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Kai Zeng

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Kai Zeng. Some problems in harmonic analysis on twsited crossed products. Commutative Algebra [math.AC]. Université Bourgogne Franche-Comté, 2023. English. NNT : 2023UBFCD048 . tel-04414142

**HAL Id: tel-04414142**

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Submitted on 24 Jan 2024

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École Doctorale Carnot-Pasteur

# THÈSE DE DOCTORAT

Présentée par:

**ZENG Kai**

en vue de l'obtention du grade de Docteur de l'Université de Bourgogne Franche-Comté

Spécialité: Mathématiques

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**Some problems in harmonic analysis on twisted crossed products**

Quelques problèmes en analyse harmonique sur les produits croisés tordus

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Soutenue publiquement le 22 december 2023 devant le jury composé de:

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# Acknowledgment

This thesis would never have been possible without the assistance and contribution of many people. I would like to express my sincere gratitude to all of them.

First of all, I would like to express my sincere gratitude to my supervisor, Quanhua XU for his enthusiastic guidance, comments, and encouragement all the time during this thesis. His guidance has helped me to learn lots of things about mathematical research and also to develop my academic career.

I would like to thank the members of the jury. It is my honor that Prof. Martijn CASPERS and Prof. Fedor SUKOCHEV have accepted to be the reviewers of this thesis. I express my gratitude to them for their careful reading of the manuscript. I would like to thank Prof. Uwe FRANZ, Prof. Eric RICARD, and Xiao XIONG for their participation in my thesis jury.

I would like to thank all my colleagues of the Laboratoire de Mathématiques de Besançon who helped me to accomplish my thesis in a happy and friendly atmosphere. Especially during the difficult period of Covid-19, I have received indorsation to create favorable conditions for my work as well as my life.

I would like to thank my chinese and foreign friends in Besançon who help me with many things in daily life and work during the past four years. Thanks for your support during the most difficult period to accomplish my doctoral degree.

Finally, I want to thank especially my family for their constant support and encouragement. They are the motivation for me to overcome difficulties and never compromise to work and life.



# Résumé

Dans cette thèse, nous considérons deux problèmes d'analyse non commutative. Ces deux problèmes sont liés aux multiplicateurs de Fourier dans les contextes non commutatifs : une classe de multiplicateurs de Fourier  $L_p$  pour le produit croisé tordu et des commutateurs de classe Schatten concernant les multiplicateurs de Fourier sur le produit croisé tordu. Ces deux objets se posent dans divers domaines de recherche mathématiques et physiques comme la géométrie non commutative et ont attiré une grande attention ces dernières années.

Du point de vue physique, puisque, en plus d'être étudié en mécanique quantique, le calcul quantifié a souvent un analogue algébrique de formes différentielles dans le cadre non commutatif ; par exemple, Connes a utilisé ce calcul quantifié pour calculer la mesure de Hausdorff des ensembles de Julia, ou ensembles limites de groupes quasi-fuchsien dans le plan. Nous nous concentrons sur l'étude de la différentiabilité quantique du produit croisé tordu, à savoir l'appartenance aux classes de Schatten des commutateurs impliquant les multiplicateurs de Fourier sur les espaces non commutatifs, et sur le calcul de la formule de trace pour les dérivées quantifiées.

Nous traitons également des multiplicateurs de Fourier sur les produits croisés tordus. Une attention particulière est portée aux multiplicateurs de Fourier  $L_p$  complètement bornés dans les cas discrets et localement compacts. Puisque les résultats de transfert entre les multiplicateurs de Schur et de Fourier complètement bornés sont bien connus pour le cas de l'algèbre de groupe de von Neumann, l'objectif principal est de combler les lacunes dans les résultats de transfert des multiplicateurs de Fourier et de Schur sur le produit croisé tordu. Les résultats susmentionnés peuvent également être étendus au niveau  $L_p$ . Nous étudions également la stabilité des propriétés d'approximation par les produits croisés tordus.

La thèse se compose de quatre chapitres. Le chapitre 1 est une introduction contenant une brève présentation des questions traitées et des résultats obtenus. Le chapitre 2 donne quelques résultats préliminaires. Dans le chapitre 3, nous étudions les multiplicateurs de Fourier sur le produit croisé tordu. Dans une situation où l'action  $\alpha$  est moyennable, nous montrons que  $\varphi$  est un multiplicateur de Fourier complètement borné sur  $L_p(\mathcal{M} \rtimes_{\alpha, \sigma} G)$  si et seulement si  $\tilde{\varphi}$  est un multiplicateur de Schur complètement borné sur  $S_p(\ell_2(G))$ , où  $\tilde{\varphi}(s, t) = \varphi(s^{-1}t)$  pour  $s, t \in G$ . En particulier, nous démontrons que les multiplicateurs de Fourier complètement bornés sur  $L_p(\mathcal{L}_\sigma(G))$  sont indépendants du cocycle. Ces résultats seront traités dans le cas discret et localement compact. D'autre part, nous démontrons que le produit croisé tordu  $\mathcal{M} \rtimes_{\alpha, \sigma} G$  hérite de l'injectivité et  $w^*$ -CBAP de  $\mathcal{M}$  sous l'hypothèse de la moyennabilité de  $\alpha$ . De plus, nous établissons que si l'algèbre de von Neumann du groupe tordu  $\mathcal{L}_\sigma(G)$  est QWEP, alors  $G$  est hyperlinéaire.

Dans le chapitre 4, nous étudions la différentiabilité quantifiée des produits croisés tordus de l'espace euclidien, à savoir l'appartenance de classe de Schatten et de Schatten faible

des commutateurs multiplicateurs de Fourier. Nous nous occupons des commutateurs sous lesquels le symbole associé des multiplicateurs de Fourier sera une fonction homogène. Dans le cas où les propriétés de l'espace de Besov des éléments connexes dans les espaces non commutatifs suggèrent l'appartenance de Schatten des commutateurs, nous engageons la théorie modulaire  $w^*$  pour traiter l'estimation de la  $p$ -norme de Schatten. Dans le cas où les propriétés de Sobolev des éléments associés dans les espaces non commutatifs suggèrent les propriétés de Schatten faibles des commutateurs, nous devons nous plonger dans le calcul pseudo-différentiel non commutatif. En outre, la formule de trace pour les commutateurs sera également calculée. De plus, nous considérerons également les résultats parallèles pour les commutateurs d'ordre supérieur.

## **Mots clefs**

Multiplicateurs de Fourier, multiplicateurs de Schur, l'algèbre de groupe de von Neumann, produit croisé tordu, classe de Schatten, espace de Besov, espace de Sobolev.

# Abstract

In this thesis, we consider two problems of noncommutative analysis. These two problems are related to the Fourier multipliers in the noncommutative settings: a class of  $L_p$  Fourier multipliers for the twisted crossed product and Schatten class commutators concerning the Fourier multipliers on twisted crossed product. These two objects arise in various mathematical and physical research domains like noncommutative geometry and have drawn wide attention in recent years.

From the physical point of view, since, in addition to being studied in quantum mechanics, quantised calculus often has an analog of the algebra of differential forms in the noncommutative setting; for instance, Connes used this quantised calculus in computing the Hausdorff measure of the Julia sets, or limit sets of Quasi-Fuchsian groups in the plane. We focus on studying the quantum differentiability of the twisted crossed product, namely the Schatten class memberships of the commutators involving the Fourier multipliers on noncommutative spaces, and calculating the trace formula for the quantised derivatives.

We also deal with the Fourier multipliers on twisted crossed products. Particular attention is paid to the completely bounded  $L_p$  Fourier multipliers in the discrete and locally compact cases. Since the transference results between the completely bounded Schur and Fourier multipliers are well known for the group von Neumann algebra case, the main goal is to fill in the gaps in the transference results of Fourier and Schur multipliers on the twisted crossed product. The aforementioned results can also be extended to the  $L_p$  level. We also study the stability of approximation properties by twisted crossed product.

The thesis consists of four chapters. Chapter 1 is an introduction containing a brief presentation of issues treated and results obtained. Chapter 2 gives some preliminary results. In Chapter 3, we study the Fourier multipliers on the twisted crossed product. In a situation where the action  $\alpha$  is amenable, we show that  $\varphi$  is a completely bounded Fourier multiplier on  $L_p(\mathcal{M} \rtimes_{\alpha, \sigma} G)$  if and only if  $\tilde{\varphi}$  is a completely bounded Schur multiplier on  $S_p(\ell_2(G))$ , where  $\tilde{\varphi}(s, t) = \varphi(s^{-1}t)$  for  $s, t \in G$ . In particular, we prove that the completely bounded Fourier multipliers on  $L_p(\mathcal{L}_\sigma(G))$  are independent of the 2-cocycle  $\sigma$ . These results will be treated in the discrete and locally compact case. On the other hand, we demonstrate that the twisted crossed product  $\mathcal{M} \rtimes_{\alpha, \sigma} G$  inherits injectivity and  $w^*$ -CBAP of  $\mathcal{M}$  under the amenability assumption of  $\alpha$ . Besides, we establish that if the twisted group von Neumann algebra  $\mathcal{L}_\sigma(G)$  is QWEP (quotient weak expectation property), then  $G$  is hyperlinear.

In Chapter 4, we study the quantised differentiability of the twisted crossed product of Euclidean space, namely, the Schatten and weak Schatten class memberships of the Fourier multiplier commutators. We deal with the commutators under which the associated symbol of the Fourier multipliers will be a homogeneous function. In the case where the Besov space properties of the related elements in the noncommutative spaces suggest the Schatten



memberships of the commutators, we engage the  $w^*$  modular theory to deal with the Schatten  $p$ -norm estimate. In the case where the Sobolev properties of the associated elements in the noncommutative spaces suggest the weak Schatten properties of the commutators, we need to delve into the noncommutative pseudo-differential calculus. Furthermore, the trace formula for the commutators will also be calculated. Moreover, we will also consider the parallel results for the higher-order commutators.

## **Keywords**

Fourier multiplier, Schur multiplier, group von Neumann algebra, twisted crossed product, Quantum euclidean space, Schatten class, Sobolev space, Besov space, pseudodifferential operator.

## **2020 Mathematics subject classification:**

Primary: 46L51, 46L52, 46L55, 46L87. Secondary: 47L25, 47L65, 43A99.

# Notations

We list below the notations that we use throughout the thesis:

- (1)  $\mathbb{R}$ ,  $\mathbb{C}$  denote respectively the set of real numbers, complex numbers.  $\mathbb{R}_+$  denotes the interval  $(0, \infty)$ .
- (2)  $\mathbb{R}^N$  denotes the  $N$ -dimensional real Euclidean space, and the typical point in  $\mathbb{R}^N$  is  $x = (x_1, x_2, \dots, x_N)$ .  $\mathbb{T}^{N-1}$  denotes the tori in the  $N$ -dimensional Euclidean space.
- (3)  $\theta$  denotes a  $d \times d$  antisymmetric matrix, namely  $\theta^* = -\theta$ .
- (4)  $\mathbb{R}_\theta^d$  denotes the  $d$ -dimensional quantum Euclidean space.
- (5)  $B_{p,q}^\alpha(\mathbb{R}_\theta^d)$  denote the Besov spaces on quantum Euclidean space.
- (6)  $W_p^m(\mathbb{R}_\theta^d)$  denote the Sobolev spaces on quantum Euclidean space respectively.  $\dot{W}_p^m(\mathbb{R}_\theta^d)$  denote the homogeneous Sobolev spaces on quantum Euclidean space.
- (7)  $\mathcal{S}(\mathbb{R}_\theta^d)$  denote the Schwartz spaces on quantum Euclidean space.
- (8)  $S_p$  and  $S_{p,\infty}$  denote the Schatten and weak-Schatten class over some Hilbert spaces  $H$ .  $MS_p$  denotes the set of Schur multipliers on  $S_p$ .
- (9)  $G$  denotes the locally compact group,  $\mathcal{L}G$  denotes the group von Neumann algebra of  $G$ , while  $C_r^*(G)$  and  $C^*(G)$  denote the reduced and full group  $C^*$ -algebra of  $G$ .  $L_p(\mathcal{L}G)$  be the noncommutative  $L_p$ -spaces on  $\mathcal{L}(G)$ .
- (10)  $M_{cb}L_p(\mathcal{L}G)$  denotes the set of all the completely bounded Fourier multipliers on  $L_p(\mathcal{L}G)$ .
- (11)  $(\mathcal{M}, G, \alpha, \sigma)$  denotes a twisted  $w^*$ -dynamical system.
- (12) The twisted crossed product is denoted as  $\mathcal{M} \rtimes_{\alpha, \sigma} G$ .
- (13) Given a function  $\varphi$  on  $G$ ,  $T_\varphi$  and  $M_{\tilde{\varphi}}$  denote the Fourier and Schur multipliers respectively.



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# Chapter 1

## Introduction

The thesis which collects some works obtained during my Ph.D is devoted to the study of some problems in harmonic analysis on twisted crossed product defined by twisted actions of a locally compact group  $G$  on a von Neumann algebra  $\mathcal{M}$ . The two chapters correspond to the work in progress in collaboration with Xiong Xiao (Harbin Institute of Technology) and Xu Quanhua.

### 1.1 Harmonic analysis on twisted crossed product

This part is motivated by [15] that is devoted to the study of harmonic analysis on quantum (or noncommutative) tori. One result of [15] states that the complete bounded Fourier multipliers on the  $L_p$ -space of a quantum  $d$ -torus  $\mathbb{T}_\theta^d$  coincide with those on  $L_p(\mathbb{T}^d)$  of the usual  $d$ -torus. This means that these multipliers are independent of the underlying skew-symmetric matrix  $\theta$ . Note that  $\mathbb{T}_\theta^d$  can be viewed as the twisted von Neumann algebra  $\mathcal{L}_\sigma(\mathbb{Z}^d)$  of the integer group  $\mathbb{Z}^d$  associated to a 2-cocycle  $\sigma$  (a notion introduced first by Mackey [60]). Recall that a 2-cocycle on a group  $G$  is a function  $\sigma : G \times G \rightarrow \mathbb{T}$  satisfying  $\sigma(r, s)\sigma(rs, t) = \sigma(s, t)\sigma(r, st)$  and  $\sigma(s, e) = \sigma(e, s) = 1$  for all  $r, s, t \in G$  ( $e$  being the unit of  $G$ ).  $\mathcal{L}_\sigma(\mathbb{Z}^d)$  is a deformation of the group von Neumann algebra  $\mathcal{L}(\mathbb{Z}^d) = L_\infty(\mathbb{T}^d)$  of  $\mathbb{Z}^d$ .

Another closely related result of [15] establishes a link between Fourier and Schur multipliers on  $L_p(\mathbb{T}_\theta^d)$  in spirit of Ricard and Neuwirth article [69]. This line of investigation has started with Eymard's pioneering work [35]. Let us go into more details. Let  $G$  be a discrete group and  $\mathcal{L}(G)$  the von Neumann algebra generated by the left regular representation  $\lambda$  of  $G$ . Given a function  $\varphi : G \rightarrow \mathbb{C}$ , the Fourier multiplier  $T_\varphi$ , initially defined on the group algebra  $\mathbb{C}[G] \cong \text{span}\lambda(G)$ , is the linear map determined by  $T_\varphi(\lambda(s)) = \varphi(s)\lambda(s)$  for all  $s \in G$ . If  $T_\varphi$  extends to a (completely) bounded map on  $\mathcal{L}(G)$  by the  $w^*$ -density of  $\mathbb{C}[G]$  in  $\mathcal{L}(G)$ ,  $\varphi$  is called a (completely) bounded Fourier multiplier on  $\mathcal{L}(G)$ . Bożejko and Fendler [9] show that  $\varphi$  is a completely bounded Fourier multiplier on  $\mathcal{L}(G)$  iff the function  $\tilde{\varphi}$  on  $G \times G$ , defined by  $\tilde{\varphi}(s, t) = \varphi(st^{-1})$ , is a completely bounded Schur multiplier on  $B(\ell_2(G))$ ; moreover, in this case, the two associated completely bounded norms are equal. Recall that given a function  $\phi$  on  $G \times G$ , the Schur multiplier  $M_\phi$  of symbol  $\phi$  sends a matrix  $(a_{s,t})_{s,t \in G}$  to  $(\phi(s, t)a_{s,t})_{s,t \in G}$ . It is well known that  $M_\phi$  is automatically completely bounded if it is bounded on  $B(\ell_2(G))$ .

Ricard and Neuwirth [69] extend Bożejko and Fendler's result to the  $L_p$ -space level. Let  $\mathcal{L}(G)$  be equipped with its canonical trace  $\tau$  that is the vector state induced by  $\delta_e$ , where  $\delta_s$  denote the Dirac mass at  $s$ . Let  $L_p(\mathcal{L}(G))$  be the noncommutative  $L_p$ -space based on  $(\mathcal{L}(G), \tau)$  for  $1 \leq p \leq \infty$  (recalling that  $L_\infty(\mathcal{L}(G)) = \mathcal{L}(G)$ ).  $L_p(\mathcal{L}(G))$  is equipped with its natural operator

space structure as introduced by Pisier [78] (see below for more details). Note that  $\mathbb{C}[G]$  is dense in  $L_p(\mathcal{L}(G))$  for  $p < \infty$ . We call  $\varphi$  a (completely) bounded Fourier multiplier on  $L_p(\mathcal{L}(G))$  if  $T_\varphi$  extends to a (completely) bounded map on  $L_p(\mathcal{L}(G))$ .

Similarly, we define (completely) bounded Schur multipliers on the Schatten  $p$ -class  $S_p(\ell_2(G))$  based on  $\ell_2(G)$ . The main result of [69] asserts that under the amenability assumption of  $G$ ,  $\varphi$  is a completely bounded Fourier multiplier on  $L_p(\mathcal{L}(G))$  iff  $\tilde{\varphi}$  is a completely bounded Schur multiplier on  $S_p(\ell_2(G))$  for  $1 \leq p \leq \infty$ . [15] extends this result to the quantum torus  $\mathbb{T}_\theta^d$ . Since Schur multipliers on  $S_p(\ell_2(\mathbb{Z}^d))$  do not depend on the skew symmetric matrix  $\theta$ , this second result of [15] implies the first one on Fourier multipliers quoted before.

In the present article, we intend to do all this in the setting of twisted group von Neumann algebras. Let  $\sigma$  be a 2-cocycle on  $G$ , and let  $\lambda_\sigma$  be the twisted version of the left regular representation  $\lambda$ . More precisely, for any  $s \in G$ ,  $\lambda_\sigma(s)$  is the unitary operator on  $\ell_2(G)$  determined by  $(\lambda_\sigma(s)\xi)(t) = \sigma(t^{-1}, s)\xi(s^{-1}t)$  for any  $\xi \in \ell_2(G)$  and  $t \in G$ . Then the twisted group von Neumann algebra  $\mathcal{L}_\sigma(G)$  is the von Neumann algebra generated by  $\lambda_\sigma(G)$ . The vector state induced by  $\delta_e$  is again a faithful trace on  $\mathcal{L}_\sigma(G)$  (see section 3.1 for more information). One of our results shows that if  $G$  is hyperlinear, then the completely bounded Fourier multipliers on  $L_p(\mathcal{L}_\sigma(G))$  are independent of  $\sigma$ . We also prove that  $\varphi$  is a completely bounded Fourier multiplier on  $L_p(\mathcal{L}_\sigma(G))$  iff  $\tilde{\varphi}$  is a completely bounded Schur multiplier on  $S_p(\ell_2(G))$ .

More generally, we will work in the setting of twisted dynamic systems in the category of von Neumann algebras. A twisted dynamic system (or more precisely,  $W^*$ -system) is a quadruple  $(\mathcal{M}, \alpha, \sigma, G)$ , where  $\mathcal{M}$  is a von Neumann algebra and  $(\alpha, \sigma)$  a twisted action of  $G$  on  $\mathcal{M}$  (see section 3.1 for the precise definition as well as all notions unexplained below). Let  $\mathcal{M} \rtimes_{\alpha, \sigma} G$  be the associated twisted crossed product. Note that twisted crossed products in the category of  $C^*$ -algebras were first studied by Zeller-Meier [107]; since then they have been extensively investigated (cf., e.g. [12, 70, 71, 83]). In a recent series of papers [2–6], Bédos and Conti have developed harmonic analysis on twisted crossed products of  $C^*$ -algebras; in particular, they obtained results on Fourier and Schur multipliers as well as on the convergence of Fourier series. The interesting work [65] by McKee, Todorov and Turowska also deals with the same subject.

However, all just quoted papers deal with either algebraic aspects of twisted crossed products or analytic aspects only at the  $C^*$ -algebraic level (i.e., at the  $L_\infty$ -level). This means that analysis for  $L_p$ -spaces with finite  $p$  has not been touched by these papers. The goal of the present article is to fill up this deficiency. We prove that the previous results on Fourier and Schur multipliers on  $L_p(\mathcal{L}_\sigma(G))$  continue to hold for  $L_p(\mathcal{M} \rtimes_{\alpha, \sigma} G)$  under the assumption that  $\mathcal{M}$  be QWEP or  $G$  be hyperlinear. Note that González-Pérez [39] has also established the link between Fourier and Schur multipliers on untwisted crossed products.

We also study the stability of approximation properties by twisted crossed products. In particular, we show that if the action  $\alpha$  is amenable, then  $\mathcal{M} \rtimes_{\alpha, \sigma} G$  inherits the injectivity and  $w^*$ -completely bounded approximation property of  $\mathcal{M}$ . Another approximation property that we consider is QWEP. Recall that  $\mathcal{L}(G)$  is QWEP iff  $G$  is hyperlinear (cf. [74]). We prove that  $\mathcal{L}_\sigma(G)$  is QWEP iff  $G$  is hyperlinear, that is, QWEP of  $\mathcal{L}_\sigma(G)$  is independent of the 2-cocycle  $\sigma$ .

Our references for operator space theory are [34, 78]. Recall that a linear map  $T : E \rightarrow F$  between two operator spaces are said to be *completely bounded* (abbreviated to cb) if  $\text{Id} \otimes T :$

$\mathbb{M}_n(E) \rightarrow \mathbb{M}_n(F)$  is bounded uniformly in  $n \in \mathbb{N}$ ; in this case, the cb-norm of  $T$  is defined to

$$\|T\|_{\text{cb}} = \sup_{n \geq 1} \|\text{Id} \otimes T : \mathbb{M}_n(E) \rightarrow \mathbb{M}_n(F)\|.$$

All noncommutative  $L_p$ -spaces in the sequel are equipped with their natural operator space structure introduced by Pisier [76, 78]. Let us briefly recall this. As a von Neumann algebra,  $L_\infty(\mathcal{M}) = \mathcal{M}$  carries its natural operator space structure. The structure on  $L_1(\mathcal{M})$  is defined as the one induced by the opposite  $(\mathcal{M})^{\text{op}}$  that is viewed as the dual of  $L_1(\mathcal{M})$ . For  $1 < p < \infty$ , the operator space structure on  $L_p(\mathcal{M})$  is given by complex interpolation  $L_p(\mathcal{M}) = \left(\mathcal{M}, L_1(\mathcal{M})\right)_{\frac{1}{p}}$ .

We will frequently use Pisier's characterization of cb maps (see [76, Lemma 1.7]): A linear map  $T : E \rightarrow F$  is cb iff  $\text{Id} \otimes T : S_p[E] \rightarrow S_p[F]$  is bounded for some  $1 \leq p \leq \infty$  (here for  $p = \infty$ ,  $S_\infty$  should be interpreted as the algebra of compact operators on  $\ell_2$ ). In this case,

$$\|T\|_{\text{cb}} = \|\text{Id} \otimes T : S_p[E] \rightarrow S_p[F]\|,$$

where  $S_p[E]$  denotes the  $E$ -valued Schatten  $p$ -class (see [76]). Applying this criterion to the special case where  $E = F = L_p(\mathcal{M})$ , we see that a map  $T$  on  $L_p(\mathcal{M})$  is cb iff  $\text{Id} \otimes T : S_p[L_p(\mathcal{M})] \rightarrow S_p[L_p(\mathcal{M})]$  is bounded, and the cb-norm of  $T$  is the norm of the latter. Note that  $S_p[L_p(\mathcal{M})] = L_p(B(\ell_2) \overline{\otimes} \mathcal{M})$ .

In the last result,  $B(\ell_2)$  can be replaced by a QWEP von Neumann algebra  $\mathcal{A}$ . Recall that a  $C^*$ -algebra  $A$  has the WEP property if for the canonical inclusions  $A \subset A^{**} \subset B(K)$  there exists a contraction  $P : B(K) \rightarrow A^{**}$  such that  $P|_A = \text{Id}$ .  $A$  is called QWEP if  $A$  is a quotient of another  $C^*$ -algebra with WEP.

We will often use the following result of Junge [53] without further reference. Let  $\mathcal{A}$  be a QWEP von Neumann algebra. If a map  $T : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})$  is cb, then  $\text{Id} \otimes T : L_p(\mathcal{A} \overline{\otimes} \mathcal{M}) \rightarrow L_p(\mathcal{A} \overline{\otimes} \mathcal{M})$  is cb too, and the two cb-norms are equal.

## 1.2 Schatten properties of commutators of singular integral operators on noncommutative euclidean space

A. Connes introduced the quantised calculus in [20] as an analogue of the algebra of differential forms in a noncommutative setting, and later explored the link with the action functional of Yang-Mills theory [21]. Connes successfully applied quantised calculus in computing the Hausdorff measure of Julia sets and limit sets of Quasi-Fuchsian groups in the plane [22, Chapter 4, Section 3.γ] (for a more recent exposition see [26, 28]).

The core ingredients of the quantised calculus, as outlined in [20], are a separable Hilbert space  $H$ , a unitary self-adjoint operator  $F$  on  $H$  and a  $C^*$ -algebra  $\mathcal{A}$  represented on  $H$  such that for all  $a \in \mathcal{A}$  the commutator  $[F, a]$  is a compact operator on  $H$ . Then the quantised differential of  $a \in \mathcal{A}$  is defined to be the operator  $\mathcal{d}a = i[F, a]$ . The compact operators on  $H$  are described by Connes as being analogous to infinitesimals, and the rate of decay of the sequence of singular values:

$$\mu(n, T) = \inf\{\|T - R\| : \text{rank}(R) \leq n\}$$

corresponds in some way to the "size" of the infinitesimal  $T$  (see [23]). In this setting one can quantify the smoothness of an element  $a \in \mathcal{A}$  in terms of the rate of decay of  $\{\mu(n, \mathcal{d}a)\}_{n=0}^\infty$ .



Of particular interest are those elements  $a \in \mathcal{A}$  which satisfy:

$$\begin{aligned} \mu(n, \mathcal{A}a) &= O((n+1)^{-1/p}) \text{ as } n \rightarrow \infty, \text{ or,} \\ \sum_{n=0}^{\infty} \mu(n, \mathcal{A}a)^p &< \infty, \text{ or,} \\ \sup_{n \geq 1} \frac{1}{\log(n+2)} \sum_{k=0}^n \mu(k, \mathcal{A}a)^p &< \infty, \end{aligned}$$

for some  $p \in (0, \infty)$ . The first condition stated above is that  $\mathcal{A}a$  is in the weak Schatten ideal  $S_{p, \infty}$ , the second condition is for  $\mathcal{A}a$  to be in the Schatten ideal  $S_p$ , and the final condition is that  $|\mathcal{A}a|^p$  is in the Macaev-Dixmier ideal  $\mathcal{M}_{1, \infty}$  [22, Chapter 4, Section 2.β] (see also [58, Example 2.6.10]).

The link between quantised calculus and geometry is discussed by Connes in [21]. A model example for quantised calculus is to take a compact Riemannian spin manifold  $M$  with Dirac operator  $D$ , and define  $H$  to be the Hilbert space of square integrable sections of the spinor bundle. The algebra  $\mathcal{A} = C(M)$  of continuous functions on  $M$  acts by pointwise multiplication on  $H$ , and one defines

$$F = \mathbb{1}_{[0, \infty)}(D) - \mathbb{1}_{(-\infty, 0)}(D).$$

One then has  $\mathcal{A}f = i[F, M_f]$ , where  $M_f$  is the operator on  $H$  of pointwise multiplication by  $f$ . In quantised calculus the immediate question is to determine the relationship between the degree of differentiability of  $f \in C(M)$  and the rate of decay of the singular values of  $\mathcal{A}f$ . In general, we have the following:

$$f \in C^\infty(M) \Rightarrow |\mathcal{A}f|^d \in \mathcal{M}_{1, \infty},$$

where  $d$  is the dimension of the manifold  $M$  [21, Theorem 3.1].

For certain special cases it is possible to obtain a far more precise understanding of the relationship between the smoothness of  $f$  and the singular values of  $\mathcal{A}f$ . The simplest example is to take the unit circle  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ , with  $\mathcal{A} = C(\mathbb{T})$ ,  $H = L_2(\mathbb{T})$  and the standard choice of  $F$  in this setting is the Hilbert transform. Then by a result of V. Peller [73, Theorem 7.3], we have that for any  $p \in (0, \infty)$ :  $\mathcal{A}f \in S_p$  if and only if  $f$  is in the Besov space  $B_{p, p}^{1/p}(\mathbb{T})$ . Peller's work has been extended to obtain even more precise relationships between  $f$  and the singular values of  $\mathcal{A}f$ , for example, L. Gheorghe [36] found necessary and sufficient conditions on  $f$  to ensure that  $\mathcal{A}f$  is in an arbitrary Riesz-Fisher space. For more details from a quantised calculus perspective, see [22, Chapter 4, Section 3.α].

In higher dimensions, the relationship between  $f$  and  $\mathcal{A}f$  has also been studied [29, 51, 89]. To illustrate the situation, consider the  $d$ -dimensional torus  $\mathbb{R}^d$ ,  $d \geq 2$ . The appropriate Dirac operator in this setting is:

$$D = \sum_{j=1}^d -i\gamma_j \otimes \partial_j,$$

where  $\partial_j$  denotes differentiation with respect to the  $j$ -th coordinate on  $\mathbb{R}^d$ , and  $\{\gamma_1, \dots, \gamma_d\}$  denotes the  $d$ -dimensional euclidean gamma matrices, which are self-adjoint  $2^{\lfloor \frac{d}{2} \rfloor} \times 2^{\lfloor \frac{d}{2} \rfloor}$  complex matrices satisfying  $\gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{j, k} 1$ . The operator  $D$  may be considered as an unbounded self-adjoint operator on the Hilbert space  $L_2(\mathbb{R}^d, \mathbb{C}^{2^{\lfloor \frac{d}{2} \rfloor}})$ . The corresponding operator  $F$  is a linear combination of Riesz transforms. The commutators of Riesz transforms

and multiplication operators are studied in classical harmonic analysis: S. Janson and T. Wolff [51] proved that for  $\dot{d}f$  to be in  $S_p$  when  $p > d$  it is necessary and sufficient that  $f$  is in the Besov space  $B_{p,p}^{\frac{d}{p}}(\mathbb{R}^d)$ . On the other hand, Janson and Wolff also proved that if  $p \leq d$  then  $\dot{d}f \in S_p$  if and only if  $f$  is a constant.

The first results [62] concerning quantum differentiability in the noncommutative euclidean space are the characterizations of the Schatten  $S_{d,\infty}$  properties of

$$\dot{d}x := \sum_{j=1}^d \gamma_j \otimes \dot{d}x_j \tag{1.2.1}$$

on noncommutative euclidean space  $\mathbb{R}_\theta^d$ . Quantum euclidean spaces were first introduced by a number of authors, including Groenewold [42] and Moyal [66], for the study of quantum mechanics in phase space. The constructions of Groenewold and Moyal were later abstracted into more general canonical commutation relation (CCR) algebras, and have since become fundamental in mathematical physics. Under the names Moyal planes or Moyal-Groenewold planes, these algebras play the role of a central and motivating example in noncommutative geometry [43], [13]. As geometrical spaces with noncommuting spatial coordinates, noncommutative euclidean spaces have appeared frequently in the mathematical physics literature [31], in the contexts of string theory [91] and noncommutative field theory [67].

Quantum euclidean spaces have also been studied as an interesting noncommutative setting for classical and harmonic analysis, and for this we refer the reader to recent work such as [56].

In (1.2.1),  $\gamma_j$ 's still denote the  $d$ -dimensional euclidean gamma matrices, and  $\dot{d}x_j := i[R_j, M_x]$ , where for  $1 \leq j \leq d$ ,  $R_j = \frac{D_j}{(-d)^{\frac{1}{2}}}$  denote the quantum counterpart of Riesz transforms on  $\mathbb{R}_\theta^d$ . One of the main results in [61] states that  $\dot{d}x_i$  has bounded extension in  $S_{d,\infty}$  for every  $1 \leq i \leq d$  iff  $x$  belongs to the homogeneous Sobolev space  $W_d^1(\mathbb{R}_\theta^d)$ .

A related direction of research concerning quantized differentials is trace formulae. As early as [21] it was known that for functions on compact manifolds, it is possible to express the Dixmier trace  $\text{tr}_\omega(|\dot{d}f|^d)$  as an integral of derivatives of  $f$ . Such trace formulae are generalized to quantum euclidean spaces case for suitable elements  $x \in \mathbb{R}_\theta^d$ ; namely, for any continuous normalised trace  $\text{tr}$  on  $S_{1,\infty}$  we have

$$\text{tr}(|\dot{d}x|^d) = c_d \int_{\mathbb{S}^{d-1}} \tau \left( \left( \sum_{j=1}^d |\partial_j x - s_j \sum_{k=1}^d s_k \partial_k x|^2 \right)^{\frac{d}{2}} \right) ds, \tag{1.2.2}$$

where  $\tau$  is the canonical trace associated to the noncommutative torus,  $c_d$  is a certain constant depending on  $d$ , and the integral is over  $s = (s_1, \dots, s_d)$  in the  $(d-1)$ -dimensional sphere  $\mathbb{S}^{d-1}$  with respect to its rotation-invariant measure  $ds$ .

From the trace formula (1.2.2) and an estimate of its right hand side integration, one immediately deduces that  $\dot{d}x_i$  has bounded extension in  $S_p$  for every  $1 \leq i \leq d$  and  $p \leq d$  iff  $x$  is a constant operator; see also [61].

## Main results

In the second part we will consider the commutators of the multiplier operator  $M_x$  with the quantum analogues of the so-called Calderón-Zygmund transforms. In the commuta-

tive setting, Calderón-Zygmund transforms are singular integral operators having kernels homogeneous of degree  $-d$  and with mean value zero on  $\mathbb{S}^{d-1}$ . By [93, Theorem II.4.2], every Calderón-Zygmund transform can be written as the Fourier multiplier  $T_\phi$  whose symbol  $\phi$  is smooth and homogeneous of degree 0. Now that no ambiguity will be caused, by homogeneity, we will identify this symbol  $\phi$  with a function in  $C^\infty(\mathbb{S}^{d-1})$  in the sequel. For  $\phi \in C^\infty(\mathbb{S}^{d-1})$  and  $x \in \mathbb{R}_\theta^d$ , we denote  $\mathbf{C}_{\phi,x} = [T_\phi, M_x]$ , which is a bounded operator on the Hilbert space  $L_2(\mathbb{R}_\theta^d)$ . More generally, if  $x$  is not necessarily bounded, say  $L_2$  integrable, we may still define  $\mathbf{C}_{\phi,x}$  on a dense subspace of  $L_2(\mathbb{R}_\theta^d)$ ; see subsection 4.1.4 for detailed interpretation.

When  $\phi(s) = s_j \in C^\infty(\mathbb{S}^{d-1})$ ,  $T_\phi = R_j = \partial_j \Delta^{-\frac{1}{2}}$  is the  $j$ -th Riesz transform on  $\mathbb{R}_\theta^d$ .

**Theorem 1.2.1.** *Let  $d < p < \infty$ . If  $x \in B_{p,p}^{\frac{d}{p}}(\mathbb{R}_\theta^d)$ , then  $\mathbf{C}_{\phi,x}$  has a bounded extension in  $S_p$  and*

$$\|\mathbf{C}_{\phi,x}\|_{S_p} \lesssim_{d,p} \left[ \sup_{s \in \mathbb{S}^{d-1}} |\phi(s)| + \sup_{s \in \mathbb{S}^{d-1}} |\nabla \phi(s)| \right] \|x\|_{B_{p,p}^{\frac{d}{p}}}.$$

*Conversely, assume additionally that  $\phi$  is not constant. If  $x \in \mathbb{R}_\theta^d$  and  $\mathbf{C}_{\phi,x} \in S_p$ , then  $x \in B_{p,p}^{\frac{d}{p}}(\mathbb{R}_\theta^d)$  and*

$$\|x\|_{B_{p,p}^{\frac{d}{p}}} \lesssim_{d,p} \left[ \sup_{s \in \mathbb{S}^{d-1}} |\phi(s)| + \sup_{s \in \mathbb{S}^{d-1}} |\nabla \phi(s)| \right] \|\mathbf{C}_{\phi,x}\|_{S_p}.$$

Here  $B_{p,p}^{\frac{d}{p}}(\mathbb{R}_\theta^d)$  denotes the homogeneous Besov spaces on noncommutative euclidean space. Since the constant term of  $x$  does not contribute to the commutator  $\mathbf{C}_{\phi,x} = [T_\phi, M_x]$ , these homogeneous Besov spaces are more appropriate than the inhomogeneous versions studied in [105] for the characterizations of the Schatten properties of  $\mathbf{C}_{\phi,x}$ .

We will also study the critical case, i.e., the  $S_{d,\infty}$  properties of  $\mathbf{C}_{\phi,x}$  for  $p \leq d$ . Again, we use the homogeneous Sobolev space  $\dot{W}_d^1(\mathbb{R}_\theta^d)$ , whose norm (module constants) is given by  $(\sum_{1 \leq j \leq d} \|\partial_j x\|_d^d)^{1/d}$ .

**Theorem 1.2.2.** *If  $x \in \dot{W}_d^1(\mathbb{R}_\theta^d)$ , then  $\mathbf{C}_{\phi,x}$  has bounded extension in  $S_{d,\infty}$ .*

We will calculate the Dixmier trace of  $|\mathbf{C}_{\phi,x}|^d$  for good enough  $x$ . The following trace formula is new even for commutators of Calderón-Zygmund transforms in the classical (euclidean space or torus) setting.

**Theorem 1.2.3.** *Let  $x \in \dot{W}_d^1(\mathbb{R}_\theta^d)$ . Then for every continuous normalised trace  $\text{Tr}_\omega$  on  $S_{1,\infty}$ , we have*

$$\text{Tr}_\omega(|\mathbf{C}_{\phi,x}|^d) = C_d \int_{\mathbb{S}^{d-1}} \tau_\theta \left( \left| \sum_{1 \leq k \leq d} \partial_{s_k} \phi \partial_k x \right|^d \right) ds.$$

*Here the integral over  $\mathbb{S}^{d-1}$  is taken with respect to the rotation-invariant measure  $ds$  on  $\mathbb{S}^{d-1}$ .*

# Chapter 2

## Preliminaries

This chapter presents some preliminary results which will be used throughout the whole thesis.

### 2.1 Group representations

Representations are vital in this thesis. There are various types of representations. The special and most used ones in this thesis are unitary representations. We give a short introduction to these representations.

In general, by a representation, we mean a group homomorphism  $\pi : G \rightarrow GL(V)$ , where  $V$  is a vector space and  $GL(V)$  is the group of linear automorphisms of  $V$ . As far as we concerned, we only consider representations on Hilbert spaces over the complex field. When  $H$  is a Hilbert space, we denote by  $B(H)$  the space of all bounded operators, by  $GL(H)$  the group of invertible operators, and by  $U(H)$  the group of all unitary operators on  $H$ . To make use of the topological structure of the group, we usually consider continuous representations.

Sometimes it is not convinient to work with the norm topology, since in sometimes the operators we are considering are not continuous under the norm topology. Thus, it is necessary to introduce some frequently used operator topology on  $B(H)$ .

Recall that the strong operator topology (WOT) on  $B(H)$  is induced by the following families of seminorm

$$\omega_\xi : T \in B(H) \mapsto \|T\xi\|, \quad \xi \in H,$$

and the weak operator topology on  $B(H)$  is the topology induced by the families of seminorm

$$\omega_{\xi,\eta} : T \in B(H) \mapsto \langle T\xi, \eta \rangle, \quad \xi, \eta \in H.$$

**Definition 2.1.1.** *We suppose that  $\pi$  is a representation of the locally compact group  $G$  on a Hilbert space  $H$ , we say that the representation  $\pi$  is continuous with respect to the SOT if the maps*

$$s \in G \mapsto \pi(s)\xi \in H$$

*are continuous with respect to all  $\xi \in H$ . Analogously, if the maps*

$$s \in G \mapsto \langle \pi(s)\xi, \eta \rangle$$

*are continuous for all  $\xi, \eta \in H$ , then we say that the representation  $\pi$  is continuous with respect to WOT.*

Throughout the thesis, all the representations we are considering are continuous.

**Definition 2.1.2.** Two representations  $(\pi_i, H_i)$ ,  $i = 1, 2$  are called similar if there is an invertible map  $T : H_1 \rightarrow H_2$  such that

$$T^{-1}\pi_2(s)T = \pi_1(s)$$

for all  $s \in G$ . Furthermore, if the operator  $T$  is unitary, we say that the two representations are unitary equivalent and write  $\pi_1 \simeq \pi_2$ .

### 2.1.1 Unitary representations

**Definition 2.1.3.** A representation  $(\pi, H)$  is called a unitary representation if it takes images in  $U(H)$ .

**Example 2.1.1.** The very basic example is the trivial representation:

$$s \in G \mapsto id_H \in U(H)$$

for all  $s \in G$ .

**Example 2.1.2.** The intrinsic example throughout this thesis is the left regular representation: the left regular representation on  $L_2(G)$  is defined by

$$(\lambda(s)f)(t) = f(s^{-1}t), \quad f \in L_2(G), \quad s, t \in G.$$

We define also the right regular representations on  $L_2(G)$ ,

$$(\rho_\sigma(s)f)(t) = \Delta(s)^{\frac{1}{2}} f(ts), \quad f \in L_2(G), \quad s, t \in G.$$

### 2.1.2 Group algebras

The left (resp. right) group von Neumann algebra  $\mathcal{L}(G)$  (resp.  $\mathcal{R}(G)$ ) is the von Neumann subalgebra of  $B(L_2(G))$  generated by the set  $\{\lambda(s) : s \in G\}$  (resp.  $\{\rho(s) : s \in G\}$ ), that is, the closure in the  $w^*$  topology of  $\text{span}\{\lambda(s) : s \in G\}$  (resp.  $\{\rho(s) : s \in G\}$ ). Since  $L_1(G)$  as  $*$ -algebra under twisted convolution and involution has the same bounded approximate identity as the usual group algebra  $L_1(G)$ ,  $\mathcal{L}(G)$  is equivalently generated by  $\{\lambda(f) : f \in L_1(G)\}$ .

The predual of  $\mathcal{L}(G)$  is denoted  $A(G)$ , called Fourier algebra. It is identified as the subspace of  $C_0(G)$  of continuous functions vanishing at infinity via  $\omega_{\xi, \eta} \mapsto \langle \lambda(\cdot)\xi, \eta \rangle, A(G, \sigma) \rightarrow C_0(G)$ . By this identification, for functions  $\varphi \in A(G, \sigma)$  and  $f \in L_1(G)$ , the duality bracket  $\langle \varphi, \lambda(f) \rangle$  is given as

$$\langle \varphi, \lambda(f) \rangle = \int_G \varphi(s)f(s) ds. \quad (2.1.1)$$

### 2.1.3 Weight on $\mathcal{L}(G)$

We follow the construction in [99] to endow  $\mathcal{L}(G)$  a weight, and  $\mathcal{R}(G)$  with its opposite weight.

We begin with  $C_c(G)$ , the space of continuous compactly supported functions on  $G$ . With the following convolution

$$\xi * \eta(s) = \int_G \xi(t)\eta(t^{-1}s)dt, \quad (2.1.2)$$

involution

$$\xi^\sharp(s) = \Delta(s^{-1})\overline{\xi(s^{-1})}, \quad (2.1.3)$$

and inner product

$$\langle \xi, \eta \rangle = \int_G \xi(s)\overline{\eta(s)}ds,$$

$C_c(G)$  is a left Hilbert algebra (see [99, Section VI.1] for definition), denoted as  $\mathfrak{M}(G)$ . If we replace the involution  $\xi^\sharp$  with

$$\xi^b(s) = \overline{\xi(s^{-1})},$$

then we get a right Hilbert algebra, denoted as  $\mathfrak{N}(G)$ .

It is evident that the completion of  $\mathfrak{M}(G)$  or  $\mathfrak{N}(G)$  is the Hilbert space  $L_2(G)$ . To each  $\xi \in \mathfrak{M}(G)$  there corresponds a unique bounded operator  $\pi_\ell(\xi) = \lambda(\xi) : \eta \mapsto \xi * \eta$  on  $L_2(G)$ , and the map  $\pi_\ell : \mathfrak{M}(G) \mapsto \pi_\ell(\xi) \in B(L_2(G))$  is a  $*$ -representation of  $\mathfrak{M}(G)$ , which is nondegenerate. Then we get the left von Neumann algebra  $\pi_\ell(\mathfrak{M}(G))''$ . Likewise, for the right Hilbert algebra  $\mathfrak{N}(G)$  and its anti  $*$ -representation  $\pi_r(\eta) = \rho_\sigma(\eta)$  on  $L_2(G)$ , we have the right von Neumann algebra  $\pi_r(\mathfrak{N}(G))''$ .

The modular operator and modular conjugation of the left Hilbert algebra  $\mathfrak{M}(G)$  are

$$\Delta\xi(s) = \Delta(s)\xi(s), \quad J\xi(s) = \Delta(s)^{-\frac{1}{2}}\overline{\xi(s^{-1})}, \quad (2.1.4)$$

which are guaranteed by [99, Lemma VI.1.5].

A vector  $\eta \in L_2(G)$  is said to be right bounded if

$$\sup\{\|\xi * \eta\|_2 : \xi \in \mathfrak{M}(G), \|\xi\|_2 \leq 1\} < \infty.$$

Denote by  $\mathfrak{B}'$  the set of all right bounded vectors in  $L_2(G)$ , and let  $\mathfrak{n}_r = \pi_r(\mathfrak{B}')$ . Define  $\mathfrak{M}(G)' = \mathfrak{B}' \cap D^b$ , where  $D^b$  is the domain of the map  $\eta \mapsto \eta^b$ . Then this  $\mathfrak{M}(G)'$  is a right Hilbert algebra. And by [99, Lemma VI.1.15],

$$\pi_r(\mathfrak{M}(G)') = \mathfrak{n}_r \cap \mathfrak{n}_r^*.$$

Similarly, starting with the right Hilbert algebra  $\mathfrak{M}(G)'$ , let  $\mathfrak{B}$  be the set of all left bounded elements in the sense

$$\sup\{\|\xi *_\sigma \eta\|_2 : \eta \in \mathfrak{M}(G, \sigma)', \|\eta\|_2 \leq 1\} < \infty, \quad (2.1.5)$$

and

$$\mathfrak{n}_\ell = \pi_\ell(\mathfrak{B}). \quad (2.1.6)$$

Likewise, define  $\mathfrak{M}(G)'' = \mathfrak{B} \cap D^\sharp$  where  $D^\sharp$  is the domain of the map  $\eta \mapsto \eta^\sharp$ . Then we have  $\pi_\ell(\mathfrak{M}(G)'') = \mathfrak{n}_\ell \cap \mathfrak{n}_\ell^*$ . Define also

$$\mathfrak{m}_\ell = \left\{ \sum_{j=1}^n y_j^* x_j : x_1, \dots, x_n, y_1, \dots, y_n \in \mathfrak{n}_\ell \right\}, \quad (2.1.7)$$

and  $\mathfrak{m}_r$  similarly.

For an arbitrary left Hilbert algebra  $\mathfrak{M}$ ,  $\mathfrak{M}''$  and  $\mathfrak{M}'$  do not agree in general. However in our case, both  $\mathfrak{M}(G)''$  and  $\mathfrak{M}(G)'$  are given by  $C_c(G)$ , a Tomita algebra, see [99, Section VI.2].

**Lemma 2.1.3.** Equipped with the complex one parameter group  $d^\alpha : \xi(s) \mapsto d(s)^\alpha \xi(s)$  of automorphisms,  $\mathfrak{M}(G)$  is a Tomita algebra. Moreover

$$\pi_\ell(\mathfrak{M}(G)'') = \mathcal{L}(G), \quad \pi_r(\mathfrak{N}(G)'') = \mathfrak{R}(G, \sigma) = \mathcal{L}(G)'.$$

Now we define  $\Phi$  on  $\mathcal{L}(G)_+$  as

$$\Phi(x^*x) = \begin{cases} \|f\|_2^2, & \text{if } x = \lambda_\sigma(f) \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.1.8)$$

By [99, Theorems VII.2.5 and VII.2.6], this  $\Phi$  is a weight on  $\mathcal{L}(G)$  associated with the full left Hilbert algebra  $\mathfrak{M}(G)$ ; this weight is the twisted version of the Plancherel weight in [99, Definition VII.3.2]. Define also  $\Psi$  on  $\mathfrak{K}(G)_+$  as

$$\Psi(y^*y) = \begin{cases} \|g\|_2^2, & \text{if } y = \rho(g) \\ +\infty, & \text{otherwise.} \end{cases}$$

Again by [99, Theorem VII.2.5], we see that  $\Psi$  is the opposite of  $\Phi$ . By the definition of  $\mathfrak{m}$  it follows that the weight  $\Phi$  (resp.  $\Psi$ ) is extended to a linear functional on  $\mathfrak{m}_\ell$  (resp.  $\mathfrak{m}_r$ ), still denoted by  $\Phi$  (resp.  $\Psi$ ).

**Proposition 2.1.4.** *The weight  $\Phi$  on  $\mathcal{L}(G)$  is tracial iff  $\Delta(s) \equiv 1$ , i.e., iff  $G$  is unimodular.*

## 2.2 Noncommutative $L_p$ space

Throughout the thesis, we mainly consider the von Neumann algebra equipped with a normal semifinite faithful trace. For this class of von Neumann algebra, the definition of noncommutative  $L_p$  space will be much neater.

Let  $\mathcal{M}$  be a von Neumann algebra equipped with a normal semifinite faithful trace  $\tau$ , and let  $S_{\mathcal{M}}^+$  be the set of all positive elements  $x$  in  $\mathcal{M}$  with  $\tau(s(x)) < \infty$ , where  $s(x)$  denotes the support of  $x$ , i.e., the smallest projection  $e$  such that  $exe = x$ . Let  $S_{\mathcal{M}} = \text{span}(S_{\mathcal{M}}^+)$ . Then every  $x \in S_{\mathcal{M}}$  has finite trace, and  $S_{\mathcal{M}}$  is a  $w^*$ -dense  $*$ -ideal of  $\mathcal{M}$ . Let  $1 \leq p < \infty$ . For any  $x \in S_{\mathcal{M}}$ , the operator  $|x|^p$  belongs to  $S_{\mathcal{M}}^+$  too (recalling  $|x| = (x^*x)^{\frac{1}{2}}$ ). We define

$$\|x\|_p = \left( \tau(|x|^p) \right)^{\frac{1}{p}}.$$

Then  $\|\cdot\|_p$  is a norm on  $S_{\mathcal{M}}$ . The completion of  $(S_{\mathcal{M}}, \|\cdot\|_p)$  is denoted by  $L_p(\mathcal{M})$ , the noncommutative  $L_p$ -space associated to  $(\mathcal{M}, \tau)$ . We refer the reader to [79] and [106] for further information on noncommutative  $L_p$ -spaces.

In particular, if  $\mathcal{M} = B(H)$  equipped with the usual trace  $\text{Tr}$  ( $H$  being a Hilbert space), the corresponding noncommutative  $L_p$ -space is the usual Schatten  $p$ -class, denoted by  $S_p(H)$ . If  $H$  is separable and  $\dim H = \infty$ ,  $S_p(H)$  is simply denoted by  $S_p$ .

If  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are two semifinite von Neumann algebras equipped with  $\tau_1$  and  $\tau_2$ , respectively, the von Neumann tensor algebra  $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$  is equipped with the tensor trace  $\tau_1 \otimes \tau_2$ . We will often consider the tensor  $\mathcal{M} \bar{\otimes} B(H)$ , equipped with  $\tau \otimes \text{Tr}$ .

We will record some frequently used results of noncommutative  $L_p$  spaces.

**Theorem 2.2.1** (Hölder inequality). *Let  $0 < p, q, r \leq \infty$  such that  $1/r = 1/p + 1/q$ . Then*

$$\|xy\|_r \leq \|x\|_p \|y\|_q, \quad x, y \in \mathcal{S}.$$

The next proposition means that the noncommutative  $L_p$  is unitary invariant.

**Proposition 2.2.2.** *Let  $x \in \mathcal{S}$  and  $a, b \in \mathcal{M}$ . Then  $\|axb\|_p \leq \|a\| \|x\|_p \|b\|$  for  $0 < p < \infty$ . Consequently,  $\|\cdot\|_p$  is unitary invariant.*

The next results is vital in the first part.

**Proposition 2.2.3.** *Given two noncommutative measure space  $(\mathcal{M}, \tau)$  and  $(\mathcal{N}, \mu)$ , if there is an isomorphism  $\pi : \mathcal{M} \rightarrow \mathcal{N}$  such that  $\mu \circ \pi = \tau$  (trace preserving), Then  $\pi$  can be extended to an isometric isomorphism from  $L_p(\mathcal{M})$  to  $L_p(\mathcal{N})$ .*





## Chapter 3

# Harmonic analysis on twisted crossed product

For the convenience of reading, this chapter is presented self-contained with the rest of the thesis. In this chapter, we study the Fourier and Schur multipliers of scalar valued symbol on the noncommutative  $L_p$  spaces of twisted crossed products.

### 3.1 Twisted crossed products

#### 3.1.1 Basic properties

In this subsection we present basic notions and properties of twisted crossed products.

**Definition 3.1.1.** A *twisted dynamical system* is a quadruple  $(A, G, \alpha, \sigma)$  with a *twisted action*  $(\alpha, \sigma)$  of  $G$  on  $\mathcal{M}$ . Here the two functions  $\alpha : G \rightarrow \text{Aut}(\mathcal{M})$  and  $\sigma : G \times G \rightarrow \mathcal{U}(\mathcal{M})$  satisfy the following conditions: for any  $s, t, r \in G$

- (i)  $\alpha_s \circ \alpha_t = \text{Ad}_{\sigma(s,t)} \circ \alpha_{st}$ ;
- (ii)  $\sigma(r, s)\sigma(rs, t) = \alpha_r(\sigma(s, t))\sigma(r, st)$ ;
- (iii)  $\sigma(e, s) = \sigma(s, e) = 1$ .

The following identity about the 2-cocycle will be often used later:

$$\sigma(s, s^{-1}) = \alpha_s(\sigma(s^{-1}, s)), \quad s \in G. \quad (3.1.1)$$

Let  $\ell_1(G, \mathcal{M})$  denote the  $\mathcal{M}$ -valued  $\ell_1$ -space indexed by  $G$ . Then  $\ell_1(G, \mathcal{M})$  becomes a Banach  $*$ -algebra with convolution and involution defined as follows: for  $f, g \in \ell_1(G, \mathcal{M})$  and  $s \in G$

$$f \star g(s) = \sum_{t \in G} f(t) \alpha_t(g(t^{-1}s)) \sigma(t, t^{-1}s), \quad (3.1.2)$$

$$f^\sharp(s) = \sigma(s, s^{-1})^* \alpha_s(f(s^{-1}))^*. \quad (3.1.3)$$

Here we have decided to omit the twisted action  $(\alpha, \sigma)$  in  $f \star g$  and  $f^\sharp$ , otherwise the notation would be too heavy. However, we will use  $\ell_1(G, \mathcal{M}, \alpha, \sigma)$  to denote the above Banach algebra if confusion is possible.

**Definition 3.1.2.** A *covariant homomorphism* of  $(A, G, \alpha, \sigma)$  is a pair  $(\rho, u)$  of a normal representation  $\rho$  of  $\mathcal{M}$  on a Hilbert space  $K$ , and a function  $u : G \rightarrow \mathcal{U}(K)$  such that

- (i)  $u(s)u(t) = \rho(\sigma(s, t))u(st)$ ,  $s, t \in G$ ;
- (ii)  $\rho(\alpha_s(a)) = u(s)\rho(a)u(s)^*$ ,  $a \in \mathcal{M}, s \in G$ .

It is easy to check that  $(\rho, u)$  gives rise to a  $*$ -representation  $\pi \times u$  of  $\ell_1(G, \mathcal{M})$ :

$$\rho \times u(f) = \sum_{s \in G} \rho(f(s))u(s), \quad f \in \ell_1(G, \mathcal{M}).$$

The covariant representation of  $(A, G, \alpha, \sigma)$  that we are interested in is the regular covariant representation. Let  $\ell_2(G, H)$  denote the  $H$ -valued  $\ell_2$ -space indexed by  $G$ . Define

$$\begin{aligned} (\pi_\alpha(a)\xi)(t) &= \alpha_{t^{-1}}(a)\xi(t), \quad \xi \in \ell_2(G, H), t \in G, \\ (\lambda_\sigma(s)\xi)(t) &= \sigma(t^{-1}, s)\xi(s^{-1}t), \quad \xi \in \ell_2(G, H), s, t \in G. \end{aligned}$$

Then  $\pi_\alpha(a)$  and  $\lambda_\sigma(s)$  are bounded operators on  $\ell_2(G, H)$  and belong to the von Neumann algebra tensor product  $\mathcal{M} \bar{\otimes} B(\ell_2(G))$ . Note that  $\pi_\alpha(a)$  is the block diagonal operator  $\alpha_{\cdot^{-1}}(a) = \text{diag}(\alpha_{t^{-1}}(a))_{t \in G}$  with entries in  $\mathcal{M}$  (relative to the canonical basis of  $\ell_2(G)$ ) and  $\lambda_\sigma(s) = \sigma(\cdot^{-1}, s) \otimes L(s)$  is unitary, where  $L(s)$  is the left translation operator on  $\ell_2(G)$ . It is easy to check that  $(\pi_\alpha, \lambda_\sigma)$  is a covariant representation of  $(A, G, \alpha, \sigma)$ ; moreover,  $\pi_\alpha$  is faithful.

**Definition 3.1.3.** The above covariant representation  $(\pi_\alpha, \lambda_\sigma)$  is called the *twisted left regular representation* of  $(A, G, \alpha, \sigma)$ . The *twisted crossed product*  $\mathcal{M} \rtimes_{\alpha, \sigma} G$  is the von Neumann algebra generated by  $\pi_\alpha(\mathcal{M})$  and  $\lambda_\sigma(G)$ :

$$\mathcal{M} \rtimes_{\alpha, \sigma} G = \left\{ \pi_\alpha(a)\lambda_\sigma(s) : a \in \mathcal{M}, s \in G \right\}'' \subset B(\ell_2(G, H)).$$

It is clear that  $\pi_\alpha \times \lambda_\sigma(\ell_1(G, \mathcal{M}, \alpha, \sigma))$  is a  $w^*$ -dense  $*$ -subalgebra of  $\mathcal{M} \rtimes_{\alpha, \sigma} G$ . Note that if  $\sigma$  is identically equal to 1,  $\mathcal{M} \rtimes_{\alpha, \sigma} G$  reduces to the usual crossed product  $\mathcal{M} \rtimes_\alpha G$ . In this case,  $\lambda_\sigma$  is simply denoted by  $\lambda$ . On the other hand, if  $\mathcal{M} = \mathbb{C}$ ,  $\mathcal{M} \rtimes_{\alpha, \sigma} G$  is the *twisted group von Neumann algebra*  $\mathcal{L}_\sigma(G)$ . The usual (untwisted) group Neumann algebra of  $G$  is denoted by  $\mathcal{L}(G)$ .

**Remark 3.1.4.** We can also define a *right regular representation*  $(\pi_\alpha^r, \lambda_\sigma^r)$  of  $(A, G, \alpha, \sigma)$  as follows:

$$\begin{aligned} (\pi_\alpha^r(a)\xi)(t) &= \alpha_t(a)\xi(t), \quad \xi \in \ell_2(G, H), t \in G, \\ (\lambda_\sigma^r(s)\xi)(t) &= \sigma(t, s)\xi(ts), \quad \xi \in \ell_2(G, H), s, t \in G. \end{aligned}$$

This leads to the right twisted crossed product  $\mathcal{M} \rtimes_{\alpha, \sigma}^r G$ . In this article, we only consider left twisted crossed products.

The above twisted crossed product is independent of the choice of a particular covariant representation. By this we mean the following property that is proved as [?, Theorem X.1.7] in the untwisted setting.

**Remark 3.1.5.** Let  $(\rho, u)$  be a covariant representation of  $(A, G, \alpha, \sigma)$  with  $\rho$  a normal faithful representation of  $\mathcal{M}$  on  $K$ . Define

$$\begin{aligned} (\rho_\alpha(a)\xi)(t) &= \rho(\alpha_{t^{-1}}(a))\xi(t), \quad \xi \in \ell_2(G, K), t \in G, \\ (\lambda_{\sigma, \rho}(s)\xi)(t) &= \rho(\sigma(t^{-1}, s))\xi(s^{-1}t), \quad \xi \in \ell_2(G, K), s, t \in G. \end{aligned}$$

Then there exists an isomorphism  $\Phi$  from  $\mathcal{M} \rtimes_{\alpha, \sigma} G$  onto  $\{\rho_\alpha(\mathcal{M}), \lambda_{\sigma, \rho}(G)\}''$  such that

$$\Phi(\pi_\alpha(a)) = \rho_\alpha(a) \quad \text{and} \quad \Phi(\lambda_\sigma(s)) = \lambda_{\sigma, \rho}(s), \quad a \in \mathcal{M}, s \in G.$$

**Convention.** To lighten notation, throughout the remainder of the article, we will set

$$\mathfrak{K} = \mathcal{M} \rtimes_{\alpha, \sigma} G \quad \text{and} \quad \mathcal{N} = \mathcal{M} \overline{\otimes} B(\ell_2(G)).$$

It is clear that  $\mathfrak{K}$  is a subalgebra of  $\mathcal{N}$  from the construction of  $\mathfrak{K}$ . The canonical embedding of  $\mathfrak{K}$  into  $\mathcal{N}$  will be denoted by  $\iota$  if one wishes to distinguish  $\mathfrak{K}$  and  $\iota(\mathfrak{K})$  in some special situations. We will use the matricial representation of  $\mathfrak{K}$  in  $\mathcal{N}$  relative to the canonical basis  $\{\delta_s\}_{s \in G}$  of  $\ell_2$  (recalling that  $\delta_s$  is the Dirac mass at  $s$ ). Let  $\{e_{s,t}\}_{s,t \in G}$  be the canonical matrix units of  $\ell_2(G)$ . Then  $\pi_\alpha(a)$  is the block diagonal matrix:

$$\pi_\alpha(a) = \sum_{s \in G} \alpha_{s^{-1}}(a) \otimes e_{s,s}, \quad a \in \mathcal{M},$$

On the other hand, it is easy to check that

$$\lambda_\sigma(r) = \sum_{s \in G} \sigma(s^{-1}, r) \otimes e_{s, r^{-1}s}, \quad r \in G.$$

Thus

$$\pi_\alpha(a)\lambda_\sigma(r) = \sum_{s \in G} \alpha_{s^{-1}}(a)\sigma(s^{-1}, r) \otimes e_{s, r^{-1}s},$$

and more generally

$$\pi_\alpha \times \lambda_\sigma(f) = \sum_{s,t \in G} \alpha_{s^{-1}}(f(st^{-1}))\sigma(s^{-1}, st^{-1}) \otimes e_{s,t}, \quad f \in \ell_1(G, \mathcal{M}). \quad (3.1.4)$$

We now extend the Fell principle to the present setting. Let  $(\mathcal{M}_1, G, \alpha_1, \sigma_1)$  and  $(\mathcal{M}_2, G, \alpha_2, \sigma_2)$  be two twisted dynamic systems, and let  $(\rho_i, u_i, K_i)$  be a covariant representation of  $(\mathcal{M}_i, G, \alpha_i, \sigma_i)$  for  $i = 1, 2$ . Define the associated tensor objects:

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2, & \alpha &= \alpha_1 \otimes \alpha_2, & \sigma &= \sigma_1 \otimes \sigma_2; \\ K &= K_1 \otimes K_2, & \rho &= \rho_1 \otimes \rho_2, & u &= u_1 \otimes u_2. \end{aligned}$$

Recall that  $\rho_\alpha$  and  $\lambda_{\sigma, \rho}$  are defined in Remark 3.1.5. The following is to be compared with [4, Theorem 4.10].

**Proposition 3.1.6.** *With the above notation, the two covariant representations  $(\rho_1 \otimes \rho_{2, \alpha_2}, u_1 \otimes \lambda_{\sigma_2, \rho_2})$  and  $(\rho_\alpha, \lambda_\sigma)$  of  $(A, G, \alpha, \sigma)$  are unitarily equivalent. More precisely, there exists a unitary operator  $U$  on  $\ell_2(G, K)$  such that*

$$U\rho_\alpha(a)U^* = \rho_1 \otimes \rho_{2, \alpha_2}(a) \quad \text{and} \quad U\lambda_{\sigma, \rho}(t)U^* = u_1(t) \otimes \lambda_{\sigma_2, \rho_2}(t), \quad a \in \mathcal{M}, t \in G.$$

*Proof.* Define  $U : \ell_2(G, K) \rightarrow \ell_2(G, K)$  by

$$(U\xi)(s) = \left( \rho_1(\sigma_1(s, s^{-1})^*) u_1(s) \otimes 1 \right) \xi(s), \quad \xi \in \ell_2(G, K), s \in G.$$

Clearly,  $U$  is unitary. Given  $a = a_1 \otimes a_2 \in \mathcal{M}$ , we calculate  $U\rho_\alpha(a)U^*$  as follows. Given  $\xi \in \ell_2(G, K), s \in G$ , applying successively  $U^*, \rho_\alpha(a)$  and  $U$ , we have

$$\begin{aligned} & \left( U\rho_\alpha(a)U^*\xi \right)(s) \\ & \mapsto \left( u_1(s)^* \rho_1(\sigma_1(s, s^{-1})) \otimes 1 \right) \xi(s) \\ & \mapsto \alpha_{1, s^{-1}}(a_1) \otimes \alpha_{2, s^{-1}}(a_2) u_1(s)^* \rho_1(\sigma_1(s, s^{-1})) \xi(s) \\ & \mapsto \left( \rho_1(\sigma_1(s, s^{-1})^*) u_1(s) \otimes 1 \right) \alpha_{1, s^{-1}}(a_1) \otimes \alpha_{2, s^{-1}}(a_2) \left( u_1(s)^* \rho_1(\sigma_1(s, s^{-1})) \otimes 1 \right) \xi(s). \end{aligned}$$

The first component in the tensor of the last operator is equal to

$$\begin{aligned} & \rho_1(\sigma_1(s, s^{-1})^*) u_1(s) \alpha_{1, s^{-1}}(a_1) u_1(s)^* \rho_1(\sigma_1(s, s^{-1})) \\ & = \rho_1(\sigma_1(s, s^{-1})^*) \rho_1(\alpha_{1, s} \circ \alpha_{1, s^{-1}}(a_1)) \rho_1(\sigma_1(s, s^{-1})) \\ & = \rho_1(\sigma_1(s, s^{-1})^*) \rho_1(\sigma_1(s, s^{-1}) a_1 \sigma_1(s, s^{-1})^*) \rho_1(\sigma_1(s, s^{-1})) = \rho_1(a_1). \end{aligned}$$

We then deduce that

$$U\rho_\alpha(a)U^* = \rho_1 \otimes \rho_{2, \alpha_2}(a), \quad a \in \mathcal{M}.$$

Similarly, we compute  $U\lambda_{\sigma, \rho}(t)U^*$ :

$$\begin{aligned} & \left( U\lambda_{\sigma, \rho}(t)U^*\xi \right)(s) \\ & = \left[ \rho_1(\sigma_1(s, s^{-1})^*) u_1(s) \rho_1(\sigma_1(s^{-1}, t)) u_1(t^{-1}s)^* \rho_1(\sigma_1(t^{-1}s, s^{-1}t)) \right] \otimes \rho_2(\sigma_2(s^{-1}, t)) \xi(t^{-1}s). \end{aligned}$$

The first tensor factor above is equal to

$$\begin{aligned} & \rho_1(\sigma_1(s, s^{-1})^*) \rho_1(\alpha_{1, s}(\sigma_1(s^{-1}, t))) u_1(s) u_1(t^{-1}s)^* \rho_1(\sigma_1(t^{-1}s, s^{-1}t)) \\ & = \rho_1(\sigma_1(s, s^{-1})^*) \rho_1(\alpha_{1, s}(\sigma_1(s^{-1}, t))) u_1(s) u_1(s^{-1}t) \\ & = \rho_1(\sigma_1(s, s^{-1})^*) \rho_1(\alpha_{1, s}(\sigma_1(s^{-1}, t))) \rho_1(\sigma_1(s, s^{-1}t)) u_1(t) \\ & = \rho_1 \left[ \sigma_1(s, s^{-1})^* \alpha_{1, s}(\sigma_1(s^{-1}, t)) \sigma_1(s, s^{-1}t) \right] u_1(t) \\ & = \rho_1 \left[ \sigma_1(s, s^{-1})^* \sigma_1(s, s^{-1}) \right] u_1(t) = u_1(t). \end{aligned}$$

Thus

$$U\lambda_\sigma(t)U^* = u_1(t) \otimes \lambda_{\sigma_2, \rho_2}(t), \quad t \in G.$$

This finishes the proof of the proposition.  $\square$

Let us record two particular instances of the above proposition for later use. The first one is the case where  $\mathcal{M}_2 = \mathbb{C}$  and  $\sigma_2 \equiv 1$ . Then  $(\mathcal{M}_1, G, \alpha_1, \sigma_1)$  becomes our usual twisted dynamic system  $(A, G, \alpha, \sigma)$  and  $\lambda_{s_2, \rho_2}$  reduces to the untwisted left regular representation  $\lambda$  on  $G$ .

**Corollary 3.1.7.** *Let  $(\rho, u)$  be a covariant representation of  $(A, G, \alpha, \sigma)$  on  $K$ . Then  $(\rho \otimes 1, u \otimes \lambda)$  is unitarily equivalent to  $(\rho_a, \lambda_{\sigma, \rho})$ .*

The second instance is where  $\mathcal{M}_1 = \mathcal{M}_2 = \mathbb{C}$  and  $\bar{\sigma}_1 = \sigma_2 = \sigma$  with  $\sigma$  a scalar valued 2-cocycle.

**Corollary 3.1.8.** *Let  $\sigma$  be a scalar valued 2-cocycle on  $G$ . Then  $\lambda_{\bar{\sigma}} \otimes \lambda_\sigma$  is unitarily equivalent to  $1 \otimes \lambda$ .*

### 3.1.2 Dual trace

Recall that  $\mathcal{M}$  is equipped with a faithful tracial normal state  $\tau$ . We view  $\mathcal{M}$  as the left multiplication algebra on  $H = L_2(\mathcal{M})$ . Then  $\tau$  is the vector state induced by the identity  $1_{\mathcal{M}}$  of  $\mathcal{M}$ , considered as an element of  $L_2(\mathcal{M})$  (recalling that  $\mathcal{M} \subset L_2(\mathcal{M})$  canonically). The *dual state*  $\widehat{\tau}$  on  $\mathcal{M} \rtimes_{\alpha, \sigma} G$  is the vector state induced by  $1_{\mathcal{M}} \otimes \delta_e$ :

$$\widehat{\tau}(x) = \langle 1_{\mathcal{M}} \otimes \delta_e, x(1_{\mathcal{M}} \otimes \delta_e) \rangle, \quad x \in \mathfrak{K}.$$

**Proposition 3.1.9.** *The dual state  $\widehat{\tau}$  on  $\mathfrak{K}$  is tracial and faithful.*

*Proof.* To show the traciality, first note that for  $f \in \ell_1(G, \mathcal{M})$

$$\widehat{\tau}(\pi_{\alpha} \times \lambda_{\sigma}(f)) = \tau(f(0)).$$

Next we calculate  $f^{\sharp} \star f(0)$ :

$$\begin{aligned} f^{\sharp} \star f(0) &= \sum_{t \in G} f^{\sharp}(t) \alpha_t(f(t^{-1})) \sigma(t, t^{-1}) \\ &= \sum_{t \in G} \sigma(t, t^{-1})^* \alpha_t(f(t^{-1})^*) \alpha_t(f(t^{-1})) \sigma(t, t^{-1}) \\ &= \sum_{t \in G} \sigma(t^{-1}, t)^* \alpha_{t^{-1}}(f(t)^* f(t)) \sigma(t^{-1}, t). \end{aligned}$$

Noting that  $\text{Ad}_{\sigma(t^{-1}, t)^*} \circ \alpha_{t^{-1}} = \alpha_t^{-1}$ , we get

$$f^{\sharp} \star f(0) = \sum_{t \in G} \alpha_t^{-1}(f(t)^* f(t)).$$

Similarly, using (3.1.1), we have

$$f \star f^{\sharp}(0) = \sum_{t \in G} f(t) f(t)^*.$$

Since  $\alpha$  preserves  $\tau$ , for  $x = \pi_{\alpha} \times \lambda_{\sigma}(f)$  we deduce

$$\widehat{\tau}(x^* x) = \sum_{t \in G} \tau(f(t) f(t)^*) = \widehat{\tau}(x x^*).$$

Then the traciality of  $\widehat{\tau}$  follows from the  $w^*$ -density of  $\pi_{\alpha} \times \lambda_{\sigma}(\ell_1(G, \mathcal{M}))$  in  $\mathfrak{K}$  and the normality of  $\widehat{\tau}$ .

We next show the faithfulness of  $\widehat{\tau}$ . Assume that  $\widehat{\tau}(x^* x) = 0$ . Let  $y = \pi_{\alpha} \times \lambda_{\sigma}(f)$  with  $f \in \ell_1(G, \mathcal{M})$ . Then

$$\|xy(1_{\mathcal{M}} \otimes \delta_e)\|^2 = \widehat{\tau}(y^* x^* xy) = \widehat{\tau}(xyy^* x^*) \leq \|y\|^2 \widehat{\tau}(xx^*) = 0.$$

Thus  $xy(1_{\mathcal{M}} \otimes \delta_e) = 0$ . An elementary calculation shows

$$y(1_{\mathcal{M}} \otimes \delta_e) = \sum_{s \in G} \pi_{\alpha}(f(s)) \lambda_{\sigma}(s)(1_{\mathcal{M}} \otimes \delta_e) = \sum_{s \in G} \alpha_{s^{-1}}(f(s)) \sigma(s^{-1}, s) \otimes \delta_s.$$

This shows that  $\{\pi_{\alpha} \times \lambda_{\sigma}(f)(1_{\mathcal{M}} \otimes \delta_e) : f \in \ell_1(G, \mathcal{M})\}$  is dense in  $\ell_2(G, L_2(\mathcal{M}))$ . It then follows that  $x = 0$ , so  $\widehat{\tau}$  is faithful.  $\square$

**Remark 3.1.10.** The above proof also shows that the map  $f \mapsto \pi_\alpha \times \lambda_\sigma(f)$  establishes a unitary from  $\ell_2(G, L_2(\mathcal{M}))$  onto  $L_2(\mathfrak{K})$ .

**Remark 3.1.11.** Note that the restriction of  $\widehat{\tau}$  to  $\pi_\alpha(\mathcal{M})$  coincides with  $\tau \circ (\pi_\alpha|_{\pi_\alpha(\mathcal{M})})^{-1}$ . In other words, if we identify  $\pi_\alpha(\mathcal{M})$  with  $\mathcal{M}$ , this restriction is just  $\tau$ . Thus there exists a unique trace preserving conditional expectation  $\mathcal{E}$  from  $\mathfrak{K}$  to  $\pi_\alpha(\mathcal{M})$ . It is easy to show that

$$\mathcal{E}(\pi_\alpha \times \lambda_\sigma(f)) = \pi_\alpha(f(0)), \quad f \in \ell_1(G, \mathcal{M}).$$

### 3.1.3 Amenability

In this subsection, we study the stability of several approximation properties by twisted crossed products. To that end, we first recall the definition of amenable actions. Given  $f, g \in \ell_1(G, \mathcal{M})$ , define the  $\mathcal{M}$ -valued inner product:

$$\langle f, g \rangle = \sum_{s \in G} f(s)^* g(s).$$

We also define a second convolution on  $\ell_1(G, \mathcal{M})$  by

$$f *_\alpha g(s) = \sum_{t \in G} f(t) \alpha_t(g(t^{-1}s)).$$

The following definition and lemma are classical in the category of  $C^*$ -algebras (see, for instance, [11, section 4.3]). We adopt them to the von Neumann algebra setting just replacing the norm convergence by the  $w^*$ -convergence.

**Definition 3.1.12.** The action  $\alpha$  is said to be *amenable* if there exists a net  $\{T_i\}_{i \in I}$  of finitely supported functions on  $G$  with values in the center  $\mathcal{Z}(\mathcal{M})$  of  $\mathcal{M}$  such that

- (i)  $T_i(s) \geq 0$  for all  $i \in I$  and  $s \in G$  and  $\sum_{s \in G} T_i(s) = 1$ ;
- (ii)  $\lim_i \langle (1 \otimes \delta_s) *_\alpha T_i - T_i, (1 \otimes \delta_s) *_\alpha T_i - T_i \rangle \xrightarrow{w^*} 0$  for all  $s \in G$ .

The following simple fact will be used later.

**Lemma 3.1.13.** Let  $T_i$  be as above and  $\check{T}_i(s) = T_i(s^{-1})$ . Then  $\lim_i (1 - T_i *_\alpha \check{T}_i(s)) = 0$  in the strong topology for any  $s \in G$ .

*Proof.* We have

$$\begin{aligned} 1 - T_i *_\alpha \check{T}_i(s) &= \sum_{t \in G} T_i(t)^2 - \sum_{t \in G} T_i(t) \alpha_t(T_i(s^{-1}t)) \\ &= \sum_{t \in G} T_i(t)^* (T_i - (1 \otimes \delta_s) *_\alpha T_i)(t) \\ &= C(T_i)^* C(T_i - (1 \otimes \delta_s) *_\alpha T_i). \end{aligned}$$

Here for a function  $f \in \ell_1(G, \mathcal{M})$ , we have used  $C(f)$  to denote the column matrix indexed by  $G$  whose entries are  $\{f(t)\}_{t \in G}$ . Now let  $\xi \in H$  and  $\widetilde{\xi} = (\xi, 0, 0, \dots) \in \ell_2(G, H)$  viewed as a

column vector. Then

$$\begin{aligned}
\| (1 - T_i *_{\alpha} \check{T}_i(s)) \xi \| &= \| C(T_i)^* C(T_i - (1 \otimes \delta_s) *_{\alpha} T_i) \check{\xi} \| \\
&\leq \| C(T_i)^* \| \| C(T_i - (1 \otimes \delta_s) *_{\alpha} T_i) \check{\xi} \| \\
&= \| C(T_i)^* C(T_i) \|^{1/2} \langle \check{\xi}, C(T_i - (1 \otimes \delta_s) *_{\alpha} T_i)^* C(T_i - (1 \otimes \delta_s) *_{\alpha} T_i) \check{\xi} \rangle^{1/2} \\
&= \langle \xi, \langle T_i - (1 \otimes \delta_s) *_{\alpha} T_i, T_i - (1 \otimes \delta_s) *_{\alpha} T_i \rangle \xi \rangle^{1/2} \rightarrow 0,
\end{aligned}$$

whence the assertion.  $\square$

If  $G$  is amenable, so is  $\alpha$ . In this case, the net  $\{T_i\}_{i \in I}$  is given by the normalized characteristic functions of a Følner net. Like for the usual crossed products, we will show the stability of the injectivity and  $w^*$ -approximation property of  $\mathcal{M}$  by twisted crossed products relative to amenable actions. We will follow the arguments in [11, Section 4.3]. However, we will do this also at the  $L_p$ -space level for the late use concerning Schur multipliers. González-Pérez [39] obtained similar results for untwisted crossed products, in particular, he already used the following map  $\Phi_p$  as a key ingredient in his paper.

Let  $T : G \rightarrow \mathcal{Z}(\mathcal{M})$  be a finitely supported function such that  $T(s) \geq 0$  and  $\sum_{s \in G} T(s)^2 = 1$ . Let  $F$  be the support of  $T$  and  $P_F$  the orthogonal projection from  $\ell_2(G)$  onto  $\ell_2(F)$ . Let  $\mathbb{M}_F$  denote the algebra of complex matrices indexed by  $F$  and  $\mathbb{M}_F(\mathcal{M}) = \mathcal{M} \otimes \mathbb{M}_F$ . Set

$$X = \sum_{s \in F} \alpha_{s^{-1}}(T(s)) \otimes e_{s,s} \in \mathcal{N} = \mathcal{M} \overline{\otimes} B(\ell_2(G)).$$

Note that  $X$  is a positive block diagonal contraction. Let  $1 \leq p \leq \infty$  and  $p'$  be the conjugate index of  $p$ . Recall that  $\iota : \mathfrak{K} \rightarrow \mathcal{N}$  is the canonical embedding. Define a map  $\Phi_p : \mathfrak{K} \rightarrow \mathcal{N}$  by

$$\Phi_p(x) = X^{\frac{1}{p}} \iota(x) X^{\frac{1}{p}}, \quad x \in \mathfrak{K}.$$

Here for  $p = \infty$ ,  $X^{\frac{1}{p}}$  is interpreted as the support projection of  $X$  which is nothing but  $P_F$ , that is,

$$\Phi_{\infty}(x) = P_F \iota(x) P_F.$$

The image of  $\Phi_p$  is contained in  $\mathbb{M}_F(\mathcal{M})$ . By (3.1.4), for  $a \in \mathcal{M}, r \in G$

$$\Phi_p(\pi_{\alpha}(a) \lambda_{\sigma}(r)) = \sum_{s \in F} \alpha_{s^{-1}}(T(s))^{\frac{1}{p}} \alpha_{s^{-1}}(a) \sigma(s^{-1}, r) \alpha_{s^{-1}r}(T(r^{-1}s))^{\frac{1}{p}} \otimes e_{s, r^{-1}s}. \quad (3.1.5)$$

In the reverse direction, define  $\Psi_1 : \mathbb{M}_F(\mathcal{M}) \rightarrow \mathfrak{K}$  by

$$\Psi_1(a) = \sum_{s, t \in F} \pi_{\alpha} \left[ \alpha_{s^{-1}}^{-1} \left( a_{s,t} \sigma(s^{-1}, st^{-1})^* \right) \right] \lambda_{\sigma}(st^{-1}), \quad a = (a_{s,t}) \in \mathbb{M}_F(\mathcal{M}),$$

and for  $1 < p \leq \infty$  (Recalling that  $p'$  denotes the conjugate index of  $p$ )

$$\Psi_p(a) = \Psi_1 \left( X^{\frac{1}{p'}} a X^{\frac{1}{p'}} \right).$$

Thus for  $a \in \mathcal{M}, s, t \in F$

$$\Psi_p(a \otimes e_{s,t}) = \pi_{\alpha} \left[ \alpha_{s^{-1}}^{-1} \left( \alpha_{s^{-1}}(T(s))^{\frac{1}{p'}} a \sigma(s^{-1}, st^{-1})^* \alpha_{s^{-1}r}(T(r^{-1}s))^{\frac{1}{p'}} \right) \right] \lambda_{\sigma}(st^{-1}). \quad (3.1.6)$$

In the following,  $\mathfrak{K}$  is equipped with the dual trace  $\widehat{\tau}$ ,  $\mathcal{N}$  and  $\mathbb{M}_F(\mathcal{M})$  with the tensor trace  $\tau \otimes \text{Tr}$ .



**Lemma 3.1.14.** *The above defined maps  $\Phi_p$  and  $\Psi_p$  satisfy the following properties:*

- (i)  $\Phi_p$  and  $\Psi_p$  are completely positive;
- (ii)  $\Phi_p$  extends to a complete contraction from  $L_p(\mathbb{K})$  into  $L_p(\mathcal{N})$ , and  $\Psi_p$  extends to a complete contraction from  $L_p(\mathbb{M}_F(\mathcal{M}))$  into  $L_p(\mathbb{K})$ ;
- (iii) The adjoint map of  $\Phi_p$  is equal to  $\Psi_p$ .
- (iv) For any  $a \in \mathcal{M}, r \in G$

$$\Psi_p \circ \Phi_p(\pi_\alpha(a)\lambda_\sigma(r)) = \pi_\alpha(T *_\alpha \check{T}(r))\pi_\alpha(a)\lambda_\sigma(r).$$

*Proof.* (i) It is clear that  $\Phi_p$  is completely positive. The complete positivity of  $\Psi_p$  will follow from that of  $\Psi_1$ . We first show that  $\Psi_1$  is positive. This is equivalent to showing that for any  $f \in \ell_1(G, \mathcal{M})$  such that the function  $s \mapsto f(s^{-1})$  is supported by  $F$  we have

$$\Psi_1\left(\sum_{s,t \in G} f(s^{-1})^* f(t^{-1}) \otimes e_{s,t}\right) \geq 0.$$

But

$$\begin{aligned} \Psi_1\left(\sum_{s,t} f(s^{-1})^* f(t^{-1}) \otimes e_{s,t}\right) &= \sum_{s,t} \pi_\alpha\left[\alpha_{s^{-1}}^{-1}\left(f(s^{-1})^* f(t^{-1}) \sigma(s^{-1}, st^{-1})^*\right)\right] \lambda_\sigma(st^{-1}) \\ &= \sum_t \sum_s \pi_\alpha\left[\alpha_{s^{-1}}^{-1}\left(f(s^{-1})^* f(s^{-1}t) \sigma(s^{-1}, t)^*\right)\right] \lambda_\sigma(t) \\ &= \sum_t \sum_s \pi_\alpha\left[\text{Ad}_{\sigma(s, s^{-1})} \circ \alpha_s\left(f(s^{-1})^* f(s^{-1}t) \sigma(s^{-1}, t)^*\right)\right] \lambda_\sigma(t) \\ &= \sum_t \sum_s \pi_\alpha\left[\sigma(s, s^{-1})^* \alpha_s\left(f(s^{-1})^* f(s^{-1}t)\right) \sigma(s, s^{-1}t)\right] \lambda_\sigma(t) \\ &= \pi_\alpha \times \lambda_\sigma(f^\# \star f) \geq 0, \end{aligned}$$

where we have used the identity  $\alpha_s(\sigma(s^{-1}, t)^*) \sigma(s, s^{-1}) = \sigma(s, s^{-1}t)$  for the next to the last equality. This shows the positivity of  $\Psi$ . To prove the  $n$ -positivity for any  $n \in \mathbb{N}$ , we use the following commutative diagram,

$$\begin{array}{ccc} \mathbb{M}_n \otimes (\mathcal{M} \otimes \mathbb{M}_F) & \xrightarrow{\cong} & (\mathbb{M}_n \otimes \mathcal{M}) \otimes \mathbb{M}_F \\ \downarrow \text{Id} \otimes \Psi_1 & & \downarrow \Psi_{1,n} \\ \mathbb{M}_n \otimes (\mathcal{M} \rtimes_{\alpha, \sigma} G) & \xrightarrow{\cong} & (\mathbb{M}_n \otimes \mathcal{M}) \rtimes_{\text{Id} \otimes \alpha, 1 \otimes \sigma} G, \end{array}$$

where  $\Psi_{1,n}$  is defined in the same way as  $\Psi_1$  just replacing  $(\mathcal{M}, \alpha, \sigma)$  by  $(\mathbb{M}_n \otimes \mathcal{M}, \text{Id} \otimes \alpha, 1 \otimes \sigma)$ . By the part already proved,  $\Psi_{1,n}$  is positive, so  $\Psi_1$  is  $n$ -positive for any  $n$ , consequently, completely positive.

(ii) As an isomorphic embedding,  $\Phi_\infty$  is completely contractive from  $\mathbb{K}$  to  $\mathcal{N}$ ; in fact,  $\Phi_\infty$  is unital from  $\mathbb{K}$  to  $\mathbb{M}_F(\mathcal{M})$ . Let us show that  $\Phi_1$  is completely contractive from  $L_1(\mathbb{K})$  to  $L_1(\mathcal{N})$ .

To that end, let  $x \in L_1(\mathbb{K})$ . Write  $x = y^* z$  with  $y, z \in L_2(\mathbb{K})$  and  $\|y\|_2 \|z\|_2 = \|x\|_1$ . Then by the Cauchy-Schwarz inequality,

$$\|\Phi_1(x)\|_{L_1(\mathcal{N})} \leq \|\iota(y)X\|_{L_2(\mathcal{N})} \|\iota(z)X\|_{L_2(\mathcal{N})}.$$

As  $L_2(\mathbb{K}) \cong \ell_2(G, L_2(\mathcal{M}))$ , there exists  $f \in \ell_2(G, L_2(\mathcal{M}))$  such that  $y = \pi_\alpha \times \lambda_s(f)$ . Then by (3.1.4), the assumption that  $T(t) \in \mathcal{Z}(\mathcal{M})$  and  $\sigma(s, t) \in \mathcal{U}(\mathcal{M})$ , we have

$$\begin{aligned} \|\iota(y)X\|_{L_2(\mathcal{N})}^2 &= \left\| \sum_{s,t \in G} \alpha_{s^{-1}}(f(st^{-1}))\sigma(s^{-1}, st^{-1})\alpha_{t^{-1}}(T(t)) \otimes e_{s,t} \right\|_{L_2(\mathcal{N})}^2 \\ &= \sum_{s,t \in G} \left\| \alpha_{s^{-1}}(f(st^{-1}))\alpha_{t^{-1}}(T(t))\sigma(s^{-1}, st^{-1}) \right\|_{L_2(\mathcal{M})}^2 \\ &= \sum_{s,t \in G} \left\| \alpha_{s^{-1}}(f(st^{-1}))\alpha_{t^{-1}}(T(t)) \right\|_{L_2(\mathcal{M})}^2. \end{aligned}$$

By the invariance of  $\tau$  under  $\alpha$ , we get

$$\begin{aligned} \sum_{s,t \in G} \left\| \alpha_{s^{-1}}(f(st^{-1}))\alpha_{t^{-1}}(T(t)) \right\|_{L_2(\mathcal{M})}^2 &= \sum_{s,t \in G} \left\| \alpha_{t^{-1}}^{-1} \circ \alpha_{s^{-1}}(f(st^{-1}))T(t) \right\|_{L_2(\mathcal{M})}^2 \\ &= \sum_{s,t \in G} \left\| \text{Ad}_{\sigma(t,t^{-1})^*} \circ \alpha_t \circ \alpha_{s^{-1}}(f(st^{-1}))T(t) \right\|_{L_2(\mathcal{M})}^2 \\ &= \sum_{s,t \in G} \left\| \text{Ad}_{\sigma(t,s^{-1})^*} \circ \alpha_{ts^{-1}}(f(st^{-1}))T(t) \right\|_{L_2(\mathcal{M})}^2 \\ &= \sum_{s,t \in G} \left\| \alpha_{s^{-1}}(f(s))T(t) \right\|_{L_2(\mathcal{M})}^2 \\ &= \sum_{s \in G} \tau \left[ \alpha_{s^{-1}}(f(s)^* f(s)) \sum_{t \in G} T(t)^2 \right] \\ &= \sum_{s \in G} \tau[\alpha_{s^{-1}}(f(s)^* f(s))] = \|y\|_{L_2(\mathbb{K})}^2. \end{aligned}$$

Thus  $\|\iota(y)X\|_{L_2(\mathcal{N})} = \|y\|_{L_2(\mathbb{K})}$ . Similarly,  $\|\iota(z)X\|_{L_2(\mathcal{N})} = \|z\|_{L_2(\mathbb{K})}$ . Hence  $\|\Phi_1(x)\|_{L_1(\mathcal{N})} \leq \|x\|_{L_1(\mathbb{K})}$ . So  $\Phi_1$  is contractive.

The case  $1 < p < \infty$  is then treated by interpolation. Indeed, the preceding arguments, with some minor modifications, remain valid for the complex powers of  $X$ :  $X^{ib}$  (corresponding to  $p = \infty$ ) and  $X^{a+ib}$  (corresponding to  $p = 1$ ) for any  $a, b \in \mathbb{R}$ . Then by the three lines lemma, we deduce that  $\Phi_p$  is contractive from  $L_p(\mathbb{K})$  to  $L_p(\mathcal{N})$ .

Repeating the previous reasoning with  $\mathbb{M}_n \otimes \mathcal{M}$  instead of  $\mathcal{M}$  for any  $n \in \mathbb{N}$ , we prove that  $\Phi_p$  is completely contractive. As in the proof of the complete positivity of  $\Psi$  in (i), this standard passage is illustrated by the following diagram:

$$\begin{array}{ccc} \mathbb{M}_n \otimes (\mathcal{M} \rtimes_{\alpha, \sigma} G) & \xrightarrow{\cong} & (\mathbb{M}_n \otimes \mathcal{M}) \rtimes_{\text{Id} \otimes \alpha, 1 \otimes \sigma} G \\ \downarrow \text{Id} \otimes \Phi_p & & \downarrow \Phi_{p,n} \\ \mathbb{M}_n \otimes (\mathcal{M} \overline{\otimes} B(\ell_2(G))) & \xrightarrow{\cong} & (\mathbb{M}_n \otimes \mathcal{M}) \overline{\otimes} B(\ell_2(G)), \end{array}$$

where  $\Phi_{p,n}$  is defined in the same way as  $\Phi_p$  just replacing  $(\mathcal{M}, \alpha, \sigma)$  by  $(\mathbb{M}_n \otimes \mathcal{M}, \text{Id} \otimes \alpha, 1 \otimes \sigma)$ .

We pass to the part on  $\Psi_p$ . First note that  $\Psi_\infty$  is unital:

$$\Psi_\infty \left( \sum_{s \in F} 1_{\mathcal{M}} \otimes e_{s,s} \right) = \Psi \left( \sum_{s \in F} \alpha_{s^{-1}}(T(s)^2) \otimes e_{s,s} \right) = \pi_\alpha \left( \sum_{s \in F} T(s)^2 \right) = 1.$$

Thus  $\Psi_\infty$  is completely contractive since it is completely positive.

Next we show that  $\Psi_1$  is completely contractive. Like for  $\Phi_1$ , we only need to show that  $\Psi_1$  is contractive. Let  $x$  be a unit vector in  $L_1(\mathbb{M}_F(\mathcal{M}))$ . We must show that  $\|\Psi_1(x)\|_{L_1(\mathbb{R})} \leq 1$ . Since the unit ball of  $L_1(\mathbb{M}_F(\mathcal{M}))$  is the closed convex hull of all tensors  $\xi \otimes \eta$  with unit vectors  $\xi, \eta \in \ell_2(F, L_2(\mathcal{M}))$ . We can assume that  $x$  itself is such a tensor. By the density of  $\mathcal{M}$  in  $L_2(\mathcal{M})$ , we can further assume that  $\xi$  and  $\eta$  takes values in  $\mathcal{M}$ . Then the matrix form of  $x$  is  $x = (\xi(s)\eta(t^*))_{s,t \in F}$ . Let  $f(s) = \xi(s^{-1})^*$  and  $g(s) = \eta(s^{-1})^*$ . Then the proof of the positivity of  $\Psi_1$  in (i) above yields the following:

$$\Psi_1(x) = \Psi_1\left(\sum_{s,t \in G} f(s^{-1})^* g(t^{-1}) \otimes e_{s,t}\right) = \pi_\alpha \times \lambda_\sigma(f \# \star g) = [\pi_\alpha \times \lambda_\sigma(f)^*][\pi_\alpha \times \lambda_\sigma(g)].$$

Therefore,

$$\|\Phi(x)\|_{L_1(\mathbb{R})} \leq \|\pi_\alpha \times \lambda_\sigma(f)\|_{L_2(\mathbb{R})} \|\pi_\alpha \times \lambda_\sigma(g)\|_{L_2(\mathbb{R})} = \|f\|_{\ell_2(F, L_2(\mathcal{M}))} \|g\|_{\ell_2(F, L_2(\mathcal{M}))} \leq 1.$$

This is the desired assertion.

As for  $\Phi_p$ , an interpolation argument then shows that  $\Psi_p$  is completely contractive for  $1 < p < \infty$ .

(iii) Let  $a, b \in \mathcal{M}$  and  $r, s, t \in F$ . Then by (3.1.5)

$$\begin{aligned} \langle \pi_\alpha(b)\lambda_\sigma(r), \Phi_p^*(a \otimes e_{s,t}) \rangle &= \langle \Phi_p(\pi_\alpha(b)\lambda_\sigma(r)), a \otimes e_{s,t} \rangle \\ &= \delta_{r^{-1}s,t} \tau\left[\left(\alpha_{s^{-1}}(T(s))^{\frac{1}{p}} \alpha_{s^{-1}}(b) \sigma(s^{-1}, r) \alpha_{s^{-1}r}(T(r^{-1}s))^{\frac{1}{p}}\right)^* a\right]. \end{aligned}$$

By the assumption that  $T$  take values in the positive part of  $\mathcal{Z}(\mathcal{M})$  and the  $\alpha$ -invariance of  $\tau$ , we have

$$\begin{aligned} &\tau\left[\left(\alpha_{s^{-1}}(T(s))^{\frac{1}{p}} \alpha_{s^{-1}}(b) \sigma(s^{-1}, r) \alpha_{s^{-1}r}(T(r^{-1}s))^{\frac{1}{p}}\right)^* a\right] \\ &= \tau\left[\alpha_{s^{-1}}(b^*) \left(\alpha_{s^{-1}}(T(s))^{\frac{1}{p}} a \sigma(s^{-1}, r)^* \alpha_{s^{-1}r}(T(r^{-1}s))^{\frac{1}{p}}\right)\right] \\ &= \tau\left[b^* \alpha_{s^{-1}}^{-1} \left(\alpha_{s^{-1}}(T(s))^{\frac{1}{p}} a \sigma(s^{-1}, r)^* \alpha_{s^{-1}r}(T(r^{-1}s))^{\frac{1}{p}}\right)\right]. \end{aligned}$$

Then by (3.1.6), we deduce that

$$\langle \pi_\alpha(b)\lambda_\sigma(r), \Phi_p^*(a \otimes e_{s,t}) \rangle = \langle \pi_\alpha(b)\lambda_\sigma(r), \Psi_p'(a \otimes e_{s,t}) \rangle.$$

Hence  $\Phi_p^* = \Psi_p'$ .

(iv) For any  $a \in \mathcal{M}, r \in G$ . Using the assumption that  $T(s) \in \mathcal{Z}(\mathcal{M})$ , we have

$$\begin{aligned} &\Psi_p \circ \Phi_p(\pi_\alpha(a)\lambda_\sigma(r)) \\ &= \Psi_1\left(\sum_{s \in G} \alpha_{s^{-1}}(T(s)) \alpha_{s^{-1}}(a) \sigma(s^{-1}, r) \alpha_{s^{-1}r}(T(r^{-1}s)) \otimes e_{s, r^{-1}s}\right) \\ &= \Psi_1\left(\sum_{s \in G} \alpha_{s^{-1}}(T(s)) \alpha_{s^{-1}}(a) \alpha_{s^{-1}r}(T(r^{-1}s)) \sigma(s^{-1}, r) \otimes e_{s, r^{-1}s}\right) \\ &= \sum_{s \in G} \pi_\alpha\left(\alpha_{s^{-1}}^{-1} \left(\alpha_{s^{-1}}(T(s)) \alpha_{s^{-1}}(a) \alpha_{s^{-1}r}(T(r^{-1}s)) \sigma(s^{-1}, r) \sigma(s^{-1}, r)^*\right)\right) \lambda_\sigma(r) \\ &= \sum_{s \in G} \pi_\alpha\left(T(s) \alpha_r(T(r^{-1}s))\right) \pi_\alpha(a) \lambda_\sigma(r) \\ &= \pi_\alpha(T *_\alpha \check{T}(r)) \pi_\alpha(a) \lambda_\sigma(r). \end{aligned}$$

Thus the proof of the lemma is complete.  $\square$

The following elementary lemma is well known. We include its proof for completeness.

**Lemma 3.1.15.** *Let  $\{a_i\}$  be a bounded net in  $\mathcal{M}$  converging strongly to 0. Then for any  $p < \infty$  and  $x \in L_p(\mathcal{M})$ , the net  $\{a_i x\}$  converges to 0 in  $L_p(\mathcal{M})$ .*

*Proof.* By the density of  $\mathcal{M}$  in  $L_p(\mathcal{M})$ , it suffices to consider  $x \in \mathcal{M}$ . The case  $p = 2$  is just the assumption (recalling that  $\mathcal{M}$  acts standardly on  $L_2(\mathcal{M})$ ). The case  $p < 2$  is an easy consequence of the case  $p = 2$  since  $\|a_i x\|_p \leq \|a_i x\|_2$ . The remaining case  $2 < p < \infty$  is dealt with by the following Hölder inequality

$$\|a_i x\|_p \leq \|a_i x\|_2^{\frac{2}{p}} \|a_i x\|_\infty^{1-\frac{2}{p}}.$$

□

Now assume that  $\alpha$  is amenable. Let  $\{T_i\}_{i \in I}$  be an approximate net as in Definition 3.1.12, and let  $\Phi_{p,i}$  and  $\Psi_{p,i}$  be the maps defined before Lemma 3.1.13 with  $T_i$  instead of  $T$ .

**Lemma 3.1.16.** *Let  $1 \leq p \leq \infty$ . Then  $\{\Psi_{p,i} \circ \Phi_{p,i}\}_{i \in I}$  converges to the identity of  $L_p(\mathfrak{K})$  in the point strong-topology ( $w^*$ -topology for  $p = \infty$ ).*

*Proof.* We know that  $\text{span}\{\pi_\alpha(a)\lambda_\sigma(r) : a \in \mathcal{M}, r \in G\}$  is dense in  $L_p(\mathfrak{K})$  for  $p < \infty$  and  $w^*$ -dense for  $p = \infty$ . Thus for  $p < \infty$ , we need only to show that for any  $x \in \text{span}\{\pi_\alpha(a)\lambda_\sigma(r) : a \in \mathcal{M}, r \in G\}$

$$\lim_i \|x - \Psi_{p,i} \circ \Phi_{p,i}(x)\|_{L_p(\mathfrak{K})} = 0.$$

By linearity, we can further assume that  $x = \pi_\alpha(a)\lambda_\sigma(r)$ . Then by Lemma 3.1.14(iii), Lemma 3.1.13 and Lemma 3.1.15, we get

$$\|x - \Psi_{p,i} \circ \Phi_{p,i}(x)\|_{L_p(\mathfrak{K})} = \|\pi_\alpha(1 - T_i *_\alpha \check{T}_i(r))\pi_\alpha(a)\lambda_\sigma(r)\|_{L_p(\mathfrak{K})} \rightarrow 0.$$

Thus the assertion is proved for  $p < \infty$ . The case  $p = \infty$  is an immediate consequence of the case  $p = 1$  for  $\Phi_{\infty,i}$  and  $\Psi_{\infty,i}$  are normal by Lemma 3.1.14 (iii). Alternately, like for the case  $p < \infty$ , one easily shows that  $\Psi_{\infty,i} \circ \Phi_{\infty,i}(x)$  strongly converges to  $x$  for any  $x \in \mathfrak{K}$ . □

An operator space  $E$  is said to have the *completely bounded approximation property* (CBAP) with constant  $\Lambda$  if the identity of  $E$  is the limit in the point strong-topology of a net of finite rank  $\Lambda$ -completely bounded maps. Similarly, if  $E$  is a dual space, it has the  $w^*$ -CBAP if the identity of  $E$  is the limit in the point  $w^*$ -topology of a net of finite rank  $\Lambda$ -completely bounded  $w^*$ -continuous maps. This is equivalent to saying that the predual  $E_*$  of  $E$  has the CBAP with constant  $\Lambda$ .

On the other hand, a von Neumann algebra  $\mathcal{M}$  is called *semidiscrete* if there exist normal completely positive contractions  $u_j : \mathcal{M} \rightarrow \mathbb{M}_{n_j}$  and  $v_j : \mathbb{M}_{n_j} \rightarrow \mathcal{M}$  such that  $v_j \circ u_j$  converges to the identity of  $\mathcal{M}$  in the point  $w^*$ -topology. It is well known that  $\mathcal{M}$  is semidiscrete iff it is injective (see [16, 18, 33]).

The following result insures the stability of the above approximation properties under twisted crossed products.

**Theorem 3.1.17.** *Assume that  $\alpha$  is amenable. Then*

- (i)  $\mathfrak{K}$  is semidiscrete iff  $\mathcal{M}$  is semidiscrete;

(ii)  $\mathfrak{K}$  has the  $w^*$ -CBAP iff  $\mathcal{M}$  has the  $w^*$ -CBAP.

*Proof.* Since  $\mathcal{M}$  is the image of a normal conditional expectation of  $\mathfrak{K}$  (see Remark 3.1.11), both “only if” parts are clear.

To show the converses, let  $\{T_i\}$  be the approximate identity in Definition 3.1.12. Let  $\Phi_{p,i}$  and  $\Psi_{p,i}$  be the maps given before Lemma 3.1.16. Let  $F_i$  be the support of  $T_i$ . By Lemma 3.1.14 and Lemma 3.1.16 (and the proof of the latter),  $\Phi_{\infty,i} : \mathfrak{K} \rightarrow \mathbb{M}_{F_i}(\mathcal{M})$  and  $\Psi_{\infty,i} : \mathbb{M}_{F_i}(\mathcal{M}) \rightarrow \mathfrak{K}$  are normal completely positive contractions and  $\Psi_{\infty,i} \circ \Phi_{\infty,i}$  converges to the identity of  $\mathfrak{K}$  in the point  $w^*$ -topology.

Now assume that  $\mathcal{M}$  is semidiscrete. Choose normal completely positive contractions  $u_j : \mathcal{M} \rightarrow \mathbb{M}_{n_j}$  and  $v_j : \mathbb{M}_{n_j} \rightarrow \mathcal{M}$  such that  $v_j \circ u_j$  converges to the identity of  $\mathcal{M}$  in the point  $w^*$ -topology. Then the composition maps

$$(\text{Id}_{\mathbb{M}_{F_i}} \otimes u_j) \circ \Phi_{\infty,i} : \mathfrak{K} \rightarrow \mathbb{M}_{F_i} \otimes \mathbb{M}_{n_j} \quad \text{and} \quad \Psi_{\infty,i} \circ (\text{Id}_{\mathbb{M}_{F_i}} \otimes v_j) : \mathbb{M}_{F_i} \otimes \mathbb{M}_{n_j} \rightarrow \mathfrak{K}$$

are normal completely positive contractions. It is easy to check that

$$\left[ \Psi_{\infty,i} \circ (\text{Id}_{\mathbb{M}_{F_i}} \otimes v_j) \right] \circ \left[ (\text{Id}_{\mathbb{M}_{F_i}} \otimes u_j) \circ \Phi_{\infty,i} \right] \rightarrow \text{Id}_{\mathfrak{K}} \quad \text{in the point } w^* \text{-topology.}$$

Therefore,  $\mathfrak{K}$  is semidiscrete. The part on  $w^*$ -CBAP is proved in the same way.  $\square$

**Remark 3.1.18.** The above proof is easily adapted to show that under the amenability of  $\alpha$  and given  $1 \leq p < \infty$ ,  $L_p(\mathfrak{K})$  has the CBAP iff  $L_p(\mathcal{M})$  has the CBAP.

**Remark 3.1.19.** As pointed out in the Introduction, twisted crossed products have been extensively investigated in the category of  $C^*$ -algebras. Let  $(\mathcal{A}, \alpha, \sigma, G)$  be a twisted dynamic  $C^*$ -system with  $\mathcal{A}$  a  $C^*$ -algebra. Let  $\mathcal{A} \rtimes_{\alpha, \sigma} G$  and  $\mathcal{A} \rtimes_{\alpha, \sigma, r} G$  be the associated full and reduced twisted crossed products, respectively (see [70, 71]). Assume that  $\alpha$  is amenable in the  $C^*$ -algebraic sense, this means that the convergence in Definition 3.1.12 now takes place in the norm topology of  $\mathcal{A}$ . Then the proof of the previous theorem can be modified to show the following:

- (i)  $\mathcal{A} \rtimes_{\alpha, \sigma} G = \mathcal{A} \rtimes_{\alpha, \sigma, r} G$ ;
- (ii)  $\mathcal{A} \rtimes_{\alpha, \sigma, r} G$  is nuclear iff  $\mathcal{A}$  is nuclear;
- (iii)  $\mathcal{A} \rtimes_{\alpha, \sigma, r} G$  is exact iff  $\mathcal{A}$  is exact.

We are also interested in the stability of QWEP under twisted crossed products. However, it is an open problem whether this property holds even for untwisted crossed products. At the time of this writing, we are even unable to show that the QWEP property of  $\mathcal{L}_\sigma(G)$  is independent of  $\sigma$ . We only have the following partial answer. Recall that the group von Neumann algebra  $\mathcal{L}(G)$  is QWEP iff  $G$  is hyperlinear (cf. [74, 84]).

**Proposition 3.1.20.** *Let  $\sigma : G \rightarrow \mathbb{T}$  be a 2-cocycle. If  $\mathcal{L}_\sigma(G)$  is QWEP, then  $G$  is hyperlinear.*

*Proof.* It is easy to see that  $\mathcal{L}_{\bar{\sigma}}(G)$  is naturally isomorphic to  $\overline{\mathcal{L}_\sigma(G)}$ , so  $\mathcal{L}_{\bar{\sigma}}(G)$  is QWEP too. Consequently,  $\mathcal{L}_{\bar{\sigma}}(G) \bar{\otimes} \mathcal{L}_\sigma(G)$  is QWEP. Corollary 3.1.8 implies that  $\mathcal{L}(G)$  is isomorphic to the von Neumann subalgebra of  $\mathcal{L}_{\bar{\sigma}}(G) \bar{\otimes} \mathcal{L}_\sigma(G)$  generated by  $\{\lambda_{\bar{\sigma}}(s) \otimes \lambda_\sigma(s) : s \in G\}$ ; it is clear that the latter subalgebra is the image of a conditional expectation. Hence,  $\mathcal{L}(G)$  is QWEP, so  $G$  is hyperlinear.  $\square$

The following open problem seems very interesting. By Theorem 3.1.25 below, it is equivalent to a similar one in the untwisted case.

**Problem 3.1.21.** Let  $\sigma : G \rightarrow \mathbb{T}$  be a 2-cocycle. Assume that  $G$  is hyperlinear. Is  $\mathcal{L}_\sigma(G)$  QWEP? More generally, let  $\mathcal{M}$  be QWEP and  $G$  hyperlinear. Is  $\mathcal{M} \rtimes_{\alpha, \sigma} G$  QWEP?

By Theorem 3.1.25 below, the first part of the above problem reduces to the following

**Problem 3.1.22.** Let  $K$  be a Hilbert space and  $\beta : G \rightarrow \text{Aut}(B(K))$  an action. Assume that  $G$  is hyperlinear. Is  $B(K) \rtimes_\beta G$  QWEP?

It is well known that every automorphism of  $B(K)$  is inner. The difficulty here is whether the action  $\beta$  can be implemented by a unitary representation  $u$  of  $G$  on  $K$  such that  $\beta_s = \text{Ad}_{u(s)}$  for every  $s \in G$ . If the answer for the latter was affirmative, then one would have  $B(K) \rtimes_\beta G \cong B(K) \overline{\otimes} \mathcal{L}(G)$ , so the above problem would have a positive solution.

### 3.1.4 Equivalence

The main result of this subsection reduces twisted crossed products to untwisted ones.

**Definition 3.1.23.** Two twisted actions  $(\alpha, \sigma)$  and  $(\beta, \nu)$  of  $G$  on  $\mathcal{M}$  are said to be *exterior equivalent* if there exists a function  $u : G \rightarrow \mathcal{U}(\mathcal{M})$  such that for all  $s, t \in G$

$$\beta_s = \text{Ad}_{u(s)} \circ \alpha_s \quad \text{and} \quad \nu(s, t) = u(s)\alpha_s(u(t))\sigma(s, t)u(st)^*.$$

It is clear that this is indeed an equivalence relation. The following two results are the von Neumann counterparts of the corresponding results for  $C^*$ -algebras in [70, section 3].

**Proposition 3.1.24.** *Let  $(\alpha, \sigma)$  and  $(\beta, \nu)$  be two exterior equivalent twisted actions of  $G$  on  $\mathcal{M}$ . Then  $\mathcal{M} \rtimes_{\alpha, \sigma} G$  and  $\mathcal{M} \rtimes_{\beta, \nu} G$  are isomorphic. More precisely, there exists a unitary operator  $U$  on  $\ell_2(G, H)$  such that*

$$U\pi_\alpha(a)U^* = \pi_\beta(a) \quad \text{and} \quad U\lambda_\sigma(s)U^* = \pi_\beta(u(s)^*)\lambda_\nu(s), \quad a \in \mathcal{M}, s \in G.$$

*Proof.* Define the required unitary  $U$  by  $(U\xi)(s) = u(s^{-1})\xi(s)$  for  $\xi \in \ell_2(G, H), s \in G$ . Then the equality  $U\pi_\alpha(a)U^* = \pi_\beta(a)$  is just a reformulation of  $\beta_s = \text{Ad}_{u(s)} \circ \alpha_s$ . Let us show the second one. For  $\xi \in \ell_2(G, H)$  and  $s, t \in G$ , we have

$$\begin{aligned} (U\lambda_\sigma(s)U^*\xi)(t) &= u(t^{-1})\sigma(t^{-1}, s)u(t^{-1}s)^*\xi(s^{-1}t) \\ &= \left[ u(t^{-1})\alpha_{t^{-1}}(u(s)^*)u(t^{-1})^* \right] \left[ u(t^{-1})\alpha_{t^{-1}}(u(s)^*)\sigma(t^{-1}, s)u(t^{-1}s)^* \right] \xi(s^{-1}t) \\ &= \beta_{t^{-1}}(u(s)^*)\nu(t^{-1}, s)\xi(s^{-1}t). \end{aligned}$$

Thus  $U\lambda_\sigma(s)U^* = \pi_\beta(u(s)^*)\lambda_\nu(s)$ . It then follows that  $U\mathcal{M} \rtimes_{\alpha, \sigma} GU^* = \mathcal{M} \rtimes_{\beta, \nu} G$ .  $\square$

The next result reduces a twisted crossed product to a untwisted one by amplification. Let  $(A, G, \alpha, \sigma)$  be a twisted dynamic system. Recall that  $\lambda(s)$  is the left translation on  $\ell_2(G)$  and we define

$$u(s) = \left( 1_{\mathcal{M}} \otimes \lambda(s) \right) \sigma(s, \cdot)^*, \quad s \in G.$$

Namely,  $(u(s)\xi)(t) = \sigma(s, s^{-1}t)^*\xi(s^{-1}t)$  for  $\xi \in \ell_2(G, H)$  and  $t \in G$ . Clearly,  $u(s) \in \mathcal{U}(\mathcal{M} \overline{\otimes} B(\ell_2(G)))$ . Next, define

$$\beta_s = \text{Ad}_{u(s)} \circ (\alpha_s \otimes \text{Id}_{B(\ell_2(G))}), \quad s \in G.$$

We claim that  $(\alpha \otimes \text{Id}_{B(\ell_2(G))}, \sigma \otimes 1_{B(\ell_2(G))})$  is exterior equivalent to  $(\beta, 1)$ . To that end, it suffices to show that

$$u(s) \left( \alpha_s \otimes \text{Id}_{B(\ell_2(G))} (u(t)) \right) \left( \sigma(s, t) \otimes 1_{B(\ell_2(G))} \right) u(st)^* = 1, \quad s, t \in G.$$

Let  $T$  be the operator on the left hand side of the above identity.  $T$  is a product of four operators. Given  $\xi \in \ell_2(G, H)$  and  $r \in G$ , we explain the successive actions of the four factors of  $T$  on  $\xi$  by the following diagram:

$$\begin{aligned} (T\xi)(r) &\mapsto \sigma(st, r)\xi(str) \\ &\mapsto \sigma(s, t)\sigma(st, r)\xi(str) \\ &\mapsto \alpha_s(\sigma(t, t^{-1}r)^*)\sigma(s, t)\sigma(st, t^{-1}r)\xi(sr) \\ &\mapsto \sigma(s, s^{-1}r)^*\alpha_s(\sigma(t, t^{-1}s^{-1}r)^*)\sigma(s, t)\sigma(st, t^{-1}s^{-1}r)\xi(r). \end{aligned}$$

The operator in the last term is equal to

$$\begin{aligned} &[\alpha_s(\sigma(t, t^{-1}s^{-1}r))\sigma(s, s^{-1}r)]^*\sigma(s, t)\sigma(st, t^{-1}s^{-1}r) \\ &= [\sigma(s, t)\sigma(st, t^{-1}s^{-1}r)]^*\sigma(s, t)\sigma(st, t^{-1}s^{-1}r) = 1. \end{aligned}$$

Therefore,  $T\xi = \xi$ , i.e.,  $T = 1$  as desired. Thus the claim is proved.

Combined the above claim and Proposition 3.1.24, we deduce the following

**Theorem 3.1.25.** *Keeping the above notation, we have*

$$(\mathcal{M} \rtimes_{\alpha, \sigma} G) \overline{\otimes} B(\ell_2(G)) \cong (\mathcal{M} \overline{\otimes} B(\ell_2(G))) \rtimes_{\beta} G.$$

*Proof.* By Proposition 3.1.24, we get

$$(\mathcal{M} \overline{\otimes} B(\ell_2(G))) \rtimes_{\alpha \otimes \text{Id}_{B(\ell_2(G))}, \sigma \otimes 1_{B(\ell_2(G))}} G \cong (\mathcal{M} \overline{\otimes} B(\ell_2(G))) \rtimes_{\beta} G.$$

However, we have canonically

$$(\mathcal{M} \overline{\otimes} B(\ell_2(G))) \rtimes_{\alpha \otimes \text{Id}_{B(\ell_2(G))}, \sigma \otimes 1_{B(\ell_2(G))}} G = (\mathcal{M} \rtimes_{\alpha, \sigma} G) \overline{\otimes} B(\ell_2(G)).$$

Thus the theorem follows.  $\square$

**Remark 3.1.26.** Theorem 3.1.25, together with its  $C^*$ -algebra version in [70], reduces the stability of approximation properties under twisted crossed products in Theorem 3.1.17 to that under usual crossed products.

## 3.2 Fourier multipliers

We now consider Fourier multipliers on twisted crossed products, in particular, on twisted group von Neumann algebras. We will always assume that  $1 \leq p \leq \infty$ .  $p'$  is the conjugate index of  $p$ . Recall that  $\mathfrak{K} = \mathcal{M} \rtimes_{\alpha, \sigma} G$ .

**Definition 3.2.1.** Given a function  $\varphi \in \ell_\infty(G)$ , define  $T_\varphi$  on  $\pi_\alpha \times \lambda_\sigma(\ell_1(G, \mathcal{M}))$  by

$$T_\varphi(\pi_\alpha \times \lambda_\sigma(f)) = \pi_\alpha \times \lambda_\sigma(\varphi f).$$

We call  $\varphi$  a (completely) bounded Fourier multiplier on  $L_p(\mathfrak{K})$  if  $T_\varphi$  extends to a (completely) bounded map on  $L_p(\mathfrak{K})$ . The Banach space of bounded (resp. cb) Fourier multipliers on  $L_p(\mathfrak{K})$  is denoted by  $M(L_p(\mathfrak{K}))$  (resp.  $M_{cb}(L_p(\mathfrak{K}))$ ).

If  $\mathcal{M} = \mathbb{C}$ , the multipliers above are just those on the twisted group von Neumann algebra  $\mathcal{L}_\sigma(G)$ . If further  $\sigma$  is trivial, they are multipliers on the usual group von Neumann algebra  $\mathcal{L}(G)$ . It is trivial that  $M(L_2(\mathbb{K})) = M_{\text{cb}}(L_2(\mathbb{K})) = \ell_\infty(G)$ . It is equally clear that  $\varphi$  is a (completely) bounded  $L_p$ -Fourier multiplier iff  $\varphi$  is a (completely) bounded  $L_{p'}$ -Fourier multiplier. Thus it suffices to consider the case  $2 < p \leq \infty$ . By interpolation, we easily show that (completely) bounded  $L_q$ -Fourier multipliers are also (completely) bounded  $L_p$ -Fourier multipliers for  $2 \leq p < q \leq \infty$ .

Motivated by the Fourier multiplier results on quantum tori in [15], we aim to reduce twisted cb multipliers to those in the untwisted and scalar setting, i.e., those on  $\mathcal{L}(G)$  independent of the von Neumann algebra  $\mathcal{M}$  and the 2-cocycle  $\sigma$ .

**Theorem 3.2.2.** *We have*

$$M_{\text{cb}}(\mathcal{L}(G)) \subset M_{\text{cb}}(\mathbb{K}) \text{ contractively.}$$

*If additionally  $\mathbb{K}$  is QWEP, then for  $2 < p < \infty$*

$$M_{\text{cb}}(L_p(\mathcal{L}(G))) \subset M_{\text{cb}}(L_p(\mathbb{K})) \text{ contractively.}$$

*Proof.* We will apply Corollary 3.1.7 to  $\rho = \pi_\alpha, u = \lambda_\sigma$  and  $K = \ell_2(G, H)$ . We find a unitary  $U$  on  $\ell_2(G, K)$  such that

$$U\rho_\alpha(a)U^* = \rho(a) \otimes 1 \quad \text{and} \quad U\lambda_{\sigma,\rho}(s)U^* = u(s) \otimes \lambda(s), \quad a \in \mathcal{M}, s \in G.$$

Let  $\mathcal{A}$  be the von Neumann subalgebra of  $\mathbb{K} \overline{\otimes} \mathcal{L}(G)$  generated by  $\{\rho(a) \otimes 1, u(s) \otimes \lambda(s), a \in \mathcal{M}, s \in G\}$ . Then  $\mathcal{A}$  is isomorphic to  $\{\rho_\alpha(\mathcal{M}), \lambda_{\sigma,\rho}(G)\}''$  by a trace preserving isomorphism. Since  $\rho$  is faithful, by Remark 3.1.5,  $\mathbb{K} \cong \{\rho_\alpha(\mathcal{M}), \lambda_{\sigma,\rho}(G)\}''$  also via a trace preserving isomorphism. Thus we deduce that there exists a trace preserving isometric isomorphism  $\Phi : \mathbb{K} \rightarrow \mathcal{A}$  such that

$$\Phi(\pi_\alpha(a)) = \rho(a) \otimes 1 \quad \text{and} \quad \Phi(\lambda_\sigma(s)) = u(s) \otimes \lambda(s), \quad a \in \mathcal{M}, s \in G.$$

This isomorphism intertwines multipliers on  $\mathbb{K}$  and those on  $\mathcal{L}(G)$ . Namely, denoting  $T_\varphi$  the Fourier multiplier of symbol  $\varphi$  on  $\mathbb{K}$  and  $\widetilde{T}_\varphi$  the one on  $\mathcal{L}(G)$ , we then have

$$\Phi \circ T_\varphi = (\text{Id}_{\mathbb{K}} \otimes \widetilde{T}_\varphi) \circ \Phi.$$

More precisely, for  $f \in \ell_1(G, \mathcal{M})$  we have

$$\Phi \left[ \sum_{s \in G} \varphi(s) \pi_\alpha(f(s)) \lambda_\sigma(s) \right] = \sum_{s \in G} \varphi(s) \rho(f(s)) u(s) \otimes \lambda(s).$$

If  $\varphi$  is a cb multiplier on  $\mathcal{L}(G)$  with cb-norm less than 1, then  $\widetilde{T}_\varphi$  is a cb contraction on  $\mathcal{L}(G)$ , so  $\text{Id}_{\mathbb{K}} \otimes \widetilde{T}_\varphi$  is contractive on  $\mathbb{K} \overline{\otimes} \mathcal{L}(G)$ . Consequently,

$$\begin{aligned} \left\| \sum_{s \in G} \varphi(s) \pi_\alpha(f(s)) \lambda_\sigma(s) \right\|_{\mathbb{K}} &= \left\| \Phi \left[ \sum_{s \in G} \varphi(s) \pi_\alpha(f(s)) \lambda_\sigma(s) \right] \right\|_{\mathcal{A}} \\ &= \left\| \sum_{s \in G} \varphi(s) \rho(f(s)) u(s) \otimes \lambda(s) \right\|_{\mathcal{A}} \\ &\leq \left\| \sum_{s \in G} \rho(f(s)) u(s) \otimes \lambda(s) \right\|_{\mathcal{A}} \\ &= \left\| \sum_{s \in G} \pi_\alpha(f(s)) \lambda_\sigma(s) \right\|_{\mathbb{K}}. \end{aligned}$$



This shows that  $T_\varphi$  is a contraction on  $\mathfrak{K}$ . Tensoring  $\mathfrak{K}$  with  $\mathbb{M}_n$  for any  $n \in \mathbb{N}$ , we then deduce the first assertion on  $p = \infty$ .

The second part is proved in a similar way. But now we require the QWEP of  $\mathfrak{K}$  to ensure that  $\text{Id}_{\mathfrak{K}} \otimes \widetilde{T}_\varphi$  is bounded on  $L_p(\mathfrak{K} \overline{\otimes} \mathcal{L}(G))$  whenever  $\widetilde{T}_\varphi$  is cb on  $L_p(\mathcal{L}(G))$ . On the other hand, since the isomorphism  $\Phi : \mathfrak{K} \rightarrow \mathcal{N}$  is trace preserving, it extends to an isometry from  $L_p(\mathfrak{K})$  onto  $L_p(\mathcal{A})$ . The rest of the argument is the same as above, so we omit the details.  $\square$

If  $\mathcal{M} = \mathbb{C}$ , the converse to the above theorem holds too.

**Theorem 3.2.3.** *We have*

$$\text{M}_{\text{cb}}(\mathcal{L}_\sigma(G)) = \text{M}_{\text{cb}}(\mathcal{L}(G)) \text{ isometrically.}$$

*If additionally  $\mathcal{L}_\sigma(G)$  is QWEP, then for  $2 < p < \infty$*

$$\text{M}_{\text{cb}}(L_p(\mathcal{L}_\sigma(G))) = \text{M}_{\text{cb}}(L_p(\mathcal{L}(G))) \text{ isometrically.}$$

Thus the cb Fourier multipliers on  $L_p(\mathcal{L}_\sigma(G))$  are independent of the 2-cocycle  $\sigma$  (under the mild condition that  $\mathcal{L}_\sigma(G)$  is QWEP). Note that this is not true for bounded Fourier multipliers (see [85]). In fact, based on Fell's absorption principle, our proof is inspired of [85].

*Proof of Theorem 3.2.3.* This proof is similar to that of the previous theorem, so only an outline is given. Now we use Corollary 3.1.8 and consider the case  $2 < p < \infty$ . By that corollary, the von Neumann subalgebra  $\mathcal{A}$  of  $\mathcal{L}_{\overline{\sigma}}(G) \overline{\otimes} \mathcal{L}(G)$  generated by  $\lambda_{\overline{\sigma}} \otimes \lambda_\sigma$  is isomorphic to  $\mathcal{L}(G)$  by a trace preserving isomorphism, so  $L_p(\mathcal{A}) \cong L_p(\mathcal{L}(G))$ . Now given  $\varphi \in \ell_\infty(G)$  let  $T_\varphi$  and  $\widetilde{T}_\varphi$  denote the associated Fourier multipliers on  $\mathcal{L}(G)$  and  $\mathcal{L}_\sigma(G)$ , respectively:

$$T_\varphi(\lambda(s)) = \varphi(s)\lambda(s) \quad \text{and} \quad \widetilde{T}_\varphi(\lambda_s(s)) = \varphi(s)\lambda_s(s), \quad s \in G.$$

Assume that  $\widetilde{T}_\varphi$  is cb on  $L_p(\mathcal{L}_\sigma(G))$ . Since  $\mathcal{L}_\sigma(G)$  is QWEP,  $\mathcal{L}_{\overline{\sigma}}(G)$  is QWEP too (see the proof of Proposition 3.1.20). Thus  $\text{Id} \otimes \widetilde{T}_\varphi$  is bounded on  $L_p(\mathcal{L}_{\overline{\sigma}}(G) \overline{\otimes} \mathcal{L}(G))$ , so is its restriction to  $L_p(\mathcal{A})$ . Then as in the proof of the previous theorem, we deduce that  $T_\varphi$  is bounded on  $L_p(\mathcal{L}(G))$ .  $\square$

### 3.3 Schur multipliers

In this section we link Fourier multipliers and Schur multipliers. Given  $\phi : G \times G \rightarrow \mathbb{C}$ , the associated Schur multiplier  $M_\phi$  is formally defined by  $M_\phi(a) = [\phi(s, t)a(s, t)]_{s, t \in G}$  for any finite matrix  $a$  indexed by  $G$ . We call  $\phi$  a (completely) bounded Schur multiplier on  $S_p(\ell_2(G))$  if  $M_\phi$  extends to a (completely) bounded map on  $S_p(\ell_2(G))$ . The resulting spaces of multipliers are denoted by  $\text{S}(S_p(\ell_2(G)))$  and  $\text{S}_{\text{cb}}(S_p(\ell_2(G)))$ , respectively. This definition also extends to the operator-valued setting. Namely, we define in the same way (completely) bounded Schur multipliers on  $L_p(\mathcal{N})$  (recalling that  $\mathcal{N} = \mathcal{M} \overline{\otimes} B(\ell_2(G))$ ), so we have the corresponding spaces  $\text{S}(L_p(\mathcal{N}))$  and  $\text{S}_{\text{cb}}(L_p(\mathcal{N}))$ . Like for Fourier multipliers, we only need to consider the case  $2 < p \leq \infty$ .

It is well known that  $\text{S}_{\text{cb}}(B(\ell_2(G))) = \text{S}(B(\ell_2(G)))$  isometrically. This follows from Grothendieck's characterization of Schur multipliers (see [77, Theorem 5.1]):  $\phi \in \text{S}(B(\ell_2(G)))$  iff there exist a Hilbert space  $K$  and  $\xi, \eta \in \ell_\infty(G, K)$  such that

$$\phi(s, t) = \langle \xi(s), \eta(t) \rangle, \quad s, t \in G.$$

Moreover, in this case,

$$\|\phi\|_{\mathcal{S}_{\text{cb}}(B(\ell_2(G)))} = \|\phi\|_{\mathcal{S}(B(\ell_2(G)))} = \inf\{\|\xi\|_{\ell_\infty(G)}\|\eta\|_{\ell_\infty(G)}\},$$

where the infimum runs over all representations of  $\phi$  as above. Consequently,

$$\mathcal{S}_{\text{cb}}(\mathcal{N}) = \mathcal{S}(\mathcal{N}) = \mathcal{S}(B(\ell_2(G))) \quad \text{isometrically.}$$

It is obvious that for  $2 < p < \infty$

$$\mathcal{S}_{\text{cb}}(L_p(\mathcal{N})) \subset \mathcal{S}_{\text{cb}}(S_p(\ell_2(G))) \quad \text{contractively.}$$

If additionally  $\mathcal{M}$  is QWEP, the converse conclusion holds too and moreover

$$\mathcal{S}_{\text{cb}}(L_p(\mathcal{N})) = \mathcal{S}_{\text{cb}}(S_p(\ell_2(G))) \quad \text{isometrically.}$$

However, it is a well known open problem of Pisier whether bounded Schur multipliers on  $S_p$  are automatically cb for  $2 < p < \infty$ .

Fourier and Schur multipliers are closely related, as shown by Bożejko and Fendler [9]:  $T_\varphi$  is cb on  $\mathcal{L}(G)$  iff  $M_{\tilde{\varphi}}$  is cb on  $B(\ell_2(G))$ , where  $\tilde{\varphi}(s, t) = \varphi(st^{-1})$ . Such a phenomenon continues to hold for  $1 < p < \infty$  under the amenability of  $G$ : Neuwirth and Ricard [69] proved that for amenable  $G$  one has  $\|T_\varphi\|_{\mathcal{M}_{\text{cb}}(L_p(\mathcal{L}(G)))} = \|M_{\tilde{\varphi}}\|_{\mathcal{S}_{\text{cb}}(S_p(\ell_2(G)))}$ .

We will study all this for  $\mathfrak{K} = \mathcal{M} \rtimes_{\alpha, \sigma} G$ . Identifying  $\mathfrak{K}$  and  $\iota(\mathfrak{K}) \subset \mathcal{N}$ , we see that the restriction of  $M_{\tilde{\varphi}}$  to  $\mathfrak{K}$  coincides with the Fourier multipliers  $T_\varphi$ . Hence,  $T_\varphi$  is (completely) bounded on  $\mathfrak{K}$  iff  $M_{\tilde{\varphi}}|_{\mathfrak{K}}$  is (completely) bounded. In particular, if  $\tilde{\varphi} \in \mathcal{S}(B(\ell_2(G)))$ , then  $\varphi \in \mathcal{M}_{\text{cb}}(\mathfrak{K})$  and  $\|\varphi\|_{\mathcal{M}_{\text{cb}}(\mathfrak{K})} \leq \|\tilde{\varphi}\|_{\mathcal{S}(B(\ell_2(G)))}$ . The first part of the following shows that the converse is true.

**Theorem 3.3.1.** *Let  $\varphi \in \ell_\infty(G)$ .*

- (i)  *$\varphi$  is a cb Fourier multiplier on  $\mathfrak{K}$  iff  $\tilde{\varphi}$  is a (cb) Schur multiplier on  $B(\ell_2(G))$ . In this case, the two associated cb-norms are equal.*
- (ii) *If  $\varphi$  is a cb Fourier multiplier on  $L_p(\mathfrak{K})$  with  $2 < p < \infty$ , then  $\tilde{\varphi}$  is a cb Schur multiplier on  $L_p(\mathcal{N})$  with  $\|\tilde{\varphi}\|_{\mathcal{S}_{\text{cb}}(L_p(\mathcal{N}))} \leq \|\varphi\|_{\mathcal{M}_{\text{cb}}(L_p(\mathfrak{K}))}$ .*

*Proof.* We will prove the missing direction of (i) and (ii) at the same time. The elements of  $L_p(\mathcal{N})$  are viewed as infinite matrices with entries in  $L_p(\mathcal{M})$ . Let  $a = \sum_{s,t \in G} a_{s,t} \otimes e_{s,t} \in L_p(\mathcal{N})$  be such a matrix with only finitely many non-vanishing entries. By the density of  $\mathcal{M}$  in  $L_p(\mathcal{M})$ , we can assume that  $a_{s,t} \in \mathcal{M}$ . On the other hand, using the faithfulness of  $\pi_\alpha$ , we also write  $a = \sum_{s,t \in G} \pi_\alpha(a_{s,t}) \otimes e_{s,t}$ . Let

$$D_1 = \sum_{s \in G} \lambda_\sigma(s) \otimes e_{s,s} \quad \text{and} \quad D_2 = \sum_{t \in G} \lambda_\sigma(t^{-1}) \otimes e_{t,t}.$$

These are block diagonal unitaries in  $\mathfrak{K} \overline{\otimes} B(\ell_2(G))$ . Now consider the element

$$\tilde{a} = D_1 a D_2 \in L_p(\mathfrak{K} \overline{\otimes} B(\ell_2)).$$

Clearly,

$$\|\tilde{a}\|_{L_p(\mathfrak{K} \overline{\otimes} B(\ell_2))} = \|a\|_{L_p(\mathcal{N})}.$$

On the other hand, since  $(\pi_a, \lambda_\sigma)$  is a covariant representation of  $(A, G, \alpha, \sigma)$ , we get

$$\begin{aligned}
\tilde{a} &= \sum_{s,t \in G} \lambda_\sigma(s) \pi_\alpha(a_{s,t}) \lambda_\sigma(t^{-1}) \otimes e_{s,t} \\
&= \sum_{s,t \in G} \pi_\alpha(\alpha_s(a_{s,t})) \lambda_\sigma(s) \lambda_\sigma(t^{-1}) \otimes e_{s,t} \\
&= \sum_{s,t \in G} \pi_\alpha(\alpha_s(a_{s,t})) \lambda_\sigma(st^{-1}) \pi_\alpha(\sigma(s, t^{-1})) \otimes e_{s,t} \\
&= \sum_{s,t \in G} \pi_\alpha[\alpha_s(a_{s,t}) \alpha_{st^{-1}}(\sigma(s, t^{-1}))] \lambda_\sigma(st^{-1}) \otimes e_{s,t}.
\end{aligned}$$

Hence

$$\begin{aligned}
T_\varphi \otimes \text{Id}_{B(\ell_2(G))}(\tilde{a}) &= \sum_{s,t \in G} \varphi(st^{-1}) \pi_\alpha[\alpha_s(a_{s,t}) \alpha_{st^{-1}}(\sigma(s, t^{-1}))] \lambda_\sigma(st^{-1}) \otimes e_{s,t} \\
&= D_1 M_{\tilde{\varphi}}(a) D_2.
\end{aligned}$$

Since  $T_\varphi$  is cb on  $L_p(\mathbb{K})$ , we then obtain

$$\begin{aligned}
\|M_{\tilde{\varphi}}(a)\|_{L_p(\mathcal{N})} &= \|D_1 M_{\tilde{\varphi}}(a) D_2\|_{L_p(\mathbb{K} \otimes B(\ell_2))} \\
&= \|T_\varphi \otimes \text{Id}_{B(\ell_2(G))}(\tilde{a})\|_{L_p(\mathbb{K} \otimes B(\ell_2))} \\
&\leq \|\varphi\|_{\text{M}_{\text{cb}}(L_p(\mathbb{K}))} \|\tilde{a}\|_{L_p(\mathbb{K} \otimes B(\ell_2))} \\
&= \|\varphi\|_{\text{M}_{\text{cb}}(L_p(\mathbb{K}))} \|a\|_{L_p(\mathcal{N})}.
\end{aligned}$$

Therefore,  $M_{\tilde{\varphi}}$  is bounded on  $L_p(\mathcal{N})$ . Applying the above argument to  $\mathbb{M}_n \otimes \mathcal{N}$  instead of  $\mathcal{N}$ , we deduce that  $M_{\tilde{\varphi}}$  is cb on  $L_p(\mathcal{N})$  with cb-norm less than or equal to  $\|\varphi\|_{\text{M}_{\text{cb}}(L_p(\mathbb{K}))}$ . The assertion is thus proved.  $\square$

The “only if” part of Theorem 3.3.1 (i) admits an alternate standard proof that we outline as follows (see [77, Chapter 6]).

Suppose that  $T_\varphi$  is cb on  $\mathbb{K}$ . By Wittstock’s factorization theorem, there exist a Hilbert space  $K$ , a  $*$ -representation  $\omega : \mathbb{K} \rightarrow B(K)$  and two bounded operators  $V_1, V_2 : H \otimes \ell_2(G) \rightarrow K$  such that

$$T_\varphi(x) = V_2^* \omega(x) V_1, \quad x \in \mathbb{K}.$$

Choosing an unit vector  $\xi \in H$ , for  $s, t \in G$  we have

$$\begin{aligned}
1 &= \langle \xi \otimes \delta_e, \xi \otimes \delta_e \rangle \\
&= \langle \lambda_\sigma(t^{-1})(\xi \otimes \delta_t), \lambda_\sigma(s^{-1})(\xi \otimes \delta_s) \rangle \\
&= \langle \lambda_\sigma(s^{-1})^* \lambda_\sigma(t^{-1})(\xi \otimes \delta_t), \xi \otimes \delta_s \rangle \\
&= \langle \pi_\alpha(\sigma(s, s^{-1})^*) \lambda_\sigma(s) \lambda_\sigma(t^{-1})(\xi \otimes \delta_t), \xi \otimes \delta_s \rangle \\
&= \langle \pi_\alpha(\sigma(s, s^{-1})^* \sigma(s, t^{-1})) \lambda_\sigma(st^{-1})(\xi \otimes \delta_t), \xi \otimes \delta_s \rangle.
\end{aligned}$$

Then

$$\begin{aligned}
\varphi(st^{-1}) &= \left\langle \varphi(st^{-1})\pi_\alpha \times \lambda_\sigma \left( \left( \sigma(s, s^{-1})^* \sigma(s, t^{-1}) \right) \otimes \delta_{st^{-1}} \right) (\xi \otimes \delta_t, \xi \otimes \delta_s) \right\rangle \\
&= \left\langle T_\varphi \left( \lambda_\sigma(s^{-1})^* \lambda_\sigma(t) \right) (\xi \otimes \delta_t, \xi \otimes \delta_s) \right\rangle \\
&= \left\langle V_2^* \omega(\lambda_\sigma(s^{-1})^* \lambda_\sigma(t^{-1})) V_1 (\xi \otimes \delta_t, \xi \otimes \delta_s) \right\rangle \\
&= \left\langle \omega(\lambda_\sigma(t^{-1})) V_1 (\xi \otimes \delta_t), \omega(\lambda_\sigma(s^{-1})) V_2 (\xi \otimes \delta_s) \right\rangle.
\end{aligned}$$

Therefore, by Grothendieck's theorem quoted previously,  $\tilde{\varphi}$  is a cb Schur multiplier on  $B(\ell_2(G))$ .

Next we aim to show the converse to Theorem 3.3.1(ii) under the amenability of the action  $\alpha$ .

**Theorem 3.3.2.** *Assume that  $\alpha$  is amenable. Let  $2 < p < \infty$  and  $\varphi \in \ell_\infty(G)$ .*

- (i) *If  $\tilde{\varphi}$  is a bounded (resp. cb) Schur multiplier on  $L_p(\mathcal{N})$ , then  $\varphi$  is a bounded (resp. cb) Fourier multiplier on  $L_p(\mathbb{R})$  with  $\|\varphi\|_{M(L_p(\mathbb{R}))} \leq \|\tilde{\varphi}\|_{S(L_p(\mathcal{N}))}$  (resp.  $\|\varphi\|_{M_{cb}(L_p(\mathbb{R}))} \leq \|\tilde{\varphi}\|_{S_{cb}(L_p(\mathcal{N}))}$ ).*
- (ii) *Assume additionally that  $\mathcal{M}$  is QWEP. If  $\tilde{\varphi}$  is a cb Schur multiplier on  $S_p(\ell_2(G))$ , then  $\varphi$  is a cb Fourier multiplier on  $L_p(\mathbb{R})$  with  $\|\varphi\|_{M_{cb}(L_p(\mathbb{R}))} \leq \|\tilde{\varphi}\|_{S_{cb}(S_p(\ell_2(G)))}$ .*

*Proof.* Let  $\Phi_{p,i}$  and  $\Psi_{p,i}$  be the maps in Lemma 3.1.16. Let  $x = \pi_\alpha \times \lambda_\sigma(f)$  with  $f \in \ell_1(G, \mathcal{M})$ . It is easy to check that

$$\Phi_{p,i}(T_\varphi(x)) = M_{\tilde{\varphi}}(\Phi_{p,i}(x)).$$

Assume that  $\tilde{\varphi}$  is a bounded Schur multiplier on  $L_p(\mathcal{N})$ . Then by Lemma 3.1.16 and Lemma 3.1.14, we get

$$\begin{aligned}
\|T_\varphi(x)\|_{L_p(\mathbb{R})} &= \lim_i \|\Psi_{p,i} \circ \Phi_{p,i}(T_\varphi(x))\|_{L_p(\mathbb{R})} \\
&\leq \lim_i \|\Phi_{p,i}(T_\varphi(x))\|_{L_p(\mathcal{N})} \\
&\leq \limsup_i \|M_{\tilde{\varphi}} \circ \Phi_{p,i}(x)\|_{L_p(\mathcal{N})} \\
&\leq \|\tilde{\varphi}\|_{S(L_p(\mathcal{N}))} \limsup_i \|\Phi_{p,i}(x)\|_{L_p(\mathcal{N})} \\
&\leq \|\tilde{\varphi}\|_{S(L_p(\mathcal{N}))} \|x\|_{L_p(\mathbb{R})}.
\end{aligned}$$

Hence,  $\varphi$  is a bounded Fourier multiplier on  $L_p(\mathbb{R})$  and  $\|\varphi\|_{M(L_p(\mathbb{R}))} \leq \|\tilde{\varphi}\|_{S(L_p(\mathcal{N}))}$ . Similarly, we show the cb version.

We know that if  $\mathcal{M}$  is QWEP, then

$$S_{cb}(S_p(\ell_2(G))) = S_{cb}(L_p(\mathcal{N})) \text{ isometrically.}$$

Thus the last assertion follows. □

Combining Theorem 3.3.1 and Theorem 3.3.2, we obtain the following

**Corollary 3.3.3.** *Assume that the action  $\alpha$  is amenable and  $\mathcal{M}$  is QWEP. Let  $\varphi \in \ell_\infty(G)$ . Then  $\varphi$  is a cb Fourier multiplier on  $L_p(\mathbb{R})$  iff  $\tilde{\varphi}$  is a cb Schur multiplier on  $S_p(\ell_2(G))$ ; in this case, the two associated cb norms are equal.*

We end this section with a result showing that there exists no nontrivial 2-cocycle on a free group  $\mathbb{F}_d$  on  $d$  generators ( $d$  may be  $\infty$ ).

**Proposition 3.3.4.** *Every scalar-valued 2-cocycle  $\sigma$  on  $\mathbb{F}_d$  is exterior equivalent to the constant function 1.*

*Proof.* Let  $\{g_1, \dots, g_d\}$  be a set of generators of  $\mathbb{F}_d$ . Using the construction of  $\mathcal{L}_\sigma(\mathbb{F}_d)$ , one easily shows that  $\lambda_\sigma(g_1), \dots, \lambda_\sigma(g_d)$  are free Haar unitaries. By this we mean that the spectral measure of each  $\lambda_\sigma(g_i)$  is normalized Haar measure on  $\mathbb{T}$  and  $\lambda_\sigma(g_1), \dots, \lambda_\sigma(g_d)$  are free in the following sense:

$$\tau(x_{i_1} \cdots x_{i_k}) = 0$$

whenever  $x_{i_1} \in \mathcal{A}_{i_1}, \dots, x_{i_k} \in \mathcal{A}_{i_k}$  with vanishing trace and  $i_1 \neq i_2 \cdots \neq i_k$ , where  $\mathcal{A}_i$  is the von Neumann subalgebra generated by  $\lambda_\sigma(g_i)$ .

It then follows that the two algebras  $\mathcal{L}(\mathbb{F}_d)$  and  $\mathcal{L}_\sigma(\mathbb{F}_d)$  are isomorphic by an isomorphism  $\rho$  mapping  $\lambda(g_i)$  to  $\lambda_\sigma(g_i)$  for  $1 \leq i \leq d$ . Writing an element  $g \in \mathbb{F}_d$  in reduced word  $g = g_{i_1}^{n_1} \cdots g_{i_k}^{n_k}$  with  $i_1 \neq i_2 \cdots \neq i_k$  and  $n_1, \dots, n_k \in \mathbb{Z} \setminus \{0\}$ , we deduce that  $\rho(\lambda(g)) = \varphi(g)\lambda_\sigma(g)$  for some function  $\varphi : \mathbb{F}_d \rightarrow \mathbb{T}$ . Consequently,  $\varphi$  is a completely contractive Fourier multiplier from  $\mathcal{L}(\mathbb{F}_d)$  to  $\mathcal{L}_\sigma(\mathbb{F}_d)$ . By the discussion before Theorem 3.3.1 and the matricial form (3.1.4) of elements in  $\mathcal{L}_\sigma(\mathbb{F}_d)$ , we see that  $(\varphi(st^{-1})\sigma(s^{-1}, st^{-1}))_{s,t \in \mathbb{F}_d}$  defines a completely contractive Schur multiplier on the subalgebra  $\mathcal{L}(\mathbb{F}_d)$  of  $B(\ell_2(\mathbb{F}_d))$ . Then by the alternate proof of the “only if” part of Theorem 3.3.1 (i), we find a Hilbert space  $K$  and two functions  $\xi, \eta : \mathbb{F}_d \rightarrow K$  such that  $\|\xi(s)\|, \|\eta(t)\| \leq 1$  and

$$\varphi(st^{-1})\sigma(s^{-1}, st^{-1}) = \langle \xi(s), \eta(t) \rangle, \quad s, t \in \mathbb{F}_d.$$

Taking  $s = t = e$  in the above equation, we get  $\langle \xi(e), \eta(e) \rangle = \phi(e) = 1$ . Hence by the equality case in the Cauchy inequality, we see that  $\xi(e) = \eta(e)$ . Next, take  $t = e$  in the above equation, we have

$$\varphi(s)\sigma(s^{-1}, s) = \langle \xi(s), \eta(e) \rangle = \langle \xi(s), \xi(e) \rangle,$$

which forces  $\xi(s) = \overline{\varphi(s)\sigma(s^{-1}, s)}\xi(e)$ .

Similarly, we have  $\eta(t) = \overline{\varphi(t^{-1})\xi(e)}$ . Combining these two expressions we deduce

$$\begin{aligned} \varphi(st^{-1})\sigma(s^{-1}, st^{-1}) &= \langle \xi(s), \eta(t) \rangle \\ &= \langle \varphi(s)\sigma(s^{-1}, s)\xi(e), \overline{\varphi(t^{-1})\xi(e)} \rangle \\ &= \varphi(s)\sigma(s^{-1}, s)\varphi(t^{-1}). \end{aligned}$$

However,  $\sigma(s^{-1}, st^{-1}) = \sigma(s^{-1}, s)\overline{\sigma(s, t^{-1})}$ . Changing  $t^{-1}$  to  $t$ , we finally obtain

$$\sigma(s, t) = \overline{\varphi(s)\varphi(t)}\varphi(st).$$

This means that  $\sigma$  is exterior equivalent to 1. □

## Chapter 4

# Schatten properties of commutators of singular integral operators on noncommutative euclidean space

### 4.1 Quantum euclidean spaces and Fourier multipliers

The original motivation for noncommutative euclidean spaces begins with the canonical commutation relations of quantum mechanics. There are various equivalent definitions for noncommutative euclidean spaces. In line with our definition on twisted crossed product in Chapter 1, we define an euclidean space to be the twisted group von Neumann algebra of  $\mathbb{R}^d$  associated with a 2-cocycle. Throughout the remainder of this chapter,  $\theta$  will be a fixed antisymmetric  $d \times d$  matrix with  $d \geq 2$ ; together with  $\theta$ , we define

$$\sigma(s, t) = \exp\left(\frac{i}{2}\langle s, \theta t \rangle\right), \quad s, t \in \mathbb{R}^d. \quad (4.1.1)$$

Then  $\sigma : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{T}$  is a 2-cocycle in the following sense

$$\sigma(s, t)\sigma(s+t, u) = \sigma(t, u)\sigma(s, t+u) \quad \text{and} \quad \sigma(0, s) = \sigma(s, 0) = 1, \quad s, t, u \in \mathbb{R}^d.$$

Note that by the antisymmetry of  $\theta$ ,  $\sigma$  further satisfies

$$\sigma(s, s) = \sigma(-s, s) = 1, \quad s \in \mathbb{R}^d. \quad (4.1.2)$$

#### 4.1.1 Basic defintions

Define the following family of unitary operators on  $L_2(\mathbb{R}^d)$ :

$$\left(\lambda_\theta(s)\xi\right)(t) = \sigma(-t, s)\xi(t-s), \quad \xi \in L_2(\mathbb{R}^d), \quad s, t \in \mathbb{R}^d. \quad (4.1.3)$$

It is routine to verify that the family  $\{\lambda_\theta(s)\}_{s \in \mathbb{R}^d}$  is strongly continuous, and satisfies

$$\lambda_\theta(s)\lambda_\theta(t) = \sigma(s, t)\lambda_\theta(s+t), \quad s, t \in \mathbb{R}^d. \quad (4.1.4)$$

The above relation is called the Weyl form of the canonical commutation relations, and its representation theory is summarised by the well-known Stone-von Neumann theorem: with the condition that  $\det(\theta) \neq 0$ , any two  $C^*$ -algebras generated by a strongly continuous unitary family  $\{\lambda_\theta(s)\}_{s \in \mathbb{R}^d}$  satisfying (4.1.4) are  $*$ -isomorphic. For more details, see [10, Section 5.2.2.2] and [46, Theorem 14.8], [96, Chapter 2, Theorem 3.1].

**Definition 4.1.1.** The *noncommutative euclidean space* associated to  $\theta$ , denoted by  $\mathbb{R}_\theta^d$ , is the von Neumann subalgebra of  $B(L_2(\mathbb{R}^d))$  generated by  $\{\lambda_\theta(s)\}_{s \in \mathbb{R}^d}$  given in (4.1.3).

Taking  $\theta = 0$ , the above definition states that  $\mathbb{R}_0^d$  is the von Neumann algebra generated by the unitary group of translations on  $\mathbb{R}^d$ , and is  $*$ -isomorphic to  $L_\infty(\mathbb{R}^d)$ . Therefore, the algebra of essentially bounded functions on euclidean space is recovered as a special case of Definition 4.1.1.

We refer the reader to [38, 56] for more details on the above approach of defining noncommutative euclidean spaces. Alternative but unitarily equivalent way can be found in the literature, see, for instance, [10, 96].

We also caution the reader that the approach taken here is the ‘‘Fourier dual’’ of the approach in [43]. In the commutative case,  $\lambda_\theta(s)$  is the operator on  $L_2(\mathbb{R}^d)$  of translation by  $s \in \mathbb{R}^d$ , and the Fourier transform provides an isomorphism with the algebra  $L_\infty(\mathbb{R}^d)$  of essentially bounded functions.

With the above definition, one easily views  $\mathbb{R}_\theta^d$  as the twisted group von Neumann algebra  $\mathcal{L}_\sigma(\mathbb{R}^d)$ , where the 2-cocycle  $\sigma$  is given by (4.1.1). Conversely, for any 2-cocycle  $\sigma$  on  $\mathbb{R}^d$ , the twisted group von Neumann algebra  $\mathcal{L}_\sigma(\mathbb{R}^d)$  is a noncommutative euclidean space.

Let us turn to the integration theory on  $\mathbb{R}^d$ . Let  $f \in L_1(\mathbb{R}^d)$ . Then  $t \mapsto f(t)\lambda_\theta(t)$  is a bounded strongly continuous function from  $\mathbb{R}^d$  to  $B(L_2(\mathbb{R}^d))$ , so the integral

$$\lambda_\theta(f) = \int_{\mathbb{R}^d} f(t)\lambda_\theta(t)dt$$

defines a bounded operator on  $L_2(\mathbb{R}^d)$ , more precisely, for any  $\xi \in L_2(\mathbb{R}^d)$  one has

$$\lambda_\theta(f)\xi = \int_{\mathbb{R}^d} f(t)\lambda_\theta(t)\xi dt.$$

Approximating  $f$  by step functions, one easily shows that  $\lambda_\theta(f)$  belongs to  $\mathbb{R}_\theta^d$ . More generally, for any bounded Borel measure  $\mu$  on  $\mathbb{R}^d$ , the following integral

$$\lambda_\theta(\mu) = \int_{\mathbb{R}^d} \lambda_\theta(t)d\mu(t)$$

defines an operator in  $\mathbb{R}_\theta^d$  too. This time, it is even easier to see that  $\lambda_\theta(\mu) \in \mathbb{R}_\theta^d$  by approximating  $\mu$  by linear combinations of Dirac measures in the  $w^*$ -topology.

Given  $f, g \in L_1(\mathbb{R}^d)$ . Then one has

$$\lambda_\theta(f)\lambda_\theta(g) = \lambda_\theta(f *_\theta g) \quad \text{and} \quad \lambda_\theta(f)^* = \lambda_\theta(f^\sharp),$$

where for  $s \in \mathbb{R}^d$

$$f *_\theta g(s) = \int_{\mathbb{R}^d} \sigma(s-t, t)f(s-t)g(t)dt = \int_{\mathbb{R}^d} \sigma(s, t)f(s-t)g(t)dt,$$

$$f^\sharp(s) = \overline{\sigma(s, -s)f(-s)} = \overline{f(-s)}.$$

Here we have used (4.1.2). Equipped with the above convolution and involution,  $L_1(\mathbb{R}^d)$  becomes an involutive Banach algebra. Then  $\mathbb{R}_\theta^d$  is the  $w^*$ -closure of  $\lambda_\theta(L_1(\mathbb{R}^d))$ . Here  $L_1(\mathbb{R}^d)$  can be replaced by any reasonable smaller space of continuous functions, for instance, the space  $C_c(\mathbb{R}^d)$  of compactly supported continuous functions on  $\mathbb{R}^d$ .

In the sequel, we will sometimes use  $\lambda_\theta(f)$  for some non integrable functions  $f$ ; for instance,  $\lambda_\theta(f)$  may be a bounded operator for  $f \in L_2(\mathbb{R}^d)$ ; whenever it is bounded,  $\lambda_\theta(f)$  belongs to  $\mathbb{R}_\theta^d$ .

Let  $f \in L_1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ . Define

$$\tau_\theta(\lambda_\theta(f)) = f(0).$$

By [62, Lemma 2.7], the functional  $\tau_\theta : L_1(\mathbb{R}^d) \cap C(\mathbb{R}^d) \rightarrow \mathbb{C}$  admits an extension to a semifinite normal faithful trace on  $\mathbb{R}_\theta^d$ . The traciality of  $\tau_\theta$  is easy to be checked. Indeed, let  $f \in L_2(\mathbb{R}^d)$  such that  $\lambda_\theta(f)$  is bounded and  $f^\sharp *_\theta f$  is continuous. Then

$$\tau_\theta(\lambda_\theta(f)^* \lambda_\theta(f)) = \int_{\mathbb{R}^d} |f(s)|^2 ds = \tau_\theta(\lambda_\theta(f) \lambda_\theta(f)^*). \quad (4.1.5)$$

More generally, let  $f, g \in L_2(\mathbb{R}^d)$  such that  $\lambda_\theta(f), \lambda_\theta(g)$  are bounded and  $f^\sharp *_\theta g$  is continuous. Then

$$\tau_\theta(\lambda_\theta(f)^* \lambda_\theta(g)) = \int_{\mathbb{R}^d} \overline{f(s)} g(s) ds. \quad (4.1.6)$$

For  $1 \leq p < \infty$ ,  $L_p(\mathbb{R}_\theta^d)$  is the noncommutative  $L_p$ -spaces associated to  $(\mathbb{R}_\theta^d, \tau_\theta)$ ; more precisely,  $L_p(\mathbb{R}_\theta^d)$  is defined as the completion of

$$\{x \in \mathbb{R}_\theta^d : \tau_\theta(|x|^p) < \infty\}$$

with the norm  $\|x\|_p = \tau_\theta(|x|^p)^{\frac{1}{p}}$ . As usual, we let  $L_\infty(\mathbb{R}_\theta^d) = \mathbb{R}_\theta^d$  with the operator norm.

**Remark 4.1.2.** By the Plancherel formula (4.1.5), the map  $f \mapsto \lambda_\theta(f)$  establishes an isometry from  $L_2(\mathbb{R}^d)$  onto  $L_2(\mathbb{R}_\theta^d)$ . Later, we will identify  $L_2(\mathbb{R}^d)$  and  $L_2(\mathbb{R}_\theta^d)$ .

### 4.1.2 Distributions and derivations

The class of Schwartz functions on  $\mathbb{R}_\theta^d$  is defined as the image of the usual Schwartz class  $\mathcal{S}(\mathbb{R}^d)$  under  $\lambda_\theta$ . That is,

$$\mathcal{S}(\mathbb{R}_\theta^d) = \{\lambda_\theta(f) : f \in \mathcal{S}(\mathbb{R}^d)\}. \quad (4.1.7)$$

The Schwartz space  $\mathcal{S}(\mathbb{R}_\theta^d)$  is equipped with the topology induced by the isomorphism  $\lambda_\theta : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}_\theta^d)$ , where  $\mathcal{S}(\mathbb{R}^d)$  is equipped with its usual Fréchet topology. On the other hand, equipped with the convolution and involution of  $L_1(\mathbb{R}^d)$ ,  $\mathcal{S}(\mathbb{R}^d)$  becomes a topological involutive algebra, so  $\mathcal{S}(\mathbb{R}_\theta^d)$  is a  $*$ -subalgebra of  $\mathbb{R}_\theta^d$ .

**Lemma 4.1.3.** *Let  $1 \leq p \leq \infty$ . Then  $\mathcal{S}(\mathbb{R}_\theta^d)$  is a norm dense subspace of  $L_p(\mathbb{R}_\theta^d)$  for  $p < \infty$  and a  $w^*$ -dense subspace of  $L_\infty(\mathbb{R}_\theta^d)$ .*

*Proof.* The assertion is clear for  $p = \infty$ . The case  $p = 2$  is equally clear by virtue of (4.1.5). Now let  $x \in \mathcal{S}(\mathbb{R}_\theta^d)$ . We need to show that  $x \in L_1(\mathbb{R}_\theta^d)$ . If  $x = yz$  with  $y, z \in \mathcal{S}(\mathbb{R}_\theta^d)$ , then the Cauchy-Schwarz inequality implies that  $x \in L_1(\mathbb{R}_\theta^d)$ . However, by [62, Proposition 2.5], all finite sums of products  $yz$  with  $y, z \in \mathcal{S}(\mathbb{R}_\theta^d)$  is a dense  $*$ -subalgebra of  $\mathcal{S}(\mathbb{R}_\theta^d)$ . It then follows that  $\mathcal{S}(\mathbb{R}_\theta^d) \subset L_1(\mathbb{R}_\theta^d)$  too. Thus  $\mathcal{S}(\mathbb{R}_\theta^d) \subset L_1(\mathbb{R}_\theta^d) \cap L_\infty(\mathbb{R}_\theta^d)$  continuously.

To show the density of  $\mathcal{S}(\mathbb{R}_\theta^d)$  in  $L_p(\mathbb{R}_\theta^d)$  for  $p < \infty$ , it suffices to show that for any  $a \in \mathbb{R}_\theta^d$  supported by a projection of finite trace, there exists a net  $(a_i)$  in  $\mathcal{S}(\mathbb{R}_\theta^d)$  such that  $a_i \rightarrow a$  in



$L_p(\mathbb{R}_\theta^d)$ . Since  $\mathcal{S}(\mathbb{R}_\theta^d)$  is a  $w^*$ -dense  $*$ -subalgebra of  $\mathbb{R}_\theta^d$ , by Kaplansky's theorem, there exists a bounded net  $(a_i)$  in  $\mathcal{S}(\mathbb{R}_\theta^d)$  strongly converging to  $a$ . Then by Lemma 2.15 of Chapter 1 (or more precisely, its proof),  $\|a_i - a\|_p \rightarrow 0$ .  $\square$

**Definition 4.1.4.** The space of *tempered distributions* on  $\mathbb{R}_\theta^d$  is the topological dual space  $\mathcal{S}'(\mathbb{R}_\theta^d)$  of  $\mathcal{S}(\mathbb{R}_\theta^d)$ , i.e., the space of continuous linear functionals on  $\mathcal{S}(\mathbb{R}_\theta^d)$ .

As in the classical case, we have the following elementary fact.

**Lemma 4.1.5.** *Every element of  $L_1(\mathbb{R}_\theta^d) + L_\infty(\mathbb{R}_\theta^d)$  defines a distribution on  $\mathbb{R}_\theta^d$ .*

*Proof.* Let  $x = x_1 + x_2$  with  $x_1 \in L_1(\mathbb{R}_\theta^d)$  and  $x_2 \in L_\infty(\mathbb{R}_\theta^d)$ . Let  $y \in \mathcal{S}(\mathbb{R}_\theta^d)$ . Then by Lemma (4.1.3),  $y \in L_1(\mathbb{R}_\theta^d) \cap L_\infty(\mathbb{R}_\theta^d)$ . Thus  $\tau_\theta(xy) = \tau_\theta(x_1y) + \tau_\theta(x_2y)$  is well defined. Since the inclusion  $\mathcal{S}(\mathbb{R}_\theta^d) \subset L_1(\mathbb{R}_\theta^d) \cap L_\infty(\mathbb{R}_\theta^d)$  is continuous, the map  $y \mapsto \tau_\theta(xy)$  defines a continuous linear functional on  $\mathcal{S}(\mathbb{R}_\theta^d)$ . This shows the assertion.  $\square$

We next pass to the definition of derivation on  $\mathcal{S}(\mathbb{R}_\theta^d)$ . For  $x = \lambda_\theta(f) \in \mathcal{S}(\mathbb{R}_\theta^d)$ , we define

$$\partial_k x = \int_{\mathbb{R}^d} s_k f(s) \lambda_\theta(s) ds.$$

More generally, for  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ , we set

$$\partial^\alpha x = \int_{\mathbb{R}^d} s^\alpha f(s) \lambda_\theta(s) ds,$$

where  $s^\alpha = s_1^{\alpha_1} \dots s_d^{\alpha_d}$ . We kindly remind the reader that, the above definition of partial derivatives coincides with that in [62], given as the commutator of  $x$  with pointwise multiplication operator acting on  $L_2(\mathbb{R}^d)$ .

By (4.1.6), we see that  $\partial_j$  is a self-adjoint operator on  $L_2(\mathbb{R}_\theta^d)$  with  $\mathcal{S}(\mathbb{R}_\theta^d)$  as definition domain. Let  $\Delta = \partial_1^2 + \dots + \partial_d^2$  be the Laplacian, it is a positive operator on  $L_2(\mathbb{R}_\theta^d)$ . We will frequently use the Bessel and Riesz operators  $(1 + \Delta)^{\frac{1}{2}}$  and  $\Delta^{\frac{1}{2}}$  which will be abbreviated as  $J$  and  $I$  respectively. More generally, for  $a \in \mathbb{R}$ , define  $J^a = (1 + \Delta)^{\frac{a}{2}}$  and  $I^a = \Delta^{\frac{a}{2}}$ .

It is clear that  $\partial^\alpha x$  belongs to  $\mathcal{S}(\mathbb{R}_\theta^d)$  too. Consequently, by duality, these partial derivations extend to all distributions. In particular,  $\partial^\alpha x$  exists as a distribution for any  $x \in L_1(\mathbb{R}_\theta^d) + L_\infty(\mathbb{R}_\theta^d)$ .

**Remark 4.1.6.** The Bessel potential  $J^a$  operates on  $\mathcal{S}'(\mathbb{R}_\theta^d)$ . A little bit more attention should be paid to the case of the Riesz potential  $I^a$ . Let

$$\mathcal{S}_0(\mathbb{R}^d) = \{x : \widehat{\partial^\alpha x}(0) = 0 \quad \forall \alpha \in \mathbb{N}_0^d\}.$$

Then  $I^a$  operates on  $\mathcal{S}_0(\mathbb{R}_\theta^d) = \lambda_\theta(\mathcal{S}_0(\mathbb{R}^d))$ , and by duality, on the dual space  $\mathcal{S}'_0(\mathbb{R}_\theta^d)$  too.

### 4.1.3 Fourier multipliers

Let  $\phi : \mathbb{R}^d \rightarrow \mathbb{C}$  be a measurable function such that  $\phi f \in L_1(\mathbb{R}^d)$  for any  $f \in \mathcal{S}(\mathbb{R}^d)$ . For any  $x = \lambda_\theta(f)$  with  $f \in \mathcal{S}(\mathbb{R}^d)$ , we define the Fourier multiplier  $T_\phi$  of symbol  $\phi$  as follows:

$$T_\phi(x) = \lambda_\theta(\phi f). \tag{4.1.8}$$

It  $T_\phi$  extends to a (completely) bounded maps on  $L_p(\mathbb{R}_\theta^d)$ , we say that  $\phi$  is a (completely) bounded Fourier multiplier on  $L_p(\mathbb{R}_\theta^d)$ . In the sequel,  $\phi$  will often be continuous on  $\mathbb{R}^d \setminus \{0\}$ .

It is often useful to express  $T_\phi$  as the convolution operator with the inverse Fourier transform  $\check{\phi}$  of  $\phi$ . For this we first introduce an action of  $\mathbb{R}^d$  on  $\mathbb{R}_\theta^d$ . For any  $s \in \mathbb{R}^d$  define

$$w_s(t) = e^{2\pi i \langle s, t \rangle}, \quad t \in \mathbb{R}^d.$$

We view  $w_s$  as a unitary operator on  $L_2(\mathbb{R}^d)$  by multiplication. Then define  $\widehat{\alpha}_s = \text{Ad}_{w_s}$ , i.e.,  $\widehat{\alpha}_s(x) = w_s x w_s^*$  for any  $x \in B(L_2(\mathbb{R}^d))$ . It is easy to check that

$$\widehat{\alpha}_s(\lambda_\theta(t)) = w_s(t) \lambda_\theta(t), \quad s, t \in \mathbb{R}^d.$$

This implies that  $\widehat{\alpha}$  leaves  $\mathbb{R}_\theta^d$  invariant, so it yields an action of  $\mathbb{R}^d$  on  $\mathbb{R}_\theta^d$ . More generally, given  $x = \lambda_\theta(f)$  with  $f \in L_1(\mathbb{R}^d)$ , we have

$$\widehat{\alpha}_s(\lambda_\theta(f)) = \lambda_\theta(w_s f).$$

This shows that  $\widehat{\alpha}$  is trace preserving, so  $\widehat{\alpha}$  extends to an isometric action of  $\mathbb{R}^d$  on  $L_p(\mathbb{R}_\theta^d)$ .

We denote  $\check{\phi}$  as the inverse Fourier transform of  $\phi$ . Now assume that  $\check{\phi} \in L_1(\mathbb{R}^d)$ . Then for  $x = \lambda_\theta(f)$  with  $f \in \mathcal{S}(\mathbb{R}^d)$ , we have

$$\begin{aligned} T_\phi(x) &= \int_{\mathbb{R}^d} \phi(s) f(s) \lambda_\theta(s) ds \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \check{\phi}(t) e^{-2\pi i \langle s, t \rangle} f(s) \lambda_\theta(s) ds dt \\ &= \int_{\mathbb{R}^d} \check{\phi}(t) \widehat{\alpha}_{-t}(x) dt. \end{aligned}$$

This suggests to define

$$\check{\phi} * x = \int_{\mathbb{R}^d} \check{\phi}(t) \widehat{\alpha}_{-t}(x) dt. \quad (4.1.9)$$

This alternate definition has an advantage:  $\check{\phi} * x$  can be defined for any  $x \in L_p(\mathbb{R}_\theta^d)$ . Since  $\widehat{\alpha}_{-t}$  is an isometry on  $L_p(\mathbb{R}_\theta^d)$ , we immediately get the following:

**Lemma 4.1.7.** *Let  $\phi$  be a function on  $\mathbb{R}^d$  such that  $\check{\phi} \in L_1(\mathbb{R}^d)$ . Then  $T_\phi$  is completely bounded on  $L_p(\mathbb{R}_\theta^d)$  for  $1 \leq p \leq \infty$  with cb-norm majorized by  $\|\check{\phi}\|_1$ .*

We need to extend Fourier multipliers or convolution in (4.1.9) to distributions. This is easy for the latter. First note that  $\widehat{\alpha}$  leaves  $\mathcal{S}(\mathbb{R}_\theta^d)$  invariant. Then taking adjoints, we see that for any  $x \in \mathcal{S}'(\mathbb{R}_\theta^d)$  and  $t \in \mathbb{R}^d$ ,  $\widehat{\alpha}_t(x)$  is again a distribution. If  $\check{\phi} \in L_1(\mathbb{R}^d)$ , (4.1.9) still defines a distribution  $\check{\phi} * x$ .

On the other hand, if  $\phi$  is a function such that  $\phi f \in \mathcal{S}(\mathbb{R}^d)$  for any  $f \in \mathcal{S}(\mathbb{R}^d)$ , then the definition (4.1.8) also extends to a distribution  $x$  by duality. These two definitions are consistent when both of them make sense for a distribution  $x$  and a function  $\phi$ ; in this case, we have  $T_\phi(x) = \check{\phi} * x$ .

**Remark 4.1.8.** Typical examples of Fourier multipliers are noncommutative analogues of partial derivatives, Bessel and Riesz potentials. The symbols of  $J^a$  and  $I^a$  will be denoted by  $J_a$  and  $I_a$ , respectively, so  $J_a(s) = (1 + |s|^2)^{\frac{a}{2}}$  and  $I_a(s) = |s|^a$  for  $s \in \mathbb{R}^d$ .

**Remark 4.1.9.** One may also define the Fourier multiplier from the perspective of functional calculus: Denote  $\nabla = (\partial_1, \dots, \partial_d)$ . For a function  $\phi$  on  $\mathbb{R}^d$ , we write the Fourier multiplier  $T_\phi$  as

$$T_\phi = \phi(\partial_1, \dots, \partial_d) = \phi(\nabla).$$

If moreover  $\phi \in C(\mathbb{R}^d \setminus \{0\})$  that is homogeneous of order 0, we may view  $\phi$  as a function in  $C(\mathbb{S}^{d-1})$ . In this case,

$$T_\phi = \phi(\partial_1, \dots, \partial_d) = \phi(\partial_1 \Delta^{-\frac{1}{2}}, \dots, \partial_d \Delta^{-\frac{1}{2}})$$

since the spectrum of the  $d$ -tuple  $(\partial_1 \Delta^{-\frac{1}{2}}, \dots, \partial_d \Delta^{-\frac{1}{2}})$  falls into the unit sphere  $\mathbb{S}^{d-1}$ .

#### 4.1.4 Commutators

In the sequel, we will fix a function  $\phi \in C^\infty(\mathbb{R}^d \setminus \{0\})$  that is homogeneous of order 0 and non identically zero.  $\phi$  can be viewed as a function in  $C^\infty(\mathbb{S}^{d-1})$ . The Fourier multiplier  $T_\phi$ , defined as in (4.1.8), is bounded on  $L_2(\mathbb{R}_\theta^d)$ .

Given  $x \in \mathbb{R}_\theta^d$ , denote by  $M_x : y \mapsto xy$  the left multiplication on  $L_2(\mathbb{R}_\theta^d)$ . Then  $M_x$  is a bounded linear operator on  $L_2(\mathbb{R}_\theta^d)$ . We now define the commutator

$$\mathbf{C}_{\phi,x} = [T_\phi, M_x].$$

This is a so-called Calderón-Zygmund transform on  $\mathbb{R}_\theta^d$ , it is bounded on  $L_2(\mathbb{R}_\theta^d)$ . When  $\phi(s) = \frac{s_j}{|s|}$ ,  $T_\phi = R_j$  is the  $j$ -th Riesz transform on  $\mathbb{R}_\theta^d$ , and  $\mathbf{C}_{\phi,x}$  is the  $j$ -th component of the quantised differential studied in [62].

As in [62], we remark that if  $x$  is not necessarily bounded we may still define  $\mathbf{C}_{\phi,x}$  on the dense subspace  $\lambda_\theta(C_c^\infty(\mathbb{R}^d))$  of  $L_2(\mathbb{R}_\theta^d)$ . Suppose that  $x \in L_p(\mathbb{R}_\theta^d)$  for some  $2 \leq p < \infty$ . Then for  $y = \lambda_\theta(g) \in \lambda_\theta(C_c^\infty(\mathbb{R}^d))$  with  $g \in C_c^\infty(\mathbb{R}^d)$  compactly supported by  $K$ , we have from [62, Theorem 3.17] that  $M_x y = M_x T_{\mathbb{1}_K} y \in L_2(\mathbb{R}_\theta^d)$ . It then follows that  $T_\phi M_x y \in L_2(\mathbb{R}_\theta^d)$ . On the other hand, since  $\phi g$  is still a compactly supported function in  $L_2(\mathbb{R}^d)$ , using the foregoing argument we have  $M_x T_\phi y \in L_2(\mathbb{R}_\theta^d)$ . Thus in that case  $\mathbf{C}_{\phi,x}(y)$  is a well-defined element in  $L_2(\mathbb{R}_\theta^d)$ .

More generally, viewing  $\mathbb{R}_\theta^d$  as a von Neumann algebra on  $L_2(\mathbb{R}_\theta^d)$  (in its standard form),  $M_x$  is just  $x$  itself. Thus if  $x$  is a measurable operator on  $(\mathbb{R}_\theta^d, \tau_\theta)$ ,  $\mathbf{C}_{\phi,x}$  is a densely defined operator on  $L_2(\mathbb{R}_\theta^d)$ .

Our main concern is to characterise the membership of  $\mathbf{C}_{\phi,x}$  in the Schatten  $p$ -class  $S_p(L_2(\mathbb{R}_\theta^d))$ . Since  $L_2(\mathbb{R}_\theta^d) \cong L_2(\mathbb{R}^d)$  unitarily via the Plancherel formula, we will always view  $\mathbf{C}_{\phi,x}$  as an operator on  $L_2(\mathbb{R}^d)$ . We will denote  $S_p(L_2(\mathbb{R}^d))$  simply by  $S_p$ .

We need to write the kernel of  $\mathbf{C}_{\phi,x}$ . To that end, we assume that  $x$  is a good operator, for instance,  $x = \lambda_\theta(f) \in L_1(\mathbb{R}_\theta^d)$  for some  $f \in L_1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ . Then a simple calculation gives

$$\tau_\theta(x \lambda_\theta(t)^*) = f(t), \quad t \in \mathbb{R}^d.$$

Put  $\widehat{x}(t) = \tau_\theta(x \lambda_\theta(t)^*)$  whenever  $x \in L_1(\mathbb{R}_\theta^d)$ .

We write formally

$$x = \int_{\mathbb{R}^d} \widehat{x}(t) \lambda_\theta(t) dt$$

whenever  $\widehat{x} \in L_1(\mathbb{R}^d)$ . Now let  $\xi \in L_2(\mathbb{R}^d)$ . Then for  $s \in \mathbb{R}_\theta^d$ , we have

$$\begin{aligned} (x\xi)(s) &= \int_{\mathbb{R}^d} \widehat{x}(t)(\lambda_\theta(t)\xi)(s)dt \\ &= \int_{\mathbb{R}^d} \widehat{x}(t)\sigma(-s,t)\xi(s-t)dt \\ &= \int_{\mathbb{R}^d} \widehat{x}(s-t)\sigma(-s,s-t)\xi(t)dt \\ &= \int_{\mathbb{R}^d} \sigma(s,t)\widehat{x}(s-t)\xi(t)dt. \end{aligned}$$

Here for the last equality, we have used (4.1.2). Thus the kernel of  $x$  is

$$[x] = [\sigma(s,t)\widehat{x}(s-t)]_{s,t \in \mathbb{R}^d}. \quad (4.1.10)$$

On the other hand, the Fourier multiplier  $T_\phi$  is a diagonal operator, or more precisely, the multiplication operator by  $\phi$  on  $L_2(\mathbb{R}^d)$ . It then follows that the kernel of the commutator  $C_{\phi,x}$  is given by

$$K_x(s,t) = \sigma(s,t)\widehat{x}(s-t)(\phi(s) - \phi(t)), \quad s,t \in \mathbb{R}^d. \quad (4.1.11)$$

## 4.2 Function spaces on quantum euclidean spaces

In this section we give the definition of Sobolev and Besov spaces on quantum euclidean spaces, which will be used to characterise the (weak) Schatten properties of commutators.

**Definition 4.2.1.** For a positive integer  $m$  and  $1 \leq p \leq \infty$ , the *Sobolev space*  $W_p^m(\mathbb{R}_\theta^d)$  is the space of  $x \in \mathcal{S}'(\mathbb{R}_\theta^d)$  such that every partial derivative of  $x$  up to order  $m$  is in  $L_p(\mathbb{R}_\theta^d)$ , equipped with the norm:

$$\|x\|_{W_p^m} = \left( \sum_{|\alpha| \leq m} \|\partial^\alpha x\|_p \right)^{\frac{1}{p}}.$$

The *homogeneous Sobolev space*  $\dot{W}_p^m(\mathbb{R}_\theta^d)$  consists of those  $x \in \mathcal{S}'(\mathbb{R}_\theta^d)$  such that every partial derivative of order  $m$  is in  $L_p(\mathbb{R}_\theta^d)$ , equipped with the seminorm:

$$\|x\|_{\dot{W}_p^m} = \left( \sum_{|\alpha|=m} \|\partial^\alpha x\|_p \right)^{\frac{1}{p}}.$$

By [62, Proposition 3.14],  $\mathcal{S}(\mathbb{R}_\theta^d)$  is norm-dense in  $W_p^m(\mathbb{R}_\theta^d)$  when  $m \geq 0$  and  $1 \leq p < \infty$ ; the density of  $\mathcal{S}(\mathbb{R}_\theta^d)$  in  $\dot{W}_p^m(\mathbb{R}_\theta^d)$  holds only when  $m \geq 0$  and  $1 < p < \infty$ , see [63].

Besov spaces are defined by using a fixed test function  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  such that

$$\begin{cases} \text{supp } \varphi \subset \{\xi : 2^{-1} \leq |\xi| \leq 2\}, \\ \varphi > 0 \text{ on } \{\xi : 2^{-1} < |\xi| < 2\}, \\ \sum_{k \in \mathbb{Z}} \varphi(2^{-k}\xi) = 1, \xi \neq 0. \end{cases} \quad (4.2.1)$$

For convenience, we assume in addition that  $\varphi$  is even (but this is not necessary for the validity of our results). The sequence  $\{\varphi(2^{-k}\cdot)\}_{k \in \mathbb{Z}}$  is a Littlewood-Paley decomposition of  $\mathbb{R}^d$ , modulo constant functions. Denote by  $\varphi_k$  the inverse Fourier transform of  $\varphi(2^{-k}\cdot)$ . Recall that for a distribution  $x$ , the convolution  $\varphi_k * x$  is defined in (4.1.9) (see the discussion after Lemma 4.1.7). Recall that the distribution space  $\mathcal{S}'_0(\mathbb{R}^d_\theta)$  is introduced in Remark 4.1.6.

**Definition 4.2.2.** Let  $1 \leq p, q \leq \infty$  and  $a \in \mathbb{R}$ . The *homogeneous Besov space* on  $\mathbb{R}^d_\theta$  is defined by

$$B_{p,q}^a(\mathbb{R}^d_\theta) = \left\{ x \in \mathcal{S}'_0(\mathbb{R}^d_\theta) : \|x\|_{B_{p,q}^a} < \infty \right\},$$

where

$$\|x\|_{B_{p,q}^a} = \left( \sum_{k \in \mathbb{Z}} 2^{qka} \|\varphi_k * x\|_p^q \right)^{\frac{1}{q}}.$$

Let  $B_{p,c_0}^a(\mathbb{R}^d_\theta)$  be the subspace of  $B_{p,\infty}^a(\mathbb{R}^d_\theta)$  consisting of all  $x$  such that  $2^{kr} \|\varphi_k * x\|_p \rightarrow 0$  as  $|k| \rightarrow \infty$ .

We state some basic properties of Besov spaces.

**Proposition 4.2.3.** Let  $1 \leq p, q \leq \infty$  and  $a \in \mathbb{R}$ . Then

- (i)  $B_{p,q}^a(\mathbb{R}^d_\theta)$  is a Banach space;
- (ii)  $\mathcal{S}_0(\mathbb{R}^d_\theta)$  is dense in  $B_{p,q}^a(\mathbb{R}^d_\theta)$  for  $1 \leq p < \infty$  and  $1 \leq q < \infty$ ;
- (iii) The dual space of  $B_{p,q}^a(\mathbb{R}^d_\theta)$  coincides isomorphically with  $B_{p',q'}^{-a}(\mathbb{R}^d_\theta)$  for  $1 \leq p < \infty$  and  $1 \leq q < \infty$ , where  $p'$  denote the conjugate exponents of  $p$ ; besides, the dual space of  $B_{p,c_0}^a(\mathbb{R}^d_\theta)$  coincides isomorphically with  $B_{p',1}^{-a}(\mathbb{R}^d_\theta)$  for  $1 \leq p < \infty$ .

*Proof.* (i) Let  $\{x_n\}_n$  be a Cauchy sequence in  $B_{p,q}^a(\mathbb{R}^d_\theta)$ . Then for each  $k \in \mathbb{Z}$ ,  $\{\varphi_k * x_n\}_n$  is a Cauchy sequence in  $L_p(\mathbb{R}^d_\theta)$ . Thus  $\varphi_k * x_n$  converges to some  $y_k$  in  $L_p(\mathbb{R}^d_\theta)$ . We define

$$y = \sum_{k \in \mathbb{Z}} y_k.$$

For every  $x \in \mathcal{S}_0(\mathbb{R}^d_\theta)$ , we write the Littlewood-Paley decomposition of  $x$ :

$$x = \sum_{j \in \mathbb{Z}} \varphi_j * x \quad (\text{convergence in } \mathcal{S}_0(\mathbb{R}^d_\theta)).$$

Then noting that  $\varphi_j * y_k = 0$  for  $j \notin \{k-1, k, k+1\}$ , we easily show that the above element  $y$  is a well-defined element in  $\mathcal{S}'_0(\mathbb{R}^d_\theta)$ . On the other hands, as  $n \rightarrow \infty$ , we have

$$\varphi_j * x_n = \varphi_j * \sum_{k=j-1}^{j+1} \varphi_k * x_n \rightarrow \varphi_j * \sum_{k=j-1}^{j+1} y_k = \varphi_j * y.$$

Thus, we conclude the proof.

(ii) For  $x \in B_{p,q}^a(\mathbb{R}^d_\theta)$ , we may choose an integer  $N \in \mathbb{N}$  large enough such that

$$\|x\|_{B_{p,q}^a}^q - \sum_{|k| \leq N} 2^{qka} \|\varphi_k * x\|_p^q$$

is small enough. Since  $\mathcal{S}(\mathbb{R}_\theta^d)$  is norm dense in  $L_p(\mathbb{R}_\theta^d)$  by Lemma 4.1.3, for  $|k| \leq N$ , we find  $y_k \in \mathcal{S}(\mathbb{R}_\theta^d)$  such that  $\|y_k - \varphi_k * x\|_p \leq \varepsilon_k$  (with very small  $\varepsilon_k > 0$ ). By Lemma 4.1.7, we also have

$$\|\varphi_j * y_k - \varphi_j * \varphi_k * x\|_p \leq \varepsilon_k.$$

We define

$$y = \sum_{|k| \leq N} \sum_{j=k-1}^{k+1} \varphi_j * y_k.$$

Then  $y \in \mathcal{S}_0(\mathbb{R}_\theta^d)$ . Using the fact that  $\varphi_j * \varphi_k = 0$  for  $j \notin \{k-1, k, k+1\}$ , we deduce  $\|\varphi_k * y - \varphi_k * x\|_p \leq \varepsilon'_k$  for  $|k| \leq N$ ; consequently,  $\|y - x\|_{B_{p,q}^a}$  is small.

(iii) Let  $y \in B_{p',q'}^{-a}(\mathbb{R}_\theta^d)$ . Define  $\ell_y(x) = \tau_\theta(xy)$  for  $x \in \mathcal{S}_0(\mathbb{R}_\theta^d)$ . Then

$$\begin{aligned} |\ell_y(x)| &= \sum_{k \in \mathbb{Z}} \tau_\theta(\varphi_k * x \sum_{j=k-1}^{k+1} \varphi_j * y) \\ &\leq \sum_{k \in \mathbb{Z}} \|\varphi_k * x\|_p \left\| \sum_{j=k-1}^{k+1} \varphi_j * y \right\|_{p'} \\ &\lesssim \|x\|_{B_{p,q}^a(\mathbb{R}_\theta^d)} \|y\|_{B_{p',q'}^{-a}(\mathbb{R}_\theta^d)}. \end{aligned}$$

By the density of  $\mathcal{S}_0(\mathbb{R}_\theta^d)$  in  $B_{p,q}^a(\mathbb{R}_\theta^d)$ , we deduce that  $y \in B_{p',q'}^{-a}(\mathbb{R}_\theta^d)$  defines a bounded linear functional on  $B_{p,q}^a(\mathbb{R}_\theta^d)$ .

For the converse, given a Banach space  $X$ , define the weighted  $\ell_q$  space  $\ell_q^a(X)$  to be the space of all sequences  $(\dots, x_{-1}, x_0, x_1, \dots)$  with  $x_k \in X$ , equipped with the norm

$$\left( \sum_{k \in \mathbb{Z}} 2^{kqa} \|x_k\|_X^q \right)^{\frac{1}{q}}.$$

If  $q = \infty$ , then we define the space  $c_0^r(X)$  to be the space of all sequences  $(\dots, x_{-1}, x_0, x_1, \dots)$  such that  $2^{ka} \|x_k\|_X \rightarrow 0$  as  $|k| \rightarrow \infty$ .

The dual space of  $\ell_q^a(X)$  is  $\ell_q^{-a}(X^*)$  for  $q < \infty$ . By definition,  $B_{p,q}^a(\mathbb{R}_\theta^d)$  embeds into  $\ell_q^a(L_p(\mathbb{R}_\theta^d))$  through the map  $x \mapsto (\dots, \varphi_{-1} * x, \varphi_0 * x, \varphi_1 * x, \dots)$ . Suppose that  $\ell$  is a continuous linear functional on  $B_{p,q}^a(\mathbb{R}_\theta^d)$ . Then by the Hahn-Banach theorem,  $\ell$  extends to a continuous functional on  $\ell_q^a(L_p(\mathbb{R}_\theta^d))$  with the same norm, so there exists an element  $y \in \ell_q^{-a}(L_{p'}(\mathbb{R}_\theta^d))$  such that

$$\ell(x) = \sum_{k \in \mathbb{Z}} \tau_\theta(y_k \varphi_k * x).$$

Let

$$y = \sum_{k \in \mathbb{Z}} (\varphi_{k-1} * y_k + \varphi_k * y_k + \varphi_{k+1} * y_k).$$

Then we have  $y \in B_{p',q'}^{-a}(\mathbb{R}_\theta^d)$  and  $\ell = \ell_y$ .

For  $B_{p,c_0}^a(\mathbb{R}_\theta^d)$ , it embeds into  $c_0^a(L_p(\mathbb{R}_\theta^d))$ , and the dual of  $c_0^a(L_p(\mathbb{R}_\theta^d))$  is  $\ell_1^{-a}(L_{p'}(\mathbb{R}_\theta^d))$ . The same argument works.  $\square$

Next, we collect the following two propositions on lifting property and interpolation for Besov spaces on  $\mathbb{R}_\theta^d$ , which are counterparts of Theorem 3.6 and Proposition 5.1 in [105]. The proofs in [105] repeat *mutatis mutandi*, so are omitted.

**Proposition 4.2.4.** *Let  $1 \leq p, q \leq \infty$ ,  $a, b \in \mathbb{R}$ . Then  $J^b$  and  $I^b$  are isomorphisms between  $B_{p,q}^a(\mathbb{R}_\theta^d)$  and  $B_{p,q}^{a-b}(\mathbb{R}_\theta^d)$ .*

**Proposition 4.2.5.** *Let  $0 < \eta < 1$ ,  $a_0, a_1 \in \mathbb{R}$  and  $p_0, p_1, q_0, q_1 \in [1, \infty]$ . Then we have the following complex interpolation formula*

$$\left( B_{p_0, q_0}^{a_0}(\mathbb{R}_\theta^d), B_{p_1, q_1}^{a_1}(\mathbb{R}_\theta^d) \right)_\eta = B_{p, q}^a(\mathbb{R}_\theta^d),$$

where

$$a = (1 - \eta)a_0 + \eta a_1, \quad \frac{1}{p} = \frac{1 - \eta}{p_0} + \frac{\eta}{p_1}, \quad \frac{1}{q} = \frac{1 - \eta}{q_0} + \frac{\eta}{q_1} \quad (\text{with } q < \infty).$$

Like in the case of quantum tori in [105, Chapter 3], one may expect various of equivalent characterisations of Besov norms on  $\mathbb{R}_\theta^d$ . However, we will not study such characterisations on  $\mathbb{R}_\theta^d$  in here. For later use, we record only the following elementary proposition.

**Proposition 4.2.6.** *Let  $\psi$  be a Schwartz function supported on the annulus  $\{s \in \mathbb{R}^d : \frac{1}{8} \leq |s| \leq 4\}$  and equal to 1 on the smaller one  $\{s \in \mathbb{R}^d : \frac{1}{4} \leq |s| \leq 2\}$ . Denote  $\psi_k$  as the inverse Fourier transform of  $\psi(2^{-k}\cdot)$ . Then we have*

$$\|x\|_{B_{p,q}^a} \approx_d \left( \sum_{k \in \mathbb{Z}} 2^{qka} \|\psi_k * x\|_p^q \right)^{\frac{1}{q}}.$$

*Proof.* For  $k \in \mathbb{Z}$ , denote  $\Delta_k = \{s \in \mathbb{R}^d : 2^{k-1} \leq |s| < 2^k\}$ . These sets  $\Delta_k$  form a partition of  $\mathbb{R}^d \setminus \{0\}$ . Put also  $\widetilde{\Delta}_k = \Delta_{k-1} \cup \Delta_k \cup \Delta_{k+1}$ . Obviously,  $\widehat{\psi}_k(s) \equiv 1$  for any  $s \in \widetilde{\Delta}_k$ .

Note that

$$\psi_k * x = \sum_{j=k-3}^{k+2} \psi_k * \varphi_j * x.$$

Since  $\psi$  is an infinitely differentiable function with compact support, we see that  $\mathcal{F}^{-1}(\psi)$  is integrable, so with Lemma 4.1.7,

$$\|\psi_k * x\|_p \leq \|\mathcal{F}^{-1}(\psi)\|_1 \sum_{j=k-3}^{k+2} \|\varphi_j * x\|_p.$$

It follows that

$$\left( \sum_{k \in \mathbb{Z}} 2^{qk\alpha} \|\psi_k * x\|_p^q \right)^{\frac{1}{q}} \leq 6 \|\mathcal{F}^{-1}(\psi)\|_1 \left( \sum_{k \in \mathbb{Z}} 2^{qk\alpha} \|\varphi_k * x\|_p^q \right)^{\frac{1}{q}},$$

whence

$$\left( \sum_{k \in \mathbb{Z}} 2^{qk\alpha} \|\psi_k * x\|_p^q \right)^{\frac{1}{q}} \lesssim \|x\|_{B_{p,q}^\alpha}.$$

For the converse, since  $\widehat{\psi}_k(\xi) \equiv 1$  for any  $\xi \in \widetilde{\Delta}_k$ , we have

$$\varphi_k * x = \varphi_k * \psi_k * x.$$

By Lemma 4.1.7 again, we have

$$\|\varphi_k * x\|_p \leq \|\mathcal{F}^{-1}(\varphi)\|_1 \|\psi_k * x\|_p.$$

It follows that

$$\left( \sum_{k \in \mathbb{Z}} 2^{qk\alpha} \|\varphi_k * x\|_p^q \right)^{\frac{1}{q}} \leq \|\mathcal{F}^{-1}(\varphi)\|_1 \left( \sum_{k \in \mathbb{Z}} 2^{qk\alpha} \|\psi_k * x\|_p^q \right)^{\frac{1}{q}},$$

which implies

$$\|x\|_{B_{p,q}^\alpha} \lesssim \left( \sum_{k \in \mathbb{Z}} 2^{qk\alpha} \|\psi_k * x\|_p^q \right)^{\frac{1}{q}}.$$

So we conclude (4.2.6).  $\square$

### 4.3 Upper bounds for commutators

We start this section with an additional assumption on  $\phi$ : Hereafter, we will assume

$$\max \left( \sup_{s \in \mathbb{S}^{d-1}} |\phi(s)|, \sup_{s \in \mathbb{S}^{d-1}} |\nabla \phi(s)| \right) \leq 1.$$

Under this assumption, all constants in the late estimates will be independent of  $\phi$ .

Let  $\mathbf{C}_{\phi,x}$  be a commutator with kernel in (4.1.11). We will show the upper estimate for  $\|\mathbf{C}_{\phi,x}\|_{S_p}$ . We need to consider a nicer operator  $I^a \mathbf{C}_{\phi,x} I^b$  for  $a, b \in \mathbb{R}$  ( $a, b$  often positive). In the Fourier transform side,  $I^a$  is the multiplication operator on  $L_2(\mathbb{R}^d)$  by  $I_a$  (recalling that  $I_a(s) = |s|^a$ ). Denoting  $K_{x,a,b}$  the kernel of  $I^a \mathbf{C}_{\phi,x} I^b$ , by (4.1.11) we have

$$K_{x,a,b}(s, t) = |s|^a \sigma(s, t) \widehat{x}(s-t) (\phi(s) - \phi(t)) |t|^b, \quad s, t \in \mathbb{R}^d. \quad (4.3.1)$$

**Convention.** For simplicity of presentation, we make the convention that for an integral operator  $T_K$  with kernel  $K$  on  $L_2(\mathbb{R}^d)$ , the notation  $\|K(s, t)\|_{S_p}$  stands also for the  $S_p$ -norm of  $T_K$ .

**Convention.** In the sequel, unless explicitly stated otherwise,  $x$  will be assumed to be a “good” operator which admits a kernel as in (4.1.10) so that all calculations are legitimate. For instance,  $x$  can be in  $\lambda_\theta(L_1(\mathbb{R}_\theta^d))$  or even in  $\mathcal{S}(\mathbb{R}_\theta^d)$ .

We will frequently use the following elementary fact without further reference.

**Fact.** Let  $\eta, \xi : \mathbb{R}^d \rightarrow \mathbb{C}$  be two bounded measurable functions with  $L_\infty$ -norm  $\leq 1$ . Let  $A$  and  $B$  be the multiplication operators on  $L_2(\mathbb{R}^d)$ . Then for  $T_K$  as above, the kernel of  $AT_K B$  is  $\eta(s)K(s, t)\xi(t)$ , so by the above convention, we have

$$\|\eta(s)K(s, t)\xi(t)\|_{S_p} = \|AT_K B\|_{S_p} \leq \|T_K\|_{S_p} = \|K(s, t)\|_{S_p}.$$

Another simple fact about  $\phi$  will be also used. Let  $s_0 \in \mathbb{R}^d \setminus \{0\}$ ,  $0 < r < \frac{|s_0|}{2}$ , and  $B(s_0, r) = \{s \in \mathbb{R}^d : |s - s_0| < r\}$ . Since the first order derivatives of  $\phi$  are homogeneous of order  $-1$ , the mean value theorem implies

$$\|\mathbb{1}_{B(s_0, r)}[\phi - \phi(s_0)]\|_{L_\infty(\mathbb{R}^d)} \leq r \sup_{|s-s_0| \leq \frac{|s_0|}{2}} |\nabla \phi(s)| \leq \frac{r}{|s_0|}. \quad (4.3.2)$$

This immediately implies

$$|\phi(s) - \phi(t)| \lesssim \min \left( 1, \frac{|s-t|}{|t|} \right), \quad s, t \in \mathbb{R}^d. \quad (4.3.3)$$



### 4.3.1 Upper bound for the $S_\infty$ -norm

The next theorem gives the upper bound estimate of  $S_\infty$ -norm of  $I^a \mathbf{C}_{\phi,x} I^b$ . In this subsection,  $x$  is a good operator with kernel so that all calculations below are legitimate.

**Theorem 4.3.1.** *Let  $a > 0, b > 0$  and  $a + b < 1$ . If  $x \in B_{\infty,\infty}^{a+b}(\mathbb{R}_\theta^d)$ , then  $I^a \mathbf{C}_{\phi,x} I^b \in S_\infty(L_2(\mathbb{R}_\theta^d))$  and*

$$\|I^a \mathbf{C}_{\phi,x} I^b\|_{S_\infty} \lesssim_{d,a,b} \|x\|_{B_{\infty,\infty}^{a+b}}.$$

*Proof.* This part is the most technical part of the chapter. Reformulated in terms of the kernel in (4.3.1), the assertion means

$$\|K_{x,a,b}\|_{S_\infty} \lesssim_{d,a,b} \|x\|_{B_{\infty,\infty}^{a+b}}.$$

In the following, assume that  $\|x\|_{B_{\infty,\infty}^{a+b}} \leq 1$ . Recall that  $\Delta_k = \{s \in \mathbb{R}^d : 2^{k-1} \leq |s| < 2^k\}$ . We claim that

$$\left\| \mathbb{1}_{\Delta_j}(s) K_{x,a,b}(s,t) \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \lesssim_{d,a,b} \begin{cases} 2^{a(j-k)} & \text{if } j \leq k, \\ 2^{b(k-j)} & \text{if } j > k. \end{cases} \quad (4.3.4)$$

Assuming the above claim, we easily prove the above desired estimate on  $K_{x,a,b}$ . Indeed, according to the partition  $\{\Delta_j\}_{j \in \mathbb{Z}}$  of  $\mathbb{R}^d$ , we write

$$K_{x,a,b}(s,t) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \mathbb{1}_{\Delta_{j+k}}(s) K_{x,a,b}(s,t) \mathbb{1}_{\Delta_k}(t).$$

The internal sum is a block diagonal operator, so by (4.3.4)

$$\left\| \sum_{k \in \mathbb{Z}} \mathbb{1}_{\Delta_{j+k}}(s) K_{x,a,b}(s,t) \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} = \sup_{k \in \mathbb{Z}} \left\| \mathbb{1}_{\Delta_{j+k}}(s) K_{x,a,b}(s,t) \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \lesssim_d 2^{-\min(a,b)|j|}.$$

Thus, since  $a, b > 0$ , we get

$$\|K_{x,a,b}\|_{S_\infty} \lesssim_d \sum_{j \in \mathbb{Z}} 2^{-\min(a,b)|j|} \lesssim_d 1.$$

We will show (4.3.4) according to the two cases  $j \leq k$  and  $j > k$ . The latter is symmetric to the former by passing to adjoints, thus we need only consider the former. We will divide this case into two sub-cases:  $j \leq k-2$  and  $k-1 \leq j \leq k$ . The first sub-case is relatively simple, while that for the second is quite subtle.

**Step 1:**  $j \leq k-2$ . Note that for  $s \in \Delta_j$  and  $t \in \Delta_k$ ,  $|s| \leq 2^j \leq 2^{k-2}$  and  $2^{k-1} \leq |t| \leq 2^k$ . Then  $2^{k-2} \leq |s-t| \leq 2^{k+1}$ . In other words,  $s-t \in \widetilde{\Delta}_k$  (recalling  $\widetilde{\Delta}_k = \Delta_{k-1} \cup \Delta_k \cup \Delta_{k+1}$ ). So we have

$$\begin{aligned} \mathbb{1}_{\Delta_j}(s) K_{x,a,b}(s,t) \mathbb{1}_{\Delta_k}(t) &= \mathbb{1}_{\Delta_j}(s) |s|^a \sigma(s,t) \widehat{x}(s-t) [\phi_i(s) - \phi_i(t)] |t|^b \mathbb{1}_{\Delta_k}(t) \\ &= \mathbb{1}_{\Delta_j}(s) |s|^a \sigma(s,t) \widehat{\psi}_k(s-t) \widehat{x}(s-t) [\phi_i(s) - \phi_i(t)] |t|^b \mathbb{1}_{\Delta_k}(t) \\ &= \mathbb{1}_{\Delta_j}(s) |s|^a \sigma(s,t) \widehat{\psi}_k * x(s-t) [\phi_i(s) - \phi_i(t)] |t|^b \mathbb{1}_{\Delta_k}(t). \end{aligned}$$

Thus

$$\begin{aligned} \left\| \mathbb{1}_{\Delta_j}(s) K_{x,a,b}(s,t) \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} &\leq \left\| \mathbb{1}_{\Delta_j}(s) |s|^a \phi_i(s) \sigma(s,t) \widehat{\psi}_k * x(s-t) |t|^b \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \\ &\quad + \left\| \mathbb{1}_{\Delta_j}(s) |s|^a \sigma(s,t) \widehat{\psi}_k * x(s-t) \phi_i(t) |t|^b \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \\ &\leq 2^{aj+bk+1} \left\| \sigma(s,t) \widehat{\psi}_k * x(s-t) \right\|_{S_\infty} \\ &= 2^{aj+bk+1} \left\| \psi_k * x \right\|_{L_\infty(\mathbb{R}_\theta^d)}. \end{aligned}$$

Since  $\|x\|_{B_{\infty,\infty}^{a+b}} \leq 1$ , by Proposition 4.2.6, we have

$$\|\psi_k * x\|_{L_\infty(\mathbb{R}_\theta^d)} \lesssim 2^{-(a+b)k}.$$

Combining the preceding inequalities, we obtain

$$\left\| \mathbb{1}_{\Delta_j}(s) K_{x,a,b}(s,t) \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \lesssim 2^{a(j-k)}.$$

Thus (4.3.4) is proved for  $j \leq k - 2$ .

**Step 2:**  $k - 1 \leq j \leq k$ . For this case we need an auxiliary partition of  $\mathbb{R}^d$ . Fix an integer  $\ell \in \mathbb{Z}$  such that

$$2^{\ell-k} < \frac{1}{4(4+2d)}. \quad (4.3.5)$$

For  $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ , let  $Q_m$  be the cube in  $\mathbb{R}^d$  with center  $2^\ell m$  and side length  $2^\ell$ , i.e.,

$$Q_m = [2^\ell m_1 - 2^{\ell-1}, 2^\ell m_1 + 2^{\ell-1}] \times \dots \times [2^\ell m_d - 2^{\ell-1}, 2^\ell m_d + 2^{\ell-1}].$$

The family  $\{Q_m\}_{m \in \mathbb{Z}^d}$  is a partition of  $\mathbb{R}^d$ . Let  $m, n \in \mathbb{Z}^d$ . We first aim to estimate

$$\begin{aligned} & \left\| \mathbb{1}_{Q_m}(s) \sigma(s,t) \varphi(2^{-\ell}(s-t)) \widehat{x}(s-t) [\phi(s) - \phi(t)] \mathbb{1}_{Q_n}(t) \right\|_{S_\infty} \\ &= \left\| \mathbb{1}_{Q_m}(s) \sigma(s,t) \widehat{\varphi_\ell * x}(s-t) [\phi(s) - \phi(t)] \mathbb{1}_{Q_n}(t) \right\|_{S_\infty} \end{aligned}$$

If  $|m - n| > 4 + d$ , for  $s \in Q_m$  and  $t \in Q_n$ ,

$$|s - t| \geq |2^\ell m - 2^\ell n| - |s - 2^\ell m| - |t - 2^\ell n| > 2^\ell(4 + d) - 2^\ell d^{\frac{1}{2}} > 2^{\ell+2},$$

so  $\varphi(2^{-\ell}(s-t)) = 0$  (recalling that  $\varphi$  is the test function given by (4.2.1)).

On the other hand, assume that  $|m - n| \leq 4 + d$  and  $Q_n \cap \Delta_k \neq \emptyset$ . Take one point  $t_0 \in Q_n \cap \Delta_k$ . Then  $Q_n \subset B(t_0, 2^{\ell-1} d^{\frac{1}{2}}) \subset B(t_0, 2^\ell(4 + 2d))$ . For  $t \in Q_m$ ,

$$\begin{aligned} |t - t_0| &\leq |t - 2^\ell m| + |2^\ell m - 2^\ell n| + |2^\ell n - t_0| \\ &\leq 2^{\ell-1} d^{\frac{1}{2}} + 2^\ell(4 + d) + 2^{\ell-1} d^{\frac{1}{2}} \leq 2^\ell(4 + 2d), \end{aligned}$$

Hence  $Q_m \subset B(t_0, 2^\ell(4 + 2d))$  as well. Now by (4.3.5),  $2^\ell(4 + 2d) < 2^{k-2} \leq \frac{|t_0|}{2}$ . Thus by (4.3.2), we deduce that

$$\begin{aligned} & \left\| \mathbb{1}_{Q_m}(s) \sigma(s,t) \widehat{\varphi_\ell * x}(s-t) [\phi(s) - \phi(t)] \mathbb{1}_{\Delta_k}(t) \mathbb{1}_{Q_n}(t) \right\|_{S_\infty} \\ &\leq \left\| \mathbb{1}_{Q_m}(s) \sigma(s,t) \widehat{\varphi_\ell * x}(s-t) [\phi(s) - \phi(t_0)] \mathbb{1}_{\Delta_k}(t) \mathbb{1}_{Q_n}(t) \right\|_{S_\infty} \\ &+ \left\| \mathbb{1}_{Q_m}(s) \sigma(s,t) \widehat{\varphi_\ell * x}(s-t) [\phi(t) - \phi(t_0)] \mathbb{1}_{\Delta_k}(t) \mathbb{1}_{Q_n}(t) \right\|_{S_\infty} \\ &\lesssim 2^{\ell-k} \left\| \sigma(s,t) \widehat{\varphi_\ell * x}(s-t) \right\|_{S_\infty} \\ &= 2^{\ell-k} \left\| \varphi_\ell * x \right\|_{L_\infty(\mathbb{R}_\theta^d)} \leq 2^{\ell-k} 2^{-\ell(a+b)}. \end{aligned} \quad (4.3.6)$$

Since  $\{Q_m\}_{m \in \mathbb{Z}^d}$  is a partition of  $\mathbb{R}^d$  and  $\{Q_n \cap \Delta_k\}_{n \in \mathbb{Z}^d}$  a partition of  $\Delta_k$ , by the above discussion, we write

$$\begin{aligned} & \sigma(s,t) \widehat{\varphi_\ell * x}(s-t) [\phi(s) - \phi(t)] \mathbb{1}_{\Delta_k}(t) \\ &= \sum_{m \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \mathbb{1}_{Q_{m+n}}(s) \sigma(s,t) \widehat{\varphi_\ell * x}(s-t) [\phi(s) - \phi(t)] \mathbb{1}_{\Delta_k}(t) \mathbb{1}_{Q_n}(t) \\ &= \sum_{|m| \leq 4+d} \sum_{n \in \mathbb{Z}^d} \mathbb{1}_{Q_{m+n}}(s) \sigma(s,t) \widehat{\varphi_\ell * x}(s-t) [\phi(s) - \phi(t)] \mathbb{1}_{\Delta_k}(t) \mathbb{1}_{Q_n}(t). \end{aligned}$$

For the internal sum that is a block diagonal operator, we have

$$\begin{aligned} & \left\| \sum_{n \in \mathbb{Z}^d} \mathbb{1}_{Q_{m+n}}(s) \sigma(s, t) \widehat{\varphi_\ell * x}(s-t) [\phi(s) - \phi(t)] \mathbb{1}_{\Delta_k}(t) \mathbb{1}_{Q_n}(t) \right\|_{S_\infty} \\ &= \sup_{n \in \mathbb{Z}^d} \left\| \mathbb{1}_{Q_{m+n}}(s) \sigma(s, t) \widehat{\varphi_\ell * x}(s-t) [\phi(s) - \phi(t)] \mathbb{1}_{\Delta_k}(t) \mathbb{1}_{Q_n}(t) \right\|_{S_\infty} \\ &\lesssim \begin{cases} 0 & \text{if } |m| > 4 + d, \\ 2^{\ell-k} 2^{-\ell(a+b)} & \text{if } |m| \leq 4 + d. \end{cases} \end{aligned}$$

Therefore, it follows that under the condition (4.3.5) we have

$$\left\| \sigma(s, t) \widehat{\varphi_\ell * x}(s-t) [\phi(s) - \phi(t)] \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \lesssim_d 2^{\ell-k} 2^{-\ell(a+b)}.$$

Consequently,

$$\left\| \mathbb{1}_{\Delta_j}(s) \sigma(s, t) \widehat{\varphi_\ell * x}(s-t) [\phi(s) - \phi(t)] \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \lesssim_d 2^{\ell-k} 2^{-\ell(a+b)}.$$

We next deal with the case  $2^{\ell-k} \geq \frac{1}{4(4+2d)}$ . If  $\ell \geq k+2$ , since  $j \leq k$ , we know from the support assumption of  $\varphi$  that  $\varphi(2^{-\ell}(s-t)) = 0$  for  $s \in \Delta_j$  and  $t \in \Delta_k$ . Hence there exists only a finite number of values of  $\ell$  to consider:  $k - \log_2[4(4+2d)] \leq \ell < k+2$ . For these  $\ell$ , we simply have

$$\begin{aligned} & \left\| \mathbb{1}_{\Delta_j}(s) \sigma(s, t) \widehat{\varphi_\ell * x}(s-t) [\phi(s) - \phi(t)] \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \\ &\leq \left\| \mathbb{1}_{\Delta_j}(s) \sigma(s, t) \widehat{\varphi_\ell * x}(s-t) \phi(s) \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \\ &\quad + \left\| \mathbb{1}_{\Delta_j}(s) \sigma(s, t) \widehat{\varphi_\ell * x}(s-t) \phi(t) \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \\ &\lesssim \left\| \varphi_\ell * x \right\|_{L_\infty(\mathbb{R}_\theta^d)} \leq 2^{-\ell(a+b)}. \end{aligned}$$

Summing over  $\ell$  all estimates obtained so far and using the condition that  $a+b < 1$ , we finally deduce that

$$\begin{aligned} & \left\| \mathbb{1}_{\Delta_j}(s) \sigma(s, t) \widehat{x}(s-t) [\phi(s) - \phi(t)] \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \\ &\leq \sum_{\ell < k+2} \left\| \mathbb{1}_{\Delta_j}(s) \sigma(s, t) \widehat{\varphi_\ell * x}(s-t) [\phi(s) - \phi(t)] \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \\ &\lesssim \sum_{\ell < k - \log_2[4(4+2d)]} 2^{\ell-k} 2^{-\ell(a+b)} + \sum_{k - \log_2[4(4+2d)] \leq \ell < k+2} 2^{-\ell(a+b)} \lesssim_d 2^{-k(a+b)}. \end{aligned}$$

Therefore, it follows that for  $k-1 \leq j \leq k$

$$\begin{aligned} & \left\| \mathbb{1}_{\Delta_j}(s) K_{x,a,b}(s, t) \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \\ &= \left\| \mathbb{1}_{\Delta_j}(s) |s|^a \sigma(s, t) \widehat{\varphi_\ell * x}(s-t) [\phi(s) - \phi(t)] |t|^b \mathbb{1}_{\Delta_k}(t) \right\|_{S_\infty} \\ &\lesssim_d 2^{ja} 2^{kb} 2^{-k(a+b)} \lesssim_d 1. \end{aligned}$$

Thus the claim (4.3.4) is completely proved, so is the theorem.  $\square$

### 4.3.2 Upper bound for the $S_1$ -norm

We next turn to the upper bound for the  $S_1$ -norm.

**Theorem 4.3.2.** *Let  $a > -\frac{d}{2}$ ,  $b > -\frac{d}{2}$  and  $a + b + d < 1$ . If  $x \in B_{1,1}^{a+b+d}(\mathbb{R}_\theta^d)$ , then  $I^a \mathbf{C}_{\phi,x} I^b \in S_1$  and*

$$\|I^a \mathbf{C}_{\phi,x} I^b\|_{S_1} \lesssim_{d,a,b} \|x\|_{B_{1,1}^{a+b+d}}. \quad (4.3.7)$$

*Proof.* By the density of  $\mathcal{S}_0(\mathbb{R}_\theta^d)$  in  $B_{1,1}^{1+b+d}(\mathbb{R}_\theta^d)$  (see Proposition 4.2.3), it suffices to show the theorem above for  $x \in \mathcal{S}_0(\mathbb{R}_\theta^d)$ . Let us justify this. Assume that (4.3.7) is valid for all  $x \in \mathcal{S}_0(\mathbb{R}_\theta^d)$ . Let  $x \in B_{1,1}^{a+b+d}(\mathbb{R}_\theta^d)$  and take a sequence  $(x_n)$  in  $\mathcal{S}_0(\mathbb{R}_\theta^d)$  converging to  $x$  in  $B_{p,p}^{a+b+d}(\mathbb{R}_\theta^d)$ . Then by (4.3.7),  $(I^a \mathbf{C}_{\phi,x_n} I^b)$  is a Cauchy sequence in  $S_1$ , so it converges in  $S_1$  to a limit that must be  $I^a \mathbf{C}_{\phi,x} I^b$ . So  $I^a \mathbf{C}_{\phi,x} I^b \in S_1$  and (4.3.7) holds.

Thus  $x$  will be assumed to be a good operator whose commutator admits a kernel as in (4.1.11). Using the Littlewood-Paley decomposition of  $x$ :

$$x = \sum_{k \in \mathbb{Z}} \varphi_k * x$$

and the definition of the  $B_{1,1}^{a+b+d}$ -norm, (4.3.7) reduces to the following

**Claim.** Assume that  $x \in L_1(\mathbb{R}_\theta^d)$  such that  $\widehat{x}$  is supported by  $B(0, R)$ . Then

$$\|I^a \mathbf{C}_{\phi,x} I^b\|_{S_1} \lesssim_{d,a,b} R^{a+b+d} \|x\|_{L_1(\mathbb{R}_\theta^d)}. \quad (4.3.8)$$

For  $n \in \mathbb{Z}^d$ , let  $Q_{Rn}$  be the cube with centre  $Rn$  and side length  $R$ , and  $\widetilde{Q}_{Rn}$  the concentric cube with side length  $3R$ . We are going to estimate

$$\| |s|^a \sigma(s, t) \widehat{x}(s-t) [\phi(s) - \phi(t)] |t|^b \mathbb{1}_{Q_{Rn}}(t) \|_{S_1} \quad (4.3.9)$$

and then calculate the sum over  $n \in \mathbb{Z}^d$ . Note that if  $s \notin \widetilde{Q}_{Rn}$  and  $t \in Q_{Rn}$ ,  $\widehat{x}(s-t) = 0$  thanks to the support assumption of  $\widehat{x}$ . So

$$(4.3.9) = \left\| \mathbb{1}_{\widetilde{Q}_{Rn}}(s) |s|^a \sigma(s, t) \widehat{x}(s-t) [\phi(s) - \phi(t)] |t|^b \mathbb{1}_{Q_{Rn}}(t) \right\|_{S_1}.$$

We divide the proof into two cases:  $|n| > 3\sqrt{d}$  and  $|n| \leq 3\sqrt{d}$ .

**Step 1:**  $|n| > 3\sqrt{d}$ . The cube  $\widetilde{Q}_{Rn}$  is contained in  $B(Rn, \frac{3}{2}\sqrt{d}R)$  and  $\frac{3}{2}\sqrt{d}R < \frac{|Rn|}{2}$ . Thus by (4.3.2), we have

$$\begin{aligned} (4.3.9) &= \left\| \mathbb{1}_{\widetilde{Q}_{Rn}}(s) |s|^a \sigma(s, t) \widehat{x}(s-t) [\phi(s) - \phi(t)] |t|^b \mathbb{1}_{Q_{Rn}}(t) \right\|_{S_1} \\ &\leq \left\| \mathbb{1}_{\widetilde{Q}_{Rn}}(s) |s|^a \sigma(s, t) \widehat{x}(s-t) [\phi(s) - \phi(Rn)] |t|^b \mathbb{1}_{Q_{Rn}}(t) \right\|_{S_1} \\ &\quad + \left\| \mathbb{1}_{\widetilde{Q}_{Rn}}(s) |s|^a \sigma(s, t) \widehat{x}(s-t) [\phi(t) - \phi(Rn)] |t|^b \mathbb{1}_{Q_{Rn}}(t) \right\|_{S_1} \\ &\lesssim_d |n|^{-1} \left\| \mathbb{1}_{\widetilde{Q}_{Rn}}(s) |s|^a \sigma(s, t) \widehat{x}(s-t) \mathbb{1}_{Q_{Rn}}(t) \right\|_{S_1}. \end{aligned} \quad (4.3.10)$$

By definition,

$$\widehat{x}(s-t) = \tau_\theta(x \lambda_\theta(s-t)^*) = \sigma(s, -t) \tau_\theta(x \lambda_\theta(t) \lambda_\theta(-s)),$$

so

$$\sigma(s, t)\widehat{x}(s-t) = \tau_\theta(\lambda_\theta(-s)x\lambda_\theta(t)).$$

We now write  $x = yz$  with  $y, z \in L_2(\mathbb{R}_\theta^d)$  such that  $\|x\|_1 = \|y\|_2\|z\|_2$ . Then

$$\mathbb{1}_{\widetilde{Q}_{Rn}}(s)|s|^a\sigma(s, t)\widehat{x}(s-t)\mathbb{1}_{Q_{Rn}}(t) = \tau_\theta\left(\left[\mathbb{1}_{\widetilde{Q}_{Rn}}(s)|s|^a\lambda_\theta(-s)y\right]\left[z\lambda_\theta(t)\mathbb{1}_{Q_{Rn}}(t)\right]\right).$$

Thus by the Cauchy-Schwarz inequality,

$$\begin{aligned} & \left\| \mathbb{1}_{\widetilde{Q}_{Rn}}(s)|s|^a\sigma(s, t)\widehat{x}(s-t)\mathbb{1}_{Q_{Rn}}(t) \right\|_{S_1} \\ & \leq \left\| \mathbb{1}_{\widetilde{Q}_{Rn}} I_a \lambda_\theta(-\cdot) y \right\|_{L_2(\mathbb{R}^d; L_2(\mathbb{R}_\theta^d))} \left\| z \lambda_\theta(\cdot) I_b \mathbb{1}_{Q_{Rn}} \right\|_{L_2(\mathbb{R}^d; L_2(\mathbb{R}_\theta^d))} \\ & = \|y\|_2 \|z\|_2 \left( \int_{\widetilde{Q}_{Rn}} |s|^{2a} ds \right)^{\frac{1}{2}} \left( \int_{Q_{Rn}} |t|^{2b} dt \right)^{\frac{1}{2}} \\ & \lesssim_d \|x\|_1 R^{a+b+d} |n|^{a+b}. \end{aligned}$$

Combining all above estimates, when  $|n| > 3\sqrt{d}$ , we arrive at

$$(4.3.9) \lesssim_d R^{a+b+d} |n|^{a+b-1} \|x\|_1.$$

Since  $a+b+d < 1$ , summing over  $|n| > 3\sqrt{d}$ , we get

$$\sum_{|n| > 3\sqrt{d}} \left\| |s|^a \sigma(s, t) \widehat{x}(s-t) [\phi(s) - \phi(t)] |t|^b \mathbb{1}_{Q_{Rn}}(t) \right\|_{S_1} \lesssim_d R^{a+b+d} \|x\|_1. \quad (4.3.11)$$

**Step 2:**  $|n| \leq 3\sqrt{d}$ . Let

$$E = \cup_{|n| \leq 3\sqrt{d}} Q_{Rn} \quad \text{and} \quad \widetilde{E} = \cup_{|n| \leq 3\sqrt{d}} \widetilde{Q}_{Rn}.$$

It suffices to estimate

$$\left\| |s|^a \sigma(s, t) \widehat{x}(s-t) [\phi(s) - \phi(t)] |t|^b \mathbb{1}_E(t) \right\|_{S_1}. \quad (4.3.12)$$

Again by the support assumption of  $\widehat{x}$

$$(4.3.12) = \left\| \mathbb{1}_{\widetilde{E}}(s) |s|^a \sigma(s, t) \widehat{x}(s-t) [\phi(s) - \phi(t)] |t|^b \mathbb{1}_E(t) \right\|_{S_1}.$$

Note that  $E \subset \widetilde{E} \subset \cup_{k \leq L} \Delta_k$  with  $L = \log_2[5R\sqrt{d}]$ . Thus, by the boundedness of  $\phi$ ,

$$\begin{aligned} (4.3.12) & \leq \sum_{j, k \leq L} \left\| \mathbb{1}_{\Delta_j}(s) |s|^a \sigma(s, t) \widehat{x}(s-t) [\phi(s) - \phi(t)] |t|^b \mathbb{1}_{\Delta_k}(t) \right\|_{S_1} \\ & \lesssim \sum_{j, k \leq L} \left\| \mathbb{1}_{\Delta_j}(s) |s|^a \sigma(s, t) \widehat{x}(s-t) |t|^b \mathbb{1}_{\Delta_k}(t) \right\|_{S_1}. \end{aligned}$$

By the arguments in Step 1, we have

$$\left\| \mathbb{1}_{\Delta_j}(s) |s|^a \sigma(s, t) \widehat{x}(s-t) |t|^b \mathbb{1}_{\Delta_k}(t) \right\|_{S_1} \leq \|x\|_1 \| |s|^a \|_{L_2(\Delta_j)} \| |t|^b \|_{L_2(\Delta_k)} \lesssim_d 2^{j(a+\frac{d}{2})+k(b+\frac{d}{2})}.$$

Since  $a, b > -\frac{d}{2}$ , summing over all  $j, k \in \mathbb{Z}$  with  $j, k \leq L$ , we get

$$(4.3.12) \lesssim_d R^{a+b+d} \|x\|_1.$$

Thus (4.3.11) is proved.

Finally, going back to (4.3.8), we have

$$\|I^a \mathbf{C}_{\phi, x} I^b\|_{S_1} \leq \sum_{n \in \mathbb{Z}} \left\| |s|^a \sigma(s, t) \widehat{x}(s-t) [\phi(s) - \phi(t)] |t|^b \mathbb{1}_{Q_{R_n}}(t) \right\|_{S_1} \lesssim_d R^{a+b+d} \|x\|_1.$$

This is (4.3.8). □

### 4.3.3 Upper bound for the $S_p$ -norm

We begin with the easiest case: the upper bound for the  $S_2$ -norm. In this case, the  $S_2$  norm of an operator  $T \in S_2(L_2(\mathbb{R}^d))$  is equal to the norm of its kernel in  $L_2(\mathbb{R}^d \times \mathbb{R}^d)$ .

**Lemma 4.3.3.** *Let  $a, b > -\frac{d}{2}$  and  $a + b + \frac{d}{2} < 1$ . If  $x \in B_{2,2}^{a+b+\frac{d}{2}}(\mathbb{R}^d)$ , then  $I^a \mathbf{C}_{\phi, x} I^b \in S_2$  and*

$$\|I^a \mathbf{C}_{\phi, x} I^b\|_{S_2} \lesssim_{d,a,b} \|x\|_{B_{2,2}^{a+b+\frac{d}{2}}}.$$

*Proof.* Like in the proof of Theorem 4.3.2, we assume that  $x$  is a good operator with kernel in (4.1.11). Then by (4.3.3)

$$\begin{aligned} \|I^a \mathbf{C}_{\phi, x} I^b\|_{S_2}^2 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| |s|^a \sigma(s, t) \widehat{x}(s-t) [\phi(s) - \phi(t)] |t|^b \right|^2 ds dt \\ &= \int_{\mathbb{R}^d} |\widehat{x}(s)|^2 \int_{\mathbb{R}^d} |s+t|^{2a} |\phi(s+t) - \phi(t)|^2 |t|^{2b} ds dt \\ &\lesssim \int_{\mathbb{R}^d} |\widehat{x}(s)|^2 \int_{\mathbb{R}^d} |s+t|^{2a} \min\left(1, \frac{|s|^2}{|t|^2}\right) |t|^{2b} ds dt \\ &= \int_{\mathbb{R}^d} |\widehat{x}(s)|^2 \left( \int_{|t| \leq |s|} |s+t|^{2a} |t|^{2b} + \int_{|t| > |s|} \frac{|s|^2}{|t|^2} |s+t|^{2a} |t|^{2b} \right) dt ds. \end{aligned}$$

Whereas  $a + b + \frac{d}{2} < 1$ , we see that

$$\int_{|t| > |s|} \frac{|s|^2}{|t|^2} |s+t|^{2a} |t|^{2b} dt \lesssim_{d,a,b} |s|^{2a+2b+d}.$$

On the other hand, by  $b > -\frac{d}{2}$ , we have

$$\int_{|t| \leq |s|} |s+t|^{2a} |t|^{2b} \lesssim_{d,b} |s|^{2a+2b+d}.$$

Putting the above estimates together, we obtain

$$\|I^a \mathbf{C}_{\phi, x} I^b\|_{S_2}^2 \lesssim_{d,a,b} \int_{\mathbb{R}^d} |\widehat{x}(s)|^2 |s|^{2a+2b+d} ds \lesssim_{d,a,b} \|x\|_{B_{2,2}^{a+b+\frac{d}{2}}}^2,$$

where the last step follows from the Plancherel formula and the fact that  $I^{a+b+\frac{d}{2}}$  is an isomorphism of  $B_{2,2}^{a+b+\frac{d}{2}}(\mathbb{R}^d)$  onto  $B_{2,2}^0(\mathbb{R}^d) = L_2(\mathbb{R}^d)$  (see Proposition 4.2.4). □

Now we are now able to prove the upper bound for the  $S_p$ -norm. We include the cases  $p \in \{1, 2, \infty\}$  already treated before just for completeness. Note that for  $p = \infty$ ,  $x$  should be additionally assumed to be good.

**Theorem 4.3.4.** *Let  $1 \leq p \leq \infty$ ,  $a + b + \frac{d}{p} < 1$  and  $a, b > \max(-\frac{d}{p}, -\frac{d}{2})$ . If  $x \in B_{p,p}^{a+b+\frac{d}{p}}(\mathbb{R}_\theta^d)$ , then  $I^a \mathbf{C}_{\phi,x} I^b$  belongs to  $B_{p,p}^{a+b+\frac{d}{p}}(\mathbb{R}_\theta^d)$  and*

$$\|I^a \mathbf{C}_{\phi,x} I^b\|_{S_p} \lesssim_{d,p,a,b} \|x\|_{B_{p,p}^{a+b+\frac{d}{p}}}.$$

*Proof.* We will use complex interpolation by the results already proved for  $p \in \{1, 2, \infty\}$ . To that end, we need first extend those results to complex powers in  $I^a$  and  $I^b$ . This is easy since for  $a, b \in \mathbb{C}$

$$\|I^a \mathbf{C}_{\phi,x} I^b\|_{S_p} = \|I^{\operatorname{Re}a} \mathbf{C}_{\phi,x} I^{\operatorname{Re}b}\|_{S_p}.$$

First consider the case  $2 < p < \infty$ . Let  $\eta = \frac{2}{p}$ . Under the conditions on  $a, b$ , one can choose  $a_i, b_i$  such that

$$\begin{aligned} a &= (1 - \eta)a_0 + \eta a_1, & b &= (1 - \eta)b_0 + \eta b_1; \\ a_0, b_0 &> 0, \quad a_0 + b_0 < 1, & a_1, b_1 &> -\frac{d}{2}, \quad a_1 + b_1 + \frac{d}{2} < 1. \end{aligned}$$

For  $z \in \{z \in \mathbb{C} : 0 \leq \operatorname{Re}z \leq 1\}$ , define

$$a(z) = a_0(1 - z) + a_1 z, \quad b(z) = b_0(1 - z) + b_1 z \quad \text{and} \quad T_z(x) = I^{a(z)} \mathbf{C}_{\phi,x} I^{b(z)}.$$

Then  $T_z$  is an analytic family of linear operators. Lemma 4.3.3 and Theorem 4.3.1 states that  $T_z$  is bounded from  $B_{\infty,\infty}^{a_0+b_0}(\mathbb{R}_\theta^d)$  to  $S_\infty$  for  $\operatorname{Re}z = 0$ , while by Lemma 4.3.3,  $T_z$  is maps  $B_{2,2}^{a_1+b_1+\frac{d}{2}}(\mathbb{R}_\theta^d)$  to  $S_2$  for  $\operatorname{Re}z = 1$ . Therefore, by complex interpolation,

$$T_\eta : \left( B_{\infty,\infty}^{a_0+b_0}(\mathbb{R}_\theta^d), B_{2,2}^{a_1+b_1+\frac{d}{2}}(\mathbb{R}_\theta^d) \right)_\eta \rightarrow \left( S_\infty, S_2 \right)_\eta \text{ is bounded.}$$

Thanks to Proposition 4.2.5, this implies that the desire assertion for  $2 < p < \infty$ . The case  $1 < p < 2$  is treated similarly, and is therefore omitted.  $\square$

Theorem 4.3.1 is not proved for all  $x \in B_{\infty,\infty}^{a_0+b_0}(\mathbb{R}_\theta^d)$  but only for good  $x$  with kernel as in (4.1.11). To make the preceding proof more rigorous, we can use the closure of  $\mathcal{S}_0(\mathbb{R}_\theta^d)$  in  $B_{\infty,\infty}^{a_0+b_0}(\mathbb{R}_\theta^d)$ . Since  $\mathcal{S}_0(\mathbb{R}_\theta^d)$  is dense in  $B_{2,2}^{a_1+b_1+\frac{d}{2}}(\mathbb{R}_\theta^d)$ , it is a classical fact from interpolation theory that Proposition 4.2.5 remains valid with this closure instead of  $B_{\infty,\infty}^{a_0+b_0}(\mathbb{R}_\theta^d)$ .

On the other hand, one cannot directly interpolate Theorem 4.3.1 and Theorem 4.3.2 because of the choice of  $a_i, b_i$  ( $i = 0, 1$ ) satisfying the required conditions. This explains why one has to consider the two cases  $p < 2$  and  $p > 2$  separately.

#### 4.3.4 Higher commutators

More generally, we may consider the  $N$ -th order commutators. Let  $\phi_1, \dots, \phi_N \in C^\infty(\mathbb{S}^{d-1})$  be  $N$  non-constant functions. Define

$$\mathbf{C}_{\phi_1, \dots, \phi_N, x} = [T_{\phi_N}, \dots, [T_{\phi_1}, M_x] \dots] \quad (4.3.13)$$

For a good operator  $x$  with kernel as in (4.1.10), the kernel of  $\mathbf{C}_{\phi_1, \dots, \phi_N, x}$  as an integral operator on  $L_2(\mathbb{R}^d)$  is

$$K_{\phi_1, \dots, \phi_N, x}(s, t) = \sigma(s, t) \widehat{x}(s - t) \prod_{i=1}^N (\phi_i(s) - \phi_i(t)). \quad (4.3.14)$$

Then for  $a, b \in \mathbb{R}$ , we have

$$\|I^a \mathbf{C}_{\phi_1, \dots, \phi_N, x} I^b\|_{S_p(L_2(\mathbb{R}^d))} = \| |s|^a K_{\phi_1, \dots, \phi_N, x}(s, t) |t|^b \|_{S_p(L_2(\mathbb{R}^d))}.$$

Theorem 4.3.4 extends to higher commutators:

**Theorem 4.3.5.** *Let  $1 \leq p \leq \infty$ . Let  $a, b \in \mathbb{R}$  such that  $a + b + \frac{d}{p} < N$  and  $a, b > \max(-\frac{d}{p}, -\frac{d}{2})$ . If  $x \in B_{p,p}^{\frac{d}{p}}(\mathbb{R}_\theta^d)$  ( $x$  is assumed additionally to be a good operator for  $p = \infty$ ), then  $\mathbf{C}_{\phi_1, \dots, \phi_N, x}$  has bounded extension in  $S_p(L_2(\mathbb{R}_\theta^d))$ . Moreover,*

$$\|\mathbf{C}_{\phi_1, \dots, \phi_N, x}\|_{S_p} \lesssim_{d,p,a,b,N} \prod_{i=1}^N \left[ \sup_{s \in \mathbb{S}^{d-1}} |\phi_i(s)| + \sup_{s \in \mathbb{S}^{d-1}} |\nabla \phi_i(s)| \right] \|x\|_{B_{p,p}^{\frac{d}{p}}}.$$

*Proof.* With minor modifications, the arguments in the previous subsections easily extend to the present setting. Let us point out only the necessary modifications. Firstly, for  $p = \infty$ , the only modification lies in the proof of (4.3.6) (and in the subsequent arguments, of course), now the factor  $2^{\ell-k}$  there is changed to  $2^{N(\ell-k)}$ , which leads to the corresponding change on  $a, b$ : now  $a + b < N$  instead of  $a + b < 1$  before. Secondly, for  $p = 1$ , the only change occurs for (4.3.9): the factor  $|n|^{-1}$  there is replaced by  $|n|^{-N}$ . Thirdly, for  $p = 2$ , in the first displayed formula in the proof of Lemma 4.3.3, we have to change  $\min\left(1, \frac{|s|^2}{|t|^2}\right)$  to  $\min\left(1, \frac{|s|^{2N}}{|t|^{2N}}\right)$ . Finally, the interpolation argument in the proof of Theorem 4.3.4 remains the same.  $\square$

## 4.4 Lower bounds for commutators and the proof of Theorem 1.2.1

This section is devoted to the converse results of those in the previous section. Let  $\phi_1, \dots, \phi_N \in C^\infty(\mathbb{S}^{d-1})$  be  $N$  functions such that

$$\sup_{s \in \mathbb{S}^{d-1}} |\phi_i(s)| + \sup_{s \in \mathbb{S}^{d-1}} |\nabla \phi_i(s)| \leq 1, \quad 1 \leq i \leq N.$$

Recall that the higher commutator  $\mathbf{C}_{\phi_1, \dots, \phi_N, x}$  in subsection 4.3.4. We will show that if  $\mathbf{C}_{\phi_1, \dots, \phi_N, x} \in S_p(L_2(\mathbb{R}_\theta^d))$ , then  $x \in B_{p,p}^{\frac{d}{p}}(\mathbb{R}_\theta^d)$  if  $\mathbf{C}_{\phi_1, \dots, \phi_N, x} \in S_p(L_2(\mathbb{R}_\theta^d))$  under the following nondegeneracy condition:

$$\forall s \in \mathbb{R}^d \setminus \{0\} \exists t \in \mathbb{R}^d \setminus \{0\} \text{ such that } \prod_{i=1}^N (\phi_i(s) - \phi_i(t)) \neq 0. \quad (4.4.1)$$

For  $N = 1$ , this condition means that  $\phi_1$  is not a constant function. We will need another associated higher one: for  $k \geq 1$  set

$$\mathbf{C}_{N,k,y} = \mathbf{C}_{\underbrace{\phi_1, \dots, \phi_N}_{k \text{ tuple}}, \underbrace{\bar{\phi}_1, \dots, \bar{\phi}_N}_{k-1 \text{ tuple}}, y'}$$



where  $\underbrace{\phi_1, \dots, \phi_N}_{k \text{ tuple}}$  means  $k$  tuple of  $(\phi_1, \dots, \phi_N)$ , that is, each  $\phi_i$  is repeated  $k$  times.

The following lemma is the counterpart of [50, Lemma 9.1] for the classical euclidean spaces. Recall that  $p'$  denotes the conjugate index of  $p$ .

**Lemma 4.4.1.** *Suppose that  $1 \leq p \leq \infty$ ,  $k \geq 1$ ,  $a, b, a_1, b_1 \in \mathbb{R}$  satisfying the following conditions*

$$b + b_1 > -d, \quad a + a_1 + b + b_1 + d < 2kN; \quad (4.4.2)$$

$$a_1, b_1 > \max\left(-\frac{d}{2}, -\frac{d}{p'}\right), \quad a_1 + b_1 + \frac{d}{p'} < (2k - 1)N. \quad (4.4.3)$$

Denote  $\gamma = -(a + a_1 + b + b_1 + d)$  and set

$$\omega(s) = |s|^\gamma \int_{\mathbb{R}^d} \prod_{i=1}^N |\phi_i(s+t) - \phi_i(t)|^{2k} |s+t|^{a+a_1} |t|^{b+b_1} dt. \quad (4.4.4)$$

Then for  $x \in \mathcal{S}(\mathbb{R}_\theta^d)$ ,

$$\|T_\omega(x)\|_{B_{p,p}^{a+b+\frac{d}{p}}} \leq C \|I^a \mathbf{C}_{\phi_1, \dots, \phi_N, x} I^b\|_{S_p}.$$

Here and late in the proof, the constant  $C$  depends on  $d$  and all involved indices.

*Proof.* It is easy to show that the integral in (4.4.4) is convergent thanks to (4.4.2). By definition, it is clear that  $\omega$  is homogenous of order zero. Let  $s_0 \neq 0$ . By (4.4.1), there exists  $t_0 \neq 0$  such that  $\phi_i(s_0) - \phi_i(t_0) \neq 0$  for every  $1 \leq i \leq N$ . Choose  $\delta > 0$  such that

$$\left| \frac{s}{|s|} - \frac{s_0}{|s_0|} \right| < \delta \implies |\phi_i(s) - \phi_i(s_0)| < \frac{1}{2} |\phi_i(s_0) - \phi_i(t_0)|, \quad 1 \leq i \leq N.$$

Let  $r > 0$  sufficiently large so that

$$\left| \frac{rs_0 + t_0}{|rs_0 + t_0|} - \frac{s_0}{|s_0|} \right| = \left| \frac{s_0 + r^{-1}t_0}{|s_0 + r^{-1}t_0|} - \frac{s_0}{|s_0|} \right| < \delta.$$

Thus

$$|\phi_i(rs_0 + t_0) - \phi_i(s_0)| < \frac{1}{2} |\phi_i(s_0) - \phi_i(t_0)|.$$

However, by the homogeneity of  $\phi_i$ , we have

$$\phi_i(s_0 + r^{-1}t_0) - \phi_i(r^{-1}t_0) = \phi_i(rs_0 + t_0) - \phi_i(t_0) = [\phi_i(s_0) - \phi_i(t_0)] + [\phi_i(rs_0 + t_0) - \phi_i(s_0)].$$

Hence,

$$|\phi_i(s_0 + r^{-1}t_0) - \phi_i(r^{-1}t_0)| \geq |\phi_i(s_0) - \phi_i(t_0)| - |\phi_i(rs_0 + t_0) - \phi_i(s_0)| > \frac{1}{2} |\phi_i(s_0) - \phi_i(t_0)|.$$

Then by continuity,  $\phi_i(s_0 + t) - \phi_i(t) \neq 0$  for  $t$  in the neighborhood of  $r^{-1}t_0$  for every  $1 \leq i \leq N$ , which clearly implies that  $\omega(s_0) \neq 0$ .

We will prove the desired inequality by duality. By (4.4.3) and Theorem 4.3.5, for any  $y \in \mathcal{S}(\mathbb{R}_\theta^d)$  we have

$$\|I^{a_1} \mathbf{C}_{N,k,y} I^{b_1}\|_{S_{p'}} \leq C \|y\|_{B_{p',p'}^{a_1+b_1+\frac{d}{p'}}}.$$

Then for  $x \in \mathcal{S}(\mathbb{R}_\theta^d)$ , by the Placherel formula, we have

$$\begin{aligned}
\langle I^a C_{\phi_1, \dots, \phi_N, x} I^b, I^{a_1} C_{N, k, y} I^{b_1} \rangle &= \tau_\theta \left[ I^a C_{\phi, x} I^b \left( I^{a_1} C_{N, k, y} I^{b_1} \right)^* \right] \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widehat{x}(s-t) \overline{\widehat{y}(s-t)} |\phi(s) - \phi(t)|^2 |s|^{a+a_1} |t|^{b+b_1} ds dt \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widehat{x}(s) \overline{\widehat{y}(s-t)} |\phi(s+t) - \phi(t)|^2 |s+t|^{a+a_1} |t|^{b+b_1} ds dt \\
&= \int_{\mathbb{R}^d} \widehat{x}(s) \overline{\widehat{y}(s-t)} |s|^{-\gamma} \omega(s) ds \\
&= \langle I^{-\gamma} T_\omega(x), y \rangle.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|\langle I^{-\gamma} T_\omega(x), y \rangle| &\leq \|I^a C_{\phi_1, \dots, \phi_N, x} I^b\|_{S_p} \|I^{a_1} C_{N, k, y} I^{b_1}\|_{S_{p'}} \\
&\leq C \|I^a C_{\phi_1, \dots, \phi_N, x} I^b\|_{S_p} \|y\|_{B_{p', p'}^{a_1+b_1+\frac{d}{p'}}}.
\end{aligned}$$

For  $p > 1$ , by Proposition 4.2.3

$$\left( B_{p', p'}^{a_1+b_1+\frac{d}{p'}}(\mathbb{R}_\theta^d) \right)^* = B_{p, p}^{-a_1-b_1-\frac{d}{p'}}(\mathbb{R}_\theta^d).$$

Then by the density of  $\mathcal{S}(\mathbb{R}_\theta^d)$  in  $B_{p', p'}^{a_1+b_1+\frac{d}{p'}}(\mathbb{R}_\theta^d)$  (see Proposition 4.2.3), we deduce

$$\|I^{-\gamma} T_\omega(x)\|_{B_{p, p}^{-a_1-b_1-\frac{d}{p'}}} \lesssim_{d, p', a_1, b_1} \|I^a C_{\phi_1, \dots, \phi_N, x} I^b\|_{S_p}.$$

However, by Proposition 4.2.4,

$$\|I^{-\gamma} T_\omega(x)\|_{B_{p, p}^{-a_1-b_1-\frac{d}{p'}}} \approx \|T_\omega(x)\|_{B_{p, p}^{a+b+\frac{d}{p}}}.$$

Thus the assertion is proved for  $p > 1$ .

For the case  $p = 1$ , we use the subspace  $B_{\infty, c_0}^{a_1+b_1+\frac{d}{p'}}(\mathbb{R}_\theta^d)$  in place of  $B_{\infty, \infty}^{a_1+b_1+\frac{d}{p'}}(\mathbb{R}_\theta^d)$ , the above argument works equally.  $\square$

In order to apply Lemma 4.4.1, we need the following Tauberian result for a general homogeneous function  $\nu$  as in [50]. Recall that a function  $\nu$  on  $\mathbb{R}^d$  is called a Fourier multiplier on a Besov space  $B_{p, q}^a(\mathbb{R}^d)$  (resp.  $B_{p, q}^a(\mathbb{R}_\theta^d)$ ) if  $T_\nu$  is bounded on  $B_{p, q}^a(\mathbb{R}^d)$  (resp.  $B_{p, q}^a(\mathbb{R}_\theta^d)$ ).

**Lemma 4.4.2.** *Suppose that  $\nu$  is a homogeneous function of order zero on  $\mathbb{R}^d \setminus \{0\}$  and never vanishes. If  $\nu$  is a Fourier multiplier on  $B_{1, 1}^{a_0}(\mathbb{R}^d)$  for some  $a_0 \in \mathbb{R}$ , then  $\nu^{-1}$  is a Fourier multiplier on  $B_{p, p}^a(\mathbb{R}_\theta^d)$  for all  $a \in \mathbb{R}$  and  $1 \leq p \leq \infty$ .*

*Proof.* Consider the Banach algebra  $\widehat{L}_1 = \{\widehat{f} : f \in L_1(\mathbb{R}^d)\}$ . Let  $\Delta' = \{s \in \mathbb{R}^d : 1 \leq |s| \leq 8\}$  and let  $I = \{\widehat{f} \in \widehat{L}_1 : f = 0 \text{ on } \Delta'\}$ . Then  $I$  is a closed ideal of  $\widehat{L}_1$  and  $\widehat{L}_1/I = \{\widehat{f}|_{\Delta'} : f \in L_1(\mathbb{R}^d)\}$  is a unital Banach algebra.

Choose  $\eta \in C^\infty(\mathbb{R}^d)$  supported in a larger annulus such that  $\eta = 1$  on  $\Delta'$ . Then  $\check{\eta} \in B_{1, 1}^{a_0}$ , so  $T_\nu(\check{\eta}) \in B_{1, 1}^{a_0}(\mathbb{R}^d)$ ; it follows that the Fourier transform of  $T_\nu(\check{\eta})$ , that is  $\nu\eta$ , belongs to  $\widehat{L}_1$ . Thus

$\nu|_{\Delta'} = \nu\eta|_{\Delta'} \in \widehat{L}_1/I$ . Since  $\nu \neq 0$  on  $\Delta'$ , we have that  $\nu^{-1}|_{\Delta'} \in \widehat{L}_1/I$ , i.e., there exists  $h \in L_1(\mathbb{R}^d)$  such that  $\widehat{h} = \nu^{-1}$  on  $\Delta'$ .

Now let  $x \in B_{p,p}^a(\mathbb{R}_\theta^d)$  and  $\widehat{y} = \nu^{-1}\widehat{x}$ . If  $\Delta'_k = \{s \in \mathbb{R}^d : 2^{k-1} \leq |s| \leq 2^{k+1}\} \subset \Delta'$ , then ( $\varphi$  being the test function in (4.2.1))

$$\widehat{\varphi_k * y} = \widehat{\varphi_k} \nu^{-1} \widehat{x} = \widehat{\varphi_k} \widehat{h} \widehat{x} = h * \widehat{\varphi_k} * x,$$

and thus

$$\|\varphi_k * y\|_p \leq \|h\|_1 \|\varphi_k * x\|_p.$$

By homogeneity, the above inequality holds for arbitrary  $\Delta'_k$ , hence  $\|y\|_{B_{p,p}^a} \leq C\|x\|_{B_{p,p}^a}$ . This means that  $\nu^{-1}$  is a Fourier multiplier on  $B_{p,p}^a(\mathbb{R}_\theta^d)$   $\square$

**Theorem 4.4.3.** *Let  $1 \leq p \leq \infty$  and  $a, b \in \mathbb{R}$ . Let  $x$  be a good operator with kernel as in (4.1.10), for instance,  $x \in \lambda_\theta(L_1(\mathbb{R}^d))$ , and  $\phi_1, \dots, \phi_N$  satisfy the nondegeneracy condition 4.4.1. If  $I^a C_{\phi_1, \dots, \phi_N, x} I^b \in S_p(L_2(\mathbb{R}_\theta^d))$ , then  $x \in B_{p,p}^{a+b+\frac{d}{p}}(\mathbb{R}_\theta^d)$  and*

$$\|x\|_{B_{p,p}^{a+b+\frac{d}{p}}} \lesssim_{d,p,a,b,N} \|I^a C_{\phi_1, \dots, \phi_N, x} I^b\|_p.$$

*Proof.* Choose first  $a_1, b_1 > 0$  such that  $a + a_1, b + b_1 > 0$ , then  $a' = b' \in (-\frac{d}{2}, \min(0, \frac{N-d}{2}))$ . Let  $a'_1 = a + a_1 - a', b'_1 = b + b_1 - b'$ . Choose  $k \in \mathbb{N}$  such that  $a_1 + b_1 + d < (2k-1)N$  and  $a + a_1 + b + b_1 + d < 2kN$ .

Applying Lemma 4.4.1 to  $p = 1$  and the indices  $a', b', a_1, b'_1$ , we get

$$\|T_\omega(x)\|_{B_{1,1}^{a'+b'+d}} \leq C \|I^{a'} C_{\phi_1, \dots, \phi_N, x} I^{b'}\|_{S_1}.$$

On the other hand, by Theorem 4.3.5,

$$\|I^{a'} C_{\phi_1, \dots, \phi_N, x} I^{b'}\|_{S_1} \leq C \|x\|_{B_{1,1}^{a'+b'+d}}.$$

Thus

$$\|T_\omega(x)\|_{B_{1,1}^{a'+b'+d}} \leq C \|x\|_{B_{1,1}^{a'+b'+d}}.$$

This means that  $\omega$  is a Fourier multiplier on  $B_{1,1}^{a'+b'+d}(\mathbb{R}_\theta^d)$ , especially for the degenerate case  $\theta = 0$  too. This enables us to apply Lemma 4.4.2; consequently,  $\omega^{-1}$  is a Fourier multiplier on  $B_{p,p}^{a+b+\frac{d}{p}}(\mathbb{R}_\theta^d)$ . Thus

$$\|x\|_{B_{p,p}^{a+b+\frac{d}{p}}} \leq C \|T_\omega(x)\|_{B_{p,p}^{a+b+\frac{d}{p}}}.$$

Now applying Lemma 4.4.1 once more, this time to  $p, a, b, \dots$  (noting that we have the same function  $\omega$  defined by (4.4.4) according to the choice of the indices), we finally deduce

$$\|x\|_{B_{p,p}^{a+b+\frac{d}{p}}} \lesssim_{d,p,a,b,N} \|I^a C_{\phi_1, \dots, \phi_N, x} I^b\|_{S_p}.$$

Thus the theorem is proved.  $\square$

*Proof of Theorem 1.2.1.* Combining Theorems 4.3.4 and 4.4.3, we conclude Theorem 1.2.1 for  $x \in \mathcal{S}(\mathbb{R}^d)$ . For general  $x \in B_{p,p}^{\frac{d}{p}}(\mathbb{R}^d)$ , by Proposition 4.2.3, we may find a sequence  $\{x_n\}_{n \geq 1} \subset \mathcal{S}(\mathbb{R}^d)$  such that  $x_n \rightarrow x$  in  $B_{p,p}^{\frac{d}{p}}(\mathbb{R}^d)$ . Since  $\{x_n\}_{n \geq 1}$  is Cauchy in  $B_{p,p}^{\frac{d}{p}}(\mathbb{R}^d)$ , by the upper bound estimate, the sequence  $\{\mathbf{C}_{\phi, x_n}\}_{n \geq 1}$  is also Cauchy in  $S_p$ , so has a limit in  $S_p$ . On the other hand, it is straightforward to verify that  $\mathbf{C}_{\phi, x_n} \eta \rightarrow \mathbf{C}_{\phi, x} \eta$  in  $L_2(\mathbb{R}^d)$  for any  $\eta \in \mathcal{S}(\mathbb{R}^d)$ . Therefore, the limit of  $\{\mathbf{C}_{\phi, x_n}\}_{n \geq 1}$  in  $S_p$  must be  $\mathbf{C}_{\phi, x}$ . Now that,

$$\lim_{n \rightarrow \infty} \|x_n\|_{B_{p,p}^{\frac{d}{p}}} = \|x\|_{B_{p,p}^{\frac{d}{p}}} \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\mathbf{C}_{\phi, x_n}\|_{S_p} = \|\mathbf{C}_{\phi, x}\|_{S_p},$$

we complete the proof of the theorem.

Conversely, we have to deduce  $x \in B_{p,p}^{\frac{d}{p}}(\mathbb{R}^d)$  from  $\mathbf{C}_{\phi, x} \in S_p$ . To that end, we will use the following simple fact: When  $x \in L_\infty(\mathbb{R}^d)$  and  $\mathbf{C}_{\phi, x} \in S_p$  with  $1 \leq p \leq \infty$ , for  $v \in L_1(\mathbb{R}^d)$ , we have

$$\|\mathbf{C}_{\phi, v * x}\|_{S_p} \leq \|v\|_1 \|\mathbf{C}_{\phi, x}\|_{S_p}. \quad (4.4.5)$$

This is the  $S_p$  analogue of [62, Equality (6.12)], and the proof is very simple since  $\mathbf{C}_{\phi, v * x}$  can be written as a convolution with  $v$ .

From [62, Lemma 3.12], we may select  $\{\psi_\varepsilon\}_{\varepsilon > 0}$  and  $\{\phi_\varepsilon\}_{\varepsilon > 0}$  such that  $\psi_\varepsilon * (\lambda_\theta(\phi_\varepsilon)x) \in \mathcal{S}(\mathbb{R}^d)$ . The upper bound (4.4.5) implies:

$$\|\mathbf{C}_{\phi, \psi_\varepsilon * (\lambda_\theta(\phi_\varepsilon)x)}\|_{S_p} \leq \|\psi_\varepsilon\|_1 \|\mathbf{C}_{\phi, \lambda_\theta(\phi_\varepsilon)x}\|_{S_p}.$$

By (4.4.5) and Theorem 4.3.4, we have

$$\begin{aligned} \|\mathbf{C}_{\phi, \psi_\varepsilon * (\lambda_\theta(\phi_\varepsilon)x)}\|_{S_p} &\leq \|\psi_\varepsilon\|_1 \left( \|\mathbf{C}_{\phi, \lambda_\theta(\phi_\varepsilon)x}\|_{S_p} + \|\lambda_\theta(\phi_\varepsilon)\|_\infty \|\mathbf{C}_{\phi, x}\|_{S_p} \right) \\ &\lesssim_{d,p} \|\psi_\varepsilon\|_1 \left( \|\lambda_\theta(\phi_\varepsilon)\|_{B_{p,p}^{\frac{d}{p}}} \|x\|_\infty + \|\lambda_\theta(\phi_\varepsilon)\|_\infty \|\mathbf{C}_{\phi, x}\|_{S_p} \right) \\ &\lesssim_{d,p} \|x\|_\infty + \|\mathbf{C}_{\phi, x}\|_{S_p}. \end{aligned}$$

It follows that  $\{\mathbf{C}_{\phi, \psi_\varepsilon * (\lambda_\theta(\phi_\varepsilon)x)}\}_{\varepsilon > 0}$  is uniformly bounded in  $S_p$  as  $\varepsilon \rightarrow 0$ . Now applying Theorem 4.4.3 to the smooth elements  $\psi_\varepsilon * (\lambda_\theta(\phi_\varepsilon)x)$ , we see that  $\{\psi_\varepsilon * (\lambda_\theta(\phi_\varepsilon)x)\}_{\varepsilon > 0}$  is uniformly bounded in  $B_{p,p}^{\frac{d}{p}}(\mathbb{R}^d)$ , so the sequence  $\{(\varphi_k * (\psi_\varepsilon * (\lambda_\theta(\phi_\varepsilon)x)))_{k \in \mathbb{Z}}\}_{\varepsilon > 0}$  is uniformly bounded in  $\ell_p^{\frac{d}{p}}(L_p(\mathbb{R}^d))$ . Since  $1 < p < \infty$ , the space  $\ell_p^{\frac{d}{p}}(L_p(\mathbb{R}^d))$  is reflexive and therefore  $\{(\varphi_k * (\psi_\varepsilon * (\lambda_\theta(\phi_\varepsilon)x)))_{k \in \mathbb{Z}}\}_{\varepsilon > 0}$  has a weak limit point in  $\ell_p^{\frac{d}{p}}(L_p(\mathbb{R}^d))$ . But one easily checks that for any  $y = (y_k)_{k \in \mathbb{Z}} \in \ell_q^{\frac{d}{p}}(L_q(\mathbb{R}^d))$  (finitely supported in the sense that only finitely many  $y_k \neq 0$ ),

$$\lim_{\varepsilon \rightarrow 0} \sum_{k \in \mathbb{Z}} (\varphi_k * (\psi_\varepsilon * (\lambda_\theta(\phi_\varepsilon)x)), y_k) = \sum_{k \in \mathbb{Z}} (\varphi_k * x, y_k)$$

It follows that the weak limit point of  $\{(\varphi_k * (\psi_\varepsilon * (\lambda_\theta(\phi_\varepsilon)x)))_{k \in \mathbb{Z}}\}_{\varepsilon > 0}$  in  $\ell_p^{\frac{d}{p}}(L_p(\mathbb{R}^d))$  must be  $(\varphi_k * x)_{k \in \mathbb{Z}}$ . Therefore,  $x \in B_{p,p}^{\frac{d}{p}}(\mathbb{R}^d)$ .  $\square$

## 4.5 The $S_{d,\infty}$ property of $C_{\phi,x}$ and the proofs of Theorems 1.2.2 and 1.2.3

In this section we consider the endpoint case of Theorem 1.2.1 when  $p = d$ . Instead of  $S_d$ , we will characterise the membership of  $C_{\phi,x}$  in the weak Schatten  $d$ -class  $S_{d,\infty}$ .

### 4.5.1 Upper estimate for $S_{d,\infty}$ property of $C_{\phi,x}$

It is shown in [63, Corollary 5.2] that  $x \in \dot{W}_d^1(\mathbb{R}_\theta^d)$  iff  $[R_j, M_x]$  has bounded extension in  $S_{d,\infty}$  for all  $1 \leq j \leq d$ . Here  $R_j$  denotes the  $j$ -th Riesz transform on  $\mathbb{R}_\theta^d$ , or from the perspective of functional calculus,  $R_j = \phi(\nabla)$  with  $\phi(\xi) = \frac{\xi_j}{|\xi|}$  and  $\nabla = (\partial_1, \dots, \partial_d)$ .

Based on the estimates of  $[R_j, M_x]$ , we are able to deduce the estimate of  $C_{\phi,x}$  for general  $\phi \in C^\infty(\mathbb{S}^{d-1})$ .

**Lemma 4.5.1.** *Assume that  $\phi \in C^\infty(\mathbb{S}^{d-1})$  and  $x \in \dot{W}_d^1(\mathbb{R}_\theta^d)$ . Then  $C_{\phi,x}$  has a bounded extension in  $S_{d,\infty}$  satisfying*

$$\|C_{\phi,x}\|_{S_{d,\infty}} \leq C_{\phi,d} \max_{1 \leq j \leq d} \|[R_j, x]\|_{S_{d,\infty}}.$$

*Proof.* We appeal to [95, Lemma 3.5], which states that: Let  $(B_1, \dots, B_d) \subset B(H)$  be a commuting self-adjoint tuple. Let  $A \in B(H)$ . Let  $h \in \mathcal{S}(\mathbb{R}^d)$ . Then for any  $1 < p < \infty$ , we have

$$\|[A, h(B_1, \dots, B_d)]\|_{S_{p,\infty}} \leq C_{d,p} \max_{1 \leq j \leq d} \|\widehat{\partial_j h}\|_1 \cdot \max_{1 \leq j \leq d} \|[A, B_j]\|_{S_{p,\infty}}. \quad (4.5.1)$$

For our purpose, we set  $p = d$ ,  $A = M_x$  and  $B_j = R_j$ . Set also  $h(t) = g(\frac{t}{|t|})\chi(|t|)$ ,  $t \in \mathbb{R}^d$  where  $\chi$  is a Schwartz function on  $\mathbb{R}$  vanishing on  $(-\infty, \frac{1}{2})$  and satisfying  $\chi(1) = 1$ . Then

$$\max_{1 \leq j \leq d} \|\widehat{\partial_j h}\|_1 \leq C_\phi.$$

and it follows from [62, Theorem 1.1] that for  $x \in \dot{W}_d^1(\mathbb{R}_\theta^d)$ ,

$$\max_{1 \leq j \leq d} \|[M_x, R_j]\|_{S_{d,\infty}} < \infty.$$

Evidently,  $h(R_1, \dots, R_d) = T_\phi$ , so (4.5.1) ensures the desired assertion.  $\square$

**Proposition 4.5.2.** *Assume that  $\phi \in C^\infty(\mathbb{S}^{d-1})$  and  $x \in \dot{W}_d^1(\mathbb{R}_\theta^d)$ . Then  $C_{\phi,x}$  has a bounded extension in  $S_{d,\infty}$  satisfying*

$$\|C_{\phi,x}\|_{S_{d,\infty}} \leq C_{\phi,d} \|x\|_{\dot{W}_d^1}.$$

*Proof.* Assume initially  $x \in \mathcal{S}(\mathbb{R}_\theta^d) \subset \dot{W}_d^1(\mathbb{R}_\theta^d)$ . The above lemma and [62, Theorem 1.1] yield

$$\|C_{\phi,x}\|_{S_{d,\infty}} \leq C_\phi \max_{1 \leq j \leq d} \|[R_j, M_x]\|_{S_{d,\infty}} \leq C_{\phi,d} \|x\|_{\dot{W}_d^1}. \quad (4.5.2)$$

For general  $x \in \dot{W}_d^1(\mathbb{R}_\theta^d)$ , by [63, Proposition 4.9], we may approximate  $x$  by a sequence  $\{x_n\}_{n \geq 1} \subset \mathcal{S}(\mathbb{R}_\theta^d)$  such that  $\lim_{n \rightarrow \infty} \|x_n - x\|_{\dot{W}_d^1} = 0$ . Then the estimate in (4.5.2) forces  $\{[T_\phi, M_{x_n}]\}_{n \geq 1}$  Cauchy in the  $S_{d,\infty}$  topology. Hence, there is a limit in  $S_{d,\infty}$ . On the other hand, for any  $\eta \in \mathcal{S}(\mathbb{R}_\theta^d) \subset L_2(\mathbb{R}_\theta^d)$ ,

$$[T_\phi, M_{x_n}]\eta \rightarrow [T_\phi, M_x]\eta$$

in  $L_2(\mathbb{R}_\theta^d)$ . So  $[T_\phi, M_{x_n}] \rightarrow [T_\phi, M_x]$  in the  $S_{d,\infty}$  topology. Since (4.5.2) holds for all  $\{x_n\}_{n \geq 1}$  with the same constant, passing  $n \rightarrow \infty$ , we conclude the proof.  $\square$

### 4.5.2 The trace formula

This part is devoted to the computation of  $\text{Tr}_\omega(|C_{\phi,x}|^d)$  for continuous normalised trace  $\text{Tr}_\omega$  on  $S_{1,\infty}$  (or especially for Dixmier trace).

Since the partial derivatives on classical euclidean spaces and on quantum euclidean spaces will appear in the same circumstance, we make the convention that,  $\partial_{\xi_k}, \partial_\xi^\alpha$  will denote the partial derivatives of real variable functions  $\phi(\xi)$  on classical euclidean spaces, while  $\partial_k, \partial^\alpha$  will denote the partial derivatives on quantum euclidean spaces defined in subsection 4.1.2.

To begin with, we replace  $T_\phi$  by another Fourier multiplier  $T_{\tilde{\phi}}$  whose symbol is smooth on the whole  $\mathbb{R}^d$ . Such  $\tilde{\phi}$  is easy to find: letting  $\chi \in C^\infty(\mathbb{R}^d)$  such that  $\chi(\xi) = 1$  for  $\{\xi \in \mathbb{R}^d : |\xi| \geq \frac{1}{2}\}$  and  $\chi(\xi) = 0$  for  $\{\xi \in \mathbb{R}^d : |\xi| \leq \frac{1}{4}\}$ . Take  $\tilde{\phi} = \chi\phi$ . We put

$$A = \frac{1}{2\pi i} \sum_{1 \leq k \leq d} T_{|\xi| \partial_{\xi_k} \tilde{\phi}} M_{\partial_k x}. \quad (4.5.3)$$

We are going to reduce the computation of  $\text{Tr}_\omega(|C_{\phi,x}|^d)$  to that of  $\text{Tr}_\omega(|A|^d (1 + \Delta)^{-\frac{d}{2}})$ .

We will engage the theory of pseudodifferential operators on  $\mathbb{R}^d$ . In the following, we collect some definitions and properties of symbol classes and pseudodifferential operators on noncommutative euclidean space. The main reference is [37].

**Definition 4.5.3.** For every  $m \in \mathbb{R}$ , the class  $S^m(\mathbb{R}^d; \mathcal{S}(\mathbb{R}_\theta^d))$  consists of all maps  $\rho \in C^\infty(\mathbb{R}^d; \mathcal{S}(\mathbb{R}_\theta^d))$  such that, for all multi-indices  $\alpha, \beta \in \mathbb{N}_0^d$ , there exists  $C_{\alpha,\beta} > 0$  such that

$$\|\partial_x^\alpha \partial_\xi^\beta \rho(\xi)\| \leq C_{\alpha,\beta} (1 + |\xi|^2)^{\frac{m-|\beta|}{2}}, \quad \forall \xi \in \mathbb{R}^d.$$

We can now define the pseudodifferential operator on  $\mathbb{R}_\theta^d$ .

**Definition 4.5.4.** Given  $f \in \mathcal{S}(\mathbb{R}^d)$  and  $\rho \in S^m(\mathbb{R}^d; \mathcal{S}(\mathbb{R}_\theta^d))$ , we set

$$P_\rho(\lambda_\theta(f)) = \int_{\mathbb{R}^d} f(\xi) \rho(\xi) \lambda_\theta(\xi) d\xi.$$

The operator  $P_\rho$  is called the pseudo-differential operator of symbol  $\rho$ .

If  $\rho \in S^m(\mathbb{R}^d; \mathcal{S}(\mathbb{R}_\theta^d))$  with  $m \in \mathbb{R}$ , then  $P_\rho$  is said to be a pseudodifferential operator of order  $m$ . The following conclusion, also quoted from [37], gives the principle of symbolic calculus for pseudodifferential operator.

**Proposition 4.5.5.** Let  $\rho_1, \rho_2$  be two symbols in  $S^{m_1}(\mathbb{R}^d; \mathcal{S}(\mathbb{R}_\theta^d)), S^{m_2}(\mathbb{R}^d; \mathcal{S}(\mathbb{R}_\theta^d))$  respectively. Then there exists a symbol  $\rho_3 \in S^{m_1+m_2}(\mathbb{R}^d; \mathcal{S}(\mathbb{R}_\theta^d))$  such that

$$P_{\rho_3} = P_{\rho_1} P_{\rho_2}.$$

Moreover,

$$\rho_3 - \sum_{|\alpha| < N_0} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \rho_1 \partial^\alpha \rho_2 \in S^{m_1+m_2-N_0}(\mathbb{R}^d; \mathcal{S}(\mathbb{R}_\theta^d)), \quad \forall N_0 \geq 0. \quad (4.5.4)$$

The following is the fundamental mapping property of the pseudodifferential operators on quantum euclidean space, see [37, Theorem 4.12].

**Proposition 4.5.6.** *Let  $\rho$  be a symbol of order 0, then  $P_\rho$  extends to a bounded operator on  $L_2(\mathbb{R}_\theta^d)$ .*

One feature of the Schwartz class  $\mathcal{S}(\mathbb{R}^d)$  is the factorisability: that is, every  $f \in \mathcal{S}(\mathbb{R}^d)$ , can be obtained as a product  $f = gh$  for  $g, h \in \mathcal{S}(\mathbb{R}^d)$ . To our best knowledge, such factorisability is not verified for  $\mathcal{S}(\mathbb{R}_\theta^d)$  for general  $\theta$ . In order to overcome this drawback, a dense subalgebra of  $\mathcal{S}(\mathbb{R}_\theta^d)$  is introduced in [62]. Namely, [62, Proposition 2.5] states that there is a dense  $*$ -subalgebra  $\mathcal{A}(\mathbb{R}_\theta^d) \subset \mathcal{S}(\mathbb{R}_\theta^d)$  such that every  $x \in \mathcal{A}(\mathbb{R}_\theta^d)$  can be expressed as a finite linear combination of products of elements of  $\mathcal{A}(\mathbb{R}_\theta^d)$ , i.e. every element  $x \in \mathcal{A}(\mathbb{R}_\theta^d)$  can be represented as  $x = \sum_{j=1}^n y_j z_j$  with each  $y_j, z_j \in \mathcal{A}(\mathbb{R}_\theta^d)$ .

With this subalgebra  $\mathcal{A}(\mathbb{R}_\theta^d)$ , we have the following

**Proposition 4.5.7.** *Let  $\rho \in S^{-m}(\mathbb{R}^d; \mathcal{S}(\mathbb{R}_\theta^d))$  with  $m > 0$ , and  $x \in \mathcal{A}(\mathbb{R}_\theta^d)$ . Then for  $n \geq 0$ , we have  $P_\rho M_x J^{-n} \in S_{\frac{d}{m+n}, \infty}$  and  $M_x P_\rho J^{-n} \in S_{\frac{d}{m+n}, \infty}$ .*

*Proof.* Without loss of generality, we may assume that  $x = yz$  for  $y, z \in \mathcal{A}(\mathbb{R}_\theta^d)$ . So we may decompose  $P_\rho M_x J^{-n}$  as

$$P_\rho M_x J^{-n} = (P_\rho J^m)(J^{-m} M_y)(M_z J^{-n})$$

Since  $P_\rho$  has order  $-m$ , Proposition 4.5.5 implies that the composition  $P_\rho M_x J^{-n}$  is of order 0 and thus has bounded extension by Proposition 4.5.6. Moreover, by [56, Theorem 7.6],  $J^{-m} M_y \in S_{\frac{d}{m}, \infty}$  and  $M_z J^{-n} \in S_{\frac{d}{n}, \infty}$  respectively. We therefore conclude  $P_\rho M_x J^{-n} \in S_{\frac{d}{m+n}, \infty}$  from the Hölder inequality. The other assertion is deduced similarly, using the decomposition  $M_x P_\rho J^{-n} = (M_x J^{-m-n})(J^{m+n} P_\rho J^{-n})$ .  $\square$

**Lemma 4.5.8.** *Let  $x \in \mathcal{A}(\mathbb{R}_\theta^d)$  and let  $A$  be defined as in (4.5.3). For any  $y \in \mathcal{A}(\mathbb{R}_\theta^d)$ , we have:*

$$M_y \mathbf{C}_{\tilde{\phi}, x} - M_y A J^{-1} \in S_{\frac{d}{2}, \infty}.$$

*Proof.* By the above proposition, we only have to show that the symbol of  $\mathbf{C}_{\tilde{\phi}, x} - A J^{-1}$  is of order  $-2$  using symbol calculus. Note that the symbol  $\tilde{\phi}$  of the Fourier multiplier  $T_{\tilde{\phi}}$  is smooth on the whole  $\mathbb{R}^d$ , so that  $T_{\tilde{\phi}}$  is viewed as a pseudodifferential operator of order 0. Now for two pseudodifferential operators  $T_{\tilde{\phi}}$  and  $M_x$  of order 0, we appeal to Lemma 4.5.5, to obtain the asymptotic expansion of the principal symbol of  $[T_{\tilde{\phi}}, M_x]$ . Indeed, the symbol of  $M_x T_{\tilde{\phi}}$  is  $\tilde{\phi}(\xi)x$ , while that of  $T_{\tilde{\phi}} M_x$  is expanded as

$$\sum_{j \geq 0} \sum_{|\alpha|=j} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \tilde{\phi}(\xi) \partial^\alpha x. \quad (4.5.5)$$

By the homogeneity of  $\phi$  and the definition of  $\tilde{\phi}$ , each  $\partial_\xi^\alpha \tilde{\phi}(\xi) \partial^\alpha x$  coincides with the homogeneous symbol  $\partial_\xi^\alpha \phi(\xi) \partial^\alpha x$  of order  $-|\alpha|$  when  $|\xi| \geq 1$ . Thus, the symbol of  $[T_{\tilde{\phi}}, M_x]$  is classical of order  $-1$ , the leading term being

$$\sum_{1 \leq k \leq d} \frac{1}{2\pi i} \partial_{\xi_k} \tilde{\phi}(\xi) \partial_k x. \quad (4.5.6)$$

Next, let us continue with the computation of symbol of  $A$  constructed in (4.5.3). Now it is easily checked that the symbol of  $A$  is classical of order 0, the principal symbol being

$$\sum_{1 \leq k \leq d} \frac{1}{2\pi i} |\xi| \partial_{\xi_k} \tilde{\phi}(\xi) \partial_k x. \quad (4.5.7)$$

Indeed. By Proposition 4.5.4, the symbol of  $A$  has the asymptotic expansion

$$\sum_{j \geq 0} \sum_{|\alpha|=j} \sum_{1 \leq k \leq d} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_{\xi}^{\alpha} (|\xi| \partial_{\xi_k} \tilde{\phi})(\xi) \partial_k x.$$

Obviously, this is a classical symbol of order 0, with principal symbol

$$\sum_{1 \leq k \leq d} |\xi| \partial_{\xi_k} \tilde{\phi}(\xi) \partial_k x.$$

So, the principal symbol of  $AJ_{\theta}^{-1}$  is of order  $-1$ , given by

$$\sum_{1 \leq k \leq d} \frac{1}{2\pi i} \partial_{\xi_k} \tilde{\phi}(\xi) \partial_k x,$$

which is the same as in (4.5.6). Hence,  $\mathbf{C}_{\tilde{\phi},x} - AJ^{-1}$  is of order  $-2$ . The desired assertion follows then from Proposition 4.5.7.  $\square$

**Lemma 4.5.9.** *Let  $x, y, A$  be as in Lemma 4.5.8. For any continuous normalised trace  $\text{Tr}_{\omega}$  on  $S_{1,\infty}$ , we have*

$$\text{Tr}_{\omega}(|M_y \mathbf{C}_{\phi,x}|^d) = \text{Tr}_{\omega}(|M_y A|^d J^{-d}).$$

*Proof.* By the fact that any continuous normalised trace  $\text{Tr}_{\omega}$  on  $S_{1,\infty}$  vanishes on  $S_1$ , we are reduced to showing that

$$|M_y \mathbf{C}_{\phi,x}|^d - |M_y A|^d J^{-d} \in S_1. \quad (4.5.8)$$

We first claim the following ‘‘symmetric’’ version of (4.5.8):

$$|M_y \mathbf{C}_{\phi,x}|^d - \left( J^{-1} |M_y A|^2 J^{-1} \right)^{\frac{d}{2}} \in S_1. \quad (4.5.9)$$

By the construction of  $\tilde{\phi}$ , we see that  $\phi - \tilde{\phi} \in L_p(\mathbb{R}^d)$  for any  $p \geq 1$ , so [62, Theorem 3.17] ensures that  $\mathbf{C}_{\phi,x} - \mathbf{C}_{\tilde{\phi},x} \in S_{\frac{d}{2},\infty}$ . By the previous lemma, we have

$$M_y \mathbf{C}_{\tilde{\phi},x} - M_y A J^{-1} \in S_{\frac{d}{2},\infty}. \quad (4.5.10)$$

Next, we proceed as the proof of [62, Lemma 6.4]. Taking the adjoint of (4.5.10), we have

$$\left( M_y \mathbf{C}_{\tilde{\phi},x} \right)^* - J^{-1} \left( M_y A \right)^* \in S_{\frac{d}{2},\infty}. \quad (4.5.11)$$

Moreover, it follows from Proposition 4.5.7 that

$$M_y A J^{-1} \in S_{d,\infty}. \quad (4.5.12)$$

Combining (4.5.10), (4.5.11) and (4.5.12), we deduce from the Hölder inequality that

$$\begin{aligned} |M_y \mathbf{C}_{\phi,x}|^2 - J^{-1} |M_y A|^2 J^{-1} &= \left( M_y \mathbf{C}_{\tilde{\phi},x} \right)^* \left( M_y \mathbf{C}_{\tilde{\phi},x} - M_y A J^{-1} \right) \\ &\quad + \left( \left( M_y \mathbf{C}_{\tilde{\phi},x} \right)^* - J^{-1} \left( M_y A \right)^* \right) M_y A J^{-1} \in S_{\frac{d}{3},\infty} \subset S_{\frac{5d}{12}}. \end{aligned}$$



When  $d = 2$ , this gives (4.5.9). If  $d > 2$ , we appeal to a result from Ricard [86, Theorem 3.4], which says that we can take a power  $\frac{1}{2}$  to each term of the preceding inclusion to get

$$|M_y \mathbf{C}_{\phi,x}| - \left( J^{-1} |M_y A|^2 J^{-1} \right)^{\frac{1}{2}} \in S_{\frac{5d}{6}, \infty}.$$

Introducing a power  $d$ , we get

$$\begin{aligned} & |M_y \mathbf{C}_{\phi,x}|^d - \left( J^{-1} |M_y A|^2 J^{-1} \right)^{d/2} \\ &= \sum_{k=0}^{d-1} |M_y \mathbf{C}_{\phi,x}|^{d-k-1} \left( |M_y \mathbf{C}_{\phi,x}| - \left( J^{-1} |M_y A|^2 J^{-1} \right)^{1/2} \right) \left( J^{-1} |M_y A|^2 J^{-1} \right)^{\frac{k}{2}} \\ &\in \sum_{k=0}^{d-1} S_{\frac{d}{d-k-1}, \infty} \cdot S_{\frac{5d}{6}} \cdot S_{\frac{d}{k}, \infty} \subset S_{\frac{5d}{5d+1}, \infty} \subset S_1. \end{aligned}$$

So we have finished the proof of (4.5.9).

It remains to deduce (4.5.8) from (4.5.9). This is done by repeating *mutatis mutandi* the proof of [62, Proposition 6.5]; the details are omitted.  $\square$

The presence of  $M_y$  is necessary in Lemma 4.5.8, so necessary in Lemma 4.5.9, due to the fact that the Bessel potential  $J^{-a}$  is not compact on  $L_2(\mathbb{R}^d)$  for  $a > 0$ .

Lemma 4.5.8 reduces the computation of  $\text{Tr}_\omega(|M_y \mathbf{C}_{\phi,x}|^d)$  to that of  $\text{Tr}_\omega(|M_y A|^d (1 - \Delta)^{-\frac{d}{2}})$ . In order to calculate the trace  $\text{Tr}_\omega(|\mathbf{C}_{\phi,x}|^d)$ , we will first calculate the trace  $\text{Tr}_\omega(|M_y A|^d (1 - \Delta)^{-\frac{d}{2}})$  by Connes' trace formula [64, Theorem 6.15] for smooth  $y$ , and then approximate  $\text{Tr}_\omega(|\mathbf{C}_{\phi,x}|^d)$  by passing  $y \rightarrow 1$  in  $\text{Tr}_\omega(|M_y \mathbf{C}_{\phi,x}|^d)$ .

Let us quote [64, Theorem 6.15] in the following. Let  $C_0(\mathbb{R}_\theta^d)$  be the norm closure of  $\mathcal{S}(\mathbb{R}_\theta^d)$  in  $B(L_2(\mathbb{R}^d))$ . For every  $g \in C(\mathbb{S}^{d-1})$ , as explained in Remark 4.1.9,  $g(\nabla \Delta^{-\frac{1}{2}})$  is the Fourier multiplier  $\lambda_\theta(f) \mapsto \lambda_\theta(g(\frac{t}{|t|})f(t))$  in  $B(L_2(\mathbb{R}_\theta^d))$ . Identifying  $L_2(\mathbb{R}_\theta^d)$  with  $L_2(\mathbb{R}^d)$  by the Plancherel formula, we may also view  $g(\nabla \Delta^{-\frac{1}{2}})$  as the pointwise multiplication operator in  $B(L_2(\mathbb{R}^d))$ . So all  $g(\nabla \Delta^{-\frac{1}{2}})$  with  $g \in C(\mathbb{S}^{d-1})$  form a commutative  $C^*$ -subalgebra of  $B(L_2(\mathbb{R}^d))$ . Set  $\Pi(C_0(\mathbb{R}_\theta^d) + \mathbb{C}, C(\mathbb{S}^{d-1}))$  to be the  $C^*$ -subalgebra of  $B(L_2(\mathbb{R}^d))$  generated by  $C_0(\mathbb{R}_\theta^d) + \mathbb{C}$  and all those  $g(\nabla \Delta^{-\frac{1}{2}})$ 's. Theorem 3.3 of [64] implies that there exists a unique norm-continuous  $*$ -homomorphism

$$\text{sym} : \Pi(C_0(\mathbb{R}_\theta^d) + \mathbb{C}, C(\mathbb{S}^{d-1})) \longrightarrow (C_0(\mathbb{R}_\theta^d) + \mathbb{C}) \otimes_{\min} C(\mathbb{S}^{d-1})$$

which maps  $x \in C_0(\mathbb{R}_\theta^d)$  to  $x \otimes 1$  and  $g(\nabla \Delta^{-\frac{1}{2}})$  to  $1 \otimes g$ . Then [64, Theorem 6.15] says that for every continuous normalised trace  $\text{Tr}_\omega$  on  $S_{1,\infty}$ , every  $x \in W_1^d(\mathbb{R}_\theta^d)$ , and every  $T \in \Pi(C_0(\mathbb{R}_\theta^d) + \mathbb{C}, C(\mathbb{S}^{d-1}))$ , we have

$$\text{Tr}_\omega(TM_x J^{-d}) = C_d \left( \tau_\theta \otimes \int_{\mathbb{S}^{d-1}} \right) (\text{sym}(TM_x)) \quad (4.5.13)$$

where  $C_d$  is a certain constant depending only on the dimension  $d$ .

**Theorem 4.5.10.** *Let  $x \in \dot{W}_d^1(\mathbb{R}_\theta^d)$  and  $\phi \in C^\infty(\mathbb{S}^{d-1})$ . Then for every continuous normalised trace  $\text{Tr}_\omega$  on  $S_{1,\infty}$ , we have*

$$\text{Tr}_\omega(|\mathbf{C}_{\phi,x}|^d) = C_d \int_{\mathbb{S}^{d-1}} \tau_\theta \left( \left| \sum_{1 \leq k \leq d} \partial_{\xi_k} \phi \partial_k x \right|^d \right) d\xi.$$

*Proof.* Firstly, we assume that  $x \in \mathcal{A}(\mathbb{R}_\theta^d)$ . For any continuous normalised trace  $\text{Tr}_\omega$  on  $S_{1,\infty}$ , and arbitrary  $y \in \mathcal{A}(\mathbb{R}_\theta^d)$ , Lemma 4.5.9 implies that

$$\text{Tr}_\omega(|M_y \mathbf{C}_{\phi,x}|^d) = \text{Tr}_\omega(|M_y A|^d J^{-d}).$$

Since  $d \geq 2$ ,

$$|M_y A|^d = |M_y A|^{d-2} (M_y A)^* M_y A$$

has the form  $TM_z$  for some  $T \in \Pi(C_0(\mathbb{R}_\theta^d) + \mathbb{C}, C(\mathbb{S}^{d-1}))$  and  $z \in W_1^d(\mathbb{R}_\theta^d)$ . So (4.5.13) is applicable, giving

$$\text{Tr}_\omega(|M_y A|^d J^{-d}) = C_d \left( \tau_\theta \otimes \int_{\mathbb{S}^{d-1}} \right) (\text{sym}(|M_y A|^d)).$$

Evidently,

$$\text{sym}(|M_y A|^d) = \left| \frac{1}{2\pi i} \sum_{1 \leq k \leq d} \partial_{\xi_k} \phi y \partial_k x \right|^d,$$

whence

$$\text{Tr}_\omega(|M_y \mathbf{C}_{\phi,x}|^d) = C_d \int_{\mathbb{S}^{d-1}} \tau_\theta \left( \left| \sum_{1 \leq k \leq d} \partial_{\xi_k} \phi y \partial_k x \right|^d \right) d\xi. \quad (4.5.14)$$

It remains to get rid of  $y \in \mathcal{A}(\mathbb{R}_\theta^d)$  and the assumption  $x \in \mathcal{A}(\mathbb{R}_\theta^d)$  using approximation argument. Firstly, we may take  $(y_n)_{n \geq 1} \subset \mathcal{A}(\mathbb{R}_\theta^d)$  such that  $y_n \rightarrow 1 \in L_\infty(\mathbb{R}_\theta^d)$  with respect to the  $w^*$ -topology. This is done by the construction of  $\mathcal{A}(\mathbb{R}_\theta^d)$  in [62, Proposition 2.5]. With this sequence  $(y_n)_{n \geq 1}$ , we note that both sides of (4.5.14) are estimated from above by  $\|x\|_{\dot{W}_d^1}$ . Passing  $n \rightarrow \infty$ , we arrive at

$$\text{Tr}_\omega(|\mathbf{C}_{\phi,x}|^d) = C_d \int_{\mathbb{S}^{d-1}} \tau_\theta \left( \left| \sum_{1 \leq k \leq d} \partial_{\xi_k} \phi \partial_k x \right|^d \right) d\xi. \quad (4.5.15)$$

Next, using the density of  $\mathcal{A}(\mathbb{R}_\theta^d)$  in  $\dot{W}_d^1(\mathbb{R}_\theta^d)$ , we find  $(x_n)_{n \geq 1} \subset \mathcal{A}(\mathbb{R}_\theta^d)$  such that  $x_n \rightarrow x$  in  $\dot{W}_d^1(\mathbb{R}_\theta^d)$ . Again, both sides of (4.5.15) are estimated from above by  $\|x\|_{\dot{W}_d^1}$ . Passing  $n \rightarrow \infty$  in

$$\text{Tr}_\omega(|\mathbf{C}_{\phi,x_n}|^d) = C_d \int_{\mathbb{S}^{d-1}} \tau_\theta \left( \left| \sum_{1 \leq k \leq d} \partial_{\xi_k} \phi \partial_k x_n \right|^d \right) d\xi,$$

we see that (4.5.15) holds for any  $x \in \dot{W}_d^1(\mathbb{R}_\theta^d)$ . This ends the proof of the theorem.  $\square$

We end this section with some remarks concerning the opposite of Proposition 4.5.2.

It is shown in [62] that, if  $[R_j, M_x] \in S_{d,\infty}$  for all  $1 \leq j \leq d$ , then  $x \in \dot{W}_d^1(\mathbb{R}_\theta^d)$  with

$$\|x\|_{\dot{W}_d^1} \leq C_d \sum_{1 \leq j \leq d} \|[R_j, M_x]\|_{S_{d,\infty}}.$$

We cannot have this estimate for a single  $[T_\phi, M_x]$ . Take  $\phi = \frac{\xi_j}{|\xi|}$  for instance: Proposition 4.5.2 and Theorem 4.5.10 ensure

$$\|\partial_j x\|_d = C_d \text{Tr}_\omega(|[R_j, M_x]|^d)^{\frac{1}{d}} \leq C_d \|[R_j, M_x]\|_{S_{d,\infty}} \leq C'_d \|x\|_{\dot{W}_d^1}.$$

But neither we can conclude

$$\|\partial_j x\|_d \approx_d \| [R_j, M_x] \|_{S_{d,\infty}}$$

nor

$$\| [T_\phi, M_x] \|_{S_{d,\infty}} \approx_d \|x\|_{\dot{W}_d^1}$$

for a single  $\phi \in C(\mathbb{S}^{d-1})$ .

Furthermore, it is shown in [62] that, if  $[R_j, M_x] \in S_p, p \leq d$  for all  $1 \leq j \leq d$ , then  $x$  is a constant. Returning to the classical euclidean setting, this assertion is weaker than the result in [51], which states that if  $[T_\phi, M_x] \in S_p, p \leq d$  for some  $\phi$ , then  $x$  is a constant. We cannot conclude such noncommutative counterpart of Janson-Wolff's result in [51] from Theorem 4.5.10 as well. This indicates the drawback of our method: when taking the Dixmier trace of the commutator, we lost some information of the original function (operator)  $x$ .

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## Résumé

Cette thèse a pour but d'étudier quelques problèmes dans l'analyse harmonique sur les produits croisés tordus qui sont définis par des actions tordues d'un groupe localement compact  $G$  sur une algèbre de von Neumann  $M$ . Elle se compose de deux parties. La première porte sur les produits croisés tordus et leurs multiplicateurs de Fourier et de Schur. Nous démontrons que la propriété d'être QWEP pour l'algèbre de von Neumann tordue d'un groupe  $G$  est indépendante du 2-cocycle sous-jacent et que les  $L_p$ -multiplicateurs de Fourier complètement bornés sur cette algèbre tordue sont aussi indépendants du 2-cocycle. Sous l'hypothèse d'une action moyennable, nous établissons plusieurs résultats de transfert entre les multiplicateurs de Fourier et de Schur sur les espaces  $L_p$  non-commutatifs du produit croisé tordu. Dans la deuxième partie, nous étudions les commutateurs de multiplicateurs de Fourier sur le produit croisé tordu d'un espace euclidien. Nous caractérisons leur appartenance à la  $p$ -classes de Schatten par celle de leurs symboles à un espace de Besov associé. Cette partie contient aussi une formule sur la trace de Dixmier qui nous donne également une caractérisation de l'appartenance de ces commutateurs à une  $p$ -classe de Schatten faible par un espace de Sobolev. En particulier, nos résultats s'appliquent au cas d'un espace euclidien quantique.

**Mots clefs:** Multiplicateurs de Fourier, multiplicateurs de Schur, l'algèbre de groupe de von Neumann, produit croisé tordu, classe de Schatten, espace de Besov, espace de Sobolev.

## Abstract

This thesis is devoted to the study of some problems in the harmonic analysis on twisted crossed products defined by twisted actions of a locally compact group  $G$  on a von Neumann algebra  $M$ . It consists of two parts. The first concerns twisted crossed products and their Fourier and Schur multipliers. We prove that the property of being QWEP for the twisted von Neumann algebra of a group  $G$  is independent of the underlying 2-cocycle and that the completely bounded  $L_p$ -Fourier multipliers on this twisted algebra are also independent of the 2-cocycle. Under the hypothesis of an amenable action, we establish several transference results between the Fourier and Schur multipliers on the noncommutative  $L_p$  spaces of the twisted crossed product. In the second part, we study Fourier multiplier commutators on the twisted crossed product of an Euclidean space. We characterize their Schatten  $p$ -class membership by that of their symbols in the associated Besov space. In addition, this part contains a formula on the Dixmier trace, which also gives us a characterization of the weak Schatten  $p$ -class membership of these commutators by a Sobolev space. In particular, our results apply to the case of quantum Euclidean spaces.

**Keywords:** Fourier multiplier, Schur multiplier, group von Neumann algebra, twisted crossed product, Quantum euclidean space, Schatten class, Sobolev space, Besov space, pseudodifferential operator.

## 2020 Mathematics subject classification:

46L51, 46L52, 46L55, 46L87, 47L25, 47L65, 43A99