Martingale paraproducts and applications to classical and noncommutative harmonic analysis

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Hao Zhang

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Soutenance prévue le 6 Juin 2023, devant le jury composé de :

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Paraproduits de martingale et applications à l'analyse harmonique classique et non commutative

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Résumé

Cette thèse a pour l'étude des paraproduits de martingales à la fois dans le cadre commutatif et noncommutatif. Il se compose de deux parties. La première concerne la décomposition bilinéaire de la multiplication ponctuelle d'éléments dans l'espace de Hardy de martingales H^1 et son dual BMO. Nous étendons également cette décomposition bilinéaire continue aux espaces de Hardy de martingales H^p $(0 < p < 1)$ et leurs espaces duaux. Nos décompositions sont basées sur des paraproduits de martingales. Comme conséquences de notre travail, nous obtenons des résultats analogues pour des martingales dyadiques sur des espaces de type homogène munis d'une mesure de dédoublement. Nos arguments reposent sur l'existence de systèmes dyadiques sur des espaces de type homogène. La deuxième partie porte principalement sur l'appartenance à la classe de Schatten des paraproduits de martingales semi-commutatifs et purement non commutatifs, en particulier pour les algèbres de Clifford et les produits tensoriels d'algèbres matricielles $\bigotimes_{k=1}^{\infty} \mathbb{M}_d$ en termes d'espaces de Besov de martingales. En utilisant la technique de la martingale dyadique de Hytönen, nous obtenons également des conditions suffisantes pour l'appartenance à la classe de Schatten et la bornitude des commutateurs à valeurs d'opérateurs concernant des opérateurs intégraux singuliers généraux. De plus, nous donnons une preuve alternative sur la caractérisation BMO de la bornitude des commutateurs concernant des opérateurs intégraux singuliers généraux dans le cadre commutatif.

Mots-clefs

Paraproduits de martingales ; Espaces de Hardy-Orlicz ; Espaces de Musielak-Orlicz ; Espaces de dédoublement ; Espaces de Besov de martingales ; Martingales noncommutatives ; Produit tensoriel ; Classe de Schatten ; Algèbre de CAR ; Algèbres matricielles ; Commutateurs ; Opérateurs intégraux singuliers ; Espaces de BMO.

Abstract

This thesis is devoted to the study of martingale paraproducts both in the commutative and noncommutative settings. It consists of two parts. The first one is about the bilinear decomposition of pointwise multiplication of elements in martingale Hardy space H^1 and its dual BMO space. We also extend this continuous bilinear decomposition to martingale Hardy spaces H^p $(0 < p < 1)$ and their dual spaces. Our decompositions are based on martingale paraproducts. As a consequence of our work, we obtain analogous results for dyadic martingales on spaces of homogeneous type equipped with a doubling measure. Our arguments are based on the existence of dyadic systems on spaces of homogeneous type. The second part focuses on the Schatten class membership of semicommutative and purely noncommutative martingale paraproducts, especially for CAR algebras and tensor product of matrix algebras $\overset{\infty}{\underset{k=1}{\otimes}} \mathbb{M}_d$ in terms of martingale Besov spaces. Using Hytönen's dyadic martingale technique, we also obtain sufficient conditions on the Schatten class membership and the boundedness of operator-valued commutators involving general singular integral operators. In addition, we give an alternative proof on the BMO characterization of the boundedness of commutators involving general singular integral operators in the commutative setting.

Keywords

Martingale paraproducts; Hardy-Orlicz spaces; Musielak-Orlicz spaces; Doubling spaces; Martingale Besov spaces; Noncommutative martingales; Tensor product; Schatten class; CAR algebra; Matrix algebras; Commutators; Singular integral operators; BMO spaces.

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Chapter 0

Introduction

In [26], David and Journé showed that for any standard singular integral operator T, T is bounded on $L_2(\mathbb{R}^n)$ if and only if $T1$ and T^*1 both belong to BMO space and T satisfies the weak boundedness property. This result is now referred to the celebrated David-Journé $T1$ theorem. It was not long before Coifman and Semmes gave a different proof of the $T1$ theorem based on dyadic martingales, which made dyadic martingales become an important tool in harmonic analysis. More precisely, the idea of Coifman and Semmes is to use dyadic martingale paraproducts. We refer the reader to [21] and [70] for more details about the dyadic proof of the $T1$ theorem. We also point out that the concept of paraproducts first emerged in [11] about the theory of paradifferential operators.

Since then, dyadic martingales have caught wide attention in harmonic analysis and dyadic martingale paraproducts have also become a crucial tool. As an extension of the work of David-Journé and Coifman-Semmes, Nazarov, Treil and Volberg applied dyadic martingale paraproducts to establish the $T1$ theorem and the Tb theorem on non-homogeneous spaces in [63], which can be perceived as a major step to complete the theory of Calderón-Zygmund operators on non-homogeneous spaces.

The deep connection between singular integral operators and dyadic martingale paraproducts has been closely investigated in various works. For example, it dates back to Figiel [29] who introduced Haar shift operators (also known as dyadic shift operators) and invoked the boundedness of martingale transforms in the study of singular integral operators. Later on Petermichl developed an explicit representation formula for the one-variable Hilbert transform [72] and Petermichl, Treil and Volberg extended this representation to the Riesz transforms [75] . In [88], Vagharshakyan successfully recovered one-dimensional Calderón-Zygmund transforms with sufficiently smooth convolution kernels by means of a properly chosen averaging of dyadic shift operators. This technique provides a bridge from the dyadic world to the non-dyadic world, where the former possesses neater structure properties and is easier to handle to some extent.

Such representations have been proved to be very powerful and widely applicable to harmonic analysis, such as the weighted inequality theory. For instance, Petermichl gave the sharp weighted bound for the Hilbert transform and the Riesz transforms, respectively in [73] and [74]. In particular, the famous A_2 conjecture, which concerns the sharp weighted bound for Calderon-Zygmund operators, has been fully resolved by Hytönen [40]. The key tool in [40] is a new dyadic representation for general singular integral operators, from which dyadic shift operators and dyadic martingale paraproducts naturally appear.

In addition, we refer the reader to [48] and [49] for the use of such representations on commutators. Hence, dyadic martingale paraproducts and dyadic shift operators have become a fundamental and useful model to study a variety of properties for singular integral operators.

Apart from close connections between martingale paraproducts and harmonic analysis, it is of independent interest to study martingale paraproducts in its own right. This is partly due to the fact that these operators are generalizations of Hankel type operators. Hankel operators are another important class of operators in function theory. It is very interesting to investigate the boundedness, compactness and Schatten class membership of Hankel operators.

Motivated by the aforementioned pioneering work, the theme of this thesis is the study of martingale paraproducts, with two different directions. The first focuses on the bilinear decompositions of the multiplication between elements in martingale Hardy spaces and their dual spaces in the commutative setting. This is a joint work with Odysseas Bakas, Zhendong Xu, and Yujia Zhai in [4]. The novelty of the work lies in the endpoint estimate of the boundedness of martingale paraproducts. Our approach is based on the atomic decompositions of martingale Hardy spaces, and thus also applies to the case for martingale Hardy space H^p with $0 < p < 1$ and its dual space. As an application, we extend these bilinear decompositions to spaces of homogeneous type.

The second direction concerns the Schatten class membership of martingale paraproducts in the noncommutative setting. This is a joint work with Zhenguo Wei in [90]. More precisely, we describe the Schatten class membership of semicommutative d-adic martingale paraproducts in terms of martingale Besov spaces. Using transference, we also obtain characterizations of Schatten class membership of purely noncommutative martingale paraproducts for CAR algebras and the tensor product of matrix algebras $\overset{\infty}{\underset{k=1}{\otimes}} \mathbb{M}_d$. We exploit the technique using dyadic representation developed by Hytönen [40] and further derive the Schatten class membership and the boundedness of operator-valued commutators involving general singular integral operators.

Our proof for operator-valued commutators relies heavily on the bilinear decompositions of the pointwise operator M_b, which is the main subject of study in the first direction. This provides a new proof of the well-known theorem [22] on the boundedness of commutators in the Euclidean setting from the perspective of martingale theory.

In the remaining part of the introduction, a detailed discussion on the background, motivations and main results will be given. After the introduction, we will give a preliminary in Chapter 1 which contains most of definitions and notation needed in this thesis, and then present our work in details. The presentation will be separated into two parts, and each part is devoted to one of the two directions highlighted above.

In the appendix, we include the proof of the boundedness of martingale paraproducts for general martingales instead of dyadic martingales. We also give another proof of the Necessity of Theorem II.2 for $p \geq 2$ by Schur multipliers.

I Multiplication between elements in martingale Hardy spaces and their dual spaces

The pointwise product of a function in the classical Hardy space $H^1(\mathbb{R}^n)$ and a function of bounded mean oscillation on $\mathbb{R}^n$ need not be in $L_1(\mathbb{R}^n)$; see e.g. §6.2 in Chapter IV in [86]. However, using Fefferman's duality theorem [28] and the fact that the pointwise product of a BMO-function and a C_0^∞-function is in $BMO(\mathbb{R}^n)$, Bonami, Iwaniec, Jones and Zinsmeister defined in [9] the product $f \times g$ of a function $f \in H^1(\mathbb{R}^n)$ and a function $g \in BMO(\mathbb{R}^n)$ as a distribution given by

$$\langle f \times g, \phi \rangle = \langle g \cdot \phi, f \rangle, \quad \phi \in C_0^\infty(\mathbb{R}^n), \tag{I.1}$$

where in the right-hand side of (I.1) the duality between $f \in H^1(\mathbb{R}^n)$ and $g \cdot \phi \in BMO(\mathbb{R}^n)$ is employed. Moreover, it is shown in [9] that for any fixed $f \in H^1(\mathbb{R}^n)$ there exist two linear continuous operators S_f from $BMO(\mathbb{R}^n)$ to $L_1(\mathbb{R}^n)$ and T_f from $BMO(\mathbb{R}^n)$ to a weighted Hardy–Orlicz space such that

$$f \times g = S_f(g) + T_f(g)$$

for all $g \in BMO(\mathbb{R}^n)$; see [9, Theorem 1.6].

In [8], using wavelet analysis, Bonami, Grellier and Ky showed that there exist two bilinear continuous operators S from $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ to $L_1(\mathbb{R}^n)$ and T from $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ to $H^{\log}(\mathbb{R}^n)$ such that

$$f \times g = S(f, g) + T(f, g)$$

for all $f \in H^1(\mathbb{R}^n)$ and for all $g \in BMO(\mathbb{R}^n)$; see [8, Theorem 1.1]. The Musielak Hardy–Orlicz space $H^{\log}(\mathbb{R}^n)$ is defined as the class consisting of all distributions h on $\mathbb{R}^n$ whose grand maximal function $\mathcal{M}h$ satisfies

$$\int_{\mathbb{R}^n} \frac{|\mathcal{M}h(x)|}{\log(e + |x|) + \log(e + |\mathcal{M}h(x)|)} dx < \infty$$

and is smaller than the weighted Hardy–Orlicz space appearing in [9]. In fact, as explained in [9], in view of the results of Nakai and Yabuta [60] on pointwise multipliers of $BMO(\mathbb{R}^n)$ and duality, the Musielak Hardy–Orlicz space $H^{\log}(\mathbb{R}^n)$ is optimal in the above decomposition.

In addition, continuous bilinear decomposition theorems for products of elements in $H^p(\mathbb{R}^n)$, for $0 < p < 1$, and their dual spaces were established in [7].

Using the theory of wavelets on spaces of homogeneous type, which was developed by Auscher and Hytönen in [1] and [2], the aforementioned results have been extended to spaces of homogeneous type by Liu, Yang and Yuan [50] and Xing, Yang and Liang [31]. More precisely, in [50] and [31], continuous bilinear decompositions for products between elements in atomic Hardy spaces $H^p_{\mathrm{at}}(\Omega)$ (in the sense of Coifman and Weiss [23]) and their dual spaces were established in the case where $p \in (\frac{n}{n+1}, 1]$. Here n is defined as the dimension of the homogeneous space Ω.

Recently in [3], Bakas, Pott, Rodríguez-López and Sola established a dyadic variant of the aforementioned results of Bonami, Grellier, and Ky; see [3, Theorem 24], which in turn was used to deduce a periodic version of [8, Theorem 1.1]; see [3, Theorem 28].

Motivated by [3], the first part of this thesis is concerned with the study of multiplication between Hardy spaces and their dual spaces for martingales on a probability space Ω. More specifically, we study multiplications between functions in the martingale Hardy space $H^1(\Omega)$ and its dual space $BMO(\Omega)$ as stated in our first result, Theorem I.1. We also investigate the case $0 < p < 1$, namely multiplication between elements in $H^p(\Omega)$ and their dual spaces, the so-called martingale Lipschitz spaces $\Lambda_1(\alpha_p)$ with $\alpha_p := \frac{1}{p} - 1$, see Theorem I.2. Since the dual space $(H^p(\Omega))^*$ could be $\{0\}$ for some irregular martingales, we will only consider regular martingales where every σ-algebra $\mathcal{F}_k$ in the corresponding filtration is generated by countably many atoms.

We would like to mention that Bonami, Jiao, Xie, Yang, and Zhou have independently obtained Theorem I.1, and derived from it interesting applications on the boundedness of operators involving commutators in [10].

Theorem I.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with the filtration $\{\mathcal{F}_k\}_{k\geq 1}$. There exist continuous bilinear operators $\Pi_1 : H^1(\Omega) \times BMO(\Omega) \to L_1(\Omega)$, $\Pi_2 : H^1(\Omega) \times BMO(\Omega) \to H^1(\Omega)$ and $\Pi_3 : H^1(\Omega) \times BMO(\Omega) \to H^\Phi(\Omega)$ such that*

$$f \cdot g = \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g)$$

for all $f \in H^1(\Omega)$ and $g \in BMO(\Omega)$, where $f \cdot g$ is in the sense of the pointwise multiplication.

In Theorem I.1, $H^\Phi(\Omega)$ is a martingale Hardy–Orlicz space defined in terms of the growth function $\Phi(t)$; see Definition 1.3.14 and (1.3.2) below. We will refer to the terms $\Pi_2(f, g)$ and $\Pi_3(f, g)$ as the martingale paraproducts.

Theorem I.1 can be regarded as an extension of [3, Theorem 24] to the general case of martingales.

For $0 < p < 1$, if $f \in H^p(\Omega)$, $g \in \Lambda_1(\alpha_p)$ and $f_0 = g_0 = 0$, then their product can be regarded as a continuous linear functional on $L_\infty(\Omega) \cap \Lambda_1(\alpha_p)$. To be more precise, for any $h \in L_\infty(\Omega) \cap \Lambda_1(\alpha_p)$, define

$$\langle f \times g, h\rangle = \langle h \cdot g, f\rangle,$$

where in the right-hand side the duality between $H^p(\Omega)$ and $\Lambda_1(\alpha_p)$ is invoked. Note that $h \cdot g$ belongs to $\Lambda_1(\alpha_p)$ since h is a pointwise multiplier on $\Lambda_1(\alpha_p)$ (see [59]).

Our following theorem establishes a continuous bilinear decomposition for products between elements in $H^p(\Omega)$ and functions in the dual space $\Lambda_1(\alpha_p)$ when $0 < p < 1$.

Theorem I.2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with the regular filtration $\{\mathcal{F}_k\}_{k\geq 1}$, where $\mathcal{F}_k$ is generated by countably many atoms for any $k \geq 1$.*

If $H^p(\Omega)$ $(0 < p < 1)$ are martingale Hardy spaces, then there exist continuous bilinear operators $\Pi_1 : H^p(\Omega) \times \Lambda_1(\alpha_p) \to L_1(\Omega)$, $\Pi_2 : H^p(\Omega) \times \Lambda_1(\alpha_p) \to H^1(\Omega)$ and $\Pi_3 : H^p(\Omega) \times \Lambda_1(\alpha_p) \to H^p(\Omega)$ such that

$$f \times g = \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g)$$

for all $f \in H^p(\Omega)$ and $g \in \Lambda_1(\alpha_p)$.

In the remaining sections of the first part, we study analogues of Theorems I.1 and I.2 for the case of dyadic martingales on spaces of homogeneous type. Such martingales were first constructed in [43]. We investigate the corresponding martingale Hardy spaces and extend Mei's results in [52] to this general setting. Compared with the probability setting, the case of spaces of homogeneous type is more difficult to deal with since backward martingales arise, and the underlying measures on homogeneous spaces may be infinite.

II Schatten class membership of noncommutative martingale paraproducts

Hankel operators were first studied by Hankel in [35], since then they have become an important class of operators. Later, Nehari characterized the boundedness of Hankel operators on the Hardy space $H^2(\mathbb{T})$ in terms of the BMO space in [64], and Hartman discussed their compactness by the VMO space in [36]. In [67], Peller obtained the Schatten p-class criterion of Hankel operators by Besov space for $1 \leq p < \infty$, while the case $0 < p < 1$ was discussed by Peller in [68] and Semmes in [84], respectively.

In harmonic analysis, commutators involving singular integral operators and multiplication operators are generalizations of Hankel type operators. So it is certainly worthwhile to study their boundedness, compactness and Schatten class membership. In [22], Coifman, Rochberg and Weiss showed the boundedness of commutators with regards to the BMO space on $\mathbb{R}^n$, which yields a new characterization of BMO. Soon after their work, Uchiyama sharpened one of their results and showed the compactness of commutators by virtue of the CMO space in [87]. The Schatten class membership of commutators was developed by Janson and Wolff in terms of Besov spaces in [45]. Afterwards, Janson and Peetre established a fairly general framework to investigate the boundedness and Schatten class of commutators in [44].

As described at the beginning of the introduction, close connections between singular integral operators and dyadic operators have been extensively explored. Thanks to such connections, dyadic operators, such as martingale paraproducts, serve as crucial tools in harmonic analysis. For instance, Petermichl discovered an explicit representation formula for the one-variable Hilbert transform as an average of dyadic shift operators to investigate Hankel operators with matrix symbol. Nazarov, Treil and Volberg in [63] proved the $T1$ and the Tb theorems based on martingale paraproducts. Later, Hytönen refined in an essential way the method of Nazarov, Treil and Volberg, and settled the well-known A_2 conjecture [40].

We would like to highlight that the dyadic operators in [40] can be considered as a particular case of martingale paraproducts. In addition to its intrinsic connection with various operators in harmonic analysis, martingale paraproducts have attracted much interest in its own right as they are martingale variants of operators of Hankel type. The boundedness of martingale paraproducts has been studied in [17]. In addition, the compactness and Schatten class have been discussed in [18] for d-adic martingales.

Motivated by all this, we aim to establish the Schatten class membership of martingale paraproducts in the noncommutative setting. Via the methodology developed by Hytönen, we also obtain the Schatten class characterization for the operator-valued

commutators involving singular integral operators and noncommutative pointwise multiplication.

At first, we introduce noncommutative martingale paraproducts. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Given a semicommutative d-adic martingale $b = (b_k)_{k\in\mathbb{Z}} \in L_2(\mathbb{R}, L_2(\mathcal{M}))$, recall that the martingale paraproduct with symbol b is defined as

$$\pi_b(f) = \sum_{k=-\infty}^{\infty} d_k b \cdot f_{k-1}, \quad \forall f = (f_k)_{k\in\mathbb{Z}} \in L_2(\mathbb{R}, L_2(\mathcal{M})),$$

where $d_k b = b_k - b_{k-1}$ for any $k \in \mathbb{Z}$. See Subsection 1.7 for the definitions of semicommutative d-adic martingales. When $d = 2$, d-adic martingales are just dyadic martingales.

When $\mathcal{M} = \mathbb{C}$, Chao and Peng described the Schatten class membership of π_b by virtue of the martingale Besov spaces. They showed the following theorem (see [18, Theorem 3.1]):

Theorem II.1. *For $0 < p < \infty$ and a locally integrable function b, $\pi_b \in S_p(L_2(\mathbb{R}))$ if and only if*

$$\sum_{k=-\infty}^{\infty} d^k \|d_k b\|_{L_p(\mathbb{R})}^p < \infty.$$

Chao and Peng's proof invoked some results about Schatten p-norms in [82]. In [80], Pott and Smith gave another proof of Theorem II.1 based on the p-John-Nirenberg inequality for $d = 2$. They also obtained an equivalent characterization of the Schatten class membership of π_b when $\mathcal{M} = B(\mathcal{H})$, still with $d = 2$.

Inspired by all this, we are concerned with the Schatten class membership of π_b for semicommutative d-adic martingales with arbitrary d and arbitrary semifinite von Neumann algebras $\mathcal{M}$.

Our first main theorem concerns the Schatten class membership of π_b for semicommutative d-adic martingales. More specifically, we use the semicommutative d-adic martingale Besov spaces $\boldsymbol{B}_p^d(\mathbb{R}, \mathcal{M})$ (see Definition 1.7.2) to characterize $\|\pi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}$:

Theorem II.2. *For $0 < p < \infty$, $\pi_b \in L_p(B(L_2(\mathbb{R})) \otimes \mathcal{M})$ if and only if $b \in \boldsymbol{B}_p^d(\mathbb{R}, \mathcal{M})$. Moreover,*

$$\|\pi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \approx_{d,p} \|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}. \tag{II.1}$$

It is much more tempting to study martingale paraproducts for purely noncommutative martingales. Let $b = (b_k)_{k\geq 1}$ be a noncommutative martingale. (see Subsection 1.6 for the definition.) The noncommutative martingale paraproduct with symbol b for any noncommutative martingale $f = (f_k)_{k\geq 1} \in L_2(\mathcal{M})$ is defined by

$$\pi_b(f) = \sum_{k=1}^{\infty} d_k b \cdot f_{k-1}.$$

However, it remains open under which circumstances π_b is bounded in $B(L_2(\mathcal{M}))$, which is also deeply related to the operator-valued $T1$ problem. The reader is referred to [38] for more details about the operator-valued $T1$ problem.

Let $\mathbb{M}_d$ be the algebra of $d \times d$ matrices equipped with the normalized trace. In particular, if $\mathcal{M} = L_\infty(\mathbb{R}) \otimes \mathbb{M}_d$, Katz employed an ingenious stopping time procedure in [47] to show

$$\|\pi_b\|_{B(L_2(\mathcal{M}))} \lesssim \log(d+1)\|b\|_{BMO(\mathbb{R},\mathbb{M}_d)}, \tag{II.2}$$

where $BMO(\mathbb{R}, \mathbb{M}_d)$ is the strong operator BMO. We refer the reader to [54] for more information for such BMO spaces. Meanwhile, Nazarov, Treil and Volberg independently obtained (II.2) in [62] by the Bellman method, and they also gave an example to show that for any $d \in \mathbb{N}$ there exists b such that

$$\|\pi_b\|_{B(L_2(\mathcal{M}))} \gtrsim \sqrt{\log(d+1)}\|b\|_{BMO(\mathbb{R},\mathbb{M}_d)}.$$

This implies that the boundedness of π_b cannot be characterized solely by $BMO(\mathbb{R}, \mathbb{M}_d)$ for infinite-dimensional $\mathcal{M}$.

In [61], Nazarov, Pisier, Treil and Volberg have proved that $\log(d+1)$ is the optimal order of the best constant in (II.2). Indeed, it has been shown that in general, $\|\pi_b\|_{B(L_2(\mathcal{M}))}$ cannot even be dominated by the operator norm $\|b\|_{\mathcal{M}}$ for infinite-dimensional $\mathcal{M}$ in [53].

Even though we do not know how to describe the boundedness of π_b, surprisingly by transference method in [77] and Theorem II.2, we get the equivalent characterization of the Schatten class membership of π_b for two families of noncommutative martingales, i.e. CAR algebra denoted by $\mathcal{C}$, and $\mathscr{M} = \bigotimes_{k=1}^{\infty} \mathbb{M}_d$.

For the CAR algebra, we obtain:

Theorem II.3. *For $0 < p < \infty$, $\pi_b \in S_p(L_2(\mathcal{C}))$ if and only if $b \in \boldsymbol{B}_p(\mathcal{C})$. Moreover,*

$$\|\pi_b\|_{S_p(L_2(\mathcal{C}))} \approx_p \|b\|_{\boldsymbol{B}_p(\mathcal{C})}.$$

Similarly, for $\mathscr{M} = \bigotimes_{k=1}^{\infty} \mathbb{M}_d$, we also have:

Theorem II.4. *For $0 < p < \infty$, $\pi_b \in S_p(L_2(\mathscr{M}))$ if and only if $b \in \boldsymbol{B}_p(\mathscr{M})$. Moreover,*

$$\|\pi_b\|_{S_p(L_2(\mathscr{M}))} \approx_{d,p} \|b\|_{\boldsymbol{B}_p(\mathscr{M})}.$$

The martingale Besov spaces $\boldsymbol{B}_p(\mathcal{C})$ and $\boldsymbol{B}_p(\mathscr{M})$ in Theorem II.3 and Theorem II.4 are defined in Definition 1.7.6 and Definition 1.7.7 respectively.

Next, we employ Theorem II.2 to give a characterization of Schatten class membership for operator-valued commutators involving singular integral operators and noncommutative pointwise multiplication, in terms of operator-valued Besov spaces. Our method is based on the dyadic representations of singular integral operators developed by Hytönen in [40] and [41]. We first provide the setup for singular integral operators.

Let $T \in B(L_2(\mathbb{R}^n))$ be a singular integral operator with a kernel $K(x, y)$, i.e. for any $f \in L_2(\mathbb{R}^n)$

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad x \notin \mathrm{supp} f.$$

We assume that $K(x,y)$ is defined for all $x \neq y$ on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies the following standard kernel estimates:

$$\begin{aligned} &|K(x,y)| \leq \frac{C}{|x-y|^n}, \\ &|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq \frac{C|x-x'|^\alpha}{|x-y|^{n+\alpha}} \end{aligned} \tag{II.3}$$

for all $x, x', y \in \mathbb{R}^n$ with $|x-y| > 2|x-x'| > 0$ and some fixed $\alpha \in (0,1]$ and constant $C > 0$.

In particular, if for any $x \neq y$

$$K(x,y) = \phi(x-y), \tag{II.4}$$

where ϕ is homogeneous of degree $-n$ with mean value zero on the unit sphere, then T is called a Calderón-Zygmund transform.

In the second part of this thesis, $T : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ will always be assumed to satisfy the above standard estimates (II.3) and to be bounded. The celebrated David-Journé $T1$ theorem in [26] asserts that for any singular integral operator T satisfying (II.3) , T is bounded on $L_2(\mathbb{R}^n)$ if and only if $T1$ and T^*1 both belong to $BMO(\mathbb{R}^n)$ and T satisfies the weak boundedness property. We recall that $BMO(\mathbb{R}^n)$ is the space consisting of all locally integrable functions b such that

$$\|b\|_{BMO(\mathbb{R}^n)} = \sup_{\substack{Q \subset \mathbb{R}^n \\ Q \text{ cube}}} \left(\frac{1}{m(Q)} \int_Q \left| b - \Big(\frac{1}{m(Q)} \int_Q b \, dm \Big) \right|^2 dm \right)^{1/2} < \infty,$$

where m is Lebesgue measure on $\mathbb{R}^n$.

Assume $b \in L_2(\mathbb{R}^n, L_2(\mathcal{M}))$, and let M_b be the pointwise multiplication by b. The operator-valued commutator is defined by $C_{T,b} = [T, M_b] = TM_b - M_bT$, that is for any $f \in L_2(\mathbb{R}^n, L_2(\mathcal{M}))$,

$$C_{T,b}(f) = T(b \cdot f) - b \cdot T(f).$$

The operator-valued Besov space $\boldsymbol{B}_p(\mathbb{R}^n, L_p(\mathcal{M}))$ is defined as the completion of all $b \in \mathcal{S}(L_\infty(\mathbb{R}^n) \otimes \mathcal{M})$ satisfying

$$\|b\|_{\boldsymbol{B}_p(\mathbb{R}^n, L_p(\mathcal{M}))} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\|b(x) - b(y)\|^p_{L_p(\mathcal{M})}}{|x-y|^{2n}} dxdy \right)^{\frac{1}{p}} < \infty, \tag{II.5}$$

with respect to the norm $\|\cdot\|_{\boldsymbol{B}_p(\mathbb{R}^n, L_p(\mathcal{M}))}$. If $\mathcal{M} = \mathbb{C}$, $\boldsymbol{B}_p(\mathbb{R}^n, L_p(\mathbb{C}))$ coincides with the classical Besov space of parameters $(p, p, n/p)$, namely the space $\Lambda^{p,q}_\alpha(\mathbb{R}^n)$ in [85, Chapter V, §5].

In the commutative setting, Janson and Wolff have obtained the following theorem (see [45, Theorem 1]):

Theorem II.5. *Let T be a Calderón-Zygmund transform with a kernel ϕ defined in* (II.4)*. Assume ϕ is $C^\infty(\mathbb{R}^n)$ except at the origin and not identically zero.*

Suppose $n \geq 2$ and $0 < p < \infty$. For $0 < p \leq n$, $C_{T,b} \in S_p(L_2(\mathbb{R}^n))$ if and only if b is constant. For $p > n$, $C_{T,b} \in S_p(L_2(\mathbb{R}^n))$ if and only if $b \in \boldsymbol{B}_p(\mathbb{R}^n, L_p(\mathbb{C}))$.

We get the analogous result for $p \geq 2$ and $n \geq 1$ in the semicommutative setting. The following theorem describes the Schatten class membership of operator-valued commutators.

Theorem II.6. *Let $T \in B(L_2(\mathbb{R}^n))$ be a singular integral operator with kernel $K(x,y)$ satisfying the standard estimates* (II.3)*. If $b \in \boldsymbol{B}_p(\mathbb{R}^n, L_p(\mathcal{M}))$ and $2 \leq p < \infty$, then $C_{T,b} \in L_p(B(L_2(\mathbb{R}^n)) \otimes \mathcal{M})$ and*

$$\|C_{T,b}\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})} \lesssim_{n,p} \Big(1 + \|T1\|_{BMO(\mathbb{R}^n)} + \|T^*1\|_{BMO(\mathbb{R}^n)}\Big)\|b\|_{\boldsymbol{B}_p(\mathbb{R}^n, L_p(\mathcal{M}))}.$$

Theorem II.6 directly implies the necessity of Theorem II.5 for $p > n \geq 2$ if we just let $\mathcal{M} = \mathbb{C}$. So we give an alternative proof of the necessity of Theorem II.5 for $p > n \geq 2$ based on martingale paraproducts. We would like to remark that Theorem II.6 is more general than Theorem II.5 because it not only concerns the semi-commutative setting, but also deals with commutators involving general singular integral operators, while [44] and [45] focus on Calderón-Zygmund transforms.

Last but not least, we attain the boundedness of commutators by martingale paraproducts. It has been shown by Coifman, Rochberg an Weiss in [22] that if $b \in BMO(\mathbb{R}^n)$ and T is a Calderón-Zygmund transform with kernel ϕ satisfying the following estimate

$$|\phi(x) - \phi(y)| \leq |x - y|, \quad \forall\ |x| = |y| = 1, \tag{II.6}$$

then for $1 < p < \infty$, $C_{T,b}$ is bounded on $L_p(\mathbb{R}^n)$. In fact, we will give a new proof to show the following theorem concerning general singular integral operators not necessarily of convolution type, which is known (for instance see [37, Theorem 1.1] or [20, Theorem 3.1]) but more general than the case considered in [22]. Our new approach is based on the boundedness of martingale paraproducts. But some new interesting martingale inequalities (Lemma 5.2.3 and Proposition 5.2.4) will be needed so as to prove the following theorem.

Theorem II.7. *Let $1 < p < \infty$ and $T \in B(L_2(\mathbb{R}^n))$ be a singular integral operator with kernel $K(x,y)$ satisfying the standard estimates* (II.3)*. If $b \in BMO(\mathbb{R}^n)$, then $C_{T,b}$ is bounded on $L_p(\mathbb{R}^n)$ and*

$$\|C_{T,b}\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)} \lesssim_{n,p} \Big(1 + \|T1\|_{BMO(\mathbb{R}^n)} + \|T^*1\|_{BMO(\mathbb{R}^n)}\Big)\|b\|_{BMO(\mathbb{R}^n)}.$$

The idea of the proof is the same as that of Theorem II.6. We also establish the boundedness of commutators involving martingale paraproducts and pointwise multiplication operator (see Proposition 5.2.4).

We would like to remark that the converse to Theorem II.7 seems to be much subtler. Coifman, Rochberg and Weiss have obtained a partial result of the "only if" part just for Riesz transforms in [22]. Uchiyama generalized this result and obtained the "only if" part for any Calderón-Zygmund transforms with the smooth estimate (II.6) in [87]. Recently, Hytönen further extended it to general "non-degenerate" singular integral operators in [42]. We refer to [42] for more details on the converse to Theorem II.7.

We can also consider the boundedness of operator-valued commutators involving general singular integral operators on $L_2(\mathbb{R}^n, L_2(\mathcal{M}))$. Denote by $BMO_{\mathcal{M}}(\mathbb{R}^n)$ the space consisting of all $\mathcal{M}$-valued functions b that are Bochner integrable on any cubes such that

$$\|b\|_{BMO_{\mathcal{M}}(\mathbb{R}^n)} = \sup_{\substack{Q\subset\mathbb{R}^n \\ Q \text{ cube}}} \left(\frac{1}{m(Q)} \int_Q \left\| b - \left(\frac{1}{m(Q)} \int_Q b \, dm \right) \right\|_{\mathcal{M}}^2 dm \right)^{1/2} < \infty.$$

The next theorem states the boundedness of operator-valued commutators for $p = 2$.

Theorem II.8. *Let $T \in B(L_2(\mathbb{R}^n))$ be a singular integral operator with kernel $K(x, y)$ satisfying the standard estimates* (II.3). *If $b \in BMO_{\mathcal{M}}(\mathbb{R}^n)$, then $C_{T,b}$ is bounded on $L_2(\mathbb{R}^n, L_2(\mathcal{M}))$ and*

$$\|C_{T,b}\|_{L_2(\mathbb{R}^n,L_2(\mathcal{M}))\to L_2(\mathbb{R}^n,L_2(\mathcal{M}))} \lesssim_n \left(1 + \|T1\|_{BMO(\mathbb{R}^n)} + \|T^*1\|_{BMO(\mathbb{R}^n)}\right) \|b\|_{BMO_{\mathcal{M}}(\mathbb{R}^n)}.$$

When T is a Riesz transform, Theorem II.8 coincides with the statement of [38, Theorem A.1], and fixes a small gap presenting in the argument of that theorem. Moreover, Theorem II.8 involves general singular integrals, which is new in the semicommutative setting and answers an open question in [38, Remark A.3].

The thesis is organized as follows. In Chapter 1, we set down notation and give some background on martingales in the commutative and noncommutative settings, spaces of homogeneous type and martingale Besov spaces. Then we divide the remaining content of the thesis into two parts separately, where detailed proofs of the above mentioned results are given.

The first part consists of Chapters 2 and 3. It is devoted to the multiplication between elements in martingale Hardy spaces and their dual spaces. In section 2.1, we prove Theorem I.1. In section 2.2, we present a characterization of martingale Lipschitz spaces $\Lambda_1(\alpha_p)$, which is of independent interest (see Theorem 2.2.4 and Remark 2.2.5 below), and then we show Theorem I.2. The remaining sections are concerned with spaces of homogeneous type. For the convenience of the reader, in section 1.8, we recall some definitions and facts regarding Hardy spaces and Lipschitz spaces on spaces of homogeneous type in the sense of Coifman and Weiss [23]. In section 3.1, we give new proofs of some results in [23] based on the martingale method and the existence of dyadic martingales on homogeneous spaces. In section 3.2, we establish analogues of Theorems I.1 and I.2 for dyadic martingales on spaces of homogeneous type; see Theorem 3.2.7 below. In the last section of the first part, we apply Theorem 3.2.7 to obtain a decomposition of products of functions in Hardy spaces and their dual spaces on spaces of homogeneous type.

The second part consists of Chapters 4 and 5. It is devoted to the Schatten class membership of noncommutative martingale paraproducts. In Section 4.1, we prove Theorem II.2. We proceed with our proof mainly by iteration. In Section 4.2 and Section 4.3, we show Theorem II.3 and Theorem II.4 respectively by transference and Theorem II.2. At the end, by virtue of Hytönen's dyadic representation for singular integral operators, we show Theorem II.6, Theorem II.7 and Theorem II.8 in Section 5.1, Section 5.2 and Section 5.3, respectively.

Throughout this thesis, the terms "homogeneous spaces" and "spaces of homogeneous type" will be used interchangeably. We will use the following notation: $A \lesssim B$ (resp.

$A \lesssim_\varepsilon B$) means that $A \leq CB$ (resp. $A \leq C_\varepsilon B$) for some absolute positive constant C (resp. a positive constant C_ε depending only on ε). $A \approx B$ or $A \approx_\varepsilon B$ means that these inequalities as well as their inverses hold.

Chapter 1

Preliminaries

In this chapter, we provide notation and background that will be used in the thesis. More precisely, Section 1.1, Section 1.2, Section 1.3 and Section 1.8 are devoted to introducing Musielak-Orlicz-type spaces, fundamental concepts and results about martingales, spaces of homogeneous type, which will needed for Chapter 2 and 3.

Section 1.4, Section 1.5, Section 1.6, and Section 1.7 are devoted to presenting semi-commutative d-adic martingales, noncommutative martingales, noncommutative L_p-spaces and martingale Besov spaces. These will be used for Chapter 4 and 5.

1.1 Notation

We will consider sums and intersections of quasi-normed spaces. For the convenience of the reader we recall these notions.

Definition 1.1.1. Let $(X_1, \|\cdot\|_{X_1}), (X_2, \|\cdot\|_{X_2})$ be two quasi-normed spaces and let X be a topological vector space X such that $X_1, X_2 \subset X$ continuously.

1. $(X_1 \cap X_2, \|\cdot\|_{X_1 \cap x_2})$ is the intersection of X_1 and X_2, where

$$\|x\|_{X_1 \cap X_2} := \max\{\|x\|_{X_1}, \|x\|_{X_2}\}$$

   for all $x \in X_1 \cap X_2$;

2. $(X_1 + X_2, \|\cdot\|_{X_1+X_2})$ is the sum of X_1 and X_2, where

$$\|x\|_{X_1+X_2} := \inf\{\|x_1\|_{X_1} + \|x_2\|_{X_2} : \ x = x_1 + x_2, \ x_1 \in X_1, \ x_2 \in X_2\}$$

   for all $x \in X_1 + X_2$.

For convenience, the sum $X_1 + X_2 + \cdots + X_n$ and the intersection $X_1 \cap X_2 + \cdots \cap X_n$ will also be denoted by $\sum_{k=1}^{n} X_k$ and $\bigcap_{k=1}^{n} X_k$, respectively.

Note that $(X_1 \cap X_2, \|\cdot\|_{X_1 \cap X_2})$ and $(X_1 + X_2, \|\cdot\|_{X_1+X_2})$ are both quasi-normed spaces. Moreover, if $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$ are Banach spaces, then $(X_1 \cap X_2, \|\cdot\|_{X_1 \cap X_2})$ and $(X_1 + X_2, \|\cdot\|_{X_1+X_2})$ are both Banach spaces.

1.2 Musielak–Orlicz-type spaces

We first recall some definitions and properties of Orlicz-type spaces and Musielak–Orlicz-type spaces. In what follows, $(\Omega, \mathcal{F}, \mu)$ denotes a σ-finite measure space.

A function $\Phi : [0,\infty) \to [0,\infty)$ is called an Orlicz function if it is strictly positive on $(0,\infty)$, non-decreasing, unbounded and $\Phi(0) = 0$. A measurable function $\Psi : \Omega \times [0,\infty) \to [0,\infty)$ is called a Musielak–Orlicz function if for all $x \in \Omega$, $\Psi(x,\cdot)$ is an Orlicz function.

The Musielak–Orlicz-type space $L^\Psi(\Omega)$ is the set consisting of all measurable functions f on Ω such that

$$\int_\Omega \Psi(x, \lambda^{-1}|f(x)|)d\mu < \infty$$

for some $\lambda > 0$. We equip $L^\Psi(\Omega)$ with the Luxembourg quasi-norm

$$\|f\|_{L^\Psi(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \Psi(x, \lambda^{-1}|f(x)|)d\mu \leqslant 1\right\}, \quad f \in L^\Psi(\Omega).$$

Let $p \in \mathbb{R}$. A Musielak–Orlicz function is said to be of uniformly lower type (respectively, upper type) p if there exists a positive constant C such that

$$\Psi(x, st) \leqslant Cs^p\Psi(x,t)$$

for all $x \in \Omega$, $t \geqslant 0$ and $s \in (0,1)$ (respectively, $s \in [1,\infty)$). In particular, if Ψ is of uniformly lower type p with $0 < p < 1$ and of uniformly upper type 1 then

$$\Psi(x, ct) \approx_c \Psi(x,t) \quad \text{for all } c > 0. \tag{1.2.1}$$

In the sequel, $\Psi(x,t)$ is always assumed to be of uniformly lower type p with $0 < p < 1$ and of uniformly upper type 1, and to be continuous in the t variable. For more information on Musielak–Orlicz spaces, we refer the reader to [8] and [95].

1.3 Martingales

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space. Given a filtration which consists of a sequence of σ-algebras

$$\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \subset \cdots \subset \mathcal{F}$$

such that $\sigma(\cup_{k=1}^\infty \mathcal{F}_k) = \mathcal{F}$, for a random variable $f \in L_1(\Omega, \mathcal{F}, P)$ and $k \in \mathbb{N}_+$, we set

$$f_k = \mathbb{E}(f \mid \mathcal{F}_k), \quad d_k f = f_k - f_{k-1},$$

where we adopt the convention that $f_0 = 0$. We shall also denote f_k by $\mathbb{E}_k(f)$. A sequence $f = \{f_k\}_{k\geq 0}$ is called a martingale with respect to $\cup_{k=1}^\infty \mathcal{F}_k$ if $f_k = \mathbb{E}(f_{k+1} \mid \mathcal{F}_k)$ for every $k \geq 1$, and $\{d_k f\}_{k\geq 1}$ is called the martingale difference sequence of $f = \{f_k\}_{k\geq 0}$. To simplify notation, we write $L_p(\Omega)$ instead of $L_p(\Omega, \mathcal{F}, P)$.

Definition 1.3.1. If $f = \{f_k\}_{k\geq 0}$ and $\{d_k f\}_{k\geq 1}$ are as above, we define:

1. the maximal function
$$f^* = \sup_{k\geqslant 0} |f_k|;$$

2. the square function

$$S(f) = \left(\sum_{k=1}^{\infty} |d_k f|^2 \right)^{1/2};$$

3. the conditional square function

$$s(f) = \left(\sum_{k=1}^{\infty} \mathbb{E}_{k-1} |d_k f|^2 \right)^{1/2}.$$

There are several types of martingale Hardy spaces, which are defined in terms of maximal functions, square functions and conditional square functions.

Definition 1.3.2. For $1 \leq p < \infty$, the martingale Hardy spaces $h^p(\Omega), H^p(\Omega), H^p_m(\Omega)$ are defined as follows

$$\begin{aligned} h^p(\Omega) &= \{f \in L_1(\Omega) : \|f\|_{h^p} := \|s(f)\|_p < \infty\}, \\ H^p(\Omega) &= \{f \in L_1(\Omega) : \|f\|_{H^p} := \|S(f)\|_p < \infty\}, \\ H^p_m(\Omega) &= \{f \in L_1(\Omega) : \|f\|_{H^p_m} := \|f^*\|_p < \infty\}, \end{aligned}$$

respectively.

For $0 < p < 1$, $h^p(\Omega)$ is defined as the completion of the space $\{f \in L_1(\Omega) : \|f\|_{h^p} := \|s(f)\|_p < \infty\}$ with respect to the norm $\|\cdot\|_{h^p}$. Similarly, $H^p(\Omega)$ is defined as the completion of the space $\{f \in L_1(\Omega) : \|f\|_{H^p} := \|S(f)\|_p < \infty\}$ with respect to the norm $\|\cdot\|_{H^p}$, and $H^p_m(\Omega)$ is defined as the completion of the space $\{f \in L_1(\Omega) : \|f\|_{H^p_m} := \|f^*\|_p < \infty\}$ with respect to the norm $\|\cdot\|_{H^p_m}$.

In general, the above three martingale Hardy spaces are different. However, for $1 \leq p < \infty$, $H^p(\Omega) = H^p_m(\Omega)$ (see [13], [27], [92]).

Definition 1.3.3. (Regular filtration) A filtration is regular if there exists a constant $C > 0$ such that for all $k \geq 2, F_k \in \mathcal{F}_k$, there exists a $G_k \in \mathcal{F}_{k-1}$ satisfying

$$F_k \subset G_k, \quad P(G_k) \leq C \cdot P(F_k).$$

In addition, a martingale $f = \{f_k\}_{k \geq 0}$ with respect to such a regular filtration is called a regular martingale.

Remark 1.3.4. Suppose that for a positive random variable $f \in L_1(\Omega)$ the corresponding martingale $\{f_k\}_{k \geqslant 0}$ is regular. Then for any $k \geq 2$

$$f_k \leqslant A \cdot f_{k-1},$$

where $A > 0$ is a constant that depends only on the constant C of Definition 1.3.3.

See [51] for more information about regular filtrations and martingales.

Remark 1.3.5. For regular filtrations, $H^p(\Omega) = h^p(\Omega) = H^p_m(\Omega)$ for all $0 < p < \infty$. See [91], [92] and [51] for more information.

An important aspect of martingale Hardy spaces is that they admit atomic decompositions. The definition of atoms in the martingale setting is given below.

Definition 1.3.6. A random variable $a : \Omega \to \mathbb{C}$ is called a martingale simple (p, q)-atom $(0 < p \leq 1, 1 \leq q \leqslant \infty)$ if there exist $k \in \mathbb{N}$ and $A \in \mathcal{F}_k$ such that

1. $\mathbb{E}_k(a) = 0$;
2. $\mathrm{supp}(a) \subset A$;
3. $\|a\|_q \leqslant P(A)^{\frac{1}{q}-\frac{1}{p}}$,

where $\frac{1}{q} := 0$ when $q = \infty$ as convention.

Definition 1.3.7. We define the martingale atomic Hardy spaces $H_{\mathrm{at}}^{p,q}(\Omega)$ for $0 < p < 1 \leqslant q \leqslant \infty$ or $p = 1,\ 1 < q \leq \infty$ as follows

$$H_{\mathrm{at}}^{p,q}(\Omega) = \left\{ f = \sum_{j=0}^{\infty} \lambda_j a^j \,:\, a^j \text{ is a simple } (p,q)\text{-atom and } \sum_{j=0}^{\infty} |\lambda_j|^p < \infty \right\},$$

equipped with the quasi-norm

$$\|f\|_{H_{\mathrm{at}}^{p,q}(\Omega)} := \inf \left\{ \Big(\sum_{j=0}^{\infty} |\lambda_j|^p \Big)^{\frac{1}{p}} : f = \sum_{j=0}^{\infty} \lambda_j a^j, \text{ where } a^j \text{ is a simple } (p,q)\text{-atom} \right\}.$$

It is well-known that $h^p(\Omega) = H_{\mathrm{at}}^{p,2}(\Omega)$ when $0 < p \leq 1$ (see [91]). In particular, if the martingale filtration is regular, then $h^p(\Omega) = H_{\mathrm{at}}^{p,q}(\Omega)$ when $0 < p \leq 1$ and $1 < q \leq \infty$. The following result is the atomic decomposition of $H^1(\Omega)$, which follows from the noncommutative result in [71]. In particular, it reveals the relationship between $H^1(\Omega)$ and $h^1(\Omega)$ and shows that $H^1(\Omega) \neq h^1(\Omega)$ for general martingales.

Theorem 1.3.8. *We have $H^1(\Omega) = h^1(\Omega) + h_d^1(\Omega)$, where h_d^1 denotes the diagonal Hardy space of martingale differences*

$$h_d^1(\Omega) = \left\{ h \in L_1(\Omega) : \|h\|_{h_d^1(\Omega)} := \sum_{k=1}^{\infty} \|d_k h\|_1 < \infty \right\}.$$

We now introduce the martingale BMO and *bmo* spaces, which are the duals of $H^1(\Omega)$ and $h^1(\Omega)$, respectively (see Theorem 1.3.12 below).

Definition 1.3.9. Assume $f, g \in L_2(\Omega)$. We say that f is a martingale BMO function if

$$\|f\|_{BMO(\Omega)} := \sup_{n \geqslant 1} \|\mathbb{E}_n |f - f_{n-1}|^2\|_\infty^{1/2} < \infty.$$

We say that g is a martingale *bmo* function if

$$\|g\|_{bmo(\Omega)} := \sup_{n \geqslant 0} \|\mathbb{E}_n |g - g_n|^2\|_\infty^{1/2} < \infty.$$

Denote by $BMO(\Omega)$ and $bmo(\Omega)$ the spaces consisting of all martingale BMO and *bmo* functions, respectively.

For regular martingales, $BMO(\Omega) = bmo(\Omega)$. The following result is the so-called martingale John–Nirenberg inequality and can be found in [32].

Theorem 1.3.10. *There exists a constant $\kappa > 0$ such that for any $f \in BMO(\Omega)$ with $\|f\|_{BMO(\Omega)} \leq \kappa$, we have*

$$\mathbb{E}\left(e^{|f|}\right) \leqslant 1.$$

Remark 1.3.11. From the martingale John–Nirenberg inequality, we have for any $1 \leq p < \infty$,

$$\|f\|_{BMO(\Omega)} \approx_p \sup_{n\geqslant 1} \|\mathbb{E}_n|f - f_{n-1}|^p\|_\infty^{1/p}.$$

However, the above John–Nirenberg inequality fails for $bmo(\Omega)$ in the general setting.

For the following duality theorem, see [32], [51], [92].

Theorem 1.3.12. $(H^1(\Omega))^* = BMO(\Omega)$ *and* $(h^1(\Omega))^* = bmo(\Omega)$.

The following proposition, which can be found in [24] and [32], is a consequence of Theorems 1.3.8 and 1.3.12 and it gives a description of the relationship between $BMO(\Omega)$ and $bmo(\Omega)$. In particular, it implies that $BMO(\Omega) \subsetneqq bmo(\Omega)$ for general martingales.

Proposition 1.3.13. *Assume f is a martingale BMO function. Then*

$$\|f\|_{BMO(\Omega)} \approx \|f\|_{bmo(\Omega)} + \sup_{k\geq 1} \|d_k f\|_\infty. \tag{1.3.1}$$

We end this section with the definition of martingale Musielak–Orlicz Hardy spaces and the generalized Hölder inequality.

Definition 1.3.14. The martingale Musielak–Orlicz Hardy space $H^\Psi(\Omega)$ (where Ψ is described in Section 1.2) is the space consisting of all martingales $f = \{f_k\}_{k\geq 0}$ such that the square function $S(f) \in L^\Psi(\Omega)$. Moreover, we define the quasi-norm

$$\|f\|_{H^\Psi(\Omega)} := \|S(f)\|_{L^\Psi(\Omega)}.$$

If Ψ is replaced by an Orlicz function Φ, the corresponding Hardy–Orlicz space $H^\Phi(\Omega)$ is defined in an analogous way.

To obtain the generalized Hölder inequality, we introduce a particular Orlicz space $L^\Phi(\Omega)$, where

$$\Phi(t) := \frac{t}{\log(e+t)}, \quad t \geq 0. \tag{1.3.2}$$

Note that Φ is an Orlicz function of uniformly lower type p $(0 < p < 1)$ and upper type 1, which guarantees that the space $L^\Phi(\Omega)$ is a quasi-normed space. Note that $L_1(\Omega) \subset L^\Phi(\Omega)$.

Remark 1.3.15. It follows from [57] that if $f = \{f_k\}_{k\geq 0}$ is a regular martingale, then the martingale Hardy–Orlicz space $H^\Phi(\Omega)$ can also be characterized by martingale maximal functions and conditional square functions. For any $f \in H^\Phi(\Omega)$ one has

$$\|f\|_{H^\Phi(\Omega)} = \|S(f)\|_{L^\Phi(\Omega)} \approx \|f^*\|_{L^\Phi(\Omega)} \approx \|s(f)\|_{L^\Phi(\Omega)}.$$

The following lemma is a variant of [9, Proposition 2.1] in the martingale setting.

Lemma 1.3.16. *Assume $(\Omega, \mathcal{F}, P)$ is a probability space, $f \in L_1(\Omega)$ and $g \in BMO(\Omega)$. Then $f \cdot g \in L^\Phi(\Omega)$ and*

$$\|f \cdot g\|_{L^\Phi(\Omega)} \lesssim \|f\|_1 \|g\|_{BMO(\Omega)}. \tag{1.3.3}$$

Proof. The proof is similar to the proof of the corresponding Euclidean result and we will only outline it here for the convenience of the reader. By [9, Lemma 2.1], one has

$$\frac{st}{M + \log(e + st)} \leq e^{t-M} + s. \tag{1.3.4}$$

for all $M \geq 0, s \geq 0, t \geq 0$.

When $\|f\|_1 = 0$ or $\|g\|_{BMO(\Omega)} = 0$, (1.3.3) trivially holds. Assume $g \in BMO(\Omega)$ with $\|g\|_{BMO(\Omega)} > 0$ and $f \in L_1(\Omega)$ with $\|f\|_1 > 0$. Let κ be the constant in Theorem 1.3.10, $M = 0$, $t = \frac{\kappa|g(x)|}{\|g\|_{BMO(\Omega)}}$ and $s = \frac{|f(x)|}{\|f\|_1}$. Then by Theorem 1.3.10 and (1.3.4), we have

$$\int_\Omega \Phi\left(\frac{|f(x) \cdot g(x)|}{\kappa^{-1}\|f\|_1\|g\|_{BMO(\Omega)}}\right) dP \leq \int_\Omega e^{\frac{\kappa|g(x)|}{\|g\|_{BMO(\Omega)}}} dP + \left\|\frac{f}{\|f\|_1}\right\|_1 \leq 2. \tag{1.3.5}$$

Hence, from (1.2.1) we conclude

$$\|f \cdot g\|_{L^\Phi(\Omega)} \lesssim \kappa^{-1}\|f\|_1\|g\|_{BMO(\Omega)},$$

which completes the proof of the lemma. □

We will refer to (1.3.3) as the generalized Hölder inequality.

1.4 d-adic martingales

Let $d \geq 2$ be a fixed integer. We are particularly interested in d-adic martingales since it is closely related to dyadic martingales on Euclidean spaces. In this section, we give a general definition of d-adic martingales. Afterwards we will present an orthonormal basis of Haar wavelets for d-adic martingales, which will be used to represent martingale paraproducts and to define martingale Besov spaces for semicommutative d-adic martingales (to be defined in Section 1.7).

Let Ω be a measure space endowed with a σ-finite measure μ. Assume that in Ω, there exists a family of measurable sets $I_{n,k}$ for $n, k \in \mathbb{Z}$ satisfying the following properties:

1. $I_{n,k}$ are pairwise disjoint for any k if n is fixed;
2. $\cup_{k\in\mathbb{Z}} I_{n,k} = \Omega$ for every n;
3. $I_{n,k} = \cup_{q=1}^d I_{n+1,kd+q-1}$ for any n, k, so each $I_{n,k}$ is a union of d disjoint subsets $I_{n+1,kd+q-1}$;
4. $\mu(I_{n,k}) = d^{-n}$ for any n, k.

Then $I_{n,k}$ are called d-adic intervals, and let $\mathcal{D}$ be the family of all such d-adic intervals. For $I \in \mathcal{D}$, let $\tilde{I}$ be the parent interval of I, and $I(j)$ the j-th subinterval of I, namely

$$(I_{n,k})(j) = I_{n+1,kd+j-1}, \quad \forall n, k \in \mathbb{Z}, 1 \leq j \leq d.$$

Denote by $\mathcal{D}_n$ the collection of d-adic intervals of length d^{-n} in $\mathcal{D}$. Given $I \in \mathcal{D}$, let $\mathcal{D}(I)$ be the collection of d-adic intervals contained in I, and $\mathcal{D}_n(I)$ the intersection of $\mathcal{D}_n$ and $\mathcal{D}(I)$. For each $n \in \mathbb{Z}$, denote by $\mathcal{F}_n$ the σ-algebra generated by the d-adic intervals $I_{n,k}$, $\forall k \in \mathbb{Z}$. Denote by $\mathcal{F}$ the σ-algebra generated by all d-adic intervals for all $I_{n,k}$, $\forall n, k \in \mathbb{Z}$.

Then $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ is a filtration associated with the measure space $(\Omega, \mathcal{F}, \mu)$. In the sense of Definition 1.3.3, $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ is regular. Denote by $L_1^{\mathrm{loc}}(\Omega)$ the family of all locally integrable functions g on Ω, that is, $g \in L_1(I_{n,k})$ for all $n, k \in \mathbb{Z}$. For a locally integrable function $g \in L_1^{\mathrm{loc}}(\Omega)$, the sequence $(g_n)_{n\in\mathbb{Z}}$ is called a d-adic martingale, where

$$g_n = \mathbb{E}(g|\mathcal{F}_n) = \sum_{k=-\infty}^{\infty} \frac{\mathbb{1}_{I_{n,k}}}{\mu(I_{n,k})} \int_{I_{n,k}} g \, d\mu.$$

The martingale differences are defined as $d_n g = g_n - g_{n-1}$ for any $n \in \mathbb{Z}$. We also denote g_n by $\mathbb{E}_n(g)$ $(n \in \mathbb{Z})$ as usual.

Remark 1.4.1. We would like to remark that d-adic martingales are slightly different from the martingales defined in the previous Section 1.3, since we are no longer working with a probability space, and the filtration for d-adic martingales is indexed by $\mathbb{Z}$.

Definition 1.4.2. Let $\omega = e^{\frac{2\pi \mathrm{i}}{d}}$ (here i is the imaginary number). For any $I = I_{n,k} \in \mathcal{D}$, define

$$h_I^i = d^{n/2} \sum_{j=0}^{d-1} \omega^{i(j+1)} \mathbb{1}_{I_{n+1,kd+j}}, \quad \forall\ 1 \le i \le d-1,$$

and $h_I^0 := d^{n/2} \mathbb{1}_I$.

Then $\{h_I^i\}_{I\in\mathcal{D}, 1\le i\le d-1}$ is an orthonormal basis on $L_2(\Omega)$ because $\forall g \in L_2(\Omega)$

$$g = \sum_{k=-\infty}^{\infty} d_k g = \sum_{k=-\infty}^{\infty} \left(\sum_{|I|=d^{-k+1}} \sum_{i=1}^{d-1} h_I^i \langle h_I^i, g\rangle \right).$$

We call $\{h_I^i\}_{I\in\mathcal{D}, 1\le i\le d-1}$ the system of Haar wavelets. Note that for any $1 \le i, j \le d-1$,

$$h_I^i \cdot h_I^j = \mu(I)^{-1/2} h_I^{\overline{i+j}}, \tag{1.4.1}$$

where $\overline{i+j}$ is the remainder in $[1, d]$ modulo d. The equality (1.4.1) is vital in our proof of Theorem II.2 and Lemma 5.1.1.

Example 1.4.3. A natural example of d-adic martingales is where $\Omega = \mathbb{R}$, $\mu = m$ and $I_{n,k}$ are defined as follows

$$I_{n,k} = [kd^{-n}, (k+1)d^{-n}), \quad \forall n, k \in \mathbb{Z}.$$

In the sequel, for simplicity of notation, we will always assume that $\Omega = \mathbb{R}$ as this does not change the d-adic martingale structure. Denote also by $|I|$ the length $m(I)$ of $I \in \mathcal{D}$.

In particular, if $d = 2^n$ for some $n \in \mathbb{N}$, we can let $\Omega = \mathbb{R}^n$, and define

$$\mathcal{D}_k = \{2^{-k}([0,1)^n + q) : q \in \mathbb{Z}^n\}, \quad \forall k \in \mathbb{Z}.$$

Then $\mathcal{D} = \{2^{-k}([0,1)^n + q) : k \in \mathbb{Z}, q \in \mathbb{Z}^n\}$ is the family of all 2^n-adic intervals. Indeed, this is the dyadic filtration on $\mathbb{R}^n$.

As in Definition 1.3.9, we define the d-adic martingale BMO space as follows:

Definition 1.4.4. The martingale BMO space of d-adic martingale denoted by $BMO^d(\mathbb{R})$ is the space of all locally integral functions b such that

$$\|b\|_{BMO^d(\mathbb{R})} = \sup_{n\in\mathbb{Z}} \left\| \mathbb{E}_n\Big(\sum_{k=n+1}^{\infty} |d_k b|^2 \Big) \right\|_\infty^{1/2} < \infty. \tag{1.4.2}$$

Remark 1.4.5. Notice that $BMO^d(\mathbb{R})$ is the martingale BMO space associated with d-adic martingale for $d \geq 2$.

For $h \in L_1^{\rm loc}(\mathbb{R})$, we define the d-adic martingale square function

$$S(h) = \left(\sum_{k\in\mathbb{Z}} |d_k h|^2 \right)^{1/2}.$$

Definition 1.4.6. The d-adic martingale Hardy space is defined by

$$H_1^d(\mathbb{R}) = \{h \in L_1(\mathbb{R}) : \|h\|_{H_1^d(\mathbb{R})} := \|S(h)\|_{L_1(\mathbb{R})} < \infty\}. \tag{1.4.3}$$

Remark 1.4.7. By the same arguments in Theorem 1.3.12, we have $(H_1^d(\mathbb{R}))^* = BMO^d(\mathbb{R})$.

1.5 Noncommutative L_p-spaces

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Denote by $\mathcal{M}_+$ the positive part of $\mathcal{M}$. Let $\mathcal{S}_+(\mathcal{M})$ be the set of all $x \in \mathcal{M}_+$ whose support projection has a finite trace, and $\mathcal{S}(\mathcal{M})$ be the linear span of $\mathcal{S}_+(\mathcal{M})$. Then $\mathcal{S}(\mathcal{M})$ is a w^*-dense $*$-subalgebra of $\mathcal{M}$. Let $x \in \mathcal{S}(\mathcal{M})$, then $|x|^p \in \mathcal{S}(\mathcal{M})$ for any $0 < p < \infty$, where $|x| := (x^*x)^{1/2}$. Define

$$\|x\|_p = (\tau(|x|^p))^{1/p}.$$

Thus $\|\cdot\|_p$ is a norm for $p \geq 1$, and a p-norm for $0 < p < 1$. The noncommutative L_p-space associated with $(\mathcal{M}, \tau)$ is the completion of $(\mathcal{S}(\mathcal{M}), \|\cdot\|_p)$ for $0 < p < \infty$ denoted by $L_p(\mathcal{M}, \tau)$. Let $L_0(\mathcal{M}, \tau)$ be the family of all measurable operators with respect to $(\mathcal{M}, \tau)$. We also write $L_p(\mathcal{M}, \tau)$ simply by $L_p(\mathcal{M})$ for short. When $p = \infty$, we set $L_\infty(\mathcal{M}) := \mathcal{M}$ equipped with the operator norm. In particular, when $p = 2$, $L_2(\mathcal{M})$ is a Hilbert space. We will view $\mathcal{M}$ as a von Neumann algebra on $L_2(\mathcal{M})$ by left multiplication, namely $\mathcal{M} \hookrightarrow B(L_2(\mathcal{M}))$ via the embedding $x \longmapsto L_x \in B(L_2(\mathcal{M}))$, where $x \in \mathcal{M}$ and $L_x(y) := x \cdot y \in L_2(\mathcal{M})$ for any $y \in L_2(\mathcal{M})$. Hence in this way, $\mathcal{M}$ is in its standard form. It is well-known that for $1 \leq p < \infty$ and $p' = \frac{p}{p-1}$

$$\Big(L_p(\mathcal{M})\Big)^* = L_{p'}(\mathcal{M}).$$

We refer the reader to [78] for a detailed exposition of noncommutative L_p-spaces.

If $\mathcal{H}$ is a Hilbert space and $\mathcal{M} = B(\mathcal{H})$ equipped with the usual trace Tr, then $L_p(\mathcal{M})$ is the Schatten p-class on $\mathcal{H}$ and denoted by $S_p(\mathcal{H})$. Denote by $\eta_1 \otimes \eta_2$ the rank 1 operator on $\mathcal{H}$ given by

$$\eta_1 \otimes \eta_2(\eta) := \eta_1 \langle \eta_2, \eta \rangle, \quad \forall \eta \in \mathcal{H},$$

where η_1 and η_2 are two vectors in $\mathcal{H}$. Then $\eta_1 \otimes \eta_2 \in \mathcal{S}(B(\mathcal{H}))$, and for any $0 < p \leq \infty$

$$\|\eta_1 \otimes \eta_2\|_{S_p(\mathcal{H})} = \|\eta_1\|_{\mathcal{H}} \|\eta_2\|_{\mathcal{H}}.$$

Now we present the tensor product of von Neumann algebras. Assume that each $\mathcal{M}_k$ $(k = 1, 2)$ is equipped with a normal semifinite faithful trace τ_k. Then the tensor product of $\mathcal{M}_1$ and $\mathcal{M}_2$ denoted by $\mathcal{M}_1 \otimes \mathcal{M}_2$ is the w^*-closure of $\text{span}\{x_1 \otimes x_2 | x_1 \in \mathcal{M}_1, x_2 \in \mathcal{M}_2\}$ in $B(L_2(\mathcal{M}_1) \otimes L_2(\mathcal{M}_2))$. Here $L_2(\mathcal{M}_1) \otimes L_2(\mathcal{M}_2)$ is the Hilbert space tensor product of $L_2(\mathcal{M}_1)$ and $L_2(\mathcal{M}_2)$.

It is well-known that there exists a unique normal semifinite faithful trace τ on the von Neumann algebra tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$ such that

$$\tau(x_1 \otimes x_2) = \tau_1(x_1)\tau_2(x_2), \quad \forall x_1 \in \mathcal{S}(\mathcal{M}_1), \forall x_2 \in \mathcal{S}(\mathcal{M}_2).$$

τ is called the tensor product of τ_1 and τ_2 and denoted by $\tau_1 \otimes \tau_2$.

Let $\mathbb{M}_d$ be the algebra of $d \times d$ matrices equipped with the usual trace Tr. Denote by $\text{tr}_d := \frac{1}{d}\text{Tr}$ the normalized trace on $\mathbb{M}_d$. For $k \geq 1$, let

$$(\mathbb{M}_d^{\otimes k}, \text{tr}_d^{\otimes k}) = \bigotimes_{i=1}^{k} (\mathbb{M}_d, \text{tr}_d)$$

be the tensor products in the sense of von Neumann algebras. We define

$$(\bigotimes_{k=1}^{\infty} \mathbb{M}_d, \bigotimes_{k=1}^{\infty} \text{tr}_d) = \bigotimes_{i=1}^{\infty} (\mathbb{M}_d, \text{tr}_d)$$

as the inductive limit of $(\mathbb{M}_d^{\otimes k}, \text{tr}_d^{\otimes k})_{k \geq 1}$, also denoted by $\bigotimes_{k=1}^{\infty} \mathbb{M}_d$ for simplicity. (see [34, Lemma 4.5] for the inductive limit)

In this thesis, we are concerned with the von Neumann algebra tensor product of $B(L_2(\mathbb{R}))$ and $\mathcal{M}$, where $B(L_2(\mathbb{R}))$ is endowed with the usual trace Tr, and $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ.

In the sequel, we will identify any left multiplication $L_x \in B(L_2(\mathcal{M}))$ and $x \in \mathcal{M}$. Then for any $T \in B(L_2(\mathbb{R}))$, $T \otimes L_x \in B(L_2(\mathbb{R})) \otimes \mathcal{M} \hookrightarrow B(L_2(\mathbb{R})) \otimes B(L_2(\mathcal{M})) = B(L_2(\mathbb{R}) \otimes L_2(\mathcal{M}))$, and

$$\|T \otimes L_x\|_{L_p(B(L_2(\mathbb{R})) \otimes \mathcal{M})} = \|T\|_{S_p(L_2(\mathbb{R}))} \|x\|_{L_p(\mathcal{M})}.$$

In the following, we write $T \otimes L_x$ as $x \cdot T$, and thus

$$(x \cdot T)(f) = T \otimes L_x(f) = x \cdot T(f), \quad \forall f \in L_2(\mathbb{R}, L_2(\mathcal{M})). \tag{1.5.1}$$

1.6 Noncommutative martingales

This section is devoted to noncommutative martingales. The reader is referred to [78] and [94]. Assume $\mathcal{M}$ is a von Neumann algebra equipped with a normal faithful semifinite trace τ and $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ such that the restriction of τ to $\mathcal{N}$ is again semifinite. Then there exists a unique map $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ satisfying the following properties:

1. $\mathcal{E}$ is a normal contractive positive projection from $\mathcal{M}$ onto $\mathcal{N}$;
2. $\mathcal{E}(axb) = a\mathcal{E}(x)b$ for any $x \in \mathcal{M}$ and $a, b \in \mathcal{N}$;
3. $\tau \circ \mathcal{E} = \tau$.

$\mathcal{E}$ is called the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{N}$. Besides, $\mathcal{E}$ extends to a contractive positive projection from $L_p(\mathcal{M})$ onto $L_p(\mathcal{N})$ for any $1 \leq p < \infty$, still denoted by $\mathcal{E}$.

Recall that a filtration of von Neumann subalgebras of $\mathcal{M}$ is an nondecreasing sequence $(\mathcal{M}_n)_{n\geq 1}$ of von Neumann subalgebras of $\mathcal{M}$ such that $\cup_n \mathcal{M}_n$ is w^*-dense in $\mathcal{M}$ and the restriction of τ to $\cup_n \mathcal{M}_n$ is also semifinite for every n. Let $\mathcal{E}_n$ be the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$. A sequence $x = (x_n) \subset L_1(\mathcal{M})$ is called a martingale with respect to $(\mathcal{M}_n)_{n\geq 1}$ if $\mathcal{E}_n(x_{n+1}) = x_n$ for every $n \geq 1$. In addition, if $x_n \in L_p(\mathcal{M})$ with $p \geq 1$, x is called an L_p-martingale with respect to $(\mathcal{M}_n)_{n\geq 1}$. Denote the martingale differences by $d_n x = x_n - x_{n-1}$ for $n \geq 1$ with the convention $x_0 = 0$.

Remark 1.6.1. Let $1 < p \leq \infty$ and $x = (x_n)$ a noncommutative martingale such that

$$\sup_n \|x_n\|_p < \infty.$$

Then there exists $x_\infty \in L_p(\mathcal{M})$ such that $x_n = \mathcal{E}_n(x_\infty)$ for every n.

We are going to introduce two particular noncommutative martingales: the CAR algebras and tensor products of matrix algebras.

1.6.1 CAR algebra

We consider the following Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

For $n \geq 1$, define

$$c_{2n-1} = \sigma_0 \otimes \cdots \sigma_0 \otimes \sigma_1 \otimes 1 \otimes 1 \cdots, \quad c_{2n} = \sigma_0 \otimes \cdots \sigma_0 \otimes \sigma_2 \otimes 1 \otimes 1 \cdots,$$

where σ_1 and σ_2 occur in the n-th position. Then $(c_n)_{n\geq 1}$ are selfadjoint unitary operators and satisfy the following canonical anticommutation relations (CAR):

$$c_j c_k + c_k c_j = 2\delta_{jk}, \quad j, k \geq 1. \tag{1.6.1}$$

The CAR algebra (Clifford algebra) denoted by $\mathcal{C}$ is the von Neumann algebra generated by $(c_n)_{n\geq 1}$. Let us give more details.

Let $\mathcal{I}$ denote the family of all finite subsets of $\mathbb{N}$. For a nonempty $A \in \mathcal{I}$, we arrange the integers of A in an increasing order and write $A = \{k_1 < k_2 < \cdots < k_n\}$. Define $\max(A) := k_n$ and

$$c_A = c_{k_1} c_{k_2} \cdots c_{k_n}.$$

If $A = \emptyset$, we set $\max(\emptyset) := 1$ and $c_A = 1$. Then c_A is unitary for any $A \in \mathcal{I}$. If A is a singleton $\{k\}$, we still use c_k instead of $c_{\{k\}}$. Let $\mathcal{C}_0$ be the family of all finite linear

combinations of $(c_A)_{A\in\mathcal{I}}$. Then $\mathcal{C}_0$ is an involutive algebra. Define τ to be the linear functional on $\mathcal{C}_0$ given by

$$\tau(x) = \alpha_\emptyset, \tag{1.6.2}$$

where $x = \sum_{A\in\mathcal{I}} \alpha_A c_A$. One can check that τ is a positive faithful tracial state on $\mathcal{C}_0$. Then the CAR algebra $\mathcal{C}$ is the von Neumann algebra of the GNS representation of τ. Note that $(c_A)_{A\in\mathcal{I}}$ is an orthonormal basis of $L_2(\mathcal{C})$. We refer the reader to [83] and [79] for more information on CAR algebra.

Let $\mathcal{C}_n$ be the von Neumann subalgebra generated by $\{c_A : \max(A) \le n\}$ for any $n \ge 1$. It is clear that $\mathcal{C}_n$ is of dimension 2^n, and $(\mathcal{C}_n)_{n\ge1}$ is a filtration of $\mathcal{C}$. Then for any $b \in L_p(\mathcal{C})$ $(1 \le p \le \infty)$,

$$d_n b = \sum_{\max(A)=n} \hat{b}(A) c_A, \quad \forall\ n \ge 1,$$

where $\hat{b}(A) = \tau(c_A^* \cdot b)$.

1.6.2 Tensor products of matrix algebras

Let $\mathscr{M}_n = \mathbb{M}_d^{\otimes n}$ be endowed with the normalized trace $\mathrm{tr}_d^{\otimes n}$. We embed $\mathscr{M}_n$ into $\mathscr{M}$ via the map $x \in \mathscr{M}_n \longmapsto x \otimes 1 \otimes 1 \otimes \cdots \in \mathscr{M}$. Then $(\mathscr{M}_n)_{n\ge1}$ is a natural filtration of $\mathscr{M}$. We will give an orthonormal basis of $L_2(\mathscr{M})$ in Section 4.3.

1.7 Semicommutative d-adic martingales and martingale Besov spaces

In this section, we are concerned with semicommutative d-adic martingales and martingale Besov spaces. Firstly, we introduce the definition of semicommutative d-adic martingales. Then we give the definitions of martingale Besov spaces for semicommutative d-adic martingales, CAR algebra and $\mathscr{M} = \bigotimes_{k=1}^{\infty} \mathbb{M}_d$.

We define the semicommutative d-adic martingales in the same way as in the commutative setting. Similarly, denote by $L_1^{\mathrm{loc}}(\mathbb{R}, L_1(\mathcal{M}))$ the family of all f such that $\mathbb{1}_{I_{n,k}} \cdot f \in L_1(\mathbb{R}, L_1(\mathcal{M}))$ for any $n, k \in \mathbb{Z}$. Then $\forall f \in L_1^{\mathrm{loc}}(\mathbb{R}, L_1(\mathcal{M}))$, the sequence $(f_n)_{n\in\mathbb{Z}}$ is called a semicommutative d-adic martingale, where

$$f_n = \mathbb{E}(f|\mathcal{F}_n) = \sum_{k=-\infty}^{\infty} \frac{\mathbb{1}_{I_{n,k}}}{\mu(I_{n,k})} \int_{I_{n,k}} f\ d\mu. \tag{1.7.1}$$

For any $f \in L_1(\mathbb{R}, L_1(\mathcal{M}))$ and $g \in L_\infty(\mathbb{R})$, define

$$\langle g, f\rangle = \int_{\mathbb{R}} \overline{g} \cdot f\ dm.$$

One can easily deduce that $\langle g, f\rangle \in L_1(\mathcal{M})$ from the triangle inequality. By a slight abuse of notation, we use the same notation $\langle\cdot,\cdot\rangle$ to denote the inner product in any given Hilbert space. Besides, by (1.7.1), the martingale differences are given by

$$d_n f = \sum_{|I|=d^{-n+1}} \sum_{i=1}^{d-1} h_I^i \otimes \langle h_I^i, f\rangle, \quad \forall f \in L_1^{\mathrm{loc}}(\mathbb{R}, L_1(\mathcal{M})) \text{ and } n \in \mathbb{Z}.$$

Note that $\langle h_I^i, f\rangle \in L_1(\mathcal{M})$. Later, we will give a more general definition of martingale differences.

Remark 1.7.1. By Remark 1.6.1, if $f \in L_p(\mathbb{R}, L_p(\mathcal{M}))$ for $1 < p \leq \infty$, then

$$\sum_{n=-k}^{k} d_n f \longrightarrow f \quad \text{as} \quad k \to \infty$$

in $L_p(\mathbb{R}, L_p(\mathcal{M}))$ (in w^*-topology for $p = \infty$).

We will utilize h_I^i to give a direct representation of π_b, which is easier to handle. It is well-known that $L_2(\mathbb{R}, L_2(\mathcal{M})) = L_2(\mathbb{R}) \otimes L_2(\mathcal{M})$. In the sequel, for any $f \in L_2(\mathbb{R})$ and $x \in L_2(\mathcal{M})$, we use "$x \cdot f$ " (or "$f \cdot x$ ") to denote $f \otimes x \in L_2(\mathbb{R}, L_2(\mathcal{M}))$ for the sake of simplicity.

Now we calculate π_b. Let $b \in L_1^{\rm loc}(\mathbb{R}, L_1(\mathcal{M}))$. For $f \in L_2(\mathbb{R}, L_2(\mathcal{M}))$, we have

$$\begin{aligned}
\pi_b(f) &= \sum_{k=-\infty}^{\infty} d_k b \cdot f_{k-1} \\
&= \sum_{k=-\infty}^{\infty} \Bigg(\sum_{|I|=d^{-k+1}} \sum_{i=1}^{d-1} h_I^i \otimes \langle h_I^i, b\rangle \Bigg) \Bigg(\sum_{|I|=d^{-k+1}} \mathbb{1}_I \otimes \left\langle \frac{\mathbb{1}_I}{|I|}, f \right\rangle \Bigg) \\
&= \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} h_I^i \otimes \langle h_I^i, b\rangle \left\langle \frac{\mathbb{1}_I}{|I|}, f \right\rangle,
\end{aligned} \tag{1.7.2}$$

which by (1.5.1) can be rewritten as

$$\pi_b(f) = \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} h_I^i \langle h_I^i, b\rangle \left\langle \frac{\mathbb{1}_I}{|I|}, f \right\rangle. \tag{1.7.3}$$

The adjoint operator of π_b is given by $\forall f \in L_2(\mathbb{R}, L_2(\mathcal{M}))$

$$\begin{aligned}
\pi_b^*(f) &= \sum_{k\in\mathbb{Z}} \mathbb{E}_{k-1}(d_k b^* d_k f) \\
&= \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \frac{\mathbb{1}_I}{|I|} \langle h_I^i, b\rangle^* \langle h_I^i, f\rangle \\
&= \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \frac{\mathbb{1}_I}{|I|} \langle b, h_I^i\rangle \langle h_I^i, f\rangle.
\end{aligned} \tag{1.7.4}$$

From (1.7.3), we can see that the martingale paraproduct π_b with symbol b is induced by the operator-valued Haar multiplier $(b_I^i)_{I\in\mathcal{D}, 1\leq i\leq d-1}$ where

$$b_I^i := \langle h_I^i, b\rangle.$$

Hence, in general, we define π_b in the following way: for any operator-valued Haar multiplier $b = (b_I^i)_{I\in\mathcal{D}, 1\leq i\leq d-1} \subset L_0(\mathcal{M})$, π_b with symbol b is defined as follows

$$\pi_b(f) = \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} h_I^i b_I^i \left\langle \frac{\mathbb{1}_I}{|I|}, f \right\rangle, \quad \forall f \in L_2(\mathbb{R}, L_2(\mathcal{M})). \tag{1.7.5}$$

Besides, the corresponding sequence of martingale differences $(d_n b)_{n\in\mathbb{Z}}$ with symbol b is given by

$$d_n b = \sum_{|I|=d^{-n+1}} \sum_{i=1}^{d-1} h_I^i \cdot b_I^i, \quad \forall n \in \mathbb{Z}. \tag{1.7.6}$$

Thus each operator-valued Haar multiplier in $L_0(\mathcal{M})$ corresponds to a sequence of martingale differences one-to-one. In the sequel, π_b is defined as in (1.7.5), and for consistency of notation, we identify b_I^i and $\langle h_I^i, b\rangle$ by a slight abuse of notation.

As mentioned before, we use Haar wavelets to define martingale Besov spaces for semicommutative d-adic martingales.

Definition 1.7.2. The martingale Besov space $\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})$ $(0<p<\infty)$ of the semicommutative d-adic martingale is the space of all operator-valued Haar multipliers $b = (\langle h_I^i, b\rangle)_{I\in\mathcal{D},1\le i\le d-1} \subset L_0(\mathcal{M})$ such that

$$\|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})} := \left(\sum_{I\in\mathcal{D}}\sum_{i=1}^{d-1}\left(\frac{\|\langle h_I^i, b\rangle\|_{L_p(\mathcal{M})}}{|I|^{1/2}}\right)^p\right)^{1/p} < \infty. \tag{1.7.7}$$

Remark 1.7.3. It is clear that when $1 \le p < \infty$, $\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})$ is a Banach space. When $0<p<1$, $\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})$ is a quasi-Banach space.

Remark 1.7.4. It is easy to see that for $x \in \mathcal{S}(\mathcal{M})$, then $b = h_I^i \otimes x \in \mathcal{S}(L_\infty(\mathbb{R})\otimes\mathcal{M})$ belongs to $\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})$ for any $0<p<\infty$. Hence, the subspace $\mathcal{S}(L_\infty(\mathbb{R})\otimes\mathcal{M})\cap\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})$ is dense in $\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})$.

Remark 1.7.5. Recall that martingale differences are defined in (1.7.6). One can verify that for any $0<p<\infty$ and $b \in \boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})$

$$\|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})} \approx_{d,p} \left(\sum_{k=-\infty}^{\infty} d^k \|d_k b\|^p_{L_p(\mathbb{R},L_p(\mathcal{M}))}\right)^{1/p}.$$

In particular, if $\mathcal{M}=\mathbb{C}$, the martingale Besov space $\boldsymbol{B}_p^d(\mathbb{R},\mathbb{C})$ is as same as that in [18].

As for the martingale Besov spaces concerning the CAR algebra and $\mathscr{M} = \bigotimes_{k=1}^{\infty} \mathbb{M}_d$, we use the martingale differences to formulate their definitions.

Definition 1.7.6. The martingale Besov space $\boldsymbol{B}_p(\mathcal{C})$ $(0<p<\infty)$ for the CAR algebra is the completion of the set consisting of all $b \in \mathcal{S}(\mathcal{C})$ such that

$$\|b\|_{\boldsymbol{B}_p(\mathcal{C})} := \left(\sum_{k=1}^{\infty} 2^k \|d_k b\|^p_{L_p(\mathcal{C})}\right)^{1/p} < \infty,$$

with respect to $\|\cdot\|_{\boldsymbol{B}_p(\mathcal{C})}$.

Definition 1.7.7. The martingale Besov space $\boldsymbol{B}_p(\mathscr{M})$ $(0<p<\infty)$ for $\mathscr{M} = \bigotimes_{k=1}^{\infty} \mathbb{M}_d$ is the completion of the set consisting of all $b \in \mathcal{S}(\mathscr{M})$ such that

$$\|b\|_{\boldsymbol{B}_p(\mathscr{M})} := \left(\sum_{k=1}^{\infty} d^{2k} \|d_k b\|^p_{L_p(\mathscr{M})}\right)^{1/p} < \infty,$$

with respect to $\|\cdot\|_{\boldsymbol{B}_p(\mathscr{M})}$.

Remark 1.7.8. When $1 \le p < \infty$, $\boldsymbol{B}_p(\mathcal{C})$ and $\boldsymbol{B}_p(\mathscr{M})$ are Banach spaces. When $0<p<1$, they are quasi-Banach spaces.

1.8 Homogeneous spaces

In this section, we introduce some fundamental concepts and important theorems for homogeneous spaces, which can be found in [23]. We begin with the definition of homogeneous spaces. Recall that d is a quasi-metric on Ω if

1. $d(x,y) \geq 0$, $\forall x, y \in \Omega$ and $d(x,y) = 0$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$, $\forall x, y \in \Omega$;
3. there exists a constant $A_0 \geq 1$ such that

$$d(x,y) \leqslant A_0(d(x,z) + d(z,y)), \quad \forall x, y, z \in \Omega. \tag{1.8.1}$$

Denote by $B(x,r) := \{y \in \Omega : d(y,x) < r\}$ the open ball centered at x with radius r. In this paper, all quasi-metric spaces are assumed to have the doubling property: there exists a positive integer $A_1 \in \mathbb{N}$ such that any ball of radius r can be covered by A_1 balls of radius $\frac{r}{2}$.

Definition 1.8.1. A σ-finite measure space $(\Omega, \mathcal{F}, \mu)$ equipped with a quasi-metric d is called a homogeneous space if μ is a Borel measure of homogeneous type:

$$0 < \mu\left(B(x,2r)\right) \leqslant 2^{C_\mu}\mu\left(B(x,r)\right) < \infty, \quad \forall x \in \Omega, r > 0, \tag{1.8.2}$$

where the constant C_μ is independent of x and r.

In [23], Coifman and Weiss defined Hardy spaces on homogeneous spaces by regarding their elements as linear functionals acting on some appropriate quasi-normed spaces. In order to state their definition, we need to introduce the notions of atoms, BMO and Lipschitz spaces on homogeneous spaces.

Definition 1.8.2. If $0 < p \leqslant 1 \leqslant q \leq \infty$ and $p < q$, we say that a function a is a (p,q)-atom if

1. $\mathrm{supp}(a) \subset B$ where B is a ball;
2. $\|a\|_q \leqslant (\mu(B))^{\frac{1}{q}-\frac{1}{p}}$;
3. $\int_\Omega a d\mu = 0$.

Definition 1.8.3. A locally integrable function f is called a BMO function if

$$\|f\|_{BMO} := \sup_B \frac{1}{\mu(B)} \int_B |f - f_B| d\mu < \infty,$$

where $f_B := \dfrac{1}{\mu(B)} \int_B f d\mu$, and the supremum runs over all balls B. Denote by $BMO(\mu)$ the BMO space consisting of all BMO functions.

Definition 1.8.4. For $\alpha > 0$, a locally integrable function l is called a Lipschitz function if

$$|l(x) - l(y)| \leqslant C_\alpha \, (\mu(B))^\alpha \text{ for any } x, y \in \Omega \text{ and any ball } B \text{ containing } x, y. \tag{1.8.3}$$

Moreover,

$$\|l\|_{\mathcal{L}_\alpha} := \inf\{C_\alpha : |l(x) - l(y)| \leqslant C_\alpha \, (\mu(B))^\alpha \, , \ \forall x, y \in B\}, \tag{1.8.4}$$

where the infimum runs over all balls B. Denote by $\mathcal{L}_\alpha(\mu)$ the space consisting of all Lipschitz functions.

It is well-known that each BMO function can be regarded as a continuous linear functional on the vector space generated by finite linear combinations of $(1, q)$-atoms for $1 < q \leq \infty$ (cf. [23]). Hence we can define the atomic Hardy space $H^{1,q}_{\rm at}(\mu)$ $(1 < q \leq \infty)$ as follows:

$$H^{1,q}_{\rm at}(\mu) = \left\{ f \in (BMO(\mu))^* : \ f = \sum_{j=0}^{\infty} \lambda_j a^j, \ a^j \text{ is a } (1,q)\text{-atom and } \sum_{j=0}^{\infty} |\lambda_j| < \infty \right\} \tag{1.8.5}$$

endowed with the norm

$$\|f\|_{H^{1,q}_{\rm at}(\mu)} := \inf \left\{ \sum_{j=0}^{\infty} |\lambda_j| : f = \sum_{j=0}^{\infty} \lambda_j a^j, \ a^j \text{ is a } (1,q)\text{-atom} \right\}.$$

Similarly, each Lipschitz function $l \in \mathcal{L}_{\alpha_p}(\mu)$ can be also regarded as a continuous linear functional of the vector space generated by finite linear combinations of (p, q)-atoms where $0 < p < 1 \leq q \leq \infty$ and $\alpha_p = \frac{1}{p} - 1$ (cf. [23]). We define the atomic Hardy spaces $H^{p,q}_{\rm at}(\mu)$ as follows:

$$H^{p,q}_{\rm at}(\mu) = \left\{ f \in \left(\mathcal{L}_{\alpha_p}(\mu)\right)^* : \ f = \sum_{j=0}^{\infty} \lambda_j a^j, \ a^j \text{ is a } (p,q)\text{-atom and } \sum_{j=0}^{\infty} |\lambda_j|^p < \infty \right\} \tag{1.8.6}$$

endowed with the quasi-norm

$$\|f\|_{H^{p,q}_{\rm at}(\mu)} := \inf \left\{ \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{\frac{1}{p}} : f = \sum_{j=0}^{\infty} \lambda_j a^j, \ a^j \text{ is a } (p,q)\text{-atom} \right\}.$$

Although the Hardy spaces vary with p and q according to the above definitions, the following theorem, which can be found in [23], shows that they actually depend only on p. Consequently, this enables us to define the Hardy spaces $H^p_{\rm at}(\mu)$ for $0 < p \leqslant 1$ to be any one of the spaces $H^{p,q}_{\rm at}(\mu)$ for $0 < p < q \leqslant \infty, 1 \leqslant q \leq \infty$.

Theorem 1.8.5. *$H^{p,q}_{\rm at}(\mu) = H^{p,\infty}_{\rm at}(\mu)$ whenever $0 < p \leq 1 \leqslant q \leqslant \infty$ and $p < q$.*

We end this section with the following duality theorem in [23].

Theorem 1.8.6. *$(H^1_{\rm at}(\mu))^* = BMO(\mu)$, and $(H^p_{\rm at}(\mu))^* = \mathcal{L}_{\alpha_p}(\mu)$ for $0 < p < 1$.*

The proofs of Theorem 1.8.5 and Theorem 1.8.6 in [23] are very technical. In Section 3.1, by employing martingale methods, we give much simpler proofs of these facts. Our approach is based on the fact that $H^p_{\rm at}(\mu)$ for $0 < p \leq 1$ is the finite sum of several dyadic martingale Hardy spaces.

Chapter 2

Bilinear decompositions on probability spaces

In this chapter, we first prove Theorem I.1. Then we will present a characterization of martingale Lipschitz spaces $\Lambda_1(\alpha_p)$, which is needed to show Theorem I.2 (see Theorem 2.2.4 below). At last, we will prove Theorem I.2. Our method is mainly based on the atomic decomposition.

2.1 Bilinear decompositions on $H^1(\Omega) \times BMO(\Omega)$

In this section we prove Theorem I.1. Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space and let $f \in H^1(\Omega),\ g \in BMO(\Omega)$. If we assume that f and g are finite martingales, then we may write their pointwise product $f \cdot g$ as follows

$$f \cdot g = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g), \tag{2.1.1}$$

where

$$\Pi_1(f,g) := \sum_{k=1}^{\infty} d_k f d_k g, \quad \Pi_2(f,g) := \sum_{k=1}^{\infty} f_{k-1} d_k g \quad \text{and} \quad \Pi_3(f,g) := \sum_{k=1}^{\infty} g_{k-1} d_k f.$$

We will estimate $\Pi_1(f,g), \Pi_2(f,g),\ \Pi_3(f,g)$ separately. To do so, we will make use of the atomic decomposition of $H^1(\Omega)$. It follows from our arguments below that the operators Π_1, Π_2 and Π_3 are well-defined (in a pointwise sense) on the product space $H^1(\Omega) \times BMO(\Omega)$. Hence, the proof of Theorem I.1 will follow from the boundedness of Π_1, Π_2 and Π_3, (2.1.1) and a limit argument.

In §2.1.4, we present a direct way to deal with $\Pi_3(f,g)$, which avoids the use of the atomic decomposition.

Proof of Theorem I.1. By Theorem 1.3.8, there always exist two functions f^h and f^d such that $f = f^h + f^d$, where $f^h \in h^1(\Omega)$ and $f^d \in h^1_d(\Omega)$. For any such decomposition of f, since $f^h \in h^1(\Omega)$, there exist $\{\lambda_j\}_{j\geqslant 1} \subset \mathbb{R}$ and simple $(1,2)$-atoms $\{a^j\}_{j\geqslant 1}$ such that

$$f^h = \sum_{j=1}^{\infty} \lambda_j a^j, \quad \|f^h\|_{h^1(\Omega)} \approx \sum_{j=1}^{\infty} |\lambda_j|, \tag{2.1.2}$$

where we assume $\mathrm{supp}(a^j) \subset A_{n_j}$ and $A_{n_j} \in \mathcal{F}_{n_j}$ with $P(A_{n_j}) > 0$ for $j \geq 1$. Then

$$\Pi_i(f,g) = \sum_{j=1}^{\infty} \lambda_j \Pi_i(a^j, g) + \Pi_i(f^d, g), \quad i = 1, 2, 3. \tag{2.1.3}$$

2.1.1 Estimates for $\Pi_1(f^h, g)$ and $\Pi_1(f^d, g)$

We are going to show that Π_1 is a bounded bilinear operator from $H^1(\Omega) \times BMO(\Omega)$ to $L^1(\Omega)$. In fact, the boundedness of Π_1 follows naturally from the duality between $H^1(\Omega)$ and $BMO(\Omega)$, i.e. Theorem 1.3.12 (see [32]). For the reader's convenience, we give a proof based on atomic decompositions.

We first focus on $\Pi_1(f^h, g)$, which can further be decomposed into atoms as described in (2.1.2). It thus suffices to consider

$$\Pi_1(a^j, g) = \sum_{k=1}^{\infty} d_k a^j d_k g,$$

which can further be localized as $d_k a^j = 1_{A_{n_j}} d_k a^j$ when $k \geq n_j + 1$ since $A_{n_j} \in \mathcal{F}_{n_j}$, namely

$$\Pi_1(a^j, g) = \sum_{k=n_j+1}^{\infty} 1_{A_{n_j}} d_k a^j d_k g.$$

Now, by applying the Cauchy-Schwarz inequality, we derive the estimate

$$\begin{aligned}
\|\Pi_1(a^j, g)\|_1 &= \mathbb{E}\left(\left| \sum_{k=n_j+1}^{\infty} 1_{A_{n_j}} d_k a^j d_k g \right| \right) \\
&\leqslant \left[\mathbb{E}\left(\sum_{k=n_j+1}^{\infty} |d_k a^j|^2 \right) \right]^{\frac{1}{2}} \left[\mathbb{E}\left(\sum_{k=n_j+1}^{\infty} 1_{A_{n_j}} |d_k g|^2 \right) \right]^{\frac{1}{2}} \\
&\leqslant \|a^j\|_2 \left[\mathbb{E}\mathbb{E}_{n_j}\left(\sum_{k=n_j+1}^{\infty} 1_{A_{n_j}} |d_k g|^2 \right) \right]^{\frac{1}{2}} \\
&\leqslant P(A_{n_j})^{-\frac{1}{2}} \left[\mathbb{E}\left(1_{A_{n_j}} \mathbb{E}_{n_j}\left(\sum_{k=n_j+1}^{\infty} |d_k g|^2 \right) \right) \right]^{\frac{1}{2}} \\
&\leqslant P(A_{n_j})^{-\frac{1}{2}} \|g\|_{bmo(\Omega)} P(A_{n_j})^{\frac{1}{2}}
\end{aligned}$$

where the fourth inequality follows from the definition of the atom. Hence, we deduce from the definition of the $bmo-$norm that

$$\|\Pi_1(a^j, g)\|_1 \leq \|g\|_{bmo(\Omega)}. \tag{2.1.4}$$

By using (2.1.4) and (2.1.3), we have by Theorem 1.3.13

$$\begin{aligned}\|\Pi_1(f,g)\|_1 &\leqslant \sum_{j=1}^{\infty} |\lambda_j| \|g\|_{bmo(\Omega)} + \left\| \sum_{k=1}^{\infty} d_k f^d d_k g \right\|_1 \\ &\lesssim \|f^h\|_{h^1} \|g\|_{bmo(\Omega)} + \left(\sup_{k\geq 1} \|d_k g\|_\infty \right) \left(\sum_{k=1}^{\infty} \|d_k f^d\|_1 \right) \\ &\lesssim \left(\|f^h\|_{h^1(\Omega)} + \|f^d\|_{h^d_1(\Omega)} \right) \|g\|_{BMO(\Omega)}.\end{aligned}$$

Since the decomposition of $f = f^h + f^d$ is arbitrary, by Theorem 1.3.8 we conclude

$$\|\Pi_1(f,g)\|_1 \lesssim \|f\|_{H^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{2.1.5}$$

2.1.2 Estimates for $\Pi_2(f^h, g)$ and $\Pi_2(f^d, g)$

We are going to show that Π_2 is a bounded bilinear operator from $H^1(\Omega) \times BMO(\Omega)$ to $H^1(\Omega)$. Arguing as in section 2.1.1, we perform the localization on each term

$$\Pi_2(a^j, g) = \sum_{k=1}^{\infty} a^j_{k-1} d_k g = \sum_{k=n_j+2}^{\infty} 1_{A_{n_j}} a^j_{k-1} d_k g.$$

It is easy to verify that

$$d_k(\Pi_2(a^j,g)) = a^j_{k-1} d_k g, \quad k \geq n_j + 2 \ \text{ and } \ d_k(\Pi_2(a^j,g)) = 0, \quad 1 \leqslant k \leqslant n_j + 1.$$

We consider the corresponding square function

$$\begin{aligned}S\left(\Pi_2(a^j,g)\right) &= \left(\sum_{k=n_j+2}^{\infty} \left(|a^j_{k-1}|^2 1_{A_{n_j}} |d_k g|^2 \right) \right)^{\frac{1}{2}} \\ &\leqslant |(a^j)^*| \left(\sum_{k=n_j+2}^{\infty} 1_{A_{n_j}} \left(|d_k g|^2 \right) \right)^{\frac{1}{2}}.\end{aligned}$$

Then by invoking the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}\|\Pi_2(a^j,g)\|_{H^1(\Omega)} &= \mathbb{E}\left[S\left(\Pi_2(a^j,g)\right) \right] \\ &\leqslant \|(a^j)^*\|_2 \left[\mathbb{E}\left(\sum_{k=n_j+2}^{\infty} 1_{A_{n_j}} (|d_k g|^2) \right) \right]^{\frac{1}{2}} \\ &\leqslant \|a^j\|_2 \left[\mathbb{E}\left(1_{A_{n_j}} \mathbb{E}_{n_j} \left(\sum_{k=n_j+2}^{\infty} |d_k g|^2 \right) \right) \right]^{\frac{1}{2}} \\ &\leqslant P(A_{n_j})^{-\frac{1}{2}} \|g\|_{BMO(\Omega)} P(A_{n_j})^{\frac{1}{2}}\end{aligned}$$

and hence,

$$\|\Pi_2(a^j,g)\|_{H^1(\Omega)} \leqslant \|g\|_{BMO(\Omega)}. \tag{2.1.6}$$

Similarly, by Theorem 1.3.13

$$\begin{aligned}\left\|\Pi_2(f^d,g)\right\|_{H^1(\Omega)} &= \mathbb{E}\left[S\left(\Pi_2(\sum_{m=1}^{\infty} d_m f^d, g)\right)\right]\\ &\leqslant \sum_{m=1}^{\infty}\mathbb{E}\left[S\left(\Pi_2(d_m f^d, g)\right)\right]\\ &= \sum_{m=1}^{\infty}\mathbb{E}\left[\mathbb{E}_m\left(\sum_{k=m+1}^{\infty}|d_m f^d|^2|d_k g|^2\right)^{\frac{1}{2}}\right]\\ &= \sum_{m=1}^{\infty}\mathbb{E}\left[|d_m f^d|\mathbb{E}_m\left(\sum_{k=m+1}^{\infty}|d_k g|^2\right)^{\frac{1}{2}}\right]\\ &\leqslant \sum_{m=1}^{\infty}\left(\|d_m f^d\|_1\left\|\mathbb{E}_m\left(\sum_{k=m+1}^{\infty}|d_k g|^2\right)\right\|_{\infty}^{\frac{1}{2}}\right)\end{aligned}$$

and hence,

$$\left\|\Pi_2(f^d,g)\right\|_{H^1(\Omega)} \leqslant \|f^d\|_{h_d^1(\Omega)}\|g\|_{BMO(\Omega)}. \tag{2.1.7}$$

By using (2.1.7), (2.1.2) and (2.1.3), we have by Theorem 1.3.8

$$\begin{aligned}\|\Pi_2(f,g)\|_{H^1(\Omega)} &\leq \left\|\Pi_2(f^h,g)\right\|_{H^1(\Omega)} + \left\|\Pi_2(f^d,g)\right\|_{H^1(\Omega)}\\ &\leqslant \sum_{j=1}^{\infty}|\lambda_j|\|g\|_{BMO(\Omega)} + \|f^d\|_{h_1^d(\Omega)}\|g\|_{BMO(\Omega)}\\ &\lesssim \left(\|f^h\|_{h^1(\Omega)} + \|f^d\|_{h_d^1(\Omega)}\right)\|g\|_{BMO(\Omega)}.\end{aligned}$$

Since the decomposition of $f = f^h + f^d$ is arbitrary, by Theorem 1.3.8 we conclude

$$\|\Pi_2(f,g)\|_{H^1(\Omega)} \lesssim \|f\|_{H^1(\Omega)}\|g\|_{BMO(\Omega)}. \tag{2.1.8}$$

2.1.3 Estimates for $\Pi_3(f^h,g)$ and $\Pi_3(f^d,g)$

We are going to show that Π_3 is a bounded bilinear operator from $H^1(\Omega)\times BMO(\Omega)$ to $H^{\Phi}(\Omega)$. To this end, we first deal with $\Pi_3(f^h,g)$. Note that

$$\begin{aligned}S(\Pi_3(f^h,g)) &= S\left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_j g_{k-1}d_k a^j\right) \leq \sum_{j=1}^{\infty}\lambda_j S\left(\sum_{k=1}^{\infty} g_{k-1}d_k a^j\right)\\ &= \sum_{j=1}^{\infty}\lambda_j\left(\sum_{k=n_j+1}^{\infty}|g_{k-1}|^2|d_k a^j|^2\right)^{\frac{1}{2}}\\ &\leq \sum_{j=1}^{\infty}\lambda_j\left(\sum_{k=n_j+1}^{\infty}|g_{k-1}-g_{n_j}|^2|d_k a^j|^2\right)^{\frac{1}{2}} + \sum_{j=1}^{\infty}\lambda_j|g_{n_j}|S(a^j)\\ &=: I_1 + I_2.\end{aligned}$$

It thus suffices to handle I_1 and I_2. For I_1, we have

$$\begin{aligned}
\mathbb{E}(I_1) &\leqslant \sum_{j=1}^{\infty} |\lambda_j| \mathbb{E}\left(\sum_{k=n_j+1}^{\infty} 1_{A_{n_j}} |g_{k-1} - g_{n_j}|^2 |d_k a^j|^2 \right)^{\frac{1}{2}} \\
&\leqslant \sum_{j=1}^{\infty} |\lambda_j| \left[\mathbb{E}\left(\sum_{k=n_j+1}^{\infty} 1_{A_{n_j}} |g_{k-1} - g|^2 |d_k a^j|^2 \right)^{\frac{1}{2}} + \mathbb{E}\left(\sum_{k=n_j+1}^{\infty} 1_{A_{n_j}} |g - g_{n_j}|^2 |d_k a^j|^2 \right)^{\frac{1}{2}} \right] \\
&\leqslant \sum_{j=1}^{\infty} |\lambda_j| \left\{ P(A_{n_j})^{\frac{1}{2}} \left[\mathbb{E}\left(\sum_{k=n_j+1}^{\infty} |g_{k-1} - g|^2 |d_k a^j|^2 \right) \right]^{\frac{1}{2}} + \mathbb{E}\left(1_{A_{n_j}} |g - g_{n_j}| S(a^j) \right) \right\} \\
&\leqslant \sum_{j=1}^{\infty} |\lambda_j| \left\{ P(A_{n_j})^{\frac{1}{2}} \left[\mathbb{E}\left(\sum_{k=n_j+1}^{\infty} |d_k a^j|^2 \mathbb{E}_k(|g_{k-1} - g|^2) \right) \right]^{\frac{1}{2}} + \|a^j\|_2 P(A_{n_j})^{\frac{1}{2}} \|g\|_{BMO(\Omega)} \right\} \\
&\leqslant 2 \sum_{j=1}^{\infty} |\lambda_j| P(A_{n_j})^{\frac{1}{2}} \|g\|_{BMO(\Omega)} \|a^j\|_2
\end{aligned}$$

and so,

$$\mathbb{E}(I_1) \lesssim \|f^h\|_{h^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{2.1.9}$$

Next, we obtain an estimate for I_2. To this end, notice that

$$\begin{aligned}
I_2 &\leq \left(\sum_{j=1}^{\infty} 1_{A_{n_j}} |\lambda_j| S(a^j) \right) \cdot |g| + \sum_{j=1}^{\infty} |\lambda_j| 1_{A_{n_j}} |g_{n_j} - g| S(a^j) \\
&=: I_3 + I_4.
\end{aligned}$$

Since a^j is a simple $(1,2)$-atom, we have $\|1_{A_{n_j}} S(a^j)\|_1 \leq 1$ and

$$\left\| \sum_{j=1}^{\infty} 1_{A_{n_j}} |\lambda_j| S(a^j) \right\|_1 \leq \sum_{j=1}^{\infty} |\lambda_j| \lesssim \|f^h\|_{h^1(\Omega)}.$$

By Lemma 1.3.16, we have

$$\|I_3\|_{L^\Phi(\Omega)} \lesssim \left\| \sum_{j=1}^{\infty} 1_{A_{n_j}} |\lambda_j| S(a^j) \right\|_1 \|g\|_{BMO(\Omega)} \lesssim \|f^h\|_{h^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{2.1.10}$$

The following estimate is implicit in the proof of (2.1.9):

$$\mathbb{E}(I_4) \leqslant \sum_{j=1}^{\infty} |\lambda_j| P(A_{n_j})^{\frac{1}{2}} \|g\|_{BMO(\Omega)} \|a^j\|_2 \lesssim \|f^h\|_{h^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{2.1.11}$$

By combining (2.1.10) and (2.1.11), we deduce that

$$\|I_2\|_{L^\Phi(\Omega)} \lesssim \|f^h\|_{h^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{2.1.12}$$

In conclusion, by (2.1.9) and (2.1.12) we get

$$\|\Pi_3(f^h, g)\|_{H^\Phi(\Omega)} \lesssim \|f^h\|_{h^1(\Omega)} \|g\|_{BMO(\Omega)}. \tag{2.1.13}$$

It remains to deal with $\Pi_3(f^d, g)$. We have

$$S(\Pi_3(f^d,g)) = \left(\sum_{k=1}^{\infty} |g_{k-1}|^2 |d_k f^d|^2\right)^{\frac{1}{2}} \leqslant \sum_{k=1}^{\infty} |g_{k-1}||d_k f^d| \\ \leq \sum_{k=1}^{\infty} |g_{k-1}-g||d_k f^d| + |g|\left(\sum_{k=1}^{\infty}|d_k f^d|\right).$$

By Lemma 1.3.16,

$$\left\| g\left(\sum_{k=1}^{\infty}|d_k f^d|\right)\right\|_{L^\Phi(\Omega)} \lesssim \left(\sum_{k=1}^{\infty}\|d_k f^d\|_1\right)\|g\|_{BMO(\Omega)} = \|f^d\|_{h^1_d(\Omega)}\|g\|_{BMO(\Omega)}. \tag{2.1.14}$$

For the remaining term, we have

$$\mathbb{E}\left(\sum_{k=1}^{\infty}|g_{k-1}-g||d_k f^d|\right) = \mathbb{E}\left(\sum_{k=1}^{\infty}|d_k f^d|\mathbb{E}_k|g_{k-1}-g|\right) \\ \leqslant \|g\|_{BMO(\Omega)}\left(\sum_{k=1}^{\infty}\|d_k f^d\|_1\right)$$

and so

$$\mathbb{E}\left(\sum_{k=1}^{\infty}|g_{k-1}-g||d_k f^d|\right) \leqslant \|f^d\|_{h^1_d(\Omega)}\|g\|_{BMO(\Omega)}. \tag{2.1.15}$$

Hence, by (2.1.14) and (2.1.15), we get

$$\|\Pi_3(f^d,g)\|_{H^\Phi(\Omega)} \lesssim \|f^d\|_{h^1_d(\Omega)}\|g\|_{BMO(\Omega)}. \tag{2.1.16}$$

By (2.1.13) and (2.1.16), we obtain

$$\|\Pi_3(f,g)\|_{H^\Phi(\Omega)} \lesssim \left(\|f^h\|_{h^1(\Omega)} + \|f^d\|_{h^1_d(\Omega)}\right)\|g\|_{BMO(\Omega)}.$$

Thus we conclude

$$\|\Pi_3(f,g)\|_{H^\Phi(\Omega)} \lesssim \|f\|_{H^1(\Omega)}\|g\|_{BMO(\Omega)}. \tag{2.1.17}$$

This completes the proof of Theorem I.1 □

2.1.4 Another method for handling $\Pi_3(f,g)$

In this subsection we present a different method for dealing with $\Pi_3(f,g)$, which is much neater and simpler than the one presented above, and it relies on the following theorem which has been shown in [32].

Theorem 2.1.1. *If $g \in BMO(\Omega)$ and $g_0 = 0$, then $(g^*)_0 \lesssim \|g\|_{BMO(\Omega)}$ and $g^* \in BMO(\Omega)$. Moreover, $\|g^*\|_{BMO(\Omega)} \lesssim \|g\|_{BMO(\Omega)}$.*

We begin with a pointwise estimate for $S(\Pi_3(f,g))$. Towards this aim, note that $d_k(\Pi_3(f,g)) = g_{k-1}d_k f$, which implies that

$$S(\Pi_3(f,g)) = \left(\sum_{k=1}^{\infty}|g_{k-1}|^2|d_k f|^2\right)^{\frac{1}{2}} \leqslant |g^*|S(f) \leqslant J_1 + J_2,$$

where

$$J_1 := |g^* - (g^*)_0| S(f) \quad \text{and} \quad J_2 := S(f)\|g\|_{BMO(\Omega)}.$$

Clearly,

$$\|J_2\|_1 \lesssim \|f\|_{H^1(\Omega)}\|g\|_{BMO(\Omega)}. \tag{2.1.18}$$

By Theorem 2.1.1, we get $g^* \in BMO(\Omega)$, and hence by Lemma 1.3.16

$$\|J_1\|_{L^\Phi(\Omega)} \lesssim \|g^*\|_{BMO(\Omega)}\|S(f)\|_1 \lesssim \|f\|_{H^1(\Omega)}\|g\|_{BMO(\Omega)}. \tag{2.1.19}$$

As $S(\Pi_3(f,g)) \leq J_1 + J_2$, by combining (2.1.18) with (2.1.19), and by the fact $L_1(\Omega) \subset L^\Phi(\Omega)$, we conclude

$$\|\Pi_3(f,g)\|_{H^\Phi(\Omega)} = \|S(\Pi_3(f,g))\|_{L^\Phi(\Omega)} \lesssim \|f\|_{H^1(\Omega)}\|g\|_{BMO(\Omega)},$$

as desired.

We would like to end this section with the comparison between our proof and the one provided in [10]. Though both arguments heavily rely on the atomic decomposition of $H^1(\Omega)$, they further use weak atom decomposition for the diagonal Hardy space while our proof proceeds more directly. Moreover, the treatment of the most technical term Π_3 is significantly simplified in this section thanks to Theorem 2.1.1.

After the submission of our paper, the authors in [10] also independently discovered a similar way to estimate Π_3 by martingale maximal functions of BMO functions. However, they utilize the John-Nirenberg inequality for the martingale maximal functions of BMO functions instead of Theorem 2.1.1.

2.2 Bilinear decompositions on $H^p(\Omega) \times \Lambda_1(\alpha_p)$ for $0 < p < 1$

In this section, we give a proof of Theorem I.2. Arguing as in the proof of Theorem I.1, it suffices to establish appropriate estimates for the bilinear operators Π_1, Π_2 and Π_3.

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space. If we consider the filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \mathcal{F}$ for all $k \geqslant 1$, then $H^p(\Omega) = L_p(\Omega)$ for $0 < p < \infty$. It is well-known that $(L_p(\Omega))^* \neq \{0\}$ if and only if the probability space $(\Omega, \mathcal{F}, P)$ contains at least one atom with non-zero measure when $0 < p < 1$. This means that $(H^p(\Omega))^* = \{0\}$ may occur. Therefore, we are only concerned with regular martingales where every $\mathcal{F}_k$ is generated by countably many atoms.

To prove Theorem I.2, we start with the following lemma, which holds for general martingales that are not necessarily regular. This will be familiar to the experts in the area, but we will enclose the proof here for the sake of completeness.

Lemma 2.2.1. *For any $0 < p < 1$, we have $L_1(\Omega) \subset H^p_m(\Omega)$.*

Proof. By Doob's maximal inequality, for any $f \in L_1(\Omega)$ and for any $\lambda > 0$ we have

$$P(f^* > \lambda) \leqslant \frac{1}{\lambda} \int_{\{f^* > \lambda\}} |f| dP. \tag{2.2.1}$$

Without loss of generality, we may assume $\|f\|_1 \leqslant 1$. Then

$$\begin{aligned}
\|f^*\|_p^p &= \int_\Omega |f^*|^p dP = p\int_0^\infty P(|f^*| > \lambda)\lambda^{p-1}d\lambda \\
&= p\int_0^1 P(f^* > \lambda)\lambda^{p-1}d\lambda + p\int_1^\infty P(f^* > \lambda)\lambda^{p-1}d\lambda \\
&\leqslant p\int_0^1 \lambda^{p-1}d\lambda + p\int_1^\infty \frac{1}{\lambda}\left(\int_{\{f^*>\lambda\}} |f|dP\right)\lambda^{p-1}d\lambda \\
&= 1 + p\int_{\{f^*>1\}} |f|\left(\int_1^{f^*} \lambda^{p-2}d\lambda\right)dP \\
&= 1 + \frac{p}{1-p}\int_{\{f^*>1\}} |f|\left(1 - |f^*|^{p-1}\right)dP \\
&\leqslant 1 + \frac{p}{1-p}\int_{\{f^*>1\}} |f|dP \leqslant \frac{1}{1-p}.
\end{aligned}$$

This implies that for any $f \in L_1(\Omega)$

$$\|f\|_{H_m^p(\Omega)} \leqslant \left(\frac{1}{1-p}\right)^{\frac{1}{p}}\|f\|_1,$$

which yields the desired result. □

For regular martingales, we have $L_1(\Omega) \subset H_m^p(\Omega) = H^p(\Omega) = h^p(\Omega)$. In what follows, the martingales are always assumed to be regular and every $\mathcal{F}_k$ is generated by countable atoms.

Corollary 2.2.2. *For $0 < p < 1$ and $1 \leq q \leq \infty$, $H^p(\Omega) = H_{\rm at}^{p,q}(\Omega)$.*

Proof. By considering the aforementioned atomic decomposition of $H^p(\Omega)$ and Definition 1.3.7, we have $H^p(\Omega) = H_{\rm at}^{p,\infty}(\Omega)$. It is easy to see $H_{\rm at}^{p,\infty}(\Omega) \subset H_{\rm at}^{p,q}(\Omega) \subset H_{\rm at}^{p,1}(\Omega)$. It thus suffices to show that $H_{\rm at}^{p,1}(\Omega) \subset H^p(\Omega)$. By Lemma 2.2.1, if a is a simple $(p,1)$-atom, then

$$\|a\|_{H^p(\Omega)} \lesssim_p \|a\|_1,$$

which implies that $a \in H^p(\Omega)$. Hence, $H_{\rm at}^{p,1}(\Omega) \subset H^p(\Omega)$ and so, $H^p(\Omega) = H_{\rm at}^{p,q}(\Omega)$. □

2.2.1 Characterization of martingale Lipschitz spaces

In this subsection, we give a characterization of martingale Lipschitz spaces that appears to be new and useful in our argument. We will first recall the definition of martingale Lipschitz spaces. For $0 < p < 1$ define

$$\Lambda_q(\alpha_p) := \left\{ f \in L_2(\Omega) : \|f\|_{\Lambda_q(\alpha_p)} = \sup_{n\geq 0}\sup_{A\in\mathcal{F}_n} P(A)^{-\frac{1}{q}-\alpha_p}\left(\int_A |f - f_n|^q dP\right)^{\frac{1}{q}} < \infty \right\}, \tag{2.2.2}$$

where $q = 1$ or $q = 2$, $\alpha_p := \frac{1}{p} - 1 > 0$.

In [91], Weisz showed that $(H^p(\Omega))^* = \Lambda_1(\alpha_p)$ and $\Lambda_1(\alpha_p) = \Lambda_2(\alpha_p)$.

Corollary 2.2.3. *For any $g \in \Lambda_1(\alpha_p)$, we have $\|g - g_0\|_\infty \lesssim_p \|g\|_{\Lambda_1(\alpha_p)}$.*

Proof. By duality and Lemma 2.2.1, for any $f \in L_2(\Omega)$,

$$|\mathbb{E}\Big(f(g-g_0)\Big)| = \Big|\mathbb{E}\left(g(f-f_0)\right)\Big| \lesssim_p \|f\|_{H^p}\|g\|_{\Lambda_1(\alpha_p)} \lesssim_p \|f\|_1\|g\|_{\Lambda_1(\alpha_p)}.$$

The above estimate together with the fact $\Big(L_1\Big(\Omega\Big)\Big)^* = L_\infty(\Omega)$ yields

$$\|g-g_0\|_\infty \lesssim_p \|g\|_{\Lambda_1(\alpha_p)},$$

which finishes the proof. □

By virtue of Corollary 2.2.3, we have the following property of martingale Lipschitz spaces.

Theorem 2.2.4. *If $g \in \Lambda_1(\alpha_p)$, we have $\|1_A \cdot |g-g_n|\|_\infty \lesssim_p P(A)^{\alpha_p}\|g\|_{\Lambda_1(\alpha_p)}$, for any $n \in \mathbb{N}$ and any $A \in \mathcal{F}_n$.*

Proof. Note that when $P(A) = 0$, the desired result holds trivially. Fix $n \in \mathbb{N}$ and $A \in \mathcal{F}_n$ with $P(A) \neq 0$. For $k \geq 0$, let $\mathcal{F}_k^A := \{B \in \mathcal{F}_{k+n} : B \subseteq A\}$. Note that the union $\mathcal{F}^A$ of all $\mathcal{F}_k^A$ is exactly $\{B \in \mathcal{F} | B \subset A\}$. Hence, if we define

$$P_A(B) := \frac{P(B)}{P(A)} \quad (B \in \mathcal{F}^A)$$

then $(A, \mathcal{F}^A, P_A)$ is a probability space. Note that for any $g \in L_1(A, \mathcal{F}^A, P_A)$ one has

$$\mathbb{E}(g|\mathcal{F}_k^A) = 1_A \cdot \mathbb{E}(g|\mathcal{F}_{k+n}).$$

Denote $\mathbb{E}(\cdot|\mathcal{F}_k^A)$ by $\mathbb{E}_k^A$. It is easy to verify $\{\mathbb{E}_k^A(g)\}_{k\geq 0}$ is also a regular martingale on $(A, \mathcal{F}^A, P_A)$. If $g \in \Lambda_1(\alpha_p)$, then for $B \in \mathcal{F}_k^A$ with $P(B) \neq 0$,

$$\begin{aligned} P_A(B)^{-1-\alpha_p}\left(\int_B |g - \mathbb{E}_k^A(g)|dP_A\right) &= P(A)^{\alpha_p}\left(P(B)^{-1-\alpha_p}\left(\int_B |g-g_{k+n}|dP\right)\right) \\ &\leq P(A)^{\alpha_p}\|g\|_{\Lambda_1(\alpha_p)} \end{aligned}$$

which implies that by Corollary 2.2.3,

$$\|1_A \cdot |g-g_n|\|_\infty = \|1_A \cdot |g - \mathbb{E}_0^A(g)|\|_\infty \lesssim_p P(A)^{\alpha_p}\|g\|_{\Lambda_1(\alpha_p)}.$$

This completes the proof of the theorem. □

Remark 2.2.5. By Theorem 2.2.4 and (2.2.2), we conclude that for $g \in \Lambda_1(\alpha_p)$ we have the characterization

$$\|g\|_{\Lambda_1(\alpha_p)} \approx_p \sup_{n\geq 0}\sup_{A\in\mathcal{F}_n} P(A)^{-\alpha_p}\|1_A \cdot |g-g_n|\|_\infty. \tag{2.2.3}$$

Note that the results in [56] can be deduced from (2.2.3).

2.2.2 Proof of Theorem I.2

As in the proof of Theorem I.1, we divide the proof into three parts. Without loss of generality, we may assume that $f_0 = g_0 = 0$.

Estimates for $\Pi_1(f,g)$ and $\Pi_3(f,g)$

The boundedness of Π_1 from $H^p(\Omega)\times\Lambda_1(\alpha_p)$ to $L_1(\Omega)$ follows directly from the duality between $H^p(\Omega)$ and $\Lambda_1(\alpha_p)$, we omit the details.

We will also prove that Π_3 is a bounded bilinear operator from $H^p(\Omega)\times\Lambda_1(\alpha_p)$ to $H^p(\Omega)$. Note that

$$S(\Pi_3(f,g))^2=\sum_{k=1}^{\infty}|g_{k-1}|^2|d_kf|^2\leqslant(g^*)^2S(f)^2. \tag{2.2.4}$$

Hence we conclude from Corollary 2.2.3 and the L_∞ boundedness of the maximal function that

$$\|\Pi_3(f,g)\|_{H^p(\Omega)}^p\lesssim\|g^*\|_\infty^p\mathbb{E}(S(f)^p)\leq\|g\|_\infty^p\|f\|_{H^p(\Omega)}^p\lesssim_p\|f\|_{H^p(\Omega)}^p\|g\|_{\Lambda_1(\alpha_p)}^p, \tag{2.2.5}$$

as desired.

Estimates for $\Pi_2(f,g)$.

We will show that Π_2 is a bounded bilinear operator from $H^p(\Omega)\times\Lambda_1(\alpha_p)$ to $H^1(\Omega)$. Note that $H^p(\Omega)=h^p(\Omega)$, and $h^p(\Omega)$ admits an atomic decomposition.

Then there exist $\{\lambda_j\}_{j\geqslant1}\subset\mathbb{R}$ and simple (p,∞)-atoms $\{a^j\}_{j\geqslant1}$ such that

$$f=\sum_{j=1}^{\infty}\lambda_ja^j,\quad\|f\|_{H^p(\Omega)}\approx_p\left(\sum_{j=1}^{\infty}|\lambda_j|^p\right)^{\frac{1}{p}}, \tag{2.2.6}$$

where we assume $\operatorname{supp}(a^j)\subset A_{n_j}$ and $A_{n_j}\in\mathcal{F}_{n_j}$ with $P(A_{n_j})>0$ for $j\geq1$. By arguing as in the corresponding case in the proof of Theorem I.1,

$$S(\Pi_2(a^j,g))=\left(\sum_{k=n_j+1}^{\infty}1_{A_{n_j}}|a_{k-1}^j|^2|d_kg|^2\right)^{\frac{1}{2}}\leqslant|(a^j)^*|\left(\sum_{k=n_j+1}^{\infty}1_{A_{n_j}}|d_kg|^2\right)^{\frac{1}{2}}. \tag{2.2.7}$$

Hence,

$$\begin{aligned}\mathbb{E}\left[S(\Pi_2(a^j,g))\right]&\leqslant\|(a^j)^*\|_\infty\left[\mathbb{E}\left(1_{A_{n_j}}\sum_{k=n_j+1}^{\infty}|d_kg|^2\right)^{\frac{1}{2}}\right]\\&\leqslant\|a^j\|_\infty\left(P(A_{n_j})\left(\mathbb{E}\sum_{k=n_j+1}^{\infty}|d_kg|^2\right)\right)^{\frac{1}{2}}\\&\leqslant P(A_{n_j})^{-\frac{1}{p}}\left(P(A_{n_j})\|g\|_{\Lambda_2(\alpha_p)}^2P(A_{n_j})^{1+2\alpha_p}\right)^{\frac{1}{2}}\\&=\|g\|_{\Lambda_2(\alpha_p)}P(A_{n_j})^{-\frac{1}{p}}P(A_{n_j})^{1+\alpha_p}\\&\leqslant\|g\|_{\Lambda_2(\alpha_p)}\lesssim_p\|g\|_{\Lambda_1(\alpha_p)},\end{aligned}$$

where the last inequality follows from the condition that $\alpha_p=\frac{1}{p}-1$. As a consequence of the above estimates, we have that

$$\|\Pi_2(f,g)\|_{H^1(\Omega)}^p\leqslant\sum_{j=1}^{\infty}|\lambda_j|^p\left[\mathbb{E}S(\Pi_2(a^j,g))\right]^p\lesssim_p\|f\|_{H^p(\Omega)}^p\|g\|_{\Lambda_1(\alpha_p)}^p. \tag{2.2.8}$$

This completes the proof of the theorem.

Chapter 3

Bilinear decompositions on homogeneous spaces

This chapter is devoted to extending Theorem I.1 and Theorem I.2 to spaces of homogeneous type. Our approach is on the basis of the existence of dyadic systems on spaces of homogeneous type developed by Hytönen and Kairema in [43]. We also show that $H^p_{\rm at}(\mu)$ for $0 < p \leq 1$ is some finite sum of several dyadic martingale Hardy spaces on homogeneous spaces.

3.1 Dyadic systems on homogeneous spaces

In this section, we start with introducing dyadic systems on homogeneous spaces, which first appeared in the work of Hytönen and Kairema [43]. With the help of these dyadic structures, we then show that $H^p_{\rm at}(\mu)$ is exactly the finite sum of martingale Hardy spaces associated with some adjacent dyadic martingales, which extends Mei's result [52] to homogeneous spaces.

The following theorem concerning the existence of dyadic structures is due to Hytönen and Kairema [43].

Theorem 3.1.1. *Let Ω denote a homogeneous space. Suppose that the constants $0 < c_0 \leqslant C_0 < \infty$ and $\delta \in (0,1)$ satisfy*

$$12A_0^3 C_0 \delta \leqslant c_0,$$

where A_0 is specified in the definition of quasi-metric, see (1.8.1).

Given a set of reference points $\{z_\alpha^k\}_\alpha$, $\alpha \in \mathcal{A}_k$ (an index set), for every $k \in \mathbb{Z}$, with the properties that

$$d(z_\alpha^k, z_\beta^k) \geqslant c_0\delta^k, \ (\alpha \neq \beta) \quad \min_\alpha d(x, z_\alpha^k) < C_0\delta^k, \ \text{for all } x \in \Omega,$$

one can construct families of sets $\tilde{Q}_\alpha^k \subseteq Q_\alpha^k \subseteq \bar{Q}_\alpha^k$, called open, half-open and closed dyadic

cubes respectively, such that:

$$\tilde{Q}_\alpha^k \text{ and } \bar{Q}_\alpha^k \text{are the interior and closure of } Q_\alpha^k; \tag{3.1.1}$$

$$\text{if } k \leqslant l, \text{ then either } Q_\beta^l \subseteq Q_\alpha^k \text{ or } Q_\beta^l \cap Q_\alpha^k = \emptyset; \tag{3.1.2}$$

$$X = \bigcup_\alpha Q_\alpha^k \text{ (disjoint union) for all } k \in \mathbb{Z}; \tag{3.1.3}$$

$$B(z_\alpha^k, c_1\delta^k) \subseteq Q_\alpha^k \subseteq B(z_\alpha^k, C_1\delta^k) =: B(Q_\alpha^k); \tag{3.1.4}$$

$$\text{if } k \leqslant l \text{ and } Q_\beta^l \subseteq Q_\alpha^k \text{ then } B(Q_\beta^l) \subseteq B(Q_\alpha^k), \tag{3.1.5}$$

where $c_1 = (3A_0^2)^{-1}c_0$ *and* $C_1 = 2A_0C_0$. *The open and closed cubes* $\tilde{Q}_\alpha^k$ *and* $\bar{Q}_\alpha^k$ *depend only on the points* z_β^l *for* $l \geqslant k$. *The half-open cubes* Q_α^k *depend on* z_β^l *for* $l \geqslant \min(k, k_0)$, *where* $k_0 \in \mathbb{Z}$ *is a preassigned number entering the construction.*

It is obvious that the construction of the above dyadic systems is not unique, and it depends on the set of the reference points $\{z_\alpha^k\}_\alpha$. We denote this dyadic system by $\mathscr{D} = \{Q_\alpha^k\}_{k,\alpha}$. Let $\mathcal{F}_k = \sigma(\{Q_\alpha^k\}_\alpha)$ be the σ-algebra generated by $\{Q_\alpha^k\}_\alpha$. Then it is clear that

$$\cdots \subset \mathcal{F}_{k-1} \subset \mathcal{F}_k \subset \cdots,$$

which implies that $\{\mathcal{F}_k\}_{k\in\mathbb{Z}}$ is a filtration generated by atoms. Let $\mathcal{F} = \sigma(\cup_{k\in\mathbb{Z}}\mathcal{F}_k)$. Note that each Q_α^k is an atom of $\mathcal{F}_k$.

Remark 3.1.2. The standard dyadic grid on the real line is a dyadic system given by

$$\mathcal{F}_k = \{[2^{-k}m, 2^{-k}(m+1)) : m \in \mathbb{Z}\} \quad \text{for all } k \in \mathbb{Z}.$$

Similarly, an example of a dyadic system on $\mathbb{R}^n$ is given by the family of standard dyadic cubes in $\mathbb{R}^n$.

Recall that, for $f \in L_1^{\mathrm{loc}}(\Omega, \mathcal{F}, \mu)$, the martingale maximal function, the square function and the conditional square function of f associated with $(\mathcal{F}_k)_{k\in\mathbb{Z}}$ are given by

$$f^* := \max_{k\in\mathbb{Z}} |f_k|, \quad S(f) := \left(\sum_{k\in\mathbb{Z}} |d_k f|^2\right)^{\frac{1}{2}} \quad \text{and} \quad s(f) := \left(\sum_{k\in\mathbb{Z}} \mathbb{E}_{k-1}|d_k f|^2\right)^{\frac{1}{2}},$$

respectively.

Let $0 < p \leq 1$. The martingale Hardy space $H^p_{m,\mathscr{D}}(\mu)$ is defined as the completion of the space consisting of all $f \in L_1^{\mathrm{loc}}(\Omega)$ such that $f^* \in L_p(\Omega)$ with respect to the quasi-norm $\|f\|_{H^p_{m,\mathscr{D}}(\mu)} := \|f^*\|_p$.

We define $H^p_{\mathscr{D}}(\mu)$ and $h^p_{\mathscr{D}}(\mu)$ by the square functions and the conditional square functions respectively, with the additional assumption that

$$\lim_{n\to-\infty} \int_\Omega \sup_{k\leq n} |f_k|^p d\mu = 0. \tag{3.1.6}$$

From (3.1.6), we have

$$\lim_{n\to-\infty} \sup_{k\leq n} |f_k| = 0.$$

Analogously, define the martingale atomic Hardy spaces $H^{p,q}_{\mathrm{at},\mathscr{D}}(\mu)$ $(0 < p < 1 \leq q \leq \infty$ or $p = 1, 1 < q \leq \infty)$ like Definition 1.3.7.

In order to show Theorem 1.8.6, we introduce the dual spaces of these atomic martingale Hardy spaces. For $0 < p < 1$, $q = 1$ or 2 and $\alpha_p = \frac{1}{p} - 1$, define

$$BMO^{\mathscr{D}}(\mu) := \left\{ f \in L_1^{\mathrm{loc}}(\Omega,\mu) \ : \ \|f\|_{BMO^{\mathscr{D}}(\mu)} := \sup_{Q\in\mathscr{D}} \frac{1}{\mu(Q)} \int_Q |f - f_Q| d\mu < \infty \right\},$$

$$\Lambda_q^{\mathscr{D}}(\alpha_p) := \left\{ f \in L_1^{\mathrm{loc}}(\Omega,\mu) \ : \ \|f\|_{\Lambda_q^{\mathscr{D}}(\alpha_p)} := \sup_{Q\in\mathscr{D}} \mu(Q)^{-\frac{1}{q}-\alpha_p} \left(\int_Q |f - f_Q|^q d\mu \right)^{\frac{1}{q}} < \infty \right\}.$$

The spaces $\Lambda_q^{\mathscr{D}}(\alpha_p)$ are called the martingale Lipschitz spaces with respect to $\mathscr{D}$. Note that $\Lambda_1^{\mathscr{D}}(\alpha_p) = \Lambda_2^{\mathscr{D}}(\alpha_p)$.

Arguing as in [91], one can show that

$$\left(H^1_{\mathrm{at},\mathscr{D}}(\mu)\right)^* = BMO^{\mathscr{D}}(\mu),$$

and for $0 < p < 1$,

$$\left(H^p_{\mathrm{at},\mathscr{D}}(\mu)\right)^* = \Lambda_q^{\mathscr{D}}(\alpha_p).$$

Remark 3.1.3. Since every simple (p,q)-atom is locally supported, by Corollary 2.2.2, we conclude that for $0 < p < 1 \le q \le \infty$ or $p = 1, 1 < q \le \infty$

$$H^{p,q}_{\mathrm{at},\mathscr{D}}(\mu) = H^{p,\infty}_{\mathrm{at},\mathscr{D}}(\mu).$$

Thus we are only concerned with $H^p_{\mathrm{at},\mathscr{D}}(\mu) := H^{p,\infty}_{\mathrm{at},\mathscr{D}}(\mu)$.

Proposition 3.1.4. *For $0 < p \leqslant 1$, the martingale Hardy spaces defined above are mutually equivalent. Namely, $H^p_{\mathscr{D}}(\mu) = H^p_{m,\mathscr{D}}(\mu) = h^p_{\mathscr{D}}(\mu) = H^p_{\mathrm{at},\mathscr{D}}(\mu)$.*

Proof. Let $p \in (0,1]$ be fixed. First, we show $H^p_{\mathscr{D}}(\mu) = H^p_{m,\mathscr{D}}(\mu)$. Suppose that $f \in H^p_{m,\mathscr{D}}(\mu)$. Then for any $n > 0$, by a well-known inequality of Burkholder–Davis–Gundy,

$$\int_\Omega \left(|f_{-n}|^2 + \sum_{k=-n+1}^{n} |d_k f|^2 \right)^{\frac{p}{2}} d\mu \lesssim \int_\Omega \sup_{-n\le k\le n} |f_k|^p d\mu \lesssim \int_\Omega (f^*)^p d\mu$$

which yields by letting $n \to \infty$ and by Fatou's lemma

$$\|S(f)\|_p \lesssim \|f^*\|_p.$$

Thus $H^p_{m,\mathscr{D}}(\mu) \subset H^p_{\mathscr{D}}(\mu)$.

Conversely, if $f \in H^p_{\mathscr{D}}(\mu)$, then for $n > 0$,

$$\int_\Omega \sup_{-n\le k\le n} |f_k|^p d\mu \lesssim \int_\Omega \left(|f_{-n}|^2 + \sum_{k=-n+1}^{n} |d_k f|^2 \right)^{\frac{p}{2}} d\mu,$$

and hence

$$\int_\Omega \sup_{-n\le k\le n} |f_k|^p d\mu \lesssim \int_\Omega \sup_{k\le -n} |f_k|^p d\mu + \int_\Omega |S(f)|^p d\mu < \infty. \tag{3.1.7}$$

Then by letting $n \to \infty$ and applying Fatou's lemma, we obtain $\|f^*\|_p < \infty$ and

$$\|f^*\|_p \lesssim \|S(f)\|_p.$$

Therefore, $H^p_{\mathscr{D}}(\mu) \subset H^p_{m,\mathscr{D}}(\mu)$ and $H^p_{m,\mathscr{D}}(\mu) = H^p_{\mathscr{D}}(\mu)$.

One shows $H^p_{m,\mathscr{D}}(\mu) = h^p_{\mathscr{D}}(\mu)$ in a completely analogous way. To show $h^p_{\mathscr{D}}(\mu) = H^p_{\mathrm{at},\mathscr{D}}(\mu)$, one can argue by mimicking the corresponding proof in [91] and [92]. We omit the details. □

The following theorem can be found in [43] and ensures that there exist enough dyadic cubes to cover all balls on homogeneous spaces.

Theorem 3.1.5. *Given a set of reference points $\{z^k_\alpha\}, k \in \mathbb{Z}, \alpha \in \mathcal{A}_k$, suppose that there exists constant $\delta \in (0,1)$ that satisfies $96A_0^6\delta \leqslant 1$. Then there exists a finite collection of families $\mathscr{D}^t, t = 1, 2, \cdots, K = K(A_0, A_1, \delta) < \infty$, where each $\mathscr{D}^t$ is a collection of dyadic cubes, associated to dyadic points $\{z^k_\alpha\}, k \in \mathbb{Z}, \alpha \in \mathcal{A}_k$, with the properties* (3.1.1)-(3.1.5) *in Theorem 3.1.1.*

In addition, the following property is satisfied:

$$\text{for every } B(x,r) \subseteq \Omega, \text{ there exist } t \text{ and } Q \in \mathscr{D}^t \text{ with } B(x,r) \subseteq Q \text{ and } \mathrm{diam}(Q) \leqslant Cr. \tag{3.1.8}$$

The constant $C < \infty$ in (3.1.8) *only depends on the quasi-metric constant A_0 and the parameter δ.*

By virtue of Proposition 3.1.4 and Theorem 3.1.5, we have the following theorem, which extends Mei's result in [52].

Theorem 3.1.6. *For $0 < p \leqslant 1$, we have*

$$H^p_{\mathrm{at}}(\mu) = \sum_{t=1}^K H^p_{\mathrm{at},\mathscr{D}^t}(\mu) = \sum_{t=1}^K H^p_{\mathscr{D}^t}(\mu) = \sum_{t=1}^K H^p_{m,\mathscr{D}^t}(\mu) = \sum_{t=1}^K h^p_{\mathscr{D}^t}(\mu). \tag{3.1.9}$$

Proof. Let $p \in (0,1]$ be fixed. In view of Proposition 3.1.4, it suffices to show $H^p_{\mathrm{at}}(\mu) = \sum_{t=1}^K H^p_{\mathrm{at},\mathscr{D}^t}(\mu)$. We prove it via comparing the atoms. Let a be a (p,∞)-atom in $H^p_{\mathrm{at}}(\mu)$. Then there exists a ball B such that

$$\mathrm{supp}(a) \subset B,\ \|a\|_\infty \leqslant (\mu(B))^{-\frac{1}{p}},\ \int_B a(x)d\mu = 0.$$

By Theorem 3.1.5, there exist t and a cube $Q \in \mathscr{D}^t$ such that $B \subset Q$, and $\mu(Q) \lesssim \mu(B)$. Then

$$\mathrm{supp}(a) \subset B \subset Q,\ \|a\|_\infty \leqslant (\mu(B))^{-\frac{1}{p}} \lesssim (\mu(Q))^{-\frac{1}{p}},\ \int_Q a d\mu = 0,$$

which implies that a is a constant multiple of a simple (p,∞)-atom in $H^p_{\mathrm{at},\mathscr{D}^t}(\mu)$. Thus

$$H^p_{\mathrm{at}}(\mu) \subset \sum_{t=1}^K H^p_{\mathrm{at},\mathscr{D}^t}(\mu). \tag{3.1.10}$$

For any $t = 1, 2, \cdots, K$ and for any given simple (p,∞)-atom b in $H^p_{\mathrm{at},\mathscr{D}^t}(\mu)$, there exists $Q \in \mathscr{D}^t$ such that

$$\mathrm{supp}(b) \subset Q,\ \|b\|_\infty \leqslant (\mu(Q))^{-\frac{1}{p}},\ \int_Q b d\mu = 0.$$

By Theorem 3.1.1, there exists a ball B such that $Q \subset B$ and $\mu(Q) \gtrsim \mu(B)$. Hence

$$\mathrm{supp}(b) \subset Q \subset B,\ \|b\|_\infty \leqslant (\mu(Q))^{-\frac{1}{p}} \lesssim (\mu(B))^{-\frac{1}{p}},\ \int_B b d\mu = 0,$$

which implies that a multiple of b is also a (p,∞)-atom in $H^p_{\mathrm{at}}(\mu)$, thus

$$\sum_{t=1}^{K} H^p_{\mathrm{at},\mathscr{D}^t}(\mu) \subset H^p_{\mathrm{at}}(\mu). \tag{3.1.11}$$

To complete the proof of the theorem, combine (3.1.10) and (3.1.11). □

Remark 3.1.7. Theorem 1.8.5 follows immediately from Corollary 2.2.2, Proposition 3.1.4 and Theorem 3.1.6, which simplifies the original proof by Coifman and Weiss in [23].

By duality and Theorem 3.1.6, we recover the following result of [43], which is an extension of a result due to Mei [52]:

$$BMO(\mu) = \bigcap_{t=1}^{K} BMO^{\mathscr{D}^t}(\mu). \tag{3.1.12}$$

We will now establish an analogous result for $\mathcal{L}_{\alpha_p}(\mu)$ $(0 < p < 1)$.

Theorem 3.1.8. *For* $0 < p < 1$,

$$\mathcal{L}_{\alpha_p}(\mu) = \bigcap_{t=1}^{K} \Lambda_2^{\mathscr{D}^t}(\alpha_p).$$

Proof. By Theorem 3.1.1, for any $Q \in \mathscr{D}^t$ (and $t = 1, 2, \cdots, K$), there exists a ball B such that $Q \subset B$ and $\mu(B) \lesssim \mu(Q)$. If $f \in \mathcal{L}_{\alpha_p}(\mu)$, then for any $x, y \in Q$, we have

$$|f(x) - f(y)| \leq \|f\|_{\mathcal{L}_{\alpha_p}(\mu)} \mu(B)^{\alpha_p} \lesssim \|f\|_{\mathcal{L}_{\alpha_p}(\mu)} \mu(Q)^{\alpha_p}.$$

We thus have

$$\begin{aligned}
\|f\|_{\Lambda_2^{\mathscr{D}^t}(\alpha_p)} &\leq \sup_{Q\in\mathscr{D}^t} (\mu(Q))^{-\frac{1}{2}-\alpha_p} \left(\mu(Q)^{-2} \int_Q \left(\int_Q |f(x) - f(y)| d\mu(y)\right)^2 d\mu(x)\right)^{\frac{1}{2}} \\
&\leqslant \sup_{Q\in\mathscr{D}^t} (\mu(Q))^{-\frac{1}{2}-\alpha_p} \left(\int_Q \|f\|^2_{\mathcal{L}_{\alpha_p}(\mu)} \mu\,(Q)^{2\alpha_p}\, d\mu\right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{\mathcal{L}_{\alpha_p}(\mu)},
\end{aligned}$$

which yields

$$\mathcal{L}_{\alpha_p}(\mu) \subset \bigcap_{t=1}^{K} \Lambda_2^{\mathscr{D}^t}(\alpha_p). \tag{3.1.13}$$

Conversely, let $f \in \bigcap_{t=1}^{K} \Lambda_2^{\mathscr{D}^t}(\alpha_p)$. For $Q \in \mathscr{D}^t$, by Theorem 2.2.4,

$$|f(x) - f_Q| \lesssim \mu(Q)^{\alpha_p} \|f\|_{\Lambda_2^{\mathscr{D}^t}(\alpha_p)} \quad \forall x \in Q,$$

which implies that for any $x, y \in Q$,

$$|f(x) - f(y)| \lesssim \mu(Q)^{\alpha_p} \|f\|_{\Lambda_2^{\mathscr{D}^t}(\alpha_p)}. \tag{3.1.14}$$

For any ball $B \subset \Omega$, by Theorem 3.1.5, there exist t and $Q \in \mathscr{D}^t$ such that $B \subset Q$ and $\mu(Q) \lesssim \mu(B)$. Then for any $x, y \in B$, by (3.1.14)

$$|f(x) - f(y)| \lesssim \mu(B)^{\alpha_p} \|f\|_{\Lambda_2^{\mathscr{D}^t}(\alpha_p)}.$$

Thus

$$\|f\|_{\mathcal{L}_{\alpha_p}} \lesssim \sum_{t=1}^{K} \|f\|_{\Lambda_2^{\mathscr{D}^t}(\alpha_p)},$$

which implies

$$\bigcap_{t=1}^{K} \Lambda_2^{\mathscr{D}^t}(\alpha_p) \subset \mathcal{L}_{\alpha_p}(\mu). \tag{3.1.15}$$

The theorem follows from (3.1.13) and (3.1.15). □

Remark 3.1.9. Theorem 3.1.6 and Theorem 3.1.8 give a simple proof of Theorem 1.8.6 originally established by Coifman and Weiss [23]:

$$(H^p_{\rm at}(\mu))^* = \left(\sum_{t=1}^{K} H^p_{{\rm at},\mathscr{D}^t}(\mu)\right)^* = \bigcap_{t=1}^{K} (H^p_{{\rm at},\mathscr{D}^t}(\mu))^* = \bigcap_{t=1}^{K} \Lambda_2^{\mathscr{D}^t}(\alpha_p) = \mathcal{L}_{\alpha_p}(\mu).$$

3.2 Bilinear decompositions for dyadic martingales on homogeneous spaces

In this section, we focus on bilinear decompositions arising in the study of products between elements in spaces of dyadic martingales on homogeneous spaces introduced in the previous section. In the setting of homogeneous spaces, due to their quasi-metrics and measures, the dyadic martingales behave worse than martingales in probability spaces and the underlying analysis is more intricate.

In §3.2.1 we prove appropriate generalized Hölder-type inequalities on homogeneous spaces (see Lemmas 3.2.2 and 3.2.4 below). We then introduce a class of pointwise multipliers of $\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$ and $BMO^{\mathscr{D}}(\mu)$; see Theorem 3.2.5 below. Using Theorem 3.2.5, we define products between dyadic martingale Hardy spaces on homogeneous spaces and their duals and then, in §3.2.2 we establish analogues of the results of Sections 2.1 and 2.2 in the setting of homogeneous spaces.

3.2.1 A generalized Hölder-type inequality

Let $0 < p \le 1$ and $\mathscr{D}$ be a dyadic system, constructed as in Theorem 3.1.1. The martingale Musielak–Orlicz Hardy spaces $H^{\Psi_p}_{\mathscr{D}}(\mu)$ consist of all measurable functions f on $(\Omega, \mathcal{F}, \mu)$ such that $s(f) \in L^{\Psi_p}(\Omega)$ where $O \in \Omega$ is a fixed point, and

$$\Psi_1(x,t) := \frac{t}{\log(e + d(x,O)) + \log(e+t)},$$
$$\Psi_p(x,t) := \frac{t}{1 + \{t[1 + \mu(B(O, d(x,O)))]\}^{1-p}} \quad (0 < p < 1).$$

Note that $L^{\Psi_p}(\Omega)$ is a quasi-normed space.

Let $M := (C_\mu + 1)\log(e + d(x, O))$. By (1.3.4) we obtain

$$\Psi_1(x, st) \lesssim (e + d(x, O))^{-(C_\mu+1)} e^t + s \lesssim w(x)e^t + s, \quad \text{for all } x \in \Omega, s, t > 0, \tag{3.2.1}$$

where $w : \Omega \to \mathbb{R}_+$ is a weight function with

$$w(x) \lesssim \min\left\{1, d(x, O)^{-(C_\mu+1)}\right\}. \tag{3.2.2}$$

Let $Q^0 \in \mathcal{F}_0$ be the dyadic cube such that $O \in Q^0$. For $g \in BMO^{\mathscr{D}}(\mu)$, define

$$\|g\|_{BMO^{\mathscr{D}}_+(\mu)} := \sup_{\alpha\in\mathcal{A}_0} \frac{|g_{Q^0_\alpha} - g_{Q^0}|}{\log(e + d(z^0_\alpha, O))} + |g_{Q^0}| + \|g\|_{BMO^{\mathscr{D}}(\mu)},$$

where $Q^0_\alpha \in \mathcal{F}_0$ is a dyadic cube with its center z^0_α and $\mathcal{A}_0$ is the index set in Theorem 3.1.1. Denote by $BMO^{\mathscr{D}}_+(\mu)$ the space consisting of all $g \in BMO^{\mathscr{D}}(\mu)$ such that $\|g\|_{BMO^{\mathscr{D}}_+(\mu)} < \infty$. It is not difficult to verify that $\|\cdot\|_{BMO^{\mathscr{D}}_+(\mu)}$ is a norm on $BMO^{\mathscr{D}}_+(\mu)$.

Remark 3.2.1. If we consider the dyadic martingales on $\mathbb{R}^n$, by taking appropriate cubes Q^0 one shows that if $g \in BMO^{\mathscr{D}}(\mu)$, then $g \in BMO^{\mathscr{D}}_+(\mu)$. Note that if $g \in BMO(\mu)$, then $g \in BMO^{\mathscr{D}}_+(\mu)$. Moreover,

$$\|g\|_{BMO^{\mathscr{D}}_+(\mu)} \lesssim \|g\|_{BMO(\mu)} + |g_{Q^0}|.$$

We now introduce the following generalized Hölder inequality for $L_1(\Omega, \mathcal{F}, \mu)$ and $BMO^{\mathscr{D}}_+(\mu)$.

Lemma 3.2.2. *If $f \in L_1(\Omega, \mathcal{F}, \mu)$ and $g \in BMO^{\mathscr{D}}_+(\mu)$, then $f \cdot g \in L^{\Psi_1}(\Omega)$. Moreover,*

$$\|fg\|_{L^{\Psi_1}(\Omega)} \lesssim \|f\|_1 \|g\|_{BMO^{\mathscr{D}}_+(\mu)}. \tag{3.2.3}$$

Proof. Without loss of generality, assume $\|f\|_1 \le 1$, $\|g\|_{BMO^{\mathscr{D}}_+(\mu)} \le 1$ and $g_{Q^0} = 0$. It suffices to show that

$$\int_\Omega \Psi_1(x, |f(x)g(x)|)d\mu \lesssim 1.$$

Let $S_k := B(O, C_0\delta^k) \setminus B(O, C_0\delta^{k+1})$ for $k < 0$ and $S_0 := B(O, C_0)$, where $\delta \in (0, 1)$ is the constant in Theorem 3.1.1. Then for each $k \le 0$, there exists a finite index subset $\mathcal{B}_k \subset \mathcal{A}_0$ such that $B(O, C_0\delta^k) \subset \bigcup_{\alpha\in\mathcal{B}_k} Q^0_\alpha$ (where $Q^0_\alpha \in \mathcal{F}_0$) and

$$\sum_{\alpha\in\mathcal{B}_k} \mu\left(Q^0_\alpha\right) = \mu\left(\bigcup_{\alpha\in\mathcal{B}_k} Q^0_\alpha\right) \le \mu\left(B(O, 2A_0C_0\delta^k)\right) \lesssim \delta^{C_\mu k}.$$

Take $s = \nu^{-1}|f(x)|, t = \nu|g(x)|$ in (3.2.1), one has

$$\begin{aligned}\int_\Omega \Psi_1(x, |f(x)g(x)|)d\mu &= \sum_{k=-\infty}^{0} \sum_{\alpha\in\mathcal{B}_k} \int_{S_k\cap Q^0_\alpha} \Psi_1(x, |f(x)g(x)|)d\mu \\ &\lesssim \sum_{k=-\infty}^{0} \sum_{\alpha\in\mathcal{B}_k} \int_{S_k\cap Q^0_\alpha} w(x)e^{\nu|g(x)|}d\mu + \nu^{-1}\|f\|_1.\end{aligned}$$

Therefore,

$$\int_\Omega \Psi_1(x, |f(x)g(x)|)d\mu \lesssim T_1 + \nu^{-1}\|f\|_1, \tag{3.2.4}$$

where

$$T_1 := \sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k}\int_{S_k\cap Q_\alpha^0} w(x)e^{\nu\left|g(x)-g_{Q_\alpha^0}\right|}e^{\nu\left|g_{Q_\alpha^0}\right|}d\mu.$$

Let $\nu := \frac{\min\{\kappa,1\}}{2} > 0$ (where κ is defined in Theorem 1.3.10), by (3.2.2) and Theorem 2.2.1, one has

$$\begin{aligned} T_1 &\lesssim \sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k}\frac{\mu(Q_\alpha^0)\left(e+d(z_\alpha^0,O)\right)^{\frac{1}{2}}}{\delta^{(k+1)(C_\mu+1)}} \\ &\lesssim \sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k}\frac{\mu(Q_\alpha^0)\delta^{\frac{k}{2}}}{\delta^{(k+1)(C_\mu+1)}} \lesssim \sum_{k=-\infty}^{0}\frac{\delta^{C_\mu k}\delta^{\frac{k}{2}}}{\delta^{C_\mu k+k}} \\ &\lesssim \sum_{k=-\infty}^{0}\delta^{-\frac{1}{2}k}, \end{aligned}$$

and hence

$$T_1 \lesssim 1. \tag{3.2.5}$$

Combine (3.2.4), (3.2.5) and the fact that $\nu^{-1}\|f\|_1 \lesssim 1$, and the proof is complete. □

We consider the case $0 < p < 1$. Define

$$\|g\|_{\Lambda_{1,+}^{\mathscr{D}}(\alpha_p)} := \sup_{\alpha\in\mathcal{A}_0}\frac{|g_{Q_\alpha^0}-g_{Q^0}|}{1+\mu\left\{B\left(O,d(z_\alpha^0,O)\right)\right\}^{\alpha_p}} + |g_{Q^0}| + \|g\|_{\Lambda_1^{\mathscr{D}}(\alpha_p)},$$

Denote by $\Lambda_{1,+}^{\mathscr{D}}(\alpha_p)$ the space consisting of all $g \in \Lambda_1^{\mathscr{D}}(\alpha_p)$ such that $\|g\|_{\Lambda_{1,+}^{\mathscr{D}}(\alpha_p)} < \infty$. It is easy to verify that $\|\cdot\|_{\Lambda_{1,+}^{\mathscr{D}}(\alpha_p)}$ is a norm on $\Lambda_{1,+}^{\mathscr{D}}(\alpha_p)$.

Remark 3.2.3. If we consider the dyadic martingales on $\mathbb{R}^n$, by taking appropriate cubes Q^0 one can show that if $g \in \Lambda_1^{\mathscr{D}}(\alpha_p)$, then $g \in \Lambda_{1,+}^{\mathscr{D}}(\alpha_p)$. Note that if $g \in \mathcal{L}_{\alpha_p}(\mu)$, then $g \in \Lambda_{1,+}^{\mathscr{D}}(\alpha_p)$. Moreover,

$$\|g\|_{\Lambda_{1,+}^{\mathscr{D}}(\alpha_p)} \lesssim \|g\|_{\mathcal{L}_{\alpha_p}(\mu)} + |g_{Q^0}|.$$

Next we present a generalized Hölder inequality for $L^p(\Omega,\mathcal{F},\mu)$ and $\Lambda_{1,+}^{\mathscr{D}}(\alpha_p)$ for $0 < p < 1$.

Lemma 3.2.4. *If $f \in L_p(\Omega,\mathcal{F},\mu)$ and $g \in \Lambda_{1,+}^{\mathscr{D}}(\alpha_p)$ for $0 < p < 1$, then $f\cdot g \in L^{\Psi_p}(\mu)$. Moreover,*

$$\|fg\|_{L^{\Psi_p}(\Omega)} \lesssim \|f\|_p\|g\|_{\Lambda_{1,+}^{\mathscr{D}}(\alpha_p)}. \tag{3.2.6}$$

Proof. Without loss of generality, assume $\|f\|_p \le 1$, $\|g\|_{\Lambda_{1,+}^{\mathscr{D}}(\alpha_p)} \le 1$ and $g_{Q^0} = 0$. It suffices to show that

$$\int_\Omega \Psi_p(x, |f(x)g(x)|)d\mu \lesssim 1.$$

Take the same family of sets $\{S_k\}_{k\le 0}$ as above. From Theorem 2.2.4, we know that for $x \in Q_\alpha^0$,

$$\begin{aligned}|g(x)| = |g(x) - g_{Q^0}| &\leqslant \left|g(x) - g_{Q_\alpha^0}\right| + \left|g_{Q_\alpha^0} - g_{Q^0}\right| \\ &\le \left(\mu(Q_\alpha^0)\right)^{\alpha_p} + \mu\left\{B\left(O, d(z_\alpha^0, O)\right)\right\}^{\alpha_p} + 1 \\ &\lesssim \mu\left(B(O, 2A_0C_0\delta^k)\right)^{\alpha_p} + 1.\end{aligned}$$

Therefore

$$\begin{aligned}\int_\Omega \Psi_p(x, |f(x)g(x)|)d\mu &= \sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k}\int_{S_k\cap Q_\alpha^0} \frac{|g(x)||f(x)|}{1+\{|g(x)||f(x)|\,[1+\mu\,(B(O,d(x,O))]\}^{1-p}}d\mu \\ &\lesssim \sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k}\int_{S_k\cap Q_\alpha^0} \frac{|g(x)|^p|f(x)|^p}{1+\mu\,(B(O,d(x,O))^{1-p}}d\mu \\ &\lesssim \sum_{k=-\infty}^{0}\sum_{\alpha\in\mathcal{B}_k}\int_{S_k\cap Q_\alpha^0} \frac{\mu\left(B(O,2A_0C_0\delta^k)\right)^{\alpha_p p}+1}{\{1+\mu\,(B(O,C_0\delta^{k+1})\}^{1-p}}|f(x)|^p d\mu \\ &\lesssim 1,\end{aligned}$$

which finishes the proof. □

We are now about to present the analogues of the results in Sections 2.1 and 2.2 concerning bilinear decompositions for dyadic martingales on homogeneous spaces. To this end, we need to define the product between martingale Hardy spaces and their dual spaces first. As in the probability setting, we regard the product in the sense of distribution as follows: for $0 < p < 1$,

$$\langle f\times g, h\rangle := \langle h\cdot g, f\rangle, \quad f\in H^p_{\mathrm{at},\mathscr{D}}(\mu),\ g\in \Lambda^{\mathscr{D}}_{1,+}(\alpha_p),$$

where h is a test function such that $h\cdot g$ is in $\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$. For $p=1$, we may define the product between $H^1_{\mathrm{at},\mathscr{D}}(\mu)$ and $BMO^{\mathscr{D}}(\mu)$ analogously. To this end, we need to introduce some pointwise multipliers of $\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$ and $BMO^{\mathscr{D}}(\mu)$.

Denote the space of test functions by $\mathcal{H}(\alpha_p)$ $(0<p\le 1)$, and a measurable function h is a test function if it satisfies the following properties:

$$|h(x)| \lesssim \frac{1}{(1+\mu(B(O,d(x,O)))^{\alpha_p})\log(e+d(x,O))}, \quad \forall x\in\Omega, \tag{3.2.7}$$

and

$$|h(y)-h(z)| \lesssim \frac{\mu(B)^{\alpha_p}}{(1+\mu[B(O,1+r+d(c_B,O))]^{\alpha_p})\log(e+r+d(c_B,O))} \tag{3.2.8}$$

whenever y, z are both contained in a ball B with center c_B and radius $r\le \frac{d(c_B,O)}{2A_0}+1$.

It is obvious that $\mathcal{H}(\alpha_p)\subset L_\infty(\Omega)$. The following theorem shows that if $h\in\mathcal{H}(\alpha_p)$, then h is a pointwise multiplier of $\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$.

Theorem 3.2.5. *For $0 < p < 1$ and any dyadic system $\mathscr{D}$, $\mathcal{H}(\alpha_p)$ is a space of pointwise multipliers of $\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$. For $p = 1$, $\mathcal{H}(0)$ is a space of pointwise multipliers of $BMO^{\mathscr{D}}_{+}(\mu)$. More precisely, for any $g \in \Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$ and $h \in \mathcal{H}(\alpha_p)$, we have*

$$\|g \cdot h\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \lesssim \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \left(\|h\|_{L_\infty(\Omega)} + 1\right),$$

and for any $g \in BMO^{\mathscr{D}}_{+}(\mu)$ and $h \in \mathcal{H}(0)$, we have

$$\|g \cdot h\|_{BMO^{\mathscr{D}}_{+}(\mu)} \lesssim \|g\|_{BMO^{\mathscr{D}}_{+}(\mu)} \left(\|h\|_{L_\infty(\Omega)} + 1\right).$$

Proof. First, we consider the case $0 < p < 1$. Assume that $g \in \Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$ and $h \in \mathcal{H}(\alpha_p)$. According to [58], it suffices to show that

$$\sup_{Q} \frac{|g_Q|}{\mu(Q)^{\alpha_p+1}} \left(\int_Q |h(x) - h_Q| dx\right) < \infty, \tag{3.2.9}$$

where Q runs over all dyadic cubes in $\mathscr{D}$.

If $Q \subset Q^0_\beta$ for some $\beta \in \mathcal{A}_0$, there exists a collection of cubes $Q = Q_0 \subset Q_1 \subset \cdots \subset Q_N = Q^0_\beta$ such that there exists a universal constant $0 < \delta^{'} < 1$ with $\mu(Q_{k-1}) \le \delta^{'} \mu(Q_k)$. Hence

$$\begin{aligned}
|g_Q - g_{Q^0_\beta}| &\le \sum_{k=1}^{N} |g_{Q_k} - g_{Q_{k-1}}| \lesssim \sum_{k=1}^{N} \mu(Q_k)^{\alpha_p} \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \\
&\lesssim \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \sum_{k=1}^{N} \int_{\mu(Q_{k-1})}^{\mu(Q_k)} t^{\alpha_p - 1} dt \\
&\lesssim \mu(Q^0_\beta)^{\alpha_p} \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)}.
\end{aligned}$$

Similarly, if $Q^0_\beta \subset Q$, we have

$$|g_Q - g_{Q^0_\beta}| \lesssim \mu(Q)^{\alpha_p} \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)}.$$

By Theorem 3.1.1, there exists a ball B, with center c_B and radius r, such that $Q \subset B$ and $\mu(B) \lesssim \mu(Q)$.

If $Q^0_\beta \subset Q$ and $r > \frac{d(O,c_B)}{2A_0} + 1$, for any $x \in B(O, r)$, we have $d(c_B, x) \le A_0(d(c_B, O) + d(O, x)) < (2A_0^2 + A_0)r$. Then $\mu(Q) \gtrsim \mu(B) \gtrsim C_\mu^{-(2A_0^2+A_0)} \mu\left(B(O, r)\right) \gtrsim 1$. Similarly, we also have $d(z^0_\beta, O) < (2A_0^2 + A_0)r$ and $\mu\left\{B\left(O, d(z^0_\beta, O)\right)\right\} \lesssim \mu(B) \lesssim \mu(Q)$. Thus

$$\begin{aligned}
\frac{|g_Q|}{\mu(Q)^{\alpha_p+1}} \left(\int_Q |h(x) - h_Q| dx\right) &\lesssim \frac{|g_Q - g_{Q^0_\beta}| + |g_{Q^0_\beta} - g_{Q^0}| + |g_{Q^0}|}{\mu(Q)^{\alpha_p}} \cdot \|h\|_{L_\infty(\Omega)} \\
&\lesssim \frac{\mu(Q)^{\alpha_p} + \mu\left\{B\left(O, d(z^0_\beta, O)\right)\right\}^{\alpha_p} + 1}{\mu(Q)^{\alpha_p}} \cdot \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \|h\|_{L_\infty(\Omega)} \\
&\lesssim \|g\|_{\Lambda^{\mathscr{D}}_{1,+}(\alpha_p)} \|h\|_{L_\infty(\Omega)}.
\end{aligned}$$

If $Q_\beta^0 \subset Q$ and $r \le \frac{d(O,c_B)}{2A_0} + 1$, for any $x \in B$, we have $d(x,O) \le A_0(d(O,c_B)+r)$, then $\mu(Q) \lesssim \mu\left(B(O, A_0(d(O,c_B)+r)\right)$. Thus

$$\begin{aligned}\frac{|g_Q|}{\mu(Q)^{\alpha_p+1}}\left(\int_Q |h(x)-h_Q|dx\right) &\lesssim \frac{|g_Q - g_{Q_\beta^0}| + |g_{Q_\beta^0} - g_{Q^0}| + |g_{Q^0}|}{\mu(Q)^{\alpha_p+1}} \frac{\mu(B)^{\alpha_p+1}}{(1+\mu[B(O,1+r+d(O,c_B))])^{\alpha_p}} \\ &\lesssim \frac{\left(\mu(Q)^{\alpha_p} + \mu\left\{B\left(O, d(z_\beta^0, O)\right)\right\}^{\alpha_p} + 1\right)\|g\|_{\Lambda_{1,+}^{\mathscr{D}}(\alpha_p)}}{(1+\mu[B(O,1+r+d(O,c_B))])^{\alpha_p}} \\ &\lesssim \|g\|_{\Lambda_{1,+}^{\mathscr{D}}(\alpha_p)}.\end{aligned}$$

If $Q \subset Q_\beta^0$, from Theorem 3.1.1, we can choose C_0 sufficiently small such that $C_1 = 2A_0C_0 \le 1$, then $r \le C_1 \le \frac{d(c_B,O)}{2A_0} + 1$. For any $x \in Q_\beta^0$, we have $d(O,x) \le A_0(d(O,z_\beta^0) + C_1)$. Then

$$\mu(Q_\beta^0) \lesssim \mu\left\{B\left(O, A_0(d(O,z_\beta^0)+C_1)\right)\right\}.$$

By a calculation similar to the one presented above, we get the desired result.

Combining the above estimates, we finish our proof for $0 < p < 1$. The case for $p = 1$ is similar. □

Remark 3.2.6. Note that in Theorem 3.2.5, the dyadic system $\mathscr{D}$ is arbitrary. Then from Theorem 3.1.8 and (3.1.12), we conclude that $\mathcal{H}(\alpha_p)$ is a space of pointwise multipliers of $\mathcal{L}_{\alpha_p}(\mu)$ and $\mathcal{H}(0)$ is a space of pointwise multipliers of $BMO(\mu)$.

3.2.2 Bilinear decompositions

Assume $f \in H_{\mathscr{D}}^1(\mu), g \in BMO_+^{\mathscr{D}}(\mu)$ or $f \in H_{\mathscr{D}}^p(\mu), g \in \Lambda_{1,+}^{\mathscr{D}}(\alpha_p)$, $0 < p < 1$.

Denote by $\mathcal{H}_{\mathscr{D},\mathrm{fin}}^p(\mu)$ $(0 < p \le 1)$ the linear space consisting of all functions which can be written as a finite sum of simple (p,∞)-atoms. Thus if $f \in \mathcal{H}_{\mathscr{D},\mathrm{fin}}^p(\mu)$, f is locally supported, $f \in L_1(\Omega) \cap L_\infty(\Omega)$ and $\int_\Omega f d\mu = 0$. Note that $\mathcal{H}_{\mathscr{D},\mathrm{fin}}^p(\mu)$ is dense in $H_{\mathscr{D}}^p(\mu)$ with respect to the norm $\|\cdot\|_{H_{\mathscr{D}}^p(\mu)}$.

In the following, we shall only consider the case where $f \in \mathcal{H}_{\mathscr{D},\mathrm{fin}}^p(\mu)$. Then $f \cdot g \in L_1(\Omega)$, and we can write

$$f \cdot g = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g), \tag{3.2.10}$$

where

$$\Pi_1(f,g) := \sum_{k=-\infty}^{\infty} d_k f d_k g, \quad \Pi_2(f,g) := \sum_{k=-\infty}^{\infty} f_{k-1} d_k g \quad \text{and} \quad \Pi_3(f,g) := \sum_{k=-\infty}^{\infty} g_{k-1} d_k f.$$

Theorem 3.2.7. *We have the following:*

1. Π_1 *is a bilinear bounded operator from* $H_{\mathscr{D}}^1(\mu) \times BMO_+^{\mathscr{D}}(\mu)$ *to* $L_1(\Omega)$, *and when* $0 < p < 1$, Π_1 *is a bilinear bounded operator from* $H_{\mathscr{D}}^p(\mu) \times \Lambda_{1,+}^{\mathscr{D}}(\alpha_p)$ *to* $L_1(\Omega)$.

2. Π_2 *is a bilinear bounded operator from* $H_{\mathscr{D}}^1(\mu) \times BMO_+^{\mathscr{D}}(\mu)$ *to* $H_{\mathscr{D}}^1(\mu)$, *and when* $0 < p < 1$, Π_2 *is a bilinear bounded operator from* $H_{\mathscr{D}}^p(\mu) \times \Lambda_{1,+}^{\mathscr{D}}(\alpha_p)$ *to* $H_{\mathscr{D}}^1(\mu)$.

3. Π_3 *is a bilinear bounded operator from* $H_{\mathscr{D}}^1(\mu) \times BMO_+^{\mathscr{D}}(\mu)$ *to* $H_{\mathscr{D}}^{\Psi_1}(\mu)$, *and when* $0 < p < 1$, Π_3 *is a bilinear bounded operator from* $H_{\mathscr{D}}^p(\mu) \times \Lambda_{1,+}^{\mathscr{D}}(\alpha_p)$ *to* $H_{\mathscr{D}}^{\Psi_p}(\mu)$.

Proof. For Π_1 and Π_2, we can argue as in the corresponding part of the proof of Theorem I.1. As for Π_3, we can also argue as in the corresponding part of the proof Theorem I.1, where in the homogeneous setting one needs to apply Lemma 3.2.2 and Lemma 3.2.4. We omit the details. □

Remark 3.2.8. For Π_1 and Π_2, the condition $H^1_{\mathscr{D}}(\mu) \times BMO^{\mathscr{D}}_+(\mu)$ and $H^p_{\mathscr{D}}(\mu) \times \Lambda^{\mathscr{D}}_{1,+}(\alpha_p)$ can be in fact replaced by $H^1_{\mathscr{D}}(\mu) \times BMO^{\mathscr{D}}(\mu)$ and $H^p_{\mathscr{D}}(\mu) \times \Lambda^{\mathscr{D}}_1(\alpha_p)$, respectively.

3.3 Applications to homogeneous spaces

In the first part of this section we show that $H^{\Psi_p}_{\mathscr{D}}(\mu)$ admits an atomic decomposition for $0 < p < 1$, which allows us to integrate several adjacent dyadic systems on homogeneous spaces.

For a given dyadic system $\mathscr{D}$ on Ω, we define the dyadic $H^{\Psi_p}_{\mathrm{at},\mathscr{D}}$-atom as follows.

Definition 3.3.1. A measurable function a is said to be an $H^{\Psi_p}_{\mathrm{at},\mathscr{D}}$-atom if

(i) $\mathrm{supp}(a) \subset Q$ where $Q \in \mathscr{D}$ is a cube;

(ii) $\int_\Omega a d\mu = 0$;

(iii) $\|a\|_\infty \leqslant \|1_Q\|^{-1}_{L^{\Psi_p}(\Omega)}$.

The atomic dyadic martingale Musielak–Orlicz Hardy spaces $H^{\Psi_p}_{\mathrm{at},\mathscr{D}}(\mu)$ $(0 < p < 1)$ are defined in a way analogous to (1.8.5) and (1.8.6). We first introduce the space $BMO^{\mathscr{D}}_{\Psi_p}(\mu)$, which is a subspace of continuous linear functionals on finite sums of atoms.

Definition 3.3.2. A locally integrable function g is said to be a dyadic $BMO^{\mathscr{D}}_{\Psi_p}(\mu)$ function associated with a dyadic system $\mathscr{D}$ if

$$\|g\|_{BMO^{\mathscr{D}}_{\Psi_p}(\mu)} := \sup_{k\in\mathbb{Z}} \sup_{Q\in\mathcal{F}_k} \frac{1}{\|1_Q\|_{L^{\Psi_p}(\Omega)}} \int_Q |g(x) - g_k(x)| dx < \infty.$$

Then we define the atomic Musielak–Orlicz martingale Hardy spaces $H^{\Psi_p}_{\mathrm{at},\mathscr{D}}(\mu)$ as follows:

$$H^{\Psi_p}_{\mathrm{at},\mathscr{D}}(\mu) :=$$
$$\left\{ f \in \left(BMO^{\mathscr{D}}_{\Psi_p}(\mu)\right)^* : \ f = \sum_{i=0}^{\infty} \lambda_i a_i, \text{ where } a_i \text{ is an } H^{\Psi_p}_{\mathrm{at},\mathscr{D}}(\mu)\text{-atom supported on a cube } Q_i. \right\},$$

where

$$\sum_{i=0}^{\infty} \int_{Q_i} \Psi_p(x, |\lambda_i| \|a_i\|_\infty) d\mu < \infty.$$

Moreover,

$$\|f\|_{H^{\Psi_p}_{\mathrm{at},\mathscr{D}}(\mu)} := \inf \left\{ \rho > 0 : \sum_{i=0}^{\infty} \int_{Q_i} \Psi_p(x, \rho^{-1}|\lambda_i| \|a_i\|_\infty) d\mu \leqslant 1 \right\}$$

Arguing as in [93], one can show that for $0 < p < 1$

$$H_{\mathscr{D}}^{\Psi_p}(\mu) = H_{\mathrm{at},\mathscr{D}}^{\Psi_p}(\mu). \tag{3.3.1}$$

We now introduce the atomic Musielak–Orlicz Hardy spaces $H_{\mathrm{at}}^{\Psi_p}(\mu)$ $(0 < p \leq 1)$ on the homogeneous space Ω. The interested reader is referred to [89] for more details. First, we present the definition of atoms for $H_{\mathrm{at}}^{\Psi_p}(\mu)$.

Definition 3.3.3. A measurable function $a(x)$ is said to be an $H_{\mathrm{at}}^{\Psi_p}(\mu)$-atom if

(i) $\mathrm{supp}(a) \subset B$ where $B \subset \Omega$ is a ball;

(ii) $\int_\Omega a d\mu = 0$;

(iii) $\|a\|_\infty \leqslant \|1_B\|^{-1}_{L^{\Psi_p}(\Omega)}$.

Definition 3.3.4. A locally integrable function g is said to be a $BMO_{\Psi_p}(\mu)$ function if

$$\|g\|_{BMO_{\Psi_p}(\mu)} := \sup_B \frac{1}{\|1_B\|_{L^{\Psi_p}(\Omega)}} \int_B |g(x) - g_B| dx < \infty,$$

where B runs over all balls in Ω.

Definition 3.3.5. The atomic Musielak–Orlicz Hardy spaces $H_{\mathrm{at}}^{\Psi_p}(\mu)$ $(0 < p \leq 1)$ are defined as follows:

$$H_{\mathrm{at}}^{\Psi_p}(\mu) :=$$
$$\left\{ f \in \left(BMO_{\Psi_p}(\mu)\right)^* : f = \sum_{i=0}^{\infty} \lambda_i a_i, \text{ where } a_i \text{ is an } H_{\mathrm{at}}^{\Psi_p}(\mu)\text{-atom supported on a ball } B_i \right\},$$

where

$$\sum_{i=0}^{\infty} \int_{B_i} \Psi_p(x, |\lambda_i| \|a_i\|_\infty) d\mu < \infty.$$

Moreover,

$$\|f\|_{H_{\mathrm{at}}^{\Psi_p}(\mu)} := \inf \left\{ \rho > 0 : \sum_{i=0}^{\infty} \int_{B_i} \Psi_p(x, \rho^{-1}|\lambda_i| \|a_i\|_\infty) d\mu \leqslant 1 \right\}.$$

Let $\mathscr{D}^t$ $(1 \leq t \leq K)$ be the adjacent systems of Theorem 3.1.5. By arguing as in the proof of Theorem 3.1.6, we have the following:

Lemma 3.3.6. *For $0 < p < 1$, $H_{\mathrm{at}}^{\Psi_p}(\mu) = H_{\mathrm{at},\mathscr{D}^1}^{\Psi_p}(\mu) + H_{\mathrm{at},\mathscr{D}^2}^{\Psi_p}(\mu) + \cdots + H_{\mathrm{at},\mathscr{D}^K}^{\Psi_p}(\mu)$.*

Proof. It suffices to show that any dyadic $H_{\mathscr{D}^t}^{\Psi_p}$-atom a is a constant multiple of an $H^{\Psi_p}(\mu)$-atom, and any $H^{\Psi_p}(\mu)$-atom b is a constant multiple of a dyadic $H_{\mathscr{D}^t}^{\Psi_p}$-atom.

If $B := B(x_0, r)$, then denote the ball $B(x_0, Dr)$ by DB for $D \geq 1$. Denote $d := d(x_0, O)$. In what follows, $C(D, p, A_0, C_\mu)$ denotes a constant that depends on D, p, A_0, C_μ and may differ from line to line. We first show that if

$$\int_B \frac{1}{1 + [1 + \mu(B(O, d(x, O)))]^{1-p}} d\mu(x) = 1,$$

then

$$\int_{DB} \frac{1}{1+[1+\mu(B(O,d(x,O)))]^{1-p}} d\mu(x) \leqslant C(D,p,A_0,C_\mu). \tag{3.3.2}$$

Notice that

$$\begin{aligned} 1 &= \int_B \frac{1}{1+[1+\mu(B(O,d(x,O)))]^{1-p}} d\mu(x) \\ &\geqslant \frac{\mu(B)}{\sup\limits_{x\in B}\{1+[1+\mu(B(O,d(x,O)))]^{1-p}\}} \\ &\geqslant \frac{\mu(B)}{1+[1+\mu(B(O,A_0(d+r)))]^{1-p}}, \end{aligned}$$

which implies

$$\mu(B) \leqslant 1+[1+\mu(B(O,A_0(d+r)))]^{1-p}.$$

If $d \leqslant 2A_0Dr$, we have

$$\begin{aligned} \mu(B) &\leqslant 1+[1+\mu(B(O,A_0(2A_0D+1)r))]^{1-p} \\ &\leqslant 1+\{1+\mu[B(x_0,A_0(A_0+1)(2A_0D+1)r)]\}^{1-p} \\ &\leqslant 1+\left\{1+[A_0(A_0+1)(2A_0D+1)]^{C_\mu}\,\mu(B)\right\}^{1-p}, \end{aligned}$$

and thus $\mu(B) \leqslant C(D,p,A_0,C_\mu)$.

Then

$$\int_{DB} \frac{1}{1+[1+\mu(B(O,d(x,O)))]^{1-p}} d\mu(x) \leqslant \mu(DB) \leqslant D^{C_\mu}\mu(B) \leqslant C(D,p,A_0,C_\mu). \tag{3.3.3}$$

If $d > 2A_0Dr$, then

$$\begin{aligned} &\int_{DB} \frac{1}{1+[1+\mu(B(O,d(x,O)))]^{1-p}} d\mu(x) \\ \leqslant &\frac{\mu(DB)}{\inf\limits_{x\in DB}\{1+[1+\mu(B(O,d(x,O)))]^{1-p}\}} \\ \leqslant &\frac{D^{C_\mu}\mu(B)}{1+[1+\mu(B(O,d/A_0-Dr))]^{1-p}}, \\ \leqslant &\frac{D^{C_\mu}\{1+\mu[B(O,[A_0+1/(2D)]d)]\}^{1-p}+D^{C_\mu}}{1+\{1+\mu[B(O,d/(2A_0))]\}^{1-p}} \\ \leqslant &\frac{D^{C_\mu}\left\{1+[(2A_0+1/D)A_0]^{C_\mu}\,\mu[B(O,d/(2A_0))]\right\}^{1-p}}{1+\{1+\mu[B(O,d/(2A_0))]\}^{1-p}}+D^{C_\mu}. \end{aligned}$$

Hence

$$\int_{DB} \frac{1}{1+[1+\mu(B(O,d(x,O)))]^{1-p}} d\mu(x) \leqslant C(D,p,A_0,C_\mu) \tag{3.3.4}$$

Combining (3.3.3) with (3.3.4), we get (3.3.2).

Assume a is an $H^{\Psi_p}(\mu)$-atom supported on B. By Theorem 3.1.5, there exist t and a cube $Q \in \mathscr{D}^t$ such that $B \subset Q$ and $\operatorname{diam}(Q) \leqslant Cr$, hence $B \subset Q \subset CB$.

Note that $\operatorname{supp}(a) \subset Q$, $\int_Q a(x)d\mu(x) = 0$ and

$$\|1_Q\|_{L^{\Psi_p}(\mu)} \leqslant \|1_{CB}\|_{L^{\Psi_p}(\mu)} \leqslant C(C, p, A_0, C_\mu)\|1_B\|_{L^{\Psi_p}(\mu)},$$

which follows from (3.3.2). Thus

$$\|a\|_\infty \leqslant \|1_B\|^{-1}_{L^{\Psi_p}(\mu)} \lesssim \|1_Q\|^{-1}_{L^{\Psi_p}(\mu)},$$

which implies a is a multiple of dyadic $H^{\Psi_p}_{\mathscr{D}^t}$-atom supported on Q.

For any $t = 1, 2 \cdots, K$, assume b is a dyadic $H^{\Psi_p}_{\mathscr{D}^t}$-atom supported on Q^k_β. By Theorem 3.1.1, there exists two balls such that $B(z^k_\beta, c_1\delta^k) \subset Q^k_\beta \subset B(z^k_\beta, C_1\delta^k)$.

Thus $\operatorname{supp}(b) \subset B(z^k_\beta, C_1\delta^k)$, $\int_{B(z^k_\beta, C_1\delta^k)} b(x)d\mu(x) = 0$ and

$$\|1_{B(z^k_\beta, C_1\delta^k)}\|_{L^{\Psi_p}(\mu)} \leqslant C\left(\frac{C_1}{c_1}, p, A_0, C_\mu\right)\|1_{B(z^k_\beta, c_1\delta^k)}\|_{L^{\Psi_p}(\mu)} \lesssim \|1_{Q^k_\beta}\|_{L^{\Psi_p}(\mu)},$$

which follows from (3.3.2). Therefore,

$$\|b\|_\infty \leqslant \|1_{Q^k_\beta}\|^{-1}_{L^{\Psi_p}(\mu)} \lesssim \|1_{B(z^k_\beta, C_1\delta^k)}\|^{-1}_{L^{\Psi_p}(\mu)},$$

which implies b is a multiple of dyadic H^{Ψ_p}-atom supported on $B(z^k_\beta, C_1\delta^k)$. □

Remark 3.3.7. In [30], Fu, Ma and Yang defined another kind of Musielak–Orlicz Hardy spaces by grand maximal function and they also proved that these Musielak–Orlicz Hardy spaces are equivalent to $H^{\Psi_p}_{\mathrm{at}}(\mu)$ with respect to the corresponding norms when $p \in (\frac{C_\mu}{C_\mu+1}, 1]$.

Let $B_1 := B(O, 1)$. Define

$$\|g\|_{BMO_+(\mu)} := |g_{B_1}| + \|g\|_{BMO(\mu)}, \quad \text{for } g \in BMO(\mu),$$

and

$$\|g\|_{\mathcal{L}_{+,\alpha}(\mu)} := |g_{B_1}| + \|g\|_{\mathcal{L}_{\alpha_p}(\mu)}, \quad \text{for } g \in \mathcal{L}_{\alpha_p}(\mu).$$

Thus $\|\cdot\|_{BMO_+(\mu)}$ and $\|\cdot\|_{\mathcal{L}_{+,\alpha_p}(\mu)}$ are quasi-norms on $BMO(\mu)$ and $\mathcal{L}_{\alpha_p}(\mu)$, respectively.

Theorem 3.3.8. *Let $0 < p < 1$ and $f \in H^p_{\mathrm{at}}(\mu)$. There exist three linear continuous operators $\Pi^f_1 : \mathcal{L}_{\alpha_p}(\mu) \to L_1(\Omega)$, $\Pi^f_2 : \mathcal{L}_{\alpha_p}(\mu) \to H^1_{\mathrm{at}}(\mu)$ and $\Pi^f_3 : \mathcal{L}_{\alpha_p}(\mu) \to H^{\Psi_p}_{\mathrm{at}}(\mu)$ such that*

$$f \cdot g = \Pi^f_1(g) + \Pi^f_2(g) + \Pi^f_3(g) \quad \textit{for all } g \in \mathcal{L}_{\alpha_p}(\mu),$$

where $\mathcal{L}_{\alpha_p}(\mu)$ is endowed with the quasi-norm $\|\cdot\|_{\mathcal{L}_{+,\alpha_p}(\mu)}$.

Proof. Let $f \in H^p_{\mathrm{at}}(\mu)$. By Theorem 3.1.6 there exist $f^t \in H^p_{\mathscr{D}^t}(\mu)$ $(t = 1, 2, \cdots, K)$ such that $f = f^1 + f^2 + \cdots + f^K$, and

$$\sum_{t=1}^K \|f^t\|_{H^p_{\mathscr{D}^t}(\mu)} \approx \|f\|_{H^p_{\mathrm{at}}(\mu)}.$$

Define $\Pi^f_i(g) := \sum_{t=1}^K \Pi_i(f^t, g)$ for $i = 1, 2, 3$ and $g \in \mathcal{L}_{\alpha_p}(\mu)$ (Π_i defined as in Theorem 3.2.7). Then

$$f \cdot g = \Pi^f_1(g) + \Pi^f_2(g) + \Pi^f_3(g).$$

By Theorem 3.2.7, Theorem 3.1.6 and Lemma 3.3.6, we have

$$\|\Pi_1^f(g)\|_1 \lesssim \sum_{t=1}^K \|\Pi_1(f^t,g)\|_1 \lesssim \sum_{t=1}^K \|f^t\|_{H^p_{\mathscr{D}^t}(\mu)}\|g\|_{\Lambda^{\mathscr{D}^t}_{1,+}(\alpha_p)} \lesssim \|f\|_{H^p_{\rm at}(\mu)}\|g\|_{\mathcal{L}_{+,\alpha_p}(\mu)},$$

$$\|\Pi_2^f(g)\|_{H^1_{\rm at}(\mu)} \lesssim \sum_{t=1}^K \|\Pi_2(f^t,g)\|_{H^1_{\mathscr{D}^t}(\mu)} \lesssim \sum_{t=1}^K \|f^t\|_{H^p_{\mathscr{D}^t}(\mu)}\|g\|_{\Lambda^{\mathscr{D}^t}_{1,+}(\alpha_p)} \lesssim \|f\|_{H^p_{\rm at}(\mu)}\|g\|_{\mathcal{L}_{+,\alpha_p}(\mu)},$$

$$\|\Pi_3^f(g)\|_{H^{\Psi_p}_{\rm at}(\mu)} \lesssim \sum_{t=1}^K \|\Pi_3(f^t,g)\|_{H^{\Psi_p}_{\mathscr{D}^t}(\mu)} \lesssim \sum_{t=1}^K \|f^t\|_{H^p_{\mathscr{D}^t}(\mu)}\|g\|_{\Lambda^{\mathscr{D}^t}_{1,+}(\alpha_p)} \lesssim \|f\|_{H^p_{\rm at}(\mu)}\|g\|_{\mathcal{L}_{+,\alpha_p}(\mu)}.$$

which finishes the proof. □

Remark 3.3.9. If the homogeneous space (Ω,μ) satisfies the reverse doubling condition, then Lemma 3.3.6 holds for $p=1$. Then we conclude the following.

Let $f\in H^1_{\rm at}(\mu)$. There exist three linear continuous operators $\Pi_1^f : BMO(\mu)\to L_1(\Omega)$, $\Pi_2^f : BMO(\mu)\to H^1_{\rm at}(\mu)$ and $\Pi_3^f : BMO(\mu)\to H^{\Psi_1}_{\rm at}(\mu)$ such that

$$f\cdot g = \Pi_1^f(g)+\Pi_2^f(g)+\Pi_3^f(g) \quad \text{for all } g\in BMO(\mu),$$

where $BMO(\mu)$ is endowed with the norm $\|\cdot\|_{BMO_+(\mu)}$.

Chapter 4

Schatten class of martingale paraproducts

This chapter is devoted to the proofs of Theorem II.2, Theorem II.3 and Theorem II.4 on the Schatten class membership of martingale paraproducts.

4.1 Proof of Theorem II.2

We divide this proof into two parts. The first aims to show the necessity of Theorem II.2, and the second focuses on the sufficiency.

Our proof of the necessity follows the pattern set up in [80]. More precisely, we use the iteration method. However, our case is more complicated since d and $\mathcal{M}$ are arbitrary in Theorem II.2. In addition, we need Lemma 4.1.2 for any $d \geq 2$ to implement the iteration procedure, while in [80] the authors only considered $d = 2$. It should be noted that for $0 < p \leq 2$, we come up with an alternative approach, which is more general than the corresponding argument in [80].

4.1.1 The Necessity of Theorem II.2

In this subsection, we are about to show that

$$b \in \boldsymbol{B}_p^d(\mathbb{R}, \mathcal{M}) \implies \pi_b \in L_p(B(L_2(\mathbb{R})) \otimes \mathcal{M}) \quad \forall\, 0 < p < \infty. \tag{$*$}$$

At first, we prove the result in the case $0 < p \leq 1$. Then we reduce the case $1 < p \leq 2$ to that $0 < p \leq 1$. Finally, we show that if $(*)$ holds for $p/2$, then so does it for p with $2 < p < \infty$. Our main ingredients are the p-John-Nirenberg inequality (Lemma 4.1.3) which appears in [80], and the decomposition of $\pi_b^* \pi_b$ (Lemma 4.1.2). As a result, the necessity of Theorem II.2 follows iteratively for all $2 < p < \infty$.

Proposition 4.1.1. *If $0 < p \leq 2$ and $b \in \boldsymbol{B}_p^d(\mathbb{R}, \mathcal{M})$, then $\pi_b \in L_p(B(L_2(\mathbb{R})) \otimes \mathcal{M})$.*

Proof. For any $I \in \mathcal{D}$ and $1 \leq i \leq d-1$, define

$$\pi_b^{I,i} = \langle h_I^i, b \rangle \cdot B^{I,i}, \tag{4.1.1}$$

where $B^{I,i} \in B(L_2(\mathbb{R}))$ is defined by

$$B^{I,i} = h_I^i \otimes \frac{\mathbb{1}_I}{|I|}. \tag{4.1.2}$$

Then we have

$$\pi_b = \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \pi_b^{I,i}. \tag{4.1.3}$$

Note that $\pi_b^{I,i} \in L_p(B(L_2(\mathbb{R})) \otimes \mathcal{M})$ and

$$\|\pi_b^{I,i}\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} = \|B^{I,i}\|_{S_p(L_2(\mathbb{R}))} \|\langle h_I^i, b\rangle\|_{L_p(\mathcal{M})} = \frac{\|\langle h_I^i, b\rangle\|_{L_p(\mathcal{M})}}{|I|^{1/2}}.$$

If $0 < p \le 1$, since $\|\cdot\|_p$ is a p-norm, one has

$$\begin{aligned}\|\pi_b\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} &\le \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \|\pi_b^{I,i}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \\ &= \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \frac{\|\langle h_I^i, b\rangle\|^p_{L_p(\mathcal{M})}}{|I|^{p/2}} = \|b\|^p_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})} < \infty\end{aligned}$$

Next, we consider the case $1 < p \le 2$. If $I \ne J$ or $i \ne j$, then $\forall g, h \in L_2(\mathbb{R})$,

$$\langle (B^{I,i})^* B^{J,j}(g), h\rangle = \langle B^{J,j}(g), B^{I,i}(h)\rangle = \overline{\left\langle \frac{\mathbb{1}_J}{|J|}, g\right\rangle}\left\langle \frac{\mathbb{1}_I}{|I|}, h\right\rangle \langle h_J^j, h_I^i\rangle = 0,$$

which implies that if $I \ne J$ or $i \ne j$,

$$(\pi_b^{I,i})^*(\pi_b^{J,j}) = 0. \tag{4.1.4}$$

So from (4.1.3) we get

$$\pi_b^*\pi_b = \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} (\pi_b^{I,i})^*(\pi_b^{I,i}). \tag{4.1.5}$$

Note that $p/2 \le 1$, hence we use (4.1.5) to estimate

$$\begin{aligned}\|\pi_b\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} &= \|\pi_b^*\pi_b\|^{p/2}_{L_{p/2}(B(L_2(\mathbb{R}))\otimes\mathcal{M})} = \left\|\sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} (\pi_b^{I,i})^*(\pi_b^{I,i})\right\|^{p/2}_{L_{p/2}(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \\ &\le \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \|(\pi_b^{I,i})^*(\pi_b^{I,i})\|^{p/2}_{L_{p/2}(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \\ &= \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \|(B^{I,i})^* B^{I,i}\|^{p/2}_{S_{p/2}(L_2(\mathbb{R}))} \|\langle h_I^i, b\rangle^* \langle h_I^i, b\rangle\|^{p/2}_{L_{p/2}(\mathcal{M})} \\ &= \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \|B^{I,i}\|^p_{S_p(L_2(\mathbb{R}))} \|\langle h_I^i, b\rangle\|^p_{L_p(\mathcal{M})} \\ &= \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \frac{\|\langle h_I^i, b\rangle\|^p_{L_p(\mathcal{M})}}{|I|^{p/2}} = \|b\|^p_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}.\end{aligned}$$

Thus the proof is completed. □

As mentioned before, we will proceed with our proof by iteration for $2 < p < \infty$. We need the following definitions.

The dyadic square function S_b of $b \in L_2(\mathbb{R}, L_2(\mathcal{M}))$ is defined by

$$S_b := \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \frac{\mathbb{1}_I}{|I|} \langle h_I^i, b\rangle^* \langle h_I^i, b\rangle. \tag{4.1.6}$$

Besides, D_b is defined as follows:

$$D_b := \sum_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{q=1}^{d} M^{I,q} \otimes N^{I,q}, \tag{4.1.7}$$

where $M^{I,q} \in B(L_2(\mathbb{R}))$ and $N^{I,q}$ are given by

$$M^{I,q} := \sum_{i=1}^{d-1} \sum_{m=1}^{d-1} \omega^{(i-m)q} h_I^m \otimes h_I^i, \tag{4.1.8}$$

$$N^{I,q} := \sum_{J\in\mathcal{D}(I(q))} \sum_{j=1}^{d-1} \langle h_J^j, b\rangle^* \langle h_J^j, b\rangle. \tag{4.1.9}$$

It is clear that $\operatorname{span}\{h_I^i : 1 \le i \le d-1\}$ is an invariant subspace of $M^{I,q}$. So for any $J, K \in \mathcal{D}$, $1 \le j, k \le d-1$ and $x, y \in L_2(\mathcal{M})$, if $J \ne K$, we have

$$\langle D_b(h_J^j \otimes x), h_K^k \otimes y\rangle = 0. \tag{4.1.10}$$

The following decomposition of $\pi_b^*\pi_b$ allows us to reduce the case p to $p/2$. It first appears in [6] for $d = 2$. We extend it for general d thanks to the good choice of the orthonormal basis $\{h_I^i\}_{I\in\mathcal{D}, 1\le i\le d-1}$.

Lemma 4.1.2.

$$\pi_b^*\pi_b = \pi_{S_b} + \pi_{S_b}^* + D_b.$$

Proof. It suffices to prove that for all $J, K \in \mathcal{D}$, $1 \le j, k \le d-1$, and $x, y \in L_2(\mathcal{M})$,

$$\langle \pi_b^*\pi_b(h_J^j \otimes x), h_K^k \otimes y\rangle = \langle (\pi_{S_b} + \pi_{S_b}^* + D_b)(h_J^j \otimes x), h_K^k \otimes y\rangle.$$

Note that if $J \subseteq K$, then for any $f \in L_2(\mathbb{R}, L_2(\mathcal{M}))$,

$$\begin{aligned} \langle \pi_f(h_J^j \otimes x), h_K^k \otimes y\rangle &= \left\langle \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} h_I^i \otimes \langle h_I^i, f\rangle \left\langle \frac{\mathbb{1}_I}{|I|}, h_J^j \right\rangle x, h_K^k \otimes y \right\rangle \\ &= \overline{\left\langle \frac{\mathbb{1}_K}{|K|}, h_J^j \right\rangle} \tau(x^* \langle f, h_K^k\rangle y) = 0. \end{aligned} \tag{4.1.11}$$

We also have

$$\langle h_K^k, S_b\rangle = \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \left\langle h_K^k, \frac{\mathbb{1}_I}{|I|} \right\rangle \langle h_I^i, b\rangle^* \langle h_I^i, b\rangle. \tag{4.1.12}$$

Now we divide the rest of the proof into three cases.

1. $J = K$. From (4.1.5) one has

$$\begin{aligned}\langle \pi_b^*\pi_b(h_J^j \otimes x), h_J^k \otimes y\rangle &= \sum_{I\in\mathcal{D}}\sum_{i=1}^{d-1}\langle \pi_b^{I,i}(h_J^j\otimes x), \pi_b^{I,i}(h_J^k\otimes y)\rangle \\ &= \sum_{I\in\mathcal{D}}\sum_{i=1}^{d-1}\overline{\left\langle \frac{\mathbb{1}_I}{|I|}, h_J^j\right\rangle}\left\langle \frac{\mathbb{1}_I}{|I|}, h_J^k\right\rangle \tau(x^*\langle h_I^i, b\rangle^*\langle h_I^i, b\rangle y) \\ &= \frac{1}{|J|}\sum_{i=1}^{d-1}\sum_{q=1}^{d}\sum_{I\in\mathcal{D}(J(q))}\omega^{(k-j)q}\tau(x^*\langle h_I^i, b\rangle^*\langle h_I^i, b\rangle y) \\ &= \left\langle \sum_{i,m=1}^{d-1}\frac{1}{|J|}\sum_{q=1}^{d}\sum_{I\in\mathcal{D}(J(q))}\omega^{(j-m)q}h_J^m\otimes\langle h_I^i, b\rangle^*\langle h_I^i, b\rangle x, h_J^k\otimes y\right\rangle.\end{aligned}$$

   By the definition (4.1.7) of D_b, we deduce

$$\begin{aligned}\langle D_b(h_J^j\otimes x), h_J^k\otimes y\rangle &= \left\langle \sum_{I\in\mathcal{D}}\frac{1}{|I|}\sum_{q=1}^{d}M^{I,q}\otimes N^{I,q}(h_J^j\otimes x), h_J^k\otimes y\right\rangle \\ &= \left\langle \sum_{I\in\mathcal{D}}\frac{1}{|I|}\sum_{q=1}^{d}\sum_{i,m=1}^{d-1}\langle h_I^i, h_J^j\rangle\omega^{(i-m)q}h_I^m\otimes N^{I,q}x, h_J^k\otimes y\right\rangle \\ &= \left\langle \sum_{i,m=1}^{d-1}\frac{1}{|J|}\sum_{q=1}^{d}\sum_{I\in\mathcal{D}(J(q))}\omega^{(j-m)q}h_J^m\otimes\langle h_I^i, b\rangle^*\langle h_I^i, b\rangle x, h_J^k\otimes y\right\rangle.\end{aligned}$$

   By (4.1.11), recall that $J = K$, and we see

$$\langle \pi_{S_b}(h_J^j\otimes x), h_K^k\otimes y\rangle = 0, \quad \langle \pi_{S_b}^*(h_J^j\otimes x), h_K^k\otimes y\rangle = 0,$$

   hence we conclude that

$$\langle \pi_b^*\pi_b(h_J^j\otimes x), h_K^k\otimes y\rangle = \langle (\pi_{S_b}+\pi_{S_b}^*+D_b)(h_J^j\otimes x), h_K^k\otimes y\rangle.$$

2. $K \subsetneqq J$ and $K\in\mathcal{D}(J(p))$ for some $1\le p\le d-1$. Then from (4.1.10)-(4.1.12),

$$\begin{aligned}\langle \pi_b^*\pi_b(h_J^j\otimes x), h_K^k\otimes y\rangle &= \sum_{I\in\mathcal{D}}\sum_{i=1}^{d-1}\overline{\left\langle \frac{\mathbb{1}_I}{|I|}, h_J^j\right\rangle}\left\langle \frac{\mathbb{1}_I}{|I|}, h_K^k\right\rangle\tau(x^*\langle h_I^i, b\rangle^*\langle h_I^i, b\rangle y) \\ &= \sum_{q=1}^{d}\sum_{I\in\mathcal{D}(K(q))}\sum_{i=1}^{d-1}\frac{\omega^{kq-pj}}{|J|^{1/2}|K|^{1/2}}\tau(x^*\langle h_I^i, b\rangle^*\langle h_I^i, b\rangle y) \\ &= \overline{\left\langle \frac{\mathbb{1}_K}{|K|}, h_J^j\right\rangle}\tau(x^*\langle S_b, h_K^k\rangle y) = \langle \pi_{S_b}(h_J^j\otimes x), h_K^k\otimes y\rangle \\ &= \langle (\pi_{S_b}+\pi_{S_b}^*+D_b)(h_J^j\otimes x), h_K^k\otimes y\rangle.\end{aligned}$$

3. $J \subsetneqq K$ and $J\in\mathcal{D}(K(p))$ for some $1\le p\le d-1$. This case is symmetric to the previous case by passing to adjoints:

$$\begin{aligned}\langle \pi_b^*\pi_b(h_J^j\otimes x), h_K^k\otimes y\rangle &= \langle \pi_{S_b}^*(h_J^j\otimes x), h_K^k\otimes y\rangle \\ &= \langle (\pi_{S_b}+\pi_{S_b}^*+D_b)(h_J^j\otimes x), h_K^k\otimes y\rangle.\end{aligned}$$

The proof is finished. □

In order to apply Lemma 4.1.2, we will show that the upper bounds of $\|S_b\|_{\boldsymbol{B}^d_{p/2}(\mathbb{R},\mathcal{M})}$ and $\|D_b\|_{L_{p/2}(B(L_2(\mathbb{R}))\otimes\mathcal{M})}$ are dominated by $\|b\|^2_{\boldsymbol{B}^d_p(\mathbb{R},\mathcal{M})}$ when $p \geq 2$. This owes to the following p-John-Nirenberg inequality appearing in [80].

Lemma 4.1.3. *Let $0 < p < \infty$. For each nonnegative sequence $(a_{I,i})_{I\in\mathcal{D},1\leq i\leq d-1}$ indexed by the dyadic intervals, we have*

$$\sum_{I\in\mathcal{D}}\left(\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)}\sum_{j=1}^{d-1}a_{J,j}\right)^p \lesssim_{d,p} \sum_{I\in\mathcal{D}}\sum_{i=1}^{d-1}\left(\frac{a_{I,i}}{|I|}\right)^p.$$

Proof. For $0 < p \leq 1$ and for all $I \in \mathcal{D}$, we have

$$\left(\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)}\sum_{j=1}^{d-1}a_{J,j}\right)^p \leq \frac{1}{|I|^p}\sum_{J\in\mathcal{D}(I)}\sum_{j=1}^{d-1}a^p_{J,j},$$

which yields

$$\begin{aligned}\sum_{I\in\mathcal{D}}\left(\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)}\sum_{j=1}^{d-1}a_{J,j}\right)^p &\leq \sum_{I\in\mathcal{D}}\frac{1}{|I|^p}\sum_{J\in\mathcal{D}(I)}\sum_{j=1}^{d-1}a^p_{J,j}\\ &= \sum_{J\in\mathcal{D}}\sum_{j=1}^{d-1}\left(\sum_{J\subseteq I}\frac{1}{|I|^p}\right)a^p_{J,j}\\ &= \sum_{J\in\mathcal{D}}\sum_{j=1}^{d-1}\left(\sum_{k=0}^{\infty}\frac{1}{(d^k|J|)^p}\right)a^p_{J,j}\\ &= \frac{d^p}{d^p-1}\sum_{J\in\mathcal{D}}\sum_{j=1}^{d-1}\left(\frac{a_{J,j}}{|J|}\right)^p.\end{aligned}$$

Let $1 < p < \infty$. For a fixed $I = I_{n,k} \in \mathcal{D}$, by the Jensen inequality and the equality

$$\sum_{m=n}^{\infty}(m-n+1)^{-2} = \frac{\pi^2}{6},$$

one has

$$\begin{aligned}\left(\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)}\sum_{j=1}^{d-1}a_{J,j}\right)^p &= \left(d^n\sum_{m=n}^{\infty}\sum_{J\in\mathcal{D}_m(I)}\sum_{j=1}^{d-1}a_{J,j}\right)^p\\ &= \left(\sum_{m=n}^{\infty}(m-n+1)^{-2}(m-n+1)^2d^{n-m}\sum_{J\in\mathcal{D}_m(I)}\sum_{j=1}^{d-1}d^m a_{J,j}\right)^p\\ &\lesssim_p \sum_{m=n}^{\infty}(m-n+1)^{2p-2}d^{p(n-m)}\left(\sum_{J\in\mathcal{D}_m(I)}\sum_{j=1}^{d-1}d^m a_{J,j}\right)^p.\end{aligned}$$

Let $p' = \frac{p}{p-1}$. When $m \geq n$, by the Hölder inequality,

$$\begin{aligned}\left(\sum_{J\in\mathcal{D}_m(I)}\sum_{j=1}^{d-1}d^m a_{J,j}\right)^p &\leq \left(\sum_{J\in\mathcal{D}_m(I)}\sum_{j=1}^{d-1}(d^m a_{J,j})^p\right)\left(\sum_{J\in\mathcal{D}_m(I)}\sum_{j=1}^{d-1}1^{p'}\right)^{p/p'}\\ &= (d-1)^{p-1}d^{(m-n)(p-1)}\sum_{J\in\mathcal{D}_m(I)}\sum_{j=1}^{d-1}(d^m a_{J,j})^p.\end{aligned}$$

We then take the summation for all $I \in \mathcal{D}$, and change the order of summations to obtain

$$\begin{aligned}\sum_{I\in\mathcal{D}}\left(\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)}\sum_{j=1}^{d-1}a_{J,j}\right)^p &\lesssim_{d,p} \sum_{n,k\in\mathbb{Z}}\sum_{m=n}^{\infty}(m-n+1)^{2p-2}d^{n-m}\sum_{J\in\mathcal{D}_m(I_{n,k})}\sum_{j=1}^{d-1}(d^m a_{J,j})^p\\ &= \sum_{n,k\in\mathbb{Z}}\sum_{m=n}^{\infty}(m-n+1)^{2p-2}d^{n-m}\sum_{j=1}^{d-1}(d^m a_{(I_{m,k}),j})^p\\ &= \sum_{m,k\in\mathbb{Z}}\sum_{n=-\infty}^{m}(m-n+1)^{2p-2}d^{n-m}\sum_{j=1}^{d-1}(d^m a_{(I_{m,k}),j})^p.\end{aligned}$$

Note that $\forall m \in \mathbb{Z}$, the sum

$$\sum_{n=-\infty}^{m}(m-n+1)^{2p-2}d^{n-m} = \sum_{l=1}^{\infty} l^{2p-2}d^{1-l}$$

is a constant only depending on d and p. Therefore,

$$\sum_{I\in\mathcal{D}}\left(\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)}\sum_{j=1}^{d-1}a_{J,j}\right)^p \lesssim_{d,p} \sum_{I\in\mathcal{D}}\sum_{i=1}^{d-1}\left(\frac{a_{I,i}}{|I|}\right)^p.$$

□

Lemma 4.1.4. *If $2 \le p < \infty$ and $b \in \boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})$, then $S_b \in \boldsymbol{B}_{p/2}^d(\mathbb{R},\mathcal{M})$ and $\|S_b\|_{\boldsymbol{B}_{p/2}^d(\mathbb{R},\mathcal{M})} \lesssim_{d,p} \|b\|^2_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}$.*

Proof. Note that by the triangle inequality and (4.1.12)

$$\begin{aligned}\|\langle h_I^i, S_b\rangle\|_{L_{p/2}(\mathcal{M})} &= \left\|\sum_{J\in\mathcal{D}}\sum_{j=1}^{d-1}\overline{\left\langle\frac{\mathbb{1}_J}{|J|},h_I^i\right\rangle}\langle h_J^j,b\rangle^*\langle h_J^j,b\rangle\right\|_{L_{p/2}(\mathcal{M})}\\ &\le \frac{1}{|I|^{1/2}}\sum_{J\subsetneq I}\sum_{j=1}^{d-1}\|\langle h_J^j,b\rangle^*\langle h_J^j,b\rangle\|_{L_{p/2}(\mathcal{M})}\\ &= \frac{1}{|I|^{1/2}}\sum_{J\subsetneq I}\sum_{j=1}^{d-1}\|\langle h_J^j,b\rangle\|^2_{L_p(\mathcal{M})}.\end{aligned}$$

By Lemma 4.1.3, we conclude

$$\|S_b\|^{p/2}_{\boldsymbol{B}_{p/2}^d(\mathbb{R},\mathcal{M})} \le \sum_{I\in\mathcal{D}}\sum_{i=1}^{d-1}\left(\frac{1}{|I|}\sum_{J\subsetneq I}\sum_{j=1}^{d-1}\|\langle h_J^j,b\rangle\|^2_{L_p(\mathcal{M})}\right)^{p/2} \lesssim_{d,p} \|b\|^p_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})},$$

as desired. □

Lemma 4.1.5. *If $2 \le p < \infty$ and $b \in \boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})$, then $D_b \in L_{p/2}(B(L_2(\mathbb{R}))\otimes\mathcal{M})$ and $\|D_b\|_{L_{p/2}(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \lesssim_{d,p} \|b\|^2_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}$.*

Proof. From (4.1.7)-(4.1.9), for any $x, y \in L_2(\mathcal{M})$, $K, L \in \mathcal{D}$, $1 \le k, l \le d-1$, one has

$$\begin{aligned}\langle h_K^k \otimes x, D_b(h_L^l \otimes y)\rangle &= \sum_{I\in\mathcal{D}} \frac{1}{|I|}\sum_{q=1}^{d}\langle h_K^k, M^{I,q}h_L^l\rangle\langle x, N^{I,q}y\rangle \\ &= \sum_{I\in\mathcal{D}} \frac{1}{|I|}\sum_{q=1}^{d}\sum_{i,m=1}^{d-1}\langle h_I^i, h_L^l\rangle\omega^{(i-m)q}\langle h_K^k, h_I^m\rangle\langle x, N^{I,q}y\rangle \\ &= \begin{cases}\left\langle x, \frac{1}{|K|}\sum\limits_{q=1}^{d}\omega^{(l-k)q}N^{K,q}y\right\rangle, & \text{if } K = L;\\ 0, & \text{if } K \ne L.\end{cases}\end{aligned}$$

This shows that D_b is a block diagonal matrix with respect to the basis $\{h_I^i\}_{I\in\mathcal{D},1\le i\le d-1}$. Denote for any $I \in \mathcal{D}$, $1 \le k, l \le d-1$

$$E_{I,k,l} := \frac{1}{|I|}\sum_{q=1}^{d}\omega^{(l-k)q}N^{I,q}.$$

Then the matrix of D_b with respect to the basis $\{h_I^i\}_{I\in\mathcal{D},1\le i\le d-1}$ is $(E_{I,k,l})_I$. Hence by the triangle inequality,

$$\begin{aligned}\|D_b\|_{L_{p/2}(B(L_2(\mathbb{R}))\otimes\mathcal{M})}^{p/2} &= \sum_{I\in\mathcal{D}}\left\|\left(E_{I,k,l}\right)_{1\le k,l\le d-1}\right\|_{L_{p/2}(\mathbb{M}_{d-1}\otimes\mathcal{M})}^{p/2} \\ &= \sum_{I\in\mathcal{D}}\left\|\left(\frac{1}{|I|}\sum_{q=1}^{d}\omega^{(l-k)q}N^{I,q}\right)_{1\le k,l\le d-1}\right\|_{L_{p/2}(\mathbb{M}_{d-1}\otimes\mathcal{M})}^{p/2} \\ &\le \sum_{I\in\mathcal{D}}\left(\sum_{q=1}^{d}\left\|\left(\frac{1}{|I|}\omega^{(l-k)q}N^{I,q}\right)_{1\le k,l\le d-1}\right\|_{L_{p/2}(\mathbb{M}_{d-1}\otimes\mathcal{M})}\right)^{p/2},\end{aligned}$$

where $\mathbb{M}_{d-1}$ is equipped with the usual trace.

However, note that $(\omega^{(l-k)q})_{1\le k,l\le d-1}$ $(1 \le q \le d)$ is a $(d-1)\times(d-1)$ matrix only depending on d, p, thus

$$\begin{aligned}&\sum_{q=1}^{d}\left\|\left(\frac{1}{|I|}\omega^{(l-k)q}N^{I,q}\right)_{1\le k,l\le d-1}\right\|_{L_{p/2}(\mathbb{M}_{d-1}\otimes\mathcal{M})} \\ =&\frac{1}{|I|}\sum_{q=1}^{d}\left\|\left(\omega^{(l-k)q}\right)_{1\le k,l\le d-1}\right\|_{S_{p/2}(\mathbb{M}_{d-1})}\|N^{I,q}\|_{L_{p/2}(\mathcal{M})} \\ \lesssim_{d,p}&\frac{1}{|I|}\sum_{q=1}^{d}\|N^{I,q}\|_{L_{p/2}(\mathcal{M})}.\end{aligned}$$

Consequently, by the triangle inequality and Lemma 4.1.3, we have

$$\begin{aligned}\|D_b\|_{L_{p/2}(B(L_2(\mathbb{R}))\otimes\mathcal{M})}^{p/2} &\lesssim_{d,p} \sum_{I\in\mathcal{D}}\left(\frac{1}{|I|}\sum_{q=1}^{d}\|N^{I,q}\|_{L_{p/2}(\mathcal{M})}\right)^{p/2} \\ &\lesssim_{d,p} \sum_{I\in\mathcal{D}}\left(\frac{1}{|I|}\sum_{q=1}^{d}\sum_{J\in\mathcal{D}(I(q))}\sum_{j=1}^{d-1}\|\langle h_J^j, b\rangle\|_{L_p(\mathcal{M})}^2\right)^{p/2} \\ &\lesssim_{d,p} \sum_{I\in\mathcal{D}}\sum_{i=1}^{d-1}\left(\frac{1}{|I|^{1/2}}\|\langle h_I^i, b\rangle\|_{L_p(\mathcal{M})}\right)^{p} = \|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}^{p}.\end{aligned}$$

This completes the proof. □

Proposition 4.1.6. *Let $2 \leq p < \infty$. Assume that for any $b \in \boldsymbol{B}^d_{p/2}(\mathbb{R}, \mathcal{M})$, the corresponding martingale paraproduct satisfies the following estimate*

$$\|\pi_b\|_{L_{p/2}(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \lesssim_{d,p} \|b\|_{\boldsymbol{B}^d_{p/2}(\mathbb{R},\mathcal{M})}.$$

Then for any $b \in \boldsymbol{B}^d_p(\mathbb{R}, \mathcal{M})$, we have

$$\|\pi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \lesssim_{d,p} \|b\|_{\boldsymbol{B}^d_p(\mathbb{R},\mathcal{M})}.$$

Proof. By direct calculations, one has

$$\begin{aligned}
\|\pi_b\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} &= \|\pi_b^*\pi_b\|^{p/2}_{L_{p/2}(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \\
&\leq \left(2\|\pi_{S_b}\|_{L_{p/2}(B(L_2(\mathbb{R}))\otimes\mathcal{M})} + \|D_b\|_{L_{p/2}(B(L_2(\mathbb{R}))\otimes\mathcal{M})}\right)^{p/2} \quad \text{(Lemma 4.1.2)} \\
&\lesssim_{d,p} \left(2\|S_b\|_{\boldsymbol{B}^d_{p/2}(\mathbb{R},\mathcal{M})} + \|D_b\|_{L_{p/2}(B(L_2(\mathbb{R}))\otimes\mathcal{M})}\right)^{p/2} \\
&\lesssim_{d,p} \|b\|^p_{\boldsymbol{B}^d_p(\mathbb{R},\mathcal{M})}. \quad \text{(Lemma 4.1.4 and Lemma 4.1.5)}
\end{aligned}$$

Thus the result is proved. □

Proof of the Necessity of Theorem II.2. We see that if $b \in \boldsymbol{B}^d_p(\mathbb{R}, \mathcal{M})$ for $0 < p \leq 2$, then $\pi_b \in L_p(B(L_2(\mathbb{R})) \otimes \mathcal{M})$ from Proposition 4.1.1. By Proposition 4.1.6, we employ iteration and finally conclude the desired result for $2 < p < \infty$. □

4.1.2 The Sufficiency of Theorem II.2

We divide the proof into two cases: $p \geq 1$ and $0 < p < 1$. Each one will be stated and proved in Propositions 4.1.8 and 4.1.9 respectively. For the first case, the proof is easier and relies on the following elementary lemma.

Lemma 4.1.7. *Let $1 \leq p < \infty$ and $T \in L_p(B(L_2(\mathbb{R})) \otimes \mathcal{M})$. $E = (E_{I,i})_{I\in\mathcal{D}, 1\leq i\leq d-1}$ is defined as the block diagonal of T, where for $I \in \mathcal{D}$ and $1 \leq i \leq d-1$, $E_{I,i} : L_2(\mathcal{M}) \to L_2(\mathcal{M})$ is given by*

$$\langle E_{I,i}x, y\rangle = \langle Th^i_I \otimes x, h^i_I \otimes y\rangle, \quad \forall x, y \in L_2(\mathcal{M}).$$

Then

$$\|T\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \geq \sum_{I\in\mathcal{D}}\sum_{i=1}^{d-1} \|E_{I,i}\|^p_{L_p(\mathcal{M})} = \|E\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}.$$

Proof. Note that E is a trace preserving conditional expectation, and thereby contractive. □

Proposition 4.1.8. *If $p \geq 1$ and $\pi_b \in L_p(B(L_2(\mathbb{R})) \otimes \mathcal{M})$, then $b \in \boldsymbol{B}^d_p(\mathbb{R}, \mathcal{M})$.*

Proof. First, define for any $I \in \mathcal{D}$, $1 \le i \le d-1$, $x \in L_2(\mathcal{M})$,

$$R : L_2(\mathbb{R}, L_2(\mathcal{M})) \to L_2(\mathbb{R}, L_2(\mathcal{M}))$$
$$h_I^i \otimes x \mapsto h_{\tilde{I}}^i \otimes x,$$

where $\tilde{I}$ is the parent interval of I. Then R is well-defined and bounded. Indeed, for any $f \in L_2(\mathbb{R}, L_2(\mathcal{M}))$,

$$\begin{aligned}
\|Rf\|_{L_2(\mathbb{R},L_2(\mathcal{M}))} &= \left\| R\left(\sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \langle h_I^i, f\rangle h_I^i \right) \right\|_{L_2(\mathbb{R},L_2(\mathcal{M}))} = \left\| \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \langle h_I^i, f\rangle h_{\tilde{I}}^i \right\|_{L_2(\mathbb{R},L_2(\mathcal{M}))} \\
&= \left\| \sum_{\tilde{I}\in\mathcal{D}} \sum_{i=1}^{d-1} \Bigg(\sum_{\substack{I\subset\tilde{I} \\ |I|=d^{-1}|\tilde{I}|}} \langle h_I^i, f\rangle \Bigg) h_{\tilde{I}}^i \right\|_{L_2(\mathbb{R},L_2(\mathcal{M}))} \\
&= \left(\sum_{\tilde{I}\in\mathcal{D}} \sum_{i=1}^{d-1} \left\| \sum_{\substack{I\subset\tilde{I} \\ |I|=d^{-1}|\tilde{I}|}} \langle h_I^i, f\rangle \right\|_{L_2(\mathcal{M})}^2 \right)^{1/2} \\
&\le \left(d \sum_{\tilde{I}\in\mathcal{D}} \sum_{i=1}^{d-1} \sum_{\substack{I\subset\tilde{I} \\ |I|=d^{-1}|\tilde{I}|}} \|\langle h_I^i, f\rangle\|_{L_2(\mathcal{M})}^2 \right)^{1/2} = \sqrt{d}\|f\|_{L_2(\mathbb{R},L_2(\mathcal{M}))}.
\end{aligned} \tag{4.1.13}$$

Now let $E = (E_{I,i})_{I\in\mathcal{D},1\le i\le d-1}$ be the block diagonal of $\pi_b R$ defined in Lemma 4.1.7. Then for $x, y \in L_2(\mathcal{M})$, we have

$$\begin{aligned}
\langle E_{I,i}x, y\rangle &= \langle \pi_b R(h_I^i \otimes x), h_I^i \otimes y\rangle = \langle \pi_b(h_{\tilde{I}}^i \otimes x), h_I^i \otimes y\rangle \\
&= \frac{\omega^{-iq}}{|\tilde{I}|^{1/2}} \langle\langle h_I^i, b\rangle x, y\rangle = \frac{\omega^{-iq}}{\sqrt{d}|I|^{1/2}} \langle\langle h_I^i, b\rangle x, y\rangle,
\end{aligned}$$

where $\tilde{I}(q) = I, 1 \le q \le d$. Thus $E_{I,i} = \frac{\omega^{-iq}}{\sqrt{d}|I|^{1/2}}\langle h_I^i, b\rangle$. Therefore, from (4.1.13) and Lemma 4.1.7, we get

$$\begin{aligned}
\|\pi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}^p &\ge \frac{1}{d^{p/2}} \|\pi_b R\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}^p \\
&\ge \frac{1}{d^{p/2}} \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \|E_{I,i}\|_{L_p(\mathcal{M}))}^p \\
&= \frac{1}{d^p} \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \left(\frac{\|\langle h_I^i, b\rangle\|_{L_p(\mathcal{M}))}}{|I|^{1/2}} \right)^p \\
&\gtrsim_{d,p} \|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}^p.
\end{aligned}$$

This yields the desired result. □

Proposition 4.1.9. *If $0 < p < 1$ and $\pi_b \in L_p(B(L_2(\mathbb{R})) \otimes \mathcal{M})$, then $b \in \boldsymbol{B}_p^d(\mathbb{R}, \mathcal{M})$.*

The remaining part of this section is devoted to the proof of Proposition 4.1.9. We will follow the arguments in [69] or [80]. To this end, define for any $m, n \in \mathbb{Z}$

$$\pi_b^{n,m} := d_{m+1}\pi_b d_{n+1}. \tag{4.1.14}$$

Recall that $d_k b = \sum_{|I|=d^{-k+1}} \sum_{i=1}^{d-1} h_I^i \langle h_I^i, b\rangle$. Thus for any $f \in L_2(\mathbb{R}, L_2(\mathcal{M}))$,

$$\pi_b^{n,m}(f) = \sum_{\substack{I\in\mathcal{D}_m\\ J\in\mathcal{D}_n}} \sum_{i,j=1}^{d-1} \left\langle \frac{\mathbb{1}_I}{|I|}, h_J^j \right\rangle h_I^i \langle h_I^i, b\rangle \langle h_J^j, f\rangle.$$

If $m \le n$, then for $I \in \mathcal{D}_m$ and $J \in \mathcal{D}_n$,

$$\left\langle \frac{\mathbb{1}_I}{|I|}, h_J^j \right\rangle = 0.$$

It thus follows that $\pi_b^{n,m} = 0$.

Let $b \in \boldsymbol{B}_p^d(\mathbb{R}, \mathcal{M})$ and $N \ge 2$ be a fixed positive integer (to be chosen later). For $k = 0, 1, \cdots, N-1$, define

$$\pi_{b,k} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \pi_b^{Nn+k,Nm+k+1} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{m} \pi_b^{Nn+k,Nm+k+1}. \tag{4.1.15}$$

In addition, we define

$$\pi_{b,k}^{(0)} = \sum_{n=-\infty}^{\infty} \pi_b^{Nn+k,Nn+k+1},$$

and

$$\pi_{b,k}^{(1)} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{m-1} \pi_b^{Nn+k,Nm+k+1}.$$

Then

$$\pi_{b,k} = \pi_{b,k}^{(0)} + \pi_{b,k}^{(1)}.$$

Indeed, $\pi_{b,k}^{(0)}$ is defined as the minor diagonal and will play an important role in our later proof. In the following, we are about to obtain the lower bound of $\|\pi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}$ by $\|\pi_{b,k}^{(0)}\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}$, which will be the dominant term. The following lemma implies that $\|\pi_{b,k}^{(1)}\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}$ is the minor term since $\|\pi_b^{n,m}\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}$ shrinks rapidly when $m > n$.

Lemma 4.1.10. *Let $b \in \boldsymbol{B}_p^d(\mathbb{R}, \mathcal{M})$. If $m > n$ and $0 < p < 1$, then*

$$\|\pi_b^{n,m}\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}^p \le (d-1) d^{(n-m)p/2} \sum_{I\in\mathcal{D}_m} \sum_{i=1}^{d-1} \left(\frac{\|\langle h_I^i, b\rangle\|_{L_p(\mathcal{M})}}{|I|^{1/2}} \right)^p.$$

Proof. Write $\pi_b^{n,m}$ in the following concrete form:

$$\begin{aligned} \pi_b^{n,m} &= \sum_{\substack{I\in\mathcal{D}_m\\ J\in\mathcal{D}_n}} \sum_{i,j=1}^{d-1} \left\langle \frac{\mathbb{1}_I}{|I|}, h_J^j \right\rangle (\pi_b^{n,m})_{I,J}^{i,j} \\ &= \sum_{J\in\mathcal{D}_n} \sum_{q=1}^{d} \sum_{I\in\mathcal{D}_m(J(q))} \sum_{i,j=1}^{d-1} \omega^{qj} \frac{1}{|J|^{1/2}} (\pi_b^{n,m})_{I,J}^{i,j}, \end{aligned} \tag{4.1.16}$$

where

$$(\pi_b^{n,m})_{I,J}^{i,j} := \langle h_I^i, b\rangle \cdot h_I^i \otimes h_J^j.$$

Hence, one has

$$\begin{aligned}\|\pi_b^{n,m}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} &\le \sum_{J\in\mathcal{D}_n}\sum_{I\in\mathcal{D}_m(J)}\sum_{i,j=1}^{d-1}\left(\frac{\|(\pi_b^{n,m})^{i,j}_{I,J}\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}}{|J|^{1/2}}\right)^p\\ &= \sum_{J\in\mathcal{D}_n}\sum_{I\in\mathcal{D}_m(J)}\sum_{i,j=1}^{d-1}\left(\frac{\|h_I^i\otimes h_J^j\|_{S_p(L_2(\mathbb{R}))}\|\langle h_I^i,b\rangle\|_{L_p(\mathcal{M})}}{|J|^{1/2}}\right)^p\\ &= \sum_{J\in\mathcal{D}_n}\sum_{I\in\mathcal{D}_m(J)}\sum_{i,j=1}^{d-1}\left(\frac{\|\langle h_I^i,b\rangle\|_{L_p(\mathcal{M})}}{|J|^{1/2}}\right)^p\\ &= (d-1)d^{(n-m)p/2}\sum_{I\in\mathcal{D}_m}\sum_{i=1}^{d-1}\left(\frac{\|\langle h_I^i,b\rangle\|_{L_p(\mathcal{M})}}{|I|^{1/2}}\right)^p.\end{aligned}$$

This finishes the proof. □

By virtue of Lemma 4.1.10, we can estimate $\|\pi_{b,k}^{(1)}\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}$ as follows:

Lemma 4.1.11. *We have* $\sum_{k=0}^{N-1}\|\pi_{b,k}^{(1)}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \le \frac{(d-1)d^{-p/2}}{d^{Np/2}-1}\|b\|^p_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}$.

Proof. By Lemma 4.1.10, one has

$$\begin{aligned}\|\pi_{b,k}^{(1)}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} &\le \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{m-1}\|\pi_b^{Nn+k,Nm+k+1}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}\\ &\le \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{m-1}(d-1)d^{(Nn-Nm-1)p/2}\sum_{I\in\mathcal{D}_{Nm+k+1}}\sum_{i=1}^{d-1}\left(\frac{\|\langle h_I^i,b\rangle\|_{L_p(\mathcal{M})}}{|I|^{1/2}}\right)^p\\ &= \sum_{m=-\infty}^{\infty}(d-1)d^{-(Nm+1)p/2}\sum_{I\in\mathcal{D}_{Nm+k+1}}\sum_{i=1}^{d-1}\left(\frac{\|\langle h_I^i,b\rangle\|_{L_p(\mathcal{M})}}{|I|^{1/2}}\right)^p\sum_{n=-\infty}^{m-1}d^{Nnp/2}\\ &= \frac{(d-1)d^{-p/2}}{d^{Np/2}-1}\sum_{m=-\infty}^{\infty}\sum_{I\in\mathcal{D}_{Nm+k+1}}\sum_{i=1}^{d-1}\left(\frac{\|\langle h_I^i,b\rangle\|_{L_p(\mathcal{M})}}{|I|^{1/2}}\right)^p.\end{aligned}$$

We then deduce

$$\begin{aligned}\sum_{k=0}^{N-1}\|\pi_{b,k}^{(1)}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} &\le \frac{(d-1)d^{-p/2}}{d^{Np/2}-1}\sum_{I\in\mathcal{D}}\sum_{i=1}^{d-1}\left(\frac{\|\langle h_I^i,b\rangle\|_{L_p(\mathcal{M})}}{|I|^{1/2}}\right)^p\\ &= \frac{(d-1)d^{-p/2}}{d^{Np/2}-1}\|b\|^p_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}.\end{aligned}$$

□

Now we come to the estimate of $\|\pi_{b,k}^{(0)}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}$. The following well-known lemma is straightforward but very helpful for us.

Lemma 4.1.12. *Let* $0<p<\infty$. *If* $\{R_i\}_{1\le i\le n}$ *are operators in* $L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})$ *satisfying* $R_i^*R_j=0,\ \ \forall 1\le i,j\le n, i\ne j$ *and* $T=\sum_{i=1}^n R_i$, *then*

$$\|T\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \ge \frac{1}{n}\sum_{i=1}^n\|R_i\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}. \tag{4.1.17}$$

Proof. This follows from the fact that

$$T^*T = \sum_{i=1}^{n} R_i^* R_i \geq R_i^* R_i.$$

□

Remark 4.1.13. It is obvious that our estimate is far from being optimal in (4.1.17), but it does not affect our later proof. See [46, Theorem 1.3] or [19, Lemma 2.1] for better constants in (4.1.17).

Lemma 4.1.14. *Let $b \in \boldsymbol{B}_p^d(\mathbb{R}, \mathcal{M})$ and $0 < p < 1$. Then*

$$\sum_{k=0}^{N-1} \|\pi_{b,k}^{(0)}\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \geq \frac{(d-1)^{p/2-1}}{d^{p/2+1}} \|b\|^p_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}.$$

Proof. From (4.1.16) we deduce

$$\begin{aligned}
\pi_b^{Nn+k,Nn+k+1} &= \sum_{J\in\mathcal{D}_{Nn+k}} \sum_{q=1}^{d} \sum_{I\in\mathcal{D}_{Nn+k+1}(J(q))} \sum_{i,j=1}^{d-1} \frac{\omega^{qj}}{|J|^{1/2}} (\pi_b^{Nn+k,Nn+k+1})_{I,J}^{i,j} \\
&= \sum_{J\in\mathcal{D}_{Nn+k}} \sum_{q=1}^{d} \sum_{i,j=1}^{d-1} \frac{\omega^{qj}}{|J|^{1/2}} (\pi_b^{Nn+k,Nn+k+1})_{J(q),J}^{i,j}.
\end{aligned}$$

Then

$$\begin{aligned}
\pi_{b,k}^{(0)} &= \sum_{n=-\infty}^{\infty} \sum_{J\in\mathcal{D}_{Nn+k}} \sum_{q=1}^{d} \sum_{i,j=1}^{d-1} \frac{\omega^{qj}}{|J|^{1/2}} (\pi_b^{Nn+k,Nn+k+1})_{J(q),J}^{i,j} \\
&= \sum_{q=1}^{d} \sum_{i=1}^{d-1} \sum_{n=-\infty}^{\infty} \sum_{J\in\mathcal{D}_{Nn+k}} \sum_{j=1}^{d-1} \frac{\omega^{qj}}{|J|^{1/2}} (\pi_b^{Nn+k,Nn+k+1})_{J(q),J}^{i,j} \\
&=: \sum_{q=1}^{d} \sum_{i=1}^{d-1} A_{q,i}.
\end{aligned}$$

Since the ranges of $\{(\pi_b^{Nn+k,Nn+k+1})_{J(q),J}^{i,j}\}_{1\leq q\leq d, 1\leq i\leq d-1}$ are mutually orthogonal, by Lemma 4.1.12 we have

$$\|\pi_{b,k}^{(0)}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \geq \frac{1}{d(d-1)} \sum_{q=1}^{d} \sum_{i=1}^{d-1} \|A_{q,i}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}.$$

When q and i are fixed, the operator $A_{q,i}$ is a block diagonal matrix with respect to the basis $\{h_J^j, h_{J(q)}^i\}_{J\in\mathcal{D}_{Nn+k}}$. Consequently, one has

$$\begin{aligned}
&\left\| \sum_{n=-\infty}^{\infty} \sum_{J\in\mathcal{D}_{Nn+k}} \sum_{j=1}^{d-1} \frac{\omega^{qj}}{|J|^{1/2}} (\pi_b^{Nn+k,Nn+k+1})_{J(q),J}^{i,j} \right\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \\
&= \sum_{n=-\infty}^{\infty} \sum_{J\in\mathcal{D}_{Nn+k}} \left\| \sum_{j=1}^{d-1} \frac{\omega^{qj}}{|J|^{1/2}} (\pi_b^{Nn+k,Nn+k+1})_{J(q),J}^{i,j} \right\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \\
&= \sum_{n=-\infty}^{\infty} \sum_{J\in\mathcal{D}_{Nn+k}} \left\| \sum_{j=1}^{d-1} \frac{\omega^{qj}}{|J|^{1/2}} h_{J(q)}^i \otimes h_J^j \right\|^p_{S_p(L_2(\mathbb{R}))} \|\langle h_{J(q)}^i, b\rangle\|^p_{L_p(\mathcal{M})}.
\end{aligned}$$

It is clear that

$$\left\|\sum_{j=1}^{d-1}\frac{\omega^{qj}}{|J|^{1/2}}h^i_{J(q)}\otimes h^j_J\right\|_{S_p(L_2(\mathbb{R}))}=\frac{(d-1)^{1/2}}{|J|^{1/2}}.$$

Combining the preceding inequalities, we obtain

$$\begin{aligned}\|\pi^{(0)}_{b,k}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}&\geq\frac{(d-1)^{p/2}}{d(d-1)}\sum_{q=1}^{d}\sum_{i=1}^{d-1}\sum_{n=-\infty}^{\infty}\sum_{J\in\mathcal{D}_{Nn+k}}\frac{1}{|J|^{p/2}}\|\langle h^i_{J(q)},b\rangle\|^p_{L_p(\mathcal{M})}\\&=\frac{(d-1)^{p/2-1}}{d^{p/2+1}}\sum_{i=1}^{d-1}\sum_{n=-\infty}^{\infty}\sum_{J\in\mathcal{D}_{Nn+k+1}}\frac{1}{|J|^{p/2}}\|\langle h^i_J,b\rangle\|^p_{L_p(\mathcal{M})}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{k=0}^{N-1}\|\pi^{(0)}_{b,k}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}&\geq\frac{(d-1)^{p/2-1}}{d^{p/2+1}}\sum_{J\in\mathcal{D}}\sum_{i=1}^{d-1}\left(\frac{\|\langle h^i_J,b\rangle\|_{L_p(\mathcal{M})}}{|J|^{1/2}}\right)^p\\&=\frac{(d-1)^{p/2-1}}{d^{p/2+1}}\|b\|^p_{\boldsymbol{B}^d_p(\mathbb{R},\mathcal{M})}.\end{aligned}$$

□

Proposition 4.1.15. *Let $b\in\boldsymbol{B}^d_p(\mathbb{R},\mathcal{M})$ and $0<p<1$. Then*

$$\|b\|_{\boldsymbol{B}^d_p(\mathbb{R},\mathcal{M})}\lesssim_{d,p}\|\pi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}.$$

Proof. From (4.1.14) and (4.1.15) we observe that

$$\pi_{b,k}=\left(\sum_{m=-\infty}^{\infty}d_{Nm+k+2}\right)\pi_b\left(\sum_{n=-\infty}^{\infty}d_{Nn+k+1}\right).$$

Note that $\sum\limits_{n=-\infty}^{\infty}d_{Nn+k+1}$ and $\sum\limits_{m=-\infty}^{\infty}d_{Nm+k+2}$ are projections with norm 1. Thus

$$\|\pi_{b,k}\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}\leq\|\pi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}.$$

By Lemmas 4.1.11, 4.1.12 and 4.1.14, we have

$$\begin{aligned}\|\pi_b\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}&\geq\frac{1}{N}\sum_{k=0}^{N-1}\|\pi_{b,k}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}\\&\geq\frac{1}{N}\sum_{k=0}^{N-1}\left(\|\pi^{(0)}_{b,k}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}-\|\pi^{(1)}_{b,k}\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}\right)\\&\geq\frac{1}{N}\left(\frac{(d-1)^{p/2-1}}{d^{p/2+1}}-\frac{(d-1)d^{-p/2}}{d^{Np/2}-1}\right)\|b\|^p_{\boldsymbol{B}^d_p(\mathbb{R},\mathcal{M})},\end{aligned}$$

which yields the desired result as long as we choose N sufficiently large. □

Now we give the proof of Proposition 4.1.9.

Proof of Proposition 4.1.9. When $0 < p < 1$, Proposition 4.1.9 follows from Proposition 4.1.15 and the standard limit argument. Indeed, for any positive integer a, we define

$$b^{(a)} := \sum_{I \in \mathcal{D}^{(a)}} \sum_{i=1}^{d} \langle h_I^i, b\rangle h_I^i,$$

where

$$\mathcal{D}^{(a)} := \{I_{n,k} \in \mathcal{D} : |n| \le a, |k| \le a\}.$$

By (4.1.5), one has for any $0 < p < \infty$

$$\|\pi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \ge \|\pi_b^{I,i}\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} = \frac{\|\langle h_I^i, b\rangle\|_{L_p(\mathcal{M})}}{|I|^{1/2}}.$$

This implies that $b^{(a)} \in \boldsymbol{B}_p^d(\mathbb{R}, \mathcal{M})$. Therefore by Proposition 4.1.15,

$$\|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})} = \lim_{a\to\infty} \|b^{(a)}\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})} \lesssim_{d,p} \|\pi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}.$$

□

Proof of the sufficiency of Theorem II.2. The desired result follows from Proposition 4.1.8 for $p \ge 1$, and from Proposition 4.1.9 for $0 < p < 1$. □

Finally, the proof of Theorem II.2 is completed.

4.2 Proof of Theorem II.3

First recall the Walsh system. Let $\mathcal{G} = \{1, -1\}^{\mathbb{N}}$ be equipped with the uniform distribution P. Recall that for any $n \ge 1$, $\varepsilon_n((\theta_k)_{k\in\mathbb{N}}) := \theta_n$, $\forall \theta = (\theta_k)_{k\in\mathbb{N}} \in \mathcal{G}$. Then $(\varepsilon_n)_{n\ge1}$ is the Rademacher sequence on $\mathcal{G}$, namely a sequence of independent identically distributed random variables on $(\mathcal{G}, P)$ such that $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = 1/2$ for all $n \in \mathbb{N}$.

Recall that $\mathcal{I}$ denotes the family of all finite subsets of $\mathbb{N}$. For a nonempty set $A \in \mathcal{I}$, we write $A = \{k_1 < k_2 < \cdots < k_n\}$ in an increasing order. Define

$$\omega_A = \varepsilon_{k_1}\varepsilon_{k_2}\cdots\varepsilon_{k_n}.$$

If $A = \emptyset$, we set $\varepsilon_A = 1$. If A is a singleton $\{k\}$, we still use ω_k instead of $\omega_{\{k\}}$. Thus $(\omega_A)_{A\in\mathcal{I}}$, called the Walsh system, is an orthonormal basis of $L_2(\mathcal{G})$. Denote by $\mathcal{G}_n$ the σ-algebra generated by $\{\omega_A : \max(A) \le n\}$. Then $(\mathcal{G}_n)_{n\ge1}$ is the filtration of $(\mathcal{G}, P)$ for the Walsh system.

We define for any $\theta \in \mathcal{G}$,

$$\begin{aligned} \sigma_\theta : \mathcal{C} &\to \mathcal{C} \\ c_i &\mapsto \varepsilon_i(\theta)c_i, \quad \forall i \in \mathbb{N}. \end{aligned}$$

Then σ_θ extends to a trace preserving automorphism of the CAR algebra $\mathcal{C}$, and consequently extends to an isometry on $L_p(\mathcal{C})$ for all $0 < p < \infty$. By virtue of σ_θ, the

CAR algebra can be transfered to the operator-valued Walsh system. For any given $b = \sum_{A\in\mathcal{I}} \hat{b}(A)c_A \in L_p(\mathcal{C})$ with $0 < p < \infty$, we define

$$\tilde{b}(\theta) := \sigma_\theta(b) = \sum_{A\in\mathcal{I}} \hat{b}(A)c_A \cdot \omega_A(\theta).$$

Then $\tilde{b} \in L_p(\mathcal{G}, L_p(\mathcal{C}))$. Hence, for any given b, define the martingale paraproduct $\pi_{\tilde{b}}$ of symbol $\tilde{b}$ associated with the Walsh system on $L_2(\mathcal{G}, L_2(\mathcal{C}))$ by

$$\begin{aligned}\pi_{\tilde{b}} : L_2(\mathcal{G}, L_2(\mathcal{C})) &\to L_2(\mathcal{G}, L_2(\mathcal{C}))\\ g &\mapsto \sum_{k=1}^{\infty} d_k\tilde{b} \cdot g_{k-1}.\end{aligned}$$

In fact, $\pi_{\tilde{b}}$ is a martingale paraproduct for semicommutative dyadic martingales.

Now we come to the proof of Theorem II.3.

Proof of Theorem II.3. Since $L_2(\mathcal{C}) \cong \ell_2(\mathcal{I})$ and $L_2(\mathcal{G}, L_2(\mathcal{C})) \cong \ell_2(\mathcal{I}, L_2(\mathcal{C}))$, we represent π_b and $\pi_{\tilde{b}}$ in the matrix form. For any $A, B \in \mathcal{I}$, note that for $k \geq 1$

$$(c_B)_{k-1} = \sum_{\max(D)\leq k-1} \tau(c_D^* \cdot c_B)c_D = \begin{cases} c_B, & \text{if } k-1 \geq \max(B);\\ 0, & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\langle c_A, \pi_b(c_B)\rangle &= \langle c_A, \sum_{k=1}^{\infty} d_k b \cdot (c_B)_{k-1}\rangle = \langle c_A, \sum_{k-1\geq\max(B)} d_k b \cdot c_B\rangle\\ &= \langle c_A, \sum_{\max(E)\geq\max(B)+1} \hat{b}(E)c_E \cdot c_B\rangle\\ &= \langle c_A c_B^*, \sum_{\max(E)\geq\max(B)+1} \hat{b}(E)c_E\rangle.\end{aligned}$$

From the CAR (1.6.1), we have

$$c_A^* = \pm c_A \quad \text{and} \quad c_A c_B = \pm c_{A\Delta B}, \quad \forall A, B \in \mathcal{I},$$

where $A\Delta B = (A \cup B)\backslash(A \cap B)$. Then

$$\begin{aligned}\langle c_A, \pi_b(c_B)\rangle &= \langle \pm c_{A\Delta B}, \sum_{\max(E)\geq\max B+1} \hat{b}(E)c_E\rangle\\ &= \begin{cases} \pm\hat{b}(A\Delta B), & \text{if } \max(A\Delta B) > \max(B);\\ 0, & \text{if } \max(A\Delta B) \leq \max(B),\end{cases}\\ &= \begin{cases} \hat{b}(A\Delta B), & \text{if } \max(A) > \max(B) \text{ and } c_A c_B^* = c_{A\Delta B};\\ -\hat{b}(A\Delta B), & \text{if } \max(A) > \max(B) \text{ and } c_A c_B^* = -c_{A\Delta B};\\ 0, & \text{if } \max(A) \leq \max(B).\end{cases}\end{aligned}$$

In the same way, one has

$$\langle \omega_A, \pi_{\tilde{b}}(\omega_B)\rangle = \begin{cases} \hat{b}(A\Delta B)c_{A\Delta B}, & \text{if } \max(A) > \max(B);\\ 0, & \text{if } \max(A) \leq \max(B).\end{cases}$$

Denote by

$$[\pi_b] := \Big((\pi_b)_{A,B}\Big)_{A,B\in\mathcal{I}}$$

the matrix form of π_b with respect to the basis $(c_A)_{A\in\mathcal{I}}$, where $(\pi_b)_{A,B} = \langle c_A, \pi_b(c_B)\rangle$. Analogously, let

$$[\pi_{\tilde{b}}] := \Big((\pi_{\tilde{b}})_{A,B}\Big)_{A,B\in\mathcal{I}}$$

be the matrix form of $\pi_{\tilde{b}}$ with respect to the basis $(\omega_A)_{A\in\mathcal{I}}$, where $(\pi_{\tilde{b}})_{A,B} = \langle \omega_A, \pi_{\tilde{b}}(\omega_B)\rangle$. By the above discussion, we see that

$$\begin{aligned}[\pi_{\tilde{b}}] &= \Big((\pi_b)_{A,B} c_A c_B^*\Big)_{A,B\in\mathcal{I}} \\ &= \begin{pmatrix} \ddots & & 0 \\ & c_A & \\ 0 & & \ddots \end{pmatrix}_{A\in\mathcal{I}} ([\pi_b]\otimes 1_{\mathcal{C}}) \begin{pmatrix} \ddots & & 0 \\ & c_B^* & \\ 0 & & \ddots \end{pmatrix}_{B\in\mathcal{I}},\end{aligned}$$

where $1_{\mathcal{C}}$ is the identity of $\mathcal{C}$. So this leads to for any $0 < p < \infty$

$$\begin{aligned}\|\pi_{\tilde{b}}\|_{L_p(B(L_2(\mathcal{G}))\otimes\mathcal{C})} &= \|[\pi_{\tilde{b}}]\|_{L_p(B(\ell_2(\mathcal{I}))\otimes\mathcal{C})} \\ &= \|[\pi_b]\|_{S_p(\ell_2(\mathcal{I}))}\cdot\|1_{\mathcal{C}}\|_{L_p(\mathcal{C})} \\ &= \|\pi_b\|_{S_p(L_2(\mathcal{C}))}.\end{aligned}$$

By Theorem II.2, we have $\pi_{\tilde{b}} \in L_p(B(L_2(\mathcal{G}))\otimes\mathcal{C})$ if and only if $\tilde{b} \in \boldsymbol{B}_p^2(\mathbb{R},\mathcal{C})$, where

$$\|\tilde{b}\|_{\boldsymbol{B}_p^2(\mathbb{R},\mathcal{C})} \approx_p \left(\sum_{k=1}^{\infty} 2^k \|d_k\tilde{b}\|^p_{L_p(\mathcal{G},L_p(\mathcal{C}))}\right)^{1/p}.$$

However, note that for any $\theta\in\mathcal{G}$ and $k\geq 1$,

$$(d_k\tilde{b})(\theta) = \Big(\sum_{\max(A)=k}\hat{b}(A)c_A\omega_A\Big)(\theta) = \sigma_\theta\Big(\sum_{\max(A)=k}\hat{b}(A)c_A\Big) = \sigma_\theta(d_k b),$$

which yields

$$\|d_k\tilde{b}\|^p_{L_p(\mathcal{G},L_p(\mathcal{C}))} = \int_{\mathcal{G}}\|(d_k\tilde{b})(\theta)\|^p_{L_p((\mathcal{C})}dP(\theta) = \int_{\mathcal{G}}\|\sigma_\theta(d_k b)\|^p_{L_p((\mathcal{C})}dP(\theta) = \|d_k b\|^p_{L_p(\mathcal{C})}.$$

Therefore, we get $\pi_b \in S_p(L_2(\mathcal{C}))$ if and only if $b\in\boldsymbol{B}_p(\mathcal{C})$ with relevant constants depending only on p. Thus Theorem II.3 is proved. □

4.3 Proof of Theorem II.4

First we construct an orthonormal basis of $\mathbb{M}_d$, which will induce an orthonormal basis in $\mathscr{M} = \bigotimes_{k=1}^{\infty} \mathbb{M}_d$. Denote by $e_{i,j}$ the matrix which has 1 in the (i,j)-th position as its only nonzero entry. Let $\sigma = (1\,2\,\cdots\,d)$ be the d-cycle, and recall $\omega = e^{2\pi \mathrm{i}/d}$. Define

$$\Omega = \left\{ U_{(i,j)} = \sum_{l=1}^{d} \omega^{i\cdot l} e_{l,\sigma^j(l)} : \quad 1 \le i,j \le d \right\}.$$

Then Ω is an orthonormal basis of $L_2(\mathbb{M}_d, \mathrm{tr}_d)$, and every element in Ω is unitary. In particular, $U_{(d,d)} = 1$. Moreover, for any $1 \le i,j,k,l \le d$

1. $U_{(i,j)}^* = \omega^{i\cdot j} U_{(\overline{-i},\overline{-j})}$;
2. $U_{(i,j)} U_{(k,l)} = \omega^{j\cdot k} U_{(\overline{i+k},\overline{j+l})}$.

Denote $\mathscr{A} = \{(k, i_k, j_k) : k \in \mathbb{N}, 1 \le i_k, j_k \le d\}$. For any nonempty finite subset $\alpha = \{(1,i_1,j_1),(2,i_2,j_2),\cdots,(n,i_n,j_n)\} \subset \mathscr{A}$, define $\max(\alpha) = n$. Besides, define $\max(\emptyset) = 1$. Let $\mathcal{J}$ be the family of all finite subsets $\alpha \subset \mathscr{A}$ with $(i_{\max(\alpha)}, j_{\max(\alpha)}) \neq (d,d)$. For any given $\alpha = \{(1,i_1,j_1),(2,i_2,j_2),\cdots,(n,i_n,j_n)\} \in \mathcal{J}$, define

$$U_\alpha = U_{(i_1,j_1)} \otimes U_{(i_2,j_2)} \otimes \cdots \otimes U_{(i_n,j_n)} \otimes 1 \otimes 1 \cdots \in \mathscr{M}.$$

In addition, we set $U_\emptyset = 1$. Then $(U_\alpha)_{\alpha\in\mathcal{J}}$ is an orthonormal basis of $L_2(\mathscr{M})$. Next, we calculate $U_\alpha U_\beta^*$. For any given $\alpha, \beta \in \mathcal{J}$, write

$$\alpha = \{(1, \tilde{i}_1, \tilde{j}_1), \cdots, (\max(\alpha), \tilde{i}_{\max(\alpha)}, \tilde{j}_{\max(\alpha)})\}$$

where $1 \le \tilde{i}_1, \tilde{j}_1, \cdots, \tilde{i}_{\max(\alpha)}, \tilde{j}_{\max(\alpha)} \le d$ and

$$\beta = \{(1, i_1, j_1), \cdots, (\max(\beta), i_{\max(\beta)}, j_{\max(\beta)})\}$$

where $1 \le i_1, j_1, \cdots, i_{\max(\beta)}, j_{\max(\beta)} \le d$. To calculate $U_\alpha U_\beta^*$, we define $\eta_{\alpha,\beta} \in \mathcal{J}$ associated with α and β as follows:

1. If $\max(\alpha) = \max(\beta)$,

$$\eta_{\alpha,\beta} = \{(1, \tilde{i}_1 - i_1, \tilde{j}_1 - j_1), \cdots, (\max(\alpha), \tilde{i}_{\max(\alpha)} - i_{\max(\alpha)}, \tilde{j}_{\max(\alpha)} - j_{\max(\alpha)})\}; \quad (4.3.1)$$

2. If $\max(\alpha) < \max(\beta)$,

$$\begin{aligned}\eta_{\alpha,\beta} = \{&(1, \tilde{i}_1 - i_1, \tilde{j}_1 - j_1), \cdots, (\max(\alpha), \tilde{i}_{\max(\alpha)} - i_{\max(\alpha)}, \tilde{j}_{\max(\alpha)} - j_{\max(\alpha)}),\\ &(\max(\alpha)+1, i_{\max(\alpha)+1}, j_{\max(\alpha)+1}), \cdots, (\max(\beta), i_{\max(\beta)}, j_{\max(\beta)})\};\end{aligned} \quad (4.3.2)$$

3. If $\max(\alpha) > \max(\beta)$,

$$\begin{aligned}\eta_{\alpha,\beta} = \{&(1, \tilde{i}_1 - i_1, \tilde{j}_1 - j_1), \cdots, (\max(\beta), \tilde{i}_{\max(\beta)} - i_{\max(\beta)}, \tilde{j}_{\max(\beta)} - j_{\max(\beta)}),\\ &(\max(\beta)+1, \tilde{i}_{\max(\beta)+1}, \tilde{j}_{\max(\beta)+1}), \cdots, (\max(\alpha), \tilde{i}_{\max(\alpha)}, \tilde{j}_{\max(\alpha)})\}.\end{aligned} \quad (4.3.3)$$

Notice that the case where $\alpha = \emptyset$ or $\beta = \emptyset$ has been included in the construction of $\eta_{\alpha,\beta}$. In (4.3.1), (4.3.2) and (4.3.3), if $\tilde{i}_k - i_k \leq 0$ (respectively $\tilde{j}_k - j_k \leq 0$), then we can substitute $\tilde{i}_k - i_k + d$ (respectively $\tilde{j}_k - j_k + d$) for $\tilde{i}_k - i_k$ (respectively $\tilde{j}_k - j_k$).

One can verify that

$$U_\alpha U_\beta^* = \lambda_{\alpha,\beta} U_{\eta_{\alpha,\beta}}, \tag{4.3.4}$$

where

$$\lambda_{\alpha,\beta} = \begin{cases} \omega^{-i_1(\tilde{j}_1 - j_1)} \cdots \omega^{-i_{\max(\alpha)}(\tilde{j}_{\max(\alpha)} - j_{\max(\alpha)})}, & \text{if } \max(\alpha) \leq \max(\beta); \\ \omega^{-i_1(\tilde{j}_1 - j_1)} \cdots \omega^{-i_{\max(\beta)}(\tilde{j}_{\max(\beta)} - j_{\max(\beta)})}, & \text{if } \max(\alpha) > \max(\beta). \end{cases}$$

This implies that $|\lambda_{\alpha,\beta}| = 1$.

Let $\upsilon = e^{2\pi \mathrm{i}/d^2}$. Let $\mathcal{R} = \{\upsilon^1, \upsilon^2, \cdots, \upsilon^{d^2}\}^{\mathbb{N}}$ be equipped with the uniform distribution. For $1 \leq i, j \leq d$, we define

$$h_{(i,j)} := \sum_{l=1}^{d^2} \upsilon^{(di+j)l} \mathbb{1}_{\{\upsilon^l\}}.$$

Similarly, for any given $\alpha = \{(1, i_1, j_1), (2, i_2, j_2), \cdots, (n, i_n, j_n)\} \in \mathcal{J}$, define

$$h_\alpha = h_{(i_1,j_1)} \otimes h_{(i_2,j_2)} \otimes \cdots \otimes h_{(i_n,j_n)} \otimes 1 \otimes 1 \cdots \in L_2(\mathcal{R}),$$

namely, for every $t = (t_m)_{m\in\mathbb{N}} \in \mathcal{R}$,

$$h_\alpha(t) := \prod_{k=1}^{n} h_{(i_k,j_k)}(t_k).$$

We also set $h_\emptyset = 1$. Then $(h_\alpha)_{\alpha\in\mathcal{J}}$ is an orthonormal basis of $L_2(\mathcal{R})$. Let $\mathscr{R}_n$ be the σ-algebra generated by $\{h_\alpha : \max(\alpha) \leq n\}$, and then $(\mathscr{R}_n)_{n\geq 1}$ is a filtration for $\mathcal{R}$. Indeed, a martingale in $L_2(\mathcal{R})$ with respect to the filtration $(\mathscr{R}_n)_{n\geq 1}$ is a d^2-adic martingale.

Define for any $t = (t_m)_{m\in\mathbb{N}} \in \mathcal{R}$, and $\forall k \in \mathbb{N}, 1 \leq i_k, j_k \leq d$,

$$\begin{aligned} \sigma_{h(t)} : \mathscr{M} &\to \mathscr{M} \\ U_{\{(k,i_k,j_k)\}} &\mapsto h_{\{(k,i_k,j_k)\}}(t_k) U_{\{(k,i_k,j_k)\}}, \end{aligned}$$

then $\sigma_{h(t)}$ extends to a trace preserving automorphism of $\mathscr{M}$, and hence extends to an isometry on $L_p(\mathscr{M})$ for all $0 < p < \infty$.

Now for any given $b = \sum_{\alpha\in\mathcal{J}} \hat{b}(\alpha) U_\alpha \in L_p(\mathscr{M})$ with $\hat{b}(\alpha) := \tau(U_\alpha^* \cdot b)$, we define

$$\tilde{b}(t) := \sigma_{h(t)}(b) = \sum_{\alpha\in\mathcal{J}} \hat{b}(\alpha) U_\alpha \cdot h_\alpha(t),$$

then $\tilde{b} \in L_p(\mathcal{R}, L_p(\mathscr{M}))$. Therefore, for any given b, define the martingale paraproduct $\pi_{\tilde{b}}$ of symbol $\tilde{b}$ on $L_2(\mathcal{R}, L_2(\mathscr{M}))$ by

$$\begin{aligned} \pi_{\tilde{b}} : L_2(\mathcal{R}, L_2(\mathscr{M})) &\to L_2(\mathcal{R}, L_2(\mathscr{M})) \\ g &\mapsto \sum_{k=1}^{\infty} d_k \tilde{b} \cdot g_{k-1}. \end{aligned}$$

In fact, $\pi_{\tilde{b}}$ is a martingale paraproduct for semicommutative d^2-adic martingales.

Now we come to the proof of Theorem II.4.

Proof of Theorem II.4. Since $L_2(\mathscr{M}) \cong \ell_2(\mathcal{J})$ and $L_2(\mathcal{R}, L_2(\mathscr{M})) \cong \ell_2(\mathcal{J}, L_2(\mathscr{M}))$, we represent π_b and $\pi_{\tilde{b}}$ in the matrix form. Note that for $k \geq 1$,

$$(U_\beta)_{k-1} = \begin{cases} U_\beta, & \text{if } k-1 \geq \max(\beta); \\ 0, & \text{otherwise.} \end{cases}$$

This implies

$$\begin{aligned} \langle U_\alpha, \pi_b(U_\beta)\rangle &= \langle U_\alpha, \sum_{k=1}^{\infty} d_k b \cdot (U_\beta)_{k-1}\rangle = \langle U_\alpha, \sum_{k-1\geq \max(\beta)} d_k b \cdot U_\beta\rangle \\ &= \langle U_\alpha, \sum_{\max(\gamma)\geq \max(\beta)+1} \hat{b}(\gamma) U_\gamma U_\beta\rangle \\ &= \langle U_\alpha U_\beta^*, \sum_{\max\gamma\geq \max(\beta)+1} \hat{b}(\gamma) U_\gamma\rangle. \end{aligned}$$

Then by (4.3.4)

$$\begin{aligned} \langle U_\alpha, \pi_b(U_\beta)\rangle &= \langle \lambda_{\alpha,\beta} U_{\eta_{\alpha,\beta}}, \sum_{\max\gamma\geq \max(\beta)+1} \hat{b}(\gamma) U_\gamma\rangle \\ &= \begin{cases} \overline{\lambda_{\alpha,\beta}} \cdot \hat{b}(\eta_{\alpha,\beta}), & \text{if } \max(\alpha) > \max(\beta); \\ 0, & \text{if } \max(\alpha) \leq \max(\beta). \end{cases} \end{aligned}$$

In the same way, one has

$$\langle h_\alpha, \pi_{\tilde{b}}(h_\beta)\rangle = \begin{cases} \hat{b}(\eta_{\alpha,\beta}) U_{\eta_{\alpha,\beta}}, & \text{if } \max(\alpha) > \max(\beta); \\ 0, & \text{if } \max(\alpha) \leq \max(\beta). \end{cases}$$

Denote by

$$[\pi_b] = \Big((\pi_b)_{\alpha,\beta}\Big)_{\alpha,\beta\in\mathcal{J}}$$

the matrix form of π_b with respect to the basis $(U_\alpha)_{\alpha\in\mathcal{J}}$, where $(\pi_b)_{\alpha,\beta} = \langle U_\alpha, \pi_b(U_\beta)\rangle$. Analogously, let

$$[\pi_{\tilde{b}}] = \Big((\pi_{\tilde{b}})_{\alpha,\beta}\Big)_{\alpha,\beta\in\mathcal{J}}$$

be the matrix form of $\pi_{\tilde{b}}$ with respect to the basis $(h_\alpha)_{\alpha\in\mathcal{J}}$, where $(\pi_{\tilde{b}})_{\alpha,\beta} = \langle h_\alpha, \pi_{\tilde{b}}(h_\beta)\rangle$.

Observing that $(\pi_{\tilde{b}})_{\alpha,\beta} = (\pi_b)_{\alpha,\beta} U_\alpha U_\beta^*$ for any $\alpha, \beta \in \mathcal{J}$, one has

$$\begin{aligned} [\pi_{\tilde{b}}] &= \Big((\pi_b)_{\alpha,\beta} U_\alpha U_\beta^*\Big)_{\alpha,\beta\in\mathcal{J}} \\ &= \begin{pmatrix} \ddots & & 0 \\ & U_\alpha & \\ 0 & & \ddots \end{pmatrix}_{\alpha\in\mathcal{J}} ([\pi_b] \otimes 1_{\mathscr{M}}) \begin{pmatrix} \ddots & & 0 \\ & U_\beta^* & \\ 0 & & \ddots \end{pmatrix}_{\beta\in\mathcal{J}}, \end{aligned}$$

where $1_{\mathscr{M}}$ is the identity of $\mathscr{M}$. So this implies that for any $0 < p < \infty$

$$\begin{aligned} \|\pi_{\tilde{b}}\|_{L_p(B(L_2(\mathcal{R}))\otimes\mathscr{M})} &= \|[\pi_{\tilde{b}}]\|_{L_p(B(\ell_2(\mathcal{J}))\otimes\mathscr{M})} \\ &= \|[\pi_b]\|_{S_p(\ell_2(\mathcal{J}))} \cdot \|1_{\mathscr{M}}\|_{L_p(\mathscr{M})} \\ &= \|\pi_b\|_{S_p(L_2(\mathscr{M}))}. \end{aligned}$$

By Theorem II.2, we have $\pi_{\tilde{b}} \in L_p(B(L_2(\mathcal{R})) \otimes \mathscr{M})$ if and only if $\tilde{b} \in \boldsymbol{B}_p^{d^2}(\mathbb{R}, \mathscr{M})$, where

$$\|\tilde{b}\|_{\boldsymbol{B}_p^{d^2}(\mathbb{R},\mathscr{M})} \approx_{d,p} \left(\sum_{k=1}^{\infty} d^{2k} \|d_k \tilde{b}\|^p_{L_p(\mathcal{R}, L_p(\mathscr{M}))} \right)^{1/p}.$$

However, note that for any $t \in \mathcal{R}$ and $k \geq 1$,

$$(d_k \tilde{b})(t) = \Big(\sum_{\max(\alpha)=k} \hat{b}(\alpha) U_\alpha h_\alpha \Big)(t) = \sigma_{h(t)}\big(\sum_{\max(\alpha)=k} \hat{b}(\alpha) U_\alpha \big) = \sigma_{h(t)}(d_k b).$$

This yields

$$\|d_k \tilde{b}\|^p_{L_p(\mathcal{R}, L_p(\mathscr{M}))} = \int_{\mathcal{R}} \|(d_k \tilde{b})(t)\|^p_{L_p(\mathscr{M})} dt = \int_{\mathcal{R}} \|\sigma_{h(t)}(d_k b)\|^p_{L_p(\mathscr{M})} dt = \|d_k b\|^p_{L_p(\mathscr{M})},$$

Therefore, we conclude that $\pi_b \in S_p(L_2(\mathscr{M}))$ if and only if $b \in \boldsymbol{B}_p(\mathscr{M})$ with relevant constants depending only on d and p. This completes the proof of Theorem II.4. $\square$

Chapter 5

Schatten class and boundedness of operator-valued commutators

The last chapter focuses on the applications of martingale paraproducts to the analysis of commutators and operator-valued commutators involving general singular integral operators, which are summarized in Theorem II.6, Theorem II.7 and Theorem II.8. Such applications are feasible thanks to the dyadic martingale technique developed by Hytönen in [40] and [41].

5.1 Proof of Theorem II.6

We first start with preparations concerning martingale paraproducts and Schatten classes, namely Lemma 5.1.1 and Proposition 5.1.4, which will be helpful in the proof of Theorem II.6. Then we introduce the key ingredient: the dyadic representation of singular integral operators by Hytönen in [40] and [41]. This representation enables the reduction to the d-adic martingale setting. Finally, we will give a proof of Theorem II.6 using the result about martingale paraproducts stated in Theorem II.2.

5.1.1 Schatten class of operator-valued commutators involving martingale paraproducts

Lemma 5.1.1. *Assume that $1 \leq p < \infty$. For any semicommutative d-adic martingale $f = (f_k)_{k\in\mathbb{Z}} \in L_2(\mathbb{R}, L_2(\mathcal{M}))$, we define*

$$\Lambda_b(f) = \sum_{k\in\mathbb{Z}} d_k b \cdot d_k f.$$

If $b \in \boldsymbol{B}_p^d(\mathbb{R}, \mathcal{M})$, then $\Lambda_b \in L_p(B(L_2(\mathbb{R})) \otimes \mathcal{M})$ and

$$\|\Lambda_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \lesssim_{d,p} \|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}.$$

Proof. We write Λ_b as follows:

$$\begin{aligned}
\Lambda_b(f) &= \sum_{k\in\mathbb{Z}} d_k b \cdot d_k f \\
&= \sum_{k\in\mathbb{Z}} \Bigg(\sum_{I\in\mathcal{D}_{k-1}} \sum_{i=1}^{d-1} \langle h_I^i, b\rangle h_I^i \Bigg) \Bigg(\sum_{J\in\mathcal{D}_{k-1}} \sum_{j=1}^{d-1} \langle h_J^j, f\rangle h_J^j \Bigg) \\
&= \sum_{k\in\mathbb{Z}} \sum_{I\in\mathcal{D}_{k-1}} \Bigg(\sum_{i=1}^{d-1} \langle h_I^i, b\rangle h_I^i \Bigg) \Bigg(\sum_{j=1}^{d-1} \langle h_I^j, f\rangle h_I^j \Bigg) \\
&= \sum_{k\in\mathbb{Z}} \sum_{I\in\mathcal{D}_{k-1}} \Bigg(\sum_{\overline{i+j}=d} \langle h_I^i, b\rangle \langle h_I^j, f\rangle \frac{\mathbb{1}_I}{|I|} + \sum_{l=1}^{d-1} \sum_{\overline{i+j}=l} \langle h_I^i, b\rangle \langle h_I^j, f\rangle \frac{h_I^l}{|I|^{1/2}} \Bigg) \\
&= \sum_{k\in\mathbb{Z}} \sum_{I\in\mathcal{D}_{k-1}} \Bigg(\sum_{\overline{i+j}=d} \langle b^*, h_I^{d-i}\rangle \langle h_I^j, f\rangle \frac{\mathbb{1}_I}{|I|} + \sum_{l=1}^{d-1} \sum_{\overline{i+j}=l} \langle h_I^i, b\rangle \langle h_I^j, f\rangle \frac{h_I^l}{|I|^{1/2}} \Bigg) \\
&= (\pi_{b^*})^*(f) + \tilde{\Lambda}_b(f),
\end{aligned} \tag{5.1.1}$$

where we have used (1.7.4), and where

$$\tilde{\Lambda}_b(f) := \sum_{k\in\mathbb{Z}} \sum_{I\in\mathcal{D}_{k-1}} \sum_{l=1}^{d-1} \sum_{\overline{i+j}=l} \langle h_I^i, b\rangle \langle h_I^j, f\rangle \frac{h_I^l}{|I|^{1/2}}.$$

By Theorem II.2, we know

$$\|(\pi_{b^*})^*\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \approx_{d,p} \|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}. \tag{5.1.2}$$

It remains to estimate $\|\tilde{\Lambda}_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}$. We represent it into the matrix form. Note that for any $S, T \in \mathcal{D}$, $1 \le s, t \le d-1$, and $x, y \in L_2(\mathcal{M})$,

$$\langle h_S^s \otimes x, \tilde{\Lambda}_b(h_T^t \otimes y)\rangle = \begin{cases} \langle x, |S|^{-1/2}\langle h_S^{s-t}, b\rangle y\rangle, & \text{if } S = T \text{ and } s \ne t; \\ 0, & \text{otherwise.} \end{cases}$$

This yields that $\tilde{\Lambda}_b$ is a block diagonal matrix with respect to the basis $\{h_I^i\}_{I\in\mathcal{D}, 1\le i\le d-1}$. (If $s-t<0$, replace $s-t$ with $s-t+d$, and still denote it by $s-t$.) For any $I \in \mathcal{D}$, $1 \le s \ne t \le d-1$, denote $|I|^{-1/2}\langle h_I^{s-t}, b\rangle$ by a_{s-t}^I, and define $a_0^I = 0$. Hence one has

$$\|\tilde{\Lambda}_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}^p = \sum_{I\in\mathcal{D}} \left\| \left(a_{s-t}^I\right)_{1\le s,t\le d-1} \right\|_{L_p(\mathbb{M}_{d-1}\otimes\mathcal{M})}^p,$$

where $\mathbb{M}_{d-1}$ is equipped with the usual trace. Let

$$B^I = \begin{pmatrix} a_0^I & a_{d-1}^I & \cdots & a_2^I & a_1^I \\ a_1^I & a_0^I & a_{d-1}^I & \cdots & a_2^I \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{d-2}^I & \cdots & a_1^I & a_0^I & a_{d-1}^I \\ a_{d-1}^I & \cdots & a_2^I & a_1^I & a_0^I \end{pmatrix} = \begin{pmatrix} & & & a_1^I \\ & \left(a_{s-t}^I\right)_{1\le s,t\le d-1} & & \vdots \\ & & & a_{d-1}^I \\ a_{d-1}^I & \cdots & a_1^I & a_0^I \end{pmatrix}.$$

By Lemma 4.1.7, we have

$$\left\| \left(a^I_{s-t} \right)_{1\le s,t\le d-1} \right\|_{L_p(\mathbb{M}_{d-1}\otimes\mathcal{M})} \le \|B^I\|_{L_p(\mathbb{M}_d\otimes\mathcal{M})},$$

which implies

$$\|\tilde{\Lambda}_b\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \le \sum_{I\in\mathcal{D}} \|B^I\|^p_{L_p(\mathbb{M}_d\otimes\mathcal{M})}.$$

Note that we can write B^I as

$$B^I = a^I_1 A + a^I_2 A^2 + \cdots + a^I_{d-1} A^{d-1},$$

where

$$A = e_{1,d} + \sum_{j=1}^{d-1} e_{j+1,j}.$$

Using the triangle inequality, one has

$$\begin{aligned}
\|\tilde{\Lambda}_b\|^p_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} &\le \sum_{I\in\mathcal{D}} \|a^I_1 A + a^I_2 A^2 + \cdots + a^I_{d-1} A^{d-1}\|^p_{L_p(\mathbb{M}_d\otimes\mathcal{M})} \\
&\lesssim_{d,p} \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \|a^I_i A^i\|^p_{L_p(\mathbb{M}_d\otimes\mathcal{M})} \le \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \|a^I_i\|^p_{L_p(\mathcal{M})} \|A\|^p_{S_p(\mathbb{M}_d)} \\
&\lesssim_{d,p} \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \|a^I_i\|^p_{L_p(\mathcal{M})} = \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \left(\frac{\|\langle h^i_I, b\rangle\|_{L_p(\mathcal{M})}}{|I|^{1/2}} \right)^p = \|b\|^p_{\boldsymbol{B}^d_p(\mathbb{R},\mathcal{M})}.
\end{aligned}$$

Combining this with (5.1.2), we obtain the desired result. □

Remark 5.1.2. Lemma 5.1.1 also holds for $0 < p < 1$ with the same proof, and we leave the details to the interested reader.

In what follows, we need to use the boundedness of the triangular projection on Schatten class. The triangular projection is defined as follows

$$\begin{aligned}
\mathcal{P} : B(\ell_2) &\longrightarrow B(\ell_2) \\
(m_{ij})_{i,j} &\longmapsto (\delta_{i>j} \cdot m_{ij})_{i,j},
\end{aligned}$$

where $\delta_{i>j} = 1$ if $i > j$, and $\delta_{i>j} = 0$ if $i \le j$. It is well-known that $\mathcal{P}$ is bounded from $S_p(\ell_2)$ to $S_p(\ell_2)$ when $1 < p < \infty$. We refer to [33].

Then for $1 < p < \infty$, we can define $\mathcal{P} \otimes Id_{L_p(\mathcal{M})}$ on the algebraic tensor product $S_p(\ell_2) \otimes L_p(\mathcal{M})$. The next lemma is well-known, and it also follows from Corollary B.1.3.

Lemma 5.1.3. *Let $1 < p < \infty$. Then $\mathcal{P} \otimes Id_{L_p(\mathcal{M})}$ extends to a bounded map on $L_p(B(\ell_2) \otimes \mathcal{M})$. Moreover,*

$$\|\mathcal{P} \otimes Id_{L_p(\mathcal{M})}\|_{L_p(B(\ell_2)\otimes\mathcal{M})\to L_p(B(\ell_2)\otimes\mathcal{M})} \lesssim \max\{p', p\}.$$

Before proving Theorem II.6, we give the following proposition, which concerns the p-Schatten class of operator-valued commutators involving martingale paraproducts and the pointwise multiplication operator M_b.

Proposition 5.1.4. *Let $1 < p < \infty$. If $a \in BMO^d(\mathbb{R})$ and $b \in \boldsymbol{B}_p^d(\mathbb{R}, \mathcal{M})$, then $[\pi_a, M_b] \in L_p(B(L_2(\mathbb{R})) \otimes \mathcal{M})$ and*

$$\|[\pi_a, M_b]\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \lesssim_{d,p} \|a\|_{BMO^d(\mathbb{R})} \|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}.$$

Proof. Let

$$R_b(f) = \sum_{k\in\mathbb{Z}} b_{k-1} \cdot d_k f, \quad \forall f \in L_2(\mathbb{R}, L_2(\mathcal{M})). \tag{5.1.3}$$

Note that for $b, f \in L_2(\mathbb{R}, L_2(\mathcal{M}))$, $M_b(f) = \pi_b(f) + \Lambda_b(f) + R_b(f)$. Thus

$$[\pi_a, M_b] = [\pi_a, \pi_b] + [\pi_a, \Lambda_b] + [\pi_a, R_b].$$

We first estimate $\|[\pi_a, R_b]\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}$. For any $f \in L_2(\mathbb{R}, L_2(\mathcal{M}))$,

$$\begin{aligned}
[\pi_a, R_b](f) &= \pi_a(R_b(f)) - R_b(\pi_a(f)) \\
&= \sum_{k\in\mathbb{Z}} d_k a \cdot \mathbb{E}_{k-1}\left(\sum_{j\in\mathbb{Z}} b_{j-1} \cdot d_j f\right) - \sum_{k\in\mathbb{Z}} b_{k-1} \cdot d_k\left(\sum_{j\in\mathbb{Z}} d_j a \cdot f_{j-1}\right) \\
&= \sum_{k\in\mathbb{Z}} d_k a \cdot \left(\sum_{j\le k-1} b_{j-1} \cdot d_j f\right) - \sum_{k\in\mathbb{Z}} b_{k-1} \cdot d_k a \cdot f_{k-1} \\
&= \sum_{k\in\mathbb{Z}} d_k a \cdot \left(\sum_{j\le k-1} b_{j-1} \cdot d_j f - b_{k-1} \cdot f_{k-1}\right) \\
&= -\sum_{k\in\mathbb{Z}} d_k a \cdot \left(\sum_{j\le k-1} d_j b \cdot d_j f\right) - \sum_{k\in\mathbb{Z}} d_k a \cdot \left(\sum_{j\le k-1} d_j b \cdot f_{j-1}\right) \\
&= -\sum_{k\in\mathbb{Z}} d_k a \cdot \left(\sum_{j\le k-1} d_j b \cdot d_j f\right) - \pi_a(\pi_b(f)) \\
&=: -\Psi_{a,b}(f) - \pi_a(\pi_b(f)).
\end{aligned}$$

Thus

$$[\pi_a, R_b] = -\Psi_{a,b} - \pi_a\pi_b. \tag{5.1.4}$$

From Theorem II.2, we know that

$$\|\pi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \lesssim_{d,p} \|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}.$$

Since π_a is bounded on $L_2(\mathbb{R}, L_2(\mathcal{M}))$, one has

$$\|\pi_a\pi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \lesssim_{d,p} \|a\|_{BMO^d(\mathbb{R})} \|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}. \tag{5.1.5}$$

To deal with $\Psi_{a,b}$, let $\mathcal{Q}_n = \Big\{(S, s) : S \in \mathcal{D}_n, 1 \le s \le d-1\Big\}$ and

$$\mathcal{Q} := \cdots \mathcal{Q}_{-2} \cup \mathcal{Q}_{-1} \cup \mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2 \cdots,$$

with such above order. Note that $L_2(\mathbb{R}, L_2(\mathcal{M})) \cong \ell_2(\mathcal{Q}, L_2(\mathcal{M}))$. Besides, we denote by

$$[\Psi_{a,b}] := \Big((\Psi_{a,b})_{(S,s),(T,t)}\Big)_{(S,s),(T,t)\in\mathcal{Q}}$$

the matrix form of $\Psi_{a,b}$ with respect to the basis $(h_S^s)_{(S,s)\in\mathcal{Q}}$, where $(\Psi_{a,b})_{(S,s),(T,t)} = \langle h_S^s, \Psi_{a,b}(h_T^t)\rangle$. Analogously, let

$$[\pi_a \Lambda_b] := \Big((\pi_a\Lambda_b)_{(S,s),(T,t)}\Big)_{(S,s),(T,t)\in\mathcal{Q}}$$

be the matrix form of $\pi_a\Lambda_b$.

Our aim is to prove that

$$[\Psi_{a,b}] = (\mathcal{P}\otimes Id_{L_p(\mathcal{M})})([\pi_a\Lambda_b]). \tag{5.1.6}$$

On the one hand, note that

$$\begin{aligned}\Psi_{a,b}(f) &= \sum_{k\in\mathbb{Z}} d_k a\cdot\left(\mathbb{E}_{k-1}\left(\sum_{j\in\mathbb{Z}} d_j b\cdot d_j f\right) - \mathbb{E}_{k-1}\left(\sum_{j\geq k} d_j b\cdot d_j f\right)\right)\\ &= \sum_{k\in\mathbb{Z}} d_k a\cdot\left(\mathbb{E}_{k-1}(\Lambda_b(f)) - \mathbb{E}_{k-1}\left(\sum_{j\geq k} d_j b\cdot d_j f\right)\right)\\ &= \pi_a(\Lambda_b(f)) - \sum_{k\in\mathbb{Z}} d_k a\cdot\mathbb{E}_{k-1}\left(\sum_{j\geq k} d_j b\cdot d_j f\right).\end{aligned} \tag{5.1.7}$$

Suppose that $S\in\mathcal{D}_m$, $T\in\mathcal{D}_n$, and $m>n$. For any $1\leq s,t\leq d-1$, note that $d_{m+1}h_S^s = h_S^s$ and $d_{n+1}h_T^t = h_T^t$, hence

$$\begin{aligned}\langle h_S^s, \Psi_{a,b}(h_T^t)\rangle - \langle h_S^s, \pi_a(\Lambda_b(h_T^t))\rangle &= -\left\langle h_S^s, \sum_{k\in\mathbb{Z}} d_k a\cdot\mathbb{E}_{k-1}\left(\sum_{j\geq k} d_j b\cdot d_j h_T^t\right)\right\rangle\\ &= -\left\langle d_{m+1}h_S^s, \sum_{k\leq n+1} d_k a\cdot\mathbb{E}_{k-1}\left(d_{n+1}b\cdot d_{n+1}h_T^t\right)\right\rangle\\ &= -\left\langle h_S^s, d_{m+1}\left(\sum_{k\leq n+1} d_k a\cdot\mathbb{E}_{k-1}\left(d_{n+1}b\cdot h_T^t\right)\right)\right\rangle\\ &= \langle h_S^s, 0\rangle = 0.\end{aligned}$$

This implies that when $S\in\mathcal{D}_m$, $T\in\mathcal{D}_n$, and $m>n$

$$(\Psi_{a,b})_{(S,s),(T,t)} = (\pi_a\Lambda_b)_{(S,s),(T,t)}. \tag{5.1.8}$$

On the other hand,

$$\begin{aligned}\Psi_{a,b}(f) &= \sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_{k-1}}\sum_{i=1}^{d-1}\langle h_I^i, a\rangle h_I^i\cdot\left(\sum_{j\leq k-1}\sum_{L\in\mathcal{D}_{j-1}}\sum_{l=1}^{d-1}\langle h_L^l, b\rangle h_L^l\cdot\sum_{Q\in\mathcal{D}_{j-1}}\sum_{q=1}^{d-1}\langle h_Q^q, f\rangle h_Q^q\right)\\ &= \sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_{k-1}}\sum_{i=1}^{d-1}\langle h_I^i, a\rangle h_I^i\cdot\left(\sum_{j\leq k-1}\sum_{L\in\mathcal{D}_{j-1}}\sum_{l=1}^{d-1}\sum_{q=1}^{d-1}\langle h_L^l, b\rangle\langle h_L^q, f\rangle h_L^{\overline{l+q}}\right)\\ &= \sum_{I\in\mathcal{D}}\sum_{i=1}^{d-1}\langle h_I^i, a\rangle\cdot\left(\sum_{I\subsetneq L}\sum_{l=1}^{d-1}\sum_{q=1}^{d-1}\langle h_L^l, b\rangle\langle h_L^q, f\rangle h_I^i h_L^{\overline{l+q}}\right)\\ &= \sum_{I\in\mathcal{D}}\sum_{i=1}^{d-1}\langle h_I^i, a\rangle\cdot\left(\sum_{I\subsetneq L}\sum_{l=1}^{d-1}\sum_{q=1}^{d-1}\langle h_L^l, b\rangle\langle h_L^q, f\rangle\left\langle\frac{\mathbb{1}_I}{|I|}, h_L^{\overline{l+q}}\right\rangle h_I^i\right),\end{aligned}$$

where the last equality follows from

$$h_I^i h_L^{\overline{l+q}} = h_I^i(\mathbb{1}_I \cdot h_L^{\overline{l+q}}) = \left\langle \frac{\mathbb{1}_I}{|I|}, h_L^{\overline{l+q}} \right\rangle h_I^i, \quad \forall I \subsetneqq L.$$

This implies that

$$(\Psi_{a,b})_{(S,s),(T,t)} = \begin{cases} \langle h_S^s, a\rangle \sum_{l=1}^{d-1} \langle h_T^l, b\rangle \left\langle \frac{\mathbb{1}_S}{|S|}, h_T^{\overline{l+t}} \right\rangle = (\pi_a \Lambda_b)_{(S,s),(T,t)}, & \text{if } S \subsetneqq T\ ; \\ 0, & \text{otherwise.} \end{cases} \tag{5.1.9}$$

Hence from (5.1.8) and (5.1.9), we conclude (5.1.6).

From Lemma 5.1.1, we know that

$$\|\Lambda_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \lesssim_{d,p} \|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}.$$

Hence from (5.1.6) and Lemma 5.1.3 one has

$$\begin{aligned} \|\Psi_{a,b}\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} &= \|[\Psi_{a,b}]\|_{L_p(B(\ell_2(\mathcal{Q}))\otimes\mathcal{M})} \\ &= \|(\mathcal{P}\otimes Id_{L_p(\mathcal{M})})([\pi_a\Lambda_b])\|_{L_p(B(\ell_2(\mathcal{Q}))\otimes\mathcal{M})} \\ &\lesssim_p \|[\pi_a\Lambda_b]\|_{L_p(B(\ell_2(\mathcal{Q}))\otimes\mathcal{M})} \\ &= \|[\pi_a\Lambda_b]\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \lesssim_{d,p} \|a\|_{BMO^d(\mathbb{R})}\|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}. \end{aligned}$$

Combining the preceding inequalities, we arrive at

$$\|[\pi_a, R_b]\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \lesssim_{d,p} \|a\|_{BMO^d(\mathbb{R})}\|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}.$$

Therefore, by the triangle inequality we deduce that

$$\begin{aligned} \|[\pi_a, M_b]\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} &\le \|[\pi_a,\pi_b]\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} + \|[\pi_a,\Lambda_b]\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \\ &\quad + \|[\pi_a, R_b]\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \\ &\lesssim_{d,p} \|a\|_{BMO^d(\mathbb{R})}\|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}, \end{aligned}$$

as desired. □

Corollary 5.1.5. *Let $1 < p < \infty$. If $a \in BMO^d(\mathbb{R})$ and $b \in \boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})$, then $[\pi_a^*, M_b] \in L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})$ and*

$$\|[\pi_a^*, M_b]\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \lesssim_{d,p} \|a\|_{BMO^d(\mathbb{R})}\|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}.$$

Proof. It is easy to verify that

$$[\pi_a^*, M_b]^* = -[\pi_a, M_{b^*}].$$

By Proposition 5.1.4 we deduce that

$$\|[\pi_a^*, M_b]\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} = \|[\pi_a, M_{b^*}]\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \lesssim_{d,p} \|a\|_{BMO^d(\mathbb{R})}\|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}.$$

□

5.1.2 Hytönen's dyadic representation

Now we introduce the dyadic system on $\mathbb{R}^n$. Recall that the standard system of dyadic cubes is

$$\mathcal{D}^0 = \{2^{-k}([0,1)^n + q) : k \in \mathbb{Z}, q \in \mathbb{Z}^n\}.$$

Let $\mathcal{D}_k^0 = \{2^{-k}([0,1)^n + q) : q \in \mathbb{Z}^n\}$ for any $k \in \mathbb{Z}$. Define $\ell(I) := 2^{-k}$ and $|I| := 2^{-nk}$ if $I \in \mathcal{D}_k^0$. Let $\omega = (\omega_j)_{j\in\mathbb{Z}} \in (\{0,1\}^n)^{\mathbb{Z}}$ and define

$$I\dot{+}\omega = I + \sum_{j:2^{-j}<\ell(I)} 2^{-j}\omega_j. \tag{5.1.10}$$

Note that by (5.1.10), $I \cap I\dot{+}\omega \neq \emptyset$ unless some coordinate of $\sum_{j:2^{-j}<\ell(I)} 2^{-j}\omega_j$ is exactly 1. Then set

$$\mathcal{D}^\omega = \{I\dot{+}\omega : I \in \mathcal{D}^0\},$$

which are obtained by translating the standard system. Indeed, in particular, if $\omega = (\omega_j)_{j\in\mathbb{Z}} \in (\{0,1\}^n)^{\mathbb{Z}}$ such that $\exists j_0 \in \mathbb{Z}$, $\forall j \le j_0$, $\omega_j = 0$, then

$$\mathcal{D}^\omega = \Big\{I + \sum_j 2^{-j}\omega_j : I \in \mathcal{D}^0\Big\}.$$

See [39] for more details on $\mathcal{D}^\omega$.

Let $\mathcal{D}_k^\omega$ be the family of all dyadic cubes with volume 2^{-nk}. For any $I \in \mathcal{D}^\omega$, let $\mathcal{D}^\omega(I)$ be the collection of dyadic cubes in $\mathcal{D}^\omega$ contained in I, and $\mathcal{D}_k^\omega(I)$ be the intersection of $\mathcal{D}_k^\omega$ and $\mathcal{D}^\omega(I)$ for any $k \in \mathbb{Z}$. In addition, we assign to the parameter set $(\{0,1\}^n)^{\mathbb{Z}}$ the natural probability measure, that is the infinite tensor product of the uniform probability measure $\sum_{\omega\in\{0,1\}^n} \frac{\delta_\omega}{2^n}$. Here δ_ω is the Dirac measure. Denote by $\mathbb{E}_\omega$ the expectation on $(\{0,1\}^n)^{\mathbb{Z}}$.

For any given cube $I = I_1 \times \cdots \times I_n \in \mathcal{D}^\omega$, let $H_{I_i}^0 := |I_i|^{-1/2}\mathbb{1}_{I_i}$ and $H_{I_i}^1 := |I_i|^{-1/2}(\mathbb{1}_{I_{i\ell}} - \mathbb{1}_{I_{ir}})$, where $\mathbb{1}_{I_{i\ell}}$ and $\mathbb{1}_{I_{ir}}$ are the left and right halves of I_i for $1 \le i \le n$. For any $\eta \in \{0,1\}^n\backslash\{0\}$, we denote by H_I^η the function on the cube $I = I_1 \times \cdots \times I_n$ which is the product of the one-variable functions:

$$H_I^\eta(x) = H_{I_1\times\cdots\times I_n}^{(\eta_1,\cdots,\eta_n)}(x_1, \cdots, x_n) := \prod_{i=1}^n H_{I_i}^{\eta_i}(x_i).$$

Hence $\{H_I^\eta\}_{I\in\mathcal{D}^\omega,\eta\in\{0,1\}^n\backslash\{0\}}$ form an orthonormal basis of $L_2(\mathbb{R}^n)$.

Let $i, j \in \mathbb{N} \cup \{0\}$. For a fixed dyadic system $\mathcal{D}^\omega$, the dyadic shift with parameters i, j is an operator of the form

$$S_\omega^{ij} f = \sum_{K\in\mathcal{D}^\omega} A_K^{ij} f, \qquad A_K^{ij} f = \sum_{\substack{I,J\in\mathcal{D}^\omega; I,J\subseteq K \\ \ell(I)=2^{-i}\ell(K) \\ \ell(J)=2^{-j}\ell(K)}} \sum_{\xi,\eta\in\{0,1\}^n\backslash\{0\}} a_{IJK}^{\xi\eta} \langle H_I^\xi, f\rangle H_J^\eta,$$

with coefficients $a_{IJK}^{\xi\eta}$ satisfying

$$|a_{IJK}^{\xi\eta}| \le \frac{\sqrt{|I||J|}}{|K|}.$$

The map $S_\omega^{i,j} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ is bounded with norm at most one. Moreover, by [40, Proposition 5.1], we see that S_ω^{ij} is of weak type $(1,1)$ with norm $O(i)$. Hence, by interpolation and duality, one has for $1 < p < \infty$

$$\|S_\omega^{ij}\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)} \lesssim_{n,p} i + j.$$

We refer the reader to [40] and [41] for more information.

Recall that in this thesis, $T : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ is always assumed to be bounded and its kernel satisfies the estimates (II.3). The following is the dyadic representation of singular integral operators discovered by Hytönen in [40] and [41] (see [41, Theorem 3.3]).

Theorem 5.1.6. *Let T be a bounded singular integral operator. Then T has a dyadic expansion, say for $f, g \in L_2(\mathbb{R}^n)$,*

$$\begin{aligned}\langle g, Tf\rangle = C_1\mathbb{E}_\omega \sum_{\substack{i,j=0\\ \max\{i,j\}>0}}^{\infty} 2^{-\max\{i,j\}\alpha/2}\langle g, S_\omega^{ij} f\rangle + C_2\mathbb{E}_\omega\langle g, S_\omega^{00} f\rangle \\ + \mathbb{E}_\omega\langle g, \pi_{T1}^\omega f\rangle + \mathbb{E}_\omega\langle g, (\pi_{T^*1}^\omega)^* f\rangle,\end{aligned} \tag{5.1.11}$$

where S_ω^{ij} is the dyadic shift of parameters (i,j) on the dyadic system $\mathcal{D}^\omega$, π_b^ω is the dyadic martingale paraproduct on the dyadic system $\mathcal{D}^\omega$ associated with the BMO-function $b \in \{T1, T^*1\}$, and C_1, C_2 are positive constants.

Remark 5.1.7. Note that S_ω^{ij} is always contractive on $L_2(\mathbb{R}^n, L_2(\mathcal{M}))$ for all w and i, j. Besides, the assumption that $T1 \in BMO(\mathbb{R}^n)$ and $T^*1 \in BMO(\mathbb{R}^n)$ also implies that π_{T1}^ω and $(\pi_{T^*1}^\omega)^*$ are still bounded on $L_2(\mathbb{R}^n, L_2(\mathcal{M}))$. This yields that T is bounded on $L_2(\mathbb{R}^n, L_2(\mathcal{M}))$, and (5.1.11) also holds for any $f, g \in L_2(\mathbb{R}^n, L_2(\mathcal{M}))$.

The dyadic system $\mathcal{D}^\omega$ on $\mathbb{R}^n$ can be regarded as the 2^n-adic system by our definition of d-adic martingales (see Subsection 1.7). So we can define the martingale Besov space $\boldsymbol{B}_p^{\omega,2^n}(\mathbb{R}^n, \mathcal{M})$ on $\mathbb{R}^n$ by virtue of H_I^η similarly as in Definition 1.7.2. More precisely, $\boldsymbol{B}_p^{\omega,2^n}(\mathbb{R}^n, \mathcal{M})$ $(0 < p < \infty)$ associated with semicommutative dyadic martingale on $\mathbb{R}^n$ is the completion of the set consisting of all $b \in \mathcal{S}(L^\infty(\mathbb{R}^n) \otimes \mathcal{M})$ such that

$$\|b\|_{\boldsymbol{B}_p^{\omega,2^n}(\mathbb{R}^n,\mathcal{M})} := \left(\sum_{I\in\mathcal{D}^\omega}\sum_{\eta\in\{0,1\}^n\setminus\{0\}}\left(\frac{\|\langle H_I^\eta, b\rangle\|_{L_p(\mathcal{M})}}{|I|^{1/2}}\right)^p\right)^{1/p} < \infty. \tag{5.1.12}$$

In addition, Theorem II.2, Lemma 5.1.1, Proposition 5.1.4 and Proposition 5.1.5 also hold for the dyadic system on $\mathbb{R}^n$ with $d = 2^n$ since our proof only depends on the martingale structure and martingale differences, and does not depend on the dimension of the Euclidean space.

The following lemma shows that $\|b\|_{\boldsymbol{B}_p^{\omega,2^n}(\mathbb{R}^n,\mathcal{M})}$ can be dominated by $\|b\|_{\boldsymbol{B}_p(\mathbb{R}^n,L_p(\mathcal{M}))}$.

Lemma 5.1.8. *Let $1 \le p < \infty$. If $b \in \boldsymbol{B}_p(\mathbb{R}^n, L_p(\mathcal{M}))$, then $b \in \boldsymbol{B}_p^{\omega,2^n}(\mathbb{R}^n, \mathcal{M})$ and $\|b\|_{\boldsymbol{B}_p^{\omega,2^n}(\mathbb{R}^n,\mathcal{M})} \lesssim_{n,p} \|b\|_{\boldsymbol{B}_p(\mathbb{R}^n,L_p(\mathcal{M}))}$.*

Proof. Without loss of generality, assume $\omega = 0$. For any given $J \in \mathcal{D}^0$ and $\eta \in \{0,1\}^n \backslash \{0\}$,

$$\begin{aligned}\frac{\|\langle H_J^\eta, b\rangle\|_{L_p(\mathcal{M})}}{|J|^{1/2}} &= \frac{1}{|J|^{1/2}}\left\|\left\langle H_J^\eta, b - \left\langle \frac{\mathbb{1}_J}{|J|}, b\right\rangle\right\rangle\right\|_{L_p(\mathcal{M})} \\ &\le \frac{1}{|J|}\int_J \left\|b(x) - \left\langle \frac{\mathbb{1}_J}{|J|}, b\right\rangle\right\|_{L_p(\mathcal{M})} dx \\ &\le \frac{1}{|J|^2}\int_J\int_J \|b(x) - b(y)\|_{L_p(\mathcal{M})} dxdy.\end{aligned}$$

Given $t \in \mathbb{Z}$, $I \in \mathcal{D}_t^0$ and $\eta \in \{0,1\}^n \backslash \{0\}$, by the Hölder inequality, one has

$$\begin{aligned}\sum_{J\in\mathcal{D}^0(I)} \frac{\|\langle H_J^\eta, b\rangle\|_{L_p(\mathcal{M})}^p}{|J|^{p/2}} &\le \sum_{J\in\mathcal{D}^0(I)} \frac{1}{|J|^2}\int_J\int_J \|b(x) - b(y)\|_{L_p(\mathcal{M})}^p dxdy \\ &= \sum_{s=t}^\infty \sum_{J\in\mathcal{D}_s^0(I)} \frac{1}{(2^{n(t-s)}|I|)^2}\int_J\int_J \|b(x) - b(y)\|_{L_p(\mathcal{M})}^p dxdy \\ &= \frac{1}{|I|^2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_I(x,y)\|b(x) - b(y)\|_{L_p(\mathcal{M})}^p dxdy,\end{aligned}$$

where

$$K_I(x,y) = \sum_{s=t}^\infty \sum_{J\in\mathcal{D}_s^0(I)} 2^{2n(s-t)}\mathbb{1}_J(x)\mathbb{1}_J(y).$$

Clearly, if $x \notin I$ or $y \notin I$, then $K_I(x,y) = 0$. On the other hand, suppose that $x, y \in I$, and $|x-y| > \sqrt{n}\ell(J)$ for some $J \in \mathcal{D}_s^0(I)$. Then $\exists 1 \le k \le n$ such that $|x_k - y_k| > \ell(J)$, where x_k is the k-th coordinate of x. We then deduce that $\mathbb{1}_J(x)\mathbb{1}_J(y) = 0$. Hence

$$K_I(x,y) \le \mathbb{1}_I(x)\mathbb{1}_I(y)\sum_{s=t}^{t+\lfloor \log_2(\sqrt{n}\ell(I)/|x-y|)\rfloor} 2^{2n(s-t)} \le \frac{(4n)^n}{4^n-1}\frac{|I|^2}{|x-y|^{2n}}\mathbb{1}_I(x)\mathbb{1}_I(y),$$

where $\lfloor\cdot\rfloor$ is the floor function.

Therefore, for a given $t \in \mathbb{Z}$, we sum up all $I \in \mathcal{D}_t^0$, and obtain

$$\sum_{I\in\mathcal{D}_t^0}\sum_{J\in\mathcal{D}^0(I)}\sum_\eta \frac{\|\langle H_J^\eta, b\rangle\|_{L_p(\mathcal{M})}^p}{|J|^{p/2}} \le \frac{2^p(4n)^n(2^n-1)}{4^n-1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\|b(x)-b(y)\|_{L_p(\mathcal{M})}^p}{|x-y|^{2n}}dxdy.$$

Finally letting $t \to -\infty$, we have

$$\|b\|_{\boldsymbol{B}_p^{0,2^n}(\mathbb{R}^n,\mathcal{M})}^p \lesssim_{n,p} \|b\|_{\boldsymbol{B}_p(\mathbb{R}^n,L_p(\mathcal{M}))}^p.$$

□

5.1.3 Proof of Theorem II.6

The following two lemmas will also be needed for the proof of Theorem II.6. Before formulating them, we introduce some definitions. Let $(e_i)_{i\in\mathbb{N}}$ be the standard orthonormal basis on ℓ_2. For any $A \in L_p(B(\ell_2)\otimes\mathcal{M})$, denote by $A_{i,j}$ the (i,j)-th entry defined as

$$A_{i,j} = \langle e_i, Ae_j\rangle \in L_p(\mathcal{M}).$$

Lemma 5.1.9. *Suppose that $(A_\gamma)_{\gamma\in\Gamma}$ is a net of operators in $L_p(B(\ell_2)\otimes\mathcal{M})$ $(1\le p<\infty)$. Let $A\in L_p(B(\ell_2)\otimes\mathcal{M})$. If for any $i,j\in\mathbb{N}$ and $x\in L_{p'}(\mathcal{M})$*

$$\lim_\gamma \tau\bigg(\Big((A_\gamma)_{i,j}-A_{i,j}\Big)x\bigg)=0,$$

then

$$\|A\|_{L_p(B(\ell_2)\otimes\mathcal{M})}\le\sup_\gamma\|A_\gamma\|_{L_p(B(\ell_2)\otimes\mathcal{M})}.$$

Proof. Note that for any projection $p\in B(\ell_2)$ with finite rank, one has for any $B\in L_{p'}(B(\ell_2)\otimes\mathcal{M})$

$$\mathrm{Tr}\otimes\tau((p\otimes 1_{\mathcal{M}})A(p\otimes 1_{\mathcal{M}})B)=\lim_\gamma \mathrm{Tr}\otimes\tau((p\otimes 1_{\mathcal{M}})A_\gamma(p\otimes 1_{\mathcal{M}})B).$$

By duality, this implies that

$$\|(p\otimes 1_{\mathcal{M}})A(p\otimes 1_{\mathcal{M}})\|_{L_p(B(\ell_2)\otimes\mathcal{M})}\le\sup_\gamma\|(p\otimes 1_{\mathcal{M}})A_\gamma(p\otimes 1_{\mathcal{M}})\|_{L_p(B(\ell_2)\otimes\mathcal{M})}.$$

Therefore, we have

$$\begin{aligned}\|A\|_{L_p(B(\ell_2)\otimes\mathcal{M})}&=\sup_{\substack{p^2=p=p^*\\ \text{finite rank}}}\|(p\otimes 1_{\mathcal{M}})A(p\otimes 1_{\mathcal{M}})\|_{L_p(B(\ell_2)\otimes\mathcal{M})}\\ &\le\sup_{\substack{p^2=p=p^*\\ \text{finite rank}}}\sup_\gamma\|(p\otimes 1_{\mathcal{M}})A_\gamma(p\otimes 1_{\mathcal{M}})\|_{L_p(B(\ell_2)\otimes\mathcal{M})}\\ &\le\sup_\gamma\|A_\gamma\|_{L_p(B(\ell_2)\otimes\mathcal{M})},\end{aligned}$$

as desired. □

Let $T\in B(L_2(\mathbb{R}^n))$. Then $T\otimes Id_{L_2(\mathcal{M})}$ extends to a bounded operator on $L_2(\mathbb{R}^n,L_2(\mathcal{M}))$. We still denote it by T for simplicity. Thus by virtue of continuity and linearity, T satisfies the following properties: for any $f\in\mathcal{S}(L^\infty(\mathbb{R}^n)\otimes\mathcal{M})$ and $x\in\mathcal{M}$

$$T(f)x=T(fx)\quad\text{and}\quad\tau(T(fx))=T(\tau(fx)).\tag{5.1.13}$$

Lemma 5.1.10. *Suppose that $(T_\gamma)_{\gamma\in\Gamma}$ is a bounded net of operators in $B(L_2(\mathbb{R}^n))$. Assume that $(T_\gamma)_{\gamma\in\Gamma}$ converges to $T\in B(L_2(\mathbb{R}^n))$ with respect to the weak operator topology. If $2\le p<\infty$ and $b\in L_p(\mathbb{R}^n,L_p(\mathcal{M}))$, then*

$$\|C_{T,b}\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}\le\sup_\gamma\|C_{T_\gamma,b}\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}.$$

Proof. First we show that for any finite cubes $I,J\subset\mathbb{R}^n$ and $x\in L_{p'}(\mathcal{M})$, one has

$$\lim_\gamma\tau\bigg(\Big\langle\mathbb{1}_I,C_{T_\gamma,b}(\mathbb{1}_J)\Big\rangle x\bigg)=\tau\bigg(\Big\langle\mathbb{1}_I,C_{T,b}(\mathbb{1}_J)\Big\rangle x\bigg).$$

Note that by (5.1.13)

$$\begin{aligned}\tau\bigg(\Big\langle\mathbb{1}_I,C_{T_\gamma,b}(\mathbb{1}_J)\Big\rangle x\bigg)&=\tau\bigg(\Big\langle\mathbb{1}_I,T_\gamma(b\mathbb{1}_J)-bT_\gamma(\mathbb{1}_J)\Big\rangle x\bigg)\\ &=\tau\Big(\Big\langle\mathbb{1}_I,T_\gamma(bx\mathbb{1}_J)\Big\rangle\Big)-\tau\Big(\Big\langle x^*b^*\mathbb{1}_J,T_\gamma(\mathbb{1}_J)\Big\rangle\Big)\\ &=\Big\langle\mathbb{1}_I,T_\gamma(\tau(bx)\mathbb{1}_J)\Big\rangle-\Big\langle\tau(x^*b^*)\mathbb{1}_J,T_\gamma(\mathbb{1}_J)\Big\rangle.\end{aligned}$$

This implies that

$$\lim_{\gamma} \tau\bigg(\Big\langle \mathbb{1}_I, C_{T_\gamma,b}(\mathbb{1}_J)\Big\rangle x\bigg) = \Big\langle \mathbb{1}_I, T(\tau(bx)\mathbb{1}_J)\Big\rangle - \Big\langle \tau(x^*b^*)\mathbb{1}_J, T(\mathbb{1}_J)\Big\rangle$$
$$= \tau\bigg(\Big\langle \mathbb{1}_I, C_{T,b}(\mathbb{1}_J)\Big\rangle x\bigg).$$

Hence, the desired result follows from Lemma 5.1.9. □

The remainder of the section is devoted to the proof of Theorem II.6.

Proof of Theorem II.6. From Proposition 5.1.4, Corollary 5.1.5 and Lemma 5.1.8, we have

$$\|[\pi^\omega_{T1}, M_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})} \lesssim_{n,p} \|T1\|_{BMO(\mathbb{R}^n)}\|b\|_{\boldsymbol{B}_p(\mathbb{R}^n,L_p(\mathcal{M}))}$$

and

$$\|[(\pi^\omega_{T^*1})^*, M_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})} \lesssim_{n,p} \|T^*1\|_{BMO(\mathbb{R}^n)}\|b\|_{\boldsymbol{B}_p(\mathbb{R}^n,L_p(\mathcal{M}))}.$$

Hence by Theorem 5.1.6, it remains to estimate $\|[S^{ij}_\omega, M_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}$ for any $i, j \in \mathbb{N}\cup\{0\}$. By the triangle inequality

$$\|[S^{ij}_\omega, M_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})} \le \|[S^{ij}_\omega, \pi_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})} + \|[S^{ij}_\omega, \Lambda_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}$$
$$+ \|[S^{ij}_\omega, R_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}.$$

Here the operators π_b, Λ_b and R_b are associated with the dyadic system on $\mathcal{D}^\omega$.

From Theorem II.2 and Lemma 5.1.1, we know that

$$\|\pi_b\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})} \lesssim_{n,p} \|b\|_{\boldsymbol{B}_p^{\omega,2^n}(\mathbb{R}^n,\mathcal{M})}, \quad \|\Lambda_b\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})} \lesssim_{n,p} \|b\|_{\boldsymbol{B}_p^{\omega,2^n}(\mathbb{R}^n,\mathcal{M})}.$$

Meanwhile, recall that $S^{ij}_\omega \in B(L_2(\mathbb{R}^n, L_2(\mathcal{M})))$ is with norm at most one. Thus, using Lemma 5.1.8, one gets

$$\|[S^{ij}_\omega, \pi_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})} \lesssim \|S^{ij}_\omega\|\|\pi_b\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M}))}$$
$$\lesssim_{n,p} \|b\|_{\boldsymbol{B}_p^{\omega,2^n}(\mathbb{R}^n,\mathcal{M})} \lesssim_{n,p} \|b\|_{\boldsymbol{B}_p(\mathbb{R}^n,L_p(\mathcal{M}))}.$$

Similarly,

$$\|[S^{ij}_\omega, \Lambda_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})} \lesssim_{n,p} \|b\|_{\boldsymbol{B}_p(\mathbb{R}^n,L_p(\mathcal{M}))}.$$

For any $i, j \in \mathbb{N}\cup\{0\}$, we will show that $\|[S^{ij}_\omega, R_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}$ increases with polynomial growth with respect to i and j uniformly on ω. Then from Theorem 5.1.6 and the triangle inequality, the desired result will follow.

Now we begin to estimate $\|[S^{ij}_\omega, R_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}$. Without loss of generality, we assume $\omega = 0$. Let $\Phi = [S^{ij}_0, R_b]$. Then

$$\Phi(f) = \sum_{K\in\mathcal{D}^0} \sum_{\substack{I,J\in\mathcal{D}^0; I,J\subseteq K \\ \ell(I)=2^{-i}\ell(K) \\ \ell(J)=2^{-j}\ell(K)}} \sum_{\xi,\eta\in\{0,1\}^n\setminus\{0\}} a^{\xi\eta}_{IJK}\langle H^\xi_I, R_b(f)\rangle H^\eta_J - \sum_{k\in\mathbb{Z}} b_{k-1}d_k(S^{ij}_0(f)). \tag{5.1.14}$$

Note that for any $s \in \mathbb{Z}$ and $\xi \in \{0,1\}^n \backslash \{0\}$, if $I \in \mathcal{D}_s^0$, then $d_{s+1} H_I^\xi = H_I^\xi$. Hence, for any $I \in \mathcal{D}_k^0$,

$$\begin{aligned}
\langle H_I^\xi, R_b(f)\rangle &= \left\langle H_I^\xi, \sum_{l\in\mathbb{Z}} b_{l-1} d_l f \right\rangle = \left\langle d_{k+1} H_I^\xi, \sum_{l\in\mathbb{Z}} b_{l-1} d_l f \right\rangle \\
&= \left\langle H_I^\xi, d_{k+1}\left(\sum_{l\in\mathbb{Z}} b_{l-1} d_l f\right)\right\rangle = \langle H_I^\xi, b_k d_{k+1} f\rangle \\
&= \langle H_I^\xi, b_k \mathbb{1}_I d_{k+1} f\rangle = \left\langle H_I^\xi, \left\langle \frac{\mathbb{1}_I}{|I|}, b\right\rangle \mathbb{1}_I d_{k+1} f\right\rangle \\
&= \left\langle \frac{\mathbb{1}_I}{|I|}, b\right\rangle \langle H_I^\xi, d_{k+1} f\rangle = \left\langle \frac{\mathbb{1}_I}{|I|}, b\right\rangle \langle H_I^\xi, f\rangle.
\end{aligned}$$

For the second term in (5.1.14), one has

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} b_{k-1} d_k(S_0^{ij}(f)) &= \sum_{k\in\mathbb{Z}} b_{k-1} d_k\left(\sum_{K\in\mathcal{D}^0} \sum_{\substack{I,J\in\mathcal{D}^0; I,J\subseteq K \\ \ell(I)=2^{-i}\ell(K) \\ \ell(J)=2^{-j}\ell(K)}} \sum_{\xi,\eta} a_{IJK}^{\xi\eta} \langle H_I^\xi, f\rangle H_J^\eta\right) \\
&= \sum_{k\in\mathbb{Z}} \sum_{K\in\mathcal{D}_{k-1-j}^0} \sum_{\substack{I,J\in\mathcal{D}^0; I,J\subseteq K \\ \ell(I)=2^{-i}\ell(K) \\ \ell(J)=2^{-j}\ell(K)}} \sum_{\xi,\eta} a_{IJK}^{\xi\eta} b_{k-1} \langle H_I^\xi, f\rangle H_J^\eta \\
&= \sum_{k\in\mathbb{Z}} \sum_{K\in\mathcal{D}_{k-1-j}^0} \sum_{\substack{I,J\in\mathcal{D}^0; I,J\subseteq K \\ \ell(I)=2^{-i}\ell(K) \\ \ell(J)=2^{-j}\ell(K)}} \sum_{\xi,\eta} a_{IJK}^{\xi\eta} \left\langle \frac{\mathbb{1}_J}{|J|}, b\right\rangle \langle H_I^\xi, f\rangle H_J^\eta \\
&= \sum_{K\in\mathcal{D}^0} \sum_{\substack{I,J\in\mathcal{D}^0; I,J\subseteq K \\ \ell(I)=2^{-i}\ell(K) \\ \ell(J)=2^{-j}\ell(K)}} \sum_{\xi,\eta} a_{IJK}^{\xi\eta} \left\langle \frac{\mathbb{1}_J}{|J|}, b\right\rangle \langle H_I^\xi, f\rangle H_J^\eta.
\end{aligned}$$

Therefore,

$$\Phi(f) = \sum_{K\in\mathcal{D}^0} \sum_{\substack{I,J\in\mathcal{D}^0; I,J\subseteq K \\ \ell(I)=2^{-i}\ell(K) \\ \ell(J)=2^{-j}\ell(K)}} \sum_{\xi,\eta} a_{IJK}^{\xi\eta} \left(\left\langle \frac{\mathbb{1}_I}{|I|}, b\right\rangle - \left\langle \frac{\mathbb{1}_J}{|J|}, b\right\rangle\right) \langle H_I^\xi, f\rangle H_J^\eta =: \sum_{K\in\mathcal{D}^0} B_K(f). \tag{5.1.15}$$

Let $b_{IJ} = \langle \frac{\mathbb{1}_I}{|I|}, b\rangle - \langle \frac{\mathbb{1}_J}{|J|}, b\rangle$.

Since B_{K_1}, B_{K_2} have orthogonal ranges when $K_1 \neq K_2$, we see

$$B_{K_1}^* B_{K_2} = 0, \quad \forall K_1 \neq K_2, K_1, K_2 \in \mathcal{D}^0,$$

which yields $\Phi^*\Phi = \sum_{K\in\mathcal{D}^0} B_K^* B_K$. Note that $\forall f \in L_2(\mathbb{R}^n, L_2(\mathcal{M}))$,

$$B_K^* B_K(f) = \sum_{\substack{I,\tilde{I},J\in\mathcal{D}^0; I,\tilde{I},J\subseteq K \\ \ell(I)=\ell(\tilde{I})=2^{-i}\ell(K) \\ \ell(J)=2^{-j}\ell(K)}} \sum_{\xi,\tilde{\xi},\eta} a_{IJK}^{\xi\eta} \overline{a_{\tilde{I}JK}^{\tilde{\xi}\eta}} b_{\tilde{I}J}^* b_{IJ} \langle H_I^\xi, f\rangle H_{\tilde{I}}^{\tilde{\xi}}, \tag{5.1.16}$$

which implies that $\Phi^*\Phi$ is a block diagonal matrix with blocks $B_K^*B_K$ for all $K \in \mathcal{D}^0$. Consequently, we have

$$\begin{aligned}\|\Phi\|^p_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})} &= \|\Phi^*\Phi\|^{p/2}_{L_{p/2}(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}\\ &= \sum_{k\in\mathbb{Z}}\sum_{K\in\mathcal{D}^0_k}\|B_K^*B_K\|^{p/2}_{L_{p/2}(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}.\end{aligned} \tag{5.1.17}$$

Now we fix $k \in \mathbb{Z}$ and $K \in \mathcal{D}^0_k$. For any $\tilde{Q}, Q \subseteq K$ satisfying $\ell(\tilde{Q}) = \ell(Q) = 2^{-i}\ell(K)$ and $\tilde{\zeta}, \zeta \in \{0,1\}^n\backslash\{0\}$,

$$\begin{aligned}\langle H^{\tilde{\zeta}}_{\tilde{Q}}, B_K^*B_KH^{\zeta}_Q\rangle &= \left\langle H^{\tilde{\zeta}}_{\tilde{Q}}, \sum_{\substack{I,\tilde{I},J\in\mathcal{D}^0;I,\tilde{I},J\subseteq K\\ \ell(I)=\ell(\tilde{I})=2^{-i}\ell(K)\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\xi,\tilde{\xi},\eta} a^{\xi\eta}_{IJK}\overline{a^{\tilde{\xi}\eta}_{\tilde{I}JK}}b^*_{\tilde{I}J}b_{IJ}\langle H^{\xi}_I, H^{\zeta}_Q\rangle H^{\tilde{\xi}}_{\tilde{I}}\right\rangle\\ &= \sum_{\substack{J\in\mathcal{D}^0;J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\eta} a^{\zeta\eta}_{QJK}\overline{a^{\tilde{\zeta}\eta}_{\tilde{Q}JK}}b^*_{\tilde{Q}J}b_{QJ}\\ &:= \sum_{\substack{J\in\mathcal{D}^0;J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\eta} W^{K,J,\eta}_{(\tilde{Q},\tilde{\zeta}),(Q,\zeta)}.\end{aligned}$$

Denote by

$$[B_K^*B_K] = \left((B_K^*B_K)_{(\tilde{Q},\tilde{\zeta}),(Q,\zeta)}\right)_{\tilde{Q},Q\in\mathcal{D}^0;\tilde{Q},Q\subseteq K,\ell(\tilde{Q})=\ell(Q)=2^{-i}\ell(K),\tilde{\zeta},\zeta\in\{0,1\}^n\backslash\{0\}}$$

the matrix form of $B_K^*B_K$ with respect to the basis $\{H^{\zeta}_Q\}_{Q\in\mathcal{D}^0;Q\subseteq K,\ell(Q)=2^{-i}\ell(K),\zeta\in\{0,1\}^n\backslash\{0\}}$, where $(B_K^*B_K)_{(\tilde{Q},\tilde{\zeta}),(Q,\zeta)} = \langle H^{\tilde{\zeta}}_{\tilde{Q}}, B_K^*B_KH^{\zeta}_Q\rangle$. We also denote the $2^{in}(2^n-1)\times 2^{in}(2^n-1)$ matrix by

$$W^{K,J,\eta} = \left(W^{K,J,\eta}_{(\tilde{Q},\tilde{\zeta}),(Q,\zeta)}\right)_{\tilde{Q},Q\in\mathcal{D}^0;\tilde{Q},Q\subseteq K,\ell(\tilde{Q})=\ell(Q)=2^{-i}\ell(K),\tilde{\zeta},\zeta\in\{0,1\}^n\backslash\{0\}}. \tag{5.1.18}$$

Then recalling that $p \geq 2$ and using the triangle inequality, one has

$$\begin{aligned}\|B_K^*B_K\|_{L_{p/2}(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})} &= \|[B_K^*B_K]\|_{L_{p/2}(\mathbb{M}_{2^{in}(2^n-1)}\otimes\mathcal{M})}\\ &\leq \sum_{\substack{J\in\mathcal{D}^0;J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\eta}\left\|W^{K,J,\eta}\right\|_{L_{p/2}(\mathbb{M}_{2^{in}(2^n-1)}\otimes\mathcal{M})}.\end{aligned}$$

Hence by (5.1.17)

$$\|\Phi\|^p_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})} \leq \sum_{k\in\mathbb{Z}}\sum_{K\in\mathcal{D}^0_k}\left(\sum_{\substack{J\in\mathcal{D}^0;J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\eta}\left\|W^{K,J,\eta}\right\|_{L_{p/2}(\mathbb{M}_{2^{in}(2^n-1)}\otimes\mathcal{M})}\right)^{p/2}. \tag{5.1.19}$$

Define the $2^{in}(2^n-1)\times 2^{in}(2^n-1)$ matrix $V^{K,J,\eta}$, the only non-zero row of which is the first row, as a row vector $\left(a^{\zeta\eta}_{QJK}b_{QJ}\right)_{Q\in\mathcal{D}^0;Q\subseteq K,\ell(Q)=2^{-i}\ell(K),\zeta\in\{0,1\}^n\backslash\{0\}}$. Thus

$$W^{K,J,\eta} = \left(V^{K,J,\eta}\right)^*V^{K,J,\eta}. \tag{5.1.20}$$

Consequently,

$$\begin{aligned}\left\|W^{K,J,\eta}\right\|_{L_{p/2}(\mathbb{M}_{2^{in}(2^n-1)}\otimes\mathcal{M})} &= \left\|V^{K,J,\eta}\left(V^{K,J,\eta}\right)^*\right\|_{L_{p/2}(\mathbb{M}_{2^{in}(2^n-1)}\otimes\mathcal{M})} \\ &= \left\| \sum_{\substack{Q\in\mathcal{D}^0;Q\subseteq K\\ \ell(Q)=2^{-i}\ell(K)\\ \zeta\in\{0,1\}^n\setminus\{0\}}} a^{\zeta\eta}_{QJK}\overline{a^{\zeta\eta}_{QJK}}b_{QJ}b^*_{QJ}\right\|_{L_{p/2}(\mathcal{M})} \\ &\le \sum_{\substack{Q\in\mathcal{D}^0;Q\subseteq K\\ \ell(Q)=2^{-i}\ell(K)\\ \zeta\in\{0,1\}^n\setminus\{0\}}} \left\|a^{\zeta\eta}_{QJK}\overline{a^{\zeta\eta}_{QJK}}b_{QJ}b^*_{QJ}\right\|_{L_{p/2}(\mathcal{M})} \\ &= \sum_{\substack{Q\in\mathcal{D}^0;Q\subseteq K\\ \ell(Q)=2^{-i}\ell(K)\\ \zeta\in\{0,1\}^n\setminus\{0\}}} \left\|a^{\zeta\eta}_{QJK}b_{QJ}\right\|^2_{L_p(\mathcal{M})}.\end{aligned} \tag{5.1.21}$$

Together with (5.1.19),

$$\|\Phi\|^p_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})} \le \sum_{k\in\mathbb{Z}}\sum_{K\in\mathcal{D}^0_k}\left(\sum_{\substack{J\in\mathcal{D}^0;J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\eta}\sum_{\substack{I\in\mathcal{D}^0;I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}}\sum_{\xi}\left\|a^{\xi\eta}_{IJK}b_{IJ}\right\|^2_{L_p(\mathcal{M})}\right)^{p/2}.$$

Note that $|a^{\xi\eta}_{IJK}| \le 2^{-(i+j)n/2}$, then we estimate

$$\sum_{\substack{J\in\mathcal{D}^0;J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\eta}\sum_{\substack{I\in\mathcal{D}^0;I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}}\sum_{\xi}\left\|a^{\xi\eta}_{IJK}b_{IJ}\right\|^2_{L_p(\mathcal{M})} \le \frac{(2^n-1)^2}{2^{(i+j)n}}\sum_{\substack{J\in\mathcal{D}^0;J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\substack{I\in\mathcal{D}^0;I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}}\|b_{IJ}\|^2_{L_p(\mathcal{M})}.$$

Since $b_{IJ} = b_{IK} - b_{JK}$, by the triangle inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\sum_{\substack{J\in\mathcal{D}^0;J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\eta}\sum_{\substack{I\in\mathcal{D}^0;I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}}\sum_{\xi}\left\|a^{\xi\eta}_{IJK}b_{IJ}\right\|^2_{L_p(\mathcal{M})} \\ &\le \frac{(2^n-1)^2}{2^{(i+j)n}}\sum_{\substack{J\in\mathcal{D}^0;J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\substack{I\in\mathcal{D}^0;I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}}2(\|b_{IK}\|^2_{L_p(\mathcal{M})}+\|b_{JK}\|^2_{L_p(\mathcal{M})}) \\ &= \frac{(2^n-1)^2}{2^{in-1}}\sum_{\substack{I\in\mathcal{D}^0;I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}}\|b_{IK}\|^2_{L_p(\mathcal{M})} + \frac{(2^n-1)^2}{2^{jn-1}}\sum_{\substack{J\in\mathcal{D}^0;J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\|b_{JK}\|^2_{L_p(\mathcal{M})}.\end{aligned} \tag{5.1.22}$$

Note that $b_{IK}\cdot \mathbb{1}_I=(b_{k+i}-b_k)\cdot \mathbb{1}_I$, and sum all I and J, one has

$$\begin{aligned}
&\sum_{\substack{J\in\mathcal{D}^0;J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\eta}\sum_{\substack{I\in\mathcal{D}^0;I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}}\sum_{\xi}\left\|a_{IJK}^{\xi\eta}b_{IJ}\right\|_{L_p(\mathcal{M})}^2\\
&\le (2^n-1)^2 2^{kn+1}\bigg(\|(b_{k+i}-b_k)\mathbb{1}_K\|_{L_2(\mathbb{R}^n,L_p(\mathcal{M}))}^2+\|(b_{k+j}-b_k)\mathbb{1}_K\|_{L_2(\mathbb{R}^n,L_p(\mathcal{M}))}^2\bigg)\\
&\le (2^n-1)^2 2^{1+2nk/p}\bigg(i\sum_{l=k+1}^{k+i}\|d_l b\cdot\mathbb{1}_K\|_{L_p(\mathbb{R}^n,L_p(\mathcal{M}))}^2+j\sum_{l=k+1}^{k+j}\|d_l b\cdot\mathbb{1}_K\|_{L_p(\mathbb{R}^n,L_p(\mathcal{M}))}^2\bigg).
\end{aligned}\tag{5.1.23}$$

Hence using the convex inequality, we obtain

$$\begin{aligned}
&\|N\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}^p\\
&\le (2^n-1)^p 2^p\sum_{k\in\mathbb{Z}}\sum_{K\in\mathcal{D}_k^0}2^{nk}\bigg(i^{p-1}\sum_{l=k+1}^{k+i}\|d_l b\cdot\mathbb{1}_K\|_{L_p(\mathbb{R}^n,L_p(\mathcal{M}))}^p+j^{p-1}\sum_{l=k+1}^{k+j}\|d_l b\cdot\mathbb{1}_K\|_{L_p(\mathbb{R}^n,L_p(\mathcal{M}))}^p\bigg)\\
&=(2^n-1)^p 2^p(i^p+j^p)\sum_{k\in\mathbb{Z}}2^{nk}\|d_k b\|_{L_p(\mathbb{R}^n,L_p(\mathcal{M}))}^p\\
&\lesssim_{n,p}(i^p+j^p)\|b\|_{\boldsymbol{B}_p^{0,2^n}(\mathbb{R}^n,\mathcal{M})}^p\le(i^p+j^p)\|b\|_{\boldsymbol{B}_p(\mathbb{R}^n,L_p(\mathcal{M}))}^p.
\end{aligned}$$

Since the above estimation is independent of the choose of ω, one has

$$\|[S_\omega^{ij},R_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}\lesssim_{n,p}(i^p+j^p)^{1/p}\|b\|_{\boldsymbol{B}_p(\mathbb{R}^n,L_p(\mathcal{M}))},$$

which yields

$$\|[S_\omega^{ij},M_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}\lesssim_{n,p}(i^p+j^p+1)^{1/p}\|b\|_{\boldsymbol{B}_p(\mathbb{R}^n,L_p(\mathcal{M}))}.$$

Therefore by Lemma 5.1.10 and the triangle inequality,

$$\begin{aligned}
&\|[T,M_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}\\
&=\left\|\left[C_1\mathbb{E}_\omega\sum_{\substack{i,j=0\\ \max\{i,j\}>0}}^{\infty}2^{-\max\{i,j\}\alpha/2}S_\omega^{ij}+C_2\mathbb{E}_\omega S_\omega^{00}+\mathbb{E}_\omega\pi_{T1}^\omega+\mathbb{E}_\omega(\pi_{T^*1}^\omega)^*,M_b\right]\right\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}\\
&\lesssim\sum_{i,j=0}^{\infty}2^{-\max\{i,j\}\alpha/2}\mathbb{E}_\omega\|[S_\omega^{ij},M_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}+\mathbb{E}_\omega\|[\pi_{T1}^\omega+(\pi_{T^*1}^\omega)^*,M_b]\|_{L_p(B(L_2(\mathbb{R}^n))\otimes\mathcal{M})}\\
&\lesssim_{n,p}\Big(1+\|T1\|_{BMO(\mathbb{R}^n)}+\|T^*1\|_{BMO(\mathbb{R}^n)}\Big)\|b\|_{\boldsymbol{B}_p(\mathbb{R}^n,L_p(\mathcal{M}))}.
\end{aligned}$$

This completes the proof of Theorem II.6. □

5.1.4 Comparison between Theorem II.5 and Theorem II.6

From our proof of Theorem II.6, we see that when $p\ge 2$ and $\mathcal{M}=\mathbb{C}$, one always has

$$\|[T,M_b]\|_{S_p(L_2(\mathbb{R}^n))}^p\lesssim_{n,p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|b(x)-b(y)|^p}{|x-y|^{2n}}dxdy.$$

However, this does not contradict with Theorem II.5 for $p\le n$ and $n\ge 2$ due to the following fact.

Proposition 5.1.11. *Let $0 < p \leq n$. Then b is constant if*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|b(x)-b(y)|^p}{|x-y|^{2n}}dxdy < \infty.$$

Proof. By changing the variables, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|b(x)-b(y)|^p}{|x-y|^{2n}}dxdy = \int_{\mathbb{R}^n} \frac{\|b(x+t)-b(x)\|^p_{L_p(\mathbb{R}^n)}}{|t|^{2n}}dt.$$

We proceed with the proof by contradiction. Assume that b is not constant. Then there exists $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $b*\varphi \in C^\infty(\mathbb{R}^n)$ is not constant either. Since by the Young inequality

$$\|\varphi * b(x+t) - \varphi * b(x)\|^p_{L_p(\mathbb{R}^n)} \leq \|\varphi\|^p_{L_1(\mathbb{R}^n)}\|b(x+t)-b(x)\|^p_{L_p(\mathbb{R}^n)},$$

we get

$$\int_{\mathbb{R}^n} \frac{\|\varphi * b(x+t) - \varphi * b(x)\|^p_{L_p(\mathbb{R}^n)}}{|t|^{2n}}dt < \infty.$$

Hence we can assume that $b \in C^\infty(\mathbb{R}^n)$, otherwise we replace b with $b*\varphi$.

Since b is not constant, there exists $\tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_n) \in \mathbb{R}^n$, such that $\nabla b(\tilde{x}) \neq 0$. Let U be a unitary matrix in $\mathbb{M}_n$ such that $\nabla b(\tilde{x}) \cdot U = (|\nabla b(\tilde{x})|, 0, \cdots, 0)$. We substitute $\tilde{b}(y) := b(y \cdot U)$ for b. So we can also assume that there exists $\tilde{x} \in \mathbb{R}^n$ with $\nabla b(\tilde{x}) = (M, 0, \cdots, 0)$ and $M > 0$.

Since $b \in C^\infty(\mathbb{R}^n)$, $\exists\ \delta > 0$ such that $\forall\ |y - \tilde{x}| < 2\delta$ with

$$|\nabla b(y) - \nabla b(\tilde{x})| \leq \frac{M}{4}.$$

Thus for any $|x - \tilde{x}| < \delta$ and $|t| < \delta$ with $|t_1| > \frac{|t|}{2}$, by the mean value theorem,

$$\begin{aligned} |b(x+t) - b(x)| &= |\nabla b(x + \theta \cdot t) \cdot t| \quad (0 < \theta < 1) \\ &\geq |\nabla b(\tilde{x}) \cdot t| - |\Big(\nabla b(x+\theta\cdot t) - \nabla b(\tilde{x})\Big) \cdot t| \\ &\geq M|t_1| - \frac{M|t|}{4} \geq \frac{M|t_1|}{2}. \end{aligned}$$

This yields that

$$\|b(x+t) - b(x)\|^p_{L_p(\mathbb{R}^n)} \gtrsim_{n,p} \delta^n M^p |t_1|^p.$$

Consequently, one has

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\|b(x+t)-b(x)\|^p_{L_p(\mathbb{R}^n)}}{|t|^{2n}}dt &\gtrsim_{n,p} \int\limits_{\substack{t\in\mathbb{R}^n, |t|<\delta \\ |t_1|>\frac{|t|}{2}}} \frac{|t_1|^p}{|t|^{2n}}dt \\ &\gtrsim_{n,p} \int_0^\delta \frac{r^p}{r^{2n}} \cdot r^{n-1}dr \\ &= \int_0^\delta \frac{1}{r^{n+1-p}}dr = \infty. \end{aligned}$$

This leads to a contradiction. □

5.2 Proof of Theorem II.7

We will follow the same route as the argument for Theorem II.6. In particular, we include theorems regarding the boundedness of commutators involving martingale paraproducts and related operators. Via Hytönen's dyadic martingale technique, we will derive Theorem II.7.

From (1.4.2) we see that

$$\|b\|_{BMO^d(\mathbb{R})} = \sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/2}} \left(\sum_{J\subseteq I} \sum_{i=1}^{d-1} |\langle h_J^i, b\rangle|^2 \right)^{1/2}.$$

In [18], Chao and Peng showed that for $1 < p < \infty$, π_b is bounded from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$ if and only if $b \in BMO^d(\mathbb{R})$. See the detailed proof in Theorem A.0.1 and Remark A.2.5 in Appendix A.

Recall that the operator Λ_b is introduced in Lemma 5.1.1.

Lemma 5.2.1. *Let $1 < p < \infty$. If $b \in BMO^d(\mathbb{R})$, then Λ_b is bounded on $L_p(\mathbb{R})$.*

Proof. We use the same notation as that in Lemma 5.1.1, and the proof of this lemma is also similar to that of Lemma 5.1.1. From (5.1.1), we write Λ_b as follows:

$$\Lambda_b(f) = (\pi_{b^*})^*(f) + \tilde{\Lambda}_b(f), \quad \forall f \in L_2(\mathbb{R}), \tag{5.2.1}$$

where

$$\tilde{\Lambda}_b(f) = \sum_{k\in\mathbb{Z}} \sum_{I\in\mathcal{D}_{k-1}} \sum_{l=1}^{d-1} \sum_{\overline{i+j}=l} \langle h_I^i, b\rangle \langle h_I^j, f\rangle \frac{h_I^l}{|I|^{1/2}}. \tag{5.2.2}$$

It has been shown in [18] that

$$\|(\pi_{b^*})^*\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})} = \|\pi_{b^*}\|_{L_{p'}(\mathbb{R})\to L_{p'}(\mathbb{R})} \approx_{d,p} \|b\|_{BMO^d(\mathbb{R})}. \tag{5.2.3}$$

It remains to estimate $\|\tilde{\Lambda}_b\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})}$.

At first, we show the boundedness of $\tilde{\Lambda}_b$ for $p = 2$. Since Λ_b is a block diagonal matrix with respect to the basis $\{h_I^i\}_{I\in\mathcal{D},1\le i\le d-1}$, one has

$$\|\tilde{\Lambda}_b\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})} = \sup_{I\in\mathcal{D}} \left\| \left(a_{s-t}^I \right)_{1\le s,t\le d-1} \right\|_{S_\infty(\mathbb{M}_{d-1})},$$

where $a_{s-t}^I = |I|^{-1/2}\langle h_I^{s-t}, b\rangle$, and $a_0^I = 0$. We have

$$\left\| \left(a_{s-t}^I \right)_{1\le s,t\le d-1} \right\|_{S_\infty(\mathbb{M}_{d-1})} \le \|B^I\|_{S_\infty(\mathbb{M}_d)},$$

which implies

$$\|\tilde{\Lambda}_b\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})} \le \sup_{I\in\mathcal{D}} \|B^I\|_{S_\infty(\mathbb{M}_d)},$$

where B^I is defined in (5.1.1). Thus by using the triangle inequality, one has

$$\begin{aligned}\|\tilde{\Lambda}_b\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})} &\le \sup_{I\in\mathcal{D}} \|a_1^I A + a_2^I A^2 + \cdots + a_{d-1}^I A^{d-1}\|_{S_\infty(\mathbb{M}_d)} \\ &\le \sup_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \|a_i^I A^i\|_{S_\infty(\mathbb{M}_d)} \le \sup_{I\in\mathcal{D}} \sum_{i=1}^{d-1} |a_i^I| \\ &= \sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/2}} \sum_{i=1}^{d-1} |\langle h_I^i, b\rangle| \le (d-1)\|b\|_{BMO^d(\mathbb{R})}.\end{aligned} \tag{5.2.4}$$

From (5.2.3) and (5.2.4), we obtain that $\tilde{\Lambda}_b$ is bounded on $L_2(\mathbb{R})$.

Next, we prove that $\tilde{\Lambda}_b$ satisfies weak type (1,1). Assume $f \in L_1(\mathbb{R})$ and let $\lambda > 0$. In the same way as in [70, Lemma 2.7], we have the Calderón-Zygmund decomposition $f = g + h$ with

1. $\|g\|_{L_\infty(\mathbb{R})} \le d\lambda$, $\|g\|_{L_1(\mathbb{R})} \le \|f\|_{L_1(\mathbb{R})}$;
2. $h = \sum_j h_j$, where $h_j = \left(f - \left\langle \frac{\mathbb{1}_{I_j}}{|I_j|}, f\right\rangle\right)\mathbb{1}_{I_j} = \sum_{J\subseteq I_j}\sum_{l=1}^{d-1}\langle h_J^l, f\rangle h_J^l$ and $\{I_j\}$ form a sequence of disjoint d-adic intervals such that $\sum_j |I_j| \le \frac{\|f\|_{L_1(\mathbb{R})}}{\lambda}$.

We see that $\tilde{\Lambda}_b$ is of strong type (2,2). In particular, (5.2.4) implies that

$$\begin{aligned}|\{|\tilde{\Lambda}_b(g)| > \lambda/2\}| &\le 4(d-1)\|b\|_{BMO^d(\mathbb{R})}\frac{\|g\|_{L_2(\mathbb{R})}^2}{\lambda^2} \\ &\le 4(d-1)\|b\|_{BMO^d(\mathbb{R})}\frac{\|g\|_{L_\infty(\mathbb{R})}\|f\|_{L_1(\mathbb{R})}}{\lambda^2} \\ &\le 4d(d-1)\|b\|_{BMO^d(\mathbb{R})}\frac{\|f\|_{L_1(\mathbb{R})}}{\lambda}.\end{aligned} \tag{5.2.5}$$

On the other hand, from (5.2.2) we deduce that $\mathrm{supp}\tilde{\Lambda}(h_j) \subseteq I_j$, and

$$|\{|\tilde{\Lambda}_b(h)| > \lambda/2\}| \le |\cup_j I_j| \le \frac{\|f\|_{L_1(\mathbb{R})}}{\lambda}. \tag{5.2.6}$$

Then from (5.2.5) and (5.2.6), we conclude that

$$\begin{aligned}|\{|\tilde{\Lambda}_b(f)| > \lambda\}| &\le |\{|\tilde{\Lambda}_b(g)| > \lambda/2\}| + |\{|\tilde{\Lambda}_b(h)| > \lambda/2\}| \\ &\le (4d(d-1)\|b\|_{BMO^d(\mathbb{R})} + 1)\frac{\|f\|_{L_1(\mathbb{R})}}{\lambda}.\end{aligned}$$

Hence $\tilde{\Lambda}_b$ is of weak type (1,1). Using interpolation and duality argument, we obtain that $\tilde{\Lambda}_b$ is bounded on $L_p(\mathbb{R})$ for $1 < p < \infty$. □

Remark 5.2.2. There is another easy proof to show that Λ_b is bounded on $L_p(\mathbb{R})$ when $1 < p \ne 2 < \infty$. By the duality between the d-adic martingale Hardy space $H_1^d(\mathbb{R})$ (see the definition below in (1.4.3)) and the d-adic martingale BMO space $BMO^d(\mathbb{R})$, we see that if $b \in BMO^d(\mathbb{R})$, then Λ_b is bounded from $H_1^d(\mathbb{R})$ to $L_1(\mathbb{R})$. Using the boundedness of Λ_b on $L_2(\mathbb{R})$ and by interpolation, we conclude that Λ_b is bounded on $L_p(\mathbb{R})$ for $1 < p \le 2$. The boundedness of Λ_b on $L_p(\mathbb{R})$ for $2 \le p < \infty$ follows from the duality.

We now provide the following useful lemma so as to prove Proposition 5.2.4.

Lemma 5.2.3. *Let $1 < p < \infty$, $f \in L_p(\mathbb{R})$ and $b \in BMO^d(\mathbb{R})$. Then*

$$\Big\| \sup_{k\in\mathbb{Z}} \Big| \mathbb{E}_{k-1}\Big(\sum_{j\geq k} d_j b \cdot d_j f \Big) \Big| \Big\|_{L_p(\mathbb{R})} \lesssim_p \|b\|_{BMO^d(\mathbb{R})} \|f\|_{L_p(\mathbb{R})}.$$

Proof. By the Hölder inequality, one has

$$\begin{aligned} \left| \mathbb{E}_{k-1}\left(\sum_{j\geq k} d_j b \cdot d_j f \right) \right| &= |\mathbb{E}_{k-1}(b - b_{k-1})(f - f_{k-1})| \\ &\leq \Big(\mathbb{E}_{k-1} |b - b_{k-1}|^q \Big)^{1/q} \cdot \Big(\mathbb{E}_{k-1} |f - f_{k-1}|^{q'} \Big)^{1/q'}, \end{aligned}$$

where $q = \frac{2p}{p-1}$ and $\frac{1}{q} + \frac{1}{q'} = 1$. From the martingale John-Nirenberg inequality in Remark 1.3.11, we have

$$\|b\|_{BMO^d(\mathbb{R})} \approx_q \sup_{k\in\mathbb{Z}} \Big\| \mathbb{E}_{k-1} |b - b_{k-1}|^q \Big\|_\infty^{1/q}. \tag{5.2.7}$$

Hence by (5.2.7),

$$\begin{aligned} &\Big\| \sup_{k\in\mathbb{Z}} \Big| \mathbb{E}_{k-1}\left(\sum_{j\geq k} d_j b \cdot d_j f \right) \Big| \Big\|_{L_p(\mathbb{R})} \\ &\leq \Big\| \sup_{k\in\mathbb{Z}} \Big(\mathbb{E}_{k-1} |b - b_{k-1}|^q \Big)^{1/q} \cdot \sup_{k\in\mathbb{Z}} \Big(\mathbb{E}_{k-1} |f - f_{k-1}|^{q'} \Big)^{1/q'} \Big\|_{L_p(\mathbb{R})} \\ &\lesssim_p \|b\|_{BMO^d(\mathbb{R})} \cdot \Big\| \sup_{k\in\mathbb{Z}} \Big(\mathbb{E}_{k-1} |f - f_{k-1}|^{q'} \Big)^{1/q'} \Big\|_{L_p(\mathbb{R})}. \end{aligned}$$

Note that $|f|^{q'} \in L_{p/q'}(\mathbb{R})$ and $p/q' > 1$, by the Doob maximal inequality,

$$\begin{aligned} \Big\| \sup_{k\in\mathbb{Z}} \Big(\mathbb{E}_{k-1} |f - f_{k-1}|^{q'} \Big)^{1/q'} \Big\|_{L_p(\mathbb{R})} &\lesssim_p \Big\| \sup_{k\in\mathbb{Z}} \Big(\mathbb{E}_{k-1} |f|^{q'} \Big)^{1/q'} \Big\|_{L_p(\mathbb{R}^n)} + \Big\| \sup_{k\in\mathbb{Z}} |f_{k-1}| \Big\|_{L_p(\mathbb{R})} \\ &\lesssim_p \|f\|_{L_p(\mathbb{R})}. \end{aligned}$$

Therefore

$$\Big\| \sup_{k\in\mathbb{Z}} \Big| \mathbb{E}_{k-1}\left(\sum_{j\geq k} d_j b \cdot d_j f \right) \Big| \Big\|_{L_p(\mathbb{R})} \lesssim_p \|b\|_{BMO^d(\mathbb{R})} \|f\|_{L_p(\mathbb{R})},$$

as desired. □

Before proving Theorem II.7, we give the following two propositions concerning the boundedness of commutators involving martingale paraproducts.

Proposition 5.2.4. *Let $1 < p < \infty$. If $a, b \in BMO^d(\mathbb{R})$, then $[\pi_a, M_b]$ is bounded on $L_p(\mathbb{R})$ and*

$$\|[\pi_a, M_b]\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})} \lesssim_{d,p} \|a\|_{BMO^d(\mathbb{R})} \|b\|_{BMO^d(\mathbb{R})}.$$

Proof. Recall that R_b is defined in (5.1.3). We will first focus on the estimate of the norm $\|[\pi_a, R_b]\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})}$. By (5.1.4) and (5.1.7), one has $\forall f \in L_p(\mathbb{R})$

$$[\pi_a, R_b](f) = -\pi_a(\Lambda_b(f)) + \sum_{k\in\mathbb{Z}} d_k a \cdot \mathbb{E}_{k-1}\bigg(\sum_{j\geq k} d_j b \cdot d_j f\bigg) - \pi_a(\pi_b(f)).$$

Define

$$V_{a,b}(f) = \sum_{k\in\mathbb{Z}} d_k a \cdot \mathbb{E}_{k-1}\bigg(\sum_{j\geq k} d_j b \cdot d_j f\bigg), \qquad \forall f \in L_p(\mathbb{R}) \tag{5.2.8}$$

and

$$\Theta_b = \pi_b + \Lambda_b. \tag{5.2.9}$$

Then by the above calculations,

$$[\pi_a, R_b] = -\pi_a\Theta_b + V_{a,b}. \tag{5.2.10}$$

By Lemma 5.2.1, one has

$$\begin{aligned}\|\pi_a\Theta_b\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})} &\leq \|\pi_a\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R})}\Big(\|\pi_b\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})} + \|\Lambda_b\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})}\Big)\\ &\lesssim_{d,p} \|a\|_{BMO^d(\mathbb{R})}\|b\|_{BMO^d(\mathbb{R})}.\end{aligned} \tag{5.2.11}$$

For any $f \in L_p(\mathbb{R})$ and $g \in L_{p'}(\mathbb{R})$,

$$\begin{aligned}\langle V_{a,b}(f), g\rangle &= \sum_{k\in\mathbb{Z}}\bigg\langle d_k a \cdot \mathbb{E}_{k-1}\bigg(\sum_{j\geq k} d_j b \cdot d_j f\bigg), g\bigg\rangle\\ &= \sum_{k\in\mathbb{Z}}\bigg\langle d_k a, d_k g \cdot \mathbb{E}_{k-1}\bigg(\sum_{j\geq k} d_j \overline{b} \cdot d_j \overline{f}\bigg)\bigg\rangle\\ &= \bigg\langle a, \sum_{k\in\mathbb{Z}} d_k g \cdot \mathbb{E}_{k-1}\bigg(\sum_{j\geq k} d_j \overline{b} \cdot d_j \overline{f}\bigg)\bigg\rangle.\end{aligned}$$

To use duality, we need to estimate the following

$$\begin{aligned}\bigg\|\sum_{k\in\mathbb{Z}} d_k g \cdot \mathbb{E}_{k-1}\bigg(\sum_{j\geq k} d_j \overline{b} \cdot d_j \overline{f}\bigg)\bigg\|_{H_1^d(\mathbb{R})} &= \bigg\|\bigg(\sum_{k\in\mathbb{Z}} |d_k g|^2 \bigg|\mathbb{E}_{k-1}\bigg(\sum_{j\geq k} d_j b \cdot d_j f\bigg)\bigg|^2\bigg)^{1/2}\bigg\|_{L_1(\mathbb{R})}\\ &\leq \bigg\|\bigg(\sum_{k\in\mathbb{Z}} |d_k g|^2\bigg)^{1/2} \cdot \sup_{k\in\mathbb{Z}}\bigg|\mathbb{E}_{k-1}\bigg(\sum_{j\geq k} d_j b \cdot d_j f\bigg)\bigg|\bigg\|_{L_1(\mathbb{R})}\\ &\leq \|S(g)\|_{L_{p'}(\mathbb{R})} \cdot \bigg\|\sup_{k\in\mathbb{Z}}\bigg|\mathbb{E}_{k-1}\bigg(\sum_{j\geq k} d_j b \cdot d_j f\bigg)\bigg|\bigg\|_{L_p(\mathbb{R})}\\ &\lesssim_p \|b\|_{BMO^d(\mathbb{R})}\|g\|_{L_{p'}(\mathbb{R})}\|f\|_{L_p(\mathbb{R})},\end{aligned} \tag{5.2.12}$$

where the third inequality is from the Hölder inequality, and the fourth is from Lemma 5.2.3. Therefore

$$\begin{aligned}|\langle V_{a,b}(f), g\rangle| &\lesssim \|a\|_{BMO^d(\mathbb{R})}\bigg\|\sum_{k\in\mathbb{Z}} d_k g \cdot \mathbb{E}_{k-1}\bigg(\sum_{j\geq k} d_j \overline{b} \cdot d_j \overline{f}\bigg)\bigg\|_{H_1^d(\mathbb{R})}\\ &\lesssim_p \|a\|_{BMO^d(\mathbb{R})}\|b\|_{BMO^d(\mathbb{R})}\|g\|_{L_{p'}(\mathbb{R})}\|f\|_{L_p(\mathbb{R})}.\end{aligned}$$

This implies that

$$\|V_{a,b}\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})} \lesssim_p \|a\|_{BMO^d(\mathbb{R})}\|b\|_{BMO^d(\mathbb{R})}. \tag{5.2.13}$$

From (5.2.11) and (5.2.13) we have

$$\|[\pi_a, R_b]\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})} \lesssim_{d,p} \|a\|_{BMO^d(\mathbb{R})}\|b\|_{BMO^d(\mathbb{R})}.$$

Recall that

$$[\pi_a, M_b] = [\pi_a, \pi_b] + [\pi_a, \Lambda_b] + [\pi_a, R_b].$$

Since π_a is bounded on $L_p(\mathbb{R})$, by the triangle inequality we deduce that

$$\|[\pi_a, M_b]\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})} \lesssim_{d,p} \|a\|_{BMO^d(\mathbb{R})}\|b\|_{BMO^d(\mathbb{R})}.$$

This completes the proof. □

Corollary 5.2.5. *Let $1 < p < \infty$. If $a, b \in BMO^d(\mathbb{R})$, then $[\pi_a^*, M_b]$ is bounded on $L_p(\mathbb{R})$ and*

$$\|[\pi_a^*, M_b]\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})} \lesssim_{d,p} \|a\|_{BMO^d(\mathbb{R})}\|b\|_{BMO^d(\mathbb{R})}.$$

Proof. Recall that

$$[\pi_a^*, M_b]^* = -[\pi_a, M_{b^*}].$$

By Proposition 5.2.4 we deduce that

$$\|[\pi_a^*, M_b]\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})} = \|[\pi_a, M_{b^*}]\|_{L_{p'}(\mathbb{R})\to L_{p'}(\mathbb{R})} \lesssim_{d,p} \|a\|_{BMO^d(\mathbb{R})}\|b\|_{BMO^d(\mathbb{R})}.$$

□

We can define the martingale BMO space $BMO^{\omega,2^n}(\mathbb{R}^n)$ on $\mathbb{R}^n$ by virtue of H_I^η similarly as in Definition 1.4.2. More precisely, $BMO^{\omega,2^n}(\mathbb{R}^n)$ associated with the dyadic system $\mathcal{D}^\omega$ on $\mathbb{R}^n$ is the space consisting of all locally integrable functions b such that

$$\|b\|_{BMO^{\omega,2^n}(\mathbb{R}^n)} := \sup_{I\in\mathcal{D}^\omega} \frac{1}{|I|^{1/2}} \left(\sum_{J\subseteq I} \sum_{\eta\in\{0,1\}^n\setminus\{0\}} |\langle H_J^\eta, b\rangle|^2 \right)^{1/2} < \infty. \tag{5.2.14}$$

Then Lemma 5.2.1, Proposition 5.2.4 and Corollary 5.2.5 also hold for the dyadic system $\mathcal{D}^\omega$. It is straightforward to verify that if $b \in BMO(\mathbb{R}^n)$, then $b \in BMO^{\omega,2^n}(\mathbb{R}^n)$ and

$$\|b\|_{BMO^{\omega,2^n}(\mathbb{R}^n)} \le \|b\|_{BMO(\mathbb{R}^n)}.$$

We come to the proof of Theorem II.7.

Proof of Theorem II.7. We use the same notation as that in the proof of Theorem II.6. From Proposition 5.2.4 and Corollary 5.2.5, we have

$$\|[\pi_{T1}^\omega, M_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)} \lesssim_{n,p} \|T1\|_{BMO(\mathbb{R}^n)}\|b\|_{BMO(\mathbb{R}^n)}$$

and

$$\|[(\pi_{T^*1}^\omega)^*, M_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)} \lesssim_{n,p} \|T^*1\|_{BMO(\mathbb{R}^n)}\|b\|_{BMO(\mathbb{R}^n)}.$$

By Theorem 5.1.6, it suffices to estimate $\|[S_\omega^{ij}, M_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}$ for any $i,j \in \mathbb{N}\cup\{0\}$. Note that by the triangle inequality

$$\begin{aligned}&\|[S_\omega^{ij}, M_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}\\ &\le \|[S_\omega^{ij}, \pi_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)} + \|[S_\omega^{ij}, \Lambda_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)} + \|[S_\omega^{ij}, R_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}.\end{aligned}$$

Here π_b, Λ_b and R_b are with respect to the dyadic system $\mathcal{D}^\omega$. From [18] and Lemma 5.2.1, we know that

$$\|\pi_b\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)} \lesssim_{n,p} \|b\|_{BMO^{\omega,2^n}(\mathbb{R}^n)}, \quad \|\Lambda_b\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)} \lesssim_{n,p} \|b\|_{BMO^{\omega,2^n}(\mathbb{R}^n)}.$$

However, S_ω^{ij} is bounded on $L_p(\mathbb{R}^n)$. Therefore one has

$$\begin{aligned}\|[S_\omega^{ij}, \pi_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)} &\lesssim \|S_\omega^{ij}\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}\|\pi_b\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}\\ &\lesssim_{n,p} (i+j)\|b\|_{BMO^{\omega,2^n}(\mathbb{R}^n)} \lesssim_{n,p} (i+j)\|b\|_{BMO(\mathbb{R}^n)}.\end{aligned}$$

Analogously, we have

$$\|[S_\omega^{ij}, \Lambda_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)} \lesssim_{n,p} (i+j)\|b\|_{BMO(\mathbb{R}^n)}.$$

It remains to estimate $\|[S_\omega^{ij}, R_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}$ for any $i,j \in \mathbb{N}\cup\{0\}$. We will show that $\|[S_\omega^{ij}, R_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}$ increases with polynomial growth with respect to i and j uniformly on ω. Then from Theorem 5.1.6 and the triangle inequality, the desired result will follow.

We first prove

$$\|[S_\omega^{ij}, R_b]\|_{L_2(\mathbb{R}^n)\to L_2(\mathbb{R}^n)} \lesssim_n \|b\|_{BMO(\mathbb{R}^n)}. \tag{5.2.15}$$

Without loss of generality, we can assume $\omega = 0$. Let $\Phi = [S_0^{ij}, R_b]$. The form of Φ and B_K ($K \in \mathcal{D}^0$) have been given in (5.1.14) and (5.1.15) respectively. From (5.1.16) we know $\Phi^*\Phi$ is a block diagonal matrix with blocks $B_K^*B_K$ for all $K \in \mathcal{D}^0$. Hence

$$\|\Phi\|_{L_2(\mathbb{R}^n)\to L_2(\mathbb{R}^n)} = \|\Phi^*\Phi\|^{1/2}_{L_2(\mathbb{R}^n)\to L_2(\mathbb{R}^n)} = \sup_{k\in\mathbb{Z}}\sup_{K\in\mathcal{D}_k^0}\|B_K^*B_K\|^{1/2}_{L_2(\mathbb{R}^n)\to L_2(\mathbb{R}^n)}. \tag{5.2.16}$$

Now we fix $k \in \mathbb{Z}$ and $K \in \mathcal{D}_k^0$. We write $B_K^*B_K$ in the matrix form $[B_K^*B_K]$ with respect to the basis $\{H_Q^\zeta\}$, where $Q \in \mathcal{D}^0, Q \subseteq K$, $\ell(Q) = 2^{-i}\ell(K)$ and $\zeta \in \{0,1\}^n\backslash\{0\}$. Then using the triangle inequality, one has

$$\begin{aligned}\|B_K^*B_K\|_{L_2(\mathbb{R}^n)\to L_2(\mathbb{R}^n)} &= \|[B_K^*B_K]\|_{S_\infty(\mathbb{M}_{2^{in}(2^n-1)})}\\ &\le \sum_{\substack{J\in\mathcal{D}^0; J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_\eta \left\|W^{K,J,\eta}\right\|_{S_\infty(\mathbb{M}_{2^{in}(2^n-1)})}\\ &\le \sum_{\substack{J\in\mathcal{D}^0; J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_\eta \left\|V^{K,J,\eta}\left(V^{K,J,\eta}\right)^*\right\|_{S_\infty(\mathbb{M}_{2^{in}(2^n-1)})},\end{aligned}$$

where $W^{K,J,\eta}, V^{K,J,\eta}$ are defined in (5.1.18) and (5.1.20). Analogously to (5.1.21), we deduce

$$\left\|V^{K,J,\eta}\left(V^{K,J,\eta}\right)^*\right\|_{S_\infty(\mathbb{M}_{2^{in}(2^n-1)})} = \left|\sum_{\substack{Q\in\mathcal{D}^0; Q\subseteq K\\ \ell(Q)=2^{-i}\ell(K)\\ \zeta\in\{0,1\}^n\backslash\{0\}}} a_{QJK}^{\zeta\eta}\overline{a_{QJK}^{\zeta\eta}}b_{QJ}\overline{b_{QJ}}\right| = \sum_{\substack{Q\in\mathcal{D}^0; Q\subseteq K\\ \ell(Q)=2^{-i}\ell(K)\\ \zeta\in\{0,1\}^n\backslash\{0\}}}\left|a_{QJK}^{\zeta\eta}b_{QJ}\right|^2,$$

where $b_{IJ} = \langle \frac{\mathbb{1}_I}{|I|}, b\rangle - \langle \frac{\mathbb{1}_J}{|J|}, b\rangle$ for any $I, J \in \mathcal{D}^0$. This implies that

$$\|\Phi\|_{L_2(\mathbb{R}^n)\to L_2(\mathbb{R}^n)} \le \sup_{k\in\mathbb{Z}} \sup_{K\in\mathcal{D}^0_k} \Bigg(\sum_{\substack{J\in\mathcal{D}^0; J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}} \sum_{\eta} \sum_{\substack{I\in\mathcal{D}^0; I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}} \sum_{\xi} \left| a^{\xi\eta}_{IJK} b_{IJ} \right|^2 \Bigg)^{1/2}.$$

Note that $|a^{\xi\eta}_{IJK}| \le 2^{-(i+j)n/2}$, $b_{IJ} = b_{IK} - b_{JK}$ and $b_{IK}\cdot \mathbb{1}_I = (b_{k+i} - b_k)\cdot \mathbb{1}_I$. Then by the Cauchy-Schwarz inequality

$$\begin{aligned}
&\sum_{\substack{J\in\mathcal{D}^0; J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}} \sum_{\eta} \sum_{\substack{I\in\mathcal{D}^0; I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}} \sum_{\xi} \left| a^{\xi\eta}_{IJK} b_{IJ} \right|^2 \\
&\le \frac{(2^n-1)^2}{2^{(i+j)n}} \sum_{\substack{J\in\mathcal{D}^0; J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}} \sum_{\substack{I\in\mathcal{D}^0; I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}} |b_{IJ}|^2 \\
&\le \frac{(2^n-1)^2}{2^{(i+j)n}} \sum_{\substack{J\in\mathcal{D}^0; J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}} \sum_{\substack{I\in\mathcal{D}^0; I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}} 2(|b_{IK}|^2 + |b_{JK}|^2) \qquad (5.2.17)\\
&= \frac{(2^n-1)^2}{2^{in-1}} \sum_{\substack{I\in\mathcal{D}^0; I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}} |b_{IK}|^2 + \frac{(2^n-1)^2}{2^{jn-1}} \sum_{\substack{J\in\mathcal{D}^0; J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}} |b_{JK}|^2 \\
&= (2^n-1)^2 2^{kn+1} \Big(\|(b_{k+i} - b_k)\mathbb{1}_K\|^2_{L_2(\mathbb{R}^n)} + \|(b_{k+j} - b_k)\mathbb{1}_K\|^2_{L_2(\mathbb{R}^n)} \Big).
\end{aligned}$$

We also notice that

$$\|(b_{k+i} - b_k)\mathbb{1}_K\|^2_{L_2(\mathbb{R}^n)} = \int_K \mathbb{E}_k(|b_{k+i}(t) - b_k(t)|^2)dt \le |K|\|b\|^2_{BMO^{0,2^n}(\mathbb{R}^n)}.$$

Hence

$$\|\Phi\|_{L_2(\mathbb{R}^n)\to L_2(\mathbb{R}^n)} \lesssim_n \|b\|_{BMO^{0,2^n}(\mathbb{R}^n)} \le \|b\|_{BMO(\mathbb{R}^n)}. \qquad (5.2.18)$$

Now we prove that Φ is of weak type (1,1). Assume $f \in L_1(\mathbb{R}^n)$ and let $\lambda > 0$. We let $A^{\xi\eta}_{IJK} = a^{\xi\eta}_{IJK} b_{IJ}$, then by (5.1.15)

$$\Phi(f) = \sum_{K\in\mathcal{D}^0} \sum_{\substack{I,J\in\mathcal{D}^0; I,J\subseteq K\\ \ell(I)=2^{-i}\ell(K)\\ \ell(J)=2^{-j}\ell(K)}} \sum_{\xi,\eta} A^{\xi\eta}_{IJK} \langle H^\xi_I, f\rangle H^\eta_J. \qquad (5.2.19)$$

Note that if $\tilde{I}$ is the parent of I, then

$$\begin{aligned}
\left| \left\langle \frac{\mathbb{1}_I}{|I|}, b \right\rangle - \left\langle \frac{\mathbb{1}_{\tilde{I}}}{|\tilde{I}|}, b \right\rangle \right| &\le \frac{1}{|I|} \int_I \left| b(t) - \left\langle \frac{\mathbb{1}_{\tilde{I}}}{|\tilde{I}|}, b \right\rangle \right| dt \\
&\le \frac{2^n}{|\tilde{I}|} \int_{\tilde{I}} \left| b(t) - \left\langle \frac{\mathbb{1}_{\tilde{I}}}{|\tilde{I}|}, b \right\rangle \right| dt \\
&\le 2^n \|b\|_{BMO^{0,2^n}(\mathbb{R}^n)}.
\end{aligned}$$

Together with the triangle inequality, this implies that

$$|A^{\xi\eta}_{IJK}| = |a^{\xi\eta}_{IJK}||b_{IJ}| \le |a^{\xi\eta}_{IJK}||b_{IK}| + |a^{\xi\eta}_{IJK}||b_{JK}| \le 2^n(i+j)\|b\|_{BMO^{0,2^n}(\mathbb{R}^n)}|a^{\xi\eta}_{IJK}|.$$

Thus the operator Φ can be written as a multiple of S_0^{ij}. Recall that S_0^{ij} is also of weak type (1,1), and hence for any $\lambda > 0$

$$|\{|\Phi(f)| > \lambda\}| \lesssim_n i(i+j)\|b\|_{BMO^{0,2^n}(\mathbb{R}^n)}\frac{\|f\|_{L_1(\mathbb{R}^n)}}{\lambda}.$$

Therefore using interpolation and duality, $\Phi = [S_0^{ij}, R_b]$ is bounded on $L_p(\mathbb{R}^n)$. Since the above estimation is independent of the choose of ω, one has

$$\|[S^{ij}_\omega, R_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)} \lesssim_{n,p} (i+j)^2\|b\|_{BMO(\mathbb{R}^n)},$$

which yields

$$\|[S^{ij}_\omega, M_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)} \lesssim_{n,p} (i+j+1)^2\|b\|_{BMO(\mathbb{R}^n)}.$$

As a consequence,

$$\begin{aligned}
&\|[T, M_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}\\
&= \left\|\left[C_1\mathbb{E}_\omega \sum_{\substack{i,j=0\\ \max\{i,j\}>0}}^{\infty} 2^{-\max\{i,j\}\alpha/2}S^{ij}_\omega + C_2\mathbb{E}_\omega S^{00}_\omega + \mathbb{E}_\omega\pi^\omega_{T1} + \mathbb{E}_\omega(\pi^\omega_{T^*1})^*, M_b\right]\right\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}\\
&\lesssim \sum_{i,j=0}^{\infty} 2^{-\max\{i,j\}\alpha/2}\mathbb{E}_\omega\|[S^{ij}_\omega, M_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)} + \mathbb{E}_\omega\|[\pi^\omega_{T1} + (\pi^\omega_{T^*1})^*, M_b]\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}\\
&\lesssim_{n,p} \Big(1 + \|T1\|_{BMO(\mathbb{R}^n)} + \|T^*1\|_{BMO(\mathbb{R}^n)}\Big)\|b\|_{BMO(\mathbb{R}^n)}.
\end{aligned}$$

This completes the proof. □

5.3 Proof of Theorem II.8

We end this thesis with the proof of Theorem II.8. Denote by $BMO^d_{\mathcal{M}}(\mathbb{R})$ the operator-valued BMO space associated with the d-adic martingale consisting of all $\mathcal{M}$-valued functions b that are Bochner integrable on any d-adic interval such that

$$\|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})} := \sup_{I\in\mathcal{D}}\left(\frac{1}{m(I)}\int_I\left\|b - \left(\frac{1}{m(I)}\int_I b\, dm\right)\right\|^2_{\mathcal{M}} dm\right)^{1/2} < \infty, \tag{5.3.1}$$

where $\mathcal{D}$ is the family of all d-adic intervals on $\mathbb{R}$.

During the proof of Theorem II.8, we also need to utilize the d-adic martingale square function defined as follows

$$S(h) = \left(\sum_{k\in\mathbb{Z}}|d_k h|^2\right)^{1/2}, \quad \forall h \in L_1(\mathbb{R}, L_1(\mathcal{M})),$$

and the d-adic martingale Hardy space $H^d_{1,\max}(\mathbb{R})$ is defined by

$$H^d_{1,\max}(\mathbb{R}) = \left\{h \in L_1(\mathbb{R}, L_1(\mathcal{M})) : \|h\|_{H^d_{1,\max}(\mathbb{R})} := \left\|\sup_{m\in\mathbb{Z}}\|\mathbb{E}_m h\|_{L_1(\mathcal{M})}\right\|_{L_1(\mathbb{R})} < \infty\right\}.$$

Bourgain and Garcia-Cuerva proved independently that $BMO^d_{\mathcal{M}}(\mathbb{R})$ embeds continuously into the dual of Hardy space $(H^d_{1,\max}(\mathbb{R}))^*$. We refer the reader to [12] for more details.

Firstly we give the following proposition and its corollary, which will be helpful in the proof of Propositions 5.3.3 that can be regarded as an analogue of Propositions 5.2.4.

Proposition 5.3.1. *Let $1 < p < \infty$ and $b \in BMO^d_{\mathcal{M}}(\mathbb{R})$. Then $\pi_b + (\pi_{b^*})^*$ is bounded on $L_p(\mathbb{R}, L_p(\mathcal{M}))$ and*

$$\|\pi_b + (\pi_{b^*})^*\|_{L_p(\mathbb{R},L_p(\mathcal{M}))\to L_p(\mathbb{R},L_p(\mathcal{M}))} \lesssim_{d,p} \|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})}.$$

Proof. For any $f \in L_p(\mathbb{R}, L_p(\mathcal{M}))$ and $g \in L_{p'}(\mathbb{R}, L_{p'}(\mathcal{M}))$, by (1.7.4)

$$\begin{aligned}\langle(\pi_b + (\pi_{b^*})^*)(f), g\rangle &= \left\langle \sum_{k\in\mathbb{Z}} d_k b \cdot f_{k-1} + \sum_{k\in\mathbb{Z}} \mathbb{E}_{k-1}(d_k b \cdot d_k f), g \right\rangle \\ &= \sum_{k\in\mathbb{Z}} \langle d_k b, d_k g \cdot f^*_{k-1}\rangle + \sum_{k\in\mathbb{Z}} \langle d_k b, g_{k-1} \cdot d_k f^*\rangle \\ &= \left\langle b, \sum_{k\in\mathbb{Z}} d_k g \cdot f^*_{k-1} + \sum_{k\in\mathbb{Z}} g_{k-1} \cdot d_k f^* \right\rangle.\end{aligned}$$

Using the same method as in [55, Theorem 1.1] or [5], we obtain that

$$|\langle(\pi_b + (\pi_{b^*})^*)(f), g\rangle| \lesssim_{d,p} \|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})}\|f\|_{L_p(\mathbb{R},L_p(\mathcal{M}))}\|g\|_{L_{p'}(\mathbb{R},L_{p'}(\mathcal{M}))}.$$

Therefore, one has

$$\|\pi_b + (\pi_{b^*})^*\|_{L_p(\mathbb{R},L_p(\mathcal{M}))\to L_p(\mathbb{R},L_p(\mathcal{M}))} \lesssim_{d,p} \|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})}.$$

□

Recall that Θ_b has been defined in (5.2.9). The following corollary is about the boundedness of Θ_b, which has been proved in [38, Proposition A.2], but it seems that the proof there contains a small gap. We give a detailed proof here.

Corollary 5.3.2. *If $b \in BMO^d_{\mathcal{M}}(\mathbb{R})$, then Θ_b is bounded on $L_2(\mathbb{R}, L_2(\mathcal{M}))$ and*

$$\|\Theta_b\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} \lesssim_d \|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})}.$$

Proof. By the triangle inequality, one has

$$\begin{aligned}&\|\Theta_b\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} \\ &\le \|\pi_b + (\pi_{b^*})^*\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} + \|\Lambda_b - (\pi_{b^*})^*\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))}.\end{aligned}$$

Using the same notation as in Lemma 5.2.1 and from (5.2.1) and (5.2.4), we have

$$\begin{aligned}\|\Lambda_b - (\pi_{b^*})^*\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} &\le \sup_{I\in\mathcal{D}} \|a^I_1 A + a^I_2 A^2 + \cdots + a^I_{d-1}A^{d-1}\|_{L_\infty(\mathbb{M}_d\otimes\mathcal{M})} \\ &\le \sup_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \|a^I_i A^i\|_{L_\infty(\mathbb{M}_d\otimes\mathcal{M})} \\ &\le \sup_{I\in\mathcal{D}} \sum_{i=1}^{d-1} \|a^I_i\|_{\mathcal{M}} = \sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/2}} \sum_{i=1}^{d-1} \|\langle h^i_I, b\rangle\|_{\mathcal{M}}.\end{aligned}$$

However,

$$\begin{aligned}\|\langle h_I^i, b\rangle\|_{\mathcal{M}} &= \left\|\left\langle h_I^i, b - \left\langle \frac{\mathbb{1}_I}{|I|}, b\right\rangle\right\rangle\right\|_{\mathcal{M}} \\ &\leq \frac{1}{|I|^{1/2}}\int_I \left\|b(x) - \left\langle \frac{\mathbb{1}_I}{|I|}, b\right\rangle\right\|_{\mathcal{M}} dx \\ &\leq \left(\int_I \left\|b(x) - \left\langle \frac{\mathbb{1}_I}{|I|}, b\right\rangle\right\|_{\mathcal{M}}^2 dx\right)^{1/2}.\end{aligned}$$

This implies

$$\|\Lambda_b - (\pi_{b^*})^*\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} \leq (d-1)\|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})}. \tag{5.3.2}$$

Therefore from (5.3.2) and Proposition 5.3.1

$$\|\Theta_b\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} \lesssim_d \|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})},$$

as desired. □

Proposition 5.3.3. *If $a \in BMO^d(\mathbb{R})$ and $b \in BMO^d_{\mathcal{M}}(\mathbb{R})$, then $[\pi_a, M_b]$ is bounded on $L_2(\mathbb{R}, L_2(\mathcal{M}))$ and*

$$\|[\pi_a, M_b]\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} \lesssim_d \|a\|_{BMO^d(\mathbb{R})}\|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})}.$$

Proof. We use the same notation as that in the proof of Proposition 5.2.4. Note that by the triangle inequality

$$\begin{aligned}&\|[\pi_a, M_b]\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} \\ &\leq \|[\pi_a, \Theta_b]\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} + \|[\pi_a, R_b]\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))}.\end{aligned}$$

Note also that from Corollary 5.3.2

$$\begin{aligned}\|\pi_a\Theta_b\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} &\leq \|\pi_a\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))}\|\Theta_b\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} \\ &\lesssim_d \|a\|_{BMO^d(\mathbb{R})}\|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})}.\end{aligned} \tag{5.3.3}$$

Hence, one has

$$\|[\pi_a, \Theta_b]\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} \lesssim_d \|a\|_{BMO^d(\mathbb{R})}\|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})}.$$

Now, we estimate $\|[\pi_a, R_b]\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))}$. From (5.2.10) we have

$$[\pi_a, R_b](f) = -\pi_a(\Theta_b(f)) + V_{a,b}(f), \quad \forall f \in L_2(\mathbb{R}, L_2(\mathcal{M})).$$

For any $f \in L_2(\mathbb{R}, L_2(\mathcal{M}))$ and $g \in L_2(\mathbb{R}, L_2(\mathcal{M}))$,

$$\begin{aligned}
\langle V_{a,b}(f), g\rangle &= \sum_{k\in\mathbb{Z}} \left\langle d_k a \cdot \mathbb{E}_{k-1}\left(\sum_{j\geq k} d_j b \cdot d_j f\right), g\right\rangle \\
&= \sum_{k\in\mathbb{Z}} \left\langle \sum_{j\geq k} d_j b \cdot d_j f, \mathbb{E}_{k-1}\left(d_k a^* \cdot d_k g\right)\right\rangle \\
&= \sum_{k\in\mathbb{Z}} \left\langle d_k b, \sum_{j\leq k} \mathbb{E}_{j-1}(d_j a^* \cdot d_j g) \cdot d_k f^*\right\rangle \qquad (5.3.4)\\
&= \left\langle b, \sum_{k\in\mathbb{Z}}\sum_{j\leq k} \mathbb{E}_{j-1}(d_j a^* \cdot d_j g) \cdot d_k f^*\right\rangle \\
&= \left\langle b, \sum_{k\in\mathbb{Z}} d_k(d_k a^* \cdot d_k g) \cdot f_{k-1}^*\right\rangle + \langle b, W_{a,f,g}\rangle,
\end{aligned}$$

where

$$W_{a,f,g} = \sum_{k\in\mathbb{Z}}\sum_{j\leq k} \mathbb{E}_{j-1}(d_j a^* \cdot d_j g) \cdot d_k f^* - \sum_{k\in\mathbb{Z}} d_k(d_k a^* \cdot d_k g) \cdot f_{k-1}^*.$$

Note that

$$\begin{aligned}
\left\langle b, \sum_{k\in\mathbb{Z}} d_k(d_k a^* \cdot d_k g) \cdot f_{k-1}^*\right\rangle &= \sum_{k\in\mathbb{Z}} \langle d_k b, d_k g \cdot d_k a^* \cdot f_{k-1}^*\rangle \\
&= \sum_{k\in\mathbb{Z}} \langle d_k g^* \cdot d_k b, d_k a^* \cdot f_{k-1}^*\rangle \\
&= \sum_{k\in\mathbb{Z}} \langle d_k(d_k g^* \cdot d_k b), d_k a^* \cdot f_{k-1}^*\rangle \\
&= \left\langle \left((\Lambda_{b^*} - (\pi_b)^*)(g)\right)^*, \pi_{a^*}(f^*)\right\rangle.
\end{aligned}$$

Then by (5.3.2),

$$\begin{aligned}
\left|\left\langle b, \sum_{k\in\mathbb{Z}} d_k(d_k a^* \cdot d_k g) \cdot f_{k-1}^*\right\rangle\right| &\leq \|(\Lambda_{b^*} - (\pi_b)^*)(g)\|_{L_2(\mathbb{R},L_2(\mathcal{M}))}\|\pi_{a^*}(f^*)\|_{L_2(\mathbb{R},L_2(\mathcal{M}))} \\
&\lesssim_d \|a\|_{BMO^d(\mathbb{R})}\|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})}\|g\|_{L_2(\mathbb{R},L_2(\mathcal{M}))}\|f\|_{L_2(\mathbb{R},L_2(\mathcal{M}))}. \qquad (5.3.5)
\end{aligned}$$

We now estimate $W_{a,f,g}$. By duality, one has

$$|\langle b, W_{a,f,g}\rangle| \lesssim \|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})}\|W_{a,f,g}\|_{H^d_{1,\max}(\mathbb{R})}.$$

We calculate directly that for any $m \in \mathbb{Z}$,

$$\begin{aligned}
&\mathbb{E}_m(W_{a,f,g}) \\
&= \sum_{k\le m}\sum_{j\le k}\mathbb{E}_{j-1}(d_j a^*\cdot d_j g)\cdot d_k f^* - \sum_{k\le m} d_k(d_k a^*\cdot d_k g)\cdot f_{k-1}^* \\
&= \sum_{j\le m}\mathbb{E}_{j-1}(d_j a^*\cdot d_j g)\cdot (f_m^* - f_{j-1}^*) - \sum_{j\le m} d_j(d_j a^*\cdot d_j g)\cdot f_{j-1}^* \\
&= \sum_{j\le m}\mathbb{E}_{j-1}(d_j a^*\cdot d_j g)\cdot f_m^* - \sum_{j\le m}(d_j a^*\cdot d_j g)\cdot f_{j-1}^* \\
&= \mathbb{E}_m\left(\sum_{j\le m}\mathbb{E}_{j-1}(d_j a^*\cdot d_j g)\right)\cdot f_m^* - \sum_{j\le m}(d_j a^*\cdot d_j g)\cdot f_{j-1}^* \\
&= \mathbb{E}_m\left(\sum_{j\in\mathbb{Z}}\mathbb{E}_{j-1}(d_j a^*\cdot d_j g)\right)\cdot f_m^* - \mathbb{E}_m\left(\sum_{j\ge m+1}\mathbb{E}_{j-1}(d_j a^*\cdot d_j g)\right)\cdot f_m^* - \sum_{j\le m}(d_j a^*\cdot d_j g)\cdot f_{j-1}^* \\
&= \mathbb{E}_m\left(\sum_{j\in\mathbb{Z}}\mathbb{E}_{j-1}(d_j a^*\cdot d_j g)\right)\cdot f_m^* - \mathbb{E}_m\left(\sum_{j\ge m+1} d_j a^*\cdot d_j g\right)\cdot f_m^* - \sum_{j\le m}(d_j a^*\cdot d_j g)\cdot f_{j-1}^*.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|W_{a,f,g}\|_{H^d_{1,\max}(\mathbb{R})} \\
&= \left\|\sup_{m\in\mathbb{Z}}\|\mathbb{E}_m(W_{a,f,g})\|_{L_1(\mathcal{M})}\right\|_{L_1(\mathbb{R})} \\
&\le \left\|\sup_{m\in\mathbb{Z}}\left\|\mathbb{E}_m\left(\sum_{j\in\mathbb{Z}}\mathbb{E}_{j-1}(d_j a^*\cdot d_j g)\right)\cdot f_m^*\right\|_{L_1(\mathcal{M})}\right\|_{L_1(\mathbb{R})} \\
&\quad + \left\|\sup_{m\in\mathbb{Z}}\left\|\mathbb{E}_m\left(\sum_{j\ge m+1} d_j a^*\cdot d_j g\right)\cdot f_m^*\right\|_{L_1(\mathcal{M})}\right\|_{L_1(\mathbb{R})} + \left\|\sup_{m\in\mathbb{Z}}\left\|\sum_{j\le m}(d_j a^*\cdot d_j g)\cdot f_{j-1}^*\right\|_{L_1(\mathcal{M})}\right\|_{L_1(\mathbb{R})} \\
&:= \text{(I)} + \text{(II)} + \text{(III)}.
\end{aligned}$$

For the term (I), from (1.7.4), we have

$$\sum_{j\in\mathbb{Z}}\mathbb{E}_{j-1}(d_j a^*\cdot d_j g) = (\pi_a)^*(g).$$

Thus

$$\begin{aligned}
\text{(I)} &\le \left\|\sup_{m\in\mathbb{Z}}\left\|\mathbb{E}_m((\pi_a)^*(g))\right\|_{L_2(\mathcal{M})}\cdot\sup_{m\in\mathbb{Z}}\|f_m\|_{L_2(\mathcal{M})}\right\|_{L_1(\mathbb{R})} \\
&\le \left\|\sup_{m\in\mathbb{Z}}\left\|\mathbb{E}_m((\pi_a)^*(g))\right\|_{L_2(\mathcal{M})}\right\|_{L_2(\mathbb{R})}\cdot\left\|\sup_{m\in\mathbb{Z}}\|f_m\|_{L_2(\mathcal{M})}\right\|_{L_2(\mathbb{R})} \qquad (5.3.6)\\
&\lesssim \|(\pi_a)^*(g)\|_{L_2(\mathbb{R},L_2(\mathcal{M}))}\|f\|_{L_2(\mathbb{R},L_2(\mathcal{M}))} \\
&\lesssim_d \|a\|_{BMO^d(\mathbb{R})}\|g\|_{L_2(\mathbb{R},L_2(\mathcal{M}))}\|f\|_{L_2(\mathbb{R},L_2(\mathcal{M}))},
\end{aligned}$$

where the first and the second inequalities are both due to the Cauchy-Schwarz inequality, and the third is from the vector-valued Doob maximal inequality for $L_2(\mathcal{M})$-valued functions.

For the term (II), one uses the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
&\sup_{m\in\mathbb{Z}}\Big\|\mathbb{E}_m\Big(\sum_{j\geq m+1} d_j a^*\cdot d_j g\Big)\cdot f_m^*\Big\|_{L_1(\mathcal{M})}\\
&=\sup_{m\in\mathbb{Z}}\Big\|\mathbb{E}_m\Big((a^*-a_m^*)(g-g_m)\Big)\cdot f_m^*\Big\|_{L_1(\mathcal{M})}\\
&\leq \sup_{m\in\mathbb{Z}}\Big\|\mathbb{E}_m\Big((a^*-a_m^*)(g-g_m)\Big)\Big\|_{L_2(\mathcal{M})}\cdot\sup_{m\in\mathbb{Z}}\|f_m\|_{L_2(\mathcal{M})}.
\end{aligned}$$

Let $r=3/2$. We have

$$\begin{aligned}
&\Big\|\mathbb{E}_m\Big((a^*-a_m^*)(g-g_m)\Big)\Big\|_{L_2(\mathcal{M})}\\
&\leq \mathbb{E}_m\Big\|(a^*-a_m^*)(g-g_m)\Big\|_{L_2(\mathcal{M})}\\
&=\mathbb{E}_m\Big(|a-a_m|\cdot\|g-g_m\|_{L_2(\mathcal{M})}\Big)\\
&\leq\Big(\mathbb{E}_m\Big(|a-a_m|^{r'}\Big)\Big)^{1/r'}\Big(\mathbb{E}_m\Big(\|g-g_m\|_{L_2(\mathcal{M})}^r\Big)\Big)^{1/r}\\
&\lesssim\|a\|_{BMO^d(\mathbb{R})}\bigg(\Big(\mathbb{E}_m\|g\|_{L_2(\mathcal{M})}^r\Big)^{1/r}+\Big(\mathbb{E}_m\|g_m\|_{L_2(\mathcal{M})}^r\Big)^{1/r}\bigg)\\
&=\|a\|_{BMO^d(\mathbb{R})}\bigg(\Big(\mathbb{E}_m\|g\|_{L_2(\mathcal{M})}^r\Big)^{1/r}+\|g_m\|_{L_2(\mathcal{M})}\bigg).
\end{aligned}$$

Hence

$$\begin{aligned}
\text{(II)}&\leq\bigg\|\sup_{m\in\mathbb{Z}}\|f_m\|_{L_2(\mathcal{M})}\bigg\|_{L_2(\mathbb{R})}\cdot\bigg\|\sup_{m\in\mathbb{Z}}\Big\|\mathbb{E}_m\Big((a^*-a_m^*)(g-g_m)\Big)\Big\|_{L_2(\mathcal{M})}\bigg\|_{L_2(\mathbb{R})}\\
&\lesssim\|f\|_{L_2(\mathbb{R},L_2(\mathcal{M}))}\|a\|_{BMO^d(\mathbb{R})}\cdot\bigg\|\sup_{m\in\mathbb{Z}}\bigg(\Big(\mathbb{E}_m\|g\|_{L_2(\mathcal{M})}^r\Big)^{1/r}+\|g_m\|_{L_2(\mathcal{M})}\bigg)\bigg\|_{L_2(\mathbb{R})}\\
&\leq\|f\|_{L_2(\mathbb{R},L_2(\mathcal{M}))}\|a\|_{BMO^d(\mathbb{R})}\cdot\bigg(\Big\|\sup_{m\in\mathbb{Z}}\mathbb{E}_m\|g\|_{L_2(\mathcal{M})}^r\Big\|_{L_{2/r}(\mathbb{R})}^{1/r}+\bigg\|\sup_{m\in\mathbb{Z}}\|g_m\|_{L_2(\mathcal{M})}\bigg\|_{L_2(\mathbb{R})}\bigg)\\
&\lesssim_r\|f\|_{L_2(\mathbb{R},L_2(\mathcal{M}))}\|a\|_{BMO^d(\mathbb{R})}\cdot\Big(\Big\|\|g\|_{L_2(\mathcal{M})}^r\Big\|_{L_{2/r}(\mathbb{R})}^{1/r}+\|g\|_{L_2(\mathbb{R},L_2(\mathcal{M}))}\Big)\\
&\lesssim\|f\|_{L_2(\mathbb{R},L_2(\mathcal{M}))}\|a\|_{BMO^d(\mathbb{R})}\|g\|_{L_2(\mathbb{R},L_2(\mathcal{M}))},
\end{aligned}\tag{5.3.7}$$

where in the first inequality we have used the Cauchy-Schwarz inequality, the second and the fourth are both from the vector-valued Doob maximal inequality, and the third is from the triangle inequality.

For the term (III), note that

$$\sum_{j\leq m}(d_j a^*\cdot d_j g)\cdot f_{j-1}^*=\sum_{j\leq m} d_j g\cdot(d_j a^*\cdot f_{j-1}^*).$$

This implies that,

$$\begin{aligned}
\text{(III)} &\le \left\| \sup_{m\in\mathbb{Z}} \left\| \left(\sum_{j\le m} |d_j g|^2 \right)^{1/2} \right\|_{L_2(\mathcal{M})} \left\| \left(\sum_{j\le m} |d_j a \cdot f_{j-1}|^2 \right)^{1/2} \right\|_{L_2(\mathcal{M})} \right\|_{L_1(\mathbb{R})} \\
&= \left\| \left\| \left(\sum_{j\in\mathbb{Z}} |d_j g|^2 \right)^{1/2} \right\|_{L_2(\mathcal{M})} \left\| \left(\sum_{j\in\mathbb{Z}} |d_j a \cdot f_{j-1}|^2 \right)^{1/2} \right\|_{L_2(\mathcal{M})} \right\|_{L_1(\mathbb{R})} \\
&\lesssim \|g\|_{L_2(\mathbb{R},L_2(\mathcal{M}))} \|\pi_a(f)\|_{L_2(\mathbb{R},L_2(\mathcal{M}))} \\
&\lesssim_d \|a\|_{BMO^d(\mathbb{R})} \|g\|_{L_2(\mathbb{R},L_2(\mathcal{M}))} \|f\|_{L_2(\mathbb{R},L_2(\mathcal{M}))},
\end{aligned} \tag{5.3.8}$$

where in the third inequality we have used the Cauchy-Schwarz inequality.

Hence from (5.3.6), (5.3.7) and (5.3.8) we deduce

$$\begin{aligned}
|\langle b, W_{a,f,g}\rangle| &\le \|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})} \|W_{a,f,g}\|_{H^d_{1,\max}(\mathbb{R})} \\
&\lesssim_d \|a\|_{BMO^d(\mathbb{R})} \|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})} \|g\|_{L_2(\mathbb{R},L_2(\mathcal{M}))} \|f\|_{L_2(\mathbb{R},L_2(\mathcal{M}))}.
\end{aligned} \tag{5.3.9}$$

Then by (5.3.4), (5.3.5) and (5.3.9), we get

$$|\langle V_{a,b}(f), g\rangle| \lesssim_d \|a\|_{BMO^d(\mathbb{R})} \|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})} \|g\|_{L_2(\mathbb{R},L_2(\mathcal{M}))} \|f\|_{L_2(\mathbb{R},L_2(\mathcal{M}))},$$

which yields

$$\|V_{a,b}\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} \lesssim_d \|a\|_{BMO^d(\mathbb{R})} \|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})}.$$

Therefore

$$\|[\pi_a, M_b]\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} \lesssim_d \|a\|_{BMO^d(\mathbb{R})} \|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})}.$$

□

Corollary 5.3.4. *If $a \in BMO^d(\mathbb{R})$ and $b \in BMO^d_{\mathcal{M}}(\mathbb{R})$, then $[\pi_a^*, M_b]$ is bounded on $L_2(\mathbb{R}, L_2(\mathcal{M}))$ and*

$$\|[\pi_a^*, M_b]\|_{L_2(\mathbb{R},L_2(\mathcal{M}))\to L_2(\mathbb{R},L_2(\mathcal{M}))} \lesssim_d \|a\|_{BMO^d(\mathbb{R})} \|b\|_{BMO^d_{\mathcal{M}}(\mathbb{R})}.$$

Proof. It is follows from Proposition 5.3.3 and

$$[\pi_a^*, M_b]^* = -[\pi_a, M_{b^*}].$$

□

We define the operator-valued martingale BMO space $BMO^{\omega,2^n}_{\mathcal{M}}(\mathbb{R}^n)$ on $\mathbb{R}^n$ similarly as in (5.3.1). More precisely, $BMO^{\omega,2^n}_{\mathcal{M}}(\mathbb{R}^n)$ associated with dyadic system $\mathcal{D}^\omega$ on $\mathbb{R}^n$ consists of all $\mathcal{M}$-valued functions b that are Bochner integrable on any d-adic interval such that

$$\|b\|_{BMO^{\omega,2^n}_{\mathcal{M}}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{D}^\omega} \left(\frac{1}{m(Q)} \int_Q \left\| b - \left(\frac{1}{m(Q)} \int_Q b \, dm \right) \right\|_{\mathcal{M}}^2 dm \right)^{1/2} < \infty. \tag{5.3.10}$$

Then Corollary 5.3.2, Proposition 5.3.3 and Corollary 5.3.4 also hold for the dyadic system $\mathcal{D}^\omega$. It is easy to verify that if $b \in BMO_{\mathcal{M}}(\mathbb{R}^n)$, then $b \in BMO^{\omega,2^n}_{\mathcal{M}}(\mathbb{R}^n)$ and

$$\|b\|_{BMO^{\omega,2^n}_{\mathcal{M}}(\mathbb{R}^n)} \le \|b\|_{BMO_{\mathcal{M}}(\mathbb{R}^n)}.$$

Thus we come to the proof of Theorem II.8.

Proof of Theorem II.8. We use the same notation as that in the proofs of Theorem II.7 and Theorem II.6. From Proposition 5.3.3 and Proposition 5.3.4, we have

$$\|[\pi^{\omega}_{T1}, M_b]\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))\to L_2(\mathbb{R}^n, L_2(\mathcal{M}))} \lesssim_n \|T1\|_{BMO(\mathbb{R}^n)}\|b\|_{BMO_{\mathcal{M}}(\mathbb{R}^n)}$$

and

$$\|[(\pi^{\omega}_{T^*1})^*, M_b]\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))\to L_2(\mathbb{R}^n, L_2(\mathcal{M}))} \lesssim_n \|T^*1\|_{BMO(\mathbb{R}^n)}\|b\|_{BMO_{\mathcal{M}}(\mathbb{R}^n)}.$$

It suffices to estimate $\|[S^{ij}_{\omega}, M_b]\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))\to L_2(\mathbb{R}^n, L_2(\mathcal{M}))}$ for any $i, j \in \mathbb{N}\cup\{0\}$. Note that by the triangle inequality

$$\begin{aligned}&\|[S^{ij}_{\omega}, M_b]\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))\to L_2(\mathbb{R}^n, L_2(\mathcal{M}))}\\ &\le \|[S^{ij}_{\omega}, \Theta_b]\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))\to L_2(\mathbb{R}^n, L_2(\mathcal{M}))} + \|[S^{ij}_{\omega}, R_b]\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))\to L_2(\mathbb{R}^n, L_2(\mathcal{M}))}.\end{aligned} \tag{5.3.11}$$

From Corollary 5.3.2 one has

$$\|[S^{ij}_{\omega}, \Theta_b]\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))\to L_2(\mathbb{R}^n, L_2(\mathcal{M}))} \lesssim_n \|b\|_{BMO_{\mathcal{M}}(\mathbb{R}^n)}.$$

Now, we estimate $\|[S^{ij}_{\omega}, R_b]\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))\to L_2(\mathbb{R}^n, L_2(\mathcal{M}))}$. Take any f with $\|f\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))} = 1$. From (5.1.15) and by the Cauchy-Schwarz inequality,

$$\begin{aligned}&\|[S^{ij}_{\omega}, R_b]f\|^2_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))}\\ &= \Big\|\sum_{K\in\mathcal{D}^{\omega}}\sum_{\substack{I,J\in\mathcal{D}^{\omega};I,J\subseteq K\\ \ell(I)=2^{-i}\ell(K)\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\xi,\eta} a^{\xi\eta}_{IJK}b_{IJ}\langle H^{\xi}_I, f\rangle H^{\eta}_J\Big\|^2_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))}\\ &= \sum_{K\in\mathcal{D}^{\omega}}\sum_{\substack{J\in\mathcal{D}^{\omega};J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\eta}\Big\|\sum_{\substack{I\in\mathcal{D}^{\omega};I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}}\sum_{\xi} a^{\xi\eta}_{IJK}b_{IJ}\langle H^{\xi}_I, f\rangle\Big\|^2_{L_2(\mathcal{M})}\\ &\le \sum_{K\in\mathcal{D}^{\omega}}\sum_{\substack{J\in\mathcal{D}^{\omega};J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}\sum_{\eta}\Big(\sum_{\substack{I\in\mathcal{D}^{\omega};I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}}\sum_{\xi}|a^{\xi\eta}_{IJK}|^2\Big)\Big(\sum_{\substack{I\in\mathcal{D}^{\omega};I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}}\sum_{\xi}\|b_{IJ}\langle H^{\xi}_I, f\rangle\|^2_{L_2(\mathcal{M})}\Big)\\ &\le \sum_{K\in\mathcal{D}^{\omega}}\sum_{\substack{J\in\mathcal{D}^{\omega};J\subseteq K\\ \ell(J)=2^{-j}\ell(K)}}(2^n-1)^2 2^{-jn}\Big(\sum_{\substack{I\in\mathcal{D}^{\omega};I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}}\sum_{\xi}\|b_{IJ}\langle H^{\xi}_I, f\rangle\|^2_{L_2(\mathcal{M})}\Big),\end{aligned} \tag{5.3.12}$$

where $b_{IJ} = \langle\frac{\mathbb{1}_I}{|I|}, b\rangle - \langle\frac{\mathbb{1}_J}{|J|}, b\rangle$. Note that if $\tilde{I}$ is the parent of I, then

$$\begin{aligned}\Big\|\Big\langle\frac{\mathbb{1}_I}{|I|}, b\Big\rangle - \Big\langle\frac{\mathbb{1}_{\tilde{I}}}{|\tilde{I}|}, b\Big\rangle\Big\|_{\mathcal{M}} &\le \frac{1}{|I|}\int_I\Big\|b(t) - \Big\langle\frac{\mathbb{1}_{\tilde{I}}}{|\tilde{I}|}, b\Big\rangle\Big\|_{\mathcal{M}}dt\\ &\le \frac{2^{n/2}}{|\tilde{I}|^{1/2}}\Big(\int_{\tilde{I}}\Big\|b(t) - \Big\langle\frac{\mathbb{1}_{\tilde{I}}}{|\tilde{I}|}, b\Big\rangle\Big\|^2_{\mathcal{M}}dt\Big)^{1/2}\\ &\le 2^{n/2}\|b\|_{BMO_{\mathcal{M}}(\mathbb{R}^n)}.\end{aligned}$$

This implies that by the triangle inequality,

$$\|b_{IJ}\|_{\mathcal{M}} \le \|b_{IK}\|_{\mathcal{M}} + \|b_{JK}\|_{\mathcal{M}} \le 2^{n/2}(i+j)\|b\|_{BMO_{\mathcal{M}}(\mathbb{R}^n)}.$$

As a consequence, one has from (5.3.12) that

$$\begin{aligned}
&\|[S_\omega^{ij}, R_b]f\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))}^2 \\
&\leq 2^n(2^n-1)^2(i+j)^2\|b\|_{BMO_{\mathcal{M}}(\mathbb{R}^n)}^2 \sum_{K\in\mathcal{D}^\omega}\Bigg(\sum_{\substack{I\in\mathcal{D}^\omega; I\subseteq K\\ \ell(I)=2^{-i}\ell(K)}}\sum_{\xi}\|\langle H_I^\xi, f\rangle\|_{L_2(\mathcal{M})}^2\Bigg) \\
&\lesssim_n (i+j)^2\|b\|_{BMO_{\mathcal{M}}(\mathbb{R}^n)}^2.
\end{aligned}$$

By (5.3.11), we have

$$\|[S_\omega^{ij}, M_b]\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))\to L_2(\mathbb{R}^n, L_2(\mathcal{M}))} \lesssim_n (i+j+1)\|b\|_{BMO_{\mathcal{M}}(\mathbb{R}^n)}.$$

Therefore by the triangle inequality, we conclude

$$\begin{aligned}
\|[T, M_b]\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))\to L_2(\mathbb{R}^n, L_2(\mathcal{M}))} &\lesssim \sum_{i,j=0}^{\infty} 2^{-\max\{i,j\}\alpha/2}\mathbb{E}_\omega\|[S_\omega^{ij}, M_b]\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))\to L_2(\mathbb{R}^n, L_2(\mathcal{M}))} \\
&\quad + \mathbb{E}_\omega\|[\pi_{T1}^\omega + (\pi_{T^*1}^\omega)^*, M_b]\|_{L_2(\mathbb{R}^n, L_2(\mathcal{M}))\to L_2(\mathbb{R}^n, L_2(\mathcal{M}))} \\
&\lesssim_n (1 + \|T1\|_{BMO(\mathbb{R}^n)} + \|T^*1\|_{BMO(\mathbb{R}^n)})\|b\|_{BMO_{\mathcal{M}}(\mathbb{R}^n)}.
\end{aligned}$$

This completes the proof. □

Appendix A: Boundedness of commutative martingale paraproducts

In this appendix, we will use the same notation as in Section 1.3. Given a martingale $\{f_k\}_{k\geq 0}$ on a fixed probability space Ω, let $f_0 = 0$ and recall that the martingale maximal function, the martingale difference and the conditional square function are defined as follows:

$$f^* = \sup_{k\geq 0} |f_k|, \quad d_k f = f_k - f_{k-1}, \quad s(f) = \Big(\sum_{k=1}^{\infty} \mathbb{E}_{k-1}|d_k f|^2\Big)^{\frac{1}{2}}.$$

For $b \in L_1(\Omega)$, recall that the martingale paraproduct with symbol b is defined as

$$\begin{aligned} \pi_b : L_2(\Omega) &\longrightarrow L_2(\Omega) \\ f &\longmapsto \pi_b(f) = \sum_{k=1}^{\infty} d_k b \cdot f_{k-1}. \end{aligned}$$

It is interesting to find the conditions that characterize the boundedness of π_b. We will show that π_b is bounded on $L_2(\Omega)$ if and only if $b \in bmo(\Omega)$. Recall that

$$\|b\|_{bmo(\Omega)} = \sup_{k\geq 1} \|\mathbb{E}_k(\sum_{j\geq k+1} |d_j b|^2)\|_{\infty}^{\frac{1}{2}} < \infty.$$

See Definition 1.3.9 for $BMO(\Omega)$ and $bmo(\Omega)$.

We are about to show the following theorem.

Theorem A.0.1. *π_b is bounded on $L_2(\Omega)$ iff $b \in bmo(\Omega)$. Moreover,*

$$\|\pi_b\|_{L_2(\Omega)\to L_2(\Omega)} \approx \|b\|_{bmo(\Omega)}.$$

In addition, if $b \in bmo(\Omega)$ and $1 < p \leq 2$, then π_b is bounded on $L_p(\Omega)$ and

$$\|\pi_b\|_{L_p(\Omega)\to L_p(\Omega)} \lesssim_p \|b\|_{bmo(\Omega)}.$$

We divide the proof of Theorem A.0.1 into two parts.

A.1 Proof of the necessity of Theorem A.0.1

Lemma A.1.1. *Let $2 \leq p < \infty$. If π_b is bounded on $L_p(\Omega)$, then $b \in bmo(\Omega)$.*

Proof. Note that $d_k(\pi_b(f)) = f_{k-1}d_k b$ for $k \geq 1$. Let $f = f_n \in L_p(\Omega)$ for $n \geq 1$, and by the Burkholder-Gundy inequality

$$\begin{aligned}\|\pi_b\|^p_{L_p(\Omega)\to L_p(\Omega)}\|f_n\|_p^p \geq \|\pi_b(f)\|_p^p &\gtrsim_p \mathbb{E}\Big(\Big(\sum_{k=1}^{\infty}|d_k b\cdot f_{k-1}|^2\Big)^{p/2}\Big)\\ &\geq \mathbb{E}\Big(\Big(\sum_{k=n+1}^{\infty}|d_k b\cdot f_{k-1}|^2\Big)^{p/2}\Big)\\ &= \mathbb{E}\Big(\Big(\sum_{k=n+1}^{\infty}|d_k b|^2\Big)^{p/2}|f_n|^p\Big)\\ &= \mathbb{E}\Big(\mathbb{E}_n\Big(\sum_{k=n+1}^{\infty}|d_k b|^2\Big)^{p/2}|f_n|^p\Big).\end{aligned}$$

This implies that

$$\Big\|\mathbb{E}_n\Big(\sum_{k=n+1}^{\infty}|d_k b|^2\Big)^{p/2}\Big\|_\infty \lesssim_p \|\pi_b\|^p_{L_p(\Omega)\to L_p(\Omega)},$$

since f_n is arbitrary and $|f_n|^p \in L_1(\Omega)$. Note that since $p \geq 2$,

$$\Big\|\mathbb{E}_n\Big(\sum_{k=n+1}^{\infty}|d_k b|^2\Big)\Big\|_\infty \leq \Big\|\mathbb{E}_n\Big(\sum_{k=n+1}^{\infty}|d_k b|^2\Big)^{p/2}\Big\|_\infty^{2/p}.$$

Hence, $\|b\|_{bmo(\Omega)} \lesssim_p \|\pi_b\|_{L_p(\Omega)\to L_p(\Omega)}$.

□

A.2 Proof of the sufficiency of Theorem A.0.1

Lemma A.2.1. *Let $1 < p \leq 2$ and $p' = p/(p-1)$. If $f \in L_p(\Omega)$ and $g \in L_{p'}(\Omega)$, then*

$$\pi_g(f) = \sum_{k=1}^{\infty} f_{k-1}d_k g \in h^1(\Omega).$$

Proof. By direct calculation, one has

$$\begin{aligned}s(\pi_g(f)) = \Big(\sum_{k=1}^{\infty}\mathbb{E}_{k-1}|f_{k-1}d_k g|^2\Big)^{\frac{1}{2}} &= \Big(\sum_{k=1}^{\infty}|f_{k-1}|^2\mathbb{E}_{k-1}|d_k g|^2\Big)^{\frac{1}{2}}\\ &\leq f^*\cdot s(g).\end{aligned}$$

The famous Burkholder inequality (cf. [14] and [15]) asserts that

$$\|s(g)\|_{p'} \lesssim_p \|g\|_{p'}.$$

This yields by the Hölder inequality,

$$\|s(\pi_g(f))\|_1 \leq \|f^*\|_p\|s(g)\|_{p'} \lesssim_p \|f\|_p\|g\|_{p'}.$$

Thus $\pi_g(f) \in h^1(\Omega)$. □

Remark A.2.2. Let $1 < p < \infty$. If $f \in L_p(\Omega)$ and $g \in L_{p'}(\Omega)$, then $\pi_g(f) \in H^1(\Omega)$. The proof is similar to the above lemma, and we will use the fact that

$$\|S(g)\|_p \approx_p \|g\|_p.$$

Lemma A.2.3. *Let $1 < p \leq 2$. If $b \in bmo(\Omega)$, then π_b is bounded on $L_p(\Omega)$. Moreover,*

$$\|\pi_b\|_{L_p(\Omega)\to L_p(\Omega)} \lesssim_p \|b\|_{bmo(\Omega)}.$$

Proof. We proceed by duality. For any $f \in L_p(\Omega)$ and $g \in L_{p'}(\Omega)$, we have by Lemma A.2.1

$$\begin{aligned}
|\langle \pi_b(f), g\rangle| &= |\sum_{k=1}^{\infty} \langle d_k b \cdot f_{k-1}, g\rangle| = |\sum_{k=1}^{\infty} \langle d_k b, \overline{f}_{k-1} \cdot g\rangle| \\
&= |\sum_{k=1}^{\infty} \langle d_k b, d_k(\overline{f}_{k-1} \cdot g)\rangle| \\
&= |\sum_{k=1}^{\infty} \langle d_k b, \overline{f}_{k-1} \cdot d_k g\rangle| = |\sum_{k=1}^{\infty} \langle d_k b, d_k(\pi_g(\overline{f}))\rangle| \\
&\lesssim \|b\|_{bmo(\Omega)} \|\pi_g(\overline{f})\|_{h^1(\Omega)} \lesssim_p \|b\|_{bmo(\Omega)} \|f\|_p \|g\|_{p'},
\end{aligned}$$

since d_k is an orthogonal projection in $L_2(\Omega)$ and $(h^1(\Omega))^* = bmo(\Omega)$. We can conclude that

$$\|\pi_b\|_{L_p(\Omega)\to L_p(\Omega)} \lesssim_p \|b\|_{bmo(\Omega)}.$$

□

Remark A.2.4. From the proof of Theorem I.1 (in particular the estimate of the term Π_2), we also see that if $b \in bmo(\Omega)$, then π_b is bounded from $h^1(\Omega)$ to $h^1(\Omega)$. Hence, if π_b is bounded on $L_2(\Omega)$, then by interpolation, we have π_b is bounded on $L_p(\Omega)$ for $1 < p \leq 2$.

Now we come to the proof of Theorem A.0.1.

Proof of Theorem A.0.1. The necessity of Theorem A.0.1 follows from Lemma A.1.1. The remaining part follows from Lemma A.2.3. □

Remark A.2.5. If $b \in BMO(\Omega)$, then π_b is bounded on $L_p(\Omega)$ for any $1 < p < \infty$ by Remark A.2.2 and by the same proof as that of Lemma A.2.3. So for regular martingales, the boundedness of π_b on $L_2(\Omega)$ implies that π_b is bounded on $L_p(\Omega)$ for any $1 < p < \infty$. However, this extrapolation property fails for irregular martingales.

Appendix B: Another proof of the Necessity of Theorem II.2 by Schur multipliers

In this appendix, we will give another proof of the Necessity of Theorem II.2 for $2 \leq p < \infty$ from [81]. This new approach is based on the boundedness of Schur multipliers, which is simpler than the former proof in Section 4.1. Moreover, our new method yields better constants in (II.1) when $p \to \infty$.

At first, we present the definition of Schur multipliers. Let $n \in \mathbb{N}$ be strictly positive. Denote by $\mathbb{M}_n(\mathcal{M})$ the tensor product of von Neumann algebras $\mathbb{M}_n(\mathbb{C})$ and $\mathcal{M}$, where $\mathbb{M}_n(\mathbb{C})$ is equipped with the usual trace, and $\mathcal{M}$ is equipped with a normal semifinite faithful trace τ. Let $\mathbb{T}$ be the unit circle of the complex plane endowed with normalized Haar measure.

Definition B.0.1. Let $A = [a_{i,j}] \in \mathbb{M}_n(\mathbb{C})$. The Schur multiplier induced by A on $\mathbb{M}_n(\mathcal{M})$ is the operator S_A defined by

$$\begin{aligned} S_A : \mathbb{M}_n(\mathcal{M}) &\longrightarrow \mathbb{M}_n(\mathcal{M}) \\ [m_{i,j}] &\longmapsto S_A([m_{i,j}]) = [a_{i,j} \cdot m_{i,j}]. \end{aligned}$$

We refer the reader to [66] and [76] for more details about Schur multipliers. We also need to use the Fourier multipliers.

Definition B.0.2. Let $\varphi : \mathbb{Z} \to \mathbb{C}$. The Fourier multiplier with symbol φ is defined by

$$F_\varphi\Big(\sum_{k\in\mathbb{Z}} a_k e^{\mathrm{i}kt}\Big) = \sum_{k\in\mathbb{Z}} \varphi(k) a_k e^{\mathrm{i}kt}, \qquad t \in [0, 2\pi]$$

for any finite sequence $(a_k)_k$ in $\mathcal{M}$.

For example, if $\varphi(k) = 1$ for all $k \geq 0$, and $\varphi(k) = -1$ for all $k < 0$, then F_φ is called the Hilbert transform, usually denoted by H. Moreover, H is bounded on $L_p(\mathbb{T}, L_p(\mathcal{M}))$ for any $1 < p < \infty$ and

$$\|H\|_{L_p(\mathbb{T},L_p(\mathcal{M}))\to L_p(\mathbb{T},L_p(\mathcal{M}))} = \cot(\pi/(2\max\{p, p'\})). \tag{B.0.1}$$

The proof of (B.0.1) is based on Cotlar equality. See [25] and [65] for more details. Next, we will show a special class of Schur multipliers by virtue of the Hilbert transform.

B.1 Transference method

This section in devoted to exploring the relationship between Schur multipliers and Fourier multipliers. In fact, using transference method in [16] and [65], we can show the boundedness of some particular Schur multipliers by Fourier multipliers.

Our main ingredient is the following lemma.

Lemma B.1.1. *Let $\varphi : \mathbb{Z} \to \mathbb{C}$. Assume that $(\alpha_i)_{i\in\mathbb{Z}}$ and $(\beta_j)_{j\in\mathbb{Z}}$ are two sequences of integers. If $A = [a_{i,j}] \in \mathbb{M}_n(\mathbb{C})$ where $a_{i,j} = \varphi(\alpha_i + \beta_j)$, then for $0 < p \leq \infty$*

$$\|S_A\|_{L_p(\mathbb{M}_n(\mathcal{M}))\to L_p(\mathbb{M}_n(\mathcal{M}))} \leq \|F_\varphi\|_{L_p(\mathbb{T},L_p(\mathbb{M}_n(\mathcal{M})))\to L_p(\mathbb{T},L_p(\mathbb{M}_n(\mathcal{M})))}.$$

Proof. We consider the following commutative diagram: $\forall z \in \mathbb{T}$ and $\forall\ M = [m_{i,j}] \in \mathbb{M}_n(\mathcal{M})$

$$\begin{array}{ccc} M = [m_{i,j}] & \xrightarrow{\Gamma_z} & \Gamma([m_{i,j}]) \\ \downarrow{\scriptstyle S_A} & & \downarrow{\scriptstyle F_\varphi} \\ S_A([m_{i,j}]) & \xrightarrow{\Gamma_z} & F_\varphi \circ \Gamma([m_{i,j}]) \end{array}$$

where Γ_z is defined as

$$\Gamma_z([m_{i,j}]) := \begin{pmatrix} \ddots & & 0 \\ & z^{\alpha_i} & \\ 0 & & \ddots \end{pmatrix}_i [m_{i,j}] \begin{pmatrix} \ddots & & 0 \\ & z^{\beta_j} & \\ 0 & & \ddots \end{pmatrix}_j.$$

Indeed, one has

$$F_\varphi \circ \Gamma_z([m_{i,j}]) = F_\varphi([z^{\alpha_i+\beta_j} \cdot m_{i,j}]) = [\varphi(\alpha_i + \beta_j) z^{\alpha_i+\beta_j} \cdot m_{i,j}],$$

and in the same way,

$$\Gamma_z \circ S_A([m_{i,j}]) = \Gamma_z([\varphi(\alpha_i + \beta_j) \cdot m_{i,j}]) = [\varphi(\alpha_i + \beta_j) z^{\alpha_i+\beta_j} \cdot m_{i,j}].$$

Note that for any $z \in \mathbb{T}$, and for $0 < p \leq \infty$

$$\|\Gamma_z([m_{i,j}])\|_{L_p(\mathbb{M}_n(\mathcal{M}))} = \|[m_{i,j}]\|_{L_p(\mathbb{M}_n(\mathcal{M}))}.$$

Hence, since $\Gamma_z \circ S_A = F_\varphi \circ \Gamma_z$, we conclude that

$$\|S_A\|_{L_p(\mathbb{M}_n(\mathcal{M}))\to L_p(\mathbb{M}_n(\mathcal{M}))} \leq \|F_\varphi\|_{L_p(\mathbb{T},L_p(\mathbb{M}_n(\mathcal{M})))\to L_p(\mathbb{T},L_p(\mathbb{M}_n(\mathcal{M})))},$$

as desired. □

Corollary B.1.2. *Assume that $A = [a_{i,j}]$ where for each $1 \leq i \leq n$, there exists j_i such that if $1 \leq j \leq j_i$, $a_{i,j} = -1$, and if $j_i < j \leq n$, $a_{i,j} = 1$. Then for $1 < p < \infty$,*

$$\|S_A\|_{L_p(\mathbb{M}_n(\mathcal{M}))\to L_p(\mathbb{M}_n(\mathcal{M}))} \leq \cot(\pi/(2\max\{p, p'\})).$$

Proof. Let $\varphi(k) = 1$ for all $k \geq 0$, and $\varphi(k) = -1$ for all $k < 0$. Choose $\beta_j = j$ for any $1 \leq j \leq n$. It suffices to let $\alpha_i = -j_i - 1$ for each $1 \leq i \leq n$. Thus $\varphi(\alpha_i + \beta_j) = a_{i,j}$. By Lemma B.1.1 and (B.0.1), we obtain the desired result. □

Corollary B.1.3. *Assume that $A = [a_{i,j}]$ where for each $1 \le i \le n$, there exists $j_{i1} \le j_{i2}$ such that if $j_{i1} \le j \le j_{i2}$, then $a_{i,j} = 1$, and otherwise $a_{i,j} = 0$. Then for $1 < p < \infty$,*

$$\|S_A\|_{L_p(\mathbb{M}_n(\mathcal{M}))\to L_p(\mathbb{M}_n(\mathcal{M}))} \le \cot(\pi/(2\max\{p, p'\})).$$

Proof. It is easy to find A_1 and A_2 in $\mathbb{M}_n(\mathbb{C})$ which satisfy the assumptions in Corollary B.1.2 such that

$$A = \frac{A_1 + \mathrm{Id}_n}{2} - \frac{A_2 + \mathrm{Id}_n}{2}.$$

This yields the desired result by Corollary B.1.2. □

Note that the estimates in Corollary B.1.2 and Corollary B.1.3 do not depend on the dimension of the matrix A, which is due to (B.0.1).

B.2 Another proof of the Necessity of Theorem II.2

We are going to prove the Necessity of Theorem II.2 for the case $p \ge 2$ with the help of Schur multipliers and by Corollary B.1.3. As mentioned before, this new method gives a better constant. However, this method fails for $0 < p \le 1$, and gives a worse constant when $p \to 1_+$. Note that our previous proof yields a universal constant as $p \to 1_+$, and see Proposition 4.1.1.

Theorem B.2.1. *For $2 \le p < \infty$, if $b \in \boldsymbol{B}_p^d(\mathbb{R}, \mathcal{M})$, then $\pi_b \in L_p(B(L_2(\mathbb{R})) \otimes \mathcal{M})$. Moreover,*

$$\|\pi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} \le d\cot(\pi/(2p)) \cdot \|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}. \tag{B.2.1}$$

Proof. Define Ξ_b as follows:

$$\Xi_b(f) = \sum_{I\in\mathcal{D}} \sum_{i=1}^{d-1} h_I^i \frac{\langle h_I^i, b\rangle}{|I|^{1/2}} \langle h_I^i, f\rangle, \quad \forall f \in L_2(\mathbb{R}, L_2(\mathcal{M})).$$

Thus one has

$$\|\Xi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} = \|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}.$$

Without loss of generality, we can assume that $\langle h_I, b\rangle$ is zero except finite I. The general case naturally follows by the standard limit argument. Then we can take $N \in \mathbb{N}$ large enough to satisfy that $\langle h_I, b\rangle \ne 0$ only if $d^{-N} \le |I| \le d^N$ and $I \subset [-d^N, d^N]$. Hence it suffices to consider $f \in L_2(\mathcal{D}_N, L_2(\mathcal{M}))$, where

$$\mathcal{D}_N = \sigma\{I_{N,k}| - d^{2N} \le k \le d^{2N} - 1\}.$$

Thus $\{h_I^i | d^{-N+1} \le |I| \le d^N, I \subset [-d^N, d^N], 1 \le i \le d-1\} \cup \{\frac{\mathbb{1}_{[-d^N,0]}}{|[-d^N,0]|^{1/2}}, \frac{\mathbb{1}_{[0,d^N]}}{|[0,d^N]|^{1/2}}\}$ and $\{\frac{\mathbb{1}_{I_{N,k}}}{|I_{N,k}|^{1/2}}| - d^{2N} \le k \le d^{2N} - 1\}$ are two orthonormal basis of $L_2(\mathcal{D}_N)$. We will write π_b and Ξ_b with respect to these two bases.

We arrange h_I^i with the order as follows:

$$\begin{aligned}
&(h^i_{I_{N-1,-d^{2N-1}}})_{1\le i\le d-1}, (h^i_{I_{N-1,-d^{2N-1}+1}})_{1\le i\le d-1}, \cdots, (h^i_{I_{N-1,d^{2N-1}-1}})_{1\le i\le d-1},\\
&(h^i_{I_{N-2,-d^{2N-2}}})_{1\le i\le d-1}, \cdots,\\
&\cdots\cdots,\\
&(h^i_{[-d^N,0]})_{1\le i\le d-1}, (h^i_{[0,d^N]})_{1\le i\le d-1}\\
&\frac{\mathbb{1}_{[-d^N,0]}}{|[-d^N,0]|^{1/2}}, \frac{\mathbb{1}_{[0,d^N]}}{|[0,d^N]|^{1/2}},
\end{aligned} \tag{B.2.2}$$

and arrange $\frac{\mathbb{1}_{I_{N,k}}}{|I_{N,k}|^{1/2}}$ from $k=-d^{2N}$ to $k=d^{2N}-1$.

Denote by $[(\pi_b)_{(I,i),J}]$ $(|J|=d^{-N})$ the matrix form of π_b where

$$\begin{aligned}
(\pi_b)_{(I,i),J} &= \begin{cases} \left\langle h_I^i, \pi_b\left(\frac{\mathbb{1}_J}{|J|^{1/2}}\right)\right\rangle, & \text{if } I\neq[-d^N,0] \text{ and } I\neq[0,d^N];\\ 0, & \text{otherwise} \end{cases}\\
&= \begin{cases} \frac{\langle h_I^i, b\rangle}{|I|^{1/2}}\langle \frac{\mathbb{1}_I}{|I|^{1/2}}, \frac{\mathbb{1}_J}{|J|^{1/2}}\rangle & \text{if } I\neq[-d^N,0] \text{ and } I\neq[0,d^N];\\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{B.2.3}$$

Note that $\|\pi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} = \|[(\pi_b)_{(I,i),J}]\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})}$. Similarly, denote by $[(\Xi_b)_{(I,i),J}]$ the matrix form of Ξ_b where

$$(\Xi_b)_{(I,i),J} = \begin{cases} \frac{\langle h_I^i, b\rangle}{|I|^{1/2}}\langle h_I^i, \frac{\mathbb{1}_J}{|J|^{1/2}}\rangle & \text{if } I\neq[-d^N,0] \text{ and } I\neq[0,d^N];\\ 0, & \text{otherwise.} \end{cases} \tag{B.2.4}$$

Thus $\|[(\Xi_b)_{(I,i),J}]\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} = \|\Xi_b\|_{L_p(B(L_2(\mathbb{R}))\otimes\mathcal{M})} = \|b\|_{\boldsymbol{B}_p^d(\mathbb{R},\mathcal{M})}$.

Define the matrix $M=[m_{(I,i),J}]$ where

$$m_{(I,i),J} = \begin{cases} \overline{\langle h_I^i, \frac{\mathbb{1}_J}{|J|^{1/2}}\rangle}\cdot\frac{|I|^{1/2}}{|J|^{1/2}} & \text{if } I\neq[-d^N,0] \text{ and } I\neq[0,d^N];\\ 0, & \text{otherwise.} \end{cases} \tag{B.2.5}$$

Then one has

$$(\pi_b)_{(I,i),J} = m_{(I,i),J}(\Xi_b)_{(I,i),J}.$$

From our construction, $m_{(I,i),J}\in\{\omega,\omega^2,\cdots,\omega^d\}$.

Therefore, it suffices to show that the Schur multiplier induced by the matrix M is bounded on $L_p(\mathbb{M}_{2\cdot d^{2N}}(\mathcal{M}))$. It is clear to see that there exist A_i $(1\le i\le d)$

$$M = \omega\cdot A_1 + \omega^2\cdot A_2 + \cdots + \omega^d\cdot A_d$$

where each A_i is in $\mathbb{M}_{2\cdot d^{2N}}(\mathbb{C})$ satisfying the assumption in Corollary B.1.3. Thus by Corollary B.1.3, we conclude

$$\|S_M\|_{L_p(\mathbb{M}_{2\cdot d^{2N}}(\mathcal{M}))\to L_p(\mathbb{M}_{2\cdot d^{2N}}(\mathcal{M}))} \le d\cot(\pi/(2p)).$$

This completes the proof. □

Remark B.2.2. In particular, for $d = 2$, there is a better way to find the Schur multiplier, which yields a better constant in (B.2.1). Let I_+ and I_- be the left and right halve of I respectively. Define the matrix $M = [m_{I,J}]$ as follows:

(1) if $J \subset I_+$, then let $m_{I,J} = 1$;

(2) if $J \subset I_-$, then let $m_{I,J} = -1$;

(3) if $J \cap I_+ = \emptyset$ and J is on the left of I_+, then let $m_{I,J} = 1$;

(4) if $J \cap I_- = \emptyset$ and J is on the right of I_-, then let $m_{I,J} = -1$.

So the matrix M is well-defined. In addition, we have

$$(\pi_b)_{I,J} = m_{I,J}(\Xi_b)_{I,J}.$$

Therefore, by this choice of Schur multiplier M, the constant in (B.2.1) can be improved to $\cot(\pi/(2p))$ by Corollary B.1.2.

Appendix C: Open problems

C.1 Problems on continuous bilinear decompositions

Problem C.1.1. Let Ω be a probability space. Do we have a continuous bilinear decomposition of pointwise multiplication between $h^1(\Omega)$ and $bmo(\Omega)$?

Problem C.1.2. Let Ω be a homogeneous space and n its dimension. Are there continuous bilinear decompositions of multiplication between atomic Hardy spaces $H^p_{\mathrm{at}}(\Omega)$ and their duals for $0 < p < \frac{n}{n+1}$?

C.2 Problems on Schatten class and boundedness of martingale paraproducts

Problem C.2.1. Are there similar results of the converse to Theorem II.6? More precisely, under which circumstances for the kernel $K(x, y)$ does $C_{T,b} \in L_p(B(L_2(\mathbb{R}^n)) \otimes \mathcal{M})$ imply $b \in \boldsymbol{B}_p(\mathbb{R}^n, L_p(\mathcal{M}))$?

Problem C.2.2. Is there any equivalent characterization of the boundedness of $C_{T,b}$ on $L_2(\mathbb{R}^n, L_2(\mathcal{M}))$?

Problem C.2.3. For semicommutative d-adic martingales,

1. under which circumstances for symbol b is the martingale paraproduct π_b bounded on $L_2(\mathbb{R}, L_2(\mathcal{M}))$?

2. does π_b have the extrapolation property? To be more specific, does the boundedness of π_b on $L_2(\mathbb{R}, L_2(\mathcal{M}))$ yield the boundedness of π_b on $L_p(\mathbb{R}, L_p(\mathcal{M}))$ $(1 < p < \infty)$?

3. if π_b does not satisfy the extrapolation property, under which circumstances is π_b bounded on $L_p(\mathbb{R}, L_p(\mathcal{M}))$ for a fixed $p \in (1, \infty)$?

Problem C.2.4. It is also tempting to investigate the boundedness of martingale paraproducts concerning CAR algebra and tensor products of matrix algebras.

Problem C.2.5. It is very interesting to establish the vector-valued variant of the boundedness of commutators, namely the vector-valued case of Theorem II.7.

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Résumé :

Cette thèse a pour l'étude des paraproduits de martingales à la fois dans le cadre commutatif et noncommutatif. Il se compose de deux parties. La première concerne la décomposition bilinéaire de la multiplication ponctuelle d'éléments dans l'espace de Hardy de martingales H^1 et son dual BMO. Nous étendons également cette décomposition bilinéaire continue aux espaces de Hardy de martingales H^p $(0 < p < 1)$ et leurs espaces duaux. Nos décompositions sont basées sur des paraproduits de martingales. Comme conséquences de notre travail, nous obtenons des résultats analogues pour des martingales dyadiques sur des espaces de type homogène munis d'une mesure de dédoublement. Nos arguments reposent sur l'existence de systèmes dyadiques sur des espaces de type homogène. La deuxième partie porte principalement sur l'appartenance à la classe de Schatten des paraproduits de martingales semi-commutatifs et purement non commutatifs, en particulier pour les algèbres de Clifford et les produits tensoriels d'algèbres matricielles $\overset{\infty}{\underset{k=1}{\otimes}} \mathbb{M}_d$ en termes d'espaces de Besov de martingales. En utilisant la technique de la martingale dyadique de Hytönen, nous obtenons également des conditions suffisantes pour l'appartenance à la classe de Schatten et la bornitude des commutateurs à valeurs d'opérateurs concernant des opérateurs intégraux singuliers généraux. De plus, nous donnons une preuve alternative sur la caractérisation BMO de la bornitude des commutateurs concernant des opérateurs intégraux singuliers généraux dans le cadre commutatif.

Mots-clés : Paraproduits de martingales ; Espaces de Hardy-Orlicz ; Espaces de Musielak-Orlicz ; Espaces de dédoublement ; Espaces de Besov de martingales ; Martingales noncommutatives ; Produit tensoriel ; Classe de Schatten ; Algèbre de CAR ; Algèbres matricielles ; Commutateurs ; Opérateurs intégraux singuliers ; Espaces de BMO.

Abstract:

This thesis is devoted to the study of martingale paraproducts both in the commutative and noncommutative settings. It consists of two parts. The first one is about the bilinear decomposition of pointwise multiplication of elements in martingale Hardy space H^1 and its dual BMO space. We also extend this continuous bilinear decomposition to martingale Hardy spaces H^p $(0 < p < 1)$ and their dual spaces. Our decompositions are based on martingale paraproducts. As a consequence of our work, we obtain analogous results for dyadic martingales on spaces of homogeneous type equipped with a doubling measure. Our arguments are based on the existence of dyadic systems on spaces of homogeneous type. The second part focuses on the Schatten class membership of semicommutative and purely noncommutative martingale paraproducts, especially for CAR algebras and tensor product of matrix algebras $\overset{\infty}{\underset{k=1}{\otimes}} \mathbb{M}_d$ in terms of martingale Besov spaces. Using Hytönen's dyadic martingale technique, we also obtain sufficient conditions on the Schatten class membership and the boundedness of operator-valued commutators involving general singular integral operators. In addition, we give an alternative proof on the BMO characterization of the boundedness of commutators involving general singular integral operators in the commutative setting.

Keywords : Martingale paraproducts; Hardy-Orlicz spaces; Musielak-Orlicz spaces; Doubling spaces; Martingale Besov spaces; Noncommutative martingales; Tensor product; Schatten class; CAR algebra; Matrix algebras; Commutators; Singular integral operators; BMO spaces.

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