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# Inverse scattering problems for the time harmonic magnetic Schrödinger operator

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# THÈSE DE DOCTORAT

présentée par

**Amal LABIDI**

pour l'obtention du

**DIPLÔME NATIONAL DE DOCTORAT**

**en Mathématiques Appliquées**

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## **Inverse scattering problems for the time harmonic magnetic Schrödinger operator**

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Soutenue le 23 décembre 2023 devant le jury composé de :

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Thèse préparée au sein du laboratoire de modélisation mathématique et numérique  
dans les sciences de l'ingénieur de l'ENIT (LAMSIN)

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Sincerely

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# Introduction

## 1.1 | Context

We are mainly interested in this thesis by the analysis of scattering and inverse scattering problems for the time harmonic magnetic Schrödinger operator. Before addressing our inverse problem, we would like to first discuss this question:

### What is an inverse problem?

The problem that may be regarded as one of the earliest inverse problems pertains to the computation of the earth's diameter by Eratosthenes in 200 B.C. For centuries, individuals have sought refuge by tapping the walls and analyzing echo, this is a distinct instance of an inverse problem. Heisenberg proposed that quantum interactions were completely characterized by their scattering matrix, which collects information about the interaction at infinity. The discovery of neutrinos by measuring the consequences of their existence is also in line with inverse problems.

Inverse problem publications have seen a sharp increase in recent years. A good example of the broad range of applications is provided by the list of inverse problems below:

- Acoustic scattering, scattering in quantum mechanics [24, 93].

- The inverse problem of geomagnetic induction [91].
- Laser tomography, ultrasound tomography, X-ray tomography [3].
- Locating cracks or mines by electrical prospecting [81].
- Radio-astronomical imaging, image analysis [76].
- The use of electrocardiography and magneto-cardiography [34].
- Determining the volatility in models for financial markets [12].

Let's assume we have a mathematical model for a physical process. This model is assumed to provide a description of the system underlying the process and its operating conditions and to explain the main quantities of the model: **input**, **system parameters**, **output**.

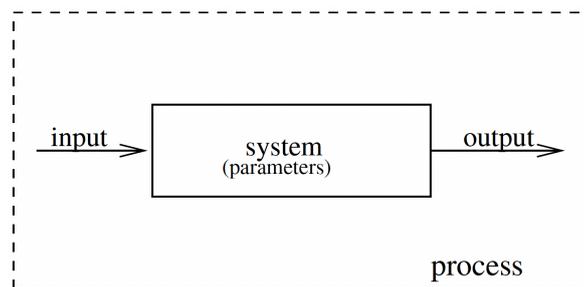


Figure 1.1: The process [7]

The description of the system is typically provided in terms of a set of equations (ordinary and/or partial differential equations, integral equations, etc.) with specific parameters. Three different sorts of problems can be distinguished in the examination of the provided physical process via the mathematical model (see Fig. 1.1).

- (I) **The direct problem:** Given the input and the system parameter, the goal is to find out the output of the model.
- (II) **The reconstruction problem:** Given the system parameters and the output, the goal is to find out which input has led to this output.

(III) **The identification problem:** Given the input and the output, the goal is to determine the system parameters which are in agreement with the relation between input and output.

A type (I) problem is referred to as a direct (or forward) problem because it is focused on a cause-and-effect sequence. In this sense, type problems (II) and (III) are referred to as "inverse problems" because they are problems of finding out unknown causes of known consequences. It is immediately obvious that treating one of the aforementioned problems necessitates dealing with the others as well. the main objective of inverse modeling is to solve inverse problems in order to have a comprehensive discussion of the model (see [7]).

A well-posed mathematical model for a physical problem must have three properties: uniqueness, existence, and stability. Accordingly, the identification of the direct and inverse problem is based on Hadamard's concept of the ill-posed problem [38]. The well-posed of the two problems is known as the direct problem, and the ill-posed is known as the inverse problem.

More precisely, we are interested in the scattering phenomena which represents the most widely used mathematical model to recover unknown physical, geophysical, or medical objects from exterior observations [49, 45]. The direct scattering problem involves determining the scattered field from the incident field and the differential equations governing the wave motion; the inverse scattering problem involves determining the nature of the scatterer, such as location, geometry, or material properties, from knowledge of the scattered field [24, 2, 41]. If the total field is viewed as the sum of an incident field and a scattered field, the scattered field can be determined from the incident field and the differential equations governing the wave motion.

Our interest is in the analysis of the inverse scattering problem for time-harmonic magnetic Schrödinger operator  $\mathcal{H}_{A,q}$  where is given by

$$\mathcal{H}_{A,q} := -(\nabla + iA)^2 + q \tag{1.1}$$

where  $A$  represents the magnetic potential and  $q$  represents the electric potential. Due to the Gauge invariance [46], the magnetic potential  $A$  cannot be uniquely determined from far

field or near field measurements outside the scatterer. So, the best one could recover is the magnetic field  $\text{curl}A$ . Let us mention that the first who studied this type of problem in the mathematical community is Sun in his paper [85]. His result was improved by other authors [11, 90, 79].

Our research focuses specifically on the stability of reconstructing the magnetic field and electric potential using measurements of the far field and near field maps. We derive logarithmic stability estimates with explicit exponents, taking into account regularity assumptions on the parameters. Prior to this analysis, we establish at first the well-posedness of the direct problem using the Lippmann-Schwinger equation and a variational approach. Additionally, we investigate sampling methods for reconstructing the geometric support of the magnetic potential. Our work concludes with a study of the interior transmission problem for the magnetic Schrödinger operator.

In the field of inverse problems, the stability estimate provides a measure of how small changes or errors in the input data affect the errors in the solution. It quantifies the relationship between the data perturbations and the resulting deviations in the solution, enabling us to assess the robustness of the inverse problem solution.

In the absence of magnetic potential, a significant number of researchers addressed the stability question, such as Hähner and Hohage in [40], who established logarithmic stability estimates based on complete aperture measurements. Stefanov, in [84], gave an explicit exponent in the logarithmic estimate and used the  $L^2$ -weighted norm for far field patterns. In [43], Isaev and Novikov demonstrated stability estimates with explicit dependence on the wave number, and we refer to [68, 70, 71, 87] for pioneering uniqueness results under various regularity assumptions. Lipschitz stability results were obtained for the time-domain acoustic wave equation in [90, 19, 73, 31, 11], which is "equivalent" to data available of all frequencies.

We are specifically interested in studying stability in the presence of a magnetic potential  $A$ . Several works in the literature addressed this question for both full and partial measurement settings. For full measurements, which corresponds with measuring the global Dirichlet to Neumann map for bounded domains, we refer to [90, 79]. While in the context of partial

measurements, we refer to [8, 9, 11, 79, 85, 90]. Recently, several studies were concerned with stability estimates for the inverse scattering problems considering both far and near fields, see [22, 40, 84, 48, 80].

To study the situation where one would like to simultaneously recover the magnetic field and the electric potential from full aperture measurements, we shall make use of geometrical optics solutions developed by Calderon, Sylvester, and Uhlmann in [11, 79, 85, 90] for various context in relation to inverse problems. We adopt a similar methodology as in [40, 84]. Tzou demonstrated in [90] a log-type stability estimate for the  $H^{-1}$  norms of the coefficients using DtN data and assuming that the magnetic potentials are in  $W^{2,\infty}$  and the electric potentials are in  $L^\infty$ . Here, we focus on stability with respect to the  $L^\infty$  norm and make a clear connection between the additional regularity that the coefficients must have and the logarithm exponent. To achieve these results, we improved technical results concerning geometric optical type solutions [87] and Helmholtz decompositions, which played an important role in obtaining the results in [10].

In complement to these theoretical results we addressed the challenging task of recovering the shape of a perturbation based on the measurements of scattered waves at a fixed frequency. We extended the Linear Sampling Method (LSM) and the Factorization Method (FM) to the case of magnetic Schrödinger operator.

These methods belong to the class of qualitative methods that aim to provide a rough estimate of the scatterer's geometry without precise quantitative information and without using a forward solver, we refer to [21, 51, 29, 37, 42, 64, 75]. These techniques were first introduced in 1996 [21]. One of these methods' primary benefits is that they don't depend on a forward problem solver. They are fast at computation because of this. Moreover, certain among them provide a proof of uniqueness for reconstructing the geometry with minimal assumptions on the material properties. However, these methods also depend on what are referred to as multistatic data, which involve a substantial number of sources and receivers. These methods exclusively furnish geometrical information about the target.

The principle of the sampling method consists in constructing an indicator function from a

measurement operator and a well-chosen test function that reveals whether a point lies inside the desired geometry. The difference between the sampling methods actually lies in the way this indicator function is constructed. We will mainly consider the cases of linear sampling method (LSM) [21, 27, 18, 14] and factorization method (FM) [53, 57, 58, 59, 60]. These approaches are interrelated and each has its advantages and disadvantages.

The linear sampling method (LSM), proposed by Kirsh and Colton [21] has probably the simplest formulation and wider range of applications (different wave models and more flexible configuration of sources and receivers). The LSM indicator function also has a direct interpretation with the so-called interior transmission problem, which plays an important role in the analysis of the method. However, the mathematical justification does not provide a full justification of the method as it does not explicitly indicate how the approximate solution can be constructed.

The factorization method (FM) was originally proposed by Kirsh in 1998 [51], and more robust from a mathematical background because the geometry is precisely characterized by a constructed indicator function. However, the analysis of the factorization method relies on a special factorization of the measurement operator, which usually corresponds to a full-aperture measurement. Using the factorization approach, one loses however the connection with the interior transmission problem.

In addition to the theoretical study of these methods in the context of magnetic Schrödinger operator, we present several validating results in 2D for simple configurations. The theoretical part required the study of the interior transmission problem with contrast produced by the refractive index and the magnetic potential. This is the objective of the last part of our work. We prove the well-posedness of the Interior Transmission Problem (ITP) and the discreteness of the set of transmission eigenvalues through the application of Fredholm theory and the upper triangular Fredholm theory. The interior transmission problem was first posed by Kirsch in [55]. A few years later, Colton and Monk in [25] used the interior transmission problem to solve the inverse scattering problem for acoustic waves in an inhomogeneous medium. Since that time, the ITP has gained a large attention in inverse scattering community (see, for

example, [13, 24, 14]).

## 1.2 | Overview and organization

The dissertation is structured as follows: After a general introduction, the manuscript is split into four chapters as follows:

**Chapter 2:** This chapter is dedicated to establishing the well-posedness of the direct problem for the magnetic Schrödinger operator. The analysis is done using first the Lippmann-Schwinger equation and then using a variational approach. We conclude the chapter with numerical examples illustrating the scattering problem solutions, using the finite element library FreeFem++.

**Chapter 3:** The purpose of this Chapter is to provide stability estimates for inverse scattering problems related to the time harmonic magnetic Schrödinger equation. Due to gauge invariance, the magnetic potential  $A$  cannot be uniquely determined from far field or near field measurements outside the scatterer. We prove logarithmic type estimates for retrieving the magnetic fields  $\text{curl}(A)$  and electric potentials from near field or far field maps. Our approach combines techniques from similar results obtained in the literature for inhomogeneous inverse scattering problems based on the use of geometrical optics solutions.

**Chapter 4:** In this chapter, we first focus on the use of sampling methods: Linear Sampling Method (LSM) and Factorization Method (FM) to solve the inverse shape problem. We provide theoretical justification of these two methods along with some numerical validating examples. We present second a comprehensive analysis of the uniqueness of the reconstruction of the shape of the domain from far field pattern.

**Chapter 5:** This chapter presents a study of the interior transmission problem (ITP) fo-



cusing on establishing the well-posedness of the (ITP), and the discreteness of the set of transmission eigenvalues through the application of Fredholm theory and the upper triangular Fredholm theory.

# Scattering problem for the magnetic Schrödinger operator

## Abstract:

*This chapter is dedicated to establishing the well-posedness of the direct problem for the magnetic Schrödinger operator. The analysis is done using first the Lippmann-Schwinger equation and then using a variational approach but we present first the magnetic Schrödinger model and the acoustic model. We conclude the chapter with numerical examples illustrating the scattering problem solutions, using the finite element library FreeFem++.*

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## 2.1 | Introduction

This chapter is organized as follows. In Section 2.2, we present the physical and mathematical models of the scattering problem for the magnetic Schrödinger operator. We discuss, in Sections 2.3 and 2.4, the existence and uniqueness of solution for the direct problem using both the variational approach and the Lippmann-Schwinger equation. In the last Section 2.5, we end this chapter by giving numerical examples using the finite element program FreeFem++. This study will be useful for the other chapters.

## 2.2 | Physical and mathematical model

### 2.2.1 | Physical model

#### 2.2.1.1 | The magnetic schrödinger model

The Schrödinger equation stands as one of the fundamental equations that govern the behavior of the nanoscopic world. It represents a key milestone in the development of quantum mechanics.

The time-dependent Schrödinger equation describing the dynamics of a quantum particle, such as an electron, of mass  $m$  moving under the influence of a three-dimensional potential in position  $\vec{x} = (x, y, z)$  at time  $t$ , can be written as:

$$-\frac{\hbar^2}{2m}\Delta\Psi(\vec{x}, t) + V\Psi(\vec{x}, t) = i\hbar\frac{\partial\Psi}{\partial t}(\vec{x}, t), \quad (2.1)$$

where

- $\Psi(\vec{x}, t)$  is the wave function,

- $H = -\frac{\hbar^2}{2m}\Delta + V$  is the Hamiltonian operator,
- $\hbar$  is the reduced Planck constant,
- $\Delta$  is the Laplacian operator,
- $V = q\mathcal{V}$  is the potential energy function,
- $q$  is the charge of particle,
- $\mathcal{V}$  is the electric potential.

Let's now take a closer look at the specific instance of time-independent potentials  $V(\vec{x}, t) = V(\vec{x})$ . The wave function can be expressed as the product of spatial and temporal components in this case, since the Hamiltonian operator is also time-independent,

$$\Psi(\vec{x}, t) = \psi(\vec{x})e^{-iEt/\hbar}, \quad (2.2)$$

where  $\psi(\vec{x})$  is the solution of time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\Delta\psi(\vec{x}) + V\psi(\vec{x}) = E\psi(\vec{x}), \quad (2.3)$$

which is of the form  $H\psi = E\psi$  with  $E$  is the total energy of the particle. This particular solution of the Schrödinger equation is called stationary state because the probability density is stationary i.e., it does not depend on time

$$|\Psi(\vec{x}, t)|^2 = |\psi(\vec{x})e^{-iEt/\hbar}|^2 = |\psi(\vec{x})|^2.$$

In the initial half of 1926, Erwin Schrödinger (1887–1961) presented the time-independent wave equation (2.3), which was a pivotal contribution and established the groundwork for wave mechanics. This significant discovery was made known in the first of his four monumental papers. Moreover, in that first paper, Schrödinger promptly utilized the equation to deduce the energy spectrum of the hydrogen atom.

Now, with the presence of the magnetic potential  $\mathcal{A}$ , the magnetic Schrödinger equation has the following form

$$\frac{1}{2m}(-i\hbar\nabla - q\mathcal{A})^2\Psi(\vec{x}, t) + V\Psi(\vec{x}, t) = i\hbar\frac{\partial\Psi}{\partial t}(\vec{x}, t). \quad (2.4)$$

The magnetic vector potential  $\mathcal{A}$  and electric potential  $\mathcal{V} = \frac{1}{q}V$  mentioned in the equation are integral components of the electromagnetic field. They are responsible for generating the electromagnetic forces that affect the behavior of particles at the quantum level. The electromagnetic field, described by the electric field  $\nabla\mathcal{V}$  and magnetic field  $\nabla \times \mathcal{A}$ .

In the case of time-independent potentials, the wave function  $\Psi(\vec{x}, t)$  can be written as (2.2) where  $\psi(\vec{x})$  is the solution of time-independent magnetic Schrödinger equation

$$\boxed{\frac{1}{2m}(-i\hbar\nabla - q\mathcal{A})^2\psi(\vec{x}) + q\mathcal{V}\psi(\vec{x}) = E\psi(\vec{x}).} \quad (2.5)$$

For further readings about the Schrödinger model, we refer to [35, 78, 93].

### 2.2.1.2 | The acoustic model

In this subsection, we provide a quick presentation of the acoustic model. Let us consider the propagation of a small amplitude acoustic wave produced by a small perturbation in a homogeneous and isotropic fluid in  $\mathbb{R}^3$ . The wave equation is then given by:

$$\partial_t^2 p(t, x) - c^2 \Delta p(t, x) = 0, \quad (2.6)$$

where

- $c$  is the speed of sound in the medium,
- $p$  represents the pressure field, where the velocity potential (according to the studied model) is derived by linearizing the fluid motion equations (Euler equation, continuity equation, and adiabatic hypothesis) [24].

When trying to solve this equation in the spectral domain, one considers harmonic waves of the form

$$p(t, x) = \Re e(e^{it\omega} u(x)),$$

where

- $\omega > 0$  represents the frequency of the time variation,
- $u$  represents the pressure field amplitude.

We then obtain that  $u$ , which takes complex values, satisfies the Helmholtz equation:

$$(\Delta + k^2 n)u = 0, \tag{2.7}$$

where

- $k = \frac{\omega}{c_0} > 0$  is the wave number,
- $c_0$  is the wave speed in the vacuum,
- $n = \frac{c_0}{c}$  is the medium refractive index.

This equation bears the name of the physicist Hermann Ludwig Ferdinand von Helmholtz (1821–1894) for his significant contributions to mathematical acoustics and electromagnetics.

For more detailed information on the derivation of the model, we refer to Chapter 2 of [24].

### 2.2.1.3 | Analogy between the Schrödinger model and acoustic model

The Schrödinger model or the acoustic model can be written in the following form:

$$-(\nabla + iA)^2 u + qu = k^2 u. \tag{2.8}$$

This corresponds to the Schrödinger model by taking:

- $A = -\frac{q}{\hbar} \mathcal{A}$ .

- $q = \frac{2m}{\hbar^2} \mathcal{Q} \mathcal{V}$ .
- $k = \frac{\sqrt{2mE}}{\hbar}$ .

It represents also the acoustic model by taking :

- $A = 0$ .
- $k = \frac{\omega}{c_0}$ .
- $q = k^2(1 - n)$ .

## 2.2.2 | Mathematical model

Let  $D \subset \mathbb{R}^3$  be a bounded open set with smooth boundary such that  $\mathbb{R}^3 \setminus D$  is connected and  $B$  be a smooth bounded a simply connected domain (typically a ball) containing  $D$  with outward normal denoted by  $\nu$ . Let  $A = (a_1, a_2, a_3) \in W^{1,\infty}(\mathbb{R}^3)^3$  be a real valued vector modeling the magnetic potential, the refractive index  $n \in L^\infty(\mathbb{R}^3, \mathbb{C})$  with non negative imaginary, positive real and  $n = 1$  outside the support  $D$  and  $q \in L^\infty(\mathbb{R}^3)$  be a complex valued function with non negative imaginary part modeling the electric potential such that  $\text{Supp}(A) \subset D$  and  $\text{Supp}(q) \subset D$ .

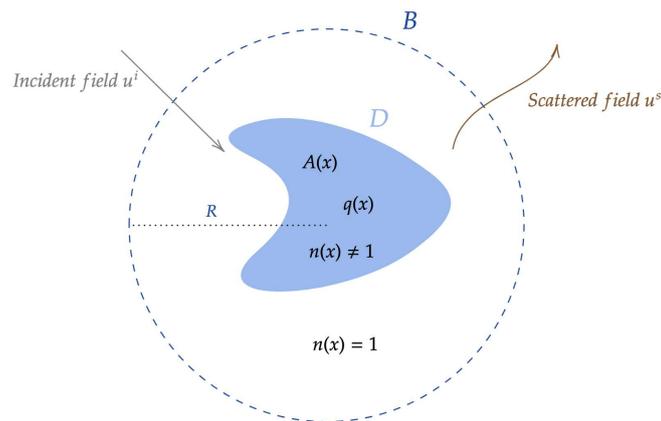


Figure 2.1: The inhomogeneous direct scattering problem

We consider the magnetic Schrödinger operator given by

$$\mathcal{H}_{A,q} := -(\nabla + iA)^2 + q = -\Delta - Q_{A,q}, \quad (2.9)$$

where  $Q_{A,q}$  is the first order operator given by

$$Q_{A,q}v := i \operatorname{div}(Av) + iA \cdot \nabla v - (|A|^2 + q)v, \quad v \in H_{\text{loc}}^1(\mathbb{R}^3). \quad (2.10)$$

The direct scattering problem is to determine the total field  $u$  knowing the incident wave  $u^i$  that satisfies the Helmholtz equation in  $\mathbb{R}^3$  which describes how waves propagate through space, and knowing that the total field satisfies with the presence of the electric potential  $q$  and magnetic potential  $A$

$$\mathcal{H}_{A,q}u_{A,q} - k^2u_{A,q} = 0 \quad \text{in } \mathbb{R}^3, \quad (2.11)$$

with  $k > 0$  is the wave number. The total field is decomposed into

$$u_{A,q} = u^i + u_{A,q}^s \quad \text{in } \mathbb{R}^3, \quad (2.12)$$

where the scattered field  $u_{A,q}^s \in H_{\text{loc}}^2(\mathbb{R}^3)$  and satisfies the Sommerfeld radiation condition at infinity

$$\lim_{r \rightarrow \infty} r \left( \partial_r u^s - iku^s \right) = 0, \quad r = |x| \quad (2.13)$$

uniformly with respect to  $\hat{x} = \frac{x}{|x|}$ . This condition was introduced by Sommerfeld in 1912 to ensure uniqueness for the solutions to the scattering problems (see [83]).

Two important families of incident fields are given by

- Plane waves:  $u^i(x) = u^i(x, d) = e^{ikx \cdot d}$ , where  $d \in \mathbb{S}^2$ , called plane wave because  $e^{i(kx \cdot d - \omega t)}$  is constant on the planes  $kx \cdot d - \omega t = cte$  where  $\omega$  is an angular frequency. They propagate with the speed  $c = \frac{\omega}{k}$  in the direction  $d \in \mathbb{S}^2$ .
- Spherical waves coming from a point source outside  $D$ . More precisely, let  $\Phi$  be the Green function given by

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y \quad (2.14)$$



which is the fundamental solution of the Helmholtz equation, i.e. satisfying  $\Delta\Phi(\cdot, y) + k^2\Phi(\cdot, y) = -\delta_y$  together with the Sommerfeld radiation condition. The spherical incident wave created by a point source at  $y$  is  $u^i(x, y) = \Phi(x, y)$ .

In the next two section, we analyze the direct scattering problem using two distinct approaches: the Lippmann-Schwinger equation and the variational approach.

## 2.3 | Analysis of the direct problem using the Lippmann-Schwinger equation

We here study the well-posedness of the direct scattering problem using the formulation of the problem as a Lippmann-Schwinger equation (see, for instance, [82, 39]) and prove a uniform bound with respect to the potentials.

**Assumption 2.3.1.** *We assume that the magnetic potential  $A = (a_1, a_2, a_3) \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$  and the electric potential  $q \in L^\infty(\mathbb{R}^3, \mathbb{C})$  with non negative imaginary part such that  $\text{Supp}(A) \subset D$  and  $\text{Supp}(q) \subset D$ . Moreover,  $A$  and  $q$  satisfies  $\|A\|_{W^{1,\infty}} \leq M$  and  $\|q\|_{L^\infty} \leq M$ , respectively, for some constant  $M > 0$ .*

The function  $v$  will be referring in this section to the incident wave (i.e.  $\Phi(\cdot, y)$  or  $u^i(\cdot, d)$ ); the associated total field is denoted by  $u = u_{A,q}$  and the scattered field  $u^s = u_{A,q}^s = u - v \in H_{\text{loc}}^2(\mathbb{R}^3)$ . Both scattering problems can then be stated as solving for  $u^s \in H_{\text{loc}}^2(\mathbb{R}^3)$  satisfying

$$-\Delta u^s - k^2 u^s = Q_{A,q}(u^s + v) \quad \text{in } \mathbb{R}^3, \quad (2.15)$$

and the Sommerfeld radiation condition (2.13). For the study of this problem, we only require that  $v \in H^1(D)$ . Convolution properties imply in particular that  $u^s$  can be represented as

$$u^s(x) = \int_D \Phi(x, y) Q_{A,q} u(y) dy, \quad x \in \mathbb{R}^3, \quad \text{with } u = u^s + v \text{ in } D. \quad (2.16)$$

Let us introduce the integral operator  $T_{A,q} : H^1(D) \rightarrow H^1(D)$  defined by

$$T_{A,q}w(x) := \int_D \Phi(x,y)Q_{A,q}w(y)dy, \quad x \in D. \quad (2.17)$$

We remark that, since  $Q_{A,q} : H^1(D) \rightarrow L^2(D)$  is continuous (by regularity assumptions on  $A$  and  $q$ ), and since the volume potential

$$w \mapsto \int_D \Phi(\cdot, y)w(y)dy,$$

continuously maps  $L^2(D)$  into  $H_{\text{loc}}^2(\mathbb{R}^3)$  (see [24]), we deduce that  $T_{A,q}$  is compact. Equation (2.16) implies in particular that the total field  $u \in H^1(D)$  and is a solution of the Lippmann-Schwinger equation

$$u - T_{A,q}u = v \quad \text{in } H^1(D). \quad (2.18)$$

Conversely, if  $u \in H^1(D)$  satisfies (2.18), then one easily verifies using the properties of volume potentials [24] that  $u_{A,q}^s$  given by (2.16) is in  $H_{\text{loc}}^2(\mathbb{R}^3)$  and is a solution of the scattering problem (2.15)-(2.13). The well-posedness of the latter is then a consequence of the following proposition.

**Proposition 2.3.2.** *The operator  $I - T_{A,q} : H^1(D) \rightarrow H^1(D)$  with  $I$  denoting the identity operator on  $H^1(D)$  is continuously invertible.*

*Proof.* The operator  $I - T_{A,q}$  is of Fredholm type with index 0. It is therefore sufficient to prove the injectivity of this operator. If  $u - T_{A,q}u = 0$ , then, from the above equivalence,  $u^s$  given by (2.16) with  $v = 0$  satisfies

$$-\Delta u^s - k^2 u^s = Q_{A,q}u^s \quad \text{in } \mathbb{R}^3.$$

Multiplying by  $\overline{u^s}$  and integrating over a ball  $B$  containing  $D$  imply after applying the Green's theorem in both sides

$$\begin{aligned} \int_B (|\nabla u^s|^2 - k^2 |u^s|^2) dx - \int_{\partial B} \partial_r u^s \overline{u^s} ds(x) \\ = \int_B (iA \cdot (\nabla u^s \overline{u^s} - \nabla \overline{u^s} u^s) - (|A|^2 + q)|u^s|^2) dx. \end{aligned}$$

Taking the imaginary part of the previous equality implies that  $\Im(\int_{\partial B} \partial_r u^s \overline{u^s} ds(x)) \geq 0$  (since  $A$  is real valued and  $\Im(q) \geq 0$ ). The Rellich lemma then implies that  $u^s = 0$  in  $\mathbb{R}^3 \setminus B$ . We now observe that

$$|\Delta u^s(x)| \leq (k^2 + \|q\|_\infty + \|A\|_\infty^2 + \|\nabla \cdot A\|_\infty) |u^s(x)| + 2\|A\|_\infty |\nabla u^s(x)|,$$

for a.e.  $x \in \mathbb{R}^3$ .

The unique continuation theorem yields  $u^s = 0$  in  $\mathbb{R}^3$  and therefore  $u = 0$  in  $D$ . This proves the injectivity of  $I - T_{A,q}$  and finishes the proof of the proposition.  $\square$

It is also possible to prove the following uniform bound.

**Proposition 2.3.3.** *The assumption 2.3.1 holds. Then there exists a constant  $C$  that only depends on  $D$ ,  $M$  and  $k$  such that  $\|(I - T_{A,q})^{-1}\| \leq C$ . Here  $\|\cdot\|$  denotes the norm in  $\mathcal{L}(H^1(D))$ .*

*Proof.* We prove the result using a contradiction argument. Let us assume that, for each  $n \in \mathbb{N}$ , there exists  $A_n \in W^{1,\infty}(\mathbb{R}^3)^3$  and  $q_n \in L^\infty(\mathbb{R}^3)$  as in the proposition such that

$$\|(I - T_{A_n, q_n})^{-1}\| \geq n.$$

This implies in particular the existence of a non trivial function  $v_n \in H^1(D)$  such that the function  $u_n \in H^1(D)$  satisfying  $u_n - T_{A_n, q_n} u_n = v_n$  in  $H^1(D)$  verifies

$$\|u_n\|_{H^1(D)} \geq n \|v_n\|_{H^1(D)}.$$

This gives for the normalized sequence  $\tilde{u}_n = \frac{u_n}{\|u_n\|_{H^1(D)}}$

$$\tilde{u}_n - T_{A_n, q_n} \tilde{u}_n = \frac{v_n}{\|u_n\|_{H^1(D)}} =: \tilde{v}_n \quad \text{in } H^1(D), \quad (2.19)$$

where  $\|\tilde{v}_n\|_{H^1(D)} \leq \frac{1}{n}$ . The associated scattered field  $\tilde{u}_n^s \in H_{\text{loc}}^2(\mathbb{R}^3)$  is defined by

$$\tilde{u}_n^s(x) = \int_D \Phi(x, y) Q_{A_n, q_n} \tilde{u}_n(y) dy, \quad x \in \mathbb{R}^3, \quad \text{with } \tilde{u}_n = \tilde{u}_n^s + \tilde{v}_n \text{ in } D. \quad (2.20)$$

Since the sequence  $(\tilde{u}_n)$  is bounded in  $H^1(D)$ , the assumptions on  $A_n$  and  $q_n$  imply that the sequence  $(Q_{A_n, q_n}(\tilde{u}_n))$  is also bounded in  $L^2(D)$ . It yields in particular, using (2.20), that the sequence  $(\tilde{u}_n^s)$  is bounded in  $H^2(D)$ . Using the Rellich-Kondrachov compactness Theorem (see [1]), we infer that an extracted subsequence that we keep denoting  $(\tilde{u}_n^s)$  is a Cauchy sequence in  $H^1(D)$ . From  $\tilde{u}_n = \tilde{u}_n^s + \tilde{v}_n$  we deduce that  $(\tilde{u}_n)$  is also a Cauchy sequence in  $H^1(D)$  and therefore converges to some  $u$  in  $H^1(D)$ . Given the boundedness of the sequences  $(A_n)$  and  $(q_n)$ , by changing the original sequence and without corrupting the contradiction argument (i.e., the  $H^1$  norm of  $\tilde{u}_n$  is equal to 1 and  $\|\tilde{v}_n\|_{H^1(D)} \rightarrow 0$  as  $n \rightarrow +\infty$ ), one can assume that  $(A_n)$  and  $(q_n)$  weak-\* converge to  $A$  and  $q$ , respectively, in  $W^{1,\infty}(\mathbb{R}^3)^3$  and  $L^\infty(\mathbb{R}^3)$ . One then easily verifies that  $Q_{A_n, q_n}(\tilde{u}_n)$  weakly converges in  $L^2(D)$  to  $Q_{A, q}(u)$ . Consequently, from (2.17), we get that  $T_{A_n, q_n} \tilde{u}_n$  strongly converges to  $T_{A, q} u$  in  $H^1(D)$ . Passing to the limit in (2.19) implies that  $u \in H^1(D)$  verifies  $u - T_{A, q} u = 0$ . The limits  $A$  and  $q$  obviously verify the hypothesis of Proposition 2.3.2 and therefore  $u = 0$ . This contradicts  $\|u\|_{H^1(D)} = \|\tilde{u}_n\|_{H^1(D)} = 1$ .  $\square$

Let us observe for later use that, thanks to (2.16), the far field associated with the scattered wave verifying (2.15) can be expressed as

$$u_{A, q}^\infty(\hat{x}) := \frac{1}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} Q_{A, q} u(y) dy, \quad \hat{x} \in \mathbb{S}^2, \quad (2.21)$$

where  $u$  is the solution of (2.18).

As a straightforward corollary of Proposition 2.3.3, the continuity properties of volume potentials and (2.21), we have the following uniform estimates for  $u^s$  solution of (2.15)-(2.13) and associated far field.

**Corollary 2.3.4.** *The assumption 2.3.1 holds. Then there exists a constant  $C$  that depends only on  $M$ ,  $D$  and  $k$  such that*

$$\|u_{A, q}^s\|_{H^2(D)} \leq C \|v\|_{H^1(D)} \quad \text{and} \quad \|u_{A, q}^\infty\|_{L^2(\mathbb{S}^2)} \leq C \|v\|_{H^1(D)}$$

for all  $v \in H^1(D)$ , where  $u_{A, q}^s \in H_{\text{loc}}^2(\mathbb{R}^3)$  and is a solution of the scattering problem (2.15)-(2.13). Moreover, for any compact  $K$  there exists a constant  $C$  that depends only on

$M, D, K$  and  $k$  such that for all  $v \in H^1(D)$

$$\|u_{A,q}^s\|_{H^2(K)} \leq C\|v\|_{H^1(D)}.$$

Armed with with above, let us define for later use the linear and continuous solution operator  $\mathcal{M}_{A,q}$  by

$$\begin{aligned} \mathcal{M}_{A,q} : H^1(D) &\rightarrow H_{\text{loc}}^2(\mathbb{R}^3), \\ v &\mapsto \mathcal{M}_{A,q}v := u_{A,q}^s, \end{aligned} \quad (2.22)$$

where  $u_{A,q}^s$  is the solution of (2.15)-(2.13).

We use this data to define the near field operator  $\mathcal{N}_{A,q} : L^2(\partial B) \rightarrow L^2(\partial B)$ , as

$$\mathcal{N}_{A,q}h(x) := \int_{\partial B} u_{A,q}^s(x,y)h(y) \, ds(y), \quad x \in \partial B \quad (2.23)$$

where  $u_{A,q}^s(\cdot, y) := \mathcal{M}_{A,q}\Phi(\cdot, y)$ ,  $y \in \partial B$ . We first remark that

$$\|\mathcal{N}_{A,q}\| \leq \|u_{A,q}^s\|_{L^2(\partial B \times \partial B)},$$

and therefore it is sufficient to study the stability of  $\mathcal{N}_{A,q} \mapsto (A, q)$  in order to infer stability results in terms of near field measurements.

We second observe that, after introducing the single-layer operator  $\mathcal{S} : L^2(\partial B) \rightarrow H^1(D)$  defined by

$$\mathcal{S}h(x) := \int_{\partial B} \Phi(x,y)h(y) \, ds(y), \quad x \in D, \quad (2.24)$$

one has by linearity and continuity properties of the mapping  $\mathcal{M}_{A,q}$  the following identity

$$\mathcal{N}_{A,q}h = (\mathcal{M}_{A,q}\mathcal{S}h)|_{\partial B}. \quad (2.25)$$

This equality states that  $\mathcal{N}_{A,q}h$  is nothing but the near field measurements on  $\partial B$  generated by an incident field  $v := \mathcal{S}h$ .

From properties and jump relations for single-layer potential (see [24])  $v = \mathcal{S}h$  with density  $h \in L^2(\partial B)$  is defined in  $\mathbb{R}^3$  and satisfies the Helmholtz equation in  $\mathbb{R}^3 \setminus \partial B$ , the Sommerfeld radiation condition (2.13), and the following continuity and jump properties across

$\partial B$ :

$$\begin{aligned} v^-(x) &= v^+(x) = v(x) && \text{on } \partial B, \\ \partial_\nu v^-(x) - \partial_\nu v^+(x) &= h(x) && \text{on } \partial B, \end{aligned} \quad (2.26)$$

where  $v^+$  and  $v^-$  respectively denote the restriction of  $v$  to  $\mathbb{R}^3 \setminus \bar{B}$  and  $B$ . In order to exploit the information encoded into the identity (2.25), one can easily check the following lemma by using (2.26).

**Lemma 2.3.5.** *Assume that Assumption 2.3.1 holds and  $h \in L^2(\partial B)$ . Set  $v = \mathcal{S}h$  and  $u^s = \mathcal{M}_{A,q}v$ . Then the total field  $u = v + u^s$  is solution to the transmission problem*

$$\begin{aligned} \mathcal{H}_{A,q}u(x) &= k^2u(x) && \text{in } \mathbb{R}^3 \setminus \partial B, \\ u^+(x) &= u^-(x) && \text{on } \partial B, \\ \partial_\nu u^-(x) - \partial_\nu u^+(x) &= h(x), && \text{on } \partial B \end{aligned} \quad (2.27)$$

together with the Sommerfeld radiation condition (2.13).

We finally point out the following lemma that will be useful what follows.

**Lemma 2.3.6.** *Assume that Assumption 2.3.1 holds. Then, we have*

$$\int_{\partial B} (\mathcal{N}_{A,q}f) g ds(x) = \int_D \mathcal{S}g Q_{A,q}(\mathcal{M}_{A,q}\mathcal{S}f + \mathcal{S}f) dx, \quad (2.28)$$

for all  $f, g \in L^2(\partial B)$ .

*Proof.* Using the definition of  $\mathcal{N}_{A,q}$ , we have

$$\int_{\partial B} (\mathcal{N}_{A,q}f) g ds(x) = \int_{\partial B} \int_{\partial B} u_{A,q}^s(x, y) f(y) g(x) ds(x) ds(y),$$

where  $u_{A,q}^s(\cdot, y)$  is the scattered field associated to  $\Phi(\cdot, y)$ ,  $y \in \partial B$ . Moreover, from (2.16), we get

$$\int_{\partial B} (\mathcal{N}_{A,q}f) g ds(x) = \int_{\partial B} \int_{\partial B} \left[ \int_D \Phi(x, t) Q_{A,q}u(t, y) dt \right] (x, y) f(y) g(x) ds(x) ds(y).$$

According to Fubini's Theorem, we find

$$\int_{\partial B} (\mathcal{N}_{A,q}f) g ds(x) = \int_D \left( \int_{\partial B} \Phi(x, t) g(x) ds(x) \right) \int_{\partial B} Q_{A,q}u(t, y) f(y) ds(y).$$

Using the fact that  $u_{A,q}(\cdot, y) = u_{A,q}^s(\cdot, y) + \Phi(\cdot, y)$ ,  $y \in \partial B$ , we obtain

$$\begin{aligned} \int_{\partial B} (\mathcal{N}_{A,q} f) g ds(x) &= \int_D \mathcal{S}g(t) Q_{A,q} \left( \int_{\partial B} u^s(t, y) f(y) ds(y) \right. \\ &\quad \left. + \int_{\partial B} \Phi(t, y) f(y) ds(y) \right) (t) dt \\ &= \int_D \mathcal{S}g(t) Q_{A,q} (\mathcal{M}_{A,q} \mathcal{S}f + \mathcal{S}f) (t) dt, \end{aligned}$$

where the operators  $\mathcal{M}_{A,q}$  and  $\mathcal{S}$  are given by (2.22) and (2.24) respectively.

The proof is completed.  $\square$

We establish now the following result that proves that the transpose operator associated with  $\mathcal{N}_{A,q}$  is equal to  $\mathcal{N}_{-A,q}$ . In order to ease the writing, we indicate two useful formulas that we shall use a few times. The first one is a consequence of the Green's theorem and states that

$$\int_B (\mathcal{H}_{A,q} u_1 u_2 - u_1 \mathcal{H}_{-A,q} u_2) dx = \int_{\partial B} (u_1 \partial_r u_2 - u_2 \partial_r u_1) ds(x), \quad (2.29)$$

for all  $u_1, u_2 \in H^2(B)$ . The second one is a classical consequence of the Green's theorem and the Rellich lemma and states that [24]

$$\int_{\partial B} (u_1 \partial_r u_2 - u_2 \partial_r u_1) ds(x) = 0, \quad (2.30)$$

for all  $u_1, u_2 \in H_{\text{loc}}^2(\mathbb{R}^3 \setminus B)$  satisfying the Helmholtz equation  $\Delta u + k^2 u = 0$  in  $\mathbb{R}^3 \setminus \bar{B}$  and the Sommerfeld radiation condition (2.13).

**Lemma 2.3.7.** *Assume that Assumption 2.3.1 holds. Let  $y, z \in \mathbb{R}^3 \setminus \bar{D}$  and set*

$$u_{A,q}^s(\cdot, y) := \mathcal{M}_{A,q} \Phi(\cdot, y), \quad \text{and} \quad u_{-A,q}^s(\cdot, z) := \mathcal{M}_{-A,q} \Phi(\cdot, z).$$

*Then we have the following reciprocity relation:*

$$u_{A,q}^s(z, y) = u_{-A,q}^s(y, z).$$

*This reciprocity implies in particular that  $(\mathcal{N}_{A,q})^t = \mathcal{N}_{-A,q}$ , i.e.,*

$$\int_{\partial B} f (\mathcal{N}_{-A,q} g) ds(x) = \int_{\partial B} (\mathcal{N}_{A,q} f) g ds(x) \quad \text{for all } f, g \in L^2(\partial B). \quad (2.31)$$

*Proof.* From (2.16)

$$u_{A,q}^s(z, y) = \int_D \Phi(z, t) Q_{A,q} (u_{A,q}^s(t, y) + \Phi(t, y)) dt.$$

On the other hand, applying (2.29) and (2.30) to  $u_1 = u_{A,q}^s(\cdot, y)$  and  $u_2 = u_{-A,q}^s(\cdot, z)$  implies

$$\int_B \left( \mathcal{H}_{A,q} u_{A,q}^s(t, y) u_{-A,q}^s(t, z) - u_{A,q}^s(t, y) \mathcal{H}_{-A,q} u_{-A,q}^s(t, z) \right) dt = 0.$$

Using (2.15) yields

$$\int_D \left( Q_{A,q} \Phi(t, y) u_{-A,q}^s(t, z) - u_{A,q}^s(t, y) Q_{-A,q} \Phi(t, z) \right) dt = 0.$$

Using the Green's theorem we obtain (since  $y, z \in \mathbb{R}^3 \setminus \bar{D}$  and  $A$  has compact support in  $D$ )

$$\int_B \left( \Phi(t, y) Q_{-A,q} u_{-A,q}^s(t, z) - Q_{A,q} u_{A,q}^s(t, y) \Phi(t, z) \right) dt = 0.$$

When then conclude, since  $\Phi(z, t) = \Phi(t, z)$ ,

$$u_{A,q}^s(z, y) = \int_D \left( \Phi(t, y) Q_{-A,q} u_{-A,q}^s(t, z) + \Phi(t, z) Q_{A,q} \Phi(t, y) \right) dt.$$

Applying the Green's theorem to the second term in the integral finally shows that

$$u_{A,q}^s(z, y) = \int_D \Phi(t, y) Q_{-A,q} (u_{-A,q}^s(t, z) + \Phi(t, z)) dt = u_{-A,q}^s(y, z).$$

Identity (2.31) is a direct consequence of the reciprocity relation and the Fubini theorem.  $\square$

## 2.4 | Analysis of the direct problem using the variational approach

We now use a variational approach to prove the well-posedness of the direct scattering problem as in the previous section. The advantage of this approach is to weaken the needed regularities for  $A$  and  $q$ .



**Assumption 2.4.1.** Let  $A \in L^\infty(\mathbb{R}^3)^3$  be a real vector valued magnetic potential such that  $\operatorname{div}(A) \in L^\infty(\mathbb{R}^3, \mathbb{R})$  and  $\operatorname{Supp}(A) \subset D$  and the refractive index  $n \in L^\infty(\mathbb{R}^3, \mathbb{C})$  with non negative imaginary part and positive real part such that  $n = 1$  outside  $D$ .

Let  $B$  be a sufficiently large ball with radius  $R$  containing  $D$ . We now introduce the Dirichlet to Neumann map  $\Lambda$ .

**Definition 2.4.2.** [15] The Dirichlet-to-Neumann map  $\Lambda$  is defined by

$$\begin{aligned} \Lambda : H^{\frac{1}{2}}(\partial B) &\longrightarrow H^{-\frac{1}{2}}(\partial B) \\ \varphi &\longmapsto \frac{\partial u^s}{\partial \nu} \Big|_{\partial B}, \end{aligned} \quad (2.32)$$

where  $u^s \in H_{loc}^1(\mathbb{R}^3 \setminus B)$  is a solution to the following system

$$\begin{cases} \Delta u^s + k^2 u^s = 0 & \text{in } \mathbb{R}^3 \setminus B, \\ u^s = \varphi & \text{on } \partial B, \\ \lim_{r \rightarrow \infty} r \left( \frac{\partial u^s}{\partial r} - i k u^s \right) = 0, \end{cases} \quad (2.33)$$

and  $\nu$  is the outward normal.

We recall that (see[15]),  $\Lambda$  is a continuous map from  $H^{\frac{1}{2}}(\partial B)$  to  $H^{-\frac{1}{2}}(\partial B)$  and verifies the following properties

$$\Im (\langle \Lambda \varphi, \varphi \rangle) > 0, \quad \forall \varphi \in H^{\frac{1}{2}}(\partial B), \varphi \neq 0, \quad (2.34)$$

and

$$\Re (\langle \Lambda \varphi, \varphi \rangle) \leq 0, \quad \forall \varphi \in H^{\frac{1}{2}}(\partial B), \quad (2.35)$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality product between  $H^{-\frac{1}{2}}(\partial B)$  and  $H^{\frac{1}{2}}(\partial B)$  that extends the  $L^2(\partial B)$  scalar product for regular functions.

Problem (2.11)-(2.13) can be equivalently formulated as a variational problem for  $u \in H^1(B)$  satisfying, for all  $v \in H^1(B)$ , the following equation

$$\int_B (\nabla + iA)u \cdot (\nabla - iA)\bar{v} \, dx - k^2 \int_B nu\bar{v} \, dx - \langle \Lambda u, v \rangle = -\langle \Lambda(u^i) - \frac{\partial u^i}{\partial \nu}, v \rangle. \quad (2.36)$$

Let us introduce the sesquilinear form  $a(\cdot, \cdot) : H^1(B) \times H^1(B) \longrightarrow \mathbf{C}$  defined by

$$a(u, v) = \int_B (\nabla + iA)u \cdot (\nabla - iA)\bar{v} \, dx - k^2 \int_B nu\bar{v} \, dx - \langle \Lambda u, v \rangle, \quad (2.37)$$

and the linear form  $\ell(\cdot) : H^1(B) \longrightarrow \mathbf{C}$  defined by

$$\ell(v) = -\langle \Lambda(u^i) - \frac{\partial u^i}{\partial v}, v \rangle = -\langle g, v \rangle. \quad (2.38)$$

Then, problem (2.36) can be rewritten in the following form

$$a(u, v) = \ell(v) \quad \forall v \in H^1(B). \quad (2.39)$$

We establish now the following result that prove the well-posedness of the direct problem.

**Theorem 2.4.3.** *Assume that Assumption 2.4.1 holds. Then, there exists a unique solution  $u \in H^1(B)$  to (2.39). Moreover, there exists a constant  $C$  independent from  $g$  such that*

$$\|u\|_{H^1(B)} \leq C \|g\|_{H^{-\frac{1}{2}}(\partial B)}. \quad (2.40)$$

*Proof.* Obviously,  $\ell$  is a bounded anti-linear functional on  $H^1(B)$ . Let  $\ell_g \in H^1(B)$  be such that  $\ell(v) = (\ell_g, v)_{H^1(B)}$ , for all  $v \in H^1(B)$ , which is uniquely provided by the Riesz representation theorem and satisfies

$$\|\ell_g\|_{H^1(B)} \leq C \|g\|_{H^{-\frac{1}{2}}(\partial B)}. \quad (2.41)$$

Now, let us define the sequilinear form

$$a_0(u, v) = \int_B \nabla u \cdot \nabla \bar{v} \, dx + \int_B u\bar{v} \, dx - \langle \Lambda u, v \rangle, \quad \forall u, v \in H^1(B), \quad (2.42)$$

and

$$\begin{aligned} b(u, v) &= a(u, v) - a_0(u, v) \\ &= \int_B i(A \cdot \nabla \bar{v} u - A \cdot \nabla u \bar{v}) \, dx + \int_B (|A|^2 - k^2 n - 1)u\bar{v} \, dx, \end{aligned} \quad (2.43)$$

for all  $w, v \in H^1(B)$ . Moreover, using the Riesz representation theorem, we can define the operators  $\mathcal{A}, \mathcal{B} : H^1(B) \longrightarrow H^1(B)$  by

$$(\mathcal{A}u, v)_{H^1(B)} = a_0(u, v),$$

and

$$(\mathcal{B}u, v)_{H^1(B)} = b(u, v),$$

respectively. Then, problem (2.39) is equivalent to the following problem: Find  $u \in H^1(B)$  such that

$$(\mathcal{A} + \mathcal{B})u = \ell_g. \quad (2.44)$$

The invertability of the operator  $(\mathcal{A} + \mathcal{B})$  is necessary to establish the existence and uniqueness of (2.44). We have

$$\begin{aligned} |a_0(u, u)| &\geq |\Re(\mathcal{A}u, u)_{H^1(B)}| \\ &\geq \|u\|_{H^1(B)}^2 - \Re(\langle \Lambda u, u \rangle). \end{aligned}$$

Using (2.35), we obtain

$$|a_0(u, u)| \geq C\|u\|_{H^1(B)}^2, \quad \forall u \in H^1(B),$$

which implies, using the Lax-Milgram theorem, that the operator  $\mathcal{A}$  is an isomorphism.

Furthermore, to prove  $\mathcal{B}$  is compact, let a sequence  $(u_\rho)_{\rho \in \mathbb{N}} \in H^1(B)$  and assume that there exist  $u \in H^1(B)$  such that

$$u_\rho \rightharpoonup u \quad \text{in } H^1(B).$$

Thanks to the compact embedding of  $H^1(B)$  into  $L^2(B)$ , we obtain  $u_\rho$  converges strongly to  $u$  in  $L^2(B)$ . Let us set  $e_\rho = u_\rho - u$ . We first observe that  $Be_\rho$  weakly converges to 0 in  $H^1(B)$  and  $e_\rho$  strongly converges to 0 in  $L^2(D)$ . Moreover,  $Be_\rho$  strongly converges to 0 in  $L^2(B)$ . From the definition of  $B$ , we have

$$\|\mathcal{B}e_\rho\|_{H^1(B)}^2 = \int_B i(A \cdot \nabla \overline{\mathcal{B}e_\rho} e_\rho - A \cdot \nabla e_\rho \overline{\mathcal{B}e_\rho}) dx + \int_B (|A|^2 - k^2 n - 1) e_\rho \overline{\mathcal{B}e_\rho} dx.$$

Since  $A \in L^\infty(B)$  then  $A\overline{B}e_\rho$  and  $Ae_\rho$  strongly converge to 0 in  $L^2(B)$ . We then can take the limit in the integrals as the product of weakly convergent sequences against strongly convergent one and obtain

$$\|\mathcal{B}e_\rho\|_{H^1(B)} \xrightarrow{\rho \rightarrow +\infty} 0,$$

which gives us the desired result. Thus, we conclude that the operator  $(\mathcal{A} + \mathcal{B})$  is Fredholm of index zero.

Finally, to prove that the operator  $(\mathcal{A} + \mathcal{B})$  is invertible, we need to show that  $(\mathcal{A} + \mathcal{B})$  is injective. We assume that we have

$$(\mathcal{A} + \mathcal{B})u = 0, \quad u \in H^1(B),$$

which implies

$$((\mathcal{A} + \mathcal{B})u, v)_{H^1(B)} = 0, \quad \forall v \in H^1(B).$$

Consequently, for  $v = u$ , we find

$$\int_B (\nabla + iA)u(\nabla - iA)\bar{u} \, dx - k^2 \int_B n|u|^2 \, dx - \langle \Lambda u, u \rangle = 0, \quad \forall u \in H^1(B).$$

Thus, we obtain

$$\Im \left( \int_B |(\nabla + iA)u|^2 \, dx - k^2 \int_B n|u|^2 \, dx - \langle \Lambda u, u \rangle \right) = 0.$$

This yields

$$k^2 \int_B \Im(n)|u|^2 \, dx + \Im(\langle \Lambda u, u \rangle) = 0, \quad \forall u \in H^1(B).$$

We know that  $\Im(n) \geq 0$ . Therefore

$$\Im(\langle \Lambda u, u \rangle) = 0, \quad \forall u \in H^1(B).$$

Using the property (2.34), we get  $u = 0$  on  $\partial B$  and by the definition of the map  $\Lambda$  (Theorem 2.4.2), we find  $\frac{\partial u}{\partial \nu} = 0$  on  $\partial B$ . By unique continuation theorems, we get  $u = 0$  in  $B \setminus D$ . The function  $u$  satisfies

$$\Delta u + i \operatorname{div}(A)u + 2iA \cdot \nabla u + (k^2 n - |A|^2)u = 0 \quad \text{in } B,$$

Since  $A \in L^\infty(B)$  and  $\operatorname{div}(A) \in L^\infty(B)$  one can apply unique continuation theorem to deduce that  $u = 0$  in  $B$ . Thus, we proved the existence of a unique solution to (2.39). Moreover, using the invertibility of  $(\mathcal{A} + \mathcal{B})$  and using (2.41), we get the inequality (2.40).  $\square$

## 2.5 | Numerical examples using FreeFem++

In this section, we give some numerical illustrations of the solution to the scattering problem using Lagrange finite elements. We apply classical finite elements discretization procedure on the variational formulation  $u \in H^1(B)$  such that

$$a(u, v) = \ell(v) \quad \forall v \in H^1(B), \quad (2.45)$$

where

$$a(u, v) := - \int_B (\nabla + iA)u \cdot (\nabla - iA)\bar{v} \, dx - k^2 \int_B nu\bar{v} \, dx - \langle \Lambda u, v \rangle, \quad (2.46)$$

for all  $w, v \in H^1(B)$  and

$$\ell(v) := - \langle \Lambda(u^i) - \frac{\partial u^i}{\partial \nu}, v \rangle = - \langle g, v \rangle, \quad \forall v \in H^1(B). \quad (2.47)$$

We modified an existing FreeFem++ program, originally designed for solving the direct Helmholtz problem for  $A = 0$  to include the case  $A \neq 0$ .

The following are numerical results obtained corresponding to inhomogeneities, in a two-dimensional setting, with index of refraction equal to  $n = 3$  in  $D$  and  $n = 1$  outside of  $D$ . We illustrate the influence of various factors such as the presence of the magnetic potential  $A$  and the frequency  $k$ . The numerical solutions are computed using  $P_1$  Lagrange finite elements.

- **Example 1:** we taking the frequency  $k = 6$ . Set  $\lambda = 2\pi/k$  as the wavelength. We choose  $D$  is disc centered at  $(0, 0)$  with radius  $r = 0.5$  and  $B$  be a disc centered also in  $(0, 0)$  with radius  $r_B = r + \pi/3$ .

We here consider an incident plane wave given by  $u^i(\vec{x}, d) = e^{ik\vec{x} \cdot d}$  where  $\vec{x} = (x, y)$  and  $d = (\cos(\theta), \sin(\theta))$  for  $\theta \in (0, 2\pi)$ :

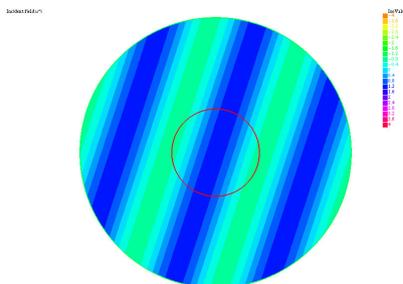


Figure 2.2: Incident field  $u^i$ .

– Influence of the presence of the magnetic potential  $A$ : for  $k = 6$

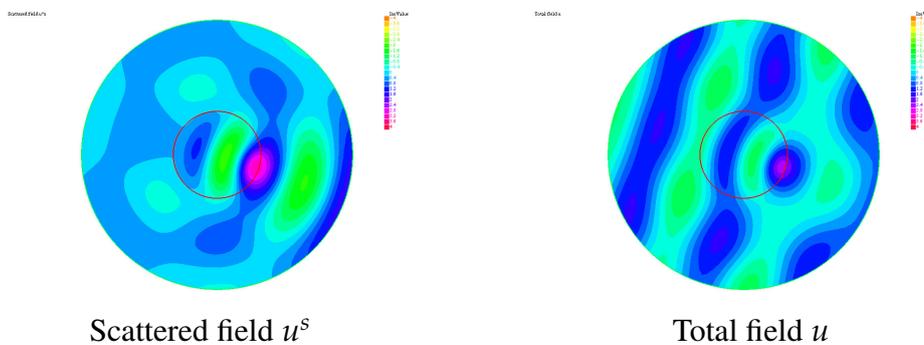


Figure 2.3: The case for  $A = 0$ .

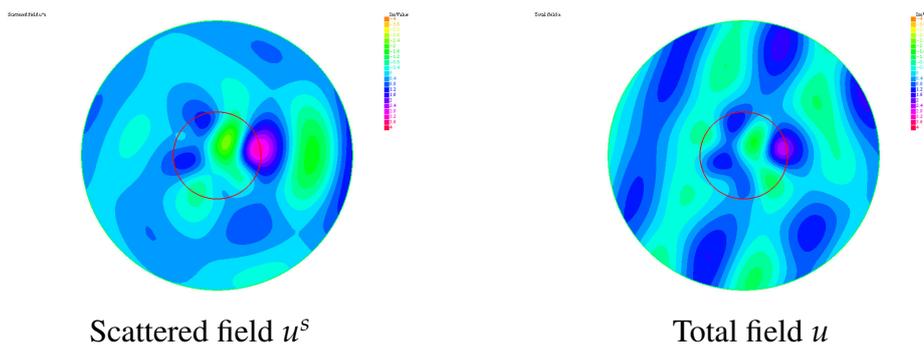


Figure 2.4: The case for  $A = 2.5(-\sin(\theta), \cos(\theta))$  in  $D$  where  $\theta \in (0, 2\pi)$  and  $A = 0$  outside  $D$  ( $\operatorname{div}(A) = 0$  in  $\mathbb{R}^3$ ).

– Influence of the frequency  $k$ :

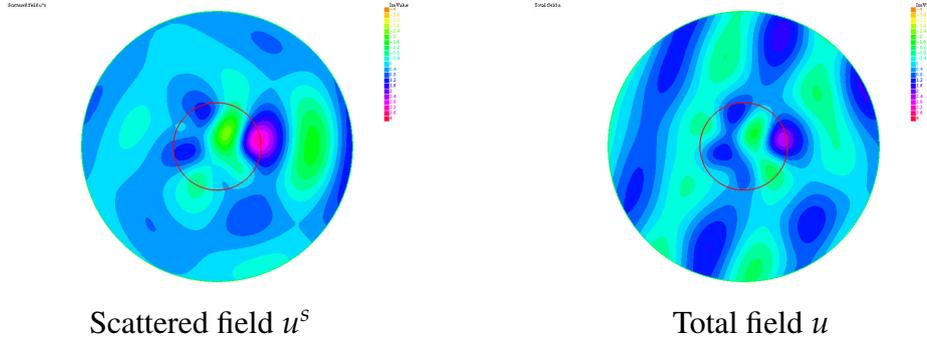


Figure 2.5: The case for  $k = 6$  and  $A = 2.5(-\sin(\theta), \cos(\theta))$  in  $D$  where  $\theta \in (0, 2\pi)$  and  $A = 0$  outside  $D$

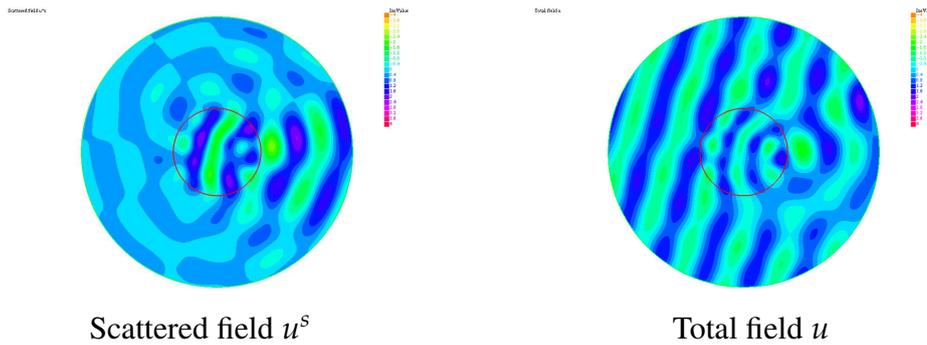


Figure 2.6: The case for  $k = 12$  and  $A = 2.5(-\sin(\theta), \cos(\theta))$  in  $D$  where  $\theta \in (0, 2\pi)$  and  $A = 0$  outside  $D$

■ **Example 2:** we change the shape of  $D$  to be a peanut as shown below.

– Influence of the presence of the magnetic potential  $A$ :

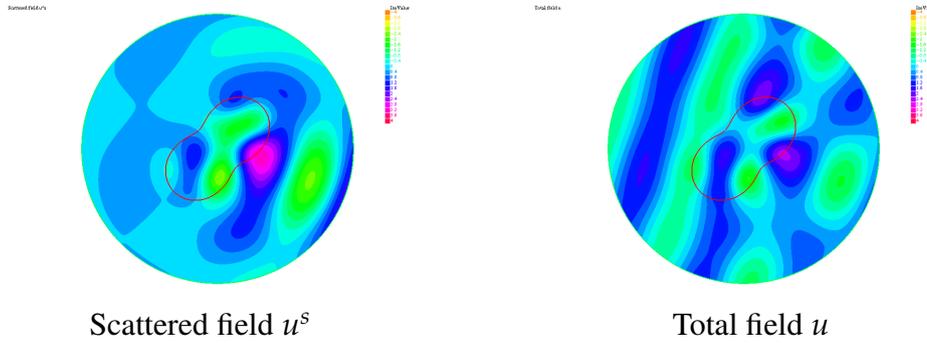


Figure 2.7: The case for  $A = 0$  and  $k = 6$ .

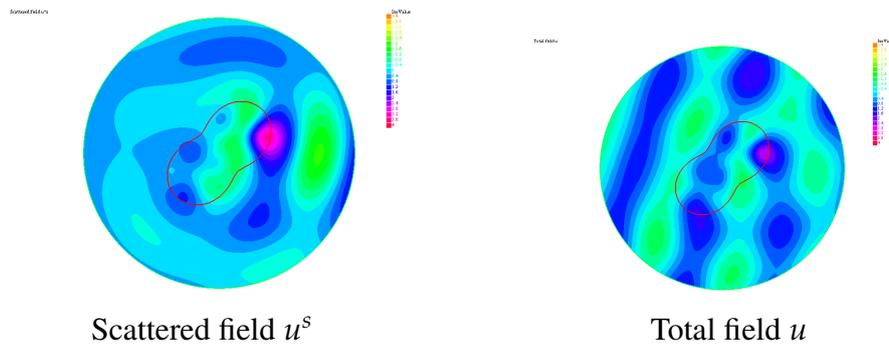


Figure 2.8: The case for  $A = 2.5(x_\theta, y_\theta)$  in  $D$  where  $x_\theta = 0.8 \cos(2\theta) \cos(\theta) - (1 + 0.4 \sin(2\theta)) \sin(\theta)$ ,  $y_\theta = 0.8 \cos(2\theta) \sin(\theta) + (1 + 0.4 \sin(2\theta)) \cos(\theta)$  and  $\theta \in (0, 2\pi)$  and  $A = 0$  outside  $D$  ( $\operatorname{div}(A) = 0$ ), and  $k = 6$ .



– Influence of the frequency  $k$ :

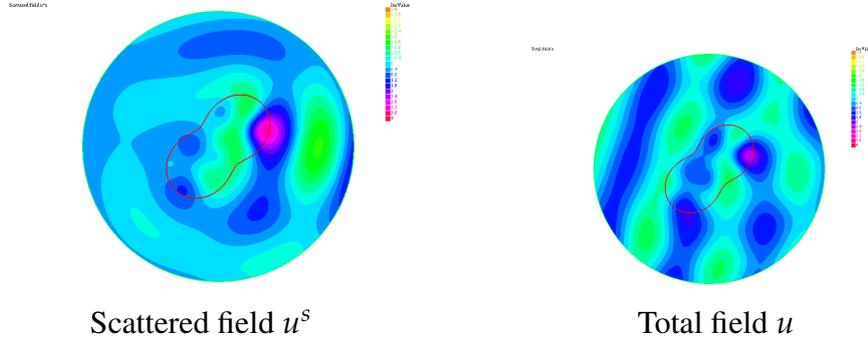


Figure 2.9: The case for  $k = 6$  for  $A = 2.5(x_\theta, y_\theta)$  in  $D$  where  $x_\theta = 0.8 \cos(2\theta) \cos(\theta) - (1 + 0.4 \sin(2\theta)) \sin(\theta)$ ,  $y_\theta = 0.8 \cos(2\theta) \sin(\theta) + (1 + 0.4 \sin(2\theta)) \cos(\theta)$  and  $\theta \in (0, 2\pi)$

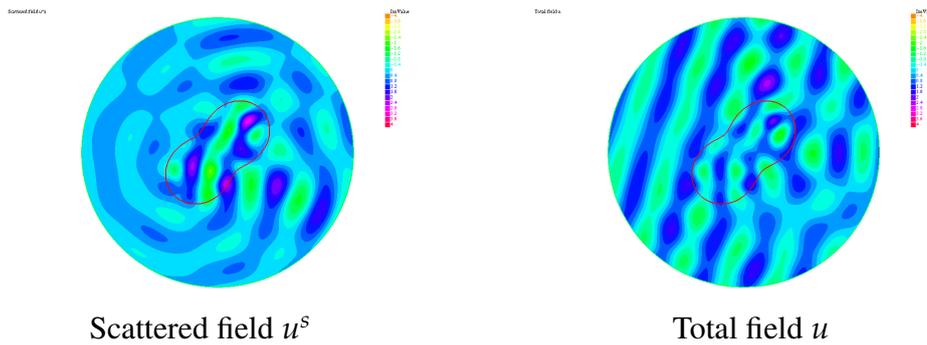


Figure 2.10: The case for  $k = 12$  for  $A = 2.5(x_\theta, y_\theta)$  in  $D$  where  $x_\theta = 0.8 \cos(2\theta) \cos(\theta) - (1 + 0.4 \sin(2\theta)) \sin(\theta)$ ,  $y_\theta = 0.8 \cos(2\theta) \sin(\theta) + (1 + 0.4 \sin(2\theta)) \cos(\theta)$  and  $\theta \in (0, 2\pi)$

# Stability results

## **Abstract:**

*This chapter is mainly extracted from the published article [10]. We derive conditional stability estimates for inverse scattering problems related to time harmonic magnetic Schrödinger equation. We prove logarithmic type estimates for retrieving the magnetic and electric potentials from near field or far field maps. Our approach combines techniques from similar results obtained in the literature for inhomogeneous inverse scattering problems based on the use of geometrical optics solutions.*

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## 3.1 | Introduction

This chapter is concerned with the inverse scattering problem of recovering the magnetic and electric potentials in the magnetic Schrödinger model from near field or far field measurements at a fixed frequency. (see for instance [82, 61]).

In the absence of the magnetic potential  $A$ , the study of the identifiability of  $q$  from full aperture measurements is one of the first foundational problems in inverse scattering theory and we refer to [68, 70, 71, 87] for pioneering uniqueness results under various (regularity) assumptions. In the presence of a magnetic potential  $A$ , we remind that there is an obstruction to uniqueness for both near field and far field settings (as has been noted in [85] for instance). In fact, the magnetic potential  $A$  cannot be uniquely determined from far field or near field measurements outside  $B$  due the gauge transformation that we will explain it in the Section 3.2. Several studies have investigated about the presence of the magnetic potential, so we refer to [8, 9, 69, 72, 90].

The outline of the chapter is as follows . We start, in Section 3.2, by presenting the Gauge invariance with numerical results. In Section 3.4, we establish the stability result first for the magnetic field in the case of near field data. We then employ a carefully designed Helmholtz decomposition to infer the stability result for the electric potential. The derivation of the results for far field data are obtained after establishing some key properties relating this data to the near field data in Section 3.5.

## 3.2 | Gauge invariance with numerical results

Gauge invariance is a concept that permeates every branch of mathematics, including differential geometry and mathematical analysis. This involves the idea that certain mathematical objects, such as functions or vector fields, remain invariant under certain transformations, so-called gauge transformations. Gauge invariance provides a powerful framework for studying scattering phenomena, independent of the choice of measuring instrument (see, for instance, [72, 32, 45, 11]).

### 3.2.1 | Gauge invariance

We recall that  $A \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$  and  $q \in L^\infty(\mathbb{R}^3, \mathbb{C})$  with non negative imaginary part such that  $\text{Supp}(A) \subset D$  and  $\text{Supp}(q) \subset D$ . For  $B$  be a ball containing  $D$  and let  $y \in \partial B$  be the location of a point source. The total field  $u(\cdot, y)$  generated by the point source satisfies

$$\mathcal{H}_{A,q}u(\cdot, y) - k^2u(\cdot, y) = \delta_y \quad \text{in } \mathbb{R}^3, \quad (3.1)$$

$$u_{A,q}(\cdot, y) = \Phi(\cdot, y) + u_{A,q}^s(\cdot, y) \quad \text{in } \mathbb{R}^3, \quad (3.2)$$

$$\lim_{r \rightarrow \infty} r \left( \partial_r u^s - iku^s \right) = 0, \quad r = |x| \quad (3.3)$$

uniformly with respect to  $\hat{x} = \frac{x}{|x|}$  and where the scattered field  $u_{A,q}^s(\cdot, y) \in H_{\text{loc}}^2(\mathbb{R}^3)$  and the incident field is given by (2.14).

The first inverse problem that we shall investigate is to recover  $A$  and  $q$  from the knowledge  $u_{A,q}^s(x, y)$  for all  $(x, y) \in \partial B \times \partial B$ . Defining the near field operator  $\mathcal{N}_{A,q} : L^2(\partial B) \rightarrow L^2(\partial B)$ , as

$$\mathcal{N}_{A,q}h(x) := \int_{\partial B} u_{A,q}^s(x, y)h(y) \, ds(y), \quad x \in \partial B, \quad (3.4)$$

where  $u_{A,q}^s(\cdot, y)$  is given by (3.2) and satisfying (3.3), the inverse problem in the near field setting can be equivalently stated as identifying  $A$  and  $q$  from the knowledge of  $\mathcal{N}_{A,q}$ .

The direct scattering problem in the far field setting formally corresponds with letting  $|y| \rightarrow \infty$  in the direction  $-d$  with  $d \in \mathbb{S}^2$  (the unit sphere of  $\mathbb{R}^3$ ) and can be phrased as

follows: Given an incident plane wave  $u^i(x, d) = e^{ikx \cdot d}$ ,  $x \in \mathbb{R}^3$ , seek a total field  $u_{A,q}(\cdot, d)$  that satisfies

$$\mathcal{H}_{A,q}u(\cdot, d) - k^2u(\cdot, d) = 0 \quad \text{in } \mathbb{R}^3, \quad (3.5)$$

$$u_{A,q}(\cdot, d) = u^i(\cdot, d) + u_{A,q}^s(\cdot, d) \quad \text{in } \mathbb{R}^3, \quad (3.6)$$

where the scattered field  $u_{A,q}^s(\cdot, d) \in H_{\text{loc}}^2(\mathbb{R}^3)$  and satisfies the Sommerfeld radiation condition. The latter implies in particular that the scattered field has the following asymptotic behavior as  $|x| \rightarrow \infty$ ,

$$u_{A,q}^s(x, d) = \frac{e^{ik|x|}}{|x|} \left( u_{A,q}^\infty(\hat{x}, d) + O\left(\frac{1}{|x|}\right) \right), \quad (3.7)$$

where  $u_{A,q}^\infty(\cdot, d)$  is the so-called far field pattern. The second inverse problem that we shall consider is the identification of  $A$  and  $q$  from the knowledge of  $u_{A,q}^\infty(\hat{x}, d)$  for all  $(\hat{x}, d) \in \mathbb{S}^2 \times \mathbb{S}^2$ .

We remind that there is an obstruction to uniqueness for both near field and far field settings (as has been noted in [85] for instance). In fact, the scattered field outside a ball  $B$  containing  $D$  is invariant under the gauge transformation of the magnetic potential. Namely, given  $\varphi \in W^{2,\infty}(\mathbb{R}^3)$  with support compactly embedded in  $B$  and letting  $\tilde{u} = u(x)e^{-i\varphi(x)}$  one easily observes that

$$\mathcal{H}_{A+\nabla\varphi,q}\tilde{u} := -(\nabla + i(A + \nabla\varphi))^2\tilde{u} + q(x)\tilde{u} = e^{-i\varphi(x)}\mathcal{H}_{A,q}u. \quad (3.8)$$

Since  $\varphi = 0$  outside  $B$ ,  $\tilde{u}$  then satisfies the same equation as  $u$ , namely (2.12) (respectively (3.6)) in the near field setting (respectively in the far field setting) with  $\mathcal{H}_{A,q}$  replaced by  $\mathcal{H}_{A+\nabla\varphi,q}$ . Let us denote by  $u_{A+\nabla\varphi,q}^s$  the scattered field associated with the potentials  $A + \nabla\varphi$  and  $q$ . From uniqueness of solutions to the above stated scattering problems one easily deduces that for all  $y \in \partial B$  and  $d \in \mathbb{S}^2$

$$u_{A+\nabla\varphi,q}^s(\cdot, y) = (e^{-i\varphi(x)} - 1)\Phi(\cdot, y) + e^{-i\varphi(x)}u_{A,q}^s(\cdot, y) \quad \text{in } \mathbb{R}^3,$$

$$u_{A+\nabla\varphi,q}^s(\cdot, d) = (e^{-i\varphi(x)} - 1)u_i(\cdot, d) + e^{-i\varphi(x)}u_{A,q}^s(\cdot, d) \quad \text{in } \mathbb{R}^3.$$

This clearly shows that  $u_{A+\nabla\varphi,q}^s(\cdot, y) = u_{A,q}^s(\cdot, y)$  and  $u_{A+\nabla\varphi,q}^s(\cdot, d) = u_{A,q}^s(\cdot, d)$  outside  $D$  and therefore, the magnetic potential  $A$  cannot be uniquely determined from far field or near field measurements outside  $B$ . It indicates that the best we can expect from the knowledge of the near field operator  $\mathcal{N}_{A,q}$  or the far field  $u_{A,q}^\infty$  is to identify  $(A, q)$  modulo a gauge transformation of  $A$ . When  $\text{Supp}(A) \subset D$  is known, the problem may be equivalently reformulated as whether the magnetic field defined by the 2-form associated with the vector  $A$ ,

$$\text{curl } A := \frac{1}{2} \sum_{i,j=1}^3 \left( \partial_{x_j} a_i - \partial_{x_i} a_j \right) dx_j \wedge dx_i, \quad (3.9)$$

and the electric potential  $q$  can be retrieved from far field or near field measurements. The uniqueness for similarly stated inverse problems has been established in [62] for  $L^\infty$  regularity of the coefficients. It has been studied in earlier works under more regularity assumptions in [69] and for small perturbations in [85, 82]. We also quote the recent uniqueness result in [66] for measurements associated with finite number of incident waves but with full frequency range.

We hereafter shall follow a similar approach as in [40, 84] to study the case when one would like to simultaneously recover  $\text{curl } A$  and  $q$  from full aperture measurements in the light of geometrical optics solutions developed in [11, 79, 85, 90] for various context in relation with the inverse problem we are interested in.

For bounded domains, the inverse problem with full aperture measurements corresponds with measuring the global Dirichlet to Neumann map. For this problem Tzou proved in [90] log-type stability estimate for  $H^{-1}$  norms of the coefficients, assuming that the magnetic potentials are in  $W^{2,\infty}$  and the electric potentials are in  $L^\infty$ . We here consider stability with respect to the  $L^\infty$  norm with explicit link between the additional needed regularity for the coefficients and the logarithm exponent. Let us finally indicate that uniqueness and log-log stability results with partial data have been also studied by many authors in the literature (see for instance [90, 19, 73, 31, 11]) but are not addressed in the present work.

### 3.2.2 | Numerical verification

In this subsection, we present the validation by numerical result pertaining to the gauge invariance by using FreeFem++ to solve the direct scattering problem with  $u^i$  a plane wave. The physical parameters are set as the frequency  $k = 6$  and the refractive index  $n = 3$  in  $D$  and  $n = 0$  outside  $D$ . Set  $\lambda = 2\pi/k$  as the wavelength. Let  $D$  is ball centered in  $(0,0)$  with radius  $r = 0.5$  and  $B$  be a ball centered also in  $(0,0)$  with radius  $r_B = r + \pi/3$ .

Moreover, we consider the magnetic potential  $A = (A_1, A_2)$  by  $A = 2.5(-\sin(\theta), \cos(\theta))$  in  $D$  where  $\theta \in (0, 2\pi)$  and  $A = (0,0)$  outside  $D$

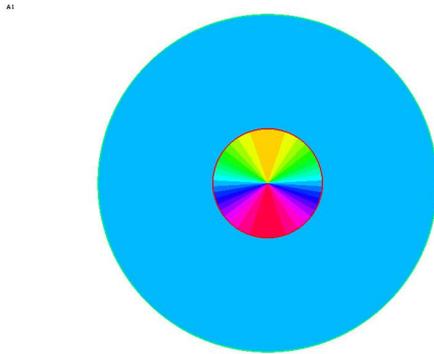


Figure 3.1:  $A_1$  (the 1<sup>st</sup> component of  $A$ )

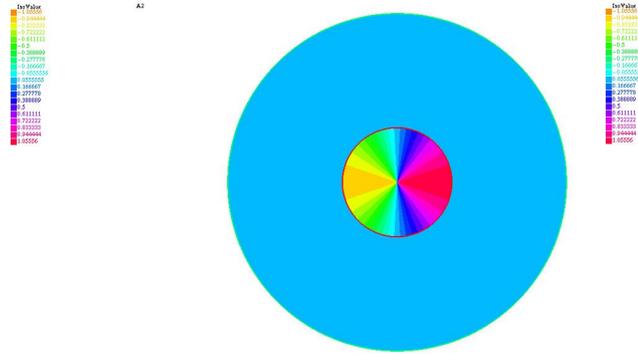


Figure 3.2:  $A_2$  (the 2<sup>nd</sup> component of  $A$ )

Let  $\varphi \in W^{2,\infty}(\mathbb{R}^3)$  with support compactly embedded in  $D$ . Numerically, we consider  $\varphi$  as a solution of

$$\begin{cases} \Delta\varphi = f & \text{in } D, \\ \varphi = 0 & \text{on } \partial D, \end{cases}$$

where  $f(x, y) = \cos(3\pi x)$ .

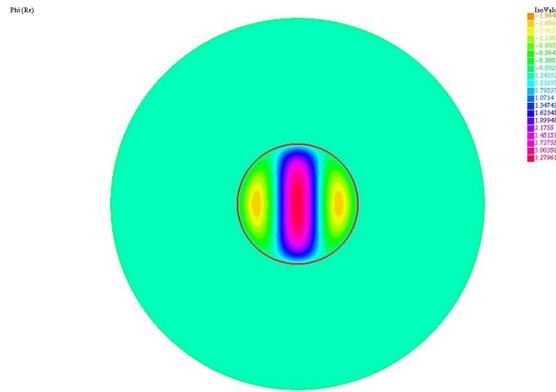


Figure 3.3: the function  $\varphi$

Now, we get the following results:

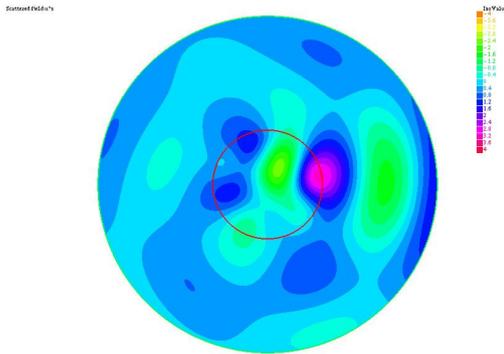


Figure 3.4: Scattered field  $u_{A,q}^s(\cdot, d)$

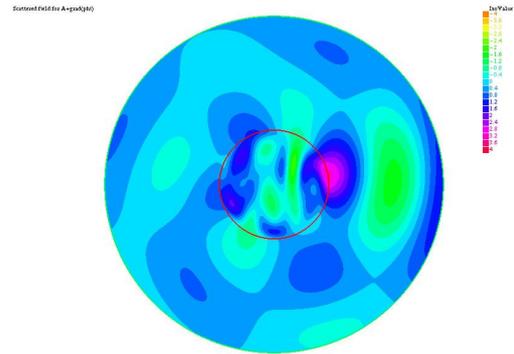
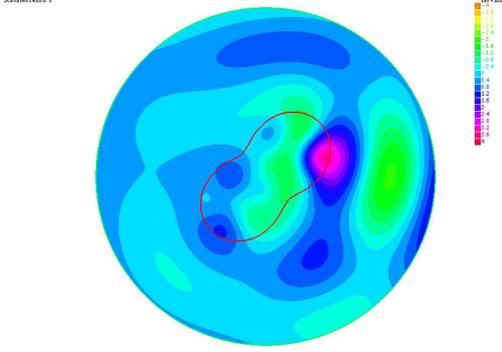
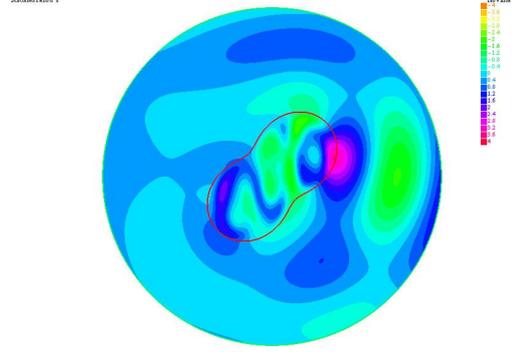


Figure 3.5: Scattered field  $u_{A+\nabla\varphi,q}^s(\cdot, d)$

$\Rightarrow$  We observe qualitatively the same solution outside  $D$  and that there is a perturbation in  $D$ .

Even if, we change the geometry of  $D$  such that  $D \subset B$ , we also validate the coincidence of the  $u_A^s$  and  $u_{A+\nabla\varphi}^s$  outside of  $D$ .




 Figure 3.6: Scattered field  $u_{A,q}^s(\cdot, d)$ 

 Figure 3.7: Scattered field  $u_{A+\nabla\varphi,q}^s(\cdot, d)$ 

### 3.3 | Main stability results

We here state the main results of this chapter concerning conditional log-stability reconstruction of the magnetic field  $\text{curl } A$  given by (3.9) and the electric potential  $q$  from knowledge of the full aperture far field measurements, i.e.,  $u_{A,q}^\infty(\hat{x}, d)$  for any  $(\hat{x}, d) \in \mathbb{S}^2 \times \mathbb{S}^2$  or from knowledge of the near field operator  $\mathcal{N}_{A,q}$ .

Let us first indicate the required conditions for admissible compactly supported magnetic potentials  $A$  and electric potentials  $q$ . Let  $M > 0$  and  $\sigma > 0$  be given. We define the class of admissible magnetic potentials  $\mathcal{A}_\sigma(M)$  by

$$\mathcal{A}_\sigma(M) := \{A \in W^{2,\infty}(\mathbb{R}^3, \mathbb{R}^3), \text{Supp}(A) \subset D, \\ \|A\|_{W^{2,\infty}} \leq M, \text{ and } \|\widehat{\text{curl} A}\|_{L_\sigma^1(\mathbb{R}^3)} \leq M\}, \quad (3.10)$$

where  $\widehat{v}$  denotes the Fourier transform of  $v$  and  $L_\tau^1(\mathbb{R}^3)$  is the weighted  $L^1(\mathbb{R}^3)$  space with norm

$$\|v\|_{L_\tau^1(\mathbb{R}^3)} := \int_{\mathbb{R}^3} (1 + |\xi|^2)^{\tau/2} |v(\xi)| d\xi.$$

Given  $M > 0$  and  $\gamma > 0$ , we define the class of admissible electric potentials  $\mathcal{Q}_\gamma(M)$  by

$$\mathcal{Q}_\gamma(M) := \{q \in L^\infty(\mathbb{R}^3, \mathbb{C}), \Im(q) \geq 0, \text{Supp}(q) \subset D, \\ \|q\|_{L^\infty(D)} \leq M \text{ and } \|\widehat{q}\|_{L_\gamma^1(\mathbb{R}^3)} \leq M\}. \quad (3.11)$$

The first main result is the following log-stability for the magnetic field  $\text{curl } A$  and the electric potential  $q$  from the near field measurements.

**Theorem 3.3.1.** *Let  $M > 0$ ,  $\sigma > 0$  and  $\gamma > 0$ . Then there exists a constant  $C > 0$  such that for any  $(A_j, q_j) \in \mathcal{A}_\sigma(M) \times \mathcal{Q}_\gamma(M)$ ,  $j = 1, 2$ , we have*

$$\begin{aligned} \|\text{curl}(A_1 - A_2)\|_{L^\infty} &\leq C(\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|^{1/2} + |\log(\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|)|^{-\frac{\sigma}{(\sigma+3)}}), \\ \|q_2 - q_1\|_{L^\infty} &\leq C(\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|^{1/2} + |\log(\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|)|^{-\frac{\gamma\sigma}{(\sigma+3)(2\gamma+3)}}), \end{aligned}$$

Here  $C$  depends only on  $B$ ,  $M$ ,  $\sigma$  and  $\gamma$ .

An exactly similar stability result can be deduced for far field measurements if one uses the following very restrictive norm on the measurements. Let

$$\Gamma := \{(\ell, m), \ell \in \mathbb{N} \cup \{0\}, m \in \llbracket -\ell, \ell \rrbracket\}, \quad (3.12)$$

and denote by  $Y_\ell^m$ ,  $(\ell, m) \in \Gamma$  the complete system of special harmonics on  $\mathbb{S}^2$ . For a far field pattern,  $u^\infty$  we denote by  $\mu_{(\ell_1, m_1; \ell_2, m_2)}$ ,  $(\ell_i, m_i) \in \Gamma$ ,  $i = 1, 2$  its Fourier coefficients given by

$$\mu_{(\ell_1, m_1; \ell_2, m_2)} := \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} u^\infty(\hat{x}, d) \overline{Y_{\ell_1}^{m_1}(\hat{x})} \overline{Y_{\ell_2}^{m_2}(d)} \, \text{ds}(\hat{x}) \, \text{ds}(d). \quad (3.13)$$

Let  $a > 0$  such that  $D \subset \{x \in \mathbb{R}^3; |x| < a\}$ . Following [40], we then introduce the following norm

$$\|u^\infty\|_{\mathcal{F}}^2 := \sum_{(\ell_1, m_1) \in \Gamma} \sum_{(\ell_2, m_2) \in \Gamma} \left(\frac{2\ell_1 + 1}{eka}\right)^{2\ell_1} \left(\frac{2\ell_2 + 1}{eka}\right)^{2\ell_2} |\mu_{(\ell_1, m_1; \ell_2, m_2)}|^2. \quad (3.14)$$

In Lemma 3.5.1 below we prove that this norm is finite for all far fields  $u_{A, q}^\infty$  with  $A \in \mathcal{A}_\sigma(M)$  and  $q \in \mathcal{Q}_\gamma(M)$ . Using Lemma 3.5.5 and Theorem 3.3.1, we immediately get the following theorem.

**Theorem 3.3.2.** *Let  $M > 0$ ,  $\sigma > 0$  and  $\gamma > 0$ . Then there exists a constant  $C > 0$  such that for any  $(A_j, q_j) \in \mathcal{A}_\sigma(M) \times \mathcal{Q}_\gamma(M)$ ,  $j = 1, 2$ , we have*

$$\|\text{curl}(A_1 - A_2)\|_{L^\infty} \leq C(\|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{\mathcal{F}}^{1/2} + |\log(\|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{\mathcal{F}})|^{-\frac{\sigma}{(\sigma+3)}}),$$

and

$$\|q_2 - q_1\|_{L^\infty} \leq C(\|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{\mathcal{F}}^{1/2} + |\log(\|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{\mathcal{F}})|^{-\frac{\gamma\sigma}{(\sigma+3)(2\gamma+3)}}).$$

Here  $C$  depends only on  $D$ ,  $a$ ,  $M$ ,  $\sigma$  and  $\gamma$ .

One can also obtain a slightly modified stability result using the  $L^2$  norm of the measurements following the method in [40]. It is summarized in the following theorem.

**Theorem 3.3.3.** *Let  $M > 0$ ,  $\sigma > 0$ ,  $\gamma > 0$  and  $\varepsilon > 0$ . Then there exist two constants  $C > 0$  and  $\delta > 0$  such that for all  $(A_j, q_j) \in \mathcal{A}_\sigma(M) \times \mathcal{Q}_\gamma(M)$ ,  $j = 1, 2$  verifying  $\|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{L^2(\mathbb{S}^2 \times \mathbb{S}^2)} < \delta$  we have*

$$\|\operatorname{curl}(A_1 - A_2)\|_{L^\infty(D)} \leq C(\log^-(\|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{L^2(\mathbb{S}^2 \times \mathbb{S}^2)})^{-\frac{\sigma}{\sigma+3} + \varepsilon},$$

$$\|q_2 - q_1\|_{L^\infty(D)} \leq C(\log^-(\|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{L^2(\mathbb{S}^2 \times \mathbb{S}^2)})^{-\frac{\gamma\sigma}{(\sigma+3)(2\gamma+3)} + \varepsilon}.$$

Here  $C$  depends only on  $D$ ,  $a$ ,  $M$ ,  $\sigma$ ,  $\varepsilon$ ,  $\delta$  and  $\gamma$ .

From Theorems 3.3.1 and 3.3.2 (or 3.3.3) we immediately derive the uniqueness corollary.

**Corollary 3.3.4.** *Let  $A_1$  and  $A_2 \in \mathcal{A}_\sigma(M)$  be two vector fields,  $q_1$  and  $q_2 \in \mathcal{Q}_\gamma(M)$  and  $B \supset D$ . Then, we have*

$$u_{A_1, q_1}^\infty(\hat{x}, d) = u_{A_2, q_2}^\infty(\hat{x}, d), \quad \forall(\hat{x}, d) \in \mathbb{S}^2 \times \mathbb{S}^2,$$

or

$$u_{A_1, q_1}^s(x, y) = u_{A_2, q_2}^s(x, y), \quad \forall(x, y) \in \partial B \times \partial B,$$

implies  $q_1 = q_2$  and  $\operatorname{curl} A_1 = \operatorname{curl} A_2$  in  $D$ .

## 3.4 | Stability analysis for near field data

The aim of this section is to prove the stability estimates given in Theorem 3.3.1. The first step will be to use the properties of the near fields to prove an orthogonality identity, which

relates the difference of potentials to the difference of near field operators. Then we will use a special family of solutions called complex geometric optics solutions to estimate the Fourier transform of the difference of the magnetic fields and the difference of the electric potentials.

Consider two pairs of potentials  $(A_j, q_j) \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3) \times L^\infty(\mathbb{R}^3, \mathbb{C})$ ,  $j = 1, 2$ , satisfying the assumption 2.3.1. We set

$$A(x) := (A_2 - A_1)(x), \quad q(x) := (q_2 - q_1)(x), \quad x \in \mathbb{R}^3, \quad (3.15)$$

and introduce the first order operator  $\mathcal{P}_{A_1, A_2, q}$  defined by

$$\mathcal{P}_{(A_1, A_2, q)} v := \operatorname{div}(Av) + iA \cdot \nabla v + (|A_2|^2 - |A_1|^2 + q)v, \quad v \in H^1(\mathbb{R}^3), \quad (3.16)$$

here we remark that the coefficients of the first order operator  $\mathcal{P}_{(A_1, A_2, q)}$  are supported in  $D$ .

### 3.4.1 | An orthogonality identity and a key integral inequality

First, we present an orthogonality identity, which relates the difference of potentials to the difference of near field operators.

**Lemma 3.4.1.** *Let  $f_1, f_2 \in L^2(\partial B)$ , and set*

$$u_1^s := \mathcal{M}_{-A_1, q_1} \mathcal{S} f_1, \quad u_2^s := \mathcal{M}_{A_2, q_2} \mathcal{S} f_2 \quad \text{in } \mathbb{R}^3, \quad (3.17)$$

and for  $j = 1, 2$ ,

$$v_j := \mathcal{S} f_j \quad \text{in } D, \quad \text{and} \quad u_j := v_j + u_j^s \quad \text{in } D. \quad (3.18)$$

Then the following identity holds true:

$$\int_{\partial B} (\mathcal{N}_{A_1, q_1} f_2 - \mathcal{N}_{A_2, q_2} f_2) f_1 \, ds(x) = \int_D \mathcal{P}_{(A_1, A_2, q)} u_1 u_2 \, dx. \quad (3.19)$$

*Proof.* Using (2.31) we first observe that

$$\int_{\partial B} (\mathcal{N}_{A_1, q_1} f_2 - \mathcal{N}_{A_2, q_2} f_2) f_1 \, ds(x) = \int_{\partial B} (\mathcal{N}_{-A_1, q_1} f_1) f_2 - (\mathcal{N}_{A_2, q_2} f_2) f_1 \, ds(x).$$

We deduce from (2.28) that

$$\int_{\partial B} (\mathcal{N}_{A_1, q_1} f_2 - \mathcal{N}_{A_2, q_2} f_2) f_1 \, ds(x) = \int_D (v_2 Q_{-A_1, q_1} u_1 - v_1 Q_{A_2, q_2} u_2) \, dx. \quad (3.20)$$

Applying the Green's theorem and (2.30) to  $u_1^s$  and  $u_2^s$  implies

$$\int_B \left( u_2^s (-\Delta u_1^s - k^2 u_1^s) - u_1^s (-\Delta u_2^s - k^2 u_2^s) \right) dx = 0,$$

which yields according to (2.15),

$$\int_D (u_2^s (Q_{-A_1, q_1} u_1) - (Q_{A_2, q_2} u_2) u_1^s) \, dx = 0.$$

Adding the left hand side of this equality to the right-hand side of (3.20) shows that

$$\int_{\partial B} (\mathcal{N}_{A_1, q_1} f_2 - \mathcal{N}_{A_2, q_2} f_2) f_1 \, ds(x) = \int_D (u_2 Q_{-A_1, q_1} u_1 - u_1 Q_{A_2, q_2} u_2) \, dx. \quad (3.21)$$

The result of the lemma follows from (3.21) after integrating by parts in the right hand side and observing that

$$Q_{-A_1, q_1} u_1 - Q_{A_2, q_2} u_1 = \mathcal{P}_{(A_1, A_2, q)} u_1.$$

This completes the proof.  $\square$

We now prove the fundamental integral inequality, which relates the difference of two magnetic potentials and electric potential in  $D$  to the difference between their corresponding near pattern fields. This integral inequality will be the starting point in the proof of the stability estimate for the corresponding inverse problem.

**Lemma 3.4.2.** *There is a constant  $C > 0$  that only depends on  $B$  and  $k$  such that*

$$\begin{aligned} & \left| \int_B [iA \cdot (u_1 \nabla u_2 - u_2 \nabla u_1) - (|A_2|^2 - |A_1|^2 + q) u_1 u_2] dx \right| \\ & \leq C \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| \|u_1\|_{H^2(B)} \|u_2\|_{H^2(B)} \end{aligned} \quad (3.22)$$

for all  $u_1 \in H^2(B)$  satisfying  $\mathcal{H}_{-A_1, q_1} u_1 = k^2 u_1$  in  $B$  and all  $u_2 \in H^2(B)$  satisfying  $\mathcal{H}_{A_2, q_2} u_2 = k^2 u_2$  in  $B$ .

*Proof.* Let  $u_j$ ,  $j = 1, 2$ , be given as in the lemma. Let  $u_j^+ \in H_{\text{loc}}^1(\mathbb{R}^3 \setminus \bar{B})$  be the outgoing solution to the following exterior Dirichlet problem (see for instance [24])

$$\begin{cases} \Delta u_j^+ + k^2 u_j^+ = 0 & \text{in } \mathbb{R}^3 \setminus \bar{B}, \\ u_j^+ = u_j & \text{on } \partial B, \end{cases}$$

and  $u_j^+$  satisfies the Sommerfeld radiation condition (3.3). Elliptic regularity infers that  $u_j^+ \in H_{\text{loc}}^2(\mathbb{R}^3 \setminus \bar{B})$  and that in particular, by trace theorems [65, Theorem 2.1 in Chapter 4],

$$\|\partial_\nu u_j^+\|_{L^2(\partial B)} \leq \tilde{C} \|u_j\|_{H^2(B)}, \quad (3.23)$$

for some constant  $\tilde{C}$  that only depends on  $B$  and  $k$ . Let us extend the functions  $u_j$  as follows

$$u_j(x) := \begin{cases} u_j^-(x) = u_j(x) & \text{if } x \in B, \\ u_j^+(x) & \text{if } x \in \mathbb{R}^3 \setminus \bar{B}, \end{cases}$$

and set

$$f_j := \partial_\nu u_j^- - \partial_\nu u_j^+ \quad \text{on } \partial B. \quad (3.24)$$

By trace theorems and (3.23) we have that  $f_j \in L^2(\partial B)$  and

$$\|f_j\|_{L^2(\partial B)}^2 \leq C \|u_j\|_{H^2(B)}^2, \quad (3.25)$$

for some constant  $C$  that only depends on  $B$  and  $k$ . One can easily check that  $u_1$  and  $u_2$  satisfy the following transmission problems, respectively

$$\begin{cases} \mathcal{H}_{-A_1, q_1} u_1 = k^2 u_1 & \text{in } \mathbb{R}^3 \setminus \partial B, \\ u_1^- = u_1^+ & \text{on } \partial B, \\ \partial_\nu u_1^- - \partial_\nu u_1^+ = f_1 & \text{on } \partial B, \end{cases} \quad \begin{cases} \mathcal{H}_{A_2, q_2} u_2 = k^2 u_2 & \text{in } \mathbb{R}^3 \setminus \partial B, \\ u_2^- = u_2^+ & \text{on } \partial B, \\ \partial_\nu u_2^- - \partial_\nu u_2^+ = f_2 & \text{on } \partial B. \end{cases}$$

Moreover,  $u_j^+$ ,  $j = 1, 2$ , satisfy the Sommerfeld radiation condition (3.3). Consider now the functions

$$v_j(x) := \mathcal{S}f_j = \int_{\partial B} \Phi(x, y) f_j(y) \, ds(y) \quad x \notin \partial B, \quad j = 1, 2.$$

Therefore,  $u_j^s := u_j - v_j, j = 1, 2$ , are the same as in Lemma 2.3.5 and then satisfies Lemma 2.3.7; i.e., it verifies (3.17)-(3.18). Consequently, identity (3.19) holds, namely,

$$\int_D \mathcal{P}_{(A_1, A_2, q)} u_1 u_2 dx = \int_{\partial B} (\mathcal{N}_{A_1, q_1} f_2 - \mathcal{N}_{A_2, q_2} f_2) f_1 ds(x).$$

In view of (3.16), we get

$$\begin{aligned} \int_B [iA \cdot (u_1 \nabla u_2 - u_2 \nabla u_1) - (|A_2|^2 - |A_1|^2 + q) u_1 u_2] dx \\ = \int_{\partial B} (\mathcal{N}_{A_1, q_1} f_2 - \mathcal{N}_{A_2, q_2} f_2) f_1 ds(x). \end{aligned} \quad (3.26)$$

Consequently,

$$\begin{aligned} \left| \int_B [iA \cdot (u_1 \nabla u_2 - u_2 \nabla u_1) - (|A_2|^2 - |A_1|^2 + q) u_1 u_2] dx \right| \\ \leq \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| \|f_1\|_{L^2(\partial B)} \|f_2\|_{L^2(\partial B)}. \end{aligned} \quad (3.27)$$

Identity (3.22) immediately follows from (3.27) and (3.25).  $\square$

### 3.4.2 | Complex geometric optics solutions-CGO

The main strategy of the proof of stability estimate on determining the magnetic field and the electric potential from the near field data is the use of complex geometrical optics solutions in (3.22) to estimate the Fourier coefficients of the difference of two magnetic fields  $\text{curl}(A_2 - A_1)$  and the difference of two potential  $q_2 - q_1$ . We therefore first outline some known results about these special solutions extracted from the literature [11, 79, 85, 90]. Let  $\omega = \omega_1 + i\omega_2$  be a vector with  $\omega_1, \omega_2 \in \mathbb{S}^2$  and  $\omega_1 \cdot \omega_2 = 0$ . We define the operator  $N_\omega := \omega \cdot \nabla$ . Since this operator can be interpreted as the  $\bar{\partial}$  operator in the complex plane defined by  $\omega_1$  and  $\omega_2$  one can construct an inverse operator that can be formally defined by

$$N_\omega^{-1}(g)(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} \left( \frac{\hat{g}(\xi)}{\omega \cdot \xi} \right) d\xi, \quad (3.28)$$

for a compactly supported distribution  $g$  (for instance). We remark that if  $\rho \in \mathbb{C}^3$  satisfies  $\rho \cdot \rho = 0$ , then  $\rho = s\omega$  with  $s = \frac{|\rho|}{\sqrt{2}}$  and  $\omega$  is as above. With this notation for  $\rho$  we

have the following Lemma, where the proof can be found in [11] with the electric potential  $q$  replaced by  $q + k^2$ . Note that this lemma requires more regularity than  $W^{1,\infty}$  for the magnetic potentials.

**Lemma 3.4.3.** *Let  $A_0 \in W^{2,\infty}(D)$  and  $q_0 \in L^\infty(D)$  such that  $\|A_0\|_{W^{2,\infty}} \leq M$ ,  $\|q_0\|_{L^\infty} \leq M$  for some positive constant  $M$ , and  $\text{Supp}(A_0), \text{Supp}(q_0) \subset D$ . There exists  $s_0 > 0$  such that, for any  $s \geq s_0$ ,  $\rho = s\omega$  satisfying  $\rho \cdot \rho = 0$ , there exist complex geometrical solutions  $u(\cdot, \rho) \in H^2(B)$  of the form*

$$u(x, \rho) = e^{ix \cdot \rho} (e^{i\varphi_0(x, \omega)} + r(x, \rho)),$$

to the equation  $\mathcal{H}_{A_0, q_0} u = k^2 u$  in  $B$ , where  $\varphi_0(x, \omega) = N_\omega^{-1}(-\omega \cdot A_0)$  and

$$\|r(\cdot, \rho)\|_{H^m(B)} \leq Cs^{m-1}, \quad 0 \leq m \leq 2 \quad \text{and} \quad \|u(\cdot, \rho)\|_{H^2(B)} \leq Cs^2 e^{\Lambda s},$$

where  $C, \Lambda$  and  $s_0$  depend only on  $B, k$  and  $M$ .

In the remainder of this section we consider two pairs of potentials  $(A_j, q_j) \in W^{2,\infty} \times L^\infty$ ,  $j = 1, 2$ , with  $\text{Supp}(A_j), \text{Supp}(q_j) \subset D$ ,  $\Im(q_j) \geq 0$  and satisfying

$$\|A_j\|_{W^{2,\infty}} \leq M, \quad \|q_j\|_{L^\infty} \leq M, \quad j = 1, 2, \quad (3.29)$$

for some  $M > 0$  fixed and set as previously

$$A(x) := (A_2 - A_1)(x), \quad q(x) := (q_2 - q_1)(x), \quad x \in \mathbb{R}^3. \quad (3.30)$$

Let  $\xi \in \mathbb{R}^3$ ,  $\omega_1, \omega_2 \in \mathbb{S}^2$  be three mutually orthogonal vectors in  $\mathbb{R}^3$ . For each  $s > |\xi|/2$ , let

$$\rho_1 = s \left( i\omega_2 + \left( \frac{\xi}{2s} - \sqrt{1 - \frac{|\xi|^2}{4s^2}} \omega_1 \right) \right) := s\omega_1^*(s), \quad (3.31)$$

$$\rho_2 = s \left( -i\omega_2 + \left( \frac{\xi}{2s} + \sqrt{1 - \frac{|\xi|^2}{4s^2}} \omega_1 \right) \right) := s\omega_2^*(s). \quad (3.32)$$



For  $s \geq s_0 > 0$  for some  $s_0$  sufficiently large (that only depends on  $B$  and  $k$ ), Lemma 3.4.3 guarantees the existence of the geometrical optics solutions:  $u_1 \in H^2(B)$  verifying  $\mathcal{H}_{-A_1, q_1} u_1 = k^2 u_1$  in  $B$  and  $u_2 \in H^2(B)$  verifying  $\mathcal{H}_{A_2, q_2} u_2 = k^2 u_2$  in  $B$  and such that

$$u_j(x) = e^{ix \cdot \rho_j} (e^{i\varphi_j(x, \omega_j^*)} + r_j(x, \rho_j)), \quad (3.33)$$

where  $r_j(\cdot, \rho_j)$ ,  $j = 1, 2$ , satisfies

$$\|r_j(\cdot, \rho_j)\|_{H^m(D)} \leq Cs^{m-1}, \quad 0 \leq m \leq 2, \quad (3.34)$$

and where  $\varphi_1(x, \omega_1^*) = N_{\omega_1^*}^{-1}(\omega_1^* \cdot A_1)$  and  $\varphi_2(x, \omega_2^*) = N_{\omega_2^*}^{-1}(-\omega_2^* \cdot A_2)$  are solutions of

$$\omega_1^* \cdot \nabla \varphi_1 = \omega_1^* \cdot A_1, \quad \omega_2^* \cdot \nabla \varphi_2 = -\omega_2^* \cdot A_2. \quad (3.35)$$

Furthermore, according to (3.33), (3.31) and (3.32), there exist  $C$  and  $\Lambda > 0$  such that

$$\|u_1 u_2\|_{L^1(B)} \leq C \quad \text{and} \quad \|u_j\|_{H^2(B)} \leq Cs^2 e^{\Lambda s} \text{ for } j = 1, 2. \quad (3.36)$$

### 3.4.3 | Stability estimate for the magnetic potential

We derive in this section a stability estimate for the magnetic fields. First, we will use Lemma 3.4.1 and the complex geometric solutions constructed as above to estimate the Fourier transform of the difference of the magnetic fields  $\text{curl } A$ . Second, we exploit the condition  $\|\widehat{\text{curl } A}\|_{L^\sigma}$ ,  $\sigma > 0$ , which is a priori bounded to prove the stability estimate.

First, we recall the following three lemmas proved in [90] on the properties of the operator  $N_\omega^{-1}$ ,  $\omega \in \mathbb{S}^2 + i\mathbb{S}^2$ , given by (3.28). The first Lemma, due to Salo [79] in the reconstruction methods and similar to the one appearing in Eskin and Ralston [33] and Sun [85], shows that a relation between a non-linear and linear Fourier transform of  $\omega \cdot A$  for a vector field  $A$ .

**Lemma 3.4.4.** *Let  $\omega = \omega_1 + i\omega_2$  with  $\omega_1, \omega_2 \in \mathbb{S}^2$  and  $\xi \in \mathbb{R}^3$ , such that  $\xi, \omega_1$  and  $\omega_2$  be three mutually orthogonal vectors in  $\mathbb{R}^3$ . Let  $A \in W^{2,\infty}(D)^3$  with  $\text{Supp}(A) \subset D$ . Then we have the following equality*

$$\int_{\mathbb{R}^3} \omega \cdot A(x) e^{ix \cdot \xi} e^{iN_\omega^{-1}(-\omega \cdot A)(x)} dx = \int_{\mathbb{R}^3} \omega \cdot A(x) e^{ix \cdot \xi} dx, \quad (3.37)$$

where  $A$  is extended by 0 outside  $D$ .

**Lemma 3.4.5.** *Let  $g \in W^{n,\infty}(\mathbb{R}^3)$ ,  $n \geq 0$ , with  $\text{Supp}(g) \subseteq D$ . Then  $N_\omega^{-1}(g) \in W^{n,\infty}(\mathbb{R}^3)$  and satisfies*

$$\|N_\omega^{-1}(g)\|_{W^{n,\infty}(\mathbb{R}^3)} \leq C \|g\|_{W^{n,\infty}(D)},$$

where  $C$  depends only on  $D$ .

Finally, we have the following result which gives the dependence of  $N_\omega^{-1}(-\omega \cdot A)$  on the parameter  $\omega$ .

**Lemma 3.4.6.** *Let  $A \in W^{2,\infty}(D)$  with  $\text{Supp}(A) \subset D$  such that  $\|A\|_{W^{2,\infty}} \leq M$  and let  $\theta, \theta' \in \mathbb{S}^2 + i\mathbb{S}^2$  such that  $|\theta - \theta'| < 1$  such that  $\Re(\theta) \cdot \Im(\theta) = \Re(\theta') \cdot \Im(\theta') = 0$ . Then, we have the following inequality*

$$\|N_\theta^{-1}(-\theta \cdot A) - N_{\theta'}^{-1}(-\theta' \cdot A)\|_{L^\infty(D)} \leq C |\theta - \theta'|, \quad (3.38)$$

where  $C$  depends only in  $D$  and  $M$ .

We have the following Lemma which is a straightforward extension and adaptation of a similar lemma in [11].

**Lemma 3.4.7.** *Let  $u_j$ ,  $j = 1, 2$ , be the functions given by (3.33) and set  $\omega = \omega_1 + i\omega_2$ . Then for any  $|\xi| \leq s$ , we have the following identity*

$$i \int_D A(x) \cdot (u_2 \nabla u_1 - u_1 \nabla u_2) dx = 2s \int_D \bar{\omega} \cdot A(x) e^{ix \cdot \xi} dx + \mathcal{R}(\xi, s), \quad (3.39)$$

with  $|\mathcal{R}(\xi, s)| \leq C \langle \xi \rangle$ , where  $C$  is independent of  $s$ ,  $\xi$  and  $M$ , with the short notation  $\langle \xi \rangle := \sqrt{|\xi|^2 + 1}$ .

*Proof.* By using (3.33), we have for  $j = 1, 2$

$$\nabla u_j = e^{ix \cdot \rho_j} (i(\rho_j + \nabla \varphi_j) e^{i\varphi_j} + i\rho_j r_j + \nabla r_j), \quad (3.40)$$

where  $\varphi_1$  and  $\varphi_2$  are given by

$$\varphi_1(x, \omega_1^*) = N_{\omega_1^*}^{-1}(\omega_1^* \cdot A_1), \quad \varphi_2(x, \omega_1^*) = N_{\omega_2^*}^{-1}(-\omega_2^* \cdot A_2), \quad (3.41)$$

where  $A_j, j = 1, 2$  are extended by 0 outside  $D$ . Therefore, direct calculation gives

$$u_2 \nabla u_1 - u_1 \nabla u_2 = i(\rho_1 - \rho_2) e^{i(\varphi_1 + \varphi_2)} e^{ix \cdot (\rho_1 + \rho_2)} + \Psi_1(x, \rho_1, \rho_2) + \Psi_2(x, \rho_1, \rho_2), \quad (3.42)$$

where  $\Psi_1$  and  $\Psi_2$  are given by

$$\begin{aligned} \Psi_1(x, \rho_1, \rho_2) &= i(\rho_1 - \rho_2) \left[ r_1 e^{i\varphi_2} + r_2 e^{i\varphi_1} + r_1 r_2 \right] e^{ix \cdot (\rho_1 + \rho_2)}, \\ \Psi_2(x, \rho_1, \rho_2) &= \left[ i(\nabla \varphi_1 - \nabla \varphi_2) e^{i(\varphi_1 + \varphi_2)} + (\nabla r_1 e^{i\varphi_2} - \nabla r_2 e^{i\varphi_1}) \right. \\ &\quad \left. + (ir_2 \nabla \varphi_1 e^{i\varphi_1} - ir_1 \nabla \varphi_2 e^{i\varphi_2}) + (r \nabla r_1 - r_1 \nabla r_2) \right] e^{ix \cdot (\rho_1 + \rho_2)}. \end{aligned}$$

Using Lemma 3.4.5 and the fact that  $A_j, j = 1, 2$ , is compactly supported in  $\mathbb{R}^3$ , we obtain

$$\|\varphi_j(\cdot, \omega_j^*)\|_{L^\infty(D)} \leq C \|A_j\|_{L^\infty(\mathbb{R}^3)} \leq CM. \quad (3.43)$$

This implies that

$$\|\Psi_1(\cdot, \rho_1, \rho_2)\|_{L^1(D)} \leq C. \quad (3.44)$$

From (3.42), we obtain

$$i \int_{\mathbb{R}^3} A \cdot (u_2 \nabla u_1 - u_1 \nabla u_2) dx = \int_{\mathbb{R}^3} A \cdot (\rho_2 - \rho_1) e^{ix \cdot \xi} e^{i(\varphi_1 + \varphi_2)} dx + \mathcal{R}_1(\xi, s), \quad (3.45)$$

where

$$\mathcal{R}_1(\xi, s) = i \int_{\mathbb{R}^3} A \cdot (\Psi_1(x, \rho_1, \rho_2) + \Psi_2(x, \rho_1, \rho_2)) dx, \quad \xi = \rho_1 + \rho_2. \quad (3.46)$$

Let now compute the first integral in the right hand side of (3.45). By using (3.31) and (3.34), we have for  $\omega = \omega_1 + i\omega_2$

$$\begin{aligned} \int_{\mathbb{R}^3} A \cdot (\rho_2 - \rho_1) e^{ix \cdot \xi} e^{i(\varphi_1 + \varphi_2)} dx &= 2s \int_{\mathbb{R}^3} \bar{\omega} \cdot A e^{ix \cdot \xi} e^{i(\varphi_1 + \varphi_2)} dx \\ &\quad + 2s \left( \sqrt{1 - \frac{|\xi|^2}{4s^2}} - 1 \right) \int_{\mathbb{R}^3} \omega_1 \cdot A e^{ix \cdot \xi} e^{i(\varphi_1 + \varphi_2)} dx, \end{aligned} \quad (3.47)$$

Let  $\psi_1 = N_{\bar{\omega}}^{-1}(\bar{\omega} \cdot A_1)$  and  $\psi_2 = N_{\bar{\omega}}^{-1}(-\bar{\omega} \cdot A_2)$ , then we have

$$\psi_1(x) + \psi_2(x) = N_{\bar{\omega}}^{-1}(-\bar{\omega} \cdot (A_2 - A_1)) = N_{\bar{\omega}}^{-1}(-\bar{\omega} \cdot A).$$

We insert  $e^{i(\psi_1+\psi_2)}$  in (3.47), then we have

$$\int_{\mathbb{R}^3} A(x) \cdot (\rho_2 - \rho_1) e^{ix \cdot \zeta} e^{i(\varphi_1+\varphi_2)} dx = \mathcal{J}(\zeta, s) + \mathcal{R}_2(\zeta, s) + \mathcal{R}_3(\zeta, s), \quad (3.48)$$

where

$$\begin{aligned} \mathcal{J}(\zeta, s) &= 2s \int_{\mathbb{R}^3} \bar{\omega} \cdot A(x) e^{ix \cdot \zeta} e^{i(\psi_1+\psi_2)} dx, \\ \mathcal{R}_2(\zeta, s) &= 2s \int_{\mathbb{R}^3} \bar{\omega} \cdot A(x) e^{ix \cdot \zeta} \left( e^{i(\varphi_1+\varphi_2)} - e^{i(\psi_1+\psi_2)} \right) dx, \\ \mathcal{R}_3(\zeta, s) &= 2s \left( \sqrt{1 - \frac{|\zeta|^2}{4s^2}} - 1 \right) \int_{\mathbb{R}^3} \omega_1 \cdot A(x) e^{ix \cdot \zeta} e^{i(\varphi_1+\varphi_2)} dx. \end{aligned}$$

By using the Lemma 3.4.4, we obtain

$$\mathcal{J}(\zeta, s) = 2s \int_{\mathbb{R}^3} \bar{\omega} \cdot A(x) e^{ix \cdot \zeta} dx. \quad (3.49)$$

On the other hand, we have

$$\mathcal{R}_2(\zeta, s) = -2s \int_{\mathbb{R}^3} e^{-ix \cdot \zeta} \bar{\omega} \cdot A \left( e^{i\varphi_1} (e^{i\varphi_2} - e^{i\psi_2}) - e^{i\psi_2} (e^{i\psi_1} - e^{i\varphi_1}) \right) dx. \quad (3.50)$$

Using the dependence of  $N_{\omega}^{-1}(-\omega \cdot A)$  on the parameter  $\omega$  given in Lemma 3.4.6 and the fact that  $\text{Supp} A \subset D$ , we get

$$\begin{aligned} |e^{i\varphi_2} - e^{i\psi_2}| &\leq C \left| N_{\omega_2^*}^{-1}(-\omega_2^* \cdot A_2) - N_{\bar{\omega}}^{-1}(-\bar{\omega} \cdot A_2) \right| \leq C |\omega_2^* - \bar{\omega}|, \\ |e^{i\psi_1} - e^{i\varphi_1}| &\leq C \left| N_{\bar{\omega}}^{-1}(\bar{\omega} \cdot A_1) - N_{-\omega_1^*}^{-1}(-\omega_1^* \cdot A_1) \right| \leq C |\bar{\omega} + \omega_1^*|. \end{aligned} \quad (3.51)$$

Taking into account (3.50), (3.51) and using that  $1 - \sqrt{1 - |\zeta|^2/s^2} \leq |\zeta|^2/4s^2$ , for all  $|\zeta| \leq 2s$ , we conclude

$$|\mathcal{R}_2(\zeta, s)| \leq Cs \frac{|\zeta|^2}{4s^2} \leq C|\zeta|. \quad (3.52)$$

By the same way, we find  $|\mathcal{R}_3(\zeta, s)| \leq C|\zeta|$ , for some positive constant which is independent of  $\zeta$  and  $s$ . The proof is completed.  $\square$

In what follows, for  $A_1$  and  $A_2 \in W^{2,\infty}(\mathbb{R}^3)$  as above, we introduce the notation

$$a_j(x) = (A_2 - A_1)(x) \cdot e_j = A(x) \cdot e_j, \quad j = 1, 2, 3, \quad x \in \mathbb{R}^3, \quad (3.53)$$

where  $(e_1, e_2, e_3)$  is the canonical basis of  $\mathbb{R}^3$  and set for  $j, \ell = 1, 2, 3$ ,

$$b_{j\ell}(x) := \frac{\partial a_\ell}{\partial x_j}(x) - \frac{\partial a_j}{\partial x_\ell}(x), \quad x \in \mathbb{R}^3, \quad (3.54)$$

the components of  $\text{curl } A$ , and

$$\hat{b}_{j\ell}(\xi) = \int_{\mathbb{R}^3} e^{ix \cdot \xi} b_{j\ell}(x) dx,$$

the associated Fourier coefficients. We then have the following estimate for the Fourier transform of the difference of the magnetic fields.

**Lemma 3.4.8.** *For any  $s \geq s_0$  and  $\xi \in \mathbb{R}^3$  such that  $|\xi| \leq s$  the following estimate holds true:*

$$|\hat{b}_{j\ell}(\xi)| \leq C \langle \xi \rangle \left( e^{\Lambda s} \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| + s^{-1} \langle \xi \rangle \right), \quad (3.55)$$

for  $j, \ell = 1, 2, 3$ , where  $C$  and  $\Lambda$  are positive constants independent of  $s$ ,  $\xi$  and  $M$ .

*Proof.* Let  $\xi \in \mathbb{R}^3$  such that  $|\xi| \leq s$ . Let  $\omega = \omega_1 + i\omega_2$ , where  $\omega_j$ ,  $j = 1, 2$  are as above and consider  $u_j$ ,  $j = 1, 2$  the solutions given by (3.33). By using (3.22) and (3.39), we get for  $|\xi| \leq s$

$$\begin{aligned} 2s \left| \int_B \bar{\omega} \cdot A e^{ix \cdot \xi} dx \right| \\ \leq C \left( \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| \|u_1\|_{H^2(B)} \|u_2\|_{H^2(B)} + \|u_1 u_2\|_{L^1(B)} + \langle \xi \rangle \right). \end{aligned} \quad (3.56)$$

Then, we obtain by (3.36)

$$\left| \int_B \bar{\omega} \cdot A(x) e^{ix \cdot \xi} dx \right| \leq C \left( e^{\Lambda s} \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| + s^{-1} \langle \xi \rangle \right). \quad (3.57)$$

The reasoning above remains valid if we change  $\omega_1$  by  $-\omega_1$  and therefore we also have

$$\left| \int_B -\omega \cdot A(x) e^{ix \cdot \xi} dx \right| \leq C \left( e^{\Lambda s} \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| + s^{-1} \langle \xi \rangle \right). \quad (3.58)$$

Outside the plane  $\xi_j e_k - \xi_\ell e_j = 0$ , we choose  $\omega_2 = \frac{\xi_j e_\ell - \xi_\ell e_j}{|\xi_j e_\ell - \xi_\ell e_j|}$  which is indeed an orthogonal unitary direction to  $\xi$ . Multiplying both sides of (3.57) and (3.58) by  $|\xi_j e_\ell - \xi_\ell e_j|$ , adding them together and using the triangular inequality to eliminate  $\omega_1$  imply

$$\left| \int_{\mathbb{R}^3} e^{ix \cdot \xi} (\xi_j a_\ell(x) - \xi_\ell a_j(x)) dx \right| \leq C \langle \xi \rangle \left( e^{\Lambda s} \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| + s^{-1} \langle \xi \rangle \right), \quad (3.59)$$

for  $\xi_j e_\ell - \xi_\ell e_j \neq 0$ . The inequality extends to all  $|\xi| \leq s$  by regularity of both sides in terms of  $\xi$  and proves (3.55).

This ends the proof.  $\square$

We are now in position to prove the main stability result for the magnetic potential from the near field data using (3.55) and regularity assumptions to estimate Fourier coefficients for large  $|\xi|$ . More precisely, we assume now that for  $j = 1, 2$ ,  $A_j \in W^{2,\infty}(D)$  with  $\|A\|_{W^{2,\infty}(D)} \leq M$ , and

$$\int_{\mathbb{R}^3} \langle \xi \rangle^\sigma |\widehat{\text{curl} A_j}(\xi)| d\xi < M \quad (3.60)$$

for some  $\sigma > 0$ , where  $\widehat{\text{curl} A_j}$  denotes the Fourier transform of  $\text{curl} A_j$ .

### End of the proof of the stability for the magnetic field

We derive now the stability estimate for the magnetic fields in  $L^\infty$  norm. Let  $s_0 > 1$  be as in Lemma 3.4.8 and  $s$  and  $R$  be two parameters satisfying  $s \geq R \geq s_0$ . From (3.55) we get

$$\begin{aligned} \int_{\mathbb{R}^3} |\hat{b}_{j\ell}(\xi)| d\xi &= \int_{\langle \xi \rangle \leq R} |\hat{b}_{j\ell}(\xi)| d\xi + \int_{\langle \xi \rangle > R} |\hat{b}_{j\ell}(\xi)| d\xi \\ &\leq CR^2 \left( e^{\Lambda s} \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| + s^{-1}R \right) + 2MR^{-\sigma}. \end{aligned}$$

Choosing  $R = s^{1/(\sigma+3)}$  we deduce that, for  $s_0$  sufficiently large (depending only on  $B$ ,  $k$ ,  $M$  and  $\sigma$ ),

$$\|b_{j\ell}\|_{L^\infty(\mathbb{R}^3)} \leq C' \left( e^{\Lambda' s} \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| + s^{-\sigma/(\sigma+3)} \right), \quad (3.61)$$

for some positive constants  $C'$  and  $\Lambda'$  and all  $s \geq s_0$ . Now if  $\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| \leq \epsilon_0$ , for some  $\epsilon_0 > 0$ , such that  $-\log(\epsilon_0) \geq 2\Lambda' s_0$ , then taking  $s = \frac{-1}{2\Lambda'} \log(\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|)$  in (3.61) implies

$$\begin{aligned} &\|b_{j\ell}\|_{L^\infty(\mathbb{R}^3)} \\ &\leq C' \left( \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|^{1/2} + \left( \frac{-1}{2\Lambda'} \log(\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|) \right)^{-\sigma/(\sigma+3)} \right). \quad (3.62) \end{aligned}$$

We also observe that this type of inequality holds true if  $\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| \geq \varepsilon_0$  since in that case we can simply write

$$\|b_{j\ell}\|_{L^\infty(\mathbb{R}^3)} \leq M \leq (M/\sqrt{\varepsilon_0})\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|^{1/2}. \quad (3.63)$$

The proof of the first estimate of Theorem 3.3.1 is then completed.

Using the above result, we are able to prove the second main result related to the stability for the electric potential.

### 3.4.4 | Stability estimate for the electric potential

In this section, we are going to use the complex geometric optics solutions and the stability estimate we already obtained for the magnetic field in order to retrieve a stability result for the electric potential. There are, however, some difficulties with this. In fact, in order to isolate the integral of the difference  $q = q_2 - q_1$  we would need to control the norm of the difference  $A = A_2 - A_1$ . Unfortunately, we can only estimate the difference of the magnetic fields  $\text{curl}(A)$ . To overcome this difficulty we will use the Helmholtz decomposition and write  $A = H - \nabla\vartheta$ , with the fact that  $\text{div} H = 0$  and we are able to estimate the norm of  $\nabla\vartheta$ .

**Lemma 3.4.9.** *Let  $p > 3$ . There exist  $\vartheta \in W^{3,p}(B) \cap H_0^1(B)$  and a positive constant  $C$  such that*

$$\|\vartheta\|_{W^{3,p}(B)} \leq C\|A\|_{W^{2,p}(D)}, \quad (3.64)$$

and

$$\|A + \nabla\vartheta\|_{W^{1,p}(B)} \leq C\|\text{curl}(A)\|_{L^p(D)}. \quad (3.65)$$

Moreover, if  $B'$  is a ball containing  $\overline{D}$  and such that  $\overline{B'} \subset B$ , then

$$\|\vartheta\|_{W^{2,p}(B \setminus B')} \leq C\|\text{curl}(A)\|_{L^p(D)}. \quad (3.66)$$

*Proof.* Let  $\vartheta$  solve the following elliptic boundary value problem in the ball  $B$

$$\begin{cases} -\Delta\vartheta = \text{div}(A) & \text{in } B, \\ \vartheta = 0 & \text{on } \partial B. \end{cases} \quad (3.67)$$

Since the source term  $\operatorname{div}(A)$  belongs to  $W^{1,p}(B)$  by the elliptic regularity (see [36, Theorem 2.5.1.1 in Chapter 2]), we have  $\vartheta \in W^{3,p}(B) \cap H_0^1(B)$ . Moreover, there exist  $C > 0$  such that

$$\|\vartheta\|_{W^{3,p}(B)} \leq C \|\operatorname{div}(A)\|_{W^{1,p}(D)} \leq C \|A\|_{W^{2,p}(D)}, \quad (3.68)$$

this ends the prove of (3.64). To prove (3.65), we consider the vector field  $H \in W^{2,p}(B)$  defined by

$$H = A + \nabla \vartheta. \quad (3.69)$$

By (3.67) and (3.69),  $H$  satisfies

$$\operatorname{div}(H) = 0, \quad \operatorname{curl}(H) = \operatorname{curl}(A) \quad \text{in } B, \quad \text{and} \quad H \wedge \nu = 0 \quad \text{on } \partial B. \quad (3.70)$$

By the  $L^p$ -div-curl estimate, we get

$$\|H\|_{W^{1,p}(B)} \leq C \|\operatorname{curl}(H)\|_{L^p(B)}, \quad (3.71)$$

and we conclude (3.65). Finally, to prove (3.66) we consider a cutoff function  $\chi_0 \in C^\infty(\mathbb{R}^n)$  such that  $\chi_0 = 1$  in  $B \setminus B'$  and  $\chi_0 = 0$  in  $\bar{D}$ . Set  $\vartheta_0 = \chi_0 \vartheta$ , we have by (3.67)

$$\begin{cases} -\Delta \vartheta_0 = [\Delta, \chi_0] \vartheta & \text{in } B, \\ \vartheta_0 = 0 & \text{on } \partial B, \end{cases} \quad (3.72)$$

where we have used  $A = 0$  outside  $D$ . Thus, we obtain

$$\|\vartheta_0\|_{W^{2,p}(B)} \leq C \|\nabla \vartheta\|_{L^p(B \setminus D)}, \quad (3.73)$$

since the first order operator  $[\Delta, \chi_0]$  is supported in  $B' \setminus D$ . Then we deduce that

$$\|\vartheta_0\|_{W^{2,p}(B)} \leq C \|A + \nabla \vartheta\|_{L^p(B \setminus D)} \leq C \|\operatorname{curl}(A)\|_{L^p(D)}. \quad (3.74)$$

This ends the proof. □

Applying now Morrey's inequality, we obtain for some positive constant  $C$

$$\|A + \nabla \vartheta\|_{L^\infty(B)} \leq C \|\operatorname{curl}(A)\|_{L^\infty(D)}, \quad (3.75)$$



and

$$\|\vartheta\|_{W^{1,\infty}(B \setminus B')} \leq C \|\operatorname{curl}(A)\|_{L^\infty(D)}. \quad (3.76)$$

We now prove the following inequality which is a slight modification of the previous estimate (3.22) given in Lemma 3.4.2. This inequality is based on the invariance of the near field operator under gauge transformation for the magnetic potential that was explained in the introduction.

**Lemma 3.4.10.** *Let  $B$  denote an open ball containing  $\bar{D}$ , and let  $\varphi \in W^{2,\infty}(B)$  with  $\operatorname{Supp}(\varphi) \subset B$ . Then there exists a constant  $C$  that only depends on  $B$  and such that*

$$\begin{aligned} & \left| \int_B e^{-i\varphi} [i(A + \nabla\varphi) \cdot (u_1 \nabla u_2 - u_2 \nabla u_1) - ((A + \nabla\varphi) \cdot (A_1 + A_2) + q)u_1 u_2] dx \right| \\ & \leq C \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| \|u_1\|_{H^2(B)} \|u_2\|_{H^2(B)}, \end{aligned} \quad (3.77)$$

for all  $u_1 \in H^2(B)$  satisfying  $\mathcal{H}_{-A_1, q_1} u_1 = k^2 u_1$  in  $B$  and all  $u_2 \in H^2(B)$  satisfying  $\mathcal{H}_{A_2, q_2} u_2 = k^2 u_2$  in the ball  $B$ .

*Proof.* Define  $\tilde{A}_1 = A_1 - \frac{1}{2}\nabla\varphi$  and  $\tilde{A}_2 = A_2 + \frac{1}{2}\nabla\varphi$ . Consider  $u_1, u_2 \in H^2(B)$  as in the lemma. Let denote  $\tilde{u}_j = e^{-i\varphi/2} u_j$ , then by (3.8) we get

$$\begin{aligned} \mathcal{H}_{-\tilde{A}_1, q_1} \tilde{u}_1 &= e^{-i\varphi/2} \mathcal{H}_{-A_1, q_1} u_1 = k^2 \tilde{u}_1, \\ \mathcal{H}_{\tilde{A}_2, q_2} \tilde{u}_2 &= e^{-i\varphi/2} \mathcal{H}_{A_2, q_2} u_2 = k^2 \tilde{u}_2, \quad \text{in } B. \end{aligned} \quad (3.78)$$

Moreover, due the gauge invariance of the scattered field and since  $\varphi|_{\partial B} = 0$ , we have from (2.23) that

$$\mathcal{N}_{\tilde{A}_j, q_j} = \mathcal{N}_{A_j, q_j}, \quad j = 1, 2. \quad (3.79)$$

With these solution at hand, we use the gauge invariance (3.79) and (3.22) with  $\tilde{A}_j$  and  $\tilde{u}_j$  instead of  $A_j$  and  $u_j$ , respectively, to obtain

$$\begin{aligned} & \left| \int_B [i\tilde{A} \cdot (\tilde{u}_1 \nabla \tilde{u}_2 - \tilde{u}_2 \nabla \tilde{u}_1) - (|\tilde{A}_2|^2 - |\tilde{A}_1|^2 + q)\tilde{u}_1 \tilde{u}_2] dx \right| \\ & \leq C \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| \|\tilde{u}_1\|_{H^2(B)} \|\tilde{u}_2\|_{H^2(B)}. \end{aligned} \quad (3.80)$$

From the identity (3.80), we get (3.77). This ends the proof.  $\square$

Lemma 3.4.10 then allows us to obtain an estimate to  $A$  added with a gradient term. By adding  $\nabla\vartheta$ , we would thus get an estimate with controlled terms. Unfortunately, we cannot directly add  $\nabla\vartheta$  because of the requirement that  $\text{Supp}(\varphi) \subset B$  in Lemma 3.4.10. We can solve this difficulty by using a cutoff argument.

Now, we will fix  $\varphi = \chi\vartheta$ , for  $\chi \in C_0^\infty(B)$  such that  $\chi = 1$  in  $\overline{B}'$  and  $\vartheta \in W^{2,\infty}(B)$  given by Lemma 3.4.9 which satisfying (3.75) and (3.76).

**Lemma 3.4.11.** *Let  $u_j$ ,  $j = 1, 2$  be the functions given by (3.33) for some  $s_0 > 0$ . Then there exist a positive constants  $C$  and  $\Lambda$  such that*

$$\left| \int_B e^{-i\varphi} q(x) u_1 u_2 dx \right| \leq C e^{2\Lambda s} \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| + s \|\text{curl}(A)\|_{L^\infty(D)} \quad (3.81)$$

for all  $s > s_0$ .

*Proof.* Let  $u_j$ ,  $j = 1, 2$  be the functions given by (3.33) for some  $s_0 > 0$ . Adding and subtracting the same terms we get

$$\begin{aligned} \int_B e^{-i\varphi} q(x) u_1 u_2 dx &= \int e^{-i\varphi} (A + \nabla\vartheta) \cdot (A_1 + A_2) u_1 u_2 dx \\ &\quad + \int_B i e^{-i\varphi} (A + \nabla\vartheta) \cdot (u_1 \nabla u_2 - u_2 \nabla u_1) dx \\ &\quad + \int_B e^{-i\varphi} (A + \nabla\vartheta) \cdot \nabla \varphi u_1 u_2 dx + \mathcal{R} \\ &:= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{R}, \end{aligned} \quad (3.82)$$

where  $\mathcal{R}$  denotes the integral

$$\begin{aligned} \mathcal{R} &= - \int_B e^{-i\varphi} [i(A + \nabla\vartheta) \cdot (u_1 \nabla u_2 - u_2 \nabla u_1) \\ &\quad + ((A + \nabla\vartheta) \cdot (A_1 + A_2) - q + (A + \nabla\vartheta) \cdot \nabla \varphi) u_1 u_2] dx. \end{aligned}$$

First, we have

$$|\mathcal{J}_2| \leq C \|A + \nabla\vartheta\|_{L^\infty(B)} (\|u_1 \nabla u_2\|_{L^1(B)} + \|u_2 \nabla u_1\|_{L^1(B)}). \quad (3.83)$$

Then, we deduce from (3.75) and (3.36) that

$$|\mathcal{J}_2| \leq Cs \|\operatorname{curl}(A)\|_{L^\infty(D)}. \quad (3.84)$$

Similarly, we have

$$|\mathcal{J}_1| \leq C \|A + \nabla \vartheta\|_{L^\infty(B)} \|u_1 u_2\|_{L^1(B)} \quad (3.85)$$

and then, we have

$$|\mathcal{J}_1| \leq C \|\operatorname{curl}(A)\|_{L^\infty(D)}. \quad (3.86)$$

Finally, by the same arguments, we have

$$|\mathcal{J}_3| \leq C \|\operatorname{curl}(A)\|_{L^\infty(D)}. \quad (3.87)$$

Now recall that  $\varphi = \chi \vartheta$  and set  $\tilde{\varphi} = (1 - \chi) \vartheta$ . Since  $\nabla \vartheta = \nabla \varphi + \nabla \tilde{\varphi}$  we obtain, by Lemma 3.4.10, that

$$\begin{aligned} |\mathcal{R}| \leq & C \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| \|u_1\|_{H^2(B)} \|u_2\|_{H^2(B)} \\ & + \left| \int_B i e^{-i\varphi} \nabla \tilde{\varphi} \cdot (u_1 \nabla u_2 - u_2 \nabla u_1) + e^{-i\varphi} (\nabla \varphi + A_1 + A_2) \cdot \nabla \tilde{\varphi} u_1 u_2 dx \right|. \end{aligned} \quad (3.88)$$

By the same arguments used previously, we get

$$|\mathcal{R}| \leq C e^{2\Lambda s} \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| + s \|\nabla \tilde{\varphi}\|_{L^\infty(B)}. \quad (3.89)$$

Since by (3.76) we have

$$\|\nabla \tilde{\varphi}\|_{L^\infty(B)} \leq \|\nabla \vartheta\|_{L^\infty(B \setminus B')} \leq C \|\operatorname{curl}(A)\|_{L^\infty(D)}, \quad (3.90)$$

we conclude (3.81) from (3.83), (3.86), (3.87) and (3.90).  $\square$

We now state the following integral identity for the electric potential which is proved in the Appendix.

**Lemma 3.4.12.** *Let  $u_j$ ,  $j = 1, 2$  be the functions given by (3.33) for some  $s_0 > 0$ . Then for all  $\xi \in \mathbb{R}^3$  and  $s \geq \max(s_0, |\xi|/2)$ , we have the following identity*

$$\int_D e^{-i\varphi} q(x) u_1 u_2 dx = \int_D q(x) e^{ix \cdot \xi} dx + \mathcal{R}'(\xi, s), \quad (3.91)$$

where  $\mathcal{R}'(\xi, s)$ , satisfy

$$|\mathcal{R}'(\xi, s)| \leq C(\|\operatorname{curl}(A)\|_{L^\infty(D)} + s^{-1} \langle \xi \rangle). \quad (3.92)$$

The constants  $C$  and  $s_0$  depend only on  $B$ ,  $M$  and  $k$ .

*Proof.* By a direct calculation, we have

$$u_1 u_2 = e^{ix \cdot \bar{\xi}} e^{i(\varphi_1 + \varphi_2)} + e^{ix \cdot \bar{\xi}} (e^{i\varphi_2} r_1 + e^{i\varphi_1} r_2 + r_1 r_2). \quad (3.93)$$

We use the identity (3.93) and we insert  $e^{ix \cdot \bar{\xi}} q(x)$ , we obtain

$$\int_D e^{-i\varphi} q(x) u_1 u_2 dx = \int_D q(x) e^{ix \cdot \bar{\xi}} dx + \mathcal{R}'_1(\xi, s) + \mathcal{R}'_2(\xi, s), \quad (3.94)$$

where

$$\begin{aligned} \mathcal{R}'_1(\xi, s) &= \int_D q(x) e^{ix \cdot \bar{\xi}} e^{i\varphi_1} (e^{i(\varphi_2 - \varphi)} - e^{-i\varphi_1}) dx, \\ \mathcal{R}'_2(\xi, s) &= \int_D e^{-i\varphi} q(x) e^{ix \cdot \bar{\xi}} (e^{i\varphi_2} r_1 + e^{i\varphi_1} r_2 + r_1 r_2) dx. \end{aligned}$$

Let  $\psi_3 = N_{\omega_2^*}^{-1}(-\omega_2^* \cdot A_1)$ . We insert  $e^{i\psi_3}$  in  $\mathcal{R}'_1$  and obtain from Lemmas 3.4.5 and 3.4.6

$$\begin{aligned} |\mathcal{R}'_1(\xi, s)| &\leq C \|e^{i(\varphi_2 - \varphi)} - e^{i\psi_3}\|_{L^\infty(D)} + \|e^{i\psi_3} - e^{-i\varphi_1}\|_{L^\infty(B)} \\ &\leq C (\|N_{\omega_2^*}^{-1}(-\omega_2^* \cdot (A_2 + \nabla \varphi)) - N_{\omega_2^*}^{-1}(-\omega_2^* \cdot A_1)\|_{L^\infty(B)} \\ &\quad + \|N_{\omega_2^*}^{-1}(-\omega_2^* \cdot A_1) - N_{-\omega_1^*}^{-1}(\omega_1^* \cdot A_1)\|_{L^\infty(B)}) \\ &\leq C (\|A + \nabla \varphi\|_{L^\infty(B)} + |\omega_2^* + \omega_1^*|) \\ &\leq C (\|A + \nabla \vartheta\|_{L^\infty(B)} + \|\vartheta\|_{W^{1,\infty}(B \setminus B')} + |\omega_2^* + \omega_1^*|). \end{aligned} \quad (3.95)$$

Using (3.75) and (3.76), we obtain

$$|\mathcal{R}'_1(\xi, s)| \leq C (\|\operatorname{curl}(A)\|_{L^\infty(B)} + s \langle \xi \rangle^{-1}). \quad (3.96)$$

Moreover from (3.34), we get  $|\mathcal{R}'_2(\xi, s)| \leq Cs^{-1}$ . Collecting this with (3.96) and (3.94) we obtain the desired result.  $\square$

This identity allows to obtain the following estimates for the Fourier coefficients

$$\hat{q}(\xi) := \int_{\mathbb{R}^3} e^{ix \cdot \xi} q(x) dx.$$

From Lemma 3.4.11 and Lemma 3.4.12 we deduce the following estimate.

**Lemma 3.4.13.** *There exists  $s_0 > 0$  such that for all  $s \geq s_0$  and  $\xi \in \mathbb{R}^3$  with  $|\xi| \leq s$  the following estimate holds true:*

$$|\hat{q}(\xi)| \leq C(e^{\Lambda s} \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| + s \|\operatorname{curl}(A)\|_{L^\infty(D)} + s^{-1} \langle \xi \rangle). \quad (3.97)$$

The constants  $s_0$ ,  $C$ , and  $\Lambda$  depend only on  $B$ ,  $M$  and  $k$ .

With the help of the previous lemma, we are now in position to prove the stability result for the electric potential under the assumption

$$\int_{\mathbb{R}^3} \langle \xi \rangle^\gamma |\hat{q}_j(\xi)| d\xi < M, \quad (3.98)$$

for some  $\gamma > 0$ .

### End of the proof of the stability estimate for the electric potential

Let  $s_0 > 1$  be as in Lemma 3.4.13 and  $s$  and  $R$  be two parameters satisfying  $s \geq R \geq s_0$ .

From (3.97) and (3.98) we get

$$\begin{aligned} \int_{\mathbb{R}^3} |\hat{q}(\xi)| d\xi &= \int_{\langle \xi \rangle \leq R} |\hat{q}(\xi)| d\xi + \int_{\langle \xi \rangle > R} |\hat{q}(\xi)| d\xi \\ &\leq CR \left( e^{\Lambda s} \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| + s \|\operatorname{curl}(A)\|_{L^\infty} + Rs^{-1} \right) + 2MR^{-\gamma}. \end{aligned}$$

Choosing  $R = s^{1/(\gamma+2)}$ , we deduce that, for  $s_0$  sufficiently large (depending only on  $B$ ,  $k$ ,  $M$  and  $\gamma$ ),

$$\|q\|_{L^\infty(\mathbb{R}^3)} \leq C' \left( e^{\Lambda' s} \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| + s^{(\gamma+3)/(\gamma+2)} \|\operatorname{curl}(A)\|_{L^\infty} + s^{-\gamma/(\gamma+2)} \right) \quad (3.99)$$

for some positive constants  $C'$  and  $\Lambda'$  and all  $s \geq s_0$ . Observe now that (3.61) implies in particular (after eventually changing the constants  $C'$ ,  $\Lambda'$  and  $s_0$ )

$$\|\operatorname{curl} A\|_{L^\infty(D)} \leq C' \left( e^{\Lambda' s^\kappa} \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| + s^{-\kappa\sigma/(\sigma+3)} \right), \quad (3.100)$$

for all  $\kappa \geq 1$ . Choosing  $\kappa$  such that

$$-\frac{\kappa\sigma}{\sigma+3} + \frac{\gamma+3}{\gamma+2} = -\frac{\gamma}{\gamma+2} \Leftrightarrow \kappa = \frac{(2\gamma+3)(\sigma+3)}{\sigma(\gamma+2)},$$

we obtain by substituting (3.100) in (3.99)

$$\|q\|_{L^\infty(\mathbb{R}^3)} \leq C' (e^{\Lambda' s^\kappa} \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| + s^{-\gamma/(\gamma+2)}), \quad (3.101)$$

with possibly different constants  $C'$  and  $\Lambda'$ .

Now if  $\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| \leq \varepsilon_0$ , for some  $\varepsilon_0 > 0$ , such that  $-\log(\varepsilon_0) \geq 2\Lambda' s_0^\kappa$ , then taking  $s^\kappa = \frac{-1}{2\Lambda'} \log(\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|)$  in (3.61) implies

$$\|q\|_{L^\infty(\mathbb{R}^3)} \leq C' (\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|^{1/2} + \left(\frac{-1}{2\Lambda'} \log(\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|)\right)^{-\gamma/\kappa(\gamma+2)}). \quad (3.102)$$

We also observe that this type of inequality holds true if  $\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| \geq \varepsilon_0$  since in that case we can simply write

$$\|q\|_{L^\infty(\mathbb{R}^3)} \leq M \leq (M/\sqrt{\varepsilon_0}) \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|^{1/2}. \quad (3.103)$$

The proof of the second part of Theorem 3.3.2 is then completed.

## 3.5 | Stability analysis for far field data

The aim of this section is to prove the stability estimates with far field measurements given in Theorems 3.3.2 and 3.3.3. First, we will introduce spherical harmonics to represent the far field pattern as a Fourier series in terms of spherical harmonics. Then, we will exploit the relation between the far field  $u_{A, q}^\infty(\hat{x}, d)$ ,  $(\hat{x}, d) \in \mathbb{S}^2 \times \mathbb{S}^2$  and the operator  $\mathcal{N}_{A, q}$ .

### 3.5.1 | Spherical harmonics

Within this section, we delve into the essential properties of both spherical harmonics and spherical Bessel functions.

Spherical harmonics are the angular part of the solution to the Laplace equation,  $\Delta u = 0$ , in the spherical coordinate system. In this system, the Laplace equation is written as follows:

$$\Delta u = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2} = 0. \quad (3.104)$$

For any  $\ell \in \mathbb{N}$  and  $m \in \{-\ell, \dots, \ell\}$ , the spherical harmonics are given by:

$$Y_\ell^m(\theta, \varphi) = (-1)^m \alpha(\ell, m) P_\ell^m(\cos \theta) e^{im\varphi}, \quad \theta \in [0, 2\pi], \quad \varphi \in [0, \pi], \quad (3.105)$$

where  $P_\ell^m$  is the associated Legendre polynomial of degree  $\ell$  and order  $m$ :

$$P_\ell^m(x) = \frac{(-1)^m}{2^m \ell!} (1-x^2)^{\frac{m}{2}} \frac{d^{\ell+m}}{dx^{\ell+m}} ((x^2-1)^\ell), \quad (3.106)$$

and  $\alpha(\ell, m)$  is the normalization function:

$$\alpha(\ell, m) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}}. \quad (3.107)$$

Spherical harmonics form an orthonormal basis in the Hilbert space  $L^2(\mathbb{S}^2)$  where  $\mathbb{S}^2$  is the unit sphere in  $\mathbb{R}^3$ . In particular, we have:

$$\int_0^{2\pi} \int_0^\pi Y_\ell^m(\theta, \varphi) \overline{Y_{\ell'}^{m'}(\theta, \varphi)} \sin \theta \, d\theta d\varphi = \delta_{\ell, \ell'} \delta_{m, m'}. \quad (3.108)$$

Given a spherical function  $g : \mathbb{S}^2 \rightarrow \mathbb{R}$ ,  $g$  can be expressed as a linear combination of spherical harmonics as follows:

$$g(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell, m} Y_\ell^m(\theta, \varphi), \quad (3.109)$$

where the coefficients  $C_{\ell, m}$ , called harmonic coefficients of order  $(\ell, m)$ , are defined as follows:

$$C_{\ell, m} = \alpha(\ell, m) \int_0^{2\pi} \int_0^\pi g(\theta, \varphi) P_\ell^m(\cos \theta) e^{-im\varphi} \sin \theta \, d\theta d\varphi. \quad (3.110)$$

For  $k > 0$ , we consider the Helmholtz equation in  $\mathbb{R}^3$ :

$$(\Delta + k^2)u(x) = 0, \quad \text{for all } x \in \mathbb{R}^3.$$

We seek a solution  $u$  of the form:

$$u(x) = f(k|x|)Y_\ell^m(\theta, \varphi) = f(kr)Y_\ell^m(\theta, \varphi), \quad r = |x|, \quad (3.111)$$

where  $(\theta, \varphi)$  denote the spherical angles of  $\hat{x} = \frac{x}{|x|}$ . The function  $f$  is then a solution of the spherical Bessel differential equation:

$$t^2 f''(t) + 2t f'(t) + (t^2 - \ell(\ell - 1))f(t) = 0. \quad (3.112)$$

This equation has two linearly independent solutions:

$$j_\ell(t) = \sum_{p=0}^{\infty} \frac{(-1)^p}{2^p p! 1 \cdot 3 \dots (2\ell + 2p - 1)} t^{\ell+2p}, \quad (3.113)$$

and

$$y_\ell(t) = -\frac{(2\ell)!}{2^\ell \ell!} \sum_{p=0}^{\infty} \frac{(-1)^p}{2^p p! (-2\ell + 1)(-2\ell + 3) \dots (-2\ell + 2p - 1)} t^{2p-\ell-1}. \quad (3.114)$$

The spherical Hankel functions of the first and second kinds are defined as:

$$h_\ell^{(1)}(t) = j_\ell(t) + iy_\ell(t), \quad (3.115)$$

and

$$h_\ell^{(2)}(t) = j_\ell(t) - iy_\ell(t). \quad (3.116)$$

Then, we have:

$$h_{\ell+1}^{(k)}(t) = -t^\ell \frac{d}{dt} (t^{-\ell} h_\ell^{(k)}(t)), \quad k = 1, 2; \quad \ell \in \mathbb{N}. \quad (3.117)$$

For a more detailed analysis of spherical harmonics and spherical Bessel functions, we refer to Lebedev [63] and to Colton and Kress [24].

## 3.5.2 | Far field pattern

We here assume that the Assumption 2.3.1 holds. Let  $B = \{x \in \mathbb{R}^3, |x| < a\}$  for some sufficiently large  $a > 0$  so that  $D \subset B$ . We recall that the far field pattern can be expressed as

$$u_{A,q}^\infty(\hat{x}, d) = \frac{1}{4\pi} \int_D e^{-ik\hat{x} \cdot y} Q_{A,q} u_{A,q}(y, d) dy. \quad (3.118)$$



We denote by  $\mu_{(\ell_1, m_1; \ell_2, m_2)}$ ,  $(\ell_i, m_i) \in \Gamma$ ,  $i = 1, 2$ , the Fourier coefficients of  $u_{A, q}^\infty$  given by

$$\mu_{(\ell_1, m_1; \ell_2, m_2)} := \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} u_{A, q}^\infty(\hat{x}, d) \overline{Y_{\ell_1}^{m_1}(\hat{x})} \overline{Y_{\ell_2}^{m_2}(d)} ds(\hat{x}) ds(d). \quad (3.119)$$

For proving the first lemma we need the following well known results about the asymptotic of spherical Bessel functions  $j_\ell$  and spherical Hankel functions of the first kind  $h_\ell^{(1)}$  [84]:

$$|j_\ell(kr)| \leq \alpha \left( \frac{ekr}{2\ell + 1} \right)^\ell \frac{1}{2\ell + 1}, \quad 0 \leq r \leq a, \quad \ell \in \mathbb{N} \cup \{0\}, \quad (3.120)$$

and

$$\left| h_\ell^{(1)}(kr) \right| \leq \alpha \left( \frac{2\ell + 1}{ekr} \right)^\ell, \quad 0 < r \leq a, \quad \ell \in \mathbb{N} \cup \{0\}, \quad (3.121)$$

where  $\alpha$  is a constant that only depend on  $a$  and  $k$ . We also recall the following equality that comes from the addition formula [24],

$$\int_{\mathbb{S}^2} Y_{\ell_2}^{m_2}(\hat{z}) \Phi(x, r\hat{z}) ds(\hat{z}) = ikj_{\ell_2}(kr) h_{\ell_2}^{(1)}(k|x|) Y_{\ell_2}^{m_2}(\hat{x}), \quad |x| > r, \quad (3.122)$$

together with the Funk-Hecke formula

$$\int_{\mathbb{S}^2} e^{-ikx \cdot \hat{z}} Y_{\ell_2}^{m_2}(\hat{z}) ds(\hat{z}) = \frac{4\pi}{i^{\ell_2}} j_{\ell_2}(k|x|) Y_{\ell_2}^{m_2}(\hat{x}), \quad x \in \mathbb{R}^3. \quad (3.123)$$

For a more detailed study about the spherical harmonics, we refer to Lebedev [63] and to Colton & Kress [24].

**Lemma 3.5.1.** *The Assumption 2.3.1 holds. Let  $\mu_{(\ell_1, m_1; \ell_2, m_2)}$  denote the Fourier coefficients of the far field patterns  $u_{A, q}^\infty$  as defined in (3.119). Then there exists a constant  $C > 0$  that only depends on  $D$ ,  $a$ ,  $k$ , and  $M$  such that*

$$\left| \mu_{(\ell_1, m_1; \ell_2, m_2)} \right|^2 \leq C \left( \frac{eka}{2\ell_1 + 1} \right)^{2\ell_1 + 3} \left( \frac{eka}{2\ell_2 + 1} \right)^{2\ell_2 + 3}, \quad (3.124)$$

and

$$\sum_{(\ell_1, m_1) \in \Gamma} \sum_{(\ell_2, m_2) \in \Gamma} \left( \frac{2\ell_1 + 1}{eka} \right)^{2\ell_1 + 1} \left( \frac{2\ell_2 + 1}{eka} \right)^{2\ell_2 + 1} \left| \mu_{(\ell_1, m_1; \ell_2, m_2)} \right|^2 \leq C.$$

*Proof.* We only need to prove (3.124). According to (3.119) and (3.118), we obtain

$$\begin{aligned}\mu_{(\ell_1, m_1; \ell_2, m_2)} &= \frac{1}{4\pi} \int_B \left( \int_{\mathbb{S}^2} Q_{A,q} u(y, d) \overline{Y_{\ell_2}^{m_2}}(d) ds(d) \right) \left( \int_{\mathbb{S}^2} e^{-ik\hat{x}\cdot y} \overline{Y_{\ell_1}^{m_1}}(\hat{x}) ds(\hat{x}) \right) dy \\ &:= \frac{1}{4\pi} \int_B w_{\ell_2, m_2}(y) v_{\ell_1, m_1}(y) dy.\end{aligned}\tag{3.125}$$

With the help of the Funk-Hecke formula (3.123) we compute

$$v_{\ell_1, m_1}(y) := \int_{\mathbb{S}^2} e^{-ik\hat{x}\cdot y} \overline{Y_{\ell_1}^{m_1}}(\hat{x}) ds(\hat{x}) = \frac{4\pi}{i^{\ell_1}} j_{\ell_1}(k|y|) \overline{Y_{\ell_1}^{m_1}}(\hat{y}), \quad y \in \mathbb{R}^3.\tag{3.126}$$

Then by (3.120), we get

$$\|v_{\ell_1, m_1}\|_{L^2(B)}^2 \leq C \int_0^a |j_{\ell_1}(kr)|^2 r^2 dr \leq C \left( \frac{eka}{2\ell_1 + 1} \right)^{2\ell_1 + 3}.\tag{3.127}$$

Using again the Funk-Hecke formula, we obtain

$$\int_{\mathbb{S}^2} u(y, d) \overline{Y_{\ell_2}^{m_2}}(d) ds(d) = T_{A,q} \left( \int_{\mathbb{S}^2} u(y, d) \overline{Y_{\ell_2}^{m_2}}(d) ds(d) \right) + (-1)^{\ell_2} v_{\ell_2, m_2},\tag{3.128}$$

and we deduce that

$$w_{\ell_2, m_2}(x) = (-1)^{\ell_2} Q_{A,q} \left( (I - T_{A,q})^{-1} v_{\ell_2, m_2} \right) (x).\tag{3.129}$$

Using the fact that  $Q_{A,q}$  is a first order operator supported in  $D$ , then we obtain from Proposition 2.3.3,

$$\|w_{\ell_2, m_2}\|_{L^2(B)} \leq C \|(I - T_{A,q})^{-1} v_{\ell_2, m_2}\|_{H^1(D)} \leq C \|v_{\ell_2, m_2}\|_{H^1(D)}.\tag{3.130}$$

We note that there is a constant  $C > 0$  such that the inequality

$$\|u\|_{H^1(D)} \leq C \|u\|_{L^2(B)}\tag{3.131}$$

holds true for all  $u \in H_{loc}^1(\mathbb{R}^3)$  satisfying the Helmholtz equation  $\Delta u + k^2 u = 0$  in  $\mathbb{R}^3$ . We can then estimate

$$\|w_{\ell_2, m_2}\|_{L^2(B)} \leq C \|v_{\ell_2, m_2}\|_{H^1(D)}^2 \leq \|v_{\ell_2, m_2}\|_{L^2(B)}^2 \leq C \left( \frac{eka}{2\ell_2 + 1} \right)^{2\ell_2 + 3},\tag{3.132}$$

we conclude, from (3.125), (3.127) and (3.132), that

$$\begin{aligned} \left| \mu_{(\ell_1, m_1; \ell_2, m_2)} \right|^2 &\leq C \|v_{\ell_1, m_1}\|_{L^2(B)}^2 \|w_{\ell_2, m_2}\|_{L^2(B)}^2 \\ &\leq C \left( \frac{eka}{2\ell_2 + 1} \right)^{2\ell_2 + 3} \left( \frac{eka}{2\ell_1 + 1} \right)^{2\ell_1 + 3}. \end{aligned} \quad (3.133)$$

This completes the proof.  $\square$

### 3.5.3 | Relation between $u_{A,q}^\infty$ and $\mathcal{N}_{A,q}$

In this subsection, we will explore the connection between the far-field pattern  $u_{A,q}^\infty(\hat{x}, d)$ ,  $(\hat{x}, d) \in \mathbb{S}^2 \times \mathbb{S}^2$  and the near-field operator  $\mathcal{N}_{A,q}$  given by (3.4) and derive the Theorem 3.3.2. We establish the following duality result where the Assumption 2.3.1 holds.

**Lemma 3.5.2.** *Let  $v_i \in H^1(B)$ ,  $i = 1, 2$  such that  $\Delta v_i + k^2 v_i = 0$  in  $B$ . Then, the identity*

$$\int_B Q_{A,q} v_1 (I - T_{-A,q})^{-1} v_2 dx = \int_B Q_{-A,q} v_2 (I - T_{A,q})^{-1} v_1 dx, \quad (3.134)$$

holds where  $Q_{A,q}$  and  $T_{A,q}$  are given by (2.10) and (2.17), respectively.

*Proof.* We associate to  $v_1$  (respectively  $v_2$ ) a total field  $u_{A,q}$  (respectively  $u_{-A,q}$ ) and a scattered field  $u_{-A,q}^s$  (respectively,  $u_{-A,q}^s$ ). Applying (2.33) to  $u_1 = u_{A,q}$  and  $u_2 = u_{-A,q}^s$  implies

$$\int_B \left( \mathcal{H}_{A,q} u_{A,q}^s u_{-A,q}^s - u_{A,q}^s \mathcal{H}_{-A,q} u_{-A,q}^s \right) dx = 0.$$

Making use of  $u_{A,q} = u_{A,q}^s + v_1$  and  $u_{-A,q} = u_{-A,q}^s + v_2$ , it follows by direct calculation

$$\int_B (Q_{A,q} v_1 u_{-A,q} - Q_{-A,q} v_2) dx = \int_B (Q_{A,q} v_1 v_2 - Q_{-A,q} v_2 v_1) dx.$$

Moreover, by integrating by parts, we get

$$\int_B (Q_{A,q} v_1 v_2 - Q_{-A,q} v_2 v_1) dx = 0.$$

Due to  $u_{A,q} = (I - T_{A,q})^{-1} v_1$  and  $u_{-A,q} = (I - T_{-A,q})^{-1} v_2$ , we obtain the desired result.  $\square$

**Proposition 3.5.3.** *For  $k > 0$  fixed, we have*

$$u_{A,q}^s(x, y) = \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|} \frac{e^{ik|y|}}{|y|} u_{A,q}^\infty(\hat{x}, -\hat{y}) + \frac{1}{|x||y|} \left( \frac{1}{|x|} + \frac{1}{|y|} \right) \Lambda(x, y), \quad x \neq y, \quad (3.135)$$

where  $\Lambda(x, y)$  is uniformly bounded as  $|x| \rightarrow \infty$  and  $|y| \rightarrow \infty$ .

*Proof.* Using the asymptotics of  $\Phi(\cdot, z)$  and  $\nabla_z \Phi(\cdot, z)$  for  $z \in D$ , we get

$$u_{A,q}^s(x, y) = \int_D Q_{-A,q} \Phi(x, z) u_{A,q}(z, y) dz \quad (3.136)$$

$$= \frac{e^{ik|x|}}{4\pi|x|} w(y, -\hat{x}) + \frac{O(1)}{|x|^2} \|u_{A,q}(\cdot, y)\|_{L^2(D)}, \quad |x| \rightarrow \infty, \quad (3.137)$$

uniformly with respect to  $y \in \mathbb{R}^3 \setminus B$ , where

$$w(y, d) := \int_D Q_{-A,q} e^{ikz \cdot d} u_{A,q}(z, y) dz.$$

Using now Lemma 3.5.2 and integrating by parts, we get

$$w(y, d) = - \int_D \Phi(z, y) Q_{-A,q} u_{-A,q}(z, d) dz, \quad (3.138)$$

and therefore  $w(\cdot, d) = u_{-A,q}^s(\cdot, d)$ . Consequently, (3.137) and (3.118) imply that for  $|x|, |y| \rightarrow \infty$

$$\begin{aligned} u_{A,q}^s(x, y) &= \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|} \frac{e^{ik|y|}}{|y|} u_{-A,q}^\infty(\hat{y}, -\hat{x}) + \frac{O(1)}{|x|^2} \|u_{A,q}(\cdot, y)\|_{L^2(D)} \\ &\quad + \frac{O(1)}{|x||y|^2} \|u_{-A,q}(\cdot, -\hat{x})\|_{L^2(D)}. \end{aligned} \quad (3.139)$$

Let us observe that according to Corollary 2.3.4 (and the asymptotic behavior of  $\Phi(x, y)$  and  $\nabla_x \Phi(x, y)$  with  $x \in D$  and  $|y| \rightarrow \infty$ ) we get that

$$\|u_{A,q}(\cdot, y)\|_{L^2(D)} = \frac{O(1)}{|y|} \quad \text{as } |y| \rightarrow \infty,$$

and  $\|u_{-A,q}(\cdot, -\hat{x})\|_{L^2(D)}$  is uniformly bounded with respect to  $\hat{x}$ . We finally obtain the desired result by noticing the reciprocity relation  $u_{A,q}^\infty(d, \theta) = u_{-A,q}^\infty(-\theta, -d)$ , where  $\theta, d \in \mathbb{S}^2$  (which is also a consequence of Lemma 3.5.2) or using Lemma 2.3.7.  $\square$

Now, let us expand the scattering amplitude  $u_{A,q}^\infty(d, \theta)$  in spherical harmonics

$$u_{A,q}^\infty(d, \theta) = \sum_{(\ell_1, m_1) \in \Gamma} \sum_{(\ell_2, m_2) \in \Gamma} \mu_{\ell_1 m_1 \ell_2 m_2} Y_{\ell_1}^{m_1}(d) Y_{\ell_1}^{m_2}(\theta), \quad (3.140)$$

where  $\mu_{\ell_1 m_1 \ell_2 m_2}$  is given by (3.119).

The following lemma makes the link between  $u_{A,q}^s(\cdot, y)$  and  $u_{A,q}^\infty(\cdot, d)$ . The proof follows similar ideas as in Stefanov [84] for  $A = 0$  but uses different arguments since we do not rely on the properties of the Green function for  $A \neq 0$  when  $x \sim y$ . Our proof would apply to more general contexts since we mainly rely on the reciprocity relation in Lemma 2.3.7.

**Lemma 3.5.4.** *The scattered field associated with point sources can be expanded as*

$$u_{A,q}^s(x, y) = -\frac{k^2}{4\pi} \sum_{\substack{(\ell_1, m_1) \in \Gamma \\ (\ell_2, m_2) \in \Gamma}} i^{\ell_1 - \ell_2} \mu_{(\ell_1, m_1; \ell_2, m_2)} h_{\ell_1}^{(1)}(k|x|) h_{\ell_2}^{(1)}(k|y|) Y_{\ell_1}^{m_1}(\hat{x}) Y_{\ell_2}^{m_2}(\hat{y}), \quad (3.141)$$

uniformly for  $|x|, |y| \geq a$ , with  $\hat{x} = x/|x|$  and  $\hat{y} = y/|y|$ .

*Proof.* Making use of the addition formula [24],

$$\Phi(x, z) = \sum_{\ell, m} \varepsilon_{\ell, m}(z) h_\ell^{(1)}(k|x|) Y_\ell^m(\hat{x}), \quad |x| > |z|, \quad (3.142)$$

$$\nabla_y \Phi(x, z) = \sum_{\ell, m} \varepsilon'_{\ell, m}(z) h_\ell^{(1)}(k|x|) Y_\ell^m(\hat{x}), \quad |x| > |z|, \quad (3.143)$$

where  $\varepsilon_{\ell, m}(z) = ikj_\ell(k|z|) Y_\ell^m(\hat{z})$  and  $\varepsilon'_{\ell, m} = ik \nabla (j_\ell(k|z|) Y_\ell^m(\hat{z}))$ , it follows from (3.136) that for  $y \in \mathbb{R}^3 \setminus D$  and uniformly for  $|x| \geq a$

$$u_{A,q}^s(x, y) = \sum_{(\ell_1, m_1) \in \Gamma} \alpha_{\ell_1 m_1}(y) h_{\ell_1}^{(1)}(k|x|) Y_{\ell_1}^{m_1}(\hat{x}).$$

Similarly, for  $x \in \mathbb{R}^3 \setminus D$  and uniformly for  $|y| \geq a$

$$u_{-A,q}^s(y, x) = \sum_{(\ell_2, m_2) \in \Gamma} \beta_{\ell_2 m_2}(x) h_{\ell_2}^{(1)}(k|y|) Y_{\ell_2}^{m_2}(\hat{y}).$$

We observe that

$$\alpha_{\ell_1 m_1}(y) h_{\ell_1}^{(1)}(ka) = \int_{\mathbb{S}^2} u_{A,q}^s(a\hat{x}, y) \overline{Y_{\ell_1}^{m_1}(\hat{x})} ds(\hat{x}).$$

Using the reciprocity relation of Lemma 2.3.7 we then get uniformly for  $|y| \geq a$

$$\alpha_{\ell_1 m_1}(y) = \sum_{(\ell_2, m_2) \in \Gamma} \gamma_{\ell_1 m_1 \ell_2 m_2} h_{\ell_2}^{(1)}(k|y|) Y_{\ell_2}^{m_2}(\hat{y}),$$

where

$$\gamma_{\ell_1 m_1 \ell_2 m_2} = \frac{1}{h_{\ell_1}^{(1)}(ka)} \int_{\mathbb{S}^2} \beta_{\ell_2 m_2}(a\hat{x}) \overline{Y_{\ell_1}^{m_1}(\hat{x})} ds(\hat{x}).$$

This yields in particular that for  $|x| \geq a$  and  $|y| \geq a$

$$u_{A,q}^s(x, y) = \sum_{\substack{(\ell_1, m_1) \in \Gamma \\ (\ell_2, m_2) \in \Gamma}} \gamma_{\ell_1 m_1 \ell_2 m_2} h_{\ell_1}^{(1)}(k|x|) Y_{\ell_1}^{m_1}(\hat{x}) h_{\ell_2}^{(1)}(k|y|) Y_{\ell_2}^{m_2}(\hat{y}). \quad (3.144)$$

Observe that from Proposition 3.5.3, we have in particular that  $u_{A,q}^s \in L^2(S_R \times S_R)$  for  $R$  sufficiently large, with  $S_R = \{|x| = R\}$ . The orthonormality of products of spherical harmonics in  $L^2(\mathbb{S}^2 \times \mathbb{S}^2)$  implies in particular that the series (3.144) is convergent in  $L^2(\mathbb{S}^2 \times \mathbb{S}^2)$  and

$$\int_{\mathbb{S}^2 \times \mathbb{S}^2} u_{A,q}^s(R\hat{x}, R\hat{y}) \overline{Y_{\ell_1}^{m_1}(\hat{x})} \overline{Y_{\ell_2}^{m_2}(\hat{y})} ds(\hat{x}) ds(\hat{y}) = \gamma_{\ell_1 m_1 \ell_2 m_2} h_{\ell_1}^{(1)}(kR) h_{\ell_2}^{(1)}(kR).$$

We recall that ([24])

$$h_{\ell}^{(1)}(r) = (-i)^{\ell+1} \frac{e^{ir}}{r} + O(r^{-2}),$$

Proposition 3.5.3 and the identity  $Y_{\ell_2}^{m_2}(-\hat{y}) = (-1)^{\ell_2} Y_{\ell_2}^{m_2}(\hat{y})$  then imply, by integrating on  $S_R \times S_R$  and letting  $R \rightarrow \infty$  that

$$\gamma_{\ell_1 m_1 \ell_2 m_2} = \frac{-k^2}{4\pi} i^{\ell_1 + \ell_2} (-1)^{\ell_2} \mu_{\ell_1 m_1 \ell_2 m_2}.$$

This yields the desired result. We remark that it also implies from (3.124) and (3.121) that the series (3.144) is absolutely convergent together with its first and second derivatives with respect to  $x$  or  $y$ .  $\square$

Then we have the following Lemma showing the Lipschitz continuity of the mapping  $u_{A,q}^\infty \mapsto \mathcal{N}_{A,q}$  when  $u_{A,q}^\infty$  is endowed with the norm (3.14).

**Lemma 3.5.5.** *We assume that the Assumption 2.3.1 holds. Then*

$$\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| \leq \alpha^2 \frac{k^2}{4\pi} \|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{\mathcal{F}}, \quad (3.145)$$

*Proof.* For  $j = 1, 2$ , denote by  $\mu_{(\ell_1, m_1; \ell_2, m_2)}^j$  the Fourier coefficients associated with  $u_{A_j, q_j}^\infty$  as above. We get from (3.141)

$$\begin{aligned} & \|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|^2 \\ & \leq \left(\frac{k^2}{4\pi}\right)^2 \sum_{\substack{(\ell_1, m_1) \in \Gamma \\ (\ell_2, m_2) \in \Gamma}} |h_{\ell_1}^{(1)}(ka)|^2 |h_{\ell_2}^{(1)}(ka)|^2 |\mu_{(\ell_1, m_1; \ell_2, m_2)}^1 - \mu_{(\ell_1, m_1; \ell_2, m_2)}^2|^2. \end{aligned}$$

The estimate then follows using (3.121).  $\square$

From (3.145) and Theorem 3.3.1 we easily derive Theorem 3.3.2. Following the same arguments as in [40] one can obtain a stability result using only the  $L^2$  norm of the far field. In fact, identity (3.141) and the uniform bound of Corollary 2.3.4 allows us to reproduce exactly the same arguments as in [40, Section 4] to state the following continuity result.

**Lemma 3.5.6.** *Let  $M > 0$  and  $0 < \theta < 1$  be given. Assume that the assumption 2.3.1 holds. Then there exists a constant  $\eta > 0$  that only depends on  $M, k, a$  and  $\theta$  and a constant  $\omega$  that only depends on  $a$  and  $k$  such that*

$$\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\| \leq \eta^2 \exp\left(-\left(-\ln \frac{\|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{L^2(\mathbb{S}^2 \times \mathbb{S}^2)}}{\omega \eta}\right)^\theta\right),$$

where  $\mathcal{N}_{A_j, q_j}$ ,  $j = 1, 2$  denote here the near field operators associated with  $B = \{x \in \mathbb{R}^3, |x| < 2a\}$ .

### 3.5.4 | Proof of the Theorem 3.3.3

We now stand poised to finish this chapter by demonstrating the last main result. Using the result of this lemma and Theorem 3.3.1 one can prove Theorem 3.3.3 as follows.

According to Lemma 3.5.6

$$\begin{aligned} -\ln (\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|) &\geq -\ln(\eta^2) + \left( -\ln \frac{\|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{L^2}}{\omega\eta} \right)^\theta \\ &\geq \frac{1}{2} \left( -\ln \frac{\|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{L^2}}{\omega\eta} \right)^\theta, \end{aligned}$$

for sufficiently small  $\|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{L^2}$  such that  $\|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{L^2} \leq \omega\eta e^{-2(\ln(\eta^2))^{\frac{1}{\theta}}}$ . Then, if we further suppose that  $\|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{L^2} \leq e/(\omega\eta)$ , then

$$\left( -\ln (\|\mathcal{N}_{A_1, q_1} - \mathcal{N}_{A_2, q_2}\|) \right)^{-\frac{\sigma}{\sigma+3}} \leq 2^{\frac{\sigma(1-\theta)}{\sigma+3}} \left( -\ln \left( \|u_{A_1, q_1}^\infty - u_{A_2, q_2}^\infty\|_{L^2} \right) \right)^{-\frac{\sigma\theta}{\sigma+3}}.$$

Using the first inequality in Theorem 3.3.1 and choosing  $\theta$  such that  $\theta \frac{\sigma}{\sigma+3} = \frac{\sigma}{\sigma+3} - \epsilon$ , where  $0 < \epsilon < \frac{\sigma}{\sigma+3}$ , yield the first inequality of Theorem 3.3.3 related to  $\|\operatorname{curl} A_1 - \operatorname{curl} A_2\|_{L^\infty(D)}$ . The estimate for  $\|q_1 - q_2\|_{L^\infty(D)}$  is derived analogously.



# Inverse geometric problem

## **Abstract:**

*We consider the inverse medium scattering problem and provide an overview of the sampling techniques employed for deducing the shape of a perturbation using measurements of scattered waves at a fixed frequency. Our primary focus is on the linear sampling method (LSM) and the factorization method (FM). We present several validating results in both 2D. Using synthetic data, we end the chapter with an alternative proof for the uniqueness of the reconstruction of the shape from far-field patterns associated with all incident plane waves.*

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## 4.1 | Introduction

The sampling methods are one of well established inversion methods for inverse scattering problems. The principle of sampling method is to define an indicator function that reveals information about the position and shape of the unknown object. The sampling methods have generated a lot of discussion in the literature because of their practical importance. Some examples include the linear sampling method [21], the factorization method [51], the Generalized Linear Sampling Method (GLSM) which combines theoretical elements of both the LSM and FM [5, 4], the MUSIC-type method [29, 37, 2], the point source and probe method [74], the enclosure method [42], and the one-shot and orthogonality sampling methods [64, 75]. The main advantage of the sampling methods is that they don't involve solving a sequence of forward inverse problems like iterative methods do, and they can handle a wide range of practical problems even when there is limited information about the scatterer. Actually, the physical characteristics of the unknown object need not be known a priori using these methods. But generally speaking, the disadvantage of these approaches is that a lot of data is required because one typically needs all measurements of the far field pattern for all observation points and all directions of incident plane waves.

The chapter is organized as follows. We recall the setting of the direct scattering problem in Section 4.2, while in Section 4.3 we present the setting of the inverse problem and demonstrate the uniqueness of the domain  $D$  from the far field pattern. Next, we introduce the near field operator and its factorizations, which are necessary for the two sampling methods on which we will concentrate: the linear sampling method (LSM) and the factorization method.

Some key properties of these operators are then proved in section 4.4 as preparations for the analysis of sampling methods.

## 4.2 | Setting of the direct scattering problem

In order to ease the reading of this chapter, we recall in this section the direct scattering problem, which was introduced in Chapter 2, and give some properties that will be helpful later.

We assume that  $A \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$  such that  $\operatorname{div}(A) \in L^\infty(\mathbb{R}^3, \mathbb{R})$  and  $\operatorname{Supp}(A) \subset D$  and  $n \in L^\infty(\mathbb{R}^3, \mathbb{C})$  with non negative imaginary part and positive real part such that  $n = 1$  outside  $D$ .

Our direct problem consists in finding the total field  $u \in H^1(D)$  and the scattered field  $u^s \in H^1(\mathbb{R}^3 \setminus D)$  which verify the following system

$$(\nabla + iA)^2 u + k^2 n u = 0 \quad \text{in } D, \quad (4.1)$$

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \quad (4.2)$$

$$u - u^s = u^i \quad \text{on } \partial D, \quad (4.3)$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial u^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} \quad \text{on } \partial D, \quad (4.4)$$

$$\lim_{r \rightarrow \infty} r \left( \frac{\partial u^s}{\partial \nu} - i k u^s \right) = 0, \quad (4.5)$$

where  $\nu$  denote the outward normal. We proved in chapter 2 that the direct scattering problem has a unique solution  $u$ .

Now, we define the function  $w$  by

$$w(x) = \begin{cases} u(x) - u^i(x) & \text{if } x \in D, \\ u^s(x) & \text{if } x \in \mathbb{R}^3 \setminus \overline{D}. \end{cases}$$

Then,  $w \in H_{loc}^1(\mathbb{R}^3)$  is the unique solution of

$$\begin{cases} (\nabla + iA)^2 w + k^2 n w = \tilde{Q}_{A,n} u^i & \text{in } \mathbb{R}^3, \\ \lim_{r \rightarrow \infty} r \left( \frac{\partial w}{\partial \nu} - i k w \right) = 0. \end{cases} \quad (4.6)$$

where  $\tilde{Q}_{A,n}$  is the first order operator given by

$$\tilde{Q}_{A,n}(w) = -iA \cdot \nabla w - i\operatorname{div}(Aw) + (|A|^2 + k^2(1-n))w. \quad (4.7)$$

**Theorem 4.2.1** ([24]). *Let  $u^s \in H_{loc}^2(\mathbb{R}^3 \setminus \bar{D})$  be an outgoing solution of the Helmholtz equation in  $\mathbb{R}^3 \setminus \bar{D}$ . Then we have the following formula*

$$u^s(x) = \int_{\partial D} \left( u^s \frac{\partial \Phi}{\partial \nu(y)}(x, \cdot) - \Phi(x, \cdot) \frac{\partial u^s}{\partial \nu} \right) ds(y), \quad \forall x \in \mathbb{R}^3 \setminus \bar{D} \quad (4.8)$$

where  $\Phi(\cdot, \cdot)$  is given by (2.14).

The following reciprocity lemma will be useful later.

**Theorem 4.2.2** (Mixed reciprocity relation). *Let  $u(\cdot, z)$  be a solution generated by a point source at  $z$ :*

$$\begin{cases} (\nabla + iA)^2 u(\cdot, z) + k^2 n u(\cdot, z) = -\delta_z, & \text{in } \mathbb{R}^3, \\ u = u_1^i + u^s; \quad u_1^i(x) = \Phi(\cdot, z), \quad z \in \mathbb{R}^3 \setminus \bar{D}, \\ \lim_{r \rightarrow \infty} r \left( \frac{\partial u^s}{\partial r} - iku^s \right) = 0, \end{cases}$$

and  $\tilde{u}(\cdot, d)$  be a solution of the dual problem generated by the plane wave of direction  $d$ ,

$$\begin{cases} (\nabla - iA)^2 \tilde{u}(\cdot, d) + k^2 n \tilde{u}(\cdot, d) = 0, & \text{in } \mathbb{R}^3, \\ \tilde{u} = u_2^i + \tilde{u}^s; \quad u_2^i(x) = e^{ikd \cdot x}, \quad d \in \mathbb{S}^2, \\ \lim_{r \rightarrow \infty} r \left( \frac{\partial \tilde{u}^s}{\partial r} - ik\tilde{u}^s \right) = 0. \end{cases}$$

Then, we have the mixed reciprocity relation

$$4\pi u^\infty(-d, z) = \tilde{u}^s(z, d), \quad z \in \mathbb{R}^3 \setminus \bar{D}, \quad d \in \mathbb{S}^2,$$

where  $u^\infty(\hat{x}, z)$  is the far field pattern of the scattered field  $u^s(x, z)$ .

*Proof.* We have  $u_j^i$  a solution of  $(\Delta + k^2)u_j^i = 0$  in  $D$  for  $j = 1, 2$ . Then, according to Green second formula, we obtain

$$\int_{\partial D} \left( u_1^i \frac{\partial u_2^i}{\partial \nu} - u_2^i \frac{\partial u_1^i}{\partial \nu} \right) ds(y) = 0. \quad (4.9)$$

In addition, we know that  $\tilde{u}^s(\cdot, d)$ ,  $u^s(\cdot, z)$  are solutions of the Helmholtz equation outside of  $D$ . Then, for  $r > 0$  large enough such that  $D \subset B(0, r)$  and according to Green second formula, we have

$$\int_{\partial D} \left( u^s(y, z) \frac{\partial \tilde{u}^s}{\partial \nu}(y, d) - \tilde{u}^s(y, d) \frac{\partial u^s}{\partial \nu}(y, z) \right) ds(y) = \lim_{r \rightarrow \infty} I_r(z, d),$$

where the integral  $I_r(z, d)$  is given by

$$I_r(z, d) = \int_{S_r} \left( u^s(y, z) \frac{\partial \tilde{u}^s}{\partial \nu}(y, d) - \tilde{u}^s(y, d) \frac{\partial u^s}{\partial \nu}(y, z) \right) ds(y).$$

Using the Cauchy-Schwartz inequality and the radiation condition, we get

$$|I_r(z, d)|^2 \xrightarrow{r \rightarrow \infty} 0.$$

This implies

$$\int_{\partial D} \left( u^s(y, z) \frac{\partial \tilde{u}^s}{\partial \nu}(y, d) - \tilde{u}^s(y, d) \frac{\partial u^s}{\partial \nu}(y, z) \right) ds(y) = 0. \quad (4.10)$$

According to the definition of the far field pattern

$$u^\infty(\hat{x}) = \frac{1}{4\pi} \int_{\partial D} \left( u^s \frac{\partial}{\partial \nu} (e^{-ik\hat{x} \cdot y}) - \frac{\partial u^s}{\partial \nu} e^{-ik\hat{x} \cdot y} \right) ds(y), \quad \forall \hat{x} \in \mathbb{S}^2, \quad (4.11)$$

we obtain for  $z \in \mathbb{R}^3 \setminus \bar{D}$  and  $d \in \mathbb{S}^2$  that

$$4\pi u^\infty(-d, z) = \int_{\partial D} \left( u^s(y, z) \frac{\partial u_2^i}{\partial \nu}(y) - u_2^i(y) \frac{\partial u^s}{\partial \nu}(y, z) \right) ds(y), \quad (4.12)$$

furthermore, due to the Theorem 4.2.1, we have

$$\tilde{u}^s(z, d) = \int_{\partial D} \left( \tilde{u}^s(y, d) \frac{\partial u_1^i}{\partial \nu}(y) - u_1^i(y) \frac{\partial \tilde{u}^s}{\partial \nu}(y, d) \right) ds(y), \quad z \in \mathbb{R}^3 \setminus \bar{D}, \quad d \in \mathbb{S}^2. \quad (4.13)$$

In the other hand, by summing (4.9) and (4.12), we get

$$4\pi u^\infty(-d, z) = \int_{\partial D} \left( u(y, z) \frac{\partial u_2^i}{\partial \nu}(y) - u_2^i(y) \frac{\partial u}{\partial \nu}(y, z) \right) ds(y), \quad (4.14)$$

and by subtracting (4.10) from (4.13), one obtains

$$-\tilde{u}^s(z, d) = \int_{\partial D} \left( u(y, z) \frac{\partial \tilde{u}^s}{\partial \nu}(y, d) - \tilde{u}^s(y, d) \frac{\partial u}{\partial \nu}(y, z) \right) ds(y), \quad (4.15)$$

Adding together (4.14) and (4.15), we have

$$\int_{\partial D} \left( u(y, z) \frac{\partial \tilde{u}}{\partial \nu}(y, d) - \tilde{u}(y, d) \frac{\partial u}{\partial \nu}(y, z) \right) ds(y) = 4\pi u^\infty(-d, z) - \tilde{u}^s(z, d).$$

Finally, using Green's formula twice and the equation satisfied by  $u$  and  $\tilde{u}$  in  $D$ , we obtain the right hand side vanishes.  $\square$

## 4.3 | Setting of the inverse problem

In this section, the inverse problem we are interested in is the reconstruction of the domain  $D$  from the knowledge of  $u^\infty(\hat{x}, d)$  for all  $(\hat{x}, d)$ , (see, for instance, [13, 15, 51, 45]).

### 4.3.1 | Definition of the far field operator

The far field operator  $F : L^2(\mathbb{S}^2) \longrightarrow L^2(\mathbb{S}^2)$  is defined, as the integral operator with the far field pattern as its kernel, as follows

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} w_{A,n}^\infty(\hat{x}, d) g(d) ds(d), \quad \hat{x} = \frac{x}{|x|}, \quad (4.16)$$

where the far field pattern is given by

$$w_{A,n}^\infty(\hat{x}) = -\frac{1}{4\pi} \int_D e^{-ik\hat{x}\cdot y} \tilde{Q}_{A,n}(u^i + w)(y) dy, \quad (4.17)$$

where  $w$  satisfies (4.6) with incident field  $u^i$ . Now, let us consider the Herglotz operator

$$\begin{aligned} \mathcal{H} : L^2(\mathbb{S}^2) &\longrightarrow L^2(D), \\ g &\longmapsto v_g|_D, \end{aligned} \quad (4.18)$$

where  $v_g$  is the Herglotz wave function defined by

$$v_g(x) = \int_{\mathbb{S}^2} e^{ikx\cdot d} g(d) ds(d), \quad x \in \mathbb{R}^3. \quad (4.19)$$

The function  $g$  is called the Herglotz kernel of  $v_g$ .

Moreover, let  $\mathcal{H}^* : L^2(D) \longrightarrow L^2(\mathbb{S}^2)$  the adjoint of the operator Herglotz  $\mathcal{H}$  be defined as follows

$$(\mathcal{H}^* \psi)(\hat{x}) = \int_D e^{-ik\hat{x} \cdot y} \psi(y) dy, \quad \psi \in L^2(D), \quad \hat{x} \in \mathbb{S}^2. \quad (4.20)$$

Let us denote by  $H_{inc}(D)$  the closure of the range of  $\mathcal{H}$  in  $L^2(D)$ . This estimate help us to consider the compact operator  $G$  as follows

$$\begin{aligned} G : H_{inc}(D) &\longrightarrow L^2(\mathbb{S}^2), \\ u_0 &\longmapsto w_{A,n}^\infty \end{aligned} \quad (4.21)$$

where  $w_{A,n}^\infty$  is the far field pattern of  $w$  solution of (4.6) with incident field  $u^i = u_0$ . Then, for all  $g \in L^2(\mathbb{S}^2)$ , we obtain

$$(G \circ H)(g) = \int_{\mathbb{S}^2} G(e^{ikx \cdot d}) g(d) ds(d) = F(g).$$

This ensures the first factorization

$$\boxed{F = G\mathcal{H}.} \quad (4.22)$$

Next, using the definition of  $w_{A,n}^\infty$  and the adjoint of the Herglotz operator  $\mathcal{H}^*$ , we can obtain that  $G = \mathcal{H}^* T$  where the operator  $T$  is defined by

$$\begin{aligned} T : L^2(D) &\longrightarrow L^2(D) \\ u_0 &\longmapsto -\frac{1}{4\pi} \tilde{Q}_{A,n}(u^i + w), \end{aligned} \quad (4.23)$$

where  $w$  is the solution of (4.6) with incident field  $u^i = u_0$ . Thus, we get the second factorization of the far field operator  $F : L^2(\mathbb{S}^2) \longrightarrow L^2(\mathbb{S}^2)$  in this form

$$\boxed{F = \mathcal{H}^* T \mathcal{H}.} \quad (4.24)$$

Moreover, let

$$F_\# := |\Re(F)| + |\Im(F)|, \quad (4.25)$$

where

$$\Re(F) := \frac{1}{2}(F + F^*) \quad , \quad \Im(F) := \frac{1}{2i}(F - F^*), \quad (4.26)$$

with  $F^*$  is the adjoint of the far field operator  $F$ .

The following interior transmission problem serves as a basis for the sampling techniques that will be discussed later and is defined by: Given  $(f, h) \in H^{\frac{3}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ , find  $(u, u_0) \in L^2(D) \times L^2(D)$  such that  $u - u_0 \in H^2(D)$  and

$$ITP(A, n) \begin{cases} (\nabla + iA)^2 u + k^2 n u = 0 & \text{in } D, \\ \Delta u_0 + k^2 u_0 = 0, & \text{in } D, \\ u - u_0 = f & \text{on } \partial D, \\ \frac{\partial u}{\partial \nu} - \frac{\partial u_0}{\partial \nu} = h & \text{on } \partial D, \end{cases} \quad (4.27)$$

where  $\nu$  denotes the outward normal on  $\partial D$ .

In the rest of this chapter, we need the following assumption that formulates the well-posedness of ITP.

**Assumption 4.3.1.** *The refractive index  $n$ , the magnetic potential  $A$  and the wave number  $k$  are such that  $ITP(A, n)$  and  $ITP(-A, n)$  are well-posed problem.*

The solvability of the interior transmission problem will be studied in Chapter 5.

### 4.3.2 | Some useful properties for sampling methods

We start now with the following Lemma which gives a characterization of the closure of the range of  $\mathcal{H}$ .

**Lemma 4.3.2** ([15]). *The operator  $\mathcal{H}$  defined by (4.18) is compact and injective. Let  $H_{inc}(D)$  be the closure of the range of  $\mathcal{H}$  in  $L^2(D)$ . Then, we have*

$$H_{inc}(D) = \{ v \in L^2(D) \text{ such that } \Delta v + k^2 v = 0 \text{ in } D \}. \quad (4.28)$$

The reciprocity lemma that follows will be helpful.



**Lemma 4.3.3.** *Let  $u_0, u_1 \in H_{inc}(D)$  and let  $w_0$  and  $w_1 \in H_{loc}^1(\mathbb{R}^3)$  be the solutions of (4.6) corresponding to  $(\nabla + iA)^2 w_0 + k^2 n w_0 = \tilde{Q}_{A,n}(u_0)$  and  $(\nabla - iA)^2 w_1 + k^2 n w_1 = \tilde{Q}_{-A,n}(u_1)$  together with the sommerfeld radiation condition. Then*

$$\int_D \tilde{Q}_{A,n} u_0 w_1 \, dx = \int_D \tilde{Q}_{-A,n} u_1 w_0 \, dx. \quad (4.29)$$

*Proof.* We have that  $w_0$  and  $w_1$  satisfy

$$(\nabla + iA)^2 w_0 + k^2 n w_0 = \tilde{Q}_{A,n} u_0 \quad \text{in } \mathbb{R}^3, \quad (4.30)$$

and

$$(\nabla - iA)^2 w_1 + k^2 n w_1 = \tilde{Q}_{-A,n} u_1 \quad \text{in } \mathbb{R}^3, \quad (4.31)$$

respectively. Assume that  $B$  is an open,  $\bar{D}$ -containing ball with radius  $R$ . Multiplying (4.30) by  $w_1$  and (4.31) by  $w_0$  and integrating over  $B$  yields

$$\begin{aligned} \int_B \Delta w_0 w_1 \, dx + i \int_B \operatorname{div}(A w_0) w_1 + iA \cdot \nabla w_0 w_1 \, dx \\ + \int_B (k^2 n - |A|^2) w_0 w_1 \, dx = \int_B \tilde{Q}_{A,n} u_0 w_1 \, dx, \end{aligned}$$

and

$$\begin{aligned} \int_B \Delta w_1 w_0 \, dx - i \int_B \operatorname{div}(A w_1) w_0 + iA \cdot \nabla w_1 w_0 \, dx \\ + \int_B (k^2 n - |A|^2) w_0 w_1 \, dx = \int_B \tilde{Q}_{-A,n} u_1 w_0 \, dx. \end{aligned}$$

Taking the difference

$$\begin{aligned} \int_B \Delta w_0 w_1 - \Delta w_1 w_0 \, dx + i \int_B \operatorname{div}(A w_0) w_1 + iA \cdot \nabla w_0 w_1 \, dx \\ + i \int_B \operatorname{div}(A w_1) w_0 + iA \cdot \nabla w_1 w_0 \, dx = \int_B \tilde{Q}_{A,n} u_0 w_1 - \tilde{Q}_{-A,n} u_1 w_0 \, dx. \end{aligned} \quad (4.32)$$

By integrating by parts, we obtain

$$\int_{|x|=R} \frac{\partial w_0}{\partial r} w_1 - \frac{\partial w_1}{\partial r} w_0 \, ds(x) = \int_B \tilde{Q}_{A,n} u_0 w_1 - \tilde{Q}_{-A,n} u_1 w_0 \, dx. \quad (4.33)$$

Due to the fact that  $w_0$  and  $w_1$  verify the Sommerfeld radiation and the following identity

$$\lim_{R \rightarrow \infty} \int_{|x|=R} |w_j|^2 \, ds(x) = \int_{\mathbb{S}^2} |w_{A,n}^{\infty,j}|^2 \, ds(\hat{x}), \quad j = 1, 2,$$

we obtain

$$\lim_{R \rightarrow \infty} \int_{|x|=R} \frac{\partial w_0}{\partial r} w_1 - \frac{\partial w_1}{\partial r} w_0 \, ds(x) = 0.$$

We finish the proof by letting  $R \rightarrow \infty$  in (4.33).  $\square$

Now, we demonstrate the key element needed to justify the sampling Methods.

**Theorem 4.3.4.** *Assume that assumption 4.3.1 holds. Then, The operator  $G : H_{inc}(D) \rightarrow L^2(\mathbb{S}^2)$ , defined by (4.21), is injective with dense range. Moreover*

$$\Phi^\infty(\cdot, z) \in \mathcal{R}(G) \quad \text{if and only if} \quad z \in D, \quad (4.34)$$

where  $\Phi^\infty(\cdot, z)$  is the far field pattern due to a point source located at the point  $z$  which is given by

$$\Phi^\infty(\hat{x}, z) = \frac{1}{4\pi} e^{-ik\hat{x} \cdot z}, \quad \hat{x} \in \mathbb{S}^2. \quad (4.35)$$

*Proof.* We start by proving that  $G$  is injective with dense range. Let  $u_0$  and  $w$  satisfy (4.6). Then, we have

$$w_{A,n}^\infty(\hat{x}) = -\frac{1}{4\pi} \int_D e^{-ik\hat{x} \cdot y} \tilde{Q}_{A,n}(u_0 + w)(y) \, dy, \quad \hat{x} \in \mathbb{S}^2.$$

Then, for  $g \in L^2(\mathbb{S}^2)$ , we get

$$\begin{aligned} (G(u_0), g)_{L^2(\mathbb{S}^2)} &= -\frac{1}{4\pi} \int_D \tilde{Q}_{A,n}(u_0 + w)(x) \overline{\int_{\mathbb{S}^2} e^{-ikd \cdot x} g(d) \, ds} \, dx \\ &= -\frac{1}{4\pi} \int_D \tilde{Q}_{A,n}(u_0 + w)(x) \overline{\mathcal{H}g}(x) \, dx. \end{aligned} \quad (4.36)$$

We assume that  $u_0 = \overline{\mathcal{H}\varphi}$  for some  $\varphi \in L^2(\mathbb{S}^2)$  and set  $w(\varphi) \equiv w$ . The equality (4.36) can be written as

$$(G(\overline{\mathcal{H}\varphi}), g)_{L^2(\mathbb{S}^2)} = k^2 \int_D \tilde{Q}_{A,n}(\overline{\mathcal{H}\varphi}) \overline{\mathcal{H}g} \, dx + k^2 \int_D \tilde{Q}_{A,n}(w(\varphi)) \overline{\mathcal{H}g} \, dx.$$

Using the Lemma 4.3.3, we obtain

$$\int_D \tilde{Q}_{A,n}(w(\varphi)) \overline{\mathcal{H}g} \, dx = \int_D \tilde{Q}_{-A,n}(w(g)) \overline{\mathcal{H}\varphi} \, dx,$$

and by integrating by parts, we find

$$\int_D \tilde{Q}_{A,n}(\overline{\mathcal{H}\varphi}) \overline{\mathcal{H}g} \, dx = \int_D \tilde{Q}_{-A,n}(\overline{\mathcal{H}g}) \overline{\mathcal{H}\varphi} \, dx.$$

Therefore, we obtain the reciprocity relation

$$(G(\overline{\mathcal{H}\varphi}), g)_{L^2(\mathbb{S}^2)} = (\tilde{G}(\overline{\mathcal{H}g}), \varphi)_{L^2(\mathbb{S}^2)}, \quad (4.37)$$

where  $\tilde{G} : H_{inc}(D) \rightarrow L^2(\mathbb{S}^2)$  is the operator defined the same as  $G$  replacing  $A$  by  $-A$ . We assume  $(G(\overline{\mathcal{H}\varphi}), g)_{L^2(\mathbb{S}^2)} = 0, \forall \varphi \in L^2(\mathbb{S}^2)$ . Using the previous reciprocity relation (4.37), we find  $\tilde{G}(\overline{\mathcal{H}g}) = 0$ . We get  $w(g) = 0$  in  $\mathbb{R}^3 \setminus D$  using Rellich's Lemma and the unique continuation principle where  $w(g)$  is solution to (4.6) with  $A$  replaced by  $-A$ . Moreover, let  $u := w(g) + \overline{\mathcal{H}g}$ , then, we obtain that  $(u, \overline{\mathcal{H}g})$  is a solution to  $ITP(-A, n)$  with  $f = h = 0$  which implies that  $\mathcal{H}g = 0$  in  $D$ . By the injectivity of  $\mathcal{H}$  (Lemma 4.3.2), we get  $g = 0$ . This proves the denseness of the range of  $G$ .

Next, we are going to prove the injectivity of  $G$ . Let  $u_0 \in H_{inc}(D)$  and let  $w \in H_{loc}^1(\mathbb{R}^3)$  be associated field via (4.6) with  $u^i = u_0$ . Assume that  $G(u_0) = 0$ . By Rellich's Lemma and the unique continuation principle, we obtain that  $w = 0$  in  $\mathbb{R}^3 \setminus D$ . Therefore, the pair  $(u, u_0)$  is a solution to (4.27) with zero data. If we set  $u := w + u_0$ , then, the assumption of the well-posedness of  $ITP(A, n)$  guarantees that  $w = u_0 = 0$ , demonstrating the injectivity of  $G$ .

We finish by proving the last part of the theorem. We note that  $\Phi^\infty(\cdot, z)$  is the far field pattern of  $u_e = \Phi(\cdot, z)$  satisfying  $\Delta u_e + k^2 u_e = -\delta_z$  in  $\mathbb{R}^3$  as well as the Sommerfeld radiation condition. Let  $z \in D$ . We take  $(u, u_0) \in L^2(D) \times L^2(D)$  to be the solution to (4.27) with

$$\begin{cases} f(x) = u_e(x, z) & x \in \partial D, \\ h(x) = \frac{\partial u_e(x, z)}{\partial \nu(x)} & x \in \partial D. \end{cases} \quad (4.38)$$

We define

$$w(x) = \begin{cases} u(x) - u_0(x) & x \in D, \\ u_e(x, z) & x \in \mathbb{R}^3 \setminus D. \end{cases} \quad (4.39)$$

According to (4.38), we have that  $w \in H^2(\mathbb{R}^3)$  and satisfies (4.6). Therefore  $G(u_0) = \Phi^\infty(\cdot, z)$ .

Next, let  $z \in \mathbb{R}^3 \setminus D$ . Suppose that there exists  $u_0 \in H_{inc}(D)$  such that  $G(u_0) = \Phi^\infty(\cdot, z)$ . We determine that  $w = u_e(\cdot, z)$  in  $\mathbb{R}^3 \setminus D$ , where  $w$  is the solution to the problem (4.6), via the Rellich's Lemma and the unique continuation principle. Given that  $w \in H^1(\mathbb{R}^3 \setminus D)$  yet  $u_e(\cdot, z) \notin H_{loc}^1(\mathbb{R}^3 \setminus D)$ , this results in a contradiction.  $\square$

The following lemma that follows will be helpful.

**Lemma 4.3.5.** *For all  $\psi \in L^2(D)$ , we have the following identity*

$$4\pi \Im (T\psi, \psi)_{L^2(D)} = k \int_D |w_{A,n}^\infty|^2 ds + k^2 \int_D \Im(n) |\psi + w|^2 dx. \quad (4.40)$$

where  $w$  is the scattered field solution of (4.6) associated with the incident field  $u^i = \psi$ .

*Proof.* Multiplying the first equation of (4.6) by  $\bar{w}$  and applying the Green Theorem, we obtain

$$\int_B (\nabla + iA)^2 w \bar{w} dx + k^2 \int_B n |w|^2 dx = \int_B \tilde{Q}_{A,n} \psi \bar{w} dx,$$

yields

$$\begin{aligned} & - \int_B |\nabla w|^2 dx + k^2 \int_B |w|^2 dx + \int_B iA \cdot \nabla w \bar{w} + i \operatorname{div}(Aw) \bar{w} dx \\ & \quad + \int_{|x|=R} \frac{\partial w}{\partial \nu} \bar{w} ds - \int_B (|A|^2 + k^2(1-n)) |w|^2 dx \\ & = - \int_B (iA \cdot \nabla \psi \bar{w} + i \operatorname{div}(A\psi) \bar{w}) dx + \int_B (|A|^2 + k^2(1-n)) \psi \bar{w} dx, \end{aligned}$$

which implies

$$\begin{aligned}
 & \int_D (|A|^2 + k^2(1-n))(\psi + w)\bar{w} \, dx - i \int_B A \cdot \nabla(\psi + w)\bar{w} \, dx \\
 & \qquad \qquad \qquad - i \int_B \operatorname{div}(A(\psi + w))\bar{w} \, dx \\
 & = - \int_B |\nabla w|^2 \, dx + k^2 \int_B |w|^2 \, dx + \int_{|x|=R} \frac{\partial w}{\partial \nu} \bar{w} \, ds.
 \end{aligned}$$

Taking the imaginary part, we find

$$\begin{aligned}
 \Im \left( \int_{|x|=R} \frac{\partial w}{\partial \nu} \bar{w} \, ds \right) &= \Im \left( \int_D (|A|^2 + k^2(1-n))(\psi + w)\bar{w} \, dx \right) \\
 &\quad - \Im \left( i \int_B A \cdot \nabla(\psi + w)\bar{w} + \operatorname{div}(A(\psi + w))\bar{w} \, dx \right),
 \end{aligned}$$

using the fact that

$$\Im \left( \int_{|x|=R} \frac{\partial w}{\partial \nu} \bar{w} \, ds \right) = k \int_{\mathbb{S}^2} |w_{A,n}^\infty|^2 \, ds, \tag{4.41}$$

we obtain

$$\begin{aligned}
 k \int_{\mathbb{S}^2} |w_{A,n}^\infty|^2 \, ds &= \Im \left( \int_D (|A|^2 + k^2(1-n))(\psi + w)\bar{w} \, dx \right) \\
 &\quad - \Im \left( i \int_B A \cdot \nabla(\psi + w)\bar{w} + \operatorname{div}(A(\psi + w))\bar{w} \, dx \right).
 \end{aligned}$$

Next, we have

$$\begin{aligned}
 (T\psi, \psi)_{L^2(D)} &= \frac{1}{4\pi} \int_D iA \cdot \nabla(\psi + w)\bar{w} \, dx + \frac{1}{4\pi} \int_D i \operatorname{div}(A(\psi + w))\bar{w} \, dx \\
 &\quad + \frac{1}{4\pi} \int_D (k^2(n-1) - |A|^2)(\psi + w)\bar{w} \, dx.
 \end{aligned}$$

Since  $(\psi + w)\bar{\psi} = |\psi + w|^2 - (\psi + w)\bar{w}$ , we find

$$\begin{aligned}
 (T\psi, \psi)_{L^2(D)} &= \frac{1}{4\pi} \int_D iA \cdot \nabla(\psi + w)\bar{\psi} \, dx + \frac{1}{4\pi} \int_D i \operatorname{div}(A(\psi + w))\bar{\psi} \, dx \\
 &\quad - \frac{1}{4\pi} \int_D (k^2(n-1) - |A|^2)(\psi + w)\bar{w} \, dx \\
 &\quad + \frac{1}{4\pi} \int_D (k^2(n-1) - |A|^2)|\psi + w|^2 \, dx,
 \end{aligned}$$

therefore, we get

$$\begin{aligned}
 4\pi \Im \left( (T\psi, \psi)_{L^2(D)} \right) &= \Im \left( \int_D iA \cdot \nabla(\psi + w) \bar{\psi} + i \operatorname{div}(A(\psi + w)) \bar{\psi} \, dx \right) \\
 &\quad + \Im \left( \int_D (|A|^2 + k^2(1-n))(\psi + w) \bar{w} \, dx \right) \\
 &\quad + k^2 \int_D \Im(n) |\psi + w|^2 \, dx \\
 &= \Im \left( \int_D iA \cdot \nabla(\psi + w) (\overline{\psi + w}) + i \operatorname{div}(A(\psi + w)) \overline{\psi + w} \, dx \right) \\
 &\quad + k \int_{\mathbb{S}^2} |w_{A,n}^\infty|^2 \, ds + k^2 \int_D \Im(n) |\psi + w|^2 \, dx.
 \end{aligned}$$

Knowing that

$$\begin{aligned}
 \int_D [iA \cdot \nabla(\psi + w) + i \operatorname{div}(A(\psi + w))] \overline{\psi + w} \, dx \\
 = -2\Im \left( \int_D A \cdot \nabla(\psi + w) \overline{(\psi + w)} \, dx \right) \in \mathbb{R},
 \end{aligned}$$

gives us the desired result.  $\square$

We have the following lemma.

**Lemma 4.3.6.** *Let  $\Im(n) \geq 0$  and  $\Re(n-1) \geq \alpha > 0$  in  $D$  ( respectively  $\Re(1-n) \geq \alpha > 0$  in  $D$ ). Then, the operator  $T$  satisfies*

$$\Im \left( (T\psi, \psi)_{L^2(D)} \right) \geq 0, \quad \forall \psi \in H_{inc}(D). \quad (4.42)$$

Moreover,  $\Im(T)$  is injective on  $H_{inc}(D)$  and we can decompose  $\Re(T)$  as  $\Re(T) = T_0 + C$  where  $C$  is compact on  $H_{inc}(D)$  and  $T_0$  satisfies

$$(T_0\psi, \psi)_{L^2(D)} \geq \beta \|\psi\|_{L^2(D)}^2, \quad (4.43)$$

for all  $\psi \in H_{inc}(D)$  and for some  $\beta > 0$ .

*Proof.* We have

$$T(\psi) = -\frac{1}{4\pi} \tilde{Q}_{A,n}(\psi + w(\psi)), \quad (4.44)$$

where  $w(\psi)$  is a solution of (4.6) with  $u^i = \psi$ . Let  $T_0 : L^2(D) \rightarrow L^2(D)$  the operator defined by

$$T_0(\psi) = \frac{k^2}{4\pi} \Re(n-1)\psi. \quad (4.45)$$

For all  $\psi \in H_{inc}(D)$ , we have

$$\left| (T_0\psi, \psi)_{L^2(D)} \right| \geq \frac{k^2}{4\pi} \left| \int_D \Re(n-1)|\psi|^2 dx \right| \geq \frac{k^2\alpha}{4\pi} \|\psi\|_{L^2(D)}^2,$$

which implies  $T_0$  is coercive. Moreover, we define  $T_1 := \Re(T) - T_0$  as follows

$$\begin{aligned} T_1 : L^2(D) &\rightarrow L^2(D) \\ \psi &\mapsto T_1(\psi) = (T'_1 + T''_1)(\psi), \end{aligned}$$

where

$$T'_1(\psi) = \frac{1}{4\pi} (iA \cdot \nabla w(\psi) + i\operatorname{div}(Aw(\psi))) + \frac{1}{4\pi} (k^2\Re(n-1) - |A|^2) w(\psi),$$

and

$$T''_1(\psi) = \frac{1}{4\pi} (iA \cdot \nabla \psi + i\operatorname{div}(A\psi) - |A|^2\psi), \quad \psi \in L^2(D).$$

Let  $\psi_j \in H_{inc}(D)$  such that  $\psi_j \rightarrow 0$  in  $L^2(D)$  and  $w_j = w(\psi_j) \in H^2(D)$ . Then, by the continuous of the mapping  $\psi_j \mapsto w(\psi_j)$  from  $L^2(D)$  into  $H^2(D)$ , we have that  $w_j \rightarrow 0$  in  $H^2(D)$  and  $\nabla w_j \rightarrow 0$  in  $H^1(D)$ . Using the Rellich's compact embedding Theorem, we find  $w_j \rightarrow 0$  in  $L^2(D)$  and  $\nabla w_j \rightarrow 0$  in  $L^2(D)$ . Then,  $w_j \rightarrow 0$  in  $H^1(D)$ . Furthermore, we obtain

$$\|T'_1(\psi_j)\| \leq C\|w_j\|_{H^1(D)} \quad \forall j \in \mathbb{N}, \quad (4.46)$$

which implies that  $T'_1$  is compact. Moreover, we have  $\psi_j \in H_{inc}(D)$  then by combining the elliptic regularity with the knowledge that  $A$  has compact support in  $D$ , we get that  $\psi \mapsto -\frac{1}{4\pi}|A|^2\psi$  is compact and that  $\operatorname{div}(A\psi_j), A \cdot \nabla \psi_j \in L^2(K)$ , for all  $K$  compact of  $D$  which means that there is a constant  $C(K)$  such that

$$\|A \cdot \nabla \psi_j\|_{L^2(K)} \leq C(K)\|\psi_j\|_{L^2(D)} \quad \forall j \in \mathbb{N}, \quad (4.47)$$

which implies that  $T_1''$  is compact. Therefore, we obtain  $T_1$  is compact. From (4.40), we have

$$\Im \left( (T\psi, \psi)_{L^2(D)} \right) \geq 0.$$

To finish the proof, we only need to demonstrate that  $\Im(T)$  is injective on  $H_{inc}(D)$ . We have  $(\Im(T)\psi, \psi)_{L^2(D)} = \Im \left( (T\psi, \psi)_{L^2(D)} \right) = 0$ , which implies

$$k \int_D |w_{A,n}^\infty|^2 ds + k^2 \int_D \Im(n) |\psi + w|^2 dx = 0.$$

Since  $k$  is not a transmission eigenvalues and  $\Im(n) \geq 0$ , we find  $w_{A,n}^\infty = 0$  and  $\psi + w = 0$  in  $D$ , which implies with the help of Rellich's Lemma that  $w = 0$  in  $\mathbb{R}^3 \setminus D$ . Moreover, the pair  $(w + \psi, w)$  is a solution to (4.27) with zero data. Then, by the assumption of the well-posedness of ITP, we deduce that  $w = \psi = 0$ .  $\square$

## 4.4 | Application to sampling methods

### 4.4.1 | The Linear Sampling Method (LSM)

Let's now describe the Linear Sampling Method (LSM), which has been first introduced in a paper by Kirsh and Colton in 1996 [21] and analyzed in a number of subsequent works, [27, 18, 14]. The LSM allows the reconstruction of the shape of an obstacle (or a local inhomogeneity) from multi-static data at a fixed frequency. This method has the simplest formulation and can be easily adapted to different settings of the data (near field data, data available on a limited aperture) and the scattering problem (inhomogeneous background). For different applications of the LSM for obstacle detection with various boundary conditions in the case of Helmholtz or Maxwell equation we refer to [13, 18, 23] and also for scattering by a partially coated crack you can see [14, 59].

The argument for the Linear Sampling Method (LSM) is based on the following theorem.

**Theorem 4.4.1.** *The operator  $F : L^2(\mathbb{S}^2) \longrightarrow L^2(\mathbb{S}^2)$  is injective with dense range. Moreover,*



- (i) If  $z \in D$  then there exists a sequence  $g_z^\alpha \in L^2(\mathbb{S}^2)$  such that  $\lim_{\alpha \rightarrow 0} \|Fg_z^\alpha - \Phi^\infty(\cdot, z)\|_{L^2(\mathbb{S}^2)} = 0$  and  $\lim_{\alpha \rightarrow 0} \|\mathcal{H}g_z^\alpha\|_{L^2(D)} < \infty$ .
- (ii) If  $z \notin D$  then for all  $g_z^\alpha \in L^2(\mathbb{S}^2)$  such that  $\lim_{\alpha \rightarrow 0} \|Fg_z^\alpha - \Phi^\infty(\cdot, z)\|_{L^2(\mathbb{S}^2)} \rightarrow 0$  and  $\lim_{\alpha \rightarrow 0} \|\mathcal{H}g_z^\alpha\|_{L^2(D)} = \infty$ .

*Proof.* Using the injectivity and the denseness of the range of the operators  $\mathcal{H}$  and  $G$ , which are provided by Lemma 4.3.2 and Theorem 4.3.4, we obtain the injectivity and the denseness of the range of  $F$ .

If  $z \in D$ , let  $u_0 \in H_{inc}(D)$  be such that  $G(u_0) = \Phi^\infty(\cdot, z)$ , which is true according to Theorem 4.3.4, exists. The first statement arises from the fact that  $F = G \circ \mathcal{H}$ , which is shown by Lemma 4.3.2, that there exists a sequence  $g_z^\alpha \in L^2(\mathbb{S}^2)$  such that  $\mathcal{H}g_z^\alpha \rightarrow u_0$  as  $\alpha \rightarrow 0$ .

If  $z \notin D$ . Let  $g_z^\alpha \in L^2(\mathbb{S}^2)$  be such that  $\|F(g_z^\alpha) - \Phi^\infty(\cdot, z)\|_{L^2(\mathbb{S}^2)} \rightarrow 0$  as  $\alpha \rightarrow 0$ . We assume that  $\|\mathcal{H}g_z^\alpha\|_{L^2(D)}$  is bounded as  $\alpha \rightarrow 0$ . We can assume that  $\mathcal{H}g_z^\alpha$  weakly converges to some  $u_0 \in H_{inc}(D)$  without suffering from generality loss. Theorem 4.3.4's final statement is in conflict with the limit  $G(u_0) = \Phi^\infty(\cdot, z)$ , which results from the fact that  $F = G \circ \mathcal{H}$ .  $\square$

The numerical implementation of the LSM consists of 3 main steps:

- Utilizing a priori data on the approximate size and location of the scatterer, the parameter  $z$  is varied over a grid in the region where the scatterer  $D$  is to be located. To ensure the effectiveness of this approach, the grid has to be "fine enough" and  $D$  needs to be within the search domain.
- This step involves using each  $z$  in the grid to approximately solve the following equation by finding  $g_z^\alpha \in L^2(D)$  such that

$$F(g_z^\alpha) \simeq \Phi^\infty(\cdot, z), \quad (4.48)$$

using Tikhonov regularization [88], which requires the solution  $g_z^\alpha \in L^2(\mathbb{S}^2)$  of the equation

$$(\alpha + FF^*)g_z^\alpha = F^*(\Phi_\infty(\cdot, z)), \quad (4.49)$$

where  $\alpha > 0$  denotes the regularization parameter, and the Morozov discrepancy principle [27]. Assuming that  $F^\delta$  is the far field operator corresponding to noisy measurements  $u_\infty^\delta$  such that

$$\|u_\infty^\delta - u_\infty\|_{L^2(\mathbb{S}^2) \times L^2(\mathbb{S}^2)} \leq \delta. \quad (4.50)$$

For each  $z$ , the parameter  $\alpha$  is chosen in such away that

$$\|F^\delta g_\alpha(\cdot, z) - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S}^2)} = \delta \|g_\alpha(\cdot, z)\|_{L^2(\mathbb{S}^2)}. \quad (4.51)$$

- The indicator function  $\mathcal{I}(z) = \frac{1}{\|g_z^\alpha\|_{L^2(D)}}$  is then plotted and enabling the extraction of the scatterer as the location of  $z$  for which  $\mathcal{I}(z)$  does not vanish.

#### 4.4.2 | The Factorization Method $F_\#$

The Factorization method has been first introduced by A. Kirsh on 1998 [51]. It is inspired from the Linear Sampling Method. Unlike LSM, which relies on an approximate solution, this approach provides a exact characterisation of the scatterer in terms of the far field operator. In [52], Kirsch studied the acoustic inhomogeneous medium case. In [54], he extended the the Factorization method to include absorbing media or limited aperture. For a thorough analysis of the Factorization method utilized to solve many inverse problems, see the monograph [56].

The following lemma gives the characterization of  $D$  in terms of the operator  $\mathcal{H}^*$ .

**Lemma 4.4.2** ([15]). *For  $z \in \mathbb{R}^3$ , we have that  $z \in D$  if and only if  $\Phi^\infty(\cdot, z)$  is in the range of  $\mathcal{H}^*$ .*

In view of the previous Lemma, Lemma 4.3.6 and Lemma 4.3.2, we can apply directly the result of the factorization method  $F_\#$  (see Theorem 2.31, [15]).

**Theorem 4.4.3.** *Let  $\Im(n) \geq 0$  and  $\Re(n-1) \geq \alpha > 0$  in  $D$  ( respectively  $\Re(1-n) \geq \alpha > 0$  in  $D$ ). Then,  $z \in D$  if and only if  $\Phi^\infty(\cdot, z)$  as in the range of  $(F_\#)^{\frac{1}{2}}$ .*

The factorization method (FM) consists also of three main steps:

- Utilizing a priori data on the approximate size and location of the scatterer, the parameter  $z$  is varied over a grid in the region where the scatterer  $D$  is sought. For this approach to be effective, the grid must be sufficiently fine and  $D$  needs to be within the search domain.
- The second step consists of using each  $z$  in the grid to approximately solve the following equation by finding  $g_{z,\#} \in L^2(D)$  such that

$$F_{\#}(g_{z,\#}) \simeq \Phi^{\infty}(\cdot, z) \quad (4.52)$$

and then computing  $g_{z,\#}$  by using Tikhonov regularization [88] along with the Morozov discrepancy principle [27].

- Finally, the indicator function  $\mathcal{I}(z) = \frac{1}{\|g_{z,\#}\|_{L^2(D)}}$  is plotted, and valuable information about the scatterer can be extracted.

### 4.4.3 | Validating Numerical Experiments

We conclude the study with numerical tests of the sampling methods presented earlier. Our focus in this section is solely on two-dimensional models.

#### 4.4.3.1 | Reconstruction using LSM

The frequency is  $k = 6$  and  $N_d$  equidistant incident directions and observation points have been used. The data have been generated synthetically by solving the forward scattering problem using a standard finite element method. Let  $g_z^{\alpha}$  be the Tikhonov regularized solution of (4.49), where the regularization parameter is computed using the Morozov discrepancy principle.

In the following numerical examples, we show the reconstruction of domain  $D$  using the indicator function

$$\mathcal{I}(z) = \frac{1}{\|g_z^{\alpha}\|_{L^2(D)}}. \quad (4.53)$$

■ For  $D$  is a ball domain:

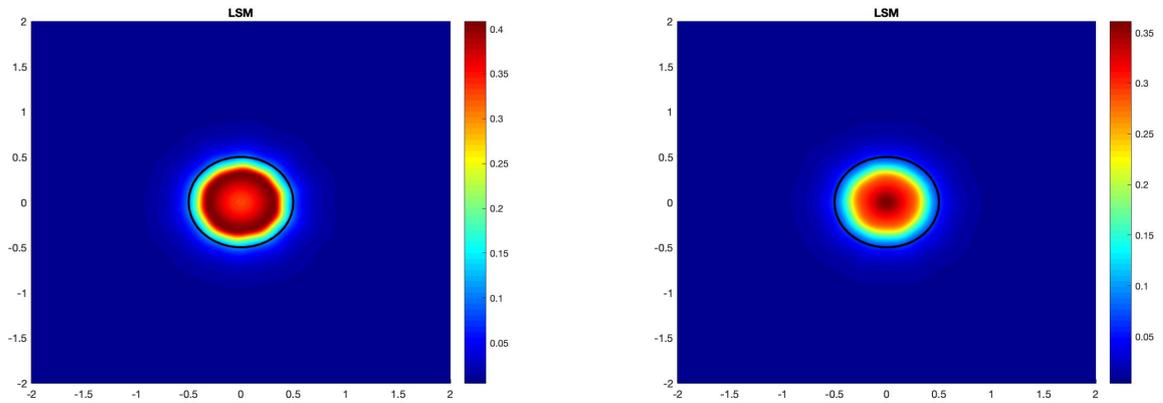


Figure 4.1: Reconstruction of the domain  $D$  for  $A = 0$  (on the left) and for  $A = 2.5(-\sin(\theta), \cos(\theta))$  in  $D$  where  $\theta \in (0, 2\pi)$  and  $A = 0$  outside  $D$  (on the right).

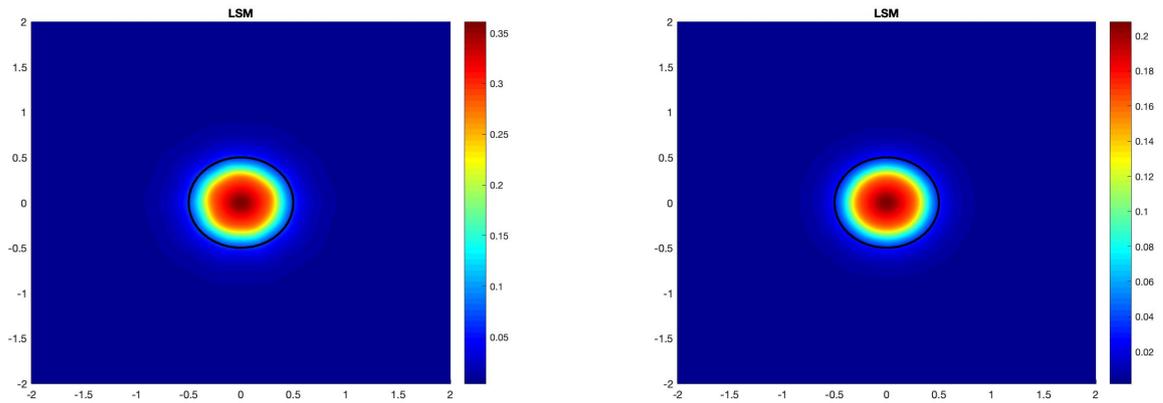


Figure 4.2: Reconstruction of the domain  $D$  with the presence of  $A$ , i.e.,  $A = 2.5(-\sin(\theta), \cos(\theta))$  in  $D$  where  $\theta \in (0, 2\pi)$  and  $A = 0$  outside  $D$ , for  $N_d = 40$  (on the left) and  $N_d = 60$  (on the right).

■ For  $D$  is a different geometry:

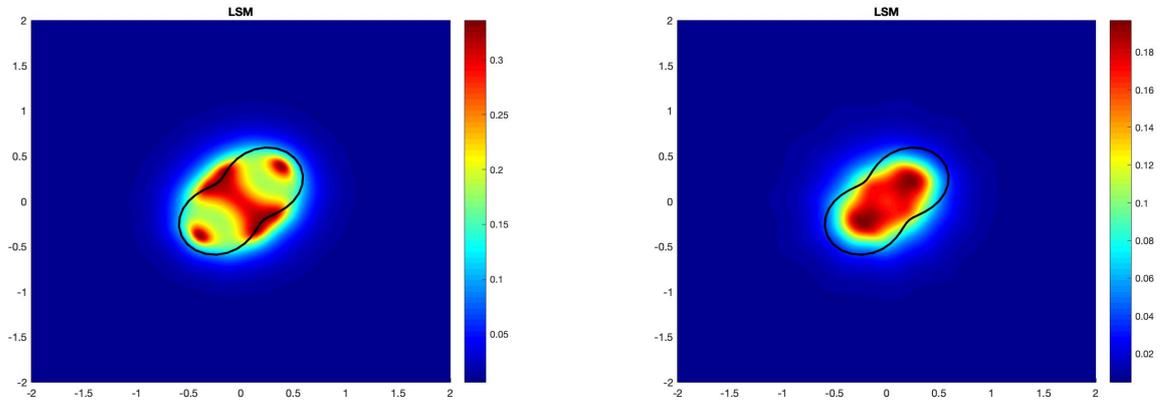


Figure 4.3: Reconstruction of the domain  $D$  for  $A = 0$  (on the left) and for  $A = 2.5(x_\theta, y_\theta)$  in  $D$  where  $x_\theta = 0.8 \cos(2\theta) \cos(\theta) - (1 + 0.4 \sin(2\theta)) \sin(\theta)$ ,  $y_\theta = 0.8 \cos(2\theta) \sin(\theta) + (1 + 0.4 \sin(2\theta)) \cos(\theta)$  and  $\theta \in (0, 2\pi)$  and  $A = 0$  outside  $D$  where  $\text{div}(A) = 0$ , (on the right).

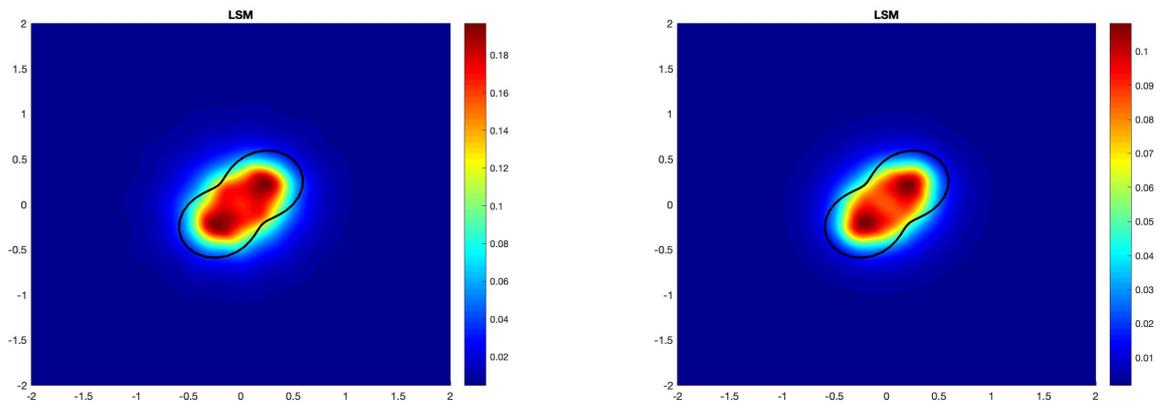


Figure 4.4: Reconstruction of the domain  $D$  with the presence of  $A$  where  $A = 2.5(x_\theta, y_\theta)$  in  $D$  where  $x_\theta = 0.8 \cos(2\theta) \cos(\theta) - (1 + 0.4 \sin(2\theta)) \sin(\theta)$ ,  $y_\theta = 0.8 \cos(2\theta) \sin(\theta) + (1 + 0.4 \sin(2\theta)) \cos(\theta)$  and  $\theta \in (0, 2\pi)$  and  $A = 0$  outside  $D$  ( $\text{div}(A) = 0$ ) for  $N_d = 40$  (on the left) and  $N_d = 60$  (on the right).

### 4.4.3.2 | Reconstruction using factorization method $F_{\#}$

The frequency is  $k = 6$  and  $N_d$  equidistant incident directions and observation points have been used. The data have been generated synthetically by solving the forward scattering problem using a standard finite element method. Let  $g_z^\alpha$  be the Tikhonov regularized solution, where the regularization parameter is equal to zero.

In the following numerical examples, we show the reconstruction of domain  $D$  using the indicator function

$$\mathcal{I}(z) = \frac{1}{\|g_{z,\#}^\alpha\|_{L^2(D)}}. \quad (4.54)$$

■ For  $D$  is a ball domain:

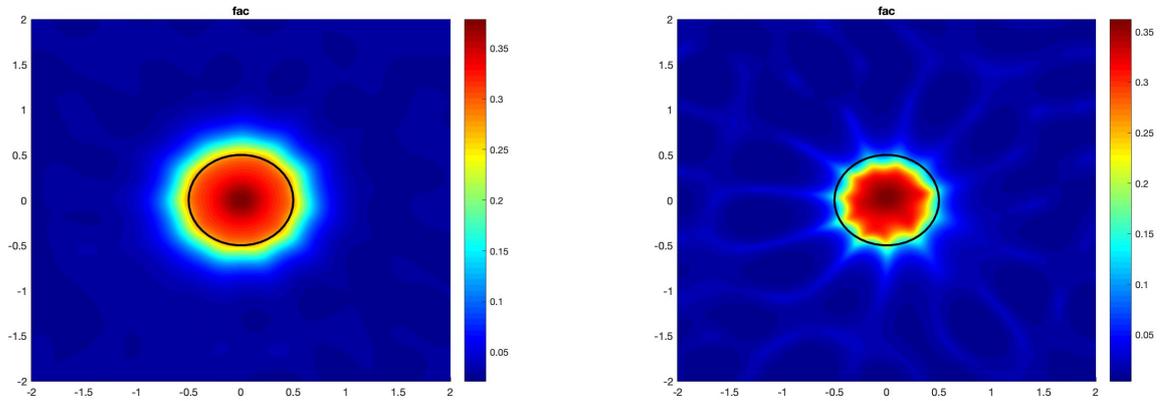


Figure 4.5: Reconstruction of the domain  $D$  for  $A = 0$  (on the left) and for  $A = 2.5(-\sin(\theta), \cos(\theta))$  in  $D$  where  $\theta \in (0, 2\pi)$  and  $A = 0$  outside  $D$  (on the right).

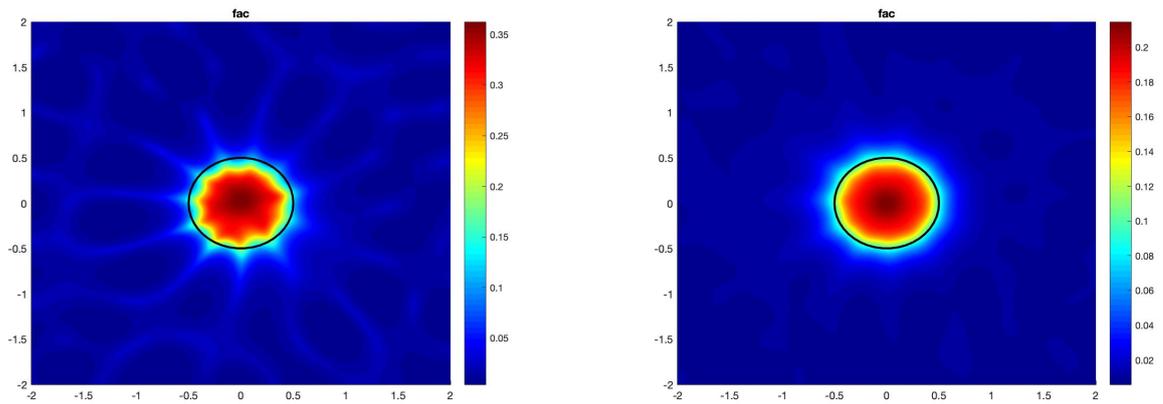


Figure 4.6: Reconstruction of the domain  $D$  with the presence of  $A$ , i.e.,  $A = 2.5(-\sin(\theta), \cos(\theta))$  in  $D$  where  $\theta \in (0, 2\pi)$  and  $A = 0$  outside  $D$ , for  $N_d = 40$  (on the left) and  $N_d = 60$  (on the right).

■ For  $D$  is a different geometry:

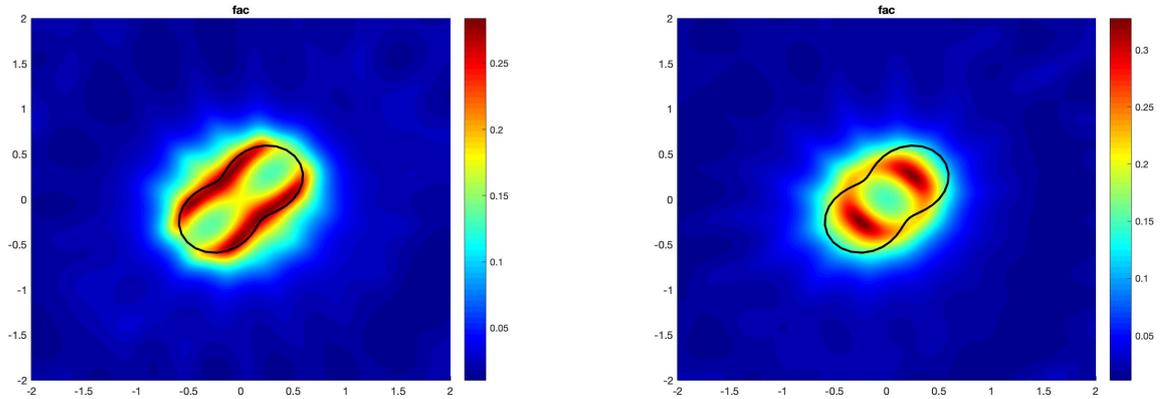


Figure 4.7: Reconstruction of the domain  $D$  for  $A = 0$  (on the left) and for  $A = 2.5(x_\theta, y_\theta)$  in  $D$  where  $x_\theta = 0.8 \cos(2\theta) \cos(\theta) - (1 + 0.4 \sin(2\theta)) \sin(\theta)$ ,  $y_\theta = 0.8 \cos(2\theta) \sin(\theta) + (1 + 0.4 \sin(2\theta)) \cos(\theta)$  and  $\theta \in (0, 2\pi)$  and  $A = 0$  outside  $D$  where  $\text{div}(A) = 0$  (on the right).

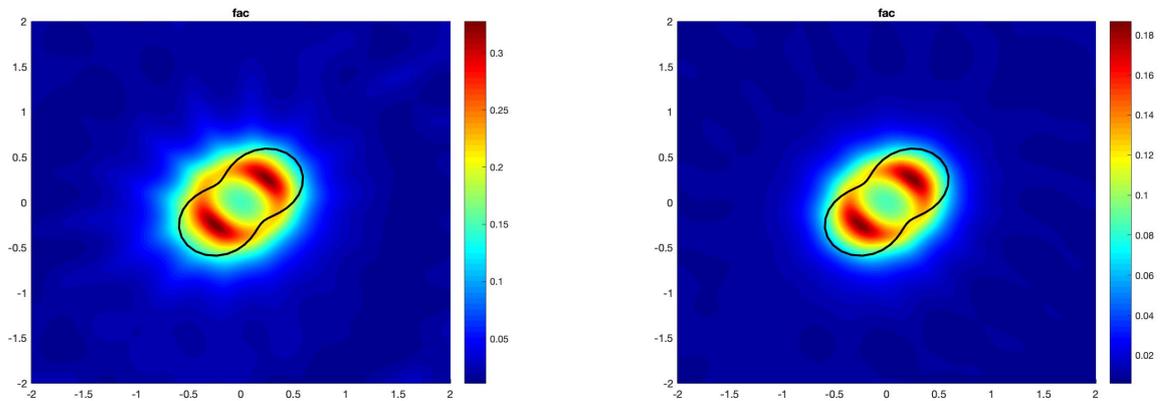


Figure 4.8: Reconstruction of the domain  $D$  with the presence of  $A$  where  $A = 2.5(x_\theta, y_\theta)$  in  $D$  where  $x_\theta = 0.8 \cos(2\theta) \cos(\theta) - (1 + 0.4 \sin(2\theta)) \sin(\theta)$ ,  $y_\theta = 0.8 \cos(2\theta) \sin(\theta) + (1 + 0.4 \sin(2\theta)) \cos(\theta)$  and  $\theta \in (0, 2\pi)$  and  $A = 0$  outside  $D$  ( $\text{div}(A) = 0$ ) for  $N_d = 40$  (on the left) and  $N_d = 60$  (on the right).



## 4.5 | Another approach for the uniqueness of the domain $D$

The factorization method proves the unique reconstruction of  $D$  from far field data. Other approach, we can demonstrate the uniqueness of the support without necessarily having the uniqueness of  $A$  and  $q$ . The idea of demonstrating uniqueness in the inverse medium scattering problem dates back to [44, 45], where it is shown that the shape of a penetrable, inhomogeneous, isotropic medium is uniquely determined by its far field pattern  $u^\infty(\cdot, d)$ ,  $d \in \mathbb{S}^2$  for all incident plane waves  $u^i := e^{ikx \cdot d}$ . Our approach follows the one in [15] for  $A = 0$ . We make the assumption that  $\text{supp}(A) \subset D$ .

Let  $f \in H^{\frac{1}{2}}(\partial D)$  and  $h \in H^{-\frac{1}{2}}(\partial D)$ . The interior transmission problem we are considering is

$$(\nabla + iA)^2 u + k^2 n u = 0 \quad \text{in } D, \quad (4.55)$$

$$\Delta v + k^2 v = 0 \quad \text{in } D, \quad (4.56)$$

$$u - v = f \quad \text{on } \partial D, \quad (4.57)$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu} = h \quad \text{on } \partial D. \quad (4.58)$$

We begin by making the following assumption, which formulates the well-posedness of a modified interior transmission problem.

**Assumption 4.5.1.**  *$A$  and  $n$  are such that the modified interior transmission problem: Given  $f \in H^{\frac{1}{2}}(\partial D)$ ,  $h \in H^{-\frac{1}{2}}(\partial D)$ ,  $\ell_1 \in L^2(D)$  and  $\ell_2 \in L^2(D)$ , find  $u \in H^1(D)$  and  $v \in H^1(D)$  satisfying*

$$(\nabla + iA)^2 u + \gamma_1 n u = \ell_1 \quad \text{in } D, \quad (4.59)$$

$$\Delta v + \gamma_2 v = \ell_2 \quad \text{in } D, \quad (4.60)$$

$$u - v = f \quad \text{on } \partial D, \quad (4.61)$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu} = h \quad \text{on } \partial D, \quad (4.62)$$

for some constants  $\gamma_1$  and  $\gamma_2$ , has a unique solution which satisfies

$$\|u\|_{H^1(D)} + \|v\|_{H^1(D)} \leq C \left( \|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)} + \|\ell_1\|_{L^2(D)} + \|\ell_2\|_{L^2(D)} \right).$$

The following lemma which will be used in the proof of the uniqueness Theorem.

**Lemma 4.5.2.** *assume that Assumption 4.5.1 holds, and let  $(v_n, u_n) \in H^1(D) \times H^1(D)$ ,  $j \in \mathbb{N}$  be a sequence of solutions to the interior transmission problem (4.55)-(4.58) with boundary data  $f_n \in H^{\frac{1}{2}}(\partial D)$ ,  $h_n \in H^{-\frac{1}{2}}(\partial D)$ . If the sequences  $(f_n)$  and  $(h_n)$  converge in  $H^{\frac{1}{2}}(\partial D)$  and  $H^{-\frac{1}{2}}(\partial D)$  respectively, and if the sequence  $(v_n)$  and  $(u_n)$  are bounded in  $H^1(D)$ , then there exists a subsequence  $(v_{n_k})$  which converges in  $H^1(D)$ .*

*Proof.* We have  $(u_n, v_n)$  a sequence of solutions to

$$\begin{aligned} (\nabla + iA)^2 u_n + k^2 n u_n &= 0 && \text{in } D, \\ \Delta v_n + k^2 v_n &= 0 && \text{in } D, \\ u_n - v_n &= f_n && \text{on } \partial D, \\ \frac{\partial u_n}{\partial \nu} - \frac{\partial v_n}{\partial \nu} &= h_n && \text{on } \partial D. \end{aligned}$$

Then, thanks to the compact embedding of  $H^1(D)$  into  $L^2(D)$ , we select  $L^2$ -convergent subsequences  $(u_{n_k})$  and  $(v_{n_k})$  which satisfies

$$\begin{aligned} (\nabla + iA)^2 u_{n_k} + \gamma_1 u_{n_k} &= (\gamma_1 - k^2 n) u_{n_k} && \text{in } D, \\ \Delta v_{n_k} + \gamma_2 v_{n_k} &= (\gamma_2 - k^2) v_{n_k} && \text{in } D, \\ u_{n_k} - v_{n_k} &= f_{n_k} && \text{on } \partial D, \\ \frac{\partial u_{n_k}}{\partial \nu} - \frac{\partial v_{n_k}}{\partial \nu} &= h_{n_k} && \text{on } \partial D, \end{aligned}$$

or we know that the assumption 4.5.1 holds, we get

$$\begin{aligned} \|v_{n_k} - v\|_{H^1(D)} &\leq C \left( \|f_{n_k} - f\|_{H^{\frac{1}{2}}(\partial D)} + \|h_{n_k} - h\|_{H^{-\frac{1}{2}}(\partial D)} + \|u_{n_k} - u\|_{L^2(D)} \right. \\ &\quad \left. + \|A\|_{L^\infty(D)} \|u_{n_k} - u\|_{H^{-\frac{1}{2}}(\partial D)} + \|v_{n_k} - v\|_{L^2(D)} \right) \\ &\xrightarrow{n_k \rightarrow \infty} 0, \end{aligned}$$

which completes the proof.  $\square$

Armed with above, we establish now the uniqueness result.

**Theorem 4.5.3.** *Let the domains  $D_1$  et  $D_2$  and the couples  $(-A_1, n_1)$  and  $(-A_2, n_2)$  are such that Assumption 4.5.1 holds. If the far field patterns  $\tilde{u}_1^\infty(\hat{x}, d)$  and  $\tilde{u}_2^\infty(\hat{x}, d)$  corresponding to  $D_1, A_1, n_1$  and  $D_2, A_2, n_2$ , respectively, coincide for all  $\hat{x} \in \mathbb{S}^2$  and  $d \in \mathbb{S}^2$ , then  $D_1 = D_2$ .*

*Proof.* Let  $G$  be the unbounded connected component of  $\mathbb{R}^3 \setminus (\overline{D}_1 \cup \overline{D}_2)$  and define  $D_1^e = \mathbb{R}^3 \setminus \overline{D}_1$  and  $D_2^e = \mathbb{R}^3 \setminus \overline{D}_2$ . We denote  $u = u_1 - u_2$ , then, we obtain

$$u^\infty(\hat{x}, d) = u_1^\infty(\hat{x}, d) - u_2^\infty(\hat{x}, d) = 0, \quad \forall \hat{x}, d \in \mathbb{S}^2,$$

therefore, according to Rellich's Lemma , we get

$$u_1^s(\cdot, d) = u_2^s(\cdot, d) \text{ in } G, \quad \forall d \in \mathbb{S}^2,$$

where  $u_j^s, j = 1, 2$  is a solution of the problem (4.1) – (4.4) with  $D_1, A_1, n_1$  and  $D_2, A_2, n_2$ , respectively, and  $u^i(x) = e^{ikx \cdot d}, d \in \mathbb{S}^2$ .

For the incident field  $u^i := \Phi(\cdot, z), z \in G$ , we denote  $u_1^s(\cdot, z)$  and  $u_2^s(\cdot, z)$  the scattered fields associated to  $D_1, -A_1, n_1$  and  $D_2, -A_2, n_2$ , respectively. Based on the mixed reciprocity relation (Theorem 4.2.2), we have

$$4\pi u_j^\infty(-d, z) = u_j^s(z, d), \quad z \in G, d \in \mathbb{S}^2, j = 1, 2,$$

where  $u^\infty(\cdot, z) = u_1^\infty(\cdot, z) - u_2^\infty(\cdot, z)$  such that  $u_1^\infty(\cdot, z)$  and  $u_2^\infty(\cdot, z)$  are the far field pattern corresponding to  $D_1, -A_1, n_1$  and  $D_2, -A_2, n_2$ , respectively, which implies

$$u^\infty(-d, z) = u_1^s(z, d) - u_2^s(z, d) = 0, \quad \forall d \in \mathbb{S}^2, z \in G.$$

Then, according to Rellich's Lemma, we obtain

$$u_1^s(\cdot, z) = u_2^s(\cdot, z), \quad z \in G.$$

Therefore, we get

$$w_1(\cdot, z) = w_2(\cdot, z), \quad \forall z \in G,$$

where  $w_j(\cdot, z)$ ,  $j = 1, 2$  is a solution of (4.6).

Now, we assume that  $\bar{D}_1 \not\subset \bar{D}_2$ . From the fact that  $D_2^\varepsilon$  is connected, we can find a point  $z \in \partial D_1$  and  $\varepsilon > 0$  with the following proprieties:

- (i)  $B(z, 8\varepsilon) \cap \bar{D}_2 = \emptyset$ ,
- (ii)  $\bar{D}_1 \cap B(z, 8\varepsilon) \subset C_z(\bar{D}_1)$ , where  $C_z(\bar{D}_1)$  connected component of  $z$  in  $\bar{D}_1$ .
- (iii) There exist points  $a_1, \dots, a_\ell \in C_z(\bar{D}_1)$  such that  $a_1, \dots, a_\ell \notin \bar{D}_1 \cap \bar{B}(z, 8\varepsilon)$ ,
- (vi) the points  $z_n = z + \frac{\varepsilon}{n} \nu(z) \in G$ ,  $\forall n \in \mathbb{N}$ , where  $\nu(z)$  is the unit normal to  $\partial D_1$  at  $z$ .

In the other hand, we have the singularity of  $\Phi(\cdot, z)$  i.e,

$$\Phi(x, z) = \frac{e^{ik|x-z|}}{4\pi|x-z|} \xrightarrow{x \rightarrow z} \infty.$$

Or we have

$$\begin{aligned} \|\Phi(\cdot, z_n)\|_{L^2(D_1)}^2 &\leq C \int_{D(z_n, \alpha\varepsilon)} \frac{1}{|x - z_n|^2} dx \\ &\leq C \int_{D(z_n, \alpha\varepsilon)} \frac{1}{|\alpha\varepsilon|^2} dx = C \frac{1}{|\alpha\varepsilon|^2} \text{Vol}(D(z_n, \alpha\varepsilon)), \quad \forall n \in \mathbb{N}, \end{aligned}$$

where  $D(z_n, \alpha\varepsilon)$  represent the disk with center  $z_n$  and radius  $\alpha\varepsilon$  with  $\alpha \in \mathbb{N}$ , such that  $D_1 \subset D(z_n, \alpha\varepsilon)$ , and we have

$$\begin{aligned} \|\nabla \Phi(\cdot, z_n)\|_{L^2(D_1)}^2 &\geq \int_{D_1} \left| \frac{\Phi(x, z_n)}{|x - z_n|} \right|^2 dx - \int_{D_1} k^2 |\Phi(x, z_n)|^2 dx \\ &\geq C \int_{D_1} \frac{1}{|x - z_n|^4} dx - c \int_{D_1} \frac{1}{|x - z_n|^2} dx \\ &\geq C \int_{D_1} \frac{1}{|x - z_n|^4} dx - m, \end{aligned}$$

or

$$\int_{D_1} \frac{1}{|x - z_n|^4} dx \text{ diverges.}$$

Then

$$\|\nabla\Phi(\cdot, z_n)\|_{L^2(D_1)}^2 \xrightarrow{n \rightarrow \infty} \infty,$$

which implies

$$\|\Phi(\cdot, z_n)\|_{H^1(D_1)} \xrightarrow{n \rightarrow \infty} \infty.$$

Now, we define  $v_n$  by

$$v_n(x) = \frac{1}{\|\Phi(\cdot, z_n)\|_{H^1(D_2)}} \Phi(x, z_n), \quad x \in \bar{D}_1 \cup \bar{D}_2,$$

we obtain, for each  $n$ , that  $v_n$  is a solution of the Helmholtz equation in  $D_1$  and  $D_2$ .

Besides, we define

$$w_j^n = \frac{w_j}{\|\Phi(\cdot, z_n)\|_{H^1(D_1)}}, \quad j = 1, 2,$$

then  $w_j^n$  is a solution of the equation

$$(\nabla + iA)^2 w_j^n + k^2 n_j w_j^n = -P_A v_n + k^2(1 - n_j)v_n \quad \text{in } \mathbb{R}^3. \quad (4.63)$$

Our goal is to prove that if  $\bar{D}_1 \not\subset \bar{D}_2$  then  $w_1(\cdot, z) = w_2(\cdot, z)$ ,  $z \in G$ , which allows the existence of a subsequence  $(v_{n_k})$  from  $(v_n)$  that implies

$$v_{n_k} \longrightarrow 0 \quad \text{in } H^1(D_1).$$

Contradiction because  $\|v_n\|_{H^1(D_1)} = 1$ . From the fact that

$$w_1(\cdot, z) = w_2(\cdot, z), \quad \forall z \in G,$$

we get

$$w_1^n(\cdot, z) = w_2^n(\cdot, z), \quad \forall z \in G.$$

In one way, we have  $\Phi(\cdot, z_n)$  with their derivatives are uniformly bounded in every compact subset of  $\mathbb{R}^3 \setminus B(z, 2\varepsilon)$  and such that

$$\|\Phi(\cdot, z_n)\|_{H^1(D_1)} \xrightarrow{n \rightarrow \infty} \infty,$$

then

$$\|v^n\|_{H^1(D_2)} \xrightarrow{n \rightarrow \infty} 0.$$

In other ways, if  $R$  is large enough such that  $\bar{D}_1 \cup \bar{D}_2 \subset B(0, R)$  and from (??) we have

$$\begin{aligned} \|w_2^n\|_{H^2(B(0,R) \cap G)} &\leq C \|v_n\|_{H^1(D_2)} \\ &\xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

therefore

$$\|w_2^n\|_{H^1(B(0,R) \cap G)} \leq C \|w_2^n\|_{H^2(B(0,R) \cap G)} \xrightarrow{n \rightarrow \infty} 0.$$

Making use of  $w_n^1 = w_n^2$  in  $G$ , it follows

$$\|w_1^n\|_{H^1(B(0,R) \cap G)} \xrightarrow{n \rightarrow \infty} 0.$$

Now, let be  $\chi \in C_0^\infty(B(z, 8\varepsilon))$  such that  $\chi(x) = 1$  in  $B(z, 7\varepsilon)$ . Or according to the trace Theorem, we get

$$\begin{aligned} \|\chi w_1^n\|_{H^{\frac{1}{2}}(\partial D_1)} &\leq C \|\chi w_1^n\|_{H^1(B(0,R) \setminus D_1)} \\ &\leq C \|w_1^n\|_{H^1(B(0,8\varepsilon) \cap (B(0,R) \cap G))} \\ &\leq C \|w_1^n\|_{H^1(G \cap B(0,R))} \\ &\xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

then

$$(\chi w_1^n) \longrightarrow 0 \quad \text{in } H^{\frac{1}{2}}(D_1). \quad (4.64)$$

and the same way for  $\partial_\nu w_1^n$ , we obtain

$$\left( \frac{\partial \chi w_1^n}{\partial \nu} \right) \longrightarrow 0 \quad \text{in } H^{\frac{1}{2}}(D_1). \quad (4.65)$$

It is enough that the condition (ii) and (vi) on  $z$  are verified to ensure  $B(z, 8\varepsilon) \cap D_1^e = B(z, 8\varepsilon) \cap G$ . Moreover, we have in the outside of  $B(z, 2\varepsilon)$

$$\|v_n\|_{H^2(B(0,R) \setminus B(z, 2\varepsilon))} \leq C, \quad \forall n \in \mathbb{N}.$$

and we have

$$\begin{aligned} \|w_1^n\|_{H^2((B(0,R) \cap D_1^e) \setminus B(z,4\varepsilon))} &\leq C \|v_n\|_{H^2(D_1 \cup D_2)} \\ &\leq C \|v_n\|_{H^2(B(0,R) \setminus B(z,2\varepsilon))}, \quad \forall n \in \mathbb{N}. \end{aligned}$$

In the one hand, using the compact embedding of  $H^2(B(0,R) \cap D_1^e)$  into  $H^1(B(0,R) \cap D_1^e)$ , then we can select a subsequence  $((1 - \chi)w_1^{n_k})$  of  $((1 - \chi)w_1^n)$ . Thus,  $((1 - \chi)w_1^{n_k})$  converges in  $H^{\frac{1}{2}}(\partial D_1)$  and  $\left(\frac{\partial((1-\chi)w_1^{n_k})}{\partial \nu}\right)$  converges in  $H^{-\frac{1}{2}}(\partial D_1)$ .

In the other hand, from (4.64) and (4.65), we obtain  $(w_1^n)$  and  $\left(\frac{\partial w_1^n}{\partial \nu}\right)$  converge in  $H^{\frac{1}{2}}(\partial D_1)$  and  $H^{-\frac{1}{2}}(\partial D_1)$  respectively. Since the functions  $w_1^n + v^{n_k}$  and  $v_{n_k}$  are solutions to the interior transmission problem (4.55)-(4.58) with  $D = D_1$ ,  $f = w_1^n$  and  $h = \frac{\partial w_1^n}{\partial \nu}$ , we get

$$\|w_1^n + v_{n_k}\|_{H^1(D_1)} \leq C_1 \text{ and } \|v_{n_k}\|_{H^1(D_1)} \leq C_2; \quad \forall n \in \mathbb{N},$$

and according to the Lemma 4.5.2, we can extract a subsequence of  $(v_{n_k})$ , denoted again by  $(v_{n_k})$ , which

$$v_{n_k} \longrightarrow v \quad \text{in } H^1(D_1),$$

with  $v \in H^1(D_1)$  is a weak solution of the Helmholtz equation.

Or due to the the uniform convergence of  $(v_{n_k})$  towards 0 outside of  $B(z, 2\varepsilon)$ , then  $v|_{D_1 \setminus B(z, 2\varepsilon)} = 0$ . Therefore, according to the Holmgren's uniqueness Theorem, we get  $v(x) = 0, \forall x \in D_1$ . Contradiction with the fact that  $\|v_{n_k}\|_{H^1(D_1)} = 1$  which implies that the hypothesis of  $\bar{D}_1 \not\subset \bar{D}_2$  is false.

We do the same work to mount the contradiction of the hypothesis  $\bar{D}_1 \not\subset \bar{D}_2$ . Then, we obtain

$$D_1 = D_2.$$

This achieves the proof. □

# The study of interior transmission problem for magnetic Schrödinger operator

## **Abstract:**

*This chapter presents a study of the interior transmission problem (ITP) for the magnetic Schrödinger operator focusing on establishing the well-posedness of the (ITP), and the discreteness of the set of transmission eigenvalues through the application of Fredholm theory and the upper triangular Fredholm theory.*

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## 5.1 | Introduction

The interior transmission problem (ITP) is by now a well known problem in inverse scattering theory that plays a fundamental role in the theoretical and numerical investigations of the inverse medium problem (see, for example, [13, 24, 14]). The ITP was first posed by Kirsch in [55]. A few years later, Colton and Monk in [25] used the interior transmission problem to solve the inverse scattering problem for acoustic waves in an inhomogeneous medium. A particular attention has been given in recent years to the study of the frequencies for which this problem has non unique solutions : the so-called transmission eigenvalues. These values can for instance be used to obtain bounds of the inclusion physical parameters [17, 26], or more importantly in non destructive testing of complex materials [16].

This chapter is organized as follows. In Section 5.2, we start by presenting the Fredholm theory in Subsection 5.2.1, which helps to study our interior transmission problem by establishing the existence and uniqueness of the solution in Subsection 5.2.2. We then explore the behavior of transmission eigenvalues and derive conditions for their existence in subsection 5.2.3. Next, in Section 5.3, we drive the analysis of our ITP by applying the upper triangular Fredholm theory in Subsection 5.3.4 and the Born approximation in Subsection 5.3.3 to demonstrate in other ways the existence and discreteness of the set of transmission eigenvalues.

## 5.2 | Study of ITP using Fredholm Theory

This section aims to investigate the solvability of the interior transmission problem. The Fredholm property of this problem and the discreteness of transmission eigenvalues were initially studied in [86]. However, our approach will align with the methodology presented by Kirsch in [50]. Kirsch revisited the findings of [86] for a real-valued refractive index and achieved identical results through a variational approach. Nevertheless, before proceeding, we will review the fundamental properties of Fredholm theory.

### 5.2.1 | Fredholm Theory

We want to briefly review the properties of the Fredholm theory that we will use in what follows for the discreteness of the set of transmission eigenvalues (see [28, 47, 86]).

**Definition 5.2.1.** *Let  $H$  be a Hilbert space. An operator  $T : H \rightarrow H$  is a Fredholm operator, if  $\text{Ker}(T)$  and  $\text{Coker}(T) := H/\text{Im}(T)$  are finite-dimensional where  $\text{Coker}(T)$  is the complement space of  $\text{Ker}(T)$ .*

*Moreover, we define the index of  $T$  by*

$$\text{ind}(T) := \dim \text{Ker}(T) - \dim \text{Coker}(T), \quad (5.1)$$

*where the codimension of  $\text{Im}(T)$  is  $\dim \text{Coker}(T)$ .*

The index indicates how many dimensions are missing for  $\text{Ker}(T)$  or  $\text{Coker}(T)$ : if the index is negative, then  $T$  is not surjective; if it is positive, then  $T$  is not injective.

We now state by the following theorem that compact perturbations do not change Fredholmness and do not change the index.

**Theorem 5.2.2** (Alternative Fredholm [47]). *Let  $T$  be Fredholm operator of index  $\text{ind}(T)$ . If  $K$  is a compact operator, then  $T + K$  is a Fredholm operator of index  $\text{ind}(T + K) = \text{ind}(T)$ .*

Using the definition of operator's index (5.1), we get the following proposition.

**Lemma 5.2.3.** *Let  $T$  a Fredholm operator of index zero . If  $T$  is injective ( $\text{Ker}(T) = \{0\}$ ) or  $T$  is surjective ( $\text{Coker}(T) = \{0\}$ ), then  $T$  is invertible.*

**Remark 5.2.4.** *If  $T$  is invertible then  $T$  is Fredholm of index zero.*

**Lemma 5.2.5** (Analytic Fredholm Theorem). *Suppose that  $R(\lambda)$  is an analytic compact operator valued function of  $\lambda$  for  $\lambda$  in an open connected set  $\Gamma$ . Then if  $I - R(\lambda_0)$  is invertible for one  $\lambda_0 \in \Gamma$ , it is invertible for all but a discrete subset  $\Lambda$  of  $\Gamma$ . Moreover, for  $\lambda \in \Lambda$ ,  $I - R(\lambda)$  is a Fredholm operator.*

## 5.2.2 | Interior transmission problem

Here, we will delve into the process of establishing the well-posedness of our interior transmission problem.

### 5.2.2.1 | Setting of the problem

Let  $D \subset \mathbb{R}^3$ , be a bounded open set with smooth boundary such that  $(\mathbb{R}^3 \setminus D)$  is connected. We assume that there is a neighborhood of the boundary  $\mathcal{N}$ , which is an open subdomain  $\mathcal{N} \subset D$  with  $\partial D \subset \overline{\mathcal{N}}$ , where we impose conditions on the contrast  $n - 1$ . In  $D \subset \mathcal{N}$ , the contrast  $n - 1$  can take on any value.

Let us denote

$$L_{\Delta}^2(D) := \{v \in L^2(D); \Delta v \in L^2(D)\}.$$

Furthermore, the Hilbert space  $H_0^2(D)$  to be the completion of  $C_0^\infty(D)$  in the norm

$$\|u\|_{L_{\Delta}^2(D)}^2 = \|u\|_{L^2(D)}^2 + \|\Delta u\|_{L^2(D)}^2. \quad (5.2)$$

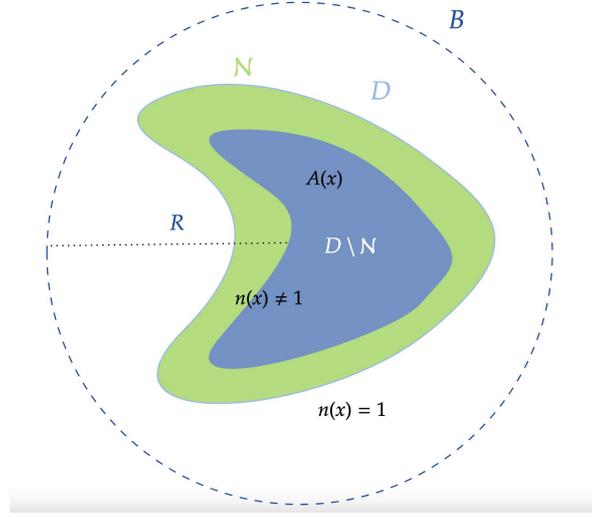


Figure 5.1: Sketch of geometry

In this section, we assume that Assumption 2.4.1 holds, i.e., letting the magnetic potential  $A \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$  such that  $\operatorname{div}(A) \in L^\infty(\mathbb{R}^3, \mathbb{R})$  and  $\operatorname{Supp}(A) \subset D$  and the refractive index  $n \in L^\infty(\mathbb{R}^3, \mathbb{C})$  with non negative imaginary, positive real and  $n = 1$  outside the support  $D$ . The interior transmission problem is to find  $(w, v) \in L^2(D) \times L^2_\Delta(D)$  such that  $w - v \in H^2(D)$  and

$$\begin{cases} (\nabla + iA)^2 w + k^2 n w = 0 & \text{in } D, \\ \Delta v + k^2 v = 0, & \text{in } D, \\ w - v = f & \text{on } \partial D, \\ \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = h & \text{on } \partial D, \end{cases} \quad (5.3)$$

for given  $(f, h) \in H^{\frac{3}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$  where  $\nu$  denotes the outward normal on  $\partial D$ .

**Definition 5.2.6.** Values of  $k \in \mathbb{C}$  for which the interior transmission problem

$$\begin{cases} (\nabla + iA)^2 w + k^2 n w = 0 & \text{in } D, \\ \Delta v + k^2 v = 0, & \text{in } D, \\ w - v = 0 & \text{on } \partial D, \\ \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial D, \end{cases} \quad (5.4)$$

has non-trivial solutions  $w$  and  $v$ , are called transmission eigenvalues.

Let  $u = \frac{1}{k^2}(w - v)$  and given  $f \in H^{\frac{3}{2}}(\partial D)$  and  $h \in H^{\frac{1}{2}}(\partial D)$ , find  $u \in H^2(D)$  and  $v \in L^2_{\Delta}(D)$  such that and

$$\begin{cases} (\nabla + iA)^2 u + k^2 nu = -\frac{1}{k^2} P_A v - (n-1)v & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ u = f & \text{on } \partial D, \\ \frac{\partial u}{\partial \nu} = h & \text{on } \partial D. \end{cases} \quad (5.5)$$

where  $P_A$  is the first order operator given by

$$P_A v = iA \cdot \nabla v + i \operatorname{div}(Av) - |A|^2 v, \quad v \in H^1_{loc}(\mathbb{R}). \quad (5.6)$$

Moreover, let  $\theta \in H^2(D)$  such that  $\theta = f$  and  $\frac{\partial \theta}{\partial \nu} = h$  on  $\partial D$ . Then,  $\theta$  satisfies

$$\|\theta\|_{H^2(D)} \leq C \left( \|f\|_{H^{\frac{3}{2}}(\partial D)} + \|h\|_{H^{\frac{1}{2}}(\partial D)} \right). \quad (5.7)$$

It's possible to transform (5.5) to the following problem: Given  $F \in L^2(D)$ , find  $u \in H^2_0(D)$  and  $v \in L^2_{\Delta}(D)$  such that

$$\begin{cases} (\nabla + iA)^2 u + k^2 nu = -\frac{1}{k^2} P_A v - (n-1)v + F & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D. \end{cases} \quad (5.8)$$

The above equations are assumed to be satisfied in the following weak sense

$$\begin{aligned} \int_D (\Delta \bar{\psi} + k^2 \bar{\psi}) v \, dx &= 0, \\ \int_D \left[ (\nabla + iA)^2 u + k^2 nu + \frac{1}{k^2} P_A v + (n-1)v \right] \bar{\varphi} \, dx &= \int_D F \bar{\varphi} \, dx, \end{aligned}$$

for all  $\psi \in H^2_0(D)$  and  $\varphi \in L^2_{\Delta}(D)$ .

Let us denote  $X(D) := H^2_0(D) \times L^2_{\Delta}(D)$ . Then, the problem (5.8) can be written in the following equivalent variant form: Find  $(u, v) \in X(D)$  such that for all  $(\psi, \varphi) \in X(D)$

$$\begin{aligned} \int_D (\Delta \bar{\psi} + k^2 \bar{\psi}) v \, dx + \int_D (\nabla + iA)^2 u \bar{\varphi} + k^2 nu \bar{\varphi} \, dx \\ + \int_D \left( \frac{1}{k^2} P_A v \bar{\varphi} + (n-1)v \varphi \right) \, dx = \int_D F \bar{\varphi} \, dx. \end{aligned} \quad (5.9)$$

For any  $k \in \mathbb{C}$ , we define the sesquilinear form  $B_k : X(D) \times X(D) \longrightarrow \mathbb{C}$  by

$$\begin{aligned} B_k(u, v; \psi, \varphi) &= \int_D (\Delta \bar{\psi} + k^2 \bar{\psi}) v \, dx + \int_D ((\nabla + iA)^2 u + k^2 nu) \bar{\varphi} \, dx \\ &\quad + \int_D \left( \frac{1}{k^2} P_A v \bar{\varphi} + (n-1) v \bar{\varphi} \right) dx, \end{aligned} \quad (5.10)$$

for all  $(u, v) \in X(D)$  and  $(\psi, \varphi) \in X(D)$ . Let define also the sesquilinear form  $\widehat{B}_k : X(D) \times X(D) \longrightarrow \mathbb{C}$  by

$$\widehat{B}_k(u, v; \psi, \varphi) = \int_D (\Delta \bar{\psi} + k^2 \bar{\psi}) v \, dx + \int_D (\Delta u + k^2 u) \bar{\varphi} \, dx + \int_D (n-1) v \bar{\varphi} \, dx, \quad (5.11)$$

for all  $(u, v) \in X(D)$  and  $(\psi, \varphi) \in X(D)$ .

### 5.2.2.2 | Well-posedness of the problem

We will now determine that the interior transmission problem (5.3) is well-posed. The Riesz representation Theorem yields the existence of bounded linear operators  $B_k, \widehat{B}_k : X(D) \longrightarrow X(D)$  such that

$$B_k(u, v; \psi, \varphi) = \langle B_k(u, v), (\psi, \varphi) \rangle_{X(D)}, \quad \text{for all } (u, v), (\psi, \varphi) \in X(D),$$

and

$$\widehat{B}_k(u, v; \psi, \varphi) = \langle \widehat{B}_k(u, v), (\psi, \varphi) \rangle_{X(D)}, \quad \text{for all } (u, v), (\psi, \varphi) \in X(D).$$

Hence, the interior transmission problem is equivalent to the following equation

$$B_k(u, v) = l, \quad (u, v) \in X(D), \quad (5.12)$$

where  $l \in X(D)$  is the Riesz representation Theorem of a bounded antilinear functional  $L : (\psi, \varphi) \longmapsto \int_D F \bar{\varphi} \, dx$  i.e.,  $L(\psi, \varphi) = \langle l, (\psi, \varphi) \rangle_{X(D)}$ , for all  $(\psi, \varphi) \in X(D)$ .

**Theorem 5.2.7.** *For any two  $k_1, k_2 \in \mathbb{C}$ , the difference  $B_{k_1} - \widehat{B}_{k_2}$  and  $B_{k_1} - B_{k_2}$  are compact.*

*Proof.* Let  $(u_j, v_j) \in X(D)$  such that

$$(u_j, v_j) \longrightarrow 0 \quad \text{in } X(D).$$

Let  $(\psi, \varphi) \in X(D)$ . Then, we have

$$\begin{aligned} (B_{k_1} - \widehat{B}_{k_2})(u_j, v_j; \psi, \varphi) &= \frac{1}{k_1^2} \int_D P_A v_j \bar{\varphi} \, dx + \int_D (k_1^2 n - k_2^2) u_j \bar{\varphi} \, dx \\ &\quad + \int_D P_A u_j \bar{\varphi} \, dx + (k_1^2 - k_2^2) \int_D v_j \bar{\psi} \, dx. \end{aligned}$$

Since  $u_j \longrightarrow 0$  in  $H_0^2(D)$  and  $\nabla u_j \longrightarrow 0$  in  $H^1(D)$ , Rellich's compact embedding Theorem implies that  $u_j \longrightarrow 0$  in  $L^2(D)$  and  $\nabla u_j \longrightarrow 0$  in  $L^2(D)$ , respectively. Then,  $u_j \longrightarrow 0$  in  $H^1(D)$ . Furthermore, we obtain

$$\left| \int_D (k_1^2 n - k_2^2) u_j \bar{\varphi} \, dx \right| \leq \|k_1^2 n - k_2^2\|_{L^\infty(D)} \|u_j\|_{L^2(D)} \|\varphi\|_{L_\Delta^2(D)}. \quad (5.13)$$

and

$$\left| \int_D P_A u_j \bar{\varphi} \, dx \right| \leq \max \left( 2\|A\|_{L^\infty(D)}, \|\operatorname{div}(A)\|_{L^\infty(D)} + \|A\|_{L^\infty(D)}^2 \right) \|u_j\|_{H^1(D)} \|\varphi\|_{L_\Delta^2(D)} \quad (5.14)$$

We know that  $v_j \longrightarrow 0$  in  $L_\Delta^2(D)$ . Let  $z_j \in H^1(D)$  with  $\Delta z_j = v_j$  in  $D$  and  $z_j = 0$  on  $\partial D$  implies that  $z_j \longrightarrow 0$  in  $H^1(D)$ . According to Rellich's compact embedding Theorem, we find  $z_j \longrightarrow 0$  in  $L^2(D)$  and the following estimates

$$\left| \int_D \bar{\psi} v_j \, dx \right| = \left| \int_D \bar{\psi} \Delta z_j \, dx \right| = \left| \int_D \Delta \bar{\psi} z_j \, dx \right| \leq \|z_j\|_{L^2(D)} \|\psi\|_{H^2(D)}, \quad (5.15)$$

Moreover, we have  $A$  has compact support in  $D$  implies that  $P_A$  is continuous and has a compact support in  $L^2(D)$ . Thus, using the fact that  $v_j \in L_\Delta^2(D)$  and the elliptic regularity, we are able to prove that

$$\|P_A v_j\|_{L^2(D)} \leq C(\|v_j\|_{L^2(D)} + \|\Delta v_j\|_{L^2(D)}) \leq C\|v_j\|_{L_\Delta^2(D)}, \quad (5.16)$$

for  $K = \operatorname{supp}(A)$  and which implies

$$\frac{1}{k_1^2} \int_D P_A v_j \bar{\varphi} \, dx \leq \frac{1}{k_1^2} \|P_A v_j\|_{L^2(D)} \|\varphi\|_{L_\Delta^2(D)} \longrightarrow 0 \quad \text{in } L^2(D). \quad (5.17)$$

Therefore, (5.13), (5.14), (5.15) and (5.17) imply

$$\begin{aligned} \left\| \left( B_{k_1} - \widehat{B}_{k_2} \right) (u_j, v_j) \right\|_{X(D)} &= \sup_{\substack{(\psi, \varphi) \in X(D) \\ (\psi, \varphi) \neq 0}} \left| \left( B_{k_1} - \widehat{B}_{k_2} \right) (u_j, v_j; \psi, \varphi) \right| \\ &\xrightarrow{j \rightarrow \infty} 0. \end{aligned}$$

This prove the compactness of  $B_{k_1} - \widehat{B}_{k_2}$ . The proof for  $(B_{k_1} - B_{k_2})$  proceeds in a similar way where

$$\begin{aligned} (B_{k_1} - B_{k_2})(u_j, v_j; \psi, \varphi) &= \left( \frac{1}{k_1^2} - \frac{1}{k_2^2} \right) \int_D P_A v_j \bar{\varphi} \, dx + (k_1^2 - k_2^2) \int_D n u_j \bar{\varphi} \, dx \\ &\quad + (k_1^2 - k_2^2) \int_D v_j \bar{\psi} \, dx, \end{aligned}$$

for  $(u_j, v_j), (\psi, \varphi) \in X(D)$ . □

Now, we need to prove the invertibility of  $\widehat{B}_k$  for some  $k \in \mathbb{C}$ . At this point, we need to assume that  $\Re(n - 1) \geq \alpha > 0$  or  $\Re(1 - n) \geq \alpha > 0$  for almost points in  $D$ . We denote

$$n_* = \inf_{\mathcal{N}} \Re(n) \quad \text{and} \quad n^* = \sup_{\mathcal{N}} \Re(n).$$

**Lemma 5.2.8.** *Let  $n \in L^\infty(D)$  such that either  $n_* > 1$  or  $0 < n_* < 1$  and  $A \in L^\infty(D, \mathbb{R}^3)$  with compact support such that  $\operatorname{div}(A) \in L^\infty(\mathbb{R}^3)$ . Then, there exist constant  $C_1 > 0$ ,  $C_2 > 0$  and  $d > 0$  such that for all  $k = i\kappa$ ,  $\kappa > 0$ , the following estimates*

$$\int_{D \setminus \mathcal{N}} |v|^2 \, dx \leq C_1 e^{-2d\kappa} \int_{\mathcal{N}} |\Re(n) - 1| |v|^2 \, dx, \quad (5.18)$$

and

$$\frac{1}{\kappa^2} \int_{D \setminus \mathcal{N}} iA \cdot \nabla v \bar{v} \, dx \leq C_2 e^{-2d\kappa} \int_{\mathcal{N}} |\Re(n) - 1| |v|^2 \, dx, \quad (5.19)$$

hold, for all  $v \in L^2(D)$  solution of  $\Delta v - \kappa^2 v = 0$  in  $D$ .

*Proof.* We pick a neighborhood  $\mathcal{N}'$  of the boundary  $\partial D$  such that  $d = \operatorname{dist}(D \setminus \mathcal{N}', \mathcal{N}') > 0$  and a function  $\rho \in C^\infty(D)$  with compact support in  $D$  such that  $\rho = 1$  in  $D \setminus \mathcal{N}'$ . Then



the Green's formula is applied to  $\rho v$  and it is noted that  $\rho v = v$  in  $D \setminus N'$ , which means  $\Delta \rho v - \kappa^2 \rho v = 0$  in  $D \setminus N'$ , results

$$\begin{aligned} \rho(x)v(x) &= - \int_D \left[ \Delta(\rho v)(y) - \kappa^2(\rho v)(y) \right] \frac{e^{-\kappa|x-y|}}{|x-y|} dy \\ &= - \int_{N'} [2\nabla \rho(y) \cdot \nabla v(y) + v(y)\Delta \rho(y)] \frac{e^{-\kappa|x-y|}}{|x-y|} dy \\ &= \int_{N'} \left[ 2\operatorname{div} \left( \nabla \rho(y) \frac{e^{-\kappa|x-y|}}{|x-y|} \right) - \Delta \rho(y) \frac{e^{-\kappa|x-y|}}{|x-y|} \right] v(y) dy. \end{aligned}$$

For  $x \in D \setminus \mathcal{N}$ , we conclude that

$$|v(x)| \leq \frac{e^{-\kappa d}}{d} \int_{N'} |\Delta \rho v(y)| dy \leq ce^{-\kappa d} \int_{N'} |v(y)| dy,$$

which implies

$$\begin{aligned} |v(x)|^2 &\leq ce^{-2\kappa d} |\mathcal{N}| \int_{\mathcal{N}} |v(y)|^2 dy \\ &\leq c\kappa^2 \frac{|\mathcal{N}|}{\beta^2 + \kappa^2 \delta} e^{-2d\kappa} \int_{\mathcal{N}} |\Re(n) - 1| |v(y)|^2 dy, \end{aligned}$$

where  $\delta = n_* - 1$  if  $n_* > 1$  or  $\delta = 1 - n^*$  if  $n^* < 1$ . Then, we find

$$\int_{D \setminus \mathcal{N}} |v(x)|^2 dx \leq C_1 e^{-2\kappa d} \int_{\mathcal{N}} |\Re(n) - 1| |v(y)|^2 dy.$$

Next, we have

$$\int_{D \setminus \mathcal{N}} iA \cdot \nabla v \bar{v} dx \leq \|A \cdot \nabla v\|_{L^2(D \setminus \mathcal{N})} \|v\|_{L^2(D \setminus \mathcal{N})}.$$

Since  $v$  satisfies  $\Delta v - \kappa^2 v = 0$  in  $D$  and  $A$  has a compact support, then, according to the elliptic regularity, there exist  $C_A$  such that  $\|A \cdot \nabla v\|_{L^2(K)} \leq C_A \|v\|_{L^2(D \setminus \mathcal{N})}$ , for  $K = \operatorname{Supp}(A) \subset\subset D \setminus \mathcal{N}$  which implies

$$\begin{aligned} \frac{1}{\kappa^2} \int_{D \setminus \mathcal{N}} iA \cdot \nabla v \bar{v} dx &\leq \frac{C_A}{\kappa^2} \int_{D \setminus \mathcal{N}} |v(y)|^2 dy \\ &\leq C_2 e^{-2d\kappa} \int_{\mathcal{N}} |\Re(n) - 1| |v(y)|^2 dy, \end{aligned}$$

where we used the estimate (5.18). This completes the proof.  $\square$

**Theorem 5.2.9.** *There exist positive constants  $\kappa_0 > 0$  and  $C > 0$  such that for all  $\kappa \geq \kappa_0$ , such that*

$$\sup_{\substack{(\psi, \varphi) \neq 0 \\ (\psi, \varphi) \in X(D)}} \frac{|\widehat{B}_{i\kappa}(u, v; \psi, \varphi)|}{\|(\psi, \varphi)\|_{X(D)}} \geq C\|(u, v)\|_{X(D)}, \quad \forall (u, v) \in X(D). \quad (5.20)$$

You can obtain the proof in [15]. Using the previous Theorem 5.2.9 implies the invertibility of  $\widehat{B}_{i\kappa}$ .

**Lemma 5.2.10.** *Let  $\kappa > 0$  be such that the condition (5.20) is valid. Then the operator  $\widehat{B}_{i\kappa} : X(D) \longrightarrow X(D)$  is invertible with bounded inverse  $\widehat{B}_{i\kappa}^{-1}$ .*

The following theorem claims the uniqueness of a solution to the interior transmission problem (5.3).

**Theorem 5.2.11.** *Let  $A \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$  such that  $\operatorname{div}(A) \in L^\infty(\mathbb{R}^3, \mathbb{R})$  and  $\operatorname{Supp}(A) \subset D$ . Let  $n \in L^\infty(\mathbb{R}^3, \mathbb{C})$  such that  $\operatorname{Supp}(n) \subset D$ ,  $\Im(n) \geq 0$  almost everywhere in  $D$  and  $\inf_{\mathcal{N}} \Re(n) > 1$  or  $0 < \sup_{\mathcal{N}} \Re(n) < 1$  for some neighborhood  $\mathcal{N}$  of the boundary  $\partial D$ . Furthermore, assume that  $k \in \mathbb{C}$  is not a transmission eigenvalue. Then, for any given  $f \in H^{\frac{3}{2}}(\partial D)$  and  $h \in H^{\frac{1}{2}}(\partial D)$ , the interior transmission problem (5.3) has a unique solution  $w \in L^2(D)$  and  $v \in L^2_\Delta(D)$  with  $w - v \in H^2(D)$  and the following a priori estimates hold*

$$\|w\|_{L^2(D)} + \|v\|_{L^2_\Delta(D)} \leq C(\|f\|_{H^{\frac{3}{2}}(\partial D)} + \|h\|_{H^{\frac{1}{2}}(\partial D)}), \quad (5.21)$$

$$\|u\|_{H^2(D)} \leq C(\|f\|_{H^{\frac{3}{2}}(\partial D)} + \|h\|_{H^{\frac{1}{2}}(\partial D)}), \quad (5.22)$$

with some positive constant  $C > 0$ .

*Proof.* For  $k \in \mathbb{C}$ , we can write  $B_k = (B_k - \widehat{B}_{i\kappa}) + \widehat{B}_{i\kappa}$  where  $\kappa > 0$  is sufficiently large. Then, by using the Theorem 5.2.7 and the Lemma 5.2.10, we obtain that  $(B_k - \widehat{B}_{i\kappa})$  is compact and  $\widehat{B}_{i\kappa}$  is invertible respectively. Thus, we can conclude that the operator  $B_k$  is Fredholm of index zero. Moreover, using the fact that  $k$  is not a transmission eigenvalue, we proved that the operator  $B_k$  is invertible .

To finish this proof, we need to demonstrate the estimates (5.21) and (5.22). Our problem (5.3) is equivalent to the problem (5.8) where  $F = -(\nabla + iA)^2\theta - k^2n\theta$  for  $\theta \in H^2(D)$  and satisfies (5.7). Then, we get

$$\|\ell\|_{X(D)}^2 \leq \int_D |F\ell_2| \, dx = \|\ell_2\|_{L^2(D)} \|F\|_{L^2(D)} \leq C\|\ell\|_{X(D)} \|\theta\|_{H^2(D)},$$

where  $\ell = (\ell_1, \ell_2) \in X(D)$  satisfies (5.12). From (5.7), we have

$$\|\ell\|_{X(D)} \leq C\|\theta\|_{H^2(D)} \leq C \left( \|f\|_{H^{\frac{3}{2}}(D)} + \|h\|_{H^{\frac{1}{3}}(D)} \right). \quad (5.23)$$

Using the fact  $B_k$  is invertible such that  $B_k^{-1}$  is continuous, we find

$$\|(u, v)\|_{X(D)} \leq C\|\ell\|_{X(D)} \leq C \left( \|f\|_{H^{\frac{3}{2}}(D)} + \|h\|_{H^{\frac{1}{3}}(D)} \right),$$

which implies the required estimates. This proof is completed.  $\square$

### 5.2.3 | Existence and discreteness of the set of transmission eigenvalues

Our next task is to establish the necessary conditions for the transmission eigenvalues in set  $\mathbb{C}$  to be discrete (possibly empty), with only  $+\infty$  as an accumulation point. To achieve this, we will begin by demonstrating the existence of a wave number  $k$  that is not a transmission eigenvalue.

The following theorem indicates that there may be only a finite number of transmission eigenvalues on the imaginary axis.

**Theorem 5.2.12.** *Let  $n \in L^\infty(\mathbb{R}^3)$  with  $\Re(n) > n_0 > 0$ ,  $\Im(n) = 0$  almost everywhere in  $D$  and  $n_* = \sup_{\mathcal{N}} \Re(n) > 1$  for some neighborhood  $\mathcal{N}$  of the boundary  $\partial D$ . Then, for sufficiently large  $\kappa > 0$ , the operator  $B_{i\kappa} : X(D) \rightarrow X(D)$  is invertible with bounded inverse.*

*Proof.* Showing that  $B_{i\kappa}$  is injective for some value of  $\kappa$  is enough, given that  $\widehat{B}_{i\kappa}$  is invertible and  $\widehat{B}_{i\kappa} - B_{i\kappa}$  is compact. We prove it by the contradiction, we assume that there exists a sequence  $\kappa_j \rightarrow \infty$  and functions  $(u_j, v_j) \in X(D)$  with  $\|(u_j, v_j)\|_{X(D)} = 1$  and  $B_{i\kappa_j}(u_j, v_j) = 0$ . Therefore,  $u_j \in H_0^2(D)$  and  $v_j \in L_\Delta^2(D)$  satisfy

$$(\nabla + iA)^2 u_j - \kappa_j^2 n u_j = \frac{1}{\kappa_j^2} P_A v_j - (n-1)v_j \quad \text{in } D. \quad (5.24)$$

and

$$\Delta v_j - \kappa_j^2 v_j = 0 \quad \text{in } D, \quad (5.25)$$

Multiplying the identity (5.24) by  $\bar{v}_j$  and integrating over  $D$ , we have

$$\int_D \Delta u_j \bar{v}_j \, dx + \int_D P_A u_j \bar{v}_j - \kappa_j^2 n u_j \bar{v}_j \, dx = \int_D \frac{1}{\kappa_j^2} P_A v_j \bar{v}_j - (n-1)|v_j|^2 \, dx,$$

using Green's second identity and the equation in (5.25) yields

$$\int_D P_A u_j \bar{v}_j - \kappa_j^2 (n-1) u_j \bar{v}_j \, dx = \int_D \frac{1}{\kappa_j^2} P_A v_j \bar{v}_j - (n-1)|v_j|^2 \, dx. \quad (5.26)$$

Multiplying (5.24) by  $\bar{u}_j$  and integrating over  $D$ , we find

$$\int_D ((\nabla + iA)^2 u) \bar{u}_j - \kappa_j^2 n |u_j|^2 \, dx = \int_D \frac{1}{\kappa_j^2} P_A v_j \bar{v}_j - (n-1)|v_j|^2 \, dx,$$

or we know that using Green's first identity together

$$\begin{aligned} \int_D ((\nabla + iA)^2 u_j) \bar{u}_j \, dx &= - \int_D ((\nabla + iA)u_j)((\nabla - iA)\bar{u}_j) \, dx \\ &= - \int_D ((\nabla + iA)u_j) \overline{((\nabla + iA)u_j)} \, dx, \end{aligned}$$

yields

$$- \int_D |(\nabla + iA)u_j|^2 \, dx - \kappa_j^2 \int_D n |u_j|^2 \, dx = \int_D \frac{1}{\kappa_j^2} P_A v_j \bar{u}_j - (n-1)v_j \bar{u}_j \, dx.$$

Moreover,  $P_A$  is self-adjoint, we have

$$\|(\nabla + iA)u_j\|_{L^2(D)}^2 + \int_D \kappa_j^2 n |u_j|^2 \, dx = -\frac{1}{\kappa_j^2} \int_D \overline{P_A u_j} v_j \, dx + \int_D (n-1)v_j \bar{u}_j \, dx, \quad (5.27)$$

from (5.26), we get

$$\|(\nabla + iA)u_j\|_{L^2(D)}^2 + \int_D \kappa_j n |u_j|^2 dx = -\frac{1}{\kappa_j^4} \int_D \overline{P_A v_j} v_j dx + \frac{1}{\kappa_j^2} \int_D (n-1) |v_j|^2 dx,$$

we have that the left side of (5.28) is positive since  $\Re(n) \geq 0$ .

$$\begin{aligned} & \|(\nabla + iA)u_j\|_{L^2(D)}^2 + \int_D \kappa_j^2 n |u_j|^2 dx \\ &= \frac{1}{\kappa_j^4} \int_D (iA \cdot \nabla \overline{u_j} v_j - iA \cdot \nabla v_j \overline{v_j}) dx + \frac{1}{\kappa_j^2} \int_D \left( n-1 + \frac{1}{\kappa_j^2} |A|^2 \right) |v_j|^2 dx \\ &= \frac{1}{\kappa_j^4} \int_D (iA \cdot \nabla \overline{u_j} v_j + \overline{iA \cdot \nabla v_j \overline{v_j}}) dx + \frac{1}{\kappa_j^2} \int_D \left( n-1 + \frac{1}{\kappa_j^2} |A|^2 \right) |v_j|^2 dx \\ &= \frac{2}{\kappa_j^4} \Re \left( \int_D iA \cdot \nabla \overline{u_j} v_j dx \right) + \frac{1}{\kappa_j^2} \int_D \left( n-1 + \frac{1}{\kappa_j^2} |A|^2 \right) |v_j|^2 dx \\ &= \frac{2}{\kappa_j^4} \Re \left( \int_{D \setminus \mathcal{N}} iA \cdot \nabla \overline{u_j} v_j dx \right) + \frac{1}{\kappa_j^2} \int_{D \setminus \mathcal{N}} \left( n-1 + \frac{1}{\kappa_j^2} |A|^2 \right) |v_j|^2 dx \\ &\quad + \frac{1}{\kappa_j^2} \int_{\mathcal{N}} (n-1) |v_j|^2 dx, \end{aligned} \tag{5.28}$$

where  $A = 0$  on  $\mathcal{N}$ . Using Lemma 5.2.8 and the fact that  $\sup_{\mathcal{N}} \Re(n) < 1$ , we obtain

$$\|(\nabla + iA)u_j\|_{L^2(D)}^2 + \int_D \kappa_j^2 n |u_j|^2 dx = \int_{\mathcal{N}} (\Re(n) - 1) |v_j|^2 dx + O(e^{-2d\kappa}) < 0,$$

for  $\kappa > 0$  sufficiently large, which contradicts the fact that  $\|(u_j, v_j)\|_{X(D)} = 1$ . This completes the proof.  $\square$

**Theorem 5.2.13.** *Assume that  $n \in L^\infty(D)$  with  $\Re(n) > n_0 > 0$ ,  $\Im(n) = 0$  almost everywhere in  $D$  and either  $\inf_{\mathcal{N}} \Re(n) > 1$  or  $\sup_{\mathcal{N}} \Re(n) < 1$  for some neighborhood  $\mathcal{N}$  of the boundary  $\partial D$ . Then the set of transmission eigenvalues is at most discrete with  $+\infty$  as the only accumulation point.*

*Proof.* We know that transmission eigenvalues are the values of  $k \in \mathbb{C}$  for which the kernel of  $B_k$  is non-trivial. Thanks to Theorem 5.2.12, we chose  $\kappa_0 > 0$  such that  $B_{i\kappa_0}$  is invertible

and write the equation  $B_k(u, v) = 0$  in the form

$$(u, v) + B_{i\kappa_0}^{-1}(B_k - B_{i\kappa_0})(u, v) = 0. \quad (5.29)$$

Using the fact that  $B_k - B_{i\kappa_0} : X(D) \rightarrow X(D)$  is compact, due to Theorem 5.2.7, we can establish the result of the theorem by invoking the analytic Fredholm Theorem.  $\square$

## 5.3 | Study of ITP using upper triangular Fredholm theory

The objective of this section is to demonstrate the existence and discreteness of the set of transmission eigenvalues with certain coercivity conditions on the contrast  $m = 1 - n$  through the application of the upper triangular Fredholm theory. This work inspired by Sylvester's paper [86], and we aim to obtain the same results with the presence of the magnetic Laplace operator.

### 5.3.1 | Setting of the problem with main results

Let  $D \subset \mathbb{R}^3$ , be a bounded open set with smooth boundary such that  $(\mathbb{R}^3 \setminus D)$  is connected and  $a > 0$  such that  $D \subset B(0, a)$ . The scattering of a time-harmonic wave in an inhomogeneous medium is modeled by the scattering problem for the magnetic Schrödinger equation. The total wave satisfies

$$(\nabla + iA)^2 u - \lambda(1 + m)u = 0 \quad \text{in } \mathbb{R}^3, \quad (5.30)$$

where the contrast,  $m$ , denotes the deviation of the index of refraction from the constant background, i.e,  $n = 1 + m$ , and  $A = (a_1, a_2, a_3)$  denotes the magnetic potential and  $\lambda = -k^2$ .

We consider the following interior transmission eigenvalues problem

$$\begin{cases} (\Delta + Q_{A,m}^\lambda - \lambda)v = 0 & \text{in } D, \\ (\Delta - \lambda)w = 0, & \text{in } D, \\ v = w; \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} & \text{on } \partial D, \end{cases} \quad (5.31)$$

where  $Q_{A,m}^\lambda$  is a first order operator given by

$$\begin{aligned} Q_{A,m}^\lambda v(x) &= i[\operatorname{div}(A(x)v(x)) + A(x) \cdot \nabla v(x)] - |A(x)|^2 v(x) - \lambda m(x)v(x) \\ &= P_A v(x) - \lambda m(x)v(x). \end{aligned} \quad (5.32)$$

Values of  $\lambda$  for which the interior transmission problem (5.31) has non-trivial solutions  $v$  and  $w$ , are called interior transmission eigenvalues.

Now, we set  $u = v - w$ . Then, the interior transmission eigenvalue problem (5.31) can be rewritten as the following problem

$$\begin{cases} (\Delta + Q_{A,m}^\lambda - \lambda)u + Q_{A,m}^\lambda w = 0 & \text{in } D, \\ (\Delta - \lambda)w = 0, & \text{in } D, \\ u \in H_0^2(D); \quad w \in L_\Delta^2(D), \end{cases} \quad (5.33)$$

Furthermore, let  $B$  be the unbounded operator represented by

$$\begin{aligned} B(\lambda) &: H_0^2(D) \times L_\Delta^2(D) \longrightarrow L^2(D) \times L^2(D), \\ B(\lambda) &:= \begin{pmatrix} \Delta + Q_{A,m}^\lambda & Q_{A,m}^\lambda \\ 0 & \Delta \end{pmatrix}. \end{aligned} \quad (5.34)$$

Therefore,  $(u, w)$  is solution of the problem (5.33) if and only if  $(u, w)$  is solution of the problem

$$(B(\lambda) - \lambda I)(u, w)^t = 0. \quad (5.35)$$

We assume that  $\lambda \neq 0$ . Then,  $(u, w)$  is solution of (5.35) if and only if  $(u, \lambda w)$  is solution of the following problem

$$\begin{cases} (\Delta + Q_{A,m}^\lambda - \lambda)u + \lambda^{-1}Q_{A,m}^\lambda w = 0 & \text{in } D, \\ (\Delta - \lambda)w = 0, & \text{in } D, \\ (u, w) \in H_0^2(D) \times L_\Delta^2(D). \end{cases} \quad (5.36)$$

**Definition 5.3.1.** Let  $m \in L^\infty(D)$  and  $A \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$  such that  $\text{supp}(A) \subset D$ . The interior transmission eigenvalues are the spectrum of the generalized eigenvalue problem

$$\tilde{B}(\lambda) - \lambda I_m := \begin{pmatrix} \Delta + P_A & \lambda^{-1} Q_{A,m}^\lambda \\ 0 & \Delta \end{pmatrix} - \lambda \begin{pmatrix} 1 + m & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.37)$$

We now delve into the main result as follows.

**Theorem 5.3.2.** Let  $A \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$  and  $m \in L^\infty(D, \mathbb{C})$  such that  $\text{supp}(A) \subset D$  and  $\text{supp}(m) \subset D$ . Assume that there are real numbers  $m^* \geq m_* > 0$  and a constant  $\theta \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$  such that

(i)  $\text{Re}(e^{i\theta} m(x)) > m_*$  in some neighborhood  $\mathcal{N}$  of  $\partial D$ , or that  $m(x)$  is real in all of  $D$ , and satisfies  $m(x) \leq -m_*$  in some neighborhood  $\mathcal{N}$  of  $\partial D$ .

(ii)  $|m(x)| < m^*$  in all of  $D$ .

(iii)  $\text{Re}(1 + m(x)) \geq \delta > 0$  in all of  $D$ .

Then, the  $\Lambda$ , set for which there exist functions  $v, w \in L^2(D)$  with  $v - w \in H_0^2(D)$  solving the system (5.31), is a discrete subset of  $\mathbb{C}^*$ , and each  $\lambda \in \Lambda$  is of finite multiplicity.

This will be derived as a corollary of the next theorem.

**Theorem 5.3.3.** Let  $A \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$  and  $m \in L^\infty(D, \mathbb{C})$  such that  $\text{supp}(A) \subset D$  and  $\text{supp}(m) \subset D$ . Suppose that there are real numbers  $m_1 \geq m_0 > 0$  and a unit complex number  $e^{i\theta}$  in the open right half plane such that

(i)  $\text{Re}(e^{i\theta} m(x)) > m_0$  in  $\mathcal{N}$ .

(ii)  $|m(x)| < m_1$  for each  $x \in D$ .

Then, the spectrum of  $\tilde{B}(\lambda)$  consists of a (possibly empty) discrete set of eigenvalues with finite dimensional generalized eigenspaces.



### 5.3.2 | A priori estimates

We are going to prove a simple version of some a priori elliptic estimates for the resolvent of  $\tilde{B}$  when  $\lambda$  is large enough.

**Proposition 5.3.4.** *Let  $A \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$  such that  $\text{supp}(A) \subset D$ . Suppose that  $\rho \in L^\infty(D)$ , that  $\Re(\rho) > \delta > 0$ , and that  $\phi \in C_0^\infty(D)$  is real valued, with  $0 \leq \phi \leq 1$  such that  $\phi(x) = 1$  outside a neighborhood  $\mathcal{N}$  of  $\partial D$ . If  $g \in L^2(D)$  and  $w$  satisfies*

$$(\Delta + P_A - \lambda\rho)w = g \quad \text{in } D, \quad (5.38)$$

then there exists a constant  $C(\phi, A, \delta)$ , such that, for sufficiently large positive  $\lambda$ ,

$$\|\phi w\|_{L^2(D)}^2 \leq \frac{C}{\lambda} \left( \|(1 - \phi)w\|_{L^2(D)}^2 + \frac{\|\phi g\|_{L^2(D)}^2}{\lambda} \right), \quad (5.39)$$

$$\|w\|_{L^2(D)}^2 \leq C \left( \|(1 - \phi)w\|_{L^2(D)}^2 + \frac{\|\phi g\|_{L^2(D)}^2}{\lambda^2} \right), \quad (5.40)$$

$$\|\nabla(\phi w)\|_{L^2(D)}^2 \leq C \left( \|w\|_{L^2(D)}^2 + \frac{\|\phi g\|_{L^2(D)}^2}{\lambda} \right). \quad (5.41)$$

*Proof.* We insert  $\int_D \phi^2 \bar{w} g dx$  in (5.38) and we use the integration by parts, so we have

$$\begin{aligned} \int_D \phi^2 \bar{w} g dx &= \int_D \left( \Delta w \bar{w} \phi^2 + P_A w \phi^2 \bar{w} - \lambda \rho \phi^2 |w|^2 \right) dx \\ &= \int_D \left[ -\nabla(\phi^2 \bar{w}) \cdot \nabla w - iA \cdot \nabla(\phi^2 \bar{w})w + iA \cdot \nabla w \bar{w} \phi^2 \right. \\ &\quad \left. - (|A|^2 + \lambda\rho)|\phi w|^2 \right] dx \\ &= - \int_D \phi \nabla w \cdot \nabla(\phi \bar{w}) dx - \int_D \phi \nabla w \cdot \nabla \phi \bar{w} dx - i \int_D A \cdot \nabla(\phi \bar{w}) \phi w dx \\ &\quad - i \int_D A \cdot \nabla \phi \phi |w|^2 dx + i \int_D A \cdot \nabla w \bar{w} \phi^2 dx - \int_D (|A|^2 + \lambda\rho) |\phi w|^2 dx. \end{aligned}$$

Using  $\phi \nabla w = \nabla(\phi w) - w \nabla \phi$  and  $\phi \nabla \phi = \frac{1}{2} \nabla(\phi^2)$ , we find

$$\begin{aligned}
 \int_D \phi^2 \bar{w} g dx &= - \int_D |\nabla(\phi w)|^2 dx + \int_D \nabla(\phi \bar{w}) \cdot \nabla \phi w dx - \int_D \nabla \phi \cdot \nabla(\phi w) \bar{w} dx \\
 &\quad + \int_D |\nabla \phi|^2 |w|^2 dx - i \int_D A \cdot \nabla(\phi \bar{w}) \phi w dx - i \int_D A \cdot \nabla \phi \phi |w|^2 dx \\
 &\quad + i \int_D A \cdot \nabla(\phi w) \bar{w} \phi dx - i \int_D A \cdot \nabla \phi \phi |w|^2 dx - \int_D (|A|^2 + \lambda \rho) |\phi w|^2 dx \\
 &= - \|\nabla(\phi w)\|_{L^2(D)}^2 + \|\nabla \phi w\|_{L^2(D)}^2 + \int_D \nabla \phi \cdot (\nabla(\bar{w} \phi) w - \nabla(w \phi) \bar{w}) dx \\
 &\quad - 2i \int_D A \cdot \nabla \phi \phi |w|^2 dx + i \int_D A \cdot (\nabla(\phi w) \bar{w} - \nabla(\phi \bar{w}) w) \phi dx \\
 &\quad - \int_D (|A|^2 + \lambda \rho) |\phi w|^2 dx \\
 &= - \|\nabla(\phi w)\|_{L^2(D)}^2 + \|\nabla \phi w\|_{L^2(D)}^2 - i \int_D A \cdot \nabla(\phi^2) |w|^2 dx \\
 &\quad + 2i \operatorname{Im} \left( \int_D \nabla \phi \cdot \nabla(\bar{w} \phi) w dx \right) - 2 \operatorname{Im} \left( \int_D A \cdot \nabla(\phi w) \bar{w} \phi dx \right) \\
 &\quad - \int_D (|A|^2 + \lambda \rho) |\phi w|^2 dx.
 \end{aligned}$$

Taking the real parts, so we get

$$\begin{aligned}
 \operatorname{Re} \left( \int_D \phi^2 \bar{w} g dx \right) &= - \|\nabla(\phi w)\|_{L^2(D)}^2 + \|\nabla \phi w\|_{L^2(D)}^2 - \int_D (|A|^2 + \lambda \operatorname{Re}(\rho)) |\phi w|^2 dx \\
 &\quad - 2 \operatorname{Im} \left( \int_D A \cdot \nabla(\phi w) \bar{w} \phi dx \right), \quad (5.42)
 \end{aligned}$$

rearranging yields

$$\begin{aligned}
 \|\nabla(\phi w)\|_{L^2(D)}^2 + \int_D (|A|^2 + \lambda \operatorname{Re}(\rho)) |\phi w|^2 dx &= \|\nabla \phi w\|_{L^2(D)}^2 - \operatorname{Re} \left( \int_D \phi^2 \bar{w} g dx \right) \\
 &\quad - 2 \operatorname{Im} \left( \int_D A \cdot \nabla(\phi w) \bar{w} \phi dx \right). \quad (5.43)
 \end{aligned}$$

Using that  $\operatorname{Re}(\rho)(x) > \delta$ ,  $\forall x \in D$ , we have

$$\begin{aligned}
 \|\nabla(\phi w)\|_{L^2(D)}^2 + \lambda \delta \|\phi w\|_{L^2(D)}^2 &\leq \|\nabla \phi w\|_{L^2(D)}^2 + 2 \|A\|_{L^\infty(D)} \|\nabla(\phi w)\|_{L^2(D)} \|\phi w\|_{L^2(D)} \\
 &\quad + \|\phi w\|_{L^2(D)} \|\phi g\|_{L^2(D)} \\
 &\leq \|\nabla \phi\|_{L^\infty(D)}^2 \|w\|_{L^2(D)}^2 + \left( 8 \|A\|_{L^\infty(D)}^2 + \frac{\lambda}{\delta} \right) \|\phi w\|_{L^2(D)}^2 \\
 &\quad + \frac{1}{2} \|\nabla(\phi w)\|_{L^2(D)}^2 + \frac{\delta}{\lambda} \|\phi g\|_{L^2(D)}^2,
 \end{aligned}$$

which yields

$$\|\nabla(\phi w)\|_{L^2(D)}^2 + \lambda \delta \|\phi w\|_{L^2(D)}^2 \leq C(\phi, A) \left( \|w\|_{L^2(D)}^2 + \frac{\|\phi g\|_{L^2(D)}}{\lambda} \right). \quad (5.44)$$

We immediately find

$$\|\nabla(\phi w)\|_{L^2(D)}^2 \leq C \left( \|w\|_{L^2(D)}^2 + \frac{\|\phi g\|_{L^2(D)}}{\lambda} \right).$$

Also, we find

$$\begin{aligned} \|\phi w\|_{L^2(D)}^2 &\leq \frac{C}{\lambda \rho} \left( \|w\|_{L^2(D)}^2 + \frac{\|\phi g\|_{L^2(D)}}{\lambda} \right) \\ &\leq \frac{C}{\lambda} \left( \|\phi w\|_{L^2(D)}^2 + \|(1-\phi)w\|_{L^2(D)}^2 + \frac{\|\phi g\|_{L^2(D)}}{\lambda} \right), \end{aligned}$$

which implies that

$$\|\phi w\|_{L^2(D)}^2 \leq \frac{C}{\lambda} \left( \|(1-\phi)w\|_{L^2(D)}^2 + \frac{\|\phi g\|_{L^2(D)}}{\lambda} \right).$$

Moreover, using (5.39), we obtain

$$\|w\|_{L^2(D)}^2 \leq C \left( \|(1-\phi)w\|_{L^2(D)}^2 + \frac{\|\phi g\|_{L^2(D)}}{\lambda^2} \right).$$

□

**Corollary 5.3.5.** *Let  $A \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$  such that  $\text{supp}(A) \subset D$ . Let  $\phi \in C_0^\infty(D)$  be real valued, with  $0 \leq \phi \leq 1$  and we suppose that  $\text{Re}(\rho) > \delta > 0$  and  $|m(x)| < m^* < \infty$ ,  $\forall x \in D$ . Moreover, for some unit complex number  $e^{i\theta}$  and some neighborhood  $\Gamma$  of  $\partial D$ ,  $\text{Re}(e^{i\theta} m(x)) > m_*$ , and that  $\phi(x) = 1$  in  $D \setminus \mathcal{N}$ . Then, for sufficiently large positive  $\lambda$ , we have*

$$\|w\|_{L^2(D)}^2 \leq C \left( \left| \int_D m |w|^2 dx \right| + \frac{\|\phi g\|_{L^2(D)}^2}{\lambda^2} \right), \quad (5.45)$$

*Proof.* We have

$$\begin{aligned}
 \|w\|_{L^2(D)}^2 &\leq \left| \int_D (1 - \phi^2) |w|^2 dx \right| + \left| \int_D \phi^2 |w|^2 dx \right| \\
 &\leq \left| \int_D \operatorname{Re} \left( \frac{e^{i\theta} m}{m_*} \right) (1 - \phi^2) |w|^2 dx \right| + \left| \int_D \phi^2 |w|^2 dx \right| \\
 &\leq \left| \operatorname{Re} \left( \frac{e^{i\theta}}{m_*} \int_D m (1 - \phi^2) |w|^2 dx \right) \right| + \left| \int_D \phi^2 |w|^2 dx \right| \\
 &\leq \left| \frac{1}{m_*} \int_D m (1 - \phi^2) |w|^2 dx \right| + \left| \int_D \phi^2 |w|^2 dx \right| \\
 &\leq \left| \frac{1}{m_*} \int_D m |w|^2 dx \right| + \frac{m_* + m^*}{m_*} \|\phi w\|_{L^2(D)}^2.
 \end{aligned}$$

Applying (5.39), we get

$$\|w\|_{L^2(D)}^2 \leq \left| \frac{1}{m_*} \int_D m |w|^2 dx \right| + \frac{m_* + m^*}{m_*} \frac{C}{\lambda} \left( \|w\|_{L^2(D)}^2 + \frac{\|\phi g\|_{L^2(D)}^2}{\lambda} \right),$$

which implies the desired estimate.  $\square$

Next, let  $\tilde{f} \in H^1(D)$  and  $g \in L^2$ . Then, for  $\lambda$  is large enough, we have the following system

$$(\Delta + P_A - \lambda)u = \tilde{f} \quad \text{in } D, \tag{5.46}$$

$$(\Delta - \lambda)w = g \quad \text{in } D, \tag{5.47}$$

$$u \in H_0^2(D), \quad w \in L_\Delta^2(D). \tag{5.48}$$

**Lemma 5.3.6.** *Let  $A \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$  such that  $\operatorname{supp}(A) \subset D$ . Let  $u$  be a solution of (5.46).*

*Then, for sufficiently large positive  $\lambda$ , there is a constant  $C$  such that the following estimate*

$$\sum_{j=0}^3 \lambda^{2-j} \|u\|_{H^j(D)}^2 \leq C \|\tilde{f}\|_{L^2(D)}^2, \tag{5.49}$$

*holds.*

*Proof.* We multiply (5.46) by  $\bar{u}$  and integrate over  $D$ . Then, we find

$$\int_D (\Delta u + P_A u - \lambda u) \bar{u} dx = \int_D \tilde{f} \bar{u} dx,$$

yields

$$\int_D \Delta u \bar{u} dx - \lambda \int_D |u|^2 dx = \int_D \tilde{f} \bar{u} dx - \int_D P_A u \bar{u} dx.$$

Integrating by parts, we get

$$\int_D |\nabla u|^2 dx + \lambda \int_D |u|^2 dx = \int_D P_A u \bar{u} dx - \int_D \tilde{f} \bar{u} dx,$$

which implies

$$\begin{aligned} \|\nabla u\|_{L^2(D)}^2 + \lambda \|u\|_{L^2(D)}^2 &\leq \|\tilde{f}\|_{L^2(D)} \|u\|_{L^2(D)} + \|P_A u\|_{L^2(D)} \|u\|_{L^2(D)} \\ &\leq \frac{1}{\varepsilon \lambda} \|\tilde{f}\|_{L^2(D)}^2 + \varepsilon \lambda \|u\|_{L^2(D)}^2 + \|P_A u\|_{L^2(D)} \|u\|_{L^2(D)}. \end{aligned}$$

Choosing  $\varepsilon = \frac{1}{2}$ , we obtain

$$\begin{aligned} \|\nabla u\|_{L^2(D)}^2 + \frac{\lambda}{2} \|u\|_{L^2(D)}^2 &\leq \frac{2}{\lambda} \|\tilde{f}\|_{L^2(D)}^2 + \frac{1}{\varepsilon'} \|P_A u\|_{L^2(D)}^2 + \varepsilon' \|u\|_{L^2(D)}^2 \\ &\leq \frac{2}{\varepsilon'} \|A\|_{L^\infty(D)}^2 \|\nabla u\|_{L^2(D)}^2 + \frac{1}{\varepsilon'} \|\operatorname{div}(A)\|_{L^\infty(D)}^2 \|u\|_{L^2(D)}^2 \\ &\quad + \frac{2}{\lambda} \|\tilde{f}\|_{L^2(D)}^2 + \varepsilon' \|u\|_{L^2(D)}^2. \end{aligned}$$

By choosing  $\varepsilon' = 4\|A\|_{L^\infty(D)}^2$ , we find

$$\|\nabla u\|_{L^2(D)}^2 + \frac{\lambda}{2} \|u\|_{L^2(D)}^2 \leq \frac{2}{\lambda} \|\tilde{f}\|_{L^2(D)}^2 + \frac{1}{2} \|\nabla u\|_{L^2(D)}^2 + \delta_1 \|u\|_{L^2(D)}^2,$$

where  $\delta_1 = \frac{\|\operatorname{div}(A)\|_{L^\infty(D)}^2}{4\|A\|_{L^\infty(D)}^2} + 4\|A\|_{L^\infty(D)}^2$ . Then, we have

$$\frac{1}{2} \|\nabla u\|_{L^2(D)}^2 + \frac{\lambda}{4} \|u\|_{L^2(D)}^2 \leq \frac{2}{\lambda} \|\tilde{f}\|_{L^2(D)}^2,$$

which implies

$$\lambda \|\nabla u\|_{L^2(D)}^2 + \lambda^2 \|u\|_{L^2(D)}^2 \leq C \|\tilde{f}\|_{L^2(D)}^2. \quad (5.50)$$

In the other side, we have

$$\begin{aligned} \|\Delta u\|_{L^2(D)}^2 &\leq \|P_A u\|_{L^2(D)}^2 + \lambda \|u\|_{L^2(D)}^2 + \|\tilde{f}\|_{L^2(D)}^2 \\ &\leq C \|u\|_{H^1(D)}^2 + \lambda \|u\|_{L^2(D)}^2 + \|\tilde{f}\|_{L^2(D)}^2 \\ &\leq C \|\tilde{f}\|_{L^2(D)}^2. \end{aligned}$$

Then, we get the desired result.  $\square$

**Lemma 5.3.7.** *Let  $\phi \in C_0^\infty(D)$  be real valued, with  $0 \leq \phi \leq 1$  and  $\phi = 1$  outside a neighborhood  $\mathcal{N}$  of  $\partial D$ . Then, for sufficiently large positive  $\lambda$ , there is a constant  $C$  such that*

$$\lambda \|\phi \nabla w\|_{L^2(D)}^2 + \lambda^2 \|\phi w\|_{L^2(D)}^2 \leq C(\|\phi g\|_{L^2(D)}^2 + \lambda \|w\|_{L^2(D)}^2), \quad (5.51)$$

and

$$\|\phi w\|_{H^1(D)}^2 \leq C\left(\frac{\|\phi g\|_{L^2(D)}^2}{\lambda} + \|w\|_{L^2(D)}^2\right) \quad (5.52)$$

hold.

*Proof.* Multiplying (5.47) by  $\phi^2 \bar{w}$  and integrating over  $D$ , we find

$$\int_D (\Delta - \lambda) w \phi^2 \bar{w} dx = \int_D g \phi^2 \bar{w} dx,$$

and integrating by parts, we get

$$\int_D -\nabla w \cdot \nabla(\phi^2 \bar{w}) dx - \lambda \int_D \phi^2 |w|^2 dx = \int_D g \phi^2 \bar{w} dx,$$

yields

$$-\int_D \nabla w \cdot \nabla(\phi^2 \bar{w}) dx - \int_D \phi^2 |\nabla w|^2 dx - \lambda \int_D \phi^2 |w|^2 dx = \int_D g \phi^2 \bar{w} dx,$$

then, we get

$$\int_D \phi^2 |\nabla w|^2 dx + \lambda \int_D \phi^2 |w|^2 dx = -\int_D g \phi^2 \bar{w} dx - 2 \int_D \bar{w} \nabla w \cdot \nabla \phi \phi dx.$$

Using Cauchy estimates, we obtain

$$\|\phi \nabla w\|_{L^2(D)}^2 + \lambda \|\phi w\|_{L^2(D)}^2 \leq \frac{2}{\lambda} \|\phi g\|_{L^2(D)}^2 + \frac{\lambda}{2} \|\phi w\|_{L^2(D)}^2 + 2 \int_D |\bar{w} \nabla w \cdot \nabla \phi \phi| dx,$$

then, we have

$$\|\phi \nabla w\|_{L^2(D)}^2 + \frac{\lambda}{2} \|\phi w\|_{L^2(D)}^2 \leq \frac{2}{\lambda} \|\phi g\|_{L^2(D)}^2 + 8 \|w \nabla \phi\|_{L^2(D)}^2 + \frac{1}{2} \|\phi \nabla w\|_{L^2(D)}^2,$$

which implies

$$\frac{1}{2} \|\phi \nabla w\|_{L^2(D)}^2 + \frac{\lambda}{2} \|\phi w\|_{L^2(D)}^2 \leq \frac{2}{\lambda} \|\phi g\|_{L^2(D)}^2 + 8 \|\nabla \phi\|_{L^\infty(D)}^2 \|w\|_{L^2(D)}^2.$$

Therefore, we find

$$\lambda \|\phi \nabla w\|_{L^2(D)}^2 + \lambda^2 \|\phi w\|_{L^2(D)}^2 \leq C(\|\phi g\|_{L^2(D)}^2 + \lambda \|w\|_{L^2(D)}^2).$$

In the other side, using (5.41) yields (5.52) which completes the proof.  $\square$

Next, we derive some a priori estimates for the resolvent of  $\tilde{B}(\lambda)$ . Let  $f, g \in L^2(D)$  and we suppose that  $f = \tilde{f} + \lambda^{-1} Q_{A,m}^\lambda w$ . Then, we have the following problem

$$(\Delta + P_A - \lambda)u + \lambda^{-1} Q_{A,m}^\lambda w = f \quad \text{in } D, \quad (5.53)$$

$$(\Delta - \lambda)w = g \quad \text{in } D, \quad (5.54)$$

$$u \in H_0^2(D), \quad w \in L_\Delta^2(D). \quad (5.55)$$

Moreover, let  $\phi = 1$  in  $\text{supp}(A)$ ,  $\text{supp}(\phi) \subset D$ . Then, using (5.51), for sufficiently large positive  $\lambda$ , the estimate (5.49) can be written as follows

$$\sum_{j=0}^2 \lambda^{2-j} \|u\|_{H^j(D)}^2 \leq C \left[ \|f\|_{L^2(D)}^2 + \lambda^{-3} \|\phi g\|_{L^2(D)}^2 + \|w\|_{L^2(D)}^2 \right]. \quad (5.56)$$

**Lemma 5.3.8.** *Let  $A \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$  such that  $\text{supp}(A) \subset D$  and  $m \in L^\infty(D)$ . Let  $(u, w)$  a solution of (5.53)-(5.54). Then, there is a constant  $C$  such that the following estimates*

$$\|w\|_{L^2(D)}^2 \leq C \left( \|f\|_{L^2(D)}^2 + \lambda^{-2} \|g\|_{L^2(D)}^2 \right), \quad (5.57)$$

and

$$\|w\|_{L_\Delta^2(D)}^2 \leq C \left( \lambda \|f\|_{L^2(D)}^2 + \|g\|_{L^2(D)}^2 \right), \quad (5.58)$$

hold.

*Proof.* We multiply (5.53) by  $\bar{w}$  and integrate over  $D$ , then we have

$$\int_D (\Delta u + P_A u - \lambda u) \bar{w} dx + \int_D \lambda^{-1} Q_{A,m}^\lambda w \bar{w} dx = \int_D f \bar{w} dx. \quad (5.59)$$

Multiplying the complex conjugate of (5.54) by  $u$  yields

$$\int_D (\Delta \bar{w} - \lambda \bar{w}) u dx = \int_D \bar{g} u dx,$$

and integrating by parts, we get

$$\int_D \bar{w}(\Delta u - \lambda u) dx = \int_D \bar{g} u dx. \quad (5.60)$$

Next, we differentiate between (5.59) and (5.60) then we find

$$\int_D (P_A u \bar{w} + \lambda^{-1} P_A w \bar{w} - m|w|^2) dx = \int_D (f \bar{w} - \bar{g} u) dx,$$

or  $P_A$  is self-adjoint, then, we get

$$\int_D m|w|^2 dx = \int_D \overline{P_A w} (\lambda^{-1} w + u) dx + \int_D (\bar{g} u - f \bar{w}) dx. \quad (5.61)$$

Using (5.51), we find

$$\begin{aligned} \left| \int_D m|w|^2 dx \right| &\leq \lambda^{-1} \|P_A w\|_{L^2(D)} \|w\|_{L^2(D)} + \|P_A w\|_{L^2(D)} \|u\|_{L^2(D)} + \|g\|_{L^2(D)} \|u\|_{L^2(D)} \\ &\quad + \|f\|_{L^2(D)} \|w\|_{L^2(D)} \\ &\leq \lambda^{-1} \|\phi w\|_{H^1(D)} \|\phi w\|_{L^2(D)} + \|u\|_{L^2(D)} \|\phi w\|_{H^1(D)} + \|g\|_{L^2(D)} \|u\|_{L^2(D)} \\ &\quad + \|f\|_{L^2(D)} \|w\|_{L^2(D)} \\ &\leq C \lambda^{-1} \|\phi w\|_{H^1(D)}^2 + C \lambda \|u\|_{L^2(D)}^2 + C_\varepsilon \lambda^{-2} \|\phi w\|_{L^2(D)}^2 + \varepsilon \|\phi w\|_{H^1(D)}^2 \\ &\quad + C_\varepsilon \lambda^{-2} \|g\|_{L^2(D)}^2 + \varepsilon \lambda^2 \|u\|_{L^2(D)}^2 + C_\varepsilon \|f\|_{L^2(D)}^2 + \varepsilon \|w\|_{L^2(D)}^2 \\ &\leq (C \lambda^{-1} + \varepsilon) \|\phi w\|_{H^1(D)}^2 + (C \lambda^{-3} + \varepsilon) \|w\|_{L^2(D)}^2 + (C \lambda + \varepsilon \lambda^2) \|u\|_{L^2(D)}^2 \\ &\quad + C_\varepsilon \|f\|_{L^2(D)}^2 + C_\varepsilon \lambda^{-2} \|g\|_{L^2(D)}^2. \end{aligned}$$

Choosing  $\varepsilon$  small enough such that  $\lambda^{-1} \leq \varepsilon$ , we obtain

$$\left| \int_D m|w|^2 dx \right| \leq C \left( \lambda^{-1} \|\phi w\|_{H^1(D)}^2 + \lambda^{-1} \|w\|_{L^2(D)}^2 + \lambda \|u\|_{L^2(D)}^2 + \|f\|_{L^2(D)}^2 + \lambda^{-2} \|g\|_{L^2(D)}^2 \right).$$

Using (5.52) and (5.56), we get

$$\left| \int_D m|w|^2 dx \right| \leq C \left( \lambda^{-2} \|g\|_{L^2(D)}^2 + \|f\|_{L^2(D)}^2 + \lambda^{-1} \|w\|_{L^2(D)}^2 \right)$$

From the Corollary 5.3.5, we find

$$\begin{aligned} \|w\|_{L^2(D)}^2 &\leq C \left( \left| \int_D m|w|^2 dx \right| + \lambda^{-2} \|g\|_{L^2(D)}^2 \right) \\ &\leq C \left( \lambda^{-2} \|g\|_{L^2(D)}^2 + \|f\|_{L^2(D)}^2 + \lambda^{-1} \|w\|_{L^2(D)}^2 \right), \end{aligned}$$



which implies

$$\|w\|_{L^2(D)}^2 \leq C \left( \|f\|_{L^2(D)}^2 + \lambda^{-2} \|g\|_{L^2(D)}^2 \right).$$

Next, we have

$$\|\Delta w\|_{L^2(D)}^2 \leq \lambda \|w\|_{L^2(D)}^2 + \|g\|_{L^2(D)}^2 \leq C \left( \lambda \|f\|_{L^2(D)}^2 + \|g\|_{L^2(D)}^2 \right),$$

yields (5.58). The proof is completed.  $\square$

Finally, using the last Lemma 5.3.8 and (5.56), we get the following estimate

$$\sum_{j=0}^2 \lambda^{2-j} \|u\|_{H^j(D)}^2 \leq C \left[ \|f\|_{L^2(D)}^2 + \lambda^{-2} \|g\|_{L^2(D)}^2 \right]. \quad (5.62)$$

### 5.3.3 | Born approximation

**Lemma 5.3.9.** [11] *Let  $T$  be a closed densely defined of a Hilbert space  $H$ . Then, the following properties are equivalent*

- (i)  $Im(T)$  is closed in  $H$ .
- (ii)  $Im(T^*)$  is closed in  $H$ .
- (iii)  $Im(T) = (Ker(T^*))^\perp$ .
- (iv)  $Im(T^*) = (Ker(T))^\perp$ .

*In particular, if  $Im(T)$  is closed and  $T^*$  is injective then  $T$  is surjective.*

**Lemma 5.3.10.** *For  $\lambda$  real, positive and large enough,  $(B(\lambda) - \lambda I)$  is injective.*

*Proof.* We need to prove that if  $(u, w) \in H_0^2(D) \times L_\Delta^2(D)$  solve

$$\begin{cases} (\Delta + Q_{A,m}^\lambda - \lambda)u + \lambda^{-1} Q_{A,m}^\lambda w = 0 & \text{in } D, \\ (\Delta - \lambda)w = 0 & \text{in } D, \end{cases} \quad (5.63)$$

then, we have  $u = w = 0$ . We argue by contradiction and we assume that  $u \neq 0$ . We normalize  $u$  with  $\|u\|_{L^2} = 1$ . Multiplying the complex conjugate of the first equation by  $u$  and integrating over  $D$  yields

$$\int_D u(\Delta - \lambda)\bar{u}dx + \lambda^{-1} \int_D u\overline{Q_{A,m}^\lambda w}dx = - \int_D u\overline{Q_{A,m}^\lambda u}dx,$$

which implies

$$\int_D |\nabla u|^2 dx + \lambda \int_D |u|^2 dx - \lambda^{-1} \int_D u\overline{P_A w}dx + \int_D u\overline{m w}dx = \int_D u\overline{P_A u}dx - \lambda \int_D \overline{m}|u|^2 dx. \quad (5.64)$$

We multiply the same equation by  $\bar{w}$  and integrate over  $D$ , then we get

$$\lambda^{-1} \int_D \overline{w}P_A w dx - \int_D m|w|^2 dx = - \int_D \overline{w}P_A u dx + \lambda \int_D m\overline{w}u dx, \quad (5.65)$$

where we have used  $\int_D \overline{w}(\Delta - \lambda)u dx = 0$ . According to  $P_A = P_A^*$ , we obtain

$$\begin{aligned} \int_D |\nabla u|^2 dx + \lambda \int_D |u|^2 dx - \lambda^{-1} \left[ \int_D m|w|^2 dx - \lambda^{-1} \int_D \overline{w}P_A w dx + \lambda \int_D m\overline{w}u dx \right] \\ + \int_D \overline{m w}u dx = -\lambda \int_D \overline{m}|u|^2 dx + \int_D \overline{u}P_A u dx. \end{aligned} \quad (5.66)$$

Therefore, we deduce that

$$\begin{aligned} \lambda^{-1} \int_D m|w|^2 dx = \int_D |\nabla u|^2 dx + \lambda \int_D |u|^2 dx - \int_D \overline{u}P_A u dx + \lambda^{-2} \int_D \overline{w}P_A w dx \\ + \lambda \int_D \overline{m}|u|^2 dx - 2i \int_D \text{Im}(m)u\overline{w}dx. \end{aligned} \quad (5.67)$$

Let  $\phi \in C_0^\infty(D)$  such that  $\phi^2 = 1$  in  $\text{Supp}(A)$  and  $\text{Re}(m) \geq -\delta$  in  $\text{Supp}(1 - \phi^2)$ . Then, we get

$$\begin{aligned} \lambda^{-1} \int_D (1 - \phi^2)m|w|^2 dx = -\lambda^{-1} \int_D \phi^2 m|w|^2 dx + \|\nabla u\|_{L^2(D)}^2 - \int_D \overline{u}P_A u dx \\ + \lambda \int_D (1 + \overline{m})|u|^2 dx + \lambda^{-2} \int_D \overline{w}P_A w dx - 2i \int_D \text{Im}(m)u\overline{w}dx. \end{aligned} \quad (5.68)$$

Further, we have

$$\left| \int_D \overline{u}P_A u dx \right| \leq \|u\|_{L^2(D)} \|P_A u\|_{L^2(D)} \leq C \|u\|_{L^2(D)}^2 + \frac{1}{2} \|\nabla u\|_{L^2(D)}^2,$$

so

$$\|\nabla u\|_{L^2(D)} - \int_D \bar{u} P_A u dx \geq -C, \quad (5.69)$$

for some positive constant  $C$  with independent of  $\lambda$ . Then, by taking the real part of (5.68), we obtain

$$\begin{aligned} \lambda^{-1} \int_D (1 - \phi^2) \operatorname{Re}(m) |w|^2 dx &\geq (\delta\lambda - C) - C\lambda^{-1} \|\phi w\|_{L^2(D)}^2 - \lambda^{-2} \|w\|_{L^2(D)} \|\phi w\|_{H^1(D)} \\ &\quad - C \|\operatorname{Im}(m)\|_{L^\infty(D)} \|w\|_{L^2(D)}, \end{aligned} \quad (5.70)$$

where we have used  $\operatorname{Re}(1 + m) \geq \delta$  in  $D$ . Moreover, applying Lemmas 5.3.7 and 5.3.8 with  $f = \lambda m$  and  $g = 0$ , we get

$$\|w\|_{L^2(D)} \leq C\lambda, \quad \|\phi w\|_{L^2(D)} \leq C\sqrt{\lambda}, \quad \|\phi w\|_{H^1(D)} \leq C\lambda, \quad (5.71)$$

we deduce that

$$\lambda^{-1} \int_D (1 - \phi^2) \operatorname{Re}(m) |w|^2 dx \geq \lambda(\delta - \varepsilon) - C. \quad (5.72)$$

Taking  $\lambda$  large,  $\varepsilon$  small, we obtain that the right-hand side of (5.72) is positive. Since  $\operatorname{Re}(m) \leq -\delta$  in  $\operatorname{Supp}(1 - \phi^2)$ , we conclude that the left-hand side of (5.72) vanishes. That is  $w = 0$  in a neighborhood of the boundary  $\partial D$ . By the unique continuation principal, we get  $u = 0$  in  $D$ . This contradicts  $\|u\|_{L^2(D)} = 1$ .  $\square$

**Lemma 5.3.11.** *We recall the Born approximation  $\tilde{B}(\lambda) : H_0^2(D) \times L_\Delta^2(D) \longrightarrow L^2(D) \times L^2(D)$  given by*

$$\tilde{B}(\lambda) := \begin{pmatrix} \Delta + P_A & \lambda^{-1} Q_{A,m}^\lambda \\ 0 & \Delta \end{pmatrix}. \quad (5.73)$$

*Then, for  $\lambda$  real, positive, and large enough,  $(\tilde{B}(\lambda) - \lambda I)$  is invertible.*

*Proof.* First, we need to prove that  $(\tilde{B}(\lambda) - \lambda I)$  is injective. Let  $(\tilde{B}(\lambda) - \lambda I)(u, w)^t = 0$ . Since

$$\|(\tilde{B}(\lambda) - \lambda I)(u, w)^t\|_{L^2(D) \times L^2(D)}^2 = \|f\|_{L^2(D)}^2 + \|g\|_{L^2(D)}^2, \quad (5.74)$$

and using the estimates (5.58) and (5.62), we find

$$\|f\|_{L^2(D)}^2 + \|g\|_{L^2(D)}^2 \geq \frac{1}{C} \left( \|u\|_{H^2(D)}^2 + \|w\|_{L^2_\Delta(D)}^2 \right), \quad (5.75)$$

which implies

$$\|(\tilde{B}(\lambda) - \lambda I)(u, w)^t\|_{L^2(D) \times L^2(D)}^2 \geq \frac{1}{C} \|(u, w)\|_{H^2(D) \times L^2_\Delta(D)}^2. \quad (5.76)$$

Then, we obtain  $u = w = 0$  in  $D$ .

Next, by the Lemma 5.3.9, it's enough to prove that  $\text{Im}(\tilde{B}(\lambda) - \lambda I)$  is closed and the adjoint of  $(\tilde{B}(\lambda) - \lambda I)$  is injective. Let  $(f_n, g_n) \in \text{Im}(\tilde{B}(\lambda) - \lambda I)$  such that  $(f_n)_n$  and  $(g_n)_n$  converge respectively in  $L^2(D)$  for some  $f, g \in L^2(D)$ . Let  $(u_n, w_n) \in H_0^2(D) \times L^2_\Delta(D)$  satisfying  $(\tilde{B}(\lambda) - \lambda I)^t(u_n, w_n) = (f_n, g_n)$  in  $D$ . Since  $(f_n, g_n)$  is Cauchy in  $L^2(D) \times L^2(D)$ ; the injectivity of  $(\tilde{B}(\lambda) - \lambda I)$  implies that the sequence  $(u_n, w_n)$  is Cauchy in  $H_0^2(D) \times L^2_\Delta(D)$  and therefore has a unique limit  $(u, w) \in H_0^2(D) \times L^2_\Delta(D)$ ; which satisfies  $(\tilde{B}(\lambda) - \lambda I)^t(u, w) = (f, g)$ . Then  $\text{Im}(\tilde{B}(\lambda) - \lambda I)$  is closed.

Since, the adjoint of  $(\tilde{B}(\lambda) - \lambda I)$  is given

$$(\tilde{B}(\lambda) - \lambda I)^* = (\tilde{B}(\lambda))^* - \lambda I = \begin{pmatrix} \Delta + P_A - \lambda & 0 \\ \lambda^{-1} Q_{A, \bar{m}}^\lambda & \Delta - \lambda \end{pmatrix}, \quad (5.77)$$

and let  $(\tilde{B}(\lambda) - \lambda I)^*(w, u)^t = 0$ . Then, we have

$$\int_D (\Delta w + P_A w - \lambda w) \bar{w} dx = 0,$$

which implies

$$\lambda \|w\|_{L^2(D)}^2 + \|\nabla w\|_{L^2(D)}^2 \leq \frac{\lambda}{2} \|w\|_{L^2(D)}^2 + \frac{2}{\lambda} \|P_A w\|_{L^2(D)}^2 \leq \frac{\lambda}{2} \|w\|_{L^2(D)}^2 + \frac{C}{\lambda} \|w\|_{H^1(D)}^2.$$

which, for  $\lambda$  positive and large enough, implies that  $w = 0$  in  $D$ . Moreover,  $(u, w)$  is also solution of (5.53)-(5.54) where  $f = P_A u$  and  $g = -P_A w$ . Therefore, using the estimate (5.62), we get

$$\sum_{j=0}^3 \lambda^{2-j} \|u\|_{H^j(D)}^2 \leq C \left( \|P_A u\|_{L^2(D)}^2 + \|P_A w\|_{L^2(D)}^2 \right) \leq C \|u\|_{H^1(D)}^2,$$

which implies  $u = 0$  in  $D$ . This ends the proof.  $\square$

**Lemma 5.3.12.** *Assume that we can write the resolvent  $(\tilde{B}(\lambda) - \lambda I)^{-1} : L^2(D) \times L^2(D) \longrightarrow L^2(D) \times L^2(D)$  in block diagonal form*

$$(\tilde{B}(\lambda) - \lambda I)^{-1} = \begin{pmatrix} R_{11}(\lambda) & R_{12}(\lambda) \\ R_{21}(\lambda) & R_{22}(\lambda) \end{pmatrix}. \quad (5.78)$$

*Then,  $R_{11}(\lambda)$ ,  $R_{12}(\lambda)$  and  $R_{22}(\lambda)$  are compact, and  $R_{21}$  is bounded, as operators from  $L^2(D)$  to itself. If  $\phi$  is a smooth function vanishing in neighborhood of  $\partial D$  and equal to 1 on  $D \subset \mathcal{N}$ , then the operator  $\phi R_{21}(\lambda)$  is compact.*

*Moreover, there exist a constant  $C > 0$  such that the following estimate*

$$\|R_{11}\| + \|R_{12}\| + \|\phi R_{21}\| + \|R_{22}\| \leq \frac{C}{\sqrt{\lambda}}. \quad (5.79)$$

*holds.*

*Proof.* We know that the operator  $R_{11}(\lambda)$  is the projection to the first component of the restriction of  $(B - \lambda I)^{-1}$  to the subspace  $L^2(D) \times \{0\}$  which means that for given  $f \in L^2(D)$ , there exist unique functions  $u \in H_0^2(D)$  and  $w \in L_\Delta^2(D)$  solving the problem (5.53)-(5.55) with  $g = 0$ , and that  $R_{11}(\lambda) : L^2(D) \longrightarrow L^2(D)$  is the mapping  $f \longmapsto u$  taking values in  $H_0^2(D)$ . Moreover,  $R_{11}(\lambda)$  is compact using the estimate (5.62) and the compact embedding of  $H_0^2(D)$  into  $L^2(D)$ . Similarly, we obtain that the operator  $R_{12}(\lambda)$ , is the projection to the first component of the restriction of  $(B - \lambda I)^{-1}$  to the subspace  $\{0\} \times L^2(D)$ , is compact where  $R_{12}(\lambda) : L^2(D) \longrightarrow L^2(D)$  is the mapping  $g \longmapsto u$  taking values in  $H_0^2(D)$ .

Next, the operator  $R_{21}(\lambda)$  is the projection to the second component of the restriction of  $(B - \lambda I)^{-1}$  to the subspace  $L^2(D) \times \{0\}$ , and that  $R_{21}(\lambda) : L^2(D) \longrightarrow L^2(D)$  is the mapping  $g \longmapsto v$  taking values in  $L_\Delta^2(D)$ . Using the estimates (5.57) and (5.51), and the compact embedding of  $L_\Delta^2(D)$  into  $L^2(D)$ ,  $\phi R_{21}(\lambda)$  is compact. Similarly, we obtain that the operator  $\phi R_{22}(\lambda)$  is compact where  $R_{22}(\lambda)$  is the projection to the second component of the restriction of  $(B - \lambda I)^{-1}$  to the subspace  $\{0\} \times L^2(D)$ , and  $R_{22}(\lambda) : L^2(D) \longrightarrow L^2(D)$  is the mapping  $g \longmapsto u$  taking values in  $L_\Delta^2(D)$ .

Let prove the the compactness of  $R_{22}(\lambda)$ . We assume that we have a sequence  $(g_n)_n \subset L^2(D)$  converges weakly to zero. Then, the sequences  $u_n = R_{12}g_n$  and  $\phi w_n = \phi R_{22}g_n$

converge strongly to zero. Using (5.61), we obtain

$$\begin{aligned} \int_D m|w_n|^2 dx &= \lambda^{-1} \int_D P_A w_n \overline{\phi w_n} dx + \int_D u_n \overline{P_A w_n} dx + \int_D \overline{g_n} u_n dx \\ &\xrightarrow[n \rightarrow \infty]{} 0. \end{aligned}$$

Moreover, we find

$$\begin{aligned} \int_D m(1 - \phi^2)|w_n|^2 dx &= \int_D \overline{g_n} u_n dx + \lambda^{-1} \int_D P_A w_n \overline{\phi w_n} dx + \int_D u_n \overline{P_A w_n} dx \\ &\quad - \int_D \phi^2 m|w_n|^2 dx \\ &\xrightarrow[n \rightarrow \infty]{} 0. \end{aligned}$$

Therefore, we have

$$\|R_{22} g_n\|_{L^2(D)}^2 = \|w_n\|_{L^2(D)}^2 \leq \|(1 - \phi)w_n\|_{L^2(D)}^2 + \|\phi w_n\|_{L^2(D)}^2 \xrightarrow[n \rightarrow \infty]{} 0.$$

Finally, according to (5.51), (5.57) and (5.62), we get the desired estimate (5.79). The proof is complete.  $\square$

### 5.3.4 | The upper triangular Fredholm theory

In this subsection, we want to briefly review the properties of the upper triangular Fredholm theory that we will use in what follows for the discreteness of the set of transmission eigenvalues (Section 5.3.5). [28, 47, 86].

We have constructed the resolvent  $(\tilde{B}(\lambda) - \lambda I)^{-1}$  for large positive  $\lambda$  and shown that it is upper triangular compact. Moreover, let us introduce the resolvent identity which is given as follows

$$R(\mu) = R(\lambda)(I - (\mu - \lambda)R(\mu))^{-1}, \quad (5.80)$$

for all complex  $\lambda$  and  $\mu$ .

**Definition 5.3.13.** [86] Suppose that  $R$  is a bounded operator mapping a Hilbert space  $H$  to itself. If the Hilbert space has a decomposition into direct sum  $H = \bigotimes_{j=1}^n H_j$ , we say  $R$  is upper

triangular compact (UTC) (with respect to this decomposition if the upper triangular blocks (including the diagonal) in the corresponding decomposition of  $R = \sum_{j,k=1}^n R_{jk}$  are compact.

We can easily check the following proposition.

**Proposition 5.3.14.** *Let  $R, \tilde{R}$  be an upper triangular compact operators. Then  $R + \tilde{R}, R - \tilde{R}$  and  $R \circ \tilde{R}$  are still UTC operators. Moreover, for any  $B$  a bounded operator,  $B \circ R$  and  $R \circ B$  are an UTCs.*

The following theorem claims that the analytic Fredholm theorem (see [77]) still holds for the upper triangular Fredholm operators.

**Lemma 5.3.15** (Upper triangular analytic Fredholm Theorem). *Suppose that  $R(\lambda)$  is an analytic UTC operator valued function of  $\lambda$  for  $\lambda$  is an open connected set  $\Gamma$ . Then, if  $\text{Ker}(I - R(\lambda_0)) = \{0\}$  or  $\text{Coker}(I - R(\lambda)) = \{0\}$  for one  $\lambda_0$ , it is invertible for that  $\lambda_0$  and for all but a discrete set  $\Lambda$  of  $\Gamma$ . Moreover, for  $\lambda \in \Lambda$ ,  $(I - R)$  is a Fredholm.*

*Proof.* Using the fact that  $R(\lambda)$  is an analytic UTC operator, we have

$$I - R(\lambda) = (I - L(\lambda)) - C(\lambda) = \begin{pmatrix} 1 & 0 \\ -R_{21} & 1 \end{pmatrix} - \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix},$$

where  $C$  is compact and  $L$  is bounded and strictly lower triangular. Moreover, for all  $\lambda$ ,  $(I - L(\lambda))$  is invertible with bounded inverse, and that inverse is an analytic function of  $\lambda$ . Therefore, we can write  $I - R(\lambda)$  as follows

$$I - R(\lambda) = (I - L(\lambda))(I - (I - L(\lambda))^{-1}C(\lambda)).$$

Since  $(I - (I - L(\lambda))^{-1}C(\lambda))$  satisfies the hypothesis of the analytic Fredholm Theorem 5.2.5, and the fact that  $(I - L(\lambda))$  is invertible, we obtain the required result for  $I - R(\lambda)$ .  $\square$

The next result presents a special case of the previous theorem.

**Theorem 5.3.16.** *Let  $\tilde{B}(\lambda)$  be a closed densely defined operator on an infinite dimensional Hilbert space and suppose that for one complex number  $\lambda_0$ ,  $(\tilde{B}(\lambda_0) - \lambda_0 I)$  is invertible and  $(\tilde{B}(\lambda_0) - \lambda_0 I)^{-1}$  is UTC. Then the spectrum of  $\tilde{B}(\lambda)$  consists of a (possibly empty) discrete set of eigenvalues with finite dimensional generalized eigenspaces.*

*Proof.* Using the resolvent identity (5.80), we find

$$(\tilde{B}(\mu) - \mu I) = (\tilde{B}(\lambda_0) - \lambda_0 I)(I - (\tilde{B}(\lambda_0) - \lambda_0 I)^{-1}(\mu - \lambda_0)), \quad \forall \mu \in \mathbb{C}.$$

Moreover, according to the fact that  $(\tilde{B}(\lambda_0) - \lambda_0 I)^{-1}$  is UTC, the UT analytic Fredholm Theorem 5.3.15 implies that  $(I - (\tilde{B}(\lambda_0) - \lambda_0 I)^{-1}(\mu - \lambda_0))$  is invertible at all but a discrete set of point  $\mu_n$  and the dimension of the kernel is finite at all such points.  $\square$

### 5.3.5 | Proof of main results

In this subsection, we show that the interior transmission eigenvalues are discrete by proving the Theorems 5.3.2 and 5.3.3. [33, 86].

*Proof of Theorem 5.3.2.* We fix  $\lambda_0$  real, positive and large enough such that  $(\tilde{B}(\lambda_0) - \lambda_0 I)$  is invertible with bounded UTC inverse. Then, we can write  $(\tilde{B}(\lambda_0) - \lambda_0 I_m)$  as follows

$$\begin{aligned} (\tilde{B}(\lambda_0) - \lambda_0 I_m) &= (\tilde{B}(\lambda_0) - \lambda_0 I) \left( I - \lambda_0 (\tilde{B}(\lambda_0) - \lambda_0 I)^{-1} \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} \right) \\ &= (\tilde{B}(\lambda_0) - \lambda_0 I) M(\lambda_0). \end{aligned}$$

Using Proposition 5.3.14 and  $(\tilde{B}(\lambda_0) - \lambda_0 I)^{-1}$  is UTC, we obtain  $M(\lambda_0)$  is an UTC and  $M(\lambda_0) = V(\lambda_0) + K(\lambda_0)$  where  $V(\lambda_0)$  is invertible and  $K(\lambda_0)$  is compact. Then, by the Fredholm alternative 5.2.2 and Remark 5.2.4,  $M(\lambda_0)$  is Fredholm of index zero. Moreover, we find  $M(\lambda_0)$  is injective. Indeed, we have  $M(\lambda_0)(u, w)^t = 0$ , for  $(u, w) \in H_0^2(D) \times L_\Delta^2(D)$ . Based on Lemma 5.3.10, we get that  $(\tilde{B}(\lambda_0) - \lambda_0 I_m)$  is injective for  $\lambda_0$  sufficiently large and since  $M(\lambda_0) = (\tilde{B}(\lambda_0) - \lambda_0 I)^{-1}(\tilde{B}(\lambda_0) - \lambda_0 I_m)^{-1}$ , we find that  $u = w = 0$  in



D. Using Lemma 5.2.3,  $M(\lambda_0)$  is invertible. Therefore, for  $\lambda_0$  large enough,  $(\tilde{B}(\lambda_0) - \lambda_0 I_m)$  is invertible and we have

$$(\tilde{B}(\lambda_0) - \lambda_0 I_m)^{-1} = \left( I - \lambda_0 (\tilde{B}(\lambda_0) - \lambda_0 I)^{-1} \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} \right)^{-1} (\tilde{B}(\lambda_0) - \lambda_0 I)^{-1}.$$

Since  $(B(\lambda_0) - \lambda_0 I)^{-1}$  is UTC and  $M(\lambda_0)^{-1}$  is a bounded operator where

$$M(\lambda_0)^{-1} = \begin{pmatrix} (1 - \lambda_0 m R_{11}(\lambda_0))^{-1} & 0 \\ \lambda_0 m (1 - \lambda_0 m R_{11}(\lambda_0))^{-1} R_{21}(\lambda_0) & 1 \end{pmatrix}, \quad (5.81)$$

we obtain, by the Proposition 5.3.14, that  $(\tilde{B}(\lambda_0) - \lambda_0 I_m)^{-1}$  is UTC and also  $(I_m^{-1} \tilde{B}(\lambda_0) - \lambda_0 I)^{-1} = I_m (\tilde{B}(\lambda_0) - \lambda_0 I_m)^{-1}$  is the UTC resolvent of  $I_m^{-1} \tilde{B}(\lambda_0)$  at  $\lambda_0$ . Using the Theorem 5.3.16 implies the desired result.  $\square$

*Proof of Theorem 5.3.3.* Let  $\tilde{B}(\lambda)$  be the operator defined in (5.73). Using the Lemma 5.3.11 and 5.3.12, if we take  $\lambda_0$  real, positive and large enough, then  $(\tilde{B}(\lambda_0) - \lambda_0 I)$  is invertible and  $(\tilde{B}(\lambda_0) - \lambda_0 I)^{-1}$  is UTC. Therefore, according to the Lemma 5.3.16, the spectrum of  $\tilde{B}(\lambda)$  is discrete and of finite multiplicity.  $\square$

## Conclusion and Perspectives

We presented in this thesis some mathematical and numerical developments for solutions to direct and inverse scattering problem for the time harmonic magnetic Schrödinger operator.

Our primary aim was to look into the inverse problem of stable determining the magnetic and electric potentials from near field and far field patterns. We first showed that the direct scattering problem is well-posed, which was accomplished by two approaches: the first one with the use of the Lippmann-Schwinger integral equation, which requires a stronger assumption of regularity for the magnetic potential  $A$  (which is bounded in the  $W^{1,\infty}$  norm) and the electric potential  $q$  (which is bounded in  $L^\infty$  norm). The second approach is the variational method, which requires less regularity on  $A$  and  $q$  ( $A$ ,  $\operatorname{div}(A)$  and  $q$  are bounded in  $L^\infty$  norm). We presented some numerical illustrations of the solutions to the scattering problem. Due to gauge invariance, the magnetic potential  $A$  cannot be uniquely determined from far field or near field measurements outside the scatterer. We proved logarithmic type estimates for recovering the magnetic fields  $\operatorname{curl}(A)$  and electric potentials from near field or far field maps based on the use of geometrical optics solutions. As a perspective of this part, we would like to study the inverse problem in the case where  $A$  has an unbounded support. We are also interested in investigating the optimality of our results.

Secondly, we addressed more specifically the inverse medium scattering problem for the magnetic Schrödinger operator. We demonstrated that the measurements of scattered waves at a fixed frequency can be used to recover the shape of an inhomogeneous scatterer. The

Linear Sampling Method (LSM) and the Factorization Method (FM) are the main topics of our review of the sampling methods used to numerically solve our problem. The LSM has the simplest formulation and can be easily adapted to different settings of the data and the scattering problem. On the other hand, the FM provides an exact characterization of the scatterer in terms of the far field operator. We obtained our results by utilizing the well-posedness of the interior transmission problem that we proved through the application of Fredholm and the upper triangular Fredholm theories. We presented several numerical validating results conducted in 2D. Throughout our studies, we assumed that the support of the magnetic potential  $A$  is strictly contained within the support of the electric potential contrast. A challenging perspective would be to remove this assumption and extend our results to the case where the both supports coincide. Another perspective is to analyze a new structure of the interior transmission problem when the electric potential contrast vanishes but  $A \neq 0$ .

Last but not least, concerning numerical aspects, we have obtained validating results in 2D and simple configurations. Our plan to extend these numerical investigations to 3D problems.

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