2Ct}{|\mathbf{k}|^2}} (1 + |\mathbf{k}|^2)^{-(3/2 + m + \varepsilon)} d\mathbf{k} \lesssim \|\mathbf{U}_{\varepsilon}(t)\|_{\mathcal{H}}^2. \quad (2.7.10)$$

that is to say, the integrand in (2.7.10) only depends on  $|\mathbf{k}|$ ,

$$I_{\varepsilon}(t) = 4\pi \int_{k_+}^{+\infty} e^{-\frac{2Ct}{|\mathbf{k}|^2}} (1 + |\mathbf{k}|^2)^{-(3/2 + m + \varepsilon)} |\mathbf{k}|^2 d|\mathbf{k}|.$$

Using the change of variable  $\xi = \sqrt{t}/|\mathbf{k}|$  in  $I_{\varepsilon}(t)$  yields, with  $|\mathbf{k}|^2 d|\mathbf{k}| = -t^{\frac{3}{2}} \xi^{-4} d\xi$

$$I_{\varepsilon}(t) = 4\pi t^{\frac{3}{2}} \int_0^{\frac{\sqrt{t}}{k_+}} e^{-2C\xi^2} \xi^{3+2m+2\varepsilon} (\xi^2 + t)^{-(3/2 + m + \varepsilon)} \frac{d\xi}{\xi^4}.$$

Then, using  $\xi^2 \leq t/k_+^2$  and thus that  $\xi^2 + t \leq (1 + k_+^2) t$  on the domain of integration gives

$$I_{\varepsilon}(t) \geq \frac{C_m^{\varepsilon}(t)}{t^{m+\varepsilon}}, \quad C_m^{\varepsilon}(t) := 4\pi (1 + k_+^2)^{-(m + \frac{3}{2} + \varepsilon)} \int_0^{\frac{\sqrt{t}}{k_+}} \frac{e^{-2C\xi^2}}{\xi^{1-2m-2\varepsilon}} d\xi.$$



As  $C_m^\varepsilon(t) > 0$  is a strictly increasing, restricting ourselves to  $t \geq 1$ , (2.7.10) leads to

$$\forall t \geq 1, \quad \frac{C_m^\varepsilon(1)}{t^{m+\varepsilon}} \leq I_\varepsilon(t) \lesssim \|\mathbf{U}_\varepsilon(t)\|_{\mathcal{H}}^2.$$

Thus,  $\gamma_m^{HF} \leq m + \varepsilon$ , for all  $\varepsilon > 0$  and this yields  $\gamma_m^{HF} \leq m$ . Thus, one concludes that  $\gamma_m^{HF} = m$ .

### Determination of $\gamma_m^{HF}$ in a critical configuration

We assume now that the Maxwell system is in a critical configuration.

The proof is very similar to the one for the non critical case and we shall only point out the differences. First the inequality (2.7.6) shows that  $\gamma_m^{HF} \geq m/2$ .

To prove the reverse inequality, we proceed as for the non critical case. The main difference lies in the choice of the initial data  $\mathbf{U}_{0,\varepsilon}$  which is now chosen as in (2.4.102), more precisely

$$\mathbf{U}_{0,\varepsilon}(\mathbf{k}) = \mathcal{F}(\mathbf{U}_0)(\mathbf{k}) = \phi_\varepsilon(|\mathbf{k}|) \mathcal{R}_{\mathbf{k}}^* \frac{\mathcal{V}_{|\mathbf{k}|}(\omega_p(|\mathbf{k}|)) \mathbf{e}_1}{|\mathcal{V}_{|\mathbf{k}|}(\omega_p(|\mathbf{k}|)) \mathbf{e}_1|}, \quad \forall \mathbf{k} \in \mathbb{R}^{3,*}, \quad (2.7.11)$$

for some  $p \in \mathcal{P}_s$  such that  $\omega_p(|\mathbf{k}|)$  satisfies (2.4.101) (see again Theorem 2.4.15) and the function  $\phi_\varepsilon$  defined in (2.7.15). In that case, we have to apply the inequality (2.4.103), instead of (2.4.100), which leads to (instead of (2.7.9))

$$e^{-\frac{2Ct}{|\mathbf{k}|^4}} (1 + |\mathbf{k}|^2)^{-(\frac{3}{2}+m+\varepsilon)} \lesssim |\mathbf{U}_\varepsilon(\mathbf{k}, t)|^2, \quad \text{for a.e. } \mathbf{k} \in \mathbb{R}^3 \mid |\mathbf{k}| \geq k_+ \text{ and } \forall t \geq 0. \quad (2.7.12)$$

the main difference with being (2.7.9) being that  $|\mathbf{k}|^2$  is replaced by  $|\mathbf{k}|^4$  in the exponential.

The rest of the calculations follows the same lines than in the non critical case modulo trivial adaptations : for instance the change of variable  $\xi = \sqrt{t}/|\mathbf{k}|$  is replaced by  $\xi = \sqrt{t}/|\mathbf{k}|^2$ . The remaining details are left to the reader.

#### 2.7.2.2 Optimality of the low frequency polynomial decay rate

We compute here the exponent  $\gamma_p^{LF}$  (defined by (2.1.40)) for  $p \geq 0$ . The approach is similar to the one used in Section 2.7.2.1 to compute  $\gamma_m^{HF}$ .

First, we underline that for any initial condition  $\mathbf{U}_0 \in \mathcal{H}_{\perp,LF}^p$  (see (2.1.38) for the definition of  $\mathcal{H}_{\perp,LF}^p$ ) which does not contain any high frequency Fourier components (since the support of  $\mathbf{U}_0$  is included in  $\overline{B(0, k_-)}$ ), we have the upper-bound (2.7.2) and (2.7.13), namely,

$$\|\mathbf{U}(t)\|_{\mathcal{H}}^2 \lesssim \int_{|\mathbf{k}| < k_-} e^{-C_1 |\mathbf{k}|^2 t} |\mathbf{U}_0(\mathbf{k})|^2 d\mathbf{k} \lesssim \frac{C(p)}{t^{p+\frac{3}{2}}} \|\mathbf{U}_0\|_{\mathcal{L}_p^N}^2. \quad (2.7.13)$$

Clearly, (2.7.13) implies that  $\gamma_p^{LF}$  exists and satisfies  $\gamma_p^{LF} \geq p + 3/2$ . To show that  $\gamma_p^{LF} \leq p + 3/2$ , we construct an initial condition  $\mathbf{U}_0 \in \mathcal{H}_{\perp,LF}^p$  such that

$$\|\mathbf{U}(t)\|_{\mathcal{H}}^2 \geq \frac{C(p)}{t^{p+\frac{3}{2}}}, \quad \text{for some } C > 0, \quad \forall t \geq 1. \quad (2.7.14)$$

To this aim, according to [Theorem 2.5.13](#), we choose  $\mathbf{U}_0$  in the form [\(2.5.34\)](#), more precisely

$$\begin{cases} \mathbb{U}_0(\mathbf{k}) = \phi(|\mathbf{k}|) \mathcal{R}_{\mathbf{k}}^* \frac{\mathcal{V}_{|\mathbf{k}|}(\omega_{0,1}(|\mathbf{k}|)) \mathbf{e}_1}{|\mathcal{V}_{|\mathbf{k}|}(\omega_{0,1}(|\mathbf{k}|)) \mathbf{e}_1}, \\ \phi(|\mathbf{k}|) = |\mathbf{k}|^p \text{ if } |\mathbf{k}| \leq k_-, = 0 \text{ otherwise} \end{cases} \quad (2.7.15)$$

Therefore, applying the inequality [\(2.5.32\)](#) of [Theorem 2.5.13](#), we get as  $|\mathbb{U}_0(\mathbf{k})| = \phi(|\mathbf{k}|)$ :

$$e^{-2C|\mathbf{k}|^2 t} |\mathbf{k}|^{2p} \lesssim |\mathbb{U}(\mathbf{k}, t)|^2, \text{ for a.e. } \mathbf{k} \in \mathbb{R}^3 \mid |\mathbf{k}| \leq k_- \text{ and } \forall t \geq 0. \quad (2.7.16)$$

By virtue of Plancherel identity:  $\|\mathbf{U}(t)\|_{\mathcal{H}}^2 = \int_0^{k_-} \|\mathbb{U}(\mathbf{k}, t)\|^2 d\mathbf{k}$ , it follows from [\(2.7.16\)](#) that

$$\int_0^{k_-} e^{-2C|\mathbf{k}|^2 t} |\mathbf{k}|^{2p} d\mathbf{k} \lesssim \|\mathbf{U}(t)\|_{\mathcal{H}}^2. \quad (2.7.17)$$

Hence, using the change of variable  $\xi = \sqrt{2Ct} \mathbf{k}$ , one obtains

$$\frac{1}{(2Ct)^{p+\frac{3}{2}}} \int_0^{\sqrt{t}k_-} |\xi|^{2p} e^{-|\xi|^2} d\xi \lesssim \|\mathbf{U}(t)\|_{\mathcal{H}}^2.$$

It leads to [\(2.7.14\)](#) and more precisely to

$$\frac{C(k_-, p)}{t^{p+\frac{3}{2}}} \lesssim \|\mathbf{U}(t)\|_{\mathcal{H}}^2 \quad \text{with } C(k_-, p) = \frac{1}{(2C)^{p+\frac{3}{2}}} \int_0^{k_-} |\xi|^{2p} e^{-|\xi|^2} d\xi, \quad \forall t \geq 1.$$

Thus,  $\gamma_p^{LF} \leq p + 3/2$  and one concludes that  $\gamma_p^{LF} = p + 3/2$ .

## Appendix B

### B.1. Spectrum and resolvent of $\mathbb{A}_{|\mathbf{k}|, \perp}$

**Proof of Proposition 2.3.2: resolvent of  $\mathbb{A}_{|\mathbf{k}|, \perp}$**

Given  $\mathbb{F} = (\mathbf{e}, \mathbf{h}, \mathbf{p}, \dot{\mathbf{p}}, \mathbf{m}, \dot{\mathbf{m}})^t \in \mathbf{C}_{\perp}^N$  and  $\omega \in \mathbb{C} \setminus S(|\mathbf{k}|) \cap \mathcal{S}_{\mathcal{T}}$ , we look at the problem

$$\text{Find } \mathbb{U} = (\mathbb{E}, \mathbb{H}, \mathbb{P}, \dot{\mathbb{P}}, \mathbb{M}, \dot{\mathbb{P}}) \in \mathbf{C}_{\perp}^N \text{ such that } (\mathbb{A}_{|\mathbf{k}|, \perp} - \omega \mathbb{I}) \mathbb{U} = \mathbb{F}, \quad (2.7.18)$$

which is equivalent to finding  $\mathbb{U} = (\mathbb{E}, \mathbb{H}, \mathbb{P}, \dot{\mathbb{P}}, \mathbb{M}, \dot{\mathbb{P}}) \in \mathbf{C}_{\perp}^N$  solution of the system

$$\begin{cases} -\frac{|\mathbf{k}| \mathbf{e}_3 \times \mathbb{H}}{\varepsilon_0} - i \sum \Omega_{e,j}^2 \dot{\mathbb{P}}_j - \omega \mathbb{E} = \mathbf{e}, & (i) \\ \frac{|\mathbf{k}| \mathbf{e}_3 \times \mathbb{E}}{\mu_0} - i \sum \Omega_{m,\ell}^2 \dot{\mathbb{M}}_{\ell} - \omega \mathbb{H} = \mathbf{h}, & (ii) \end{cases} \quad (2.7.19)$$

$$\begin{cases} i \dot{\mathbb{P}}_j - \omega \mathbb{P}_j = \mathbf{p}_j, & (i) \\ -i \alpha_{e,j} \dot{\mathbb{P}}_j - i \omega_{e,j}^2 \mathbb{P}_j + i \mathbb{E} - \omega \dot{\mathbb{P}}_j = \dot{\mathbf{p}}_j, & (ii) \end{cases} \quad (2.7.20)$$

$$\begin{cases} i\dot{\mathbb{M}}_\ell - \omega\mathbb{M}_\ell = \ell, & (i) \\ -i\alpha_{m,\ell}\dot{\mathbb{M}}_\ell - i\omega_{m,\ell}^2\mathbb{M}_\ell + i\mathbb{H} - \omega\dot{\mathbb{M}}_\ell = \mathbf{m}_\ell. & (ii) \end{cases} \quad (2.7.21)$$

We want to show that this problem admits a unique solution  $\mathbb{U}$  defining  $\mathbb{U} = R_{|\mathbf{k}|}(\omega)\mathbb{F}$ . The proof of Proposition 2.3.2 is thus purely computational and reduces to solving explicitly the system (2.7.19, 2.7.20, 2.7.21). We provide some details for the ease of the reader.

Substituting the expression of  $\dot{\mathbb{P}}_j$  and  $\dot{\mathbb{M}}_\ell$  given by (2.7.20)(i) and (2.7.21)(i), i.e.

$$\dot{\mathbb{P}}_j = -i\omega\mathbb{P}_j - i\mathbf{p}, \quad \dot{\mathbb{M}}_\ell = -i\omega\mathbb{M}_\ell - i\mathbf{m} \quad (2.7.22)$$

into (2.7.20)(ii) and (2.7.21)(ii), we get, by definition of  $q_{e,j}(\omega)$  and  $q_{m,\ell}(\omega)$  and by definition (2.3.6) of the operators  $\mathbb{A}_{p,j}(\omega), \mathbb{A}_{m,\ell}(\omega)$  (the division by  $q_{e,j}(\omega)$  and  $q_{m,\ell}(\omega)$  is allowed because  $\omega \notin \mathcal{P}$ ),

$$\mathbb{P}_j = -\frac{\mathbb{E}}{q_{e,j}(\omega)} + \mathbb{A}_{p,j}(\omega)\mathbb{F} \quad \text{and} \quad \mathbb{M}_\ell = -\frac{\mathbb{H}}{q_{m,\ell}(\omega)} + \mathbb{A}_{m,\ell}(\omega)\mathbb{F}. \quad (2.7.23)$$

Thus, going back to  $\dot{\mathbb{P}}_j$  and  $\dot{\mathbb{M}}_\ell$ , via (2.7.20)(i) and (2.7.21)(i), by definition (2.3.6) of  $\dot{\mathbb{A}}_{p,j}(\omega), \dot{\mathbb{A}}_{m,\ell}(\omega)$

$$\dot{\mathbb{P}}_j = \frac{i\omega\mathbb{E}}{q_{e,j}(\omega)} + \dot{\mathbb{A}}_{p,j}(\omega)\mathbb{F} \quad \text{and} \quad \dot{\mathbb{M}}_\ell = -\frac{i\omega\mathbb{H}}{q_{m,\ell}(\omega)} + \dot{\mathbb{A}}_{m,\ell}(\omega)\mathbb{F}. \quad (2.7.24)$$

Substituting  $\dot{\mathbb{P}}_j$ , from (2.7.24), into (2.7.19)(ii) we get, by definitions (2.1.5, 2.3.6) of  $\mu(\omega)$  and  $\mathbb{A}_m(\omega)$ :

$$|\mathbf{k}|\mathbf{e}_3 \times \mathbb{E} - \omega\mu(\omega)\mathbb{H} = \mu_0 \left( \mathbf{h} + i \sum \Omega_{m,\ell}^2 \dot{\mathbb{A}}_{m,\ell}(\omega)\mathbb{F} \right) = -\mathbb{A}_m(\omega)\mathbb{F}. \quad (2.7.25)$$

Similarly, from  $\dot{\mathbb{M}}_\ell$  in (2.7.24), we obtain, by definitions (2.1.5, 2.3.6) of  $\varepsilon(\omega)$  and  $\mathbb{A}_e(\omega)$ ,

$$-|\mathbf{k}|\mathbf{e}_3 \times \mathbb{H} - \omega\varepsilon(\omega)\mathbb{E} = \varepsilon_0 \left( \mathbf{e} + i \sum \Omega_{e,j}^2 \dot{\mathbb{A}}_{p,j}(\omega)\mathbb{F} \right) = -\mathbb{A}_e(\omega)\mathbb{F}. \quad (2.7.26)$$

In order to eliminate  $\mathbb{H}$  between (2.7.25) and (2.7.26), we perform the combination

$$|\mathbf{k}|\mathbf{e}_3 \times (2.7.25) - \omega\mu(\omega)(2.7.26)$$

This gives, since  $-|\mathbf{k}|\mathbf{e}_3 \times (|\mathbf{k}|\mathbf{e}_3 \times \mathbb{E}) = |\mathbf{k}|^2\mathbb{E}$ , (as  $\mathbb{U} \in \mathbf{C}_\perp^N$ ,  $\mathbf{e}_3$  and  $\mathbb{E}$  are orthogonal)

$$(|\mathbf{k}|^2 - \mathcal{D}(\omega))\mathbb{E} = \omega\mu(\omega)\mathbb{A}_e(\omega)\mathbb{F} - |\mathbf{k}|\mathbf{e}_3 \times \mathbb{A}_m(\omega)\mathbb{F}. \quad (2.7.27)$$

As  $\omega \notin S(|\mathbf{k}|)$ , one has  $\mathcal{D}(\omega) - |\mathbf{k}|^2 \neq 0$  and therefore, by definition (2.3.10) of  $\mathcal{S}_{|\mathbf{k}|}(\omega)$ ,

$$\mathbb{E} = \frac{|\mathbf{k}|\mathbf{e}_3 \times \mathbb{A}_m(\omega)\mathbb{F} - \omega\mu(\omega)\mathbb{A}_e(\omega)\mathbb{F}}{\mathcal{D}(\omega) - |\mathbf{k}|^2} = \mathcal{S}_{|\mathbf{k}|}(\omega)\mathbb{F}. \quad (2.7.28)$$

Thus, since  $\omega\mu(\omega) \neq 0$  (as  $\omega \notin \mathcal{Z}_m \cup \{0\}$ ), from (2.7.25) and by definition (2.3.7) of  $\mathbb{A}_h(\omega)$  :

$$\mathbb{H} = \frac{1}{\omega\mu(\omega)} \left( |\mathbf{k}|\mathbf{e}_3 \times \mathcal{S}_{|\mathbf{k}|}(\omega)\mathbb{F} + \mathbb{A}_m(\omega)\mathbb{F} \right). \quad (2.7.29)$$

Finally, substituting (2.7.27, 2.7.29) in (2.7.23, 2.7.24) we get the expression of the other

$$\begin{cases} \mathbb{P}_j = -\frac{\mathcal{S}_{|\mathbf{k}|}(\omega) \mathbb{F}}{q_{e,j}(\omega)} + \mathbb{A}_{p,j}(\omega) \mathbb{F}, & \mathbb{M}_\ell = \mathbb{A}_{m,\ell}(\omega) \mathbb{F} - \frac{\mathbb{A}_m(\omega) \mathbb{F} + |\mathbf{k}| \mathbf{e}_3 \times \mathcal{S}_{|\mathbf{k}|}(\omega) \mathbb{F}}{\omega \mu(\omega) q_{m,\ell}(\omega)}, \\ \dot{\mathbb{P}}_j = \frac{i\omega \mathcal{S}_{|\mathbf{k}|}(\omega) \mathbb{F}}{q_{e,j}(\omega)} + \dot{\mathbb{A}}_{p,j}(\omega) \mathbb{F}, & \dot{\mathbb{M}}_\ell = \dot{\mathbb{A}}_{m,\ell}(\omega) \mathbb{F} + i \frac{\mathbb{A}_m(\omega) \mathbb{F} + |\mathbf{k}| \mathbf{e}_3 \times \mathcal{S}_{|\mathbf{k}|}(\omega) \mathbb{F}}{\mu(\omega) q_{m,\ell}(\omega)}. \end{cases} \quad (2.7.30)$$

The reader will then easily verify that the formulas (2.7.27, 2.7.29, 2.7.30) lead to the expression (2.3.9, 2.3.10, 2.3.10, 2.3.12) given in Proposition 2.3.2 for the resolvent  $R_{|\mathbf{k}|}(\omega)$ .

### Spectrum $\mathbb{A}_{|\mathbf{k}|,\perp}$ : proof of Proposition 2.3.1

From Proposition 2.3.2, we already know that

$$\sigma(\mathbb{A}_{|\mathbf{k}|,\perp}) \subset S(|\mathbf{k}|) \cup \mathcal{S}_{\mathcal{T}}.$$

To prove the Proposition 2.3.1, it suffices to show that

- (i)  $\sigma(\mathbb{A}_{|\mathbf{k}|,\perp}) \cap \mathcal{S}_{\mathcal{T}} = \emptyset$ .
- (ii)  $S(|\mathbf{k}|) \subset \sigma(\mathbb{A}_{|\mathbf{k}|,\perp})$  and for any  $\omega \in S(|\mathbf{k}|)$ ,  $\dim \text{Ker} (\mathbb{A}_{|\mathbf{k}|,\perp} - \omega \text{Id}) = 2$ .

**Preambulus.** For both steps, we shall use the fact that  $\mathbb{U} = (\mathbb{E}, \mathbb{H}, (\mathbb{P}_j), (\dot{\mathbb{P}}_j), (\mathbb{M}_\ell), (\dot{\mathbb{M}}_\ell)) \in \text{Ker} (\mathbb{A}_{|\mathbf{k}|,\perp} - \omega \text{Id})$  means that solution of (2.7.19, 2.7.20, 2.7.21) with  $\mathbb{F} = (\mathbf{e}, \mathbf{h}, \mathbf{p}, \dot{\mathbf{p}}, \mathbf{m}, \dot{\mathbf{m}}) = 0$ . Therefore, proceeding as for obtaining (2.7.23) and (2.7.24), before division by  $q_{e,j}(\omega)$  and  $q_{m,\ell}(\omega)$ , we deduce from (2.7.20, 2.7.21) that

$$\forall i, j, \quad q_{e,j}(\omega) \mathbb{P}_j = -\mathbb{E}, \quad q_{m,\ell}(\omega) \mathbb{M}_\ell = -\mathbb{H}, \quad q_{e,j}(\omega) \dot{\mathbb{P}}_j = i\omega \mathbb{E}, \quad q_{m,\ell}(\omega) \dot{\mathbb{M}}_\ell = -i\omega \mathbb{H}. \quad (2.7.31)$$

**Proof of Step (i).** Assume that  $\omega \in \mathcal{S}_{\mathcal{T}} = \mathcal{P}_e \cup \mathcal{P}_m \cup \mathcal{Z}_m \cup \{0\}$ .

(a) Let  $\omega \in \mathcal{P}_e$  and  $\mathbb{U} \in \text{Ker} (\mathbb{A}_{|\mathbf{k}|,\perp} - \omega \text{Id})$ , for some  $j_0$ ,  $q_{e,j_0}(\omega) = 0$ , thus, by (2.7.31),  $\mathbb{E} = 0$ . Thus, by (2.7.31) and assumption (H<sub>1</sub>) that yields  $q_{e,j}(\omega) \neq 0$  for  $j \neq j_0$ , we conclude that  $\mathbb{P}_j = 0$  and  $\dot{\mathbb{P}}_j = 0$  for any  $j \neq j_0$ .

Next, according to assumption (H<sub>1</sub>) again,

- Either  $q_{m,\ell}(\omega) \neq 0$  for any  $m$ , we conclude from (2.7.31)  $\mathbb{M} = 0, \dot{\mathbb{M}} = 0$ . Thus, going back to (2.7.19)(ii) with  $\mathbf{h} = 0$ , we have  $\mathbb{H} = 0$  (since  $\omega \neq 0$ ). Then, from (2.7.19)(i) with  $\mathbf{e} = 0$ , we have  $\dot{\mathbb{P}}_{j_0} = 0$ , thus  $\mathbb{P}_{j_0} = 0$  since, by (2.7.22) for  $\mathbf{p} = 0$ ,  $-i\omega \mathbb{P}_{j_0} = \dot{\mathbb{P}}_{j_0}$  and  $\omega \neq 0$  (as  $0 \notin \mathcal{P}_e$ ).
- Or there exists a unique  $\ell_0$  for which  $q_{m,\ell}(\omega) = 0$ . In that case (2.7.31) implies that  $\mathbb{H} = 0$  and  $\mathbb{M}_\ell = \dot{\mathbb{M}}_\ell = 0$  for any  $\ell \neq \ell_0$ . Finally, going back to (2.7.19) (i) and (ii) with  $\mathbf{e} = \mathbf{h} = 0$ , we have  $\dot{\mathbb{P}}_{j_0} = \dot{\mathbb{M}}_{\ell_0} = 0$ , thus  $\mathbb{P}_{\ell_0} = \mathbb{M}_{\ell_0} = 0$  since (2.7.22) for  $\mathbf{m} = \mathbf{p} = 0$  gives  $-i\omega \mathbb{P}_{j_0} = \dot{\mathbb{P}}_{j_0}$  and  $-i\omega \mathbb{M}_{\ell_0} = \dot{\mathbb{M}}_{\ell_0}$  with  $\omega \neq 0$ .

In all cases,  $\mathbb{U} = 0$ , thus  $\omega \notin \sigma(\mathbb{A}_{|\mathbf{k}|,\perp})$ .

(b) In a symmetric manner, we prove that if  $\omega \in \mathcal{P}_m$ ,  $\omega \notin \sigma(\mathbb{A}_{|\mathbf{k}|,\perp})$ .

Now, for the rest of the proof, we point out that for  $\omega \notin \mathcal{P}$ , one shows by proceeding exactly as

for obtaining (2.7.25) and (2.7.26) that

$$(1) \quad |\mathbf{k}| \mathbf{e}_3 \times \mathbb{E} - \omega \mu(\omega) \mathbb{H} = 0, \quad (2) \quad -|\mathbf{k}| \mathbf{e}_3 \times \mathbb{H} - \omega \varepsilon(\omega) \mathbb{E} = 0. \quad (2.7.32)$$

More precisely, one checks that, as soon as (2.7.31) holds and  $\omega \notin \mathcal{P}$ , (2.7.32) is equivalent to (2.7.19) for  $\mathbb{F} = 0$ .

(c) It remains to look at  $\omega \in \mathcal{Z}_m \cup \{0\}$ , i. e.  $\omega \mu(\omega) = 0$ . By (H<sub>2</sub>), one has  $\mathcal{P} \cap (\mathcal{Z}_m \cup \{0\}) = \emptyset$ . Thus, from (2.7.32)(1) and since  $\mathbb{E} \cdot \mathbf{e}_3 = 0$ , we deduce that  $\mathbb{E} = 0$  and it follows with (2.7.32)(2) that  $\mathbb{H} = 0$  (as  $\mathbb{H} \cdot \mathbf{e}_3 = 0$ ). Then, going back to (2.7.31), since  $\omega \notin \mathcal{P}$ , we deduce that  $\mathbb{P} = \dot{\mathbb{P}} = 0$  and  $\mathbb{M} = \dot{\mathbb{M}} = 0$ , thus  $\mathbb{U} = 0$ . This proves that  $\omega \notin \sigma(\mathbb{A}_{|\mathbf{k}|, \perp})$ .

**Proof of Step (ii).** Let  $\omega \in S(|\mathbf{k}|)$  and  $\mathbb{U} \in \ker(\mathbb{A}_{|\mathbf{k}|, \perp} - \omega \text{Id})$ . First note that, since by assumption  $\mathbf{k} \neq 0$  and thus  $\omega \mu(\omega) \neq 0$  (since  $\omega$  satisfies the dispersion relation (2.3.4)), according to (2.7.32)(a), we have

$$\mathbb{H} = \frac{|\mathbf{k}| \mathbf{e}_3 \times \mathbb{E}}{\omega \mu(\omega)}, \quad (2.7.33)$$

so that, it yields with (2.7.31):

$$\mathbb{P}_j = -\frac{\mathbb{E}}{q_{e,j}(\omega)}, \quad \dot{\mathbb{P}}_j = \frac{i\omega \mathbb{E}}{q_{e,j}(\omega)}, \quad \mathbb{M}_\ell = -\frac{|\mathbf{k}| \mathbf{e}_3 \times \mathbb{E}}{\omega \mu(\omega) q_{m,\ell}(\omega)}, \quad \dot{\mathbb{M}}_\ell = \frac{i\omega |\mathbf{k}| \mathbf{e}_3 \times \mathbb{E}}{\omega \mu(\omega) q_{m,\ell}(\omega)}. \quad (2.7.34)$$

Of course (2.7.33) and (2.7.34) means that  $\dim \text{Ker}(\mathbb{A}_{|\mathbf{k}|, \perp} - \omega \text{Id}) \leq \dim \mathbf{C}_\perp = 2$ .

To prove (ii), it suffices to check that  $\forall \mathbb{E} \in \mathbf{C}_\perp$ ,  $\mathbb{U} = (\mathbb{E}, \mathbb{H}, \mathbb{P}, \dot{\mathbb{P}}, \mathbb{M}, \dot{\mathbb{M}}) \in \ker(\mathbb{A}_{|\mathbf{k}|, \perp} - \omega \text{Id})$ , with  $(\mathbb{H}, \mathbb{P}, \dot{\mathbb{P}}, \mathbb{M}, \dot{\mathbb{M}})$  given by (2.7.33, 2.7.34) is a solution to (2.7.19, 2.7.20, 2.7.21) with  $\mathbb{F} = 0$ . This is equivalent (since  $\omega \notin \mathcal{P}$ ) to checking (2.7.31, 2.7.32). This is rather immediate since (2.7.34) is nothing but (2.7.31) with  $\mathbb{H}$  given by (2.7.33) while (2.7.33) is nothing but (2.7.32)(1). It remains to check (2.7.32)(2). However, substituting (2.7.33) into the left hand side of (2.7.32)(2) gives, as  $-|\mathbf{k}| \mathbf{e}_3 \times (|\mathbf{k}| \mathbf{e}_3 \times \mathbb{E}) = |\mathbf{k}|^2 \mathbb{E}$  for  $\mathbb{E}$  in  $\mathbf{C}_\perp$ ,

$$(\mathcal{D}(\omega) - |\mathbf{k}|^2) \mathbb{E} = 0, \quad (2.7.35)$$

which is true for any  $\mathbb{E} \in \mathbf{C}_\perp$  since  $\mathcal{D}(\omega) = |\mathbf{k}|^2$ .

## B.2. Technical result for the analysis of the dispersion curves

We are interested in the solution of the parametrized (by  $\zeta \in \mathbb{C}$ ) nonlinear equations in  $\omega \in \mathbb{C}$  of the form ( $z \in \mathbb{C}$  being given):

$$\text{Find } \omega \in \mathbb{C} \text{ such that } (\omega - z)^m g(\omega) = \zeta, \quad \text{with } g \text{ analytic near } z \text{ and } g(z) \neq 0, \quad (2.7.36)$$

and more precisely to the fact that (2.7.36) defines implicitly  $\omega$  in function of  $\zeta$ , via  $m$  branches of solutions  $\omega_n(\zeta)$ , for small  $|\zeta|$ .

**Lemma B. 1**

Let  $\mathcal{G}$  be an analytic function on a domain  $\Omega \subset \mathbb{C}$  and  $z \in \Omega$  a zero of multiplicity  $\mathbf{m} \in \mathbb{N}^*$  of  $\mathcal{G}$ . Thus,  $\mathcal{G}$  can be rewritten as

$$\mathcal{G}(\omega) = (\omega - z)^{\mathbf{m}} g(\omega) \text{ with } g \text{ analytic on a vicinity of } z \text{ and } g(z) = A \neq 0. \quad (2.7.37)$$

Then, there exist an open neighborhood  $U$  of  $\zeta = 0$  and  $\mathbf{m}$  distinct analytic functions  $\zeta \mapsto \omega_n(\zeta)$  for  $n = 1 \dots, m$  defined on  $U$  such that

$$\mathcal{G}(\omega_n(\zeta)) = \zeta^{\mathbf{m}} \quad \text{on } U \quad (2.7.38)$$

Moreover the functions  $\omega_n(\zeta)$  have the following Taylor expansion

$$\omega_n(\zeta) = z + a_n^{-1} \zeta - \frac{a_n^{-2} g'(z)}{\mathbf{m} g(z)} \zeta^2 + O(\zeta^3), \text{ as } \zeta \rightarrow 0, \text{ where } a_n = |A|^{1/\mathbf{m}} e^{i \frac{\theta}{\mathbf{m}}} e^{2i \frac{n\pi}{\mathbf{m}}}, \quad (2.7.39)$$

$\theta \in (-\pi, \pi]$  is the principal value of the argument of  $A$  and the complex coefficients  $a_n$  are the  $\mathbf{m}_p$  distinct roots of the polynomial equation  $X^{\mathbf{m}} = A$ .

Furthermore, one can find an open neighborhood  $D$  of  $z$  included in  $\Omega$  such that the set of solutions of  $G(\omega) = \zeta^{\mathbf{m}}$  on  $D$  for  $\zeta \in U$  is exactly given by  $\{\omega_n(\zeta) \text{ for } n = 1, \dots, \mathbf{m}\}$ .

*Proof.* We want to find branches of solutions  $\zeta \rightarrow \omega_n(\zeta)$  of the equation

$$\mathcal{G}(\omega) = \zeta^{\mathbf{m}}, \quad (2.7.40)$$

where  $\zeta$  is a parameter that lies on an open neighborhood of 0. To this aim, fix  $1 \leq n \leq \mathbf{m}$  and make the following change of unknown  $\omega \rightarrow \eta$ , for  $\zeta \neq 0$ :

$$\eta = -1 + (\omega - z) a_n \zeta^{-1} \iff \omega = z + a_n^{-1} \zeta (1 + \eta) \quad (2.7.41)$$

where the coefficient  $a_n$  is the root of the polynomial equation  $X^{\mathbf{m}} = A$  defined in (2.7.39). Therefore by replacing (2.7.41) into (2.7.40) and using the fact that  $g(z) = A = a_n^{\mathbf{m}}$ , we notice that on an open neighborhood of  $\zeta = 0$ , finding an analytic branches  $\zeta \mapsto \omega_n(\zeta)$  of solutions of (2.7.40) is equivalent to find an analytic branch  $\zeta \mapsto \eta_n(\zeta)$  satisfying

$$H_n(\zeta, \eta) = 0 \quad \text{with} \quad H_n(\zeta, \eta) := (1 + \eta)^{\mathbf{m}} g(z + a_n^{-1} \zeta (1 + \eta)) - g(z), \quad (2.7.42)$$

with  $\omega_n$  given in term of  $\eta_n$  by (2.7.41). As  $g$  is analytic on a vicinity of  $z$ , by the Hartogs Theorem (see [?], Theorem 36.8 page 271), the function  $H_n(\zeta, \eta)$  is well-defined and analytic on a neighbourhood of  $(0, 0)$  in  $\mathbb{C} \times \mathbb{C}$  and satisfies  $H_n(0, 0) = 0$ . We then want to use the analytic implicit function Theorem (see [?], Theorem 7.6 page 34) to solve  $H_n(\zeta, \eta) = 0$ . Thus, one has first to check that  $\partial_{\eta} H_n(0, 0) \neq 0$ . Indeed, from (2.7.42), we compute

$$\partial_{\eta} H_n(\zeta, \eta) = \mathbf{m} (1 + \eta)^{\mathbf{m}-1} g(z + a_n^{-1} \zeta (1 + \eta)) + (1 + \eta)^{\mathbf{m}} a_n^{-1} \zeta g'(z + a_n^{-1} \zeta (1 + \eta))$$

so that in particular

$$\partial_{\eta} H_n(0, 0) = \mathbf{m} g(z) \neq 0. \quad (2.7.43)$$

Hence, the analytic implicit function Theorem proves the existence of two open neighbourhoods of the origin  $U_n \subset \mathbb{C}$  (the  $\zeta$ -complex plane) and  $V_n \subset \mathbb{C}$  (the  $\eta$ -complex plane) and of an analytic

function  $\eta_n(\zeta) : U_n \rightarrow V_n$  satisfying  $\eta(0) = 0$  such that

$$\{(\zeta, \eta) \in U_n \times V_n \mid H_n(\zeta, \eta_n) = 0\} = \{(\zeta, \eta_n(\zeta)), \zeta \in U_n\}.$$

Moreover, one has

$$\eta'(0) = -\frac{\partial_\zeta H_n(0, 0)}{\partial_\eta H_n(0, 0)} = -\frac{a_n^{-1} g'(z)}{\mathbf{m} g(z)}. \quad (2.7.44)$$

Thus  $\zeta \mapsto \omega_n(\zeta) := z + a_n^{-1} \zeta (1 + \eta_n(\zeta))$  is analytic branch of solution of (2.7.40) on  $U_n$ . Since  $\eta_n(0) = 0$ , as  $\zeta \rightarrow 0$ ,  $\eta_n(\zeta) = \zeta \eta'_n(0) + O(\zeta^2)$ , which, put in the definition of  $\omega_n(\zeta)$ , gives (2.7.39).

We point out that, as the coefficients  $a_n^{-1}$  in (2.7.39) are all distinct, we have constructed  $\mathbf{m}$  distinct analytic functions  $\zeta \rightarrow \omega_n(\zeta)$  for  $n = 1, \dots, \mathbf{m}$  defined on the open neighborhood  $\tilde{U}$ , defined as the intersection of the sets  $U_n$  for  $n \in \{1, \dots, \mathbf{m}\}$ , which satisfy (2.7.38) and (2.7.39).

The last point is proven with Rouché Theorem (see for e.g. Theorem 10.43 page 225 of [?]). As  $g$  does not vanish at the vicinity of  $z$ , we can find  $r > 0$  such that, if  $D$  is the open disk of radius  $r$  and center  $z$ ,  $\bar{D} \subset \Omega$  and  $g$  does not vanish in  $\bar{D}$ . Hence, by (2.7.37), the only zero of  $G(\omega)$  in  $D$  is  $z$ , with multiplicity  $\mathbf{m}$ . Let us define the function  $G_\zeta : \omega \mapsto G(\omega) - \zeta^m$  which is analytic on  $\Omega$  and  $\partial D$  the circle of center  $z$  and radius  $r$ . Then, one has:

$$\forall \omega \in \partial D, \quad |G_\zeta(\omega) - G(\omega)| = |\zeta|^m \text{ and by (2.7.37), } |G(\omega)| \geq r^m \min_{\omega \in \partial D} |g(\omega)| > 0.$$

Thus for  $|\zeta| < R = r \min_{\omega \in \partial D} |g(\omega)|^{1/m}$ , one gets

$$\forall |\zeta| < R, \quad \forall \omega \in \partial D, \quad |G_\zeta(\omega) - G(\omega)| < |G(\omega)|.$$

Therefore by the Rouché Theorem,  $G$  and  $G_\zeta$  have the same number of zeros (counted with multiplicity) in  $D$ . Hence  $G_\zeta$  has  $\mathbf{m}$  zeros on  $D(z, r)$  for  $|\zeta| < R$ . Let  $D(0, R)$  be the open disk of center 0 and radius  $R$ . Thus, if we define  $U$  as  $\tilde{U} \cap D(0, R)$ , then there are no other solutions of  $G(\omega) = \zeta^m$  on  $D$  and  $\zeta \in U$  than  $\omega_n(\zeta)$  for  $n \in \{1, \dots, \mathbf{m}\}$ .  $\square$

**Remark 2.** The result of Lemma 1 is related to Puiseux series. Puiseux series have been derived in [?, ?, ?] for the perturbation analysis of eigenvalues of non-selfadjoint matrices. However the results of [?] do not apply directly here since in some situations we are perturbing derogatory eigenvalues (i.e. non simple eigenvalues). Moreover, we need to compute explicitly the two first coefficients of the Puiseux series which is not done in [?], Section 1.2 or in [?], see Corollary 4.2.9.

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## Conclusion and future perspectives of the Part I

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In this first part, we have analyzed in details the dissipation effects due the absorption mechanisms present in generalised Drude-Lorentz medium, via the large time behaviour of the corresponding Cauchy problem. The main conclusion is that the decay of the solution is weaker (polynomial decay) than the one observed with more standard dissipation effects such as the electric conduction, which leads to exponential decay.

These results have been obtained under the assumption that the whole propagation domain. A rather natural, but much more difficult, question is whether our results can be extended to the case where absorption is present only in a strict subdomain of the domain of propagation. This question has in particular be addressed in the case of standard dissipative media (conductive media) concerning the decay of the local energy (i. e. the energy in a bounded domain while the problem is posed typically in the whole space, see [9, 79], in the continuation of similar results in non dissipative media [11].



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## PART II

GUIDED WAVES BY A SLAB OF METAMATERIAL  
EMBEDDED IN A DIELECTRIC MEDIUM

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## CHAPTER 3

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# Mathematical formulation of the problem

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## 3.1 Physical and mathematical framework

### 3.1.1 The Drude model in a perfect lens

In this work, we consider in the real-three-dimensional space  $\mathbb{R}^3$ , a metamaterial filling an infinite rectangular plate of width  $2L$  ( $L > 0$ ), namely,

$$\mathcal{R} := \{\vec{\mathbf{x}} := (x, y, z) \in \mathbb{R}^3 \mid -L < y < L\} = \mathbb{R} \times (-L, L) \times \mathbb{R},$$

whose behavior is given by a Drude model [14,16,48,86], and the complementary domain,  $\mathbb{R}^3/\overline{\mathcal{R}}$ , is composed by a dielectric, for instance, the vacuum (see Figure 3.1). The triplet  $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$  designates the canonical basis of  $\mathbb{R}^3$ .

For any vector  $\vec{\mathbf{x}} \in \mathbb{R}^3$  and instant  $t > 0$ , we denote

$$\begin{aligned} \blacktriangleright \mathbf{E}(\vec{\mathbf{x}}, t) \text{ the electric field, } & \blacktriangleright \mathbf{D}(\vec{\mathbf{x}}, t) \text{ the electric induction,} \\ \blacktriangleright \mathbf{H}(\vec{\mathbf{x}}, t) \text{ the magnetic field, } & \blacktriangleright \mathbf{B}(\vec{\mathbf{x}}, t) \text{ the magnetic induction.} \end{aligned} \tag{3.1.1}$$

Assuming the presence of a source current density  $\mathbf{J}_s$ , the time evolution in the whole space of  $(\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B})$  are governed by the macroscopic Maxwell's equations (in the following we use the

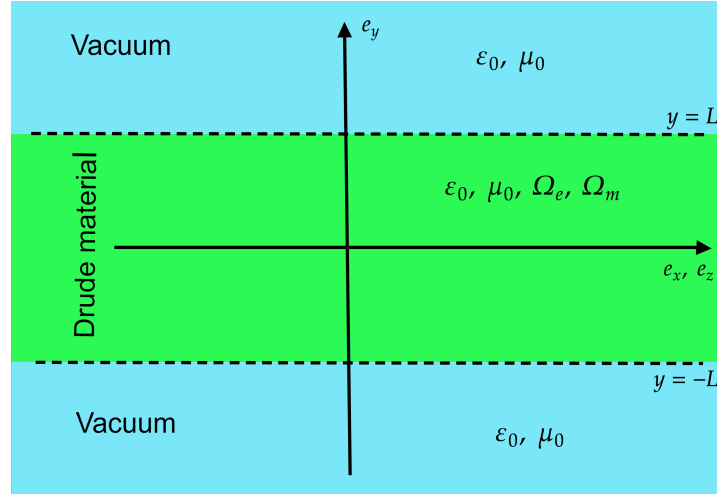


Figure 3.1: Description of the infinite slab filled by a Drude material in the vacuum.

notation **Curl** to refer to the classic curl operation in  $\mathbb{R}^3$ )

$$\text{(Maxwell's equations)} \quad \left\{ \begin{array}{l} \partial_t \mathbf{D} - \mathbf{Curl} \mathbf{H} = -\mathbf{J}_s, \\ \partial_t \mathbf{B} + \mathbf{Curl} \mathbf{E} = 0, \end{array} \right. \quad \text{in } \mathbb{R}^3 \times [0, \infty). \quad (3.1.2a)$$

$$(3.1.2b)$$

which are complemented by the constitutive laws characterizing the material in which electromagnetic waves propagate by relating the electric (or magnetic) field and the corresponding induction, they are written in following form

$$\text{(Constitutive laws)} \quad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \text{and} \quad \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}, \quad \text{in } \mathbb{R}^3 \times [0, \infty), \quad (3.1.3a)$$

involving two additional unknowns, the electric and magnetic polarizations, respectively denoted as,  $\mathbf{P}$  and  $\mathbf{M}$ . The positive constants  $\varepsilon_0$  and  $\mu_0$  stand, respectively, for the permittivity and the permeability of the vacuum.

In the vacuum,  $\mathbf{P} = \mathbf{M} \equiv 0$  thus the Maxwell's equations turns into

$$\text{(Vacuum)} \quad \left\{ \begin{array}{l} \varepsilon_0 \partial_t \mathbf{E} - \mathbf{Curl} \mathbf{H} = -\mathbf{J}_s, \\ \mu_0 \partial_t \mathbf{H} + \mathbf{Curl} \mathbf{E} = 0, \end{array} \right. \quad \text{in } \mathbb{R}^3 / \overline{\mathcal{R}} \times [0, \infty). \quad (3.1.4a)$$

$$(3.1.4b)$$

On the other side, for a homogeneous nondissipative Drude material, the constitutive laws are characterized by relating the polarization fields  $\mathbf{P}$  and  $\mathbf{M}$  to  $\mathbf{E}$  and  $\mathbf{H}$  as following

$$\partial_t \mathbf{P} = \mathbf{J}, \quad \partial_t \mathbf{J} = \varepsilon_0 \Omega_e^2 \mathbf{E}, \quad \text{and} \quad \partial_t \mathbf{M} = \mathbf{K}, \quad \partial_t \mathbf{K} = \mu_0 \Omega_m^2 \mathbf{H},$$

where the two unknowns  $\mathbf{J}$  and  $\mathbf{K}$  are called usually the induced electric and magnetic currents. Both parameters,  $\Omega_e$  and  $\Omega_m$ , are positive constants which characterize the behavior of a Drude material. By eliminating the  $\mathbf{D}$ ,  $\mathbf{B}$ ,  $\mathbf{P}$  and  $\mathbf{M}$ , we obtain the time-dependent Maxwell's equations

in a Drude material, namely,

$$\begin{aligned}
 \text{(Drude material)} \quad & \begin{cases} \varepsilon_0 \partial_t \mathbf{E} - \mathbf{Curl} \mathbf{H} + \mathbf{J} = -\mathbf{J}_s, & \partial_t \mathbf{J} = \varepsilon_0 \Omega_e^2 \mathbf{E}, \\ \mu_0 \partial_t \mathbf{H} + \mathbf{Curl} \mathbf{E} + \mathbf{K} = 0, & \partial_t \mathbf{K} = \mu_0 \Omega_m^2 \mathbf{H}, \end{cases} & \text{in } \mathcal{R} \times [0, \infty).
 \end{aligned} \tag{3.1.5a}$$

$$\tag{3.1.5b}$$

The equations (3.1.4) and (3.1.5) must be supplemented by the usual transmission conditions on the boundary

$$[\mathbf{e}_y \times \mathbf{E}(\cdot, t)]_{y=\pm L} = 0, \quad [\mathbf{e}_y \times \mathbf{H}(\cdot, t)]_{y=\pm L} = 0, \quad \text{for all } t \geq 0, \tag{3.1.6}$$

the latter express the continuity of the tangential electric and magnetic fields through the interfaces  $y = -L$  and  $y = L$ . The notation  $[f]_{y=y_0}$  designates the gap of a quantity across  $y = y_0$ , i.e.,

$$[f]_{y=y_0} := \lim_{h \rightarrow 0^+} [f(y_0 + h) - f(y_0 - h)],$$

where the limits must be understood in the trace sense, in the framework of the classic Sobolev spaces adapted to Maxwell's equations.

For every instant  $t > 0$ , we assume the fields  $\mathbf{E}(\cdot, t)$ ,  $\mathbf{H}(\cdot, t)$ ,  $\mathbf{J}(\cdot, t)$ ,  $\mathbf{K}(\cdot, t)$ ,  $\mathbf{Curl} \mathbf{E}(\cdot, t)$  and  $\mathbf{Curl} \mathbf{H}(\cdot, t)$ , satisfying (3.1.4), (3.1.5) and (3.1.6), are square integrable functions. Under this premise, it naturally leads to introduce the following function spaces

$$\mathbf{L}^2(\mathcal{O}) := \left\{ \mathbf{u} : \mathcal{O} \longrightarrow \mathbb{R}^d \mid \int_{\mathcal{O}} |\mathbf{u}(\vec{\mathbf{x}})|^2 d\vec{\mathbf{x}} < \infty \right\}, \tag{3.1.7}$$

where  $|\cdot|$  and  $\mathcal{O}$  are, respectively, the euclidean norm and an open set of  $\mathbb{R}^d$ ,  $d \in \mathbb{N}$ . In particular, if  $d = 3$ , in addition we introduce

$$\mathbf{H}_{\mathbf{Curl}}(\mathcal{O}) := \{ \mathbf{u} \in \mathbf{L}^2(\mathcal{O}) \mid \mathbf{Curl} \mathbf{u} \in \mathbf{L}^2(\mathcal{O}) \}. \tag{3.1.8}$$

Therefore, for a fixed instant  $t > 0$ , and given  $\mathbf{J}_s(\cdot, t) \in \mathbf{L}^2(\mathbb{R}^3)$ , we search for solutions fields

$$(\mathbf{E}(\cdot, t), \mathbf{H}(\cdot, t), \mathbf{J}(\cdot, t), \mathbf{K}(\cdot, t)) \in (\mathbf{H}_{\mathbf{Curl}}(\mathbb{R}^3))^2 \times (\mathbf{L}^2(\mathcal{R}))^2.$$

**Remark 3.1.1.** Since the space  $\mathbf{H}_{\mathbf{Curl}}(\mathbb{R}^3)$  is characterized in the following way (cf. [60])

$$\mathbf{u} \in \mathbf{H}_{\mathbf{Curl}}(\mathbb{R}^3) \iff \begin{cases} \mathbf{u}|_{\mathcal{R}} \in \mathbf{H}_{\mathbf{Curl}}(\mathcal{R}), \\ \mathbf{u}|_{\mathbb{R}^3/\overline{\mathcal{R}}} \in \mathbf{H}_{\mathbf{Curl}}(\mathbb{R}^3/\overline{\mathcal{R}}), \\ [\mathbf{e}_y \times \mathbf{u}]_{y=-L} = 0, \\ [\mathbf{e}_y \times \mathbf{u}]_{y=L} = 0, \end{cases} \tag{3.1.9}$$

thus, the transmission conditions (3.1.6) for the electric and magnetic fields are implicitly implied for elements belonging to the functional space  $\mathbf{H}_{\mathbf{Curl}}(\mathbb{R}^3)$ .

Finally, in order to rewrite the system equations presented in (3.1.4), (3.1.5) and (3.1.6) in a unified evolution problem, hence we introduce the operators, *extension-by-zero* and *restriction*

to  $\mathcal{R}$  for 3D fields as following:

$$\begin{aligned} \mathbf{\Pi}_3 &:= L^2(\mathcal{R}) \longrightarrow L^2(\mathbb{R}^3) \\ f &\longmapsto f \text{ in } \mathcal{R} \\ &\quad \mathbf{0} \text{ in } \mathbb{R}^3/\overline{\mathcal{R}} \end{aligned} \quad \text{and} \quad \begin{aligned} \mathbf{R}_3 &:= L^2(\mathbb{R}^3) \longrightarrow L^2(\mathcal{R}) \\ F &\longmapsto F|_{\mathcal{R}} \end{aligned}.$$

Under the latter setting, we reformulate (3.1.4), (3.1.5) and (3.1.6) in the evolution problem

$$\begin{cases} \varepsilon_0 \partial_t \mathbf{E} - \mathbf{Curl} \mathbf{H} + \mathbf{\Pi}_3 \mathbf{J} = -\mathbf{J}_s, & \text{in } \mathbb{R}^3 \times [0, \infty), & (3.1.10a) \\ \mu_0 \partial_t \mathbf{H} + \mathbf{Curl} \mathbf{E} + \mathbf{\Pi}_3 \mathbf{K} = 0, & \text{in } \mathbb{R}^3 \times [0, \infty), & (3.1.10b) \\ \partial_t \mathbf{J} = \varepsilon_0 \Omega_e^2 \mathbf{R}_3 \mathbf{E}, & \text{in } \mathcal{R} \times [0, \infty), & (3.1.10c) \\ \partial_t \mathbf{K} = \mu_0 \Omega_m^2 \mathbf{R}_3 \mathbf{H}, & \text{in } \mathcal{R} \times [0, \infty), & (3.1.10d) \end{cases}$$

and where  $(\mathbf{E}, \mathbf{H}, \mathbf{J}, \mathbf{K}) \in (\mathbf{H}_{\mathbf{Curl}}(\mathbb{R}^3))^2 \times (L^2(\mathcal{R}))^2$ .

### 3.1.2 2D Maxwell's equation. TE and TM equations

In the following, we exploit the invariance-by-translation properties of the established medium, under this basis we will restrict ourselves to the study of the transverse electric equations (transverse magnetic equations), i.e., when the electric (magnetic) field is orthogonal to the plane of propagation, this fact will simplify our 3D Maxwell's equations system into a 2D model. For this purpose, the following hypothesis is assumed:

◆( $H_1$ ) The source current density  $\mathbf{J}_s$  does not depend on the variable  $z$ .

As a consequence, the whole setting of our problem is invariant with respect the  $z$ -variable. Then we can make the simplification  $\partial_z \equiv 0$  (seen as linear operator). Following this premise, we split every field  $\mathbf{U} \in \{\mathbf{E}, \mathbf{H}, \mathbf{J}, \mathbf{K}, \mathbf{J}_s\}$  (only depending on  $(x, y)$ ) into  $\mathbf{U}_{\perp} := (U_x, U_y)^{\top}$  the transverse component to the invariant direction  $\mathbf{e}_z$ , and  $U_{\parallel} := U_z$  the parallel component to this direction. The  $\mathbf{Curl}$  operator can be reformulated as

$$\mathbf{Curl} \mathbf{U} = \begin{pmatrix} \mathbf{curl} U_{\parallel} \\ \mathbf{curl} \mathbf{U}_{\perp} \end{pmatrix},$$

where  $\mathbf{curl}$  and  $\mathbf{curl}$  are, respectively, the 2D *curl operators* of scalar and vector fields, namely

$$\mathbf{curl} u := \begin{pmatrix} \partial_y u \\ -\partial_x u \end{pmatrix} \quad \text{and} \quad \mathbf{curl} \mathbf{u} := \partial_x u_y - \partial_y u_x, \quad \text{where } \mathbf{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

**Remark 3.1.2.** Note that for any field  $\mathbf{U} = (\mathbf{U}_{\perp}^{\top}, U_{\parallel})^{\top}$  satisfying (3.1.6), where the equations are understood in the sense of distributions, assumes implicitly the 2D version of the transmission conditions, namely; the transverse component  $\mathbf{U}_{\perp}$  imposes the transmission condition:

$$[U_x]_{y=-L} = 0 \quad \text{and} \quad [U_x]_{y=L} = 0, \quad (3.1.11)$$

and in turn, the parallel component  $U_{\parallel}$ :

$$[U_{\parallel}]_{y=-L} = 0 \quad \text{and} \quad [U_{\parallel}]_{y=L} = 0. \quad (3.1.12)$$

**Remark 3.1.3.** Also observe that the condition  $\mathbf{U} \in \mathbf{H}_{\text{Curl}}(\mathbb{R}^3)$  implies

$$\begin{cases} U_{\parallel} \in L^2(\mathbb{R}^2) & \text{and} & \mathbf{curl} U_{\parallel} \in \mathbf{L}^2(\mathbb{R}^2) & \text{(or equivalently, } U_{\parallel} \in L^2(\mathbb{R}^2) & \text{and } \nabla U_{\parallel} \in \mathbf{L}^2(\mathbb{R}^2)), \\ \mathbf{U}_{\perp} \in \mathbf{L}^2(\mathbb{R}^2) & \text{and} & \mathbf{curl} \mathbf{U}_{\perp} \in \mathbf{L}^2(\mathbb{R}^2), \end{cases}$$

which naturally leads us to introduce the function spaces

$$H^1(\mathbb{R}^2) := \{u \in L^2(\mathbb{R}^2) \mid \nabla u \in \mathbf{L}^2(\mathbb{R}^2)\} \quad \text{and} \quad \mathbf{H}_{\text{curl}}(\mathbb{R}^2) := \{\mathbf{u} \in \mathbf{L}^2(\mathbb{R}^2) \mid \mathbf{curl} \mathbf{u} \in \mathbf{L}^2(\mathbb{R}^2)\},$$

thus  $\mathbf{U} \in \mathbf{H}_{\text{Curl}}(\mathbb{R}^3)$  imposes the conditions  $U_{\parallel} \in H^1(\mathbb{R}^2)$  and  $\mathbf{U}_{\perp} \in \mathbf{H}_{\text{curl}}(\mathbb{R}^2)$ .

**Remark 3.1.4.** The respective belongings of  $U_{\parallel}$  and  $\mathbf{U}_{\perp}$  to the spaces  $H^1(\mathbb{R}^2)$  and  $\mathbf{H}_{\text{curl}}(\mathbb{R}^2)$  link us back to the 2D transmission conditions (see Remark 3.1.2), more specifically,  $U_{\parallel} \in H^1(\mathbb{R}^2)$  contains the transmission condition (3.1.12) and  $\mathbf{U}_{\perp} \in \mathbf{H}_{\text{curl}}(\mathbb{R}^2)$  contains (3.1.11).

Finally, let us define the infinite slab  $Q := \mathbb{R} \times (-L, L)$ , the operators *extension-by-zero* and *restriction* to  $Q$  for 2D fields:

$$\begin{aligned} \mathbf{\Pi}_2 := L^2(Q) &\longrightarrow L^2(\mathbb{R}^2) & \text{and} & \mathbf{R}_2 := L^2(\mathbb{R}^2) &\longrightarrow L^2(Q), \\ \mathbf{f} &\longmapsto \mathbf{f} \text{ in } Q & & \mathbf{F} &\longmapsto \mathbf{F}|_Q, \\ & \mathbf{0} \text{ in } \mathbb{R}^3/\overline{Q} & & & \end{aligned} \quad (3.1.13)$$

and the operators *extension-by-zero* and *restriction* to  $Q$  for scalar fields:

$$\begin{aligned} \Pi_2 := L^2(Q) &\longrightarrow L^2(\mathbb{R}^2) & \text{and} & R_2 := L^2(\mathbb{R}^2) &\longrightarrow L^2(Q), \\ f &\longmapsto f \text{ in } Q & & F &\longmapsto F|_Q, \\ & 0 \text{ in } \mathbb{R}^3/\overline{Q} & & & \end{aligned} \quad (3.1.14)$$

Under this setting, we notice that the evolution system (3.1.10) decouples into two new independent evolution systems. The first one involves the unknowns

$$(\mathbf{E}_{\perp}(\cdot, t), H_{\parallel}(\cdot, t), \mathbf{J}_{\perp}(\cdot, t), K_{\parallel}(\cdot, t)) \in \mathbf{H}_{\text{curl}}(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(Q) \times L^2(Q)$$

$$(TE) \begin{cases} \varepsilon_0 \partial_t \mathbf{E}_{\perp} - \mathbf{curl} H_{\parallel} + \mathbf{\Pi}_2 \mathbf{J}_{\perp} = -\mathbf{J}_{s\perp}, & \text{in } \mathbb{R}^2 \times [0, \infty), & (3.1.15a) \\ \mu_0 \partial_t H_{\parallel} + \mathbf{curl} \mathbf{E}_{\perp} + \Pi_2 K_{\parallel} = 0, & \text{in } \mathbb{R}^2 \times [0, \infty), & (3.1.15b) \\ \partial_t \mathbf{J}_{\perp} = \varepsilon_0 \Omega_e^2 \mathbf{R}_2 \mathbf{E}_{\perp}, & \text{in } Q \times [0, \infty), & (3.1.15c) \\ \partial_t K_{\parallel} = \mu_0 \Omega_m^2 R_2 H_{\parallel}, & \text{in } Q \times [0, \infty). & (3.1.15d) \end{cases}$$

The latter is called *(TE)* for *transverse electric* inasmuch as the electric field  $\mathbf{E}_{\perp}$  is transverse to the invariant direction for the problem setting, i.e.,  $\mathbf{e}_z$ . The second system evokes the unknowns

$$(E_{\parallel}(\cdot, t), \mathbf{H}_{\perp}(\cdot, t), J_{\parallel}(\cdot, t), \mathbf{K}_{\perp}(\cdot, t)) \in H^1(\mathbb{R}^2) \times \mathbf{H}_{\text{curl}}(\mathbb{R}^2) \times L^2(Q) \times L^2(Q) :$$

$$(TM) \begin{cases} \varepsilon_0 \partial_t E_{\parallel} - \mathbf{curl} \mathbf{H}_{\perp} + \Pi_2 J_{\parallel} = -J_{s\parallel}, & \text{in } \mathbb{R}^2 \times [0, \infty), & (3.1.16a) \\ \mu_0 \partial_t \mathbf{H}_{\perp} + \mathbf{curl} E_{\parallel} + \mathbf{\Pi}_2 \mathbf{K}_{\perp} = 0, & \text{in } \mathbb{R}^2 \times [0, \infty), & (3.1.16b) \\ \partial_t J_{\parallel} = \varepsilon_0 \Omega_e^2 R_2 E_{\parallel}, & \text{in } Q \times [0, \infty), & (3.1.16c) \\ \partial_t \mathbf{K}_{\perp} = \mu_0 \Omega_m^2 \mathbf{R}_2 \mathbf{H}_{\perp}, & \text{in } Q \times [0, \infty). & (3.1.16d) \end{cases}$$

Analogously, we denominate the latter as  $(TM)$  for transverse magnetic. Given the mathematics similarity of the systems  $(TE)$  and  $(TM)$ , we chose arbitrarily to study the system  $(TM)$ . Before entering to the functional analysis terrain, we agree on simplify the notation for  $(TM)$  by renaming the unknowns

$$E = E_{\parallel}, \quad \mathbf{H} = \mathbf{H}_{\perp} = \begin{pmatrix} H_x \\ H_y \end{pmatrix}, \quad J = J_{\parallel}, \quad \mathbf{K} = \mathbf{K}_{\perp} = \begin{pmatrix} K_x \\ K_y \end{pmatrix}, \quad J_s = J_{s\parallel}. \quad (3.1.17)$$

In this sense, the  $TM$  transmission problem reads as

$$(E(\cdot, t), \mathbf{H}(\cdot, t), J(\cdot, t), \mathbf{K}(\cdot, t)) \in H^1(\mathbb{R}^2) \times \mathbf{H}_{\text{curl}}(\mathbb{R}^2) \times L^2(Q) \times \mathbf{L}^2(Q) :$$

$$(TM) \begin{cases} \varepsilon_0 \partial_t E - \text{curl } \mathbf{H} + \Pi_2 J = -J_s, & \text{in } \mathbb{R}^2 \times [0, \infty), & (3.1.18a) \\ \mu_0 \partial_t \mathbf{H} + \mathbf{curl } E + \Pi_2 \mathbf{K} = 0, & \text{in } \mathbb{R}^2 \times [0, \infty), & (3.1.18b) \\ \partial_t J = \varepsilon_0 \Omega_e^2 R_2 E, & \text{in } Q \times [0, \infty), & (3.1.18c) \\ \partial_t \mathbf{K} = \mu_0 \Omega_m^2 \mathbf{R}_2 \mathbf{H}, & \text{in } Q \times [0, \infty). & (3.1.18d) \end{cases}$$

### 3.1.3 Reformulation of the TM equations as a Schrödinger evolution equation

In this subsection, we reformulate the  $TM$  transmission problem as a Schrödinger evolution equation

$$\frac{d}{dt} \mathbf{U} + \mathbb{A} \mathbf{U} = \mathbf{F}. \quad (3.1.19)$$

For this purpose we assume that  $\mathbb{A}$  is an unbounded operator on the Hilbert space

$$\mathcal{H}_{2D} := L^2(\mathbb{R}^2) \times \mathbf{L}^2(\mathbb{R}^2) \times L^2(Q) \times \mathbf{L}^2(Q) \quad (3.1.20)$$

(note that  $\mathbf{L}^2(\mathcal{O}) = (L^2(\mathcal{O}))^2$  for  $\mathcal{O} \in \{\mathbb{R}^2, Q\}$ ), the latter endowed with the inner product defined for all  $\mathbf{U} = (E, \mathbf{H}, J, \mathbf{K})^{\top}$  and  $\mathbf{U}' = (E', \mathbf{H}', J', \mathbf{K}')^{\top}$  in  $\mathcal{H}_{2D}$  by

$$\langle \mathbf{U}, \mathbf{U}' \rangle_{2D} := \varepsilon_0 \langle E, E' \rangle_{\mathbb{R}^2} + \mu_0 \langle \mathbf{H}, \mathbf{H}' \rangle_{\mathbb{R}^2} + (\varepsilon_0 \Omega_e^2)^{-1} \langle J, J' \rangle_Q + (\mu_0 \Omega_m^2)^{-1} \langle \mathbf{K}, \mathbf{K}' \rangle_Q, \quad (3.1.21)$$

where  $\langle u, v \rangle_{\mathcal{O}} := \int_{\mathcal{O}} u \cdot \bar{v} \, dx \, dy$  denotes the classical  $L^2$  inner product, for  $\mathcal{O} \in \{\mathbb{R}^2, Q\}$ .

Finally, one obtains (3.1.19) by setting  $\mathbf{F} := (-\varepsilon_0^{-1} J_s, \mathbf{0}, 0, \mathbf{0})$  and by defining the operator  $\mathbb{A}$  as

$$\mathbb{A} \mathbf{U} := \mathcal{A} \mathbf{U}, \quad \text{for all } \mathbf{U} \in D(\mathbb{A}) := H^1(\mathbb{R}^2) \times \mathbf{H}_{\text{curl}}(\mathbb{R}^2) \times L^2(Q) \times \mathbf{L}^2(Q) \subset \mathcal{H}_{2D}, \quad (3.1.22)$$

where  $\mathcal{A}$  is the following matrix differential operator (derivatives in the distributional sense):

$$\mathcal{A} := \mathbf{i} \begin{pmatrix} 0 & \varepsilon_0^{-1} \text{curl} & -\varepsilon_0^{-1} \Pi_2 & 0 \\ -\mu_0^{-1} \mathbf{curl} & 0 & 0 & -\mu_0^{-1} \Pi_2 \\ \varepsilon_0 \Omega_e^2 R_2 & 0 & 0 & 0 \\ 0 & \mu_0 \Omega_m^2 \mathbf{R}_2 & 0 & 0 \end{pmatrix}. \quad (3.1.23)$$

One again, we recall that the belongings  $E \in H^1(\mathbb{R}^2)$  and  $\mathbf{H} \in \mathbf{H}_{\text{curl}}(\mathbb{R}^2)$  contains the transmission conditions in [Remark 3.1.2](#).

**Proposition 3.1.5**

The operator  $\mathbb{A} : D(\mathbb{A}) \subset \mathcal{H}_{2D} \longrightarrow \mathcal{H}_{2D}$  is self-adjoint.

*Proof.* This proof is similar to the ones done in Proposition 2.1 of [\[16\]](#) or Proposition 2 of [\[17\]](#). We recall it here for the reading. We first prove that  $\mathbb{A}$  is symmetric. Let  $\mathbf{U} = (E, \mathbf{H}, J, \mathbf{K})^\top$ ,  $\mathbf{U}' = (E', \mathbf{H}', J', \mathbf{K}')^\top \in D(\mathbb{A})$ , then it follows from [\(3.1.21\)](#), [\(3.1.22\)](#) and [\(3.1.23\)](#):

$$\begin{aligned} \langle \mathbb{A} \mathbf{U}, \mathbf{U}' \rangle_{2D} = i \left[ \langle \text{curl } \mathbf{H}, E' \rangle_{\mathbb{R}^2} - \langle \Pi_2 J, E' \rangle_{\mathbb{R}^2} - \langle \mathbf{curl } E, \mathbf{H}' \rangle_{\mathbb{R}^2} \right. \\ \left. - \langle \Pi_2 \mathbf{K}, \mathbf{H}' \rangle_{\mathbb{R}^2} + \langle R_2 E, J' \rangle_Q + \langle \mathbf{R}_2 \mathbf{H}, \mathbf{K}' \rangle_Q \right]. \end{aligned} \quad (3.1.24)$$

By parts integration it is known that

$$\forall (u, \mathbf{v}) \in H^1(\mathbb{R}^2) \times \mathbf{H}_{\text{curl}}(\mathbb{R}^2) \quad \langle u, \text{curl } \mathbf{v} \rangle_{\mathbb{R}^2} = \langle \mathbf{curl } u, \mathbf{v} \rangle \quad (3.1.25)$$

and by the fact  $\Pi_2^* = R_2$  and  $\Pi_2^* = \mathbf{R}_2$ , respectively, for the classic inner products (weighted by 1) in  $L^2(\mathbb{R}^2)$  and  $\mathbf{L}^2(\mathbb{R}^2)$ , then [\(3.1.24\)](#) rewrites as

$$\begin{aligned} \langle \mathbb{A} \mathbf{U}, \mathbf{U}' \rangle_{2D} = i \left[ \langle \mathbf{H}, \mathbf{curl } E' \rangle_{\mathbb{R}^2} - \langle J, R_2 E' \rangle_{\mathbb{R}^2} - \langle E, \mathbf{curl } \mathbf{H}' \rangle_{\mathbb{R}^2} \right. \\ \left. - \langle \mathbf{K}, \mathbf{R}_2 \mathbf{H}' \rangle_{\mathbb{R}^2} + \langle E, \Pi_2 J' \rangle_Q + \langle \mathbf{H}, \Pi_2 \mathbf{K}' \rangle_Q \right] = \langle \mathbf{U}, \mathbb{A} \mathbf{U}' \rangle_{2D}, \end{aligned} \quad (3.1.26)$$

meaning that  $\mathbb{A}$  is a symmetric operator. By classic results  $D(\mathbb{A})$  is dense in  $\mathcal{H}_{2D}$ , so one can define the adjoint operator  $\mathbb{A}^*$  whose domain is defined as

$$D(\mathbb{A}^*) := \left\{ \mathbf{W} \in \mathcal{H}_{2D} \mid \exists \mathbf{W}^* \in \mathcal{H}_{2D} \forall \mathbf{V} \in D(\mathbb{A}) : \langle \mathbf{W}, \mathbb{A} \mathbf{V} \rangle_{2D} = \langle \mathbf{W}^*, \mathbf{V} \rangle_{2D} \right\}. \quad (3.1.27)$$

Since  $\mathbb{A}$  is a symmetric operator, then it only rests to prove that  $D(\mathbb{A}) = D(\mathbb{A}^*)$  to conclude that  $\mathbb{A}$  is a self-adjoint operator. Moreover, the inclusion  $D(\mathbb{A}) \subseteq D(\mathbb{A}^*)$  is immediately followed from the symmetry, we show the inclusion  $D(\mathbb{A}^*) \subseteq D(\mathbb{A})$ . Let be  $\mathbf{W} = (E_{\mathbf{W}}, \mathbf{H}_{\mathbf{W}}, J_{\mathbf{W}}, \mathbf{K}_{\mathbf{W}})^\top \in D(\mathbb{A}^*)$ , we have to prove that  $(E_{\mathbf{W}}, \mathbf{H}_{\mathbf{W}}) \in H^1(\mathbb{R}^2) \times \mathbf{H}_{\text{curl}}(\mathbb{R}^2)$ . Consider the adjoint  $\mathbf{W}^* = (E_{\mathbf{W}}^*, \mathbf{H}_{\mathbf{W}}^*, J_{\mathbf{W}}^*, \mathbf{K}_{\mathbf{W}}^*)^\top \in \mathcal{H}_{2D}$  which holds

$$\forall \mathbf{V} \in D(\mathbb{A}) : \langle \mathbf{W}, \mathbb{A} \mathbf{V} \rangle_{2D} = \langle \mathbf{W}^*, \mathbf{V} \rangle_{2D}. \quad (3.1.28)$$

By choosing  $\mathbf{V} = (E, \mathbf{0}, 0, \mathbf{0})^\top \in D(\mathbb{A})$  with  $E \in \mathcal{D}(\mathbb{R}^2) := \mathcal{C}_c^\infty(\mathbb{R}^2)$ , then [\(3.1.28\)](#) is equivalent to

$$\langle \mathbf{H}_{\mathbf{W}}, \mathbf{curl } E \rangle_{\mathbb{R}^2} = \langle J_{\mathbf{W}}, R_2 E \rangle_Q + i \varepsilon_0 \langle E_{\mathbf{W}}^*, E \rangle_{\mathbb{R}^2},$$

and since  $\Pi_2 = R_2^*$  and using integration by parts, we deduce

$$\begin{aligned} \langle \mathbf{curl } \mathbf{H}_{\mathbf{W}}, E \rangle &= \langle \mathbf{H}_{\mathbf{W}}, \mathbf{curl } E \rangle_{\mathbb{R}^2} \\ &= \langle \Pi_2 J_{\mathbf{W}}, E \rangle_{\mathbb{R}^2} + \langle i \varepsilon_0 E_{\mathbf{W}}^*, E \rangle_{\mathbb{R}^2} \\ &= \langle \Pi_2 J_{\mathbf{W}} + i \varepsilon_0 E_{\mathbf{W}}^*, E \rangle_{\mathbb{R}^2} \end{aligned} \quad (3.1.29)$$

where  $\langle \cdot, \cdot \rangle$  stands for the duality product in  $\mathcal{D}(\mathbb{R}^2)' \times \mathcal{D}(\mathbb{R}^2)$ , we observe that [\(3.1.29\)](#) implies



that

$$\operatorname{curl} \mathbf{H}_{\mathbf{W}} = \Pi_2 J_{\mathbf{W}} + i\varepsilon_0 E_{\mathbf{W}}^* \quad \text{in } \mathcal{D}(\mathbb{R}^2)', \quad (3.1.30)$$

and provided that  $\Pi_2 J_{\mathbf{W}} + i\varepsilon_0 E_{\mathbf{W}}^* \in L^2(\mathbb{R}^2)$ , then

$$\operatorname{curl} \mathbf{H}_{\mathbf{W}} = \Pi_2 J_{\mathbf{W}} + i\varepsilon_0 E_{\mathbf{W}}^* \quad \text{in } L^2(\mathbb{R}^2), \quad (3.1.31)$$

which implies that  $\mathbf{H}_{\mathbf{W}} \in \mathbf{H}_{\operatorname{curl}}(\mathbb{R}^2)$ . In the same way, by choosing  $\mathbf{V} = (0, \mathbf{H}, 0, \mathbf{0})^\top \in D(\mathbb{A})$  with  $\mathbf{H} \in \mathcal{D}(\mathbb{R}^2) := (\mathcal{D}(\mathbb{R}^2))^2$ , one can show that  $E_{\mathbf{W}} \in H^1(\mathbb{R}^2)$ . We conclude that  $\mathbf{W} \in D(\mathbb{A})$ .  $\square$

### 3.2 Fourier decomposition of the TM Hamiltonian $\mathbb{A}$ : the reduced Hamiltonians $\mathbb{A}_k$ .

Due to the invariance of our medium with respect to the  $x$ -axis, we can allow us to reduce the spectral analysis of the operator  $\mathbb{A}$  to the spectral theory of a family of self-adjoint operators  $(\mathbb{A}_k)_{k \in \mathbb{R}}$  whose domains are functions depending only on the variable  $y$ . This perspective will reduce the analysis of the guide modes into a scalar problem.

Let be  $\mathcal{G}$  the 1D Fourier transform along the  $x$ -axis, in other words

$$(\mathcal{F}u)(k) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} u(x) \exp(-ikx) dx, \quad \forall u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \quad (3.2.1)$$

which can be extended to a isometric transformation from  $L^2(\mathbb{R}_x)$  to  $L^2(\mathbb{R}_k)$ . For functions of both variables  $x$  and  $y$ , we still denote by  $\mathcal{G}$  be the partial Fourier transform in the  $y$ -direction. Fubini's theorem assures us that the partial Fourier transform for fields of both variables  $x$  and  $y$ ,  $\mathbf{U} \in \mathcal{H}_{2D}$ , is such that

$$\widehat{\mathbf{U}}(k, \cdot) := \mathcal{F}\mathbf{U}(k, \cdot) \in \mathcal{H}_{1D}, \quad \text{for a.e } k \in \mathbb{R}, \quad \text{where}$$

$$\mathcal{H}_{1D} := L^2(\mathbb{R}) \times \mathbf{L}^2(\mathbb{R}) \times L^2((-L, L)) \times L^2((-L, L)).$$

The Hilbert space  $\mathcal{H}_{1D}$  is endowed with the inner product  $\langle \cdot, \cdot \rangle_{1D}$  defined in an analogous way as the expression (3.1.21) except that  $L^2$  inner products are now defined on one-dimensional domains,  $\mathbb{R}$  (instead of  $\mathbb{R}^2$ ) and  $(-L, L)$  (instead of  $Q$ ).

By applying  $\mathcal{G}$  to our *TM* transmission problem (3.1.18) leads us to introduce a family of transmission problems indexed by  $k \in \mathbb{R}$  which read as

$$\widehat{\mathbf{U}}(\cdot, t) = \left( \widehat{E}(\cdot, t), \widehat{\mathbf{H}}(\cdot, t), \widehat{J}(\cdot, t), \widehat{\mathbf{K}}(\cdot, t) \right) \in H^1(\mathbb{R}) \times \mathbf{H}_{\operatorname{curl}_k}(\mathbb{R}) \times L^2(-L, L) \times \mathbf{L}^2(-L, L) :$$

$$(TM_k) \begin{cases} \varepsilon_0 \partial_t \widehat{E} - \operatorname{curl}_k \widehat{\mathbf{H}} + \Pi \widehat{J} = -\widehat{J}_s, & \text{in } \mathbb{R} \times [0, \infty), & (3.2.2a) \\ \mu_0 \partial_t \widehat{\mathbf{H}} + \operatorname{curl}_k \widehat{E} + \Pi \widehat{\mathbf{K}} = 0, & \text{in } \mathbb{R} \times [0, \infty), & (3.2.2b) \\ \partial_t \widehat{J} = \varepsilon_0 \Omega_e^2 R \widehat{E}, & \text{in } (-L, L) \times [0, \infty), & (3.2.2c) \\ \partial_t \widehat{\mathbf{K}} = \mu_0 \Omega_m^2 \mathbf{R} \widehat{\mathbf{H}}, & \text{in } (-L, L) \times [0, \infty). & (3.2.2d) \end{cases}$$

$$\text{where } \operatorname{curl}_k \mathbf{u} := \left( \frac{d\mathbf{u}}{dy}, -ik\mathbf{u} \right)^\top, \quad \operatorname{curl}_k \mathbf{u} := ik u_y - \frac{d u_x}{dy}, \quad \text{for } \mathbf{u} = (u_x, u_y)^\top,$$

and the operators  $\Pi, \mathbf{\Pi}$ , and  $R, \mathbf{R}$  are defined as in (3.1.13) and (3.1.14) but over the domain

$(-L, L)$  corresponding to the  $y$ -variable. Finally

$$\mathbf{H}_{\text{curl}_k}(\mathbb{R}) := \{\mathbf{u} \in \mathbf{L}^2(\mathbb{R}) \mid \text{curl}_k \mathbf{u} \in L^2(\mathbb{R})\} = H^1(\mathbb{R}) \times L^2(\mathbb{R}).$$

Analogously to the 2D *TM* transmission problem (3.1.18), one translates (3.2.2) into a Schrödinger evolution equation, by introducing a family of operators  $(\mathbb{A}_k)_{k \in \mathbb{R}}$  in  $\mathcal{H}_{1D}$  related to  $\mathbb{A}$  by

$$\mathcal{F}(\mathbb{A}U)(\cdot, k) = \mathbb{A}_k(\widehat{U})(\cdot, k), \quad \text{for a.e } k \in \mathbb{R}, \quad (3.2.3)$$

$$\mathbb{A}_k \widehat{U} := \mathcal{A}_k \widehat{U}, \quad \text{for all } \widehat{U} \in D(\mathbb{A}_k) := H^1(\mathbb{R}) \times \mathbf{H}_{\text{curl}_k}(\mathbb{R}) \times L^2(-L, L) \times \mathbf{L}^2(-L, L) \subset \mathcal{H}_{1D}, \quad (3.2.4)$$

where  $\mathcal{A}_k$  is deduced from the definition of  $\mathcal{A}$  (3.1.23) by replacing the  $x$ -derivative by the product  $ik$ , i.e.,

$$\mathcal{A}_k := i \begin{pmatrix} 0 & \varepsilon_0^{-1} \text{curl}_k & -\varepsilon_0^{-1} \Pi & 0 \\ -\mu_0^{-1} \mathbf{curl}_k & 0 & 0 & -\mu_0^{-1} \mathbf{\Pi} \\ \varepsilon_0 \Omega_e^2 R & 0 & 0 & 0 \\ 0 & \mu_0 \Omega_m^2 \mathbf{R} & 0 & 0 \end{pmatrix}. \quad (3.2.5)$$

Note once more time that the transmission conditions (3.1.6) are satisfied insofar as

$$[\widehat{E}]_{y=-L} = 0 = [\widehat{E}]_{y=+L} \quad \text{and} \quad [\widehat{H}]_{y=-L} = 0 = [\widehat{H}]_{y=+L}, \quad (3.2.6)$$

the which is summarized in the belonging  $(\widehat{E}, \widehat{H}) \in H^1(\mathbb{R}) \times \mathbf{H}_{\text{curl}_k}(\mathbb{R})$ .

Under this basis, it comes naturally:

**Proposition 3.2.1**

The operator  $\mathbb{A}_k : D(\mathbb{A}_k) \subset \mathcal{H}_{1D} \longrightarrow \mathcal{H}_{1D}$  is self-adjoint.

*Proof.* Similar proof as in Proposition 3.1.5. □

### 3.3 Spectral theory of the reduced Hamiltonian operators $\mathbb{A}_k$

#### 3.3.1 General structure of the spectrum of $\mathbb{A}_k$

Since  $\mathbb{A}_k$  is a self-adjoint operator, we know that its whole *spectrum*,  $\sigma(\mathbb{A}_k)$  is a real subset composed uniquely by its *point spectrum* ( $\sigma_p(\mathbb{A}_k)$ ) and the *continuum spectrum* ( $\sigma_c(\mathbb{A}_k)$ ).

For the *point spectrum*, we look for the real values  $\omega \in \mathbb{R}$  such that

$$\mathbb{A}_k \widehat{U} = \omega \widehat{U}, \quad \text{for some } \widehat{U} = \left( \widehat{E}, \widehat{H}, \widehat{J}, \widehat{K} \right) \in D(\mathbb{A}_k) / \{0\}, \quad (3.3.1)$$

or equivalently, by replacing the definition of  $\mathbb{A}_k$  (see (3.2.4) and (3.2.5)),

$$\begin{cases} i \varepsilon_0^{-1} \left( \text{curl}_k \widehat{H} - \Pi \widehat{J} \right) = \omega \widehat{E}, & (3.3.2a) \\ i \mu_0^{-1} \left( -\mathbf{curl}_k \widehat{E} - \mathbf{\Pi} \widehat{K} \right) = \omega \widehat{H}, & (3.3.2b) \\ i \varepsilon_0 \Omega_e^2 R \widehat{E} = \omega \widehat{J}, & (3.3.2c) \\ i \mu_0 \Omega_m^2 \mathbf{R} \widehat{H} = \omega \widehat{K}. & (3.3.2d) \end{cases}$$

In this sense, every non trivial solution  $\widehat{U} = (\widehat{E}, \widehat{H}, \widehat{J}, \widehat{K})$  of (3.3.2) belonging to  $D(\mathbb{A}_k)$  is an eigenfunction of  $\mathbb{A}_k$  associated to the eigenvalue  $\omega \in \mathbb{R}$ . Inasmuch as  $\mathbb{R}^2$  is an unbounded domain, as the reduced Hamiltonians does not have compact resolvent, those are not completely formed by punctual spectrum. Only the eigenfunctions cannot form by themselves a family basis (also called *modal family*). In order to complete this modal family, we need to take in consideration other proper functions: *generalized eigenfunctions*, which consist by relaxing the belonging condition to  $D(\mathbb{A}_k)$ . Therefore we are interested into solving (3.3.2) inside a functional space bigger than the solutions of finite energy.

### 3.3.2 Modal decomposition of $\mathbb{A}$ : Guided modes and point spectrum of $\mathbb{A}_k$

The core idea of a *modal decomposition* consists on representing at every instant the solution of an evolution equation (e.g. (3.1.19)) as the superposition of elements in a family of functions depending only on the spatial variable, the elements of this family are called *free vibration modes* (or *proper modes*, or simply *mode*). In this sense, the solution  $U$  of (3.1.19) can be rewritten as the superposition (an infinite sum for a countable *modal family*, or an integral for the continuous case) where each one of the terms is the product of the form

$$\Re(\tilde{U}(x, y) e^{-i\omega t}), \quad (3.3.3)$$

in other words, the product of a *mode*  $\tilde{U}(x, y)$  and a function depending only on the time, allowing the variable separation between the spatial variable with the time variable.

In addition, the geometry of our problem leads us to find a particular subfamily of *proper modes*, called *guided modes*, which are given by the form inspired in a *variable separations*

$$\tilde{U}(x, y) = \widehat{U}(y) e^{ikx}, \quad (3.3.4)$$

the function  $\widehat{U}(y)$  is called a *reduced mode*.

Note that any solution to the homogeneous evolution equation (3.1.19) of the form (3.3.3) must satisfy

$$\mathbb{A} \tilde{U} = \omega \tilde{U}, \quad (3.3.5)$$

meaning that to find the (non trivial) *free vibration modes* of our problem implies to find the eigenfunctions of the operator  $\mathbb{A}$ .

Similarly, by replacing  $\partial_x$  for the multiplication by  $ik$  in the definition of  $\mathbb{A}$ , then one can notice that a *reduced proper mode* defining a guide mode as in (3.3.4), results into an eigenfunction of  $\mathbb{A}_k$  associated to the eigenvalue  $\omega$  :

$$\mathbb{A}_k \widehat{U} = \omega \widehat{U}. \quad (3.3.6)$$

(The guided modes are the modes localized in the slab material representing evanescent solutions out of the slab, conducting us to find  $L^2(\mathbb{R}_y)$  solutions.) Therefore, to find the *guided modes* of the whole Hamiltonian  $\mathbb{A}$  rewrites as to study  $\sigma_p(\mathbb{A}_k), k \in \mathbb{R}$ . The *reduced modes* can be divided into two types: the *reduced proper modes* (or *reduced eigenfunctions*) and the *reduced generalized modes* (or *reduced generalized eigenfunctions*). The *reduced proper modes* are the solutions  $\widehat{U}$  to (3.3.6) in  $D(\mathbb{A}_k)$ , these are solutions of finite energy and they describe the point spectrum  $\sigma_p(\mathbb{A}_k)$ . Whereas the *reduced generalized modes* are bounded solutions of (3.3.6), they do not decay sufficiently at infinity to belong to  $D(\mathbb{A}_k)$ , thus they represent solutions of infinite energy. It can be formally proved in subsequent discussions that the *reduced generalized modes* characterize the continuous spectrum  $\sigma_c(\mathbb{A}_k)$ . This last spectrum consists of intervals for which (3.3.6) has no trivial solutions in  $D(\mathbb{A}_k)$  but admits bounded solutions.

### 3.3.3 Point spectrum of $\mathbb{A}_k$ : reduction to a scalar problem

Insofar, the Fourier transformation (or in more humble means the variable separation) has allowed us to introduced the reduced Hamiltonians  $\mathbb{A}_k$ . Thus the study of the spectrum  $\sigma(\mathbb{A}_k)$  allows us to reduce the problem into a one-dimensional problem, the latter enables the explicit computation of solutions of the system (3.3.2). For this purpose, we express the vector solution  $\widehat{\mathbf{U}} = (\widehat{E}, \widehat{\mathbf{H}}, \widehat{J}, \widehat{\mathbf{K}}) \in D(\mathbb{A}_k)$  in terms of the single scalar variable  $\widehat{E}$ . In order to write  $\widehat{\mathbf{H}}$  and  $\widehat{\mathbf{K}}$  in terms of  $\widehat{E}$ , we need to assume that  $\omega$  in (3.3.2) is non zero, we will treat first the particular case  $\omega = 0$ . We assume in the whole analysis that  $k \neq 0$ .

**CASE  $\omega = 0$ .**

By making  $\omega = 0$  in (3.3.2) it is obtained that

$$\begin{cases} \Pi \widehat{J} = \text{curl}_k \widehat{\mathbf{H}}, & (3.3.7a) \\ \Pi \widehat{\mathbf{K}} = -\text{curl}_k \widehat{E}, & (3.3.7b) \\ R \widehat{E} = 0, & (3.3.7c) \\ \mathbf{R} \widehat{\mathbf{H}} = 0. & (3.3.7d) \end{cases}$$

From (3.3.7a) and (3.3.7d) we have

$$\widehat{J} = R \Pi \widehat{J} = R \text{curl}_k \widehat{\mathbf{H}} = \text{curl}_k \mathbf{R} \widehat{\mathbf{H}} = 0, \quad (3.3.8)$$

and analogously from (3.3.7b) and (3.3.7c),  $\widehat{\mathbf{K}} = 0$ . Hence all the fields  $\widehat{E}, \widehat{\mathbf{H}}, \widehat{J}, \widehat{\mathbf{K}}$  are 0 in  $(-L, L)$  (in the Drude material). On the other hand in  $\mathbb{R}/(-L, L)$  (the vacuum), the equations (3.3.7a), (3.3.7b) and the transmission conditions on the interface (3.2.6) imply that

$$\begin{cases} \text{curl}_k \widehat{E} = 0, & \text{in } \mathbb{R}/(-L, L), \\ \widehat{E}(-L) = 0 = \widehat{E}(L), \end{cases} \quad \begin{cases} \text{curl}_k \widehat{\mathbf{H}} = 0, & \text{in } \mathbb{R}/(-L, L), \\ \widehat{\mathbf{H}}(-L) = 0 = \widehat{\mathbf{H}}(L), \end{cases} . \quad (3.3.9)$$

Since we look for solutions  $\widehat{\mathbf{U}}$  of (3.3.7) in  $D(\mathbb{A}_k)$  it yields

$$\begin{cases} \widehat{E}|_{\mathbb{R}/(-L, L)} = 0, \\ \widehat{\mathbf{H}}|_{\mathbb{R}/(-L, L)} = \mathbf{grad}_k \widehat{\phi}, \widehat{\phi} \in H_0^1(\mathbb{R}/(-L, L)) \end{cases} , \quad (3.3.10)$$

$$\text{where } \mathbf{grad}_k u = \left( ik u, \frac{d}{dy} u \right)^\top ,$$

we conclude that  $\widehat{\mathbf{U}}$  takes the form

$$\widehat{\mathbf{U}} = (0, \widetilde{\Pi} \mathbf{grad}_k \widehat{\phi}, 0, 0)^\top, \quad (3.3.11)$$

where  $\widetilde{\Pi}$  is *extension-by-zero* over  $\mathbb{R}/(-L, L)$  :

$$\begin{aligned} \widetilde{\Pi} &:= \mathbf{L}^2(\mathbb{R}/(-L, L)) \longrightarrow \mathbf{L}^2(\mathbb{R}) \\ \mathbf{f} &\longmapsto \mathbf{f} \text{ in } \mathbb{R}/(-L, L) \quad . \\ &\quad \mathbf{0} \text{ in } (-L, L) \end{aligned} \quad (3.3.12)$$

We assume for the rest of the computations that  $\omega \neq 0$ . From (3.3.2c) and (3.3.2d) it follows

$$\widehat{\mathbf{J}} = \frac{i\varepsilon_0 \Omega_e^2}{\omega} R \widehat{\mathbf{E}}, \quad \widehat{\mathbf{K}} = \frac{i\mu_0 \Omega_m^2}{\omega} \mathbf{R} \widehat{\mathbf{H}}, \quad (3.3.13)$$

the last ones being substituted in (3.3.2a) and (3.3.2b) combined with the identities  $\Pi R f = \mathbb{1}_{(-L,L)} f$  and  $\mathbf{\Pi} \mathbf{R} \mathbf{f} = \mathbb{1}_{(-L,L)} \mathbf{f}$  (where  $\mathbb{1}_{(-L,L)}$  is the indicator function on  $(-L, L)$ ) yields to

$$\begin{cases} \operatorname{curl}_k \widehat{\mathbf{H}} = -i\omega\varepsilon_0 \left(1 - \frac{\Omega_e^2}{\omega^2} \mathbb{1}_{(-L,L)}\right) \widehat{\mathbf{E}}, \\ \mathbf{curl}_k \widehat{\mathbf{E}} = i\omega\mu_0 \left(1 - \frac{\Omega_m^2}{\omega^2} \mathbb{1}_{(-L,L)}\right) \widehat{\mathbf{H}}, \end{cases} \quad (3.3.14a)$$

$$\quad (3.3.14b)$$

We can observe that in order to express  $\widehat{\mathbf{H}}$  in terms of  $\widehat{\mathbf{E}}$ , it is necessary to have  $\left(1 - \frac{\Omega_m^2}{\omega^2} \mathbb{1}_{(-L,L)}\right) \neq 0$  in  $\mathbb{R}$ , which is only possible if  $\omega \neq \pm\Omega_m$ . We examine this particular case as follows.

**CASE**  $\omega \in \{-\Omega_m, \Omega_m\}$ .

In this case and from (3.3.14b), we obtain that  $\mathbf{curl}_k \widehat{\mathbf{E}} = \mathbf{0}$  in  $(-L, L)$ , which implies  $\widehat{\mathbf{E}} = 0$  in  $(-L, L)$  (Drude material) and, recalling (3.3.2c),  $\widehat{\mathbf{J}} = 0$ . On the other hand, in  $\mathbb{R}/(-L, L)$  (vacuum), by combining (3.3.14a) and (3.3.14b) it follows

$$k^2 \widehat{\mathbf{E}} - \frac{d^2}{dy^2} \widehat{\mathbf{E}} = \operatorname{curl}_k \mathbf{curl}_k \widehat{\mathbf{E}} = i\omega\mu_0 \operatorname{curl}_k \widehat{\mathbf{H}} = \varepsilon_0 \mu_0 \omega^2 \widehat{\mathbf{E}}, \quad \text{in } \mathbb{R}/(-L, L), \quad (3.3.15)$$

and from the transmission condition (3.2.6), we have  $\widehat{\mathbf{E}}(-L) = 0 = \widehat{\mathbf{E}}(L)$ . Then we have that

$$\begin{cases} -\frac{d^2}{dy^2} \widehat{\mathbf{E}} + (k^2 - \varepsilon_0 \mu_0 \Omega_m^2) \widehat{\mathbf{E}} = 0, \text{ in } \mathbb{R}/(-L, L), \\ \widehat{\mathbf{E}}(-L) = 0 = \widehat{\mathbf{E}}(L). \end{cases} \quad (3.3.16)$$

Since we are still looking for solutions  $\widehat{\mathbf{E}} \in H^1(\mathbb{R})$ , and the equation (3.3.16) does not have non trivial solutions in  $H^1(\mathbb{R}/(-L, L))$ , we conclude that  $\widehat{\mathbf{E}} = 0$  in  $\mathbb{R}/(-L, L)$ . Hence  $\widehat{\mathbf{E}} \equiv 0$ .

Now we study the magnetic field  $\widehat{\mathbf{H}}$ . The nullity of the electric field  $\widehat{\mathbf{E}}$  and (3.3.14b) shows that  $\widehat{\mathbf{H}} = 0$  in  $\mathbb{R}/(-L, L)$  (vacuum). On the other side in  $(-L, L)$  (Drude material), from (3.3.14a) combined with the transmission condition in the interface  $\widehat{\mathbf{H}}(-L) = 0 = \widehat{\mathbf{H}}(L)$  (since  $\widehat{\mathbf{H}} = 0$  in  $\mathbb{R}/(-L, L)$ ), we get

$$\begin{cases} \operatorname{curl}_k \widehat{\mathbf{H}} = 0, & \text{in } (-L, L), \\ \widehat{\mathbf{H}}(-L) = 0 = \widehat{\mathbf{H}}(L), \end{cases}, \quad (3.3.17)$$

thus  $\widehat{\mathbf{H}}|_{(-L,L)} = \mathbf{grad}_k \hat{\phi}$  for some  $\hat{\phi} \in H_0^1((-L, L))$ , and thus, recalling (3.3.2d),  $\widehat{\mathbf{K}} = \pm i\mu_0 \Omega_m \mathbf{grad}_k \hat{\phi}$  according to  $\omega = \pm\Omega_m$ . Finally in this case,  $\widehat{\mathbf{U}}$  takes the form

$$\widehat{\mathbf{U}} = (0, \mathbf{\Pi} \mathbf{grad}_k \hat{\phi}, 0, \pm i\mu_0 \Omega_m \mathbf{grad}_k \hat{\phi})^\top, \quad \hat{\phi} \in H_0^1((-L, L)). \quad (3.3.18)$$

All these results for  $\omega \in \{-\Omega_m, 0, \Omega_m\}$  can be summarized in the following proposition:

**Proposition 3.3.1**

For all  $k \in \mathbb{R}^*$ , the values  $-\Omega_m, 0$  and  $\Omega_m$  are eigenvalues of infinite multiplicity of  $\mathbb{A}_k$  whose respective associated eigenspaces  $\text{Ker}(\mathbb{A}_k + \Omega_m)$ ,  $\text{Ker}(\mathbb{A}_k)$  and  $\text{Ker}(\mathbb{A}_k - \Omega_m)$  are given by

$$\begin{aligned} \text{Ker}(\mathbb{A}_k \pm \Omega_m) &= \left\{ (0, \mathbf{\Pi} \mathbf{grad}_k \hat{\phi}, 0, \pm i \mu_0 \Omega_m \mathbf{grad}_k \hat{\phi})^\top \mid \hat{\phi} \in H_0^1((-L, L)) \right\}, \\ \text{Ker}(\mathbb{A}_k) &= \left\{ (0, \tilde{\mathbf{\Pi}} \mathbf{grad}_k \hat{\phi}, 0, 0)^\top \mid \hat{\phi} \in H_0^1(\mathbb{R}/(-L, L)) \right\}. \end{aligned} \quad (3.3.19)$$

Moreover, the orthogonal space  $(\text{Ker}(\mathbb{A}_k) \oplus \text{Ker}(\mathbb{A}_k + \Omega_m) \oplus \text{Ker}(\mathbb{A}_k - \Omega_m))^\perp$  coincides with

$$\mathcal{H}_{1D}(\text{div}_k 0) := \left\{ \left( \widehat{E}, \widehat{H}, \widehat{J}, \widehat{K} \right)^\top \in \mathcal{H}_{1D} \mid \begin{array}{l} \text{div}_k \widehat{H} = 0 \text{ in } \mathbb{R}/\{-L, L\} \text{ and} \\ \text{div}_k \widehat{K} = 0 \text{ in } (-L, L) \end{array} \right\}. \quad (3.3.20)$$

*Proof.* We have previously seen the contentions

$$\begin{aligned} \text{Ker}(\mathbb{A}_k \pm \Omega_m) &\subseteq \left\{ (0, \mathbf{\Pi} \mathbf{grad}_k \hat{\phi}, 0, \pm i \mu_0 \Omega_m \mathbf{grad}_k \hat{\phi})^\top \mid \hat{\phi} \in H_0^1((-L, L)) \right\}, \\ \text{Ker}(\mathbb{A}_k) &\subseteq \left\{ (0, \tilde{\mathbf{\Pi}} \mathbf{grad}_k \hat{\phi}, 0, 0)^\top \mid \hat{\phi} \in H_0^1(\mathbb{R}/(-L, L)) \right\}. \end{aligned}$$

Conversely, by stepping back the previous computations, one deduce that for all  $\hat{\phi} \in H_0^1((-L, L))$ , then the vector  $(0, \mathbf{\Pi} \mathbf{grad}_k \hat{\phi}, 0, \pm i \mu_0 \Omega_m \mathbf{grad}_k \hat{\phi})^\top$  (resp.  $(0, \tilde{\mathbf{\Pi}} \mathbf{grad}_k \hat{\phi}, 0, 0)^\top$ ) belongs to  $D(\mathbb{A}_k)$  and satisfies (3.3.2) for  $\omega = \pm \Omega_m$  (resp.  $\omega = 0$ ), in other words

$$(0, \mathbf{\Pi} \mathbf{grad}_k \hat{\phi}, 0, \pm i \mu_0 \Omega_m \mathbf{grad}_k \hat{\phi})^\top \in \text{Ker}(\mathbb{A}_k \mp \Omega_m) \text{ (resp. } (0, \tilde{\mathbf{\Pi}} \mathbf{grad}_k \hat{\phi}, 0, 0)^\top \in \text{Ker}(\mathbb{A}_k))$$

. This proves the identities (3.3.19).

On the other hand, one can notice the following equivalences derived from (3.3.19)

$$\begin{aligned} &\left( \widehat{E}, \widehat{H}, \widehat{J}, \widehat{K} \right)^\top \in \mathbf{Ker}(\mathbb{A}_k)^\perp \\ \iff &\int_{\mathbb{R} \setminus (-L, L)} \widehat{H} \cdot \overline{\mathbf{grad}_k \hat{\phi}} \, dx = 0, \quad \forall \hat{\phi} \in H_0^1(\mathbb{R}/(-L, L)) \\ \iff &\text{div}_k \widehat{H} = 0 \text{ in } H^{-1}(\mathbb{R} \setminus (-L, L)). \end{aligned} \quad (3.3.21)$$

Similarly,

$$\begin{aligned} &\left( \widehat{E}, \widehat{H}, \widehat{J}, \widehat{K} \right)^\top \in \mathbf{Ker}(\mathbb{A}_k \pm \Omega_m)^\perp \\ \iff &\int_{(-L, L)} (\widehat{H} \mp i \mu_0 \Omega_m \widehat{K}) \cdot \overline{\mathbf{grad}_k \hat{\phi}} \, dx = 0, \quad \forall \hat{\phi} \in H_0^1(-L, L) \\ \iff &\text{div}_k (\widehat{H} \mp i \mu_0 \Omega_m \widehat{K}) = 0 \text{ in } H^{-1}(-L, L), \end{aligned}$$

Henceforth

$$\begin{aligned}
 & \left( \widehat{E}, \widehat{H}, \widehat{J}, \widehat{K} \right)^\top \in \left( \text{Ker}(\mathbb{A}_k + \Omega_m) \oplus \text{Ker}(\mathbb{A}_k - \Omega_m) \right)^\perp \\
 \iff & \text{div}_k(\widehat{H} - i\mu_0\Omega_m\widehat{K}) = 0 \text{ in } H^{-1}(-L, L) \text{ and } \text{div}_k(\widehat{H} + i\mu_0\Omega_m\widehat{K}) = 0 \text{ in } H^{-1}(-L, L) \\
 \iff & \text{div}_k\widehat{H} = \text{div}_k\widehat{K} = 0 \text{ in } H^{-1}(-L, L).
 \end{aligned} \tag{3.3.22}$$

The result follows by combining (3.3.21) and (3.3.22).  $\square$

**CASE**  $\omega \notin \{-\Omega_m, 0, \Omega_m\}$ .

We introduce for  $y \in \mathbb{R} \setminus \{-L, L\}$

$$\varepsilon(y, \omega) := \varepsilon_0 \left( 1 - \frac{\Omega_e^2}{\omega^2} \mathbb{1}_{(-L, L)}(y) \right), \quad \mu(y, \omega) := \mu_0 \left( 1 - \frac{\Omega_m^2}{\omega^2} \mathbb{1}_{(-L, L)}(y) \right), \tag{3.3.23}$$

which are piece-wise functions, constants on the domains  $(-L, L)$  and  $\mathbb{R}/(-L, L)$  for every  $\omega \neq 0$ . Hence we can rewrite (3.3.14) as

$$\begin{cases} \text{curl}_k \widehat{H} = -i\omega \varepsilon(\cdot, \omega) \widehat{E}, \\ \mathbf{curl}_k \widehat{E} = i\omega \mu(\cdot, \omega) \widehat{H}. \end{cases} \tag{3.3.24a}$$

$$\tag{3.3.24b}$$

If in addition, we assume  $\omega \notin \{-\Omega_m, \Omega_m\}$ , then it follows that  $\mu(\omega) \neq 0$ , hence

$$\begin{cases} \text{curl}_k \left( \frac{\mathbf{curl}_k \widehat{E}}{\mu(\cdot, \omega)} \right) - \omega^2 \varepsilon(\cdot, \omega) \widehat{E} = 0, \\ \widehat{H} = -\frac{i}{\omega \mu(\cdot, \omega)} \mathbf{curl}_k \widehat{E}. \end{cases} \tag{3.3.25a}$$

$$\tag{3.3.25b}$$

Thus, from (3.3.2c), (3.3.2d) and (3.3.25b):

$$\widehat{U} = \mathbb{V}_{k, \omega} \widehat{E}, \quad \mathbb{V}_{k, \omega} u = \left( u, -\frac{i}{\omega \mu(\cdot, \omega)} \mathbf{curl}_k u, \frac{i\varepsilon_0 \Omega_e^2}{\omega} R u, \frac{\mu_0 \Omega_m^2}{\omega^2 \mu(\cdot, \omega)} \mathbf{R} \mathbf{curl}_k u \right)^\top \tag{3.3.26}$$

Therefore  $\widehat{U}$  is completely described in terms of  $\widehat{E}$ . Our problem reduces to find the electromagnetic field  $\widehat{E}$ , the last drives us to rewrite (3.3.25a) into the scalar Sturm-Liouville differential equation

$$-\frac{d}{dy} \left( \frac{1}{\mu(\cdot, \omega)} \frac{d}{dy} \widehat{E} \right) + \frac{1}{\mu(\cdot, \omega)} (k^2 - \omega^2 \varepsilon(\cdot, \omega) \mu(\cdot, \omega)) \widehat{E} = 0. \tag{3.3.27}$$

In order to guarantee the regularity  $\widehat{U} = \mathbb{V}_{k, \omega} \widehat{E} \in D(\mathbb{A}_k)$ , then we need the belonging  $\widehat{E} \in H^1(\mathbb{R})$ . Such solution  $\widehat{E} \in H^1(\mathbb{R})$  of the equation (3.3.27) must then satisfy the transmission condition on the interface  $\{-L, L\}$  (see (3.2.6)), namely,

$$[\widehat{E}]_{y=\pm L} = 0 \quad \text{and} \quad \left[ \mu(\cdot, \omega)^{-1} \frac{d}{dy} \widehat{E} \right]_{y=\pm L} = 0. \tag{3.3.28}$$

By stepping back the computations we notice that the reciprocal is also true, in other words, we therefore have the following proposition:

**Proposition 3.3.2**

Let be  $\omega \in \mathbb{R} \setminus \{-\Omega_m, 0, \Omega_m\}$ , then  $\widehat{E} \in H^1(\mathbb{R})$  is solution of the ODE (3.3.27) with interface conditions (3.3.28) if and only if  $\widehat{U} = \mathbb{V}_{k,\omega} \widehat{E} \in \mathbf{Ker}(\mathbb{A}_k - \omega)$ . In that case, if such solution  $\widehat{E}$  is non trivial, then  $\omega \in \sigma_p(\mathbb{A}_k)$  with  $\widehat{U} = \mathbb{V}_{k,\omega} \widehat{E}$  as eigenfunction associated to the eigenvalue  $\omega$ .

**3.3.4 Towards the solution of the scalar problem: derivation of the dispersion relations**

In the following, we will consider the functions

$$\varepsilon(y, \omega) := \varepsilon_0 \left( 1 - \frac{\Omega_e^2}{\omega^2} \mathbb{1}_{(-L, L)}(y) \right), \quad \mu(y, \omega) := \mu_0 \left( 1 - \frac{\Omega_m^2}{\omega^2} \mathbb{1}_{(-L, L)}(y) \right), \quad (3.3.29)$$

defined for  $\omega \in \mathbb{R} \setminus \{-\Omega_m, 0, \Omega_m\}$ .

We are interested in solving the Hemholtz equation for  $\omega$  and  $k \in \mathbb{R}$

$$-\frac{d^2}{dy^2} \widehat{E} + (k^2 - \omega^2 \varepsilon(\cdot, \omega) \mu(\cdot, \omega)) \widehat{E} = 0, \quad \text{in } \mathbb{R}/\{-L, L\}. \quad (3.3.30)$$

The equation (3.3.30) is complemented with the *transmission conditions* on the interface,

$$[\widehat{E}]_{y=\pm L} = 0 \quad \text{and} \quad \left[ \mu^{-1}(\cdot, \omega) \frac{d}{dy} \widehat{E} \right]_{y=\pm L} = 0. \quad (3.3.31)$$

We consider the piece-wise function

$$k^2 - \omega^2 \varepsilon(y, \omega) \mu(y, \omega) = \begin{cases} k^2 - \varepsilon_0 \mu_0 \omega^2, & \text{for } |y| \geq L, \\ k^2 - \varepsilon(\omega) \mu(\omega) \omega^2, & \text{for } |y| < L, \end{cases} \quad (3.3.32)$$

where the functions  $\varepsilon(\omega)$  and  $\mu(\omega)$  stand, respectively, for the frequency dependent *electric permittivity* and the frequency dependent *magnetic permeability*, and these are given by

$$\varepsilon(\omega) := \varepsilon_0 \left( 1 - \frac{\Omega_e^2}{\omega^2} \right), \quad \mu(\omega) := \mu_0 \left( 1 - \frac{\Omega_m^2}{\omega^2} \right). \quad (3.3.33)$$

In order to express the solutions to the Helmholtz equation (3.3.30), we need to introduce the principal branch of the *complex square root*  $\sqrt{\cdot}$ , for  $z \in \mathbb{C}$  :

$$\sqrt{z} = |z|^{\frac{1}{2}} \exp\left(i \frac{\theta}{2}\right), \quad \text{where } z = |z| \exp(i\theta), \quad \theta \in (-\pi, \pi]. \quad (3.3.34)$$

In this sense, the last consideration yields to the usual square root for real positive numbers and extend the definition in  $\mathbb{R}^-$  by the limit form the half-plane  $\mathbb{C}^+$ , namely,  $\sqrt{a} = a^{\frac{1}{2}}$  is the real positive square root for  $a \geq 0$  and  $\sqrt{a} = i|a|^{\frac{1}{2}}$  for  $a < 0$ . Therefore, we define

$$\theta_{k,\omega}(y) := \begin{cases} \theta_{k,\omega}^D = \sqrt{k^2 - \varepsilon(\omega) \mu(\omega) \omega^2}, & |y| < L, \\ \theta_{k,\omega}^V = \sqrt{k^2 - \varepsilon_0 \mu_0 \omega^2}, & |y| \geq L. \end{cases} \quad (3.3.35)$$

Now we are able to express the general solution for the equation (3.3.30).



For this purpose, we notice, since (3.3.30) is a second order ODE governed by the operator  $-\frac{d^2}{dy^2} + \theta_{k,\omega}$ , which is invariant with respect to even distributions and odd distributions in  $\mathcal{D}'(\mathbb{R})$ , hence it is implied that

$$\forall \widehat{E} \in \mathcal{D}'(\mathbb{R}), \quad \widehat{E} \text{ is solution of (3.3.30)} \iff \widehat{E}^e \text{ and } \widehat{E}^o \text{ are solutions of (3.3.30)}, \quad (3.3.36)$$

where  $\widehat{E}^e$  and  $\widehat{E}^o$  stand, respectively, for the even and odd parts distributions of  $\widehat{E}$ . Moreover, one can verify that  $\widehat{E}^e$  and  $\widehat{E}^o$  satisfy the transmission conditions (3.3.31) provided  $\widehat{E}$  realizes them as well. This suggests to look for the even and odd functions satisfying (3.3.30). We recall that finding solutions in  $H^1(\mathbb{R})$  is equivalent to search for eigenvalues for  $\mathbb{A}_k$ , meanwhile bounded solutions correspond to generalized eigenvalues. Let us begin the analysis for the finite energy solutions which are described by the belonging to  $H^1(\mathbb{R})$ .

**Even solutions**  $\widehat{E} \in H^1(\mathbb{R})$

If we look for the pair solutions to (3.3.30) belonging to  $H^1(\mathbb{R})$ , then we have the following problem in the half-line  $[0, \infty)$  :

$$\left\{ \begin{array}{l} -\frac{d^2}{dy^2} \widehat{E}^e + \theta_{k,\omega}^2(\cdot) \widehat{E}^e = 0, \text{ in } (0, L) \cup (L, \infty), \\ \frac{d}{dy} \widehat{E}^e|_{y=0} = 0, \end{array} \right. , \quad (3.3.37)$$

in order to find finite energy solutions, we require the condition  $\theta_{k,\omega} > 0$  in the unbounded domain  $(L, \infty)$ , which implies  $\theta_{k,\omega}^V > 0$ , or

$$k^2 > \varepsilon_0 \mu_0 \omega^2. \quad (3.3.38)$$

In this case, the solutions of (3.3.37) with finite energy define a one-dimensional space generated by the canonical solution

$$v_{k,\omega}^e(y) := \begin{cases} \cosh(\theta_{k,\omega}^D y), & y \in [0, L), \\ A_{k,\omega}^e \exp(-\theta_{k,\omega}^V y), & y \in [L, \infty) \end{cases} \quad (3.3.39)$$

where  $\exp(\cdot)$  and  $\cosh(\cdot)$  stand, respectively, for the complex exponential and hyperbolic cosine functions.  $A_{k,\omega}^e$  is a value to determinate in order to  $v_{k,\omega}^e$  verifies the transmission conditions (3.3.31). The solution  $v_{k,\omega}^e$  given by (3.3.39) is canonical in the sense that it holds:

$$v_{k,\omega}^e(0) = 1 \quad \text{and} \quad \left( \frac{1}{\mu(\cdot, \omega)} \frac{d}{dy} v_{k,\omega}^e \right) (0) = 0. \quad (3.3.40)$$

The condition  $[v_{k,\omega}^e]_{y=L} = 0$  gives us directly the value of  $V_{k,\omega}^e$  :

$$\begin{aligned} \cosh(\theta_{k,\omega}^D L) &= A_{k,\omega}^e \exp(-\theta_{k,\omega}^V L) \\ \implies A_{k,\omega}^e &= \exp(\theta_{k,\omega}^V L) \cosh(\theta_{k,\omega}^D L). \end{aligned} \quad (3.3.41)$$

Meanwhile, by substituting the value of  $A_{k,\omega}^e$  in  $\left[ \mu(\cdot, \omega)^{-1} \frac{d}{dy} v^e \right]_{y=L} = 0$ , we obtain the implicit relation between  $k$  and  $\omega$

$$\frac{1}{\mu(\omega)} \theta_{k,\omega}^D \sinh(\theta_{k,\omega}^D L) = -\frac{1}{\mu_0} \theta_{k,\omega}^V V_{k,\omega}^e \exp(-\theta_{k,\omega}^V L). \quad (3.3.42)$$

Hence, by combining (3.3.41) and (3.3.42), we obtain the *dispersion equation for even modes* -the equation relating the wavenumber  $k$  with the frequency  $\omega$ -, namely,

$$\theta_{k,\omega}^D \tanh(\theta_{k,\omega}^D L) = -\frac{\mu(\omega)}{\mu_0} \theta_{k,\omega}^V. \quad (3.3.43)$$

Finally, the even solutions in  $H^1(\mathbb{R})$  to (3.3.30)-(3.3.31) are generated by the basis

$$V_{k,\omega}^e(y) := v_{k,\omega}^e(|y|), \quad y \in \mathbb{R}. \quad (3.3.44)$$

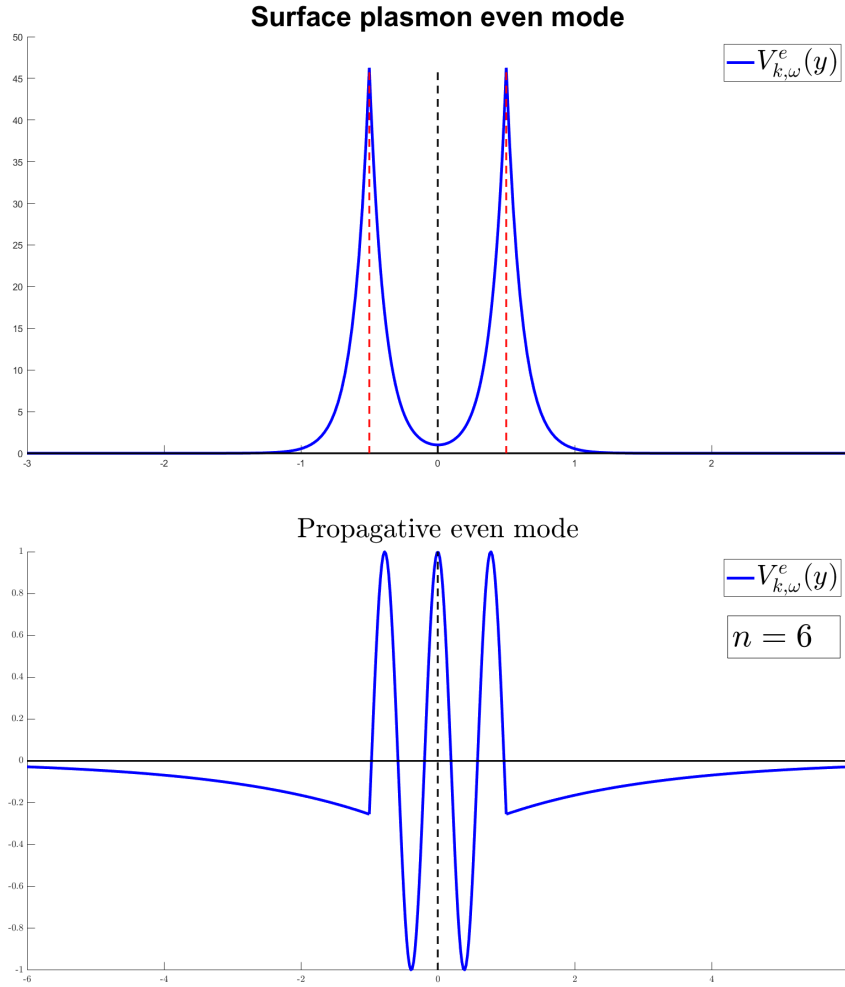


Figure 3.2: Graphic representation of the even canonical mode  $V_{k,\omega}^o$  : (top) evanescent mode  $\theta_{k,\omega}^D \in \mathbb{R}^+$ , (bottom) propagative mode  $\theta_{k,\omega}^D \in i\mathbb{R}^+$ .

### Odd solutions $\widehat{E} \in H^1(\mathbb{R})$

Similarly, for the odd solutions in  $H^1(\mathbb{R})$  to (3.3.30), we consider the problem in the half-line  $[0, \infty)$  :

$$\begin{cases} -\frac{d^2}{dy^2} \widehat{E}^o + \theta_{k,\omega}^2(\cdot) \widehat{E}^o = 0, & \text{in } (0, L) \cup (L, \infty), \\ \widehat{E}^o(0) = 0, \end{cases}, \quad (3.3.45)$$

once again, to find finite energy solutions, we require the condition (3.3.38). Thus, (3.3.45) defines a one-dimensional solution space given by the canonical solution

$$v_{k,\omega}^o(y) := \begin{cases} \frac{\mu(\omega)}{\theta_{k,\omega}^D} \sinh(\theta_{k,\omega}^D y), & y \in [0, L], \\ A_{k,\omega}^o \exp(-\theta_{k,\omega}^V y), & y \in [L, \infty), \end{cases} \quad \text{if } \theta_{k,\omega}^D \neq 0, \quad (3.3.46)$$

$$v_{k,\omega}^o(y) := \begin{cases} \mu(\omega) y, & y \in [0, L], \\ A_{k,\omega}^o \exp(-\theta_{k,\omega}^V y), & y \in [L, \infty), \end{cases} \quad \text{if } \theta_{k,\omega}^D = 0. \quad (3.3.47)$$

In both cases, (3.3.38) holds in order to have finite energy solutions. The expressions (3.3.46) and (3.3.47) are said to be canonical solutions of (3.3.45) in the sense that they satisfy:

$$v_{k,\omega}^o(0) = 0 \quad \text{and} \quad \left( \frac{1}{\mu(\cdot, \omega)} \frac{d}{dy} v_{k,\omega}^o \right) (0) = 1, \quad (3.3.48)$$

and where  $A_{k,\omega}^o$  is obtained by the transmission condition  $[v_{k,\omega}^o]_{y=L} = 0$ :

$$A_{k,\omega}^o = \begin{cases} \frac{\mu(\omega)}{\theta_{k,\omega}^D} \exp(\theta_{k,\omega}^V L) \sinh(\theta_{k,\omega}^D L), & \text{if } \theta_{k,\omega}^D \neq 0, \\ \mu(\omega) \exp(\theta_{k,\omega}^V L) L, & \text{if } \theta_{k,\omega}^D = 0. \end{cases} \quad (3.3.49)$$

In this case, we obtain a different (but dual) implicit relation between  $k$  and  $\omega$ , derived from the transmission condition  $[\mu(\cdot, \omega)^{-1} \frac{d}{dy} v_{k,\omega}^o]_{y=L} = 0$ : if  $\theta_{k,\omega}^D \neq 0$  then one has

$$\mu(\omega)^{-1} \frac{\mu(\omega)}{\theta_{k,\omega}^D} \left( \theta_{k,\omega}^D \cosh(\theta_{k,\omega}^D L) \right) = -\frac{1}{\mu_0} \theta_{k,\omega}^V A_{k,\omega}^o \exp(-\theta_{k,\omega}^V L),$$

by a similar argument as in the even case solution, one can infer that  $\sinh(\theta_{k,\omega}^D L) \neq 0$  (otherwise  $v_{k,\omega}^o$  is not a canonical solution as stated), hence one arrives to

$$\theta_{k,\omega}^D \coth(\theta_{k,\omega}^D L) = -\frac{\mu(\omega)}{\mu_0} \theta_{k,\omega}^V, \quad (3.3.50)$$

in the case when  $\theta_{k,\omega}^D = 0$  one simply obtains

$$\frac{1}{L} = -\frac{\mu(\omega)}{\mu_0} \theta_{k,\omega}^V. \quad (3.3.51)$$

Finally, the odd solutions in  $H^1(\mathbb{R})$  to (3.3.30)-(3.3.31) are generated by the basis

$$V_{k,\omega}^o(y) := \text{sgn}(y) v_{k,\omega}^o(|y|). \quad (3.3.52)$$

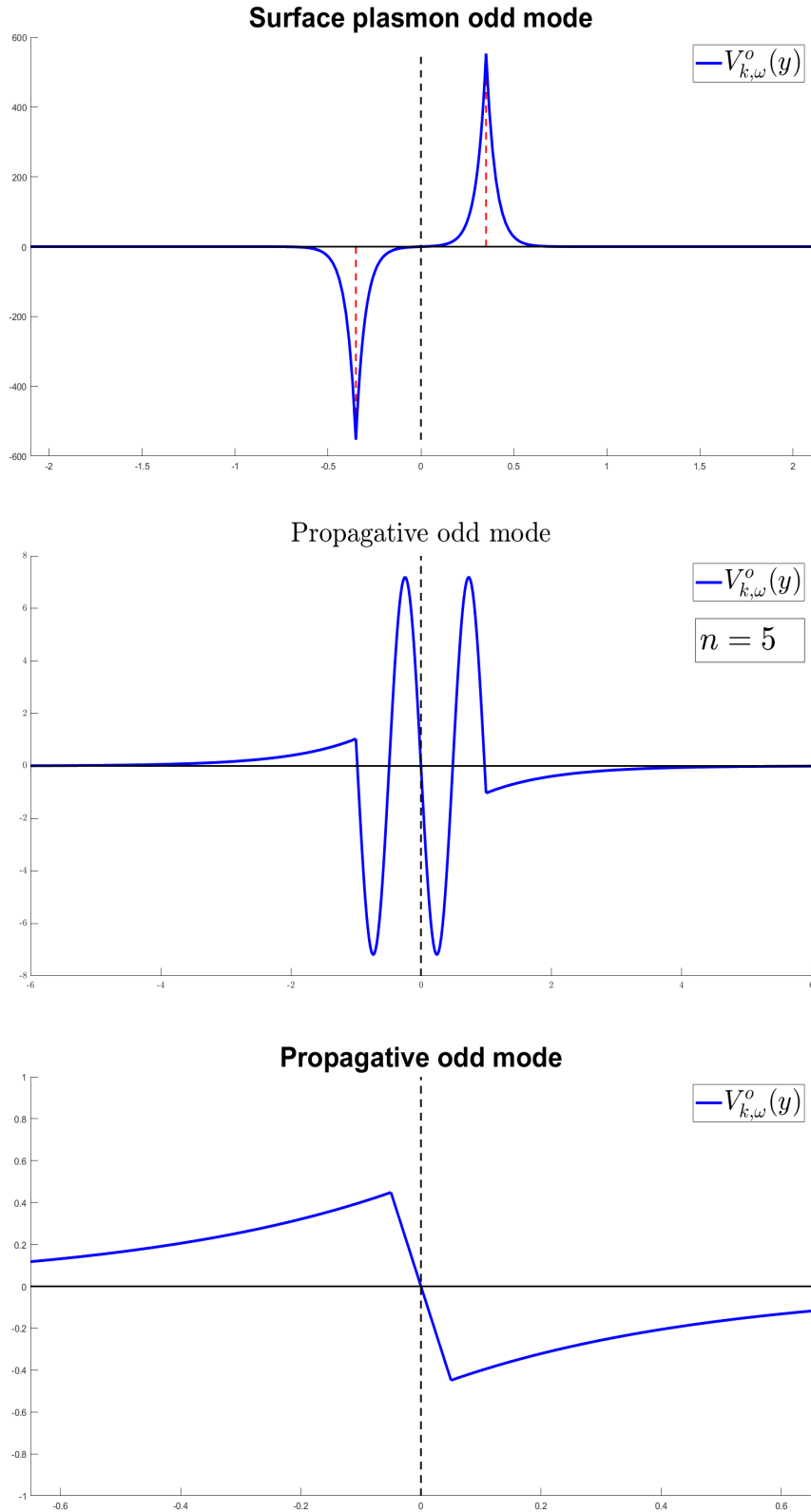


Figure 3.3: Graphic representation of the odd canonical mode  $V_{k,\omega}^o$  : (top) evanescent mode  $\theta_{k,\omega}^D \in \mathbb{R}^+$ , (center) propagative mode  $\theta_{k,\omega}^D \in i\mathbb{R}^+$ , (bottom) lineal mode  $\theta_{k,\omega}^D = 0$ .

In this sense, for all  $k \in \mathbb{R} \setminus \{0\}$ , we introduce the following sets:

$$\sigma_{\text{even}}(k) := \left\{ \omega \in \mathbb{R} \setminus \{-\Omega_m, 0, \Omega_m\} \mid \theta_{k,\omega}^V \in \mathbb{R}^+ \quad \text{and} \quad \theta_{k,\omega}^D \tanh(\theta_{k,\omega}^D L) = -\frac{\mu(\omega)}{\mu_0} \theta_{k,\omega}^V \right\} \quad (3.3.53)$$

$$\sigma_{\text{odd}}(k) := \sigma_{\text{odd}}^*(k) \cup \sigma_{\text{odd}}^0(k),$$

$$\sigma_{\text{odd}}^*(k) := \left\{ \omega \in \mathbb{R} \setminus \{-\Omega_m, 0, \Omega_m\} \mid \theta_{k,\omega}^V \in \mathbb{R}^+ \quad \text{and} \quad \theta_{k,\omega}^D \coth(\theta_{k,\omega}^D L) = -\frac{\mu(\omega)}{\mu_0} \theta_{k,\omega}^V \right\},$$

$$\sigma_{\text{odd}}^0(k) := \left\{ \omega \in \mathbb{R} \setminus \{-\Omega_m, 0, \Omega_m\} \mid \theta_{k,\omega}^V \in \mathbb{R}^+, \quad \theta_{k,\omega}^D = 0 \quad \text{and} \quad \frac{1}{L} = -\frac{\mu(\omega)}{\mu_0} \theta_{k,\omega}^V \right\}. \quad (3.3.54)$$

Now we can establish the following result

**Proposition 3.3.3**

For all  $k \in \mathbb{R} \setminus \{0\}$  we have that

$$\sigma_p(\mathbb{A}_k) \setminus \{-\Omega_m, 0, \Omega_m\} = \sigma_{\text{even}}(k) \cup \sigma_{\text{odd}}(k) \quad (3.3.55)$$

*Proof.* If  $\omega \in \sigma_p(\mathbb{A}_k) \setminus \{-\Omega_m, 0, \Omega_m\}$ , then there exists  $\widehat{\mathbf{E}} \in H^1(\mathbb{R}) \setminus \{0\}$  such that  $\widehat{\mathbf{E}}$  is solution of (3.3.30) and (3.3.31), hence  $\widehat{\mathbf{E}}^e$  and  $\widehat{\mathbf{E}}^o$  are also solutions to (3.3.30) and (3.3.31). Since  $\widehat{\mathbf{E}}$  is not identically zero then  $\widehat{\mathbf{E}}^e$  or  $\widehat{\mathbf{E}}^o$ , one of them, is also not identically zero. If  $\widehat{\mathbf{E}}^e \neq 0$ , then from the fact that  $\widehat{\mathbf{E}}^e$  satisfies (3.3.30) and (3.3.31) it follows that  $\widehat{\mathbf{E}}^e := \lambda V_{k,\omega}^e$  for some  $\lambda \in \mathbb{C} \setminus \{0\}$  (where  $V_{k,\omega}^e$  is defined as in (3.3.44)) and the pair  $(k, \omega)$  satisfies (3.3.38) and (3.3.43), in other words,  $\omega \in \sigma_{\text{even}}(k)$ . Similarly,  $\omega \in \sigma_{\text{odd}}(k)$  in the case  $\widehat{\mathbf{E}}^o \neq 0$ . Therefore we have the subset relation

$$\sigma_p(\mathbb{A}_k) \setminus \{-\Omega_m, 0, \Omega_m\} \subseteq \sigma_{\text{even}}(k) \cup \sigma_{\text{odd}}(k).$$

Conversely, if  $\omega \in \sigma_{\text{even}}(k)$  then  $\omega \in \mathbb{C} \setminus \{-\Omega_m, 0, \Omega_m\}$  and  $\widehat{\mathbf{E}} := V_{k,\omega}^e$  (see again (3.3.44)) is solution of (3.3.30) and (3.3.31) belonging to  $H^1(\mathbb{R})$ . Hence  $\omega \in \sigma_p(\mathbb{A}_k)$ , this show the contention  $\sigma_{\text{even}}(k) \subseteq \sigma_p(\mathbb{A}_k) \setminus \{-\Omega_m, 0, \Omega_m\}$ . Analogously one proves  $\sigma_{\text{odd}}(k) \subseteq \sigma_p(\mathbb{A}_k) \setminus \{-\Omega_m, 0, \Omega_m\}$ .  $\square$

## CHAPTER 4

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# Existence and analysis of the guided modes

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## 4.1 Preliminaries for solving the dispersion relations

The approach in what follows is to find, for all  $k \in \mathbb{R} \setminus \{0\}$ , all the values  $\omega \in \mathbb{R}$  for which the function  $v^e(\cdot)$  described in (3.3.39) forms a canonical basis for (3.3.37). The latter fact leads us to want to solve the *dispersion equation* (3.3.43) which is complemented with the *finite energy condition*

$$\theta_{k,\omega}^V \in \mathbb{R}^+, \quad \text{equivalently,} \quad \varepsilon_0 \mu_0 \omega^2 < k^2. \quad (4.1.1)$$

The usual point of view for solving this kind of dispersion relation is to consider the wavenumber  $k$  and look for the solutions  $\omega$  (the frequency) of (3.3.43) (we shall see that they are infinitely many) as functions of  $k$ , giving the so-called dispersion curves. Of course, due to the high non linearity of the equation (3.3.43), its resolution - even the existence of solutions - is not obvious. The point of view we shall follow is slightly different that simply seeing (3.3.43) as an equation in  $\omega$ : it is essentially guided by technical reasons (that we try to explain below) and inspired by the computations made by C. H. Wilcox in [93] (see also the related works of Y. Dermanjean, J.C. Guillot and R. Weder in stratified media [29–32, 88–90]).

Before entering in the technique used for the resolution of (3.3.43) we introduce the notion of dimensionless terms of our problem, this will allow us to deepen in the analysis in a more comfortable way and to rescue important properties of the solutions related to the parameters of the Drude model and the medium:  $\Omega_e, \Omega_m, L$ .

### 4.1.1 Nondimensionalization/Scaling: dimensionless dispersion equations

In order to avoid dimensional disproportions, we introduce the analogous adimensional terms of  $k, \omega, L$  and the ratio linking  $\Omega_e$  with  $\Omega_m$  which read as follows

$$\kappa := kL, \quad \underline{\omega} := \sqrt{\varepsilon_0 \mu_0} \omega L, \quad \underline{\Omega}_m = \sqrt{\varepsilon_0 \mu_0} \Omega_m L, \quad \rho = \frac{\Omega_e}{\Omega_m}, \quad (4.1.2)$$

The ratio  $\rho$  between the two frequencies  $\Omega_e$  and  $\Omega_m$  is a characteristics of the Drude material while  $\underline{\Omega}_m$  is a dimensionless half-thickness of the internal layer : it is the ratio between  $L$  and the wavelength  $\lambda_m = \varepsilon_0 \mu_0 / \Omega_m$  of a plane wave propagating at frequency  $\Omega_m$  in the vacuum. The dimensionless wavenumber  $\kappa$  is the ratio between the half-thickness  $L$  and the  $x$ -wave length corresponding to the  $x$ -wavenumber  $k$ . The dimensionless frequency  $\underline{\omega}$  represents the ratio between  $L$  and the wavelength  $\lambda = \varepsilon_0 \mu_0 / \omega$  of a plane wave propagating in the vacuum at frequency  $\omega$ . Let us introduce the *relative dimensionless permittivity and permeability*, respectively as,

$$\varepsilon_r(\underline{\omega}) = 1 - \frac{\rho^2 \underline{\Omega}_m^2}{\underline{\omega}^2}, \quad \mu_r(\underline{\omega}) = 1 - \frac{\underline{\Omega}_m^2}{\underline{\omega}^2}. \quad (4.1.3)$$

We also introduce the analogous dimensionless terms associated to  $\theta_{k,\omega}^V, \theta_{k,\omega}^D$ , namely,

$$\xi_{\kappa,\underline{\omega}}^V := \sqrt{\kappa^2 - \underline{\omega}^2}, \quad \xi_{\kappa,\underline{\omega}}^D := \sqrt{\kappa^2 - \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2}. \quad (4.1.4)$$

Using the above quantities, the *finite energy condition* (3.3.38) becomes into,

$$\underline{\omega}^2 < \kappa^2, \quad (4.1.5)$$

and (3.3.43) implies the *dimensionless dispersion equation* for even eigenfunctions:

$$\xi_{\kappa,\underline{\omega}}^D \tanh \xi_{\kappa,\underline{\omega}}^D = -\mu_r(\underline{\omega}) \xi_{\kappa,\underline{\omega}}^V, \quad (4.1.6)$$

analogously, for odd eigenfunctions, (3.3.50) rewrites as:

$$\xi_{\kappa, \underline{\omega}}^D \coth \xi_{\kappa, \underline{\omega}}^D = -\mu_r(\underline{\omega}) \xi_{\kappa, \underline{\omega}}^V. \quad (4.1.7)$$

#### 4.1.2 Spectral zones $\mathcal{N}^-$ and $\mathcal{N}^+$

Since (4.1.6) and (4.1.7) are invariant with respect to the sign of  $\underline{\omega}$  or  $\kappa$ , that is

$$\begin{aligned} (\kappa, \underline{\omega}) \text{ is solution of (4.1.6) ( (4.1.7))} \\ \iff \end{aligned} \quad (4.1.8)$$

$$(a_1 \kappa, a_2 \underline{\omega}) \text{ is solution of (4.1.6) ((4.1.7)), for all } a_1, a_2 \in \{-1, 1\}.$$

Henceforth, it suffices to consider the solutions of (4.1.6) in the first quadrant of the  $\kappa\tau$ -plane satisfying (4.1.5). We therefore restrict our study inside the set

$$\mathcal{N} := \{(\kappa, \underline{\omega}) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid \underline{\omega} < \kappa\}. \quad (4.1.9)$$

In this sense, our goal traduces to characterize the sets

$$\mathcal{D}_{\text{even}} = \{(\kappa, \underline{\omega}) \in \mathcal{N} \mid (\kappa, \underline{\omega}) \text{ satisfies (4.1.6)}\}, \quad (4.1.10)$$

$$\mathcal{D}_{\text{odd}} = \{(\kappa, \underline{\omega}) \in \mathcal{N} \mid (\kappa, \underline{\omega}) \text{ satisfies (4.1.7)}\}. \quad (4.1.11)$$

Since  $\xi_{\kappa, \underline{\omega}}^D \in \mathbb{R}^+ \cup i\mathbb{R}^+ \cup \{0\}$ , we esteem convenient to decompose  $\mathcal{N}$  in three regions as follows:

$$\mathcal{N} = \mathcal{N}^- \cup \mathcal{N}^0 \cup \mathcal{N}^+,$$

$$\begin{aligned} \mathcal{N}^- &:= \left\{ (\kappa, \underline{\omega}) \in \mathcal{N} \mid \xi_{\kappa, \underline{\omega}}^D \in i\mathbb{R}^+ \right\} = \left\{ (\kappa, \underline{\omega}) \in \mathcal{N} \mid \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2 > \kappa^2 \right\}, \\ \mathcal{N}^+ &:= \left\{ (\kappa, \underline{\omega}) \in \mathcal{N} \mid \xi_{\kappa, \underline{\omega}}^D \in \mathbb{R}^+ \right\} = \left\{ (\kappa, \underline{\omega}) \in \mathcal{N} \mid \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2 < \kappa^2 \right\}, \\ \mathcal{N}^0 &:= \left\{ (\kappa, \underline{\omega}) \in \mathcal{N} \mid \xi_{\kappa, \underline{\omega}}^D = 0 \right\} = \left\{ (\kappa, \underline{\omega}) \in \mathcal{N} \mid \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2 = \kappa^2 \right\}. \end{aligned} \quad (4.1.12)$$

According to [14, 16], it is easy to describe the regions  $\mathcal{N}^-$ ,  $\mathcal{N}^+$  and  $\mathcal{N}^0$  with the help of the graph of a function  $\Phi(\kappa)$ . More precisely, let us define the critical (dimensionless) wavenumber and frequency

$$\kappa_c^2 := \frac{\rho^2 \underline{\Omega}_m^2}{1 + \rho^2}, \quad \underline{\omega}_c := \kappa_c, \quad (4.1.13)$$

$\Phi(\kappa) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is the piecewise smooth function defined by

$$\Phi(\kappa) := \begin{cases} \kappa, & \text{if } \kappa \leq \kappa_c, \\ \frac{1}{\sqrt{2}} \left( \kappa^2 + \underline{\Omega}_m^2 (\rho^2 + 1) - [(\kappa^2 + \underline{\Omega}_m^2 (\rho^2 - 1))^2 + 4 \underline{\Omega}_m^2 \kappa^2]^{\frac{1}{2}} \right)^{\frac{1}{2}}, & \text{if } \kappa \geq \kappa_c. \end{cases} \quad (4.1.14)$$

It is easy to see that  $\Phi(\kappa)$  is strictly increasing on  $[0, \kappa_c]$ , strictly decreasing on  $[\kappa_c, +\infty)$  and thus reaches its maximum at  $\kappa = \kappa_c$  where  $\Phi(\kappa_c) = \kappa_c$ . Moreover, for large values of  $\kappa$ , one has

$$\Phi(\kappa) \sim \frac{\Phi_\infty}{\kappa} \quad \text{with} \quad \Phi_\infty := \rho \underline{\Omega}_m^2. \quad (4.1.15)$$



All these properties are visible in [Figure 4.1](#) where the graph of  $\Phi$  is presented.

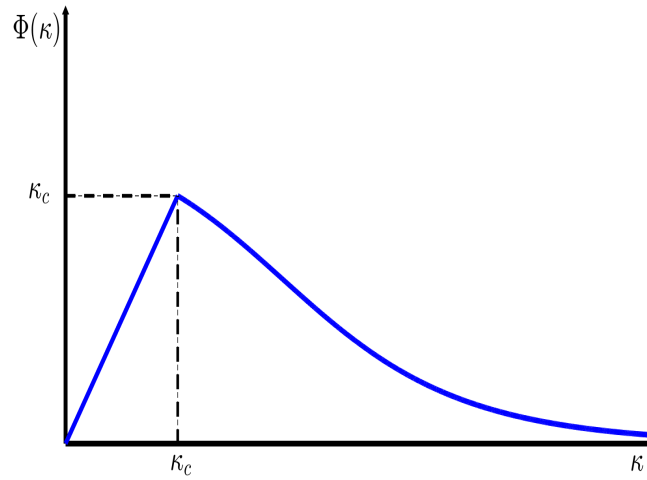


Figure 4.1: Graphic of  $\Phi(\cdot)$ .

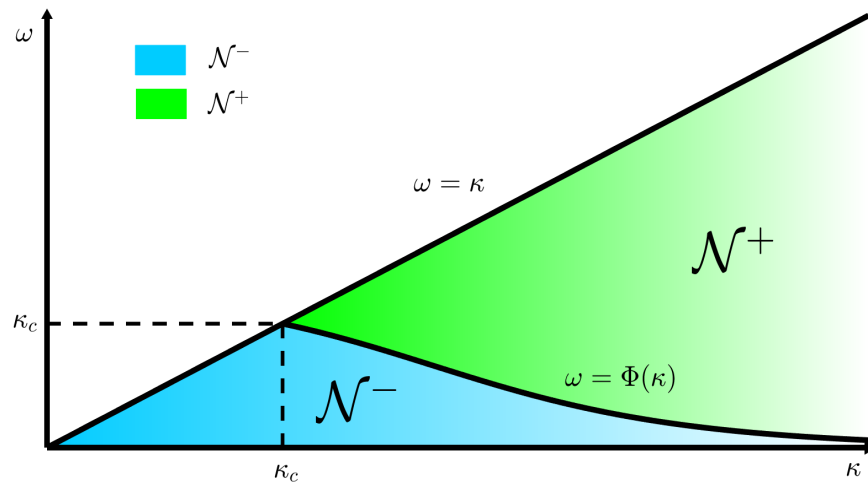


Figure 4.2: Representation of the regions  $\mathcal{N}^-$  and  $\mathcal{N}^+$ .

Then the sets  $\mathcal{N}^-$  and  $\mathcal{N}^+$  are characterized by (see [Figure 4.2](#))

$$\begin{aligned}
 \mathcal{N}^- &= \left\{ (\kappa, \underline{\omega}) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid \underline{\omega} < \Phi(\kappa) \right\}, \\
 \mathcal{N}^+ &= \left\{ (\kappa, \underline{\omega}) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid \kappa > \kappa_c, \Phi(\kappa) < \underline{\omega} < \kappa \right\}, \\
 \mathcal{N}^0 &= \left\{ (\kappa, \underline{\omega}) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid \kappa > \kappa_c, \underline{\omega} = \Phi(\kappa) \right\}.
 \end{aligned} \tag{4.1.16}$$

**Lemma 4.1.1**

$$\mathcal{D}_{\text{even}} \cap \mathcal{N}^0 = \emptyset,$$

$$\mathcal{D}_{\text{odd}} \cap \mathcal{N}^0 = \left\{ (\sqrt{\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2}, \underline{\omega}) \mid (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = (\mu_r(\underline{\omega}))^{-2} \right\}.$$

*Proof.* Let  $(\kappa, \underline{\omega}) \in \mathcal{D}_{\text{even}} \cap \mathcal{N}^0$ , hence  $(\kappa, \underline{\omega})$  satisfies

$$\begin{cases} \xi_{\kappa, \underline{\omega}}^D \tanh \xi_{\kappa, \underline{\omega}}^D = -\mu_r(\underline{\omega}) \xi_{\kappa, \underline{\omega}}^V, \\ \xi_{\kappa, \underline{\omega}}^D = 0, \end{cases} \quad (4.1.17)$$

the last system implies that  $\mu_r(\underline{\omega}) \xi_{\kappa, \underline{\omega}}^V = 0$ , and since  $\xi_{\kappa, \underline{\omega}}^V > 0$  ( $(\kappa, \underline{\omega}) \in \mathcal{N}$ ) then  $\mu_r(\underline{\omega}) = 0$ . Finally, from  $\xi_{\kappa, \underline{\omega}}^D = 0$  we deduce

$$\kappa^2 = \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2 = 0, \quad (4.1.18)$$

which is a contradiction since  $0 < \underline{\omega} < \kappa$  by definition of  $\mathcal{D}_{\text{even}}$ . This proves the first identity.

On the other hand

$$\begin{aligned} (\kappa, \underline{\omega}) \in \mathcal{D}_{\text{odd}} \cap \mathcal{N}^0 &\iff \xi_{\kappa, \underline{\omega}}^D = \sqrt{\kappa^2 - \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2} = 0 \quad \text{and} \quad \xi_{\kappa, \underline{\omega}}^D \coth \xi_{\kappa, \underline{\omega}}^D = -\mu_r(\underline{\omega}) \xi_{\kappa, \underline{\omega}}^V \\ &\iff \kappa = \sqrt{\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2} \quad \text{and} \quad 1 = (x \coth x)|_{x=0} = -\mu_r(\underline{\omega}) \xi_{\kappa, \underline{\omega}}^V \\ &\iff \kappa = \sqrt{\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2} \quad \text{and} \quad (\mu_r(\underline{\omega}))^{-2} = \kappa^2 - \underline{\omega}^2 \\ &\iff \kappa = \sqrt{\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2} \quad \text{and} \quad (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = (\mu_r(\underline{\omega}))^{-2} \end{aligned}$$

In the last series of equivalences we have used the fact that  $x \coth x$  is continuously extended in  $x = 0$ , since the limit of  $x \coth x$  when  $x \rightarrow 0$  is equal to 1, and we have also used that if  $\underline{\omega}$  satisfies  $(\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = (\mu_r(\underline{\omega}))^{-2}$  then

$$\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) > 1 \implies \underline{\omega} < \frac{\rho \underline{\Omega}_m}{\sqrt{1 + \rho^2}} < \underline{\Omega}_m \implies -\mu_r(\underline{\omega}) > 0. \quad (4.1.19)$$

This concludes the proof.  $\square$

Consequently  $\mathcal{D}_{\text{even}}$  and  $\mathcal{D}_{\text{odd}}$  are decomposed as

$$\begin{aligned} \mathcal{D}_{\text{even}} &= \mathcal{D}_{\text{even}}^- \cup \mathcal{D}_{\text{even}}^+, & \text{where } \mathcal{D}_{\text{even}}^s &:= \mathcal{D}_{\text{even}} \cap \mathcal{N}^s, \quad s \in \{-, +\}, \\ \mathcal{D}_{\text{odd}} &= \mathcal{D}_{\text{odd}}^- \cup \mathcal{D}_{\text{odd}}^0 \cup \mathcal{D}_{\text{odd}}^+, & \text{where } \mathcal{D}_{\text{odd}}^s &:= \mathcal{D}_{\text{odd}} \cap \mathcal{N}^s, \quad s \in \{-, 0, +\}. \end{aligned}$$

**Remark 4.1.2.** In subsequent sections we will prove that for all the values of  $\rho, \underline{\Omega}_m > 0$ ,  $\mathcal{D}_{\text{odd}}^0$  consists of exactly one element.

#### 4.1.2.1 Reformulation of the *dispersion equations* (4.1.6)-(4.1.7) in the zone $\mathcal{N}^+$

In the following, we will try introduce the approach for solving (4.1.6) for values  $(\kappa, \underline{\omega}) \in \mathcal{N}^+$ , meaning that  $\xi_{\kappa, \underline{\omega}}^D \in \mathbb{R}^+$ . We first start our analysis for  $\mathcal{D}_{\text{even}}$ . Let us note that if  $(\kappa, \underline{\omega}) \in \mathcal{D}_{\text{even}}^+$ ,

then from (4.1.5) and (4.1.6) we have that

$$\mu_r(\underline{\omega}) = -\frac{\xi_{\kappa,\underline{\omega}}^D \tanh \xi_{\kappa,\underline{\omega}}^D}{\xi_{\kappa,\underline{\omega}}^V} < 0 \quad (\text{since } \xi_{\kappa,\underline{\omega}}^D \in \mathbb{R}^+), \quad (4.1.20)$$

moreover  $\mu_r(\underline{\omega}) = 1 - \frac{\underline{\Omega}_m^2}{\underline{\omega}^2} < 0$  implies  $\underline{\omega}^2 < \underline{\Omega}_m^2$  and consequently  $\underline{\omega} \in (0, \underline{\Omega}_m)$  (since  $\underline{\omega} \in \mathbb{R}^+$ ). Henceforth

$$\mathcal{D}_{\text{even}}^+ = \left\{ (\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \underline{\Omega}_m) \mid \text{satisfies (4.1.6)} \right\} \cap \mathcal{N}^+.$$

Due to non-polynomial nature of (4.1.6) (appearance of trigonometric complex functions), and in order to avoid the manipulation of square roots appearing in the definition of  $\xi_{\kappa,\underline{\omega}}^V$  and  $\xi_{\kappa,\underline{\omega}}^D$  (see (4.1.4)), therefore, it appears convenient to retype (4.1.4) and (4.1.6) under the form of a system of 3 equations with 4 variables  $(\kappa, \underline{\omega}, \xi^V, \xi^D) \in \mathbb{R}^+ \times (0, \underline{\Omega}_m) \times \mathbb{R}^+ \times \mathbb{R}^+$  :

$$\begin{cases} \xi^D \tanh \xi^D = -\mu_r(\underline{\omega}) \xi^V, & (4.1.21a) \\ (\xi^V)^2 = \kappa^2 - \underline{\omega}^2, & (4.1.21b) \\ (\xi^D)^2 = \kappa^2 - \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2. & (4.1.21c) \end{cases}$$

The usual approach is to fix any of the four variables (usually one fixes the wavenumber  $\kappa$ ) and solve the system for the other three as unknowns. We make some few manipulations as follow: we first eliminate the wavenumber  $\kappa$  in (4.1.21c) by using (4.1.21b) in order to obtain the next equivalent system,

$$\begin{cases} \xi^D \tanh \xi^D = -\mu_r(\underline{\omega}) \xi^V, & (4.1.22a) \\ \kappa^2 = \underline{\omega}^2 + (\xi^V)^2, & (4.1.22b) \\ (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = (\xi^V)^2 - (\xi^D)^2. & (4.1.22c) \end{cases}$$

Finally we eliminate the variable  $\xi^V$  by replacing its value from (4.1.22a):

$$\begin{cases} (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = ((\mu_r(\underline{\omega}))^{-2} \tanh^2 \xi^D - 1) (\xi^D)^2, & (4.1.23a) \\ \kappa^2 = \underline{\omega}^2 + (\mu_r(\underline{\omega}))^{-2} (\xi^D \tanh \xi^D)^2. & (4.1.23b) \end{cases}$$

Summarizing, we have seen so far that  $(\kappa, \underline{\omega}) \in \mathcal{D}_{\text{even}}^+$  implies that  $(\kappa, \underline{\omega}, \xi^D)$  is solution of (4.1.23) for some  $\xi^D \in \mathbb{R}^+$ .

Conversely, if  $(\kappa, \underline{\omega}, \xi^D) \in \mathbb{R}^+ \times (0, \underline{\Omega}_m) \times \mathbb{R}^+$  is solution of (4.1.23), then by stepping back the calculus lately made, we arrive to

$$\xi^D = \xi_{\kappa,\underline{\omega}}^D, \quad \text{and} \quad (\xi_{\kappa,\underline{\omega}}^D \tanh \xi_{\kappa,\underline{\omega}}^D)^2 = (\mu_r(\underline{\omega}) \xi_{\kappa,\underline{\omega}}^V)^2,$$

where  $\xi_{\kappa,\underline{\omega}}^D$  and  $\xi_{\kappa,\underline{\omega}}^V$  are defined as in (4.1.4), consequently

$$\xi_{\kappa,\underline{\omega}}^D \tanh \xi_{\kappa,\underline{\omega}}^D = |\xi_{\kappa,\underline{\omega}}^D \tanh \xi_{\kappa,\underline{\omega}}^D| = |\mu_r(\underline{\omega}) \xi_{\kappa,\underline{\omega}}^V| = -\mu_r(\underline{\omega}) \xi_{\kappa,\underline{\omega}}^V, \quad (4.1.24)$$

where we have used that  $\tanh(\cdot) > 0$  in  $\mathbb{R}_*^+$  and  $\mu_r(\cdot) < 0$  in  $(0, \underline{\Omega}_m)$ . We have therefore prove the following.

**Proposition 4.1.3**

$(\kappa, \underline{\omega}) \in \mathcal{D}_{\text{even}}^+$  if and only if  $(\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \underline{\Omega}_m)$  and for some  $\tau > 0$ ,  $(\kappa, \underline{\omega}, \tau)$  satisfies the system

$$\begin{cases} (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = ((\mu_r(\underline{\omega}))^{-2} \tanh^2 \tau - 1) \tau^2, & (4.1.25a) \\ \kappa^2 = \omega^2 + (\mu_r(\underline{\omega}))^{-2} (\tau \tanh \tau)^2. & (4.1.25b) \end{cases}$$

In this case the value of  $\tau$  is given by  $\tau = \xi_{\kappa, \underline{\omega}}^D \left( = \sqrt{\kappa^2 - \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2} \right)$ .

**Remark 4.1.4.** In the statement of [Proposition 4.1.3](#), we have intentionally suggested as unknown " $\tau$ " instead of  $\xi^D$  as a manner to introduce to the a parametric characterization which allows us to think  $\mathcal{D}_+^{\text{even}}$  as a parametric curve  $\{(\kappa(\tau), \underline{\omega}(\tau)) \mid \tau > 0\}$ , this fact will be justified and refined in following sections.

The characterization of the set  $\mathcal{D}_{\text{odd}}^+$  follows the same principle as for  $\mathcal{D}_{\text{even}}^+$ . Indeed, one can easily repeat the previous calculus by exchanging the role of  $\tanh(\cdot)$  by  $\coth(\cdot)$ , in order to obtain the following analogous characterization.

**Proposition 4.1.5**

$(\kappa, \underline{\omega}) \in \mathcal{D}_{\text{odd}}^+$  if and only if  $(\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \underline{\Omega}_m)$  and for some  $\tau > 0$ ,  $(\kappa, \underline{\omega}, \tau)$  satisfies the system

$$\begin{cases} (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = ((\mu_r(\underline{\omega}))^{-2} \coth^2 \tau - 1) \tau^2, & (4.1.26a) \\ \kappa^2 = \omega^2 + (\mu_r(\underline{\omega}))^{-2} (\tau \coth \tau)^2. & (4.1.26b) \end{cases}$$

In this case the value of  $\tau$  is given by  $\tau = \xi_{\kappa, \underline{\omega}}^D \left( = \sqrt{\kappa^2 - \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2} \right)$ .

**4.1.2.2 Reformulation of the dispersion equations (4.1.6)-(4.1.7) in the zone  $\mathcal{N}^-$** 

It is possible to give a similar characterization for values  $(\kappa, \underline{\omega}) \in \mathcal{D}^-$  with some changes.

First, we notice that if  $(\kappa, \underline{\omega}) \in \mathcal{N}^-$ , then  $\xi_{\kappa, \underline{\omega}}^D \in i\mathbb{R}^+$  meaning that  $\kappa^2 < \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2$ , which combined with [\(4.1.5\)](#) leads us to

$$\underline{\omega}^2 < \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2. \quad (4.1.27)$$

The last inequality, applied to the Drude model (see [\(4.1.3\)](#)) yields

$$\begin{aligned} \underline{\omega}^2 &< \left(1 - \frac{\underline{\Omega}_m^2}{\underline{\omega}^2}\right) \left(1 - \frac{\rho^2 \underline{\Omega}_m^2}{\underline{\omega}^2}\right) \underline{\omega}^2 = \underline{\omega}^2 - (1 + \rho^2) \underline{\Omega}_m^2 + \frac{\rho^2 \underline{\Omega}_m^4}{\underline{\omega}^2} \\ \implies \underline{\omega} &< \frac{\rho}{\sqrt{1 + \rho^2}} \underline{\Omega}_m = \kappa_c, \end{aligned} \quad (4.1.28)$$

hence  $\underline{\omega} \in (0, \kappa_c) \subset (0, \underline{\Omega}_m)$  (since  $\kappa_c < \underline{\Omega}_m$ ). Therefore

$$\mathcal{D}_{\text{even}}^- = \left\{ (\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \kappa_c) \mid \text{satisfies (4.1.6)} \right\} \cap \mathcal{N}^-.$$

On another hand, for  $(\kappa, \underline{\omega}) \in \mathcal{N}^-$  we define

$$\eta_{\kappa, \underline{\omega}} := \sqrt{\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2 - \kappa^2} \in \mathbb{R}^+ \quad (\text{and consequently } \xi_{\kappa, \underline{\omega}}^D = i\eta_{\kappa, \underline{\omega}}). \quad (4.1.29)$$

In this sense, if  $(\kappa, \underline{\omega}) \in \mathcal{D}_{\text{even}}^-$ , by substituting (4.1.29) in (4.1.6) we obtain

$$\begin{aligned} -\mu_r(\underline{\omega}) \xi_{\kappa, \underline{\omega}}^V &= \xi_{\kappa, \underline{\omega}}^D \tanh \xi_{\kappa, \underline{\omega}}^D \\ &= (i \eta_{\kappa, \underline{\omega}}) \tanh(i \eta_{\kappa, \underline{\omega}}) \\ &= (i \eta_{\kappa, \underline{\omega}}) (i \tan \eta_{\kappa, \underline{\omega}}) = -\eta_{\kappa, \underline{\omega}} \tan \eta_{\kappa, \underline{\omega}}, \end{aligned}$$

thus the dispersion equation rewrites as

$$\eta_{\kappa, \underline{\omega}} \tan \eta_{\kappa, \underline{\omega}} = \mu_r(\underline{\omega}) \xi_{\kappa, \underline{\omega}}^V. \quad (4.1.30)$$

From the latter, a new (important) observation is about the sign of  $\tan \eta_{\kappa, \underline{\omega}}$ , indeed, since  $\underline{\omega} \in (0, \kappa_c) \subset (0, \underline{\Omega}_m)$  and consequently  $\mu_r(\underline{\omega}) < 0$ , then (4.1.30) implies  $\eta_{\kappa, \underline{\omega}} \tan \eta_{\kappa, \underline{\omega}} < 0$ , hence

$$\tan \eta_{\kappa, \underline{\omega}} < 0.$$

The latter suggest to study the solutions  $(\kappa, \underline{\omega}) \in \mathcal{D}_{\text{even}}^-$  depending on the belonging of  $(\eta_{\kappa, \underline{\omega}}, \tan \eta_{\kappa, \underline{\omega}})$  to each one of the branches of the graph of the real tangent function,  $\tan(\cdot)$ .

Using the fact that the positive domain where the real tangent function defines a disjoint union of intervals, namely,

$$\tan^{-1}\{\mathbb{R}^-\} \cap \mathbb{R}^+ = \bigcup_{n \in \mathbb{N}} \left( (n - \frac{1}{2}) \pi, n \pi \right), \quad (4.1.31)$$

then the property,  $\tan \eta_{\kappa, \underline{\omega}} < 0$ , yields to express  $\eta_{\kappa, \underline{\omega}}$  of the form:

$$\eta_{\kappa, \underline{\omega}} = n \pi - \arctan \tau, \quad \text{for some } \tau > 0. \quad (4.1.32)$$

In here, we have parametrized each point in the interval  $((n - \frac{1}{2}) \pi, n \pi)$  into the form

$$\{n \pi - \arctan \tau \mid \tau > 0\}.$$

This lead us to define the family of functions

$$\eta_{2n} : [0, +\infty) \longrightarrow \left( (n - \frac{1}{2}) \pi, n \pi \right], \quad \eta_{2n}(\tau) := n \tau - \arctan \tau, \quad n \in \mathbb{N}, \quad (4.1.33)$$

in such a way that (4.1.32) retypes as

$$\eta_{\kappa, \underline{\omega}} = \eta_{2n}(\tau), \quad \text{for some } \tau > 0. \quad (4.1.34)$$

By substituting the latter in (4.1.30) and by noticing that

$$\tan \eta_{2n}(\tau) = -\tau, \quad \text{for all } n \in \mathbb{N}, \text{ and } \tau > 0, \quad (4.1.35)$$

then we obtain

$$-\tau \eta_{2n}(\tau) = \mu_r(\underline{\omega}) \xi_{\kappa, \underline{\omega}}^V. \quad (4.1.36)$$

Summarizing,  $(\kappa, \underline{\omega}) \in \mathcal{D}_{\text{even}}^+$  satisfies, for some  $n \in \mathbb{N}$ , (4.1.36) and

$$\eta_n(\tau) = \sqrt{\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2 - \kappa^2}. \quad (4.1.37)$$

From here, we begin the same approach as the reformulation of  $\mathcal{D}_{\text{even}}^+$ . We see the couple of equations, (4.1.36) and (4.1.37), as a system of 4 variables  $(\kappa, \underline{\omega}, \xi^V, \tau) \in \mathbb{R}^+ \times (0, \kappa_c) \times \mathbb{R}^+ \times \mathbb{R}^+$

satisfying

$$\begin{cases} -\tau \eta_{2n}(\tau) = \mu_r(\underline{\omega}) \xi^V, \\ (\xi^V)^2 = \kappa^2 - \underline{\omega}^2, \\ \eta_{2n}^2(\tau) = \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2 - \kappa^2, \end{cases} \quad (4.1.38)$$

and by an analogous algebraic manipulation previously made, we arrive to the system

$$\begin{cases} (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = \left( (\mu_r(\underline{\omega}))^{-2} \tau^2 + 1 \right) \eta_{2n}^2(\tau), \\ \kappa^2 = \omega^2 + (\mu_r(\underline{\omega}))^{-2} (\tau \eta_{2n}(\tau))^2. \end{cases} \quad (4.1.39a)$$

Conversely, if for some  $n \in \mathbb{N}$ ,  $(\kappa, \underline{\omega}, \tau) \in \mathbb{R}^+ \times (0, \kappa_c) \times \mathbb{R}^+$  is solution of (4.1.39), one can verify that

$$\eta_{2n}(\tau) = \eta_{\kappa, \underline{\omega}} \quad \text{and} \quad (\tau \eta_{2n}(\tau))^2 = (\mu_r(\underline{\omega}) \xi_{\kappa, \underline{\omega}}^V)^2. \quad (4.1.40)$$

Since  $\eta_{2n}(\cdot)$  satisfies (4.1.35) and from the first equation of (4.1.40), we infer that  $\tan \eta_{\kappa, \underline{\omega}} = -\tau < 0$ , and by substituting the latter in the second equation of (4.1.40) and taking square root we get (once more, since  $\mu_r(\underline{\omega}) < 0$  in  $(0, \underline{\Omega}_m)$ )

$$-\eta_{\kappa, \underline{\omega}} \tan \eta_{\kappa, \underline{\omega}} = |\eta_{\kappa, \underline{\omega}} \tan \eta_{\kappa, \underline{\omega}}| = |\mu_r(\underline{\omega}) \xi_{\kappa, \underline{\omega}}^V| = -\mu_r(\underline{\omega}) \xi_{\kappa, \underline{\omega}}^V, \quad (4.1.41)$$

which is equivalent to (4.1.6) with the consideration  $\xi_{\kappa, \underline{\omega}}^D = i \eta_{\kappa, \underline{\omega}}$ , this shows that  $(\kappa, \underline{\omega}) \in \mathcal{D}_{\text{even}}^-$ .

All the above can be condensed in the following proposition:

**Proposition 4.1.6**

$(\kappa, \underline{\omega}) \in \mathcal{D}_{\text{even}}^-$  if and only if  $(\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \kappa_c)$  and for some  $n \in \mathbb{N}$  and  $\tau > 0$ ,  $(\kappa, \underline{\omega}, \tau)$  satisfies the system

$$\begin{cases} (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = \left( (\mu_r(\underline{\omega}))^{-2} \tau^2 + 1 \right) \eta_{2n}^2(\tau), \\ \kappa^2 = \omega^2 + (\mu_r(\underline{\omega}))^{-2} (\tau \eta_{2n}(\tau))^2. \end{cases} \quad (4.1.42a)$$

$$\quad (4.1.42b)$$

In this case the value of  $\tau$  is given by  $\tau = -\tan(i \xi_{\kappa, \underline{\omega}}^D)$ .

For the characterization of the set  $\mathcal{D}_{\text{odd}}^-$ , we see that we cannot simply change the role of  $\tan(\cdot)$  by  $\cot(\cdot)$  in (4.1.30) and consequently the role of  $\arctan(\cdot)$  by  $\operatorname{arccot}(\cdot)$  in (4.1.33), this occurs because, while the identity

$$\tanh(ix) = i \tan x, \quad x \in \mathbb{R}$$

occurs, in the other hand we have that

$$\coth(ix) = -i \cot x, \quad x \in \mathbb{R},$$

so the appearance of the negative sign slightly modifies the computation. Nevertheless, we will obtain a resemblance to Proposition 4.1.6. Indeed, we recall that for  $(\kappa, \underline{\omega}) \in \mathcal{D}_{\text{odd}}^- \subset \mathcal{N}^-$  we have  $\xi_{\kappa, \underline{\omega}}^D = i \eta_{\kappa, \underline{\omega}}$ , where  $\eta_{\kappa, \underline{\omega}}$  is defined in (4.1.29). Since

$$(i \eta_{\kappa, \underline{\omega}}) \coth(i \eta_{\kappa, \underline{\omega}}) = \eta_{\kappa, \underline{\omega}} \cot \eta_{\kappa, \underline{\omega}},$$

therefore, (4.1.7) rewrites as

$$\eta_{\kappa, \underline{\omega}} \cot \eta_{\kappa, \underline{\omega}} = -\mu_r(\underline{\omega}) \xi_{\kappa, \underline{\omega}}^V. \quad (4.1.43)$$

By using the identity

$$\tan(x + \frac{\pi}{2}) = -\cot x, \quad x \in \mathbb{R},$$

we arrive to the corresponding reformulation of the *dispersion equation* in  $\mathcal{D}_{\text{odd}}^-$ ,

$$\eta_{\kappa, \underline{\omega}} \tan(\eta_{\kappa, \underline{\omega}} + \frac{\pi}{2}) = \mu_r(\underline{\omega}) \xi_{\kappa, \underline{\omega}}^V. \quad (4.1.44)$$

Notice the similitude between and (4.1.30) and (4.1.44), indeed, (4.1.44) is a shifted version of (4.1.30) inside the argument of the tangent function, the latter fact lead us to consider the following function:

$$\eta_{2n+1} : [0, +\infty) \longrightarrow (n\pi, (n + \frac{1}{2})\pi], \quad \eta_{2n+1}(\tau) := (n + \frac{1}{2})\pi - \arctan \tau, \quad n \in \mathbb{N} \cup \{0\}, \quad (4.1.45)$$

which satisfies

$$\tan(\eta_{2n+1}(\tau) + \frac{\pi}{2}) = -\tau.$$

From here we can replicate exactly the same reasoning made for arriving to [Proposition 4.1.6](#). The following result is therefore obtained:

**Proposition 4.1.7**

$(\kappa, \underline{\omega}) \in \mathcal{D}_{\text{odd}}^-$  if and only if  $(\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \kappa_c)$  and for some  $n \in \mathbb{N} \cup \{0\}$  and  $\tau > 0$ ,  $(\kappa, \underline{\omega}, \tau)$  satisfies the system

$$\begin{cases} (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = ((\mu_r(\underline{\omega}))^{-2} \tau^2 + 1) \eta_{2n+1}^2(\tau), & (4.1.46a) \\ \kappa^2 = \omega^2 + (\mu_r(\underline{\omega}))^{-2} (\tau \eta_{2n+1}(\tau))^2. & (4.1.46b) \end{cases}$$

In this case the value of  $\tau$  is given by  $\tau = \tan(\eta_{\kappa, \underline{\omega}} + \frac{\pi}{2})$ .

**Remark 4.1.8.** We can standardize the definition of the functions defined in (4.1.33) and (4.1.45) as follows: for all  $n \in \mathbb{N}$ ,

$$\eta_n : [0, +\infty) \longrightarrow \left( \frac{n-1}{2} \pi, \frac{\pi}{2} n \right], \quad \eta_n(\tau) := \frac{\pi}{2} n - \arctan \tau, \quad n \in \mathbb{N}, \quad (4.1.47)$$

therefore the definition of (4.1.47) coincides with (4.1.33) (resp. (4.1.45)) when  $n = 2m$  is a positive even number (resp.  $n = 2m + 1$  is positive odd number).

### 4.1.3 The dimensionless dispersion systems $(\mathcal{DS}_n)_{n \geq 0}$ and the dispersion curves $(\mathcal{C}_n)_{n \geq 0}$

In the previous section we presented characterizations of the sets  $\mathcal{D}_{\text{even}}^-, \mathcal{D}_{\text{even}}^+, \mathcal{D}_{\text{odd}}^-, \mathcal{D}_{\text{odd}}^+$  as the solutions  $(\kappa, \underline{\omega})$  in a certain domain resolving a system (among a family of more manageable systems) by including the existence of a positive parameter  $\tau > 0$ . This approach is inspired in the previous work by C. Wilcox (see [93]). This approach suggests to see the dispersion solution sets  $\mathcal{D}_{\text{even}}, \mathcal{D}_{\text{odd}}$  as the countable union of *dispersion curves* designed, respectively, by the even and odd positive numbers, each of them characterized by what we will call a *dispersion system*. This idea can be formally expressed as follows, first we introduced the *even dispersion systems*

Dispersion system  $(\mathcal{DS}_0)$

$$(\mathcal{DS}_0) : \left\{ \begin{array}{l} (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = A_0(\tau) (\mu_r(\underline{\omega}))^{-2} + B_0(\tau), \\ \kappa^2 = \omega^2 + A_0(\tau) (\mu_r(\underline{\omega}))^{-2}, \end{array} \right. \quad (4.1.48a)$$

$$(4.1.48b)$$

where the functions  $A_0$  and  $B_0$  are defined in  $(0, +\infty)$  as

$$A_0(\tau) := (\tau \tanh \tau)^2, \quad B_0(\tau) := -\tau^2, \quad \tau > 0. \quad (4.1.49)$$

Dispersion system  $(\mathcal{DS}_{2n})$

$$(\mathcal{DS}_{2n}) : \left\{ \begin{array}{l} (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = A_{2n}(\tau) (\mu_r(\underline{\omega}))^{-2} + B_{2n}(\tau), \\ \kappa^2 = \omega^2 + A_{2n}(\tau) (\mu_r(\underline{\omega}))^{-2}. \end{array} \right. \quad (4.1.50a)$$

$$(4.1.50b)$$

where  $n \in \mathbb{N}$  and the functions  $A_{2n}$  and  $B_{2n}$  are defined in  $(0, +\infty)$  as

$$A_{2n}(\tau) := (\tau \eta_{2n}(\tau))^2, \quad B_{2n}(\tau) := \eta_{2n}^2(\tau), \quad \tau > 0. \quad (4.1.51)$$

In this sense, [Propositions 4.1.3](#) and [4.1.6](#) can be rewritten as the following identities,

$$\mathcal{D}_{\text{even}}^+ = \mathcal{C}_0, \quad \mathcal{D}_{\text{even}}^- = \bigcup_{n \in \mathbb{N}} \mathcal{C}_{2n}, \quad (4.1.52)$$

where the *even dispersion curves* are defined as

$$\begin{aligned} \mathcal{C}_0 &:= \left\{ (\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \underline{\Omega}_m) \mid (\kappa, \underline{\omega}, \tau) \text{ satisfies } (\mathcal{DS}_0) \text{ for some } \tau > 0 \right\}, \\ \mathcal{C}_{2n} &:= \left\{ (\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \underline{\Omega}_m) \mid (\kappa, \underline{\omega}, \tau) \text{ satisfies } (\mathcal{DS}_{2n}) \text{ for some } \tau > 0 \right\}, \quad n \in \mathbb{N}. \end{aligned} \quad (4.1.53)$$

Therefore  $\mathcal{D}_{\text{even}}$  is the countable union of dispersion curves indexed by the non negative even numbers,

$$\mathcal{D}_{\text{even}} = \bigcup_{n \in \mathbb{N}_0} \mathcal{C}_{2n}. \quad (4.1.54)$$

One can expect that decomposition of the set  $\mathcal{D}_{\text{odd}}$  into  $\mathcal{D}_{\text{odd}}^+$  and  $\mathcal{D}_{\text{odd}}^-$  follows the same principle as for  $\mathcal{D}_{\text{even}}$ . However, the latter would mean that  $\mathcal{D}_{\text{odd}}^-$  reassembles a countable collection of dispersion curves indexed by the positive odd numbers  $\{2n+1 \mid n \in \mathbb{N}\}$ , meanwhile  $\mathcal{D}_{\text{odd}}^+$  forms an extra dispersion curve! Even if this fact does not present any mathematical contradiction, we want to keep consistency within the presentation of  $\mathcal{D}_{\text{odd}}$ , more precisely, we aim to present  $\mathcal{D}_{\text{odd}}$  as the countable union of dispersion curves indexed by the positive odd numbers (with no extra curve). For this purpose, we need to be careful in defining the odd dispersion curves, we proceed by defining the *odd dispersion systems* as follows:

Dispersion system  $(\mathcal{DS}_1^+)$

$$(\mathcal{DS}_1^+) : \left\{ \begin{array}{l} (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = A_1^+(\tau) (\mu_r(\underline{\omega}))^{-2} + B_1^+(\tau), \\ \kappa^2 = \omega^2 + A_1^+(\tau) (\mu_r(\underline{\omega}))^{-2}, \end{array} \right. \quad (4.1.55a)$$

$$(4.1.55b)$$



where the functions  $A_1^+$  and  $B_1^+$  are defined in  $(0, +\infty)$  as

$$A_1^+(\tau) := (\tau \coth \tau)^2, \quad B_1^+(\tau) := -\tau^2, \quad \tau > 0. \quad (4.1.56)$$

Dispersion system  $(\mathcal{DS}_1^-)$

$$(\mathcal{DS}_1^-) : \begin{cases} (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = A_1^-(\tau) (\mu_r(\underline{\omega}))^{-2} + B_1^-(\tau), & (4.1.57a) \\ \kappa^2 = \omega^2 + A_1^-(\tau) (\mu_r(\underline{\omega}))^{-2}, & (4.1.57b) \end{cases}$$

where the functions  $A_1^-$  and  $B_1^-$  are defined in  $(0, +\infty)$  as

$$A_1^-(\tau) := (\tau \eta_1(\tau))^2, \quad B_1^-(\tau) := \eta_1^2(\tau), \quad \tau > 0. \quad (4.1.58)$$

Dispersion system  $(\mathcal{DS}_{2n+1})$

$$(\mathcal{DS}_{2n+1}) : \begin{cases} (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = A_{2n+1}(\tau) (\mu_r(\underline{\omega}))^{-2} + B_{2n+1}(\tau), & (4.1.59a) \\ \kappa^2 = \omega^2 + A_{2n+1}(\tau) (\mu_r(\underline{\omega}))^{-2}, & (4.1.59b) \end{cases}$$

where  $n \in \mathbb{N}$  and the functions  $A_{2n+1}$  and  $B_{2n+1}$  are defined in  $(0, +\infty)$  as

$$A_{2n+1}(\tau) := (\tau \eta_{2n+1}(\tau))^2, \quad B_{2n+1}(\tau) := \eta_{2n+1}^2(\tau), \quad \tau > 0. \quad (4.1.60)$$

We define also the *odd dispersion curves* as

$$\begin{aligned} \mathcal{C}_1^+ &:= \left\{ (\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \underline{\Omega}_m) \mid (\kappa, \underline{\omega}, \tau) \text{ satisfies } (\mathcal{DS}_1^+) \text{ for some } \tau > 0 \right\}, \\ \mathcal{C}_1^- &:= \left\{ (\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \underline{\Omega}_m) \mid (\kappa, \underline{\omega}, \tau) \text{ satisfies } (\mathcal{DS}_1^-) \text{ for some } \tau > 0 \right\}, \\ \mathcal{C}_{2n+1} &:= \left\{ (\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \underline{\Omega}_m) \mid (\kappa, \underline{\omega}, \tau) \text{ satisfies } (\mathcal{DS}_{2n+1}) \text{ for some } \tau > 0 \right\}, \quad n \in \mathbb{N}, \end{aligned} \quad (4.1.61)$$

therefore [Propositions 4.1.5](#) and [4.1.7](#) rewrite as

$$\mathcal{D}_{\text{odd}}^+ = \mathcal{C}_1^+, \quad \mathcal{D}_{\text{odd}}^- = \left( \bigcup_{n \in \mathbb{N}} \mathcal{C}_{2n+1} \right) \cup \mathcal{C}_1^- \quad (4.1.62)$$

Note that we decomposed the first *odd dispersion system* in two parts, namely,  $(\mathcal{DS}_1^+)$  corresponding to solutions in the zone  $\mathcal{N}^+$ , and  $(\mathcal{DS}_1^-)$  for solutions in  $\mathcal{N}^-$ . This is issued from the fact, as it will be later proved, that these two systems form separately  $C^\infty$  curves but concatenated by adding a point (which coincides to be the unique solution  $(\kappa, \underline{\omega})$  of [\(4.1.7\)](#) such that  $\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2 = \kappa^2$ , that is  $\xi_{\kappa, \underline{\omega}}^D = 0$ , in other words, the unique element of  $\mathcal{D}_{\text{odd}}^0$ ) form a single  $C^1$  curve solution resulting in what we will name the *dispersion curve*  $\mathcal{C}_1$ . In other words, we will consider the following system

Dispersion system  $(\mathcal{DS}_1)$

$$(\mathcal{DS}_1) : \begin{cases} (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = A_1(\tau) (\mu_r(\underline{\omega}))^{-2} + B_1(\tau), & (4.1.63a) \\ \kappa^2 = \omega^2 + A_1(\tau) (\mu_r(\underline{\omega}))^{-2}, & (4.1.63b) \end{cases}$$

$$\left| \begin{array}{l} A_1(s) := \begin{cases} A_1^-(\tan(\frac{\pi}{2}s)), & 0 < s < 1, \\ 1, & s = 1, \\ A_1^+(s-1), & s > 1, \end{cases} = \begin{cases} \left(\frac{\pi}{2}(1-s)\tan(\frac{\pi}{2}s)\right)^2, & 0 < s < 1, \\ 1, & s = 1, \\ \left((s-1)\coth(s-1)\right)^2, & s > 1, \end{cases} \\ B_1(s) := \begin{cases} B_1^-(\tan(\frac{\pi}{2}s)), & 0 < s < 1, \\ B_1^+(s-1), & s \geq 1, \end{cases} = \begin{cases} \left(\frac{\pi}{2}(1-s)\right)^2, & 0 < s < 1, \\ -(s-1)^2, & s \geq 1. \end{cases} \end{array} \right. \quad (4.1.64)$$

The definitions of  $A_1$  and  $B_1$  arise from smoothly mapping ( $C^\infty$ ) the intervals  $(0, 1)$  and  $(1, +\infty)$  onto  $(0, +\infty)$ , respectively, given by the following diffeomorphisms

$$\begin{array}{ccc} (0, 1) & \longrightarrow & (0, +\infty) \\ s & \mapsto & \tau = \tan\left(\frac{\pi}{2}s\right) \end{array} \quad \text{and} \quad \begin{array}{ccc} (1, +\infty) & \longrightarrow & (0, +\infty) \\ s & \mapsto & \tau = s - 1 \end{array}.$$

In this sense,  $A_1$  gathers the information of  $A_1^-$  and  $A_1^+$  inside, respectively, the intervals  $(0, 1)$  and  $(1, +\infty)$ . Similarly for  $B_1, B_1^-$  and  $B_1^+$ . On this basis, we can equivalently redefine  $C_1^-$  and  $C_1^+$  as

$$\begin{aligned} C_1^- &:= \left\{ (\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \underline{\Omega}_m) \mid (\kappa, \underline{\omega}, \tau) \text{ satisfies } (\mathcal{DS}_1) \text{ for some } \tau \in (0, 1) \right\}, \\ C_1^+ &:= \left\{ (\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \underline{\Omega}_m) \mid (\kappa, \underline{\omega}, \tau) \text{ satisfies } (\mathcal{DS}_1) \text{ for some } \tau > 1 \right\}. \end{aligned} \quad (4.1.65)$$

In addition we define the dispersion curve  $\mathcal{C}_1$  as

$$\mathcal{C}_1 := \left\{ (\kappa, \underline{\omega}) \in \mathbb{R}^+ \times (0, \underline{\Omega}_m) \mid (\kappa, \underline{\omega}, \tau) \text{ satisfies } (\mathcal{DS}_1) \text{ for some } \tau > 0 \right\}, \quad (4.1.66)$$

Finally,  $\mathcal{D}_{\text{odd}}$  is the countable union of dispersion curves indexed by the positive odd numbers,

$$\mathcal{D}_{\text{odd}} = \bigcup_{n \in \mathbb{N}_0} \mathcal{C}_{2n+1}. \quad (4.1.67)$$

**Remark 4.1.9.** Thanks to [Remark 4.1.8](#), we can regroup the definition of the *dispersion systems* for natural numbers  $n \geq 2$  as follows

Dispersion system  $(\mathcal{DS}_n)$  for  $n \geq 2$

$$(\mathcal{DS}_n) : \left\{ \begin{array}{l} (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = A_n(\tau) (\mu_r(\underline{\omega}))^{-2} + B_n(\tau), \\ \kappa^2 = \omega^2 + A_n(\tau) (\mu_r(\underline{\omega}))^{-2}, \end{array} \right. \quad (4.1.68a)$$

$$\kappa^2 = \omega^2 + A_n(\tau) (\mu_r(\underline{\omega}))^{-2}, \quad (4.1.68b)$$

where  $n \geq 2$  and the functions  $A_n$  and  $B_n$  are defined in  $(0, +\infty)$  as

$$A_n(\tau) := \tau^2 \eta_n(\tau)^2, \quad B_n(\tau) := \eta_n(\tau)^2, \quad \tau > 0. \quad (4.1.69)$$

## 4.2 General technique and tools for the parametric resolutions of the dispersion equations

### 4.2.1 A general family (4.2.2) of $\tau$ -dependent systems

In great part of the work which follows, we enter to the domain of the study of the solutions of the parametric curves consisting of points,  $(\kappa(\tau), \underline{\omega}(\tau))$  satisfying the *dispersion equation* (4.1.6). Among all the properties we will derive for the parametric curves, there are specially two of great interest leading a technique developed in this work consisting in basic tools —basic real analysis— : the first one which is also the first result we state for those curves is —as it should be— the existence and uniqueness of smooth curves; secondly, we pay attention to the *critical points*, that is, the points where the derivatives of the parametric curves nullify, giving as consequence a complete study of the intervals of decaying or increasing.

We present now the main general technique used when treating this topic. In first instance, we notice from the expressions of the systems  $(\mathcal{DS}_n)_{n \geq 2}$ ,  $(\mathcal{DS}_1^-)$ ,  $(\mathcal{DS}_1^+)$ ,  $(\mathcal{DS}_1)$  and  $(\mathcal{DS}_0)$  (defined, respectively, in (4.1.68), (4.1.57), (4.1.55), (4.1.63) and (4.1.48)) the appearance of two recurrent functions defined as follows:

**Definition 4.2.1** (Definiiton of  $u$  and  $v$ ). We define the continuously differentiable functions on  $(0, \underline{\Omega}_m)$

$$\begin{aligned} u(\underline{\omega}) &:= (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^2 = \frac{\rho^2 \underline{\Omega}_m^4}{\underline{\omega}^2} - (1 + \rho^2) \underline{\Omega}_m^2, \\ v(\underline{\omega}) &:= \mu_r^{-2}(\underline{\omega}) = \frac{\underline{\omega}^4}{(\underline{\Omega}_m^2 - \underline{\omega}^2)^2}. \end{aligned} \tag{4.2.1}$$

By using the last defined functions, we can rewrite the *dispersion systems*  $(\mathcal{DS}_n)_{n \geq 2}$ ,  $(\mathcal{DS}_1^-)$ ,  $(\mathcal{DS}_1^+)$ ,  $(\mathcal{DS}_1)$  and  $(\mathcal{DS}_0)$  into a general abstract form

$$\begin{cases} u(\underline{\omega}) = A(\tau) v(\underline{\omega}) + B(\tau), & (4.2.2a) \\ \kappa^2 = \underline{\omega}^2 + A(\tau) v(\underline{\omega}). & (4.2.2b) \end{cases}$$

Note that if  $(\kappa, \underline{\omega})$  solves (4.2.2a) then (4.2.2b) is equivalent to

$$\kappa^2 = \mathcal{J}(\underline{\omega}) - B(\tau), \tag{4.2.3}$$

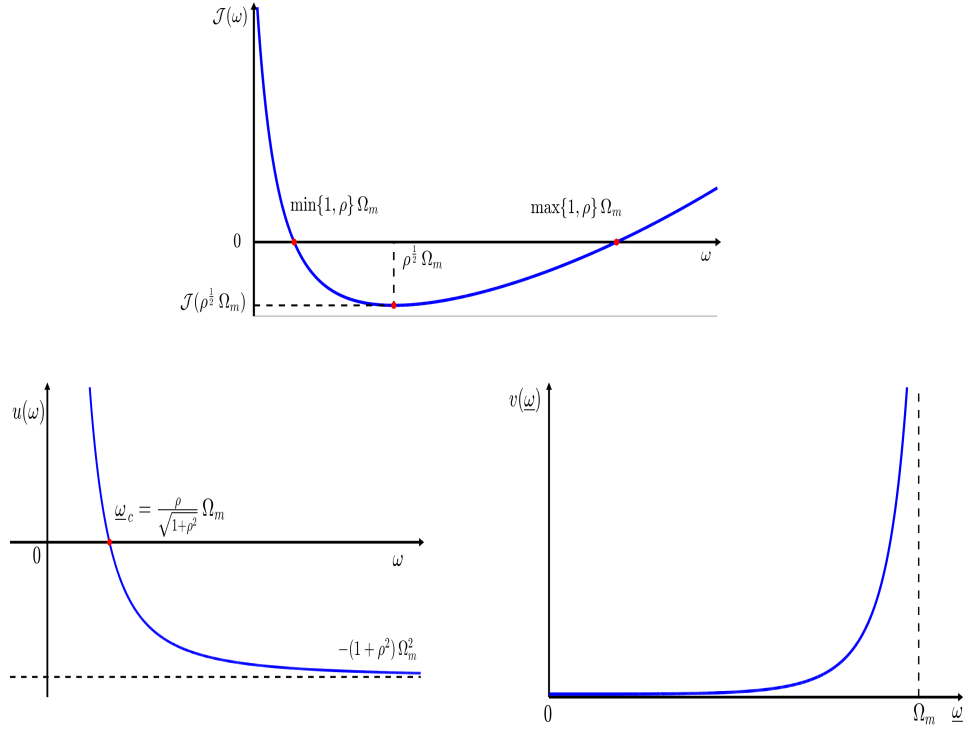
where we have defined

$$\mathcal{J} := \underline{\omega}^2 + u(\underline{\omega}) = \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) \underline{\omega}^2 = \underline{\omega}^2 + \frac{\rho^2 \underline{\Omega}_m^4}{\underline{\omega}^2} - (1 + \rho^2) \underline{\Omega}_m^2.$$

In some results it will be more convenient to work with the equivalent expression (4.2.3).

### 4.2.2 A general existence and uniqueness result for (4.2.2)

The proof existence of solutions for a system of the form (4.2.2) resides on the *Implicit Function Theorem*, whose hypothesis can be verified under specific assumptions for  $A(\cdot)$  and  $B(\cdot)$ , moreover, if these hypothesis are verified for every  $\tau > 0$ , then it is guaranteed the existence of a global and smooth curve solution  $\{(\kappa(\tau), \underline{\omega}(\tau)) \mid \tau > 0\}$ . For this purpose, we first give some important properties for  $u(\cdot)$ ,  $v(\cdot)$  and  $\mathcal{J}(\cdot)$ .


 Figure 4.3: Graphic representation of the functions  $\mathcal{J}(\cdot)$  (top)  $u(\cdot)$  (bottom left) and  $v(\cdot)$  (bottom right).

**Lemma 4.2.2**

The following properties hold for  $u(\cdot)$ ,  $v(\cdot)$  and  $\mathcal{J}(\cdot)$ .

(a)

$$\begin{aligned} \lim_{\underline{\omega} \rightarrow 0} u(\underline{\omega}) &= +\infty, & \lim_{\underline{\omega} \rightarrow 0} v(\underline{\omega}) &= 0, \\ \lim_{\underline{\omega} \rightarrow \underline{\Omega}_m} u(\underline{\omega}) &= -\underline{\Omega}_m^2, & \lim_{\underline{\omega} \rightarrow \underline{\Omega}_m} v(\underline{\omega}) &= +\infty; \end{aligned}$$

(b)  $v'(\underline{\omega}) > 0$ , for all  $\underline{\omega} \in (0, \underline{\Omega}_m)$ ;

(c)  $u'(\underline{\omega}) < 0$ , for all  $\underline{\omega} > 0$ ;

(d)  $\mathcal{J}'(\underline{\omega}) < 0$  if  $\underline{\omega} \in (0, \rho^{\frac{1}{2}} \underline{\Omega}_m)$ , while  $\mathcal{J}'(\underline{\omega}) > 0$  if  $\underline{\omega} > \rho^{\frac{1}{2}} \underline{\Omega}_m$ .

*Proof.* The property (a) follows directly from the definition of  $u(\cdot)$  and  $v(\cdot)$  in (4.2.1).

By differentiating  $v(\cdot)$  in (4.2.1) we have,

$$v'(\underline{\omega}) = -2\mu_r^{-3}(\underline{\omega})\mu_r'(\underline{\omega}) > 0, \quad \text{since } \mu_r'(\underline{\omega}) > 0 \text{ and } \mu_r(\underline{\omega}) < 0 \text{ in } (0, \underline{\Omega}_m), \quad (4.2.4)$$

while for  $u(\cdot)$  we compute

$$u(\underline{\omega}) = \frac{\rho^2 \underline{\Omega}_m^4}{\underline{\omega}^2} - (1 + \rho^2) \underline{\Omega}_m^2, \quad \text{thus } u'(\underline{\omega}) = -\frac{2\rho^2 \underline{\Omega}_m^4}{\underline{\omega}^3}, \quad (4.2.5)$$

from which (b) and (c) follow immediately. Finally, for  $\underline{\omega} > 0$  we have

$$\mathcal{J}'(\underline{\omega}) = 2\underline{\omega} - \frac{2\rho^2 \underline{\Omega}_m^4}{\underline{\omega}^3} < 0 \iff 2\underline{\omega} < \frac{2\rho^2 \underline{\Omega}_m^4}{\underline{\omega}^3} \iff \underline{\omega}^4 < \rho^2 \underline{\Omega}_m^4 \iff \underline{\omega} < \rho^{\frac{1}{2}} \underline{\Omega}_m. \quad (4.2.6)$$

□

Now we are able to establish the following theorem:

**Theorem 4.2.3**

Let be  $I \subseteq \mathbb{R}$  an open interval and  $m \in \mathbb{N} \cup \{\infty\}$ . Suppose that  $A, B \in C^m(I)$  such that  $A(\tau) > 0$  for all  $\tau \in I$ . Then there exists unique  $C^m(I)$ -functions

$$\underline{\omega} : I \longrightarrow (0, \underline{\Omega}_m) \quad \text{and} \quad \tilde{\kappa} : I \longrightarrow \mathbb{R}^+$$

such that for each  $\tau \in I$ ,  $(\tilde{\kappa}(\tau), \underline{\omega}(\tau))$  is the unique solution of the system

$$\begin{cases} u(\underline{\omega}) = A(\tau) v(\underline{\omega}) + B(\tau), & (4.2.7a) \\ \kappa^2 = \underline{\omega}^2 + A(\tau) v(\underline{\omega}). & (4.2.7b) \end{cases}$$

Conversely, if there exists  $C^m$ -functions  $\underline{\omega} : I \longrightarrow (0, \underline{\Omega}_m)$  and  $\tilde{\kappa} : I \longrightarrow \mathbb{R}^+$ , such that  $(\tilde{\kappa}(\tau), \underline{\omega}(\tau))$  solves (4.2.7) for every  $\tau \in I$ , then  $A, B \in C^m(I)$ .

*Proof.* Let us define the function  $F : (0, \underline{\Omega}_m) \times I \longrightarrow \mathbb{R}$  given by

$$F(\underline{\omega}, \tau) = A(\tau) v(\underline{\omega}) + B(\tau) - u(\underline{\omega}). \quad (4.2.8)$$

From the hypothesis we have that  $F \in C^m((0, \underline{\Omega}_m) \times I)$  and thanks to (4.2.2) we verify that

$$\partial_{\underline{\omega}} F(\underline{\omega}, \tau) = A(\tau) v'(\underline{\omega}) - u'(\underline{\omega}) > 0, \quad \text{for all } (\underline{\omega}, \tau) \in (0, \underline{\Omega}_m) \times \mathbb{R}^+. \quad (4.2.9)$$

Let be  $\tau_0 \in I$  fixed and consider the function  $F(\cdot, \tau_0) : (0, \underline{\Omega}_m) \longrightarrow \mathbb{R}$ . From Lemma 4.2.2 one can deduce that, since  $A(\tau_0) > 0$ ,

$$\lim_{\underline{\omega} \rightarrow 0^+} F(\underline{\omega}, \tau_0) = -\infty \quad \text{and} \quad \lim_{\underline{\omega} \rightarrow \underline{\Omega}_m^-} F(\underline{\omega}, \tau_0) = +\infty, \quad (4.2.10)$$

then by continuity, there exists  $\underline{\omega}_{\tau_0} \in (0, \underline{\Omega}_m)$  such that  $F(\underline{\omega}_{\tau_0}, \tau_0) = 0$ , this value for  $\underline{\omega}_{\tau_0}$  is unique since (4.2.9) guarantees that  $F(\cdot, \tau_0)$  is an increasing function. Hence, (4.2.9) and the Implicit Function Theorem show that the function defined as  $\underline{\omega}(\tau_0) := \underline{\omega}_{\tau_0}$  is  $C^m(I)$  and satisfies  $F(\underline{\omega}(\tau_0), \tau_0) = 0$ , for  $\tau_0 \in I$ . Because of the way  $F$  was defined, we get that  $(\underline{\omega}(\tau_0), \tau_0)$  satisfy (4.2.7a). Finally, by defining

$$\tilde{\kappa}(\tau) := \sqrt{(\underline{\omega}(\tau))^2 + A(\tau) v(\underline{\omega}(\tau))}, \quad \tau \in I. \quad (4.2.11)$$

On the other hand, if  $(\tilde{\kappa}(\tau), \underline{\omega}(\tau))$  solves (4.2.7) for every  $\tau \in I$ , then it follows that

$$\begin{aligned} A(\tau) &= v(\underline{\omega}(\tau))^{-1} (\tilde{\kappa}^2(\tau) - \underline{\omega}^2(\tau)), \\ B(\tau) &= u(\underline{\omega}(\tau)) - A(\tau) v(\underline{\omega}(\tau)) \end{aligned}$$

hence  $A, B \in C^m(I)$  provided that  $\underline{\omega}, \tilde{\kappa} \in C^m(I)$ , concluding the proof of this theorem. □

### 4.2.3 Monotonicity result for the solution to the solution of (4.2.2)

In what follows, we assume the hypothesis and notation of the statement and the proof of [Theorem 4.2.3](#). [Theorem 4.2.3](#) enable us to see the solutions of the abstract system (4.2.2) as a parametric curve

$$\tilde{\mathcal{C}} := \{(\tilde{\kappa}(\tau), \tilde{\omega}(\tau)) \mid \tau > 0\}. \quad (4.2.12)$$

The curve  $\tilde{\mathcal{C}}$  possess a certain degree of smoothness provided enough regularity on the functions  $A(\cdot)$  and  $B(\cdot)$ . In particular we give notorious interest to the first derivatives of  $\tilde{\omega}$  and  $\tilde{\kappa}$ , both of them guaranteed by assuming  $A, B \in C^1(I)$ .

Remember that our ultimate goal is describing the collection of *dispersion curves*,  $\{\mathcal{C}_n\}_{n \in \mathbb{N}_0}$ , as graphs over  $\kappa$ . The last goal is reached whenever it is possible to invert the function  $\tau \mapsto \tilde{\kappa}(\tau)$  on the interval  $I$ , in this case, the curve  $\tilde{\mathcal{C}}$  takes the form of the graph

$$\tilde{\mathcal{C}} = \{(\kappa, \tilde{\omega}^*(\kappa)) \mid \kappa \in \tilde{\kappa}[I]\}, \quad (4.2.13)$$

where  $\tilde{\omega}^* = \tilde{\omega} \circ \{\tilde{\kappa}\}^{-1}$  and  $\{\tilde{\kappa}\}^{-1}$  is the inverse function of  $\tilde{\kappa}(\cdot)$  which maps the interval  $\tilde{\kappa}[I]$  onto  $I$ . The continuity of  $\kappa \mapsto \{\tilde{\kappa}\}^{-1}(\kappa)$  is guaranteed by the continuity of  $\tau \mapsto \tilde{\kappa}(\tau)$ , which leads in turn the continuity of the function  $\kappa \mapsto \tilde{\omega}^*(\kappa)$ . On the other hand, the function  $\tilde{\omega}^*$  has the same smoothness as  $\tilde{\omega}$  and  $\tilde{\kappa}$  as long as  $\tilde{\kappa}' \neq 0$ , moreover, the *Inverse Function Theorem* allows us to easily compute the first derivative as

$$\frac{d}{d\kappa} \tilde{\omega}^*(\kappa) = \frac{\tilde{\omega}'(\tau)}{\tilde{\kappa}'(\tau)}, \quad \text{where } \tilde{\kappa}(\tau) = \kappa \text{ and } \tilde{\kappa}'(\tau) \neq 0. \quad (4.2.14)$$

The differentiability of  $\tilde{\omega}^*$  at  $\kappa \in \tilde{\kappa}[I]$  for which  $(\tilde{\kappa})'(\tau) = 0$  ( $\tilde{\kappa}(\tau) = \kappa$ ) must be studied by a limit computation, in other words, one must verify the existence of the limit

$$\lim_{\tau' \rightarrow \tau} \frac{\tilde{\omega}'(\tau')}{\tilde{\kappa}'(\tau')}, \quad \tau' \in I. \quad (4.2.15)$$

The *Implicit Function Theorem* allows us to implicitly calculate the derivatives of  $\tilde{\omega}$  and  $\tilde{\kappa}$ . Indeed, the first derivative of  $\tilde{\omega}$  can be found as:

$$\tilde{\omega}'(\tau) = -\frac{\partial_\tau F(\tilde{\omega}(\tau), \tau)}{\partial_\omega F(\tilde{\omega}(\tau), \tau)} = -\frac{A'(\tau) v(\tilde{\omega}(\tau)) + B'(\tau)}{A(\tau) v'(\tilde{\omega}(\tau)) - u'(\tilde{\omega}(\tau))}, \quad (4.2.16)$$

while (4.2.2b) gives us

$$\tilde{\kappa}'(\tau) = \frac{1}{2\tilde{\kappa}(\tau)} \left( [2\tilde{\omega}(\tau) + A(\tau) v'(\tilde{\omega}(\tau))] \tilde{\omega}'(\tau) + A'(\tau) v(\tilde{\omega}(\tau)) \right), \quad (4.2.17)$$

where the denominator never degenerates given the hypothesis of [Theorem 4.2.3](#). One can also derive an alternative expression for  $\tilde{\kappa}'(\tau)$  in terms of  $\tilde{\omega}'(\tau)$  using (4.2.3):

$$\tilde{\kappa}'(\tau) = \frac{1}{2\tilde{\kappa}(\tau)} (\mathcal{J}'(\tilde{\omega}(\tau)) \tilde{\omega}'(\tau) - B'(\tau)). \quad (4.2.18)$$

By substituting (4.2.16) into (4.2.17) (one obtains the same result by substituting (4.2.16) in (4.2.18)) and by reducing terms, then we obtain

$$\tilde{\kappa}'(\tau) = -\frac{1}{2\tilde{\kappa}(\tau)} \frac{A'(\tau) v(\tilde{\omega}(\tau)) \mathcal{J}'(\tilde{\omega}(\tau)) + [2\tilde{\omega}(\tau) + A(\tau) v'(\tilde{\omega}(\tau))] B'(\tau)}{A(\tau) v'(\tilde{\omega}(\tau)) - u'(\tilde{\omega}(\tau))}, \quad (4.2.19)$$

where it was used the fact  $\mathcal{J}'(\underline{\omega}) = 2\underline{\omega} + u'(\underline{\omega})$ . The two expressions for the first derivative of  $\tilde{\kappa}(\tau)$ , namely, (4.2.17) and (4.2.18) will be subsequently used. On another hand, (4.2.16) provides a first manner to localize the critical points of  $\tilde{\omega}(\tau)$ , indeed,

$$\tilde{\omega}'(\tau) = 0 \iff A'(\tau)v(\tilde{\omega}(\tau)) + B'(\tau) = 0, \quad (4.2.20)$$

in this sense, we can distinguish two scenarios depending if  $(A'(\tau), B'(\tau))$  nullifies:

- If  $(A'(\tau), B'(\tau)) = (0, 0)$ , then  $\tau$  is actually a critical point of  $\tilde{\omega}(\tau)$ .
- If  $(A'(\tau), B'(\tau)) \neq (0, 0)$  then, if  $\tau$  is a critical point of  $\tilde{\omega}(\tau)$ , necessarily  $A'(\tau) \neq 0$  and  $B'(\tau) \neq 0$  (remember that  $v(\tilde{\omega}(\tau)) > 0$ ), hence

$$\tilde{\omega}'(\tau) = 0 \iff v(\tilde{\omega}(\tau)) = \alpha(\tau), \quad \text{where } \alpha(\tau) := -\frac{B'(\tau)}{A'(\tau)}, \quad (4.2.21)$$

in this case, since  $v(\cdot)$  is an increasing function in  $(0, \underline{\Omega}_m)$  (see Lemma 4.2.2), then

$$\tilde{\omega}(\tau) = v^{\{-1\}}(\alpha(\tau)), \quad (4.2.22)$$

where  $v^{\{-1\}}(\cdot)$  stands for the inverse function of  $v(\cdot)$  in  $(0, \underline{\Omega}_m)$ . On the other hand, substituting  $v(\tilde{\omega}(\tau)) = \alpha(\tau)$  in (4.2.7a) yields to

$$u(\tilde{\omega}(\tau)) = \beta(\tau), \quad \text{where } \beta(\tau) := A(\tau)\alpha(\tau) + B(\tau) = -\frac{A(\tau)B'(\tau)}{A'(\tau)} + B(\tau). \quad (4.2.23)$$

Once again, since  $u(\cdot)$  is monotone in  $(0, \underline{\Omega}_m)$  we consider its inverse,  $u^{\{-1\}}(\cdot)$ , hence (4.2.23) implies

$$\tilde{\omega}(\tau) = u^{\{-1\}}(\beta(\tau)). \quad (4.2.24)$$

Comparing (4.2.22) and (4.2.24) we obtain a necessary condition for a critical point  $\tau \in I$ , namely

$$v^{\{-1\}}(\alpha(\tau)) = u^{\{-1\}}(\beta(\tau)). \quad (4.2.25)$$

In fact, we can also prove that is a sufficient condition, for that we explore when the derivative (4.2.16) is negative or positive. For this purpose it is important to consider the sign  $A'(\cdot)$ , as established in the following theorem.

#### Theorem 4.2.4

Under the hypothesis and notation of Theorem 4.2.3. If  $\alpha(\cdot) > 0$  and  $A'(\cdot) > 0$  in  $I$ , then the following statements are true for all  $\tau \in I$ :

- if  $\tilde{\omega}'(\tau) = 0$ , then  $v^{\{-1\}}(\alpha(\tau)) = u^{\{-1\}}(\beta(\tau))$ ;
- if  $\tilde{\omega}'(\tau) < 0$ , then  $v^{\{-1\}}(\alpha(\tau)) < u^{\{-1\}}(\beta(\tau))$ ;
- if  $\tilde{\omega}'(\tau) > 0$ , then  $v^{\{-1\}}(\alpha(\tau)) > u^{\{-1\}}(\beta(\tau))$ .

*Proof.* The proof of (a) has been treated in the latter discussion. We prove hereafter (b). Assume that  $\tilde{\omega}'(\tau) < 0$ , by using (4.2.16) and the fact  $\partial_{\underline{\omega}} F(\tilde{\omega}(\tau), \tau) = A(\tau)v'(\tilde{\omega}(\tau)) - u'(\tilde{\omega}(\tau)) > 0$ , hence it follows that

$$A'(\tau)v(\tilde{\omega}(\tau)) + B'(\tau) > 0.$$

The latter together with the hypothesis  $A'(\cdot) > 0$  in  $\mathbb{R}^+$  leads to

$$\alpha(\tau) = -\frac{B'(\tau)}{A'(\tau)} < v(\underline{\omega}(\tau)), \quad (4.2.26)$$

and since  $v(\cdot)$  is strictly increasing, so it is  $v^{\{-1\}}(\cdot)$ , therefore

$$v^{\{-1\}}(\alpha(\tau)) < \underline{\omega}(\tau). \quad (4.2.27)$$

Since  $\underline{\omega}(\tau)$  satisfies (4.2.7a) and using (4.2.26), we have, according to (4.2.23),

$$u(\underline{\omega}(\tau)) = A(\tau)v(\underline{\omega}(\tau)) + B(\tau) > -\frac{A(\tau)B'(\tau)}{A'(\tau)} + B(\tau) = \beta(\tau). \quad (4.2.28)$$

This time we use that  $u^{\{-1\}}(\cdot)$  is strictly decreasing ( $u(\cdot)$  is strictly decreasing) to deduce from the latter that

$$\underline{\omega}(\tau) < u^{\{-1\}}(\beta(\tau)). \quad (4.2.29)$$

From (4.2.27) and (4.2.29) it follows the conclusion of (b). The proof of (c) is analogous.  $\square$

One can return to our Drude model by recurring to the definition of  $u(\cdot), v(\cdot)$  given in (4.2.1), from which we deduce the expressions for the inverse of these functions in the domain  $(0, \underline{\Omega}_m)$ , namely,

$$u^{\{-1\}}(z) = \left( \frac{\rho^2 \underline{\Omega}_m^4}{z + (1 + \rho^2) \underline{\Omega}_m^2} \right)^{\frac{1}{2}}, \quad v^{\{-1\}}(z) = \left( \frac{\underline{\Omega}_m^2}{z^{-\frac{1}{2}} + 1} \right)^{\frac{1}{2}}. \quad (4.2.30)$$

Therefore, for a critical point  $\tau \in I$  of  $\underline{\omega}(\cdot)$ , the equation (4.2.25) yields the following

$$\left( \frac{\underline{\Omega}_m^2}{\alpha^{-\frac{1}{2}}(\tau) + 1} \right)^{\frac{1}{2}} = \left( \frac{\rho^2 \underline{\Omega}_m^4}{\beta(\tau) + (1 + \rho^2) \underline{\Omega}_m^2} \right)^{\frac{1}{2}}$$

This implies

$$\frac{\alpha^{-\frac{1}{2}}(\tau) + 1}{\underline{\Omega}_m^2} = \frac{\beta(\tau) + (1 + \rho^2) \underline{\Omega}_m^2}{\rho^2 \underline{\Omega}_m^4} \quad \text{thus} \quad \rho^2 \alpha^{-\frac{1}{2}}(\tau) = 1 + \frac{\beta(\tau)}{\underline{\Omega}_m^2}.$$

In this sense, we have the following corollary.

#### Corollary 4.2.5

Under the assumptions of [Theorem 4.2.4](#), then the followings are true for  $\tau > 0$  :

- (a) if  $\underline{\omega}'(\tau) = 0$ , then  $S_{\rho, \underline{\Omega}_m}(\tau) = 0$ ;
- (b) if  $\underline{\omega}'(\tau) < 0$ , then  $S_{\rho, \underline{\Omega}_m}(\tau) < 0$ ;
- (c) if  $\underline{\omega}'(\tau) > 0$ , then  $S_{\rho, \underline{\Omega}_m}(\tau) > 0$ ;

where  $S_{\rho, \underline{\Omega}_m} : (0, +\infty) \longrightarrow \mathbb{R}$  is defined as

$$S_{\rho, \underline{\Omega}_m}(\tau) := 1 + \frac{\beta(\tau)}{\underline{\Omega}_m^2} - \rho^2 \alpha^{-\frac{1}{2}}(\tau). \quad (4.2.31)$$



From the last corollary, we give interest to study the zeros (if any) of the function  $S_{\rho, \underline{\Omega}_m}(\cdot)$ , in order to treat this problem, it will be useful to consider the critical points of  $S_{\rho, \underline{\Omega}_m}(\cdot)$  and classify them as local minima or maxima. For this purpose, we present the following result links the critical points of  $S_{\rho, \underline{\Omega}_m}(\cdot)$  and  $\alpha(\cdot)$ .

**Proposition 4.2.6**

Under the assumptions of [Theorem 4.2.4](#), then  $S_{\rho, \underline{\Omega}_m}(\cdot)$  and  $\alpha(\cdot)$  have the same intervals of increase (resp. decrease). In particular, they have the same critical points, moreover, those have the same nature of local minima, local maxima or saddle point.

*Proof.* By derivating,  $S_{\rho, \underline{\Omega}_m}(\cdot)$  and using the definition of  $\alpha(\cdot)$  and  $\beta(\cdot)$  in, respectively, [\(4.2.21\)](#) and [\(4.2.23\)](#), we obtain the following:

$$\begin{aligned} S'_{\rho, \underline{\Omega}_m}(\tau) &= \frac{\beta'(\tau)}{\underline{\Omega}_m^2} + \frac{\rho^2}{2} \alpha^{-\frac{3}{2}}(\tau) \alpha'(\tau) \\ &= \frac{A'(\tau) \alpha(\tau) + A(\tau) \alpha'(\tau) + B'(\tau)}{\underline{\omega}_m^2} + \frac{\rho^2}{2} \alpha^{-\frac{3}{2}}(\tau) \alpha'(\tau) \\ &= \frac{-B'(\tau) + A(\tau) \alpha'(\tau) + B'(\tau)}{\underline{\omega}_m^2} + \frac{\rho^2}{2} \alpha^{-\frac{3}{2}}(\tau) \alpha'(\tau) \\ &= \left( \frac{A(\tau)}{\underline{\omega}_m^2} + \frac{\rho^2}{2} \alpha^{-\frac{3}{2}}(\tau) \right) \alpha'(\tau), \end{aligned}$$

since  $\alpha(\cdot) > 0$ , it follows that  $S'_{\rho, \underline{\Omega}_m}(\tau)$  and  $\alpha'(\tau)$  have the same sign, therefore they are simultaneously increasing (resp. decreasing) in the same intervals. In the same way,  $S'_{\rho, \underline{\Omega}_m}(\tau) = 0$  if and only if  $\alpha'(\tau) = 0$ , hence they share the same critical points. Furthermore, if  $\tau^*$  is a critical point of  $S_{\rho, \underline{\Omega}_m}(\cdot)$  and  $\alpha(\cdot)$ , by computing the second derivative of  $S_{\rho, \underline{\Omega}_m}(\tau^*)$ , we have that:

$$\begin{aligned} S''_{\rho, \underline{\Omega}_m}(\tau^*) &= \left( \frac{A(\tau)}{\underline{\omega}_m^2} + \frac{\rho^2}{2} \alpha^{-\frac{3}{2}}(\tau) \right)' \Big|_{\tau=\tau^*} \alpha'(\tau^*) + \left( \frac{A(\tau^*)}{\underline{\omega}_m^2} + \frac{\rho^2}{2} \alpha^{-\frac{3}{2}}(\tau^*) \right) \alpha'(\tau^*) \\ &= \left( \frac{A(\tau^*)}{\underline{\omega}_m^2} + \frac{\rho^2}{2} \alpha^{-\frac{3}{2}}(\tau^*) \right) \alpha''(\tau^*), \quad (\text{since } \alpha'(\tau^*) = 0). \end{aligned}$$

Thus  $S''_{\rho, \underline{\Omega}_m}(\tau^*)$  and  $\alpha''(\tau^*)$  have the same sign, provided that  $A(\cdot) \geq 0$  and  $\alpha(\cdot) > 0$ .  $\square$

### 4.3 Study of the curves $(\mathcal{C}_n)_{n \geq 2}$ .

As said previously in [Section 4.1.3](#), our aim is to prove that the sets  $(\mathcal{C}_n)_{n \geq 2}$  are the graphs of a  $C^\infty$ -function on the variable  $\kappa$ .

The outline of the rest of this section is as follows:

- In [Section 4.3.1](#) we prove that each  $\mathcal{C}_n$  constitutes a  $C^\infty$ -curve,  $\{(\kappa_n(\tau), \underline{\omega}_n(\tau)) \mid \tau > 0\}$ .
- In [Section 4.3.2](#), we study the monotonicity of the functions  $\tau \mapsto \underline{\omega}_n(\tau)$ .
- In [Section 4.3.3](#), we study the monotonicity of the functions  $\tau \mapsto \kappa_n(\tau)$ .
- In [Section 4.3.4](#), we study the behaviour at  $\infty$  of the functions  $\tau \mapsto \underline{\omega}_n(\tau)$  and  $\tau \mapsto \kappa_n(\tau)$ .
- In [Section 4.3.6](#), we give a quite complete ("geometrical") description of the dispersion curves.

### 4.3.1 Existence and uniqueness of the solution $(\kappa_n(\tau), \underline{\omega}_n(\tau))$ of $(\mathcal{DS}_n)$

#### Theorem 4.3.1: Existence of the parametric curves solutions to $(\mathcal{DS}_n)_{n \geq 2}$

Let be  $\rho, \underline{\Omega}_m > 0$  and  $n \geq 2$  fixed. There exists a unique couple of  $C^\infty$  functions  $\underline{\omega}_n : [0, \infty) \rightarrow (0, \kappa_c)$  and  $\kappa_n : [0, \infty) \rightarrow \mathbb{R}^+$  such that for any  $\tau \geq 0$ ,  $(\underline{\omega}_n(\tau), \kappa_n(\tau))$  is solution (parameterized by  $\tau$ ) of  $(\mathcal{DS}_n)$  (defined in (4.1.68)-(4.1.69)). Consequently,  $\mathcal{C}_n$  is fully described as

$$\mathcal{C}_n := \{(\kappa_n(\tau), \underline{\omega}_n(\tau)) \mid \tau > 0\}. \quad (4.3.1)$$

Moreover, the initial point of the curve  $\mathcal{C}_n$  is given by  $(\kappa_n(0), \underline{\omega}_n(0))$

$$\underline{\omega}_n(0) = \kappa_n(0) = \kappa_{c,n} := \rho \underline{\Omega}_m^2 \left( \left( \frac{\pi}{2} n \right)^2 + (1 + \rho^2) \underline{\Omega}_m^2 \right)^{-\frac{1}{2}}. \quad (4.3.2)$$

In particular, the sequence  $(\kappa_n(0))_{n \geq 2}$  forms a strictly decreasing sequence that tends to 0 when  $n \rightarrow +\infty$  as  $2\rho \underline{\Omega}_m^2 / \pi n$ .

*Proof.* The existence of the  $C^\infty([0, \infty))$  functions  $\underline{\omega}_n : [0, \infty) \rightarrow (0, \kappa_c)$  and  $\kappa_n : [0, \infty) \rightarrow \mathbb{R}^+$  solving the *dispersion system*  $(\mathcal{DS}_n)$  (see (4.1.68)) is a direct application of [Theorem 4.2.3](#) applied to the functions  $A_n$  and  $B_n$  defined in (4.1.69), which are  $C^\infty$  in  $[0, +\infty)$ .

Finally, by taking  $\tau = 0$  in (4.1.42b) yields to  $\underline{\omega}_n(0) = \kappa_n(0)$  and (4.1.42a) applied in  $\tau = 0$  gives  $u(\underline{\omega}_n(0)) = \eta_n(0)^2 = \left(\frac{\pi}{2} n\right)^2$ , which shows the announced value for  $\underline{\omega}_n(0)$  in virtue of the formula for the inverse of  $u(\cdot)$  (see (4.2.30)).  $\square$

### 4.3.2 Study of the monotonicity of $\underline{\omega}_n(\tau)$

In what follows we want to use the results presented in [Section 4.2.3](#). First note that for  $n \geq 2$

$$A'_n(\tau) = 2\tau \eta_n(\tau) (\eta_n(\tau) + \tau \eta'_n(\tau)) = 2\tau \eta_n(\tau) \left( \eta_n(\tau) - \frac{\tau}{1 + \tau^2} \right) > 0, \quad \tau > 0, \quad (4.3.3)$$

since  $\eta_n(\tau) > \frac{\tau}{1 + \tau^2} > 0$  for  $\tau > 0$  (see [Lemma C. 1](#)). Secondly, we introduce the functions  $\alpha_n$  and  $\beta_n$  defined using  $A_n$  and  $B_n$ , see (4.1.69), (as  $(\alpha, \beta)$  were defined from  $(A, B)$  in (4.2.21) and (4.2.23)). That is to say, for  $\alpha_n := -B'_n/A'_n$ , as  $A_n(\tau) := \tau^2 \eta_n(\tau)^2$  and  $B_n(\tau) := \eta_n(\tau)^2$ ,

$$\alpha_n(\tau) = -\frac{2\eta_n(\tau) \eta'_n(\tau)}{2\tau \eta_n(\tau) (\eta_n(\tau) + \tau \eta'_n(\tau))} = -\frac{\eta'_n(\tau)}{\tau (\eta_n(\tau) + \tau \eta'_n(\tau))}$$

or equivalently, as  $\eta'_n = -(1 + \tau^2)^{-1}$

$$\alpha_n(\tau) = (\tau + \tau^3)^{-1} \left( \eta_n(\tau) - \frac{\tau}{1 + \tau^2} \right)^{-1} \quad (4.3.4)$$

while, for  $\beta_n(\tau) := A_n(\tau) \alpha_n(\tau) + B_n(\tau)$ , we compute that

$$\beta_n(\tau) = \eta_n(\tau)^3 \left( \eta_n(\tau) - \frac{\tau}{1 + \tau^2} \right)^{-1} \quad (4.3.5)$$

Let us remind that [Lemma C. 1](#) guarantees us that  $\alpha_n(\cdot)$  is a positive function in  $\mathbb{R}^+$ .

With these elements, we can proceed as previously presented in [Section 4.2.3](#) to study the

monotonicity of  $\underline{\omega}_n(\cdot)$  ( $n \in \mathbb{N}$ ). For this purpose, we need to study the real zeros of the function

$$S_{n,\rho,\underline{\Omega}_m}(\tau) := 1 + \frac{\beta_n(\tau)}{\underline{\Omega}_m^2} - \rho^2 (\alpha_n(\tau))^{-\frac{1}{2}}. \quad (4.3.6)$$

**Proposition 4.3.2**

For all  $n \geq 2$  and  $\rho, \underline{\Omega}_m > 0$ , the function  $S_{n,\rho,\underline{\Omega}_m}(\cdot)$  has a unique positive zero,  $\tau_n > 0$ , such that  $S_{n,\rho,\underline{\Omega}_m}(\cdot)$  is positive (resp. negative) in  $(0, \tau_n)$  (resp.  $(\tau_n, +\infty)$ ).

*Proof.* From Proposition 4.2.6, one deduces that  $S_{n,\rho,\underline{\Omega}_m}(\cdot)$  is strictly decreasing in  $\mathbb{R}^+$ , provided that  $\alpha_n(\cdot)$  is strictly decreasing in  $\mathbb{R}^+$  (see Lemma C. 1). With the aid of Lemma C. 1 we can also compute

$$\lim_{\tau \rightarrow 0^+} S_{n,\rho,\underline{\Omega}_m}(\tau) = 1 + \frac{(\frac{\pi}{2}n)^2}{\underline{\Omega}_m^2} > 0 \quad \text{and} \quad \lim_{\tau \rightarrow +\infty} S_{n,\rho,\underline{\Omega}_m}(\tau) = -\infty. \quad (4.3.7)$$

Therefore, the Intermediate Value Theorem guarantee us that  $S_{n,\rho,\underline{\Omega}_m}(\cdot)$  has a unique positive zero, whom we denote as  $\tau_n$ . Since  $S_{n,\rho,\underline{\Omega}_m}(\cdot)$  is strictly decreasing, it easily follows that  $S_{n,\rho,\underline{\Omega}_m}(\cdot)$  is positive (resp. negative) in  $(0, \tau_n)$  (resp.  $(\tau_n, +\infty)$ ).  $\square$

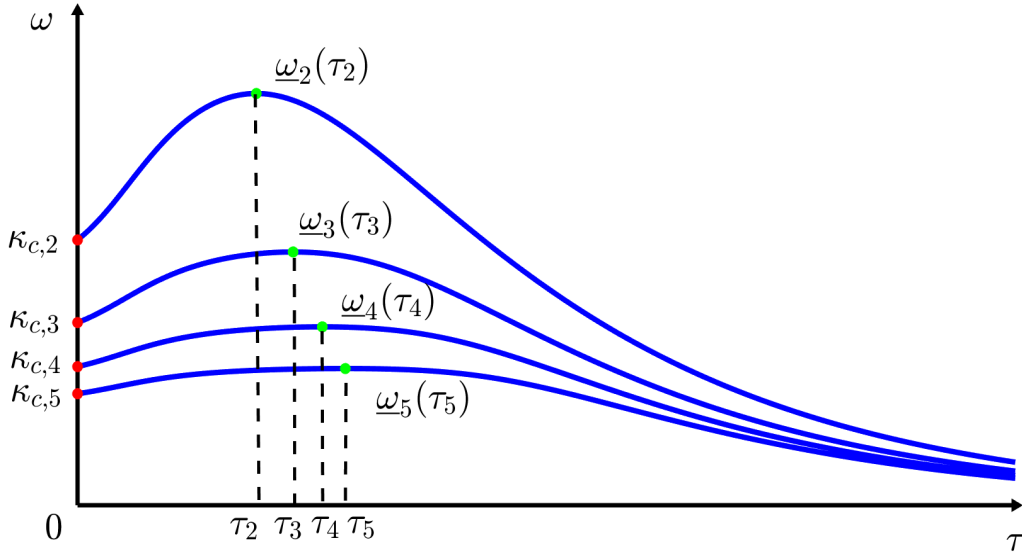


Figure 4.4: Graphic representation of the function  $\underline{\omega}_n(\cdot)$  for  $n = 2, 3, 4, 5$ .

The following corollary is a direct consequence of Corollary 4.2.5 and Proposition 4.3.2.

**Corollary 4.3.3**

Under the notation of Proposition 4.3.2. For all  $n \geq 2$  and  $\rho, \underline{\Omega}_m > 0$ ,  $\tau_n > 0$  is the unique positive critical point of  $\underline{\omega}_n(\cdot)$ . Moreover,  $\underline{\omega}_n(\cdot) > 0$  (resp.  $\underline{\omega}'_n(\cdot) < 0$ ) in the interval  $(0, \tau_n)$  (resp.  $(\tau_n, +\infty)$ ).

### 4.3.3 Study of the monotonicity of the function $\kappa_n(\tau)$

This analysis is essential to show that the curve  $\mathcal{C}_n$  is the graph of a function of  $\kappa$  (this will be done in section 4.3.6).

#### Lemma 4.3.4

The function  $\tau \mapsto \kappa_n(\tau)$  is a strictly increasing function. Moreover,  $\kappa'_n(\tau) > 0$  for all  $\tau > 0$ .

*Proof.* By using (4.2.19) one can compute

$$\kappa'_n = -\frac{1}{2\kappa_n} \frac{(2\underline{\omega}_n + u'(\underline{\omega}_n))(\tau^2 \eta_n^2)' v(\underline{\omega}_n) + (\eta_n^2)'(2\underline{\omega}_n + \tau^2 \eta_n^2 v'(\underline{\omega}_n))}{\tau^2 \eta_n^2 v'(\underline{\omega}_n) - u'(\underline{\omega}_n)}. \quad (4.3.8)$$

Thanks to Lemma 4.2.2 and the fact that  $\underline{\omega}_n \in (0, \kappa_c) \subset (0, \underline{\Omega}_m)$  we see that the denominator of the right hand side in (4.3.8) is positive. Thus, we deduce that the sign  $\kappa'_n$  is the opposite of the sign of

$$g_n^1 + g_n^2, \quad g_n^1 := (2\underline{\omega}_n + u'(\underline{\omega}_n))(\tau^2 \eta_n^2)' v(\underline{\omega}_n), \quad g_n^2 := (\eta_n^2)'(2\underline{\omega}_n + \tau^2 \eta_n^2 v'(\underline{\omega}_n)).$$

Since  $v(\cdot) > 0$ , the function  $g_n^1$  is negative because

$$\begin{cases} 2\underline{\omega}_n + u'(\underline{\omega}_n) = \mathcal{J}'(\underline{\omega}_n) < 0, \text{ by Lemma 4.2.2 (d) since } 0 < \underline{\omega}_n < \kappa_c < \rho^{\frac{1}{2}} \underline{\Omega}_m \\ (\tau^2 \eta_n^2)' = 2\tau \eta_n (\eta_n + \tau \eta'_n) = 2\tau \eta_n (\eta_n - \frac{\tau}{1+\tau^2}) > 0, \text{ (see Lemma C.1 (i)).} \end{cases} \quad (4.3.9)$$

The function  $g_n^2$  is negative because

$$\begin{cases} (\eta_n^2)' = 2\eta_n \eta'_n = -\frac{2\eta_n}{1+\tau^2} < 0, \\ 2\underline{\omega}_n + \tau^2 \eta_n^2 v'(\underline{\omega}_n) > 0, \text{ since } v'(\underline{\omega}_n) > 0 \text{ by Lemma 4.2.2 (a).} \end{cases} \quad (4.3.10)$$

We deduce that  $\kappa'_n > 0$  which concludes the proof.  $\square$

### 4.3.4 Asymptotic behaviour of the functions $\underline{\omega}_n(\tau)$ and $\kappa_n(\tau)$ when $\tau \rightarrow +\infty$

#### Preliminary informal asymptotic analysis of $\underline{\omega}_n$

Below,  $\underline{\omega}$  stands for  $\underline{\omega}_n(\tau)$ , so that  $\underline{\omega}$  satisfies (4.1.68a), i. e.

$$u(\underline{\omega}) = (1 + v(\underline{\omega}) \tau^2) \eta_n^2(\tau) \quad (4.3.11)$$

Since  $\eta_n(\tau) \rightarrow \frac{n-1}{2} \pi$  when  $\tau \rightarrow +\infty$ , for large  $\tau$ , we can replace the above equation by the "tangent" equation

$$u(\underline{\omega}) = \left( \frac{n-1}{2} \pi \right)^2 (1 + v(\underline{\omega}) \tau^2). \quad (4.3.12)$$

We first see that, when  $\tau \rightarrow +\infty$ , if (4.3.12) holds,  $\underline{\omega}$  cannot remain bounded away from 0. Indeed, in such a case,  $v(\underline{\omega})$  remains as well bounded away from 0 (see Lemma 4.2.2), hence the right hand side of (4.3.12) would blow up when  $\tau \rightarrow +\infty$  while the left hand side would remain bounded (since we already know that  $u(\underline{\omega})$  is bounded far from 0). Therefore,  $\underline{\omega} \rightarrow 0$  as  $\tau \rightarrow \infty$ .

This suggests to replace (4.3.12) by another approximate equation obtained by replacing  $u(\underline{\omega})$  and  $v(\underline{\omega})$  by their equivalent for small  $\omega$ , namely

$$u(\underline{\omega}) \sim \frac{\rho^2 \underline{\Omega}_m^4}{\underline{\omega}^2}, \quad v(\underline{\omega}) \sim \frac{\underline{\omega}^4}{\underline{\Omega}_m^4}.$$

This leads to the approximate equation

$$\frac{\rho^2 \underline{\Omega}_m^4}{\underline{\omega}^2} = \left( \frac{n-1}{2} \pi \right)^2 \left( 1 + \frac{\underline{\omega}^4 \tau^2}{\underline{\Omega}_m^4} \right). \quad (4.3.13)$$

When  $\tau \rightarrow +\infty$ ,  $\underline{\omega}^4 \tau^2$  can not remain bounded since the left hand side of (4.3.13) blows up.

Therefore, 1 can be neglected with respect to  $\frac{\underline{\omega}^4 \tau^2}{\underline{\Omega}_m^4}$ , which leads to the other approximate equation

$$\frac{\rho^2 \underline{\Omega}_m^4}{\underline{\omega}^2} = \left( \frac{n-1}{2} \pi \right)^2 \frac{\underline{\omega}^4 \tau^2}{\underline{\Omega}_m^4} \quad (4.3.14)$$

which gives

$$\underline{\omega} = \Omega_n \tau^{-\frac{1}{3}}, \quad \text{where} \quad \Omega_n := \left( \frac{2 \rho \underline{\Omega}_m^4}{\pi (n-1)} \right)^{\frac{1}{3}}. \quad (4.3.15)$$

The above (formal) analysis suggests that the function  $\underline{\omega}_n(\tau)$  satisfies

$$\underline{\omega}_n(\tau) \sim \Omega_n \tau^{-\frac{1}{3}} \quad (\tau \rightarrow +\infty)$$

This is what we are going to prove rigorously using the implicit function theorem. Note that (4.3.15) suggests to introduce  $X = \tau^{\frac{1}{3}} \omega$  as a new unknown of the equation (4.3.11).

**Rigorous analysis.** The rigorous proof of the announced result will be obtained by combining the use of the proposed change of variable with the implicit function theorem.

The precise result is the following :

#### Lemma 4.3.5

Let be  $n \geq 2$ . The function  $\tau \mapsto \underline{\omega}_n(\tau)$  has the following asymptotic behavior

$$\begin{cases} \underline{\omega}_n(\tau) = \Omega_n \tau^{-\frac{1}{3}} - \gamma_n \tau^{-1} + o(\tau^{-1}), & (\tau \rightarrow \infty), \\ \Omega_n := \left( \frac{2 \rho \underline{\Omega}_m^4}{(n-1) \pi} \right)^{\frac{1}{3}}, \quad \gamma_n := \frac{\left( \frac{n-1}{2} \pi \right)^2 + (1 + 3\rho^2) \underline{\Omega}_m^2}{3 \rho (n-1) \pi} \end{cases} \quad (4.3.16)$$

The function  $\tau \mapsto \kappa_n(\tau)$  has the following asymptotic behavior

$$\begin{cases} \kappa_n(\tau) = K_n \tau^{\frac{1}{3}} - \delta_n \tau^{-\frac{1}{3}} + o(\tau^{-\frac{1}{3}}), & \text{when } \tau \rightarrow +\infty, \\ K_n := \frac{n-1}{2} \pi \Omega_n^2 \underline{\Omega}_m^{-2}, \quad \delta_n = \Omega_n \frac{\left( \frac{n-1}{2} \pi \right)^2 + \underline{\Omega}_m^2}{3 \rho \underline{\Omega}_m^2} \end{cases} \quad (4.3.17)$$

*Proof.* (i) Study of  $\underline{\omega}_n(\tau)$ . First note that the form of the approximate equation (4.3.14) suggests

to multiply the equation (4.3.11) by  $\underline{\omega}^2$ , which leads to as

$$U(\underline{\omega}) = (\underline{\omega}^2 + V(\underline{\omega}) \tau^2) \eta_n(\tau)^2, \quad (4.3.18)$$

where we have defined the two functions

$$\begin{cases} U(\underline{\omega}) := (\varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) - 1) \underline{\omega}^4 = \rho^2 \underline{\Omega}_m^4 - (1 + \rho^2) \underline{\Omega}_m^2 \underline{\omega}^2, \\ V(\underline{\omega}) := \underline{\omega}^2 v(\underline{\omega}) = \frac{\underline{\omega}^6}{(\underline{\omega}^2 - \underline{\Omega}_m^2)^2} \end{cases} \quad (4.3.19)$$

We introduce, as suggested above, the new unknown  $X = \tau^{\frac{1}{3}} \underline{\omega}$  as well as the new parameter  $\varepsilon := \tau^{-\frac{2}{3}}$ , so that  $\underline{\omega}^2 = \varepsilon X^2$  and  $\tau = \varepsilon^{-\frac{3}{2}}$ , (4.3.18) becomes, using (4.3.19) for  $U$  and  $V$ ,

$$\rho^2 \underline{\Omega}_m^4 - (1 + \rho^2) \underline{\Omega}_m^2 \varepsilon X^2 = \left( \varepsilon X^2 + \frac{X^6}{(\underline{\Omega}_m^2 - \varepsilon X^2)^2} \right) \eta_n(\varepsilon^{-\frac{3}{2}})^2. \quad (4.3.20)$$

We wish to study (4.3.20) as an equation in  $X$  parametrized by  $\varepsilon$ , when  $\varepsilon \rightarrow 0$  (which corresponds to  $\tau \rightarrow +\infty$ ). In order to apply the implicit function theorem, we introduce the function

$$\mathcal{F}_n(X, \varepsilon) := \rho^2 \underline{\Omega}_m^4 - (1 + \rho^2) \underline{\Omega}_m^2 \varepsilon X^2 - \left( \varepsilon X^2 + \frac{X^6}{(\underline{\Omega}_m^2 - \varepsilon X^2)^2} \right) \eta_n^2(\varepsilon^{-\frac{3}{2}}), \quad \varepsilon X^2 \neq \underline{\Omega}_m^2, \quad (4.3.21)$$

in such a way that

$$(X, \varepsilon) \text{ satisfies (4.3.20)} \iff \mathcal{F}_n(X, \varepsilon) = 0. \quad (4.3.22)$$

It is easy to see that  $\mathcal{F}_n(X, \varepsilon)$  is of class  $C^1$  (details are omitted) in the open set defined by  $\varepsilon X^2 \neq \underline{\Omega}_m^2$  and that

$$\mathcal{F}_n(X, 0) = \rho^2 \underline{\Omega}_m^4 - \left( \frac{n-1}{2} \pi \right)^2 \frac{X^6}{\underline{\Omega}_m^4}.$$

In particular,

$$\mathcal{F}_n(X, 0) = 0, \quad X > 0 \iff X = \Omega_n = \left( \frac{2 \rho \underline{\Omega}_m^4}{(n-1) \pi} \right)^{\frac{1}{3}}.$$

We can apply the implicit function theorem near  $(\varepsilon = 0, X = \Omega_n)$  since

$$\partial_X \mathcal{F}_n(X, 0) := -6 \left( \frac{n-1}{2} \pi \right)^2 \frac{X^5}{\underline{\Omega}_m^4} < 0, \quad \forall X \neq 0. \quad (4.3.23)$$

From the implicit function theorem, we deduce that, for  $|\varepsilon|$  small enough, the equation  $\mathcal{F}_n(X, \varepsilon) = 0$  admits, in the neighborhood of  $X_n$ , a unique solution  $X_n(\varepsilon)$  and that the function  $\varepsilon \mapsto X_n(\varepsilon)$  is of class  $C^1$  in the neighbourhood of  $\varepsilon = 0$ , with in particular

$$X_n'(0) = -\partial_X \mathcal{F}_n(X_n, 0)^{-1} \partial_\varepsilon \mathcal{F}_n(X_n, 0). \quad (4.3.24)$$

Of course, in view of the global existence and uniqueness of the solution  $\underline{\omega}_n(\tau) > 0$  of the equation (4.3.11), which is linked to the equation  $\mathcal{F}_n(X, \varepsilon) = 0$  through the change of variables  $\varepsilon := \tau^{-\frac{2}{3}}$ , and  $X = \tau^{\frac{1}{3}} \underline{\omega}$ , by identification, we deduce that for large enough  $\tau$  we have

$$\underline{\omega}_n(\tau) = \tau^{-\frac{1}{3}} X_n(\tau^{-\frac{2}{3}}).$$

Using  $X_n(\varepsilon) = \Omega_n + X'_n(0) \varepsilon + o(\varepsilon)$ , we deduce

$$\underline{\omega}_n(\tau) = \tau^{-\frac{1}{3}} \Omega_n + \tau^{-1} X'_n(0) + o(\tau^{-1}), \quad (\tau \rightarrow +\infty) \quad (4.3.25)$$

To complete the proof, it remains to compute  $X'_n(0)$  via (4.3.24). Using the expansion of  $\eta_n(\tau)$  at infinity:

$$\eta_n(\tau) = \frac{n-1}{2} \pi + \tau^{-1} - \frac{\tau^{-2}}{2} + O(\tau^{-3}), \quad \tau \rightarrow +\infty$$

we deduce that  $\eta_n(\varepsilon^{-\frac{3}{2}}) = \frac{n-1}{2} \pi + O(\varepsilon^{\frac{3}{2}})$ . Moreover

$$(\underline{\Omega}_m^2 - \varepsilon X^2)^{-2} = \underline{\Omega}_m^{-4} \left(1 - \varepsilon \frac{X^2}{\underline{\Omega}_m^2}\right)^{-2} = \underline{\Omega}_m^{-4} \left(1 + 2\varepsilon \frac{X^2}{\underline{\Omega}_m^2} + O(\varepsilon^2)\right)$$

thus, we can expand  $\mathcal{F}_n(X, \varepsilon)$  for small  $\varepsilon$  as

$$\mathcal{F}_n(X, \varepsilon) = \rho^2 \underline{\Omega}_m^4 - \left(\frac{n-1}{2} \pi\right)^2 \frac{X^6}{\underline{\Omega}_m^4} - (1 + \rho^2) \underline{\Omega}_m^2 \varepsilon X^2 - \left(\frac{n-1}{2} \pi\right)^2 \varepsilon \left(X^2 + 2 \frac{X^8}{\underline{\Omega}_m^6}\right) + O(\varepsilon^{\frac{3}{2}})$$

which shows that

$$\partial_\varepsilon \mathcal{F}_n(X, 0) = -(1 + \rho^2) \underline{\Omega}_m^2 X^2 - \frac{1}{4} (n-1)^2 \pi^2 X^2 \left(1 + 2 \frac{X^6}{\underline{\Omega}_m^6}\right).$$

For  $X = \Omega_n$ ,  $\frac{1}{4} (n-1)^2 \pi^2 \frac{X_n^6}{\underline{\Omega}_m^6} = \underline{\Omega}_m^2 \rho^2$ , thus

$$\partial_\varepsilon \mathcal{F}_n(\Omega_n, 0) = - \left[ \left(\frac{n-1}{2} \pi\right)^2 + (1 + 3\rho^2) \underline{\Omega}_m^2 \right] \Omega_n^2. \quad (4.3.26)$$

On the other hand, cf. (4.3.23), we have

$$\partial_X \mathcal{F}_n(\Omega_n, 0) = -6 \left(\frac{n-1}{2} \pi\right)^2 \frac{\Omega_n^5}{\underline{\Omega}_m^4} = -3\rho \Omega_n^2 (n-1) \pi, \quad (4.3.27)$$

where we have used  $\frac{n-1}{2} \pi \frac{\Omega_n^3}{\underline{\Omega}_m^4} = \rho$  for the second equality.

Substituting (4.3.26) and (4.3.23) gives

$$X'_n(0) = - \frac{\left(\frac{n-1}{2} \pi\right)^2 + (1 + 3\rho^2) \underline{\Omega}_m^2}{3\rho(n-1)\pi}. \quad (4.3.28)$$

Hence, thanks to (4.3.25), we obtain the expansion (4.3.16).

(i) Study of  $\kappa_n(\tau)$ . We use (4.1.50b), i. e.

$$\kappa_n(\tau)^2 = \underline{\omega}_n(\tau)^2 + v(\underline{\omega}_n(\tau)) \tau^2 \eta_n(\tau)^2. \quad (4.3.29)$$

We first study the behaviour of  $v(\underline{\omega}_n(\tau))$ . Since  $v(\underline{\omega}) = \frac{\underline{\omega}^4}{(\underline{\omega}^2 - \underline{\Omega}_m^2)^2}$

$$v(\underline{\omega}) = \underline{\omega}^4 \underline{\Omega}_m^{-4} + 2 \underline{\omega}^6 \underline{\Omega}_m^{-6} + O(\underline{\omega}^8) \quad (\underline{\omega} \rightarrow 0) \quad (4.3.30)$$

On the other hand, from (4.3.16), we can write

$$\underline{\omega}_n(\tau) = \Omega_n \tau^{-\frac{1}{3}} \left( 1 - \gamma_n^* \tau^{-\frac{2}{3}} + o(\tau^{-\frac{2}{3}}) \right),$$

with

$$\gamma_n^* := \Omega_n^{-1} \gamma_n \quad (4.3.31)$$

we deduce

$$\begin{cases} \underline{\omega}_n(\tau)^4 = \Omega_n^4 \tau^{-\frac{4}{3}} \left( 1 - 4 \gamma_n^* \tau^{-\frac{2}{3}} + o(\tau^{-\frac{2}{3}}) \right) \\ \underline{\omega}_n(\tau)^6 = \Omega_n^6 \tau^{-\frac{6}{3}} \left( 1 - 6 \gamma_n^* \tau^{-\frac{2}{3}} + o(\tau^{-\frac{2}{3}}) \right) \end{cases}$$

that we can substitute into (4.3.30) to obtain

$$v(\underline{\omega}_n(\tau)) = \Omega_n^4 \underline{\Omega}_m^{-4} \tau^{-\frac{4}{3}} \left( 1 - 4 \gamma_n^* \tau^{-\frac{2}{3}} + 2 \Omega_n^2 \underline{\Omega}_m^{-2} \tau^{-\frac{2}{3}} + o(\tau^{-\frac{2}{3}}) \right).$$

Thus, as  $\eta_n(\tau)^2 = \left(\frac{n-1}{2} \pi\right)^2 + O(\tau^{-1})$

$$v(\underline{\omega}_n(\tau)) \tau^2 \eta_n(\tau)^2 = \left(\frac{n-1}{2} \pi\right)^2 \Omega_n^4 \underline{\Omega}_m^{-4} \tau^{\frac{2}{3}} \left( 1 - 2(2 \gamma_n^* - \Omega_n^2 \underline{\Omega}_m^{-2}) \tau^{-\frac{2}{3}} + o(\tau^{-\frac{2}{3}}) \right)$$

Substituting the above in (4.3.29), we get,  $\underline{\omega}_n(\tau)^2 = o(1)$ ,

$$\kappa_n(\tau)^2 = \left(\frac{n-1}{2} \pi\right)^2 \Omega_n^4 \underline{\Omega}_m^{-4} \tau^{\frac{2}{3}} \left( 1 - 2(2 \gamma_n^* - \Omega_n^2 \underline{\Omega}_m^{-2}) \tau^{-\frac{2}{3}} + o(\tau^{-\frac{2}{3}}) \right), \quad (4.3.32)$$

which leads to

$$\kappa_n(\tau) = \frac{n-1}{2} \pi \Omega_n^2 \underline{\Omega}_m^{-2} \tau^{\frac{1}{3}} \left( 1 - (2 \gamma_n^* - \Omega_n^2 \underline{\Omega}_m^{-2}) \tau^{-\frac{2}{3}} + o(\tau^{-\frac{2}{3}}) \right). \quad (4.3.33)$$

By substituting the values of  $\gamma_n, \Omega_n, \gamma_n^*$  (see (4.3.16) and (4.3.31)) we obtain that

$$\begin{aligned} 2 \gamma_n^* - \Omega_n^2 \underline{\Omega}_m^{-2} &= \Omega_n^{-1} (2 \gamma_n - \Omega_n^3 \underline{\Omega}_m^{-2}) \\ &= \Omega_n^{-1} \left( 2 \frac{\left(\frac{n-1}{2} \pi\right)^2 + (1 + 3\rho^2) \underline{\Omega}_m^2}{3 \rho (n-1) \pi} - \frac{2 \rho \underline{\Omega}_m^2}{(n-1) \pi} \right) \\ &= 2 \Omega_n^{-1} \frac{\left(\frac{n-1}{2} \pi\right)^2 + \underline{\Omega}_m^2}{3 \rho (n-1) \pi}, \end{aligned} \quad (4.3.34)$$

The expansion of  $\kappa_n(\tau)$  follows easily. □

### 4.3.5 Characterization of the curves $(\mathcal{C}_n)$ as graphs

#### Theorem 4.3.6

For all  $n \geq 2$ , each one of the curves  $\mathcal{C}_n$  is the graph of a  $C^\infty$ -function on  $(\kappa_{c,n}, +\infty)$ . More precisely,

$$\mathcal{C}_n = \{(\kappa, \underline{\omega}_n^*(\kappa)) \mid \kappa > \kappa_{c,n}\}, \quad \underline{\omega}_n^* := \underline{\omega}_n \circ \{\kappa_n\}^{-1} \quad (4.3.35)$$

where  $\kappa_n = \kappa_n(\tau)$  is an invertible function whose inverse is given by  $\{\kappa_n\}^{-1} : [\kappa_{c,n}, +\infty) \rightarrow [0, +\infty)$ .



*Proof.* Let  $n \geq 2$  fixed. From [Lemma 4.3.4](#), we know that  $\kappa_n(\cdot)$  is an invertible function (since it is strictly increasing) which maps  $\kappa_n(0) = \kappa_{c,n}$  (see [Theorem 4.3.1](#)) and such that  $\kappa_n(\tau) \rightarrow +\infty$  as  $\tau \rightarrow +\infty$ , hence  $\kappa_n$  maps the domain  $[0, +\infty)$  onto  $[\kappa_{c,n}, +\infty)$ , thus its inverse function is well defined:  $\{\kappa_n\}^{-1} : [\kappa_{c,n}, +\infty) \rightarrow [0, +\infty)$ . Since  $\kappa_n$  is  $C^\infty$  and  $\kappa_n'(\cdot) > 0$  in  $(0, +\infty)$ , then the *Inverse Function Theorem* guarantees that  $\{\kappa_n\}^{-1}$  is  $C^\infty$  in  $(\kappa_{c,n}, +\infty)$ .

Finally, from [Theorem 4.3.1](#) it follows that

$$\begin{aligned} (\kappa, \underline{\omega}) \in \mathcal{C}_n &\iff (\kappa_n(\tau), \underline{\omega}_n(\tau)) = (\kappa, \underline{\omega}) \text{ for some } \tau > 0, \\ &\iff \kappa > \kappa_{c,n}, \quad \tau = \{\kappa_n\}^{-1}(\kappa) \text{ and } \underline{\omega} = \underline{\omega}_n(\tau) \\ &\iff \kappa > \kappa_{c,n} \text{ and } \underline{\omega} = \underline{\omega}_n(\{\kappa_n\}^{-1}(\kappa)) = \underline{\omega}_n^*(\kappa). \end{aligned}$$

□

### 4.3.6 Geometric properties of the curves $(\mathcal{C}_n)$

#### Theorem 4.3.7

For all  $n \geq 2, \rho, \Omega_m > 0$ ,  $\underline{\omega}_n^*$  has the following properties:

- (a)  $\underline{\omega}_n^*$  is continuously differentiable at  $\kappa = \kappa_{c,n}$  and  $(\underline{\omega}_n^*)'(\kappa_{c,n}) = 1$ ;
- (b)  $\underline{\omega}_n^*$  has a unique critical point,  $\kappa_n^* \in (\kappa_{c,n}, \infty)$ , i.e.,  $(\underline{\omega}_n^*)'(\kappa_n^*) = 0$ , moreover,  $(\underline{\omega}_n^*)'(\kappa) > 0$  (resp.  $(\underline{\omega}_n^*)'(\kappa) < 0$ ) if  $\kappa \in [\kappa_{c,n}, \kappa_n^*)$  (resp.  $\kappa \in (\kappa_n^*, +\infty)$ );
- (c) for all  $\kappa > \kappa_{c,n}$ ,  $0 < \underline{\omega}_{n+1}^*(\kappa) < \underline{\omega}_n^*(\kappa) < \Phi(\kappa)$ ;
- (d)  $\underline{\omega}_n^*(\kappa) = \Phi_\infty \kappa^{-1} - a_n \kappa^{-3} + o(\kappa^{-3})$ , as  $\kappa \rightarrow +\infty$ , where  $\Phi_\infty$  was defined in [\(4.1.15\)](#) and  $a_n := \delta_n \Omega_n K_n^2 + \gamma_n K_n^3 > 0$  (see the definitions of  $\gamma_n, K_n$  and  $\delta_n$  in [Lemma 4.3.5](#)).

*Proof.* (a) From chain's rule derivation and the inverse derivative formula one gets that

$$(\underline{\omega}_n^*)'(\kappa) = \frac{\underline{\omega}_n'(\tau)}{\kappa_n'(\tau)}, \quad \text{where } \kappa_n(\tau) = \kappa. \quad (4.3.36)$$

Substituting in [\(4.3.36\)](#) the expression for the derivatives of  $\underline{\omega}_n$  and  $\kappa_n$  deduced respectively, from [\(4.2.16\)](#) and [\(4.3.8\)](#), thus we obtain

$$(\underline{\omega}_n^*)'(\kappa) = 2\kappa \frac{(\eta_n^2)' + (\tau^2 \eta_n^2)' v(\underline{\omega}_n^*)}{(2\underline{\omega}_n^* + u'(\underline{\omega}_n^*))(\tau^2 \eta_n^2)' v(\underline{\omega}_n^*) + (\eta_n^2)'(2\underline{\omega}_n^* + \tau^2 \eta_n^2 v'(\underline{\omega}_n^*))}, \quad (4.3.37)$$

where  $\kappa_n(\tau) = \kappa, \underline{\omega}_n^* = \underline{\omega}_n(\tau) = \underline{\omega}_n^*(\kappa)$  and  $\eta_n := \eta_n(\tau)$ . Since  $\kappa_n(0) = \kappa_{c,n}$ , hence we need evaluate the right-hand expression of [\(4.4.60\)](#) at  $\tau = 0, \underline{\omega}_n^* = \underline{\omega}_n(0) = \kappa_{c,n}$ , by using the fact  $(\eta_n^2)'|_{\tau=0} = (2\eta_n \eta_n')|_{\tau=0} = -\pi n \neq 0$  and  $(\tau^2 \eta_n^2)'|_{\tau=0} = (2\tau \eta_n (\eta_n + \tau \eta_n'))|_{\tau=0} = 0$  thus we get

$$(\underline{\omega}_n^*)'(\kappa_{c,n}) = 2\kappa_{c,n} \left( \frac{(\eta_n^2)'|_{\tau=0}}{2\kappa_{c,n} (\eta_n^2)'|_{\tau=0}} \right) = 1. \quad (4.3.38)$$

- (b) From [\(4.3.36\)](#) and [Lemma 4.3.4](#), it follows that  $(\underline{\omega}_n^*)'(\kappa)$  and  $\underline{\omega}_n'(\tau)$  have the same sign, where  $\kappa_n(\tau) = \kappa$ . Therefore, this clause is a direct consequence of [Corollary 4.3.3](#) and in this case  $\kappa_n^* = \kappa_n(\tau_n^*)$ .
- (c) The relations  $0 < \underline{\omega}_{n+1}^*(k)$  and  $\underline{\omega}_n^*(k) < \Phi(k)$  are derived directly from the fact that  $\mathcal{C}_{n+1}, \mathcal{C}_N \subset \mathcal{N}^-$  and [\(4.1.16\)](#). For the last inequality, we notice since  $\kappa > \kappa_{c,n} > \kappa_{c,n+1}$ ,

then  $\kappa = \kappa_n(\tau) = \kappa_{n+1}(s)$  for some  $\tau, s > 0$ . Since  $(\kappa, \underline{\omega}_n^*(\kappa)) = (\kappa_n(\tau), \underline{\omega}_n(\tau))$  satisfies (4.1.68a)-(4.1.68b) for  $\tau$ , by subtracting, we obtain

$$\eta_n^2(\tau) = u(\underline{\omega}_n(\tau)) + \underline{\omega}_n^2(\tau) - \kappa_n^2(\tau) = \mathcal{J}(\underline{\omega}_n^*(\kappa)) - \kappa^2, \quad (4.3.39)$$

analogously

$$\eta_{n+1}^2(s) = \mathcal{J}(\underline{\omega}_{n+1}^*(\kappa)) - \kappa^2, \quad (4.3.40)$$

since the ranges of the functions  $\eta_n^2$  and  $\eta_{n+1}^2$  are, respectively, the intervals

$$\left( \left( (n - \frac{1}{2})\pi \right)^2, (n\pi)^2 \right] \quad \text{and} \quad \left( \left( (n + \frac{1}{2})\pi \right)^2, ((n + 1)\pi)^2 \right],$$

then one deduce that  $\eta_n^2(\tau) < \eta_{n+1}^2(s)$  and therefore

$$\mathcal{J}(\underline{\omega}_n^*(\kappa)) - \kappa^2 < \mathcal{J}(\underline{\omega}_{n+1}^*(\kappa)) - \kappa^2, \quad (4.3.41)$$

consequently  $\mathcal{J}(\underline{\omega}_n^*(\kappa)) < \mathcal{J}(\underline{\omega}_{n+1}^*(\kappa))$ , finally since  $\mathcal{J}$  is strictly decreasing in  $(0, \kappa_c)$  (remember that  $\kappa_c < \rho^{\frac{1}{2}} \underline{\Omega}_m$  and see Lemma 4.2.2) and  $\underline{\omega}_n^*(\kappa), \underline{\omega}_{n+1}^*(\kappa) \in (0, \kappa_c)$ , then it follows that  $\underline{\omega}_{n+1}^*(\kappa) < \underline{\omega}_n^*(\kappa)$ .

- (d) We use the estimation (4.3.17) proved in Lemma 4.3.5, altogether with Lemma C. 4 applied to  $\kappa_n$  seen as a function on  $\tau^{\frac{1}{3}}$  :

$$\tau^{\frac{1}{3}} = K_n^{-1} \kappa + \delta_n \kappa^{-1} + o(\kappa^{-1}), \quad \text{as } \kappa \longrightarrow +\infty, \quad (4.3.42)$$

and we have abbreviate  $\tau = \{\kappa_n\}^{-1}(\kappa)$ , hence

$$\begin{aligned} \tau^{-\frac{1}{3}} &= (K_n^{-1} \kappa + \delta_n \kappa^{-1} + o(\kappa^{-1}))^{-1} \\ &= (K_n^{-1} \kappa)^{-1} (1 + \delta_n K_n \kappa^{-2} + o(\kappa^{-2}))^{-1} \\ &= K_n \kappa^{-1} (1 - \delta_n K_n \kappa^{-2} + o(\kappa^{-2})) \\ &= K_n \kappa^{-1} - \delta_n K_n^2 \kappa^{-3} + o(\kappa^{-3}), \quad \text{as } \kappa \longrightarrow +\infty, \end{aligned} \quad (4.3.43)$$

analogously one proves

$$\begin{aligned} \tau^{-1} &= \left( K_n^{-1} \kappa + \delta_n \kappa^{-1} + o(\kappa^{-1}) \right)^{-3} \\ &= K_n^3 \kappa^{-3} - 3 \delta_n K_n^4 \kappa^{-5} + o(\kappa^{-5}), \quad \text{as } \kappa \longrightarrow +\infty. \end{aligned} \quad (4.3.44)$$

Finally, by substituting (4.3.43) and (4.3.44) in (4.3.16):

$$\begin{aligned} \underline{\omega}_n^*(\kappa) &= \Omega_n \left( K_n \kappa^{-1} - \delta_n K_n^2 \kappa^{-3} + o(\kappa^{-3}) \right) - \gamma_n \left( K_n^3 \kappa^{-3} + o(\kappa^{-3}) \right) \\ &= \Omega_n K_n \kappa^{-1} - (\delta_n \Omega_n K_n^2 + \gamma_n K_n^3) \kappa^{-3} + o(\kappa^{-3}) \end{aligned}$$

We notice that  $\Omega_n K_n = \frac{n-1}{2} \pi \Omega_n^3 \underline{\Omega}_m^{-2} = \rho \underline{\Omega}_m^2 = \Phi_\infty$ . Thus the clause (d) follows.  $\square$

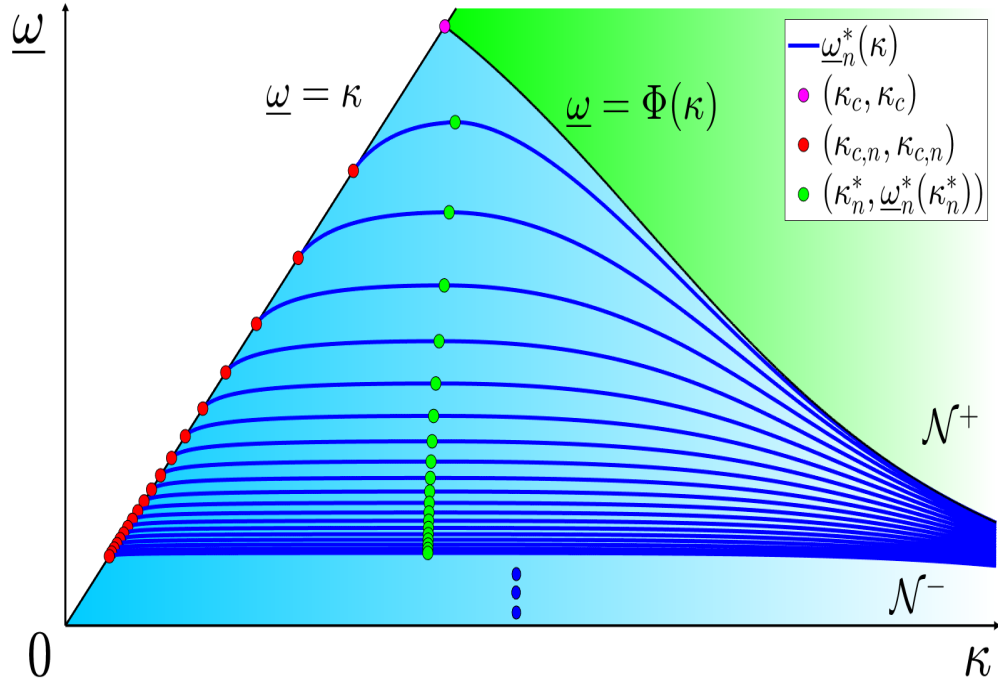


Figure 4.5: Graphic representation of the function  $\underline{\omega}_n^*(\cdot)$  for  $n = 2, \dots, 20$ . (a) The functions  $\underline{\omega}_n^*$  (blue graphs) are smooth in  $\kappa_{c,n}$  (red points) and the respective graphs are tangent to the identity line  $\underline{\omega} = \kappa$ . (b) The green points represents the maximum of each function  $\underline{\omega}_n^*$ . (c) The graphs are *well-ordered* in decreasing order,  $\underline{\omega}_2^*$  on the top and  $\underline{\omega}_n^*(0)$  at the bottom. (d) All the functions  $\underline{\omega}_n^*$  tend asymptotically to  $\underline{\omega} = \Phi(\kappa)$ .

## 4.4 Study of the curve $\mathcal{C}_1$

### 4.4.1 Description of the set $\mathcal{C}_1^- = \mathcal{C}_1 \cap \mathcal{N}^-$

#### 4.4.1.1 Existence and uniqueness of the solution $(\kappa_1^-(\tau), \underline{\omega}_1^-(\tau))$ of $(\mathcal{DS}_1^-)$

##### Theorem 4.4.1: Existence of the parametric curves in $\mathcal{D}_{\text{odd}}^-$

There exists a unique couple of  $C^1$  functions  $\underline{\omega}_1^- : [0, \infty) \rightarrow (0, \kappa_c)$  and  $\kappa_1^- : [0, \infty) \rightarrow \mathbb{R}^+$  such that for any  $\tau \geq 0$ ,  $(\kappa_1^-(\tau), \underline{\omega}_1^-(\tau))$  is solution (parameterized by  $\tau$ ) of  $(\mathcal{DS}_1^+)$  (4.1.57)-(4.1.58). Consequently,  $\mathcal{C}_1^-$  is fully described as

$$\mathcal{C}_1^- := \{(\kappa_1^-(\tau), \underline{\omega}_1^-(\tau)), \tau \in \mathbb{R}^+\}. \quad (4.4.1)$$

The initial point of the curve  $\mathcal{C}_n$  is given by

$$\underline{\omega}_1^-(0) = \kappa_1^-(0) = \kappa_{c,1} := \rho \underline{\Omega}_m^2 \left( \frac{\pi^2}{4} + (1 + \rho^2) \underline{\Omega}_m^2 \right)^{-\frac{1}{2}}. \quad (4.4.2)$$

*Proof.* The proof is analogous to the one of [Theorem 4.3.1](#), indeed, as a consequence of [Theorem 4.2.3](#) with the choice of  $C^\infty$  functions

$$A_1^+(\tau) := \tau^2 \eta_1^2(\tau), \quad B_1^+(\tau) := \eta_1^2(\tau). \quad (4.4.3)$$

□

#### 4.4.1.2 Study of the monotonicity of $\underline{\omega}_1^-(\tau)$

In this part we are interested into studying the increasing and decreasing intervals for the function  $\underline{\omega}_1^-(\cdot)$ . For this purpose we present the functions

$$\begin{aligned}\alpha_1^-(\tau) &:= -\frac{(B_1^-)'(\tau)}{(A_1^-)'(\tau)} = -\frac{2\eta_1(\tau)\eta_1'(\tau)}{2\tau\eta_1(\tau)(\eta_1(\tau) + \tau\eta_1'(\tau))} = -\frac{\eta_1'(\tau)}{\tau(\eta_1(\tau) + \tau\eta_1'(\tau))} \\ &= \left( (\tau + \tau^3) \left( \eta_1(\tau) - \frac{\tau}{1 + \tau^2} \right) \right)^{-1},\end{aligned}\quad (4.4.4)$$

$$\begin{aligned}\beta_1^-(\tau) &:= A_1^-(\tau)\alpha_1^-(\tau) + B_1^-(\tau) = \left( 1 + \frac{\tau^2}{(\tau + \tau^3) \left( \eta_1(\tau) - \frac{\tau}{1 + \tau^2} \right)} \right) \eta_1^2(\tau) \\ &= \frac{\eta_1^3(\tau)}{\eta_1(\tau) - \frac{\tau}{1 + \tau^2}}.\end{aligned}\quad (4.4.5)$$

Next we introduce the following function

$$S_{1,\rho,\underline{\Omega}_m}^-(\tau) := 1 + \frac{\beta_1^-(\tau)}{\underline{\Omega}_m^2} - \rho^2 \alpha_1^-(\tau)^{-\frac{1}{2}} \quad (4.4.6)$$

that satisfy the following properties :

#### Proposition 4.4.2

Let be  $\rho, \underline{\Omega}_m > 0$ , then the following are true for the zeros of  $S_{1,\rho,\underline{\Omega}_m}^-(\cdot)$  :

- (a) If  $1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} \geq 0$ , then  $S_{1,\rho,\underline{\Omega}_m}^-(\cdot) > 0$ .
- (b) If  $1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} < 0$ , then  $S_{1,\rho,\underline{\Omega}_m}^-(\cdot)$  has a unique positive zero,  $\tau_1^- > 0$ .

Moreover  $S_{1,\rho,\underline{\Omega}_m}^-(\cdot)$  is positive (resp. negative) in  $(0, \tau_1^-)$  (resp.  $(\tau_1^-, +\infty)$ ).

*Proof.* We know from [Lemma C. 1](#) that  $\alpha_1^-(\cdot)$  is strictly decreasing, therefore  $S_{1,\rho,\underline{\Omega}_m}^-(\cdot)$  is also decreasing (see [Corollary 4.2.5](#)) and satisfies

$$\lim_{\tau \rightarrow 0} S_{1,\rho,\underline{\Omega}_m}^-(\tau) = 1 + \frac{(\pi/2)^2}{\underline{\Omega}_m^2} > 0 \quad \text{and} \quad \lim_{\tau \rightarrow +\infty} S_{1,\rho,\underline{\Omega}_m}^-(\tau) = 1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}}, \quad (4.4.7)$$

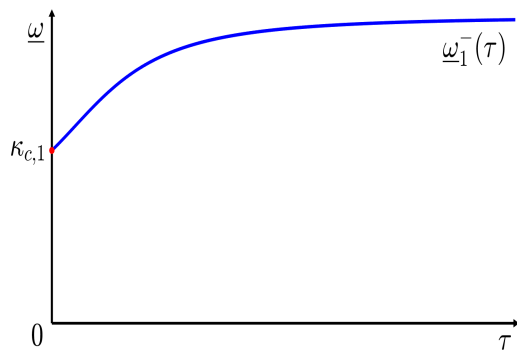
hence  $S_{1,\rho,\underline{\Omega}_m}^-(\cdot)$  has a positive zero if and only if  $1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} < 0$ , in this case, this zero is unique given that  $S_{1,\rho,\underline{\Omega}_m}^-(\cdot)$  is strictly decreasing. The clauses (a) and (b) follow directly. □

As a consequence of [Corollary 4.2.5](#) and [Proposition 4.4.2](#) we have:

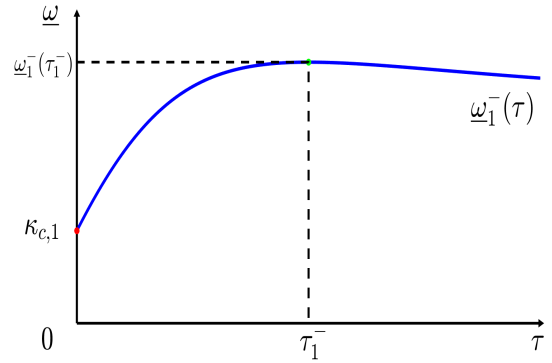
**Corollary 4.4.3**

Let be  $\rho, \underline{\Omega}_m > 0$ , then under the notation of [Proposition 4.4.2](#), the following are true for the zeros of  $S_{n,\rho,\underline{\Omega}_m}^-(\cdot)$ .

- (a) If  $1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} \geq 0$ , then  $(\underline{\omega}_1^-)'(\cdot) > 0$  in  $\mathbb{R}^+$ .
- (b) If  $1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} < 0$ , then  $\tau_1^- > 0$  is the unique positive critical point of  $\underline{\omega}_1^-(\cdot)$ .  
Moreover,  $(\underline{\omega}_1^-)'(\cdot) > 0$  (resp.  $(\underline{\omega}_1^-)'(\cdot) < 0$ ) in the interval  $(0, \tau_1^-)$  (resp.  $(\tau_1^-, +\infty)$ ).



(a)  $\underline{\Omega}_m = 1.0000, \rho = 0.3500,$   
 $1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} = 2.4000.$



(b)  $\underline{\Omega}_m = 1.0000, \rho = 2.0998,$   
 $1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} = -1.1000.$

Figure 4.6: Graphic representation of the function  $\underline{\omega}_1^-(\cdot)$ .

**4.4.1.3 Study of the monotonicity of the function  $\kappa_1^-(\tau)$**

Similarly as for the functions  $\kappa_n(\cdot), n \geq 2$  we can prove in exactly the same manner that  $\kappa_1^-(\cdot)$  is a strictly increasing function, this result is important again for establishing the character of graphs of function of the curves  $\mathcal{C}_1^-$  :

**Lemma 4.4.4**

The function  $\tau \mapsto \kappa_1^-(\tau)$  is a strictly increasing function.

*Proof.* Analogous proof as [Lemma 4.3.4](#). □

**4.4.2 Description of the set  $\mathcal{C}_1^+ = \mathcal{C}_1 \cap \mathcal{N}^+$**

**4.4.2.1 Existence and uniqueness of the solution  $(\kappa_1^+(\tau), \underline{\omega}_1^+(\tau))$  of  $(\mathcal{DS}_1^+)$**

**Theorem 4.4.5: Existence of the unique parametric curve in  $\mathcal{D}_{\text{even}}^+$** 

There exists a unique couple of  $C^\infty$  functions  $\underline{\omega}_1^+ : [0, \infty) \rightarrow (0, \underline{\Omega}_m)$  and  $\kappa_1^+ : [0, \infty) \rightarrow \mathbb{R}^+$  such that for any  $\tau \geq 0$ ,  $(\underline{\omega}_1^+(\tau), \kappa_1^+(\tau))$  is solution (parameterized by  $\tau$ ) of  $(\mathcal{DS}_1^+)$  (defined in (4.1.55)-(4.1.56)). Consequently,  $\mathcal{C}_1^+$  is fully described as

$$\mathcal{C}_1^+ = \{(\underline{\omega}_1^+(\tau), \kappa_1^+(\tau)) \mid \tau > 0\}. \quad (4.4.8)$$

Moreover, this curve has as initial point:

$$\begin{cases} \underline{\omega}_1^+(0) & \text{is the unique solution in } (0, \kappa_c) \text{ of the equation } u(\underline{\omega}) = v(\underline{\omega}), \\ \kappa_1^+(0) & = \left( \mathcal{J}(\underline{\omega}_1^+(0)) \right)^{\frac{1}{2}}. \end{cases} \quad (4.4.9)$$

*Proof.* The existence of the  $C^\infty$  ( $[0, \infty)$ ) functions  $\underline{\omega}_1^+ : [0, \infty) \rightarrow (0, \underline{\Omega}_m)$  and  $\kappa_1^+ : [0, \infty) \rightarrow \mathbb{R}^+$  solving the *dispersion system*  $(\mathcal{DS}_1^+)$  (see (4.1.55)) is a direct application of [Theorem 4.2.3](#) applied to the functions  $A_1^+$  and  $B_1^+$  defined in (4.1.56), which are  $C^\infty$  in  $[0, +\infty)$ . Finally, since  $A_1^+(\cdot)$  is continuously defined at  $\tau = 0$ , indeed,  $\lim_{\tau \rightarrow 0} A_1^+(\tau) = 1$ , then by making  $\tau = 0$  in (4.1.55), we notice that

$$u(\underline{\omega}_1^+(0)) = v(\underline{\omega}_1^+(0)) \quad \text{and} \quad (\kappa_1^+(0))^2 = (\underline{\omega}_1^+(0))^2 + v(\underline{\omega}_1^+(0)), \quad (4.4.10)$$

but the equation  $u(\underline{\omega}) = v(\underline{\omega})$  has a unique solution in  $(0, \underline{\Omega}_m)$  since  $u'(\cdot) < 0$  and  $v'(\cdot) > 0$  (see [Lemma 4.2.2](#)), therefore the value of  $\underline{\omega}_1^+(0)$  is specifically determined by this solution, hence

$$(\kappa_1^+(0))^2 = (\underline{\omega}_1^+(0))^2 + v(\underline{\omega}_1^+(0)) = (\underline{\omega}_1^+(0))^2 + u(\underline{\omega}_1^+(0)) = \mathcal{J}(\underline{\omega}_1^+(0)). \quad (4.4.11)$$

□

**4.4.2.2 Study of the monotonicity of  $\tau \mapsto \underline{\omega}_1^+(\tau)$** 

With no more delay, we introduce the functions

$$\alpha_1^+(\tau) := -\frac{(B_1^+)'(\tau)}{(A_1^+)'(\tau)} = (\coth \tau (\coth \tau - \tau \operatorname{csch}^2 \tau))^{-1}, \quad (4.4.12)$$

$$\beta_1^+(\tau) := A_1^+(\tau) \alpha_1^+(\tau) + B_1^+(\tau) = (\tau^3 \operatorname{csch}^2 \tau) (\coth \tau - \tau \operatorname{csch}^2 \tau)^{-1}, \quad (4.4.13)$$

$$S_{1,\rho,\underline{\Omega}_m}^+(\tau) := 1 + \frac{\beta_1^+(\tau)}{\underline{\Omega}_m^2} - \rho^2 (\alpha_1^+(\tau))^{-\frac{1}{2}}. \quad (4.4.14)$$

**Proposition 4.4.6**

Let be  $\rho, \underline{\Omega}_m > 0$ , then exactly one of the following occurs:

- (a) If  $\rho \leq 1$ , then  $S_{1,\rho,\underline{\Omega}_m}^+(\cdot) > 0$  in  $\mathbb{R}^+$
- (b) If  $\rho > 1$  and  $1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} \leq 0$ , then  $S_{1,\rho,\underline{\Omega}_m}^+(\cdot) < 0$  in  $\mathbb{R}^+$
- (c) If  $\rho > 1$  and  $1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} > 0$ , then  $S_{1,\rho,\underline{\Omega}_m}^+(\cdot)$  has a unique zero  $\tau_1^+ \in (0, \infty)$ .

This zero satisfies  $S_{1,\rho,\underline{\Omega}_m}^+(\cdot) > 0$  ( $S_{1,\rho,\underline{\Omega}_m}^+(\cdot) < 0$ ) in  $(0, \tau_1^+)$  (resp. in  $(\tau_1^+, +\infty)$ ).

*Proof.* [Lemma C. 2](#) guarantee us that  $\alpha_1^+(\cdot)$  is strictly decreasing, hence from [Proposition 4.2.6](#) it follows that  $S_{1,\rho,\underline{\Omega}_m}^+(\cdot)$  is also strictly decreasing. In addition, [Lemma C. 2](#) give us

$$\begin{cases} X_0 := \lim_{\tau \rightarrow 0} S_{1,\rho,\underline{\Omega}_m}^+(\tau) = 1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} \\ X_\infty := \lim_{\tau \rightarrow +\infty} S_{1,\rho,\underline{\Omega}_m}^+(\tau) = 1 - \rho^2. \end{cases} \quad (4.4.15)$$

Case (a) :  $\rho \leq 1$ . In this case,  $X_\infty \geq 0$  and since  $S_{1,\rho,\underline{\Omega}_m}^+(\cdot)$  is strictly decreasing, it follows that  $S_{1,\rho,\underline{\Omega}_m}^+(\cdot) > 0$  in  $\mathbb{R}^+$ , proving (a).

Case (b) :  $\rho > 1$  and  $1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} \leq 0$ .

In this case  $X_0 \leq 0$ , then  $S_{1,\rho,\underline{\Omega}_m}^+(\tau) < 0$  for  $\tau > 0$  (since the  $S_{1,\rho,\underline{\Omega}_m}^+(\cdot)$  is strictly decreasing) proving the clause (b).

Case (c) :  $\rho > 1$  and  $1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} > 0$ .

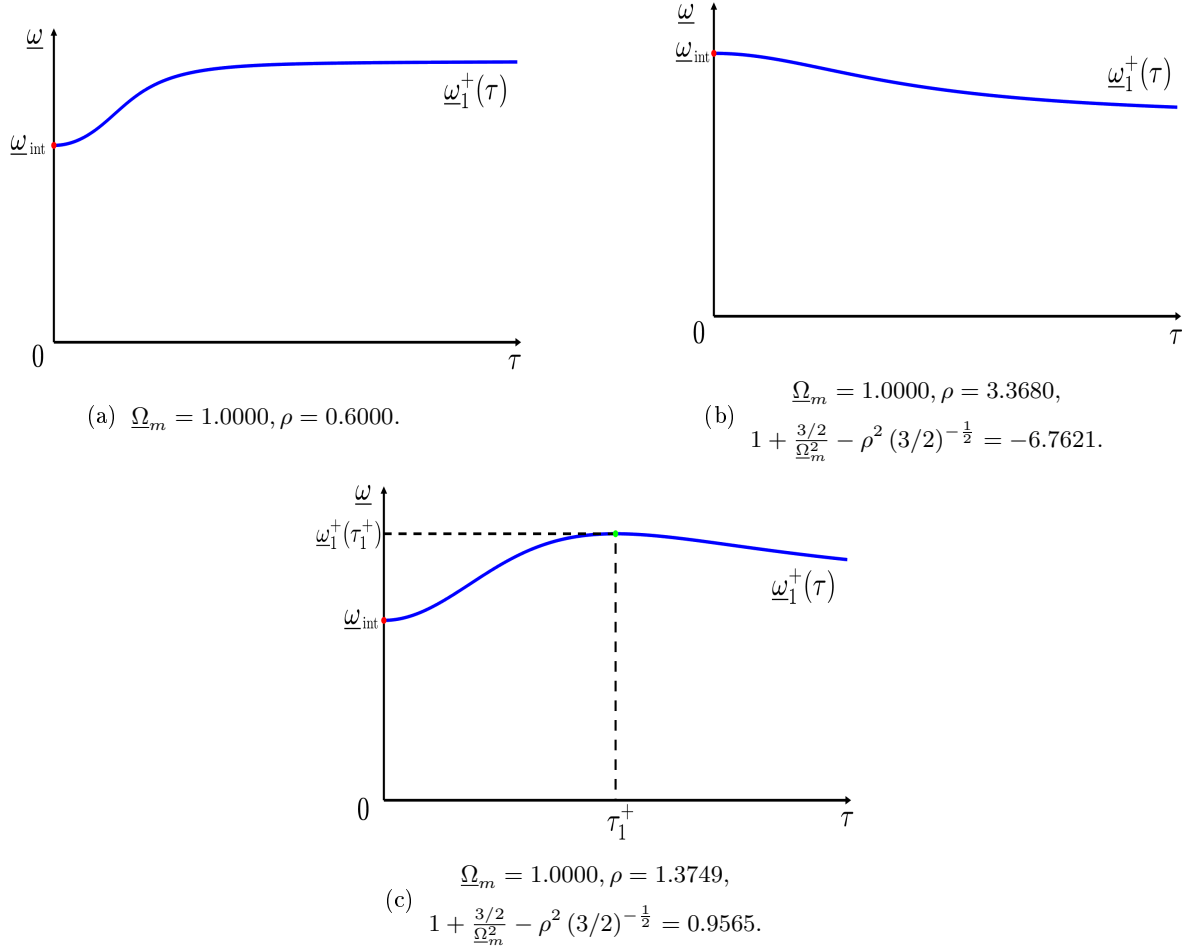
In this case  $X_0 > 0$  and  $X_\infty < 0$  since  $\rho > 1$ . Then the continuity and monotonicity of  $S_{1,\rho,\underline{\Omega}_m}^+(\cdot)$  and the *Intermediate Value Theorem* guarantee that  $S_{1,\rho,\underline{\Omega}_m}^+$  maps the interval  $(0, +\infty)$  onto  $(X_\infty, X_0)$ , in particular, since  $0 \in (X_\infty, X_0)$ , then  $S_{1,\rho,\underline{\Omega}_m}^+$  has a unique zero in  $(0, +\infty)$  which proves the clauses (c).  $\square$

**Corollary 4.4.7**

Let be  $\rho, \underline{\Omega}_m > 0$ . Under the notation of [Proposition 4.4.6](#), then one of the following is true:

- (a) If  $\rho \leq 1$ , then  $(\underline{\omega}_1^+)'(\cdot) > 0$  in  $(0, +\infty)$ .
- (b) If  $\rho > 1$  and  $1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} \leq 0$ , then  $(\underline{\omega}_1^+)'(\cdot) < 0$  in  $(0, +\infty)$ .
- (c) If  $\rho > 1$  and  $1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} > 0$ , then  $\tau_1^+ > 0$  is the unique critic point of  $\underline{\omega}_1^+(\cdot)$ , moreover,  $(\underline{\omega}_1^+)'(\cdot) > 0$  (resp.  $(\underline{\omega}_1^+)'(\cdot) < 0$ ) in  $(0, \tau_1^+)$  (resp.  $(\tau_1^+, +\infty)$ ).

**4.4.2.3 Study of the monotonicity of the function  $\kappa_1^+(\tau)$**


 Figure 4.7: Graphic representation of the function  $\underline{\omega}_1^+(\cdot)$ .

**Lemma 4.4.8**

The function  $\tau \mapsto \kappa_1^+(\tau)$  is a strictly increasing function.

*Proof.* We shall verify that for any  $\tau > 0$ ,  $(\kappa_1^+)'(\tau) > 0$ . Two cases are distinguished:

- (i) If  $\rho \leq 1$ , then [Corollary 4.4.7](#) assures us that  $(\underline{\omega}_1^+)'(\cdot) > 0$  in  $(0, +\infty)$ , on the other hand from [\(4.2.17\)](#) we have that

$$(\kappa_1^+)'(\tau) = \frac{[2\underline{\omega}_1^+(\tau) + A_1^+(\tau)v'(\underline{\omega}_1^+(\tau))](\underline{\omega}_1^+)'(\tau) + (A_1^+)'(\tau)v(\underline{\omega}_1^+(\tau))}{2\kappa_1^+(\tau)},$$

the which is compose of only positive terms, hence it follows that  $(\kappa_1^+)'(\tau) > 0$  in  $(0, +\infty)$ .

- (ii) If  $\rho > 1$ , then for all  $\tau > 0$  we have that  $0 < \underline{\omega}_1^+(\tau) < \underline{\Omega}_m < \rho^{\frac{1}{2}} \underline{\Omega}_m$ , henceforth  $\mathcal{J}'(\underline{\omega}_1^+(\tau)) < 0$  (see [Lemma 4.2.2](#)), then by using [\(4.2.19\)](#) one gets

$$(\kappa_1^+)'(\tau) = -\frac{1}{2\kappa_1^+(\tau)} \frac{(A_1^+)'(\tau)v(\underline{\omega}_1^+) \mathcal{J}'(\underline{\omega}_1^+) + [2\underline{\omega}_1^+ + A_1^+(\tau)v'(\underline{\omega}_1^+)](B_1^+)'(\tau)}{A(\tau)v'(\underline{\omega}_1^+) - u'(\underline{\omega}_1^+)},$$

where  $\underline{\omega}_1^+ = \underline{\omega}_1^+(\tau)$ , and since  $\mathcal{J}'(\underline{\omega}_1^+) < 0$  and  $(B_1^+)'(\tau) = -2\tau < 0$ , then one can proceed



as in [Lemma 4.3.4](#) to conclude that  $(\kappa_1^+)'(\tau) > 0$ .

□

#### 4.4.2.4 Asymptotic behaviour of the functions $\underline{\omega}_1^+(\tau)$ and $\kappa_1^+(\tau)$ when $\tau \rightarrow \infty$

We start this part with a heuristic deduction of the analysis behaviour of  $\underline{\omega}_1^+(\tau)$  as  $\tau \rightarrow \infty$ . For simplicity, given  $\tau > 0$ , we mean by  $\underline{\omega}_1^+ = \underline{\omega}_1^+(\tau)$ .

The first natural thought is to determinate if  $\underline{\omega}_1^+ \rightarrow \underline{\omega}_\infty$  as  $\tau \rightarrow \infty$ , for some  $\underline{\omega}_\infty \in \mathbb{R}$ . We will suppose that is the case. For the next computations, we recall that  $\underline{\omega}_1^+(\cdot)$  satisfies [\(4.1.57a\)](#). First notice, since the range of  $\underline{\omega}_1^+(\cdot)$  is  $(0, \underline{\Omega}_m)$ , then  $\underline{\omega}_\infty \in [0, \underline{\Omega}_m]$ .

If we assume that  $\underline{\omega}_\infty = 0$ , then from [\(4.1.57a\)](#) and the following limits

$$\lim_{\tau \rightarrow +\infty} A_1^+(\tau) = 1, \quad \lim_{\tau \rightarrow +\infty} B_1^+(\tau) = -\infty,$$

we can therefore notice that the left-side of the equation [\(4.1.57a\)](#) tends to  $+\infty$  (since  $\underline{\omega} = 0$  is a pole of  $u(\cdot)$ , see [Lemma 4.2.2](#)).

On the other hand, the right-side of the equation tends to  $-\infty$  (since  $\underline{\omega} = 0$  is a zero of  $v(\cdot)$ ), this shows that in fact it must occur that  $\underline{\omega}_\infty > 0$ . The latter fact yields to the fact that  $u(\underline{\omega})$  must be bounded as  $\tau \rightarrow \infty$ , so necessarily we need that

$$v(\underline{\omega}_1^+) \coth^2 \tau - 1 = u(\underline{\omega}_1^+) \tau^{-2} \rightarrow 0 \quad \text{as } \tau \rightarrow \infty, \quad (4.4.16)$$

which implies in turn that

$$v(\underline{\omega}_1^+) \sim \tanh^2 \tau \rightarrow 1 \quad \text{as } \tau \rightarrow \infty, \quad (4.4.17)$$

therefore  $v(\underline{\omega}_\infty) = 1$ , hence

$$\left(1 - \frac{\underline{\Omega}_m^2}{\underline{\omega}_\infty^2}\right)^{-2} = v(\underline{\omega}_\infty) = 1 \quad \implies \quad \left|1 - \frac{\underline{\Omega}_m^2}{\underline{\omega}_\infty^2}\right| = 1 \quad \implies \quad \frac{\underline{\Omega}_m^2}{\underline{\omega}_\infty^2} - 1 = 1 \quad (\underline{\omega}_\infty \leq \underline{\Omega}_m)$$

which yields  $\underline{\omega}_\infty = \frac{\underline{\Omega}_m}{\sqrt{2}}$ .

Inspired from the asymptotic expressions for the functions  $\underline{\omega}_1^+(\cdot)$ ,  $n \geq 1$ , we want to find an expression an approximation of the form

$$\underline{\omega}(\tau) \sim \frac{\underline{\Omega}_m}{\sqrt{2}} + \Omega_0 \tau^{-\alpha}, \quad \text{as } \tau \rightarrow \infty, \text{ for some } \alpha > 0. \quad (4.4.18)$$

By computation the first order approximation of  $v(\cdot)$  at  $\underline{\omega} = \frac{\underline{\Omega}_m}{\sqrt{2}}$  :

$$\begin{cases} u(\underline{\omega}_1^+) \sim u\left(\frac{\underline{\Omega}_m}{\sqrt{2}}\right) = \underline{\Omega}_m^2 (\rho^2 - 1), & \text{as } \tau \rightarrow +\infty \\ v(\underline{\omega}_1^+) \sim v\left(\frac{\underline{\Omega}_m}{\sqrt{2}}\right) + v'\left(\frac{\underline{\Omega}_m}{\sqrt{2}}\right) \left(\underline{\omega}_1^+ - \frac{\underline{\Omega}_m}{\sqrt{2}}\right) = 1 + \frac{8\sqrt{2}}{\underline{\Omega}_m} \left(\underline{\omega}_1^+ - \frac{\underline{\Omega}_m}{\sqrt{2}}\right) & \text{as } \tau \rightarrow +\infty, \end{cases} \quad (4.4.19)$$

substituting [\(4.4.18\)](#), [\(4.4.19\)](#) and  $\coth \tau \sim 1$  when  $\tau \rightarrow \infty$  in [\(4.1.57a\)](#), we obtain

$$\underline{\Omega}_m^2 (\rho^2 - 1) = \frac{8\sqrt{2}}{\underline{\Omega}_m} \left(\underline{\omega}_1^+ - \frac{\underline{\Omega}_m}{\sqrt{2}}\right) \tau^2, \quad (4.4.20)$$

which gives us

$$\omega_1^+ = \frac{\Omega_m}{\sqrt{2}} + \frac{\Omega_m^3 (\rho^2 - 1)}{8\sqrt{2}} \tau^{-2}. \quad (4.4.21)$$

Our next goal is to prove formally the last asymptotic behavior, for this purpose we will need to the following values:

$$\begin{cases} \mu_r \left( \frac{\Omega_m}{\sqrt{2}} \right) = -1, \\ \mu_r' \left( \frac{\Omega_m}{\sqrt{2}} \right) = \left( \frac{2\Omega_m^2}{\omega^3} \right)_{\omega=\frac{\Omega_m}{\sqrt{2}}} = \frac{4\sqrt{2}}{\Omega_m}, \\ \mu_r'' \left( \frac{\Omega_m}{\sqrt{2}} \right) = \left( -\frac{6\Omega_m^2}{\omega^4} \right)_{\omega=\frac{\Omega_m}{\sqrt{2}}} = -\frac{24}{\Omega_m^2}, \end{cases}$$

- We therefore compute for  $v(\cdot)$  and its first two derivatives,

$$\begin{cases} v \left( \frac{\Omega_m}{\sqrt{2}} \right) = (\mu_r^{-2}(\omega))|_{\omega=\frac{\Omega_m}{\sqrt{2}}} = 1, \\ v' \left( \frac{\Omega_m}{\sqrt{2}} \right) = (-2\mu_r^{-3}(\omega)\mu_r'(\omega))|_{\omega=\frac{\Omega_m}{\sqrt{2}}} = \frac{8\sqrt{2}}{\Omega_m}, \\ v'' \left( \frac{\Omega_m}{\sqrt{2}} \right) = (6\mu_r^{-4}(\omega)(\mu_r'(\omega))^2 - 2\mu_r^{-3}(\omega)\mu_r''(\omega))|_{\omega=\frac{\Omega_m}{\sqrt{2}}} = \frac{144}{\Omega_m^2}, \end{cases} \quad (4.4.22)$$

- while for  $u(\cdot)$  and its derivative we get

$$\begin{cases} u \left( \frac{\Omega_m}{\sqrt{2}} \right) = \left( \frac{\Omega_m^4 \rho^2}{\omega^2} - \Omega_m^2 (1 + \rho^2) \right) \Big|_{\omega=\frac{\Omega_m}{\sqrt{2}}} = (\rho^2 - 1) \Omega_m^2, \\ u' \left( \frac{\Omega_m}{\sqrt{2}} \right) = \left( -\frac{2\Omega_m^4 \rho^2}{\omega^3} \right) \Big|_{\omega=\frac{\Omega_m}{\sqrt{2}}} = -4\sqrt{2} \Omega_m \rho^2. \end{cases} \quad (4.4.23)$$

#### Lemma 4.4.9

The function  $\tau \mapsto \omega_1^+(\tau)$  has the following asymptotic behavior

$$\begin{cases} \omega_1^+(\tau) = \frac{\Omega_m}{\sqrt{2}} + \Omega_\infty \tau^{-2} + \gamma_\infty \tau^{-4} + o(\tau^{-4}), \quad (\tau \rightarrow \infty), \\ \Omega_\infty := \frac{\Omega_m^3 (\rho^2 - 1)}{8\sqrt{2}}, \quad \gamma_\infty := \frac{1}{64\sqrt{2}} (17\rho^2 - 9)(\rho^2 - 1) \Omega_m^5 \end{cases} \quad (4.4.24)$$

The function  $\tau \mapsto \kappa_1^+(\tau)$  has the following asymptotic behavior

$$\begin{cases} \kappa_1^+(\tau) = \tau + K_\infty \tau^{-1} + \delta_\infty \tau^{-3} + o(\tau^{-3}), \quad \text{when } \tau \rightarrow +\infty, \\ K_\infty := \frac{1}{2} \left( \rho^2 - \frac{1}{2} \right) \Omega_m^2, \quad \delta_\infty = \frac{\sqrt{2}}{2} (1 - 4\rho^2) \Omega_m \Omega_\infty. \end{cases} \quad (4.4.25)$$

*Proof.* We consider the variable change  $\varepsilon = \tau^{-2}$ , and we notice that the analysis of (4.1.57a) around  $\tau = \infty$ , is equivalent to the one around  $\varepsilon = 0$  of the following equation

$$u(X)\varepsilon + 1 = v(X) \coth^2 \left( \varepsilon^{-\frac{1}{2}} \right). \quad (4.4.26)$$

We show that (4.4.26) has a smooth solution  $\varepsilon \mapsto X_1(\varepsilon)$  around  $\varepsilon = 0$ . We base our proof by means of the Implicit Function Theorem. Let us consider the function

$$\mathcal{G}_1(X, \varepsilon) := u(X)\varepsilon + 1 - v(X) \coth^2(\varepsilon^{-\frac{1}{2}}), \quad (X, \varepsilon) \in (0, \underline{\Omega}_m) \times [0, \infty). \quad (4.4.27)$$

Since  $\varepsilon \mapsto \coth(\varepsilon^{-\frac{1}{2}})$  defines a smooth function on around a positive neighborhood of  $\varepsilon > 0$ , moreover, one can verify

$$\left\{ \begin{array}{l} \lim_{\varepsilon \rightarrow 0^+} \coth(\varepsilon^{-\frac{1}{2}}) = 1, \\ \lim_{\varepsilon \rightarrow 0^+} \left( \coth(\varepsilon^{-\frac{1}{2}}) \right)' = - \lim_{\varepsilon \rightarrow 0^+} \frac{\operatorname{csch}^2(\varepsilon^{-\frac{1}{2}})}{2\varepsilon^{\frac{3}{2}}} = 0, \\ \lim_{\varepsilon \rightarrow 0^+} \left( \coth(\varepsilon^{-\frac{1}{2}}) \right)'' = \lim_{\varepsilon \rightarrow 0^+} - \frac{\left( 3\varepsilon^{-\frac{1}{2}} - 2\coth(\varepsilon^{-\frac{1}{2}}) \right) \operatorname{csch}^2(\varepsilon^{-\frac{1}{2}})}{4\varepsilon^3} = 0. \end{array} \right. \quad (4.4.28)$$

Consequently, the former limits yield:

$$\left\{ \begin{array}{l} \lim_{\varepsilon \rightarrow 0^+} \coth^2(\varepsilon^{-\frac{1}{2}}) = 1, \\ \lim_{\varepsilon \rightarrow 0^+} \left( \coth^2(\varepsilon^{-\frac{1}{2}}) \right)' = 0, \\ \lim_{\varepsilon \rightarrow 0^+} \left( \coth^2(\varepsilon^{-\frac{1}{2}}) \right)'' = 0, \end{array} \right. \quad (4.4.29)$$

which permits to show that  $\mathcal{G}_1$  is smooth at  $(X, \varepsilon) = \left( \frac{\underline{\Omega}_m}{\sqrt{2}}, 0 \right)$ . Furthermore, one has

$$\mathcal{G}_1\left(\frac{\underline{\Omega}_m}{\sqrt{2}}, 0\right) = 0,$$

and, as  $\partial_X \mathcal{G}_1\left(\frac{\underline{\Omega}_m}{\sqrt{2}}, 0\right) = [u'(X)\varepsilon - v'(X) \coth^2(\varepsilon^{-\frac{1}{2}})] \Big|_{(X, \varepsilon) = \left(\frac{\underline{\Omega}_m}{\sqrt{2}}, 0\right)}$ , one computes that

$$\partial_X \mathcal{G}_1\left(\frac{\underline{\Omega}_m}{\sqrt{2}}, 0\right) = -v'\left(\frac{\underline{\Omega}_m}{\sqrt{2}}\right) = -\frac{8\sqrt{2}}{\underline{\Omega}_m} \neq 0.$$

Hence the *Implicit Function Theorem* give us the existence of a smooth function  $\varepsilon \mapsto X_1(\varepsilon)$  around a positive neighborhood of  $\varepsilon = 0$  such that

$$\mathcal{G}_1(X_1(\varepsilon), \varepsilon) = 0 \quad \text{and} \quad X_1(0) = \frac{\underline{\Omega}_m}{\sqrt{2}}. \quad (4.4.30)$$

Using the second order Taylor approximation around  $\varepsilon = 0$ , we deduce that

$$X_1(\varepsilon) = X_1(0) + X_1'(0)\varepsilon + \frac{X_1''(0)}{2}\varepsilon^2 + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0^+. \quad (4.4.31)$$

now we are interested into compute  $X_1'(0)$  and  $X_1''(0)$ . For this purpose, we use the following

expression for the two first derivatives derived from the *implicit differentiation*:

$$\left\{ \begin{array}{l} X_1'(\varepsilon) = -\frac{\partial_\varepsilon \mathcal{G}_1(X_1(\varepsilon), \varepsilon)}{\partial_X \mathcal{G}_1(X_1(\varepsilon), \varepsilon)}, \\ \\ X_1''(\varepsilon) = -\frac{\begin{pmatrix} \partial_\varepsilon^2 \mathcal{G}_1(X_1(\varepsilon), \varepsilon) \\ 2 \partial_X \partial_\varepsilon \mathcal{G}_1(X_1(\varepsilon), \varepsilon) \\ \partial_X^2 \mathcal{G}_1(X_1(\varepsilon), \varepsilon) \end{pmatrix} \cdot \begin{pmatrix} X_1'(\varepsilon)^2 \\ X_1'(\varepsilon) \\ 1 \end{pmatrix}}{\partial_X \mathcal{G}_1(X_1(\varepsilon), \varepsilon)}. \end{array} \right. \quad (4.4.32a)$$

By computing the first partial derivatives of  $\mathcal{G}_1$  we obtain:

$$\left\{ \begin{array}{l} \partial_X \mathcal{G}_1(X, \varepsilon) = u'(X) \varepsilon - v'(X) \coth^2 \left( \varepsilon^{-\frac{1}{2}} \right), \\ \\ \partial_\varepsilon \mathcal{G}_1(X, \varepsilon) = u(X) - v(X) \left( \coth^2 \left( \varepsilon^{-\frac{1}{2}} \right) \right)', \end{array} \right. \quad (4.4.33a)$$

consequently, the second order derivatives hold

$$\left\{ \begin{array}{l} \partial_X^2 \mathcal{G}_1(X, \varepsilon) = u''(X) \varepsilon - v''(X) \coth^2 \left( \varepsilon^{-\frac{1}{2}} \right), \\ \\ \partial_X \partial_\varepsilon \mathcal{G}_1(X, \varepsilon) = \partial_\varepsilon \partial_X \mathcal{G}_1(X, \varepsilon) = u'(X) - v'(X) \left( \coth^2 \left( \varepsilon^{-\frac{1}{2}} \right) \right)', \\ \\ \partial_\varepsilon^2 \mathcal{G}_1(X, \varepsilon) = v(X) \left( \coth^2 \left( \varepsilon^{-\frac{1}{2}} \right) \right)'' . \end{array} \right. \quad (4.4.34a)$$

Hence, with the aid of (4.4.22), (4.4.23) and (4.4.29), we evaluate (4.4.33) and (4.4.34) at  $(X, \varepsilon) = \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right)$  as follows

$$\left\{ \begin{array}{l} \partial_X \mathcal{G}_1 \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right) = -v' \left( \frac{\Omega_m}{\sqrt{2}} \right) = -\frac{8\sqrt{2}}{\Omega_m}, \\ \\ \partial_\varepsilon \mathcal{G}_1 \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right) = u \left( \frac{\Omega_m}{\sqrt{2}} \right) = (\rho^2 - 1) \Omega_m^2 \\ \\ \partial_X^2 \mathcal{G}_1 \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right) = -v'' \left( \frac{\Omega_m}{\sqrt{2}} \right) = -\frac{144}{\Omega_m^2}, \\ \\ \partial_X \partial_\varepsilon \mathcal{G}_1 \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right) = \partial_\varepsilon \partial_X \mathcal{G}_1 \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right) = u' \left( \frac{\Omega_m}{\sqrt{2}} \right) = -4\sqrt{2} \Omega_m \rho^2, \\ \\ \partial_\varepsilon^2 \mathcal{G}_1 \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right) = 0. \end{array} \right.$$

Consequently by substituting the latter in (4.4.32) evaluated at  $(X, \varepsilon) = \left(\frac{\underline{\Omega}_m}{\sqrt{2}}, 0\right)$ , it yields

$$\left\{ \begin{array}{l} X_1'(0) = -\frac{(\rho^2 - 1)\underline{\Omega}_m^2}{-8\sqrt{2}/\underline{\Omega}_m} = \frac{(\rho^2 - 1)\underline{\Omega}_m^3}{8\sqrt{2}}, \\ X_1''(0) = -\frac{\begin{pmatrix} -\frac{144}{\underline{\Omega}_m^2} \\ -8\sqrt{2}\underline{\Omega}_m\rho^2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} (X_1'(0))^2 \\ X_1'(0) \\ 1 \end{pmatrix}}{-8\sqrt{2}/\underline{\Omega}_m} \\ = -\frac{\frac{9(\rho^2 - 1)^2\underline{\Omega}_m^4}{8} + \rho^2(\rho^2 - 1)\underline{\Omega}_m^4}{8\sqrt{2}/\underline{\Omega}_m} \\ = -\frac{1}{64\sqrt{2}}(17\rho^2 - 9)(\rho^2 - 1)\underline{\Omega}_m^5. \end{array} \right. \quad (4.4.35)$$

The unicity of smooth solutions of  $(\mathcal{DS}_1^+)$  indicates that  $\underline{\omega}_1^+(\tau) = X_1(\varepsilon) = X_1(\tau^{-2})$ , therefore from (4.4.31) and (4.4.35) we obtain the asymptotic expression (4.4.24).

For the asymptotic analysis for  $\kappa_1^+(\cdot)$ , first we derive from (4.1.57a) and (4.1.57b) the equivalent definition for  $\kappa_1^+(\cdot)$ , namely

$$(\kappa_1^+(\tau))^2 = \tau^2 + \mathcal{J}(\underline{\omega}_1^+(\tau)), \quad \mathcal{J}(\underline{\omega}) := \underline{\omega}^2 \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega}) = \underline{\omega}^2 + \frac{\underline{\Omega}_m^4 \rho^2}{\underline{\omega}^2} - \underline{\Omega}_m^2 (1 + \rho^2), \quad (4.4.36)$$

by computing the Taylor expansion of first order for  $\mathcal{J}(\cdot)$  at  $\underline{\omega} = \frac{\underline{\Omega}_m}{\sqrt{2}}$ ,

$$\begin{aligned} \mathcal{J}(\underline{\omega}) &= \mathcal{J}\left(\frac{\underline{\Omega}_m}{\sqrt{2}}\right) + \mathcal{J}'\left(\frac{\underline{\Omega}_m}{\sqrt{2}}\right) \left(\underline{\omega} - \frac{\underline{\Omega}_m}{\sqrt{2}}\right) + o\left(\underline{\omega} - \frac{\underline{\Omega}_m}{\sqrt{2}}\right) \\ &= \left(\rho^2 - \frac{1}{2}\right) \underline{\Omega}_m^2 + \sqrt{2}(1 - 4\rho^2) \underline{\Omega}_m \left(\underline{\omega} - \frac{\underline{\Omega}_m}{\sqrt{2}}\right) + o\left(\underline{\omega} - \frac{\underline{\Omega}_m}{\sqrt{2}}\right), \end{aligned}$$

and by using (4.4.24) until an order of  $\tau^{-2}$  we obtain

$$\mathcal{J}(\underline{\omega}_1^+(\tau)) = \left(\rho^2 - \frac{1}{2}\right) \underline{\Omega}_m^2 + \sqrt{2}(1 - 4\rho^2) \underline{\Omega}_m \Omega_0 \tau^{-2} + o(\tau^{-2}), \quad \text{as } \tau \rightarrow \infty. \quad (4.4.37)$$

The information in (4.4.36) and (4.4.37) allow us to deduce that

$$(\kappa_1^+(\tau))^2 = \tau^2 \left(1 + \left(\rho^2 - \frac{1}{2}\right) \underline{\Omega}_m^2 \tau^{-2} + \sqrt{2}(1 - 4\rho^2) \underline{\Omega}_m \Omega_0 \tau^{-4} + o(\tau^{-4})\right), \quad (4.4.38)$$

and hence by taking square root

$$\begin{aligned} \kappa_1^+(\tau) &= \tau \left(1 + \frac{1}{2} \left(\rho^2 - \frac{1}{2}\right) \underline{\Omega}_m^2 \tau^{-2} + \frac{\sqrt{2}}{2} (1 - 4\rho^2) \underline{\Omega}_m \Omega_0 \tau^{-4} + o(\tau^{-4})\right) \\ &= \tau + \frac{1}{2} \left(\rho^2 - \frac{1}{2}\right) \underline{\Omega}_m^2 \tau^{-1} + \frac{\sqrt{2}}{2} (1 - 4\rho^2) \underline{\Omega}_m \Omega_0 \tau^{-3} + o(\tau^{-3}). \end{aligned} \quad (4.4.39)$$

□

### 4.4.3 Full description of the curve $\mathcal{C}_1$

#### 4.4.3.1 Existence and uniqueness of the solution $(\kappa_1(s), \underline{\omega}_1(s))$ of $(\mathcal{DS}_1)$

In this section we will show that  $\mathcal{C}_1$  is the continuous and smooth concatenation of the curves  $\mathcal{C}_1^-$  and  $\mathcal{C}_1^+$  by adding a point, namely, the unique point in the set  $\mathcal{C}_1 \cap \mathcal{N}^0$ .

#### Theorem 4.4.10

There exists a unique couple of  $C^1$  functions  $\underline{\omega}_1 : [0, \infty) \rightarrow (0, \underline{\Omega}_m)$  and  $\kappa_1 : [0, \infty) \rightarrow \mathbb{R}^+$  such that for any  $s \geq 0$ ,  $(\kappa_1(s), \underline{\omega}_1(s))$  is solution (parameterized by  $s$ ) of  $(\mathcal{DS}_1)$  (4.1.63)-(4.1.64). Consequently,  $\mathcal{C}_1$  is fully described as

$$\mathcal{C}_1 := \{(\kappa_1(s), \underline{\omega}_1(s)), s \in \mathbb{R}^+\}. \quad (4.4.40)$$

The pairs of functions  $(\kappa_1^-(\cdot), \underline{\omega}_1^-(\cdot))$  and  $(\kappa_1^+(\cdot), \underline{\omega}_1^+(\cdot))$  are related to  $(\kappa_1(\cdot), \underline{\omega}_1(\cdot))$  as

$$\begin{cases} (\kappa_1, \underline{\omega}_1)|_{(0,1)} = (\kappa_1^- \circ \nu^-, \underline{\omega}_1^- \circ \nu^-), \\ (\kappa_1, \underline{\omega}_1)|_{(1,+\infty)} = (\kappa_1^+ \circ \nu^+, \underline{\omega}_1^+ \circ \nu^+), \end{cases} \quad (4.4.41)$$

where  $\nu^-$  and  $\nu^+$  are the  $C^\infty$ -diffeomorphisms defined, respectively, in  $(0, 1)$  and  $(1, \infty)$  onto  $(0, \infty)$ :

$$\begin{cases} \nu^- := (0, 1) \rightarrow (0, +\infty) \\ s \mapsto \tan\left(\frac{\pi}{2}s\right) \end{cases} \quad \text{and} \quad \begin{cases} \nu^+ := (1, +\infty) \rightarrow (0, +\infty) \\ s \mapsto s - 1 \end{cases}. \quad (4.4.42)$$

The initial point of the curve  $\mathcal{C}_1$  is given by

$$\underline{\omega}_1(0) = \kappa_1(0) = \kappa_{c,1} := \rho \underline{\Omega}_m^2 \left(\frac{\pi^2}{4} + (1 + \rho^2) \underline{\Omega}_m^2\right)^{-\frac{1}{2}}. \quad (4.4.43)$$

On the other hand,  $\mathcal{C}_1$  intersects the curve  $\mathcal{N}_0$  in exactly one point, which is given by  $(\kappa_1(1), \underline{\omega}_1(1)) = (\kappa_{\text{int}}, \underline{\omega}_{\text{int}})$ , where the value of  $(\kappa_{\text{int}}, \underline{\omega}_{\text{int}})$  is defined as

$$\begin{cases} \underline{\omega}_{\text{int}} \text{ is the unique solution in } (0, \kappa_c) \text{ of the equation } u(\underline{\omega}) = v(\underline{\omega}), \\ \kappa_{\text{int}} = (\mathcal{J}(\underline{\omega}_{\text{int}}))^{\frac{1}{2}}. \end{cases} \quad (4.4.44)$$

*Proof.* The existence of the  $C^1$  ( $[0, \infty)$ ) functions  $\underline{\omega}_1 : [0, \infty) \rightarrow (0, \underline{\Omega}_m)$  and  $\kappa_1 : [0, \infty) \rightarrow \mathbb{R}^+$  solving the *dispersion system*  $(\mathcal{DS}_1)$  (see (4.1.63)) is a direct application of [Theorem 4.2.3](#) applied to the functions  $A_1$  and  $B_1$  defined in (4.1.64), which are  $C^1$  in  $[0, +\infty)$ .

By the definitions given in (4.1.58) and (4.1.64), we notice that  $A_1 = A_1^- \circ \nu^-$  and  $B_1 = B_1^- \circ \nu^-$  in  $(0, 1)$ , it therefore follows that for all  $s \in (0, 1)$ ,  $(\kappa_1(s), \underline{\omega}_1(s))$  is solution of (4.1.57) for  $\tau = \nu^-(s) \in (0, +\infty)$ , hence by the uniqueness of such solution (see [Theorem 4.4.1](#)), then we conclude that

$$(\kappa_1(s), \underline{\omega}_1(s)) = (\kappa_1^-(\tau), \underline{\omega}_1^-(\tau)) = (\kappa_1^-(\nu^-(s)), \underline{\omega}_1^-(\nu^-(s))), \quad 0 < s < 1.$$

A very similar argument proves that

$$(\kappa_1(s), \underline{\omega}_1(s)) = (\kappa_1^+(\nu^+(s)), \underline{\omega}_1^+(\nu^+(s))), \quad s > 1.$$

From [Theorem 4.4.1](#), we know that  $(\kappa_1^-(\cdot), \underline{\omega}_1^-(\cdot))$  is continuously defined at  $\tau = 0$  on the right, hence it follows that

$$\lim_{s \rightarrow 0^+} (\kappa_1(s), \underline{\omega}_1(s)) = \lim_{s \rightarrow 0^+} (\kappa_1^-(\nu^-(s)), \underline{\omega}_1^-(\nu^-(s))) = \lim_{\tau \rightarrow 0^+} (\kappa_1^-(\tau), \underline{\omega}_1^-(\tau)) = (\kappa_{c,1}, \underline{\omega}_{c,1}),$$

hence  $(\kappa_1(\cdot), \underline{\omega}_1(\cdot))$  is continuously extended to  $s = 0$  on the right and proving [\(4.4.43\)](#).

Similarly, one has

$$\lim_{s \rightarrow 1^+} (\kappa_1(s), \underline{\omega}_1(s)) = \lim_{s \rightarrow 1^+} (\kappa_1^+(\nu^+(s)), \underline{\omega}_1^+(\nu^+(s))) = \lim_{\tau \rightarrow 0^+} (\kappa_1^+(\tau), \underline{\omega}_1^+(\tau)) = (\kappa_{\text{int}}, \underline{\omega}_{\text{int}}),$$

and since  $(\kappa_1(\cdot), \underline{\omega}_1(\cdot))$  is continuous, then [\(3.3.2\)](#) follows.  $\square$

**Remark 4.4.11.** The identities [\(4.4.41\)](#) imply that  $(\kappa_1(\cdot), \underline{\omega}_1(\cdot))$  is  $C^\infty$  in the intervals  $(0, 1)$  and  $(1, +\infty)$ .

#### 4.4.3.2 Study of the monotonicity of $\underline{\omega}_1(\tau)$

One would expect to proceed as it has been previously treated in the [Sections 4.3.2, 4.4.1.2](#) and [4.4.2.2](#). However we find a new particularity given by the nature of the functions  $A_1(\cdot)$  and  $B_1(\cdot)$ , namely  $A_1'(1) = 0 = B_1'(1)$ . The former fact implies that

$$\underline{\omega}_1'(1) = 0 = \kappa_1'(1), \tag{4.4.45}$$

in this sense we need to evaluate  $\lim_{s \rightarrow 1} \underline{\omega}_1'(s)/\kappa_1'(s)$ .

#### Lemma 4.4.12

For all  $\rho > 0, \underline{\Omega}_m > 0$  the limit  $\lim_{s \rightarrow 1} \underline{\omega}_1'(s)/\kappa_1'(s)$  exists, moreover, we have the following

$$\lim_{s \rightarrow 1} \frac{\underline{\omega}_1'(s)}{\kappa_1'(s)} = \frac{2 \kappa_{\text{int}} \left(1 - \frac{2}{3} v(\underline{\omega}_{\text{int}})\right)}{2 \underline{\omega}_{\text{int}} + v'(\underline{\omega}_{\text{int}}) - \frac{2}{3} v(\underline{\omega}_{\text{int}}) \mathcal{J}'(\underline{\omega}_{\text{int}})}, \tag{4.4.46}$$

$$\text{sign} \left( \lim_{s \rightarrow 1} \frac{\underline{\omega}_1'(s)}{\kappa_1'(s)} \right) = \text{sign} (S_{1,\rho,\underline{\Omega}_m}), \tag{4.4.47}$$

where we have defined the quantity

$$S_{1,\rho,\underline{\Omega}_m} := 1 + \frac{3/2}{\underline{\Omega}_m^2} - \rho^2 (3/2)^{-\frac{1}{2}}. \tag{4.4.48}$$

*Proof.* We first calculate the limits  $\lim_{\tau \rightarrow +\infty} (\underline{\omega}_1^-)'(\tau)/(\kappa_1^-)'(\tau)$  and  $\lim_{\tau \rightarrow +0^+} (\underline{\omega}_1^+)'(\tau)/(\kappa_1^+)'(\tau)$ .

From [\(4.2.18\)](#) applied to [\(4.1.57\)](#) we obtain

$$\begin{aligned} \lim_{\tau \rightarrow +\infty} \frac{(\underline{\omega}_1^-)'(\tau)}{(\kappa_1^-)'(\tau)} &= \lim_{\tau \rightarrow +\infty} 2 \kappa_1^-(\tau) \frac{(\underline{\omega}_1^-)'(\tau)}{\mathcal{J}'(\underline{\omega}_1^-(\tau)) ((\underline{\omega}_1^-)'(\tau) - (B_1^-)'(\tau))} \\ &= \lim_{\tau \rightarrow +\infty} 2 \kappa_1^-(\tau) \frac{(\underline{\omega}_1^-)'(\tau)/(B_1^-)'(\tau)}{\mathcal{J}'(\underline{\omega}_1^-(\tau)) ((\underline{\omega}_1^-)'(\tau)/(B_1^-)'(\tau)) - 1}, \end{aligned} \tag{4.4.49}$$

thus by computing

$$\begin{aligned}
 \lim_{\tau \rightarrow +\infty} \frac{(\underline{\omega}_1^-)'(\tau)}{(B_1^-)'(\tau)} &= \lim_{\tau \rightarrow +\infty} \left( -\frac{1}{(B_1^-)'(\tau)} \frac{(A_1^-)'(\tau) v(\underline{\omega}_1^-(\tau)) + (B_1^-)'(\tau)}{A_1^-(\tau) v'(\underline{\omega}_1^-(\tau)) - u'(\underline{\omega}_1^-(\tau))} \right) \\
 &= \lim_{\tau \rightarrow +\infty} \frac{(\alpha_1^-(\tau))^{-1} v(\underline{\omega}_1^-(\tau)) - 1}{A_1^-(\tau) v'(\underline{\omega}_1^-(\tau)) - u'(\underline{\omega}_1^-(\tau))} \\
 &= \frac{\frac{2}{3} v(\underline{\omega}_{\text{int}}) - 1}{v'(\underline{\omega}_{\text{int}}) - u'(\underline{\omega}_{\text{int}})} \tag{4.4.50}
 \end{aligned}$$

where  $\alpha_1^-(\cdot)$  was defined in (4.4.4) and they were used the limits  $\lim_{\tau \rightarrow +\infty} \underline{\omega}_1^-(\tau) = \underline{\omega}_{\text{int}}$  and  $\lim_{\tau \rightarrow +\infty} \alpha_1^-(\tau) = 3/2$  (see Lemma C.1). Thus, by substituting (4.4.53) in (4.4.49) one obtains

$$\begin{aligned}
 \lim_{\tau \rightarrow +\infty} \frac{(\underline{\omega}_1^-)'(\tau)}{(\kappa_1^-)'(\tau)} &= 2 \kappa_{\text{int}} \frac{\frac{\frac{2}{3} v(\underline{\omega}_{\text{int}}) - 1}{v'(\underline{\omega}_{\text{int}}) - u'(\underline{\omega}_{\text{int}})}}{\mathcal{J}'(\underline{\omega}_{\text{int}}) \left( \frac{\frac{2}{3} v(\underline{\omega}_{\text{int}}) - 1}{v'(\underline{\omega}_{\text{int}}) - u'(\underline{\omega}_{\text{int}})} \right) - 1} \\
 &= \frac{2 \kappa_{\text{int}} \left( \frac{2}{3} v(\underline{\omega}_{\text{int}}) - 1 \text{Big} \right)}{\frac{2}{3} v(\underline{\omega}_{\text{int}}) \mathcal{J}'(\underline{\omega}_{\text{int}}) - \mathcal{J}'(\underline{\omega}_{\text{int}}) - v'(\underline{\omega}_{\text{int}}) + u'(\underline{\omega}_{\text{int}})} \\
 &= \frac{2 \kappa_{\text{int}} \left( 1 - \frac{2}{3} v(\underline{\omega}_{\text{int}}) \right)}{2 \underline{\omega}_{\text{int}} + v'(\underline{\omega}_{\text{int}}) - \frac{2}{3} v(\underline{\omega}_{\text{int}}) \mathcal{J}'(\underline{\omega}_{\text{int}})}. \tag{4.4.51}
 \end{aligned}$$

On the other hand, in an analogous way we have

$$\begin{aligned}
 \lim_{\tau \rightarrow 0^+} \frac{(\underline{\omega}_1^+)'(\tau)}{(\kappa_1^+)'(\tau)} &= \lim_{\tau \rightarrow 0^+} \left( 2 \kappa_1^+(\tau) \frac{(\underline{\omega}_1^+)'(\tau)}{\mathcal{J}'(\underline{\omega}_1^+(\tau)) (\underline{\omega}_1^+)'(\tau) - (B_1^+)'(\tau)} \right) \\
 &= \lim_{\tau \rightarrow 0^+} \left( 2 \kappa_1^+(\tau) \frac{(\underline{\omega}_1^+)'(\tau)/(B_1^+)'(\tau)}{\mathcal{J}'(\underline{\omega}_1^+(\tau)) ((\underline{\omega}_1^+)'(\tau)/(B_1^+)'(\tau)) - 1} \right). \tag{4.4.52}
 \end{aligned}$$

$$\begin{aligned}
 \lim_{\tau \rightarrow 0^+} \frac{(\underline{\omega}_1^+)'(\tau)}{(B_1^+)'(\tau)} &= \lim_{\tau \rightarrow 0^+} \left( -\frac{1}{(B_1^+)'(\tau)} \frac{(A_1^+)'(\tau) v(\underline{\omega}_1^+(\tau)) + (B_1^+)'(\tau)}{A_1^+(\tau) v'(\underline{\omega}_1^+(\tau)) - u'(\underline{\omega}_1^+(\tau))} \right) \\
 &= \lim_{\tau \rightarrow 0^+} \frac{(\alpha_1^+(\tau))^{-1} v(\underline{\omega}_1^+(\tau)) - 1}{A_1^+(\tau) v'(\underline{\omega}_1^+(\tau)) - u'(\underline{\omega}_1^+(\tau))} \\
 &= \frac{\frac{2}{3} v(\underline{\omega}_{\text{int}}) - 1}{v'(\underline{\omega}_{\text{int}}) - u'(\underline{\omega}_{\text{int}})} \tag{4.4.53}
 \end{aligned}$$

And hence

$$\lim_{\tau \rightarrow 0^+} \frac{(\underline{\omega}_1^+)'(\tau)}{(\kappa_1^+)'(\tau)} = \frac{2 \kappa_{\text{int}} \left( 1 - \frac{2}{3} v(\underline{\omega}_{\text{int}}) \right)}{2 \underline{\omega}_{\text{int}} + v'(\underline{\omega}_{\text{int}}) - \frac{2}{3} v(\underline{\omega}_{\text{int}}) \mathcal{J}'(\underline{\omega}_{\text{int}})}. \tag{4.4.54}$$



From (4.4.41), one can deduce

$$\begin{cases} \frac{\omega_1'(s)}{\kappa_1'(s)} = \frac{(\omega_1^-)(\nu^-(s))(\nu^-)'(s)}{(\kappa_1^-)(\nu^-(s))(\nu^-)'(s)} = \frac{(\omega_1^-)(\nu^-(s))}{(\kappa_1^-)(\nu^-(s))}, & s \in (0, 1) \\ \frac{\omega_1'(s)}{\kappa_1'(s)} = \frac{(\omega_1^+)(\nu^+(s))(\nu^+)'(s)}{(\kappa_1^+)(\nu^+(s))(\nu^+)'(s)} = \frac{(\omega_1^+)(\nu^+(s))}{(\kappa_1^+)(\nu^+(s))}, & s \in (1, +\infty), \end{cases} \quad (4.4.55)$$

which together with (4.4.51) and (4.4.54) leads to

$$\begin{cases} \lim_{s \rightarrow 1^-} \frac{\omega_1'(s)}{\kappa_1'(s)} = \lim_{\tau \rightarrow +\infty} \frac{(\omega_1^-)'(\tau)}{(\kappa_1^-)'(\tau)} = \frac{2\kappa_{\text{int}} \left(1 - \frac{2}{3}v(\omega_{\text{int}})\right)}{2\omega_{\text{int}} + v'(\omega_{\text{int}}) - \frac{2}{3}v(\omega_{\text{int}})\mathcal{J}'(\omega_{\text{int}})}, \\ \lim_{s \rightarrow 1^+} \frac{\omega_1'(s)}{\kappa_1'(s)} = \lim_{\tau \rightarrow 0^+} \frac{(\omega_1^+)'(\tau)}{(\kappa_1^+)'(\tau)} = \frac{2\kappa_{\text{int}} \left(1 - \frac{2}{3}v(\omega_{\text{int}})\right)}{2\omega_{\text{int}} + v'(\omega_{\text{int}}) - \frac{2}{3}v(\omega_{\text{int}})\mathcal{J}'(\omega_{\text{int}})}, \end{cases} \quad (4.4.56)$$

proving (4.4.46). For the last part, as  $\omega_{\text{int}} \in (0, \kappa_c) \subset (0, \rho^{\frac{1}{2}}\Omega_m)$  (put reference), then  $\mathcal{J}'(\omega_{\text{int}}) < 0$  and  $v(\omega_{\text{int}}), v'(\omega_{\text{int}}) > 0$  (see Lemma 4.2.2), therefore

$$\text{sign} \left( \frac{2\kappa_{\text{int}} \left(1 - \frac{2}{3}v(\omega_{\text{int}})\right)}{2\omega_{\text{int}} + v'(\omega_{\text{int}}) - \frac{2}{3}v(\omega_{\text{int}})\mathcal{J}'(\omega_{\text{int}})} \right) = \text{sign} \left( 1 - \frac{2}{3}v(\omega_{\text{int}}) \right) \quad (4.4.57)$$

We next notice the following , since  $v(\kappa_{\text{int}}) = u(\kappa_{\text{int}})$  :

$$\begin{aligned} 1 - \frac{2}{3}v(\omega_{\text{int}}) > 0 &\iff v(\omega_{\text{int}}) < \frac{3}{2} \quad \text{and} \quad u(\omega_{\text{int}}) < \frac{3}{2} \\ &\iff \left| 1 - \frac{\Omega_m^2}{\omega_{\text{int}}^2} \right|^{-2} < \frac{3}{2} \quad \text{and} \quad \frac{\rho^2 \Omega_m^4}{\omega_{\text{int}}^2} - (1 + \rho^2)\Omega_m^2 < \frac{3}{2} \\ &\iff (3/2)^{-\frac{1}{2}} + 1 < \frac{\Omega_m^2}{\omega_{\text{int}}^2} \quad \text{and} \quad \frac{\rho^2 \Omega_m^2}{\omega_{\text{int}}^2} < 1 + \rho^2 + \frac{3/2}{\Omega_m^2} \\ &\implies \rho^2 (3/2)^{-\frac{1}{2}} + \rho^2 < 1 + \rho^2 + \frac{3/2}{\Omega_m^2} \\ &\iff 1 + \frac{3/2}{\Omega_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} > 0, \end{aligned}$$

similarly one can prove

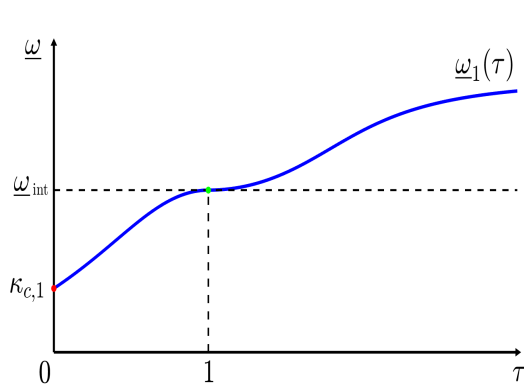
$$\begin{aligned} 1 - \frac{2}{3}v(\omega_{\text{int}}) = 0 &\implies 1 + \frac{3/2}{\Omega_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} = 0, \\ 1 - \frac{2}{3}v(\omega_{\text{int}}) < 0 &\implies 1 + \frac{3/2}{\Omega_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} < 0. \end{aligned}$$

Hence  $\text{sign} \left( 1 - \frac{2}{3}v(\omega_{\text{int}}) \right) = \text{sign} \left( 1 + \frac{3/2}{\Omega_m^2} - \rho^2 (3/2)^{-\frac{1}{2}} \right)$  which finishes the proof.  $\square$

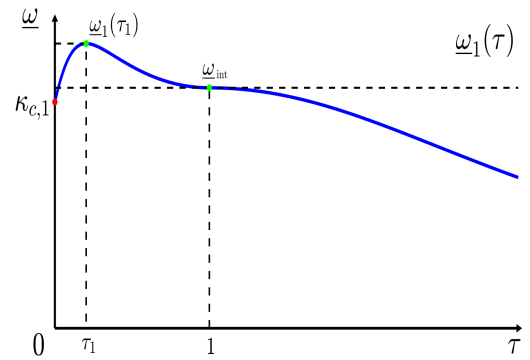
**Corollary 4.4.13**

Let be  $\rho, \underline{\Omega}_m > 0$ . The set containing the critical points of  $\underline{\omega}_1$  (i.e.,  $\{\tau > 0 \mid \underline{\omega}'_1(\tau) = 0\}$ ) is of the form  $\{1, \tau_1\}$ , moreover, the following statements are true for the critical points of  $\underline{\omega}_1$  :

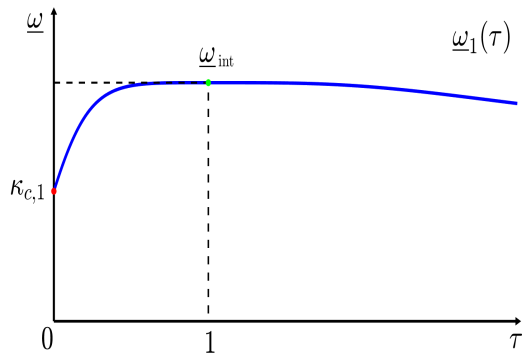
- (a) If  $\rho \leq 1$ , then  $\tau_1 = 1$ , in other words,  $\tau = 1$  is the unique critical point of  $\underline{\omega}_1$ , in this case,  $\underline{\omega}'_1(\tau) > 0$  for  $\tau \neq 1$ .
- (b) If  $\rho > 1$ , then  $\underline{\omega}'_1(\tau) > 0$  if  $\tau \in (0, \tau_1) \setminus \{1\}$  and  $\underline{\omega}'_1(\tau) < 0$  if  $\tau \in (\tau_1, +\infty) \setminus \{1\}$ , furthermore:
  - (b.1) If  $S_{1,\rho,\underline{\Omega}_m} < 0$ , then  $\tau_1 < 1$  and  $\kappa_1(\tau_1) < \kappa_{\text{int}}$ .
  - (b.2) If  $S_{1,\rho,\underline{\Omega}_m} = 0$ , then  $\tau_1 = 1$  and  $\kappa_1(\tau_1) = \kappa_{\text{int}}$ .
  - (b.3) If  $S_{1,\rho,\underline{\Omega}_m} > 0$ , then  $\tau_1 > 1$  and  $\kappa_1(\tau_1) > \kappa_{\text{int}}$ .



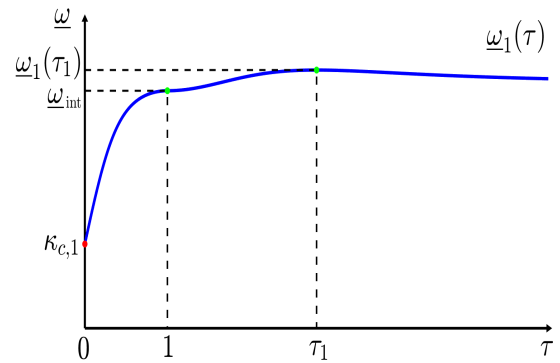
(a)  $\underline{\Omega}_m = 1.0000, \rho = 0.6000$ .



(b)  $\underline{\Omega}_m = 1.0000, \rho = 2.2748,$   
 $S_{1,\rho,\underline{\Omega}_m} = -1.7250$ .



(c)  $\underline{\Omega}_m = 1.0000, \rho = 1.7498,$   
 $S_{1,\rho,\underline{\Omega}_m} = 0$ .



(d)  $\underline{\Omega}_m = 2.0000, \rho = 1.3749,$   
 $S_{1,\rho,\underline{\Omega}_m} = 0.9565$ .

Figure 4.8: Graphic representation of the function  $\underline{\omega}_1(\cdot)$ .

**4.4.3.3 Study of the monotonicity of  $\kappa_1(\tau)$**

**Lemma 4.4.14**

The function  $\tau \mapsto \kappa_1(\tau)$  is a strictly increasing function. Moreover,  $\kappa_1' > 0$  in  $\mathbb{R} \setminus \{1\}$  and  $\kappa_1'(1) = 0$ .

*Proof.* It follows directly since [Lemma 4.4.4](#), [Lemma 4.4.8](#) and [\(4.4.41\)](#) establish that  $\kappa_1$  is a composition of strictly increasing functions on the intervals  $(0, 1)$  and  $(1, +\infty)$  (indeed,  $\nu^-$  and  $\nu^+$  are clearly strictly increasing). The value  $\kappa_1'(1) = 0$  has been indicated previously in [\(4.4.45\)](#),  $\square$

**4.4.4 Characterization of the curve  $(\mathcal{C}_1)$  as a graph**

**Theorem 4.4.15**

The curve  $\mathcal{C}_1$  is the graph of a  $C^1$ -function on  $\kappa$ . More precisely,

$$\mathcal{C}_1 = \{(\kappa, \underline{\omega}_1^*(\kappa)) \mid \kappa > \kappa_{c,1}\}, \quad \underline{\omega}_1^* := \underline{\omega}_1 \circ \{\kappa_1\}^{-1} \tag{4.4.58}$$

where  $\kappa_1 = \kappa_1(\tau)$  is an invertible function whose inverse is given by a  $C^1$ -function, namely,  $\{\kappa_1\}^{-1} : [\kappa_{c,1}, +\infty) \rightarrow [0, +\infty)$ .

*Proof.* From [Lemma 4.4.14](#), we know that  $\kappa_1(\cdot)$  is an invertible function (since it is strictly increasing) which maps  $\kappa_1(0) = \kappa_{c,1}$  (see [Theorem 4.4.10](#)) and such that  $\kappa_1(s) = \kappa_1^+(s-1) \rightarrow +\infty$  as  $s \rightarrow +\infty$  (see [Lemma 4.4.9](#)), hence  $\kappa_1$  has as domain and as image, respectively, the intervals  $[0, +\infty)$  and  $[\kappa_{c,1}, +\infty)$ , therefore its inverse function is well defined,  $\{\kappa_1\}^{-1} : [\kappa_{c,1}, +\infty) \rightarrow [0, +\infty)$ . Hence from [Theorem 4.3.1](#) it follows that

$$\begin{aligned} (\kappa, \underline{\omega}) \in \mathcal{C}_1 &\iff (\kappa_1(s), \underline{\omega}_1(s)) = (\kappa, \underline{\omega}) \quad \text{for some } \sigma > 0, \\ &\iff \kappa > \kappa_{c,1}, \quad s = \{\kappa_1\}^{-1}(\kappa) \quad \text{and} \quad \underline{\omega} = \underline{\omega}_1(\tau) \\ &\iff \kappa > \kappa_{c,1} \quad \text{and} \quad \underline{\omega} = \underline{\omega}_1(\{\kappa_1\}^{-1}(\kappa)) = \underline{\omega}_1^*(\kappa). \end{aligned}$$

From [Lemma 4.4.14](#), it follows that the inverse function  $\{\kappa_1\}^{-1}$  is derivable in  $(0, +\infty)$  except at  $s = 1$  (where  $\kappa_1(1) = \kappa_{\text{int}}$ ), hence  $\underline{\omega}_1^*$  is derivable on the intervals  $(\kappa_{c,1}, \kappa_{\text{int}})$ ,  $(\kappa_{\text{int}}, +\infty)$ . On another hand, [Lemma 4.4.12](#) assures us that  $(\underline{\omega}_1^*)'(\kappa_{\text{int}}) = \lim_{s \leftarrow 1} \underline{\omega}_1'(s)/\kappa_1'(s)$  exists, which finishes the proof.  $\square$

**4.4.5 Geometric properties of the curve  $(\mathcal{C}_1)$**

**Theorem 4.4.16**

For all  $\rho, \Omega_m > 0$ ,  $\underline{\omega}_1^*$  has the following properties:

- (a)  $\underline{\omega}_1^*$  is continuously differentiable at  $\kappa = \kappa_{c,1}$  and  $(\underline{\omega}_1^*)'(\kappa_{c,1}) = 1$ ;
- (b) The followings are true for the critical points of  $\underline{\omega}_1^*$  :
  - (b.1) if  $\rho \leq 1$ , then  $\underline{\omega}_1^*$  does not have critical points, moreover,  $(\underline{\omega}_1^*)'(\cdot) > 0$  in  $(0, +\infty)$ ,
  - (b.2) if  $\rho > 1$ , then  $\underline{\omega}_1^*$  has exactly one critical point,  $\kappa_1^*$ , which is a global maximal in  $(\kappa_{c,1}, +\infty)$ , that is to say,  $(\underline{\omega}_1^*)'(\kappa) > 0$  (resp.  $(\underline{\omega}_1^*)'(\kappa) < 0$ ) if  $\kappa \in (\kappa_{c,1}, \kappa_1^*)$  (resp.  $\kappa \in (\kappa_1^*, +\infty)$ ), moreover,
    - if  $S_{1,\rho,\Omega_m} < 0$ , then  $\kappa_1^* < \kappa_{\text{int}}$ ,
    - if  $S_{1,\rho,\Omega_m} = 0$ , then  $\kappa_1^* = \kappa_{\text{int}}$ ,
    - if  $S_{1,\rho,\Omega_m} > 0$ , then  $\kappa_1^* > \kappa_{\text{int}}$ .
- (c) for all  $\kappa > \kappa_{c,1}$ ,  $\underline{\omega}_2^*(\kappa) < \underline{\omega}_1^*(\kappa)$ ;
- (d) [Asymptotic behavior for long frequencies]

$$\underline{\omega}_1^*(\kappa) = \frac{\Omega_m}{\sqrt{2}} + \Omega_\infty \kappa^{-2} + (2\Omega_\infty K_\infty + \gamma_\infty) \kappa^{-4} + o(\kappa^{-4}), \quad \text{as } \kappa \rightarrow +\infty,$$

where  $\Omega_\infty, K_\infty$  and  $\gamma_\infty$  were defined in (4.4.24) and (4.4.25) (see Lemma 4.4.9).

*Proof.* (a) From the definition of  $\underline{\omega}_1^*$ , the chain's rule derivation and the inverse derivative formula one gets that

$$(\underline{\omega}_1^*)'(\kappa) = \frac{\omega_1'(s)}{\kappa_1'(s)} = \frac{(\underline{\omega}_1^-)'(\tau)}{(\kappa_1^-)'(\tau)}, \quad \text{where } \kappa_1(s) = \kappa = \kappa_1^-(\tau), \text{ for } \kappa_{c,1} < \kappa < \kappa_{\text{int}}. \quad (4.4.59)$$

Substituting in (4.3.36) the expression for the derivatives of  $\underline{\omega}_1^-$  and  $\kappa_1^-$  deduced, respectively, from (4.2.16) and (4.2.19), thus we obtain

$$(\underline{\omega}_1^*)'(\kappa) = 2\kappa \left( \frac{(\eta_1^2)' + (\tau^2 \eta_1^2)' v(\underline{\omega}_1^*)}{(\tau^2 \eta_1^2)' v(\underline{\omega}_1^*) \mathcal{J}(\underline{\omega}_1^*) + (\eta_1^2)'(2\underline{\omega}_1^* + \tau^2 \eta_1^2 v'(\underline{\omega}_1^*))} \right), \quad (4.4.60)$$

where  $\kappa_1^-(\tau) = \kappa \in (\kappa_{c,1}, \kappa_{\text{int}})$ ,  $\underline{\omega}_1^* = \underline{\omega}_1^-(\tau) = \underline{\omega}_1^*(\kappa)$  and  $\eta_1 := \eta_1(\tau)$ . Since  $\kappa_1^-(0) = \kappa_{c,1}$ , hence we need to evaluate the right-hand expression of (4.4.60) at  $\tau = 0$ ,  $\underline{\omega}_1^* = \underline{\omega}_1^-(0) = \kappa_{c,1}$ , by using the fact  $(\eta_1^2)'|_{\tau=0} = (2\eta_1 \eta_1')|_{\tau=0} = -\pi \neq 0$  and  $(\tau^2 \eta_1^2)'|_{\tau=0} = (2\tau \eta_1 (\eta_1 + \tau \eta_1'))|_{\tau=0} = 0$  thus we get

$$(\underline{\omega}_1^*)'(\kappa_{c,1}) = 2\kappa_{c,1} \left( \frac{(\eta_1^2)'|_{\tau=0}}{2\kappa_{c,1} (\eta_1^2)'|_{\tau=0}} \right) = 1. \quad (4.4.61)$$

- (b) From Lemma 4.4.14 and the identity (4.4.59) we derive that  $(\underline{\omega}_1^*)'(\kappa)$  and  $\underline{\omega}_1'(s)$  have the same sign, where  $\kappa = \kappa_1(s)$  and  $s \in \mathbb{R} \setminus \{1\}$ . From Lemma 4.4.12, we know that

$$\text{sign}((\underline{\omega}_1^*)'(\kappa_{\text{int}})) = \text{sign} \left( \lim_{s \rightarrow 1} \frac{(\underline{\omega}_1)'(s)}{(\kappa_1)'(s)} \right) = \text{sign}(S_{1,\rho,\Omega_m}),$$

thus, this clause follows directly as a consequence of Corollary 4.4.13.

(c) We consider two cases:

(c.1) If  $\kappa_{c,1} < \kappa < \kappa_{\text{int}}$ , then we can write  $\underline{\omega}_1^*(\kappa) = \underline{\omega}_1^-(\tau)$  where  $\tau > 0$  is such that  $\kappa = \kappa_1^-(\tau)$ . Notice that  $\kappa > \kappa_{c,1} > \kappa_{c,2}$ , henceforth  $\kappa = \kappa_2(s)$  for some  $s > 0$ . From here, we can use exactly the same reasoning as in the clause (c) of [Theorem 4.3.7](#) to conclude that

$$\underline{\omega}_2^*(\kappa) = \underline{\omega}_2(s) < \underline{\omega}_1^-(\tau) < \underline{\omega}_1^*(\kappa).$$

(c.2) If  $\kappa_{\text{int}} \leq \kappa$ , then  $\underline{\omega}_1^*(\kappa) = \underline{\omega}_1^+(\tau)$  for some  $\tau \geq 0$ , then  $(\kappa, \underline{\omega}_1^*(\kappa)) \in \mathcal{C}_1^+ \subset \mathcal{N}^+$  which implies that  $\Phi(\kappa) \leq \underline{\omega}_1^*(\kappa)$ , on the other hand  $(\kappa, \underline{\omega}_2^*(\kappa)) \in \mathcal{C}_2^+ \subset \mathcal{N}^-$  gives us  $\underline{\omega}_2^*(\kappa) < \Phi(\kappa)$ , those relations give us the desired inequality.

(d) We use the estimation [\(4.4.25\)](#) proved in [4.4.9](#), altogether with [Lemma C. 4](#) applied to  $\kappa = \kappa_1^+$  seen as a function on  $\tau$  :

$$\tau = \kappa - K_\infty \kappa^{-1} + o(\kappa^{-1}), \quad \text{as } \kappa \longrightarrow +\infty, \quad (4.4.62)$$

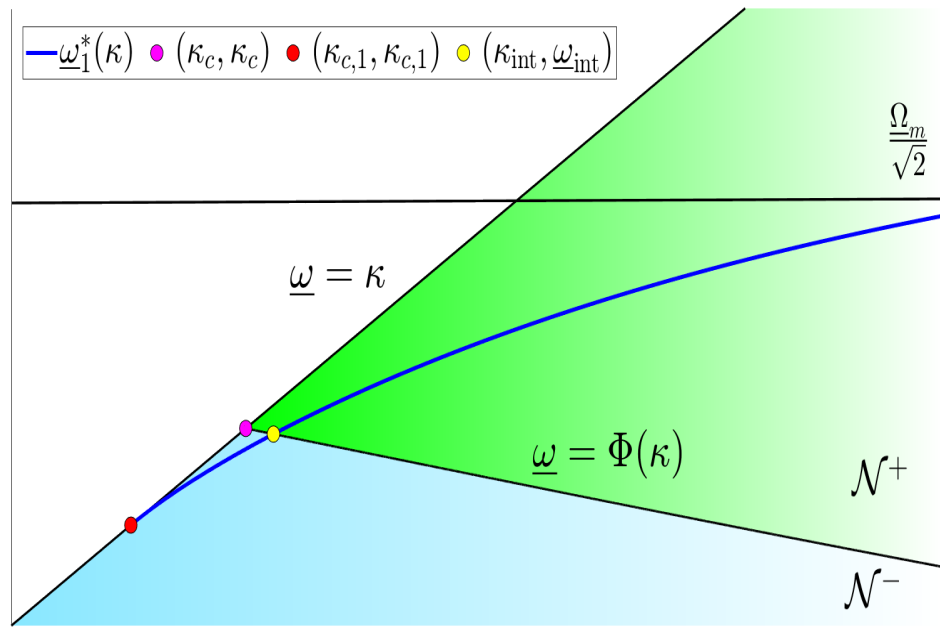
and we have abbreviate  $\tau = \{\kappa_1^+\}^{-1}(\kappa)$ , hence

$$\begin{aligned} \tau^{-2} &= \left( \kappa - K_\infty \kappa^{-1} + o(\kappa^{-1}) \right)^{-2} \\ &= \kappa^{-2} \left( 1 - K_\infty \kappa^{-2} + o(\kappa^{-2}) \right)^{-2} \\ &= \kappa^{-2} \left( 1 + 2K_\infty \kappa^{-2} + o(\kappa^{-2}) \right) \\ &= \kappa^{-2} + 2K_\infty \kappa^{-4} + o(\kappa^{-4}), \quad \text{as } \kappa \longrightarrow +\infty, \end{aligned} \quad (4.4.63)$$

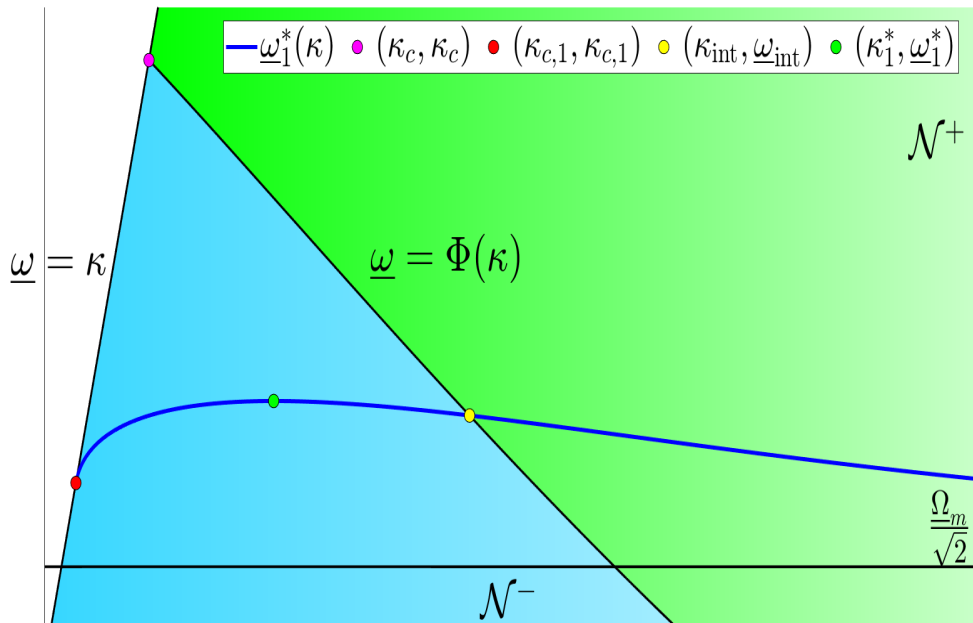
Finally, by substituting [\(4.4.63\)](#) in [\(4.4.24\)](#) and since  $\tau^{-4} = \kappa^{-4} + o(\kappa^{-4})$  as  $\kappa \longrightarrow +\infty$ , one obtains

$$\underline{\omega}_1^*(\kappa) = \frac{\Omega_m}{\sqrt{2}} + \Omega_\infty \kappa^{-2} + (2\Omega_\infty K_\infty + \gamma_\infty) \kappa^{-4} + o(\kappa^{-4}).$$

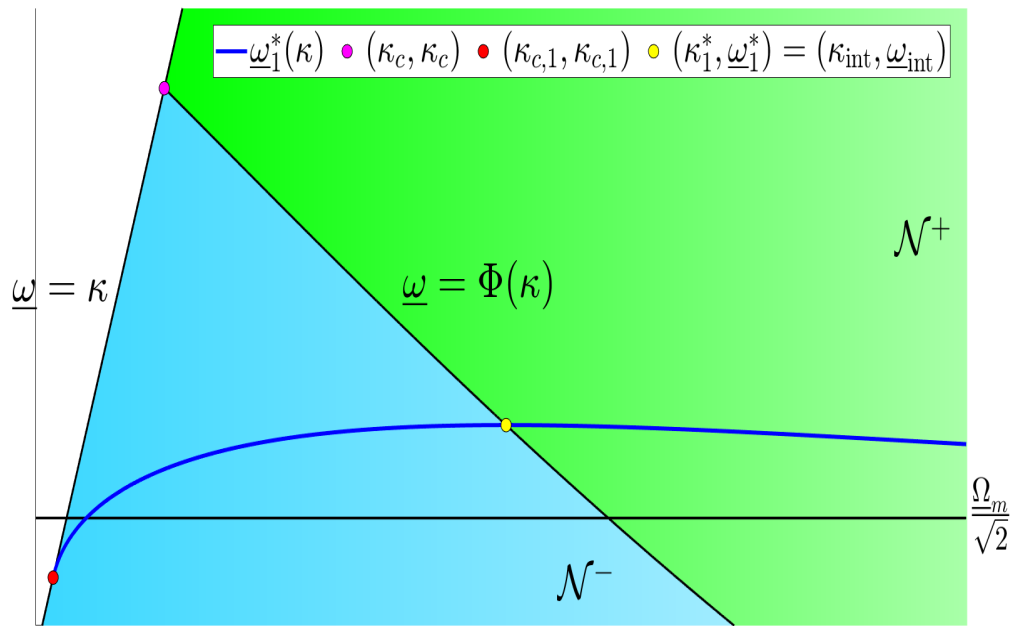
□



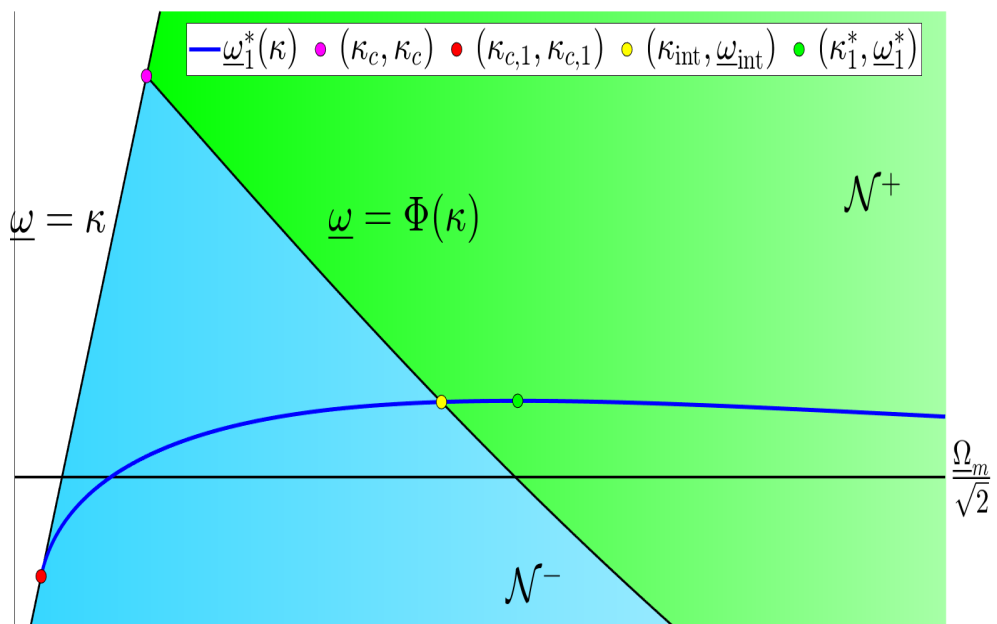
(a)  $\underline{\Omega}_m = 1.0000, \rho = 0.6000.$



(b)  $\underline{\Omega}_m = 1.0000, \rho = 2.0406,$   
 $S_{1,\rho,\underline{\Omega}_m}(\tau_c) = -0.900.$



(c)  $\underline{\Omega}_m = 1.0000, \rho = 1.7498,$   
 $S_{1,\rho,\underline{\Omega}_m}(\tau_c) = 0.$



(d)  $\underline{\Omega}_m = 2.0000, \rho = 1.6784,$   
 $S_{1,\rho,\underline{\Omega}_m}(\tau_c) = 0.2000.$

Figure 4.9: Graphic representation of the function  $\underline{\omega}_1^*(\cdot)$ .

## 4.5 Study of the curve $\mathcal{C}_0$

In this section, we will prove that the set  $\mathcal{D}_{\text{even}}^+$  is composed by a unique dispersion curve named  $\mathcal{C}_0$ , for this purpose, we follow the same technique developed for the description of the dispersion curves  $\{\mathcal{C}_n\}_{n \geq 2}$ .

### 4.5.1 Existence and uniqueness of the solution $(\kappa_0(\tau), \underline{\omega}_0(\tau))$ of $(\mathcal{DS}_0)$

#### Theorem 4.5.1: Existence of the unique parametric curve in $\mathcal{D}_{\text{even}}^+$

There exists a unique couple of  $C^\infty$  functions  $\underline{\omega}_0 : [0, \infty) \rightarrow (0, \underline{\Omega}_m)$  and  $\kappa_0 : [0, \infty) \rightarrow \mathbb{R}^+$  such that for any  $\tau \geq 0$ ,  $(\underline{\omega}_0(\tau), \kappa_0(\tau))$  is solution (parameterized by  $\tau$ ) of  $(\mathcal{DS}_0)$  (defined in (4.1.48)-(4.1.49)). Consequently,  $\mathcal{C}_0$  is fully described as

$$\mathcal{C}_0 = \{(\underline{\omega}_0(\tau), \kappa_0(\tau)) \mid \tau > 0\}. \quad (4.5.1)$$

Moreover, this curve has as initial point

$$\underline{\omega}_0(0) = \kappa_0(0) = \kappa_c := \rho \underline{\Omega}_m \left(1 + \rho^2\right)^{-\frac{1}{2}}. \quad (4.5.2)$$

*Proof.* The existence of the  $C^\infty$  ( $[0, \infty)$ ) functions  $\underline{\omega}_0 : [0, \infty) \rightarrow (0, \underline{\Omega}_m)$  and  $\kappa_0 : [0, \infty) \rightarrow \mathbb{R}^+$  solving the *dispersion system*  $(\mathcal{DS}_{2n})$  (see (4.1.50)) is a direct application of [Theorem 4.2.3](#) applied to the functions  $A_0$  and  $B_0$  defined in (4.1.49), which are  $C^\infty$  in  $[0, +\infty)$ . Finally by making  $\tau = 0$  in (4.1.48), we notice that  $\underline{\omega}_0(0) = \kappa_0(0)$  and

$$\frac{\rho^2 \underline{\Omega}_m^4}{\underline{\omega}_0(0)^2} - (1 + \rho^2) \underline{\Omega}_m^2 = 0,$$

by resolving the latter for  $\underline{\omega}_0(0) > 0$  we obtain (4.5.2).  $\square$

### 4.5.2 Study of the monotonicity of $\underline{\omega}_0(\tau)$

In what follows we want to use the results presented in [Section 4.2.3](#). In the first instance, we note that

$$A'_0(\tau) = 2\tau \tanh(\tau) (\tanh \tau + \tau \operatorname{sech}^2 \tau) > 0, \quad \tau > 0. \quad (4.5.3)$$

Secondly, we present the functions  $\alpha_0$  and  $\beta_0$  defined using  $A_0$  and  $B_0$  (see (4.2.21) and (4.2.23)) as follows

$$\begin{aligned} \alpha_0(\tau) &:= -\frac{B'_0}{A'_0} = \frac{2\tau}{2\tau \tanh \tau (\tanh \tau + \tau \operatorname{sech}^2 \tau)} \\ &= \left( \tanh \tau (\tanh \tau + \tau \operatorname{sech}^2 \tau) \right)^{-1}, \end{aligned} \quad (4.5.4)$$

$$\begin{aligned} \beta_0(\tau) &:= A_0(\tau) \alpha_0(\tau) + B_0(\tau) = \frac{\tau^2 \tanh \tau}{\tanh \tau + \tau \operatorname{sech}^2 \tau} - \tau^2 \\ &= -\frac{\tau^3 \operatorname{sech}^2 \tau}{\tanh \tau + \tau \operatorname{sech}^2 \tau}. \end{aligned} \quad (4.5.5)$$

[Lemma C. 2](#) guarantee us that  $\alpha_0(\cdot)$  is a positive function in  $\mathbb{R}^+$ .



With these elements, we can proceed as previously presented in [Section 4.2.3](#) to study the monotonicity of  $\underline{\omega}_0(\cdot)$ . For this purpose, we need to study the real zeros of the function

$$S_{0,\rho,\underline{\Omega}_m}(\tau) := 1 + \frac{\beta_0(\tau)}{\underline{\Omega}_m^2} - \rho^2 (\alpha_0(\tau))^{-\frac{1}{2}}. \quad (4.5.6)$$

From [Lemma C. 2](#), it is deduced that  $\alpha_0(\cdot)$  has a global minimum at  $\tau = \tau_c$ , then [Proposition 4.2.6](#) guarantee us that  $S_{0,\rho,\underline{\Omega}_m}(\cdot)$  has a also a minimum in  $\tau = \tau_c$ , with this in mind we establish the following proposition which describes the zeros of  $S_{0,\rho,\underline{\Omega}_m}(\cdot)$ .

**Proposition 4.5.2**

Given  $\rho, \underline{\Omega}_m > 0$ , then exactly one of the following is true:

- (a) If  $\rho \geq 1$ , then  $S_{0,\rho,\underline{\Omega}_m}(\cdot) > 0$  has a unique zero,  $\tau_0 > 0$ , such that  $S_{0,\rho,\underline{\Omega}_m}(\tau) > 0$  if and only if  $\tau \in (0, \tau_0)$ .
- (b) If  $\rho < 1$  and  $S_{0,\rho,\underline{\Omega}_m}(\tau_c) > 0$ , then  $S_{0,\rho,\underline{\Omega}_m}(\cdot)$  is strictly positive in  $\mathbb{R}^+$ .
- (c) If  $\rho < 1$  and  $S_{0,\rho,\underline{\Omega}_m}(\tau_c) = 0$ , then  $S_{0,\rho,\underline{\Omega}_m}(\cdot)$  is non negative and  $\tau_c$  is its unique zero.
- (d) If  $\rho < 1$  and  $S_{0,\rho,\underline{\Omega}_m}(\tau_c) < 0$ , then  $S_{0,\rho,\underline{\Omega}_m}(\cdot)$  has exactly two zeros,  $\tau_0^{(1)} < \tau_0^{(2)}$  such that  $S_{0,\rho,\underline{\Omega}_m}(\tau) > 0$  if and only if  $\tau \in (0, \tau_0^{(1)}) \cup (\tau_0^{(2)}, +\infty)$ .

*Proof.* From [Proposition 4.2.6](#), one deduces that  $S_{0,\rho,\underline{\Omega}_m}(\cdot)$  and  $\alpha_0(\cdot)$  have the same critical points (in this case this is unique and equal to  $\tau_c$  which is a global minimum (see [Lemma C. 2](#)), therefore

$$S_{0,\rho,\underline{\Omega}_m}(\cdot) \text{ is strictly decreasing (resp. strictly increasing) in } (0, \tau_c) \text{ (resp. } (\tau_c, +\infty)). \quad (4.5.7)$$

In addition, with the aid of [Lemma C. 2](#) we can also compute

$$\lim_{\tau \rightarrow 0^+} S_{0,\rho,\underline{\Omega}_m}(\tau) = 1 > 0 \quad \text{and} \quad \lim_{\tau \rightarrow +\infty} S_{0,\rho,\underline{\Omega}_m}(\tau) = \begin{cases} 1 - \rho^2 \leq 0, & \text{if } \rho \geq 1, \\ 1 - \rho^2 > 0, & \text{if } \rho < 1, \end{cases} \quad (4.5.8)$$

- (a) If  $\rho \geq 1$ , then from the fact that  $\beta_0(\cdot) < 0$  by definition and  $\alpha_0(\tau_c) < 1$ , one can deduce that

$$S_{0,\rho,\underline{\Omega}_m}(\tau_c) = 1 + \frac{\beta_0(\tau_c)}{\underline{\Omega}_m^2} - \rho^2 (\alpha_0(\tau_c))^{-\frac{1}{2}} < 0.$$

By combining, [\(4.5.7\)](#) and [\(4.5.8\)](#) with the latter proven, we deduce that there exists a unique  $\tau_0 \in (0, \tau_c)$  such that  $S_{0,\rho,\underline{\Omega}_m}(\tau_0) = 0$ , while  $S_{0,\rho,\underline{\Omega}_m}(\cdot) < 0$  in  $(\tau_0, +\infty)$ , showing this clause.

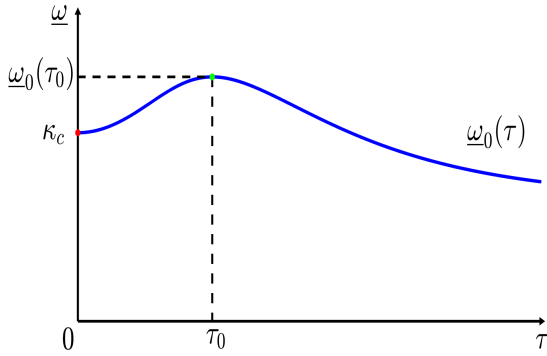
- (b) In this case we have  $\min\{S_{0,\rho,\underline{\Omega}_m}(\tau) \mid \tau > 0\} = S_{0,\rho,\underline{\Omega}_m}(\tau_c) > 0$ , henceforth  $S_{0,\rho,\underline{\Omega}_m}(\cdot)$  is strictly positive in  $\mathbb{R}^+$ .
- (c) Inhere we have  $\min\{S_{0,\rho,\underline{\Omega}_m}(\tau) \mid \tau > 0\} = S_{0,\rho,\underline{\Omega}_m}(\tau_c) = 0$ , then the clause follows directly.
- (d) For the last clause, if  $S_{0,\rho,\underline{\Omega}_m}(\tau_c) < 0$  then from [\(4.5.7\)](#) and [\(4.5.8\)](#) it follows that there exists a unique  $\tau_0^{(1)} \in (0, \tau_c)$  such that  $S_{0,\rho,\underline{\Omega}_m}(\tau_0^{(1)}) = 0$ . On another hand, since  $\rho < 1$ , then one deduces that there exists a unique  $\tau_0^{(2)} \in (\tau_c, +\infty)$  such that  $S_{0,\rho,\underline{\Omega}_m}(\tau_0^{(2)}) = 0$ .

These facts imply (retake again (4.5.7)) that  $S_{0,\rho,\underline{\Omega}_m}(\cdot) > 0$  in  $(0, \tau_0^{(1)}) \cup (\tau_0^{(2)}, +\infty)$ , while  $S_{0,\rho,\underline{\Omega}_m}(\cdot) < 0$  in  $(\tau_0^{(1)}, \tau_0^{(2)})$ , finishing the proof. □

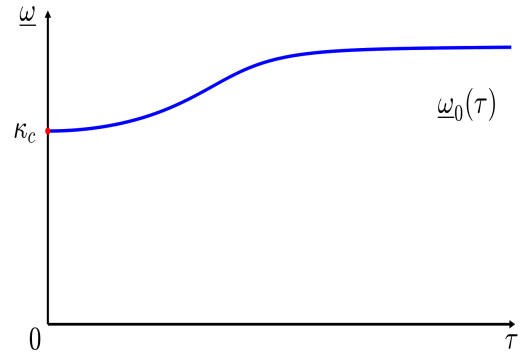
**Corollary 4.5.3**

Under the notation of Proposition 4.5.2. Given  $\rho, \underline{\Omega}_m > 0$ , then exactly one of the following happens:

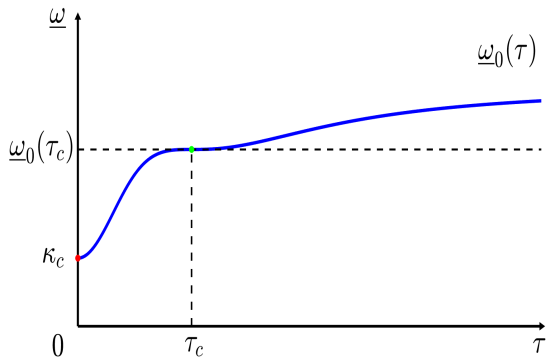
- (a) If  $\rho \geq 1$ , then  $\underline{\omega}_0(\cdot) > 0$  has a unique critical point,  $\tau_0 > 0$  such that  $\underline{\omega}'_0(\tau) > 0$  (resp.  $\underline{\omega}'_0(\tau) < 0$ ) in  $(0, \tau_0)$  (resp.  $(\tau_0, +\infty)$ ).
- (b) If  $\rho < 1$  and  $S_{0,\rho,\underline{\Omega}_m}(\tau_c) > 0$ , then  $\underline{\omega}'_0(\cdot) > 0$  in  $\mathbb{R}^+$ .
- (c) If  $\rho < 1$  and  $S_{0,\rho,\underline{\Omega}_m}(\tau_c) = 0$ , then  $\underline{\omega}'_0(\tau) > 0$  in  $\tau \neq \tau_c$  and  $\underline{\omega}'_0(\tau_c) = 0$ .
- (d) If  $\rho < 1$  and  $S_{0,\rho,\underline{\Omega}_m}(\tau_c) < 0$ , then  $\underline{\omega}'_0(\cdot)$  has exactly two critical points,  $\tau_0^{(1)} < \tau_0^{(2)}$  such that  $\underline{\omega}'_0(\tau) > 0$  (resp.  $\underline{\omega}'_0(\tau) < 0$ ) in  $(0, \tau_0^{(1)}) \cup (\tau_0^{(2)}, +\infty)$  (resp.  $(\tau_0^{(1)}, \tau_0^{(2)})$ ).



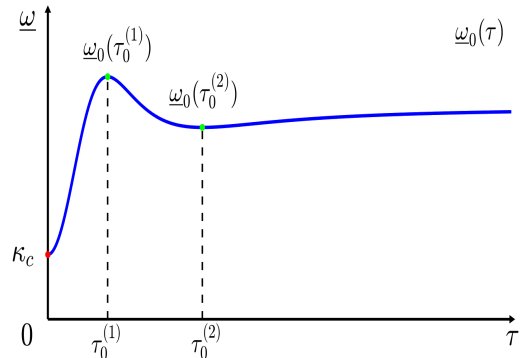
(a)  $\underline{\Omega}_m = 1.0000, \rho = 1.1000$ .



(b)  $\underline{\Omega}_m = 1.0000, \rho = 0.5566$ ,  
 $S_{0,\rho,\underline{\Omega}_m}(\tau_c) = 0.1438$ .



(c)  $\underline{\Omega}_m = 1.0000, \rho = 0.9527$ ,  
 $S_{0,\rho,\underline{\Omega}_m}(\tau_c) = 0$ .



(d)  $\underline{\Omega}_m = 2.0000, \rho = 0.9571$ ,  
 $S_{0,\rho,\underline{\Omega}_m}(\tau_c) = -0.0831$ .

Figure 4.10: Graphic representation of the function  $\underline{\omega}_0(\cdot)$ .

### 4.5.3 Study of the monotonicity of $\kappa_0(\tau)$

#### 4.5.3.1 A first result under restrictive conditions

In this section we want to study the increasing monotonicity of the function  $\tau \mapsto \kappa_0(\tau)$  for all the values  $\rho, \underline{\Omega}_m > 0$ . We will see that for some cases of  $\rho, \underline{\Omega}_m > 0$ , this monotonicity property is true, the other cases are not still proved.

In the same manner when the monotonicity of the functions  $\kappa_n(\cdot)$  was studied, we are interested in analysing the sign of the derivative  $(\kappa_0)'(\cdot)$ , one first approach is to compute this value by using the *Implicit Function Theorem* (see resemblances with the proof in [Lemma 4.3.4](#)). Since  $\underline{\omega}_0(\tau)$  satisfies

$$F_0(\underline{\omega}_0(\tau), \tau) = 0, \quad \text{where } F_0(\underline{\omega}, \tau) := (\tau^2 \tanh^2 \tau) v(\underline{\omega}) - \tau^2 - u(\underline{\omega}),$$

then by differentiating we have

$$\underline{\omega}'_0(\tau) = -\frac{\partial_\tau F_0(\underline{\omega}_0(\tau), \tau)}{\partial_{\underline{\omega}} F_0(\underline{\omega}_0(\tau), \tau)} = \frac{2\tau - (\tau^2 \tanh^2 \tau)' v(\underline{\omega}_0(\tau))}{(\tau^2 \tanh^2 \tau)' v'(\underline{\omega}_0(\tau)) - u'(\underline{\omega}_0(\tau))}, \quad (4.5.9)$$

and from [\(4.1.48b\)](#) one deduces

$$2\kappa_0(\kappa_0)' = (\tau^2 \tanh^2 \tau)' v(\underline{\omega}_0) + \left(2\underline{\omega}_0 + (\tau^2 \tanh^2 \tau)' v'(\underline{\omega}_0)\right) \underline{\omega}'_0 \quad (4.5.10)$$

and by replacing the value of  $\underline{\omega}'_0$  obtained in [\(4.5.9\)](#) we deduce that

$$\begin{aligned} 2\kappa_0(\kappa_0)' &= \frac{2\tau \left(2\underline{\omega}_0 + (\tau^2 \tanh^2 \tau)' v'(\underline{\omega}_0)\right) - (\tau^2 \tanh^2 \tau)' v(\underline{\omega}_0) (u'(\underline{\omega}_0) + 2\underline{\omega}_0)}{v'(\underline{\omega}_0) (\tau^2 \tanh^2 \tau)' - u'(\underline{\omega}_0)} \\ &= \frac{2\tau \left(2\underline{\omega}_0 + (\tau^2 \tanh^2 \tau)' v'(\underline{\omega}_0)\right) - (\tau^2 \tanh^2 \tau)' v(\underline{\omega}_0) \mathcal{J}'(\underline{\omega}_0)}{v'(\underline{\omega}_0) (\tau^2 \tanh^2 \tau)' - u'(\underline{\omega}_0)} \end{aligned} \quad (4.5.11)$$

Thanks to [Lemma 4.2.2](#) and the fact that  $\underline{\omega}_0 \in (0, \underline{\Omega}_m)$  we see that the denominator of the right hand side in [\(4.5.11\)](#) is positive. Therefore  $(\kappa_0)'$  and  $s_1 + s_2$  are of the same sign, where

$$s_1 := 2\tau \left(2\underline{\omega}_0 + (\tau^2 \tanh^2 \tau)' v'(\underline{\omega}_0)\right) \quad \text{and} \quad s_2 := -(\tau^2 \tanh^2 \tau)' v(\underline{\omega}_0) (u'(\underline{\omega}_0) + 2\underline{\omega}_0) \quad (4.5.12)$$

The function  $s_1$  is positive since is composed by positive terms including  $v'(\underline{\omega})$  (see [\(4.2.2\)](#)).

On the other side, since  $v(\underline{\omega}) > 0$  and  $(\tau^2 \tanh^2 \tau)' \geq 0$ , then the sign of  $s_2$  is delimited by the sign of  $u'(\underline{\omega}_0) + 2\underline{\omega}_0$ , additionally we know that the condition

$$0 < \underline{\omega} < \rho^{1/2} \underline{\Omega}_m, \quad (4.5.13)$$

implies  $u'(\underline{\omega}) + 2\underline{\omega} < 0$  (see [Lemma 4.2.2](#)). Therefore, if we have  $\underline{\omega}_0 < \rho^{1/2} \underline{\Omega}_m$ , then it will occur in turn that  $s_2 > 0$ . The problem is that we cannot assure [\(4.5.13\)](#) for all the cases  $\rho > 0$ , indeed, we know that for all the values of  $\rho > 0$ ,  $\underline{\omega}_0 \rightarrow \frac{\underline{\Omega}_m}{\sqrt{2}}$ , therefore for values  $\rho < \frac{1}{2}$ , the condition [\(4.5.13\)](#) is not true for large positive values of  $\tau$ .

Nevertheless, one easy way to guarantee [\(4.5.13\)](#) is considering the particular case  $\rho \geq 1$ , indeed, since  $\underline{\omega}_0 \in (0, \underline{\Omega}_m)$  the condition [\(4.5.13\)](#) follows directly, consequently  $s_2 > 0$  and finally  $(\kappa_0)' > 0$ . We have proved the following:

**Proposition 4.5.4**

If  $\rho \geq 1$ , then  $\kappa_0(\cdot)$  is a strictly increasing function.

We want now to explore what happens when  $\rho < 1$ . Coming back to (4.5.10), since the terms  $(\tau^2 \tanh^2 \tau)'$ ,  $v(\underline{\omega})$ ,  $v'(\underline{\omega})$  are all positive, then one can notice that  $\omega'_0 \geq 0$  implies  $(\kappa_0)' \geq 0$ , moreover, in this case of  $(\kappa_0)' = 0$  requires necessarily that  $(\tau^2 \tanh^2 \tau)'v(\underline{\omega}) = 0$  which is only valid when  $\tau = 0$ , therefore we can conclude that  $(\kappa_0)'(\tau) > 0$  for  $\tau > 0$ . This analysis and Corollary 4.5.3 lead us to the following result:

**Proposition 4.5.5**

If  $0 < \rho < 1$  and  $\underline{\Omega}_m > 0$  are such that  $\rho^{-2} \left(1 - \frac{\beta_{\max}}{\underline{\Omega}_m^2}\right) \geq (\alpha_{\min})^{-\frac{1}{2}}$ , then  $\kappa_0(\cdot)$  is a strictly increasing function.

**4.5.3.2 A new alternative approach for the general case**
**4.5.3.3 Introduction to some ideas**

We consider  $\underline{\Omega}_m, \rho > 0$  fixed. Let us consider the following functions

$$\mathcal{J}(\underline{\omega}) := \underline{\omega}^2 \varepsilon(\underline{\omega}) \mu(\underline{\omega}), \quad (4.5.14a)$$

$$\tau(\kappa, \underline{\omega}) := \sqrt{\kappa^2 - \mathcal{J}(\underline{\omega})}, \quad (4.5.14b)$$

$$s(\kappa, \underline{\omega}) := \sqrt{\kappa^2 - \underline{\omega}^2}, \quad (4.5.14c)$$

$$F(\kappa, \underline{\omega}) := \tau(\kappa, \underline{\omega}) \tanh \tau(\kappa, \underline{\omega}), \quad (4.5.14d)$$

$$G(\kappa, \underline{\omega}) := -\mu_r(\underline{\omega}) s(\kappa, \underline{\omega}), \quad (4.5.14e)$$

$$H(\kappa, \underline{\omega}) := F(\kappa, \underline{\omega}) - G(\kappa, \underline{\omega}). \quad (4.5.14f)$$

In addition we will consider the function

$$a_0(\tau) := \tau \tanh \tau, \quad \tau > 0, \quad (4.5.15)$$

hence  $F(\kappa, \underline{\omega}) = a_0(\tau(\kappa, \underline{\omega}))$ . Given  $\kappa > 0$  fixed, we will write  $f_\kappa(\underline{\omega}) = f(\kappa, \underline{\omega})$ , where  $f = f(\kappa, \underline{\omega})$  and  $\kappa_{<\underline{\Omega}_m}$  will stand for  $\min\{\kappa, \underline{\Omega}_m\}$ . In that sense, the function  $s_\kappa$  (and consequently, also  $G_\kappa$ ) is well defined in the open interval  $(0, \kappa)$ , and  $\tau_\kappa$  (and consequently, also  $F_\kappa$ ) is well defined in  $(\Phi(\kappa), \kappa_{<\underline{\Omega}_m})$  where

$$\Phi(\kappa) := \frac{1}{\sqrt{2}} \left( \kappa^2 + \underline{\Omega}_m^2 (\rho^2 + 1) - [(\kappa^2 + \underline{\Omega}_m^2 (\rho^2 - 1))^2 + 4 \underline{\Omega}_m^2 \kappa^2]^{\frac{1}{2}} \right)^{\frac{1}{2}}, \quad \text{for } \kappa \geq \kappa_c := \frac{\rho \underline{\Omega}_m}{\sqrt{1 + \rho^2}}, \quad (4.5.16)$$

stands for the minimal root of the equation  $\mathcal{J}(\underline{\omega}) = \kappa^2$ . We finally define the following interval  $I_\kappa := (\Phi(\kappa), \kappa_{<\underline{\Omega}_m})$ . The functions  $\tau_\kappa(\cdot)$ ,  $s_\kappa(\cdot)$ ,  $F_\kappa(\cdot)$ ,  $G_\kappa(\cdot)$  are then all positives in the interval  $I_\kappa$ , for all  $\kappa \geq \kappa_c$ .

**Objective:** We want to show that for all  $\kappa \geq \kappa_c$ , there exists a unique  $\underline{\omega}_\kappa \in I_\kappa$  such that  $H_\kappa(\underline{\omega}_\kappa) = 0$  and  $H'_\kappa(\underline{\omega}_\kappa) > 0$ .

As a state of art, we proceed as usually, by analyzing the existence of zeros for  $H_\kappa(\cdot)$  and

intervals of monoticity.

**Proposition 4.5.6: Existence**

For all  $\kappa \geq \kappa_c$  there exists  $\underline{\omega}^* \in I_\kappa$  such that  $H_\kappa(\underline{\omega}^*) = 0$ .

*Proof.* This is a direct consequence of the Intermediate Value Theorem by noticing that

$$H(\kappa, \Phi(k)) = 0 - G(\kappa, \Phi(\kappa)) < 0 \quad \text{and} \quad H(\kappa, \kappa_{<\underline{\Omega}_m}) = F(\kappa, \kappa_{<\underline{\Omega}_m}) - 0 > 0. \quad (4.5.17)$$

□

**Lemma 4.5.7: Derivatives**

Let be  $\kappa > 0$  fixed.

- (i)  $F_\kappa$  has a maximum at  $\underline{\omega} = \underline{\omega}_{cc} := \rho^{\frac{1}{2}} \underline{\Omega}_m$ , moreover,  $F_\kappa$  is increasing for  $\underline{\omega} < \underline{\omega}_{cc}$  and decreasing for  $\underline{\omega} > \underline{\omega}_{cc}$ .
- (ii)  $G_\kappa$  is decreasing for  $\underline{\omega} < \ell$ .

*Proof.* By computing  $F'_\kappa$  we have

$$F'_\kappa(\underline{\omega}) = -\frac{a'_0(\tau_\kappa(\underline{\omega}))}{2\tau_\kappa(\underline{\omega})} \mathcal{J}'(\underline{\omega}), \quad (4.5.18)$$

since  $\mathcal{J}$  has an unique critical point which is minimum, namely,  $\underline{\omega}_{cc}$ , then (i) follows immediately. On the other hand, by computing  $G'_\kappa$ ,

$$G'_\kappa(\underline{\omega}) := -\left( (-\mu(\underline{\omega})) \frac{\underline{\omega}}{s_k(\underline{\omega})} + \frac{2\underline{\Omega}_m^2}{\underline{\omega}^3} s_k(\underline{\omega}) \right), \quad (4.5.19)$$

and since  $-\mu_r(\underline{\omega})$  is positive in  $(0, \underline{\Omega}_m) (\supset I_\kappa)$ , then we have (ii). □

**Corollary 4.5.8**

For all  $\kappa \geq \kappa_c$ ,  $H_\kappa(\cdot)$  is strictly increasing in the interval  $(\Phi(\kappa), \underline{\omega}_{cc})$ . Consequently,  $H_\kappa(\cdot)$  has at most one zero in  $(\Phi(\kappa), \underline{\omega}_{cc})$ , in that case, if  $H_\kappa(\underline{\omega}^*) = 0$ , then  $H'_\kappa(\underline{\omega}^*) > 0$ .

**Remark 4.5.9.** Corollary 4.5.8 does not answer completely our objective because of the following:

- (I) We cannot guarantee the existence of zeros of  $H_\kappa(\cdot)$  inside the interval  $(\Phi(\kappa), \underline{\omega}_{cc})$ .
- (II) Even if  $H_\kappa(\cdot)$  had a zero belonging to the interval  $(\Phi(\kappa), \underline{\omega}_{cc})$ , we cannot exclude the possibility of any other zero in  $I_\kappa / (\Phi(\kappa), \underline{\omega}_{cc})$ .
- (III) If  $\underline{\omega}^* \in I_\kappa / (\Phi(\kappa), \underline{\omega}_{cc})$  satisfies  $H_\kappa(\underline{\omega}^*) = 0$ , it is not readily seen that  $H'_\kappa(\underline{\omega}^*) > 0$ .

Remark 4.5.9 leads us to consider second derivatives in order to gather more information:

**Lemma 4.5.10: Second derivatives**

- (i)  $F_k$  is a strictly concave function on  $I_k$ .
- (ii) If  $\underline{\Omega}_m \leq \kappa$ , then  $G_\kappa$  is a strictly convex on  $I_\kappa$ .
- (iii) If  $\kappa < \underline{\Omega}_m$ , then  $G_\kappa$  has a unique inflection point in  $(0, \kappa_{<\underline{\Omega}_m})$ , namely,

$$\underline{\omega}_{\text{in}}(\kappa) := \left( \frac{9 \underline{\Omega}_m^2 \kappa^2 - \underline{\Omega}_m \kappa^2 \sqrt{3(11 \underline{\Omega}_m^2 - 8 \kappa^2)}}{2(2 \underline{\Omega}_m^2 + \kappa^2)} \right)^{\frac{1}{2}}. \quad (4.5.20)$$

*Proof.* We consider the function

$$b(\tau) := \frac{a'(\tau)}{2\tau}, \quad \tau > 0, \quad (4.5.21)$$

then by differentiating (4.5.18) we get

$$F_\kappa''(\underline{\omega}) = \frac{b'(\tau_\kappa(\underline{\omega}))}{2\tau_\kappa(\underline{\omega})} (\mathcal{J}'(\underline{\omega}))^2 - b(\tau_\kappa(\underline{\omega})) \mathcal{J}''(\underline{\omega}), \quad (4.5.22)$$

so in order to prove that  $F_\kappa$  is concave, we demonstrate that  $F_\kappa'' < 0$ , and since  $b(\tau) = \frac{1}{2} \left( \frac{\tanh \tau}{\tau} + \text{sech}^2 \tau \right) > 0$ , for  $\tau > 0$ , and  $\mathcal{J}''(\underline{\omega}) > 0$ , then it suffices to show that  $b'(\tau) \leq 0$ , for  $\tau > 0$ , indeed,

$$b'(\tau) = \frac{1}{2} \left( \frac{\tau \text{sech}^2 \tau - \tanh \tau}{\tau^2} - 2 \tanh \tau \text{sech}^2 \tau \right) \leq 0, \quad \tau > 0,$$

since  $\tau \text{sech}^2 \tau \leq \tanh \tau$ .

On the other hand, by differentiating twice  $G_\kappa$  we obtain

$$\begin{aligned} G_\kappa''(\underline{\omega}) &= (-\mu(\underline{\omega}))'' s_\kappa(\underline{\omega}) + 2(-\mu(\underline{\omega}))' s_\kappa'(\underline{\omega}) + (-\mu(\underline{\omega})) s_\kappa''(\underline{\omega}) \\ &= \frac{(2 \underline{\Omega}_m^2 + \kappa^2) \underline{\omega}^4 - 9 \underline{\Omega}_m^2 \kappa^2 \underline{\omega}^2 + 6 \underline{\Omega}_m^2 \kappa^4}{\underline{\omega}^4 s_\kappa^3(\underline{\omega})}, \end{aligned}$$

therefore the zeroes of  $G_\kappa''$  are given by the positives roots of  $P_\kappa(\underline{\omega}) := (2 \underline{\Omega}_m^2 + \kappa^2) \underline{\omega}^4 - 9 \underline{\Omega}_m^2 \kappa^2 \underline{\omega}^2 + 6 \underline{\Omega}_m^2 \kappa^4$ , namely,

$$\underline{\omega}_{1,2} = \left( \frac{9 \underline{\Omega}_m^2 \kappa^2 \pm \underline{\Omega}_m \kappa^2 \sqrt{3(11 \underline{\Omega}_m^2 - 8 \kappa^2)}}{2(2 \underline{\Omega}_m^2 + \kappa^2)} \right)^{\frac{1}{2}}, \quad (4.5.23)$$

provided that  $11 \underline{\Omega}_m^2 - 8 \kappa^2 \geq 0$ . Since

$$\left( \frac{9 \underline{\Omega}_m^2 \kappa^2 + \underline{\Omega}_m \kappa^2 \sqrt{3(11 \underline{\Omega}_m^2 - 8 \kappa^2)}}{2(2 \underline{\Omega}_m^2 + \kappa^2)} \right)^{\frac{1}{2}} > \min\{\kappa, \underline{\Omega}_m\} = \kappa_{<\underline{\Omega}_m}, \quad (4.5.24)$$

then we focus in the smallest root

$$\underline{\omega}_{\text{in}}(\kappa) := \left( \frac{9 \underline{\Omega}_m^2 \kappa^2 - \underline{\Omega}_m \kappa^2 \sqrt{3(11 \underline{\Omega}_m^2 - 8 \kappa^2)}}{2(2 \underline{\Omega}_m^2 + \kappa^2)} \right)^{\frac{1}{2}}.$$

For the case  $\kappa \geq \underline{\Omega}_m$ , and provided  $11 \underline{\Omega}_m^2 - 8 \kappa^2 \geq 0$ , then we have

$$\underline{\omega}_{\text{in}}(\kappa) \geq \left( \frac{9 \underline{\Omega}_m^2 \kappa^2 - \underline{\Omega}_m \kappa^2 \sqrt{3(11 \underline{\Omega}_m^2 - 8 \underline{\Omega}_m^2)}}{2(2 \kappa^2 + \kappa^2)} \right)^{\frac{1}{2}} = \underline{\Omega}_m \quad (4.5.25)$$

meaning that  $G''_{\kappa}$  has no roots in the interval  $I_{\kappa} = (\Phi(\kappa), \underline{\Omega}_m)$ , and since

$$G''_{\kappa}(\underline{\Omega}_m) = \frac{2 \underline{\Omega}_m^2 (\kappa^2 - \underline{\Omega}_m^2) (3 \kappa^2 - \underline{\Omega}_m^2)}{\underline{\Omega}_m^4 s_{\kappa}^4(\underline{\Omega}_m)} > 0,$$

it follows that  $G''_{\kappa} > 0$  in  $I_{\kappa}$ , showing (ii).

On the other hand, if  $\kappa < \underline{\Omega}_m$ , then  $\underline{\omega}_{\text{in}}(\kappa)$  is well defined and one can verify that

$$P_{\kappa}(0) = 6 \underline{\Omega}_m^2 \kappa^4 > 0, \quad P_{\kappa}(\kappa) = \kappa^4 (\kappa^2 - \underline{\Omega}_m^2) < 0, \quad (4.5.26)$$

and since  $\underline{\omega}_{\text{in}}(\kappa)$  is the smallest root of  $P_{\kappa}$ , then necessarily  $\underline{\omega}_{\text{in}}(\kappa) \in (0, \kappa)$ .  $\square$

The main reason why we have introduced the second derivatives is for the use of the following lemma (whose proof is oriented by a geometrical argument):

**Lemma 4.5.11**

Let be  $I = (a, b) \subset \mathbb{R}$  an interval and  $f \in C^2(I)$  such that  $f'' < 0$  in  $I$ . Suppose that  $f(a) < 0$  and  $f(b) > 0$ , therefore  $f$  has a unique zero in  $I$ , furthermore, if  $z \in I$  is the zero, then  $f'(z) > 0$ .

*Proof.* The existence of a zero is a direct consequence of the hypothesis and the *Intermediate Value Theorem*. For proving the uniqueness, we proceed by contradiction, let us suppose that  $a < z_0 < z_1 < b$  are such that  $f(z_0) = 0 = f(z_1)$ . Since  $f'' < 0$ , meaning that  $f$  is strictly concave, we therefore have  $f((1-t)z_0 + tz_1) > (1-t)f(z_0) + tf(z_1) > 0$  for all  $t \in (0, 1)$ , in consequence  $f(x) > 0$  for all  $x \in (z_0, z_1)$ , which contradicts the assumption  $f(z_1) = 0$ , proving uniqueness. Finally, considering  $z \in I$  the zero of  $f$ , the Mean Value Theorem guarantee us that exists  $x_0 \in (z, b)$  such that  $f'(x_0) = \frac{f(b) - f(z)}{b - z} > 0$ , and since  $f'(\cdot)$  is strictly decreasing ( $f'' < 0$ ), it follows that  $f'(z) > f'(x_0) > 0$ , finishing the proof.  $\square$

**Proposition 4.5.12**

If  $\kappa \in (\kappa_c, \underline{\omega}_{cc}] \cup [\underline{\Omega}_m, +\infty)$ , then there exists a unique  $\underline{\omega}_{\kappa} \in I_{\kappa}$  such that  $H_{\kappa}(\underline{\omega}_{\kappa}) = 0$ , moreover, it holds  $H'_{\kappa}(\underline{\omega}_{\kappa}) > 0$ .

*Proof.*  $\blacksquare$  If  $\kappa \in (\kappa_c, \underline{\omega}_{cc}]$ , then  $I_{\kappa} \subseteq (\Phi(\kappa), \underline{\omega}_{cc})$ , the result therefore follows from [Corollary 4.5.8](#).

$\blacksquare$  If  $\kappa \geq \underline{\Omega}_m$ , then [Lemma 4.5.10](#) guarantee us that  $H_{\kappa}(\cdot)$  is a strictly concave function on  $I_{\kappa}$  such that  $H_{\kappa}(\Phi(\kappa)) < 0 < H_{\kappa}(\underline{\Omega}_m)$ , hence we can apply [Lemma 4.5.11](#) to conclude.

□

#### 4.5.3.4 Approach using Complex Analysis

In what follows, we desire to use Hurwitz' theorem in Complex Analysis:

##### Theorem 4.5.13: Hurwitz' theorem

Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of holomorphic functions on a connected open set  $U$  that converge uniformly on compact subsets of  $U$  to a holomorphic function  $f$  which is not constantly zero on  $U$ . If  $f$  has a zero of order  $m \in \mathbb{N}$  at  $z_0$  then for every small  $\delta > 0$  such that  $B(z_0, \delta) \subseteq U$  and for sufficiently large  $n \in \mathbb{N}$  (depending on  $\delta$ ),  $f_n$  has precisely  $m$  zeroes in  $B(z_0, \delta)$ , including multiplicity. Furthermore, these zeroes converge to  $z_0$  as  $n \rightarrow \infty$ .

We use the last theorem in order to prove the following abstract result

##### Lemma 4.5.14

Let be  $\kappa_c > 0$  and  $I^-, I^+ : [0, \infty) \rightarrow \mathbb{R}$  continuous functions on  $[\kappa_c, \infty)$  such that

- a)  $I^-(\kappa) < I^+(\kappa)$  for all  $\kappa > \kappa_c$ ,
- b)  $I^-(\cdot)$  is monotonous decreasing and  $I^+(\cdot)$  is monotonous increasing.

Let be the domain *indexed complex domain*

$$X := \left\{ (\kappa, z) \in [\kappa_c, +\infty) \times \mathbb{C} \mid I^-(\kappa) < \Re(z) < I^+(\kappa) \right\}$$

and consider  $H \in C(X; \mathbb{C})$ . Suppose that for all  $\kappa > \kappa_c$ ,  $H(\kappa, \cdot) : U_\kappa \rightarrow \mathbb{C}$  is a holomorphic function not constantly zero, where

$$U_\kappa := \{z \in \mathbb{C} \mid I^-(\kappa) < \Re(z) < I^+(\kappa)\}. \quad (4.5.27)$$

We say that  $\kappa > 0$  satisfies the property  $\mathcal{P}$  if and only if

$$H(\kappa, \cdot) \text{ has a unique zero on } U_\kappa \text{ and this is real and simple.} \quad (4.5.28)$$

If  $\kappa_0 > \kappa_c$  is such that  $\kappa$  satisfies the property  $\mathcal{P}$  for all  $\kappa_c < \kappa < \kappa_0$  and  $\mathcal{Z}(H(\kappa_0, \cdot)) \cap U_{\kappa_0} \neq \emptyset$ , then  $\kappa_0$  satisfies also the property  $\mathcal{P}$ .

*Proof.* We first prove that the zero of  $H(\kappa_0, \cdot)$  in  $U_{\kappa_0}$  is unique. For all  $\kappa \in (0, \kappa_0)$  we denote as  $z_\kappa$  the unique zero of  $H(\kappa, \cdot)$  in  $U_\kappa$ . We prove that for any zero of  $H(\kappa_0, \cdot)$ ,  $z' \in U_{\kappa_0}$ , it holds

$$\lim_{\kappa \rightarrow \kappa_0^-} z_\kappa = z'. \quad (4.5.29)$$

Let  $\delta > 0$  such that  $B(z', \delta) \subseteq U_{\kappa_0}$ , hence

$$I^-(\kappa_0) < \Re(z') - \delta < \Re(z') + \delta < I^+(\kappa_0). \quad (4.5.30)$$

By continuity and monotonicity of  $I^-(\cdot)$  and  $I^+(\cdot)$ , there exists  $\kappa^* \in (0, \kappa_0)$  such that

$$I^-(\kappa) < I^-(\kappa_0) + \frac{\delta}{2} \quad \text{and} \quad I^+(\kappa_0) < I^+(\kappa) + \frac{\delta}{2}, \quad \text{for all } \kappa \in (\kappa^*, \kappa_0). \quad (4.5.31)$$



Combining (4.5.30) and (4.5.31), we obtain

$$I^-(\kappa) < \Re(z') - \frac{\delta}{2} < \Re(z') + \frac{\delta}{2} < I^+(\kappa), \quad \text{for all } \kappa \in (\kappa^*, \kappa_0). \quad (4.5.32)$$

(4.5.32) shows that

$$B\left(z', \frac{\delta}{2}\right) \subset U_{\kappa^*} \subseteq U_{\kappa}, \quad \text{for all } \kappa \in [\kappa^*, \kappa_0]. \quad (4.5.33)$$

Now we define what it will be the common domain for the functions  $H(\kappa, \cdot)$ ,  $\kappa^* \leq \kappa \leq \kappa_0$ . Let be  $M > 0$  big enough such that

$$B\left(z', \frac{\delta}{2}\right) \subset U := \{z \in \mathbb{C} \mid I^-(\kappa^*) < \Re(z) < I^+(\kappa^*), \quad |\Im(z)| < M\}, \quad (4.5.34)$$

hence  $U$  is a bounded open of  $\mathbb{C}$  such that  $U \subset U_{\kappa^*} \subseteq U_{\kappa}$ , for all  $\kappa \in [\kappa^*, \kappa_0]$ . Hence we consider the family of holomorphic functions

$$\{H(\kappa, \cdot) \mid \kappa \in [\kappa^*, \kappa_0]\},$$

defined on the domain  $U$ . Since  $U$  is bounded, then  $R := [\kappa^*, \kappa_0] \times U$  is compactly embedded in  $X$ . As a consequence of our hypothesis, we have that  $H$  is uniformly continuous in  $R$ , which implies in turn that  $\{H(\kappa, \cdot)\}_{\kappa^* \leq \kappa < \kappa_0}$  converges uniformly to  $H(\kappa_0, \cdot)$  on compact subsets of  $U$ , as  $\kappa \rightarrow \kappa_0$ . Then we can use Hurwitz' Theorem, given any neighborhood  $V$  of  $z'$  in  $U$ , and any sequence  $(\kappa_n) \subset (\kappa^*, \kappa_0)$  such that  $\kappa_n \rightarrow \kappa_0$  as  $n \rightarrow +\infty$ , there exists a further subsequence  $(\kappa_{n_j})$  such that

$$\lim_{j \rightarrow +\infty} z_{\kappa_{n_j}} = z', \quad (4.5.35)$$

In here, we have used that  $H(\kappa, \cdot)$  has unique zeros in  $U$ . This proves (4.5.29), which proves in turn the uniqueness of the zero of  $H(\kappa_0, \cdot)$  and that  $z'$  is real (since  $(z_\kappa) \subset \mathbb{R}$ ). Moreover, from Hurwitz' Theorem, it can be seen that the unique zero of  $H(\kappa_0, \cdot)$  is simple, since the number of zeros (with multiplicities) of  $H(\kappa, \cdot)$  ( $\kappa < \kappa_0$ ) is 1.  $\square$

#### 4.5.3.5 General proof for the monotony of $\kappa_0(\cdot)$

Our objective is to prove the following proposition:

**Proposition 4.5.15: Strictly increasing of  $\kappa'_0$**

$(\kappa_0)'(\cdot) > 0$  in  $\mathbb{R}^+$ , for all  $\rho, \underline{\Omega}_m > 0$ .

First we present the setting for using Lemma 4.5.14. We define  $I^-, I^+ : [\kappa_c, \infty) \rightarrow \mathbb{R}$  as

$$I^-(\kappa) := \Phi(\kappa), \quad I^+(\kappa) := \min\{k, \underline{\Omega}_m\}, \quad \kappa \geq \kappa_c, \quad (4.5.36)$$

where  $\kappa_c$  and  $\Phi$  are defined as in (4.5.16). Let be  $X$  the complex domain

$$X := \left\{ (\kappa, z) \in [\kappa_c, +\infty) \times \mathbb{C} \mid I^-(\kappa) < \Re(z) < I^+(\kappa) \right\}. \quad (4.5.37)$$

And define  $H : X \rightarrow \mathbb{C}$  as in (4.5.14f). It is clear that  $H \in C(X; \mathbb{C})$ . One can notice that for all  $k > \kappa_c$ ,  $H(k, \cdot)$  is a non-constantly holomorphic function on  $\mathbb{C} \setminus \{\Phi(\kappa), \kappa\}$ , hence is holomorphic on  $U_\kappa$  defined as in (4.5.27).

**Lemma 4.5.16**

The following are true for  $(\kappa, z) \in X$  :

- (i)  $\Re(\sqrt{\kappa^2 - z^2}) > 0$ ;
- (ii) If  $H(\kappa, z) = 0$ , then  $z \in \mathbb{R}$ ;
- (ii)  $\mathcal{Z}(H(k, \cdot)) \cap U_\kappa \neq \emptyset$ .

*Proof.* (i) First note that  $\Re(z) > 0$  since  $I^-(\kappa) = \Phi(\kappa) > 0$ . Proceeding by contradiction, and since we are considering the principal branch of the square root, we suppose that  $\Re(\sqrt{\kappa^2 - z^2}) = 0$ , in other words,  $z^2 = \kappa^2$ , which implies that  $z = \Re(z) \in \mathbb{R}^+$  and hence contradicts the fact that  $\Re(z) < I^+(\kappa)$ .

(ii) Let us consider  $k := \kappa/L$  and  $\omega := z/(\sqrt{\varepsilon_0 \mu_0} L)$ , from the definition of  $H$  we have

$$\begin{aligned} H(\kappa, z) = 0 &\iff \xi_{\kappa, z}^D \tanh \xi_{\kappa, z}^D = -\mu_r(\underline{\omega}) \xi_{\kappa, z}^V \\ &\iff \theta_{k, \omega}^D \tanh(\theta_{k, \omega}^D L) = -\frac{\mu(\omega)}{\mu_0} \theta_{k, \omega}^V, \end{aligned}$$

and from the first clause, we have that  $\Re(\theta_{k, \omega}^V) > 0$ , hence  $\omega \in \sigma_{\text{even}}(k) \subseteq \mathbb{R}$ , this implies that  $z \in \mathbb{R}$ .

(iii) It follows directly from [Proposition 4.5.6](#). □

**Lemma 4.5.17**

For all  $\kappa > \kappa_c$ , there exists a unique  $\underline{\omega}_\kappa \in (I^-(\kappa), I^+(\kappa))$  such that  $H(\kappa, \underline{\omega}_\kappa) = 0$ , moreover, it also holds  $\partial_{\underline{\omega}} H(\kappa, \underline{\omega}_\kappa) \neq 0$ .

*Proof.* We say that  $s > \kappa_c$  satisfies the property  $(\mathcal{Q})$  if and only if there exists a unique  $\underline{\omega}_s \in (I^-(s), I^+(s))$  such that  $H(s, \underline{\omega}_s) = 0$  and  $\partial_{\underline{\omega}} H(s, \underline{\omega}_s) \neq 0$ . We therefore define the following set

$$\mathcal{I} := \{\kappa > \kappa_c \mid s \text{ satisfies the property } (\mathcal{Q}) \text{ for all } s \in (\kappa_c, \kappa]\}, \quad (4.5.38)$$

one deduces from [Proposition 4.5.12](#) that  $(\kappa_c, \underline{\omega}_{\kappa_c}) \subseteq \mathcal{I}$ , in particular,  $\mathcal{I} \neq \emptyset$ , and since the manner it was defined, it follows that  $\mathcal{I}$  is an interval. It only rest to prove that  $\mathcal{I}$  does not have an upper bound, indeed, arguing by contradiction, if  $\mathcal{I}$  is upper bounded then we can consider  $\kappa_0 := \sup \mathcal{I} < +\infty$ . The latter yields to  $(\kappa_c, \kappa_0) \subseteq \mathcal{I}$ , in other words, for all  $\kappa_c < \kappa < \kappa_0$ ,  $\kappa$  satisfies the property  $(\mathcal{Q})$ .

We first prove that  $\kappa_0$  satisfies the property  $(\mathcal{Q})$ . For this purpose, we see verify the hypothesis for using [Lemma 4.5.14](#), in fact, it only remains to prove that the property  $(\mathcal{Q})$  implies the property  $(\mathcal{P})$  as in [Lemma 4.5.14](#), indeed, [Lemma 4.5.17](#) implies that for all  $\kappa \in (\kappa_c, \kappa_0)$ ,  $H(\kappa, \cdot)$  has exclusively real zeros, and this is unique and simple provided the property  $(\mathcal{Q})$ . Hence,  $H(\kappa_0, \cdot)$  has a unique zero  $(I^-(\kappa_0), I^+(\kappa_0))$ ,  $\underline{\omega}_{\kappa_0}$ , which is simple, in other words,  $\partial_{\underline{\omega}} H(\kappa, \underline{\omega}_{\kappa_0}) \neq 0$ .

The former information yields  $\kappa_0 \in \mathcal{I}$  and consequently,  $\kappa_0 = \max \mathcal{I}$ . The fact that  $\kappa_0$  satisfies

the property  $(\mathcal{Q})$  give us the necessary hypothesis for apply the *Implicit Function Theorem* for real fuctions, therefore the property  $(\mathcal{Q})$  can be extended in open neighborhood of  $\kappa_0$ , which contradicts the maximality of  $\kappa_0$ , this proves that  $\mathcal{I} = (\kappa_c, +\infty)$  finishing the proof.  $\square$

**Lemma 4.5.18: Monotonicity of  $\kappa_0$**

The following are true for  $\tau > 0$  :

- (i)  $\kappa_0(\tau) > \kappa_c$  and  $I^-(\kappa_0(\tau)) < \underline{\omega}_0(\tau) < I^+(\kappa_0(\tau))$ ,
- (ii)  $H(\kappa_0(\tau), \underline{\omega}_0(\tau)) = 0$ ,
- (iii)  $\text{sign}(\partial_{\underline{\omega}} H(\kappa_0(\tau), \underline{\omega}_0(\tau))) = \text{sign}(\kappa'_0(\tau))$ , consequently  $\kappa'_0(\tau) > 0$ .

*Proof.* (i) It follows from the fact that  $(\kappa_0(\tau), \underline{\omega}_0(\tau)) \in \mathcal{C}_0 \subset \mathcal{N}^+$  and (4.1.16).

(ii) It follows from the definition of  $H$  and  $(\kappa_0(\tau), \underline{\omega}_0(\tau)) \in \mathcal{C}_0 \subset \mathcal{D}_{\text{even}}^+$ .

(iii) For simplicity, we make  $\kappa = \kappa_0(\tau)$  and  $\underline{\omega} = \underline{\omega}_0(\tau)$ . From (4.2.19) and since  $\kappa > 0$  and  $A_0(\tau) v'(\underline{\omega}) - u'(\underline{\omega}) > 0$ , then we derive that

$$\text{sign}(\kappa'_0(\tau)) = -\text{sign}(A'_0(\tau) v(\underline{\omega}) \mathcal{J}'(\underline{\omega}) + (2\underline{\omega} + A_0(\tau) v'(\underline{\omega})) B'_0(\tau)). \quad (4.5.39)$$

On another hand,

$$\begin{aligned} \partial_{\underline{\omega}} H(\kappa, \underline{\omega}) &= \partial_{\underline{\omega}} F(\kappa, \underline{\omega}) - \partial_{\underline{\omega}} G(\kappa, \underline{\omega}) \\ &= a'_0(\tau(\kappa, \underline{\omega})) \partial_{\underline{\omega}} \tau(\kappa, \underline{\omega}) + \mu'_r(\underline{\omega}) s(\kappa, \underline{\omega}) + \mu_r(\underline{\omega}) \partial_{\underline{\omega}} s(\kappa, \underline{\omega}), \\ &= -\frac{a'_0(\tau(\kappa, \underline{\omega})) \mathcal{J}'(\underline{\omega})}{2\tau(\kappa, \underline{\omega})} + \mu'_r(\underline{\omega}) s(\kappa, \underline{\omega}) - \frac{2\underline{\omega} \mu_r(\underline{\omega})}{2s(\kappa, \underline{\omega})}, \end{aligned} \quad (4.5.40)$$

and since  $(\kappa, \underline{\omega}) = (\kappa_0(\tau), \underline{\omega}_0(\tau))$  solves (4.1.48), then we can notice

$$\tau = \tau(\kappa, \underline{\omega}) \quad \text{and} \quad s(\kappa, \underline{\omega}) = \sqrt{A_0(\tau) v(\underline{\omega})} = |a_0(\tau) \mu_r^{-1}(\underline{\omega})| = -a_0(\tau) \mu_r^{-1}(\underline{\omega}), \quad (4.5.41)$$

where we have used that  $A_0(\tau) = a_0^2(\tau)$  and  $v(\underline{\omega}) = \mu_r^{-2}(\underline{\omega})$ , then by replacing the latter in (4.5.40) we obtain

$$\begin{aligned} \partial_{\underline{\omega}} H(\kappa, \underline{\omega}) &= -\frac{a'_0(\tau) \mathcal{J}'(\underline{\omega})}{2\tau} - \frac{a_0(\tau) \mu'_r(\underline{\omega})}{\mu_r(\underline{\omega})} + \frac{\underline{\omega} \mu_r^2(\underline{\omega})}{a_0(\tau)} \\ &= -\frac{a'_0(\tau) \mathcal{J}'(\underline{\omega})}{2\tau} - \frac{2a_0(\tau) \mu'_r(\underline{\omega}) \mu_r^2(\underline{\omega})}{2\mu_r^3(\underline{\omega})} + \frac{\underline{\omega} \mu_r^2(\underline{\omega})}{a_0(\tau)} \\ &= -\frac{a'_0(\tau) \mathcal{J}'(\underline{\omega})}{-B'_0(\tau)} + \frac{a_0(\tau) v'(\underline{\omega})}{2v(\underline{\omega})} + \frac{\underline{\omega}}{a_0(\tau) v(\underline{\omega})}, \end{aligned} \quad (4.5.42)$$

where it was used that  $B'_0(\tau) = -2\tau$  and  $v'(\underline{\omega}) = -\frac{2\mu'_r(\underline{\omega})}{\mu_r^3(\underline{\omega})}$ . Finally,

$$\begin{aligned} \partial_{\underline{\omega}} H(\kappa, \underline{\omega}) &= -\left( \frac{2a_0(\tau) a'_0(\tau) v(\underline{\omega}) \mathcal{J}'(\underline{\omega}) + (2\underline{\omega} + a_0^2(\tau) v'(\underline{\omega})) B'_0(\tau)}{-2a_0(\tau) B'_0(\tau) v(\underline{\omega})} \right), \\ &= -\left( \frac{A'_0(\tau) v(\underline{\omega}) \mathcal{J}'(\underline{\omega}) + (2\underline{\omega} + A_0(\tau) v'(\underline{\omega})) B'_0(\tau)}{-2a_0(\tau) B'_0(\tau) v(\underline{\omega})} \right), \end{aligned} \quad (4.5.43)$$

where again it was used the fact that  $A_0 = a_0^2$  and consequently  $A'_0 = 2a_0 a'_0$ , since  $a_0(\tau), -B_0(\tau) > 0$  for  $\tau > 0$  and  $v(\underline{\omega}) > 0$ , then we deduce that from (4.5.39) and (4.5.43) that

$$\begin{aligned} \text{sign}(\partial_{\underline{\omega}} H(\kappa_0(\tau), \underline{\omega}_0(\tau))) &= \text{sign}(\partial_{\underline{\omega}} H(\kappa, \underline{\omega})) \\ &= -\text{sign}(A'_0(\tau) v(\underline{\omega}) \mathcal{J}'(\underline{\omega}) + (2\underline{\omega} + A_0(\tau) v'(\underline{\omega})) B'_0(\tau)) \\ &= \text{sign}(\kappa'_0(\tau)), \end{aligned}$$

and from Lemma 4.5.17 and the clauses (i) and (ii) we conclude that  $\partial_{\underline{\omega}} H(\kappa_0(\tau), \underline{\omega}_0(\tau)) > 0$  and therefore  $\kappa'_0(\tau) > 0$ . □

#### 4.5.4 Asymptotic behaviour of the functions $\underline{\omega}_0(\tau)$ and $\kappa_0(\tau)$ when $\tau \rightarrow \infty$

Inspired in the results obtained in 4.4.2.4, an heuristic argument will lead us to consider an asymptotic behavior

$$\underline{\omega}_0(\tau) \sim \frac{\underline{\Omega}_m}{\sqrt{2}} + \underline{\Omega}_{-2} \tau^{-2} + o(\tau^{-2}), \quad \text{as } \tau \rightarrow +\infty. \quad (4.5.44)$$

##### Lemma 4.5.19

The function  $\tau \mapsto \underline{\omega}_0(\tau)$  has the following asymptotic behavior

$$\begin{cases} \underline{\omega}_0(\tau) = \frac{\underline{\Omega}_m}{\sqrt{2}} + \underline{\Omega}_\infty \tau^{-2} + \gamma_0 \tau^{-4} + o(\tau^{-4}), & (\tau \rightarrow \infty), \\ \underline{\Omega}_\infty := \frac{\underline{\Omega}_m^3 (\rho^2 - 1)}{8\sqrt{2}}, \quad \gamma_0 := \frac{1}{64\sqrt{2}} (17\rho^2 - 9) (\rho^2 - 1) \underline{\Omega}_m^5 \end{cases} \quad (4.5.45)$$

The function  $\tau \mapsto \kappa_0(\tau)$  has the following asymptotic behavior

$$\begin{cases} \kappa_0(\tau) = \tau + K_\infty \tau^{-1} + \delta_\infty \tau^{-3} + o(\tau^{-3}), & \text{when } \tau \rightarrow +\infty, \\ K_\infty := \frac{1}{2} \left( \rho^2 - \frac{1}{2} \right) \underline{\Omega}_m^2, \quad \delta_\infty = \frac{\sqrt{2}}{2} (1 - 4\rho^2) \underline{\Omega}_m \Omega_\infty. \end{cases} \quad (4.5.46)$$

*Proof.* We consider the variable change  $\varepsilon = \tau^{-2}$ , and we notice that the analysis of (4.1.48a) around  $\tau = \infty$ , is equivalent to the one around  $\varepsilon = 0$  of the following equation

$$u(X) \varepsilon + 1 = v(X) \tanh^2 \left( \varepsilon^{-\frac{1}{2}} \right). \quad (4.5.47)$$

We show that (4.5.47) has a smooth solution  $\varepsilon \mapsto X_0(\varepsilon)$  around  $\varepsilon = 0$ . We base our proof by means of the Implicit Function Theorem. Let us consider the function

$$\mathcal{G}_0(X, \varepsilon) := u(X) \varepsilon + 1 - v(X) \tanh^2 \left( \varepsilon^{-\frac{1}{2}} \right), \quad (X, \varepsilon) \in (0, \underline{\Omega}_m) \times [0, \infty). \quad (4.5.48)$$

Since  $\varepsilon \mapsto \tanh \left( \varepsilon^{-\frac{1}{2}} \right)$  defines a smooth function on around a positive neighborhood of  $\varepsilon > 0$ ,

moreover, one can verify

$$\begin{cases} \lim_{\varepsilon \rightarrow 0^+} \tanh\left(\varepsilon^{-\frac{1}{2}}\right) = 1, \\ \lim_{\varepsilon \rightarrow 0^+} \left(\tanh\left(\varepsilon^{-\frac{1}{2}}\right)\right)' = -\lim_{\varepsilon \rightarrow 0^+} \frac{\operatorname{sech}^2\left(\varepsilon^{-\frac{1}{2}}\right)}{2\varepsilon^{\frac{3}{2}}} = 0, \\ \lim_{\varepsilon \rightarrow 0^+} \left(\tanh\left(\varepsilon^{-\frac{1}{2}}\right)\right)'' = \lim_{\varepsilon \rightarrow 0^+} \frac{\left(3\varepsilon^{\frac{1}{2}} - 2\tanh\left(\varepsilon^{-\frac{1}{2}}\right)\right)\operatorname{sech}^2\left(\varepsilon^{-\frac{1}{2}}\right)}{4\varepsilon^3} = 0, \end{cases} \quad (4.5.49)$$

consequently

$$\begin{cases} \lim_{\varepsilon \rightarrow 0^+} \tanh^2\left(\varepsilon^{-\frac{1}{2}}\right) = 1, \\ \lim_{\varepsilon \rightarrow 0^+} \left(\tanh^2\left(\varepsilon^{-\frac{1}{2}}\right)\right)' = 0, \\ \lim_{\varepsilon \rightarrow 0^+} \left(\tanh^2\left(\varepsilon^{-\frac{1}{2}}\right)\right)'' = 0, \end{cases} \quad (4.5.50)$$

then  $\mathcal{G}_0$  is smooth at  $(X, \varepsilon) = \left(\frac{\Omega_m}{\sqrt{2}}, 0\right)$ , furthermore, this point verifies  $\mathcal{G}_0\left(\frac{\Omega_m}{\sqrt{2}}, 0\right) = 0$  and

$$\begin{aligned} \partial_X \mathcal{G}_0\left(\frac{\Omega_m}{\sqrt{2}}, 0\right) &= \left(u'(X)\varepsilon - v'(X)\tanh^2\left(\varepsilon^{-\frac{1}{2}}\right)\right)\Big|_{(X, \varepsilon) = \left(\frac{\Omega_m}{\sqrt{2}}, 0\right)} \\ &= -v'\left(\frac{\Omega_m}{\sqrt{2}}\right) = -\frac{8\sqrt{2}}{\Omega_m} \neq 0, \end{aligned}$$

hence the *Implicit Function Theorem* give us the existence of a smooth function  $\varepsilon \mapsto X(\varepsilon)$  around a positive neighborhood of  $\varepsilon = 0$  such that

$$\mathcal{G}_0(X_0(\varepsilon), \varepsilon) = 0 \quad \text{and} \quad X_0(0) = \frac{\Omega_m}{\sqrt{2}}. \quad (4.5.51)$$

Using the second order Taylor approximation around  $\varepsilon = 0$ , we deduce that

$$X_0(\varepsilon) = X_0(0) + X_0'(0)\varepsilon + \frac{X_0''(0)}{2}\varepsilon^2 + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0^+. \quad (4.5.52)$$

now we are interested into find  $X'(0)$  and  $X''(0)$ . For this purpose, we use the following expression for the two first derivatives derived from the *implicit differentiation*:

$$\begin{cases} X_0'(\varepsilon) = -\frac{\partial_\varepsilon \mathcal{G}_0(X(\varepsilon), \varepsilon)}{\partial_X \mathcal{G}_0(X(\varepsilon), \varepsilon)}, \end{cases} \quad (4.5.53a)$$

$$\begin{cases} X_0''(\varepsilon) = -\frac{\begin{pmatrix} \partial_\varepsilon^2 \mathcal{G}_0(X_0(\varepsilon), \varepsilon) \\ 2\partial_X \partial_\varepsilon \mathcal{G}_0(X_0(\varepsilon), \varepsilon) \\ \partial_X^2 \mathcal{G}_0(X_0(\varepsilon), \varepsilon) \end{pmatrix} \cdot \begin{pmatrix} (X_0'(\varepsilon))^2 \\ X_0'(\varepsilon) \\ 1 \end{pmatrix}}{\partial_X \mathcal{G}_0(X_0(\varepsilon), \varepsilon)}. \end{cases} \quad (4.5.53b)$$

By computing the first partial derivatives of  $\mathcal{G}_0$  we obtain:

$$\begin{cases} \partial_X \mathcal{G}_0(X, \varepsilon) = u'(X) \varepsilon - v'(X) \tanh^2(\varepsilon^{-\frac{1}{2}}), & (4.5.54a) \\ \partial_\varepsilon \mathcal{G}_0(X, \varepsilon) = u(X) - v(X) \left( \tanh^2(\varepsilon^{-\frac{1}{2}}) \right)', & (4.5.54b) \end{cases}$$

consequently, the second order derivatives hold

$$\begin{cases} \partial_X^2 \mathcal{G}_0(X, \varepsilon) = u''(X) \varepsilon - v''(X) \tanh^2(\varepsilon^{-\frac{1}{2}}), & (4.5.55a) \\ \partial_X \partial_\varepsilon \mathcal{G}_0(X, \varepsilon) = \partial_\varepsilon \partial_X \mathcal{G}_0(X, \varepsilon) = u'(X) - v'(X) \left( \tanh^2(\varepsilon^{-\frac{1}{2}}) \right)', & (4.5.55b) \\ \partial_\varepsilon^2 \mathcal{G}_0(X, \varepsilon) = v(X) \left( \tanh^2(\varepsilon^{-\frac{1}{2}}) \right)''. & (4.5.55c) \end{cases}$$

Hence, with the aid of (4.4.22), (4.4.23) and (4.5.50), we evaluate (4.5.54) and (4.5.55) at  $(X, \varepsilon) = \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right)$  as follows

$$\begin{cases} \partial_X \mathcal{G}_0 \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right) = -v' \left( \frac{\Omega_m}{\sqrt{2}} \right) = -\frac{8\sqrt{2}}{\Omega_m}, \\ \partial_\varepsilon \mathcal{G}_0 \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right) = u \left( \frac{\Omega_m}{\sqrt{2}} \right) = (\rho^2 - 1) \Omega_m^2 \\ \partial_X^2 \mathcal{G}_0 \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right) = -v'' \left( \frac{\Omega_m}{\sqrt{2}} \right) = -\frac{144}{\Omega_m^2}, \\ \partial_X \partial_\varepsilon \mathcal{G}_0 \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right) = \partial_\varepsilon \partial_X \mathcal{G}_0 \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right) = u' \left( \frac{\Omega_m}{\sqrt{2}} \right) = -4\sqrt{2} \Omega_m \rho^2, \\ \partial_\varepsilon^2 \mathcal{G}_0 \left( \frac{\Omega_m}{\sqrt{2}}, 0 \right) = 0. \end{cases}$$

From here, the remaining of the proof is identical to the proof of [Lemma 4.4.9](#).  $\square$

#### 4.5.5 Characterization of the curves $(\mathcal{C}_0)$ as graphs

##### **Theorem 4.5.20**

The curve  $\mathcal{C}_0$  is the graph of a  $C^\infty$ -function on  $\kappa$ . More precisely,

$$\mathcal{C}_0 = \{(\kappa, \underline{\omega}_0^*(\kappa)) \mid \kappa > \kappa_c\}, \quad \underline{\omega}_0^* := \underline{\omega}_0 \circ \{\kappa_0\}^{-1} \quad (4.5.56)$$

where  $\kappa_0 = \kappa_0(\tau)$  is an invertible function whose inverse is given by a  $C^\infty$ -function, namely,  $\{\kappa_0\}^{-1} : [\kappa_c, +\infty) \rightarrow [0, +\infty)$ .

*Proof.* The proof is similar as the proof of [Theorem 4.4.15](#).  $\square$

#### 4.5.6 Geometric properties of the curves $(\mathcal{C}_0)$

**Theorem 4.5.21**

For all  $\rho, \underline{\Omega}_m > 0$ ,  $\underline{\omega}_0^*$  has the following properties:

- (a)  $\underline{\omega}_0^*$  is continuously differentiable at  $\kappa = \kappa_c$  and  $(\underline{\omega}_0^*)'(\kappa_c) = 1$ ;
- (b) exactly one of the following holds:
  - (b.1) if  $\rho \geq 1$ , then  $\underline{\omega}_0^*$  has a unique critical point  $\kappa_0^* \in (\kappa_c, +\infty)$  such that  $(\underline{\omega}_0^*)'(\kappa) > 0$  (resp.  $(\underline{\omega}_0^*)'(\kappa) < 0$ ) in  $(\kappa_c, \kappa_0^*)$  (resp.  $(\kappa_0^*, +\infty)$ );
  - (b.2) if  $\rho < 1$  and  $S_{0,\rho,\underline{\Omega}_m}(\tau_c) > 0$  then  $(\underline{\omega}_0^*)'(\kappa) > 0$  for all  $\kappa > \kappa_c$ ;
  - (b.3) if  $\rho < 1$  and  $S_{0,\rho,\underline{\Omega}_m}(\tau_c) = 0$ , then  $(\underline{\omega}_0^*)'(\kappa) > 0$  for  $\kappa \neq \kappa_0(\tau_c)$  and  $(\underline{\omega}_0^*)'(\kappa_0(\tau_c)) = 0$ ;
  - (b.4) if  $\rho < 1$  and  $S_{0,\rho,\underline{\Omega}_m}(\tau_c) < 0$ , then  $\underline{\omega}_0(\cdot)$  has exactly two critical points,  $\kappa_0^{(1)} < \kappa_0^{(2)}$  such that  $(\underline{\omega}_0^*)'(\kappa) > 0$  (resp.  $(\underline{\omega}_0^*)'(\kappa) < 0$ ) in  $(\kappa_c, \kappa_0^{(1)}) \cup (\kappa_0^{(2)}, +\infty)$  (resp.  $(\kappa_0^{(1)}, \kappa_0^{(2)})$ );
- (c) for all  $\kappa \geq \kappa_c$ ,  $\underline{\omega}_1^*(\kappa) < \underline{\omega}_0^*(\kappa) < \underline{\Omega}_m$ ;
- (d) [Asymptotic behavior for long frequencies]

$$\underline{\omega}_1^*(\kappa) = \frac{\underline{\Omega}_m}{\sqrt{2}} + \Omega_\infty \kappa^{-2} + (2\Omega_\infty K_\infty + \gamma_\infty) \kappa^{-4} + o(\kappa^{-4}), \quad \text{as } \kappa \rightarrow +\infty,$$

where  $\Omega_\infty, K_\infty$  and  $\gamma_\infty$  were defined in (4.5.45) and (4.5.46) (see Lemma 4.5.19).

*Proof.* (a) From chain's rule derivation and the inverse derivative formula one gets that

$$(\underline{\omega}_0^*)'(\kappa) = \frac{\underline{\omega}'_0(\tau)}{\kappa'_0(\tau)}, \quad \text{where } \kappa_0(\tau) = \kappa. \quad (4.5.57)$$

From (4.2.18) and replacing  $B'_0(\tau) = -2\tau$  we have that

$$2\kappa_0(\tau)\kappa'_0(\tau) = 2\tau + \mathcal{J}'(\underline{\omega}_0(\tau))\underline{\omega}'_0(\tau). \quad (4.5.58)$$

Let us first notice that, since  $(\tau^2 \tanh^2 \tau)'|_{\tau=0} = 0$ , then from (4.5.9) we have  $\underline{\omega}'_0(0) = 0$ , and consequently from (4.5.58),  $\kappa'_0(0) = 0$ . Then we need to study (4.5.57) by taking the limit when  $\tau \rightarrow 0^+$ . By substituting (4.5.58) in (4.5.57) we obtain

$$(\underline{\omega}_0^*)'(\kappa) = 2\kappa_0(\tau) \frac{\underline{\omega}'_0(\tau)}{2\tau + \mathcal{J}'(\underline{\omega}_0(\tau))\underline{\omega}'_0(\tau)}, \quad (4.5.59)$$

From (4.5.9) we can deduce that

$$\lim_{\tau \rightarrow 0^+} \frac{\underline{\omega}'_0(\tau)}{2\tau} = \lim_{\tau \rightarrow 0^+} \frac{1 - \tanh \tau (\tanh \tau + \tau \operatorname{sech}^2 \tau)' v(\underline{\omega}_0(\tau))}{(\tau^2 \tanh^2 \tau)' v(\underline{\omega}_0(\tau)) - u'(\underline{\omega}_0(\tau))} = \frac{1}{-u'(\underline{\omega}_0(0))} \neq 0 \quad (4.5.60)$$

where the last limit is well determined since  $u' < 0$  (see (4.2.2)). Henceforth

$$\lim_{\kappa \rightarrow \kappa_c^+} (\underline{\omega}_0^*)'(\kappa) = \lim_{\tau \rightarrow 0^+} 2\kappa_0(\tau) \frac{1}{2\tau/\underline{\omega}'_0(\tau) + \mathcal{J}'(\underline{\omega}_0(\tau))} = 2\kappa_c \frac{1}{-u'(\kappa_c) + \mathcal{J}'(\kappa_c)} = 1, \quad (4.5.61)$$

where it was used that  $\underline{\omega}_0(0) = \kappa_c$  and  $u'(\underline{\omega}) = \mathcal{J}'(\underline{\omega}) - 2\underline{\omega}$ .

- (b) From (4.5.57) and Lemma 4.5.18, it follows that  $(\underline{\omega}_n^*)'(\kappa)$  and  $\underline{\omega}'_n(\tau)$  have the same sign, where  $\kappa_0(\tau) = \kappa$ . Therefore, this clause is a direct consequence of 4.5.3 and in this case  $\tau_c$  is a critical point of  $\underline{\omega}_0(\cdot)$  if and only if  $k = \kappa_0(\tau_c)$  is a critical point of  $\underline{\omega}_0^*(\cdot)$ .
- (c) The relation  $\underline{\omega}_0^*(k) < \underline{\Omega}_m$  is readily seen from the definition of  $\mathcal{C}_0$  and Theorem 4.5.20. For the last inequality, consider  $\kappa \geq \kappa_c$ .

First, if  $\kappa < \kappa_{\text{int}}$ , then  $s := \{\kappa_1\}^{-1}(\kappa) < \{\kappa_1\}^{-1}(\kappa_{\text{int}}) = 1$  and  $\underline{\omega}_1(s) = \underline{\omega}_1^*(\kappa)$ . The fact that  $s < 1$  implies that  $\underline{\omega}_1(s) = \underline{\omega}_1^-(\nu^-(\tau))$  for some  $\tau > 0$  (see Theorem 4.4.10), hence

$$(\kappa, \underline{\omega}_0^*(k)) = (\kappa_1^-(\tau), \underline{\omega}_1^-(\tau)) \in \mathcal{C}_1^- \subset \mathcal{N}^-,$$

which implies that

$$\underline{\omega}_1^*(k) < \Phi(\kappa), \quad (4.5.62)$$

but  $(\kappa, \underline{\omega}_0^*(\kappa)) \in \mathcal{C}_0 \subset \mathcal{N}^+$ , which leads to

$$\Phi(\kappa) < \underline{\omega}_0^*(\kappa). \quad (4.5.63)$$

From (4.5.62) and (4.5.63) follows that  $\underline{\omega}_1^*(\kappa) < \underline{\omega}_0^*(\kappa)$ .

If  $\kappa = \kappa_{\text{int}}$ , then it follows necessarily that

$$(\kappa, \underline{\omega}_0^*(k)) = (\kappa_{\text{int}}, \underline{\omega}_{\text{int}}) \in \mathcal{N}_0,$$

hence  $\underline{\omega}_0^*(k) = \Phi(\kappa) < \underline{\omega}_0^*(\kappa)$ .

Finally, we treat the case  $\kappa_{\text{int}} < \kappa$ . This case is equivalent to prove that the curves  $\mathcal{C}_0$  and  $\mathcal{C}_1^+$  does not intersect themselves, indeed, we have that starting from the depart point  $\kappa = \kappa_{\text{int}}$  one gets (see the proven above)

$$\underline{\omega}_1^*(\kappa_{\text{int}}) < \underline{\omega}_0^*(\kappa_{\text{int}}),$$

hence if for some  $\kappa_* > \kappa_{\text{int}}$  it happens that  $\underline{\omega}_1^*(\kappa_*) < \underline{\omega}_0^*(\kappa_*)$  then we can use the *Intermediate Value Theorem* to prove that there exists  $\kappa_{**} \in (\kappa_{\text{int}}, \kappa_*)$  such that  $\underline{\omega}_1^*(\kappa_{**}) = \underline{\omega}_0^*(\kappa_{**})$ , meaning that  $\mathcal{C}_0$  and  $\mathcal{C}_1^+$  intersect themselves in some point. We will prove that the latter cannot happen. We proceed by a contradiction reasoning, let us assume that there exists  $k_{**} > \kappa_{\text{int}}$  such that  $\underline{\omega}_{**} := \underline{\omega}_1^*(k_{**}) = \underline{\omega}_0^*(k_{**})$ , then we can write  $\kappa = \kappa_0(\tau)$  for some  $\tau > 0$  and  $\kappa = \kappa_1^+(s)$  for some  $s > 0$ . The latter implies that  $(\kappa, V)$  satisfies (4.1.55) for  $s > 0$  and  $(\kappa, \underline{\omega}_{**})$  satisfies (4.1.48) for  $\tau > 0$ :

$$\left| \begin{array}{l} (\varepsilon_r(\underline{\omega}_{**}) \mu_r(\underline{\omega}_{**}) - 1) \underline{\omega}_{**}^2 = s^2 \coth^2 s (\mu_r(\underline{\omega}_{**}))^{-2} - s^2, \end{array} \right. \quad (4.5.64a)$$

$$\left| \begin{array}{l} \kappa^2 = \underline{\omega}_{**}^2 + s^2 \coth^2 s (\mu_r(\underline{\omega}_{**}))^{-2}, \end{array} \right. \quad (4.5.64b)$$

$$\left| \begin{array}{l} (\varepsilon_r(\underline{\omega}_0) \mu_r(\underline{\omega}_0) - 1) \underline{\omega}_{**}^2 = \tau^2 \tanh^2 \tau (\mu_r(\underline{\omega}_{**}))^{-2} - \tau^2, \end{array} \right. \quad (4.5.65a)$$

$$\left| \begin{array}{l} \kappa^2 = \underline{\omega}_{**}^2 + \tau^2 \tanh^2 \tau (\mu_r(\underline{\omega}_0))^{-2}. \end{array} \right. \quad (4.5.65b)$$

By combining, (4.5.64a) and (4.5.64b) we obtain

$$\kappa^2 = \mathcal{F}(\underline{\omega}_{**}) + s^2,$$

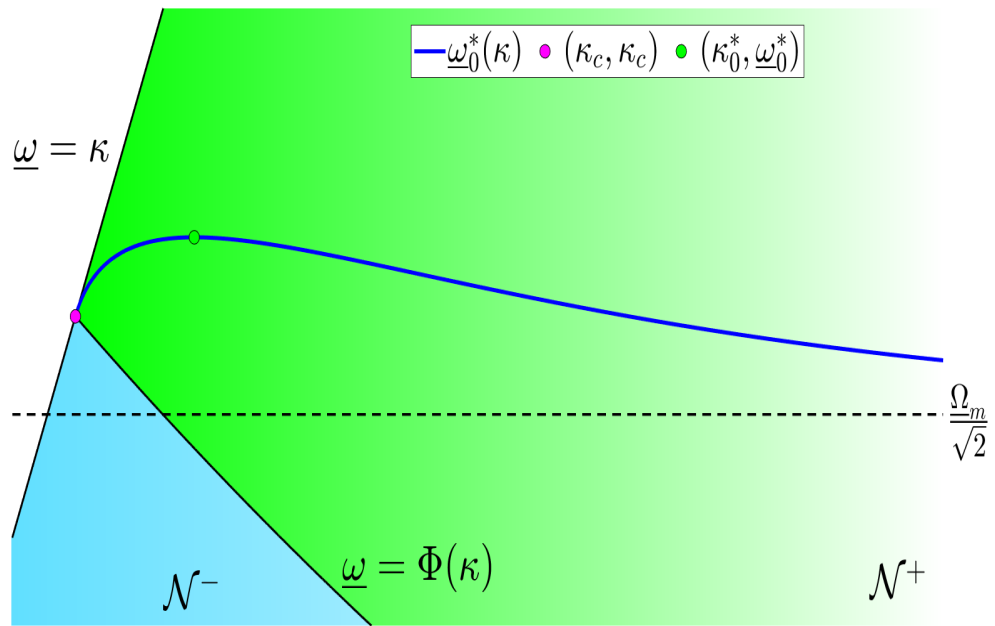


where we remind that  $\mathcal{F}(\underline{\omega}) := \underline{\omega}^2 \varepsilon_r(\underline{\omega}) \mu_r(\underline{\omega})$ , , we derive analogously that

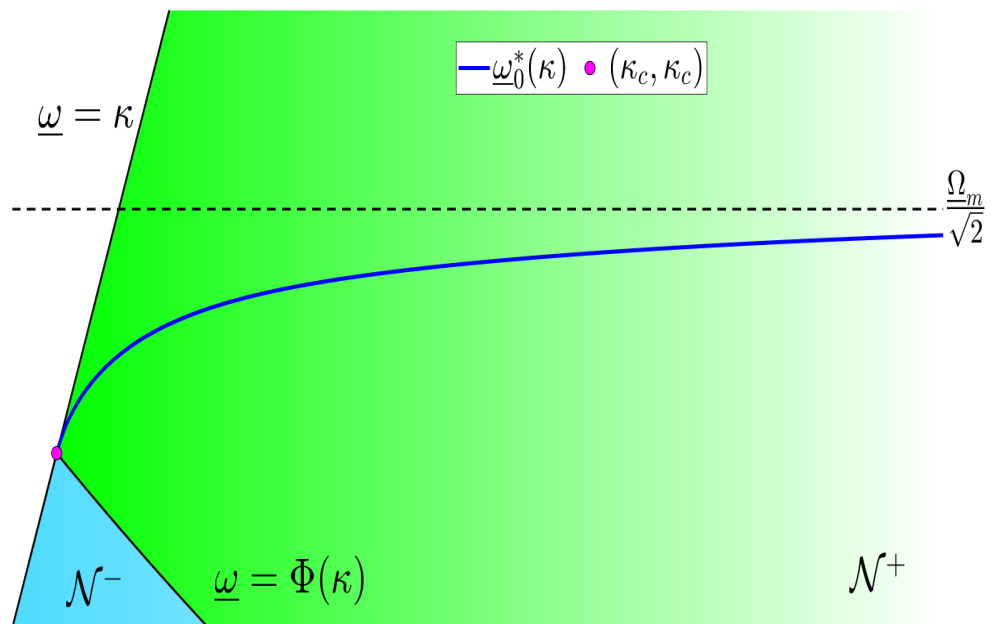
$$\kappa^2 = \mathcal{F}(\underline{\omega}_{**}) + \tau^2.$$

The latter two equations imply that  $s = \tau$ , hence by using (4.5.64a) and (4.5.65a), one deduces that  $\coth^2 s = \tanh^2 s$  which is impossible since  $\tanh x < 1 < \coth x$  for  $x > 0$ . This finishes the proof of this clause.

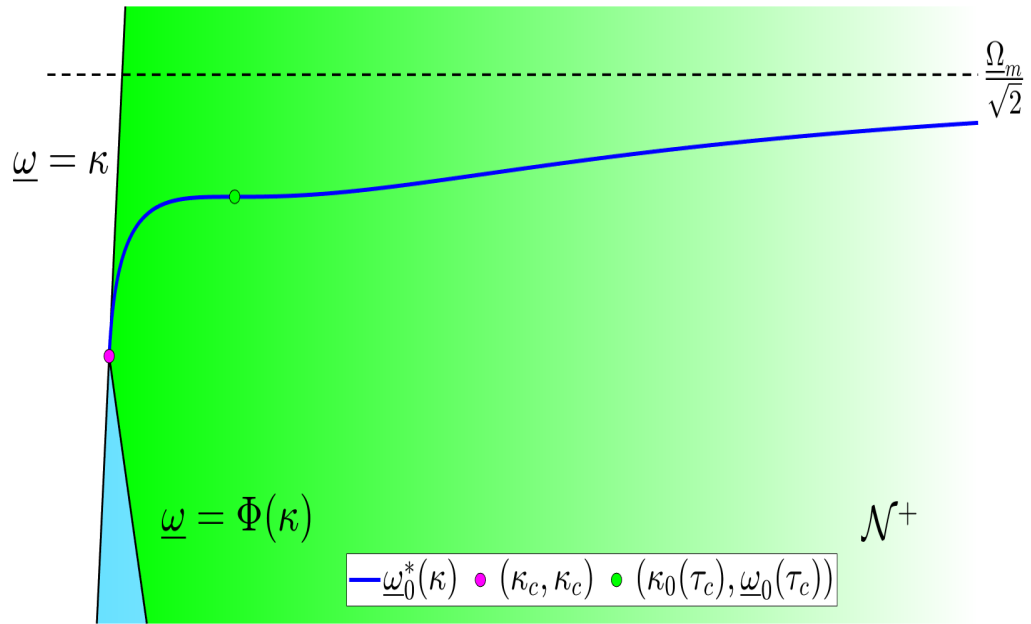
- (d) By using the asymptotic behavior of  $\underline{\omega}_0$  and  $\kappa_0$  as  $\tau \rightarrow +\infty$  (see Lemma 4.5.19), one can replicate the proof of the clause (d) in Theorem 4.4.16 to conclude this one. □



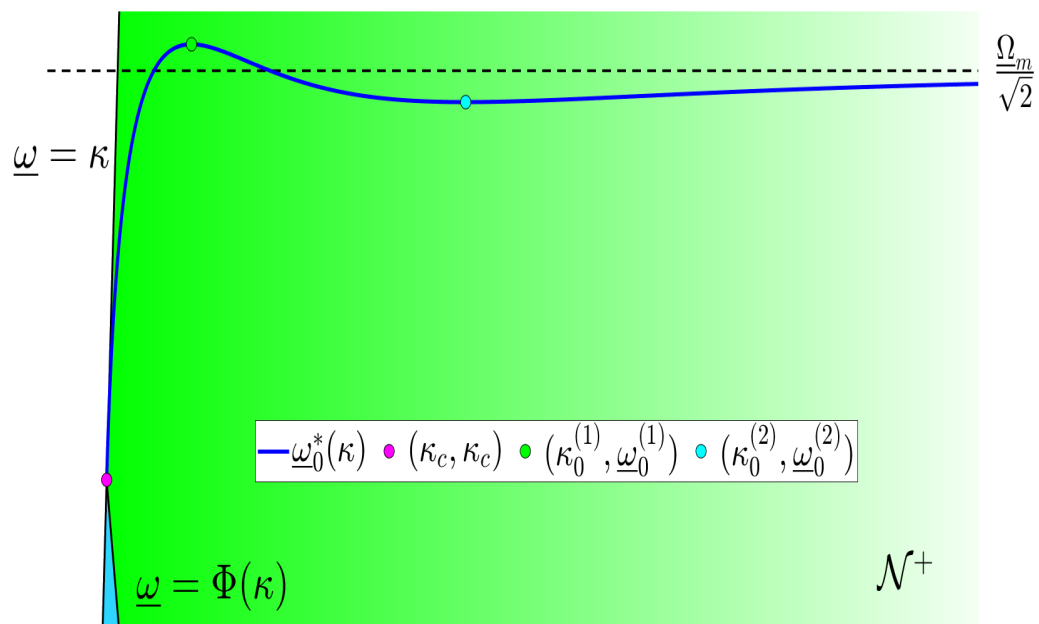
(a)  $\underline{\Omega}_m = 1.0000, \rho = 1.1000$ .



(b)  $\underline{\Omega}_m = 1.0000, \rho = 0.5566,$   
 $S_{1,\rho,\underline{\Omega}_m}(\tau_c) = 0.1438.$



(c)  $\Omega_m = 1.0000, \rho = 0.9527,$   
 $S_{1,\rho,\Omega_m}(\tau_c) = 0.$



(d)  $\Omega_m = 2.0000, \rho = 0.9571,$   
 $S_{1,\rho,\Omega_m}(\tau_c) = -0.0831.$

Figure 4.11: Graphic representation of the function  $\underline{\omega}_0^*(\cdot)$ .

## Appendix C

### C.1. Real analysis tools

Define

$$\begin{aligned}\eta_1(\tau) &:= \frac{\pi}{2} - \arctan \tau, & \tau \geq 0, \\ \alpha_1^-(\tau) &:= \left( (\tau + \tau^3) \left( \eta_1(\tau) - \frac{\tau}{1+\tau^2} \right) \right)^{-1}, & \tau > 0, \\ \beta_1^-(\tau) &:= \frac{\eta_1^3(\tau)}{\eta_1(\tau) - \frac{\tau}{1+\tau^2}}, & \tau \geq 0,\end{aligned}\tag{4.5.66}$$

and for  $p > 1$  define the following functions

$$\begin{aligned}\eta_p(\tau) &:= \frac{p\pi}{2} - \arctan \tau, & \tau \geq 0, \\ \alpha_p(\tau) &:= \left( (\tau + \tau^3) \left( \eta_p(\tau) - \frac{\tau}{1+\tau^2} \right) \right)^{-1}, & \tau > 0, \\ \beta_p(\tau) &:= \frac{\eta_p^3(\tau)}{\eta_p(\tau) - \frac{\tau}{1+\tau^2}}, & \tau \geq 0.\end{aligned}\tag{4.5.67}$$

#### Lemma C. 1

The following properties hold for all  $p > 1$ :

- (i)  $\eta_p(\tau) \geq \eta_1(\tau) > \frac{3\tau}{1+3\tau^2} \geq \frac{\tau}{1+\tau^2}$  for all  $\tau \geq 0$ .
- (ii)  $\alpha_1^-$  and  $\alpha_p$  are strictly decreasing positive functions in  $\mathbb{R}^+$ .
- (iii)  $\lim_{\tau \rightarrow 0^+} \alpha_1^-(\tau) = \lim_{\tau \rightarrow 0^+} \alpha_p(\tau) = +\infty$ .
- (iv)  $\lim_{\tau \rightarrow +\infty} \alpha_1^-(\tau) = \frac{3}{2}$  and  $\lim_{\tau \rightarrow +\infty} \alpha_p(\tau) = 0$ .
- (v)  $\beta_1^-(0) = \left(\frac{\pi}{2}\right)^2$  and  $\beta_p(0) = \left(\frac{p\pi}{2}\right)^2$ .
- (vi)  $\lim_{\tau \rightarrow +\infty} \beta_1^-(\tau) = \frac{3}{2}$  and  $\lim_{\tau \rightarrow +\infty} \beta_p(\tau) = \left(\frac{(p-1)\pi}{2}\right)^2$ .

*Proof.* (i) An elementary computation shows that  $\frac{3\tau}{1+3\tau^2} \geq \frac{\tau}{1+\tau^2}$  for  $\tau \geq 0$ . On the other side, the relation  $\eta_p(\cdot) \geq \eta_1(\cdot)$  derives directly from the definition. Let us consider the function  $a(\tau) := \eta_1(\tau) - \frac{3\tau}{1+3\tau^2}$ . It is easy to note that

$$a(0) = \frac{\pi}{2} > 0 \quad \text{and} \quad \lim_{\tau \rightarrow +\infty} a(\tau) = 0.$$

Moreover, by differentiating  $a(\cdot)$  we obtain

$$\begin{aligned}a'(\tau) &= -\frac{1}{1+\tau^2} - \frac{3-9\tau^2}{(1+3\tau^2)^2} = -\frac{(1+3\tau^2)^2 + (1+\tau^2)(3-9\tau^2)}{(1+\tau^2)(1+3\tau^2)^2} \\ &= -\frac{(1+6\tau^2+9\tau^4) + (3-6\tau^2-9\tau^4)}{(1+\tau^2)(1+3\tau^2)^2} = -\frac{4}{(1+\tau^2)(1+3\tau^2)^2} < 0, \quad \text{for all } \tau \geq 0,\end{aligned}$$

hence  $a(\cdot)$  defines a strictly decreasing and consequently we have that  $a(\cdot) > 0$ , concluding

the proof of this clause.

(ii) Let us define  $p \geq 1$ ,

$$\begin{aligned} Q_p(\tau) &:= \alpha_p^{-1}(\tau) \\ &= (\tau + \tau^3) \left( \eta_p(\tau) - \frac{\tau}{1 + \tau^2} \right), \end{aligned}$$

so this clause is equivalent to prove that  $Q_p$  is a strictly increasing positive function in  $\mathbb{R}^+$ . Observe that

$$\begin{aligned} Q_p(\tau) &= (\tau + \tau^3) \left( \frac{(p-1)\pi}{2} + \eta_1(\tau) - \frac{\tau}{1 + \tau^2} \right) \\ &= \frac{(p-1)\pi}{2} (\tau + \tau^3) + Q_1(\tau), \end{aligned}$$

then it only suffices to prove the result for  $Q_1(\cdot)$ . Indeed, by inspecting the derivative of  $Q_1(\cdot)$  and using the proven in the clause (i), we have

$$\begin{aligned} Q_1'(\tau) &= (1 + 3\tau^2) \left( \eta_1(\tau) - \frac{\tau}{1 + \tau^2} \right) + (\tau + \tau^3) \left( -\frac{1}{1 + \tau^2} - \frac{1 - \tau^2}{(1 + \tau^2)^2} \right) \\ &= (1 + 3\tau^2) \left( \eta_1(\tau) - \frac{\tau}{1 + \tau^2} \right) - (\tau + \tau^3) \left( \frac{2}{(1 + \tau^2)^2} \right) \\ &= (1 + 3\tau^2) \left( \eta_1(\tau) - \frac{\tau}{1 + \tau^2} \right) - \frac{2\tau}{1 + \tau^2} \\ &> (1 + 3\tau^2) \left( \frac{3\tau}{1 + 3\tau^2} - \frac{\tau}{1 + \tau^2} \right) - \frac{2\tau}{1 + \tau^2} \\ &= (1 + 3\tau^2) \frac{(3\tau)(1 + \tau^2) - \tau(1 + 3\tau^2)}{(1 + 3\tau^2)(1 + \tau^2)} - \frac{2\tau}{1 + \tau^2} \\ &= (1 + 3\tau^2) \frac{2\tau}{(1 + 3\tau^2)(1 + \tau^2)} - \frac{2\tau}{1 + \tau^2} = 0, \end{aligned}$$

where the inequality is strict for  $\tau > 0$ . This proves that  $Q_1$  is strictly increasing in  $\mathbb{R}^+$  and since  $Q_1(0) = 0$ , hence it is positive for  $\tau > 0$ .

(iii) For  $p \geq 1$ , it is clear that  $Q_p$  is continuously well defined at  $\tau = 0$  and  $Q_p(0) = 0$ , hence it follows this clause.

(iv) Note that  $\lim_{\tau \rightarrow +\infty} \left( \eta_1(\tau) - \frac{\tau}{1 + \tau^2} \right) = \lim_{\tau \rightarrow +\infty} \frac{1}{\tau + \tau^3} = 0$  and since  $\left( \frac{1}{\tau + \tau^3} \right)' = -\frac{1 + 3\tau^2}{(\tau + \tau^3)^2} \neq 0$  in  $\mathbb{R}^+$ , then we can use L'Hôpital's rule to the function  $\left( \eta_1(\tau) - \frac{\tau}{1 + \tau^2} \right) / \left( \frac{1}{\tau + \tau^3} \right)$  having

$$\begin{aligned} \lim_{\tau \rightarrow +\infty} Q_1(\tau) &= \lim_{\tau \rightarrow +\infty} \left( \eta_1(\tau) - \frac{\tau}{1 + \tau^2} \right)' / \left( \frac{1}{\tau + \tau^3} \right)' \\ &= \lim_{\tau \rightarrow +\infty} \left( -\frac{2}{(1 + \tau^2)^2} \right) / \left( -\frac{1 + 3\tau^2}{(\tau + \tau^3)^2} \right) \\ &= \lim_{\tau \rightarrow +\infty} \frac{2\tau^2}{1 + 3\tau^2} = \frac{2}{3}. \end{aligned}$$

The result for  $p > 1$  follows directly from the fact that  $Q_p(\tau) = \frac{(p-1)\pi}{2} (\tau + \tau^3) + Q_1(\tau)$ .

(v) One can readily compute from the definition and since for every  $p \geq 1$ ,  $\eta_p^2(0) = \left(\frac{p\pi}{2}\right)^2$ .

(vi) For  $p > 1$  one has  $\lim_{\tau \rightarrow +\infty} \eta_p(\tau) = \frac{(p-1)\pi}{2} > 0$ , then it easily follows

$$\lim_{\tau \rightarrow +\infty} \frac{\eta_p^3(\tau)}{\eta_p(\tau) - \frac{\tau}{1+\tau^2}} = \frac{((p-1)\pi/2)^3}{(p-1)\pi/2} = \left(\frac{(p-1)\pi}{2}\right)^2.$$

On the other hand, we can repeatedly apply again L'Hôpital's rule to the quotient  $\frac{\eta_1^3(\tau)}{\eta_1(\tau) - \frac{\tau}{1+\tau^2}}$

since  $\lim_{\tau \rightarrow +\infty} \eta_1^3(\tau) = \lim_{\tau \rightarrow +\infty} \left(\eta_1(\tau) - \frac{\tau}{1+\tau^2}\right) = 0$  and  $\left(\eta_1(\tau) - \frac{\tau}{1+\tau^2}\right)' = -\frac{2}{(1+\tau^2)^2} \neq 0$  :

$$\begin{aligned} \lim_{\tau \rightarrow +\infty} \frac{\eta_1^3(\tau)}{\eta_1(\tau) - \frac{\tau}{1+\tau^2}} &= \lim_{\tau \rightarrow +\infty} \frac{-3\eta_1^2(\tau) \frac{1}{1+\tau^2}}{\left(-\frac{2}{(1+\tau^2)^2}\right)} \\ &= \lim_{\tau \rightarrow +\infty} 3\eta_1^2(\tau) \left/ \left(\frac{2}{1+\tau^2}\right)\right. \\ &= \lim_{\tau \rightarrow +\infty} 6\eta_1(\tau) \left(-\frac{1}{1+\tau^2}\right) \left/ \left(\frac{-4\tau}{(1+\tau^2)^2}\right)\right. \\ &= \lim_{\tau \rightarrow +\infty} 3\eta_1(\tau) \left/ \left(\frac{2\tau}{1+\tau^2}\right)\right. \\ &= \lim_{\tau \rightarrow +\infty} 3 \left(-\frac{1}{1+\tau^2}\right) \left/ \left(-\frac{2(\tau^2-1)}{(1+\tau^2)^2}\right)\right. \\ &= \lim_{\tau \rightarrow +\infty} \frac{3(1+\tau^2)}{2(\tau^2-1)} = \frac{3}{2}. \end{aligned}$$

□

Let us consider the functions defined in  $\mathbb{R}^+ = (0, +\infty)$  :

$$\begin{aligned} \alpha_0(\tau) &:= (\tanh \tau (\tanh \tau + \tau \operatorname{sech}^2 \tau))^{-1}, \quad \tau > 0, \\ \beta_0(\tau) &:= -\frac{\tau^3 \operatorname{sech}^2 \tau}{\tanh \tau + \tau \operatorname{sech}^2 \tau}, \quad \tau > 0, \\ \alpha_1^+(\tau) &:= (\coth \tau (\coth \tau - \tau \operatorname{csch}^2 \tau))^{-1}, \quad \tau > 0, \\ \beta_1^+(\tau) &:= \frac{\tau^3 \operatorname{csch}^2 \tau}{\coth \tau - \tau \operatorname{csch}^2 \tau}, \quad \tau > 0. \end{aligned} \tag{4.5.68}$$

**Lemma C. 2**

The following properties hold:

- (i)  $\alpha_0$  is positive in  $\mathbb{R}^+$  and has an unique positive critical point,  $\tau_c > 0$ . Moreover  $\alpha'_0(\cdot) < 0$  in  $(0, \tau_c)$  and  $\alpha'_0(\cdot) > 0$  in  $(\tau_c, +\infty)$ . In this case,  $\alpha_0(\tau_c) < 1$ .
- (ii)  $\lim_{\tau \rightarrow 0^+} \alpha_0(\tau) = +\infty$  and  $\lim_{\tau \rightarrow +\infty} \alpha_0(\tau) = 1$ .
- (iii)  $\lim_{\tau \rightarrow 0^+} \beta_0(\tau) = 0$  and  $\lim_{\tau \rightarrow +\infty} \beta_0(\tau) = 0$ .
- (iv)  $\alpha_1^+$  is a strictly decreasing positive function in  $\mathbb{R}^+$ .
- (v)  $\lim_{\tau \rightarrow 0^+} \alpha_1^+(\tau) = \frac{3}{2}$  and  $\lim_{\tau \rightarrow +\infty} \alpha_1^+(\tau) = 1$ .
- (vi)  $\lim_{\tau \rightarrow 0^+} \beta_1^+(\tau) = \frac{3}{2}$  and  $\lim_{\tau \rightarrow +\infty} \beta_1^+(\tau) = 0$ .

*Proof.* (i) Since  $\tanh \tau (\tanh \tau + \tau \operatorname{sech}^2 \tau) > 0$  ( $\tau > 0$ ), then  $\alpha_0$  is positive in  $\mathbb{R}^+$ , by computing the first derivative,

$$\begin{aligned}
 (\alpha_0)'(\tau) &= -\frac{\operatorname{sech}^2 \tau (\tanh \tau + \tau \operatorname{sech}^2 \tau) + \tanh \tau (\operatorname{sech}^2 \tau + \operatorname{sech}^2 \tau - 2\tau \operatorname{sech}^2 \tau \tanh \tau)}{\tanh^2 \tau (\tanh \tau + \tau \operatorname{sech}^2 \tau)^2} \\
 &= -\frac{\operatorname{sech}^2 \tau (3 \tanh \tau + \tau \operatorname{sech}^2 \tau - 2\tau \tanh^2 \tau)}{\tanh^2 \tau (\tanh \tau + \tau \operatorname{sech}^2 \tau)^2} \\
 &= -\frac{\operatorname{sech}^2 \tau (\tau + 3 \tanh \tau - 3\tau \tanh^2 \tau)}{\tanh^2 \tau (\tanh \tau + \tau \operatorname{sech}^2 \tau)^2} \\
 &= -\frac{\tau \operatorname{sech}^2 \tau}{\tanh^2 \tau (\tanh \tau + \tau \operatorname{sech}^2 \tau)^2} a(\tau), \tag{4.5.69}
 \end{aligned}$$

where it was used the identity  $\operatorname{sech}^2(\cdot) + \tanh^2(\cdot) = 1$  and we have defined

$$a(\tau) := 1 + 3 \frac{\tanh \tau}{\tau} - 3 \tanh^2 \tau.$$

Since

$$\frac{\operatorname{sech}^2 \tau}{\tau \tanh^2 \tau (\tanh \tau + \tau \operatorname{sech}^2 \tau)^2} > 0, \quad \tau > 0,$$

then the sign of  $(\alpha_0)'(\tau)$  depends entirely of the sign of  $a(\tau)$ , we focus in this last function. One can prove using L'Hôpital's rule, that  $\lim_{\tau \rightarrow 0} a(\tau) = 4$  and it is readily seen that

$\lim_{\tau \rightarrow +\infty} a(\tau) = -2$ . Moreover, by inspecting the derivative we have

$$a'(\tau) = -3 \left( \frac{\tanh \tau - \tau \operatorname{sech}^2 \tau}{\tau^2} + 2 \tanh \tau \operatorname{sech}^2 \tau \right) < 0, \quad \text{for } \tau > 0,$$

the last derivative being negative since the inequality  $\sinh(2\tau) > 2\tau, \tau > 0$  implies

$$\begin{aligned} \sinh(2\tau) > 2\tau &\implies 2 \sinh \tau \cosh \tau > 2\tau \\ &\implies \tanh \tau = (2 \sinh \tau \cosh \tau) \frac{\operatorname{sech}^2 \tau}{2} > \frac{2\tau \operatorname{sech}^2 \tau}{2} = \tau \operatorname{sech}^2 \tau. \end{aligned}$$

This shows that  $a(\cdot)$  is strictly decreasing in  $\mathbb{R}^+$ , thus it has an unique positive zero, let us name it  $\tau_c > 0$ , by adding the last fact with (4.5.69) we conclude the proof of this clause:

$$\begin{cases} (\alpha_0)'(\tau) < 0 &\iff a(\tau) > 0 &\iff \tau \in (0, \tau_c), \\ (\alpha_0)'(\tau) = 0 &\iff a(\tau) = 0 &\iff \tau = \tau_c, \\ (\alpha_0)'(\tau) > 0 &\iff a(\tau) < 0 &\iff \tau \in (\tau_c, +\infty). \end{cases}$$

Finally, in order to prove that  $\alpha(\tau_c) < 1$ , we notice first that

$$\begin{aligned} \alpha_0(\tau_c) < 1 &\iff \tanh \tau_c (\tanh \tau_c + \tau_c \operatorname{sech}^2 \tau_c) > 1 \\ &\iff \tau_c \tanh \tau_c \operatorname{sech}^2 \tau_c > 1 - \tanh^2 \tau_c = \operatorname{sech}^2 \tau_c \\ &\iff r := \tau_c \tanh \tau_c > 1. \end{aligned}$$

So it only remains to show that  $r > 1$ , for that, since  $a(\tau_c) = 0$ , it follows that

$$\tau_c^2 + 3r - 3r^2 = 0,$$

therefore

$$3r(r-1) = \tau_c^2 > 0 \implies r > 1.$$

(ii) Since  $\lim_{\tau \rightarrow 0^+} \tanh \tau (\tanh \tau + \tau \operatorname{sech}^2 \tau) = 0^+$ , it follows  $\lim_{\tau \rightarrow 0^+} \alpha_0(\tau) = +\infty$ .

On the other hand, as  $\operatorname{sech} \tau \sim 2 \exp(-\tau)$  as  $\tau \rightarrow +\infty$ ,

$$\lim_{\tau \rightarrow +\infty} \tau \operatorname{sech}^2 \tau = 0,$$

and since  $\lim_{\tau \rightarrow +\infty} \tanh(\tau) = 1$ , then it is readily seen that  $\lim_{\tau \rightarrow +\infty} \alpha_0(\tau) = 1$ .

(iii) From the limit  $\lim_{\tau \rightarrow 0} (\tau / \tanh \tau) = 1$ , one derives

$$\lim_{\tau \rightarrow 0} \beta_0(\tau) = \lim_{\tau \rightarrow 0} \frac{-(\tau / \tanh \tau) (\tau \operatorname{sech} \tau)^2}{1 + (\tau / \tanh \tau) \operatorname{sech}^2 \tau} = -\frac{(1)(0)}{1 + (1)(1)} = 0.$$

On another hand, since and  $\operatorname{sech} \tau \sim 2 \exp(-\tau)$  as  $\tau \rightarrow +\infty$ , then  $\lim_{\tau \rightarrow +\infty} \tau \operatorname{sech}^2 \tau = \lim_{\tau \rightarrow +\infty} \tau^3 \operatorname{sech}^2 \tau = 0$ . The last added to the fact  $\lim_{\tau \rightarrow +\infty} \tanh \tau = 1$  lead us to  $\lim_{\tau \rightarrow +\infty} \beta_0(\tau) = 0$ .

(iv) The inequality  $\sinh(2\tau) > 2\tau$  ( $\tau > 0$ ) assures us that

$$\begin{aligned} \sinh(2\tau) > 2\tau &\implies \sinh \tau \cosh \tau = \frac{\sinh(2\tau)}{2} > \tau \\ &\implies \coth \tau = \frac{\cosh \tau}{\sinh \tau} = \sinh \tau \cosh \tau \operatorname{csch}^2 \tau > \tau \operatorname{csch}^2 \tau. \end{aligned}$$



This guarantees us that  $\alpha_1^+$  is positive in  $\mathbb{R}^+$ . On another hand, by inspecting its derivative we have

$$\begin{aligned} (\alpha_1^+)'(\tau) &= -\frac{-\operatorname{csch}^2 \tau (\coth \tau - \tau \operatorname{csch}^2 \tau) + \coth \tau (-\operatorname{csch}^2 \tau - \operatorname{csch}^2 \tau + 2\tau \coth \tau \operatorname{csch}^2 \tau)}{\coth^2 \tau (\coth \tau - \tau \operatorname{csch}^2 \tau)^2} \\ &= -\frac{\operatorname{csch}^2 \tau (\tau \operatorname{csch}^2 \tau + 2\tau \coth^2 \tau - 3 \coth \tau)}{\coth^2 \tau (\coth \tau - \tau \operatorname{csch}^2 \tau)^2} \\ &= -\frac{\operatorname{csch}^2 \tau (3\tau \coth^2 \tau - 3 \coth \tau - \tau)}{\coth^2 \tau (\coth \tau - \tau \operatorname{csch}^2 \tau)^2} \\ &= -\frac{\operatorname{csch}^2 \tau}{(\coth \tau - \tau \operatorname{csch}^2 \tau)^2} b(\tau), \end{aligned}$$

where it was used the identity  $\coth^2(\cdot) - \operatorname{csch}^2(\cdot) = 1$  and we have defined

$$b(\tau) := \frac{3\tau \coth^2 \tau - 3 \coth \tau - \tau}{\coth^2 \tau} = 3(\tau - \tanh \tau) - \tau \tanh^2 \tau.$$

Since

$$\frac{\operatorname{csch}^2 \tau}{\coth \tau (\coth \tau - \tau \operatorname{csch}^2 \tau)^2} > 0, \quad \tau > 0,$$

then it suffices to show that  $b(\cdot)$  is a positive function in  $\mathbb{R}^+$ .

$$\begin{aligned} b'(\tau) &= 3(1 - \operatorname{sech}^2 \tau) - (\tanh^2 \tau + 2\tau \tanh \tau \operatorname{sech}^2 \tau) \\ &= 3 \tanh^2 \tau - \tanh^2 \tau - 2\tau \tanh \tau \operatorname{sech}^2 \tau \\ &= 2 \tanh \tau (\tanh \tau - \tau \operatorname{sech}^2 \tau) > 0, \quad \tau > 0 \end{aligned}$$

where we have used the inequality  $\tanh \tau > \tau \operatorname{sech}^2 \tau$  ( $\tau > 0$ ), proved in the clause (i). Therefore,  $b(\cdot)$  is strictly increasing in  $\mathbb{R}^+$  and since  $b(0) = 0$ , it follows our result.

(v) We use repeatedly L'Hôpital's rule as follows

$$\begin{aligned} \lim_{\tau \rightarrow 0} \alpha_1^+(\tau) &= \lim_{\tau \rightarrow 0} \frac{\sinh^3 \tau}{\sinh^3 \tau} \frac{1}{\coth \tau (\coth \tau - \tau \operatorname{csch}^2 \tau)} \\ &= \lim_{\tau \rightarrow 0} \frac{\sinh^3 \tau}{\cosh \tau (\frac{1}{2} \sinh(2\tau) - \tau)} \\ &= \lim_{\tau \rightarrow 0} \frac{1}{\cosh \tau} \cdot \lim_{\tau \rightarrow 0} \frac{\sinh^3 \tau}{\frac{1}{2} \sinh(2\tau) - \tau} \\ &= \lim_{\tau \rightarrow 0} \frac{3 \sinh^2 \tau \cosh(\tau)}{\cosh(2\tau) - 1} \\ &= \lim_{\tau \rightarrow 0} \cosh \tau \cdot \lim_{\tau \rightarrow 0} \frac{6 \sinh \tau \cosh \tau}{2 \sinh(2\tau)} = \lim_{\tau \rightarrow 0} \frac{3 \sinh(2\tau)}{2 \sinh(2\tau)} = \frac{3}{2}. \end{aligned}$$

On the another hand, by using  $\lim_{\tau \rightarrow +\infty} \tau \operatorname{csch}^2 \tau = 0$  and  $\lim_{\tau \rightarrow +\infty} \coth \tau = 1$ , we compute

$$\lim_{\tau \rightarrow +\infty} \alpha_1^+(\tau) = \lim_{\tau \rightarrow +\infty} \frac{1}{\coth \tau (\coth \tau - \tau \operatorname{csch}^2 \tau)} = 1.$$

(vi) Using L'Hôpital's rule,

$$\begin{aligned} \lim_{\tau \rightarrow 0} \beta_1^+(\tau) &= \lim_{\tau \rightarrow 0} \left( \frac{\sinh^2 \tau}{\sinh^2 \tau} \right) \left( \frac{\tau^3 \operatorname{csch}^2 \tau}{\coth \tau - \tau \operatorname{csch}^2 \tau} \right) \\ &= \lim_{\tau \rightarrow 0} \frac{\tau^3}{\frac{1}{2} \sinh(2\tau) - \tau} \\ &= \lim_{\tau \rightarrow 0} \frac{3\tau^2}{\cosh(2\tau) - 1} \\ &= \lim_{\tau \rightarrow 0} \frac{6\tau}{2 \sinh(2\tau)} = \lim_{\tau \rightarrow 0} \frac{6}{4 \cosh(2\tau)} = \frac{6}{4} = \frac{3}{2}. \end{aligned}$$

Once again, using the fact that  $\sinh(2\tau) \sim e^{2\tau}/2$  as  $\tau \rightarrow +\infty$ , we have that

$$\lim_{\tau \rightarrow +\infty} \sinh(2\tau)/\tau^3 = +\infty,$$

hence

$$\begin{aligned} \lim_{\tau \rightarrow +\infty} \beta_1^+(\tau) &= \lim_{\tau \rightarrow +\infty} \frac{\tau^3}{\frac{1}{2} \sinh(2\tau) - \tau} \\ &= \lim_{\tau \rightarrow +\infty} \frac{1}{\frac{1}{2} \sinh(2\tau)/\tau^3 - 1/\tau^2} = 0. \end{aligned}$$

□

**Lemma C. 3**

The following piece-wise functions are  $C^1$  in  $\mathbb{R}^+$ .

$$\left| \begin{aligned} A_1(s) &:= \begin{cases} A_1^-(\tan(\frac{\pi}{2}s)), & 0 < s < 1, \\ 1, & s = 1, \\ A_1^+(s-1), & s > 1, \end{cases} = \begin{cases} \left(\frac{\pi}{2}(1-s)\tan(\frac{\pi}{2}s)\right)^2, & 0 < s < 1, \\ 1, & s = 1, \\ \left((s-1)\coth(s-1)\right)^2, & s > 1, \end{cases} \\ B_1(s) &:= \begin{cases} B_1^-(\tan(\frac{\pi}{2}s)), & 0 < s < 1, \\ B_1^+(s-1), & s \geq 1, \end{cases} = \begin{cases} \left(\frac{\pi}{2}(1-s)\right)^2, & 0 < s < 1, \\ -(s-1)^2, & s \geq 1. \end{cases} \end{aligned} \right. \quad (4.5.70)$$

*Proof.* It is readily seen that  $A_1(\cdot)$  and  $B_1(\cdot)$  are  $C^1$  in the intervals  $(0, 1)$  and  $(1, +\infty)$ . Hence, it only remains to prove that they are continuous and differentiable at  $s = 1$ . It is not difficult to see that  $B_1(\cdot)$  is continuous at  $s = 1$  and  $B_1(1) = 0$ , moreover

$$B_1'(s) := \begin{cases} -\frac{\pi^2}{2}(1-s), & 0 < s < 1, \\ -2(s-1), & s > 1, \end{cases} \quad (4.5.71)$$

which is continuously extended at  $s = 1$  as  $B_1'(1) = 0$ . On the other side, by using L'Hôpital's

rule:

$$\begin{aligned} \lim_{s \rightarrow 1^-} \frac{\pi}{2} (1-s) \tan\left(\frac{\pi}{2}s\right) &= \lim_{s \rightarrow \frac{\pi}{2}^-} \frac{\left(\frac{\pi}{2} - s\right) \sin s}{\cos s} \\ &= \lim_{s \rightarrow \frac{\pi}{2}^-} \frac{-\sin s + \left(\frac{\pi}{2} - s\right) \cos s}{-\sin s} = 1 \\ \lim_{s \rightarrow 1^+} (s-1) \coth(s-1) &= \lim_{s \rightarrow 0^+} \frac{s \cosh s}{\sinh s} \\ &= \lim_{s \rightarrow 0^+} \frac{\cosh s + s \sinh s}{\cosh s} = 1, \end{aligned}$$

which proves that  $\lim_{s \rightarrow 1^-} A_1(s) = 1 = \lim_{s \rightarrow 1^+} A_1(s)$ , hence  $A_1(\cdot)$  is continuous at  $s = 1$ . Now by differentiating,

$$A_1'(s) := \begin{cases} 2\sqrt{A_1(s)} \left(\frac{\pi}{s} (1-s) \tan\left(\frac{\pi}{s}\right)\right)', & 0 < s < 1, \\ 2\sqrt{A_1(s)} \left((s-1) \coth(s-1)\right)', & s > 1, \end{cases} \quad (4.5.72)$$

then we proceed to calculate the following limits

$$\begin{aligned} \lim_{s \rightarrow 1^-} \left(\frac{\pi}{s} (1-s) \tan\left(\frac{\pi}{s}\right)\right)' &= \lim_{s \rightarrow 1^-} \frac{\pi}{2} \left(\frac{\pi}{2} (1-s) \sec^2\left(\frac{\pi}{2}s\right) - \tan\left(\frac{\pi}{2}s\right)\right)' \\ &= \lim_{s \rightarrow \frac{\pi}{2}^-} \frac{\pi}{2} \left(\left(\frac{\pi}{2} - s\right) \sec^2 s - \tan s\right)' \\ &= \lim_{s \rightarrow \frac{\pi}{2}^-} \frac{\pi}{2} \left(\frac{\left(\frac{\pi}{2} - s\right) - \frac{1}{2} \sin 2s}{\cos^2 s}\right)' \\ &= \lim_{s \rightarrow \frac{\pi}{2}^-} \frac{\pi}{2} \left(\frac{1 + \cos 2s}{\sin 2s}\right)' \\ &= \lim_{s \rightarrow \frac{\pi}{2}^-} \frac{\pi}{2} \left(\frac{2 \sin 2s}{2 \cos 2s}\right)' = 0, \\ \lim_{s \rightarrow 1^+} \left((s-1) \cosh(s-1)\right)' &= \lim_{s \rightarrow 1^+} \left(\coth(s-1) - (s-1) \operatorname{csch}^2(s-1)\right)' \\ &= \lim_{s \rightarrow 0^+} \left(\coth s - s \operatorname{csch}^2 s\right)' \\ &= \lim_{s \rightarrow 0^+} \frac{\frac{1}{2} \sinh 2s - s \cosh^2 s}{\sinh^2 s} \\ &= \lim_{s \rightarrow 0^+} \frac{\cosh 2s - \cosh^2 s - s \sinh 2s}{\sinh 2s} \\ &= \lim_{s \rightarrow 0^+} \frac{-2s \cosh 2s}{2 \cosh 2s} = 0, \end{aligned}$$

hence  $\lim_{s \rightarrow 1^-} A_1'(s) = 0 = \lim_{s \rightarrow 1^+} A_1'(s)$ , finishing this proof.  $\square$

**Lemma C. 4**

Let  $R > 0$  and  $y = y(x)$  an invertible and continuous real function defined for  $x > R$ . Let  $a, b \in \mathbb{R}$  such that  $a > 0$ . If  $y$  has the following asymptotic behavior

$$y(x) := ax + bx^{-1} + o(x^{-1}), \quad \text{as } x \longrightarrow +\infty, \quad (4.5.73)$$

hence, the inverse function  $x = x(y)$  holds

$$x(y) = a^{-1}y - by^{-1} + o(y^{-1}) \quad (4.5.74)$$

*Proof.* For simplicity, we use  $x := x(y)$  and  $y := y(x)$ . From (4.5.73) we derive that  $y \longrightarrow +\infty$  as  $x \longrightarrow +\infty$ , hence,  $x$  and  $y$  are increasing since they are both invertible and continuous. Then (4.5.74) is equivalent to prove

$$xy - a^{-1}y^2 + b = o(1), \quad \text{as } y \longrightarrow +\infty, \quad (4.5.75)$$

which in turn is equivalent to

$$xy - a^{-1}y^2 + b = o(1), \quad \text{as } x \longrightarrow +\infty. \quad (4.5.76)$$

We now prove the desired asymptotic behavior. From (4.5.73), one deduces

$$\begin{aligned} y^2 &= a^2 x^2 \left( 1 + \frac{b}{a} x^{-2} + o(x^{-2}) \right)^2 \\ &= a^2 x^2 \left( 1 + \frac{2b}{a} x^{-2} + o(x^{-2}) \right) \\ &= a^2 x^2 + 2ab + o(1), \quad \text{as } x \longrightarrow +\infty. \end{aligned}$$

Thus,

$$\begin{aligned} xy - a^{-1}y^2 + b &= x(ax + bx^{-1} + o(x^{-1})) - a^{-1}(a^2 x^2 + 2ab + o(1)) + b \\ &= (ax^2 + b) - (ax^2 + 2b) + b + o(1) \\ &= o(1), \quad \text{as } x \longrightarrow +\infty. \end{aligned}$$

□

## CHAPTER 5

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# Comparisons with two previous related works

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In the following, we present two previous works related to ours, we aim to enlighten the main differences with respect their results to ours and highlight the new features obtained in this new study.

### 5.1 Comparison with the "classical three layers media case" [93]

The first of these comparisons begins, in chronological order, with the work made by C. Wilcox in 1976 [93], on what we refer ourselves as *the classical case* standing for the study of wave propagation in stratified media (see [29–31, 44, 88–90, 93, 94] for the case of acoustic waves, [42, 90] for electromagnetic waves and [32, 43, 81] in the domain of elasticity). In [93] this article a spectral analysis for the Pekeris Operator acting on a class of functions  $u = u(y)$  defined for  $y > 0$  and vanishing at  $y = 0$  is given, the Pekeris differential operator in one dimension is defined by

$$Au = -c^2(y) \mu(y) - \frac{d}{dy} (\mu(y)^{-1} \frac{d}{dy} u), \quad (5.1.1)$$

where  $y \in \mathbb{R}$ , and the functions  $c = c(y)$  and  $\mu = \mu(y)$  satisfy

$$c(y) = \begin{cases} c_1, & 0 \leq y < L, \\ c_2, & y \geq L, \end{cases} \quad (5.1.2)$$

$$\mu(y) = \begin{cases} \mu_1, & 0 \leq y < L, \\ \mu_2, & y \geq L. \end{cases} \quad (5.1.3)$$

In [93], the resolvent and spectral family of  $A$  is constructed by using Fourier analysis (on the variable  $y$  dimensions larger than one) to reduce  $A$  to an ordinary differential operator, in this study the problem is complemented with boundary conditions at  $y = 0$  equal to 0. Hence, the

determination of the reduced eigenfunctions (of finite energy) lead to the dispersion equation

$$\xi = -\frac{\mu_2}{\mu_1} \eta \cot(\eta L), \quad (5.1.4)$$

where  $\xi := (k^2 - \frac{\omega^2}{c_2^2})^{\frac{1}{2}}$  and  $\eta = (\frac{\omega^2}{c_1^2} - k^2)^{\frac{1}{2}}$  are both real and positive. Once more,  $k$  is the spectral variable while  $\omega$  is the frequency or eigenvalue associated. Note that  $\xi, \eta \in \mathbb{R}_*^+$  implies that  $c_1 < c_2$ . The dispersion relation (5.1.4) has a parametric representation for the solutions given by a countable family of dispersion curves of the form:

$$\begin{cases} \omega = \frac{c_1 c_2}{h (c_2^2 - c_1^2)^{\frac{1}{2}}} \left(1 + \left(\frac{\rho_2^2}{\rho_1^2}\right) \tau^2\right)^{\frac{1}{2}} \left[(n - \frac{1}{2}) \pi + \arctan \tau\right], \\ k = \frac{c_1 c_2}{h (c_2^2 - c_1^2)^{\frac{1}{2}}} \left(\frac{1}{c_2^2} + \frac{1}{c_1^2} \left(\frac{\rho_2^2}{\rho_1^2}\right) \tau^2\right)^{\frac{1}{2}} \left[(n - \frac{1}{2}) \pi + \arctan \tau\right], \end{cases} \quad \tau \geq 0, \quad (5.1.5)$$

where  $n \in \mathbb{N}_*$ . The following properties of  $\omega_n = \omega_n(k)$  can be derived from (5.1.5) (see Figure 5.1).

- $\omega_n(k)$  is analytic in  $(k_n, +\infty)$ , where  $k_n := (2n - 1) k_1$ ,  $k_1 := \frac{c_1}{(c_2^2 - c_1^2)^{\frac{1}{2}}} \frac{\pi}{2L}$ , and  $\omega'_n(k) > 0$  for  $k \geq k_n$ .
- $c_1 k < \omega_n(k) < c_2 k$  for all  $k > k_n$ .
- $\omega_n(k_n) = c_2 k_n$  and  $\omega'_n(k_n) = c_2$ .
- $\omega_n(k) \sim c_1 k$  as  $k \rightarrow +\infty$ .

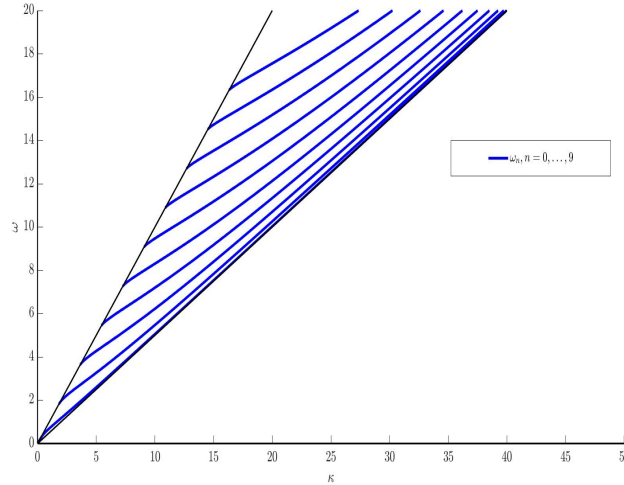


Figure 5.1: Slab of a dielectric configuration.

The results of [93] can be interpreted in our same context, as the study of the transmission problem between a slab of a dielectric on a second dielectric with permittivities and permeabilities, respectively,  $\varepsilon_1, \mu_1 > 0$  and  $\varepsilon_2, \mu_2 > 0$ , and whose speed wave propagations are given by, respectively,

$$c_1 := (\varepsilon_1 \mu_1)^{-\frac{1}{2}}, \quad c_2 := (\varepsilon_2 \mu_2)^{-\frac{1}{2}}. \quad (5.1.6)$$

In this sense, the existence of finite energy guided modes is assured only when  $c_1 < c_2$ , and are evanescent (in our Drude model, we talk about plasmon surfaces since we face a dispersive

material). In contrast to our model, we exhibit the existence of an infinite number of dispersion curves linked to propagative guided modes, while only two curves dispersion are related to the evanescent ones. Moreover, in our study we have showed that the dispersion curves are not in general monotone, indeed, the dispersion curves  $\omega_n(k), n \geq 2$  have all a maximum (where the group velocity vanishes). On another hand, the asymptotic computation has show that the dispersion curves in our analysis have a finite limite (0 for the propagative curves and  $\frac{\Omega_m}{\sqrt{2}}$  for the plasmonic curves), in contrast to the unbounded growth for *the classical case*.

## 5.2 Comparison with the case of a two layer media (vacuum/Drude material) [14, 15]

The second comparison is devoted to the work made by M. Cassier, C. Hazard and P. Joly [14,15]. In the latter, a geometry of a two layer configuration between a Drude material (as in our case) and a dielectric (the vacuum) is considered (see Figure 5.2). The search of finite energy guided

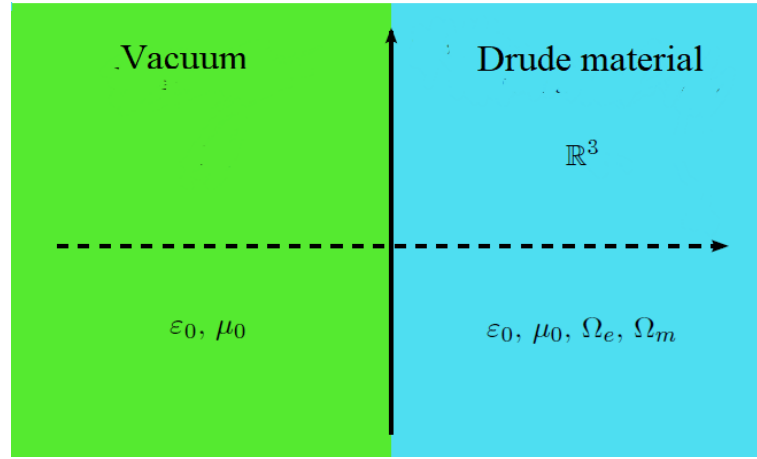


Figure 5.2: Two layer configuration.

modes leads to a dispersion relation linking the frequency parameter  $\lambda$  to the spectral variable  $k$ , this dispersion relation is of polynomial parameter and can be resolved ( $\lambda$  as a function on  $k$ ) in order to obtain the unique solution given by

$$\lambda(k) = \begin{cases} \sqrt{\Lambda(k)}, & \text{if } \Omega_m \neq \Omega_e, \\ \frac{\Omega_m}{\sqrt{2}}, & \text{if } \Omega_m = \Omega_e, \end{cases} \quad (5.2.1)$$

where  $\Lambda(k)$  is defined as follows

$$\Lambda(k) = \begin{cases} \frac{\Omega_m^2}{2} + \frac{k^2 \Omega_m^2}{\varepsilon_0 \mu_0 (\Omega_m^2 - \Omega_e^2)} - \Omega_m^2 \sqrt{\frac{k^4}{(\varepsilon_0 \mu_0 (\Omega_m^2 - \Omega_e^2))^2} + \frac{1}{4}}, & \text{if } \Omega_m > \Omega_e, \\ \frac{\Omega_m^2}{2} + \frac{k^2 \Omega_m^2}{\varepsilon_0 \mu_0 (\Omega_m^2 - \Omega_e^2)} + \Omega_m^2 \sqrt{\frac{k^4}{(\varepsilon_0 \mu_0 (\Omega_m^2 - \Omega_e^2))^2} + \frac{1}{4}}, & \text{if } \Omega_e > \Omega_m. \end{cases} \quad (5.2.2)$$

In [14,15], it is proved that  $\lambda(k)$  takes values on  $[k_c, +\infty)$  ( $k_c := \sqrt{\varepsilon_0 \mu_0} \frac{\Omega_e \Omega_m}{\sqrt{\Omega_e^2 + \Omega_m^2}}$ ) and is analytic, moreover,

- $\lambda(k)$  is a constant function equal to  $\lambda_p := \frac{\Omega_m}{\sqrt{2}} = \frac{\Omega_e}{\sqrt{2}}$ , if  $\Omega_e = \Omega_m$ ,
- $\lambda(k)$  is strictly increasing and  $\lambda(k) \uparrow \frac{\Omega_m}{\sqrt{2}}$  as  $k \uparrow +\infty$ , if  $\Omega_e < \Omega_m$ ,
- $\lambda(k)$  is strictly decreasing and  $\lambda(k) \downarrow \frac{\Omega_m}{\sqrt{2}}$  as  $k \uparrow +\infty$ , if  $\Omega_m < \Omega_e$ .

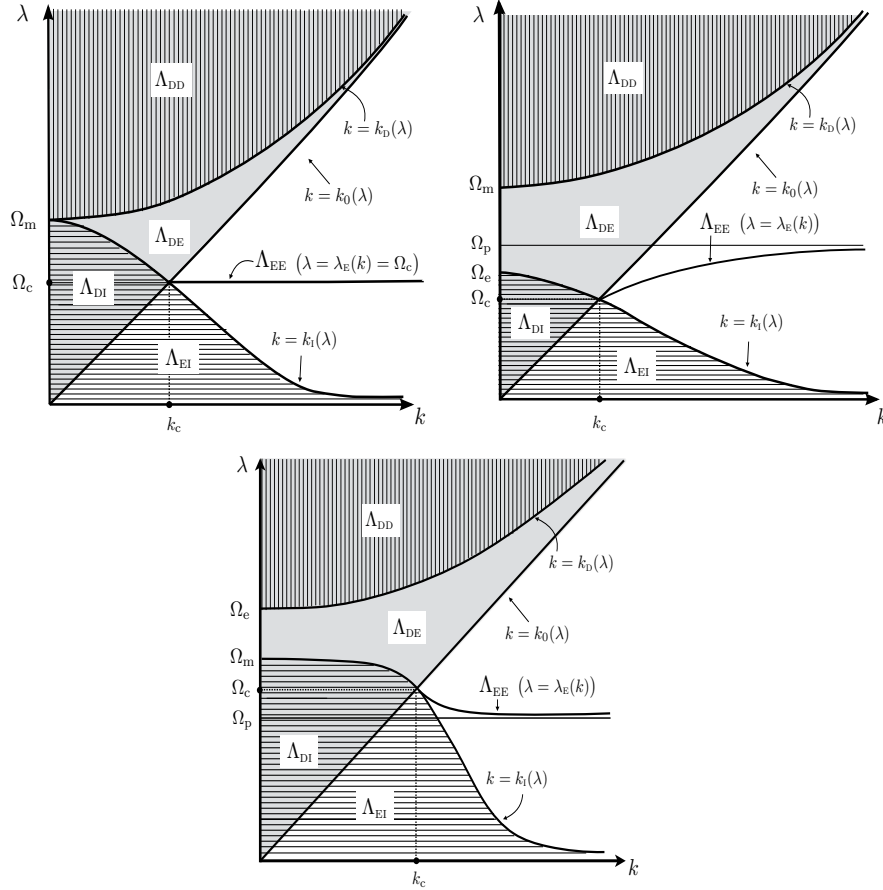


Figure 5.3: Dispersion curves for the two layer configuration. From left to right and top to bottom, the cases:  $\Omega_e = \Omega_m$ ,  $\Omega_e < \Omega_m$  and  $\Omega_m < \Omega_e$ .

See Figure 5.3 for an illustration of  $\lambda(k)$ . We can appreciate that for the same kind of meta-material (a Drude material), the geometry has an important role: in our slab configuration we have found an infinite number of dispersion curves, two of them related to the evanescent guided modes (surface plasmons) and an infinity of them of propagative order in the dielectric, also we have seen that in the case  $\Omega_e = \Omega_m$  (case  $\rho = 1$ ) no one of this evanescent guided waves is constant on any interval, indeed, we have proved that the dispersion curves have at most two critical points. The latter considerations is quite important: the fact that in the bi-layered configuration exists a constant plasmonic wave equal to  $\lambda_p$  in the case  $\Omega_e = \Omega_m$ , implies that this value  $\lambda_p$  is an eigenvalue of the operator  $\mathbb{A}$ , furthermore, it is proven that  $\sigma_p(\mathbb{A}) = \{\pm\Omega_m, 0, \pm\lambda_p\}$ . In the opposite case,  $\Omega_e \neq \Omega_m$ , we have  $\sigma_p(\mathbb{A}) = \{\pm\Omega_m, 0\}$ . In both cases, the punctual spectrum is formed by eigenvalues of infinite multiplicity, all of the latter is proven using the proper tools of direct integrals of Hilbert spaces and operators [33, 76, 87]. In a very resumed explanation, the reason for which the constant plasmonic curve  $\lambda \equiv \lambda_p$  is in the punctual spectrum of the whole Hamiltonian  $\mathbb{A}$  in the case  $\Omega_e = \Omega_p$ , it is mainly the fact of the existence of open intervals on the spectral parameter ( $k \in \mathbb{R}$ ) (and hence set of positive



measure) whose reduced Hamiltonians ( $\mathbb{A}_k$ ) contain  $\lambda_p$  in the punctual spectrum ( $\lambda_p \in \mathbb{A}_k$  for  $k \in (k_c, +\infty)$ ). This consideration is never achieved in our study, in any case for the parameters  $\Omega_e$  and  $\Omega_m$ , the dispersion curves (propagative or plasmonic) have intervals for the which they are constants, in other words, the phase velocity never vanishes on intervals of the spectral frequency  $k \in \mathbb{R}$ .

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## Conclusion and future perspectives of the Part II

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The spectral study performed in the second part of this work , has allowed us to described completely the guided modes associated to a slab of Drude metamaterial embedded in the vacuum. This model constitutes a mathematical modeling of the geometry of the perfect flat lens described by J. Pendry in [72]. We have showed the existence of an infinity number of those modes for every wavelength number  $k \in \mathbb{R}$  and the existence of a threshold critical wavelength number  $k_c$  for the which the problem stars to exhibits the existence of plasmonic surface waves. Moreover, we have seen that there exists a second threshold critical wavelength number  $k_{\text{int}} > k_c$  such that the problem have two plasmonic surface waves.

Finally, we want to comment some future perspectives to work on the subject. Since the geometry represents an interesting aspect, our first aim is to extend the guided waves study to others structures studied in physics. We mention two of these structures. The first of this geometries is the case of a cylindrical lens [59] which consists of a three layered media in a *cylindrical geometry* (see Figure 5.4), this geometry enters in category of separable geometry, just as well as the two configurations treated before: the first studied in [14, 15] (the two-layered media) and later in this work (perfect flat lense or the slab of metamaterial). The second kind of geometry to explore is more challenging. In the harmonic regime, if the interfaces delimiting the dispersive negative medium is not smooth and presents corners, in a given critical frequency range (which depends on the corner), energy can accumulate near this singularity, so that the energy seems to leak at the corner, given raise to the black-hole effect [6, 46, 70, 77]

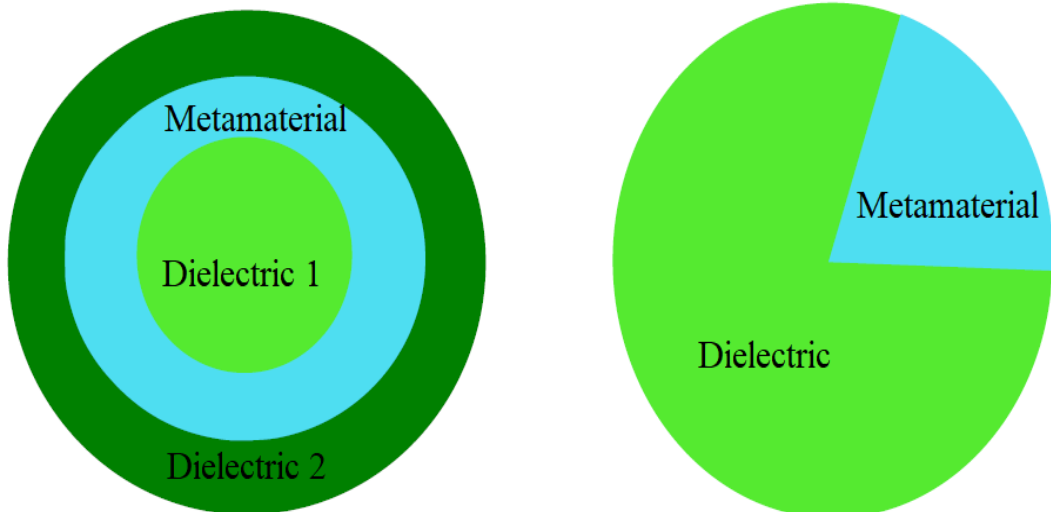


Figure 5.4: Metamaterial cylindrical configuration (left) and metamaterial configuration with corners (right).

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**Titre** : Étude de deux problèmes de propagation d'ondes en milieu électromagnétique dispersif: 1) Stabilité en temps long dans un milieu de Drude-Lorentz; 2) Transmission entre une couche de métamatériau et un diélectrique.

**Mots clés** : Milieux dispersifs électromagnétiques, stabilité en temps long, fonctions de Lyapunov, analyse spectrale, relation de dispersion, ondes guidées.

**Résumé** : Cette thèse traite de deux problèmes indépendants liés aux phénomènes de propagation des ondes dans les milieux dispersifs. Dans la première partie, nous étudions le comportement en temps long des solutions des équations de Maxwell dans des milieux dissipatifs généralisés de Drude-Lorentz. Plus précisément, nous souhaitons quantifier les pertes dans de tels milieux à l'aide du taux de décroissance de l'énergie électromagnétique pour le problème de Cauchy correspondant. Cette première partie est elle-même composée de deux approches. La première, l'approche par fonctions de Lyapunov en fréquence, consiste à obtenir une inégalité différentielle (en temps) pour certaines fonctionnelles de la solution, les fonctions de Lyapunov  $\mathcal{L}_k$ , où  $k$  désigne la fréquence spatiale. Les estimations de stabilité sont ensuite obtenues par l'intégration en temps de l'inégalité différentielle. En développant cette méthode, nous obtenons un résultat de stabilité polynomiale sous des hypothèses de dissipation fortes. La deuxième approche, l'approche modale, exploite les propriétés spectrales de l'opérateur hamiltonien apparaissant dans le problème de Cauchy. Cette dernière approche améliore

la première en autorisant des hypothèses de dissipation faibles. Dans la deuxième partie du travail, nous nous intéressons au problème de transmission d'une couche de métamatériau de Drude non dissipatif dans un milieu diélectrique. Dans ce contexte, nous considérons les équations de Maxwell temporelles bidimensionnel en polarisation TM et nous les reformulons en une équation de Schrödinger dont le Hamiltonien,  $\mathbb{A}$ , est un opérateur autoadjoint non borné. La transformation de Fourier nous permet de travailler avec des Hamiltoniens réduits  $\mathbb{A}_k, k \in \mathbb{R}$ . Enfin, nous nous intéressons au spectre ponctuel du Hamiltonien réduit qui est lié aux modes guidés du problème original. Cette étude débouche sur une relation de dispersion dont la difficulté réside dans son caractère hautement non linéaire par rapport au paramètre spectral. Nous prouvons l'existence d'une infinité dénombrable de branches de solutions pour la relation de dispersion : les courbes de dispersion. Nous donnons une analyse précise de ces courbes et mettons en lumière, notamment, l'existence d'ondes guidées correspondant à des palsmons surface.

**Title** : Study of two wave propagation problems in electromagnetic dispersive media: 1) Long-time stability analysis in Drude-Lorentz media; 2) Transmission between a slab of metamaterial and a dielectric.

**Keywords** : Electromagnetic dispersive media, long-time stability, Lyapunov functions, spectral analysis, dispersion relation, guided waves

**Abstract** : This PhD thesis addresses two independent problems related to wave propagation phenomena in dispersive media. In the first part, we investigate the long-time behavior of solutions of Maxwell's equations in dissipative generalized Drude-Lorentz media. More precisely, we wish to quantify the loss in such media in terms of the decay rate of the electromagnetic energy for the corresponding Cauchy problem. This first part is in turn composed by two approaches. The first one, namely, the frequency dependent Lyapunov approach, consists in deriving a differential inequality (in time) for certain functionals of the solution, the Lyapunov functions  $\mathcal{L}_k$ , where  $k$  is the spatial frequency. The stability estimates are then obtained from the time integration of the differential inequality. By developing this method, we obtain a polynomial stability result under strong dissipative assumptions. The second approach, the modal approach, exploits the spectral properties of the Hamiltonian operator appearing in the Cauchy problem. This last approach

ameliorates the first one by considering weak dissipation assumptions. In the second part of the work, we are interested in the transmission problem of a slab of non-dissipative Drude metamaterial within a dielectric. In this context, we consider the TM two dimensional time-dependent Maxwell's equations and we reformulate it into a Schrödinger equation whose Hamiltonian,  $\mathbb{A}$ , is an unbounded self-adjoint operator. Fourier transform allow us to work with the reduced Hamiltonians  $\mathbb{A}_k, k \in \mathbb{R}$ . Finally, we are interested in the point spectrum of the reduced Hamiltonian which is related to the guided modes of the original problem. This study leads to a dispersion relation whose difficulty lies in its highly non-linear character with respect to the spectral parameter. We prove the existence of a countable infinity of solution branches for the dispersion relation: the so-called dispersion curves. We give a precise analysis of these curves and enlighten the existence of guided waves which correspond to surface plasmons.