



HAL
open science

Algebraic Area of Lattice Random Walks and Exclusion Statistics

Li Gan

► **To cite this version:**

Li Gan. Algebraic Area of Lattice Random Walks and Exclusion Statistics. Mathematical Physics [math-ph]. Université Paris-Saclay, 2023. English. NNT : 2023UPASP169 . tel-04413106

HAL Id: tel-04413106

<https://theses.hal.science/tel-04413106v1>

Submitted on 23 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Algebraic Area of Lattice Random Walks and Exclusion Statistics

*Aire algébrique de marches aléatoires et
statistique d'exclusion*

Thèse de doctorat de l'université Paris-Saclay

École doctorale n°564, Physique en Île-de-France (PIF)
Spécialité de doctorat : Physique

Graduate School : Physique. Référent : Faculté des sciences d'Orsay

Thèse préparée dans l'unité de recherche **Laboratoire de Physique Théorique et
Modèles Statistiques (Université Paris-Saclay, CNRS)**, sous la direction de
Stéphane OUVRY, Directeur de Recherche CNRS

Thèse soutenue à Paris-Saclay, le 07 décembre 2023, par

Li GAN

Composition du jury

Membres du jury avec voix délibérative

Clément SIRE Directeur de Recherche CNRS, LPT (Université Paul Sabatier)	Président
Brian HOPKINS Professeur, Saint Peter's University	Rapporteur & Examineur
Stephan WAGNER Professeur, Uppsala University	Rapporteur & Examineur
Cyril BANDERIER Chargé de Recherche CNRS, LIPN (Université de Paris Nord)	Examineur
Cyril FURTLHNER Chargé de Recherche Inria, LRI (Université Paris-Saclay)	Examineur
Valentina ROS Chargée de Recherche CNRS, LPTMS (Université Paris-Saclay)	Examinatrice

Titre : Aire algébrique de marches aléatoires et statistique d'exclusion

Mots clés : aire algébrique, marches aléatoires, statistique d'exclusion

Résumé : La thèse se concentre sur l'énumération de marches aléatoires fermées sur réseau avec une aire algébrique donnée, en lien avec les statistiques d'exclusion quantique et la combinatoire de chemins de Dyck et Motzkin généralisés. Dans le chapitre 1, nous rappelons la notion d'aire algébrique d'une marche aléatoire sur réseau, le modèle de Hofstadter, les statistiques d'exclusion et leurs connexions. Dans le chapitre 2, nous nous intéressons aux coefficients du déterminant séculaire du Hamiltonien de Hofstadter qui sont interprétés en termes de fonctions de partition d'exclusion avec paramètre d'exclusion $g = 2$. L'énumération de l'aire algébrique est obtenue en termes des coefficients de cluster associés. Nous étudions ensuite les marches aléatoires fermées sur le réseau en nid d'abeille et établissons une correspondance avec un système

de particules obéissant à un mélange de statistiques d'exclusion $g = 1$ (fermions) et $g = 2$. Dans le chapitre 3, nous relierons la combinatoire de chemins de Dyck et Motzkin généralisés périodiques aux coefficients de cluster de particules obéissant à des statistiques d'exclusion généralisées, et obtenons des expressions explicites pour compter les chemins de Dyck et Motzkin avec un nombre fixe de pas de chaque type. Dans le chapitre 4, nous étendons le concept d'aire algébrique aux marches fermées sur un réseau cubique et associons l'énumération aux coefficients de cluster de trois types de particules obéissant à des statistiques d'exclusion $g = 1$, $g = 1$ et $g = 2$, respectivement, avec la contrainte que les nombres de particules d'exclusion $g = 1$ des deux types soient égaux.

Title: Algebraic Area of Lattice Random Walks and Exclusion Statistics

Keywords: Algebraic Area, Lattice Random Walks, Exclusion Statistics

Abstract: The thesis focuses on the enumeration of closed lattice random walks according to their algebraic area, with connections to quantum exclusion statistics, as well as the combinatorics of generalized Dyck and Motzkin paths. In Chapter 1, we review the concept of algebraic area of a lattice random walk, the Hofstadter model, exclusion statistics, and their connections. In Chapter 2, we focus on the coefficients of the secular determinant of the Hofstadter Hamiltonian which are interpreted in terms of partition functions with exclusion parameter $g = 2$. The algebraic area enumeration is obtained in terms of the associated cluster coefficients. We then study the closed random walks on the honeycomb lattice and establish

a correspondence to a system of particles obeying a mixture of $g = 1$ (fermions) and $g = 2$ exclusion statistics. In Chapter 3, we relate the combinatorics of periodic generalized Dyck and Motzkin paths to the cluster coefficients of particles obeying generalized exclusion statistics, and obtain explicit expressions for counting Dyck and Motzkin paths with a fixed number of steps of each kind. In Chapter 4, we extend the algebraic area concept to closed cubic lattice walks and map the enumeration onto the cluster coefficients of three types of particles obeying $g = 1$, $g = 1$, and $g = 2$ exclusion statistics, respectively, with the constraint that the numbers of $g = 1$ exclusion particles of the two types are equal.

Acknowledgments

It always seems impossible until it's done.

— Nelson Mandela

I would like to express my heartfelt gratitude to my supervisor, Stéphane Ouvry, for his patient and forward-thinking guidance, unwavering support, and constant encouragement during these three years of my PhD. His rigorous and inquisitive mindset towards research has had a profound impact on shaping my studies and influencing my future career.

Also, a big thank you to our collaborator, Alexios P. Polychronakos, for his invaluable assistance, insightful perspectives, and innovative ideas, which have broadened the horizons of my research, challenging me to explore new avenues and approach problems from fresh angles.

I would like to extend my appreciation to all the jury members (Cyril Banderier, Cyril Furtlehner, Brian Hopkins, Valentina Ros, Clément Sire, and Stephan Wagner) for their comments, suggestions, critiques, and encouragement, especially the two rapporteurs (Brian Hopkins and Stephan Wagner) for their fruitful feedback that has enhanced the quality of my thesis. I would also like to express my sincere thanks to my mentor, Valentina Ros, and my tutor, Didina Serban, for their guidance and advice throughout the past three years.

A special acknowledgment goes out to all members of LPTMS, including directors Emmanuel Trizac and Alberto Rosso, as well as the secretariat members Claudine Le Vaou, Delphine Hannoy, Karolina Kolodziej, and the IT engineer Zhiqiang Qin for their contributions. A shout-out to Veronique Terras, the Deputy Director of the Doctoral School, and Sabine Hoarau, the secretary, for their consistent support. Additionally, I would like to express my thanks to Marine Valois at the University library and Thérèse Toro at the Department of Doctoral Studies for their assistance.

I cannot overstate my deepest appreciation for my family and friends, whose staunch support has been a constant source of strength throughout this PhD journey.

Lastly, I would like to acknowledge the financial support of the China Scholarship Council and the assistance from the French government, which has been instrumental in supporting me as an international student.

Publication list

The original results of this thesis are based on the following three publications:

- L. Gan, S. Ouvry, and A. P. Polychronakos, “Algebraic area enumeration of random walks on the honeycomb lattice,” *Phys. Rev. E* **105**, 014112 (2022).
- L. Gan, S. Ouvry, and A. P. Polychronakos, “Combinatorics of generalized Dyck and Motzkin paths,” *Phys. Rev. E* **106**, 044123 (2022).
- L. Gan, “Algebraic area of cubic lattice walks,” *Phys. Rev. E* **108**, 054104 (2023).

Synthèse en français

Les marches aléatoires discrètes et leur équivalent continu, les courbes browniennes, sont des processus stochastiques qui trouvent des applications dans des domaines aussi variés que la physique statistique, la physique des polymères, la théorie des probabilités, la biologie, les réseaux sociaux, la finance, etc. Dans le domaine des marches aléatoires, une classe spécifique de problèmes a fait l'objet d'investigations répétées, à savoir déterminer le nombre de marches aléatoires fermées sur un réseau qui partent et reviennent au même point d'une longueur n donnée, avec une aire algébrique donnée.

L'aire algébrique enclose par une marche est pondérée par ses *nombre d'enroulement* : si la marche se déplace dans le sens trigonométrique autour d'une région, l'aire qu'elle enclôt compte positivement ; sinon, elle compte négativement. Lorsque la marche s'enroule autour d'une région plus d'une fois, l'aire est comptée avec multiplicité, comme illustré dans la Figure 1.

Dans le chapitre 1, nous commençons par le problème original d'énumération de marches aléatoires fermées sur un réseau carré avec une aire algébrique donnée. Nous introduisons deux opérateurs de saut sur le réseau u, v dans les directions droite et vers le haut, ainsi que u^{-1}, v^{-1} dans les

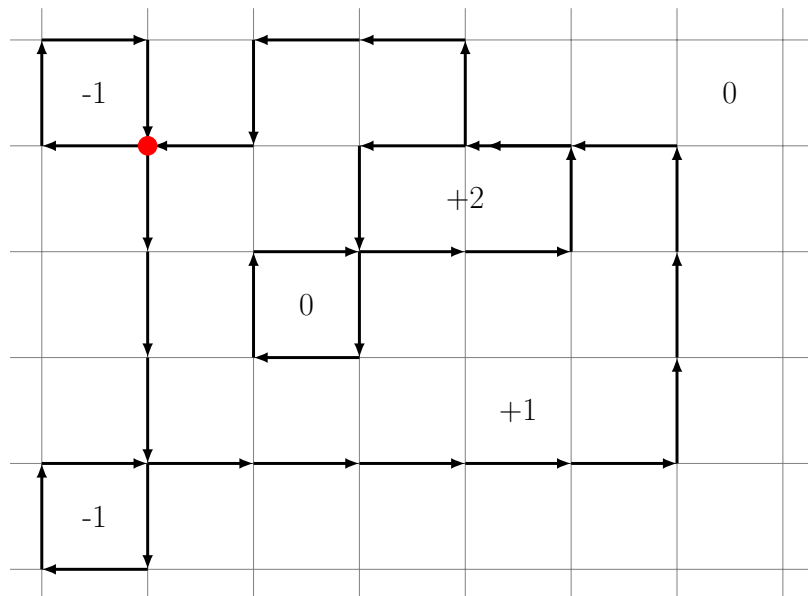


Figure 1: Une marche aléatoire fermée sur un réseau carré de longueur $n = 36$, commençant et se terminant au même point rouge, avec des secteurs d'enroulement $m = +2, +1, 0, -1$ et des nombres de cellules de réseau par secteur d'enroulement 2, 14, 1, 2, respectivement. Le nombre d'enroulement 0 à l'intérieur de la marche résulte d'une superposition d'enroulements $+1$ et -1 . En tenant compte des secteurs d'enroulement non nuls, nous obtenons une aire algébrique $A = (+2) \times 2 + (+1) \times 14 + (-1) \times 2 = 16$. Notez la double flèche sur un lien horizontal, qui indique que la marche est passée deux fois par ce lien, ici dans la même direction à gauche.

directions gauche et vers le bas. Ces opérateurs satisfont la relation non commutative

$$v u = Q u v,$$

ce qui revient à dire que la marche “droite-haut-gauche-bas” qui entoure la cellule de réseau unité dans le sens antihoraire enclôt une aire 1, c’est-à-dire $v^{-1}u^{-1}vu = Q$. L’aire algébrique A enclose par une marche peut donc être calculée en réduisant les opérateurs de saut correspondants à Q^A en utilisant cette relation non commutative. En conséquence, la partie indépendante de u et v dans

$$(u + u^{-1} + v + v^{-1})^{\mathbf{n}} = \sum_A C_{\mathbf{n}}(A) Q^A + \dots$$

fournit les nombres $C_{\mathbf{n}}(A)$ qui comptent les marches fermées de longueur \mathbf{n} enfermant une aire algébrique A . À condition que Q soit interprété comme $Q = e^{2\pi i \Phi / \Phi_0}$ où $\Phi = Ba^2$ représente le flux d’un champ magnétique externe perpendiculaire au réseau B à travers la cellule unité de pas a (fixé à 1 ci-après) et où $\Phi_0 = h/e$ est le quantum de flux (h est la constante de Planck et e est la charge de la particule), l’opérateur

$$H = u + u^{-1} + v + v^{-1}$$

devient le Hamiltonien modélisant une particule chargée sautant sur un réseau carré en présence de B , connu sous le nom de modèle de Hofstadter [1]. Ce modèle a une énorme signification mathématique et physique. Son spectre d’énergie, une structure de papillon notable, est un rare exemple de fractale émergeant de la mécanique quantique. Le papillon de Hofstadter joue également un rôle important dans la théorie de l’effet quantique de Hall et les nombres quantiques topologiques.

On peut aller plus loin en considérant que le flux ϕ/ϕ_0 est rationnel, c’est-à-dire $Q = e^{2\pi i p/q}$ avec p, q deux entiers premiers entre eux, et obtenir une représentation matricielle des opérateurs u et v comme des matrices « d’horloge » et de « décalage » de taille $q \times q$ où $k_x, k_y \in [0, 2\pi/q]$ sont les quasimoments dans les directions x et y . La sélection de la partie indépendante de u et v de $(u + u^{-1} + v + v^{-1})^{\mathbf{n}}$ se traduit dans le monde quantique par le calcul de la trace quantique de $H^{\mathbf{n}}$ définie par

$$\mathbf{Tr} H^{\mathbf{n}} = \frac{1}{q} \int_0^{2\pi} \int_0^{2\pi} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} \text{tr} H^{\mathbf{n}},$$

avec la normalisation $\mathbf{Tr} I = 1$ où I est la matrice identité $q \times q$. L’intervalle k_x et k_y a été étendu à l’intervalle $[0, 2\pi]$ pour simplifier l’expression. Notez que le fait que $\mathbf{Tr} u^m v^n = \delta_{m,0} \delta_{n,0}$ indique que seuls les termes avec un nombre égal de u et u^{-1} , v et v^{-1} survivent, correspondant à toutes les marches fermées. En particulier, l’intégration sur k_x et k_y élimine les termes indésirables contenant u^{qm} et v^{qn} avec $m, n \neq 0$, qui correspondent à des marches ouvertes mais peuvent être fermées par q -périodicité. En utilisant la trace quantique, nous obtenons la fonction génératrice pour le dénombrement de l’aire algébrique

$$\mathbf{Tr} H^{\mathbf{n}} = \sum_A C_{\mathbf{n}}(A) Q^A,$$

qui revient donc à calculer les traces quantiques de la puissance \mathbf{n} -ième de H . Ignorer les éléments de matrice dans les coins de H simplifie le calcul à la trace usuelle $\text{tr} H^{\mathbf{n}}$, avec une normalisation appropriée. Dans les chapitres 2 et 3, nous présentons deux approches pour calculer $\text{tr} H^{\mathbf{n}}$. La première approche repose sur le calcul du déterminant séculaire $\det(I - zH)$ et sa relation avec les statistiques

d'exclusion, tandis que la deuxième approche implique le calcul direct de la trace usuelle et sa relation avec l'énumération de chemins de Dyck.

Avant d'entrer en profondeur dans chaque approche, nous présentons les statistiques d'exclusion. La mécanique statistique des systèmes à plusieurs corps obéit aux lois décrites par les statistiques de Maxwell-Boltzmann. Les statistiques quantiques sont plus subtiles en raison des comportements différents de deux types de particules appelées bosons et fermions qui suivent respectivement les statistiques de Bose-Einstein et de Fermi-Dirac. Pour les statistiques de Bose-Einstein, plus d'une particule peut occuper un état, tandis que pour les statistiques de Fermi-Dirac, une seule particule peut occuper un état, connu sous le nom de principe d'exclusion de Pauli. Les statistiques d'exclusion sont une généralisation du principe d'exclusion de Pauli. Le paramètre statistique g est défini comme le taux auquel le nombre d'états disponibles diminue à mesure que davantage de particules sont ajoutées. Pour les bosons et les fermions, g prend les valeurs 0 et 1, respectivement, puisque l'ajout d'une particule réduit le nombre d'états disponibles de g . $g \geq 2$ signifie plus exclusif que le principe d'exclusion pour les fermions. Une connexion intéressante sera discutée entre une classe générale de marches aléatoires sur des réseaux et les statistiques d'exclusion, le paramètre statistique g dépendant du type de marches.

Maintenant, nous introduisons en détail les deux approches pour calculer $\text{tr } H^n$. La première approche présentée au chapitre 2 repose sur le calcul du déterminant séculaire $\det(I - zH)$. Précisément, les coefficients du déterminant séculaire du Hamiltonien de Hofstadter peuvent être réinterprétés comme fonctions de partition d'exclusion avec paramètre d'exclusion $g = 2$. Le dénombrement de l'aire algébrique est ensuite obtenu en termes des coefficients de cluster associés, à savoir (2.13), où les coefficients combinatoires $c(l_1, l_2, \dots, l_j)$ dans (2.10) étiquetés par les compositions (c'est-à-dire les partitions ordonnées) l_1, l_2, \dots, l_j de n , jouent un rôle important. En utilisant la même méthode, nous étudions les marches aléatoires fermées sur un réseau en nid d'abeille dans la section 2.2, où le Hamiltonien est la somme des trois opérateurs de saut dans les trois directions, à savoir la matrice $2q \times 2q$

$$\begin{pmatrix} 0 & u+v+Q^{1/2}vu^{-1} \\ u^{-1}+v^{-1}+Q^{-1/2}uv^{-1} & 0 \end{pmatrix}.$$

Nous démontrons que le dénombrement des marches aléatoires sur un réseau en nid d'abeille avec une aire algébrique donnée dérive d'un système de statistiques d'exclusion $g = 2$ avec un spectre « Hofstadter dilué », ou, de manière équivalente, d'un mélange de statistiques d'exclusion $g = 1$ (fermions) et $g = 2$. En calculant les fonctions de partition et les coefficients de cluster pertinents, nous arrivons à des expressions explicites de dénombrement de l'aire algébrique (A.1) et (2.26).

Nous notons que l'exclusion $g = 2$ peut être généralisée à une exclusion arbitraire g [2], en introduisant les nouveaux coefficients combinatoires $c_g(l_1, l_2, \dots, l_j)$ étiquetés par les g -compositions de $n = l_1 + l_2 + \dots + l_j$ avec $\mathbf{n} = gn$. $g = 2$ reproduit les compositions standard et $c(l_1, l_2, \dots, l_j)$ dans (2.10). Cependant, une interprétation combinatoire directe de $c_g(l_1, l_2, \dots, l_j)$ manquait.

Dans le chapitre 3, nous contournons le déterminant séculaire et abordons directement la trace matricielle $\text{tr } H^n$. Nous mettons en relation l'expression de la trace avec des chemins de Dyck généralisés périodiques, qui sont des chemins 2D composés d'une série de pas vers le haut $(1, g-1)$ montant de $g-1$ étages et des pas usuels vers le bas $(1, -1)$ descendant d'1 étage (voir la Figure 2). Par définition, chaque pas ne peut ni descendre sous l'étage 1 ni monter au-delà de l'étage $j+g-1$. $g = 2$

reproduit les chemins de Dyck périodiques usuels composés de pas $(1, 1)$ et $(1, -1)$. Plus précisément, nous démontrons que

- $gn c_g(l_1, l_2, \dots, l_j)$ compte le nombre de chemins de Dyck généralisés périodiques de longueur $\mathbf{n} = gn$ avec l_1, l_2, \dots, l_j pas vers le haut provenant respectivement de l'étage $1, 2, \dots, j$. l_1, l_2, \dots, l_j est une g -composition de n .
- $l_i c_g(l_1, l_2, \dots, l_j)$ compte le nombre de ces chemins commençant à l'étage $i = 1, 2, \dots, j$ avec un pas *vers le haut*, et $(l_{i-g+1} + \dots + l_{i-1}) c_g(l_1, l_2, \dots, l_j)$ compte le nombre de ces chemins commençant à l'étage $i = 2, 3, \dots, j+g-1$ avec un pas *vers le bas*.

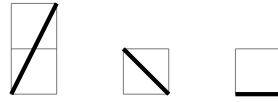


Figure 2: Illustration des pas $(1, g-1)$, $(1, -1)$ et $(1, 0)$ dans des chemins généralisés de Dyck et de Motzkin pour $g = 3$.

Nous étendons ensuite nos résultats au dénombrement de chemins de Motzkin généralisés périodiques qui incluent également des pas $(1, 0)$ sur l'horizontale (voir la Figure 2), en reliant de tels chemins à des matrices correspondant à un mélange d'exclusion pour des particules ayant soit des statistiques fermioniques $g = 1$ ou d'exclusion g . Les expressions dérivées pour les coefficients combinatoires $c_{1,g}$ correspondants qui comptent ces chemins avec un nombre fixe de pas horizontaux, vers le haut ou vers le bas pour chaque étage sont étiquetées par une $(1, g)$ -composition généralisée¹ du nombre de pas \mathbf{n} . L'extension à d'autres classes de chemins, correspondant à d'autres généralisations des statistiques quantiques d'exclusion, semble réalisable grâce à notre méthode.

Dans le chapitre 4, nous étendons le concept d'aire algébrique aux marches fermées sur un réseau cubique. L'aire algébrique est définie comme la somme des aires algébriques de la marche projetée sur les plans xy, yz, zx le long des directions $-z, -x, -y$ (voir la Figure 3). Nous obtenons une formule explicite pour dénombrer les marches aléatoires fermées sur un réseau cubique avec une longueur et une aire algébrique spécifiées. Cette formule de dénombrement peut être mise en correspondance avec les coefficients de cluster de trois types de particules qui obéissent à des statistiques d'exclusion quantique avec des paramètres statistiques $g = 1, g = 1$ et $g = 2$, respectivement, soumis à la contrainte que les nombres de particules d'exclusion $g = 1$ (fermions) des deux types sont égaux. Nous introduisons trois opérateurs de saut U, V, W le long des directions x, y, z , ainsi que U^{-1}, V^{-1}, W^{-1} le long des directions $-x, -y, -z$. Ces opérateurs satisfont aux relations non commutatives

$$VU = QUV, \quad WV = QVW, \quad UW = QWU.$$

¹Concernant les notations, une $(1, g)$ -composition ne doit pas être confondue avec un pas vers le haut $(1, g-1)$ dans un chemin généralisé de Dyck. La première notation réfère à une généralisation des g -compositions correspondant à un mélange d'exclusion $g = 1$ et g , tandis que la seconde notation signifie un pas montant de $g-1$ étages.

En exprimant la phase $Q = e^{2\pi i\phi/\phi_0}$ en termes du flux ϕ à travers la cellule de réseau unitaire sur chacun des trois plans cartésiens en unité du quantum de flux ϕ_0 , l'opérateur Hermitien

$$H = U + V + W + U^{-1} + V^{-1} + W^{-1}$$

représente un Hamiltonien de type Hofstadter qui décrit une particule chargée sautant sur un réseau cubique couplé à un champ magnétique $\mathbf{B} = (1, 1, 1)$, comme indiqué dans la définition de l'aire algébrique. Lorsque le flux magnétique est rationnel, c'est-à-dire $\phi/\phi_0 = p/q$ avec p et q étant premiers entre eux, ainsi $Q = e^{2\pi ip/q}$, nous choisissons la représentation en termes de u et v comme

$$U = u \otimes I, \quad V = v \otimes I, \quad W = (v^{-1}u^{-1}) \otimes u.$$

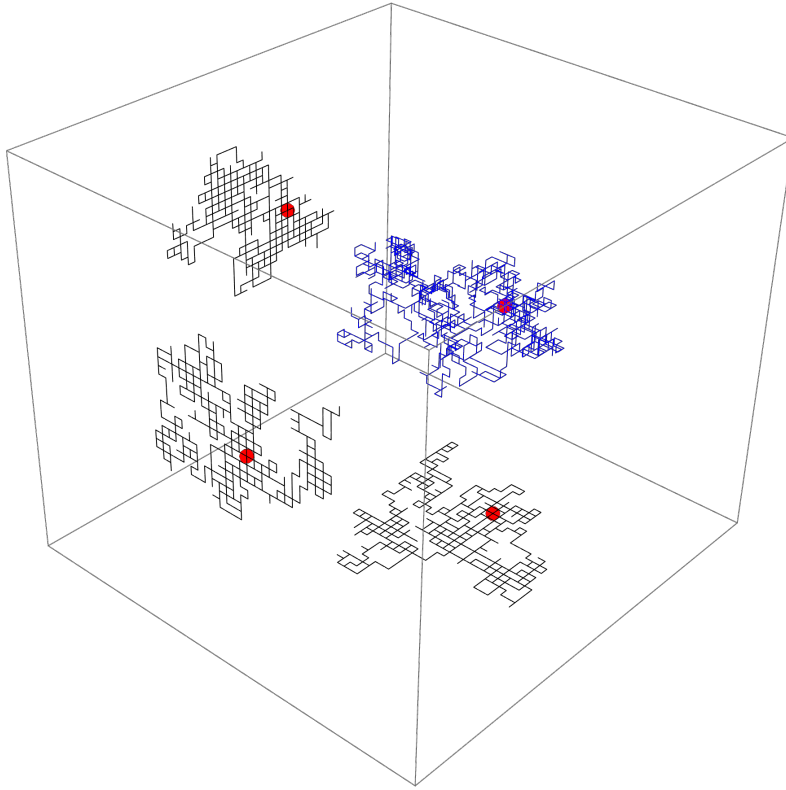


Figure 3: Une marche aléatoire fermée 3D sur un réseau cubique (bleue) ainsi que les trois marches 2D aléatoires projetées correspondantes (noires), commençant et se terminant aux points marqués en rouge.

En résumé, dans cette thèse, nous nous sommes concentrés sur le dénombrement des marches fermées sur un réseau selon leur aire algébrique, avec des connexions aux statistiques d'exclusion quantique, ainsi qu'à la combinatoire de chemins de Dyck et de Motzkin généralisés (également connus sous le nom de chemins de Łukasiewicz). La relation dévoilée entre les marches aléatoires fermées et les statistiques d'exclusion ne fait qu'effleurer la surface de ce sujet énigmatique. Diverses perspectives pour de futures recherches existent, la plus évidente étant le lien entre les marches de

diverses propriétés et sur divers réseaux et les statistiques d'exclusion et chemins de Łukasiewicz correspondants. Des marches sur d'autres réseaux tels que le réseau kagomé et le réseau de plus grande dimension peuvent être étudiées avec des méthodes similaires. Une question pertinente qui se pose est de savoir s'il est possible de classer systématiquement les marches aléatoires en fonction du paramètre d'exclusion g .

Dans les annexes C et D, nous présentons des résultats originaux qui ne sont pas inclus dans nos publications [3, 4, 5], ni, à notre connaissance, dans aucune autre publication. Dans l'annexe C, nous dérivons le dénombrement des marches aléatoires fermées sur un réseau triangulaire avec une aire algébrique donnée, où les six directions sont permises à chaque pas. Par ailleurs, il est à noter qu'en plus des formules explicites pour $C_n(A)$ dérivées dans cette thèse, nous pouvons également utiliser des algorithmes récursifs pour calculer numériquement $C_n(A)$ pour de petites valeurs de n . Par exemple, l'énumération de l'aire algébrique des marches sur un réseau carré peut être calculée numériquement jusqu'à $n = 140$ en utilisant le cluster au LPTMS. On pourra se référer à l'annexe D pour plus de détails.

De plus, la relation de commutation $vu = Quv$ apparaît également dans des modèles exactement résolubles. Explorer sa connexion avec les sujets abordés dans cette thèse serait d'un grand intérêt. Par exemple, une famille de chaînes quantiques à plusieurs spins avec un spectre d'énergie dans une forme (para)fermionique libre [6] a récemment été réanalysée dans [7] et les énergies propres ont été obtenues à partir des coefficients du déterminant séculaire de l'Hamiltonien $g = 2$ pertinent pour le modèle considéré, indiquant une connexion avec la statistique d'exclusion qui mériterait une investigation approfondie.

Contents

Synthèse en français	v
1 Introduction	1
1.1 Algebraic area of random walks	1
1.2 Hofstadter model	2
1.3 Exclusion statistics	5
2 Algebraic area enumeration of random walks on the square and honeycomb lattice	7
2.1 Square lattice walks algebraic area enumeration	7
2.1.1 Hofstadter Hamiltonian and Kreft coefficients	7
2.1.2 Algebraic area enumeration on the square lattice	10
2.2 Honeycomb lattice walks algebraic area enumeration	12
2.2.1 Honeycomb Hofstadter Hamiltonian and Kreft coefficients	12
2.2.2 Diluted spectral functions and algebraic area enumeration	14
2.2.3 Undiluted spectral functions and algebraic area enumeration	15
2.2.4 Algebraic area enumeration on the honeycomb lattice	18
2.3 Conclusion and perspectives	20
Article 1: Algebraic area enumeration of random walks on the honeycomb lattice	21
3 Combinatorics of generalized Dyck and Motzkin paths	37
3.1 Introduction	37
3.2 g -exclusion statistics and generalized Dyck paths	38
3.3 $(1, g)$ -exclusion statistics and generalized Motzkin paths	43
3.4 Conclusion and perspectives	49
Article 2: Combinatorics of generalized Dyck and Motzkin paths	51
4 Algebraic area enumeration of random walks on the cubic lattice	65
4.1 Introduction	65
4.2 Cubic Hofstadter Hamiltonian	67
4.3 $(1, 1, 2)$ -exclusion statistics	69
4.4 Cubic lattice walks algebraic area enumeration	71
4.5 Conclusion and perspectives	72
Article 3: Algebraic area of cubic lattice walks	75
5 Epilogue	85
Appendix A: Algebraic area enumeration of closed honeycomb lattice walks	87

Appendix B: Remarks on the b_n in (2.24)	89
Appendix C: Algebraic area enumeration of closed triangular lattice walks	91
Appendix D: Recurrence relation for enumeration of closed lattice walks	95
D.1 Square lattice walks	95
D.2 Honeycomb lattice walks	95
D.3 Triangular lattice walks	96
D.4 Kagome lattice walks	97
D.5 Cubic lattice walks	98
Bibliography	99

1 - Introduction

1.1 Algebraic area of random walks

Random walks and their continuous counterpart, Brownian curves, are stochastic processes that find application in a diverse range of fields such as statistical physics, polymer physics, probability theory, biology, social networks, finance, etc. In the domain of random walks, a specific class of problems has been the subject of ongoing investigation, that is, to determine the number of closed lattice random walks that start and end at a given point of a given length with a given algebraic area.

We start with the original algebraic area enumeration problem for closed walks on a square lattice: Among the $\binom{n}{n/2}^2$ closed n -step walks, how many of them enclose a given algebraic area A ? Note that, for closed walks, n is necessarily even, $n = 2n$.

The algebraic area enclosed by a walk is weighted by its *winding numbers*: If the walk moves around a region in a counterclockwise direction, its area counts as positive, otherwise negative; if the walk winds more than once, the area is counted with multiplicity, as illustrated in Figure 1.1. These regions inside the walk are referred to as winding sectors.

Let S_m denote the arithmetic area of the m -winding sector inside a walk, which is defined as the

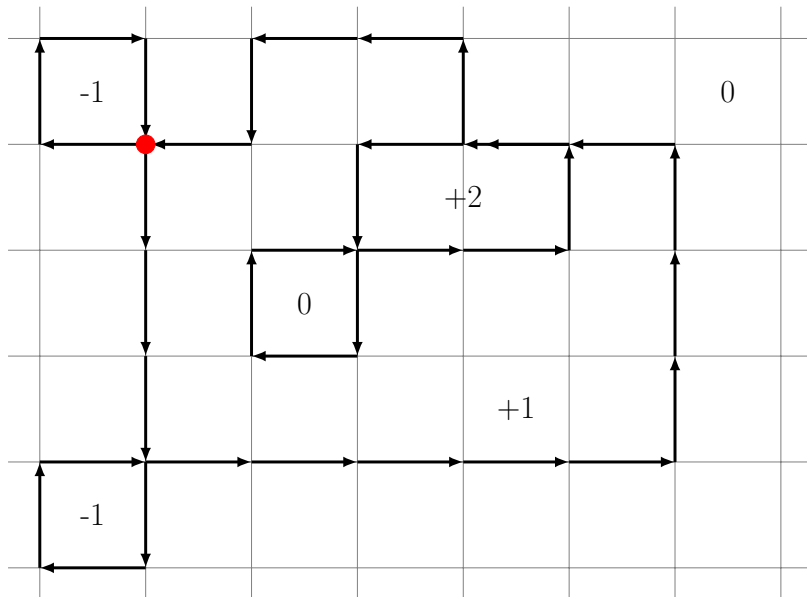


Figure 1.1: A closed walk of length $n = 36$, starting and ending at the same bullet (red) point with winding sectors $m = +2, +1, 0, -1$ and various numbers of lattice cells per winding sectors 2, 14, 1, 2, respectively. The 0-winding number inside the walk arises from a superposition of a +1 and a -1 winding. Taking into account the nonzero winding sectors, we end up with an algebraic area $A = (+2) \times 2 + (+1) \times 14 + (-1) \times 2 = 16$. Note the double arrow on the horizontal link which indicates that the walk has moved twice on this link, here in the same left direction.

total number of lattice cells it encloses with winding number m (where m can be positive or negative). The algebraic area is then the sum of the weighted winding numbers of all the winding sectors, i.e.,

$$A = \sum_{m=-\infty}^{\infty} m S_m.$$

To count closed random walks on a square lattice with a given length and algebraic area, we begin by introducing two lattice hopping operators u, v in the right and up directions, as well as u^{-1}, v^{-1} in the left and down directions. These operators satisfy the noncommutative relation

$$v u = Q u v, \tag{1.1}$$

which amounts to saying that the “right-up-left-down” walk that encircles the unit lattice cell in a counterclockwise direction encloses an area 1, i.e., $v^{-1}u^{-1}vu = Q$. Note that (1.1) is also referred to by other names, including quantum torus algebra, noncommutative 2-tori algebra, Weyl commutation relation, Weyl braiding relation, and Q -commutativity.

The algebraic area A enclosed by a walk can thus be computed by reducing the corresponding hopping operators to Q^A using the commutation relations (1.1). As a consequence, the u - and v -independent part in

$$(u + u^{-1} + v + v^{-1})^{\mathbf{n}} = \sum_A C_{\mathbf{n}}(A) Q^A + \dots \tag{1.2}$$

provides the numbers $C_{\mathbf{n}}(A)$ which count the closed walks of length \mathbf{n} enclosing an algebraic area A . For example, $(u + u^{-1} + v + v^{-1})^4 = 28 + 4Q + 4Q^{-1} + \dots$ indicates that among the $\binom{4}{2}^2 = 36$ closed walks of length 4, $C_4(0) = 28$ enclose an area $A = 0$ and $C_4(1) = C_4(-1) = 4$ enclose an area $A = \pm 1$.

1.2 Hofstadter model

Provided that Q is interpreted as $Q = e^{2\pi i \Phi / \Phi_0}$, where $\Phi = Ba^2$ represents the flux of an external magnetic field B through the unit lattice cell with the lattice spacing a (set to 1 hereafter) and $\Phi_0 = h/e$ is the flux quantum (h is the Planck constant and e is the electric charge of the particle), the operator

$$H = u + u^{-1} + v + v^{-1}$$

becomes the Hamiltonian modeling a charged particle hopping on a square lattice in the presence of an external perpendicular magnetic field, which is known as the Hofstadter model¹ [1]. The Hofstadter model contains tremendous mathematical and physical significance. Its energy spectrum, a self-similar pattern referred to as the “butterfly” (as illustrated in Figure 1.2), represents a rare example of fractal structures emerging from quantum mechanics [9]. Its intriguing fractal properties have been a focal point of various theoretical studies, both in pure mathematics (see, e.g., [10]) and in theoretical physics, such as the quantum Hall effect [11] and topological quantum numbers [12]. It has been shown that the gaps within the butterfly spectrum can be labeled by integers, which have a

¹It is also called the Harper-Hofstadter model or Azbel-Harper-Hofstadter model. See an account of its historical development in [8].

topological origin and are known as Chern numbers. These Chern numbers represent the quantum numbers associated with the quantized Hall conductivity. Along with experimental breakthroughs [13], the quantum Hall effect is now the basis for defining the unit of electrical resistance.

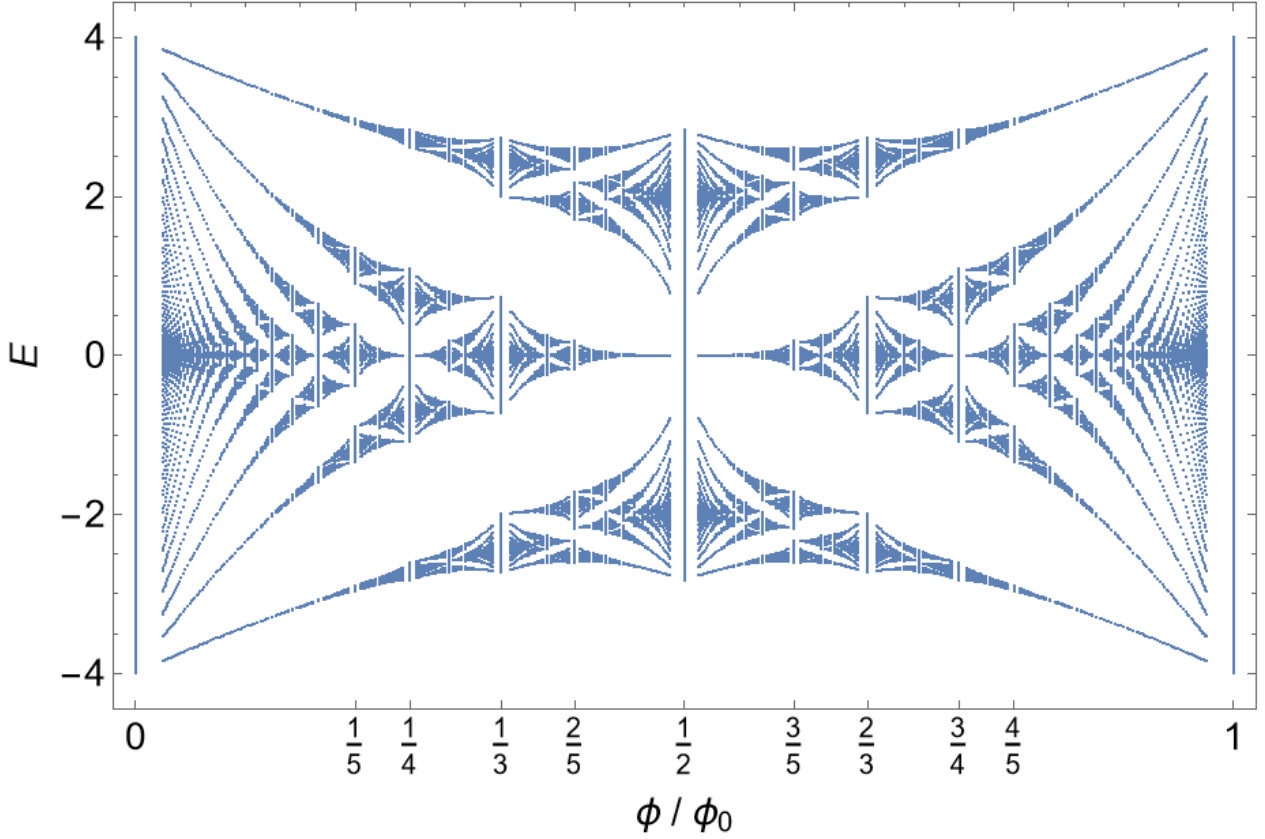


Figure 1.2: Hofstadter's butterfly: Energy spectrum of the Hofstadter model for various rational values of the magnetic flux $\phi/\phi_0 \in [0, 1]$, obtained by diagonalizing the Hamiltonian matrix H in (2.1), with quasimomenta $k_x, k_y \in [0, 2\pi/q]$. In this figure, we take ϕ/ϕ_0 to be p/q , where p and q are coprime positive integers with $p \leq q$ and $q \leq 40$. For each $\phi/\phi_0 = p/q$, the energy spectrum exhibits q bands (allowed values of energies). For an even q , the two central bands touch one another. The white regions representing forbidden energy values, known as gaps, resemble the four wings of a butterfly. Hofstadter's butterfly illustrates self-similar characteristics: Zooming into the butterfly fractal reveals identical patterns at all scales.

To derive the matrix representation of u and v , we start by writing the hopping operators (also called magnetic translator operators) in quantum mechanics as

$$u = e^{i(p_x - eA_x)/\hbar}, \quad v = e^{i(p_y - eA_y)/\hbar},$$

where $p_x = -i\hbar\partial_x$ and $p_y = -i\hbar\partial_y$ are the two components of the momentum operator; $A_x = -By$ and $A_y = 0$ are those of the vector potential A (defined by $\mathbf{B} = \nabla \times \mathbf{A}$) in the Landau gauge; $\hbar = h/(2\pi)$. The relation (1.1) can be verified by utilizing the canonical commutation relation $[x, p_x] = [y, p_y] = i\hbar$ and the Baker-Campbell-Hausdorff formula.

We then introduce the wave function $\Psi_{m,n}$ that describes the quantum state at lattice site (m, n) . The hopping operators act on $\Psi_{m,n}$ as

$$u\Psi_{m,n} = e^{ienB/\hbar}\Psi_{m+1,n}, \quad v\Psi_{m,n} = \Psi_{m,n+1}.$$

Since $[H, p_x] = 0$, we can write all eigenstates of H as those of p_x , namely $\Psi_{m,n} = e^{imk_x}\Phi_n$, where k_x is the component of the quasimomentum in the x direction. Substituting these into the Schrödinger equation $H\Psi = E\Psi$ yields Harper's equation (a special case of the almost Mathieu operator)

$$\Phi_{n+1} + \Phi_{n-1} + (Q^n e^{ik_x} + Q^{-n} e^{-ik_x})\Phi_n = E\Phi_n,$$

which determines the Hofstadter spectrum E .

We can go a step further by considering the rational flux ϕ/ϕ_0 , i.e., $Q = e^{2\pi ip/q}$ with p, q two coprime integers. Thanks to the q -periodicity of the Schrödinger equation in the vertical direction, we can choose Bloch states

$$\Phi_n = e^{ink_y}\tilde{\Phi}_n, \quad \tilde{\Phi}_{n+q} = \tilde{\Phi}_n,$$

where k_y is the component of the quasimomentum in the y direction. Then, the action of u, v becomes

$$u|n\rangle = e^{ik_x}Q^n|n\rangle, \quad v|n\rangle = e^{ik_y}|n-1\rangle,$$

where the bra-ket notation $|n\rangle$ stands for the eigenstate of the position operator located at the position $n = 1, 2, \dots, q$, and by convention $|0\rangle = |q\rangle$. Therefore, we obtain the representation of the operators u and v as $q \times q$ matrices

$$u = e^{ik_x} \begin{pmatrix} Q & 0 & 0 & \cdots & 0 & 0 \\ 0 & Q^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & Q^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Q^{q-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad v = e^{ik_y} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \quad (1.3)$$

u and v are often referred to as "clock" and "shift" matrices: Quantum states correspond to the q "time" positions on a circular clock; u "reads" the time when applied on them, while v "shifts" the time by one unit. These two matrices, without the phase factors, are also called Sylvester's generalized Pauli matrices and form the cornerstone of quantum mechanical dynamics in finite-dimensional vector spaces. We recall that $k_x, k_y \in [0, 2\pi/q]$ are the quasimomenta in the x and y directions. Note that $u^q = e^{iqk_x}$ and $v^q = e^{iqk_y}$.

With the $q \times q$ matrices u and v , selecting as in (1.2) the u, v -independent part of $(u+u^{-1}+v+v^{-1})^n$ translates in the quantum world to computing the quantum trace of H^n defined by

$$\mathbf{Tr} H^n = \frac{1}{q} \int_0^{2\pi} \int_0^{2\pi} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} \text{tr} H^n, \quad (1.4)$$

which corresponds to summing over the q bands of the spectrum (yielding the usual matrix trace $\text{tr} H^n$) and over the scattering states labeled by k_x and k_y in a continuum normalization and the matrix normalization $\mathbf{Tr} I = 1$, where I is the $q \times q$ identity matrix. The range of k_x and k_y has been

harmlessly extended to the full interval $[0, 2\pi]$ to simplify the expression. Note that the fact that $\text{Tr } u^m v^n = \delta_{m,0} \delta_{n,0}$ indicates that only terms with an equal number of u and u^{-1} , v and v^{-1} survive, corresponding to all closed walks. In particular, the integration over k_x and k_y eliminates the unwanted terms containing u^{qm} and v^{qn} with $m, n \neq 0$, which correspond to open walks but can be closed by q -periodicity. Using the quantum trace, we obtain the generating function for the algebraic area enumeration²

$$\text{Tr } H^n = \sum_A C_n(A) Q^A. \quad (1.5)$$

In Chapters 2 and 3, we will introduce two approaches for computing $\text{Tr } H^n$. The first approach relies on the computation of the secular determinant $\det(I - zH)$ and its relation to exclusion statistics, while the second approach involves a direct computation of the usual trace.

Studying a charged particle hopping on other planar lattices in the presence of a perpendicular magnetic field is also of interest. We shall call such models and associated Hamiltonians Hofstadter-like. A case of particular physical and mathematical interest is the honeycomb lattice. It arises naturally in the form of graphene and carbon nanotubes, and many of its quantum properties have been extensively studied (see, for example, [15, 16, 17]). The honeycomb lattice is also relevant in graph theory [18] and various physical models [19, 20, 21]. The Hofstadter-like model for a particle hopping on the honeycomb lattice pierced by a perpendicular magnetic field was introduced in [22, 23]. The effect of lattice defects on its spectrum was investigated in [24] and its butterfly-like spectrum was obtained in [25]. See [26] for a review of 2D Hofstadter-like models, including the triangular lattice [27, 28].

The Hofstadter model has also been explored in higher dimensions. For example, the energy spectrum with external magnetic field $\mathbf{B} = (1, 1, 1)$ on a cubic lattice was initially investigated in [29]. The 3D Hofstadter model was studied earlier in [30], and the general case of the uniform magnetic field was explored in [31], with an experimental scheme proposed in [32]. Hofstadter models have also been studied on other 3D lattices, such as the tetragonal monoatomic and double-atomic lattice [33], and in 4D [34] as well.

1.3 Exclusion statistics

The statistical mechanics of many-body systems obeys laws described by Maxwell–Boltzmann statistics. Quantum statistics is more subtle because of the different behaviors of two kinds of particles called bosons and fermions that follow Bose–Einstein statistics and Fermi–Dirac statistics, respectively. For Bose–Einstein statistics, more than one particle can occupy the same state, while for Fermi–Dirac statistics, no two particles can occupy the same state, which is known as the Pauli exclusion principle. Exclusion statistics, initially proposed by Haldane [35], is a generalization of the Pauli exclusion principle. The statistical parameter g is defined as the rate at which the number of available states decreases as more particles are added. For bosons and fermions, g takes on the value 0 and 1, respectively, since adding one particle reduces the number of available states by g . Values $g \geq 2$ mean more exclusive than the exclusion principle for fermions.

²This was also obtained by another approach in [14].

In the following chapters, an interesting connection will be discussed between a general class of random walks on planar lattices and exclusion statistics, with the statistical parameter g depending on the type of walks. Figure 1.3 shows three examples of 2D lattice random walks: the square lattice walk corresponds to $g = 2$ exclusion, the Kreweras-like chiral walk on a triangular lattice corresponds to $g = 3$ exclusion, and the honeycomb lattice walk corresponds to a mixture of $g = 1$ and $g = 2$ exclusion, with an appropriate spectral function.

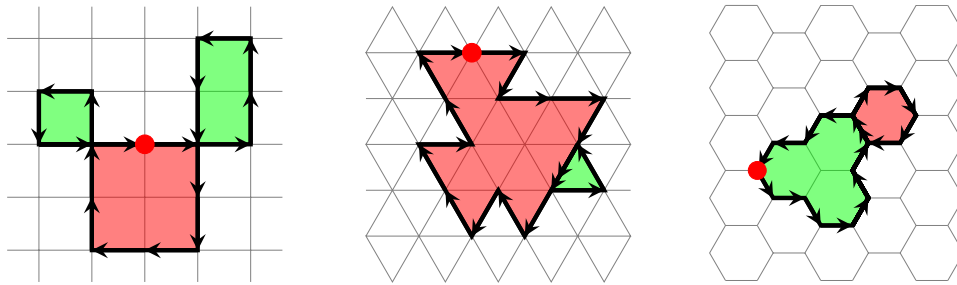


Figure 1.3: A closed square lattice walk, chiral triangular lattice walk, and honeycomb lattice walk of length 18, starting and ending at the bullet (red) point, with algebraic area -1 , -14 , and 2 , respectively. The region inside the walk, i.e., the winding sector, is colored green if its area is positive, otherwise it is colored red. In the chiral triangular lattice walk, only three of the possible six directions are allowed at each step.

At the end of this chapter, we note that the mathematical formulas in this thesis have been verified with Wolfram Mathematica 13.3. The documentation can be accessed via the link

<https://drive.google.com/drive/folders/1ovFGf5H96qu3L1800lwWLonKfXponMbK>

or by scanning the QR code



2 - Algebraic area enumeration of random walks on the square and honeycomb lattice

In Section 2.1, we review the coefficients of the secular determinant of the Hofstadter Hamiltonian which are reinterpreted in terms of $g = 2$ exclusion partition functions. The algebraic area enumeration is then obtained in terms of the associated cluster coefficients. Using the same method, we study the closed random walks on the honeycomb lattice in Section 2.2. We establish a correspondence to an exclusion statistics system and calculate the relevant partition functions and cluster coefficients, arriving at explicit algebraic area enumeration expressions. Some open questions are discussed in Section 2.3.

2.1 Square lattice walks algebraic area enumeration

2.1.1 Hofstadter Hamiltonian and Kreft coefficients

In Section 1.2, we obtained the representation of the hopping operators u and v as $q \times q$ matrices in (1.3). This was done under the assumption that the flux is rational, that is, $Q = e^{2\pi ip/q}$ where p and q are coprime. Therefore, the Hofstadter Hamiltonian becomes the $q \times q$ matrix

$$H = u + u^{-1} + v + v^{-1} = \begin{pmatrix} \omega_1 & e^{ik_y} & 0 & \cdots & 0 & e^{-ik_y} \\ e^{-ik_y} & \omega_2 & e^{ik_y} & \cdots & 0 & 0 \\ 0 & e^{-ik_y} & \omega_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_{q-1} & e^{ik_y} \\ e^{ik_y} & 0 & 0 & \cdots & e^{-ik_y} & \omega_q \end{pmatrix}, \quad (2.1)$$

where $\omega_k = Q^k e^{ik_x} + Q^{-k} e^{-ik_x} = 2 \cos(k_x + 2k\pi p/q)$. Its spectrum follows from the zeros of the secular determinant

$$\det(I - zH) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z_n z^{2n} - 2[\cos(qk_x) + \cos(qk_y)]z^q,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. As we shall see, the coefficient Z_n , named the Kreft coefficient [36], which does not depend on k_x and k_y , is at the core of the lattice walks algebraic area enumeration. To determine Z_n , we need to eliminate the spurious ‘‘umklapp’’ cosine term $-2[\cos(qk_x) + \cos(qk_y)]z^q$. This term arises from the effects of momentum periodicity on the Hofstadter model and will disappear if the upper right and lower left corners of H vanish. To archive this, let us use an alternative form of the Hofstadter Hamiltonian involving a different but equivalent representation of the operators u and v . We perform the transformation on u and v that leaves their commutation relation invariant

$$u \rightarrow -uv, \quad v \rightarrow v.$$

Denoting $f_k = (1 - Q^k e^{ik_x}) e^{ik_y}$, the new Hamiltonian

$$H' = -uv - v^{-1}u^{-1} + v + v^{-1} = \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 & \bar{f}_q \\ \bar{f}_1 & 0 & f_2 & \cdots & 0 & 0 \\ 0 & \bar{f}_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & f_{q-1} \\ f_q & 0 & 0 & \cdots & \bar{f}_{q-1} & 0 \end{pmatrix} \quad (2.2)$$

describes the same random walks but on a deformed square lattice (see Figure 2.1). Let us set $f_q = 0$, i.e., $k_x = 0$, so that both corners of H' vanish. Denote the matrix $H'|_{k_x=0}$ as H_2 . The secular determinant reads

$$d_q := \det(I - zH_2) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z_n z^{2n}, \quad (2.3)$$

where the umklapp cosine term disappears, as expected. The form of H_2 provides an iterative procedure for calculating the Z_n . We temporarily treat Q as a free parameter, independent of q , and introduce the spectral function

$$s_k = f_k \bar{f}_k = (1 - Q^k)(1 - Q^{-k}), \quad k = 1, 2, \dots, q-1. \quad (2.4)$$

The expansion of d_q in terms of the bottom row yields the recursion

$$\begin{aligned} d_0 &= d_1 = 1, \\ d_q &= d_{q-1} - z^2 s_{q-1} d_{q-2}, \quad q \geq 2. \end{aligned} \quad (2.5)$$

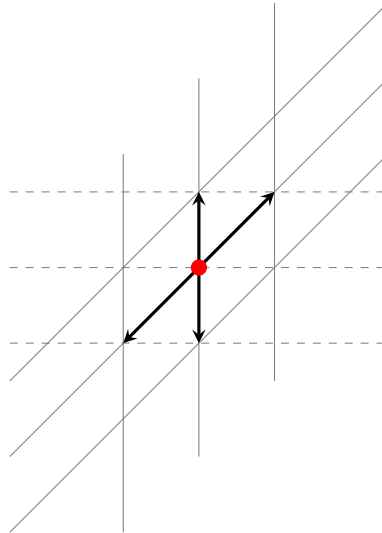


Figure 2.1: Four possible directions at each step of a random walk on the deformed square lattice (solid gray lines) after the transformation. The standard square lattice is shown by dashed gray lines.

Expanding d_q as a polynomial in z and solving the corresponding recursion relation for its coefficients, we obtain the Kreft coefficient [36] (see Appendix 1 of Article 1 for details)

$$Z_n = \sum_{k_1=1}^{q-2n+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+2n-2} s_{k_2+2n-4} \cdots s_{k_{n-1}+2} s_{k_n}, \quad Z_0 = 1 \quad (2.6)$$

as a trigonometric multiple nested sum with $+2$ shifts among the spectral functions s_k which now return to their actual form $s_k = 4 \sin^2(k\pi p/q)$.

In statistical mechanics, Z_n can be interpreted as the n -body partition function for n particles in a one-body spectrum ϵ_k ($k = 1, 2, \dots, q-1$) with Boltzmann factor $e^{-\beta\epsilon_k} = s_k$, where β is the inverse temperature. The $+2$ shifts indicate that these particles obey $g = 2$ exclusion statistics, i.e., no two particles can occupy adjacent quantum states. We recall that the statistical parameter $g = 0$ for bosons, $g = 1$ for fermions, and $g \geq 2$ means a stronger exclusion than fermions.

Note that the physics interpretation of Z_n can be inferred directly from the recursion (2.5) as well: Interpreting $-z^2$ as the fugacity $x = e^{\beta\mu}$ with μ the chemical potential, the secular determinant d_q can be interpreted as an expansion of a grand canonical partition function \mathcal{Z}_{q-1} of noninteracting particles in $q-1$ quantum levels $\epsilon_1, \dots, \epsilon_{q-1}$, obeying the exclusion principle that no two particles can occupy adjacent levels, namely

$$\mathcal{Z}_{q-1} = \mathcal{Z}_{q-2} + x s_{q-1} \mathcal{Z}_{q-3}$$

in terms of the last level ϵ_{q-1} being empty (first term) or occupied (second term). Then (2.3) identifies Z_n as the n -body partition function for particles occupying these $q-1$ quantum states, with gaps of 2 between successive terms reproducing $g = 2$ exclusion.

To relate the quantum trace $\mathbf{Tr} H^{2n}$ to the Z_n in (2.6), first we introduce the n -th cluster coefficients b_n through

$$\ln \left(\sum_{n=0}^{\lfloor q/2 \rfloor} Z_n x^n \right) = \sum_{n=1}^{\infty} b_n x^n \quad (2.7)$$

with x the fugacity. Setting $x = -z^2$ and using the identity

$$\ln \det(I - zH_2) = \text{tr} \ln(I - zH_2) = - \sum_{n=1}^{\infty} \frac{z^n}{n} \text{tr} H_2^n,$$

we establish a connection between the matrix trace $\text{tr} H_2^{2n}$ and the cluster coefficient b_n , that is,

$$\text{tr} H_2^{2n} = 2n(-1)^{n+1} b_n. \quad (2.8)$$

The b_n can be calculated directly in terms of the spectral function to be

$$b_n = (-1)^{n+1} \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} c(l_1, l_2, \dots, l_j) \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}, \quad (2.9)$$

where the combinatorial coefficients

$$c(l_1, l_2, \dots, l_j) = \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2} l_2 \frac{\binom{l_2+l_3}{l_2}}{l_2+l_3} \cdots l_{j-1} \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1}+l_j} = \frac{1}{l_1} \prod_{i=2}^j \binom{l_{i-1}+l_i-1}{l_i} \quad (2.10)$$

are labeled by the compositions l_1, l_2, \dots, l_j of n . The compositions of n are ordered partitions of n , i.e., the ways of writing n as the sum of a sequence of positive integers. For example, there are four compositions of 3, namely $3, 2 + 1, 1 + 2, 1 + 1 + 1$, which contribute to b_3 the terms

$$b_3 = \frac{1}{3} \sum_{k=1}^{q-1} s_k^3 + \sum_{k=1}^{q-2} s_k^2 s_{k+1} + \sum_{k=1}^{q-2} s_k s_{k+1}^2 + \sum_{k=1}^{q-3} s_k s_{k+1} s_{k+2}.$$

In general, there are 2^{n-1} compositions of n .

With the expression of $\text{tr } H_2^{2n}$ in (2.8), the quantum trace can be obtained through (1.4). Noting that $\text{tr } H_2^{2n}$ does not contain k_x and k_y , we have, within a periodicity of q (i.e., for $\mathbf{n} < q$) $\mathbf{Tr } H_2^{2n} = (1/q) \text{tr } H_2^{2n}$. By leaving q as a free parameter that can be made arbitrarily large, the quantum trace can be calculated for all n .

Since H_2 and H describe the same random walks on a square lattice, differing only in whether the lattice is deformed or not, their quantum traces, which correspond to the enumeration of closed square lattice walks, remain identical. Therefore, we have

$$\mathbf{Tr } H^{2n} = \mathbf{Tr } H_2^{2n} = \frac{1}{q} \text{tr } H_2^{2n} = 2n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} c(l_1, l_2, \dots, l_j) \frac{1}{q} \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}. \quad (2.11)$$

2.1.2 Algebraic area enumeration on the square lattice

Based on (1.5), (2.11), and the fact that the trigonometric sum $\frac{1}{q} \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}$ in (2.11) can be computed and read [37, 38]¹

$$\begin{aligned} \frac{1}{q} \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j} &= \sum_{A=-\lfloor (l_1+l_2+\dots+l_j)^2/4 \rfloor}^{\lfloor (l_1+l_2+\dots+l_j)^2/4 \rfloor} Q^A \sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \cdots \sum_{k_j=-l_j}^{l_j} \binom{2l_1}{l_1 + A + \sum_{i=3}^j (i-2)k_i} \\ &\times \binom{2l_2}{l_2 - A - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{2l_i}{l_i + k_i}, \end{aligned} \quad (2.12)$$

where the summation variable A is the algebraic area of square lattice walks, we deduce the desired number of closed random walks on a square lattice with given length $2n$ and given algebraic area A

¹For $j = 1$ and $j = 2$, the nested multiple sum

$$\sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \cdots \sum_{k_j=-l_j}^{l_j} \binom{2l_1}{l_1 + A + \sum_{i=3}^j (i-2)k_i} \binom{2l_2}{l_2 - A - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{2l_i}{l_i + k_i}$$

is understood to be, respectively,

$$\binom{2l_1}{l_1} \delta_{A,0} \text{ and } \binom{2l_1}{l_1 + A} \binom{2l_2}{l_2 - A}.$$

(A is between $-\lfloor n^2/4 \rfloor$ and $\lfloor n^2/4 \rfloor$) as

$$C_{2n}(A) = 2n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2} l_2 \frac{\binom{l_2+l_3}{l_2}}{l_2+l_3} \cdots l_{j-1} \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1}+l_j} \sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \cdots \sum_{k_j=-l_j}^{l_j} \left(l_1 + A + \sum_{i=3}^j (i-2)k_i \right) \\ \times \binom{2l_2}{l_2 - A - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{2l_i}{l_i + k_i}. \quad (2.13)$$

Table 2.1 lists the first few values of $C_{2n}(A)$.

	$n = 2$	4	6	8	10	12	14	16	18
$A = 0$	4	28	232	2156	21944	240280	2787320	33820044	424925872
± 1		8	144	2016	26320	337560	4337088	56267456	739225296
± 2			24	616	11080	174384	2582440	37139616	526924440
± 3				96	3120	67256	1220464	20255488	319524480
± 4				16	840	23928	525224	10030216	176290488
± 5					160	7272	203952	4579520	90612576
± 6					40	2400	80752	2072736	45522456
± 7						528	27440	870080	21840912
± 8						144	9800	368208	10416744
± 9						24	3024	146112	4797504
± 10							840	56128	2171448
± 11							224	20672	956016
± 12							56	7520	417456
± 13								2176	168624
± 14								704	69120
± 15								192	26784
± 16								32	9576
± 17									3168
± 18									1080
± 19									288
± 20									72
Total count	4	36	400	4900	63504	853776	11778624	165636900	2363904400

Table 2.1: $C_n(A)$ up to $n = 18$ for closed square lattice walks of length n .

We note that, since (see a combinatorial proof at the end of Section 3.2)

$$2n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} c(l_1, l_2, \dots, l_j) = \binom{2n}{n}, \quad (2.14)$$

and in the limit $q \rightarrow \infty$, i.e., $Q \rightarrow 1$ [37, 2],

$$\frac{1}{q} \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j} \rightarrow \binom{2(l_1 + l_2 + \cdots + l_j)}{l_1 + l_2 + \cdots + l_j}, \quad (2.15)$$

the total number of closed square lattice walks is recovered to be

$$2n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} c(l_1, l_2, \dots, l_j) \binom{2(l_1 + l_2 + \cdots + l_j)}{l_1 + l_2 + \cdots + l_j} = \binom{2n}{n}^2 = \left(\frac{\mathbf{n}}{2}\right)^2,$$

as expected².

In the continuum limit, in which the lattice spacing $a \rightarrow 0$, closed square lattice walks become closed Brownian curves (also called Brownian loops). Using Lévy's law [39] for the distribution of the algebraic area enclosed by Brownian loops after a time t , that is, $P(A) = \pi/[2t \cosh^2(\pi A/t)]$, and with the scaling $\mathbf{n}a^2 = 2t$, we obtain the asymptotic behavior of $C_{\mathbf{n}}(A)$ as the walk length $\mathbf{n} = 2n \rightarrow \infty$,

$$C_{\mathbf{n}}(A) \sim \frac{\pi}{\mathbf{n} \cosh^2(2\pi A/\mathbf{n})} \left(\frac{\mathbf{n}}{2}\right)^2, \quad (2.16)$$

where $A = 0, \pm 1, \pm 2, \dots$ is dimensionless. The asymptotics (2.16) has been checked numerically for \mathbf{n} up to 140. See [40] for details. However, deriving (2.16) directly from (2.13) is nontrivial and remains an open problem.

2.2 Honeycomb lattice walks algebraic area enumeration

We saw in Section 2.1 that the algebraic area enumeration of walks on the square lattice is directly related to the quantum mechanics of an underlying Hofstadter system. It invokes statistical mechanical concepts that provide a physical context for the calculations and results. In this section, we will follow the same approach to obtain an explicit algebraic area enumeration for closed walks on the honeycomb lattice.

2.2.1 Honeycomb Hofstadter Hamiltonian and Kreft coefficients

We consider a charged particle hopping on a honeycomb lattice coupled to a constant perpendicular magnetic field. As illustrated in Figure 2.2, the honeycomb lattice is bipartite. It contains two types of sites, marked by black and white circles. Each hop moves a particle from one site to the other type of site. Thus, we can define unique hopping operators U , V , and W associated with each of the three bond directions, respectively. When the particle hops around a unit lattice cell in a counterclockwise direction, it accumulates a phase Q due to the magnetic field. Therefore, the operators satisfy the "honeycomb algebra"

$$U^2 = V^2 = W^2 = I, \quad (UVW)^2 = Q.$$

²We can also derive the total number $\sum_A C_{\mathbf{n}}(A)$, by setting $Q = 1$, from the u, v -independent part in the expansion of $(u + u^{-1} + v + v^{-1})^{\mathbf{n}}$. Since $vu = uv$, applying the binomial theorem, we have $(u + u^{-1} + v + v^{-1})^{\mathbf{n}} = (u + v)^{\mathbf{n}}(1 + u^{-1}v^{-1})^{\mathbf{n}} = \sum_{i=0}^{\mathbf{n}} \sum_{j=0}^{\mathbf{n}} \binom{\mathbf{n}}{i} \binom{\mathbf{n}}{j} u^i v^{\mathbf{n}-i} (vu)^{-j}$. Setting $i = j = \mathbf{n}/2$ yields $\left(\frac{\mathbf{n}}{2}\right)^2$.

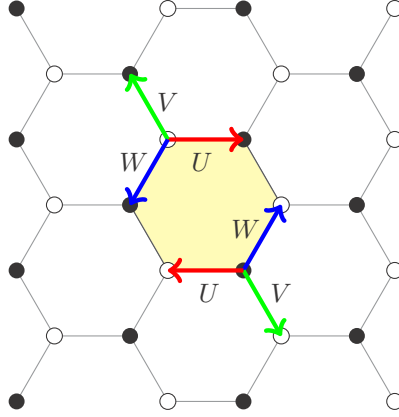


Figure 2.2: Hopping operators U, V, W on the honeycomb lattice with $U^2 = V^2 = W^2 = I$ and $(UVW)^2 = Q$.

In the case of a rational flux, $Q = e^{2\pi ip/q}$ with p and q coprime, U, V , and W can be written as $2q \times 2q$ matrices (see Appendix 3 of Article 1 for derivation)

$$U = \begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & v \\ v^{-1} & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & Q^{1/2}vu^{-1} \\ Q^{-1/2}uv^{-1} & 0 \end{pmatrix}$$

with u, v given in (1.3) and $Q^{\pm 1/2}$ understood to stand for $e^{\pm \pi ip/q}$. Therefore, the honeycomb Hamiltonian reads

$$H_{2q} = U + V + W = \begin{pmatrix} 0 & u+v+Q^{1/2}vu^{-1} \\ u^{-1}+v^{-1}+Q^{-1/2}uv^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}. \quad (2.17)$$

Define the $q \times q$ matrix

$$H_q = AA^\dagger = \begin{pmatrix} 1 + \omega_2\bar{\omega}_2 & \omega_2 & 0 & \cdots & 0 & \bar{\omega}_1 \\ \bar{\omega}_2 & 1 + \omega_3\bar{\omega}_3 & \omega_3 & \cdots & 0 & 0 \\ 0 & \bar{\omega}_3 & 1 + \omega_4\bar{\omega}_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 + \omega_q\bar{\omega}_q & \omega_q \\ \omega_1 & 0 & 0 & \cdots & \bar{\omega}_q & 1 + \omega_1\bar{\omega}_1 \end{pmatrix} \quad (2.18)$$

with $\omega_k = Q^{-k}(1 + e^{-ik_x}Q^{\frac{1}{2}-k})e^{-i(k_x-k_y)}$. The spectrum of H_q is the square of the honeycomb Hamil-

tonian H_{2q} spectrum.³ The secular determinant is given by

$$\begin{aligned} \det(I - zH_{2q}) &= \det(I - z^2H_q) \\ &= \sum_{n=0}^q (-1)^n Z_n z^{2n} + 2\{-(-1)^p[\cos(qk_y - 2qk_x) + \cos(qk_x)] + (-1)^q[\cos(qk_y - qk_x) + 1]\}z^{2q}. \end{aligned} \quad (2.19)$$

To determine the honeycomb Kreft coefficients Z_n , like in the Hofstadter (square lattice) case in Section 2.1, we set both corners $\omega_1 = \bar{\omega}_1 = 0$, i.e., $e^{-ik_x} = -Q^{1/2}$, so that the Hamiltonian becomes

$$H_q|_{\omega_1=0} = \begin{pmatrix} \tilde{s}_1 & f_1 & 0 & \cdots & 0 & 0 \\ \bar{f}_1 & \tilde{s}_2 & f_2 & \cdots & 0 & 0 \\ 0 & \bar{f}_2 & \tilde{s}_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{s}_{q-1} & f_{q-1} \\ 0 & 0 & 0 & \cdots & \bar{f}_{q-1} & \tilde{s}_q \end{pmatrix}, \quad (2.20)$$

where $f_k = -Q^{-\frac{1}{2}-k} e^{ik_y} (1 - Q^{-k})$ and $\tilde{s}_k = 1 + f_k \bar{f}_k = 1 + (1 - Q^k)(1 - Q^{-k})$. The secular determinant reduces to

$$d_q := \det(I - z^2H_q|_{\omega_1=0}) = \sum_{n=0}^q (-1)^n Z_n z^{2n},$$

which has again the suggestive form of a grand partition function, with Z_n the n -body partition function and $x = -z^2$ the fugacity. Expanding d_q in terms of its bottom row, we obtain the recursion relation

$$\begin{aligned} d_0 &= 1, \quad d_j = 0 \text{ for } j < 0, \\ d_q &= (1 - \tilde{s}_q z^2) d_{q-1} - z^4 s_{q-1} d_{q-2}, \quad q \geq 1 \end{aligned} \quad (2.21)$$

with $s_k = f_k \bar{f}_k = 4 \sin^2(k\pi p/q)$ as in (2.4) and $\tilde{s}_k = 1 + s_k$.

2.2.2 Diluted spectral functions and algebraic area enumeration

In this subsection, we will show that the recursion (2.21) still admits a $g = 2$ exclusion statistics interpretation although it is different from (2.5).

Let us consider a set of $2q$ energy levels S_1, S_2, \dots, S_{2q} given by

$$S_{2k-1} = 1, \quad S_{2k} = s_k$$

with $s_k = 4 \sin^2(k\pi p/q)$ as in (2.4). That is, s_k is diluted by additional zero-energy levels between successive levels, in a "tooth comb" pattern, i.e., $1, s_1, 1, s_2, \dots, 1, s_q$. The truncated grand partition

³This can be derived from the fact that $H_{2q}^2 = \begin{pmatrix} H_q & 0 \\ 0 & A^\dagger A \end{pmatrix}$ and H_q and $A^\dagger A$ have identical spectra since $\det(I - z^2 H_q) = \begin{vmatrix} I_q - z^2 H_q & -zA \\ 0 & I_q \end{vmatrix} = \begin{vmatrix} I_q & -zA \\ -zA^\dagger & I_q \end{vmatrix} \begin{vmatrix} I_q & 0 \\ zA^\dagger & I_q \end{vmatrix} = \det(I - zH_{2q}) \det I_{2q} = \det(I - zH_{2q}) = \begin{vmatrix} I_q & -zA \\ -zA^\dagger & I_q \end{vmatrix} \begin{vmatrix} I_q & zA \\ 0 & I_q \end{vmatrix} = \begin{vmatrix} I_q & 0 \\ -zA^\dagger & I_q - z^2 A^\dagger A \end{vmatrix} = \det(I - z^2 A^\dagger A)$.

function \mathcal{Z}_r of $g = 2$ exclusion particles for levels S_1, S_2, \dots, S_r with fugacity parameter x , is given by the recursion relations depending on the last level r being empty or filled

$$\begin{aligned} r = 2k: & \quad \mathcal{Z}_{2k} = \mathcal{Z}_{2k-1} + x s_k \mathcal{Z}_{2k-2}, \\ r = 2k-1: & \quad \mathcal{Z}_{2k-1} = \mathcal{Z}_{2k-2} + x \mathcal{Z}_{2k-3}. \end{aligned}$$

Substituting the $r = 2k$ relation $\mathcal{Z}_{2k-1} = \mathcal{Z}_{2k} - x s_k \mathcal{Z}_{2k-2}$ into the $r = 2k-1$ relation yields

$$\mathcal{Z}_{2k} = (1 + x + x s_k) \mathcal{Z}_{2k-2} - x^2 s_{k-1} \mathcal{Z}_{2k-4},$$

which can be checked to be identical to the recursion (2.21) with $x = -z^2$ and $\mathcal{Z}_{2q} = d_q$. We can then derive the honeycomb Kreft coefficients, i.e., n -body partition functions Z_n , and the associated cluster coefficients b_n with the diluted spectrum S_1, S_2, \dots, S_{2q} as

$$Z_n = \sum_{k_1=1}^{2q-2n+2} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} S_{k_1+2n-2} S_{k_2+2n-4} \cdots S_{k_{n-1}+2} S_{k_n}, \quad (2.22)$$

$$b_n = (-1)^{n+1} \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} c(l_1, l_2, \dots, l_j) \sum_{k=1}^{2q-j+1} S_k^{l_1} S_{k+1}^{l_2} \cdots S_{k+j-1}^{l_j} \quad (2.23)$$

with the same Hofstadter combinatorial factors $c(l_1, l_2, \dots, l_j)$ given in (2.10). The new trigonometric sums $\frac{1}{q} \sum_{k=1}^{2q-j+1} S_k^{l_1} S_{k+1}^{l_2} \cdots S_{k+j-1}^{l_j}$ now entering the definition of the b_n can be obtained using the same tools [38] as for the usual trigonometric sums in (2.9).

Finally, following the same steps as in Section 2.1 regarding the number $C_n(A)$ of closed random walks of length $2n$ enclosing an algebraic area A on the honeycomb lattice, i.e., considering on the one hand

$$\sum_A C_{2n}(A) Q^A = \mathbf{Tr} H_{2q}^{2n} = \frac{1}{2q} \text{tr} H_{2q}^{2n},$$

which is the analog of (1.5) for the honeycomb Hamiltonian (2.17) (where the factor $1/q$ is replaced by $1/(2q)$ in view of the normalization over $2q$ states), and on the other hand

$$\text{tr} H_{2q}^{2n} = 2n(-1)^{n+1} b_n,$$

which generalizes (2.8), the expressions above directly lead to an algebraic area enumeration similar to the square lattice walks enumeration (2.13). See Appendix A for its expression.

2.2.3 Undiluted spectral functions and algebraic area enumeration

In this subsection, we consider d_q in terms of the original Hofstadter spectral functions s_k and derive an alternative algebraic area enumeration formula.

From the recursion (2.21), we can iteratively derive the honeycomb Kreft coefficients Z_n in (2.22)

but now expressed in terms of \tilde{s}_k and s_k (see Appendix 4 of Article 1 for details). Explicitly,

$$\begin{aligned}
Z_1 &= + \sum_{i=1}^q \tilde{s}_i, \\
Z_2 &= + \sum_{i=1}^{q-1} \sum_{j=1}^i \tilde{s}_{i+1} \tilde{s}_j - \sum_{i=1}^{q-1} s_i, \\
Z_3 &= + \sum_{i=1}^{q-2} \sum_{j=1}^i \sum_{k=1}^j \tilde{s}_{i+2} \tilde{s}_{j+1} \tilde{s}_k - \sum_{i=1}^{q-2} \sum_{j=1}^i \tilde{s}_{i+2} s_j - \sum_{i=1}^{q-2} \sum_{j=1}^i s_{i+1} \tilde{s}_j, \\
Z_4 &= + \sum_{i=1}^{q-3} \sum_{j=1}^i \sum_{k=1}^j \sum_{l=1}^k \tilde{s}_{i+3} \tilde{s}_{j+2} \tilde{s}_{k+1} \tilde{s}_l - \sum_{i=1}^{q-3} \sum_{j=1}^i \sum_{k=1}^j \tilde{s}_{i+3} \tilde{s}_{j+2} s_k \\
&\quad - \sum_{i=1}^{q-3} \sum_{j=1}^i \sum_{k=1}^j \tilde{s}_{i+3} s_{j+1} \tilde{s}_k - \sum_{i=1}^{q-3} \sum_{j=1}^i \sum_{k=1}^j s_{i+2} \tilde{s}_{j+1} \tilde{s}_k + \sum_{i=1}^{q-3} \sum_{j=1}^i s_{i+2} s_j.
\end{aligned}$$

Studying the above nested sums, we can infer some general rules for their structure. The Z_n are combinations of all possible nested sums of products of \tilde{s}_k and $-s_k$ distributed over all $k = 1, 2, \dots, q$ in a natural alphabetical ordering inferred from their nested indices i, j, k, \dots, r such that


- The rightmost factor is either $-s_r$ or \tilde{s}_r .
- Any factor multiplying $-s_l$ immediately on its left obeys $g = 2$ exclusion, i.e., $\sum_k \sum_l s_k s_l$ or $-\sum_k \sum_l \tilde{s}_k s_l$ where $k - l \geq 2$.
- Any factor multiplying \tilde{s}_l immediately on its left obeys $g = 1$ exclusion, i.e., $-\sum_k \sum_l s_k \tilde{s}_l$ or $\sum_k \sum_l \tilde{s}_k \tilde{s}_l$ where $k - l \geq 1$.
- The leftmost factor is either $-s_{i+n-2}$ or \tilde{s}_{i+n-1} with summation range $\sum_{i=1}^{q-(n-1)}$.

It follows that products will have n_1 factors \tilde{s}_l and n_2 factors $-s_l$ such that $n_1 + 2n_2 = n$.

These rules admit a simple physical interpretation: consider a system of one-body levels $k = 1, 2, \dots, q$ with fermions in level k having Boltzmann factor \tilde{s}_k and two-fermion bound states⁴ in levels $k, k + 1$ having Boltzmann factor $-s_k$. Then, Z_n is the n -fermion partition function with all possible bound states. The honeycomb lattice secular determinant can, therefore, be described as the grand partition function of a mixture of $g = 1$ and $g = 2$ exclusion particles.

From these rules and the definition of the cluster coefficients b_n via

$$\ln \left(\sum_{n=0}^q Z_n x^n \right) = \sum_{n=1}^{\infty} b_n x^n,$$

⁴A two-fermion bound state in levels $k, k + 1$ refers to a bound state that consists of two fermions, with one occupying level k and the other occupying level $k + 1$, or graphically, . According to the Pauli exclusion principle, no two fermions can occupy the same level, thus another two-fermion bound state cannot overlap the existing bound state. Therefore, the two-fermion bound states behave effectively as $g = 2$ exclusion particles.

we get the b_n in terms of single sums of products of s_k (up to terms involving s_q which vanish anyway; see Appendix B for details) with a form a bit more complicated than in the Hofstadter case⁵

$$\begin{aligned}
b_1 &= \sum_{k=1}^{q-1} s_k + \sum_{k=1}^q s_k^0, \\
-b_2 &= \frac{1}{2} \sum_{k=1}^{q-1} s_k^2 + 2 \sum_{k=1}^{q-1} s_k + \frac{1}{2} \sum_{k=1}^q s_k^0, \\
b_3 &= \frac{1}{3} \sum_{k=1}^{q-1} s_k^3 + 2 \sum_{k=1}^{q-1} s_k^2 + \sum_{k=1}^{q-2} s_k s_{k+1} + 3 \sum_{k=1}^{q-1} s_k + \frac{1}{3} \sum_{k=1}^q s_k^0, \\
-b_4 &= \frac{1}{4} \sum_{k=1}^{q-1} s_k^4 + 2 \sum_{k=1}^{q-1} s_k^3 + \sum_{k=1}^{q-2} s_k s_{k+1}^2 + \sum_{k=1}^{q-2} s_k^2 s_{k+1} + 5 \sum_{k=1}^{q-1} s_k^2 + 4 \sum_{k=1}^{q-2} s_k s_{k+1} + 4 \sum_{k=1}^{q-1} s_k + \frac{1}{4} \sum_{k=1}^q s_k^0,
\end{aligned}$$

etc. Note that the b_n in (2.23) are now expressed in terms of s_k . We infer, in general, that

$$b_n = (-1)^{n+1} \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n'=0,1,2,\dots,n \\ j \leq \min(n', n-n'+1)}} c_n(l_1, l_2, \dots, l_j) \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}, \quad (2.24)$$

where the new n -dependent combinatorial coefficients $c_n(l_1, l_2, \dots, l_j)$

$$\begin{aligned}
c_n(0) &= \frac{1}{n}, \quad c_n(l_1) = \frac{1}{l_1} \binom{n+l_1-1}{2l_1-1}, \\
c_n(l_1, l_2, \dots, l_j) &= \frac{1}{l_1 l_2 \cdots l_j} \sum_{m_1=0}^{\min(l_1, l_2)} \sum_{m_2=0}^{\min(l_2, l_3)} \cdots \sum_{m_{j-1}=0}^{\min(l_{j-1}, l_j)} \left(\prod_{i=1}^{j-1} m_i \binom{l_i}{m_i} \binom{l_{i+1}}{m_i} \right) \binom{n + \sum_{i=1}^j l_i - \sum_{i=1}^{j-1} m_i - 1}{2 \sum_{i=1}^j l_i - 1}. \quad (2.25)
\end{aligned}$$

are labeled by the compositions of $n' = 0, 1, 2, \dots, n$ with a number of parts $j \leq \min(n', n - n' + 1)$. By convention, the unique composition of $n' = 0$ has only one part and the trigonometric sum becomes $\sum_{k=1}^q s_k^0$. For example, when $n = 5$, the sequences (l_1, l_2, \dots, l_j) corresponding to the summation index in (2.24) are: (0), (1), (2), (1, 1), (3), (2, 1), (1, 2), (1, 1, 1), (4), (3, 1), (2, 2), (1, 3), (5).

⁵It is also possible to rewrite b_n in terms of the Boltzmann factors of fermions \tilde{s}_k and bound states $-s_k$. For example,

$$\begin{aligned}
b_1 &= \sum_{k=1}^q \tilde{s}_k, & -b_2 &= \frac{1}{2} \sum_{k=1}^q \tilde{s}_k^2 + \sum_{k=1}^{q-1} s_k, & b_3 &= \frac{1}{3} \sum_{k=1}^q \tilde{s}_k^3 + \sum_{k=1}^{q-1} \tilde{s}_k s_k + \sum_{k=1}^{q-1} s_k \tilde{s}_{k+1}, \\
-b_4 &= \frac{1}{4} \sum_{k=1}^q \tilde{s}_k^4 + \sum_{k=1}^{q-1} \tilde{s}_k^2 s_k + \sum_{k=1}^{q-1} \tilde{s}_k s_k \tilde{s}_{k+1} + \sum_{k=1}^{q-1} s_k \tilde{s}_{k+1}^2 + \sum_{k=1}^{q-2} s_k s_{k+1} + \frac{1}{2} \sum_{k=1}^{q-1} s_k^2.
\end{aligned}$$

We will discuss this in Section 3.3.

2.2.4 Algebraic area enumeration on the honeycomb lattice

Following the steps in Section 2.1 and remembering that the spectrum of H_q is the square of that of the honeycomb Hamiltonian H_{2q} , the generating function for the number $C_{2n}(A)$ of closed walks of length $2n$ enclosing an algebraic area A can as well be given in terms of the matrix trace of H_q^n weighted by $1/q$, i.e.,

$$\sum_A C_{2n}(A) Q^A = \mathbf{Tr} H_q^n = \frac{1}{q} \text{tr} H_q^n,$$

where

$$\text{tr} H_q^n = n(-1)^{n+1} b_n.$$

We arrive at the conclusion that on the honeycomb lattice the $C_{2n}(A)$'s are

$$\begin{aligned} C_{2n}(A) = n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n'=0,1,2,\dots,n \\ j \leq \min(n', n-n'+1)}} c_n(l_1, l_2, \dots, l_j) \sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \cdots \sum_{k_j=-l_j}^{l_j} \left(l_1 + A + \sum_{i=3}^j (i-2)k_i \right) \\ \times \left(l_2 - A - \sum_{i=3}^j (i-1)k_i \right) \prod_{i=3}^j \binom{2l_i}{l_i + k_i} \end{aligned} \quad (2.26)$$

with the $c_n(l_1, l_2, \dots, l_j)$'s given in (2.25) and the algebraic area bounded⁶ by $\lfloor (n^2+3)/12 \rfloor$. We present some enumeration examples in Table 2.2.

	$2n = 2$	4	6	8	10	12	14	16	18
$A = 0$	3	15	87	543	3543	23859	164769	1162719	8363895
± 1			6	96	1080	10560	96096	839040	7143210
± 2					30	726	11130	138720	1537668
± 3						24	798	15648	237714
± 4							42	1536	33246
± 5								96	3834
± 6									252
± 7									18
Total count	3	15	93	639	4653	35169	272835	2157759	17319837

Table 2.2: $C_{2n}(A)$ up to $2n = 18$ for closed honeycomb lattice walks of length $2n$.

We note that since the number of compositions of n' with j parts is $\binom{n'-1}{j-1}$, the total number of compositions of $n' = 0, 1, 2, \dots, n$ with a number of parts $j \leq \min(n', n - n' + 1)$ appearing in (2.24) is

$$1 + \sum_{n'=1}^n \sum_{j=1}^{\min(n', n-n'+1)} \binom{n'-1}{j-1} = 1 + \sum_{j=1}^{\lfloor (n+1)/2 \rfloor} \sum_{n'=j}^{n-j+1} \binom{n'-1}{j-1} = \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-j+1}{j} = F_{n+2},$$

⁶The minimal perimeter of a polyhex with A cells is $2\lceil \sqrt{12A-3} \rceil$ [41]. The maximum A for walks of length $2n$ is then $\lfloor (n^2+3)/12 \rfloor$.

where F_n refers to the n -th Fibonacci number. Also note that by ignoring the n -dependent binomial $\binom{n+\sum_{i=1}^j l_i - \sum_{i=1}^{j-1} m_i - 1}{2\sum_{i=1}^j l_i - 1}$ in the sums (2.25), we recover the $c(l_1, l_2, \dots, l_j)$ in (2.10), that is, thanks to the identity

$$\frac{1}{l_1 l_2} \sum_{m=0}^{\min(l_1, l_2)} m \binom{l_1}{m} \binom{l_2}{m} = \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2},$$

we have

$$\frac{1}{l_1 l_2 \cdots l_j} \sum_{m_1=0}^{\min(l_1, l_2)} \sum_{m_2=0}^{\min(l_2, l_3)} \cdots \sum_{m_{j-1}=0}^{\min(l_{j-1}, l_j)} \prod_{i=1}^{j-1} m_i \binom{l_i}{m_i} \binom{l_{i+1}}{m_i} = \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2} l_2 \frac{\binom{l_2+l_3}{l_2}}{l_2+l_3} \cdots l_{j-1} \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1}+l_j}.$$

We also find

$$n \sum_{l=0}^n c_n(l) = F_{2n+1} + F_{2n-1} - 1,$$

where again a Fibonacci counting appears, and

$$n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n' \\ j \leq \min(n', n-n'+1)}} c_n(l_1, l_2, \dots, l_j) = \binom{n}{n'}^2,$$

from which we infer

$$n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n'=0,1,2,\dots,n \\ j \leq \min(n', n-n'+1)}} c_n(l_1, l_2, \dots, l_j) = \binom{2n}{n}.$$

Last but not least, again using (2.15), the total number of closed honeycomb lattice walks of length $2n$ is recovered as

$$\begin{aligned} & n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n'=0,1,2,\dots,n \\ j \leq \min(n', n-n'+1)}} c_n(l_1, l_2, \dots, l_j) \binom{2(l_1 + l_2 + \cdots + l_j)}{l_1 + l_2 + \cdots + l_j} \\ &= \sum_{n'=0}^n \left(n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n' \\ j \leq \min(n', n-n'+1)}} c_n(l_1, l_2, \dots, l_j) \binom{2(l_1 + l_2 + \cdots + l_j)}{l_1 + l_2 + \cdots + l_j} \right) \\ &= \sum_{n'=0}^n \binom{n}{n'}^2 \binom{2n'}{n'}, \end{aligned}$$

as expected⁷.

⁷By setting $Q = 1$, the total number $\sum_A C_{2n}(A)$ can also be derived from the u, v -independent part in the expansion of $H_q^n = [(u + v + v u^{-1})(u^{-1} + v^{-1} + u v^{-1})]^n = (1 + u^{-1}v + u^{-2}v)^n (1 + u v^{-1} + u^2 v^{-1})^n = \sum_{n'=0}^n \sum_{i=0}^{n'} \sum_{j=0}^n \sum_{k=0}^j \binom{n}{n'} \binom{n'}{i} \binom{n}{j} \binom{j}{k} u^{-2n'+i+2j-k} v^{n'-j}$. Setting $-2n' + i + 2j - k = n' - j = 0$, i.e., $i = k$ and $j = n'$ yields $\sum_{n'=0}^n \binom{n}{n'}^2 \sum_{k=0}^{n'} \binom{n'}{k}^2 = \sum_{n'=0}^n \binom{n}{n'}^2 \binom{2n'}{n'}$, where $\sum_{k=0}^{n'} \binom{n'}{k}^2 = \sum_{k=0}^{n'} \binom{n'}{k} \binom{n'}{n'-k} = \binom{2n'}{n'}$ by the Chu-Vandermonde identity. The total number was also obtained by another approach in [42].

2.3 Conclusion and perspectives

We reviewed the algebraic area enumeration of random walks on the square lattice and obtained analogous results for the honeycomb lattice. We demonstrated that the area counting of honeycomb lattice walks derives from a $g = 2$ exclusion statistics system with a “diluted Hofstadter” spectrum, or, alternatively, from a mixture of $g = 1$ (fermions) and $g = 2$ exclusion statistics.

There are several directions for potential future investigations. An immediate area of interest within honeycomb lattice walks involves exploring the generating function for $\text{tr } H_{2q}^{2n}$ and the asymptotic behavior of $C_{2n}(A)$ in (2.26) as the walk length $2n \rightarrow \infty$. See [43, 40] for square lattice walks.

It is shown in [2] that a $g = 2$ Hamiltonian can be written in the form $F(u)v + v^{-1}G(u)$, where $F(u)$ and $G(u)$ are scalar functions of u . The honeycomb Hamiltonian H_{2q} is apparently not of this form. However, the expression of a walk in terms of a Hamiltonian is not unique: Alternative versions corresponding to modular transformations on the lattice, or, equivalently, alternative realizations of u and v , can exist. We expect that an alternative realization of the honeycomb Hamiltonian H_{2q} that makes its connection to $g = 2$ statistics and the diluted spectral function S_k manifest does exist and is related to the form given in Section 2.2.1 by a unitary transformation. The identification of this transformation and the alternative form of H_{2q} is an interesting open question.

Further, an anisotropic Hamiltonian with general hopping amplitudes, e.g., $H = aU + bV + cW$ for honeycomb Hamiltonian, is of physical, but also mathematical, interest. The corresponding generating function of lattice walks would depend on these parameters and would “count” the number of moves in the three different lattice directions U, V , and W separately. The calculation of this generalized generating function through traces of powers of the Hamiltonian appears to be within reach using the methods and techniques discussed in this chapter and constitutes a subject for further investigation.

Article 1: Algebraic area enumeration of random walks on the honeycomb lattice


Li Gan, Stéphane Ouvry, and Alexios P. Polychronakos
Phys. Rev. E **105**, 014112 (2022)

Algebraic area enumeration of random walks on the honeycomb lattice

Li Gan^{1,*}, Stéphane Ouvry^{1,†} and Alexios P. Polychronakos^{2,‡}

¹*LPTMS, CNRS, Université Paris-Sud, Université Paris-Saclay, 91405 Orsay Cedex, France*

²*Physics Department, the City College of New York, New York 10031, USA
and The Graduate Center of CUNY, New York, New York 10016, USA*

 (Received 25 July 2021; revised 3 November 2021; accepted 3 December 2021; published 13 January 2022)

We study the enumeration of closed walks of given length and algebraic area on the honeycomb lattice. Using an irreducible operator realization of honeycomb lattice moves, we map the problem to a Hofstadter-like Hamiltonian and show that the generating function of closed walks maps to the grand partition function of a system of particles with exclusion statistics of order $g = 2$ and an appropriate spectrum, along the lines of a connection previously established by two of the authors. Reinterpreting the results in terms of the standard Hofstadter spectrum calls for a mixture of $g = 1$ (fermion) and $g = 2$ exclusion particles whose properties merit further studies. In this context we also obtain some unexpected Fibonacci sequences within the weights of the combinatorial factors appearing in the counting of walks.

DOI: [10.1103/PhysRevE.105.014112](https://doi.org/10.1103/PhysRevE.105.014112)

I. INTRODUCTION

Random walks on lattices emerge in the study of various problems of physical interest. The dynamics of electrons (or quasiparticles) on an atomic lattice can be well approximated by their hopping to the ground-state levels of different atoms in the lattice: Hopping to excited states would introduce extra effective discrete degrees of freedom but such transitions are generally energetically suppressed; likewise, hopping to atoms beyond the few near neighbors of the atom presently binding the electron is also suppressed. As a consequence, the entire dynamical process can be described by a random lattice walk. Percolation processes, Brownian-type diffusion processes, and various other statistical processes can also be modeled as random walks.

The algebraic area enumeration of closed random walks on two-dimensional lattices is a topic with rich mathematical and physical implications. Indeed, it is well known that the algebraic area of a walk introduces in the quantum case an interaction of the particle performing the walk with a constant magnetic field perpendicular to the plane of motion. The algebraic area is defined as the total oriented area spanned by the walk as it traces the lattice. A unit lattice cell enclosed in a counterclockwise (positive) direction has an area $+1$, whereas when enclosed in a clockwise (negative) direction it has an area -1 . The total algebraic area is the area enclosed by the walk weighted by its winding number: If the walk winds around more than once, then the area is counted with multiplicity. Figure 1 represents examples of closed random walks on the square, triangular, and honeycomb lattices.

In the case of the square lattice, the algebraic area enumeration is embedded in the quantum dynamics of the Hofstadter model [1] which describes the motion of an electron hopping on a square lattice in a uniform perpendicular magnetic field, with its spin frozen and thus nondynamical. The generating function for the number $C_{2n}(A)$ of closed walks of length $2n$ (necessarily even) enclosing an algebraic area A is given in terms of the trace of the Hofstadter Hamiltonian H_γ ,

$$\sum_A C_{2n}(A) Q^A = \text{Tr} H_\gamma^{2n}, \quad (1)$$

where $\gamma = 2\pi\phi/\phi_0$ stands for the flux per plaquette in units of the flux quantum, $Q = e^{i\gamma}$, and H_γ is the Hofstadter Hamiltonian,

$$H_\gamma = u + u^{-1} + v + v^{-1}.$$

The unitary operators u and v are unit magnetic translations (hopping operators) in the x and y directions of the square lattice and satisfy the magnetic translations algebra

$$v u = Q u v \quad (2)$$

due to the perpendicular magnetic field piercing the lattice. Terms contributing to the trace in (1) must involve an equal number of u and u^{-1} and of v and v^{-1} . Such terms represent closed walks, each power of H_γ representing one step. Because of the commutation rules of u and v (2), the power of the total factor of Q for such walks can be seen to be equal to the algebraic area A of the walk, $v^{-1}u^{-1}vu = Q$ corresponding to a walk around an elementary plaquette. In quantum mechanics the trace becomes a sum of the expectation value of H_γ over all quantum states, with an appropriate normalization.

In Ref. [2] the question of enumerating all walks of given length and area was studied, and an explicit algebraic area enumeration was obtained in terms of a sum over compositions (that is, partitions where the order of terms matters) of the integer n which is half the walk length. In Ref. [3] and

*li.gan92@gmail.com

†stephane.ouvry@u-psud.fr

‡apolychronakos@ccny.cuny.edu

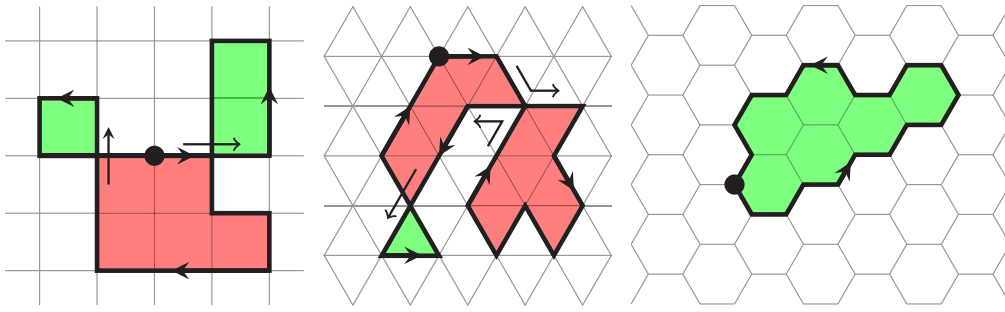


FIG. 1. Closed random walks of length $2n = 20$ on the square, triangular, and honeycomb lattice with algebraic area -2 , -12 , and 6 , respectively.

Ref. [4], an interpretation of this enumeration was given in terms of the statistical mechanics of particles obeying quantum *exclusion statistics* with exclusion parameter g ($g = 0$ for bosons, $g = 1$ for fermions, and higher g means a stronger exclusion beyond Fermi). The square lattice enumeration was found to be governed by $g = 2$ exclusion together with a Hofstadter-induced spectral function $s_k := e^{-\beta\epsilon_k}$ accounting for the one-body quantum spectrum ϵ_k . Other, different types of lattice walks were governed by higher values of g and, in general, other types of spectral functions. Explicit examples of such enumerations were given, in particular for Kreweras-like chiral walks on a triangular lattice [3], corresponding to yet another quantum Hofstadter-like model (chiral and non-Hermitian, though) and $g = 3$ exclusion. This particular chiral model is to be distinguished from the triangular lattice Hofstadter-like model originally proposed in Ref. [5]. Its butterfly structure—among other Hofstadter-like models—has been studied in Ref. [6].

A case of particular physical and mathematical interest is the honeycomb lattice. It arises naturally in the form of graphene and carbon nanotubes, and many of its quantum properties have been extensively studied (see, for example, Refs. [7–9]). The honeycomb lattice is also relevant in graph theory [10] and various physical models [11–13]. The quantum model for a particle hopping on the honeycomb lattice pierced by a perpendicular magnetic field was introduced in Refs. [14,15]. The effect of lattice defects on its spectrum was investigated in Ref. [16] and its butterfly-like spectrum was obtained in Ref. [17].

In this work we address the question of the algebraic area enumeration of closed random walks on the honeycomb lattice: Can this enumeration be explicitly obtained, and does it fall in the category described in Ref. [3] and Ref. [4], i.e., does it correspond to a system of particles with a particular exclusion statistics? We will show that, indeed, the honeycomb enumeration can be interpreted in terms of particles with $g = 2$ exclusion on a single-particle level spectrum identical to the one for the Hofstadter model, i.e., with the same spectral function, but “diluted” by additional zero-energy levels between successive levels, in a “toothcomb” pattern. Alternatively, it can be interpreted in terms of the (undiluted) Hofstadter level spectrum but with a statistical mixture of $g = 1$ and $g = 2$ exclusion particles. This last system can, in turn, be interpreted as a system of fermions with the possibility that two fermions on neighboring levels can form a bound state of modified energy. The physical properties of these systems and

the mapping between their physical observables need further exploration. As a by-product of our analysis we will obtain some unexpected Fibonacci sequences, either for the number of compositions entering the enumeration or for the sum of the coefficients weighting particular compositions, the occurrence of which remains to be better understood.

The paper is structured as follows: In Sec. II we review the Hofstadter model on the square lattice, where the coefficients of the secular determinant of the Hofstadter Hamiltonian [18] are reinterpreted in terms of $g = 2$ exclusion partition functions. The algebraic area enumeration is then obtained in terms of the associated cluster coefficients. In Sec. III we study the honeycomb lattice, establish its correspondence to an exclusion statistics system, and calculate the relevant partition functions and cluster coefficients, arriving at an explicit algebraic area enumeration expression. Some open questions and possible physical applications are exposed under Conclusions.

II. SQUARE LATTICE WALKS ALGEBRAIC AREA ENUMERATION

From now on we consider the flux γ per lattice cell to be rational, i.e., $\phi/\phi_0 = p/q$ with p and q coprime, so $Q = \exp(2i\pi p/q)$.

A. Hofstadter Hamiltonian

When the magnetic flux is rational the magnetic translations algebra has a finite-dimensional irreducible representation in which u and v are represented by the $q \times q$ matrices [19]

$$\begin{aligned}
 u &= e^{ik_x} \begin{pmatrix} Q & 0 & 0 & \dots & 0 & 0 \\ 0 & Q^2 & 0 & \dots & 0 & 0 \\ 0 & 0 & Q^3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & Q^{q-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & Q^q \end{pmatrix}, \\
 v &= e^{ik_y} \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{3}
 \end{aligned}$$

(u and v are often referred to as “clock” and “shift” matrices: quantum states correspond to the q “time” positions on a circular clock; u “reads” the time when applied on them, while v “shifts” the time by one unit); k_x and k_y are the quasimomenta (remnants of Bloch momenta) in the x and y directions and are related to the Casimirs of the u, v algebra

$$u^q = e^{iqk_x}, v^q = e^{iqk_y}.$$

The Casimirs make clear that the relevant range of k_x and k_y is $[0, 2\pi/q]$. Indeed, shifting either k_x or k_y by $2\pi/q$ in (3) amounts to performing a unitary transformation on

u, v :

$$k_x \rightarrow k_x + \frac{2\pi}{q} \Leftrightarrow u \rightarrow v^r u v^{-r},$$

$$k_y \rightarrow k_y + \frac{2\pi}{q} \Leftrightarrow v \rightarrow u^{-r} v u^r, \text{ with } rp = 1 \pmod{q}.$$

Since u^q and v^q perform translations by q lattice units in the x or the y direction, and they are set to a phase, this representation corresponds to making the lattice $q \times q$ periodic, with quantum states picking up a phase $e^{iqk_{x,y}}$ on going around each period. Because of this structure, the algebra of u, v is often called the “quantum torus” algebra, and we will refer to it by this name in the sequel.

In this representation the Hofstadter Hamiltonian becomes the $q \times q$ matrix,

$$H_q = \begin{bmatrix} Qe^{ik_x} + Q^{-1}e^{-ik_x} & e^{ik_y} & 0 & \dots & 0 & e^{-ik_y} \\ e^{-ik_y} & Q^2e^{ik_x} + Q^{-2}e^{-ik_x} & e^{ik_y} & \dots & 0 & 0 \\ 0 & e^{-ik_y} & () & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & () & e^{ik_y} \\ e^{ik_y} & 0 & 0 & \dots & e^{-ik_y} & Q^q e^{ik_x} + Q^{-q} e^{-ik_x} \end{bmatrix},$$

whose spectrum follows from the zeros of the secular determinant $\det(1 - zH_q)$, where z denotes the inverse energy. H_q has q eigenvalues, which, on varying k_x and k_y , become q bands. The evolution of these bands as the magnetic flux $2\pi p/q$ takes nearby values but with drastically different q gives rise to the fractal spectral flow known as the “Hofstadter butterfly.”

The secular determinant $\det(1 - zH_q)$ has been shown [18] to rewrite as

$$\det(1 - zH_q) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z(n) z^{2n} - 2[\cos(qk_x) + \cos(qk_y)]z^q, \tag{4}$$

where the $Z(n)$ ’s are given by the nested trigonometric sums

$$Z(n) = \sum_{k_1=0}^{q-2n} \sum_{k_2=0}^{k_1} \dots \sum_{k_n=0}^{k_{n-1}} 4 \sin^2 \left[\frac{\pi(k_1 + 2n - 1)p}{q} \right] \times 4 \sin^2 \left[\frac{\pi(k_2 + 2n - 3)p}{q} \right] \dots$$

$$\times 4 \sin^2 \left[\frac{\pi(k_{n-1} + 3)p}{q} \right] 4 \sin^2 \left[\frac{\pi(k_n + 1)p}{q} \right] \tag{5}$$

with $Z(0) = 1$.

As we shall see, $Z(n)$ in (5) is at the core of the lattice walks algebraic area enumeration. To recover (5) let us use an alternative form of the Hofstadter Hamiltonian involving a different but equivalent representation of the operators u and v , namely $-uv$ and v (this corresponds to performing a modular transformation on the lattice that leaves it invariant). They still satisfy the same quantum torus algebra

$$v(-uv) = Q(-uv)v,$$

albeit with a different Casimir $(-uv)^q = -e^{iq(k_x+k_y)}$, and lead to the new Hamiltonian

$$H'_q = -uv - (uv)^{-1} + v + v^{-1},$$

i.e.,

$$H'_q = \begin{bmatrix} 0 & (1 - Qe^{ik_x})e^{ik_y} & 0 & \dots & 0 & (1 - Q^{-q}e^{-ik_x})e^{-ik_y} \\ (1 - Q^{-1}e^{-ik_x})e^{-ik_y} & 0 & (1 - Q^2e^{ik_x})e^{ik_y} & \dots & 0 & 0 \\ 0 & (1 - Q^{-2}e^{-ik_x})e^{-ik_y} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & (1 - Q^{(q-1)}e^{ik_x})e^{ik_y} \\ (1 - Q^q e^{ik_x})e^{ik_y} & 0 & 0 & \dots & (1 - Q^{-(q-1)}e^{-ik_x})e^{-ik_y} & 0 \end{bmatrix},$$

or, denoting $\omega(k) = (1 - Q^k e^{ik_x})e^{iky}$,

$$H'_q = \begin{bmatrix} 0 & \omega(1) & 0 & \cdots & 0 & \bar{\omega}(q) \\ \bar{\omega}(1) & 0 & \omega(2) & \cdots & 0 & 0 \\ 0 & \bar{\omega}(2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \omega(q-1) \\ \omega(q) & 0 & 0 & \cdots & \bar{\omega}(q-1) & 0 \end{bmatrix}.$$

Its secular determinant is the same as that of H_q given in (4) but for the new Casimirs, that is,

$$\begin{aligned} \det(1 - zH'_q) &= \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z(n) z^{2n} - \left[\prod_{j=1}^q \omega(j) + \prod_{j=1}^q \bar{\omega}(j) \right] z^q \\ &= \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z(n) z^{2n} - 2[\cos(qk_y) - \cos(qk_x + qk_y)] z^q. \end{aligned} \tag{6}$$

Let us set $\omega(q) = 0$, which makes the cosine term in (6) vanish and the matrix H'_q tridiagonal,

$$H'_q|_{\omega(q)=0} = \begin{bmatrix} 0 & (1 - Q^{1-q})e^{ik_y} & 0 & \cdots & 0 & 0 \\ (1 - Q^{q-1})e^{-ik_y} & 0 & (1 - Q^{2-q})e^{ik_y} & \cdots & 0 & 0 \\ 0 & (1 - Q^{q-2})e^{-ik_y} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & (1 - Q^{-1})e^{ik_y} \\ 0 & 0 & 0 & \cdots & (1 - Q)e^{-ik_y} & 0 \end{bmatrix}.$$

This form provides an iterative procedure for calculating the $Z(n)$'s. Putting aside for a moment that $Q = \exp(2i\pi p/q)$ and leaving it as a free parameter, independent of q , we introduce the spectral function

$$s_k = (1 - Q^k)(1 - Q^{-k}), \quad k = 1, 2, \dots, q. \tag{7}$$

Denoting the secular determinant $\det[1 - zH'_q|_{\omega(q)=0}] = d_q$, its expansion in terms of the first row yields

$$d_q = d_{q-1} - z^2 s_{q-1} d_{q-2}, \quad q \geq 2, \tag{8}$$

where, by convention, $d_0 = d_1 = 1$. Expanding d_q as a polynomial in z and solving the corresponding recursion relation for its coefficients, we obtain (see subsection 1 in the Appendix)

$$Z(n) = \sum_{k_1=1}^{q-2n+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+2n-2} s_{k_2+2n-4} \cdots s_{k_{n-1}+2} s_{k_n}, \tag{9}$$

which, on restoring Q to its actual value $\exp(2i\pi p/q)$, i.e., the spectral function s_k to its actual form $s_k = 4 \sin^2(\pi k p/q)$, gives (5).

The recursion (8) is at the root of the connection between square lattice walks and $g = 2$ exclusion statistics. Interpreting the spectral function s_k as the Boltzmann factor for a one-body level $e^{-\beta\epsilon_k}$ and $-z^2$ as the fugacity $x = e^{\beta\mu}$, (8) can be interpreted as an expansion of a grand partition function \mathcal{Z}_{q-1} —here identified with d_q —of noninteracting particles in $q - 1$ quantum levels $\epsilon_1, \dots, \epsilon_{q-1}$, obeying the exclusion principle that no two particles can occupy adjacent levels, namely

$$\mathcal{Z}_{q-1} = \mathcal{Z}_{q-2} + x s_{q-1} \mathcal{Z}_{q-3}$$

in terms of the last level ϵ_{q-1} being empty (first term) or occupied (second term). Then (6) identifies $Z(n)$ as the n -body partition function for particles occupying these $q - 1$ quantum states, with gaps of 2 between successive terms reproducing $g = 2$ exclusion.

B. Algebraic area enumeration on the square lattice

As already stressed, when $Q = \exp(2i\pi p/q)$ the algebraic area counting (1),

$$\sum_A C_{2n}(A) Q^A = \frac{1}{q} \text{Tr} H_q^{2n}, \tag{10}$$

involves a trace over a finite number q of quantum states. To normalize the contribution of each walk to Q^A and reproduce the left-hand side of (10), a factor of $1/q$ must be included in the normalization. Also, when $2n \geq q$ the trace involves extra terms arising from the Casimirs k_x, k_y similarly to the cosine terms in (4), corresponding to open walks that close only up to periods (q, q) on the lattice (“umklapp” on the quantum torus). These spurious contributions can be eliminated by integrating the Casimirs k_x and k_y over $[0, 2\pi]$ which makes all factors of e^{iqk_x} and e^{iqk_y} vanish. So the definition of the trace in (10) is

$$\text{Tr} H_q^{2n} = \frac{1}{(2\pi)^2} \int_0^{2\pi} dk_x \int_0^{2\pi} dk_y \text{tr} H_q^{2n},$$

which corresponds to summing over the q bands of the spectrum and over the scattering states labeled by k_x, k_y , in a continuum normalization (we harmlessly extended the range of k_x, k_y to the full interval $[0, 2\pi]$ to simplify the expression).

To relate this trace to the $Z(n)$'s in (5) or, equivalently, in (9), we make use of the fact that $\det(1 - zH_q)$ is interpreted

as a grand partition function and the $Z(n)$ as n -body partition functions. These lead to cluster coefficients $b(n)$ defined via the expansion of the grand potential

$$\log \left[\sum_{n=0}^{\infty} Z(n)x^n \right] = \sum_{n=1}^{\infty} b(n)x^n \quad (11)$$

with x the fugacity. Using the identity

$$\log \det(1 - zH_q) = \text{tr} \log(1 - zH_q) = - \sum_{n=1}^{\infty} \frac{z^n}{n} \text{tr} H_q^n,$$

setting $x = -z^2$ in (11), keeping in mind that trivially $\text{tr} H_q^{2n+1} = 0$, and comparing the two expressions we reach the conclusion [2,3] that the trace in (10) for $2n$ is nothing but the cluster coefficient $b(n)$ up to a trivial factor

$$\text{Tr} H_q^{2n} = 2n(-1)^{n+1}b(n). \quad (12)$$

The cluster coefficients can be directly calculated in terms of the spectral function. One obtains

$$b(n) = (-1)^{n+1} \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} c(l_1, l_2, \dots, l_j) \sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}, \quad (13)$$

where the $c(l_1, l_2, \dots, l_j)$'s are labeled by the compositions of the integer n with

$$c(l_1, l_2, \dots, l_j) = \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2} l_2 \frac{\binom{l_2+l_3}{l_2}}{l_2+l_3} \cdots l_{j-1} \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1}+l_j}. \quad (14)$$

Further, the trigonometric sums $\frac{1}{q} \sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}$ can also be computed [2,4]

$$\begin{aligned} \frac{1}{q} \sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1} &= \sum_{A=-\infty}^{\infty} \cos \left(\frac{2A\pi p}{q} \right) \sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \cdots \sum_{k_j=-l_j}^{l_j} \left(l_1 + A + \sum_{i=3}^j (i-2)k_i \right) \left(l_2 - A - \sum_{i=3}^j (i-1)k_i \right) \\ &\times \prod_{i=3}^j \binom{2l_i}{l_i + k_i}. \end{aligned} \quad (15)$$

Using (12), (13), (14), and (15), we deduce the desired algebraic area counting

$$\sum_A C_{2n}(A)Q^A = \frac{1}{q} \text{Tr} H_q^{2n} = 2n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} c(l_1, l_2, \dots, l_j) \frac{1}{q} \sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1},$$

i.e.,

$$\begin{aligned} C_{2n}(A) &= 2n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2} l_2 \frac{\binom{l_2+l_3}{l_2}}{l_2+l_3} \cdots l_{j-1} \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1}+l_j} \sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \cdots \sum_{k_j=-l_j}^{l_j} \left(l_1 + A + \sum_{i=3}^j (i-2)k_i \right) \\ &\times \left(l_2 - A - \sum_{i=3}^j (i-1)k_i \right) \prod_{i=3}^j \binom{2l_i}{l_i + k_i}. \end{aligned} \quad (16)$$

We also note that, since

$$\sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} c(l_1, l_2, \dots, l_j) = \frac{\binom{2n}{n}}{2n},$$

and, when $q \rightarrow \infty$ [2,3],

$$\frac{1}{q} \sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1} \rightarrow \binom{2(l_1 + l_2 + \dots + l_j)}{l_1 + l_2 + \dots + l_j}, \quad (17)$$

the overall closed square lattice walks counting

$$\begin{aligned} 2n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} c(l_1, l_2, \dots, l_j) \binom{2(l_1 + l_2 + \dots + l_j)}{l_1 + l_2 + \dots + l_j} \\ = \binom{2n}{n}^2 \end{aligned}$$

is recovered as it should (see subsection 2 in the Appendix for some enumeration examples).

III. HONEYCOMB LATTICE WALKS ALGEBRAIC AREA ENUMERATION

We saw that the algebraic area enumeration of walks on the square lattice is directly related to the quantum mechanics of an underlying particle system and invokes statistical mechanical concepts that put the calculations and results in a physical context. We plan to follow the same route to obtain an explicit algebraic area enumeration for closed walks on the honeycomb lattice.

A. Honeycomb Hamiltonian

Consider a particle hopping on a honeycomb lattice pierced by a constant magnetic field (see Fig. 2). The honeycomb lattice is bipartite and each individual hop moves the particle to a site of the other part, so unitary operators representing such translations act off-diagonally in the two sublattices. This also means that we can define a unique hopping operator for each of the three orientations of links, irrespective of the direction of the move, since the action of such operators is uniquely determined by the sublattice on which they act. Therefore, we define three operators U , V , and W generating the hops in each direction and such that when the particle hops around a honeycomb cell it picks up a phase Q due to the magnetic field. They satisfy the ‘‘honeycomb algebra’’

$$U^2 = V^2 = W^2 = 1, (UVW)^2 = Q. \quad (18)$$

U , V , and W are both unitary and Hermitian. The Hofstadter-like Hamiltonian follows as

$$H_{\text{honeycomb}} = aU + bV + cW,$$

with $a, b, c \in \mathbb{R}^+$ transition amplitudes. The physical Hilbert space consists of the irreducible representations of the honeycomb algebra. As in the square lattice case, the quasimomenta are encoded in the Casimirs of the algebra.

In the case of a rational flux $Q = \exp(2i\pi p/q)$ with p and q coprime, the irreducible representation of U , V , and W for generic quasimomenta (Casimirs) becomes $2q$ -dimensional

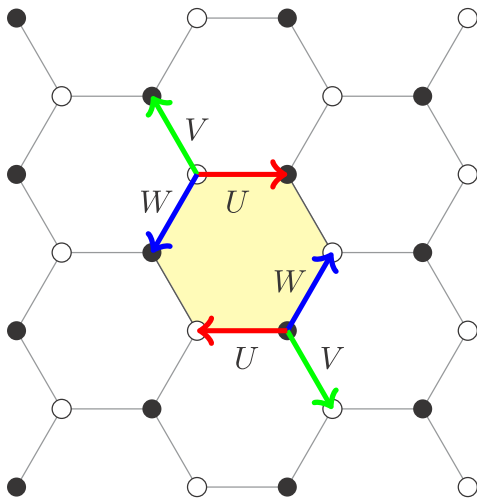


FIG. 2. Hopping operators U , V , and W on the honeycomb lattice with $U^2 = V^2 = W^2 = 1$ and $(UVW)^2 = Q$.

(see subsection 3 in the Appendix) and can be realized as

$$U = \begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & v \\ v^{-1} & 0 \end{pmatrix},$$

$$W = \begin{pmatrix} 0 & Q^{1/2}vu^{-1} \\ Q^{-1/2}uv^{-1} & 0 \end{pmatrix} \quad (19)$$

with u, v given in (3) and $Q^{\pm 1/2}$ understood to stand for $\exp(\pm i\pi p/q)$. The Casimirs of this algebra for rational flux can be written as

$$C_1 := (UV)^q + (VU)^q = -2(-1)^q \cos[q(k_x - k_y)],$$

$$C_2 := (VW)^q + (WV)^q = 2(-1)^p \cos(qk_x),$$

$$C_3 := (WU)^q + (UW)^q = 2(-1)^p \cos[q(k_y - 2k_x)],$$

where the second expressions evaluate them in the specific realization (19).

C_1 , C_2 , and C_3 are not independent, but satisfy

$$C_1^2 + C_2^2 + C_3^2 + (-1)^q C_1 C_2 C_3 = 4,$$

leading to two independent Casimirs, as expected for a two-dimensional lattice, encoded in the phases u^q, v^q . From the definitions of U , V , and W in Fig. 2 we see that C_2 generates translations by one plaquette width in the vertical direction (up or down depending on the sublattice), while C_1 and C_3 generate translations in directions at angles $\pm 2\pi/3$ from the vertical. $C_2 = 2(-1)^p \cos(qk_x)$ then indicates that k_x is actually the pseudomomentum in the vertical direction, whereas the values of C_1 and C_3 imply that $3k_x - 2k_y$ is the pseudomomentum in the horizontal direction.

For an isotropic lattice, $a = b = c = 1$, the honeycomb Hamiltonian reduces to

$$H_{2q} = \begin{pmatrix} 0 & u+v+Q^{1/2}vu^{-1} \\ u^{-1}+v^{-1}+Q^{-1/2}uv^{-1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}. \quad (20)$$

As expected, it is block off-diagonal. Its square, however, is block-diagonal

$$H_{2q}^2 = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} H_q & 0 \\ 0 & \tilde{H}_q \end{pmatrix},$$

where $H_q = AA^\dagger$ and $\tilde{H}_q = A^\dagger A$ have identical spectra equal to the square of the honeycomb Hamiltonian spectrum. Denoting

$$\omega(k) = Q^{-k} (1 + e^{-ik_x} Q^{\frac{1}{2}-k}) e^{-i(k_x - k_y)},$$

H_q can be rewritten as

$$H_q = \begin{bmatrix} 1 + \omega(2)\bar{\omega}(2) & \omega(2) & 0 & \dots & 0 & \bar{\omega}(1) \\ \bar{\omega}(2) & 1 + \omega(3)\bar{\omega}(3) & \omega(3) & \dots & 0 & 0 \\ 0 & \bar{\omega}(3) & () & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & () & \omega(q) \\ \omega(1) & 0 & 0 & \dots & \bar{\omega}(q) & 1 + \omega(1)\bar{\omega}(1) \end{bmatrix} \quad (21)$$

with secular determinant

$$\begin{aligned} \det(1 - zH_{2q}) &= \det(1 - z^2H_q) \\ &= \sum_{n=0}^q (-1)^n Z(n) z^{2n} + \left[(-1)^q \prod_{j=1}^q \omega(j)\bar{\omega}(j) - \prod_{j=1}^q \omega(j) - \prod_{j=1}^q \bar{\omega}(j) \right] z^{2q} \\ &= \sum_{n=0}^q (-1)^n Z(n) z^{2n} + 2\{(-1)^p [\cos(qk_y - 2qk_x) + \cos(qk_x)] + (-1)^q [\cos(qk_y - qk_x) + 1]\} z^{2q}. \end{aligned} \quad (22)$$

B. Honeycomb coefficients $Z(n)$

Our aim is to find for the $Z(n)$ in (22) an expression analogous to the one in (5) or (9) obtained in the Hofstadter case. To this end, we reduce the honeycomb matrix (21) to a tridiagonal form by making both corners $\omega(1)$ and $\bar{\omega}(1)$ vanish, i.e., by

setting $e^{-ik_x} = -Q^{\frac{1}{2}}$ so that $\omega(k)$ becomes

$$\omega(k)|_{\omega(1)=0} = -Q^{\frac{1}{2}-k} (1 - Q^{1-k}) e^{ik_y},$$

and

$$H_q \Big|_{\omega(1)=0} = \begin{bmatrix} 1 + (1-Q^{-1})(1-Q) & -Q^{-\frac{3}{2}}(1-Q^{-1})e^{ik_y} & 0 & \dots & 0 & 0 \\ -Q^{\frac{3}{2}}(1-Q)e^{-ik_y} & 1 + (1-Q^{-2})(1-Q^2) & -Q^{-\frac{5}{2}}(1-Q^{-2})e^{ik_y} & \dots & 0 & 0 \\ 0 & -Q^{\frac{5}{2}}(1-Q^2)e^{-ik_y} & () & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & () & -Q^{\frac{1}{2}-q}(1-Q^{-(q-1)})e^{ik_y} \\ 0 & 0 & 0 & \dots & -Q^{q-\frac{1}{2}}(1-Q^{q-1})e^{-ik_y} & 1 + (1-Q^{-q})(1-Q^q) \end{bmatrix}.$$

This also eliminates the z^{2q} umklapp term in (22), i.e., the secular determinant reduces to

$$\det[1 - z^2H_q|_{\omega(1)=0}] = \sum_{n=0}^q (-1)^n Z(n) z^{2n}.$$

This has again the suggestive form of a grand partition function, with $Z(n)$ the n -body partition function and $x = -z^2$ the fugacity. However, the analogy with the Hofstadter (square lattice) case is imperfect, since H_q in (21) has a nonvanishing diagonal. As a result, the exclusion statistics connection is not straightforward. Nevertheless, we will proceed as before: We will consider Q as a free parameter and denote $d_q = \det[1 - z^2H_q|_{\omega(1)=0}]$. Then expanding d_q in terms of its bottom row we obtain the recursion relation

$$d_q = \{1 - [1 + (1 - Q^q)(1 - Q^{-q})]z^2\}d_{q-1} - z^4(1 - Q^{q-1}) \times (1 - Q^{-(q-1)})d_{q-2}, \quad q \geq 1,$$

i.e.,

$$d_q = [1 - (1 + s_q)z^2]d_{q-1} - z^4s_{q-1}d_{q-2}, \quad (23)$$

with $d_0 = 1$, $d_j = 0$ for $j < 0$, and s_k as in (7). From (23) we can iteratively derive the $Z(n)$'s in (22) (see subsection 4 in the Appendix).

The recursion relation (23) is distinct from (8) but still admits a simple $g = 2$ exclusion statistics interpretation. Consider a set of $2q$ energy levels with spectral parameters S_r , $r = 1, 2, \dots, 2q$ given by

$$S_{2k-1} = 1, \quad S_{2k} = s_k,$$

that is, s_k ‘‘diluted’’ by unit insertions: $1, s_1, 1, s_2, \dots, 1, s_q$, and consider the grand partition function of $g = 2$ exclusion particles in the above spectrum with fugacity parameter x . Calling \mathcal{Z}_r the truncated grand partition function for levels S_1, S_2, \dots, S_r and expanding it in terms of the last level r being empty or filled, we obtain the recursion relations

$$\begin{aligned} r = 2k: \quad \mathcal{Z}_{2k} &= \mathcal{Z}_{2k-1} + xs_k\mathcal{Z}_{2k-2}, \\ r = 2k-1: \quad \mathcal{Z}_{2k-1} &= \mathcal{Z}_{2k-2} + x\mathcal{Z}_{2k-3}. \end{aligned}$$

From the $r = 2k$ relation we can express the odd functions \mathcal{Z}_{2k-1} in terms of even ones, $\mathcal{Z}_{2k-1} = \mathcal{Z}_{2k} - xs_k\mathcal{Z}_{2k-2}$. Substituting this expression in the $r = 2k - 1$ relation and rearranging we obtain

$$\mathcal{Z}_{2k} = (1 + x + xs_k)\mathcal{Z}_{2k-2} - x^2s_{k-1}\mathcal{Z}_{2k-4}.$$

This is identical to the recursion (23) on putting $x = -z^2$ and identifying $\mathcal{Z}_{2k} = d_k$. Moreover, \mathcal{Z}_{2k} satisfies the same

initial conditions as d_k , namely $Z_0 = 1$, $Z_{2k} = 0$ for $k < 0$. Therefore, $d_q = Z_{2q}$.

It follows that the expressions for the n -body partition functions $Z(n)$ and the cluster coefficients $b(n)$ are identical to the corresponding expressions (9) and (13) for square lattice walks but now, instead of the spectrum s_k , one has to consider the diluted spectrum S_k , $k = 1, \dots, 2q$ (but note that $S_{2q} = s_q = 0$, so the levels effectively end at $S_{2q-1} = 1$)

$$Z(n) = \sum_{k_1=1}^{2q-2n+2} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} S_{k_1+2n-2} S_{k_2+2n-4} \cdots S_{k_{n-1}+2} S_{k_n},$$

$$b(n) = (-1)^{n+1} \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} c(l_1, l_2, \dots, l_j) \\ \times \sum_{k=1}^{2q-j+1} S_{k+j-1}^{l_j} \cdots S_{k+1}^{l_2} S_k^{l_1}$$

with the same Hofstadter combinatorial factors $c(l_1, l_2, \dots, l_j)$ given in (14). The new diluted trigonometric sums $\frac{1}{q} \sum_{k=1}^{2q-j+1} S_{k+j-1}^{l_j} \cdots S_{k+1}^{l_2} S_k^{l_1}$ now entering the definition of the $b(n)$'s have to be computed. They can be obtained using the same tools [4] as for the usual

trigonometric sums (15) (see subsection 5 in the Appendix for an explicit expression).

Finally, following the same steps as in Sec. II B regarding the number $C_{2n}(A)$ of closed random walks of length $2n$ enclosing an algebraic area A on the honeycomb lattice, i.e., considering on the one hand

$$\sum_A C_{2n}(A) Q^A = \frac{1}{2q} \text{Tr} H_{2q}^{2n},$$

which is the analog of (10) for the honeycomb Hamiltonian (20) [where the factor $1/q$ is replaced by $1/(2q)$ in view of the normalization over $2q$ states], and on the other hand

$$\text{Tr} H_{2q}^{2n} = 2n(-1)^{n+1} b(n),$$

which generalizes (12), the expressions above directly lead to an algebraic area enumeration similar to the square lattice walks enumeration (16).

In the sequel, we will consider d_q in terms of the original (undiluted) Hofstadter spectrum s_k . In that case, the $g = 2$ exclusion interpretation does not hold anymore and has to be traded for a mixture of $g = 2$ and $g = 1$ statistics, as we are going to show in detail.

C. Modified statistics for the spectral function s_k

If we insist on keeping s_k as the spectral function, then the first few $Z(n)$ can be written in the form

$$Z(1) = + \sum_{i=1}^q (1 + s_i),$$

$$Z(2) = + \sum_{i=1}^{q-1} \sum_{j=1}^i (1 + s_{i+1})(1 + s_j) - \sum_{i=1}^{q-1} s_i,$$

$$Z(3) = + \sum_{i=1}^{q-2} \sum_{j=1}^i \sum_{k=1}^j (1 + s_{i+2})(1 + s_{j+1})(1 + s_k) - \sum_{i=1}^{q-2} \sum_{j=1}^i (1 + s_{i+2}) s_j - \sum_{i=1}^{q-2} \sum_{j=1}^i s_{i+1} (1 + s_j),$$

$$Z(4) = + \sum_{i=1}^{q-3} \sum_{j=1}^i \sum_{k=1}^j \sum_{l=1}^k (1 + s_{i+3})(1 + s_{j+2})(1 + s_{k+1})(1 + s_l) - \sum_{i=1}^{q-3} \sum_{j=1}^i \sum_{k=1}^j (1 + s_{i+3})(1 + s_{j+2}) s_k \\ - \sum_{i=1}^{q-3} \sum_{j=1}^i \sum_{k=1}^j (1 + s_{i+3}) s_{j+1} (1 + s_k) - \sum_{i=1}^{q-3} \sum_{j=1}^i \sum_{k=1}^j s_{i+2} (1 + s_{j+1})(1 + s_k) + \sum_{i=1}^{q-3} \sum_{j=1}^i s_{i+2} s_j,$$

etc.

Studying the above nested sums we can infer some general rules for their structure. The $Z(n)$'s are combinations of all possible nested sums of products of $1 + s_k$ and $-s_k$ distributed over all $k = 1, 2, \dots, q$ in a natural alphabetical ordering inferred from their nested indices i, j, k, \dots, r such that

- (i) The rightmost factor is either $-s_r$ or $1 + s_r$.
- (ii) Any factor multiplying $-s_l$ immediately on its left obeys $g = 2$ exclusion, i.e., $\sum_k \sum_l s_k s_l$ or $-\sum_k \sum_l (1 + s_k) s_l$ where $k - l \geq 2$.
- (iii) Any factor multiplying $1 + s_l$ immediately on its left obeys $g = 1$ exclusion, i.e., $-\sum_k \sum_l s_k (1 + s_l)$ or $\sum_k \sum_l (1 + s_k)(1 + s_l)$ where $k - l \geq 1$.

- (iv) The leftmost factor is either $-s_{i+n-2}$ or $1 + s_{i+n-1}$ with summation range $\sum_{i=1}^{q-(n-1)}$.

It follows that products will have n_1 factors $1 + s_l$ and n_2 factors $-s_l$ such that $n_1 + 2n_2 = n$.

These rules admit a simple physical interpretation: Consider a system of one-body levels $k = 1, 2, \dots, q$ with fermions in level k having Boltzmann factor $1 + s_k$ and two-fermion bound states in levels $k, k + 1$ having Boltzmann factor $-s_k$. Then $Z(n)$ is the n -fermion partition function with all possible bound states. The two-fermion bound states behave effectively as $g = 2$ exclusion particles. The honeycomb lattice secular determinant can, therefore, be described as the grand partition function of a mixture of $g = 1$ and $g = 2$ exclusion particles.

From these rules and the definition (11) we get the $b(n)$'s in terms of single sums of products of s_k (up to terms involving s_q which vanish anyway) with a form a bit more complicated than in the Hofstadter case

$$\begin{aligned}
 b(1) &= \sum_{k=1}^{q-1} s_k + \sum_{k=1}^q s_k^0, \\
 -b(2) &= \frac{1}{2} \sum_{k=1}^{q-1} s_k^2 + 2 \sum_{k=1}^{q-1} s_k + \frac{1}{2} \sum_{k=1}^q s_k^0, \\
 b(3) &= \frac{1}{3} \sum_{k=1}^{q-1} s_k^3 + 2 \sum_{k=1}^{q-1} s_k^2 + \sum_{k=1}^{q-2} s_{k+1} s_k + 3 \sum_{k=1}^{q-1} s_k \\
 &\quad + \frac{1}{3} \sum_{k=1}^q s_k^0, \\
 -b(4) &= \frac{1}{4} \sum_{k=1}^{q-1} s_k^4 + 2 \sum_{k=1}^{q-1} s_k^3 + \sum_{k=1}^{q-2} s_{k+1}^2 s_k + \sum_{k=1}^{q-2} s_{k+1} s_k^2 \\
 &\quad + 5 \sum_{k=1}^{q-1} s_k^2 + 4 \sum_{k=1}^{q-2} s_{k+1} s_k + 4 \sum_{k=1}^{q-1} s_k + \frac{1}{4} \sum_{k=1}^q s_k^0,
 \end{aligned} \tag{24}$$

etc. Note that it is again possible to rewrite these expressions in terms of the Boltzmann factors of fermions $1+s_k$ and bound states $-s_k$; e.g.,

$$\begin{aligned}
 b(1) &= \sum_{k=1}^q (1 + s_k), \\
 -b(2) &= \frac{1}{2} \sum_{k=1}^q (1 + s_k)^2 + \sum_{k=1}^{q-1} s_k, \\
 b(3) &= \frac{1}{3} \sum_{k=1}^q (1 + s_k)^3 + \sum_{k=1}^{q-1} s_k (1 + s_k) + \sum_{k=1}^{q-1} (1 + s_{k+1}) s_k, \\
 -b(4) &= \frac{1}{4} \sum_{k=1}^q (1 + s_k)^4 + \sum_{k=1}^{q-1} s_k (1 + s_k)^2 \\
 &\quad + \sum_{k=1}^{q-1} (1 + s_{k+1}) s_k (1 + s_k) + \sum_{k=1}^{q-1} (1 + s_{k+1})^2 s_k \\
 &\quad + \sum_{k=1}^{q-2} s_{k+1} s_k + \frac{1}{2} \sum_{k=1}^{q-1} s_k^2,
 \end{aligned}$$

etc. The form of these expressions satisfy the physical interpretation discussed before since it identifies them as cluster coefficients of a mixture of $g=1$ fermions and $g=2$ bound states particles. Pure fermionic terms $(1+s_k)^n/n$ are the familiar fermion cluster coefficients, while pure $g=2$ terms (arising only for even n) are the exclusion-2 cluster coefficients found in Ref. [2]. Mixed terms consist of $g=2$ cluster terms, involving factors $-s_k$, with fermions accumulating on levels k and $k+1$ for each such factor with appropriate multiplicities.

Coming back to the algebraic area enumeration we focus on the $b(n)$'s in (24) expressed solely in terms of the s_k 's to infer in general that

$$\begin{aligned}
 b(n) &= (-1)^{n+1} \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n'=0, 1, 2, \dots, n \\ j \leq \min(n', n-n'+1)}} c_n(l_1, l_2, \dots, l_j) \\
 &\quad \times \sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}.
 \end{aligned} \tag{25}$$

The combinatorial coefficients $c_n(l_1, l_2, \dots, l_j)$ appearing in (25) are labeled by the compositions of $n' = 0, 1, 2, \dots, n$ with a number of parts $j \leq \min(n', n - n' + 1)$ (by convention the unique composition of $n' = 0$ has only one part and the trigonometric sum becomes $\sum_{k=1}^q s_k^0$). Since the number of compositions of an integer n' with j parts is $\binom{n'-1}{j-1}$, the total number of such compositions is

$$\begin{aligned}
 1 + \sum_{n'=1}^n \sum_{j=1}^{\min(n', n-n'+1)} \binom{n'-1}{j-1} &= 1 + \sum_{j=1}^{\lfloor (n+1)/2 \rfloor} \sum_{n'=j}^{n-j+1} \binom{n'-1}{j-1} \\
 &= \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-j+1}{j} \\
 &= F_{n+2}.
 \end{aligned}$$

Note that the Fibonacci number F_{n+2} is also the number of compositions of $(n+1)$ with only parts 1 and 2.

We obtain for the $c_n(l_1, l_2, \dots, l_j)$'s

$$\begin{aligned}
 c_n(l_1, l_2, \dots, l_j) &= \frac{1}{l_1 l_2 \dots l_j} \sum_{m_1=0}^{\min(l_1, l_2)} \sum_{m_2=0}^{\min(l_2, l_3)} \cdots \sum_{m_{j-1}=0}^{\min(l_{j-1}, l_j)} \\
 &\quad \times \left(\prod_{i=1}^{j-1} m_i \binom{l_i}{m_i} \binom{l_{i+1}}{m_i} \right) \\
 &\quad \times \binom{n + \sum_{i=1}^j l_i - \sum_{i=1}^{j-1} m_i - 1}{2 \sum_{i=1}^j l_i - 1},
 \end{aligned} \tag{26}$$

and also note that by ignoring the n -dependent binomial $\binom{n + \sum_{i=1}^j l_i - \sum_{i=1}^{j-1} m_i - 1}{2 \sum_{i=1}^j l_i - 1}$ in the sums (26) one recovers the $c(l_1, l_2, \dots, l_j)$ in (14), that is, thanks to the identity

$$\frac{1}{l_1 l_2} \sum_{m=0}^{\min(l_1, l_2)} m \binom{l_1}{m} \binom{l_2}{m} = \frac{\binom{l_1+l_2}{l_1}}{l_1 + l_2},$$

one has

$$\begin{aligned}
 &\frac{1}{l_1 l_2 \dots l_j} \sum_{m_1=0}^{\min(l_1, l_2)} \sum_{m_2=0}^{\min(l_2, l_3)} \cdots \sum_{m_{j-1}=0}^{\min(l_{j-1}, l_j)} \prod_{i=1}^{j-1} m_i \binom{l_i}{m_i} \binom{l_{i+1}}{m_i} \\
 &= \frac{\binom{l_1+l_2}{l_1}}{l_1 + l_2} \frac{\binom{l_2+l_3}{l_2}}{l_2 + l_3} \cdots \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1} + l_j}.
 \end{aligned}$$

We find

$$n \sum_{l=0}^n c_n(l) = F_{2n+1} + F_{2n-1} - 1,$$

where again a Fibonacci counting appears, and

$$n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n' \\ j \leq \min(n', n-n'+1)}} c_n(l_1, l_2, \dots, l_j) = \binom{n}{n'}^2,$$

from which we infer

$$n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n'=0,1,2,\dots,n \\ j \leq \min(n', n-n'+1)}} c_n(l_1, l_2, \dots, l_j) = \binom{2n}{n}.$$

Last, again using (17), the counting of closed honeycomb lattice walks of length $2n$ is, as it should, recovered as

$$\begin{aligned} n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n'=0,1,2,\dots,n \\ j \leq \min(n', n-n'+1)}} c_n(l_1, l_2, \dots, l_j) \binom{2(l_1 + l_2 + \dots + l_j)}{l_1 + l_2 + \dots + l_j} &= \sum_{n'=0}^n \left\{ n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n' \\ j \leq \min(n', n-n'+1)}} c_n(l_1, l_2, \dots, l_j) \binom{2(l_1 + l_2 + \dots + l_j)}{l_1 + l_2 + \dots + l_j} \right\} \\ &= \sum_{n'=0}^n \binom{n}{n'}^2 \binom{2n'}{n'}. \end{aligned}$$

D. Algebraic area enumeration on the honeycomb lattice

Remembering that the spectrum of H_q is the square of that of the honeycomb Hamiltonian H_{2q} , the generating function for the number $C_{2n}(A)$ of closed walks of length $2n$ enclosing an algebraic area A can as well be given in terms of the trace of H_q^n weighted by $1/q$, i.e.,

$$\sum_A C_{2n}(A) Q^A = \frac{1}{q} \text{Tr } H_q^n,$$

where now, following again the steps of Sec. II B,

$$\text{Tr } H_q^n = (-1)^{n+1} n b(n).$$

We arrive at the conclusion that on the honeycomb lattice the $C_{2n}(A)$'s are

$$\begin{aligned} C_{2n}(A) &= n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n'=0,1,2,\dots,n \\ j \leq \min(n', n-n'+1)}} c_n(l_1, l_2, \dots, l_j) \sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \dots \sum_{k_j=-l_j}^{l_j} \binom{2l_1}{l_1 + A + \sum_{i=3}^j (i-2)k_i} \binom{2l_2}{l_2 - A - \sum_{i=3}^j (i-1)k_i} \\ &\times \prod_{i=3}^j \binom{2l_i}{l_i + k_i} \end{aligned}$$

with the $c_n(l_1, l_2, \dots, l_j)$'s given in (26) and the algebraic area bounded¹ by $\lfloor (n^2 + 3)/12 \rfloor$.

A few examples of $\frac{1}{q} \text{Tr } H_q^n$ are listed below, and the corresponding $C_{2n}(A)$ are listed in Table I:

$$\frac{1}{q} \text{Tr } H_q = 3,$$

$$\frac{1}{q} \text{Tr } H_q^2 = 15,$$

$$\frac{1}{q} \text{Tr } H_q^3 = 3 \left(29 + 2 \cos \frac{2\pi p}{q} \right),$$

$$\frac{1}{q} \text{Tr } H_q^4 = 3 \left(181 + 32 \cos \frac{2\pi p}{q} \right),$$

$$\frac{1}{q} \text{Tr } H_q^5 = 3 \left(1181 + 360 \cos \frac{2\pi p}{q} + 10 \cos \frac{4\pi p}{q} \right),$$

$$\frac{1}{q} \text{Tr } H_q^6 = 3 \left(7953 + 3520 \cos \frac{2\pi p}{q} + 242 \cos \frac{4\pi p}{q} \right.$$

$$\left. + 8 \cos \frac{6\pi p}{q} \right),$$

¹The sequence OEIS A135711 states that the minimal perimeter of a polyhex with A cells is $2\lceil \sqrt{12A - 3} \rceil$. The maximum A for walks of length $2n$ is then $\lfloor (n^2 + 3)/12 \rfloor$.

TABLE I. $C_{2n}(A)$ up to $2n = 14$ for honeycomb lattice walks of length $2n$.

	$2n = 2$	4	6	8	10	12	14
$A = 0$	3	15	87	543	3543	23 859	164 769
± 1			6	96	1080	10 560	96 096
± 2					30	726	11 130
± 3						24	798
± 4							42
Total counting	3	15	93	639	4653	35 169	272 835

$$\frac{1}{q} \text{Tr} H_q^7 = 3 \left(54\,923 + 32\,032 \cos \frac{2\pi p}{q} + 3710 \cos \frac{4\pi p}{q} + 266 \cos \frac{6\pi p}{q} + 14 \cos \frac{8\pi p}{q} \right).$$

IV. CONCLUSIONS

We demonstrated that the area counting of honeycomb walks derives from an exclusion statistics $g = 2$ system with a “diluted Hofstadter” spectrum or, equivalently, from a mixture of $g = 2$ and $g = 1$ statistics. This fact calls for a more detailed justification: In previous work [3,4], two of the authors had shown that lattice walks that map to exclusion statistics are of the general form

$$H = f(u)v + v^{1-g}g(u)$$

with u, v the quantum torus matrices and $f(u), g(u)$ scalar functions. The honeycomb Hamiltonian is apparently not of this form. However, the expression of a walk in terms of a Hamiltonian is not unique: Alternative versions corresponding to modular transformations on the lattice, or, equivalently, alternative realizations of the quantum torus algebra, can exist. We expect that an alternative realization of the honeycomb Hamiltonian H_{2q} that makes its connection to $g = 2$ statistics and the diluted spectral function S_k manifest does exist and is related to the form given in Sec. III A by a unitary transformation. The identification of this transformation and the alternative form of H_{2q} is an interesting open question.

Further, the anisotropic honeycomb Hamiltonian with general transition amplitudes a, b , and c is of physical, but also mathematical, interest. The corresponding generating function of lattice walks would depend on these parameters and would “count” the number of moves in the three different lattice directions U, V , and W separately. The calculation of

TABLE II. $C_{2n}(A)$ up to $2n = 10$ for square lattice walks of length $2n$.

	$2n = 2$	4	6	8	10
$A = 0$	4	28	232	2156	21 944
± 1		8	144	2016	26 320
± 2			24	616	11 080
± 3				96	3120
± 4				16	840
± 5					160
± 6					40
Total counting	4	36	400	4900	63 504

this generalized generating function through traces of powers of the Hamiltonian appears to be within reach using the methods and techniques of this paper and constitutes a subject for further investigation.

There are various physical systems for which the analysis and results of this paper may be relevant, the quantum mechanics of an actual particle hopping on the honeycomb lattice sites being the most immediate example. It should be noted that the original formulation of such a system involves a wave function with values on each site of the full lattice and six hopping operators, one for each of the three direction in each sublattice. Our formulation in terms of three Hermitian operators U, V , and W and a finite Hilbert space is a lot more economical, and yet equivalent to the original one: It reduces the system to specific Bloch sectors, encoded in the Casimirs of the algebra of the operators, and uses a unique operator for the hopping in each direction, which acts off-diagonally in the two sublattices. It therefore reduces the problem of identifying quantum states to its bare bones.

The case of a particle on the honeycomb lattice with no magnetic field is well studied; less so the one with a magnetic field and a “butterfly” spectrum. The calculation of propagators in this case, and in particular of the propagator for identical initial and final lattice points, is of physical interest, since its value would indicate the rate of diffusion of a quantum mechanical particle initially on a single site. The calculation of this propagator in the path integral formulation involves sums of quantities precisely of the form calculated above. For a continuous time system a particular scaling limit has to be taken, distinct from the continuum scaling that would lead to a particle on a plane. The calculation of this and similar propagators using results in the present work remains an open question.

The relation of the area counting problem to the quantum dynamics of a charged particle on the lattice was exploited in Ref. [20] to calculate moments of the area distribution of walks on a square lattice using propagator techniques. A similar calculation for the case of the honeycomb lattice would be of interest. Moments of the area can also be computed from the results in this paper, either using the explicit area counting, or performing an expansion in $1/q$ of the Q -dependent area generating function. Both calculations are nontrivial. These and related issues are interesting topics for future research.

ACKNOWLEDGMENTS

L.G. acknowledges the financial support of China Scholarship Council (No. 202009110129). A.P. acknowledges the hospitality of LPTMS, Université Paris-Saclay, during the early phases of this work.

APPENDIX

1. $Z(n)$ for square lattice walks

We denote $Z(n)$ as $Z_q(n)$ to include its dependence on q .

Substituting $d_q = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z_q(n) z^{2n}$ into (8) and equating the coefficient of z^{2n} on both sides, we get

$$\begin{aligned} Z_q(n) &= Z_{q-1}(n) + s_{q-1} Z_{q-2}(n-1) \\ &= Z_{q-2}(n) + s_{q-2} Z_{q-3}(n-1) + s_{q-1} Z_{q-2}(n-1) \end{aligned}$$

$$= \dots$$

$$= Z_1(n) + \sum_{m=0}^{q-2} s_{m+1} Z_m(n-1).$$

Since $Z_m(n-1) = 0$ for $n-1 > \lfloor m/2 \rfloor$, i.e., $m < 2n-2$, we obtain

$$Z_q(n) = \sum_{m=2n-2}^{q-2} s_{m+1} Z_m(n-1)$$

with $Z_q(0) = 1$. Thus,

$$Z_q(1) = \sum_{m=0}^{q-2} s_{m+1} Z_m(0) = \sum_{k_1=1}^{q-1} s_{k_1},$$

$$Z_q(2) = \sum_{m=2}^{q-2} s_{m+1} Z_m(1)$$

$$= \sum_{m=2}^{q-2} \sum_{k_1=1}^{m-1} s_{m+1} s_{k_1} = \sum_{k_1=1}^{q-3} \sum_{k_2=1}^{k_1} s_{k_1+2} s_{k_2},$$

$$Z_q(3) = \sum_{m=4}^{q-2} s_{m+1} Z_m(2)$$

$$= \sum_{m=4}^{q-2} \sum_{k_1=1}^{m-3} \sum_{k_2=1}^{k_1} s_{m+1} s_{k_1+2} s_{k_2}$$

$$= \sum_{k_1=1}^{q-5} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} s_{k_1+4} s_{k_2+2} s_{k_3},$$

etc. Formula (9) can then be proven by induction, where we check

$$Z_q(n+1) = \sum_{m=2n}^{q-2} s_{m+1} Z_m(n)$$

$$= \sum_{m=2n}^{q-2} \sum_{k_1=1}^{m-2n+1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} \dots \sum_{k_n=1}^{k_{n-1}} s_{m+1} s_{k_1+2n-2} \dots s_{k_{n-1}+2} s_{k_n}$$

$$= \sum_{k_1=1}^{q-2n-1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} \sum_{k_4=1}^{k_3} \dots \sum_{k_{n+1}=1}^{k_n} s_{k_1+2n} s_{k_2+2n-2} \dots s_{k_n+2} s_{k_{n+1}}.$$

2. Examples of algebraic area enumeration of random walks on the square lattice

A few examples of $\frac{1}{q} \text{Tr} H_q^{2n}$ and the corresponding $C_{2n}(A)$'s are listed below and in Table II.

$$\frac{1}{q} \text{Tr} H_q^2 = 4,$$

$$\frac{1}{q} \text{Tr} H_q^4 = 4 \left(7 + 2 \cos \frac{2\pi p}{q} \right),$$

$$\frac{1}{q} \text{Tr} H_q^6 = 4 \left(58 + 36 \cos \frac{2\pi p}{q} + 6 \cos \frac{4\pi p}{q} \right),$$

$$\frac{1}{q} \text{Tr} H_q^8 = 4 \left(539 + 504 \cos \frac{2\pi p}{q} + 154 \cos \frac{4\pi p}{q} + 24 \cos \frac{6\pi p}{q} + 4 \cos \frac{8\pi p}{q} \right),$$

$$\frac{1}{q} \text{Tr} H_q^{10} = 4 \left(5486 + 6580 \cos \frac{2\pi p}{q} + 2770 \cos \frac{4\pi p}{q} + 780 \cos \frac{6\pi p}{q} + 210 \cos \frac{8\pi p}{q} + 40 \cos \frac{10\pi p}{q} + 10 \cos \frac{12\pi p}{q} \right).$$

3. Irreducible representations of the honeycomb algebra

Define three new operators u, v , and σ as

$$\sigma = Q^{-1/2} UVW, u = U\sigma, v = V\sigma$$

$$\Rightarrow U = u\sigma, V = v\sigma, W = Q^{1/2} v\sigma u. \quad (A1)$$

From the honeycomb algebra (18) we see that σ, u , and v are all unitary and satisfy

$$vu = Quv, u\sigma = \sigma u^{-1}, v\sigma = \sigma v^{-1}, \sigma^2 = 1. \quad (A2)$$

Since U, V , and W can be uniquely expressed in terms of σ, u , and v , it is sufficient to derive the irreducible representation ("irrep" for short) of u, v , and σ .

Operators u and v satisfy the quantum torus algebra and have a q -dimensional irrep if $Q = \exp(2i\pi p/q)$. However, σ can be embedded within this irrep only for specific values of the Casimirs $u^q = e^{i\phi}$ and $v^q = e^{i\theta}$. Indeed, assuming σ acts within this irrep,

$$e^{i\phi} = u^q = \sigma u^q \sigma = (\sigma u \sigma)^q = u^{-q} = e^{-i\phi} \Rightarrow e^{i\phi} = e^{-i\phi}.$$

So ϕ can only be 0 or $\pi \pmod{2\pi}$ and similarly for θ . For $\theta, \phi \in \{0, \pi\}$ we can show that the irrep of (A2) is unique up to unitary transformations, and up to the algebra automorphism $\sigma \rightarrow -\sigma$, and is given by the action on basis states $|n\rangle$

$$\begin{aligned} u|n\rangle &= e^{i(\phi+2\pi pn)/q}|n\rangle, & n &= 0, 1, \dots, q-1, \\ v|n\rangle &= e^{i\theta/q}|n-1\rangle, & |-1\rangle &\equiv |q-1\rangle, \\ \sigma|n\rangle &= e^{i\theta(2n-r)/q}|r-n\rangle, & rp + \phi/\pi &= 0 \pmod{q}. \end{aligned} \quad (\text{A3})$$

The ‘‘pivot’’ r in the inversion action of σ is $r = 0$, if $\phi = 0$, or the primary solution of the Diophantine equation $kq - rp = 1$, if $\phi = \pi$. The momenta $qk_x = \phi$ and $qk_y = \theta$ in this irrep are quantized as

$$k_x = \frac{\pi n_x}{q}, \quad k_y = \frac{\pi n_y}{q}, \quad n_x, n_y \in \mathbb{Z}.$$

For either θ or $\phi \notin \{0, \pi\}$ the irrep of (A2) must decompose into more than one q -dimensional irreps of the quantum torus algebra u, v with σ mixing the irreps. The minimal irrep of the full algebra (A2) involves 2 irreps of the torus algebra, all other situations being reducible. Representing all operators in block diagonal form in the space of the two irreps $u_i, v_i, i = 1, 2$, with Casimirs $u_i^q = e^{i\phi_i}, v_i^q = e^{i\theta_i}$,

$$u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \quad \sigma = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix},$$

and implementing the relations $\sigma u^q \sigma = u^{-q}, \sigma v^q \sigma = v^{-q}$ leads to

$$\begin{aligned} (e^{i\phi_1} - e^{-i\phi_1})A &= (e^{i\phi_2} - e^{-i\phi_2})C = (e^{i\phi_1} - e^{-i\phi_2})B = 0, \\ (e^{i\theta_1} - e^{-i\theta_1})A &= (e^{i\theta_2} - e^{-i\theta_2})C = (e^{i\theta_1} - e^{-i\theta_2})B = 0. \end{aligned}$$

If all of $\phi_1, \phi_2, \theta_1, \theta_2$ are 0 or π , then the representation is reducible, as we will soon demonstrate. If ϕ_1, θ_1 are 0 or π , but not both of ϕ_2, θ_2 are, then the above relations imply $C = B = 0$ and thus $\sigma^2 = 1$ is impossible, and similarly if ϕ_2, θ_2 are both 0 or π . Therefore, both ϕ_1, θ_1 and ϕ_2, θ_2 must have at least one angle $\neq 0, \pi$. The above relations then imply $A = C = 0$, and $\sigma^2 = 1$ implies $B^\dagger B = 1$. The last equalities above, then, require $\phi_1 = -\phi_2, \theta_1 = -\theta_2$. Further, a unitary transformation

$$S = \begin{pmatrix} B^\dagger & 0 \\ 0 & 1 \end{pmatrix}, \quad u \rightarrow SuS^{-1}, \quad v \rightarrow SvS^{-1}, \quad \sigma \rightarrow S\sigma S^{-1}$$

eliminates B in σ , and $\sigma u \sigma = u^{-1}, \sigma v \sigma = v^{-1}$ imply $u_1 = u_2^{-1}, v_1 = v_2^{-1}$. Altogether, the irrep of the honeycomb algebra for two arbitrary Casimirs $\phi = \phi_1 = -\phi_2, \theta = \theta_1 = -\theta_2$, is given by the $2q$ -dimensional matrices

$$u = \begin{pmatrix} u_o & 0 \\ 0 & u_o^{-1} \end{pmatrix}, \quad v = \begin{pmatrix} v_o & 0 \\ 0 & v_o^{-1} \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\text{A4})$$

where u_o and v_o are the basic q -dimensional quantum torus irreps with Casimirs $e^{i\phi}$ and $e^{i\theta}$. Finally, from (A1) we obtain the corresponding irreducible forms for U, V , and W

$$\begin{aligned} U &= \begin{pmatrix} 0 & u_o \\ u_o^{-1} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & v_o \\ v_o^{-1} & 0 \end{pmatrix}, \\ W &= Q^{1/2} \begin{pmatrix} 0 & v_o u_o^{-1} \\ v_o^{-1} u_o & 0 \end{pmatrix}. \end{aligned}$$

We conclude with a demonstration that the above representation becomes reducible if $\phi, \theta \in \{0, \pi\}$. In that case, as we demonstrated before in (A3), there is a $q \times q$ matrix σ_o (to be distinguished from the $2q \times 2q$ matrix σ in (A4) above) satisfying (A2) for the matrices u_o and v_o . Performing the unitary transformation

$$S_o = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\sigma_o \\ \sigma_o & 1 \end{pmatrix}$$

on all matrices, and using $\sigma_o u_o \sigma_o = u_o^{-1}, \sigma_o v_o \sigma_o = v_o^{-1}$, we obtain

$$u = \begin{pmatrix} u_o & 0 \\ 0 & u_o^{-1} \end{pmatrix}, \quad v = \begin{pmatrix} v_o & 0 \\ 0 & v_o^{-1} \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_o & 0 \\ 0 & -\sigma_o \end{pmatrix},$$

or

$$\begin{aligned} U &= \begin{pmatrix} u_o \sigma_o & 0 \\ 0 & -\sigma_o u_o \end{pmatrix}, \quad V = \begin{pmatrix} v_o \sigma_o & 0 \\ 0 & -\sigma_o v_o \end{pmatrix}, \\ W &= Q^{1/2} \begin{pmatrix} v_o u_o^{-1} \sigma_o & 0 \\ 0 & -\sigma_o v_o u_o^{-1} \end{pmatrix} \end{aligned}$$

reducing to the direct sum of two q -dimensional irreps.

4. $Z(n)$ for honeycomb lattice walks

We denote $Z(n)$ as $Z_q(n)$ to include its dependence on q .

Substituting $d_q = \sum_{n=0}^q (-1)^n Z_q(n) z^{2n}$ into (23) and equating the coefficient of z^{2n} on both sides, we get

$$\begin{aligned} Z_q(n) &= Z_{q-1}(n) + (1 + s_q)Z_{q-1}(n-1) - s_{q-1}Z_{q-2}(n-2) \\ &= Z_{q-2}(n) + (1 + s_{q-1})Z_{q-2}(n-1) + (1 + s_q) \\ &\quad \times Z_{q-1}(n-1) - s_{q-2}Z_{q-3}(n-2) - s_{q-1} \\ &\quad \times Z_{q-2}(n-2) \\ &= \dots \\ &= Z_1(n) + \sum_{m=1}^{q-1} (1 + s_{m+1})Z_m(n-1) \\ &\quad - \sum_{m=0}^{q-2} s_{m+1}Z_m(n-2). \end{aligned}$$

Since $Z_m(n) = 0$ for $n > m$, we obtain

$$Z_q(n) = \sum_{m=n-1}^{q-1} (1 + s_{m+1})Z_m(n-1) - \sum_{m=n-2}^{q-2} s_{m+1}Z_m(n-2)$$

with $Z_q(0) = 1$ and $Z_q(j) = 0$ for $j < 0$. Thus,

$$\begin{aligned} Z_q(1) &= \sum_{m=0}^{q-1} (1 + s_{m+1})Z_m(0) \\ &= \sum_{k_1=1}^q (1 + s_{k_1}), \\ Z_q(2) &= \sum_{m=1}^{q-1} (1 + s_{m+1})Z_m(1) - \sum_{m=0}^{q-2} s_{m+1}Z_m(0) \\ &= \sum_{m=1}^{q-1} \sum_{k_1=1}^m (1 + s_{m+1})(1 + s_{k_1}) - \sum_{m=0}^{q-2} s_{m+1} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k_1=1}^{q-1} \sum_{k_2=1}^{k_1} (1 + s_{k_1+1})(1 + s_{k_2}) - \sum_{k_1=1}^{q-1} s_{k_1}, & - \sum_{m=2}^{q-1} \sum_{k_1=1}^{m-1} (1 + s_{m+1})s_{k_1} - \sum_{m=1}^{q-2} \sum_{k_1=1}^m s_{m+1}(1 + s_{k_1}) \\
 Z_q(3) &= \sum_{m=2}^{q-1} (1 + s_{m+1})Z_m(2) - \sum_{m=1}^{q-2} s_{m+1}Z_m(1) &= \sum_{k_1=1}^{q-2} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} (1 + s_{k_1+2})(1 + s_{k_2+1})(1 + s_{k_3}) \\
 &= \sum_{m=2}^{q-1} \sum_{k_1=1}^{m-1} \sum_{k_2=1}^{k_1} (1 + s_{m+1})(1 + s_{k_1+1})(1 + s_{k_2}) & - \sum_{k_1=1}^{q-2} \sum_{k_2=1}^{k_1} (1 + s_{k_1+2})s_{k_2} - \sum_{k_1=1}^{q-2} \sum_{k_2=1}^{k_1} s_{k_1+1}(1 + s_{k_2}).
 \end{aligned}$$

Likewise $Z_q(5)$ would read

$$\begin{aligned}
 Z_q(5) &= + \sum_{i=1}^{q-4} \sum_{j=1}^i \sum_{k=1}^j \sum_{l=1}^k \sum_{m=1}^l (1 + s_{i+4})(1 + s_{j+3})(1 + s_{k+2})(1 + s_{l+1})(1 + s_m) \\
 &- \sum_{i=1}^{q-4} \sum_{j=1}^i \sum_{k=1}^j \sum_{l=1}^k (1 + s_{i+4})(1 + s_{j+3})(1 + s_{k+2})s_l - \sum_{i=1}^{q-4} \sum_{j=1}^i \sum_{k=1}^j \sum_{l=1}^k (1 + s_{i+4})(1 + s_{j+3})s_{k+1}(1 + s_l) \\
 &- \sum_{i=1}^{q-4} \sum_{j=1}^i \sum_{k=1}^j \sum_{l=1}^k (1 + s_{i+4})s_{j+2}(1 + s_{k+1})(1 + s_l) - \sum_{i=1}^{q-4} \sum_{j=1}^i \sum_{k=1}^j \sum_{l=1}^k s_{i+3}(1 + s_{j+2})(1 + s_{k+1})(1 + s_l) \\
 &+ \sum_{i=1}^{q-4} \sum_{j=1}^i \sum_{k=1}^j (1 + s_{i+4})s_{j+2}s_k + \sum_{i=1}^{q-4} \sum_{j=1}^i \sum_{k=1}^j s_{i+3}(1 + s_{j+2})s_k + \sum_{i=1}^{q-4} \sum_{j=1}^i \sum_{k=1}^j s_{i+3}s_{j+1}(1 + s_k).
 \end{aligned}$$

5. Diluted trigonometric sums

The diluted trigonometric sums $\sum_{k=1}^{2q-j+1} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}$ can be computed and read

$$\begin{aligned}
 &\frac{1}{q} \sum_{k=1}^{2q-j+1} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1} \\
 &= \sum_{A=-\infty}^{\infty} \cos\left(\frac{2A\pi p}{q}\right) \left[\sum_{k_5=-l_5}^{l_5} \sum_{k_7=-l_7}^{l_7} \cdots \sum_{k_{2[(j-1)/2]+1}=-l_{2[(j-1)/2]+1}}^{l_{2[(j-1)/2]+1}} \left(l_1 + A + \sum_{\substack{i=5 \\ i \text{ odd}}}^{2[(j-1)/2]+1} (i-3)k_i/2 \right) \right. \\
 &\quad \times \left(l_3 - A - \sum_{\substack{i=5 \\ i \text{ odd}}}^{2[(j-1)/2]+1} (i-1)k_i/2 \right) \prod_{\substack{i=5 \\ i \text{ odd}}}^{2[(j-1)/2]+1} \left(l_i + k_i \right) \\
 &\quad \left. + \sum_{k_6=-l_6}^{l_6} \sum_{k_8=-l_8}^{l_8} \cdots \sum_{k_{2\lfloor j/2 \rfloor}=-l_{2\lfloor j/2 \rfloor}}^{l_{2\lfloor j/2 \rfloor}} \left(l_2 + A + \sum_{\substack{i=6 \\ i \text{ even}}}^{2\lfloor j/2 \rfloor} (i-4)k_i/2 \right) \left(l_4 - A - \sum_{\substack{i=6 \\ i \text{ even}}}^{2\lfloor j/2 \rfloor} (i-2)k_i/2 \right) \prod_{\substack{i=6 \\ i \text{ even}}}^{2\lfloor j/2 \rfloor} \left(l_i + k_i \right) \right].
 \end{aligned}$$

[1] D. R. Hofstadter, Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields, *Phys. Rev. B* **14**, 2239 (1976).
 [2] S. Ouvry and S. Wu, The algebraic area of closed lattice random walks, *J. Phys. A: Math. Theor.* **52**, 255201 (2019).
 [3] S. Ouvry and A. P. Polychronakos, Exclusion statistics and lattice random walks, *Nucl. Phys. B* **948**, 114731 (2019).
 [4] S. Ouvry and A. P. Polychronakos, Lattice walk area combinatorics, some remarkable trigonometric sums and Apéry-like numbers, *Nucl. Phys. B* **960**, 115174 (2020).
 [5] F. H. Claro and G. H. Wannier, Magnetic subband structure of electrons in hexagonal lattices, *Phys. Rev. B* **19**, 6068 (1979).
 [6] F. Yılmaz and M. Ö. Oktel, Hofstadter butterfly evolution in the space of two-dimensional Bravais lattices, *Phys. Rev. A* **95**, 063628 (2017).

- [7] K. S. Novoselov, Z. Jiang, Y. Zhang, S. V. Morozov, H. L. Stormer, U. Zeitler, J. C. Maan, G. S. Boebinger, P. Kim, and A. K. Geim, Room-temperature quantum Hall effect in graphene, *Science* **315**, 1379 (2007).
- [8] K. I. Bolotin, F. Ghahari, M. D. Shulman, H. L. Stormer, and P. Kim, Observation of the fractional quantum Hall effect in graphene, *Nature* **462**, 196 (2009).
- [9] E. Perfetto, J. González, F. Guinea, S. Bellucci, and P. Onorato, Quantum Hall effect in carbon nanotubes and curved graphene strips, *Phys. Rev. B* **76**, 125430 (2007).
- [10] M.-H. Kang, Toroidal fullerenes with the Cayley graph structures, *Discr. Math.* **311**, 2384 (2011).
- [11] A. Kitaev, Anyons in an exactly solved model and beyond, *Ann. Phys.* **321**, 2 (2006).
- [12] M. A. Levin and X.-G. Wen, String-net condensation: A physical mechanism for topological phases, *Phys. Rev. B* **71**, 045110 (2005).
- [13] I. F. Herbut, Interactions and Phase Transitions on Graphene's Honeycomb Lattice, *Phys. Rev. Lett.* **97**, 146401 (2006).
- [14] R. Rammal, Landau level spectrum of Bloch electrons in a honeycomb lattice, *J. Phys. France* **46**, 1345 (1985).
- [15] Ch. Kreft and R. Seiler, Models of the Hofstadter-type, *J. Math. Phys.* **37**, 5207 (1996).
- [16] A. L. C. Pereira and P. A. Schulz, Additional levels between Landau bands due to vacancies in graphene: Towards defect engineering, *Phys. Rev. B* **78**, 125402 (2008).
- [17] A. Agazzi, J.-P. Eckmann, and G. M. Graf, The colored Hofstadter butterfly for the honeycomb lattice, *J. Stat. Phys.* **156**, 417 (2014).
- [18] Ch. Kreft, Explicit computation of the discriminant for the Harper equation with rational flux, SFB 288 Preprint No. 89, TU-Berlin, 1993 (unpublished).
- [19] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1950), Chap. IV, p. 279.
- [20] T. Epelbaum, F. Gelis, and B. Wu, From lattice Quantum Electrodynamics to the distribution of the algebraic areas enclosed by random walks on \mathbb{Z}^2 , *Ann. Inst. Henri Poincaré D* **3**, 381 (2016).

3 - Combinatorics of generalized Dyck and Motzkin paths

3.1 Introduction

As discussed in the first two chapters, enumerating \mathbf{n} -step closed lattice walks according to their algebraic area amounts to computing the quantum traces of the \mathbf{n} -th power of Hamiltonian matrices H . Ignoring the spurious umklapp matrix elements in the corners of H simplifies the computation to the matrix trace $\text{tr } H^{\mathbf{n}}$, with an appropriate normalization. In Chapter 2, calculating $\text{tr } H^{\mathbf{n}}$ relies on the secular determinants of these matrices. By interpreting these determinants as grand canonical partition functions for systems of particles obeying exclusion statistics in an appropriate one-body spectrum and expressing the partition functions in terms of their corresponding cluster coefficients, we obtain the sought-after traces. In [2], $g = 2$ exclusion is generalized to arbitrary $g \geq 2$ exclusion, introducing the new combinatorial coefficients $c_g(l_1, l_2, \dots, l_j)$ labeled by the g -compositions of $n = l_1 + l_2 + \dots + l_j$ with $\mathbf{n} = gn$, which is defined later on in Section 3.2. Letting $g = 2$ reproduces the standard compositions and $c(l_1, l_2, \dots, l_j)$ in (2.10). However, a direct combinatorial interpretation of $c_g(l_1, l_2, \dots, l_j)$ was missing.

In this chapter, we bypass the secular determinant and tackle directly the matrix trace $\text{tr } H^{\mathbf{n}}$. We relate the expression for the trace to periodic generalized $[g-1, -1]$ Dyck paths, which are 2D paths starting and ending at the same floor and consisting of a series of up steps $(1, g-1)$ that ascend $g-1$ floors and down steps $(1, -1)$ that descend 1 floor. By construction, each step can neither go below floor 1 nor go beyond floor $j+g-1$. Letting $g = 2$ reproduces the usual periodic Dyck paths. More precisely, we demonstrate that $gn c_g(l_1, l_2, \dots, l_j)$ counts the number of all possible such paths with l_1 up steps from floor 1, l_2 up steps from floor 2, \dots , l_j up steps from floor j , for a total of $\mathbf{n} = gn$ steps inside the strip between floors 1 and $j+g-1$. In fact, we obtain an even more detailed enumeration of these paths by providing the count of paths starting with an up step from a given i -th floor among the $j+g-1$ floors and similarly of paths starting with a down step.

We then further extend our results to the enumeration of periodic generalized $[g-1, 0, -1]$ Motzkin paths that also include the steps $(1, 0)$ on the horizontal (see Figure 3.1), by relating such paths to matrices corresponding to mixed exclusion for particles having either fermionic $g = 1$ or g -exclusion statistics. The derived expressions for the corresponding combinatorial coefficients $c_{1,g}$ counting such paths with a fixed number of horizontal, up, or down steps for each floor are labeled by a further generalized $(1, g)$ -composition¹ of the number of steps \mathbf{n} . The extension to other classes of paths, corresponding to other generalizations of quantum exclusion statistics, appears attainable through our method.

By “periodic generalized Dyck or Motzkin paths”, we always mean generalized Dyck or Motzkin bridges (including excursions that start and end at the first floor) with the constraint for the path to be in a strip of a given width (see Figure 3.2). Meanders and more general paths are not relevant to this thesis.

¹Regarding notations, a $(1, g)$ -composition should not be confused with an up step $(1, g-1)$ in a generalized Dyck path. The first notation refers to a generalization of g -compositions corresponding to a mixed exclusion $g = 1$ and g , while the second notation signifies an up step going up $g-1$ floors.

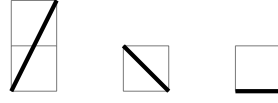


Figure 3.1: Illustration of the steps $(1, g-1)$, $(1, -1)$, and $(1, 0)$ in generalized Dyck and Motzkin paths for $g = 3$.

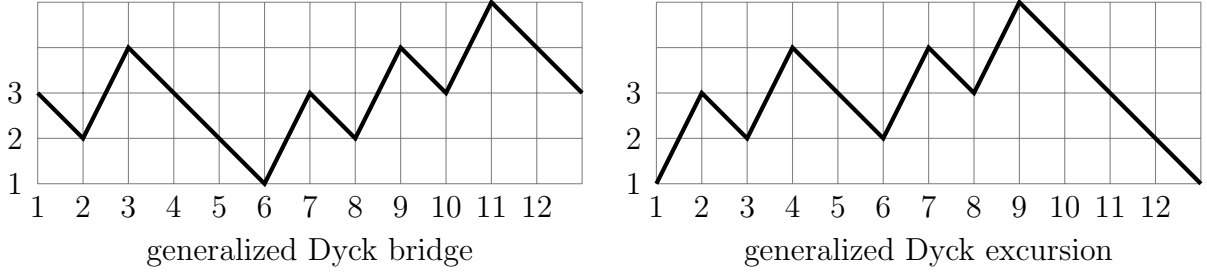


Figure 3.2: A generalized Dyck bridge and a generalized Dyck excursion for $g = 3$, which starts from the third floor and the first floor, respectively, of length $\mathbf{n} = 12$ with $l_1 = 1$ up step from the first floor, $l_2 = 2$ up steps from the second floor, and $l_3 = 1$ up step from the third floor. We refer to them as periodic generalized Dyck paths.

3.2 g -exclusion statistics and generalized Dyck paths

In Section 2.1, the Hamiltonian $H'|_{k_x=0}$ in (2.2) is a particular case of the general class of $g = 2$ exclusion matrices

$$H_2 = \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 & 0 \\ g_1 & 0 & f_2 & \cdots & 0 & 0 \\ 0 & g_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-1} & 0 \end{pmatrix}, \quad (3.1)$$

with secular determinant

$$\det(I - zH_2) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z_n z^{2n}. \quad (3.2)$$

In general, for g -exclusion the Hamiltonian is [2]

$$H_g = F(u)v + v^{1-g}G(u), \quad (3.3)$$

where $F(u)$ and $G(u)$ are scalar functions of u , which amounts to the g -exclusion matrix² (again ig-

²Indeed the Hofstadter Hamiltonian is a $g = 2$ Hamiltonian since $H = -uv - v^{-1}u^{-1} + v + v^{-1} = (I - u)v + v^{1-2}(I - u^{-1})$. An example of a $g = 3$ Hamiltonian involves closed chiral walks on a triangular lattice [2], as illustrated in Figure 1.3. The Hamiltonian is given by $H = U + V + QU^{-1}V^{-1}$. Only three out of the possible six directions are allowed at each step. By choosing the representation $U = -iuv$ and $V = iu^{-1}v$, the Hamiltonian can be expressed as $H = i(-u + u^{-1})v + v^{1-3}$, leading to a $g = 3$ -exclusion matrix. See [38] for its algebraic area enumeration.

noring the spurious umklapp matrix elements in the corners)

$$H_g = \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & f_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ g_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & g_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-g+1} & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (3.4)$$

where $g - 1$ zeros appear between the f_k and g_k subdiagonals. Its secular determinant

$$\det(I - zH_g) = \sum_{n=0}^{\lfloor q/g \rfloor} (-1)^n Z_n z^{gn} \quad (3.5)$$

yields

$$Z_n = \sum_{k_1=1}^{q-gn+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} s_{k_1+gn-g} s_{k_2+gn-2g} \cdots s_{k_{n-1}+g} s_{k_n} \quad (3.6)$$

with

$$s_k = g_k f_k f_{k+1} \cdots f_{k+g-2}, \quad k = 1, 2, \dots, q-g+1. \quad (3.7)$$

As in the $g = 2$ case, Z_n admits an interpretation as the partition function of n exclusion g particles in the one-body spectrum implied by the spectral function $s_k = e^{-\beta \epsilon_k}$, with one-body levels labeled by $k = 1, 2, \dots, q-g+1$.

From

$$\ln \left(\sum_{n=0}^{\lfloor q/g \rfloor} Z_n x^n \right) = \sum_{n=1}^{\infty} b_n x^n,$$

we infer

$$b_n = (-1)^{n+1} \sum_{\substack{l_1, l_2, \dots, l_j \\ g\text{-composition of } n}} c_g(l_1, l_2, \dots, l_j) \sum_{k=1}^{q-j-g+2} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}, \quad (3.8)$$

where

$$c_g(l_1, l_2, \dots, l_j) = \frac{1}{l_1} \prod_{i=2}^j \binom{l_{i-g+1} + \cdots + l_i - 1}{l_i} \quad (3.9)$$

with the convention $l_i = 0$ if $i \leq 0$. Finally,

$$\text{tr } H_g^{\mathbf{n}=gn} = gn(-1)^{n+1} b_n = gn \sum_{\substack{l_1, l_2, \dots, l_j \\ g\text{-composition of } n}} c_g(l_1, l_2, \dots, l_j) \sum_{k=1}^{q-j-g+2} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}. \quad (3.10)$$

These expressions generalize (2.6), (2.9), (2.10), and (2.11) to g -exclusion statistics, where now in (3.8) and (3.10) we have to sum over the g -compositions of the integer n , obtained by inserting at will inside the usual compositions (i.e., the 2-compositions) no more than $g - 2$ zeros in succession, i.e., obtained by allowing up to $g - 2$ consecutive integers in the composition to vanish. For example, there are nine $g = 3$ -compositions of $n = 3$, namely $3, 2+1, 1+2, 1+1+1, 2+0+1, 1+0+2, 1+0+1+1, 1+1+0+1$, and $1+0+1+0+1$. In general, there are g^{n-1} such g -compositions of the integer n (see [44] for an analysis of these extended compositions, also called multicompositions).

To seek a combinatorial interpretation of the coefficients $c_g(l_1, l_2, \dots, l_j)$ in (3.9) appearing in the trace the \mathbf{n} -th power of the g -exclusion matrix H_g in (3.4), let us consider directly this trace and denote by h_{ij} the matrix elements of H_g^T . The trace of H_g^n becomes

$$\mathrm{tr} H_g^n = \sum_{k_1=1}^q \sum_{k_2=1}^q \cdots \sum_{k_n=1}^q h_{k_1 k_2} h_{k_2 k_3} \cdots h_{k_n k_1}. \quad (3.11)$$

The structure of the g -exclusion matrix (3.4) implies that (3.11) is a sum of products of \mathbf{n} factors $h_{k_i k_{i+1}}$ with indices such that $k_{i+1} - k_i = g - 1$ or -1 . We map the sequence of indices $k_1, k_2, \dots, k_{n-1}, k_n, k_1$ to the heights (i.e., floors) of a periodic generalized $[g - 1, -1]$ Dyck path of length \mathbf{n} starting and ending at floor k_1 and consisting of a series of up steps $(1, g - 1)$ and down steps $(1, -1)$. The path is constrained within the strip ranging from floor 1 to g . Evaluating the trace (3.11) amounts to summing the corresponding products over all such periodic paths. We note that periodic paths must have n up steps and $n(g - 1)$ down steps for a total length $gn = \mathbf{n}$.

To group together terms with the same weight $h_{k_1 k_2} \cdots h_{k_n k_1}$, for each path we denote by l_1, l_2, \dots, l_j the number of up steps starting at floor $k, k+1, \dots, k+j-1$ ($k = 1, 2, \dots, q-j-g+2$ is the lowest floor reached by the path). Clearly $l_1 + l_2 + \cdots + l_j = n$, and at most $g - 2$ successive l_i can vanish, since steps of size $g - 1$ can skip $g - 2$ floors, so l_1, \dots, l_j is a g -composition of n (Figure 3.3 depicts the $g = 3$ composition $3, 0, 1, 1$. Here, k is set to 1). Further, each up step $k_i \rightarrow k_i + g - 1$ necessarily implies down steps $k_i + g - 1 \rightarrow k_i + g - 2, \dots, k_i + 1 \rightarrow k_i$, so factors in each term in (3.11) corresponding to each up step $k_i \rightarrow k_i + g - 1$ contribute the combination $h_{k_i, k_i + g - 1} h_{k_i + g - 1, k_i + g - 2} \cdots h_{k_i + 1, k_i} = g k_i f_{k_i + g - 2} \cdots f_{k_i} = s_{k_i}$, where we used (3.4) and (3.7). Altogether, the sum in (3.11) rewrites as

$$\mathrm{tr} H_g^n = \sum_{k=1}^{q-j-g+2} \sum_{\substack{l_1, l_2, \dots, l_j \\ g\text{-composition of } n}} C_g(l_1, l_2, \dots, l_j) s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j},$$

where $C_g(l_1, \dots, l_j)$ is the number of periodic generalized Dyck paths of length $\mathbf{n} = gn$ with l_1 up steps originating from the first floor, l_2 from the second floor, etc.³ The sum over k ensures that paths of all starting indices k_1 from 1 to q in (3.11) are included. Comparing this expression with (3.8), we see that $C_g(l_1, l_2, \dots, l_j) = gn c_g(l_1, l_2, \dots, l_j)$. Therefore, $gn c_g(l_1, l_2, \dots, l_j)$ admits the combinatorial interpretation of the number of periodic generalized $[g - 1, -1]$ Dyck paths of length $\mathbf{n} = gn$ with l_1, \dots, l_j up steps originating from floors $1, \dots, j$ as defined above.

³Recall that l_1, l_2, \dots, l_j were initially defined as the number of up steps originating from floors $k, k + 1, \dots, k + j - 1$. Since we can always vertically shift the paths by $1 - k$ floors or alternatively set $k = 1$, which leaves the number $C_g(l_1, l_2, \dots, l_j)$ unchanged, for the convenience of the following combinatorial study, l_1, l_2, \dots, l_j in $C_g(l_1, l_2, \dots, l_j)$ can be harmlessly redefined as the number of up steps originating from floors $1, 2, \dots, j$.

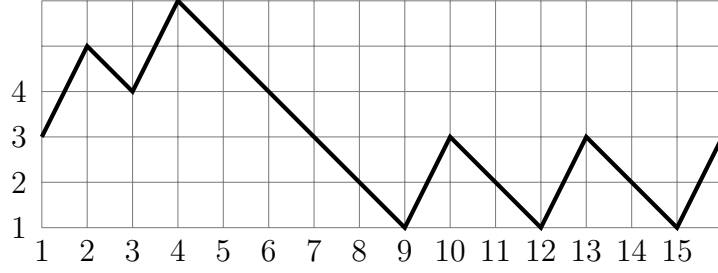


Figure 3.3: A periodic generalized $[2, -1]$ Dyck path of length 15 for the $g = 3$ -composition $3, 0, 1, 1$. The path starts from the third floor with an up step.

Furthermore, we can prove that $l_i c_g(l_1, l_2, \dots, l_j)$ counts the number of such periodic generalized $[g - 1, -1]$ Dyck paths of length $\mathbf{n} = 2n$ starting from the i -th floor with an up step. (The result $l_1 c_2(l_1, l_2, \dots, l_j)$ for $g = 2$ and $i = 1$ was also derived in [45, 46].) Clearly, the sum of the above counts for all $i = 1, 2, \dots, j$ reproduces the total count $n c_g(l_1, l_2, \dots, l_j)$ of paths starting with an up step. The remaining count $n(g - 1) c_g(l_1, l_2, \dots, l_j)$ corresponds to paths starting with a down step.

A comprehensive and detailed proof for the above counting formulas for $g = 2$ and arbitrary g can be found in Section IV.A and IV.B of Article 2, respectively. Here, we present an outline of the proof. First, we prove that the sequence of floors from which the n up steps start fully determines the path (see Figure 3.4). It follows that it is sufficient to count all possible sequences of floors from which each up step starts, given the constraint that l_1 up steps originating from the first floor, l_2 up steps originating from the second floor, and so on. Now, let us calculate the number of paths starting with an up step from the lowest floor $i = 1$. By looking at all successive sets of g floors from the top, it can be shown that the multiplicity of paths for a fixed set of up steps starting from floors $1, 2, \dots, g - 1$ is

$$\binom{l_{j-g+1} + \dots + l_j - 1}{l_j} \binom{l_{j-g} + \dots + l_{j-1} - 1}{l_{j-1}} \dots \binom{l_1 + \dots + l_g - 1}{l_g} = \prod_{k=g}^j \binom{l_{k-g+1} + \dots + l_k - 1}{l_k}$$

The place of up steps that start in the bottom $g - 1$ floors is arbitrary with the expectation that the first up step occurs on floor 1, for a multiplicity of

$$\binom{l_1 + \dots + l_{g-1} - 1}{l_1 - 1, l_2, \dots, l_{g-1}} = \prod_{k=1}^{g-1} \binom{l_{k-g+1} + \dots + l_k - 1}{l_k}.$$

Multiplying these factors together gives the number of paths starting with an up step from the first floor as $l_1 c_g(l_1, \dots, l_j)$.

The total number of paths with l_1, l_2, \dots, l_j up steps originating from the floor $1, 2, \dots, j$, respectively, can therefore be obtained by circularly permuting the gn steps of paths starting at the bottom, which produces $gn l_1 c_g(l_1, \dots, l_j)$ paths. However, each time an up step from the first floor occurs first, it reproduces the set of paths starting at the bottom. Since there are l_1 such steps, this results in an overcounting by a factor l_1 . Correcting for this, we recover the total number of paths as $gn c_g(l_1, l_2, \dots, l_j)$.

The count of paths starting with an up step from floor i can be obtained with a similar argument. Cyclically permuting these paths reproduces, again, all possible paths, but with an overcounting by a

factor of l_i , since each time that an up step at floor i becomes first it reproduces the full set. Therefore, we obtain a count of $l_i c_g(l_1, \dots, l_j)$.

The number of paths starting with a down step from the i -th floor, with $i = 2, 3, \dots, j + g - 1$, can be reproduced graphically (see Figure 3.5): (i) consider all periodic paths starting with an up step from either the $(i - 1)$ -th or the $(i - 2)$ -th, or \dots the $(i - g + 1)$ -th floor, and cut them at the last occurrence of a down step from the i -th floor and (ii) exchange the two pieces. This establishes a one-to-one correspondence between the two sets of paths, and gives the number of paths starting with a down step from floor $i = 2, 3, \dots, j + g - 1$ as $(l_{i-g+1} + \dots + l_{i-1}) c_g(l_1, \dots, l_j)$.

Finally, we note that the number of all periodic generalized $[g - 1, -1]$ Dyck paths is obtained by summing c_g over all g -compositions and yields the relation

$$gn \sum_{\substack{l_1, \dots, l_j \\ g\text{-composition of } n}} c_g(l_1, \dots, l_j) = [x^0](x^{g-1} + x^{-1})^{gn} = \binom{gn}{n},$$

which, when $g = 2$, proves (2.14).

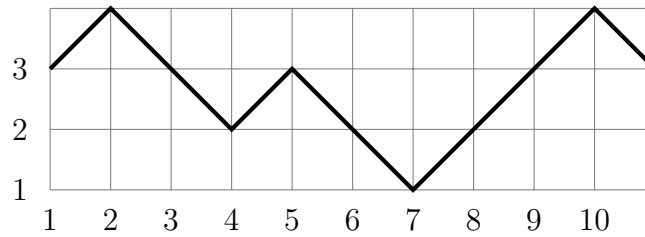


Figure 3.4: A periodic $[1, -1]$ Dyck path (i.e., $g = 2$) of length 10 starting at $i = 3$, characterized by the sequence $i_s = 3, 2, 1, 2, 3$ (for $s = 1, 2, 3, 4, 5$) of floors from which the up steps start. We can uniquely reconstruct the path from this sequence by producing the sequence of up step positions $p_s = i - i_s + 2s - 1 = 1, 4, 7, 8, 9$ and filling the gaps with down steps at positions 2, 3, 5, 6, 10. In general, $p_s = i - i_s + g(s - 1) + 1$.

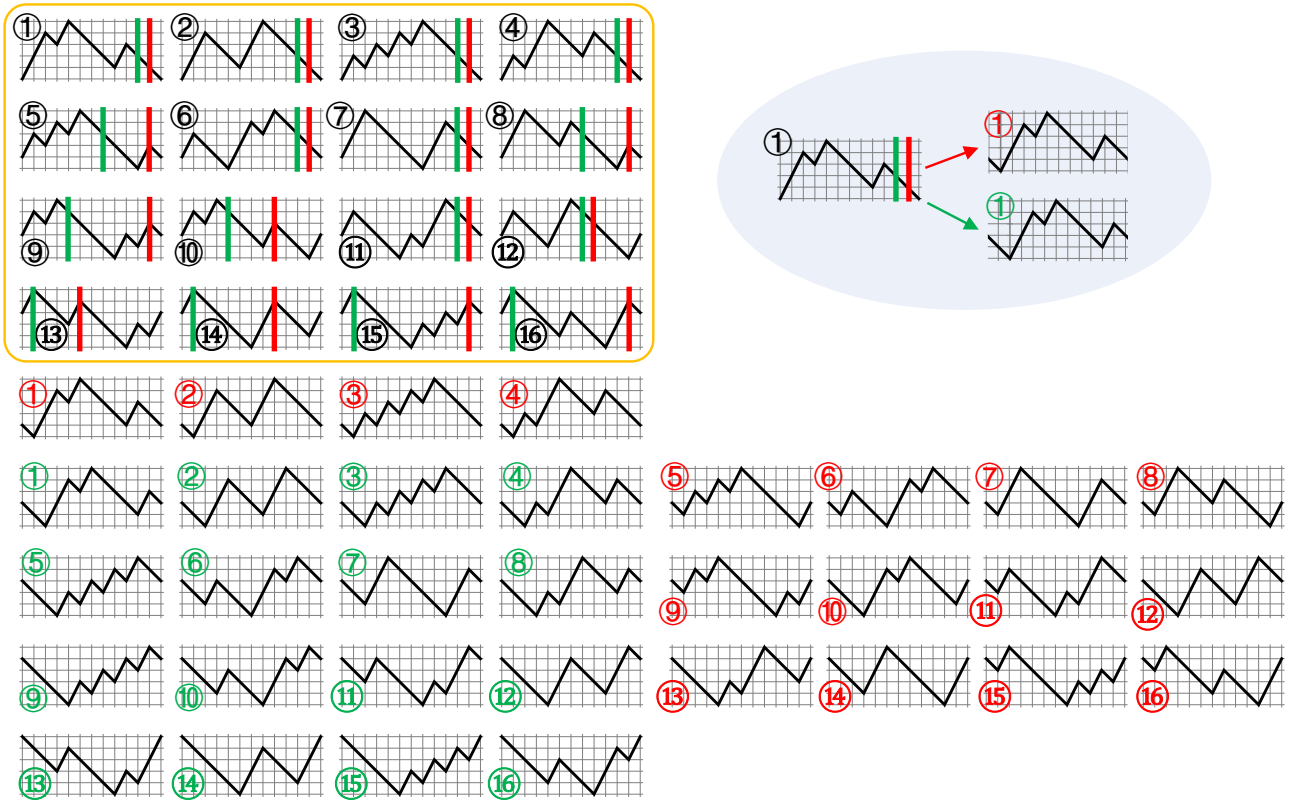


Figure 3.5: Example of the “cutting and exchanging” argument for the $g = 3$ -composition $1, 1, 1, 1$. To obtain the periodic generalized $[2, -1]$ Dyck paths starting with a down step from floor i , where $i = 2, 3, 4, 5, 6$, we take the generalized Dyck paths starting with an up step from either floor $i - 1$ or $i - 2$, cut them at the last occurrence of a down step from floor i , and exchange the two pieces. The first four rows of paths in the orange rectangle box correspond to all the paths, numbered in circles, starting with an up step from floor $i - 1$, respectively. The green and red vertical lines indicate the position of the last occurrence of a down step from floor $i + 1$ and i , respectively. By performing the described cutting and exchanging, we obtain the following five rows of paths starting with a down path from each floor i .

3.3 $(1, g)$ -exclusion statistics and generalized Motzkin paths

In Section 2.2, we addressed the algebraic area enumeration of closed walks on a honeycomb lattice. Again, a Hofstadter-like Hamiltonian was central to the enumeration, rewritten as a $2q \times 2q$ matrix, which was subsequently reduced to a $q \times q$ matrix whose secular determinant can be interpreted as the grand canonical partition function of a mixture of $g = 1$ and $g = 2$ exclusion particles.

The essence of the enumeration was encapsulated in the general form of the $(1, 2)$ -exclusion matrix

$$H_{1,2} = \begin{pmatrix} \tilde{s}_1 & f_1 & 0 & \cdots & 0 & 0 \\ g_1 & \tilde{s}_2 & f_2 & \cdots & 0 & 0 \\ 0 & g_2 & \tilde{s}_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{s}_{q-1} & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-1} & \tilde{s}_q \end{pmatrix}. \quad (3.12)$$

In addition to the two subdiagonals f_k and g_k , a hallmark of $g = 2$ exclusion, $H_{1,2}$ also has a nonvanishing \tilde{s}_k main diagonal, a hallmark of $g = 1$ statistics, i.e., Fermi-Dirac statistics, as it indeed describes particles obeying a mixture of the two statistics $g = 1$ and $g = 2$.

The generalization to a mixture of $g = 1$ and g exclusion leads to the $(1, g)$ -exclusion matrix (again, assuming zero umklapp matrix elements at the off-diagonal corners) which encapsulates the associate algebraic area enumeration

$$H_{1,g} = \begin{pmatrix} \tilde{s}_1 & f_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \tilde{s}_2 & f_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ g_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & g_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-g+1} & 0 & 0 & \cdots & \tilde{s}_q \end{pmatrix}. \quad (3.13)$$

The secular determinant reads

$$\det(I - zH_{1,g}) = \sum_{n=0}^q (-z)^n Z_n. \quad (3.14)$$

Similar to the discussion of $H_{1,2}$ in Section 2.2, here Z_n can be interpreted as the n -body partition function for particles in a one-body spectrum ϵ_k ($k = 1, 2, \dots, q$) with fermions occupying one-body energy level k with Boltzmann factor $e^{-\beta\epsilon_k} = \tilde{s}_k$ and g -fermion bound states with g particles occupying g successive one-body energy levels $k, k+1, \dots, k+g-1$ with Boltzmann factor⁴

$$e^{-\beta\epsilon_{k, \dots, k+g-1}} = (-1)^{g-1} s_k := (-1)^{g-1} g_k f_k f_{k+1} \cdots f_{k+g-2} \quad (3.15)$$

behaving effectively as g -exclusion particles. The secular determinant $\det(I - zH_{1,g})$ becomes a grand partition function with $-z$ playing the role of the fugacity parameter. For example, for $g = 2$ and $q = 5$

$$\begin{aligned} Z_4 = & \tilde{s}_4 \tilde{s}_3 \tilde{s}_2 \tilde{s}_1 + \tilde{s}_5 \tilde{s}_3 \tilde{s}_2 \tilde{s}_1 + \tilde{s}_5 \tilde{s}_4 \tilde{s}_2 \tilde{s}_1 + \tilde{s}_5 \tilde{s}_4 \tilde{s}_3 \tilde{s}_1 + \tilde{s}_5 \tilde{s}_4 \tilde{s}_3 \tilde{s}_2 \\ & + \tilde{s}_4 \tilde{s}_3 (-s_1) + \tilde{s}_5 \tilde{s}_3 (-s_1) + \tilde{s}_5 \tilde{s}_4 (-s_1) + \tilde{s}_4 \tilde{s}_1 (-s_2) + \tilde{s}_5 \tilde{s}_1 (-s_2) + \tilde{s}_5 \tilde{s}_4 (-s_2) \\ & + \tilde{s}_2 \tilde{s}_1 (-s_3) + \tilde{s}_5 \tilde{s}_1 (-s_3) + \tilde{s}_5 \tilde{s}_2 (-s_3) + \tilde{s}_2 \tilde{s}_1 (-s_4) + \tilde{s}_3 \tilde{s}_1 (-s_4) + \tilde{s}_3 \tilde{s}_2 (-s_4) \\ & + (-s_3)(-s_1) + (-s_4)(-s_1) + (-s_4)(-s_2) \end{aligned}$$

⁴Note that in the pure $g = 2$ case we took the Boltzmann factors of exclusion particles (bound states) as $+s_k$ and compensated by absorbing the negative sign in the fugacity $-z^2$. In the mixed $1, g$ case we have no such flexibility.

can be readily interpreted in Figure 3.6 as the four-body partition function for four particles, either individual fermions or two-fermion bound states, occupying in all possible ways the five one-body levels ϵ_k , $k = 1, \dots, 5$.

Note that when all \tilde{s}_k in (3.13) are set to 0, the Z_{gn+i} ($i = 1, 2, \dots, g-1$) vanish and the Z_{gn} reduces to the n -body partition function (3.6) for g -exclusion particles, that is,

$$\det(I - zH_{1,g}) = \sum_{n=0, \text{ divisible by } g}^q (-z)^n Z_n = \sum_{n=0}^{\lfloor q/g \rfloor} (-z)^{gn} Z_{gn} = \det(I - zH_g), \quad (3.16)$$

where in the last step we identified $(-1)^{(g-1)n} Z_{gn}$ to the Z_n for g -exclusion appearing in $\det(I - zH_g)$ and given in (3.5).

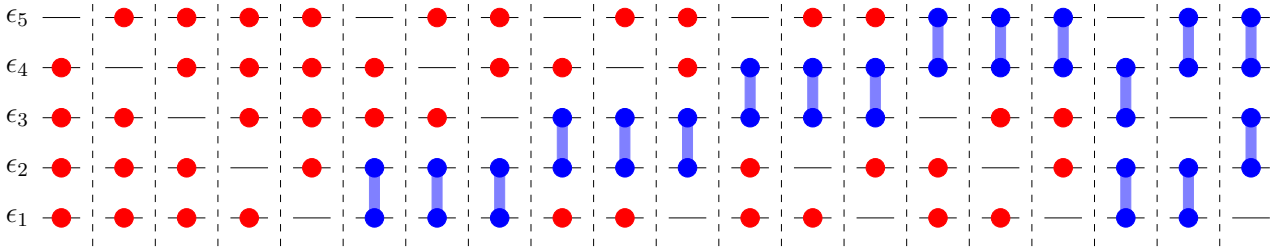


Figure 3.6: Z_4 for $q = 5$: All possible occupancies of the five one-body levels by four particles with either fermions (red) or two-fermion bound states (blue).

From

$$\ln \left(\sum_{n=0}^q Z_n x^n \right) = \sum_{n=1}^{\infty} b_n x^n, \quad (3.17)$$

we infer the associated cluster coefficients

$$b_{\mathbf{n}} = (-1)^{n+1} \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j \\ (1,g)\text{-composition of } \mathbf{n}}} c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j) \sum_{k=1}^{q-j-g+2} \tilde{s}_k^{\tilde{l}_1} s_k^{l_1} \tilde{s}_{k+1}^{\tilde{l}_2} s_{k+1}^{l_2} \dots \quad (3.18)$$

We define the sequence of integers $\tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_{j+g-1}; l_1, l_2, \dots, l_j$, $j \geq 0$, as a $(1, g)$ -composition of \mathbf{n} if they satisfy the conditions

$$\begin{aligned} \mathbf{n} &= (\tilde{l}_1 + \tilde{l}_2 + \dots + \tilde{l}_{j+g-1}) + g(l_1 + l_2 + \dots + l_j) \\ \tilde{l}_i &\geq 0; l_i \geq 0, l_1, l_j > 0, \text{ at most } g-2 \text{ successive vanishing } l_i. \end{aligned} \quad (3.19)$$

That is, the l_j 's are the usual g -compositions of integers $1, 2, \dots, \lfloor \mathbf{n}/g \rfloor$ and the \tilde{l}_i 's are additional nonnegative integers. (We also include the trivial composition $\tilde{l}_1 = \mathbf{n}$.) For example, there are seven $(1, 3)$ -compositions of 5, namely

$$\begin{aligned} j = 0: & (5); \\ j = 1: & (2, 0, 0; 1), (1, 1, 0; 1), (1, 0, 1; 1), (0, 2, 0; 1), (0, 1, 1; 1), (0, 0, 2; 1), \end{aligned}$$

and five $(1, 4)$ -compositions of 5, namely,

$$\begin{aligned} j = 0: & (5); \\ j = 1: & (1, 0, 0, 0; 1), (0, 1, 0, 0; 1), (0, 0, 1, 0; 1), (0, 0, 0, 1; 1). \end{aligned}$$

The $c_{1,g}(\tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_{j+g-1}; l_1, l_2, \dots, l_j)$ in (3.18) read

$$c_{1,g}(\tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_{j+g-1}; l_1, l_2, \dots, l_j) = \frac{(\tilde{l}_1 + l_1 - 1)!}{\tilde{l}_1! l_1!} \prod_{k=2}^{j+g-1} \binom{\tilde{l}_k + \sum_{i=k-g+1}^k l_i - 1}{\sum_{i=k-g+1}^{k-1} l_i - 1, \tilde{l}_k, l_k} \quad (3.20)$$

with $l_i = 0$ for $i \leq 0$ or $i > j$ as usual. It is clear that when $\tilde{l}_i = 0$ only the standard g -composition survives so that the coefficients $c_{1,g}$ in (3.20) go over to c_g in (3.9). Equivalently, when $\tilde{s}_i = 0$, terms with nonvanishing \tilde{l}_i in (3.18) drop and we recover the g -exclusion cluster coefficients. Note that the inverse of a composition, defined by inverting the order of the \tilde{l}_i and of the l_i : $\tilde{l}_i \rightarrow \tilde{l}_{j+g-i}$, $l_i \rightarrow l_{j+1-i}$, leaves $c_{1,g}$ invariant.

Finally,

$$\text{tr } H_{1,g}^{\mathbf{n}} = \mathbf{n}(-1)^{\mathbf{n}+1} b_{\mathbf{n}} = \mathbf{n} \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j \\ (1,g)\text{-composition of } \mathbf{n}}} c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j) \sum_{k=1}^{q-j-g+2} \tilde{s}_k^{\tilde{l}_1} s_k^{l_1} \tilde{s}_{k+1}^{\tilde{l}_2} s_{k+1}^{l_2} \dots \quad (3.21)$$

We will now discuss the combinatorial interpretation of $(1, g)$ -compositions and $c_{1,g}$.

$(1, g)$ -compositions already have a combinatorial interpretation, deriving from their relation to cluster coefficients of $(1, g)$ -exclusion statistics. Specifically, $(1, g)$ -compositions correspond to all *distinct connected* arrangements of \mathbf{n} particles on a one-body spectrum, either alone or in g -fermion bound states. That is, they correspond to all the possible ways to place particles and bound states such that they cannot be separated into two or more mutually non-overlapping groups (see Figure 3.7). If the arrangement covers $j+g-1$ consecutive one-body levels $k+i-1$, where $i = 1, \dots, j+g-1$, then \tilde{l}_i is the number of single particles on one-body level $k+i-1$ and l_i is the number of g -fermion bound states that extend over the g levels $k+i-1$ to $k+i+g-2$, where $i = 1, \dots, j$.

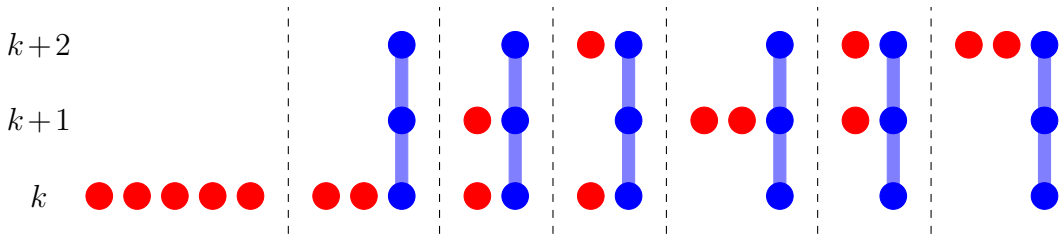


Figure 3.7: Seven $(1, 3)$ -compositions of 5: (5) , $(2, 0, 0; 1)$, $(1, 1, 0; 1)$, $(1, 0, 1; 1)$, $(0, 2, 0; 1)$, $(0, 1, 1; 1)$, $(0, 0, 2; 1)$, illustrated by fermions (red) and three-fermion bound states (blue).

Based on the above combinatorial interpretation, by counting all possibilities of a given number of g -fermion bound states occupying a given height and filling single fermions into these levels, the

number of $(1, g)$ -compositions of a given integer \mathbf{n} can be derived as

$$N_{1,g}(\mathbf{n}) = 1 + \sum_{k=0}^{\lfloor \mathbf{n}/g \rfloor - 1} \sum_{m=0}^{(g-1)k} \binom{k}{m}_g \binom{\mathbf{n} + m - gk - 1}{m + g - 1}, \quad (3.22)$$

where the g -nomial coefficient is defined as

$$\binom{k}{m}_g = [x^m](1 + x + x^2 + \cdots + x^{g-1})^k = [x^m] \left(\frac{1 - x^g}{1 - x} \right)^k = \sum_{j=0}^{\lfloor m/g \rfloor} (-1)^j \binom{k}{j} \binom{k + m - gj - 1}{k - 1}.$$

For $g = 2$, it reduces to the binomial coefficient $\binom{k}{m}_2 = \binom{k}{m}$. Equivalently, the generating function of the $N_{1,g}(\mathbf{n})$ is

$$\sum_{\mathbf{n}=0}^{\infty} x^{\mathbf{n}} N_{1,g}(\mathbf{n}) = \frac{(1-x)^{g-2}(1+x^{g-1}-x^g) - x^{g-1}}{(1-x)^{g-1}(1+x^{g-1}-x^g) - x^{g-1}}.$$

To provide a combinatorial interpretation for the multiplicity coefficients $c_{1,g}$, we refer back to the trace of the \mathbf{n} -th power of $H_{1,g}$. In terms of the matrix elements h_{ij} of $H_{1,g}^T$ in (3.13), this trace can be written as

$$\text{tr } H_{1,g}^{\mathbf{n}} = \sum_{k_1=1}^q \sum_{k_2=1}^q \cdots \sum_{k_{\mathbf{n}}=1}^q h_{k_1 k_2} h_{k_2 k_3} \cdots h_{k_{\mathbf{n}} k_1}. \quad (3.23)$$

The structure of the $(1, g)$ -exclusion matrix (3.13) implies that (3.23) is a sum of products of \mathbf{n} factors $h_{k_i k_{i+1}}$ with indices such that $k_{i+1} - k_i = g - 1, 0$, or -1 . We map the sequence of indices $k_1, k_2, \dots, k_{\mathbf{n}-1}, k_{\mathbf{n}}, k_1$ to the heights (i.e., floors) of a periodic generalized $[g - 1, 0, -1]$ Motzkin path ("bridge") of length \mathbf{n} starting and ending at floor k_1 , with vertical steps up by $g - 1$ floors or down by 1 floor as well as horizontal steps (see Figure 3.8 for an example). Evaluating the trace (3.23) amounts to summing the corresponding products over all such periodic paths. We note that periodic paths must have $g - 1$ down steps for each up step.

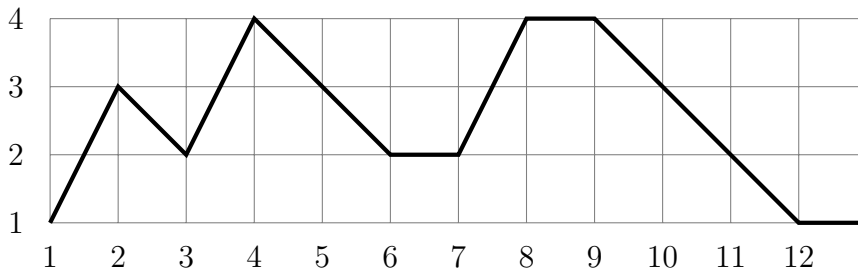


Figure 3.8: A periodic generalized $[2, 0, -1]$ Motzkin path of length $\mathbf{n} = 12$ corresponding to the $(1, 3)$ -composition $1, 1, 0, 1; 1, 2$, starting with an up step from the first floor.

As in the g -exclusion case, to group together terms with the same weight $h_{k_1 k_2} \cdots h_{k_{\mathbf{n}} k_1}$ we need to consider paths with a fixed number of transitions per floor. For each path that reaches a lowest floor k and highest floor from which an up step starts $k + j - 1$, and thus highest floor reached $k + j + g - 2$, we denote by l_1, l_2, \dots, l_j the number of up steps from floors $k, k + 1, \dots, k + j - 1$ and by $\tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_{j+g-1}$

the number of horizontal steps at floors $k, k+1, \dots, k+j-g-2$. Clearly, $\tilde{l}_1 + \dots + \tilde{l}_{j+g-1} + g(l_1 + \dots + l_j) = \mathbf{n}$, and at most $g-2$ successive l_i can vanish, since up steps can skip $g-2$ floors. Therefore, $\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j$ is a $(1, g)$ -composition of \mathbf{n} . As before, each up step $k_i \rightarrow k_i + g - 1$ necessarily implies down steps $k_i + g - 1 \rightarrow k_i + g - 2, \dots, k_i + 1 \rightarrow k_i$, so factors in each term in (3.23) corresponding to each up step $k_i \rightarrow k_i + g - 1$ contribute the full combination $h_{k_i, k_i + g - 1} h_{k_i + g - 1, k_i + g - 2} \dots h_{k_i + 1, k_i} = g_{k_i} f_{k_i + g - 2} \dots f_{k_i} = s_{k_i}$.

Altogether, the sum in (3.23) rewrites as

$$\mathrm{tr} H_{1,g}^{\mathbf{n}} = \sum_{k=1}^{q-j-g+2} \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j \\ (1,g)\text{-composition of } \mathbf{n}}} C_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j) \tilde{s}_k^{\tilde{l}_1} s_k^{l_1} \tilde{s}_{k+1}^{\tilde{l}_2} s_{k+1}^{l_2} \dots,$$

where $C_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j)$ is the number of periodic generalized Motzkin paths of length \mathbf{n} with \tilde{l}_1 horizontal steps and l_1 up steps originating from the first floor, \tilde{l}_2 and l_2 from the second floor, etc. The sum over k ensures that paths of all possible starting at floor k_1 in (3.23) are included. Comparing with (3.21), we see that

$$C_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j) = \mathbf{n} c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j).$$

Therefore, $\mathbf{n} c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j)$ admits the interpretation of the number of periodic generalized $[g-1, 0, -1]$ Motzkin paths with horizontal and up steps as defined above.

The number of such paths starting with an up step, respectively a horizontal step, from floor i can also be deduced as $l_i c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j)$, respectively $\tilde{l}_i c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j)$, while the total number of paths starting from floor i is

$$\left(\tilde{l}_i + \sum_{k=i-g+1}^i l_k \right) c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j).$$

The result $(\tilde{l}_1 + l_1) c_{1,2}$ for $i = 1$ Motzkin excursions was also derived in [47]. Finally, the number of paths starting at floor i with a down step can be deduced as

$$\left(\sum_{k=i-g+1}^{i-1} l_k \right) c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j).$$

The proof of the above counting formulas can be obtained similarly to the case of g -exclusion and generalized Dyck paths. A simple method is based on cyclic permutations as outlined in Section 3.2. Consider first paths that start with an up step from the first floor. A combinatorial argument entirely analogous to the one in Section 3.2 yields the result $l_1 c_{1,g}$ for the number of such paths, and by periodic permutation and reduction by an overcounting factor of l_1 the total number of paths is obtained as $\mathbf{n} c_{1,g}$. A repetition of the periodic argument from floor i , then, produces the results $l_i c_{1,g}$ and $\tilde{l}_i c_{1,g}$ for the number of paths starting up or horizontally from floor i , and a ‘‘cutting and exchanging’’ argument similar to the ones in Section 3.2 gives the number of paths starting down from floor i .

Finally, we note that the number of all unrestricted periodic generalized $[g-1, 0, -1]$ Motzkin paths is obtained by summing $c_{1,g}$ over all $(1, g)$ -compositions and yields the relation

$$\sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j \\ (1,g)\text{-composition of } \mathbf{n}}} c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j) = [x^0](x^{g-1} + 1 + x^{-1})^{\mathbf{n}} = \sum_{k=0}^{\lfloor \mathbf{n}/g \rfloor} \binom{\mathbf{n}}{gk} \binom{gk}{k}.$$

3.4 Conclusion and perspectives

We have established a connection between the enumeration of lattice walks according to their algebraic area, quantum exclusion statistics, and the combinatorics of generalized Dyck and Motzkin paths (also known as Łukasiewicz paths). The key common quantities are the coefficients $c_g(l_1, \dots, l_j)$ and $c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j)$ labeled by the g -compositions and the $(1, g)$ -compositions of the length of the walks. These coefficients appear as essential building blocks of the algebraic area partition function of walks on square or honeycomb lattices, the cluster coefficients g -exclusion and $(1, g)$ -exclusion statistical systems, and the counting of generalized paths with specific number of steps from each visited floor. The connection of Dyck paths and $g = 2$ exclusion statistics was established in [48] and used to calculate the length and area generating function for such paths, and the method was extended to Motzkin paths in [49].

The concept of exclusion statistics and related compositions naturally generalizes to (g', g) and more general (g_1, g_2, \dots, g_n) statistics and compositions, and the statistical mechanical properties of these systems and mathematical properties of their compositions are of interest. It would also be worthwhile to derive the corresponding combinatorial quantities c_{g_1, \dots, g_n} and study their relevance for generalized Łukasiewicz paths.

In a different direction, it is known that Dyck and Motzkin paths appear in various contexts in physics and mathematics. In physics, they appear in percolation processes, interfaces between fluids of different surface tension, and other statistical systems such as long polymer molecules in solution, where the generalized weighted paths are introduced to study the interactions with the boundary and the polymer is “adsorbed” when the attractive force is sufficiently strong (see, e.g., [50]). It would be challenging to solve it in this framework. Further, Dyck and Motzkin paths can be mapped to spin-1/2 and spin-1 chains. For example, in [51] a spin-1 frustration-free Hamiltonian was constructed using Motzkin paths. Finally, in knot theory, the Temperley–Lieb algebra can have a representation based on Dyck paths, while, if empty vertices (vertices not incident to an edge) are allowed, Motzkin paths become relevant [52]. The extension of this connection to more general paths, and the meaning of the c_g and $c_{1,g}$ coefficients in this context, are nontrivial issues that deserve further study.

Article 2: Combinatorics of generalized Dyck and Motzkin paths

Li Gan, Stéphane Ouvry, and Alexios P. Polychronakos
Phys. Rev. E **106**, 044123 (2022)

Combinatorics of generalized Dyck and Motzkin paths

Li Gan^{1,*}, Stéphane Ouvry^{1,†} and Alexios P. Polychronakos^{2,‡}¹LPTMS, CNRS, Université Paris-Saclay, 91405 Orsay Cedex, France²Physics Department, the City College of New York, New York 10031, USA
and The Graduate Center of CUNY, New York, New York 10016, USA

(Received 19 July 2022; accepted 16 September 2022; published 14 October 2022)

We relate the combinatorics of periodic generalized Dyck and Motzkin paths to the cluster coefficients of particles obeying generalized exclusion statistics, and obtain explicit expressions for the counting of paths with a fixed number of steps of each kind at each vertical coordinate. A class of generalized compositions of the integer path length emerges in the analysis.

DOI: [10.1103/PhysRevE.106.044123](https://doi.org/10.1103/PhysRevE.106.044123)

I. INTRODUCTION

Enumerating closed random walks on various lattices according to their algebraic area amounts to computing traces of the \mathbf{n} th power of quantum Hamiltonians, where \mathbf{n} is the length of the walks. This approach was initiated in the study of random walks on the square lattice by relating the problem of their enumeration to the Hofstadter model of a charged particle hopping on the lattice in the presence of a magnetic field [1], and we shall call such Hamiltonians Hofstadter-like. The generating function of walks weighted by their length and algebraic area maps to the secular determinant of the Hamiltonian. For specific choices of the parameter dual to the area, these Hamiltonians can be reduced to finite-size matrices whose near-diagonal structure depends crucially on the type of walks considered.

Previous work on the subject relied on the computation of the secular determinants of these matrices [2,3]. Progress in this direction was achieved by interpreting these determinants as grand partition functions for systems of particles obeying generalized g -exclusion quantum statistics in an appropriate one-body spectrum [4] (g is a positive integer, $g = 0$ being boson statistics, $g = 1$ fermion statistics). Expressing these partition functions in terms of their corresponding cluster coefficients yields, in turn, the sought-after traces. In this process, new combinatorial coefficients $c_g(l_1, l_2, \dots, l_j)$ appear, labeled by the g -compositions of $n = l_1 + l_2 + \dots + l_j$ with $\mathbf{n} = gn$ ($g = 2$ reproducing standard compositions). However, a direct combinatorial interpretation of these coefficients was missing.

In this work we bypass the secular determinant and instead tackle directly the trace of the \mathbf{n} th power of the matrices devised to enumerate closed walks on various lattices according to their algebraic area. We relate the expression for the trace to periodic generalized Dyck paths (or Łukasiewicz paths) on

a square lattice with gn horizontal unit steps to the right going vertically either $g - 1$ units up or one unit down per step and never dipping below vertical coordinate 0 ($g = 2$ reproducing the usual periodic Dyck paths). By “periodic generalized Dyck paths” here and in the sequel we always mean generalized Dyck bridges (and excursions) with the constraint for the path to be in a strip of a given width (see Fig. 1). Meanders and more general paths are not relevant in this paper. More precisely calling “floor i ” the level at vertical coordinate $i - 1$, we demonstrate that $gn c_g(l_1, l_2, \dots, l_j)$ counts the number of all possible such paths with l_1 up steps from floor 1, l_2 up steps from floor 2, \dots , l_j up steps from floor j , for a total of $\mathbf{n} = gn$ steps inside the strip between floors 1 and $j + g - 1$. In fact, we obtain an even more detailed enumeration of these paths by providing the count of paths starting with an up step from a given i th floor among the $j + g - 1$ floors, and similarly of paths starting with a down step.

We further extend our results to the enumeration of generalized periodic Motzkin paths that can also move by horizontal unit steps, by relating such paths to matrices corresponding to mixed $(1, g)$ -exclusion statistics for particles having either fermionic $g = 1$ or g -exclusion statistics. The derived expressions for the corresponding combinatorial coefficients $c_{1,g}$ counting such paths with a fixed number of horizontal, up, or down steps for each floor are labeled by a further generalized $(1, g)$ -composition of the number of steps \mathbf{n} . The extension to other classes of paths, corresponding to other generalizations of quantum exclusion statistics, appears to be within reach of our method.

II. SQUARE LATTICE WALKS: THE HOFSTADTER MODEL

We start with the original algebraic area enumeration problem for closed walks on a square lattice: Among the $\binom{\mathbf{n}}{\mathbf{n}/2}$ closed \mathbf{n} -steps walks that one can draw, how many of them enclose a given algebraic area A ? Note that, for closed walks, \mathbf{n} is necessarily even, $\mathbf{n} = 2n$.

The algebraic area enclosed by a walk is weighted by its *winding numbers*: If the walk moves around a region in a

*li.gan92@gmail.com

†stephane.ouvry@u-psud.fr

‡apolychronakos@ccny.cuny.edu

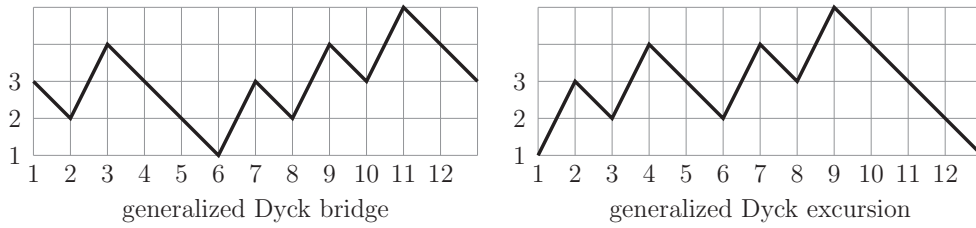


FIG. 1. A generalized Dyck bridge and a generalized Dyck excursion for $g = 3$, which starts from the first floor, of length $\mathbf{n} = 12$ with $l_1 = 1$ up step from the first floor, $l_2 = 2$ up steps from the second floor, and $l_3 = 1$ up step from the third floor. We refer to them as periodic generalized Dyck paths.

counterclockwise (positive) direction its area counts as positive, otherwise negative; if the walk winds around more than once, the area is counted with multiplicity (see Fig. 2). These regions inside the walk are called winding sectors. Calling S_m the arithmetic area of the m -winding sector inside a walk (i.e., the total number of lattice cells it encloses with winding number m , where m can be positive or negative) the algebraic area is

$$A = \sum_{m=-\infty}^{\infty} mS_m.$$

Counting the number of closed walks of length \mathbf{n} on the square lattice enclosing an algebraic area A can be achieved by introducing two lattice hopping operators u and v in the right and up directions obeying

$$v u = Q u v,$$

and, as a consequence, such that the u and v independent part in

$$(u + u^{-1} + v + v^{-1})^{\mathbf{n}} = \sum_A C_{\mathbf{n}}(A) Q^A + \dots \quad (1)$$

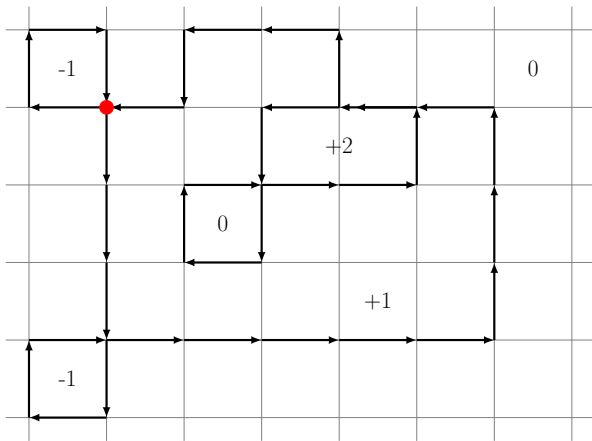


FIG. 2. A closed walk of length $\mathbf{n} = 36$ starting from and returning to the same bullet (red) point with winding sectors $m = +2, +1, 0, -1$ and various numbers of lattice cells per winding sectors, respectively 2, 14, 1, 2. The 0-winding number inside the walk arises from a superposition of $+1$ and -1 windings. Taking into account the nonzero winding sectors we end up with an algebraic area $A = (+2) \times 2 + (+1) \times 14 + (-1) \times 2 = 16$. Note the double arrow on the horizontal link which indicates that the walk has moved twice on this link, here in the same left direction.

counts the number $C_{\mathbf{n}}(A)$ of walks enclosing area A . For example, $(u + u^{-1} + v + v^{-1})^4 = 28 + 4Q + 4Q^{-1} + \dots$ tells that among the $\binom{4}{2}^2 = 36$ closed walks of length 4, $C_4(0) = 28$ enclose an area $A = 0$ and $C_4(1) = C_4(-1) = 4$ enclose an area $A = \pm 1$.

Provided that Q is interpreted as $Q = e^{i2\pi\Phi/\Phi_0}$ where Φ is the flux of an external magnetic field through the unit lattice cell and Φ_0 the flux quantum,

$$H = u + u^{-1} + v + v^{-1}$$

becomes the Hamiltonian for a quantum particle hopping on a square lattice and coupled to a perpendicular magnetic field, i.e., the Hofstadter model [1]. Selecting in (1) the u, v independent part of $(u + u^{-1} + v + v^{-1})^{\mathbf{n}}$ translates in the quantum world to focusing on the trace of $H^{\mathbf{n}}$ with the normalization $\text{Tr} I = 1$, where I is the identity operator. It follows that

$$\text{Tr} H^{\mathbf{n}} = \sum_A C_{\mathbf{n}}(A) Q^A, \quad (2)$$

i.e., the trace gives the generating function of walks weighted by their algebraic area.

When the flux is rational, $Q = e^{i2\pi p/q}$ with p, q coprime integers, the lattice operators u and v can be represented by $q \times q$ matrices

$$u = e^{ik_x} \begin{pmatrix} Q & 0 & 0 & \dots & 0 & 0 \\ 0 & Q^2 & 0 & \dots & 0 & 0 \\ 0 & 0 & Q^3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & Q^{q-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & Q^q \end{pmatrix},$$

$$v = e^{ik_y} \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

where k_x and k_y are quasimomenta in the x and y directions. It follows that H becomes a $q \times q$ matrix as well, and computing the trace $\text{Tr} H^{\mathbf{n}}$ amounts to taking the matrix trace and integrating over k_x and k_y and dividing by $(2\pi)^2 q$ for a proper normalization.

One way to evaluate this trace is to compute the secular determinant of H , namely $\det(I - zH)$. To do so one first

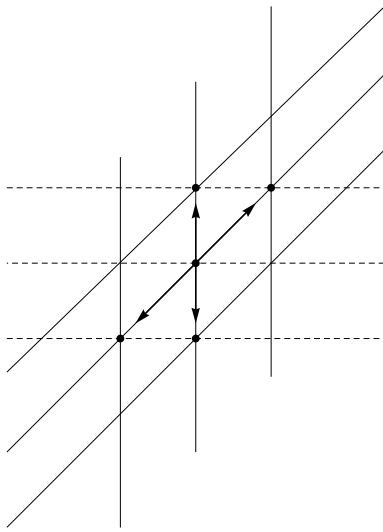


FIG. 3. The deformed square lattice walk steps after the modular transformation.

performs on u and v the modular transformation

$$u \rightarrow -uv, \quad v \rightarrow v,$$

which preserves the relation $vu = Quv$ and the corresponding traces. It amounts to looking at lattice walks on the deformed square lattice of Fig. 3.

The Hofstadter matrix becomes

$$H = -uv - v^{-1}u^{-1} + v + v^{-1} = \begin{pmatrix} 0 & \omega_1 & 0 & \cdots & 0 & \bar{\omega}_q \\ \bar{\omega}_1 & 0 & \omega_2 & \cdots & 0 & 0 \\ 0 & \bar{\omega}_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \omega_{q-1} \\ \omega_q & 0 & 0 & \cdots & \bar{\omega}_{q-1} & 0 \end{pmatrix} \quad (3)$$

with $\omega_k = (1 - Q^k e^{ik_x})e^{iky}$. Its secular determinant reads

$$\det(I - zH) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z(n) z^{2n} - 2[\cos(qk_y) - \cos(qk_x + qk_y)]z^q \quad (4)$$

with coefficients $Z(n)$ which rewrite as trigonometric multiple nested sums [2]

$$Z(n) = \sum_{k_1=1}^{q-2n+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+2n-2} s_{k_2+2n-4} \cdots s_{k_{n-1}+2} s_{k_n}, \quad (5)$$

where $s_k = (1 - Q^k)(1 - Q^{-k}) = 4 \sin^2(\pi k p/q)$ [by definition $Z(0) = 1$].

From the $Z(n)$'s in (5) the algebraic area enumeration can proceed [3,4] via

$$\log \left\{ \sum_{n=0}^{\lfloor q/2 \rfloor} Z(n) z^n \right\} = \sum_{n=1}^{\infty} b(n) z^n. \quad (6)$$

The $b(n)$'s rewrite as linear combinations of trigonometric simple sums

$$b(n) = (-1)^{n+1} \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} c_2(l_1, l_2, \dots, l_j) \sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}, \quad (7)$$

where

$$c_2(l_1, l_2, \dots, l_j) = \binom{l_1+l_2}{l_1} l_2 \binom{l_2+l_3}{l_2} \cdots l_{j-1} \binom{l_{j-1}+l_j}{l_{j-1}} \quad (8)$$

is labeled by the compositions l_1, l_2, \dots, l_j of n , i.e., the 2^{n-1} ordered partitions of n ; for example, for $n = 3$ one has the four composition $3 = 2 + 1 = 1 + 2 = 1 + 1 + 1$.

Finally, thanks to the identity $\log \det(I - zM) = \text{tr} \log(I - zM)$, where tr stands for the usual trace of the matrix M , we can show that the sought-after trace reduces to

$$\text{Tr } H^{n=2n} = 2n(-1)^{n+1} \frac{1}{q} b(n)$$

so that

$$\text{Tr } H^{n=2n} = 2n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} c_2(l_1, l_2, \dots, l_j) \times \frac{1}{q} \sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}. \quad (9)$$

The trigonometric simple sum $\sum_{k=1}^{q-j} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}$ in (9) remains to be computed, which in turn yields the desired algebraic area enumeration via (2).

Looking at the structure of (5) one realizes that, if s_k is interpreted as a spectral function (Boltzmann factor),

$$s_k = e^{-\beta \epsilon_k},$$

where β is the inverse temperature and ϵ_k a one-body spectrum labeled by an integer k , then $Z(n)$ is the n -body partition function for n particles with one-body spectrum ϵ_k and obeying $g = 2$ exclusion statistics (no two particles can occupy two adjacent quantum states)¹ and (4) identifies $\det(1 - zH)$ as the grand-canonical partition function for exclusion-2 particles in this spectrum with fugacity parameter $-z^2$. Exclusion statistics is a purely quantum concept which describes the statistical mechanical properties of identical particles. Usual particles

¹For example in the three-body case one has

$$Z(3) = \sum_{k_1=1}^{q-5} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} s_{k_1+4} s_{k_2+2} s_{k_3}$$

i.e., for $q = 7$

$$Z(3) = s_5 s_3 s_1 + s_6 s_3 s_1 + s_6 s_4 s_1 + s_6 s_4 s_2 = \sum_{6 \geq k_1 \geq k_2+2, k_2 \geq k_3+2} s_{k_1} s_{k_2} s_{k_3},$$

where indeed no adjacent one-body quantum states contribute. This is the hallmark of $g = 2$ exclusion with the $+2$ shifts in the nested multiple sums.

are either bosons ($g = 0$) or fermions ($g = 1$). Square lattice walks invoke statistics $g = 2$, beyond Fermi exclusion. In this context, (6) identifies the $b(n)$'s as nothing but the cluster coefficients of the $Z(n)$'s.

III. g -EXCLUSION

We can go a step further by setting the quasimomenta k_x and k_y to zero, since (4) makes it evident that they do not appear in the $Z(n)$'s. This sets the corners of the Hofstadter matrix (3) to zero,² so that H becomes a particular case of the general class of $g = 2$ exclusion matrices

$$H_2 = \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 & 0 \\ g_1 & 0 & f_2 & \cdots & 0 & 0 \\ 0 & g_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-1} & 0 \end{pmatrix}, \quad (10)$$

whose secular determinant $\det(I - zH_2) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z(n) z^{2n}$ leads to the $Z(n)$'s and the $b(n)$'s in (5), (7), (8), and (9) with $s_k = g_k f_k$ as spectral function.³ So the enumeration of square lattice walks according to their algebraic area is captured by the $g = 2$ exclusion matrix (10), whose hallmark is a vanishing diagonal flanked by two nonvanishing subdiagonals f_k and g_k .

The generalization to $g = 3$ exclusion leads to the natural matrix form of H ,

$$H_3 = \begin{pmatrix} 0 & f_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & f_2 & 0 & \cdots & 0 & 0 & 0 \\ g_1 & 0 & 0 & f_3 & \cdots & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & f_{q-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & f_{q-1} \\ 0 & 0 & 0 & 0 & \cdots & g_{q-2} & 0 & 0 \end{pmatrix}.$$

Now two vanishing diagonal and subdiagonals appear between the f_k and g_k subdiagonals (i.e., there is an extra vanishing subdiagonal below the vanishing diagonal). Computing its secular determinant $\det(I - zH_3)$ yields

$$Z(n) = \sum_{k_1=1}^{q-3n+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} s_{k_1+3n-3} s_{k_2+3n-6} \cdots s_{k_{n-1}+3} s_{k_n}$$

with spectral function $s_k = g_k f_k f_{k+1}$. Clearly $Z(n)$ is the partition function of n particles of exclusion statistics $g = 3$ in the one-body spectrum implied by s_k .

In general, for g -exclusion the Hamiltonian is

$$H_g = F(u)v + v^{1-g}G(u),$$

where $F(u)$ and $G(u)$ are scalar functions of u , and amounts to a g -exclusion matrix⁴ (again ignoring the spurious umklapp matrix elements in the corners)

$$H_g = \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & f_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ g_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & g_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-g+1} & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (11)$$

where now $g - 1$ zeros appear between the f_k and g_k subdiagonals. Its secular determinant

$$\det(I - zH_g) = \sum_{n=0}^{\lfloor q/g \rfloor} (-1)^n Z(n) z^{gn} \quad (12)$$

yields

$$Z(n) = \sum_{k_1=1}^{q-gn+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} s_{k_1+gn-g} s_{k_2+gn-2g} \cdots s_{k_{n-1}+g} s_{k_n}$$

with

$$s_k = g_k f_k f_{k+1} \cdots f_{k+g-2}, \quad (13)$$

where $k = 1, 2, \dots, q - g + 1$. Again, as in the $g = 2, 3$ cases, $Z(n)$ admits an interpretation as the partition function of n exclusion g particles in the one-body spectrum implied by the spectral function $s_k = e^{-\beta \epsilon_k}$, with one-body levels labeled by the integer k . From

$$\log \left\{ \sum_{n=0}^{\lfloor q/g \rfloor} Z(n) z^n \right\} = \sum_{n=1}^{\infty} b(n) z^n$$

one infers

$$b(n) = (-1)^{n+1} \sum_{\substack{l_1, l_2, \dots, l_j \\ g\text{-composition of } n}} c_g(l_1, l_2, \dots, l_j) \times \sum_{k=1}^{q-j-g+2} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1} \quad (14)$$

with

$$c_g(l_1, l_2, \dots, l_j) = \frac{1}{l_1} \prod_{i=2}^j \binom{l_i - g + 1 + \cdots + l_i - 1}{l_i} \quad (15)$$

²These particular matrix elements contribute to spurious umklapp terms in (4) and can be ignored.

³The parameter g of g -exclusion should not be confused with the function g_k .

⁴Indeed the Hofstadter Hamiltonian is a $g = 2$ Hamiltonian

$$H = -uv - v^{-1}u^{-1} + v + v^{-1} = (1 - u)v + v^{1-2}(1 - u^{-1}).$$

with the convention $l_i = 0$ if $i \leq 0$. Finally

$$\begin{aligned} \text{tr } H_g^{\mathbf{n}=gn} &= gn(-1)^{n+1}b(n) \\ &= gn \sum_{\substack{l_1, l_2, \dots, l_j \\ g\text{-composition of } n}} c_g(l_1, l_2, \dots, l_j) \\ &\quad \times \sum_{k=1}^{q-j-g+2} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}. \end{aligned} \quad (16)$$

These expressions generalize (5), (7), (8) and (9) to g -exclusion statistics, where now in (14) and (16) one has to sum over the g -compositions of the integer n , obtained by inserting at will inside the usual compositions (i.e., the 2-compositions) no more than $g-2$ zeros in succession, i.e., obtained by allowing up to $g-2$ consecutive integers in the composition to vanish. For example, one has nine $g=3$ -compositions of $n=3$, namely $n=3=2+1=1+2=1+1+1=2+0+1=1+0+2=1+0+1+1=1+1+0+1=1+0+1+0+1$. In general there are g^{n-1} such g -compositions of the integer n (see Ref. [5] for an analysis of these extended compositions, also called multicompositions).

IV. DYCK PATH COMBINATORICS

We now turn to giving a combinatorial interpretation to the numbers $c_g(l_1, l_2, \dots, l_j)$ in (15), l_1, l_2, \dots, l_j being a g -composition of n . Specifically, we address the question: Is there a class of objects whose counting would be determined by these numbers?

We recall that $c_g(l_1, l_2, \dots, l_j)$ was obtained by considering the secular determinant (12) and the resulting n -body partition functions $Z(n)$ of the g -exclusion matrix (11), and then by turning to the associated cluster coefficient $b(n)$ in (14). On the other hand one sees that these coefficients appear in the trace of the \mathbf{n} th power of the g -exclusion matrix H_g . Let us consider directly this trace and denote h_{ij} the matrix elements of H_g^T . The matrix trace of $H_g^{\mathbf{n}}$ becomes

$$\text{tr } H_g^{\mathbf{n}} = \sum_{k_1=1}^q \sum_{k_2=1}^q \cdots \sum_{k_n=1}^q h_{k_1 k_2} h_{k_2 k_3} \cdots h_{k_n k_1}. \quad (17)$$

The structure of the g -exclusion matrix (11) implies that (17) is a sum of products of \mathbf{n} factors $h_{k_i k_{i+1}}$ with indices such that $k_{i+1} - k_i$ take values $g-1$ or -1 .

We map the sequence of indices $k_1, k_2, \dots, k_{n-1}, k_n, k_1$ to the heights of a periodic generalized Dyck path of length \mathbf{n} starting and ending at height k_1 , with vertical steps up by $g-1$ units or down by 1 unit, denoted as a $[g-1, -1]$ Dyck path (Fig. 4 depicts an example of a $g=3$ path). Evaluating the trace (17) amounts to summing the corresponding products over all such periodic paths, an expression clearly evoking a path integral. We note that periodic paths must have n up steps and $n(g-1)$ down steps for a total length $gn = \mathbf{n}$.

To group together terms with the same weight $h_{k_1 k_2} \cdots h_{k_n k_1}$, for each path we denote by l_1, l_2, \dots, l_j the number of up steps starting at one-body level $k, k+1, \dots, k+j-1$ (k is the lowest one-body level reached by the path). Clearly $l_1 + l_2 + \cdots + l_j = n$, and at

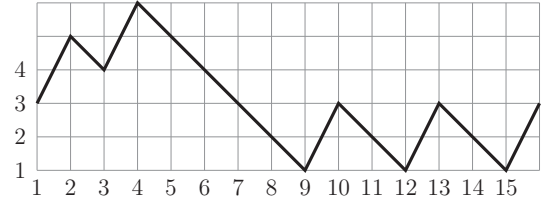


FIG. 4. A periodic generalized Dyck path of length 15 for the $g=3$ composition 3,0,1,1. The path starts from the third floor with an up step.

most $g-2$ successive l_i can vanish, since steps of size $g-1$ can skip $g-2$ levels, so l_1, \dots, l_j is a g -composition of n (Fig. 4 depicts the $g=3$ composition 3,0,1,1). Further, each up step $k_i \rightarrow k_i + g-1$ necessarily implies down steps $k_i + g-1 \rightarrow k_i + g-2, \dots, k_i + 1 \rightarrow k_i$, so factors in each term in (17) corresponding to each up step $k_i \rightarrow k_i + g-1$ contribute the combination

$$h_{k_i, k_i+g-1} h_{k_i+g-1, k_i+g-2} \cdots h_{k_i+1, k_i} = g_{k_i} f_{k_i+g-2} \cdots f_{k_i} = s_{k_i},$$

where we used (11) and (13). Altogether, the sum in (17) rewrites as

$$\text{tr } H_g^{\mathbf{n}} = \sum_{k=1}^{q-j-g+2} \sum_{\substack{l_1, l_2, \dots, l_j \\ g\text{-composition of } n}} C_g(l_1, l_2, \dots, l_j) s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1},$$

where $C_g(l_1, \dots, l_j)$ is the number of periodic generalized Dyck paths of length gn with l_1 up steps originating from the first floor, l_2 from the second floor, and so on. The sum over k ensures that paths of all starting indices k_1 in (17) are included. (Note that the values of indices $k, k+1, \dots$ from where up steps can originate map to one-body levels in the exclusion interpretation.) Comparing this expression with (14), we see that

$$C_g(l_1, l_2, \dots, l_j) = gn c_g(l_1, l_2, \dots, l_j).$$

Therefore, $gn c_g(l_1, l_2, \dots, l_j)$ admits the combinatorial interpretation of the number of generalized periodic Dyck paths with l_1, \dots, l_j up steps from the first, second, etc. floors as defined above.

A. $g=2$

We focus on the simplest nontrivial case $g=2$ and derive the combinatorics. The combinatorial interpretation of $c_2(l_1, l_2, \dots, l_j)$ was already hinted at in Ref. [3], where it was remarked that $nc_2(l_1, l_2, \dots, l_j)$ counts the number of closed random walks of length $\mathbf{n} = 2n$ on a one-dimensional (1D) lattice starting toward the right, containing l_1 right-left steps on top of each other followed by l_2 right-left steps on top of each other, and so on, as shown for $\mathbf{n} = 6$ in Fig. 5.

It is easy to see that such closed walks map to periodic Dyck paths starting with an up step, by rotating the lattice by $\pi/2$ and performing one horizontal step to the right with each walk step, as in Fig. 6.

We conclude that $nc_2(l_1, l_2, \dots, l_j)$ counts the total number of periodic Dyck paths of length $\mathbf{n} = 2n$ starting with an up step and having l_1 up steps originating from the first floor, l_2 from the second floor, and so on. The remaining count

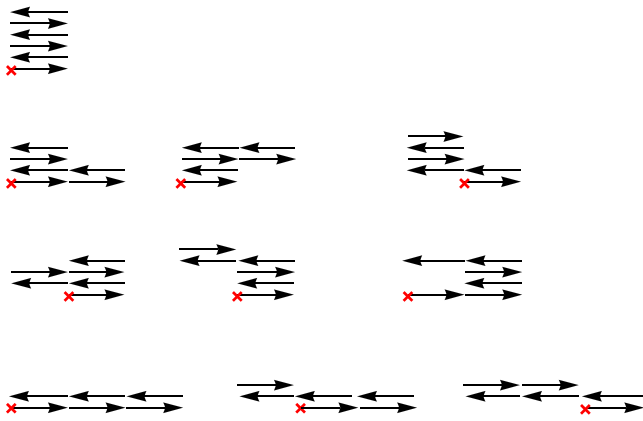


FIG. 5. The 10 closed 1D lattice walks of length $n = 6$ starting to the right. Their counts are, from top, $3c_2(3) = 1$, $3c_2(2, 1) = 3$, $3c_2(1, 2) = 3$, and $3c_2(1, 1, 1) = 3$. The four sets of walks correspond to the four compositions of $n = 3$, namely $3 = 2 + 1 = 1 + 2 = 1 + 1 + 1$. The walks have been spread in the vertical direction for clarity, and to demonstrate their correspondence with Dyck paths. Red crosses denote the starting and ending point of each walk.

$nc_2(l_1, l_2, \dots, l_j)$ corresponds to paths starting with a down step, since for $g = 2$ the two sets of paths map to each other through reflection with respect to the horizontal, for a total of $2nc_2(l_1, l_2, \dots, l_j)$ paths.

We can also infer the more granular Dyck path counting that

$$l_i c_2(l_1, l_2, \dots, l_j) = \prod_{k=1}^{i-1} \binom{l_k + l_{k+1} - 1}{l_k} \times \prod_{k=i}^{j-1} \binom{l_k + l_{k+1} - 1}{l_{k+1}} \quad (18)$$

counts the number of periodic Dyck paths of length $n = 2n$ starting from the i th floor with an up step and having l_1 up steps originating from the first floor, l_2 steps from the second floor, and so on. (The result $l_1 c_2(l_1, l_2, \dots, l_j)$ for $i = 1$ was also derived in Refs. [6,7].) Clearly the sum of the above counts for all $i = 1, 2, \dots, j$ reproduces the total count of starting up paths $nc_2(l_1, l_2, \dots, l_j)$.

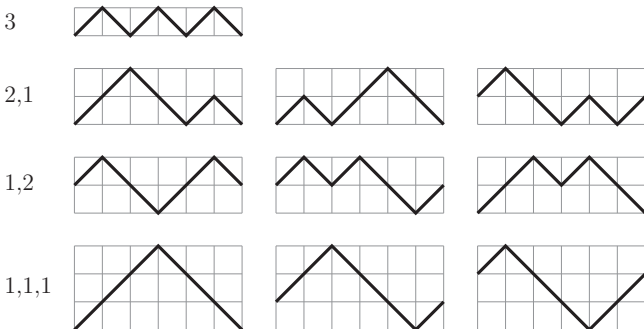


FIG. 6. The 10 periodic Dyck paths of length $n = 6$ starting with an up step. They are in one-to-one correspondence with the 1D walks of Fig. 5.

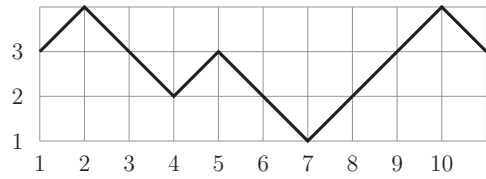


FIG. 7. A periodic Dyck path of length 10 starting at $i = 3$, characterized by the sequence $i_s = 3, 2, 1, 2, 3$ of floors from which the up steps start. One can uniquely reconstruct the path from this sequence by producing the sequence of up step positions $p_s = i - i_s + 2s - 1 = 1, 4, 7, 8, 9$ and filling the gaps with down steps at positions 2, 3, 5, 6, 10.

To give a proof of (18), we remark that the sequence of floors where an up step starts fully determines the path. This is obvious since there is a unique way to “fill in” the remaining down steps to form a path, and can be made explicit by the relation

$$p_s = i - i_s + 2s - 1, \quad s = 1, \dots, n,$$

where $p_s \in [1, 2n]$ is the step of the path at which the s th up step occurs and $i_s \in [1, j]$ is the floor at which it occurs. (p_s, i_s) thus determine both the position and floor at which each up step occurs, fully fixing the path. This is illustrated in Fig. 7 for a periodic path of length 10.

It follows that, to enumerate all possible periodic Dyck paths starting from a given floor i with an up step, it is sufficient to count all the possible sequences of floors where an up step starts, given the constraint that l_1 up steps are on the first floor, l_2 steps on the second floor, and so on. Note, however, that admissible floor sequences satisfy the additional constraint $i_{s+1} \leq i_s + 1$ (arising from $p_{s+1} > p_s$) as well as the starting condition at floor i , namely $i_1 = i$.

To count these configurations efficiently, we start from the top two floors j and $j - 1$. Since these floors are above the floor i of the starting step, the first up step among the l_{j-1} and l_j steps on these floors must necessarily be on the $j - 1$ floor. Now notice that all floor j up steps that are between any two floor $j - 1$ steps *must be connected to the left floor $j - 1$ up step and to each other* (see Fig. 8). Therefore, the configuration of floor j steps is fully fixed by the floor $j - 1$ steps and by the distribution of j floor steps between them. The number of different ways the l_j steps can be distributed among the remaining $l_{j-1} - 1$ steps after the first one is $\binom{l_{j-1} + l_j - 1}{l_j}$.

Moving to the next two floors, $j - 1$ and $j - 2$, we can repeat the above argument between the l_{j-2} and l_{j-1} up steps. Each floor $j - 1$ up step comes with a fixed set of floor j up steps attached and constitutes one compact unit. The distribution between the l_{j-2} steps and the l_{j-1} units, with the first step again on the $j - 2$ floor, fully fixes the positions of the l_{j-1} units, and there are $\binom{l_{j-2} + l_{j-1} - 1}{l_{j-1}}$ such configurations. The argument can be repeated as long as all steps are above i , that is, down to floors i and $i + 1$, giving an overall multiplicity of paths with fixed up steps on floors $i, i - 1, \dots, 1$

$$C_{\text{above } i} = \prod_{k=i}^{j-1} \binom{l_k + l_{k+1} - 1}{l_{k+1}}.$$

Once we dip below i the situation changes. For floors k and $k - 1$, $k \leq i$, the first up step could be either on floor k or on

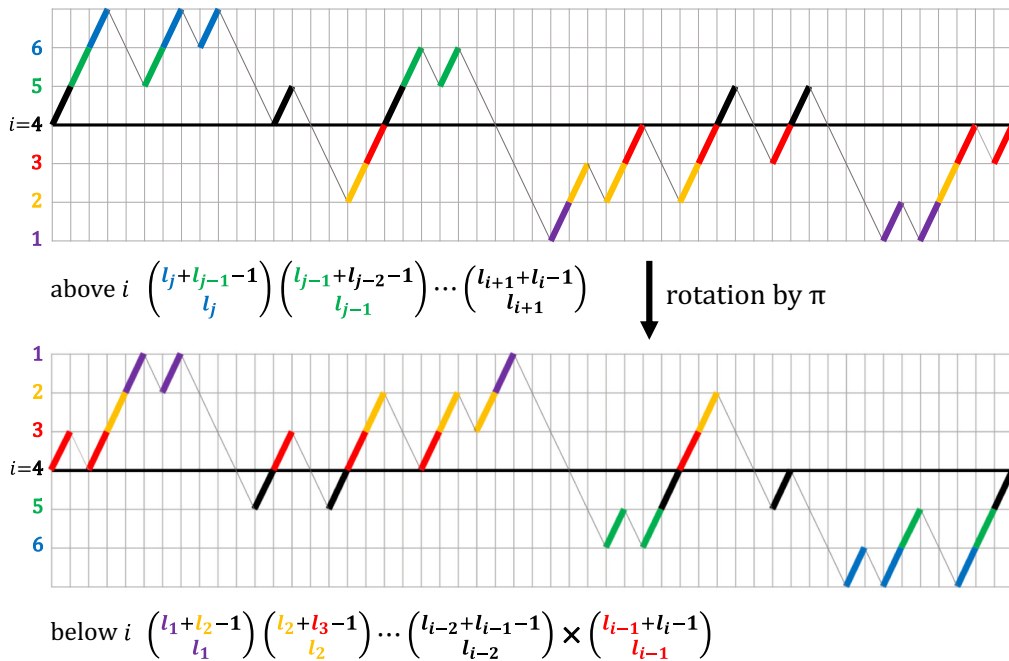


FIG. 8. A periodic Dyck path starting and ending at floor $i = 4$. The first (blue) up step on floor 6 is connected to the (green) up step on floor 5 at its left, and the next two floor 6 up steps are connected to each other and the floor 5 up step to their left. The steps on floor 5 become “units” with the floor 6 steps attached to them, and are attached to floor 4 (black) up steps to their left. Rotating by π , (purple) up steps on floor 1 are connected to (orange) up steps on floor 2 to their left, and similarly for steps on floors 2 and 3 (red). The ordering of up steps on the remaining floors 4 and 3 (except the starting step) is unrestricted.

$k - 1$, and the configuration of floor k units in *not* fixed by the floor k up steps. To deal with floors below i , we rotate the path by π , which inverts floors as well as the direction of the path but leaves up steps as up steps. The bottom floors 1 and 2 now effectively become top floors, and the situation is similar to floors j and $j - 1$. The first up step is necessarily at floor 2, and a similar argument as before gives the multiplicity of paths with fixed up steps on floor 2 as $\binom{l_1+l_2-1}{l_1}$. Repeating the above argument for higher floors, as long as all steps are below i , that is, up to floors $i - 2$ and $i - 1$, we get an overall multiplicity of paths with fixed up steps on floors $i - 1, i, \dots, j$

$$C_{\text{below } i} = \prod_{k=1}^{i-2} \binom{l_k + l_{k+1} - 1}{l_k}.$$

Note that we cannot extend the argument to floors $i - 1$ and i since up steps originating at floor i lie *above* i (and thus, upon inversion, below it).

The product of the above two factors gives the multiplicity of paths with fixed up steps on floors $i - 1$ and i (the only two left fixed by above- i or below- i considerations). The full multiplicity of paths can be determined by also considering the relative placement of the l_{i-1} and l_i up steps on these two floors. They can, in principle, be distributed at will, except that we have fixed the first up step to be on floor i . The remaining $l_i - 1$ and l_{i-1} steps can be distributed in $\binom{l_{i-1}+l_i-1}{l_{i-1}}$ ways, which contributes the missing factor $k = i - 1$ in the product for $C_{\text{below } i}$. Combining the factors reproduces (18).

Note that if we relax the condition that paths start with an up step from the starting floor i , then all l_i and l_{i-1} steps can be distributed at will and contribute a multiplicity $\binom{l_{i-1}+l_i}{l_{i-1}}$. This differs by a factor $(l_{i-1} + l_i)/l_i$ from the previous result and gives the result

$$C_i = (l_i + l_{i-1}) c_2(l_1, l_2, \dots, l_j)$$

for the number of paths starting and ending at floor i . (This result was also derived in [7].) Summing over $i = 1, 2, \dots, j + 1$ (with $l_0 = l_{j+1} = 0$) reproduces the full number of paths $2n c_2(l_1, l_2, \dots, l_j)$.

We also note that the case of paths starting with a down step at floor $i = 2, \dots, j + 1$ can be dealt with in a similar way, with the difference that now the first up step on floors $i - 1$ and i happens at floor $i - 1$, so their relative arrangement has a multiplicity $\binom{l_{i-1}+l_i-1}{l_i}$, which yields, as expected, the result $l_{i-1} c_2(l_1, l_2, \dots, l_j)$. This can also be obtained graphically by (i) cutting the periodic Dyck paths starting with an up step from the $(i - 1)$ th floor at the last occurrence of a down step from the i th floor and (ii) interchanging the two pieces (see Fig. 9).

This establishes a one-to-one mapping between paths starting with an up step from floor $i - 1$ and paths starting with a down step from floor i and shows that $l_{i-1} c_2(l_1, l_2, \dots, l_j)$ also counts the number of these paths, consistent with the analytical result as well as the counting $(l_i + l_{i-1}) c_2(l_1, l_2, \dots, l_j)$ for (up- or down-starting) periodic paths starting at floor i derived before, and the total number $2n c_2(l_1, l_2, \dots, l_j)$ of such periodic paths.

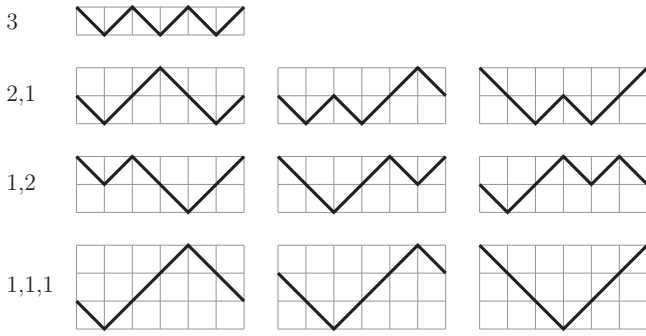


FIG. 9. The 10 periodic Dyck paths of length $n = 6$ starting with a down step. Their counts are $3c_2(3) = 1$, $3c_2(2, 1) = 3$, $3c_2(1, 2) = 3$ and $3c_2(1, 1, 1) = 3$.

B. General g

These results can be generalized to g -exclusion paths. Consider periodic generalized $[g - 1, -1]$ Dyck paths of length $n = gn$ with n up steps, each going up $g - 1$ floors, and $(g - 1)n$ down steps, each going down 1 floor, and thus confined between the 1st and $(j + g - 1)$ th floor. The number of paths starting with an up step from the i th floor is

$$\begin{aligned}
 & l_i c_g(l_1, l_2, \dots, l_j) \\
 &= \frac{(l_{i-g+1} + \dots + l_i - 1)!}{l_{i-g+1}! \dots l_{i-1}! (l_i - 1)!} \prod_{k=1}^{i-g} \binom{l_k + \dots + l_{k+g-1} - 1}{l_k} \\
 &\quad \times \prod_{k=i-g+2}^{j-g+1} \binom{l_k + \dots + l_{k+g-1} - 1}{l_{k+g-1}} \quad (19a)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{(l_i + \dots + l_{i+g-2} - 1)!}{(l_i - 1)! l_{i+1}! \dots l_{i+g-2}!} \prod_{k=1}^{i-1} \binom{l_k + \dots + l_{k+g-1} - 1}{l_k} \\
 &\quad \times \prod_{k=i}^{j-g+1} \binom{l_k + \dots + l_{k+g-1} - 1}{l_{k+g-1}} \quad (19b)
 \end{aligned}$$

with the conventions $l_i = 0$ for $i < 1$ or $i > j$, and $\prod_{k=m}^n (\dots) = 1$ for $n < m$ understood.

The proof of this formula can be achieved with a method similar to the one for $g = 2$, appropriately generalized. Again, the sequence i_s of floors with up steps fixes the path, the positions p_s of up steps in the path being given by

$$p_s = i - i_s + g(s - 1) + 1.$$

Similarly to the $g = 2$ case, the up steps on the top floor j are fully determined by their relative position with respect to the up steps on the $g - 1$ floors below it $j - 1, \dots, j - g + 1$ (which can only be followed by down steps), with the first up step always occurring in one of these lower floors, for a multiplicity of $\binom{l_{j-g+1} + \dots + l_j - 1}{l_j}$. Up steps on floors $j - g + 1, \dots, j - 1$ now constitute units with any possible floor j steps attached to them, and the argument can be repeated for all successive sets of g floors $k, k - 1, \dots, k - g + 1$, accumulating multiplicity factors $\binom{l_{k-g+1} + \dots + l_k - 1}{l_k}$ down to $k = i + 1$.

The same argument can be used for steps below i after a π rotation, starting from the bottom g floors $1, \dots, g$ and for all higher floors $k, k + 1, \dots, k + g - 1$, accumulating multiplicity factors $\binom{l_k + \dots + l_{k+g-1} - 1}{l_k}$ up to $k = i - g$ (up steps that connect to floor k below i downwards, as the rotation by π argument requires, originate from floor $k + g - 1$, and $k = i - g$ is the highest floor for which this step originates below floor i).

The multiplicities picked up for steps below and above floor i reproduce the products in (19a). The combination of the two reduction processes leaves a common set of fixed steps on floors $i - g + 1, \dots, i$. The relative placement of these steps can be chosen at will, with the constraint that the first up step is from floor i , giving a multiplicity of

$$\binom{l_{i-g+1} + \dots + l_i - 1}{l_{i-g+1}, \dots, l_{i-1}, l_i - 1} = \frac{(l_{i-g+1} + \dots + l_i - 1)!}{l_{i-g+1}! \dots l_{i-1}! (l_i - 1)!}$$

reproducing the remaining factor in (19a). For $i \leq g$ there are no steps below i to consider and the reduction above i may need to terminate at a floor higher than $i + 1$, and for $j \leq g$ all up steps can arise in any arrangement, and these cases are captured by the conventions below (19b). The rewriting in (19b) minimized the use of these conventions.

The number of paths starting from floor i with either an up or down step can also be derived. In this case, there is no requirement that the first step among floors $i - g + 1, \dots, i$ must be on floor i , and the relative placement of up steps is unrestricted. Their multiplicity is $\binom{l_{i-g+1} + \dots + l_i}{l_{i-g+1}, \dots, l_i}$, which gives the result for the number of paths $(l_{i-g+1} + \dots + l_i) c_g(l_1, \dots, l_j)$. Correspondingly, the number of paths starting with a down step from floor i is deduced by subtracting $l_i c_g(l_1, \dots, l_j)$ from the full count, yielding $(l_{i-g+1} + \dots + l_{i-1}) c_g(l_1, \dots, l_j)$, and the total number of paths is obtained by summing over all i as $gn c_g(l_1, \dots, l_j)$.

The derivation of the counting formula can be substantially simplified using the following alternative approach based on cyclic permutations that bypasses the subtleties around the starting floor i .

We first calculate the number of paths starting from the lowest floor $i = 1$. Now all steps originate above i , and the reduction argument applies down to floors $1, \dots, g$, giving the multiplicity of paths for a fixed set of up steps on floors 1 through $g - 1$ as the product of factors $\binom{l_k + \dots + l_{k+g-1} - 1}{l_{k+g-1}}$ for $k = 1, \dots, j - g + 1$. The placement of up steps that start in the bottom $g - 1$ floors is arbitrary with the exception that the first up step occurs on floor 1, for a multiplicity of $\binom{l_1 + \dots + l_{g-1} - 1}{l_1, l_2, \dots, l_{g-1}}$. Altogether, the number of paths starting at the bottom floor is

$$\begin{aligned}
 & \frac{(l_1 + \dots + l_{g-1} - 1)!}{(l_1 - 1)! l_2! \dots l_{g-1}!} \prod_{k=1}^{j-g+1} \binom{l_k + \dots + l_{k+g-1} - 1}{l_{k+g-1}} \\
 &= l_1 c_g(l_1, \dots, l_j). \quad (20)
 \end{aligned}$$

The number of all possible paths can be obtained by circularly permuting the gn steps of paths starting at the bottom, which produces $gn l_1 c_g(l_1, \dots, l_j)$ paths. However, each time an up step from the 1st floor occurs first, it reproduces the set of paths starting at the bottom. Since there are l_1 such steps, this

results in an overcounting by a factor l_1 . Correcting for this, we recover the total number of paths as $gn c_g(l_1, l_2, \dots, l_j)$.

The count of paths starting with an up step at floor i can be obtained with a similar argument. Cyclically permuting these paths reproduces, again, all possible paths, but with an overcounting by a factor of l_i , since each time that an up step at floor i becomes first it reproduces the full set. Therefore, we obtain a count of $l_i c_g(l_1, \dots, l_j)$ as obtained before. The number of paths starting with a down step from the i th floor, with $i = 2, 3, \dots, j + g - 1$, can also be reproduced graphically: (i) consider all periodic paths starting with an up step from either the $(i - 1)$ th or the $(i - 2)$ th, or ... the $(i - g + 1)$ th floor, and cut them at the last occurrence of a down step from the i th floor and (ii) interchange the two pieces.

This, again, establishes a one-to-one correspondence between the two sets of paths, and gives the number of paths starting with a down step from floor $i = 2, \dots, j + g - 1$ and l_1, l_2, \dots, l_j up steps from each floor as $(l_{i-g+1} + \dots + l_{i-1}) c_g(l_1, \dots, l_j)$, as obtained before.

V. A GENERALIZATION: (1, g)-EXCLUSION STATISTICS AND GENERALIZED MOTZKIN PATHS

A. (1,2)-exclusion statistics

In Ref. [8] we tackled the algebraic area enumeration of closed walks on a honeycomb lattice. Again a Hofstadter-like Hamiltonian was central to the enumeration, rewritten as a $2q \times 2q$ matrix, which was subsequently reduced to a $q \times q$ matrix. The essence of the enumeration was encapsulated in the exclusion matrix

$$H_{1,2} = \begin{pmatrix} \tilde{s}_1 & f_1 & 0 & \dots & 0 & 0 \\ g_1 & \tilde{s}_2 & f_2 & \dots & 0 & 0 \\ 0 & g_2 & \tilde{s}_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \tilde{s}_{q-1} & f_{q-1} \\ 0 & 0 & 0 & \dots & g_{q-1} & \tilde{s}_q \end{pmatrix}.$$

In addition to the two subdiagonals f_k and g_k , a hallmark of $g = 2$ exclusion, $H_{1,2}$ also has a nonvanishing \tilde{s}_k main diagonal, a hallmark of $g = 1$ statistics, i.e., Fermi statistics, as it indeed describes particles obeying a mixture of the two statistics $g = 1$ and $g = 2$.

The secular determinant reads

$$\det(I - zH_{1,2}) = \sum_{n=0}^q (-z)^n Z(n), \tag{21}$$

where $Z(n)$ can be interpreted as the n -body partition function for particles in a one-body spectrum $\epsilon_1, \epsilon_2, \dots, \epsilon_k, \dots, \epsilon_q$ with fermions occupying one-body energy level k with Boltzmann factor $e^{-\beta\epsilon_k} = \tilde{s}_k$ and two-fermion bound states occupying one-body energy levels $k, k + 1$ with Boltzmann factor $e^{-\beta\epsilon_{k,k+1}} = -g_k f_k := -s_k$.⁵ Since the two-fermion

⁵In the pure $g = 2$ case we took the Boltzmann factors of exclusion particles (bound states) as $+s_k$ and compensated by absorbing the negative sign in the fugacity $-z^2$. In the mixed 1, g case we have no such flexibility, although the alternative, more symmetric choice $e^{-\beta\epsilon_k} = -\tilde{s}_k, e^{-\beta\epsilon_{k,k+1}} = -s_k$ and fugacity $+z$ could have been made.

bound states behave effectively as $g = 2$ exclusion particles, we end up with a mixture of $g = 1$ and $g = 2$ exclusion statistics, where $\det(I - zH_{1,2})$ becomes a grand partition function with $-z$ playing the role of the fugacity parameter. For example, for $q = 5$

$$\begin{aligned} Z(4) = & \tilde{s}_4 \tilde{s}_3 \tilde{s}_2 \tilde{s}_1 + \tilde{s}_5 \tilde{s}_3 \tilde{s}_2 \tilde{s}_1 + \tilde{s}_5 \tilde{s}_4 \tilde{s}_2 \tilde{s}_1 + \tilde{s}_5 \tilde{s}_4 \tilde{s}_3 \tilde{s}_1 + \tilde{s}_5 \tilde{s}_4 \tilde{s}_3 \tilde{s}_2 \\ & + \tilde{s}_4 \tilde{s}_3 (-s_1) + \tilde{s}_5 \tilde{s}_3 (-s_1) + \tilde{s}_5 \tilde{s}_4 (-s_1) + \tilde{s}_4 \tilde{s}_1 (-s_2) \\ & + \tilde{s}_5 \tilde{s}_1 (-s_2) + \tilde{s}_5 \tilde{s}_4 (-s_2) + \tilde{s}_2 \tilde{s}_1 (-s_3) + \tilde{s}_5 \tilde{s}_1 (-s_3) \\ & + \tilde{s}_5 \tilde{s}_2 (-s_3) + \tilde{s}_2 \tilde{s}_1 (-s_4) + \tilde{s}_3 \tilde{s}_1 (-s_4) + \tilde{s}_3 \tilde{s}_2 (-s_4) \\ & + (-s_3)(-s_1) + (-s_4)(-s_1) + (-s_4)(-s_2) \end{aligned}$$

can be readily interpreted in Fig. 10 as the four-body partition function for four particles, either individual fermions or two-fermion bound states, occupying in all possible ways the five one-body levels $\epsilon_k, k = 1, \dots, 5$. Clearly when all \tilde{s}_k are set to 0 in (21), the $Z(2n + 1)$'s vanish and the $Z(2n)$'s reduce to the n -body partition functions (5) for $g = 2$ exclusion particles, that is,

$$\begin{aligned} \det(I - zH_{1,2}) &= \sum_{n=0, \text{ even}}^q (-z)^n Z(n) \\ &= \sum_{n=0}^{\lfloor q/2 \rfloor} z^{2n} Z(2n) = \det(I - zH_2), \tag{22} \end{aligned}$$

where in the last step we identified $(-1)^n Z(2n)$ to the $Z(n)$ for 2-exclusion appearing in $\det(I - zH_2)$ and given in (5). From

$$\log \left\{ \sum_{n=0}^q Z(n) z^n \right\} = \sum_{n=1}^{\infty} b(n) z^n \tag{23}$$

implying

$$\text{tr } H_{1,2}^n = \mathbf{n}(-1)^{n+1} b(\mathbf{n}),$$

one infers

$$\begin{aligned} b(\mathbf{n}) &= (-1)^{n+1} \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j \\ (1,2)\text{-composition of } \mathbf{n}}} c_{1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j) \\ &\times \sum_{k=1}^{q-j} \tilde{s}_k^{\tilde{l}_1} s_k^{l_1} \tilde{s}_{k+1}^{\tilde{l}_2} s_{k+1}^{l_2} \dots \end{aligned} \tag{24}$$

with

$$\begin{aligned} c_{1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j) &= \frac{\binom{\tilde{l}_1+l_1}{\tilde{l}_1}}{\tilde{l}_1+l_1} l_1 \frac{\binom{l_1+\tilde{l}_2+l_2}{l_1, \tilde{l}_2, l_2}}{l_1+\tilde{l}_2+l_2} \dots l_j \frac{\binom{l_j+\tilde{l}_{j+1}}{l_j}}{l_j+\tilde{l}_{j+1}} \\ &= \frac{(\tilde{l}_1+l_1-1)!}{\tilde{l}_1! l_1!} \prod_{k=2}^{j+1} \binom{l_{k-1}+\tilde{l}_k+l_k-1}{l_{k-1}-1, \tilde{l}_k, l_k} \end{aligned} \tag{25}$$

with the usual convention $l_k = 0$ for $k > j$. We note that setting all \tilde{l}_i 's to zero reduces $c_{1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j)$ in (25) to the standard 2-exclusion $c_2(l_1, \dots, l_j)$ already discussed in (8). Likewise, setting $\tilde{s}_k = 0$ in (24) eliminates all terms with nonzero \tilde{l}_i 's and (25) effectively reduces to (8).

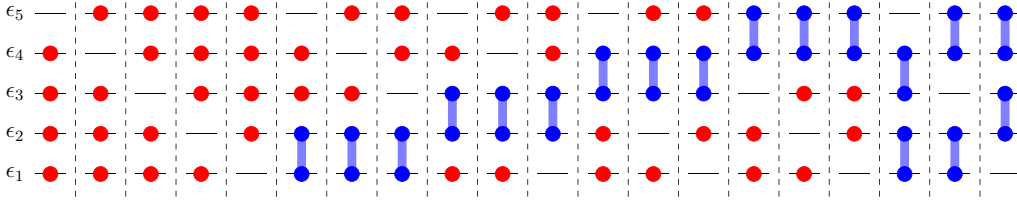


FIG. 10. $Z(4)$ for $q = 5$: All possible occupancies of the five one-body levels by four particles with either fermions (red) or two-fermion bound states (blue).

We define the sequence of integers $\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j$, $j \geq 0$, labeling $c_{1,2}$ in (25) as a $(1,2)$ -composition of the integer \mathbf{n} if they satisfy the defining conditions

$$\mathbf{n} = (\tilde{l}_1 + \tilde{l}_2 + \dots + \tilde{l}_{j+1}) + 2(l_1 + l_2 + \dots + l_j),$$

$$\tilde{l}_i \geq 0, \quad l_i > 0. \quad (26)$$

That is, the l_i 's are the usual compositions of integers $1, 2, \dots, \lfloor \mathbf{n}/2 \rfloor$, while the \tilde{l}_i 's are additional non-negative integers. (For $j = 0$, we have the trivial composition $\tilde{l}_1 = \mathbf{n}$.) For example, there are six $(1,2)$ -compositions of 4: (4), (2, 0; 1), (1, 1; 1), (0, 2; 1), (0, 0; 2), (0, 0, 0; 1, 1), which contribute to $b(4)$ the terms

$$-b(4) = \frac{1}{4} \sum_{k=1}^q \tilde{s}_k^4 + \sum_{k=1}^{q-1} \tilde{s}_k^2 s_k + \sum_{k=1}^{q-1} \tilde{s}_k s_k \tilde{s}_{k+1} + \sum_{k=1}^{q-1} s_k \tilde{s}_{k+1}^2$$

$$+ \frac{1}{2} \sum_{k=1}^{q-1} s_k^2 + \sum_{k=1}^{q-2} s_k s_{k+1}.$$

Note that the inverse of a composition, defined as $\tilde{l}_{j+1}, \dots, \tilde{l}_1; l_j, \dots, l_1$ leaves $c_{1,2}$ invariant.

B. $(1, g)$ -exclusion statistics

For a mixture of $g = 1$ and g exclusion the associated algebraic area enumeration is encapsulated in the $(1, g)$ -exclusion matrix (again assuming zero “umklapp” matrix elements at the off-diagonal corners)

$$H_{1,g} = \begin{pmatrix} \tilde{s}_1 & f_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \tilde{s}_2 & f_2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ g_1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & g_2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & f_{q-1} \\ 0 & 0 & 0 & \dots & g_{q-g+1} & 0 & 0 & \dots & \tilde{s}_q \end{pmatrix}. \quad (27)$$

Following the same route as in the g -exclusion case, i.e., computing the secular determinant $\det(I - zH_{1,g})$, leads now to a mixture of fermions with Boltzmann factors $e^{-\beta\epsilon_k} = \tilde{s}_k$ and g -fermion bound states with g particles occupying g successive one-body levels $k, k+1, \dots, k+g-1$ with Boltzmann factors

$$e^{-\beta\epsilon_{k, \dots, k+g-1}} = (-1)^{g-1} s_k := (-1)^{g-1} g_k f_k f_{k+1} \dots f_{k+g-2} \quad (28)$$

behaving effectively as g -exclusion particles. The associated cluster coefficients are

$$b(\mathbf{n}) = (-1)^{n+1} \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j \\ (1,g)\text{-composition of } \mathbf{n}}} c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j)$$

$$\times \sum_{k=1}^{q-j-g+2} \tilde{s}_k^{\tilde{l}_1} s_k^{l_1} \tilde{s}_{k+1}^{\tilde{l}_2} s_{k+1}^{l_2} \dots \quad (29)$$

We define the sequence of integers $\tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_{j+g-1}; l_1, l_2, \dots, l_j$, $j \geq 1$, as a $(1, g)$ -composition of \mathbf{n} if they satisfy the conditions

$$\mathbf{n} = (\tilde{l}_1 + \tilde{l}_2 + \dots + \tilde{l}_{j+g-1}) + g(l_1 + l_2 + \dots + l_j)$$

$$\tilde{l}_i \geq 0; l_i \geq 0, l_1, l_j > 0, \text{ at most } g-2 \text{ successive vanishing } l_i. \quad (30)$$

That is, the l_j 's are the usual g -compositions of integers $1, 2, \dots, \lfloor \mathbf{n}/g \rfloor$ and the \tilde{l}_i 's are additional non-negative integers. (We also include the trivial composition $\tilde{l}_1 = \mathbf{n}$.) For example, there are seven $(1,3)$ -compositions of 5

$j = 0$: (5); $j = 1$: (2, 0, 0; 1), (1, 1, 0; 1), (1, 0, 1; 1), (0, 2, 0; 1), (0, 1, 1; 1), (0, 0, 2; 1)

and five $(1,4)$ -compositions of 5

$j = 0$: (5); $j = 1$: (1, 0, 0, 0; 1), (0, 1, 0, 0; 1), (0, 0, 1, 0; 1), (0, 0, 0, 1; 1)

The $c_{1,g}(\tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_{j+g-1}; l_1, l_2, \dots, l_j)$ in (29) read

$$c_{1,g}(\tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_{j+g-1}; l_1, l_2, \dots, l_j)$$

$$= \frac{(\tilde{l}_1 + l_1 - 1)!}{\tilde{l}_1! l_1!} \prod_{k=2}^{j+g-1} \binom{\tilde{l}_k + \sum_{i=k-g+1}^k l_i - 1}{\sum_{i=k-g+1}^{k-1} l_i - 1, \tilde{l}_k, l_k}$$

with $l_i = 0$ for $i \leq 0$ or $i > j$ as usual. It is clear that when $\tilde{l}_i = 0$ only the standard g -composition survives so that the coefficients $c_{1,g}$ in (31) go over to c_g in (15). Equivalently, when $\tilde{s}_i = 0$, terms with non vanishing \tilde{l}_i in (29) drop and we recover the g -exclusion cluster coefficients.

Finally, we define the inverse of a composition by inverting the order of the \tilde{l}_i and of the l_i : $\tilde{l}_i \rightarrow \tilde{l}_{j+g-i}$, $l_i \rightarrow l_{j+1-i}$. Inverse compositions produce the same coefficient $c_{1,g}$.

C. Combinatorial interpretation

$(1, g)$ -compositions already have a combinatorial interpretation, deriving from their relation to cluster coefficients of $(1, g)$ -exclusion statistics. Specifically, $(1, g)$ -compositions correspond to all *distinct connected* arrangements of \mathbf{n} particles on a one-body spectrum, either alone or in g -bound states; that is, to all the possible ways to place particles and bound

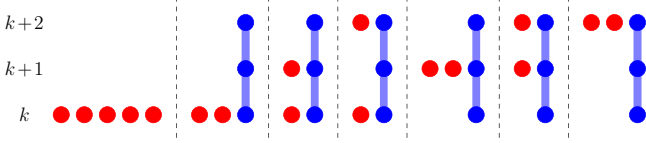


FIG. 11. Seven (1,3)-compositions of 5: (5), (2, 0, 0; 1), (1, 1, 0; 1), (1, 0, 1; 1), (0, 2, 0; 1), (0, 1, 1; 1), (0, 0, 2; 1), illustrated by fermions (red) and three-fermion bound states (blue).

states such that they cannot be separated into two or more mutually non-overlapping groups (see Fig. 11). If the arrangement covers $j + g - 1$ consecutive one-body levels $k + i - 1$, $i = 1, \dots, j + g - 1$, then \tilde{l}_i is the number of single particles on one-body level $k + i - 1$ and l_i is the number of g -bound states that extend over the g levels $k + i - 1$ to $k + i + g - 2$ ($i = 1, \dots, j$).

Note that the above configurations are forbidden by (1, g)-exclusion statistics. They constitute the “connected” components of the grand partition function, and their exponentiation, with appropriate coefficients $(-1)^{n+1} z^n c_{1,g}(\tilde{l}_1, \dots; l_1, \dots)$, produces the correct exclusion grand partition function, each term “correcting” the overcounting arising from the exponentiation of lower order terms. For pure fermion statistics $g = 1$, all particles must occupy the same level, leading to the trivial composition $\tilde{l}_1 = \mathbf{n}$ and the fermionic cluster coefficients $(-1)^{n+1}/\mathbf{n}$.

To give a combinatorial interpretation to the multiplicity coefficients $c_{1,g}(\tilde{l}_1, \dots; l_1, \dots)$ we revert to the trace of the \mathbf{n} th power of $H_{1,g}$. In terms of the matrix elements h_{ij} of $H_{1,g}^\top$ in (27) this trace is

$$\text{tr } H_{1,g}^{\mathbf{n}} = \sum_{k_1=1}^q \sum_{k_2=1}^q \cdots \sum_{k_{\mathbf{n}}=1}^q h_{k_1 k_2} h_{k_2 k_3} \cdots h_{k_{\mathbf{n}} k_1}. \quad (31)$$

The structure of the (1, g)-exclusion matrix (27) implies that (31) is a sum of products of \mathbf{n} factors $h_{k_i k_{i+1}}$ with indices such that $k_{i+1} - k_i$ take values $g - 1, 0$, or -1 . We map the

$$\text{tr } H_{1,g}^{\mathbf{n}} = \sum_{k=1}^{q-j-g+2} \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j \\ (1,g)\text{-composition of } \mathbf{n}}} C_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j) \tilde{s}_k^{\tilde{l}_1} s_k^{l_1} \tilde{s}_{k+1}^{\tilde{l}_2} s_{k+1}^{l_2} \cdots,$$

where $C_{1,g}(\tilde{l}_1, \dots; l_1, \dots)$ is the number of periodic generalized Motzkin paths of length \mathbf{n} with \tilde{l}_i horizontal steps and l_1 up steps originating from the first floor, \tilde{l}_2 and l_2 from the second floor, and so on [the sum over k ensures that paths of all possible starting level k_1 in (31) are included]. Comparing with

$$\text{tr } H_{1,g}^{\mathbf{n}} = \mathbf{n} \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j \\ (1,g)\text{-composition of } \mathbf{n}}} c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j) \sum_{k=1}^{q-j-g+2} \tilde{s}_k^{\tilde{l}_1} s_k^{l_1} \tilde{s}_{k+1}^{\tilde{l}_2} s_{k+1}^{l_2} \cdots$$

we see that

$$C_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j) = \mathbf{n} c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j).$$

Therefore, $\mathbf{n} c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j)$ admits the interpretation of the number of periodic generalized $[g - 1, 0, -1]$

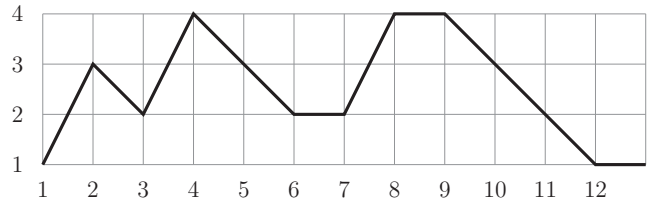


FIG. 12. A periodic generalized Motzkin path of length $\mathbf{n} = 12$ corresponding to the (1,3)-composition 1, 1, 0, 1; 1, 2, starting with an up step from the first floor.

sequence of indices $k_1, k_2, \dots, k_{\mathbf{n}-1}, k_{\mathbf{n}}, k_1$ to the heights of a periodic generalized $[g - 1, 0, -1]$ Motzkin path (“bridge”) of length \mathbf{n} starting and ending at height k_1 , with vertical steps up by $g - 1$ units or down by 1 unit as well as horizontal steps (see Fig. 12 for an example). Evaluating the trace (31) amounts to summing the corresponding products over all such periodic paths. We note that periodic paths must have $g - 1$ down steps for each up step.

As in the g -exclusion case, to group together terms with the same weight $h_{k_1 k_2} \cdots h_{k_{\mathbf{n}} k_1}$ we need to consider paths with a fixed number of transitions per level. For each path that reaches a lowest one-body energy level k and highest level from which an up step starts $k + j - 1$, and thus highest level reached $k + j + g - 2$, we denote by l_1, l_2, \dots, l_j the number of up steps from levels $k, k + 1, \dots, k + j - 1$ and by $\tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_{j+g-1}$ the number of horizontal steps at levels $k, k + 1, \dots, k + j + g - 2$. Clearly $\tilde{l}_1 + \cdots + \tilde{l}_{j+g-1} + g(l_1 + \cdots + l_j) = \mathbf{n}$, and at most $g - 2$ successive l_i can vanish, since up steps can skip $g - 2$ floors. Therefore, $\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j$ is a (1, g)-composition of \mathbf{n} . As before, each up step $k_i \rightarrow k_i + g - 1$ necessarily implies down steps $k_i + g - 1 \rightarrow k_i + g - 2, \dots, k_i + 1 \rightarrow k_i$, so factors in each term in (31) corresponding to each up step $k_i \rightarrow k_i + g - 1$ contribute the full combination $h_{k_i, k_i+g-1} h_{k_i+g-1, k_i+g-2} \cdots h_{k_i+1, k_i} = g_{k_i} f_{k_i+g-2} \cdots f_{k_i} = s_{k_i}$.

Altogether, the sum in (31) rewrites as

Motzkin paths with horizontal and up steps as defined above.

The number of such paths starting with an up step, respectively a horizontal step, from floor i can also be deduced as $l_i c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j)$, respectively $\tilde{l}_i c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j)$, while the total number of

paths starting from floor i is

$$\left(\tilde{l}_i + \sum_{k=i-g+1}^i l_k \right) c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j).$$

(The result $(\tilde{l}_1 + l_1) c_{1,2}$ for $i = 1$ Motzkin excursions was also derived in Ref. [9].) Finally, the number of paths starting at floor i with a down step can be deduced as

$$\left(\sum_{k=i-g+1}^{i-1} l_k \right) c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j).$$

The proof of the above counting formulas can be obtained similarly to the case of g -exclusion and generalized Dyck paths. The simplest method is the one outlined at the end of Sec. IV B based on cyclic permutations. Consider first paths that start with an up step from the first floor. A combinatorial argument entirely analogous to the one in Sec. IV B yields the result $l_1 c_{1,g}$ for the number of such paths, and by periodic permutation and reduction by an overcounting factor of l_1 the total number of paths obtains as $\mathbf{n} c_{1,g}$. A repetition of the periodic argument from floor i , then, produces the results $l_i c_{1,g}$ and $\tilde{l}_i c_{1,g}$ for the number of paths starting up or horizontally from floor i , and a ‘‘cutting and exchanging’’ argument gives the number of paths starting down from floor i . The details are similar to the ones in Sec. IV B and are left as an exercise.

We conclude by giving the number of $(1, g)$ -compositions of a given integer \mathbf{n}

$$N_{1,g}(\mathbf{n}) = 1 + \sum_{k=0}^{\lfloor \mathbf{n}/g \rfloor - 1} \sum_{m=0}^{(g-1)k} \binom{k}{m}_g \binom{\mathbf{n} + m - gk - 1}{m + g - 1}, \quad (32)$$

where the g -nomial coefficient is defined as

$$\begin{aligned} \binom{k}{m}_g &= [x^m](1 + x + x^2 + \dots + x^{g-1})^k = [x^m] \left(\frac{1 - x^g}{1 - x} \right)^k \\ &= \sum_{j=0}^{\lfloor m/g \rfloor} (-1)^j \binom{k}{j} \binom{k + m - gj - 1}{k - 1}. \end{aligned}$$

For $g = 2$ it reduces to the standard binomial coefficient $\binom{k}{m}_2 = \binom{k}{m}$. So (32) becomes the triple sum

$$\begin{aligned} N_{1,g}(\mathbf{n}) &= 1 + \sum_{k=0}^{\lfloor \mathbf{n}/g \rfloor - 1} \sum_{m=0}^{(g-1)k} \sum_{j=0}^{\lfloor m/g \rfloor} (-1)^j \binom{k}{j} \binom{k + m - gj - 1}{k - 1} \\ &\quad \times \binom{\mathbf{n} + m - gk - 1}{m + g - 1}. \end{aligned}$$

Equivalently, the generating function of the $N_{1,g}(\mathbf{n})$'s is

$$\begin{aligned} \sum_{\mathbf{n}=0}^{\infty} x^{\mathbf{n}} N_{1,g}(\mathbf{n}) &= \frac{(1-x)^{g-2}(1+x^{g-1}-x^g) - x^{g-1}}{(1-x)^{g-1}(1+x^{g-1}-x^g) - x^{g-1}} \\ &= \frac{1}{1-x} \left[1 + \frac{x^g}{(1-x)^{g-1}(1+x^{g-1}-x^g) - x^{g-1}} \right]. \end{aligned} \quad (33)$$

In the second line, the term $1/(1-x)$ reproduces the trivial compositions ($\tilde{l}_1 = \mathbf{n}$) while the other term reproduces all the nontrivial ones. Finally, the number of all unrestricted periodic $[g-1, 0, -1]$ generalized Motzkin paths is obtained by summing $c_{1,g}$ over all $1, g$ -compositions and yields the relation

$$\begin{aligned} \mathbf{n} \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j \\ (1,g)\text{-composition of } \mathbf{n}}} c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j) \\ = [x^0](x^{g-1} + 1 + x^{-1})^{\mathbf{n}} = \sum_{k=0}^{\lfloor \mathbf{n}/g \rfloor} \binom{\mathbf{n}}{gk} \binom{gk}{k}. \end{aligned}$$

VI. CONCLUSIONS

We have established a connection between the enumeration of lattice walks according to their algebraic area, quantum exclusion statistics, and the combinatorics of generalized Dyck and Motzkin paths (also known as Łukasiewicz paths). The key common quantities are the coefficients $c_g(l_1, \dots, l_j)$ and $c_{1,g}(\tilde{l}_1, \dots, \tilde{l}_{j+g-1}; l_1, \dots, l_j)$ labeled by the g -compositions and the $(1, g)$ -compositions of the length of the walks. These coefficients appear as essential building blocks of the algebraic area partition function of walks on square or honeycomb lattices, the cluster coefficients g -exclusion and $(1, g)$ -exclusion statistical systems, and the counting of generalized paths with specific number of steps from each visited floor. The connection of Dyck paths and $g = 2$ exclusion statistics was established in Ref. [10] and used to calculate the length and area generating function for such paths, and the method was extended to Motzkin paths in Ref. [11]. To the best of our knowledge, the full threefold connection between walks, statistics, and paths, as well as the explicit expressions of c_g and $c_{1,g}$ for $g > 2$ and their relevance to Łukasiewicz path counting, were put forward for the first time in the present work.

There are various directions for possible future investigation. The most immediate one is along the lines already laid out in this work, that is, in the connection of walks of various properties and on various lattices and corresponding paths. For instance, the enumeration of open walks on the square lattice according to their algebraic area was recently achieved [12], with Dyck path combinatorics again playing a key role. Walks on other lattices, such as the kagomé lattice, and of different properties can be investigated with similar methods.

The concept of exclusion statistics and related compositions naturally generalizes to (g', g) and more general (g_1, g_2, \dots, g_n) statistics and compositions, and the statistical mechanical properties of these systems and mathematical properties of their compositions are of interest. It would also be worthwhile to derive the corresponding combinatorial quantities c_{g_1, \dots, g_n} and study their relevance for generalized Łukasiewicz paths.

In a different direction, it is known that Dyck and Motzkin paths appear in various contexts in physics and mathematics. In physics, they appear in percolation processes, interfaces between fluids of different surface tension, and other statistical systems such as long polymer molecules in solution, where the generalized weighted paths are introduced to study the

interactions with the boundary and the polymer is “adsorbed” when the attractive force is sufficiently strong (see, e.g., Ref. [13]). It would be challenging to solve it in this framework. Further, Dyck and Motzkin paths can be mapped to spin-1/2 and spin-1 chains. For example, in Ref. [14] a spin-1 frustration-free Hamiltonian was constructed using Motzkin paths. In a related direction, a family of multispin quantum chains with a free-(para)fermionic eigenspectrum [15] was recently reanalyzed in Ref. [16] and the eigenenergies were obtained via the roots of a polynomial with coefficients similar to the $Z(n)$ in the present paper, indicating a connection with exclusion statistics that warrants further investigation. Finally, in knot theory, the Temperley–Lieb algebra can

have a representation based on Dyck paths, while, if empty vertices (vertices not incident to an edge) are allowed, Motzkin paths become relevant [17]. The extension of this connection to more general paths, and the meaning of the c_g and $c_{1,g}$ coefficients in this context, are nontrivial issues that deserve further study.

ACKNOWLEDGMENTS

L.G. acknowledges the financial support of China Scholarship Council (No. 202009110129). The work of A.P. was supported in part by NSF under Grant No. NSF-PHY-2112729 and by a PSC-CUNY grant.

-
- [1] D. R. Hofstadter, Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields, *Phys. Rev. B* **14**, 2239 (1976).
 - [2] Ch. Krefl, Explicit computation of the discriminant for the Harper equation with rational flux, SFB 288 Preprint No. 89, TU-Berlin, 1993 (unpublished).
 - [3] S. Ouvry and S. Wu, The algebraic area of closed lattice random walks, *J. Phys. A: Math. Theor.* **52**, 255201 (2019).
 - [4] S. Ouvry and A. P. Polychronakos, Exclusion statistics and lattice random walks, *Nucl. Phys. B* **948**, 114731 (2019); Lattice walk area combinatorics, some remarkable trigonometric sums and Apéry-like numbers, **960**, 115174 (2020).
 - [5] B. Hopkins and S. Ouvry, Combinatorics of multicompositions, in *Proceedings of the Combinatorial and Additive Number Theory Conference* (Springer, Berlin, 2020).
 - [6] C. Krattenthaler, Permutations with restricted patterns and Dyck paths, *Adv. Appl. Math.* **27**, 510 (2001).
 - [7] G. M. Cicuta, M. Contedini and L. Molinari, Enumeration of simple random walks and tridiagonal matrices, *J. Phys. A: Math. Gen.* **35**, 1125 (2002).
 - [8] L. Gan, S. Ouvry, and A. P. Polychronakos, Algebraic area enumeration of random walks on the honeycomb lattice, *Phys. Rev. E* **105**, 014112 (2022).
 - [9] R. Oste and J. Van der Jeugt, Motzkin paths, Motzkin polynomials and recurrence relations, *Electron. J. Combin.* **22**, P2.8 (2015).
 - [10] S. Ouvry and A. P. Polychronakos, Hamiltonian and exclusion statistics approach to discrete forward-moving paths, *Phys. Rev. E* **104**, 014143 (2021).
 - [11] A. P. Polychronakos, Length and area generating functions for height-restricted Motzkin meanders, *Phys. Rev. E* **105**, 024102 (2022).
 - [12] S. Ouvry and A. P. Polychronakos, Algebraic area enumeration for open latticewalks, [arXiv:2206.12428](https://arxiv.org/abs/2206.12428).
 - [13] R. Brak, G. K. Iliev, and T. Prellberg, An infinite family of adsorption models and restricted Lukasiewicz paths, *J. Stat. Phys.* **145**, 669 (2011).
 - [14] S. Bravyi, L. Caha, R. Movassagh, D. Nagaj, and P. W. Shor, Criticality without Frustration for Quantum Spin-1 Chains, *Phys. Rev. Lett.* **109**, 207202 (2012).
 - [15] R. J. Baxter, A simple solvable Z_N Hamiltonian, *Phys. Lett. A* **140**, 155 (1989); Superintegrable chiral Potts model: Thermodynamic properties, an inverse model and a simple associated Hamiltonian, *J. Stat. Phys.* **57**, 1 (1989); Transfer matrix functional relations for the generalized $\tau_2(t_q)$ model, **117**, 1 (2004).
 - [16] P. Fendley, Free parafermions, *J. Phys. A: Math. Theor.* **47**, 075001 (2014); F. C. Alcaraz and R. A. Pimenta, Free fermionic and parafermionic quantum spin chains with multispin interactions, *Phys. Rev. B* **102**, 121101(R) (2020).
 - [17] E. Posner, K. Hatch, and M. Ly, Presentation of the Motzkin monoid, [arXiv:1301.4518](https://arxiv.org/abs/1301.4518).

4 - Algebraic area enumeration of random walks on the cubic lattice

In this chapter, we extend the algebraic area enumeration to closed cubic lattice walks and map this problem onto the cluster coefficients of three types of particles obeying $g = 1$, $g = 1$, and $g = 2$ exclusion statistics, respectively, with the constraint that the numbers of $g = 1$ exclusion particles of the two types are equal.

4.1 Introduction

In the context of 2D Hofstadter-like models, the algebraic area can be expressed as $\frac{1}{2} \oint (\mathbf{r} \times d\mathbf{r}) \cdot \mathbf{B}$, where the magnetic field $\mathbf{B} = (0, 0, 1)$ and the integral is along the closed walk in the xy plane. In a 3D Hofstadter-like model, the magnetic field can be arbitrary in 3D, for example, $\mathbf{B} = (1, 1, 1)$. This leads to the algebraic area being defined for cubic lattice walks as the algebraic area sum of three projected walks (see Figure 4.1). Precisely, we define the algebraic area of cubic lattice walks as the sum of the algebraic areas of the walk projected onto the xy, yz, zx planes along the $-z, -x, -y$ directions.

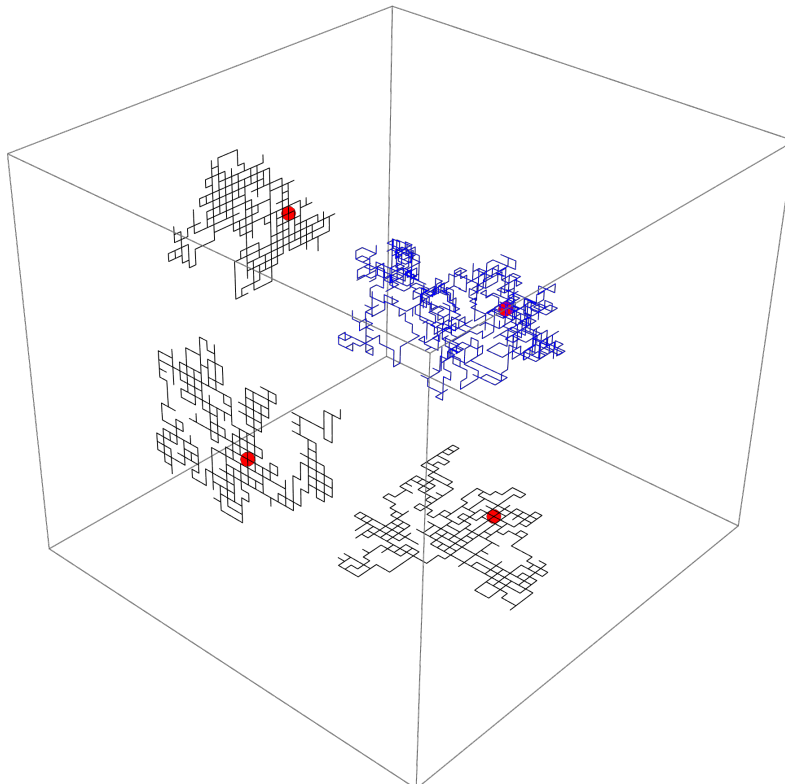


Figure 4.1: A 3D closed random walk on a cubic lattice (blue) along with the corresponding three projected 2D random walks (black) starting and ending at the red bullet points.

To count the number $C_{2n}(A)$ of closed random walks on a cubic lattice with a given length $2n$ (necessarily even) and algebraic area A , we begin by introducing three lattice hopping operators U, V, W along the x, y, z directions, as well as U^{-1}, V^{-1}, W^{-1} along the $-x, -y, -z$ directions. These operators satisfy the noncommutative 3-tori algebra [53]

$$VU = QU V, WV = QV W, UW = QW U, \quad (4.1)$$

which amounts to saying that the planar walks that go around the unit lattice cell on the Cartesian planes in a counterclockwise direction enclose an algebraic area 1, i.e., $V^{-1}U^{-1}VU = Q$, $W^{-1}V^{-1}WV = Q$, and $U^{-1}W^{-1}UW = Q$. The algebraic area A enclosed by a cubic lattice walk can thus be computed by reducing the hopping operators to Q^A using the commutation relations (4.1). See Figure 4.2 for the closed six-step cubic lattice walk $UW^{-1}V^{-1}U^{-1}WV = Q$ as an example. Another example involves enumerating closed four-step walks. By taking the u - and v -independent part in the expansion of $(U + V + W + U^{-1} + V^{-1} + W^{-1})^4 = 6(11 + 2Q + 2Q^{-1}) + \dots$, only terms with an equal number of U and U^{-1} , V and V^{-1} , W and W^{-1} survive, yielding the count of algebraic area: $C_4(0) = 66$ walks enclose an algebraic area $A = 0$, $C_4(1) = 12$ walks enclose an algebraic area $A = 1$, and $C_4(-1) = 12$ walks enclose an algebraic area $A = -1$.

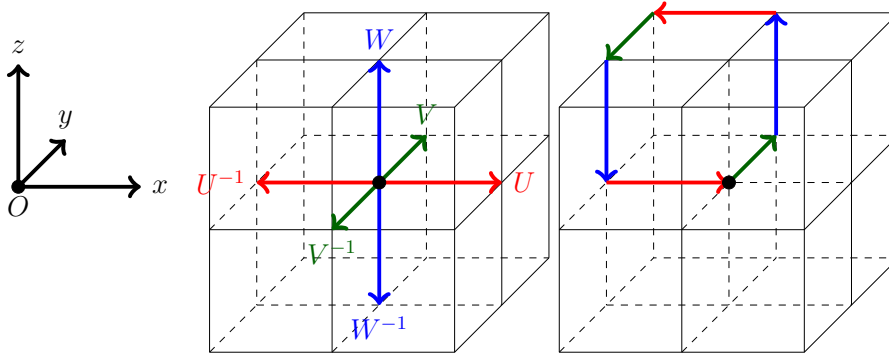


Figure 4.2: (left) 3D Cartesian coordinate system; (middle) six lattice hopping operators in a cubic lattice; (right) closed six-step cubic lattice walk $UW^{-1}V^{-1}U^{-1}WV$ whose algebraic area is given by $A = 1 + 1 + (-1) = 1$. Using the commutation relations (4.1), $UW^{-1}V^{-1}U^{-1}WV$ is simplified to Q^1 , as expected.

By expressing the phase $Q = e^{2\pi i \phi / \phi_0}$ in terms of the flux ϕ through the unit lattice cell on each of the three Cartesian planes in unit of the flux quantum ϕ_0 , the Hermitian operator

$$H = U + V + W + U^{-1} + V^{-1} + W^{-1} \quad (4.2)$$

represents a Hamiltonian that describes a charged particle hopping on a cubic lattice coupled to a magnetic field $\mathbf{B} = (1, 1, 1)$, as indicated in the definition of the algebraic area for a closed cubic lattice walk.

4.2 Cubic Hofstadter Hamiltonian

From now on, we assume that the magnetic flux on each Cartesian plane is rational, i.e., $\phi/\phi_0 = p/q$ with p and q being coprime, thus $Q = e^{2\pi ip/q}$. To obtain the finite-dimensional representation of U, V, W , we use the $q \times q$ "clock" and "shift" matrices u and v as defined in (1.3). Due to the open lattice walk $UVW \neq I$, it is not possible to represent the operators U, V, W as $u, v, v^{-1}u^{-1}$, respectively, even though they satisfy the algebra (4.1). To address this, we introduce an additional vector space with dimension q' , in which U and V act as identity operators, while W does not. Consequently, we obtain the representation of (4.1) as $qq' \times qq'$ matrices

$$U = u \otimes I, \quad V = v \otimes I, \quad W = (v^{-1}u^{-1}) \otimes u',$$

where u' is an arbitrary $q' \times q'$ matrix that is not proportional to I .

To obtain a Hamiltonian that can be utilized for simplifying the quantum trace $\mathbf{Tr} H^n$ to the usual matrix trace $\text{tr} H^n$, a similar approach to the square lattice walk, i.e., the Hofstadter scenario in Section 2.1 is required to eliminate the umklapp terms. Therefore, we perform the algebra-preserving transformation

$$u \rightarrow -u^{-1}v, \quad v \rightarrow v^{-1}, \quad u' \rightarrow -u'.$$

This leads to the new Hamiltonian

$$H' = (-u^{-1}v - v^{-1}u + v + v^{-1}) \otimes I + u \otimes u' + u^{-1} \otimes u'^{-1},$$

which describes walks on a deformed cubic lattice. Note that $-u^{-1}v - v^{-1}u + v + v^{-1}$ corresponds to the $g = 2$ Hofstadter Hamiltonian associated to the square lattice walks discussed in Section 2.1. By setting $u' = u$ and $k_x = 0$, we have

$$\mathbf{Tr} H^{2n} = \mathbf{Tr} H'^{2n} = \frac{1}{q^2} \text{tr} H'^{2n}. \quad (4.3)$$

As the relation (1.5) remains valid, enumerating closed cubic lattice walks amounts to computing the matrix trace $\text{tr} H'^{2n}$, which we will now perform.

Introduce

$$\tilde{s}_{k,k'} = Q^{k+k'} + Q^{-k-k'}, \quad f_k = (1 - Q^{-k})e^{ik_y}$$

and $q \times q$ diagonal matrices

$$\tilde{\mathbf{s}}_k = \text{diag}(\tilde{s}_{k,1}, \tilde{s}_{k,2}, \dots, \tilde{s}_{k,q}), \quad \mathbf{f}_k = f_k I_q, \quad \bar{\mathbf{f}}_k = \bar{f}_k I_q, \quad \mathbf{0} = 0 I_q.$$

H' can be expressed as a block tridiagonal matrix

$$H' = \begin{pmatrix} \tilde{\mathbf{s}}_1 & \mathbf{f}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{f}_1 & \tilde{\mathbf{s}}_2 & \mathbf{f}_2 & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{f}_2 & \tilde{\mathbf{s}}_3 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \tilde{\mathbf{s}}_{q-1} & \mathbf{f}_{q-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{f}_{q-1} & \tilde{\mathbf{s}}_q \end{pmatrix},$$

which has the same form as the (1, 2)-exclusion matrix $H_{1,2}$ in (3.12), but now the elements are block matrices. We apply the trace computation strategy described in Section 3.3 to get

$$\frac{1}{q^2} \text{tr } H'^{2n} = 2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j \\ (1,2)\text{-composition of } 2n}} c_{1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j) \frac{1}{q} \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \dots s_{k+j-1}^{l_j} \frac{1}{q} \sum_{k'=1}^q \tilde{s}_{k,k'}^{\tilde{l}_1} \tilde{s}_{k+1,k'}^{\tilde{l}_2} \dots \tilde{s}_{k+j,k'}^{\tilde{l}_{j+1}} \quad (4.4)$$

with the (1, 2)-composition defined in (3.19), $c_{1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j)$ in (3.20), and $s_k = f_k \bar{f}_k = 4 \sin^2(k\pi p/q)$. The second trigonometric sum in (4.4) can be expanded, using the binomial theorem, as

$$\begin{aligned} & \frac{1}{q} \sum_{k'=1}^q \tilde{s}_{k,k'}^{\tilde{l}_1} \tilde{s}_{k+1,k'}^{\tilde{l}_2} \dots \tilde{s}_{k+j,k'}^{\tilde{l}_{j+1}} \\ &= \frac{1}{q} \sum_{\tilde{l}'_i + \tilde{l}''_i = \tilde{l}_i} \mathbf{Q}^{(\tilde{l}'_2 - \tilde{l}''_2) + 2(\tilde{l}'_3 - \tilde{l}''_3) + \dots + j(\tilde{l}'_{j+1} - \tilde{l}''_{j+1}) + k[(\tilde{l}'_1 + \dots + \tilde{l}'_{j+1}) - (\tilde{l}''_1 + \dots + \tilde{l}''_{j+1})]} \sum_{k'=1}^q \mathbf{Q}^{k'[(\tilde{l}'_1 + \dots + \tilde{l}'_{j+1}) - (\tilde{l}''_1 + \dots + \tilde{l}''_{j+1})]} \\ & \times \binom{\tilde{l}_1}{\tilde{l}'_1} \binom{\tilde{l}_2}{\tilde{l}'_2} \dots \binom{\tilde{l}_{j+1}}{\tilde{l}'_{j+1}} \end{aligned}$$

with $\tilde{l}'_i, \tilde{l}''_i \geq 0$, $i = 1, \dots, j+1$. Since $\sum_{k'=1}^q \mathbf{Q}^{k'[(\tilde{l}'_1 + \dots + \tilde{l}'_{j+1}) - (\tilde{l}''_1 + \dots + \tilde{l}''_{j+1})]}$ is nonvanishing only when $\tilde{l}'_1 + \dots + \tilde{l}'_{j+1} = \tilde{l}''_1 + \dots + \tilde{l}''_{j+1}$, we obtain

$$\frac{1}{q} \sum_{k'=1}^q \tilde{s}_{k,k'}^{\tilde{l}_1} \tilde{s}_{k+1,k'}^{\tilde{l}_2} \dots \tilde{s}_{k+j,k'}^{\tilde{l}_{j+1}} = \sum_{\tilde{l}'_i + \tilde{l}''_i = \tilde{l}_i} \mathbf{Q}^{(\tilde{l}'_2 - \tilde{l}''_2) + 2(\tilde{l}'_3 - \tilde{l}''_3) + \dots + j(\tilde{l}'_{j+1} - \tilde{l}''_{j+1})} \binom{\tilde{l}_1}{\tilde{l}'_1} \binom{\tilde{l}_2}{\tilde{l}'_2} \dots \binom{\tilde{l}_{j+1}}{\tilde{l}'_{j+1}}.$$

Absorbing the binomial product $\binom{\tilde{l}_1}{\tilde{l}'_1} \binom{\tilde{l}_2}{\tilde{l}'_2} \dots \binom{\tilde{l}_{j+1}}{\tilde{l}'_{j+1}}$ into $c_{1,2}$ and changing the notation $\tilde{l}'_i \rightarrow \tilde{l}_i$, $\tilde{l}''_i \rightarrow \tilde{l}''_i$, we have

$$\begin{aligned} \frac{1}{q^2} \text{tr } H'^{2n} &= 2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ (1,1,2)\text{-composition of } 2n \\ \tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}}} c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) \\ & \times \mathbf{Q}^{(\tilde{l}_2 - \tilde{l}'_2) + 2(\tilde{l}_3 - \tilde{l}'_3) + \dots + j(\tilde{l}_{j+1} - \tilde{l}'_{j+1})} \frac{1}{q} \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \dots s_{k+j-1}^{l_j} \end{aligned} \quad (4.5)$$

with the combinatorial coefficients

$$c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) = \frac{(\tilde{l}_1 + \tilde{l}'_1 + l_1 - 1)!}{\tilde{l}_1! \tilde{l}'_1! l_1!} \prod_{k=2}^{j+1} \binom{l_{k-1} + \tilde{l}_k + \tilde{l}'_k + l_k - 1}{l_{k-1} - 1, \tilde{l}_k, \tilde{l}'_k, l_k}.$$

By convention $l_k = 0$ for $k > j$. The (1, 1, 2)-composition can be seen as an extension of (1, g)-composition in Section 3.3. Formally, we define the sequence of integers $\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j$ as a (1, 1, 2)-composition of $2n$ if they satisfy the conditions

$$2n = (\tilde{l}_1 + \dots + \tilde{l}_{j+1}) + (\tilde{l}'_1 + \dots + \tilde{l}'_{j+1}) + 2(l_1 + \dots + l_j), \quad \tilde{l}_i, \tilde{l}'_i \geq 0, \quad l_i > 0,$$

that is, l_i 's are the usual compositions of $1, 2, \dots, n$ and $\tilde{l}_i, \tilde{l}'_i$'s are nonnegative integers. We also include, with constraint $\tilde{l}_1 = \tilde{l}'_1$, the trivial composition $(n; n; 0)$.

4.3 (1, 1, 2)-exclusion statistics

In Section 4.2, we derived $\text{tr } H'^{2n}$, where the $(1, 1, 2)$ -composition appears in the summation and strongly implies a connection with a mixture of $g = 1$, $g = 1$, and $g = 2$ exclusion, which we refer to as $(1, 1, 2)$ -exclusion statistics. In this section, we will rewrite $\text{tr } H'^{2n}$ in its standard form that consists solely of $(1, 1, 2)$ -compositions, combinatorial coefficients, and trigonometric sums, and discuss the associated combinatorial interpretations.

Introduce $t_k = Q^k$. Given that for $\tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}$

$$\frac{1}{q} \sum_{k=1}^{q-j} t_k^{\tilde{l}_1} \bar{t}_k^{\tilde{l}'_1} s_k^{l_1} t_{k+1}^{\tilde{l}_2} \bar{t}_{k+1}^{\tilde{l}'_2} s_{k+1}^{l_2} \dots = Q^{(\tilde{l}_2 - \tilde{l}'_2) + 2(\tilde{l}_3 - \tilde{l}'_3) + \dots + j(\tilde{l}_{j+1} - \tilde{l}'_{j+1})} \frac{1}{q} \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \dots s_{k+j-1}^{l_j},$$

the trace in (4.5) can be rewritten in its standard form

$$\frac{1}{q^2} \text{tr } H'^{2n} = 2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ (1,1,2)\text{-composition of } 2n \\ \tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}}} c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) \frac{1}{q} \sum_{k=1}^{q-j} t_k^{\tilde{l}_1} \bar{t}_k^{\tilde{l}'_1} s_k^{l_1} t_{k+1}^{\tilde{l}_2} \bar{t}_{k+1}^{\tilde{l}'_2} s_{k+1}^{l_2} \dots, \quad (4.6)$$

which indicates indeed a $(1, 1, 2)$ -exclusion statistics, as expected. Therefore,

$$\frac{1}{q} \text{tr } H'^{2n} = -2n b_{2n}.$$

That is, $\text{tr } H'^{2n}$ is equivalent, up to a trivial factor, to the cluster coefficient b_{2n} associated with the $2n$ -body partition function for particles in a one-body spectrum ϵ_k ($k = 1, \dots, q$) obeying a mixture of three statistics: fermions with Boltzmann factor $e^{-\beta\epsilon_k} = t_k$, fermions of another type with Boltzmann factor $e^{-\beta\epsilon_k} = \bar{t}_k$, and two-fermion bound states occupying one-body energy levels k and $k+1$ with Boltzmann factor $e^{-\beta\epsilon_{k,k+1}} = -s_k$ behaving effectively as $g = 2$ exclusion particles. Unlike the usual cluster coefficient, b_{2n} is *constrained* by the requirement that the numbers of the two types of fermions are equal, implying $\text{tr } H'^{2n+1} = 0$, thus $\text{Tr } H^{2n+1} = 0$, as expected. Note that setting $t_k = \bar{t}_k = 0$ in (4.6) eliminates all terms with nonzero $\tilde{l}_i, \tilde{l}'_i$'s and (4.6) effectively reduces to (2.11).

We will now discuss the combinatorial interpretation of $(1, 1, 2)$ -compositions and $c_{1,1,2}$. The former one with the constraint $\tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}$ can be derived from their relation to cluster coefficients of $(1, 1, 2)$ -exclusion statistics. Precisely, $(1, 1, 2)$ -compositions of $2n$ with constraints correspond to all distinct connected arrangements of $2n$ particles on a one-body spectrum, consisting of two types of fermions (with equal numbers) and two-fermion bound states. In other words, they represent all the possible ways to place two types of particles and bound states on the spectrum such that they cannot be separated into two or more mutually non-overlapping groups. For example, as shown in Figure 4.3, there are seven $(1, 1, 2)$ -compositions of 4 with $\tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}$,

which contribute to

$$\begin{aligned}
 -b'_4 = \frac{1}{4q} \text{tr } H'^4 &= \frac{3}{2} \sum_{k=1}^q t_k^2 \bar{t}_k^2 + 2 \sum_{k=1}^{q-1} t_k \bar{t}_k s_k + \sum_{k=1}^{q-1} t_k s_k \bar{t}_{k+1} + \sum_{k=1}^{q-1} \bar{t}_k s_k t_{k+1} \\
 &+ 2 \sum_{k=1}^{q-1} s_k t_{k+1} \bar{t}_{k+1} + \frac{1}{2} \sum_{k=1}^{q-1} s_k^2 + \sum_{k=1}^{q-2} s_k s_{k+1}.
 \end{aligned}$$

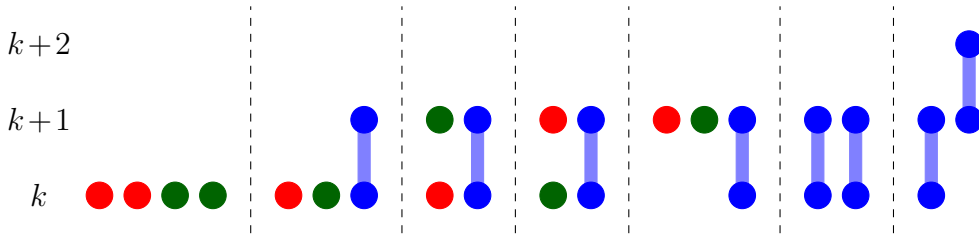


Figure 4.3: Seven $(1, 1, 2)$ -compositions of 4 with $\tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}$: $(2; 2; 0)$, $(1, 0; 1, 0; 1)$, $(1, 0; 0, 1; 1)$, $(0, 1; 1, 0; 1)$, $(0, 1; 0, 1; 1)$, $(0, 0; 0, 0; 2)$, $(0, 0; 0; 0, 0; 1, 1)$, illustrated by two types of fermions (red, green) and two-fermion bound states (blue).

Analogously to the derivation of $N_{1,g}(\mathbf{n})$ in (3.22), we obtain the number of $(1, 1, 2)$ -compositions of $2n$ with $\tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}$

$$N_{1,1,2}(n) = 1 + \sum_{k=0}^{n-1} \sum_{m=0}^k \binom{k}{m} \binom{n+m-k}{m+1}^2$$

with the convention $N_{1,1,2}(0) = 1$. Equivalently, the generating function of the $N_{1,1,2}(n)$ is

$$\sum_{n=0}^{\infty} x^n N_{1,1,2}(n) = \frac{1-x}{\sqrt{x^4 - 2x^3 + 7x^2 - 6x + 1}}.$$

Following the argument in Sections 3.2 and 3.3, $2n c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j)$ admits an interpretation as the number of periodic generalized Motzkin paths of length $2n$ with \tilde{l}_i horizontal steps, \tilde{l}'_i horizontal steps of another type, and l_i up steps originating from the first floor (see Figure 4.4 for an example).

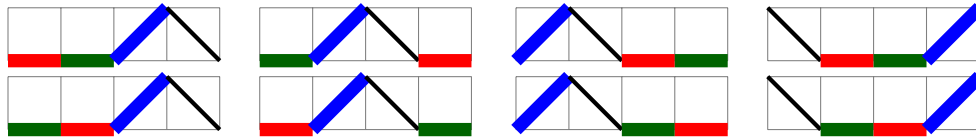


Figure 4.4: All the $4c_{1,1,2}(1, 0; 1, 0; 1) = 8$ periodic generalized Motzkin paths of length $2n = 4$ with $\tilde{l}_1 = 1$ horizontal step (red), $\tilde{l}'_1 = 1$ horizontal step of another type (green), and $l_1 = 1$ up step (blue) originating from the first floor.

4.4 Cubic lattice walks algebraic area enumeration

Based on (1.5), (4.3), (4.5), and (2.12), we deduce the desired number of closed random walks on a cubic lattice with given length $2n$ and algebraic area A (see Table 4.1 for a few examples)

$$\begin{aligned}
 C_{2n}(A) = & 2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ (1,1,2)\text{-composition of } 2n \\ \tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}}} \frac{(\tilde{l}_1 + \tilde{l}'_1 + l_1 - 1)!}{\tilde{l}_1! \tilde{l}'_1! l_1!} \prod_{k=2}^{j+1} \binom{l_{k-1} + \tilde{l}_k + \tilde{l}'_k + l_k - 1}{l_{k-1} - 1, \tilde{l}_k, \tilde{l}'_k, l_k} \sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \dots \sum_{k_j=-l_j}^{l_j} \\
 & \binom{2l_1}{l_1 + A - \sum_{i=2}^{j+1} (i-1)(\tilde{l}_i - \tilde{l}'_i) + \sum_{i=3}^j (i-2)k_i} \binom{2l_2}{l_2 - A + \sum_{i=2}^{j+1} (i-1)(\tilde{l}_i - \tilde{l}'_i) - \sum_{i=3}^j (i-1)k_i} \\
 & \times \prod_{i=3}^j \binom{2l_i}{l_i + k_i}. \tag{4.7}
 \end{aligned}$$

	$2n = 2$	4	6	8	10	12	14	16
$A = 0$	6	66	948	16626	338616	7672212	187381956	4830722994
± 1		24	756	19392	483420	12250440	319317012	8569537728
± 2			144	6744	230340	7095780	210620508	6185694816
± 3			12	1584	82980	3290664	115564092	3823229856
± 4				336	27000	1370952	57073380	2139867672
± 5				48	7740	528120	26372724	1124938272
± 6					1980	193524	11711952	569269968
± 7					420	65736	4987668	279376896
± 8					60	20952	2059344	134095632
± 9						5952	813204	62801664
± 10						1512	307608	28736784
± 11						288	109452	12790656
± 12						24	36204	5526240
± 13							11004	2304960
± 14							2940	925248
± 15							588	352032
± 16							84	127296
± 17								42528
± 18								12960
± 19								3360
± 20								672
± 21								96
Total count	6	90	1860	44730	1172556	32496156	936369720	27770358330

Table 4.1: $C_{2n}(A)$ up to $2n = 16$ for closed cubic lattice walks of length $2n$.

We note that, since

$$2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ (1,1,2)\text{-composition of } 2n \\ \tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1} \\ l_1 + \dots + l_j = k}} c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) = \binom{2n}{n} \binom{n}{k}^2,$$

from which we infer

$$2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ (1,1,2)\text{-composition of } 2n \\ \tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}}} c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) = \sum_{k=0}^n \binom{2n}{n} \binom{n}{k}^2 = \binom{2n}{n}^2,$$

and in the limit $q \rightarrow \infty$, i.e., $Q \rightarrow 1$ [37, 2],

$$\frac{1}{q} \sum_{k=1}^{q-j} t_k^{\tilde{l}_1} \tilde{t}_k^{\tilde{l}'_1} s_k^{l_1} t_{k+1}^{\tilde{l}_2} \tilde{t}_{k+1}^{\tilde{l}'_2} s_{k+1}^{l_2} \cdots \rightarrow \binom{2(l_1 + \dots + l_j)}{l_1 + \dots + l_j},$$

we recover the total number to be

$$\begin{aligned} & 2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ (1,1,2)\text{-composition of } 2n \\ \tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}}} c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) \binom{2(l_1 + \dots + l_j)}{l_1 + \dots + l_j} \\ &= \sum_{k=0}^n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ (1,1,2)\text{-composition of } 2n \\ \tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1} \\ l_1 + \dots + l_j = k}} 2n c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) \binom{2k}{k} \\ &= \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}, \end{aligned}$$

as expected¹.

4.5 Conclusion and perspectives

We have derived the number of closed walks of fixed length on a cubic lattice according to the algebraic area by an extension of the operator method used for the enumeration of closed planar lattice walks. A connection is made with the (1, 1, 2)-exclusion statistics.

¹By setting $Q = 1$, the total number $C_{2n}(A)$ can also be derived from the U, V, W -independent part in the expansion of $(U + V + W + U^{-1} + V^{-1} + W^{-1})^{2n}$. Using the binomial theorem and the total number of closed square lattice walks, we have $[U^0 V^0 W^0](U + V + W + U^{-1} + V^{-1} + W^{-1})^{2n} = [U^0 V^0 W^0] \sum_{k=0}^n \binom{2n}{2k} (U + V + U^{-1} + V^{-1})^{2k} (W + W^{-1})^{2n-2k} = \sum_{k=0}^n \binom{2n}{2k} \binom{2k}{k}^2 \binom{2n-2k}{n-k} = \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}$.

In the continuum limit, in which the lattice spacing $a \rightarrow 0$, closed cubic lattice walks become 3D Brownian loops. The probability distribution of the enclosed algebraic area A for a Brownian loop after a time t is given by

$$P'(A) = \frac{\pi}{2\sqrt{3}t} \frac{1}{\cosh^2[\pi A/(\sqrt{3}t)]}. \quad (4.8)$$

Note that this distribution is simply the Fourier transform of the partition function of a charged particle moving in continuous 3D space coupled to a uniform magnetic field $\mathbf{B} = (1, 1, 1)$. By aligning the magnetic field with the z direction through a change of coordinates, we obtain the standard Landau levels plus free motion in the z direction. This explains why (4.8) coincides with Lévy's law for 2D Brownian loops [39], up to a rescaling of A due to the normalization of \mathbf{B} . With the scaling $\mathbf{n}a^2 = 3t$, we infer from (4.8) the asymptotics for (4.7) as the walk length $\mathbf{n} = 2n \rightarrow \infty$

$$C_{\mathbf{n}}(A) \sim \frac{\sqrt{3}\pi}{2\mathbf{n} \cosh^2(\sqrt{3}\pi A/\mathbf{n})} \binom{\mathbf{n}}{\mathbf{n}/2} \sum_{k=0}^{\mathbf{n}/2} \binom{\mathbf{n}/2}{k}^2 \binom{2k}{k}, \quad (4.9)$$

where $A = 0, \pm 1, \pm 2, \dots$ is dimensionless. The asymptotics (4.9) has been checked numerically for \mathbf{n} up to 50. However, deriving (4.9) directly from (4.7) is nontrivial and remains an open problem.

It is natural to extend the definition of the algebraic area for a cubic lattice walk to the sum of projection areas with *arbitrary* weights, which is equivalent to specifying an arbitrary magnetic field \mathbf{B} . For instance, when $\mathbf{B} = (0, 0, 1)$, the algebraic area is defined as the area of the walk projected onto the xy plane. The counting for closed \mathbf{n} -step cubic lattice walks enclosing an algebraic area A under this definition turns out to be

$$C'_{\mathbf{n}}(A) = \sum_{l=0}^{\mathbf{n}/2} \binom{\mathbf{n}}{2l, \mathbf{n}/2 - l, \mathbf{n}/2 - l} C_{2l}(A),$$

where $C_{2l}(A)$ is the number of closed $2l$ -step square lattice walks enclosing an algebraic area A in (2.13) (not to be confused with the $C_{2n}(A)$ in (4.7) for cubic lattice walks). Similarly, as $\mathbf{n} \rightarrow \infty$,

$$C'_{\mathbf{n}}(A) \sim \frac{3\pi}{2\mathbf{n} \cosh^2(3\pi A/\mathbf{n})} \binom{\mathbf{n}}{\mathbf{n}/2} \sum_{k=0}^{\mathbf{n}/2} \binom{\mathbf{n}/2}{k}^2 \binom{2k}{k}.$$

The methodology used to define the algebraic area for cubic lattice walks can be extended to other 3D lattices, such as deformed triangular and honeycomb lattices (see Figure 4.5). However, the associated enumeration formulas and their connection with quantum exclusion statistics remain unresolved issues that require further study. Additionally, exploring the algebraic area enumeration for *open* random walks on various 3D lattices would also be of interest (see [54, 55] for open walks on a square lattice).

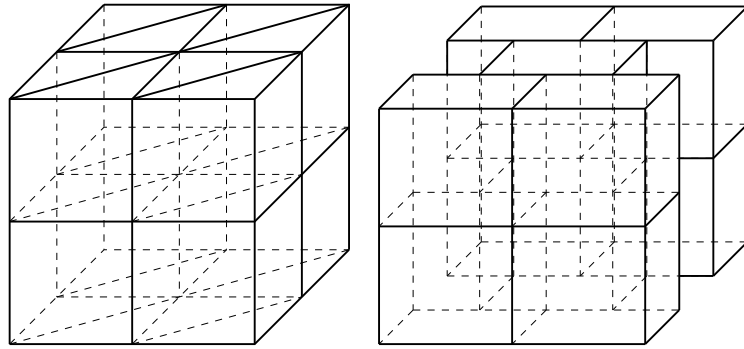



Figure 4.5: Deformed triangular and honeycomb lattices in 3D.

Article 3: Algebraic area of cubic lattice walks

Li Gan

Phys. Rev. E **108**, 054104 (2023)

Algebraic area of cubic lattice walks

Li Gan (甘立) *

LPTMS, CNRS, Université Paris-Saclay, 91405 Orsay Cedex, France



(Received 25 July 2023; accepted 9 October 2023; published 6 November 2023)

We obtain an explicit formula to enumerate closed random walks on a cubic lattice with a specified length and algebraic area. The algebraic area of a closed cubic lattice walk is defined as the sum of the algebraic areas obtained from the walk's projection onto the three Cartesian planes. This enumeration formula can be mapped onto the cluster coefficients of three types of particles that obey quantum exclusion statistics with statistical parameters $g = 1$, $g = 1$, and $g = 2$, respectively, subject to the constraint that the numbers of $g = 1$ (fermions) exclusion particles of two types are equal.

DOI: [10.1103/PhysRevE.108.054104](https://doi.org/10.1103/PhysRevE.108.054104)

I. INTRODUCTION

The algebraic area of a planar closed random walk is defined as the area swept by the walk, weighted by the winding number in each winding sector. The area is considered positive if the walk moves around the sector in a counterclockwise direction. In the continuous case, the probability distribution of the algebraic area A enclosed by closed Brownian curves after a time t is given by Lévy's stochastic area formula (also known as Lévy's law) [1]

$$P(A) = \frac{\pi}{2t} \frac{1}{\cosh^2(\pi A/t)}. \quad (1)$$

In the discrete case, a series of explicit algebraic area enumeration formulas [2–4] for closed random walks on various lattices have recently been obtained from the Kreft coefficients [5] encoding the Schrödinger equation of quantum Hofstadter-like models [6] that describe a charged particle hopping on planar lattices coupled to a perpendicular magnetic field. Essentially the enumeration amounts to calculating the trace of the power of the Hofstadter-like Hamiltonian and has an interpretation in terms of the statistical mechanics of particles that obey exclusion statistics with an integer exclusion parameter g ($g = 0$ for bosons, $g = 1$ for fermions, $g \geq 2$ for stronger exclusion than fermions). Figure 1 shows three examples of two-dimensional (2D) lattice random walks: the square lattice walk corresponds to the $g = 2$ exclusion, the Kreweras-like chiral walk on a triangular lattice corresponds to the $g = 3$ exclusion, and the honeycomb lattice walk corresponds to a mixture of the $g = 1$ and $g = 2$ exclusions, with an appropriate spectrum. Note that, in the context of Hofstadter-like model, the algebraic area can be expressed as $\frac{1}{2} \oint (\mathbf{r} \times d\mathbf{r}) \cdot \mathbf{B}$, where $\mathbf{B} = (0, 0, 1)$ and the integral is along the closed walk in the xy plane.

In this article, we extend the concept of algebraic area to closed cubic lattice walks by defining it as the sum of the algebraic areas of the walk projected onto the xy , yz , zx planes along the $-z$, $-x$, $-y$ directions. To count closed random

walks on a cubic lattice with a given length and algebraic area, we begin by introducing three lattice hopping operators U , V , W along the x , y , z directions, as well as U^{-1} , V^{-1} , W^{-1} along the $-x$, $-y$, $-z$ directions. These operators satisfy the noncommutative three-tori algebra [7]

$$VU = QUV, \quad WV = QVW, \quad UW = QWU, \quad (2)$$

with Q a central element (that is, Q commutes with all operators). It amounts to saying that the planar walks that go around the unit lattice cell on the Cartesian planes in a counterclockwise direction enclose an algebraic area 1, i.e., $V^{-1}U^{-1}VU = Q$, $W^{-1}V^{-1}WV = Q$, and $U^{-1}W^{-1}UW = Q$. The algebraic area A enclosed by a cubic lattice walk can thus be computed by reducing the corresponding hopping operators to Q^A using the commutation relations (2). See Fig. 2 for the closed six-step cubic lattice walk $UW^{-1}V^{-1}U^{-1}WV = Q$ as an example. Another example involves enumerating closed four-step walks. By taking the trace of $(U + V + W + U^{-1} + V^{-1} + W^{-1})^4 = 6(11 + 2Q + 2Q^{-1}) + \dots$, only terms with an equal number of U and U^{-1} , V and V^{-1} , W and W^{-1} survive, yielding the count of algebraic area¹: 66 walks enclose an algebraic area $A = 0$, 12 walks enclose an algebraic area $A = 1$, and 12 walks enclose an algebraic area $A = -1$.

By expressing the phase $Q = \exp(2\pi i\phi/\phi_0)$ in terms of the flux ϕ through the unit lattice cell on each of the three Cartesian planes in unit of the flux quantum ϕ_0 , the Hermitian operator

$$H = U + V + W + U^{-1} + V^{-1} + W^{-1} \quad (3)$$

represents a Hamiltonian that describes a charged particle hopping on a cubic lattice coupled to a magnetic field $\mathbf{B} = (1, 1, 1)$, as indicated in the definition of the algebraic area for a cubic lattice walk. The energy spectrum with $\mathbf{B} = (1, 1, 1)$

¹By symmetry, we can focus on the walks that start with a step along the x direction (i.e., U). There are 11 walks that enclose an algebraic area 0: $U^{-1}UU^{-1}U$, $UU^{-1}U^{-1}U$, $V^{-1}VU^{-1}U$, $VV^{-1}U^{-1}U$, $W^{-1}WU^{-1}U$, $WW^{-1}U^{-1}U$, $U^{-1}U^{-1}UU$, $U^{-1}V^{-1}VU$, $U^{-1}V^{-1}U$, $U^{-1}W^{-1}WU$, and $U^{-1}WW^{-1}U$, two walks that enclose an algebraic area 1: $V^{-1}U^{-1}VU$ and $WU^{-1}W^{-1}U$, and two walks that enclose an algebraic area -1 : $VU^{-1}V^{-1}U$ and $W^{-1}U^{-1}WU$.

*li.gan92@gmail.com

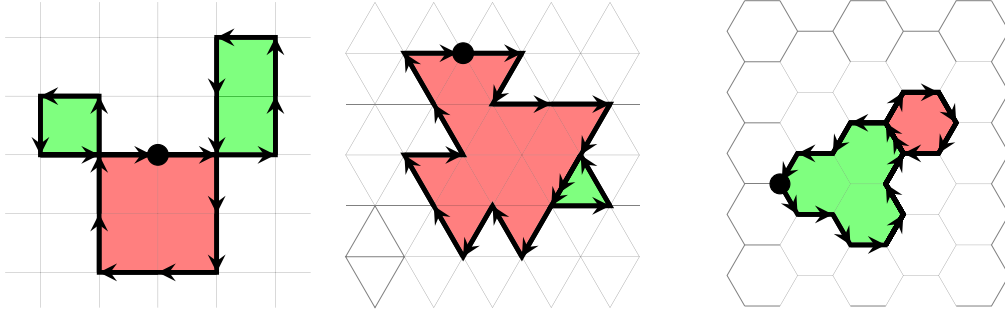


FIG. 1. A closed square lattice walk, chiral triangular lattice walk, and honeycomb lattice walk of length 18, starting and ending at the bullet point, with algebraic area -1 , -14 , and 2 , respectively. The region inside the walk, i.e., the winding sector, is colored green if its area is positive, otherwise it is colored red. In the chiral triangular lattice walk, only three of the possible six directions are allowed at each step, namely, in directions with angles 0 , $2\pi/3$ and $4\pi/3$ with respect to the horizontal axis.

on a cubic lattice was initially investigated in [8]. The three-dimensional (3D) Hofstadter model was studied earlier in [9], and the general case of the uniform magnetic field was explored in [10], with an experimental scheme proposed in [11]. Hofstadter models have also been studied on other 3D lattices, such as the tetragonal monoatomic and double-atomic lattice [12], and in four dimensions [13] as well.

As with the case of a planar lattice, the trace of H^{2n} provides the generating function for the number $C_{2n}(A)$ of closed random walks of length $2n$ (necessarily even) on a cubic lattice enclosing an algebraic area A . Specifically,

$$\sum_A C_{2n}(A)Q^A = \text{Tr} H^{2n}, \quad (4)$$

with the normalization $\text{Tr} I = 1$, where I denotes the identity operator.

The article is organized as follows. Assuming that the flux is rational, we use the finite-dimensional representation of the algebra (2) to derive the trace of H^{2n} , establish its connection with quantum exclusion statistics ($g = 1, g = 1, g = 2$), and provide a combinatorial interpretation based on the combinatorial coefficients $c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j)$ labeled by the $(1,1,2)$ -compositions. In Sec. III, we present the explicit formula for $C_{2n}(A)$, as well as its asymptotics as $n \rightarrow \infty$, and discuss potential generalizations and applications.

II. ALGEBRAIC AREA ENUMERATION OF CUBIC LATTICE WALKS

A. Hamiltonian

From now on, we assume that the magnetic flux on each Cartesian plane is rational, i.e., $\phi/\phi_0 = p/q$ with p and q being coprime, thus $Q = \exp(2\pi i p/q)$. To obtain the finite-dimensional representation of U, V, W , we introduce the $q \times q$ “clock” and “shift” matrices

$$u = e^{ik_x} \begin{pmatrix} Q & 0 & 0 & \dots & 0 & 0 \\ 0 & Q^2 & 0 & \dots & 0 & 0 \\ 0 & 0 & Q^3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & Q^{q-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

$$v = e^{ik_y} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

which satisfy $vu = Quv$ and contribute to the Hofstadter Hamiltonian $u + v + u^{-1} + v^{-1}$ for square lattice walks.

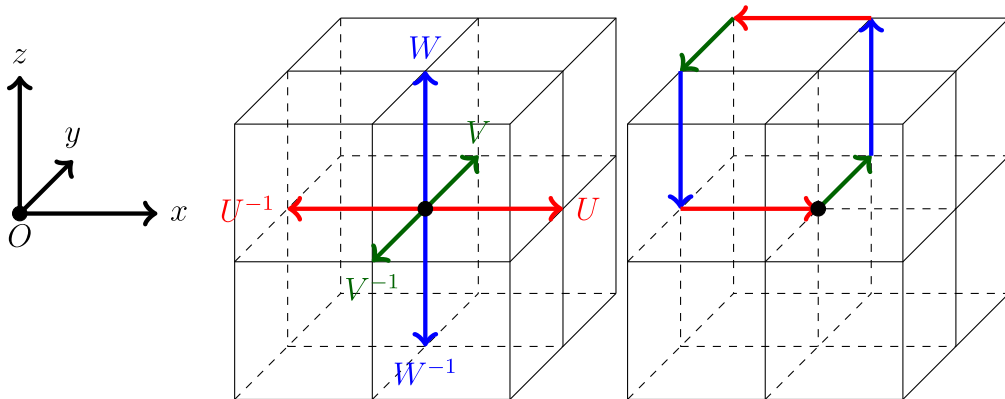


FIG. 2. (left) Three-dimensional Cartesian coordinate system; (middle) six lattice hopping operators in a cubic lattice; (right) closed 6-step cubic lattice walk $UW^{-1}V^{-1}U^{-1}WV$ whose algebraic area is given by $A = 1 + 1 + (-1) = 1$. Using the commutation relations (2), $UW^{-1}V^{-1}U^{-1}WV$ is simplified to Q^1 as expected.

Here, k_x and k_y denote the quasimomenta in the x and y directions. In the quantum trace, integration over k_x and k_y eliminates the unwanted terms containing u^q and v^q which correspond to open walks but can be closed by q periodicity. Another way to achieve this is by setting $k_x = k_y = 0$ and considering walks of length less than q .

Because of the open walk $UVW \neq I$, it is not possible to represent the operators U, V, W as $u, v, v^{-1}u^{-1}$, respectively, even though they satisfy the algebra (2). To address this, we introduce an additional vector space with dimension q' , in which U and V act as identity operators, while W does not. Consequently, we obtain the representation of (2) as $qq' \times qq'$ matrices

$$U = u \otimes I_{q'}, \quad V = v \otimes I_{q'}, \quad W = (v^{-1}u^{-1}) \otimes u',$$

where u' is an arbitrary $q' \times q'$ matrix that is not proportional to $I_{q'}$. Again, the quasimomenta are set to be zero. The sought-after quantum trace of the $qq' \times qq'$ Hamiltonian matrix (3) reduces to the usual trace up to a normalization factor, that is,

$$\text{Tr } H^{2n} = \frac{1}{qq'} \text{tr } H^{2n}.$$

Let u' be diagonal and equal to $u|_{q \rightarrow q'}$ [therefore $Q \rightarrow Q' = \exp(2\pi i p/q')$ in u']. Performing the algebra-preserving transformation $u \rightarrow -u^{-1}v, v \rightarrow v^{-1}, u' \rightarrow -u'$ leads to the new Hamiltonian that describes walks on a deformed cubic lattice

$$H' = H_2 \otimes I_{q'} + u \otimes u' + u^{-1} \otimes u'^{-1},$$

where the Hofstadter Hamiltonian associated to the usual square lattice walks is

$$H_2 = -u^{-1}v - v^{-1}u + v + v^{-1}$$

$$= \begin{pmatrix} 0 & \bar{f}_1 & 0 & \cdots & 0 & 0 \\ f_1 & 0 & \bar{f}_2 & \cdots & 0 & 0 \\ 0 & f_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \bar{f}_{q-1} \\ 0 & 0 & 0 & \cdots & f_{q-1} & 0 \end{pmatrix},$$

with $f_k = 1 - Q^k$. Note that H_2 is a $g = 2$ matrix in the sense that its secular determinant $\det(I_q - zH_2) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z_n z^{2n}$ captures the Kreft coefficient [5]

$$Z_n = \sum_{k_1=1}^{q-2n+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} s_{k_1+2n-2} s_{k_2+2n-4} \cdots s_{k_{n-1}+2} s_{k_n},$$

$$Z_0 = 1,$$

as a trigonometric multiple nested sum with +2 shifts among the spectral functions $s_k := f_k \bar{f}_k = 4 \sin^2(k\pi p/q)$. In statistical mechanics, Z_n can be interpreted as the n -body partition function for n particles in a one-body spectrum ϵ_k ($k = 1, 2, \dots, q-1$) with Boltzmann factor $e^{-\beta \epsilon_k} = s_k$. The +2 shifts indicate that these particles obey $g = 2$ exclusion statistics, i.e., no two particles can occupy adjacent quantum states. By introducing cluster coefficients b_n via $\log(\sum_{n=0}^{\lfloor q/2 \rfloor} Z_n x^n) = \sum_{n=1}^{\infty} b_n x^n$ with fugacity $x = -z^2$, and using the identity $\log \det(I_q - zH_2) = \text{tr} \log(I_q - zH_2) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \text{tr } H_2^n$ we establish a connection between the generating function for algebraic area enumeration of square lattice

walks and the cluster coefficients with $g = 2$ exclusion statistics, that is,

$$\text{Tr } H_2^{2n} = \frac{1}{q} \text{tr } H_2^{2n} = 2n(-1)^{n+1} \frac{1}{q} b_n$$

$$= 2n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} c_2(l_1, l_2, \dots, l_j) \frac{1}{q} \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}, \tag{5}$$

where $c_2(l_1, l_2, \dots, l_j) = \frac{1}{l_1} \prod_{i=2}^j \binom{l_{i-1} + l_i - 1}{l_i}$. As we will see in Sec. II B, the algebraic area enumeration for cubic lattice walks can also be mapped onto cluster coefficients with appropriate exclusion parameters and spectral functions.

Now come back to the Hamiltonian H' . Introduce $\tilde{s}_{k,k'} = Q^k Q^{k'} + Q^{-k} Q^{-k'}$ and $q' \times q'$ diagonal matrices $\tilde{s}_k = \text{diag}(\tilde{s}_{k,1}, \tilde{s}_{k,2}, \dots, \tilde{s}_{k,q'})$, $\mathbf{f}_k = f_k I_{q'}$, $\bar{\mathbf{f}}_k = \bar{f}_k I_{q'}$, $\mathbf{0} = 0 I_{q'}$. H' can be expressed as a $qq' \times qq'$ block tridiagonal matrix

$$H' = \begin{pmatrix} \tilde{\mathbf{s}}_1 & \bar{\mathbf{f}}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{f}_1 & \tilde{\mathbf{s}}_2 & \bar{\mathbf{f}}_2 & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{f}_2 & \tilde{\mathbf{s}}_3 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \tilde{\mathbf{s}}_{q-1} & \bar{\mathbf{f}}_{q-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{f}_{q-1} & \tilde{\mathbf{s}}_q \end{pmatrix}.$$

Applying the trace computation techniques described in [14] we obtain

$$\frac{1}{qq'} \text{tr } H^{2n} = 2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j \\ (1,2)\text{-composition of } 2n}} c_{1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j)$$

$$\times \frac{1}{q} \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}$$

$$\times \frac{1}{q'} \sum_{k'=1}^{q'} \tilde{s}_{k,k'}^{\tilde{l}_1} \tilde{s}_{k+1,k'}^{\tilde{l}_2} \cdots \tilde{s}_{k+j,k'}^{\tilde{l}_{j+1}}, \tag{6}$$

with

$$c_{1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j)$$

$$= \frac{(\tilde{l}_1 + l_1 - 1)!}{\tilde{l}_1! l_1!} \prod_{k=2}^{j+1} \binom{l_{k-1} + \tilde{l}_k + l_k - 1}{l_{k-1} - 1, \tilde{l}_k, l_k}.$$

By convention $l_k = 0$ for $k > j$. We define the sequence of integers $\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j, j \geq 0$, as a (1,2)-composition of $2n$ if they satisfy the conditions

$$2n = (\tilde{l}_1 + \cdots + \tilde{l}_{j+1}) + 2(l_1 + \cdots + l_j), \tilde{l}_i \geq 0, l_i > 0,$$

i.e., l_i 's are the usual compositions of $1, 2, \dots, n$ and \tilde{l}_i 's are additional nonnegative integers. For $j = 0$ we have the trivial composition $\tilde{l}_1 = 2n$.

As q' is arbitrary, for simplicity of calculation we set $q' = q$ in the sequel. The second trigonometric sum in (6) is expanded

to be

$$\frac{1}{q} \sum_{k'=1}^q \tilde{s}_{k,k'}^{\tilde{l}_1} \tilde{s}_{k+1,k'}^{\tilde{l}_2} \cdots \tilde{s}_{k+j,k'}^{\tilde{l}_{j+1}} = \frac{1}{q} \sum_{\tilde{l}'_i + \tilde{l}''_i = \tilde{l}_i} Q^{(\tilde{l}_2 - \tilde{l}'_2) + 2(\tilde{l}_3 - \tilde{l}'_3) + \cdots + j(\tilde{l}_{j+1} - \tilde{l}'_{j+1}) + k(\tilde{l}'_1 + \cdots + \tilde{l}'_{j+1}) - (\tilde{l}''_1 + \cdots + \tilde{l}''_{j+1})}$$

$$\times \sum_{k'=1}^q Q^{k(\tilde{l}'_1 + \cdots + \tilde{l}'_{j+1}) - (\tilde{l}''_1 + \cdots + \tilde{l}''_{j+1})} \binom{\tilde{l}_1}{\tilde{l}'_1} \binom{\tilde{l}_2}{\tilde{l}'_2} \cdots \binom{\tilde{l}_{j+1}}{\tilde{l}'_{j+1}},$$

with $\tilde{l}'_i, \tilde{l}''_i \geq 0, i = 1, \dots, j + 1$. Since $\sum_{k'=1}^q Q^{k(\tilde{l}'_1 + \cdots + \tilde{l}'_{j+1}) - (\tilde{l}''_1 + \cdots + \tilde{l}''_{j+1})}$ is nonvanishing only when $\tilde{l}'_1 + \cdots + \tilde{l}'_{j+1} = \tilde{l}''_1 + \cdots + \tilde{l}''_{j+1}$ we obtain

$$\frac{1}{q} \sum_{k'=1}^q \tilde{s}_{k,k'}^{\tilde{l}_1} \tilde{s}_{k+1,k'}^{\tilde{l}_2} \cdots \tilde{s}_{k+j,k'}^{\tilde{l}_{j+1}} = \sum_{\tilde{l}'_i + \tilde{l}''_i = \tilde{l}_i} Q^{(\tilde{l}_2 - \tilde{l}'_2) + 2(\tilde{l}_3 - \tilde{l}'_3) + \cdots + j(\tilde{l}_{j+1} - \tilde{l}'_{j+1})} \binom{\tilde{l}_1}{\tilde{l}'_1} \binom{\tilde{l}_2}{\tilde{l}'_2} \cdots \binom{\tilde{l}_{j+1}}{\tilde{l}'_{j+1}}.$$

Finally, by recognizing that the binomial product $\binom{\tilde{l}_1}{\tilde{l}'_1} \binom{\tilde{l}_2}{\tilde{l}'_2} \cdots \binom{\tilde{l}_{j+1}}{\tilde{l}'_{j+1}}$ can be absorbed into $c_{1,2}$, as well as changing the notation $\tilde{l}'_i \rightarrow \tilde{l}_i, \tilde{l}''_i \rightarrow \tilde{l}'_i$, we arrive at

$$\frac{1}{q^2} \text{tr } H^{2n} = 2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ (1,1,2)\text{-composition of } 2n \\ \tilde{l}_1 + \cdots + \tilde{l}_{j+1} = \tilde{l}'_1 + \cdots + \tilde{l}'_{j+1}}} c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) Q^{(\tilde{l}_2 - \tilde{l}'_2) + 2(\tilde{l}_3 - \tilde{l}'_3) + \cdots + j(\tilde{l}_{j+1} - \tilde{l}'_{j+1})} \frac{1}{q} \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}, \tag{7}$$

with the combinatorial coefficients

$$c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) = \frac{(\tilde{l}_1 + \tilde{l}'_1 + l_1 - 1)!}{\tilde{l}_1! \tilde{l}'_1! l_1!} \prod_{k=2}^{j+1} \binom{l_{k-1} + \tilde{l}_k + \tilde{l}'_k + l_k - 1}{l_{k-1} - 1, \tilde{l}_k, \tilde{l}'_k, l_k}.$$

By convention $l_k = 0$ for $k > j$. We define the sequence of integers $\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j$ as a (1,1,2)-composition of $2n$ if they satisfy the conditions

$$2n = (\tilde{l}_1 + \cdots + \tilde{l}_{j+1}) + (\tilde{l}'_1 + \cdots + \tilde{l}'_{j+1}) + 2(l_1 + \cdots + l_j), \tilde{l}_i, \tilde{l}'_i \geq 0, l_i > 0,$$

i.e., l_i 's are the usual compositions of $1, 2, \dots, n$ and $\tilde{l}_i, \tilde{l}'_i$'s are nonnegative integers. We also include, with constraint $\tilde{l}_1 = \tilde{l}'_1$, the trivial composition $(n; n; 0)$. A combinatorial interpretation of the (1,1,2)-composition and $c_{1,1,2}$ will be discussed in Sec. II C.

B. (1,1,2)-exclusion statistics

Now we take a step further by defining $t_k = Q^k$. Given that for $\tilde{l}_1 + \cdots + \tilde{l}_{j+1} = \tilde{l}'_1 + \cdots + \tilde{l}'_{j+1}$

$$\frac{1}{q} \sum_{k=1}^{q-j} t_k^{\tilde{l}_1} \tilde{t}_k^{\tilde{l}'_1} s_k^{l_1} \tilde{s}_{k+1}^{\tilde{l}_2} \tilde{t}_{k+1}^{\tilde{l}'_2} s_{k+1}^{l_2} \cdots = Q^{(\tilde{l}_2 - \tilde{l}'_2) + 2(\tilde{l}_3 - \tilde{l}'_3) + \cdots + j(\tilde{l}_{j+1} - \tilde{l}'_{j+1})} \frac{1}{q} \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j},$$

we rewrite (7) in its standard form that consists solely of compositions, a combinatorial coefficient, and a trigonometric sum, as follows:

$$\frac{1}{q^2} \text{tr } H^{2n} = 2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ (1,1,2)\text{-composition of } 2n \\ \tilde{l}_1 + \cdots + \tilde{l}_{j+1} = \tilde{l}'_1 + \cdots + \tilde{l}'_{j+1}}} c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) \frac{1}{q} \sum_{k=1}^{q-j} t_k^{\tilde{l}_1} \tilde{t}_k^{\tilde{l}'_1} s_k^{l_1} \tilde{s}_{k+1}^{\tilde{l}_2} \tilde{t}_{k+1}^{\tilde{l}'_2} s_{k+1}^{l_2} \cdots, \tag{8}$$

which indicates a mixture of $g = 1, g = 1$, and $g = 2$ exclusion. We call it (1,1,2)-exclusion statistics. Therefore,

$$\text{Tr } H^{2n} = \frac{1}{q^2} \text{tr } H^{2n} = \frac{1}{q^2} \text{tr } H^{2n} = -\frac{2n}{q} b'_{2n}. \tag{9}$$

That is, $\text{Tr } H^{2n}$ is equivalent, up to a trivial factor, to the cluster coefficient b'_{2n} associated with the $2n$ -body

partition function for particles in a one-body spectrum ϵ_k ($k = 1, \dots, q$) obeying a mixture of three statistics: fermions with Boltzmann factor $e^{-\beta\epsilon_k} = t_k$, fermions of another type with Boltzmann factor $e^{-\beta\epsilon_k} = \tilde{t}_k$, and two-fermion bound states occupying one-body energy levels k and $k + 1$ with Boltzmann factor $e^{-\beta\epsilon_{k,k+1}} = -s_k$ behaving effectively as $g = 2$ exclusion particles. b'_{2n} is constrained by the requirement that

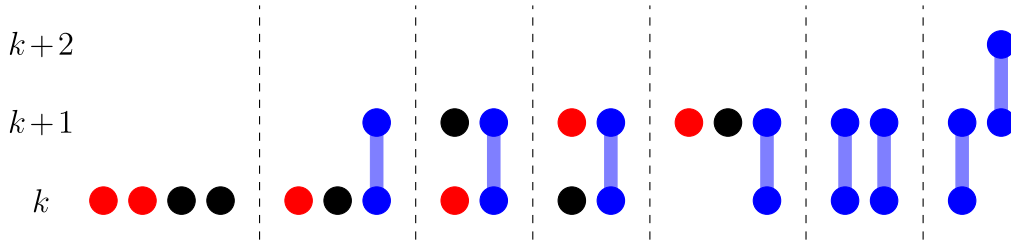


FIG. 3. Seven (1,1,2)-compositions of 4 with $\tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}$: (2; 2; 0), (1, 0; 1, 0; 1), (1, 0; 0, 1; 1), (0, 1; 1, 0; 1), (0, 1; 0, 1; 1), (0, 0; 0, 0; 2), (0, 0; 0, 0, 0, 0; 1, 1), illustrated by two types of fermions (red, black) and two-fermion bound states (blue).

the numbers of the two types of fermions are equal, implying $\text{Tr } H^{2n+1} = 0$ as expected. Note that setting $t_k = \tilde{t}_k = 0$ in (8) eliminates all terms with nonzero $\tilde{l}_i, \tilde{l}'_i$'s and (9) effectively reduces to (5).

C. Combinatorial interpretation

The (1,1,2)-compositions with the constraint $\tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}$ have a combinatorial interpretation, which can be derived from their relation to cluster coefficients of (1,1,2)-exclusion statistics. Specifically, (1,1,2)-compositions of $2n$ with constraints correspond to all distinct connected arrangements of $2n$ particles on a one-body spectrum, consisting of two types of fermions (with equal numbers) and two-fermion bound states. In other words, they represent all the possible ways to place two types of particles and bound states on the spectrum such that they cannot be separated into two or more mutually nonoverlapping groups. For example, as shown in Fig. 3, there are seven (1,1,2)-compositions of 4 with $\tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}$, which contribute to

$$-b'_4 = \frac{1}{4q} \text{tr } H^4 = \frac{3}{2} \sum_{k=1}^q t_k^2 \tilde{t}_k^2 + 2 \sum_{k=1}^{q-1} t_k \tilde{t}_k s_k$$

$$+ \sum_{k=1}^{q-1} t_k s_k \tilde{t}_{k+1} + \sum_{k=1}^{q-1} \tilde{t}_k s_k t_{k+1} + 2 \sum_{k=1}^{q-1} s_k t_{k+1} \tilde{t}_{k+1} + \frac{1}{2} \sum_{k=1}^{q-1} s_k^2 + \sum_{k=1}^{q-2} s_k s_{k+1}.$$

Following the argument in [14], $2n c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j)$ admits an interpretation as the number of periodic generalized Motzkin paths of length $2n$ with \tilde{l}_i horizontal steps (1,0), \tilde{l}'_i horizontal steps (1,0) of another type, and l_i up steps (1,1) originating from the i th floor (see Fig. 4 for an example). By ‘‘periodic generalized Motzkin paths’’ we refer to generalized Motzkin bridges (and excursions) that start and end on the same floor.

III. CONCLUSION

Based on (4), (7), (9), and the fact that the trigonometric sum $\frac{1}{q} \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \dots s_{k+j-1}^{l_j}$ can be computed [2,3], we deduce the desired counting for closed random walks on a cubic lattice with given length $2n$ and algebraic area A

$$C_{2n}(A) = 2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ \text{(1,1,2)-composition of } 2n \\ \tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}}} \frac{(\tilde{l}_1 + \tilde{l}'_1 + l_1 - 1)!}{\tilde{l}_1! \tilde{l}'_1! l_1!} \prod_{k=2}^{j+1} \binom{l_{k-1} + \tilde{l}_k + \tilde{l}'_k + l_k - 1}{l_{k-1} - 1, \tilde{l}_k, \tilde{l}'_k, l_k} \sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \dots \sum_{k_j=-l_j}^{l_j} \left(l_1 + A - \sum_{i=2}^{j+1} (i-1)(\tilde{l}_i - \tilde{l}'_i) + \sum_{i=3}^j (i-2)k_i \right) \left(l_2 - A + \sum_{i=2}^{j+1} (i-1)(\tilde{l}_i - \tilde{l}'_i) - \sum_{i=3}^j (i-1)k_i \right) \prod_{i=3}^j \binom{2l_i}{l_i + k_i}. \tag{10}$$

Note that the enumeration can be computed recursively as well. See Appendix A for further details and several examples of $C_{2n}(A)$. In Appendix B, we present some combinatorial results for (1,1,2)-compositions and $c_{1,1,2}$, where the overall counting of closed $2n$ -step cubic lattice walks is recovered to be $\binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}$.

In the continuum limit, in which the lattice spacing $a \rightarrow 0$, closed cubic lattice walks become 3D closed Brownian curves. The probability distribution of the enclosing an algebraic area A for a closed Brownian curve after a time t is

given by

$$P'(A) = \frac{\pi}{2\sqrt{3}t} \frac{1}{\cosh^2[\pi A / (\sqrt{3}t)]}. \tag{11}$$

Note that this distribution is simply the Fourier transform of the partition function of a charged particle moving in continuous 3D space coupled to a uniform magnetic field $\mathbf{B} = (1, 1, 1)$. By aligning the magnetic field with the z direction through a change of coordinates, we obtain the standard Landau levels plus free motion in the z direction. This

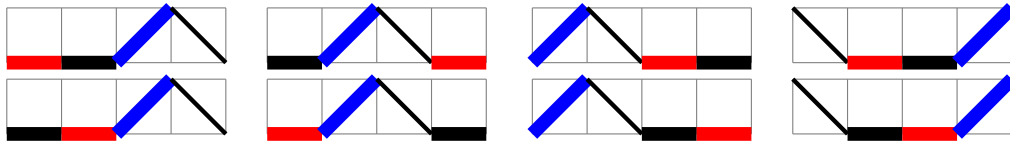


FIG. 4. All the $4c_{1,1,2}(1, 0; 1, 0; 1) = 8$ periodic generalized Motzkin paths of length $2n = 4$ with $\tilde{l}_1 = 1$ horizontal step (red), $\tilde{l}'_1 = 1$ horizontal step of another type (black), and $l_1 = 1$ up step (blue) originating from the first floor.

explains why (11) coincides with Lévy’s law (1) for 2D closed Brownian curves, up to a rescaling of A due to the normalization of \mathbf{B} . With the scaling $n a^2 = 3t$, we infer from (11) the asymptotics for (10) as the walk length $\mathbf{n} = 2n \rightarrow \infty$

$$C_{\mathbf{n}}(A) \sim \frac{\sqrt{3}\pi}{2\mathbf{n} \cosh^2(\sqrt{3}\pi A/\mathbf{n})} \binom{\mathbf{n}}{\mathbf{n}/2} \sum_{k=0}^{\mathbf{n}/2} \binom{\mathbf{n}/2}{k}^2 \binom{2k}{k}, \tag{12}$$

where $A = 0, \pm 1, \pm 2, \dots$ is dimensionless. The asymptotics (12) has been checked numerically for \mathbf{n} up to 50. However, deriving (12) directly from (10) is nontrivial and remains an open problem.

It is natural to extend the definition of the algebraic area for a cubic lattice walk to the sum of projection areas with arbitrary weights, which is equivalent to specifying an arbitrary magnetic field \mathbf{B} . For instance, when $\mathbf{B} = (0, 0, 1)$, the algebraic area is defined as the area of the walk projected onto the xy plane. The counting for closed \mathbf{n} -step cubic lattice walks enclosing an algebraic area A under this definition turns out to be

$$C'_{\mathbf{n}}(A) = \sum_{l=0}^{\mathbf{n}/2} \binom{\mathbf{n}}{2l, \mathbf{n}/2 - l, \mathbf{n}/2 - l} C_{2l, \text{sq}}(A),$$

where $C_{2l, \text{sq}}(A)$ is the number of closed $2l$ -step square lattice walks enclosing an algebraic area A . Similarly, as $\mathbf{n} \rightarrow \infty$,

$$C'_{\mathbf{n}}(A) \sim \frac{3\pi}{2\mathbf{n} \cosh^2(3\pi A/\mathbf{n})} \binom{\mathbf{n}}{\mathbf{n}/2} \sum_{k=0}^{\mathbf{n}/2} \binom{\mathbf{n}/2}{k}^2 \binom{2k}{k}.$$

The methodology used to define an algebraic area for a cubic lattice walk can be extended to other 3D lattices, such as deformed triangular and honeycomb lattices (see Fig. 5). However, the associated enumeration formulas and their connection with quantum exclusion statistics remain unresolved

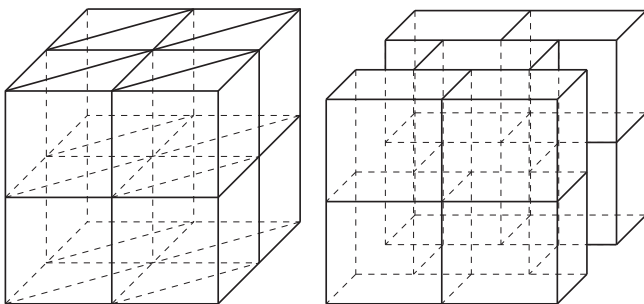


FIG. 5. Deformed triangular and honeycomb lattices in three dimensions.

issues that require further study. Additionally, exploring the algebraic area enumeration for open random walks on various 3D lattices would also be of interest (see [15,16] for open walks on a square lattice).

In addition to the algebraic area enumeration, the explicit expression for $\text{tr} H^{2n}$ can also be regarded as a term, up to a factor, in the expansion of the partition function $\text{tr}(e^{-\beta H})$ for the 3D Hofstadter model, which provides a reference from a different perspective for investigating the spectrum and its intriguing fractal properties. Furthermore, our results have potential implications for the study of spin models with complex frustration graphs. In a frustration graph, each vertex corresponds to a Hamiltonian term. The edges connect all the terms that do not commute. Unconnected terms mutually commute. Analyzing the 3D Hofstadter Hamiltonian (3) offers insights into solving the spectrum of spin models whose frustration graphs contain even holes (see [17] for free-fermionic systems, each with an even-hole-free and claw-free frustration graph).

ACKNOWLEDGMENTS

The author would like to express gratitude to Stéphane Ouvry and Alexios P. Polychronakos for valuable discussions, and to the anonymous referees for their useful comments and suggestions. The author also acknowledges the financial support of China Scholarship Council (Grant No. 2020091110129).

APPENDIX A: RECURRENCE RELATION FOR ENUMERATION OF CUBIC LATTICE WALKS

Consider an \mathbf{n} -step cubic lattice walk that consists of m_1 steps in the direction $(1, 0, 0)$, m_2 steps in the

TABLE I. $C_{2n}(A)$ up to $2n = 10$ for cubic lattice walks of length $2n$.

	$2n = 2$	4	6	8	10
$A = 0$	6	66	948	16 626	338 616
± 1		24	756	19 392	483 420
± 2			144	6744	230 340
± 3			12	1584	82 980
± 4				336	27 000
± 5				48	7740
± 6					1980
± 7					420
± 8					60
Total counting	6	90	1860	44 730	1 172 556

direction $(-1, 0, 0)$, l_1 steps in the direction $(0,1,0)$, l_2 steps in the direction $(0, -1, 0)$, r_1 steps in the direction $(0,0,1)$, r_2 steps in the direction $(0, 0, -1)$, where $m_1 + m_2 + l_1 + l_2 + r_1 + r_2 = \mathbf{n}$. If the walk is open, we can close it by adding a straight line that connects the endpoint to the starting point. Let $C_{m_1, m_2, l_1, l_2, r_1, r_2}(A)$ denote the number of such walks that enclose an algebraic area A . The generating function $Z_{m_1, m_2, l_1, l_2, r_1, r_2}(\mathbf{Q}) = \sum_A C_{m_1, m_2, l_1, l_2, r_1, r_2}(A) \mathbf{Q}^A$ can be computed by the recursion

$$\begin{aligned} Z_{m_1, m_2, l_1, l_2, r_1, r_2}(\mathbf{Q}) &= \mathbf{Q}^{(l_2 - l_1 + r_1 - r_2)/2} Z_{m_1 - 1, m_2, l_1, l_2, r_1, r_2}(\mathbf{Q}) \\ &+ \mathbf{Q}^{(l_1 - l_2 + r_2 - r_1)/2} Z_{m_1, m_2 - 1, l_1, l_2, r_1, r_2}(\mathbf{Q}) \\ &+ \mathbf{Q}^{(m_1 - m_2 + r_2 - r_1)/2} Z_{m_1, m_2, l_1 - 1, l_2, r_1, r_2}(\mathbf{Q}) \end{aligned}$$

$$\begin{aligned} &+ \mathbf{Q}^{(m_2 - m_1 + r_1 - r_2)/2} Z_{m_1, m_2, l_1, l_2 - 1, r_1, r_2}(\mathbf{Q}) \\ &+ \mathbf{Q}^{(m_2 - m_1 + l_1 - l_2)/2} Z_{m_1, m_2, l_1, l_2, r_1 - 1, r_2}(\mathbf{Q}) \\ &+ \mathbf{Q}^{(m_1 - m_2 + l_2 - l_1)/2} Z_{m_1, m_2, l_1, l_2, r_1, r_2 - 1}(\mathbf{Q}), \end{aligned}$$

with $Z_{0,0,0,0,0,0}(\mathbf{Q}) = 1$ and $Z_{m_1, m_2, l_1, l_2, r_1, r_2}(\mathbf{Q}) = 0$ whenever $\min(m_1, m_2, l_1, l_2, r_1, r_2) < 0$.

For closed walks of length $\mathbf{n} = 2n$, we have

$$\sum_A C_{2n}(A) \mathbf{Q}^A = \sum_{m=0}^n \sum_{l=0}^{n-m} Z_{m, m, l, l, n-m-l, n-m-l}(\mathbf{Q}). \quad (\text{A1})$$

Table I lists some examples of $C_{2n}(A)$.

APPENDIX B: COMBINATORIAL RESULTS FOR (1,1,2)-COMPOSITIONS AND $c_{1,1,2}$

By considering the combinatorial interpretation of cluster coefficient b'_{2n} as fermions of two types and two-fermion bound states, we can derive the counting of (1,1,2)-compositions of $2n$ with $\tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}$ to be

$$\begin{aligned} N_{1,1,2}(n) &= 1 + \sum_{k=0}^{n-1} \sum_{m=0}^k \binom{k}{m} \binom{n+m-k}{m+1}^2 \\ &= 1, 2, 7, 27, 108, 443, \dots, \end{aligned}$$

with the convention $N_{1,1,2}(0) = 1$. Equivalently, the generating function of the $N_{1,1,2}(n)$'s is

$$\sum_{n=0}^{\infty} x^n N_{1,1,2}(n) = \frac{1-x}{\sqrt{x^4 - 2x^3 + 7x^2 - 6x + 1}}.$$

We have

$$2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ \text{(1,1,2)-composition of } 2n \\ \tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1} \\ l_1 + \dots + l_j = k}} c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) = \binom{2n}{n} \binom{n}{k}^2,$$

from which we infer

$$2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ \text{(1,1,2)-composition of } 2n \\ \tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}}} c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) = \sum_{k=0}^n \binom{2n}{n} \binom{n}{k}^2 = \binom{2n}{n}^2.$$

In the limit $q \rightarrow \infty$, i.e., $\mathbf{Q} \rightarrow 1$ [2,3]

$$\frac{1}{q} \sum_{k=1}^{q-j} t_k^{\tilde{l}_1} \bar{t}_k^{\tilde{l}'_1} s_k^{l_1} t_{k+1}^{\tilde{l}_2} \bar{t}_{k+1}^{\tilde{l}'_2} s_{k+1}^{l_2} \dots \rightarrow \binom{2(l_1 + \dots + l_j)}{l_1 + \dots + l_j},$$

we recover the overall counting to be

$$\begin{aligned} &2n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ \text{(1,1,2)-composition of } 2n \\ \tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1}}} c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) \binom{2(l_1 + \dots + l_j)}{l_1 + \dots + l_j} \\ &= \sum_{k=0}^n \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j \\ \text{(1,1,2)-composition of } 2n \\ \tilde{l}_1 + \dots + \tilde{l}_{j+1} = \tilde{l}'_1 + \dots + \tilde{l}'_{j+1} \\ l_1 + \dots + l_j = k}} 2n c_{1,1,2}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; \tilde{l}'_1, \dots, \tilde{l}'_{j+1}; l_1, \dots, l_j) \binom{2k}{k} = \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}, \end{aligned}$$

which is indeed $[x^0 y^0 z^0](x + y + z + x^{-1} + y^{-1} + z^{-1})^{2n} = 6, 90, 1860, 44730, 1172556, \dots$ (see the [OEIS sequence A002896](#)).

- [1] P. Lévy, *Processus Stochastiques et Mouvement Brownien* (Gauthier-Villard, Paris, 1965); Wiener's random function, and other Laplacian random functions, in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* (University of California Press, Berkeley, CA, 1951).
- [2] S. Ouvry and S. Wu, The algebraic area of closed lattice random walks, *J. Phys. A: Math. Theor.* **52**, 255201 (2019).
- [3] S. Ouvry and A. P. Polychronakos, Exclusion statistics and lattice random walks, *Nucl. Phys. B* **948**, 114731 (2019); Lattice walk area combinatorics, some remarkable trigonometric sums and Apéry-like numbers, **960**, 115174 (2020); Algebraic area enumeration for lattice paths, [arXiv:2110.09394](https://arxiv.org/abs/2110.09394).
- [4] L. Gan, S. Ouvry, and A. P. Polychronakos, Algebraic area enumeration of random walks on the honeycomb lattice, *Phys. Rev. E* **105**, 014112 (2022).
- [5] C. Kreft, Explicit computation of the discriminant for the Harper equation with rational flux, SFB 288 Preprint No. 89, TU-Berlin, 1993 (unpublished).
- [6] D. R. Hofstadter, Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields, *Phys. Rev. B* **14**, 2239 (1976).
- [7] E. Bédos, An introduction to 3D discrete magnetic Laplacians and noncommutative 3-tori, *J. Geom. Phys.* **30**, 204 (1999).
- [8] Y. Hasegawa, Generalized flux states on 3-dimensional lattice, *J. Phys. Soc. Jpn.* **59**, 4384 (1990).
- [9] G. Montambaux and M. Kohmoto, Quantized Hall effect in three dimensions, *Phys. Rev. B* **41**, 11417 (1990).
- [10] M. Koshino, H. Aoki, K. Kuroki, S. Kagoshima, and T. Osada, Hofstadter butterfly and integer quantum Hall effect in three dimensions, *Phys. Rev. Lett.* **86**, 1062 (2001); M. Koshino and H. Aoki, Integer quantum Hall effect in isotropic three-dimensional crystals, *Phys. Rev. B* **67**, 195336 (2003).
- [11] D.-W. Zhang, R.-B. Liu, and S.-L. Zhu, Generalized Hofstadter model on a cubic optical lattice: From nodal bands to the three-dimensional quantum Hall effect, *Phys. Rev. A* **95**, 043619 (2017).
- [12] J. Brüning, V. V. Demidov, and V. A. Geyler, Hofstadter-type spectral diagrams for the Bloch electron in three dimensions, *Phys. Rev. B* **69**, 033202 (2004).
- [13] F. Di Colandrea, A. D'Errico, M. Maffei, H. M. Price, M. Lewenstein, L. Marrucci, F. Cardano, A. Dauphin, and P. Massignan, Linking topological features of the Hofstadter model to optical diffraction figures, *New J. Phys.* **24**, 013028 (2022).
- [14] L. Gan, S. Ouvry, and A. P. Polychronakos, Combinatorics of generalized Dyck and Motzkin paths, *Phys. Rev. E* **106**, 044123 (2022).
- [15] J. Desbois, Algebraic area enclosed by random walks on a lattice, *J. Phys. A: Math. Theor.* **48**, 425001 (2015).
- [16] S. Ouvry and A. P. Polychronakos, Algebraic area enumeration for open lattice walks, *J. Phys. A: Math. Theor.* **55**, 485005 (2022).
- [17] S. J. Elman, A. Chapman, and S. T. Flammia, Free fermions behind the disguise, *Commun. Math. Phys.* **388**, 969 (2021).

5 - Epilogue

In this thesis, we focus on the enumeration of closed lattice random walks according to their algebraic area, with connections to quantum exclusion statistics, as well as the combinatorics of generalized Dyck and Motzkin paths (also known as Łukasiewicz paths).

The unveiled relationship between closed random walks and exclusion statistics merely scratches the surface of this enigmatic subject. Various avenues for future investigation exist, with the most immediate relating to the connection of walks of various properties and on various lattices with the corresponding exclusion statistics and Łukasiewicz paths. For example, the enumeration of open walks on the square lattice according to their algebraic area was recently achieved [55], with Dyck path combinatorics again playing a key role. Walks on other lattices, such as the kagome lattice, hyperbolic lattice, and higher dimensional lattice, and of different properties can be investigated with similar methods. A pertinent question that arises is whether it is possible to systematically classify random walks based on the exclusion parameter g .

In Appendices C and D, we present additional original results that are not included in Articles 1, 2, 3, or any other known publications to the best of our knowledge. In Appendix C, we derive the algebraic area enumeration of closed random walks on a triangular lattice. In contrast to the closed chiral triangular lattice walks shown in Figure 1.3 where only three specific directions are allowed at each step, in this case, all six directions at each step are permissible (see Figure C.1). It is worth noting that, besides the explicit formulas for $C_{\mathbf{n}}(A)$ derived in this thesis, we can also employ recursive algorithms to numerically compute $C_{\mathbf{n}}(A)$ for small \mathbf{n} . For example, the algebraic area enumeration of square lattice walks can be computed up to $\mathbf{n} = 140$ using the computer cluster at LPTMS. See Appendix D for further details.

Furthermore, the commutation relation (1.3) also appears in exactly solvable models. Exploring its connection with the subjects discussed in this thesis would be of great interest. For example, a family of multispin quantum chains with a free-(para)fermionic eigenspectrum [6] was recently reanalyzed in [7] and the eigenenergies were obtained via the roots of a polynomial with coefficients similar to the Z_n in (2.6), indicating a connection with exclusion statistics that warrants further investigation.

Appendix A: Algebraic area enumeration of closed honeycomb lattice walks

Using the diluted spectral functions in Subsection 2.2.2, the number $C_{2n}(A)$ of closed honeycomb lattice walks of length $2n$ enclosing an algebraic area A is derived to be

$$\begin{aligned}
 C_{2n}(A) = 2n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n}} & \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2} l_2 \frac{\binom{l_2+l_3}{l_2}}{l_2+l_3} \cdots l_{j-1} \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1}+l_j} \left[\sum_{k_5=-l_5}^{l_5} \sum_{k_7=-l_7}^{l_7} \cdots \sum_{k_{2\lceil j/2 \rceil+1}=-l_{2\lceil j/2 \rceil+1}}^{l_{2\lceil j/2 \rceil+1}} \right. \\
 & \left(\binom{2l_1}{l_1+A+\sum_{i=5, i \text{ odd}}^{2\lceil j/2 \rceil+1} (i-3)k_i/2} \right) \left(\binom{2l_3}{l_3-A-\sum_{i=5, i \text{ odd}}^{2\lceil j/2 \rceil+1} (i-1)k_i/2} \right) \prod_{i=5, i \text{ odd}}^{2\lceil j/2 \rceil+1} \binom{2l_i}{l_i+k_i} \\
 & + \sum_{k_6=-l_6}^{l_6} \sum_{k_8=-l_8}^{l_8} \cdots \sum_{k_{2\lfloor j/2 \rfloor}=-l_{2\lfloor j/2 \rfloor}}^{l_{2\lfloor j/2 \rfloor}} \left(\binom{2l_2}{l_2+A+\sum_{i=6, i \text{ even}}^{2\lfloor j/2 \rfloor} (i-4)k_i/2} \right) \left(\binom{2l_4}{l_4-A-\sum_{i=6, i \text{ even}}^{2\lfloor j/2 \rfloor} (i-2)k_i/2} \right) \\
 & \times \prod_{i=6, i \text{ even}}^{2\lfloor j/2 \rfloor} \binom{2l_i}{l_i+k_i} \left. \right], \tag{A.1}
 \end{aligned}$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions.

Appendix B: Remarks on the b_n in (2.24)

We note that by leaving s_k as an arbitrary spectral function, all terms involving s_q in b_n in (2.24) are given by

$$(-1)^{n+1} \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{composition of } n'=0,1,2,\dots,n \\ j \leq \min(n', n-n'+1)}} c'_n(l_1, l_2, \dots, l_j) s_{q-j+1}^{l_1} s_{q-j+2}^{l_2} \cdots s_q^{l_j},$$

where the combinatorial coefficients c'_n are different from c_n in (2.25) and read

$$c'_n(0) = \frac{1}{n}, \quad c'_n(l_1) = \frac{1}{l_1} \binom{n-1}{l_1-1},$$

$$c'_n(l_1, l_2, \dots, l_j) = \frac{1}{l_1 l_2 \cdots l_j} \sum_{m_1=0}^{\min(l_1, l_2)} \sum_{m_2=0}^{\min(l_2, l_3)} \cdots \sum_{m_{j-1}=0}^{\min(l_{j-1}, l_j)} \left(\prod_{i=1}^{j-1} m_i \binom{l_i}{m_i} \binom{l_{i+1}}{m_i} \right) \binom{n + \sum_{i=1}^{j-1} l_i - \sum_{i=1}^{j-1} m_i - 1}{2 \sum_{i=1}^{j-1} l_i + l_j - 1}.$$

Nevertheless, in the honeycomb lattice case, we can ignore the difference arising from c'_n since $s_q = 0$ and $c'_n(0) = c_n(0)$.

Appendix C: Algebraic area enumeration of closed triangular lattice walks

In this appendix, we tackle the enumeration problem of closed random walks on a triangular lattice that enclose a specified algebraic area. The corresponding quantum Hamiltonian is given by

$$H = U + U^{-1} + V + V^{-1} + QU^{-1}V^{-1} + Q^{-1}VU,$$

where the six hopping operators $U, Q^{-1}VU, V, U^{-1}, QU^{-1}V^{-1}, V^{-1}$ represent hopping in directions with angles $0, \pi/6, \dots, 5\pi/6$, respectively, with respect to the horizontal axis (see Figure C.1). Similar to the commutation relation (1.1) in the Hofstadter case, the operators U and V satisfy

$$VU = Q^2UV,$$

where Q^2 arises from the closed walk $V^{-1}U^{-1}VU$ that encloses an algebraic area of $+2$.

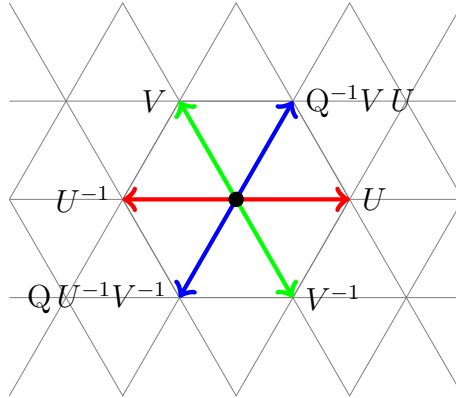


Figure C.1: Hopping operators $U, Q^{-1}VU, V, U^{-1}, QU^{-1}V^{-1}, V^{-1}$ on the triangular lattice with $VU = Q^2UV$.

When the flux is rational, we choose the representation in terms of u and v in (1.3)

$$U = u^2, \quad V = v.$$

Then, H becomes a tridiagonal matrix with nonvanishing corners. By setting both corners to zero, namely $e^{-2ik_x} = -Q$, the Hamiltonian H has a similar form as $H_q|_{\omega_1} = 0$ in (2.20), but now with different elements

$$\tilde{s}_k = -Q^{2k-1} - Q^{-2k+1} = s_{2k-1} - 2, \quad f_k = e^{ik_y}(1 - Q^{2k}),$$

where $s_k = 4 \sin^2(k\pi p/q)$ is the standard Hofstadter spectral function. This leads to the spectral functions (Boltzmann factors) \tilde{s}_k and $-f_k \bar{f}_k = -s_{2k}$ in the triangular Kreft coefficients. Following the

same route as in Section 2.2, the associated cluster coefficient b_n can be written in terms of s_{2k-1} and s_{2k} , namely

$$b_n = (-1)^{n+1} \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j \\ (1,2)\text{-composition of } n=0,1,2, \dots, n}} c_n(\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j) \sum_{k=1}^{q-j} s_{2k-1}^{\tilde{l}_1} s_{2k}^{l_1} s_{2k+1}^{\tilde{l}_2} s_{2k+2}^{l_2} \cdots s_{2k+2j-1}^{\tilde{l}_{j+1}} \quad (C.1)$$

with

$$c_n(0) = \frac{(-2)^n}{\mathbf{n}}, \quad c_n(\tilde{l}_1) = \frac{1}{\tilde{l}_1} (-2)^{n-\tilde{l}_1} \binom{\mathbf{n}-1}{\tilde{l}_1-1},$$

$$c_n(\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j) = \frac{(\tilde{l}_1 + l_1 - 1)!}{\tilde{l}_1! l_1!} \prod_{k=2}^{j+1} \binom{l_{k-1} + \tilde{l}_k + l_k - 1}{l_{k-1} - 1, \tilde{l}_k, l_k} (-2)^{n - (\tilde{l}_1 + \dots + \tilde{l}_{j+1}) - 2(l_1 + \dots + l_j)}$$

$$\times \binom{\mathbf{n}-1}{\tilde{l}_1 + \dots + \tilde{l}_{j+1} + 2(l_1 + \dots + l_j) - 1}$$

with the convention $l_k = 0$ for $k > j$.

We define the sequence of integers $\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j$, $j \geq 0$, labeling c_n in (C.1) as a $(1, 2)$ -composition of the integer n if they satisfy the defining conditions

$$n = (\tilde{l}_1 + \tilde{l}_2 + \dots + \tilde{l}_{j+1}) + 2(l_1 + l_2 + \dots + l_j), \quad \tilde{l}_i \geq 0, \quad l_i > 0.$$

That is, the l_i 's are the usual compositions of integers $1, 2, \dots, \lfloor n/2 \rfloor$, while the \tilde{l}_i 's are additional nonnegative integers (for $j = 0$, we have the trivial composition $\tilde{l}_1 = n$). For example, the $(1, 2)$ -compositions of $n = 0, 1, 2, 3, 4$ read

$$\begin{aligned} n = 0: & (0); \\ n = 1: & (1); \\ n = 2: & (2), (0, 0; 1); \\ n = 3: & (3), (1, 0; 1), (0, 1; 1); \\ n = 4: & (4), (2, 0; 1), (1, 1; 1), (0, 2; 1), (0, 0; 2), (0, 0, 0; 1, 1), \end{aligned}$$

which contribute to b_4 the terms

$$\begin{aligned} -b_4 = & 4 \sum_{k=1}^q s_{2k-1}^0 - 8 \sum_{k=1}^q s_{2k-1} + 6 \sum_{k=1}^q s_{2k-1}^2 + 12 \sum_{k=1}^{q-1} s_{2k} - 2 \sum_{k=1}^q s_{2k-1}^3 - 6 \sum_{k=1}^{q-1} s_{2k-1} s_{2k} - 6 \sum_{k=1}^{q-1} s_{2k} s_{2k+1} \\ & + \frac{1}{4} \sum_{k=1}^q s_{2k-1}^4 + \sum_{k=1}^{q-1} s_{2k-1}^2 s_{2k} + \sum_{k=1}^{q-1} s_{2k-1} s_{2k} s_{2k+1} + \sum_{k=1}^{q-1} s_{2k} s_{2k+1}^2 + \frac{1}{2} \sum_{k=1}^{q-1} s_{2k}^2 + \sum_{k=1}^{q-2} s_{2k} s_{2k+2}. \end{aligned}$$

Note that the inverse of a composition, defined as $\tilde{l}_{j+1}, \dots, \tilde{l}_1; l_j, \dots, l_1$ leaves c_n invariant. Also, note that the n -independent part of c_n leads to the standard combinatorial coefficients $c_{1,2}$ of the Hamiltonian of a mixture of $g = 1$ and $g = 2$ exclusion, which is discussed in Section 3.3.

Using

$$\sum_A C_{\mathbf{n}}(A) Q^A = \mathbf{Tr} H^{\mathbf{n}} = \frac{1}{q} \operatorname{tr} H^{\mathbf{n}} = \frac{1}{q} \mathbf{n} (-1)^{\mathbf{n}} b_{\mathbf{n}}$$

and the identity

$$\sum_{k=1}^{q-j} s_{2k-1}^{\tilde{l}_1} s_{2k}^{l_1} s_{2k+1}^{\tilde{l}_2} s_{2k+2}^{l_2} \cdots s_{2k+2j-1}^{\tilde{l}_{j+1}} = \sum_{k=1}^{q-2j-1} s_k^{\tilde{l}_1} s_{k+1}^{l_1} \cdots s_{k+2j}^{\tilde{l}_{j+1}},$$

the number $C_{\mathbf{n}}(A)$ of closed triangular lattice walks of length \mathbf{n} enclosing an algebraic area¹ A is derived to be (see Table C.1 for some examples)

$$\begin{aligned} C_{\mathbf{n}}(A) = & \mathbf{n} \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j \\ (1,2)\text{-composition of } n=0,1,2, \dots, \mathbf{n}}} \frac{(\tilde{l}_1 + l_1 - 1)!}{\tilde{l}_1! l_1!} \prod_{k=2}^{j+1} \binom{l_{k-1} + \tilde{l}_k + l_k - 1}{l_{k-1} - 1, \tilde{l}_k, l_k} (-2)^{\mathbf{n}-n} \binom{\mathbf{n}-1}{n-1} \sum_{k_3=-\tilde{l}_2}^{\tilde{l}_2} \sum_{k_4=-l_2}^{l_2} \cdots \sum_{k_{2j+1}=-\tilde{l}_{j+1}}^{\tilde{l}_{j+1}} \\ & \times \binom{2\tilde{l}_1}{\tilde{l}_1 + A + \sum_{i=3}^{2j+1} (i-2)k_i} \binom{2l_1}{l_1 - A - \sum_{i=3}^{2j+1} (i-1)k_i} \prod_{i=2}^{j+1} \binom{2\tilde{l}_i}{\tilde{l}_i + k_{2i-1}} \prod_{i=2}^j \binom{2l_i}{l_i + k_{2i}}. \end{aligned}$$

Note that from

$$\mathbf{n} \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j \\ (1,2)\text{-composition of } n \\ l_1 + \dots + l_j = n'}} c_{\mathbf{n}}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j) = (-2)^{\mathbf{n}-n} \binom{\mathbf{n}}{n} \binom{n}{2n'} \binom{2n'}{n'},$$

we recover the total number of closed triangular lattice walks of length \mathbf{n} to be

$$\begin{aligned} & \mathbf{n} \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j \\ (1,2)\text{-composition of } n=0,1,2, \dots, \mathbf{n}}} c_{\mathbf{n}}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j) \binom{2(\tilde{l}_1 + \dots + \tilde{l}_{j+1} + l_1 + \dots + l_j)}{\tilde{l}_1 + \dots + \tilde{l}_{j+1} + l_1 + \dots + l_j} \\ &= \sum_{n=0}^{\mathbf{n}} \sum_{n'=0}^{\lfloor n/2 \rfloor} \left(\mathbf{n} \sum_{\substack{\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j \\ (1,2)\text{-composition of } n \\ l_1 + \dots + l_j = n'}} c_{\mathbf{n}}(\tilde{l}_1, \dots, \tilde{l}_{j+1}; l_1, \dots, l_j) \binom{2n - 2n'}{n - n'} \right) \\ &= \sum_{n=0}^{\mathbf{n}} \sum_{n'=0}^{\lfloor n/2 \rfloor} (-2)^{\mathbf{n}-n} \binom{\mathbf{n}}{n} \binom{n}{2n'} \binom{2n'}{n'} \binom{2n - 2n'}{n - n'} \\ &= \sum_{n=0}^{\mathbf{n}} \sum_{k=0}^n (-2)^{\mathbf{n}-n} \binom{\mathbf{n}}{n} \binom{n}{k}^3, \end{aligned}$$

as expected².

¹ A is bounded by $\operatorname{round}(n^2/6) - (0 \text{ if } \mathbf{n} = 0 \pmod 6, 1 \text{ else})$ [56]. $\operatorname{round}(x)$ returns the integer that is closest to x and rounds half-integers towards the nearest even integer.

²By setting $Q = 1$, the total number $C_{\mathbf{n}}(A)$ can also be derived from the U, V -independent part in the expansion of $(U + U^{-1} + V + V^{-1} + U^{-1}V^{-1} + VU)^{\mathbf{n}} = [(1 + U)(1 + V)(1 + U^{-1}V^{-1}) - 2]^{\mathbf{n}} = \sum_{n=0}^{\mathbf{n}} (-2)^{\mathbf{n}-n} \binom{\mathbf{n}}{n} \sum_{i=0}^n \sum_{j=0}^n \sum_{k=0}^n \binom{n}{i} \binom{n}{j} \binom{n}{k} U^i V^j (VU)^{-k}$. Setting $i = j = k$ yields the desired number.

	$n = 2$	3	4	5	6	7	8	9	10	11
$A = 0$	6		66		1020		19890		449976	
± 1		12		300		6888		164124		4124340
± 2			24		840		23904		654840	
± 3				60		2604		85944		2617428
± 4					168		8568		317940	
± 5						504		29628		1215456
± 6					12		1968		114360	
± 7						84		8496		475200
± 8							432		37560	
± 9								1980		167244
± 10							48		10380	
± 11								432		55308
± 12									2700	
± 13									36	15972
± 14										540
± 15										4356
± 16										60
± 17										924
± 18										
± 19										132
Total count	6	12	90	360	2040	10080	54810	290640	1588356	8676360

Table C.1: $C_n(A)$ up to $n = 11$ for closed triangular lattice walks of length n .

Appendix D: Recurrence relation for enumeration of closed lattice walks

D.1 Square lattice walks

Consider an n -step square lattice walk having m_1 steps right, m_2 steps left, l_1 steps up, and l_2 steps down, with $m_1 + m_2 + l_1 + l_2 = n$. If the walk is open, we can close it by adding a straight line that connects the endpoint to the starting point. Let $C_{m_1, m_2, l_1, l_2}(A)$ denote the number of such walks that enclose an algebraic area A . The generating function $Z_{m_1, m_2, l_1, l_2}(Q) = \sum_A C_{m_1, m_2, l_1, l_2}(A) Q^A$ can be computed by the recurrence relation

$$Z_{m_1, m_2, l_1, l_2}(Q) = Q^{(l_2 - l_1)/2} Z_{m_1 - 1, m_2, l_1, l_2}(Q) + Q^{(l_1 - l_2)/2} Z_{m_1, m_2 - 1, l_1, l_2}(Q) \\ + Q^{(m_1 - m_2)/2} Z_{m_1, m_2, l_1 - 1, l_2}(Q) + Q^{(m_2 - m_1)/2} Z_{m_1, m_2, l_1, l_2 - 1}(Q)$$

with $Z_{0,0,0,0}(Q) = 1$ and $Z_{m_1, m_2, l_1, l_2}(Q) = 0$ whenever $\min(m_1, m_2, l_1, l_2) < 0$. For closed walks of length $n = 2n$, we have

$$\sum_A C_{2n}(A) Q^A = \sum_{m=0}^n Z_{m, m, n-m, n-m}(Q).$$

Note that the algebraic area of an open walk can alternatively be defined as that of a closed walk by adding the minimum necessary number of steps first in the vertical directions, followed by the horizontal directions. This alternative definition gives rise to a similar, yet distinct algorithm that can be employed [57, 58].

D.2 Honeycomb lattice walks

We transform the honeycomb lattice into a topologically equivalent brick-wall lattice, as depicted in Figure D.1. The horizontal and vertical lattice spacing are set to $1/2$ and 1 , respectively, ensuring that the unit cell area is 1 .

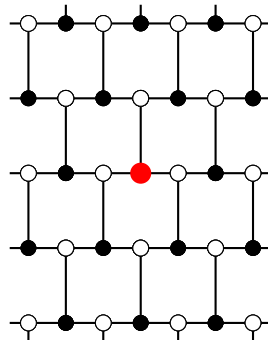


Figure D.1: Deformed honeycomb lattice. Two types of sites are colored black and white. The red bullet point indicates both the starting and ending point.

Consider an \mathbf{n} -step closed walk on this lattice, starting and ending at the red bullet point. The walk consists of m_1 steps right, m_2 steps left, l_1 steps up, and l_2 steps down, with $m_1 + m_2 + l_1 + l_2 = \mathbf{n}$. The generating function obeys the recurrence relation

$$\begin{aligned} Z_{m_1, m_2, l_1, l_2}(\mathbb{Q}) &= \mathbb{Q}^{(l_2 - l_1)/4} Z_{m_1 - 1, m_2, l_1, l_2}(\mathbb{Q}) + \mathbb{Q}^{(l_1 - l_2)/4} Z_{m_1, m_2 - 1, l_1, l_2}(\mathbb{Q}) \\ &+ [m_1 - m_2 \text{ and } l_1 - l_2 \text{ have opposite parity}] \mathbb{Q}^{(m_1 - m_2)/4} Z_{m_1, m_2, l_1 - 1, l_2}(\mathbb{Q}) \\ &+ [m_1 - m_2 \text{ and } l_1 - l_2 \text{ have same parity}] \mathbb{Q}^{(m_2 - m_1)/4} Z_{m_1, m_2, l_1, l_2 - 1}(\mathbb{Q}), \end{aligned}$$

where the Iverson bracket $[P]$ is 1 if P is true, 0 otherwise. Again, $Z_{0,0,0,0}(\mathbb{Q}) = 1$ and $Z_{m_1, m_2, l_1, l_2}(\mathbb{Q}) = 0$ whenever $\min(m_1, m_2, l_1, l_2) < 0$. For closed walks of length $\mathbf{n} = 2n$, we have

$$\sum_A C_{2n}(A) \mathbb{Q}^A = \sum_{m=0}^n Z_{m, m, n-m, n-m}(\mathbb{Q}).$$

D.3 Triangular lattice walks

We transform the triangular lattice into a topologically equivalent lattice, as depicted in Figure D.2. The horizontal and vertical lattice spacing are set to $\sqrt{2}$ so that the unit triangular cell area is 1.

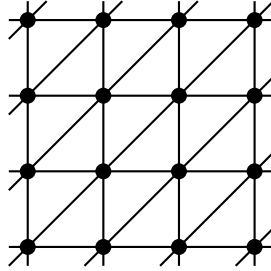


Figure D.2: Deformed triangular lattice.

Consider an \mathbf{n} -step closed walk on this lattice. The walk consists of m_1 steps right, m_2 steps left, l_1 steps up, l_2 steps down, r_1 steps up-right, and r_2 steps down-left with $m_1 + m_2 + l_1 + l_2 + r_1 + r_2 = \mathbf{n}$. The generating function obeys the recurrence relation

$$\begin{aligned} Z_{m_1, m_2, l_1, l_2, r_1, r_2}(\mathbb{Q}) &= \mathbb{Q}^{l_2 + r_2 - l_1 - r_1} Z_{m_1 - 1, m_2, l_1, l_2, r_1, r_2}(\mathbb{Q}) + \mathbb{Q}^{l_1 + r_1 - l_2 - r_2} Z_{m_1, m_2 - 1, l_1, l_2, r_1, r_2}(\mathbb{Q}) \\ &+ \mathbb{Q}^{m_1 + r_1 - m_2 - r_2} Z_{m_1, m_2, l_1 - 1, l_2, r_1, r_2}(\mathbb{Q}) + \mathbb{Q}^{m_2 + r_2 - m_1 - r_1} Z_{m_1, m_2, l_1, l_2 - 1, r_1, r_2}(\mathbb{Q}) \\ &+ \mathbb{Q}^{m_1 - m_2 + l_2 - l_1} Z_{m_1, m_2, l_1, l_2, r_1 - 1, r_2}(\mathbb{Q}) + \mathbb{Q}^{m_2 - m_1 + l_1 - l_2} Z_{m_1, m_2, l_1, l_2, r_1, r_2 - 1}(\mathbb{Q}) \end{aligned}$$

with $Z_{0,0,0,0,0,0}(\mathbb{Q}) = 1$ and $Z_{m_1, m_2, l_1, l_2, r_1, r_2}(\mathbb{Q}) = 0$ whenever $\min(m_1, m_2, l_1, l_2, r_1, r_2) < 0$. For closed walks of length \mathbf{n} , we have

$$\sum_A C_{\mathbf{n}}(A) \mathbb{Q}^A = \sum_{\substack{m_1 + m_2 + l_1 + l_2 + r_1 + r_2 = \mathbf{n} \\ m_1 + r_1 = m_2 + r_2 \\ l_1 + r_1 = l_2 + r_2}} Z_{m_1, m_2, l_1, l_2, r_1, r_2}(\mathbb{Q}).$$

D.4 Kagome lattice walks

We transform the kagome lattice into a topologically equivalent lattice, as depicted in Figure D.3. The spacing between each black site and its respective nearest white and green sites is set to $\sqrt{2}$ so that the unit triangular cell area is 1.

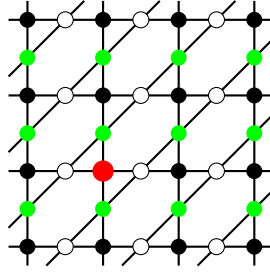


Figure D.3: Deformed kagome lattice. Three types of sites are colored black, white, and green. The red bullet point indicates both the starting and ending point.

Consider an \mathbf{n} -step closed walk on this lattice, starting and ending at the red bullet point. The walk consists of m_1 steps right, m_2 steps left, l_1 steps up, l_2 steps down, r_1 steps up-right, and r_2 steps down-left with $m_1 + m_2 + l_1 + l_2 + r_1 + r_2 = \mathbf{n}$. The generating function obeys the recurrence relation

$$\begin{aligned} & Z_{m_1, m_2, l_1, l_2, r_1, r_2}(\mathbb{Q}) \\ = & [l_1 + r_1 - l_2 - r_2 \text{ is even}] \left(\mathbb{Q}^{l_2 + r_2 - l_1 - r_1} Z_{m_1 - 1, m_2, l_1, l_2, r_1, r_2}(\mathbb{Q}) + \mathbb{Q}^{l_1 + r_1 - l_2 - r_2} Z_{m_1, m_2 - 1, l_1, l_2, r_1, r_2}(\mathbb{Q}) \right) \\ & + [m_1 + r_1 - m_2 - r_2 \text{ is even}] \left(\mathbb{Q}^{m_1 + r_1 - m_2 - r_2} Z_{m_1, m_2, l_1 - 1, l_2, r_1, r_2}(\mathbb{Q}) + \mathbb{Q}^{m_2 + r_2 - m_1 - r_1} Z_{m_1, m_2, l_1, l_2 - 1, r_1, r_2}(\mathbb{Q}) \right) \\ & + [m_1 + r_1 - m_2 - r_2 \text{ and } l_1 + r_1 - l_2 - r_2 \text{ have opposite parity}] \left(\mathbb{Q}^{m_1 - m_2 + l_2 - l_1} Z_{m_1, m_2, l_1, l_2, r_1 - 1, r_2}(\mathbb{Q}) \right. \\ & \left. + \mathbb{Q}^{m_2 - m_1 + l_1 - l_2} Z_{m_1, m_2, l_1, l_2, r_1, r_2 - 1}(\mathbb{Q}) \right), \end{aligned}$$

where $[\]$ denotes the Iverson bracket defined in Appendix D.2, $Z_{0,0,0,0,0,0}(\mathbb{Q}) = 1$ and $Z_{m_1, m_2, l_1, l_2, r_1, r_2}(\mathbb{Q}) = 0$ whenever $\min(m_1, m_2, l_1, l_2, r_1, r_2) < 0$. For closed walks of length \mathbf{n} , we have

$$\sum_A C_{\mathbf{n}}(A) \mathbb{Q}^A = \sum_{\substack{m_1 + m_2 + l_1 + l_2 + r_1 + r_2 = \mathbf{n} \\ m_1 + r_1 = m_2 + r_2 \\ l_1 + r_1 = l_2 + r_2}} Z_{m_1, m_2, l_1, l_2, r_1, r_2}(\mathbb{Q}).$$

See Table D.1 for several examples of $C_{\mathbf{n}}(A)$. We also find the total number is³

$$\sum_A C_{\mathbf{n}}(A) = \frac{1}{3} \left((-2)^{\mathbf{n}} + 2 \sum_{k=0}^{\lfloor \mathbf{n}/2 \rfloor} \sum_{j=0}^k \binom{\mathbf{n}}{2k} \binom{k}{j}^2 \binom{2j}{j} \right).$$

³From [59], the total number of \mathbf{n} -step closed kagome lattice walks can be derived from the u, v -independent part in the expansion of $\frac{1}{3} \text{tr} H^{\mathbf{n}}$ with $H = \begin{pmatrix} 0 & v + uv & u^{-1} + v \\ v^{-1} + v^{-1}u^{-1} & 0 & u^{-1} + u^{-1}v^{-1} \\ u + v^{-1} & u + vu & 0 \end{pmatrix}$ and $vu = uv$, which amounts to calculating the u, v -independent part, up to a factor, in expansion of the sum of the \mathbf{n} -th power of the three eigenvalues: $-2, 1 \pm (3 + u + v + uv + u^{-1} + v^{-1} + u^{-1}v^{-1})^{1/2}$.

	$n = 2$	3	4	5	6	7	8	9	10	11	12
$A = 0$	4	28		244		2412		25804		290956	
± 1		4	60		756		9216		112420		
± 2				16		416		7560		119680	
± 3							76		2816		
± 4										656	
± 5								36		1276	
± 6				4		96		1860		33120	
± 7						28		756		14608	
± 8							80		2480	51984	
± 9								120		4400	
± 10									100		4560
± 11										44	
± 12											36
± 13										44	
± 14									20		1104
± 15										176	
± 16											672
\vdots											
± 22											16
Total count	4	4	28	60	264	784	3004	10204	37824	135784	502784

Table D.1: $C_n(A)$ up to $n = 12$ for kagome lattice walks of length n .

D.5 Cubic lattice walks

Consider an n -step cubic lattice walk that consists of m_1 steps in the direction $(1, 0, 0)$, m_2 steps in the direction $(-1, 0, 0)$, l_1 steps in the direction $(0, 1, 0)$, l_2 steps in the direction $(0, -1, 0)$, r_1 steps in the direction $(0, 0, 1)$, r_2 steps in the direction $(0, 0, -1)$, where $m_1 + m_2 + l_1 + l_2 + r_1 + r_2 = n$. Again, if the walk is open, we can close it by adding a straight line that connects the endpoint to the starting point. Let $C_{m_1, m_2, l_1, l_2, r_1, r_2}(A)$ denote the number of such walks that enclose an algebraic area A . The generating function $Z_{m_1, m_2, l_1, l_2, r_1, r_2}(Q) = \sum_A C_{m_1, m_2, l_1, l_2, r_1, r_2}(A) Q^A$ can be computed by the recurrence relation

$$\begin{aligned}
Z_{m_1, m_2, l_1, l_2, r_1, r_2}(Q) &= Q^{(l_2 - l_1 + r_1 - r_2)/2} Z_{m_1 - 1, m_2, l_1, l_2, r_1, r_2}(Q) + Q^{(l_1 - l_2 + r_2 - r_1)/2} Z_{m_1, m_2 - 1, l_1, l_2, r_1, r_2}(Q) \\
&\quad + Q^{(m_1 - m_2 + r_2 - r_1)/2} Z_{m_1, m_2, l_1 - 1, l_2, r_1, r_2}(Q) + Q^{(m_2 - m_1 + r_1 - r_2)/2} Z_{m_1, m_2, l_1, l_2 - 1, r_1, r_2}(Q) \\
&\quad + Q^{(m_2 - m_1 + l_1 - l_2)/2} Z_{m_1, m_2, l_1, l_2, r_1 - 1, r_2}(Q) + Q^{(m_1 - m_2 + l_2 - l_1)/2} Z_{m_1, m_2, l_1, l_2, r_1, r_2 - 1}(Q).
\end{aligned}$$

Again, $Z_{0,0,0,0,0,0}(Q) = 1$ and $Z_{m_1, m_2, l_1, l_2, r_1, r_2}(Q) = 0$ whenever $\min(m_1, m_2, l_1, l_2, r_1, r_2) < 0$.

For closed walks of length $n = 2n$, we have

$$\sum_A C_{2n}(A) Q^A = \sum_{m=0}^n \sum_{l=0}^{n-m} Z_{m, m, l, l, n-m-l, n-m-l}(Q).$$

Bibliography

- [1] D. R. Hofstadter, "Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields," *Phys. Rev. B* **14**, 2239 (1976).
- [2] S. Ouvry and A. P. Polychronakos, "Exclusion statistics and lattice random walks," *Nucl. Phys. B* **948**, 114731 (2019).
- [3] L. Gan, S. Ouvry, and A. P. Polychronakos, "Algebraic area enumeration of random walks on the honeycomb lattice," *Phys. Rev. E* **105**, 014112 (2022).
- [4] L. Gan, S. Ouvry, and A. P. Polychronakos, "Combinatorics of generalized Dyck and Motzkin paths," *Phys. Rev. E* **106**, 044123 (2022).
- [5] L. Gan, "Algebraic area of cubic lattice walks," *Phys. Rev. E* **108**, 054104 (2023).
- [6] R. J. Baxter, "A simple solvable Z_N Hamiltonian," *Phys. Lett. A* **140**, 155 (1989) ; "Superintegrable chiral Potts model: Thermodynamic properties, an inverse model and a simple associated Hamiltonian," *J. Stat. Phys.* **57**, 1 (1989); "Transfer matrix functional relations for the generalized $\tau_2(t_q)$ model," *J. Stat. Phys.* **117**, 1 (2004).
- [7] P. Fendley, "Free parafermions," *J. Phys. A: Math. Theor.* **47**, 075001 (2014); F. C. Alcaraz and R. A. Pimenta, "Free fermionic and parafermionic quantum spin chains with multispin interactions," *Phys. Rev. B* **102**, 121101(R) (2020).
- [8] F. Harper, S. H. Simon, and R. Roy, "Perturbative approach to flat Chern bands in the Hofstadter model," *Phys. Rev. B* **90**, 075104 (2014).
- [9] A. H. MacDonald, "Landau-level subband structure of electrons on a square lattice," *Phys. Rev. B* **28**, 6713 (1983).
- [10] I. I. Satija, "A tale of two fractals: The Hofstadter butterfly and the integral Apollonian gaskets," *Eur. Phys. J. Spec. Top.* **225**, 2533 (2016).
- [11] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, "Quantized Hall conductance in a two-dimensional periodic potential," *Phys. Rev. Lett.* **49**, 405 (1982).
- [12] M. Kohmoto, "Topological invariant and the quantization of the Hall conductance," *Ann. Phys.* **160**, 343 (1985).
- [13] K. v. Klitzing, G. Dorda, and M. Pepper, "New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance," *Phys. Rev. Lett.* **45**, 494 (1980).
- [14] J. Bellissard, C. J. Camacho, A. Barello, and F. Claro, "Exact random walk distributions using non-commutative geometry," *J. Phys. A: Math. Gen.* **30**, L707 (1997).

- [15] K. S. Novoselov, Z. Jiang, Y. Zhang, S. V. Morozov, H. L. Stormer, U. Zeitler, J. C. Maan, G. S. Boebinger, P. Kim, and A. K. Geim, "Room-temperature quantum Hall effect in graphene," *Science* **315**, 1379 (2007).
- [16] K. I. Bolotin, F. Ghahari, M. D. Shulman, H. L. Stormer, and P. Kim, "Observation of the fractional quantum Hall effect in graphene," *Nature* **462**, 196 (2009).
- [17] E. Perfetto, J. González, F. Guinea, S. Bellucci, and P. Onorato, "Quantum Hall effect in carbon nanotubes and curved graphene strips," *Phys. Rev. B* **76**, 125430 (2007).
- [18] M.-H. Kang, "Toroidal fullerenes with the Cayley graph structures," *Discr. Math.* **311**, 2384 (2011).
- [19] A. Kitaev, "Anyons in an exactly solved model and beyond," *Ann. Phys.* **321**, 2 (2006).
- [20] M. A. Levin and X.-G. Wen, "String-net condensation: A physical mechanism for topological phases," *Phys. Rev. B* **71**, 045110 (2005).
- [21] I. F. Herbut, "Interactions and phase transitions on graphene's honeycomb lattice," *Phys. Rev. Lett.* **97**, 146401 (2006).
- [22] R. Rammal, "Landau level spectrum of Bloch electrons in a honeycomb lattice," *J. Phys. France* **46**, 1345 (1985).
- [23] C. Kreft and R. Seiler, "Models of the Hofstadter-type," *J. Math. Phys.* **37**, 5207 (1996).
- [24] A. L. C. Pereira and P. A. Schulz, "Additional levels between Landau bands due to vacancies in graphene: Towards defect engineering," *Phys. Rev. B* **78**, 125402 (2008).
- [25] A. Agazzi, J.-P. Eckmann, and G. M. Graf, "The colored Hofstadter butterfly for the honeycomb lattice," *J. Stat. Phys.* **156**, 417 (2014).
- [26] J. Bellissard, C. Kreft, and R. Seiler, "Analysis of the spectrum of a particle on a triangular lattice with two magnetic fluxes by algebraic and numerical methods," *J. Phys. A: Math. Gen.* **24**, 2329 (1991).
- [27] F. H. Claro and G. H. Wannier, "Magnetic subband structure of electrons in hexagonal lattices," *Phys. Rev. B* **19**, 6068 (1979).
- [28] J. Bellissard, C. Kreft, and R. Seiler, "Analysis of the spectrum of a particle on a triangular lattice with two magnetic fluxes by algebraic and numerical methods," *J. Phys. A: Math. Gen.* **24**, 2329 (1991).
- [29] Y. Hasegawa, "Generalized flux states on 3-dimensional lattice," *J. Phys. Soc. Jpn.* **59**, 4384 (1990).
- [30] G. Montambaux and M. Kohmoto, "Quantized Hall effect in three dimensions," *Phys. Rev. B* **41**, 11417 (1990).

- [31] M. Koshino, H. Aoki, K. Kuroki, S. Kagoshima, and T. Osada, "Hofstadter butterfly and integer quantum Hall effect in three dimensions," *Phys. Rev. Lett.* **86**, 1062 (2001); M. Koshino and H. Aoki, "Integer quantum Hall effect in isotropic three-dimensional crystals," *Phys. Rev. B* **67**, 195336 (2003).
- [32] D.-W. Zhang, R.-B. Liu, and S.-L. Zhu, "Generalized Hofstadter model on a cubic optical lattice: From nodal bands to the three-dimensional quantum Hall effect," *Phys. Rev. A* **95**, 043619 (2017).
- [33] J. Brüning, V. V. Demidov, and V. A. Geyler, "Hofstadter-type spectral diagrams for the Bloch electron in three dimensions," *Phys. Rev. B* **69**, 033202 (2004).
- [34] F. Di Colandrea, A. D'Errico, M. Maffei, H. M. Price, M. Lewenstein, L. Marrucci, F. Cardano, A. Dauphin, and P. Massignan, "Linking topological features of the Hofstadter model to optical diffraction figures," *New J. Phys.* **24**, 013028 (2022).
- [35] F. D. M. Haldane, " 'Fractional statistics' in arbitrary dimensions: A generalization of the Pauli principle," *Phys. Rev. Lett.* **67**, 937 (1991).
- [36] C. Kreft, "Explicit computation of the discriminant for the Harper equation with rational flux," *SFB 288 Preprint No. 89*, TU-Berlin, 1993 (unpublished).
- [37] S. Ouvry and S. Wu, "The algebraic area of closed lattice random walks," *J. Phys. A: Math. Theor.* **52**, 255201 (2019).
- [38] S. Ouvry and A. P. Polychronakos, "Lattice walk area combinatorics, some remarkable trigonometric sums and Apéry-like numbers," *Nucl. Phys. B* **960**, 115174 (2020).
- [39] P. Lévy, *Processus stochastiques et mouvement brownien* (Gauthier-Villard, Paris, 1965); P. Lévy, "Wiener's random function, and other Laplacian random functions," *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* (University of California Press, 1951).
- [40] S. Ouvry and A. P. Polychronakos, "Signed area enumeration for lattice walks," [arXiv:2110.09394v3](https://arxiv.org/abs/2110.09394v3).
- [41] G. Malen, É. Roldán, and R. Toalá-Enríquez, "Extremal $\{p, q\}$ -animals," *Ann. Comb.* **27**, 169 (2023).
- [42] B. Vidakovic, "All roads lead to Rome-even in the honeycomb world," *Am. Stat.* **48**, 234 (1994).
- [43] S. Ouvry, S. Wagner, and S. Wu, "On the algebraic area of lattice walks and the Hofstadter model," *J. Phys. A: Math. Theor.* **49**, 495205 (2016).
- [44] B. Hopkins and S. Ouvry, "Combinatorics of multicompositions," in *Proceedings of the Combinatorial and Additive Number Theory conference* (Springer, Berlin, 2020).
- [45] C. Krattenthaler, "Permutations with restricted patterns and Dyck paths," *Adv. Appl. Math.* **27**, 510 (2001).
- [46] G. M. Cicuta, M. Contedini, and L. Molinari, "Enumeration of simple random walks and tridiagonal matrices," *J. Phys. A: Math. Gen.* **35**, 1125 (2002).

- [47] R. Oste and J. Van der Jeugt, "Motzkin paths, Motzkin polynomials and recurrence relations," *Electron. J. Combin.* **22**, P2.8 (2015).
- [48] S. Ouvry and A. P. Polychronakos, "Hamiltonian and exclusion statistics approach to discrete forward-moving paths," *Phys. Rev. E* **104**, 014143 (2021).
- [49] A. P. Polychronakos, "Length and area generating functions for height-restricted Motzkin meanders," *Phys. Rev. E* **105**, 024102 (2022).
- [50] R. Brak, G. K. Iliev, and T. Prellberg, "An infinite family of adsorption models and restricted Lukasiewicz paths," *J. Stat. Phys.* **145**, 669 (2011).
- [51] S. Bravyi, L. Caha, R. Movassagh, D. Nagaj, and P. W. Shor, "Criticality without frustration for quantum spin-1 chains," *Phys. Rev. Lett.* **109**, 207202 (2012).
- [52] E. Posner, K. Hatch, and M. Ly, "Presentation of the Motzkin monoid," [arXiv:1301.4518](https://arxiv.org/abs/1301.4518).
- [53] E. Bédos, "An introduction to 3D discrete magnetic Laplacians and noncommutative 3-tori," *J. Geom. Phys.* **30**, 204 (1999).
- [54] J. Desbois, "Algebraic area enclosed by random walks on a lattice," *J. Phys. A: Math. Theor.* **48**, 425001 (2015).
- [55] S. Ouvry and A. P. Polychronakos, "Algebraic area enumeration for open lattice walks," *J. Phys. A: Math. Theor.* **55**, 485005 (2022).
- [56] W. C. Yang and R. R. Meyer, "Maximal and minimal polyiamonds," *Math Prog Technical Reports, University of Wisconsin–Madison* (2002).
- [57] C. Béguin, A. Valette, and A. Zuk, "On the spectrum of a random walk on the discrete Heisenberg group and the norm of Harper's operator," *J. Geom. Phys.* **21**, 337 (1997).
- [58] S. Mashkevich and S. Ouvry, "Area distribution of two-dimensional random walks on a square lattice," *J. Stat. Phys.* **137**, 71 (2009).
- [59] P. Kerdelhué and J. Royo-Letelier, "On the low lying spectrum of the magnetic Schrödinger operator with kagome periodicity," *Rev. Math. Phys.* **26**, 1450020 (2014).