



**HAL**  
open science

# Unique continuation for wave and Schrödinger operators and applications to control theory

Spyridon Filippas

► **To cite this version:**

Spyridon Filippas. Unique continuation for wave and Schrödinger operators and applications to control theory. Analysis of PDEs [math.AP]. Université Paris-Saclay, 2023. English. NNT : 2023UPASM035 . tel-04417784

**HAL Id: tel-04417784**

**<https://theses.hal.science/tel-04417784v1>**

Submitted on 25 Jan 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Prolongement unique pour des opérateurs d'onde et de Schrödinger et applications à la théorie du contrôle

*Unique continuation for wave and Schrödinger operators and  
applications to control theory*

**Thèse de doctorat de l'université Paris-Saclay**

École doctorale n° 574 mathématiques Hadamard (EDMH)

Spécialité de doctorat : Mathématiques appliquées

Graduate School : Mathématiques. Référent : Faculté des sciences d'Orsay

Thèse préparée dans l'unité de recherche **Laboratoire de Mathématiques d'Orsay (Université Paris-Saclay, CNRS)** sous la direction de **Matthieu LÉAUTAUD**, Professeur (**Université Paris-Saclay**), et le co-encadrement de **Camille LAURENT**, Chargé de Recherche (**CNRS, Sorbonne Université**)

**Thèse soutenue à Paris-Saclay, le 20 décembre 2023, par**

**Spyridon FILIPPAS**

## Composition du jury

Membres du jury avec voix délibérative

<b>Nicolas BURQ</b> Professeur des Universités, Université Paris-Saclay	Président
<b>David DOS SANTOS FERREIRA</b> Professeur des Universités, Université de Lorraine	Rapporteur & Examineur
<b>Sylvain ERVEDOZA</b> Directeur de Recherche, CNRS & Université de Bordeaux	Rapporteur & Examineur
<b>Lucie BAUDOUIN</b> Directrice de Recherche, CNRS & Université de Toulouse	Examinatrice
<b>Lauri OKSANEN</b> Professeur des Universités, Université d'Helsinki	Examineur
<b>Luc ROBBIANO</b> Professeur des Universités, Université Versailles Saint-Quentin-en-Yvelines	Examineur

**Titre :** Prolongement unique pour des opérateurs d'onde et de Schrödinger et applications à la théorie du contrôle

**Mots clés :** prolongement unique, inégalités de Carleman, théorie du contrôle, équations d'onde, équations de Schrödinger

**Résumé :** La problématique du prolongement unique consiste à retrouver toute l'onde à partir d'une observation partielle et a des applications à la théorie du contrôle. Dans une première partie de cette thèse nous nous intéressons à des propriétés de prolongement unique quantitatif pour des ondes dans un milieu singulier. Nous démontrons un résultat de stabilité logarithmique pour des opérateurs d'onde dont la métrique présente une discontinuité à travers une interface. Nous ne faisons aucune hypothèse sur le signe du saut ou sur la géométrie de l'interface. L'ingrédient clef de notre preuve est une inégalité de Carleman locale près de l'interface. Par un argument de propagation nous en déduisons une estimée de stabilité glo-

bale. Nous montrons ensuite que l'estimée obtenue est optimale dans certaines géométries. Pour ce faire, nous construisons pour un opérateur elliptique à coefficients discontinus à travers une interface, des fonctions propres se concentrant exponentiellement près de l'interface.

Dans une deuxième partie nous démontrons des résultats de prolongement unique locaux et globaux sous des hypothèses géométriques minimales pour des opérateurs de Schrödinger à coefficients d'ordre inférieur Gevrey 2 par rapport au temps et bornés en espace. On relaxe ainsi pour ces opérateurs l'hypothèse d'analyticité en temps du théorème de Tataru-Hörmander-Robbiano-Zuily.

**Title :** Unique continuation for wave and Schrödinger operators and applications to control theory

**Keywords :** unique continuation, Carleman inequalities, control theory, wave equations, Schrödinger equations

**Abstract :** The problem of unique continuation consists in recovering the whole wave from a partial observation and has applications to control theory. In the first part of this thesis we investigate quantitative unique continuation properties for waves in a singular medium. We prove a logarithmic stability result for wave operators whose metric exhibits a jump discontinuity across an interface. We make no assumption about the sign of the jump or the geometry of the interface. The key ingredient of our proof is a local Carleman inequality near the interface. Using a propagation argument, we derive a global stability estimate. We then show that the es-

timate obtained is optimal in certain geometries. To this end, we construct eigenfunctions for an elliptic operator with discontinuous coefficients across an interface, concentrating exponentially near the interface.

In the second part, we prove local and global unique continuation results under minimal geometric assumptions for Schrödinger operators with Gevrey 2 lower order coefficients with respect to time and bounded in space. This relaxes the time analyticity assumption of the Tataru-Hörmander-Robbiano-Zuily theorem for these operators.

# Remerciements

Mes tous premiers remerciements vont à mes deux directeurs de thèse, Matthieu et Camille. Plus le temps passe, plus j'apprécie la chance que j'ai eue de vous avoir à mes côtés tout au long de cette aventure. Votre vaste culture mathématique m'a tout de suite impressionné, et je vous suis infiniment reconnaissant de l'avoir partagée avec moi avec tant de patience et de bienveillance. Au delà de ces qualités, votre rigueur et votre intégrité scientifique sont pour moi une vraie source d'inspiration et vous m'avez transmis des valeurs qui dépassent de loin le cadre strictement mathématique. J'espère qu'un jour, ma façon de faire de la recherche sera à la hauteur de ce que m'avez appris. Je vous remercie également de m'avoir donné la possibilité de collaborer avec vous, ce qui m'a permis de réaliser que lorsqu'on travaille avec les bonnes personnes, la recherche devient encore plus belle.

Je remercie chaleureusement David Dos Santos Ferreira et Sylvain Ervedoza d'avoir accepté la tâche de rapporter cette thèse ainsi que de leurs lectures attentives de ce manuscrit. C'est un vrai honneur pour moi de savoir qu'ils se sont intéressés à mon travail.

Je remercie également Lucie Baudouin, Nicolas Burq, Luc Robbiano, et Lauri Oksanen d'avoir accepté de faire partie de mon jury, j'en suis tout autant honoré.

Cette thèse met une fin à mon parcours Orcéen qui a commencé il y a presque dix ans, quand j'ai commencé ma L1. Pendant toutes ces années j'ai pu profiter d'un excellent environnement de travail et j'ai eu la chance d'avoir des enseignants extraordinaires grâce auxquels ma passion pour les mathématiques n'a fait que croître. Je les remercie tous.

Je tiens à remercier Stéphane Nonnenmacher qui a joué un double rôle dans mon cursus : à la fois en tant qu'enseignant-chercheur avec son cours passionnant d'analyse semiclassique mais aussi en tant que directeur de l'École Doctorale. Je le remercie pour sa disponibilité et bienveillance tout au long de ces trois ans.

En M1 j'ai eu la chance de suivre le cours de Distributions de Patrick Gérard. Je le remercie d'avoir été un enseignant exemplaire qui m'a profondément inspiré et motivé. Je le remercie aussi des discussions que nous avons eues qui m'ont beaucoup aidé à prendre des décisions sur mon parcours et de m'avoir orienté vers Matthieu pour mon stage de M2.

Je remercie Alano Ancona d'avoir si bien encadré mon projet de M1 et pour toutes les discussions que nous avons pu avoir par la suite. Sa disponibilité et sa gentillesse combinées avec ses profondes connaissances venant de tous les domaines de Mathématiques m'ont permis d'élargir considérablement ma culture mathématique et je lui en suis reconnaissant.

Toute cette aventure parisienne n'aurait jamais vu le jour si Georges Alexopoulos ne m'avait pas proposé de venir à Paris. Je le remercie de m'avoir offert cette opportunité qui a radicalement changé ma vie et de la confiance qu'il m'a accordée.

I also want to thank Lauri Oksanen for giving me the opportunity to continue my mathematical adventure in Finland. Kiitos Lauri!

Une thèse ne s'écrit pas toute seule et j'ai eu la chance d'avoir autour de moi de nombreuses personnes qui ont rendu ce voyage si spécial. Merci à Carlo, Balthazar, Anthony, François, Juan, Polina, Luigi, Nir, Ioann, Dawei, Antoine, Marc pour toutes les discussions (mathématiques ou pas), pour nos pauses cafés (parfois démesurées) et pour tous les moments de rire et de convivialité qui ont été pour moi d'une aide précieuse pendant ma thèse. J'ai une pensée particulière pour mes amis Antonio et Monica avec lesquels j'ai partagé une très grande partie de ce voyage mais pas que... Merci pour tout.

Je ne pourrai oublier mes fidèles acolytes de la vie "normale" en dehors du labo. Merci Yanis, Aurélien et Wanzhen pour les verres, les dîners, les nuits où l'on a arpenté les rues de Paris et toutes les expériences (parfois loufoques) qu'on a vécues ensemble. Votre amitié m'est inestimable.

Ευχαριστώ το φίλο μου το Σπύρο, που είναι σταθερά στα μετόπισθεν (και όχι μόνο), τις αγαπημένες μου ξαδέρφες Αγγελίνα και Ελένη και τα φιλαράκια μου Γεώργιο, Σταμάτη και Οδυσσέα, για όλα όσα έχουμε περάσει μαζί αλλά και για όσα έρχονται.

Τίποτα από όλα αυτά δεν θα είχε γίνει δίχως την άνευ ορίων στήριξη και αγάπη των γονιών μου και του αδερφού μου. Σας ευχαριστώ για όλα.

Τέλος, ένα ευχαριστώ για εσένα, Χριστίνα. Για όλες τις στιγμές που στάθηκες δίπλα μου, ισορροπώντας με, δίνοντας μου αγάπη και κουράγιο. Α! και για όλα τα μπαλέτα...

# Table des matières

<b>1</b>	<b>Introduction</b>	<b>9</b>
1.1	Préliminaires . . . . .	9
1.1.1	Généralités sur le prolongement unique . . . . .	9
1.1.2	Prolongement unique et contrôle . . . . .	11
1.1.3	L'idée de Carleman . . . . .	13
1.1.4	Trois résultats fondamentaux de prolongement unique . . . . .	14
1.1.5	Structure de cette thèse . . . . .	19
1.2	Ondes dans un milieu singulier . . . . .	19
1.2.1	Résultats antérieurs et motivations . . . . .	19
1.2.2	Problème étudié et principaux résultats obtenus . . . . .	23
1.2.3	Outils et idée de preuve . . . . .	26
1.2.4	Perspectives . . . . .	31
1.3	Modes propres pour un problème de transmission . . . . .	32
1.3.1	Résultats antérieurs et motivations . . . . .	32
1.3.2	Principaux résultats obtenus . . . . .	33
1.3.3	Outils et idée de preuve . . . . .	35
1.3.4	Perspectives . . . . .	39
1.4	Opérateurs de Schrödinger . . . . .	40
1.4.1	Résultats antérieurs et motivations . . . . .	40
1.4.2	Principaux résultats obtenus . . . . .	42
1.4.3	Outils et idée de preuve . . . . .	45
1.4.4	Perspectives . . . . .	49
<b>I</b>	<b>On unique continuation for waves in singular media</b>	<b>51</b>
<b>2</b>	<b>Waves in singular media</b>	<b>53</b>
2.1	Introduction . . . . .	53
2.1.1	Setting and statement of main results . . . . .	54
2.1.2	Strategy of the proof and organization of the paper . . . . .	57
2.1.3	Some notations and definitions . . . . .	59
2.2	The Carleman estimate . . . . .	60
2.2.1	General transmission conditions . . . . .	61
2.2.2	Local setting in a neighborhood of the interface . . . . .	62
2.2.3	Proof of Theorem 2.2.2 from Proposition 2.2.6 . . . . .	65
2.3	Proof of Proposition 2.2.6 for a toy model . . . . .	68

2.3.1	Factorization and first estimates . . . . .	68
2.3.2	End of the proof for the toy model . . . . .	75
2.4	Proof of Proposition 2.2.6 for the general case . . . . .	77
2.4.1	Notation, microlocal regions and first estimates . . . . .	77
2.4.2	Microlocal estimates in the non-elliptic regions . . . . .	86
2.4.3	Microlocal estimate in the elliptic region . . . . .	89
2.4.4	Patching together microlocal estimates: End of the proof of Proposition 2.2.6 . . . . .	97
2.4.5	Convexification: A perturbation argument . . . . .	100
2.5	The quantitative estimates . . . . .	105
2.5.1	Some definitions and statement of the local estimate . . . . .	105
2.5.2	Proof of Theorem 2.5.1 . . . . .	106
2.5.3	Propagation of information and applications . . . . .	117
2.A	A few facts on pseudodifferential calculus . . . . .	126
2.A.1	Differential operators . . . . .	126
2.A.2	Standard tangential classes . . . . .	127
2.A.3	Tangential classes with a large parameter . . . . .	128
2.B	Some lemmata used in the quantitative estimates . . . . .	129
<b>3</b>	<b>Whispering galleries</b> . . . . .	<b>133</b>
3.1	Introduction . . . . .	133
3.1.1	Optimality of unique continuation results for operators with jumps . . . . .	134
3.1.2	Domain and self-adjointness of the operator . . . . .	136
3.2	The case of an annulus . . . . .	137
3.2.1	Reduction to a semiclassical Schrödinger operator . . . . .	137
3.2.2	The Agmon estimate . . . . .	140
3.2.3	Finer information on the concentration of the eigenfunctions . . . . .	144
3.2.4	Back to the two dimensional annulus . . . . .	147
3.3	The disk and other surfaces of revolution . . . . .	148
3.3.1	Maximally vanishing eigenfunctions . . . . .	148
3.3.2	A result assuming separation of variables only locally . . . . .	151
<b>II</b>	<b>On unique continuation for Schrödinger operators</b> . . . . .	<b>153</b>
<b>4</b>	<b>Uniqueness for Schrödinger operators</b> . . . . .	<b>155</b>
4.1	Introduction and main results . . . . .	155
4.1.1	Background and results . . . . .	155
4.1.2	Application to controllability and observability . . . . .	159
4.1.3	Idea of the proof, structure of the paper . . . . .	160
4.2	The Carleman estimate . . . . .	162
4.2.1	Toolbox of Riemannian geometry . . . . .	162
4.2.2	The Carleman weight . . . . .	164
4.2.3	Proof of the subelliptic estimate . . . . .	168
4.2.4	Choice of weight function via convexification . . . . .	175
4.3	Conjugation with a partially Gevrey function . . . . .	177
4.3.1	Gevrey functions and Banach valued symbols . . . . .	177

4.3.2	The conjugated operator . . . . .	179
4.4	The uniqueness theorem . . . . .	189
4.4.1	Adding partially Gevrey lower order terms . . . . .	189
4.4.2	Using the Carleman estimate . . . . .	191
4.4.3	Reducing the regularity of the solution . . . . .	193
4.A	Tools . . . . .	195
4.A.1	A technical lemma . . . . .	195
4.A.2	Integration by parts formulæ . . . . .	198





# Chapitre 1

## Introduction

### 1.1 Préliminaires

#### 1.1.1 Généralités sur le prolongement unique

Ce mémoire gravite autour du problème du *prolongement unique* pour certaines solutions d'équations aux dérivées partielles. La question que l'on se pose est la suivante. Soit  $P$  un opérateur différentiel dans un ouvert  $\Omega \subset \mathbb{R}^n$  et considérons  $\omega \subset \Omega$  un ouvert que l'on imagine petit. Supposons maintenant que  $u$  est une solution de l'équation aux dérivées partielles  $Pu = 0$  sur  $\Omega$ . On veut alors savoir si une observation partielle de la solution  $u$  sur le petit ouvert  $\omega$  suffit pour déterminer la solution sur *tout* l'ouvert  $\Omega$ . Autrement dit :

$$\begin{cases} Pu = 0 \text{ sur } \Omega \\ u = 0 \text{ sur } \omega \end{cases} \stackrel{\text{A-t-on?}}{\implies} u = 0 \text{ sur } \Omega.$$

Telle que nous l'avons posée ci-dessus, la question de prolongement unique est une question de nullité : si l'on observe la solution nulle sur le petit ouvert  $\omega$  notre solution est-elle la solution nulle ? Il faut alors remarquer que comme nous avons affaire à un problème *linéaire*, cette question de nullité est en fait équivalente à une question d'unicité, à savoir :

$$\begin{cases} Pu_1 = Pu_2 = 0 \text{ sur } \Omega \\ u_1 = u_2 \text{ sur } \omega \end{cases} \stackrel{\text{A-t-on?}}{\implies} u_1 = u_2 \text{ sur } \Omega.$$

Admettons maintenant que  $P$  satisfait effectivement à cette propriété qualitative de prolongement unique. Une question naturelle est alors de savoir si nous pouvons la *quantifier*. C'est à dire

$$\begin{cases} Pu \text{ petit sur } \Omega \\ u \text{ petit sur } \omega \end{cases} \implies u \text{ petit sur } \Omega.$$

Plus précisément, quantifier cette propriété de prolongement unique revient à montrer une estimée de stabilité satisfaite par la solution  $u$

$$\|u\|_{\Omega} \lesssim \phi(\|u\|_{\omega}, \|Pu\|_{\Omega}, \|u\|_{\Omega}), \quad (1.1.1)$$

avec une fonction  $\phi$  telle que

$$\phi(a, b, c) \xrightarrow{a, b \rightarrow 0} 0, \quad \text{avec } c \text{ borné.}$$

Les deux questions que nous avons présentées sont de nature *globale*. On fait une hypothèse globale (la nullité sur tout l'ouvert d'observation  $\omega$ ) et on obtient un résultat global (la nullité de la solution partout). On peut alors se demander si la propriété d'unicité est satisfaite localement par  $P$ . Pour préciser la propriété désirée on a besoin d'un point  $x_0 \in \Omega$  et d'une hypersurface  $S = \{\Psi = 0\}$  avec  $\Psi : \Omega \rightarrow \mathbb{R}$  telle que  $\Psi(x_0) = 0$  et  $\nabla\Psi(x_0) \neq 0$ . La question de l'unicité locale est alors de savoir si pour une solution  $u$  de  $Pu = 0$  le fait d'être nulle sur *un côté* de l'hypersurface  $S$  implique que  $u$  est nulle sur tout un voisinage de  $x_0$ . Plus précisément on veut savoir s'il existe un voisinage  $V$  de  $x_0$  tel que l'implication suivante soit vraie

$$\begin{cases} Pu = 0 \text{ sur } V \\ u = 0 \text{ sur } V \cap \{\Psi > 0\} \end{cases} \implies u = 0 \text{ sur un voisinage de } x_0. \quad (1.1.2)$$

Si la réponse est affirmative on veut l'accompagner avec une estimée de stabilité qui quantifie cette fois la petitesse locale de la solution  $u$  autour de  $x_0$ .

Une stratégie possible afin d'obtenir un résultat d'unicité globale est alors de propager l'unicité en utilisant des résultats locaux. En ce qui concerne le résultat qualitatif, on voit aussitôt que pour qu'une telle approche soit efficace on aura besoin d'une famille assez large d'hypersurfaces pour lesquelles la propriété d'unicité locale est satisfaite par  $P$ . En effet, plus on a d'hypersurfaces admissibles plus on aura des directions possibles pour propager l'unicité et obtenir ainsi un résultat global satisfaisant. Pour la partie quantitative du problème on a une difficulté additionnelle : non seulement il faut que la propriété d'unicité soit satisfaite par une certaine famille d'hypersurfaces, mais il faut aussi que l'estimée quantitative locale puisse être itérée de sorte que l'estimée de stabilité obtenue à la fin soit intéressante (c'est à dire que l'estimée locale ne dégénère pas trop après l'avoir appliquée un certain nombre de fois).

Donnons maintenant deux exemples de nature assez différente où la propriété du prolongement unique est satisfaite :

- Le premier est le laplacien plat  $\Delta$  agissant sur  $C^2(\mathbb{R}^2)$ . Supposons que  $u$  satisfasse  $\Delta u = 0$  sur  $\Omega$  que l'on suppose connexe et que  $u = 0$  sur un petit ouvert  $\omega$  non vide. Le fait que  $u$  résolve  $\Delta u = 0$  implique qu'elle est analytique réelle. En particulier, elle satisfait au principe des zéros isolés et comme elle s'annule sur un ouvert non vide on doit forcément avoir  $u = 0$  sur  $\Omega$ .
- Le deuxième exemple concerne l'opérateur des ondes  $\square = \partial_t^2 - \Delta$  sur  $\mathbb{R}^{1+n}$ . Soit  $u$  solution de  $(\partial_t^2 - \Delta)u = 0$  sur  $\mathbb{R}^{1+n}$  avec données initiales  $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ . On définit alors l'énergie  $E = E(t)$  par

$$E(t) = \frac{1}{2} \|\partial_t u(t)\|_{L^2}^2 + \frac{1}{2} \|\nabla_x u(t)\|_{L^2}^2,$$

et on constate que pour  $u$  assez régulière (par exemple  $u \in C^2(\mathbb{R} \times \mathbb{R}^n)$ ) l'équation  $(\partial_t^2 - \Delta)u = 0$  implique grâce à une intégration par parties que  $\frac{d}{dt}E(t) = 0$  ce qui donne  $E(t) = E(0)$ . Cela montre que si les données initiales sont nulles en  $t = 0$  alors notre solution est nulle partout, ce qui correspond à une propriété d'unicité globale. On peut légèrement modifier la définition de l'énergie pour rendre cet argument local. Si l'on pose

$$E_r(t) = \frac{1}{2} \|\partial_t u(t)\|_{L^2(B(0,r))}^2 + \frac{1}{2} \|\nabla_x u(t)\|_{L^2(B(0,r))}^2$$

alors on peut montrer (voir [Eva98, Chapter 2.4], [Ler19, Chapter 1.2], [LL20, Chapter 1.2]) que pour tout  $r_0 > 0$  et  $t \in [0, r_0]$  nous avons

$$E_{r_0-t}(t) \leq E_{r_0}(0),$$

ce qui implique que si  $u(0) = \partial_t(0) = 0$  pour  $|x| \leq r_0$  alors  $u = 0$  dans le cône

$$C_{r_0} = \{(t, x) \in \mathbb{R}^{1+n} | t \in [0, r_0] \text{ and } |x| \leq r_0 - t\}.$$

Nous avons donc obtenu une propriété d'unicité locale satisfaite par l'opérateur  $\partial_t^2 - \Delta$  à travers l'hypersurface  $\{t = 0\}$ . Cette propriété est une manifestation de la *vitesse finie de propagation* pour les ondes. On peut par exemple consulter [Eva98, Chapter 2.4] pour plus de détails. En plus, on a une estimée linéaire qui quantifie cette propriété d'unicité. Ce résultat très fort est lié au fait que l'équation des ondes est *bien posée* quand les données initiales sont sur l'hypersurface  $\{t = 0\}$ , ou plus généralement sur une hypersurface de type espace (c'est-à-dire une hypersurface  $S = \{\Psi = 0\}$  telle que  $|\partial_t \Psi| > |\nabla_x \Psi|$ ), et est dû à ce qu'on appelle l'hyperbolicité de l'opérateur des ondes par rapport à l'hypersurface  $\{t = 0\}$ , voir par exemple [Ler19, Chapter 1.3]. On en conclut que, au moins pour ce qui concerne les ondes, la question du prolongement unique est surtout pertinente quand nos "données" sont sur une hypersurface autre que  $\{t = 0\}$  (ou plus généralement de type espace). Il est alors intéressant de découvrir que la question du prolongement unique pour des hypersurfaces qui ne sont pas de type espace a des conséquences importantes pour des problèmes venant de la théorie du contrôle des équations aux dérivées partielles.

### 1.1.2 Prolongement unique et contrôle

Un système de contrôle est un système dynamique sur lequel on peut agir en modifiant certains de ses paramètres : les contrôles. Etant donné deux états, la question principale est de savoir s'il est possible de choisir son contrôle de sorte que la solution du système passe d'un état à l'autre. Les problèmes de contrôle d'équations aux dérivées partielles linéaires peuvent souvent s'écrire sous la forme abstraite :

$$\begin{cases} \frac{d}{dt}y(t) = Ay(t) + Bg, & t \in [0, T], \\ y(0) = y_0 \in H, \end{cases} \quad (1.1.3)$$

où

- L'opérateur  $A$  est le générateur infinitésimal d'un semi-groupe fortement continu  $S(t) \in \mathcal{L}(H)$ ,  $t \in [0, \infty)$  sur un espace de Hilbert  $H$ .
- L'opérateur  $B \in \mathcal{L}(E; H)$  est l'opérateur de contrôle et l'espace de Hilbert  $E$  est l'espace des contrôles.
- $g \in L^2((0, T); E)$  est le contrôle à choisir.

Nous donnons quelques définitions et résultats classiques pour lesquels nous suivons [Lio88, Cor07, TW09].

**Définition 1.** Soit  $T > 0$ . On dit que le système (1.1.3) est *exactement contrôlable* en temps  $T$  si pour tout  $y_0 \in H$  et  $y_1 \in H$  il existe un contrôle  $g \in L^2((0, T); E)$  tel que la solution de (1.1.3) satisfasse  $y(T) = y_1$ .

Bien entendu, le contrôlabilité exacte est une propriété très forte et en général difficile à satisfaire. Nous avons alors une autre propriété moins exigeante, celle de la contrôlabilité approchée.

**Définition 2.** Soit  $T > 0$ . On dit que le système (1.1.3) est *approximativement contrôlable* en temps  $T$  si pour tout  $y_0 \in H$ ,  $y_1 \in H$  et toute précision  $\varepsilon > 0$  il existe un contrôle  $g \in L^2((0, T); E)$  tel que la solution de (1.1.3) satisfasse  $\|y(T) - y_1\|_H \leq \varepsilon$ .

On peut remarquer que si  $S(t), t \in \mathbb{R}$  est un *groupe* (ceci est par exemple le cas pour les ondes) on peut remplacer dans les définitions ci-dessus la cible  $y_1 \in H$  par  $y_1 = 0$ . En effet, supposons que le système est exactement contrôlable à zéro (respectivement approximativement contrôlable à zéro). Soient  $y_0, y_1 \in H$ . Il existe alors  $g$  tel que la solution de

$$\frac{d}{dt}z(t) = Az + Bg, \quad z(0) = y(0) - S(-T)y_1$$

satisfasse

$$z(T) = 0, \quad (\text{respectivement } \|z(T)\|_H \leq \varepsilon).$$

On remarque que la solution de (1.1.3) est donnée par  $y(t) = z(t) + S(t - T)y_1$  et satisfait alors  $y(T) - y_1 = z(T)$ , c'est à dire  $y(T) = y_1$  (respectivement  $\|y(T) - y_1\|_H \leq \varepsilon$ ).

En pratique, un résultat abstrait d'existence peut être difficile à démontrer. La méthode HUM (Hilbert Uniqueness Method) de S. Dolecki et D.L. Russell [DR77] et J.-L. Lions [Lio88] transforme par dualité ce problème et fournit des critères très utiles pour la contrôlabilité (approchée ou exacte). On a le résultat suivant (voir ([Cor07, Chapter 2.3]) pour plus de détails) :

**Théorème 1.1.1.** *Soit  $T > 0$ . Alors :*

- *Le système (1.1.3) est exactement contrôlable en temps  $T > 0$  si et seulement s'il existe  $c > 0$  avec*

$$\int_0^T \|B^*S(t)^*z_0\|_E^2 dt \geq c \|z_0\|_H^2, \quad z_0 \in D(A^*),$$

- *Le système (1.1.3) est approximativement contrôlable en temps  $T$  si et seulement si pour tout  $z \in H$  on a l'implication suivante :*

$$(B^*S(\cdot)^*z_0 = 0 \text{ sur } L^2((0, T); E)) \implies (z_0 = 0).$$

La première propriété correspond à l'*observabilité* pour le système (1.1.3) et décrit le fait qu'on arrive à observer une partie de l'énergie totale du système depuis le petit ouvert  $\omega$ . Le deuxième critère n'est rien d'autre qu'une propriété de prolongement unique! La raison c'est que le prolongement unique exprime une propriété d'unicité (ou d'injectivité) qui par dualité se transforme en une propriété de densité.

Comme on va s'intéresser à l'équation des ondes, précisons ce que la méthode HUM nous dit dans ce cas précis.

Pour  $\mathcal{M}$  l'adhérence d'un ouvert borné de  $\mathbb{R}^n$  et  $(u_0, u_1) \in L^2 \times H^{-1}$  on considère le système :

$$\begin{cases} (\partial_t^2 - \Delta)u = \chi_\omega g & (0, T) \times \text{sur Int}(\mathcal{M}), \\ u = 0, & \text{sur } (0, T) \times \partial\mathcal{M}, \\ (u(0), \partial_t u(0)) = (u_0, u_1), & \text{sur } \mathcal{M}. \end{cases} \quad (1.1.4)$$

Ici  $\chi_\omega$  est une fonction lisse telle que  $\omega = \{\chi_\omega \neq 0\}$  et  $\omega$  est la région du contrôle. On considère alors le problème adjoint suivant, avec  $(w_0, w_1) \in H_0^1 \times L^2$  :

$$\begin{cases} (\partial_t^2 - \Delta)w = 0 & \text{sur } (0, T) \times \text{Int}(\mathcal{M}) \\ w = 0 & \text{sur } (0, T) \times \partial\mathcal{M} \\ (w, \partial_t w)|_{t=T} = (w_0, w_1) & \text{sur } \mathcal{M}. \end{cases} \quad (1.1.5)$$

On peut multiplier (1.1.4) par une solution  $w$  de (1.1.5), intégrer sur  $(0, T) \times \mathcal{M}$  et effectuer une intégration par parties pour trouver l'identité suivante :

$$\langle \partial_t u(T), w_0 \rangle_{H^{-1}, H_0^1} - (u(T), w_1)_{L^2} = \langle f, \chi_\omega w \rangle_{L^2((0, T); H^{-1}), L^2((0, T); H_0^1)}.$$

On note maintenant que  $H_0^1 \times L^2$  et  $L^2 \times H^{-1}$  sont en dualité de la façon suivante : si  $(u_0, u_1) \in L^2 \times H^{-1}$  et  $(w_0, w_1) \in H_0^1 \times L^2$  alors

$$\langle u, w \rangle_{L^2 \times H^{-1}, H_0^1 \times L^2} = \langle u_1, w_0 \rangle_{H^{-1}, H_0^1} - (u_0, w_1)_{L^2}.$$

Le deuxième point du Théorème 1.1.1 se traduit alors comme suit :

**Théorème 1.1.2** (Lien entre contrôle approché et prolongement unique pour les ondes). *Le système (1.1.4) est approximativement contrôlable dans  $L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$  si et seulement si la propriété suivante est satisfaite :*

$$\left( w \text{ résout (1.1.5), } (w_0, w_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M}), w = 0 \text{ sur } (0, T) \times \omega \right) \implies w = 0 \text{ sur } \mathcal{M}.$$

On voit donc qu'un résultat de prolongement unique qualitatif fournit l'existence d'un sous espace dense de  $L^2 \times H^{-1}$  de données contrôlables. En effet, la propriété du contrôle approché nous dit que nous pouvons approcher de façon arbitrairement précise tout élément de  $L^2 \times H^{-1}$  par une solution de (1.1.4) quitte à bien choisir son contrôle  $g$ . Un résultat de prolongement unique *quantitatif* permet alors d'obtenir, via la dualité, une estimation du *coût* du contrôle  $g$  (voir [Rob95]). Plus précisément cela permet d'obtenir une estimée sur  $\|g\|_{L^2((0, T) \times \omega)}$  en fonction de la précision  $\varepsilon$ .

### 1.1.3 L'idée de Carleman

Maintenant que nous avons donné quelques motivations pour étudier le problème du prolongement unique décrivons un des outils clés pour l'obtention de tels résultats : les inégalités de Carleman. Revenons alors au premier exemple d'opérateur que nous avons vu, celui du laplacien plat sur  $\mathbb{R}^2$ . On a vu que l'analyticité et les propriétés très fortes qui en découlent ont permis de montrer assez facilement un résultat d'unicité globale pour  $\Delta$ . On peut alors se demander si ce résultat est toujours vrai si l'on perturbe  $\Delta$  en ajoutant un potentiel  $V \in C^\infty$ . Comme il s'agit d'une perturbation très faible (au moins d'un point de vue différentiel) nous avons envie de dire que le résultat d'unicité reste valable pour

$\Delta + V$  aussi. Effectivement c'est encore le cas, toutefois il n'y a plus de propriété d'analyticité et il faut alors un argument de nature très différente. C'est ici qu'intervient l'idée de Carleman.

Nous considérons le problème d'unicité à travers l'hypersurface  $\{\Psi = \Psi(x_0)\}$  comme nous l'avons introduit au début de la Section 1.1.1. En 1939, T. Carleman introduit dans [Car39] une approche nouvelle qui consiste à montrer des inégalités à poids dans  $L^2$  avec un grand paramètre  $\tau > 0$  :

$$C \left\| e^{\tau\phi} Pu \right\|_{L^2} \geq \left\| e^{\tau\phi} u \right\|_{L^2}, \quad \tau \geq \tau_0, \quad (1.1.6)$$

où la constante  $C$  est indépendante de  $\tau$  et  $u \in C_0^\infty(B(x_0, r))$  avec  $r > 0$  petit. La fonction  $\phi$  est étroitement liée à l'hypersurface à travers laquelle on veut propager l'unicité. L'idée intuitive est que le poids  $e^{\tau\phi}$  renforce l'ensemble où la solution est nulle et permet de propager l'unicité de  $\{\phi > 0\}$  à  $\{\phi < 0\}$ . Si par exemple on a  $Pu = 0$  sur  $\{\phi \geq 0\}$  alors le membre de droite de cette inégalité tend vers 0 pour  $\tau \rightarrow +\infty$  et par conséquent on aura que  $u = 0$  sur  $\{\phi > 0\}$ .

Bien entendu cet exemple est extrêmement simplifié puisque notre solution  $u$  de  $Pu = 0$  n'a aucune raison d'être à support compact. Il faudra donc appliquer l'inégalité à  $v = \chi u$  avec  $\chi$  une fonction plateau bien choisie qui localise autour du point  $x_0 \in \{\Psi = 0\}$ , puis écrire

$$Pv = P(\chi u) = \chi Pu + [P, \chi]u = [P, \chi]u,$$

puisque  $Pu = 0$ . Si l'on suppose que  $u = 0$  sur  $\{\Psi > 0\}$  on aura alors

$$\text{supp}[P, \chi]u \subset \text{supp}(\nabla\chi) \cap \text{supp}(u) \subset \text{supp}(\nabla\chi) \cap \{\Psi < 0\} \subset \{\phi \leq -\eta\},$$

pour  $\eta > 0$  petit et  $\phi$  bien choisie. On voit ici que les lignes de niveau de  $\phi$  doivent être très proche des lignes de niveau de  $\Psi$  sans qu'elles soient exactement les mêmes. En effet, pour que la propriété ci-dessus soit vérifiée il faut que les lignes de niveau de  $\phi$  soient un peu courbées par rapport à celles de l'hypersurface  $\{\Psi = \Psi(x_0)\}$ . Ceci correspond à la convexification géométrique du poids. On obtient ensuite

$$\left\| e^{\tau\phi} \chi u \right\|_{L^2} \leq C e^{-\eta\tau} \|[P, \chi]u\|_{L^2},$$

ce qui donne

$$\left\| e^{\tau(\phi+\eta)} \chi u \right\|_{L^2} \leq C_u, \quad \tau \geq \tau_0.$$

Cette dernière inégalité implique finalement que  $\chi u = 0$  sur  $\{\phi + \eta > 0\}$ . En effet, dans le cas contraire on peut intégrer sur un compact où  $|\chi u| > c$  et faire tendre  $\tau$  vers plus infini pour obtenir une contradiction. Grâce aux choix de  $\chi$  et  $\phi$  cela permet de conclure que  $u$  est nulle sur tout un voisinage de  $x_0$ . Pour plus de détails la stratégie de Carleman on peut consulter [Ler19, Chapter 1.5], [LL20, Chapter 1.2]

Remarquons en particulier que comme cette preuve est basée sur une inégalité, elle peut donner un bon point de départ pour l'obtention d'un résultat quantitatif également (voir par exemple les travaux de [Bah87, Rob95, LR95]).

#### 1.1.4 Trois résultats fondamentaux de prolongement unique

Nous donnons maintenant trois résultats très importants de prolongement unique dans l'ordre chronologique. On commence avec quelques définitions.

**Définition 3.** Soit  $m \in \mathbb{N}$  et  $\Omega \subset \mathbb{R}^n$  un ouvert. On considère l'opérateur différentiel  $P = \sum_{|\alpha| \leq m} p_\alpha(x) D^\alpha$  avec  $p_\alpha \in C_b^\infty(\Omega)$ . Nous définissons alors son *symbole principal* comme le polynôme

$$p_m(x, \xi) = \sum_{|\alpha|=m} p_\alpha(x) \xi^\alpha.$$

Les coefficients  $p_\alpha$  avec  $|\alpha| = m$  sont appelés les coefficients principaux de  $P$ .

**Définition 4.** Soit  $P$  un opérateur différentiel d'ordre  $m$  sur un ouvert  $\Omega$ ,  $x_0 \in \Omega$ , et  $S = \{\Psi = 0\}$  une hypersurface avec  $\Psi(x_0) = 0$ . On dit alors que l'hypersurface  $S$  est non-caractéristique par rapport à  $P$  en  $x_0$  si  $p_m(x_0, \nabla \Psi(x_0)) \neq 0$ .

La condition de non-caractéristicité est une condition assez naturelle. Pour un opérateur elliptique toutes les hypersurfaces sont non-caractéristiques et pour l'opérateur des ondes  $\partial_t^2 - \Delta$  les hypersurfaces non-caractéristiques sont celles qui ne sont pas tangentes aux cônes d'ondes  $|x - x_0| = |t - t_0|$ . On note maintenant qu'on peut construire à l'aide des formules explicites (voir [Eva98]) une solution  $u$  de  $(\partial_t^2 - \Delta)u = 0$  sur  $\mathbb{R}_t \times \mathbb{R}^3$  qui satisfait (voir [LL20, Section 1.1.2] pour plus de détails) :

$$\text{supp } u \cap \{t \geq 0\} = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, t - r \leq |x| \leq t + r\},$$

pour  $r > 0$ . On voit alors qu'on ne peut pas prolonger la solution  $u$  à travers une hypersurface tangente au cône  $|x| = t + r$  dans les  $t > 0$ , qui est une hypersurface caractéristique. Cela suggère qu'un résultat de prolongement unique à travers toute hypersurface non-caractéristique serait en quelque sorte optimal pour des opérateurs de type ondes.

Le premier résultat qu'on présente a une conclusion très satisfaisante mais au prix d'une hypothèse très forte sur les coefficients de l'opérateur.

**Théorème 1.1.3** (Théorème de John-Holmgren). *Soit  $P$  un opérateur différentiel ayant tous ses coefficients analytiques dans un voisinage de  $x_0$ . Alors la propriété de prolongement unique locale (1.1.2) est satisfaite à travers toute hypersurface  $S$  non-caractéristique pour  $P$  en  $x_0$ .*

On peut consulter [Hör63, Theorem 5.3.1] pour un énoncé plus précis et une preuve. Au vu de l'argument d'analyticité qu'on a employé pour montrer la propriété de prolongement unique satisfaite par le laplacien plat il n'est pas étonnant qu'on ait ici une hypothèse d'analyticité.

Cette hypothèse est très restrictive et pour l'affaiblir il faudra une autre approche : celle de Carleman. Avant de présenter le résultat, esquissons la stratégie (de façon très simplifiée) pour l'obtention d'une inégalité de Carleman comme (1.1.6). En posant  $v = e^{\tau\phi}u$  on se ramène d'abord à l'étude de l'opérateur conjugué  $P_\phi = e^{\tau\phi}Pe^{-\tau\phi}$  et il suffit de montrer que  $C \|P_\phi v\|_{L^2} \geq \|v\|_{L^2}$ . On considère alors la partie autoadjointe et anti-autoadjointe de l'opérateur  $P_\phi : P_R = \frac{P_\phi + P_\phi^*}{2}$ ,  $P_I = \frac{P_\phi - P_\phi^*}{2i}$  de sorte que  $P_\phi = P_R + iP_I$  avec  $P_R$  et  $P_I$  des opérateurs formellement autoadjoints. Cela permet d'écrire pour  $v$  à support compact :

$$\begin{aligned} \|P_\phi v\|_{L^2}^2 &= \|P_R v\|_{L^2}^2 + \|P_I v\|_{L^2}^2 + i([P_R, P_I]v, v)_{L^2} \\ &= \|P_R v\|_{L^2}^2 + \|P_I v\|_{L^2}^2 + 2 \text{Re}(P_R v, P_I v)_{L^2}, \end{aligned}$$

où  $[P_R, P_I]$  est le commutateur entre  $P_R$  et  $P_I$ . Les deux premiers termes dans l'égalité ci-dessus sont positifs mais le dernier peut très bien être négatif. On voit alors qu'il serait



difficile de chercher à obtenir une estimée sur  $P_\phi$  sans une hypothèse de positivité sur le commutateur  $i[P_R, P_I]$  là où  $\|P_R v\|_{L^2}^2, \|P_I v\|_{L^2}^2$  sont nuls/petits. Notons que le symbole principal du commutateur  $i[P_R, P_I]$  est donné par  $\{p_R, p_I\}$  où  $p_R, p_I$  sont les symboles principaux de  $P_R, P_I$  et  $\{\cdot, \cdot\}$  le crochet de Poisson. Cette propriété de positivité qui fait intervenir les symboles  $p_R, p_I$ , c'est à dire le symbole principal de  $P$  et  $\phi$ , est encodée dans la condition de *pseudoconvexité*. Donnons sa définition précise :

**Définition 5.** Soit  $\Omega \ni x_0$  un ouvert et  $P$  un opérateur différentiel d'ordre 2 dont le symbole principal est réel. On dit que l'hypersurface orientée  $S = \{\Psi = \Psi(x_0)\}$  est fortement pseudoconvexe par rapport à  $P$  au point  $x_0$  si pour tout  $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}_+$ ,

$$\begin{aligned} \{p_2, \{p_2, \Psi\}\}(x_0, \xi) &> 0, \text{ si } p_2(x_0, \xi) = \{p_2, \xi\} = 0 \text{ et } \xi \neq 0; \\ \frac{1}{i\tau} \{\overline{p_\Psi}, p_\Psi\}(x_0, \xi, \tau) &> 0, \text{ si } p_\Psi(x_0, \xi, \tau) = \{p_\Psi, \Psi\}(x_0, \xi, \tau) = 0 \text{ et } \tau > 0, \end{aligned}$$

avec  $p_\Psi(x_0, \xi, \tau) = p_2(x_0, \xi + i\tau \nabla \Psi)$

Notons que dans la définition ci-dessus la deuxième condition résulte toujours de la première dans le cas d'un opérateur différentiel d'ordre 2 à symbole principal réel. Nous avons toutefois choisi de la présenter ainsi pour la comparer avec la notion de pseudoconvexité en  $\xi_t = 0$  donnée dans la Définition 6.

La condition ci-dessus est une condition *géométrique* qui porte sur l'hypersurface. Il faut néanmoins noter qu'elle n'est pas suffisante pour obtenir une estimée de Carleman avec  $\Psi$  comme poids. Le lien entre l'hypersurface  $S = \{\Psi = 0\}$  est le poids  $\phi$  dans l'estimée de Carleman est donné par  $\phi = e^{\lambda \Psi}$  avec  $\lambda > 0$  positif grand. Ceci correspond à la convexification analytique, voir [LL20, Chapitre 2.2.1] pour plus de détails.

Nous sommes en mesure d'énoncer le résultat suivant dû à Hörmander :

**Théorème 1.1.4** (Théorème d'unicité de Hörmander). *Soit  $P$  un opérateur différentiel d'ordre 2 dont le symbole principal est réel. On suppose que ses coefficients principaux sont  $C^\infty$  et que tous ses coefficients sont bornés. Alors la propriété d'unicité locale (1.1.2) est satisfaite à travers toute hypersurface orientée fortement pseudoconvexe par rapport à  $P$ .*

Pour un énoncé précis on renvoie aux notes [LL20, Theorem 2.3.2] et pour la version plus générale de ce théorème au livre [Hör94, Theorem 28.3.4]. Ce qui est très important est que le théorème de Hörmander donne une condition *géométrique*, la pseudoconvexité, sur l'hypersurface suffisante pour le prolongement unique. Quant à la régularité des coefficients elle est  $C^\infty$  (voire  $C^1$ ) ce qui est une grande amélioration par rapport à l'analyticité qui est exigée dans le théorème de Holmgren. En plus, la preuve est basée sur une inégalité de Carleman et elle est donc stable par des perturbations d'ordre inférieur. En effet, on peut déjà constater dans l'esquisse qu'on a faite ci-dessus que c'est la partie principale de l'opérateur qui donne la positivité. En particulier les coefficients non principaux de  $P$  peuvent être dans  $L^\infty$ .

Dans le cas des opérateurs d'ordre deux à symbole principal réel la condition de pseudoconvexité peut s'interpréter géométriquement comme une condition sur la façon dont la projection sur l'espace physique du flot Hamiltonien associé au symbol principal de  $P$  rencontre l'hypersurface en question. L'apparition du flot Hamiltonien vient du fait déjà mentionné que le symbole principal de  $i[P_R, P_I]$  est donné par  $\{p_R, p_I\}$ .

Revenons maintenant aux deux exemples mentionnés : celui des opérateurs elliptiques et celui des ondes. Si  $P$  est elliptique alors toute hypersurface est pseudoconvexe par

rapport à  $P$ , en effet la définition de l'ellipticité implique que les deux conditions de pseudoconvexité sont vides. Le théorème de Hörmander est alors optimal pour ces opérateurs. Concernant les ondes, ce théorème est satisfait pour toutes les hypersurfaces de type espace mais aussi par certaines hypersurfaces de type temps. Toutefois, un inconvénient important du théorème de Hörmander est que la notion de pseudoconvexité est sensible à l'orientation de l'hypersurface. Cela se traduit par le fait que si l'on peut propager l'unicité de  $\{\Psi > 0\}$  à  $\{\Psi < 0\}$  (par exemple) on ne pourra pas a priori la propager de  $\{\Psi < 0\}$  à  $\{\Psi > 0\}$ . Ceci contraste avec la notion d'hypersurface non-caractéristique qui est indépendante de l'orientation. En particulier le théorème de Hörmander ne permet pas d'atteindre toutes les hypersurfaces non-caractéristiques pour l'opérateur des ondes et on peut se demander s'il est possible de l'améliorer. Cependant, les constructions de Alinhac et Alinhac-Baouendi limitent considérablement nos attentes. En effet, dans [Ali83, AB79, AB95] les auteurs montrent que le résultat de Hörmander est presque optimal dans le sens suivant : une violation de la condition de pseudoconvexité pour  $P$  (qu'on peut même prendre à coefficients constants) et l'hypersurface non pseudoconvexe  $S$  peut donner des contres-exemples d'unicité à travers  $S$  pour l'opérateur  $P + V$  avec  $V = V(t, x) \in C^\infty$  ! Plus précisément, nous avons le résultat suivant qui est un cas particulier de [AB95, Theorem] :

**Théorème 1.1.5.** *Soit  $n \geq 2$  et  $P = D_t^2 - \sum_{j=1}^n D_{x_j}^2$ . Il existe un ouvert  $U \subset \mathbb{R}^{n+1}$  avec  $0 \in U$  et  $V, u \in C^\infty(U; \mathbb{C})$  avec*

$$(P + V)u = 0 \text{ sur } U, \text{ et } \text{supp } u = \{x_1 \geq 0\} \cap U.$$

Cela signifie que la propriété d'unicité n'est pas satisfaite pour  $P + V$  à travers l'hypersurface  $\{x_1 = 0\}$ , qui n'est pas pseudoconvexe par rapport à  $P + V$  en 0.

Toutefois, le théorème de Holmgren nous donne l'unicité à travers toute hypersurface non-caractéristique pour  $P + V$  si  $V$  est analytique au lieu de  $C^\infty$ . Cela suggère qu'une bonne hypothèse de régularité sur les coefficients pourrait être l'analyticité par rapport à la variable du temps  $t$ .

Dans cette direction Robbiano obtient dans [Rob91] un résultat d'unicité crucial qui est ensuite généralisé par Hörmander dans [Hör92]. C'est alors Tataru qui obtient dans [Tat95] un théorème qui permet d'interpoler entre celui de Holmgren et de Hörmander. Ce théorème est ensuite généralisé/simplifié dans [Hör97], [RZ98] et finalement dans [Tat99b]. Nous nous focalisons dans un cas particulier de ce théorème qui nous sera utile pour l'équation des ondes. Nous avons la définition :

**Définition 6.** Soit  $\Omega \ni x_0$  un ouvert et  $P$  un opérateur différentiel d'ordre 2 sur un ouvert de  $\mathbb{R} \times \mathbb{R}^n$  dont le symbole principal est réel. On dit que l'hypersurface orientée  $S = \{\Psi = \Psi(x_0)\}$  est fortement pseudoconvexe en  $\xi_t = 0$  par rapport à  $P$  au point  $x_0$  si pour tout  $(\xi_t, \xi_x, \tau) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^+$ ,

$$\begin{aligned} \{p_2, \{p_2, \Psi\}\}(x_0, \xi) &> 0, \text{ si } p_2(x_0, \xi) = \{p_2, \xi\} = \xi_t = 0 \text{ et } \xi \neq 0; \\ \frac{1}{i\tau} \{\overline{p_\Psi}, p_\Psi\}(x_0, \xi, \tau) &> 0, \text{ si } p_\Psi(x_0, \xi, \tau) = \{p_\Psi, \Psi\}(x_0, \xi, \tau) = \xi_t = 0 \text{ et } \tau > 0, \end{aligned}$$

avec  $p_\Psi(x_0, \xi, \tau) = p_2(x_0, \xi + i\tau\nabla\Psi)$ .

Nous pouvons alors énoncer le résultat suivant :

**Théorème 1.1.6** (Théorème de Tataru-Hörmander-Robbiano-Zuily). *Soit  $P$  un opérateur différentiel d'ordre 2 sur un ouvert de  $\mathbb{R} \times \mathbb{R}^n$  dont le symbole principal est réel. On suppose que **tous** les coefficients de  $P$  sont analytiques par rapport à la variable  $t$  et bornés par rapport à  $x$ . On suppose aussi que les coefficients principaux sont  $C^1$  par rapport à  $x$ . Alors la propriété d'unicité locale (1.1.2) est satisfaite à travers toute **hypersurface orientée fortement pseudoconvexe** en  $\xi_t = 0$  par rapport à  $P$ .*

Outre les articles originaux cités ci-dessus on renvoie à [LL23b, Chapter 3] pour une présentation claire et pédagogique de ce théorème dans le cas où  $P$  est l'opérateur des ondes avec des coefficients indépendants du temps.

Expliquons maintenant le contenu de ce théorème. On a une hypothèse d'analyticité mais cette fois ci uniquement par rapport à la variable du temps. Il s'agit d'une hypothèse naturellement satisfaite par plusieurs problèmes d'évolution intéressants (en particulier le cas où les coefficients sont indépendants du temps). La conclusion est essentiellement la même que dans le théorème de Hörmander mais le gain est ici le fait qu'on doit vérifier l'hypothèse de pseudoconvexité seulement en  $\xi_t = 0$ . Ici  $\xi_t$  est la variable duale de  $t$ , la condition de pseudoconvexité étant une condition sur l'espace des phases. Cela donne une condition moins restrictive et par conséquent une famille strictement plus grande que celle des hypersurfaces pseudoconvexes "partout". Le théorème général porte en fait sur des opérateurs partiellement analytiques et stipule que plus on a des coefficients analytiques, plus on peut restreindre la condition de pseudoconvexité. Dans les cas extrêmes où tous les coefficients de  $P$  sont analytiques on retrouve alors le théorème de Holmgren, et quand on ne fait aucune hypothèse d'analyticité on retrouve celui de Hörmander. Le point clef est que si  $P$  est un opérateur de type ondes alors on peut montrer que les deux conditions de la Définition 6 sont vides si l'hypersurface  $S$  est non-caractéristique par rapport à  $P$ . Cela signifie que  $S$  est pseudoconvexe en  $\xi_t = 0$  pour  $P$ . Par conséquent, le Théorème 1.1.6 implique que si  $P$  a tous ses coefficients analytiques en temps, on peut propager l'unicité à travers toute hypersurface non-caractéristique pour  $P$ .

Revenons maintenant aux applications à la théorie du contrôle. Comme nous l'avons vu le prolongement unique équivaut à un résultat de contrôle approché. Concernant l'équation des ondes, la vitesse finie de propagation implique qu'étant donné un ouvert  $\omega \subset \mathcal{M}$ , il existe un temps  $T_{opt}$  tel que le système (1.1.4) ne soit pas approximativement contrôlable dès que  $T \leq T_{opt}$ . Le temps  $T_{opt}$  est une quantité géométrique naturelle donnée par  $T_{opt} = 2\mathcal{L}(\mathcal{M}, \omega)$  où  $\mathcal{L}(\mathcal{M}, \omega) := \sup_{x \in \mathcal{M}} \text{dist}(x, \omega)$  est la "plus grande distance" entre  $\omega$  et un point de  $\mathcal{M}$ .

Le Théorème 1.1.6 permet alors de montrer que 1.1.4 est approximativement contrôlable en temps  $T > T_{opt}$ , ce qui donne un résultat de contrôle optimal par rapport au temps durant lequel le contrôle doit agir.

La preuve du Théorème 1.1.6 est aussi basée sur une inégalité de Carleman mais elle contient en plus une nouvelle idée cruciale due à Tataru [Tat95]. Pour une inégalité de Carleman classique on utilise le fait élémentaire suivant : si on a une estimée du type

$$\int |e^{\tau\phi} u|^2 dx \leq C, \quad \forall \tau \geq \tau_0,$$

uniformément en  $\tau \geq \tau_0$  alors  $u = 0$  presque partout sur  $\{\phi > 0\}$ . L'observation clef de Tataru est que si l'on remplace l'inégalité ci-dessus par

$$\int |e^{-\frac{\delta|D_t|^2}{\tau}} e^{\tau\phi} u|^2 dx \leq C, \quad \forall \tau \geq \tau_0,$$

pour un  $\delta > 0$  alors on a toujours la même conclusion, à savoir que  $u = 0$  presque partout sur  $\{\phi > 0\}$ . Ici  $e^{-\frac{\delta|D_t|^2}{\tau}}$  est le multiplicateur de Fourier associé à la multiplication par  $e^{-\frac{\delta|\xi_t|^2}{\tau}}$ . Il s'agit donc d'un opérateur qui régularise et qui localise dans un régime "basses fréquences", qui dépend de la taille de  $\xi_t$  devant  $\tau$ . On voit aussitôt l'intérêt de ce résultat : en plus du poids classique  $e^{\tau\phi}$  de Carleman, le nouveau poids  $e^{-\frac{\delta|D_t|^2}{\tau}}$  permet de se focaliser en quelque sorte uniquement en basses fréquences, et l'observation clef de Tataru ci-dessus nous dit qu'une estimée qui est localisée en basses fréquences peut être suffisante pour propager l'unicité. Cela explique alors le fait de pouvoir restreindre la condition de pseudoconvexité en basses fréquences (en  $\xi_t$ ) également.

Pour pouvoir exploiter cette observation clef il faut aussi démontrer une nouvelle estimée de Carleman, qui aura maintenant la forme suivante

$$C \left\| e^{-\frac{\delta|D_t|^2}{\tau}} e^{\tau\phi} P u \right\|_{L^2} \geq \left\| e^{-\frac{\delta|D_t|^2}{\tau}} e^{\tau\phi} u \right\|_{L^2}, \quad \tau \geq \tau_0. \quad (1.1.7)$$

Ce nouveau poids non-local engendre un certain nombre de difficultés techniques, une des plus importante étant celle de l'opérateur conjugué. En effet, la conjugaison formelle " $e^{-\frac{\delta|D_t|^2}{\tau}} e^{\tau\phi} P e^{\frac{\delta|D_t|^2}{\tau}} e^{-\tau\phi}$ " nous oblige à considérer l'opérateur  $e^{\frac{\delta|D_t|^2}{\tau}}$  qui n'est même pas défini sur l'espace de Schwartz. C'est ici que l'hypothèse d'analyticité intervient, permettant de définir un "opérateur conjugué" approché.

### 1.1.5 Structure de cette thèse

Cette thèse comprend trois chapitres. A chacun de ces chapitres correspond un paragraphe dans la suite de cette introduction. Ces trois paragraphes présentent quelques résultats fondamentaux antérieurs, les résultats principaux que nous avons obtenus ainsi que la stratégie de preuve avec ses idées et ses outils clef. Nous concluons avec quelques perspectives et problèmes ouverts en lien avec les résultats présentés.

## 1.2 Sur le prolongement unique quantitatif pour des ondes dans un milieu singulier

Nous nous intéressons dans cette partie au prolongement unique quantitatif pour des ondes se propageant dans un milieu non-homogène. Avant de donner plus de détails sur ce problème nous commençons par présenter quelques résultats antérieurs.

### 1.2.1 Résultats antérieurs et motivations

Nous avons présenté dans la Section 1.1.4 trois résultats *qualitatifs* fondamentaux, le troisième étant une généralisation des deux précédents dans le cadre des opérateurs d'ordre deux à symbole principal réel. Mais qu'en est-il du problème *quantitatif*? Dans le contexte du Théorème de Holmgren des estimées de stabilité locale de la forme (1.1.1) ont été obtenues par John [Joh60] avec  $\phi(a, b, c) = (a + b)^\delta c^{1-\delta}$  (estimations de type Hölder) dans le cas où  $P$  est elliptique et  $\phi(a, b, c) = c \left( \log(1 + \frac{c}{a+b}) \right)^{-1}$  dans le cas général.

Concernant le théorème de Hörmander, Bahouri [Bah87] montre des résultats locaux de stabilité Hölder, puis Lebeau et Robbiano démontrent pour  $P$  elliptique d'ordre

deux dans [Rob95, LR95] des estimées globales de type Hölder. Passons maintenant au cas où  $P$  est l'opérateur des ondes. Nous notons par  $T_{opt}$  le temps optimal pour que l'équation des ondes soit approximativement contrôlable. Dans le cas où  $P$  est à coefficients analytiques Lebeau montre dans [Leb92a] une estimée quantitative globale avec  $\phi(a, b, c) = c \left( \log(1 + \frac{c}{a+b}) \right)^{-1}$  et ce dès que  $T > T_{opt}$ . Dans le cadre  $C^\infty$  pour les ondes le premier résultat est démontré par Robbiano dans [Rob95] pour un temps  $T > CT_{opt}$  et avec  $\phi(a, b, c) = c \left( \log(1 + \frac{c}{a+b}) \right)^{-1/2}$ . Ce résultat a été amélioré par Phung [Phu10] en  $\phi(a, b, c) = c \left( \log(1 + \frac{c}{a+b}) \right)^{-(1-\epsilon)}$ , toujours pour un temps  $T > CT_{opt}$ . Afin de montrer un résultat quantitatif d'unicité optimal par rapport au temps il semble naturel de s'attaquer à la quantification du théorème qualitatif général Théorème 1.1.6. Pour faire cela, Tataru propose dans ses notes [Tat99a] une stratégie qui devrait donner une estimée de stabilité avec  $\phi(a, b, c) = c \left( \log(1 + \frac{c}{a+b}) \right)^{-(1-\epsilon)}$ . Mais le résultat optimal n'est démontré finalement qu'en 2015 par Laurent et Léautaud dans [LL19] où les auteurs quantifient de façon optimale le théorème de Tataru-Hörmander-Robbiano-Zuily dans le contexte général des opérateurs à coefficients partiellement analytiques. Notons aussi qu'un résultat quantitatif moins général et presque optimal mais plus précis par rapport aux applications aux problèmes inverses a été démontré dans [BKL16] en même temps que [LL19].

### Le théorème de Laurent-Léautaud pour l'opérateur des ondes

Avant de préciser le résultat de [LL19] fixons quelques notations. On considère une variété riemannienne compacte (avec ou sans bord)  $\mathcal{M}$  munie d'une métrique  $g$  lisse. On note  $P$  l'opérateur  $P = \partial_t^2 - \Delta_g$  où  $\Delta_g$  est l'opérateur de Laplace-Beltrami sur  $\mathcal{M}$ . Pour  $E \subset \mathcal{M}$  on rappelle que  $\mathcal{L}(\mathcal{M}, E) := \sup_{x \in \mathcal{M}} \text{dist}_g(x, E)$ .

On considère l'équation des ondes avec conditions de Dirichlet sur  $\partial\mathcal{M}$  et données initiales  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  :

$$\begin{cases} Pu = 0 & \text{sur } (0, T) \times \mathcal{M} \\ u = 0 & \text{sur } (0, T) \times \partial\mathcal{M} \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{sur } \mathcal{M}. \end{cases} \quad (1.2.1)$$

Nous avons alors le résultat suivant :

**Théorème 1.2.1** (Théorème 1.1 dans [LL19]). *Pour tout sous-ensemble ouvert non vide  $\omega$  de  $\mathcal{M}$  et tout  $T > 2\mathcal{L}(\mathcal{M}, \omega)$ , il existe  $C, \kappa, \mu_0$  tels que pour tout  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  et  $u$  satisfaisant (1.2.1) on a, pour tout  $\mu \geq \mu_0$ ,*

$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq Ce^{\kappa\mu} \|u\|_{L^2((0, T) \times \omega)} + \frac{C}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}. \quad (1.2.2)$$

*Si de plus  $\partial\mathcal{M} \neq \emptyset$  et  $\Gamma$  est un sous-ensemble ouvert non vide de  $\partial\mathcal{M}$ , pour tout  $T > 2\mathcal{L}(\mathcal{M}, \Gamma)$ , il existe  $C, \kappa, \mu_0 > 0$  tels que pour tout  $(u_0, u_1)$  dans  $H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  et  $u$  satisfaisant (1.2.1), on a pour tout  $\mu \geq \mu_0$ ,*

$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq Ce^{\kappa\mu} \|\partial_{\nu_\Gamma} u\|_{L^2((0, T) \times \Gamma)} + \frac{C}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}.$$

Supposons maintenant que l'on observe depuis le petit ouvert  $\omega$  pendant un temps  $T > 2\mathcal{L}(\mathcal{M}, \omega)$  et qu'on ne voit rien, c'est à dire  $\|u\|_{L^2((0,T)\times\omega)} = 0$ . On fait alors tendre  $\mu \rightarrow \infty$  dans l'inégalité ci-dessus pour trouver que  $u_0$  et  $u_1$  sont toutes les deux nulles. Le problème (1.2.1) étant bien posé cela implique que  $u = 0$  partout et nous avons alors retrouvé la propriété de prolongement unique qualitative. En plus le temps  $T > 2\mathcal{L}(\mathcal{M}, \omega)$  est le temps optimal dicté par la vitesse finie de propagation (on peut consulter [Rus71a, Rus71b] pour des contres-exemples explicites). On note que dans [LL19] c'est la norme  $H^1((0, T) \times \omega)$  qui est utilisée dans le membre de droite de (1.2.2) au lieu de la norme  $L^2((0, T) \times \omega)$ . Le fait qu'on puisse la remplacer par cette norme moins forte est expliqué dans [LL22, Chapter 5].

En choisissant une valeur de  $\mu$  qui minimise le second membre de l'inégalité ci-dessus on peut reformuler le Théorème 1.2.1 comme suit (on écrit seulement le cas intérieur) :

**Théorème 1.2.2** (Corollary 1.2 dans [LL19]). *Sous les hypothèses du Théorème 1.2.1 il existe  $C > 0$  tel que pour tout  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  avec  $(u_0, u_1) \neq (0, 0)$  on a :*

$$\begin{aligned} \|(u_0, u_1)\|_{H^1 \times L^2} &\leq C e^{C\Lambda} \|u\|_{L^2((0,T)\times\omega)}, \\ \|(u_0, u_1)\|_{L^2 \times H^{-1}} &\leq C \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\log\left(1 + \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\|u\|_{L^2((0,T)\times\omega)}}\right)}. \end{aligned}$$

$$\text{où } \Lambda = \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\|(u_0, u_1)\|_{L^2 \times H^{-1}}}.$$

Ici la quantité  $\Lambda$  doit être interprétée comme la fréquence typique des données. La première inégalité de ce théorème ressemble à une estimée d'observabilité. En effet, elle nous dit qu'on arrive à observer depuis  $\omega$  une partie (exponentiellement petite) de l'énergie totale des données initiales. La différence par rapport à une vraie estimée d'observabilité comme nous l'avons introduite au Théorème 1.1.1 est que la constante  $\Lambda$  dépend des données initiales. Elle est néanmoins uniforme si l'on se restreint à de données initiales dans un régime basse fréquence.

Notons aussi que l'estimée avec l'exponentielle dans le théorème ici est équivalente au fait que nous avons une stabilité logarithmique  $\phi(a, b, c) = c \left(\log\left(1 + \frac{c}{a+b}\right)\right)^{-1}$ . Il a été démontré par Lebeau dans le contexte analytique [Leb92a] qu'en général cette dépendance est optimale. Plus précisément, elle est optimale dès que la Condition de Contrôle Géométrique n'est pas satisfaite par  $(\omega, T)$ . Cela signifie qu'il existe un rayon d'optique géométrique qui n'intersecte pas  $[0, T] \times \bar{\omega}$  et n'a que des réflexions transverses à  $[0, T] \times \partial\mathcal{M}$ .

Comme démontré par Robbiano dans [Rob95] un résultat de prolongement unique quantitatif permet d'obtenir une estimation du coût du contrôle approché. Le résultat quantitatif de Laurent-Léautaud donne alors le théorème suivant :

**Théorème 1.2.3** (Théorème 1.3 dans [LL19]). *Soit  $\mathcal{M}$  et  $\omega \subset \mathcal{M}$  comme dans les théorèmes précédents. Alors pour tout  $T > 2\mathcal{L}(\mathcal{M}, \omega)$  il existe  $C, c > 0$  tels que pour tout  $\epsilon > 0$  et tout  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ , il existe  $f \in L^2((0, T) \times \omega)$  avec*

$$\|f\|_{L^2((0,T)\times\omega)} \leq C e^{c/\epsilon} \|(u_0, u_1)\|_{H_0^1 \times L^2}$$



tel que la solution de

$$\begin{cases} Pu = \mathbb{1}_\omega f & \text{sur } (0, T) \times \mathcal{M} \\ u = 0 & \text{sur } (0, T) \times \partial\mathcal{M} \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{sur } \mathcal{M}, \end{cases}$$

satisfait

$$\left\| (u, \partial_t u)|_{t=T} \right\|_{L^2 \times H^{-1}} \leq \epsilon \|(u_0, u_1)\|_{H_0^1 \times L^2}.$$

En d'autres termes, si nous agissons sur la région  $\omega$  pendant un temps  $T > 2\mathcal{L}(\mathcal{M}, \omega)$ , nous pouvons faire passer l'énergie de notre solution de taille 1 (dans l'espace  $H^1 \times L^2$ ) à  $\epsilon$  proche de 0 (dans l'espace  $L^2 \times H^{-1}$ ). De plus, cela s'accompagne d'une estimation de l'énergie du contrôle qui est de l'ordre de  $e^{c/\epsilon}$ . Puisque le Théorème quantitatif 1.2.1 traite le cas du bord aussi, on a un résultat analogue pour le coût du contrôle frontière.

L'ingrédient crucial pour la preuve du Théorème 1.2.1 est l'obtention d'une estimée quantitative locale "propageable". Cela signifie, qu'il faut une estimée quantitative dont la forme est préservée lorsqu'on l'applique successivement. Comme nous l'avons déjà remarqué, les inégalités de Carleman étant elles-mêmes en quelque sorte de nature quantitative elles constituent un bon point de départ pour l'obtention d'une telle estimée. Néanmoins, dans le contexte du Théorème 1.1.6 l'inégalité de Carleman utilisée a la forme (1.1.7) et le poids microlocal qui l'accompagne rend cette tâche très difficile. Dans [LL19] les auteurs expliquent une forme optimale d'estimée locale quantitative adaptée à l'inégalité de Carleman ci-dessus. Elle s'écrit comme

$$Ce^{\kappa\mu} (\|M^\mu \theta_\mu u\|_{H^1} + \|Pu\|) + Ce^{-\kappa'\mu} \|u\|_{H^1} \geq \|M^\mu \chi_\mu u\|_{H^1}, \quad \mu \geq \mu_0, \quad (1.2.3)$$

où  $M^\mu$  est un opérateur qui régularise et localise en des fréquences  $\xi_t$  plus petites que  $\mu$ . La fonction  $\theta$  est un cut-off qui localise dans le domaine d'observation et  $\chi$  localise sur un voisinage du point qui nous intéresse. Toutefois dans l'estimée ci-dessus ce ne sont pas les fonctions  $\chi$  et  $\theta$  qui interviennent mais des versions régularisées analytiques de ces fonctions, qui dépendent également du paramètre  $\mu$ . Le fait de régulariser et localiser en basses fréquences permet d'avoir une erreur exponentiellement petite dans le membre de gauche et c'est cela qui rend cette estimée "propageable". Par contre le fait de travailler avec des fonctions qui ne localisent pas exactement (puisqu'elles sont analytiques) rend la preuve de ce résultat très technique.

Dans une deuxième partie de [LL19] il est expliqué comment on peut propager l'estimée locale (1.2.3) en passant par des hypersurfaces admissibles (non-caractéristiques). A la fin de la propagation il faut aussi contrôler les hautes fréquences, ce qui explique le passage d'erreur de taille  $e^{-c\mu}$  à une erreur en  $1/\mu$  dans le Théorème 1.2.1.

## Milieux non-homogènes

Motivés par plusieurs applications physiques, on veut étudier la question du prolongement unique quantitatif dans le cas où la métrique  $g$  présente une discontinuité à travers une interface présentant un saut. En effet, dans de nombreux contextes les ondes se propagent dans des milieux singuliers et donc en présence de coefficients non lisses. C'est le cas par exemple des ondes sismiques [Sym83] ou des ondes acoustiques [YDdH<sup>+</sup>17, AdHG17,

[CdHKU19] qui se propagent dans la croûte terrestre. Les modèles proposant de décrire de tels phénomènes utilisent des métriques discontinues et plus précisément des métriques qui sont régulières par morceaux mais qui présentent des sauts le long de certaines hypersurfaces. Nous pouvons donner comme exemple la discontinuité de Mohorovičić entre la croûte terrestre et le manteau. Un autre exemple se présente en imagerie médicale. Le cerveau humain [FdLKBH01, MCCP17] a deux composantes principales : la matière blanche et la matière grise. Celles-ci ont des conductivités électriques très différentes et les modèles décrivant la situation sont très similaires à l'exemple précédent.

La question du prolongement unique quantitatif à travers une interface semble être bien comprise dans le contexte elliptique/parabolique. L'un des premiers résultats (dans le cas parabolique) est démontré dans [DOP02] où l'opérateur  $\partial_t - \operatorname{div}(c\nabla \cdot)$  est étudié avec une hypothèse de monotonie imposée sur le coefficient scalaire  $c = c(x)$  : l'observation devrait avoir lieu dans la région où le coefficient  $c$  est plus petit. Dans cet article, une estimation globale de Carleman est prouvée. Ensuite, dans le cas elliptique, dans [LR10], un résultat similaire a été obtenu mais sans aucune restriction sur le signe du saut du coefficient. Ces techniques ont été étendues au contexte parabolique dans [LR11]. Une nouvelle approche a été introduite par Le Rousseau-Lerner [LRL13] pour montrer une estimée de Carleman dans le cas elliptique anisotrope ( $-\operatorname{div}(A(x)\nabla \cdot)$ , avec  $A$  une matrice dont les coefficients présentent un saut à travers une interface). Nous donnerons plus de détails dans le paragraphe 1.2.3 sur la stratégie employée dans [LRL13]. D'autres travaux intéressants dans le contexte elliptique incluent [DCFL<sup>+</sup>17, FLVW16] où le problème d'affaiblir la régularité des coefficients principaux loin de l'interface est traité.

La question du contrôle *exact* des ondes avec des sauts à travers une interface a déjà été abordée dans le livre [Lio88, Chapitre 6]. Un résultat de contrôlabilité est prouvé pour l'opérateur  $\partial_t^2 - \operatorname{div}(c\nabla \cdot)$  avec  $c$  un coefficient constant par morceaux sous une hypothèse géométrique sur l'hypersurface de saut et une condition de signe sur le saut. L'une des premières estimations de Carleman a été prouvée dans le cadre discontinu dans [BMO07]. Avec la même hypothèse sur le coefficient et en supposant que l'interface est convexe, les auteurs prouvent des estimations quantitatives linéaires de stabilité. Récemment, dans [BGM21], des résultats quantitatifs ont également été prouvés pour des interfaces qui sont des perturbations des surfaces convexes. D'autres travaux connexes sont [Gag18] et [BCDCG20].

### 1.2.2 Problème étudié et principaux résultats obtenus

Nous commençons par fixer le cadre. Soit  $(\mathcal{M}, g)$  une variété riemannienne compacte munie d'une métrique lisse. On considère  $S$  une sous-variété de codimension 1 de  $\mathcal{M}$  sans bord. On suppose de plus que  $\mathcal{M} \setminus S = \Omega_- \cup \Omega_+$  avec  $\Omega_- \cap \Omega_+ = \emptyset$ . On considère aussi un coefficient scalaire  $c(x) = \mathbb{1}_{\Omega_-} c_-(x) + \mathbb{1}_{\Omega_+} c_+(x)$  avec  $c_{\pm} \in C^\infty(\overline{\Omega_{\pm}})$  tel que  $0 < c_{\min} < c(x) < c_{\max}$  uniformément sur  $\Omega_- \cup \Omega_+$  de sorte que la condition d'ellipticité soit vérifiée. Nous allons travailler avec l'opérateur des ondes "discontinu"  $P_c$  défini par

$$P_c = \partial_t^2 - \operatorname{div}_g(c(x)\nabla_g), \quad \text{sur } \mathbb{R}_t \times \Omega_- \cup \Omega_+.$$

Cet opérateur permet de modéliser la propagation d'une onde scalaire dans un milieu non-homogène. En effet, on peut imaginer que la géométrie globale est décrite par la métrique lisse  $g$  et que le coefficient scalaire  $c$  correspond au carré de la vitesse de propagation



d'une onde à l'intérieur de  $\mathcal{M}$ . Le saut de  $c$  correspondant alors au fait que la vitesse de propagation est différente selon que l'onde se trouve dans  $\Omega_-$  ou  $\Omega_+$ .

On considère pour  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  le problème d'évolution suivant :

$$\begin{cases} P_c u = 0 & \text{sur } (0, T) \times \Omega_- \cup \Omega_+ \\ u|_{S_-} = u|_{S_+} & \text{sur } (0, T) \times S \\ (c\partial_\nu u)|_{S_-} = (c\partial_\nu u)|_{S_+} & \text{sur } (0, T) \times S \\ u = 0 & \text{sur } (0, T) \times \partial\mathcal{M} \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{sur } \mathcal{M}, \end{cases} \quad (1.2.4)$$

où  $\partial_\nu$  est un champs de vecteurs non nul défini sur un voisinage de  $S$ , normal à  $S$  (pour la métrique  $g$ ), qui pointe vers  $\Omega_+$  et normalisé pour  $g$ . On écrit  $u|_{S_\pm}$  pour les traces de  $u|_{\Omega_\pm}$  sur l'hypersurface  $S$ .

On remarque qu'il y a deux équations supplémentaires dans notre système par rapport à l'équation des ondes (1.2.1). Il s'agit de conditions de transmission naturelles que nous imposons près de l'interface. Ces conditions impliquent que l'opérateur elliptique sous-jacent est auto-adjoint sur  $L^2(\mathcal{M})$  et on peut montrer en utilisant des méthodes classiques (par exemple avec le théorème de Hille-Yosida) que le système (1.2.4) est bien posé.

Nous introduisons aussi la quantité  $\mathcal{L}_c(\mathcal{M}, \omega) = \sup_{x \in \mathcal{M}} \text{dist}_c(x, \omega)$  qui est la "plus grande distance" du sous-ensemble  $\omega$  à un point de  $\mathcal{M}$ , où  $\text{dist}_c$  est une fonction de distance adaptée à  $(\mathcal{M}, g, c)$ . Nous avons alors le résultat suivant, qui donne la même estimée quantitative que celle obtenue par Laurent-Léautaud [LL19] dans le cas d'une métrique lisse :

**Théorème 1.2.4** (Théorème 1.1 dans [Fil22]). *Pour tout sous-ensemble ouvert non vide  $\omega$  de  $\mathcal{M}$  et tout  $T > 2\mathcal{L}_c(\mathcal{M}, \omega)$ , il existe  $C, \kappa, \mu_0$  tels que pour tout  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  et  $u$  satisfaisant (1.2.4) on a, pour tout  $\mu \geq \mu_0$ ,*

$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq C e^{\kappa\mu} \|u\|_{L^2((0, T) \times \omega)} + \frac{C}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}.$$

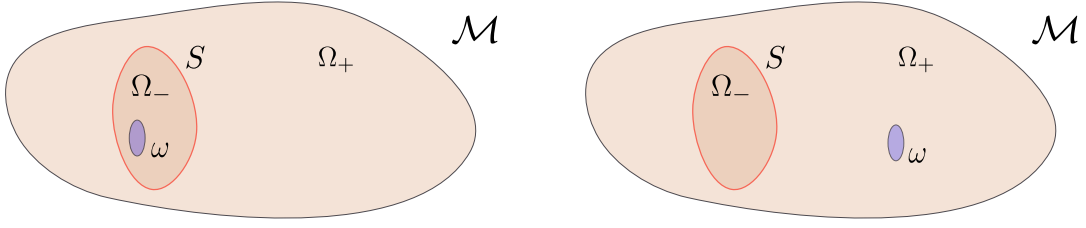
*Si de plus  $\partial\mathcal{M} \neq \emptyset$  et  $\Gamma$  est un sous-ensemble ouvert non vide de  $\partial\mathcal{M}$ , pour tout  $T > 2\mathcal{L}_c(\mathcal{M}, \Gamma)$ , il existe  $C, \kappa, \mu_0 > 0$  tels que pour tout  $(u_0, u_1)$  dans  $H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  et  $u$  satisfaisant (1.2.4), on a*

$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq C e^{\kappa\mu} \|\partial_{\nu_T} u\|_{L^2((0, T) \times \Gamma)} + \frac{C}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}.$$

Puisque nous montrons la même estimée quantitative que celle obtenue par Laurent-Léautaud [LL19] nous obtenons aussi les conséquences qui en découlent concernant l'estimée d'observabilité approchée ainsi que le coût du contrôle approchée. Ce sont les deux théorèmes suivants dont les conclusions sont identiques à leurs homologues dans le cadre lisse.

**Théorème 1.2.5.** *Sous les hypothèses du Théorème 1.2.4 il existe  $C > 0$  tel que pour tout  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  avec  $(u_0, u_1) \neq (0, 0)$  on a :*

$$\begin{aligned} \|(u_0, u_1)\|_{H^1 \times L^2} &\leq C e^{C\Lambda} \|u\|_{L^2((0, T) \times \omega)}, \\ \|(u_0, u_1)\|_{L^2 \times H^{-1}} &\leq C \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\log \left( 1 + \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\|u\|_{L^2((0, T) \times \omega)}} \right)}, \end{aligned}$$

(a) L'observation a lieu à l'intérieur de  $\Omega_-$ .(b) L'observation a lieu à l'intérieur de  $\Omega_+$ . Si  $c_- < c_+$ , une partie de l'onde peut rester piégée à l'intérieur de  $\Omega_-$ . Néanmoins, le prolongement unique quantitatif et ses conséquences sont toujours valables.

$$\text{où } \Lambda = \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\|(u_0, u_1)\|_{L^2 \times H^{-1}}}.$$

**Théorème 1.2.6** (Coût du contrôle approché pour les ondes dans un milieu non-homogène). *Soit  $\mathcal{M}$  et  $\omega \subset \mathcal{M}$  comme dans le Théorèmes 1.2.4. Alors pour tout  $T > 2\mathcal{L}_c(\mathcal{M}, \omega)$  il existe  $C, c > 0$  tels que pour tout  $\epsilon > 0$  et tout  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ , il existe  $f \in L^2((0, T) \times \omega)$  avec*

$$\|f\|_{L^2((0, T) \times \omega)} \leq C e^{c/\epsilon} \|(u_0, u_1)\|_{H_0^1(\mathcal{M}) \times L^2(\mathcal{M})}$$

tel que la solution de

$$\begin{cases} P_c u = \mathbf{1}_\omega f & \text{sur } (0, T) \times \mathcal{M} \\ u|_{S_-} = u|_{S_+} & \text{sur } (0, T) \times S \\ (c\partial_\nu u)|_{S_-} = (c\partial_\nu u)|_{S_+} & \text{sur } (0, T) \times S \\ u = 0 & \text{sur } (0, T) \times \partial\mathcal{M} \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{sur } \mathcal{M}, \end{cases}$$

satisfasse

$$\|(u, \partial_t u)|_{t=T}\|_{L^2 \times H^{-1}} \leq \epsilon \|(u_0, u_1)\|_{H_0^1 \times L^2}.$$

Il est important de remarquer que nous n'avons fait aucune hypothèse sur le signe de  $c_- - c_+$ . Par conséquent, l'ouvert d'observation  $\omega$  peut être choisi aussi bien dans  $\Omega_-$  que dans  $\Omega_+$ . Expliquons pourquoi cela est intéressant. Supposons, pour fixer les idées, que  $c_- < c_+$  soient deux constantes et  $g = Id$ . Nous pouvons alors interpréter  $\sqrt{c_-}$  et  $\sqrt{c_+}$  comme la vitesse de propagation d'une onde se propageant dans deux milieux isotropes  $\Omega_-$  et  $\Omega_+$  avec des indices de réfraction différents,  $n_-$  et  $n_+$  respectivement (rappelons que  $n_\pm = 1/\sqrt{c_\pm}$ ). Imaginons qu'une onde commence à se propager en partant d'une région située à l'intérieur de  $\Omega_-$ . On a  $\sqrt{\frac{c_-}{c_+}} = \frac{n_+}{n_-}$  et donc l'hypothèse  $c_- < c_+$  se traduit par  $n_- > n_+$ . La loi de Snell-Descartes stipule que lorsqu'une onde passe d'un milieu à indice de réfraction élevé à un milieu dont l'indice de réfraction est plus petit, il existe un angle critique à partir duquel il y a *réflexion interne totale*, c'est-à-dire qu'il n'y a pas de réfraction du tout. Au niveau de l'optique géométrique, c'est-à-dire dans le régime des hautes fréquences, une telle onde reste piégée à l'intérieur de  $\Omega_-$ . On s'attend donc à ce que, au moins en régime haute fréquence, aucune information ne se propage de  $\Omega_-$  à  $\Omega_+$ ,

suisant les lois de l'optique géométrique. Les Théorèmes 1.2.4 et 1.2.5 montrent néanmoins que l'intensité de l'onde dans  $\Omega_+$  sera au moins exponentiellement petite en fonction de la fréquence.

Notons enfin qu'en ajoutant des hypothèses géométriques sur l'interface ainsi que sur l'ordre des vitesses  $c_-, c_+$  (à savoir que si l'observation a lieu dans  $\Omega_+$  alors  $c_+ > c_-$ ) des estimées d'observabilité exacte peuvent être démontrées. Voir par exemple [Lio88, Chapitre 6], [BMO07], [BGM21], [Gag18].

### 1.2.3 Outils et idée de preuve

La preuve du Théorème 1.2.4 repose sur une inégalité de Carleman locale pour l'opérateur  $P_c$ . Loin de l'interface les coefficients de  $P_c$  sont lisses et nous sommes dans le cadre de [LL19]. L'idée est alors la suivante : Dans [LL19] les auteurs donnent une méthode qui permet de passer d'une estimée de Carleman locale adaptée à l'opérateur  $\partial_t^2 - \Delta_g$  à une estimée locale *quantitative* que l'on peut propager. Notre but est alors d'utiliser la machinerie développée par [LL19] afin d'obtenir également une estimée locale quantitative "propageable" près de l'interface. Le point clef est alors l'obtention d'une inégalité de Carleman contenant le poids microlocal  $e^{-\frac{\delta|D_t|^2}{2\tau}}$  pour l'opérateur "discontinu"  $P_c = \partial_t^2 - \operatorname{div}_g(c(x)\nabla_g)$ .

On travaille dans un petit voisinage  $V$  d'un point  $x_0 \in S$  où  $S$  est l'interface et l'on se place en coordonnées géodésiques normales  $(t, x', x_n)$  de sorte que l'interface devienne  $S = \{x_n = 0\}$  et  $\Omega_+ \cap V = \{x_n > 0\}$ ,  $\Omega_- \cap V = \{x_n < 0\}$ . On note  $H_- = \mathbb{1}_{\{x_n < 0\}}$  et  $H_+ = \mathbb{1}_{\{x_n > 0\}}$ . Pour une fonction  $u$  sur  $\mathbb{R} \times V$  on écrit  $u_-$  pour  $H_-u$  et  $u_+$  pour  $H_+u$ . Le coefficient  $c$  étant discontinu on doit être particulièrement attentif avec le domaine de  $P_c$ . Nous définissons l'espace  $\mathcal{W}$  comme l'espace des fonctions  $u$  qui s'écrivent

$$u = H_-u_- + H_+u_+, \quad u_{\pm} \in C_0^\infty(\mathbb{R} \times V)$$

et satisfaisant les conditions de transmission suivantes :

$$\begin{aligned} u_-|_{x_n=0} &= u_+|_{x_n=0}, \\ c_+ \partial_{x_n} u_+|_{x_n=0} &= c_- \partial_{x_n} u_-|_{x_n=0}. \end{aligned} \tag{1.2.5}$$

Les conditions de transmission impliquent que  $P_c u \in L^2$  pour  $u \in \mathcal{W}$ . Nous montrons l'inégalité suivante :

**Théorème 1.2.7** (Estimée de Carleman locale pour  $P_c$ ). *Dans le contexte géométrique présenté ci-dessus, supposons que  $(t_0, x_0) \in \mathbb{R}_t \times V$ . Il existe alors un poids approprié  $\phi$  et des constantes positives  $C, \tau_0, \delta, d, r_0$  telles que.*

$$\begin{aligned} C \left( \left\| e^{-\frac{\delta|D_t|^2}{2\tau}} e^{\tau\phi} P_c u \right\|_{L^2(\Omega_- \cup \Omega_+)}^2 + e^{-d\tau} \left\| e^{\tau\phi} u \right\|_{H^1(\Omega_- \cup \Omega_+)}^2 \right) \\ \geq \tau^3 \left\| e^{-\frac{\delta|D_t|^2}{2\tau}} e^{\tau\phi} u \right\|_{L^2(\Omega_- \cup \Omega_+)}^2 + \tau \left\| \nabla e^{-\frac{\delta|D_t|^2}{2\tau}} e^{\tau\phi} u \right\|_{L^2(\Omega_- \cup \Omega_+)}^2, \end{aligned} \tag{1.2.6}$$

pour tout  $u \in \mathcal{W}$  avec  $\operatorname{supp} u \subset B((t_0, x_0), r_0)$  et  $\tau \geq \tau_0$ .

En fait, pour des raisons techniques liées au fait que cette inégalité ne sera pas appliquée à notre solution  $u$  mais à  $v = \chi u$  avec  $\chi$  un cut-off, on est obligés de montrer une estimée pour des fonctions satisfaisant des conditions de transmission plus générales que celles de  $\mathcal{W}$ .

Le poids  $\phi$  est bien adapté à la situation géométrique au sens suivant : pour  $|x_n|$  petit, nous avons  $\{x_n = 0\} = \{\phi = 0\}$  et les deux côtés de l'interface sont donnés par  $\{\phi > 0\}$  et  $\{\phi < 0\}$ . Nous avons également (pour  $|x_n|$  suffisamment petit) que  $\phi' > 0$ , et par conséquent  $\phi$  est plus grand dans  $\Omega^+$  :  $\Omega^+$  est la région d'observation et l'estimation de Carleman propagera l'unicité de  $\Omega^+$  à  $\Omega^-$ .

L'inégalité ci-dessus est pratiquement du même type que celle démontrée dans les travaux de Tataru, Hörmander, Robbiano-Zuily [Tat95, Tat99b, Hör97, RZ98] ainsi que dans [LL19] (en présence d'un bord de Dirichlet) dans le cas des coefficients réguliers. Cela indique qu'elle fournit un bon point de départ afin d'obtenir une estimée locale quantitative à l'aide des techniques de [LL19].

Nous écrivons

$$P^\pm := \partial_t^2 - \operatorname{div}(c_\pm \nabla \cdot).$$

Si nous supposons que  $\phi$  est analytique par rapport à  $t$  nous pouvons définir un opérateur conjugué  $P_{c,\phi}^\pm$  donné par la relation

$$e^{-\frac{\delta |D_t|^2}{2\tau}} e^{\tau\phi} P^\pm = P_{c,\phi}^\pm e^{-\frac{\delta |D_t|^2}{2\tau}} e^{\tau\phi}.$$

L'étape clef pour la preuve du Théorème 1.2.7 est alors l'obtention d'une estimée sous-elliptique qui porte sur l'opérateur conjugué  $P_{c,\phi}^\pm$ .

Écrivons d'abord les conditions de transmission modifiées, qui correspondent aux conditions de transmission (1.2.5) après conjugaison :

$$\begin{aligned} v_{-|x_n=0} &= v_{+|x_n=0} \\ c_+(D_n v_+ + i\tau\phi'_{|x_n=0^+} v_+)_{|x_n=0} &= c_-(D_n v_- + i\tau\phi'_{|x_n=0^-} v_-)_{|x_n=0}. \end{aligned} \quad (1.2.7)$$

On note par  $w_{|x_n=0}$  la trace de  $w$  sur l'hypersurface  $\{x_n = 0\}$ . L'estimation sous-elliptique clef est le contenu de la proposition suivante :

**Proposition 1.2.1.** *Dans le contexte du Théorème 1.2.7 il existe des constantes positives  $C$ ,  $\tau_0$ ,  $r_0$  telles que*

$$\begin{aligned} C \left( & \left\| H_- P_{c,\phi}^- v_- \right\|_{L^2(\mathbb{R}^{n+1})}^2 + \left\| H_+ P_{c,\phi}^+ v_+ \right\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \left\| H_+ D_t v_+ \right\|_{L^2(\mathbb{R}^{n+1})}^2 \right. \\ & \left. + \tau \left\| H_- D_t v_- \right\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \left| (D_t v_+)_{|x_n=0} \right|_{L^2(\mathbb{R}^n)}^2 + \tau \left| (D_t v_-)_{|x_n=0} \right|_{L^2(\mathbb{R}^n)}^2 \right) \\ & \geq \tau^3 \left\| v \right\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \left\| H_+ \nabla v_+ \right\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \left\| H_- \nabla v_- \right\|_{L^2(\mathbb{R}^{n+1})}^2 \\ & + \tau^3 \left| v_+ \right|_{L^2(\mathbb{R}^n)}^2 + \tau^3 \left| v_- \right|_{L^2(\mathbb{R}^n)}^2 + \tau \left| (\nabla v_+)_{|x_n=0} \right|_{L^2(\mathbb{R}^n)}^2 + \tau \left| (\nabla v_-)_{|x_n=0} \right|_{L^2(\mathbb{R}^n)}^2, \end{aligned}$$

pour  $v$  satisfaisant les conditions de transmission (1.2.7) et tel que  $\operatorname{supp} v \subset B((t_0, x_0), r_0)$  et  $\tau \geq \tau_0$ .

La preuve du Théorème 1.2.7 repose sur l'idée suivante : Comme nous l'avons déjà remarqué, le poids microlocal  $e^{-\frac{\delta|D_t|^2}{2\tau}}$  localise en basses fréquences en  $\xi_t$ , qui est un régime où l'opérateur  $P_{c,\phi}^\pm$  n'est pas loin d'être elliptique dans le sens où son symbole devient positif si  $\xi_t = 0$ . Loin de la zone elliptique (c'est à dire la région microlocale où le symbole de  $P_{c,\phi}^\pm$  est elliptique) on espère exploiter la forme du poids de Carleman afin d'absorber les termes venant de la zone non-elliptique comme erreurs admissibles. La forme de l'estimée sous-elliptique permet de voir cela de façon plus précise. En effet, les termes en  $D_t v$  dans le membre gauche de l'estimée de la Proposition 1.2.1 sont des erreurs admissibles grâce au fait que l'action de  $e^{-\frac{\delta|D_t|^2}{2\tau}}$  sur  $D_t v$  donne une erreur exponentiellement petite et par conséquent admissible au vu du Théorème 1.2.7. Une fois focalisés sur la zone elliptique nous exploitons le fait que nous avons des résultats très précis concernant les estimées de Carleman pour des opérateurs elliptiques avec des sauts et c'est plus précisément la technique de Le Rousseau-Lerner [LRL13] sur laquelle nous nous basons pour traiter cette région.

### La stratégie de Le Rousseau-Lerner

Comme déjà mentionné, étant donné un opérateur différentiel  $P$ , pour obtenir une estimation de Carleman avec un poids  $\phi$ , on travaille avec l'opérateur conjugué  $P_\phi$  défini par  $P_\phi := e^{\tau\phi} P e^{-\tau\phi}$ . Dans le cadre elliptique traité dans [LRL13], les auteurs utilisent une idée qui remonte à Calderón [Cal58]. Présentons la stratégie employée dans le cas d'un modèle jouet. On considère l'interface  $\{x_n = 0\}$  et un coefficient constant par morceaux  $c = H_- c_- + H_+ c_+$  avec  $c_\pm > 0$  où  $H_- = \mathbb{1}_{\{x_n < 0\}}$  et  $H_+ = \mathbb{1}_{\{x_n > 0\}}$ . On travaille alors avec l'opérateur elliptique  $P = -\operatorname{div}(c\nabla)$  et on écrit  $P^\pm = -\operatorname{div}(c_\pm \nabla)$ . On définit l'opérateur conjugué comme  $P_\phi^\pm = e^{\tau\phi} P^\pm e^{-\tau\phi}$ . Dans [LRL13] les auteurs factorisent l'opérateur conjugué  $P_\phi^\pm$  comme un produit de deux opérateurs d'ordre 1 de chaque côté de l'interface et prouvent des estimées pour chacun de ces facteurs. Nous écrivons  $(x_n, x')$  pour les variables physiques et  $(\xi_n, \xi')$  pour les variables duales via la transformée de Fourier. En considérant la transformée de Fourier partielle par rapport à la variable  $x'$ ,  $\mathcal{F}_{x' \rightarrow \xi'}$ , et dans le cas d'un poids  $\phi$  dépendant uniquement de  $x_n$  on a la factorisation suivante :

$$P_\phi^\pm = c_\pm (D_n + i(\tau\phi' + |\xi'|)) (D_n + i(\tau\phi' - |\xi'|)).$$

Ici, avec un petit abus de notation, nous avons identifié l'opérateur avec son symbole dans la variable tangentielle à l'interface  $x'$ . Cela permet de travailler avec la variable normale  $x_n$  et de traiter les autres comme des paramètres. Nous définissons alors les facteurs d'ordre un :

$$e_\pm = H_\pm(\tau\phi' + |\xi'|), \quad f_\pm = H_\pm(\tau\phi' - |\xi'|)$$

dans le but d'obtenir de bonnes estimations unidimensionnelles pour les opérateurs de premier ordre  $D_n + i e_\pm$  et  $D_n + i f_\pm$ . Le principe général est que le signe de  $e_\pm, f_\pm$  détermine la qualité des estimations unidimensionnelles. En effet, puisque  $\phi' > 0$  près de l'interface les facteurs  $e_\pm$  sont positifs. Cela nous donne une bonne première estimée que l'on doit ensuite combiner avec l'estimée obtenue pour les deux autres termes  $f_\pm$ . La subtilité consiste au fait que le signe de  $f_\pm$  n'est pas constant alors que nous avons besoin d'une condition de positivité au moins sur un des deux côtés de l'interface afin de pouvoir contrôler tous les termes dans l'estimée de Carleman. C'est ici qu'intervient la condition sur le poids  $\phi$ . Le choix de  $\phi$  est fait de manière à ce que la propriété clef suivante soit satisfaite : si l'on

est dans une région de l'espace  $V \subset \mathbb{R}^+ \times \mathbb{R}^{n-1} \ni (\tau, \xi')$  où  $f_+ < 0$  alors cela implique que  $f_- > 0$  sur  $V$ . Autrement dit, au moins un de ces deux termes aura "un peu de positivité". Il se trouve que cela est suffisant, grâce aussi aux conditions de transmission, afin de contrôler tous les termes voulus.

Dans [LRL13] il est ensuite démontré que cette propriété sur les signes de  $f_\pm$  peut être satisfaite si l'on impose une condition géométrique sur le poids. Plus précisément, il faut demander que le saut  $\phi'|_{x_n=0^+} - \phi'|_{x_n=0^-}$  de la dérivée normale de  $\phi$  à travers l'interface soit suffisamment grand.

### Régions microlocales

Revenons maintenant au cas de l'opérateur des ondes  $P_c$ . Pour simplifier on suppose que  $P_c = \partial_t^2 - \operatorname{div}(c\nabla)$  avec  $c$  constant par morceaux. Pour  $\phi$  qui dépend uniquement de  $x_n$  l'opérateur conjugué devient, après avoir identifié les variables tangentielles à l'interface avec leur transformée de Fourier

$$P_{c,\phi}^\pm = c_\pm \left( (D_n + i\tau\phi')^2 + |\xi'|^2 - \frac{1}{c_\pm} |\xi_t|^2 \right).$$

Pour  $\epsilon > 0$  petit, on considère alors les régions suivantes de  $\mathbb{R}^n \ni (\xi', t)$  variables duales de  $(x', \xi_t)$  :

1.  $\mathcal{E}_\epsilon^\pm : \left[ |\xi'|^2 - c_\pm^{-1} |\xi_t|^2 \geq \epsilon (|\xi'|^2 + |\xi_t|^2) \right]$ . Ce sont les régions elliptiques. Cela signifie que nous pouvons considérer la racine de  $|\xi'|^2 - c_\pm^{-1} |\xi_t|^2$  et écrire

$$\begin{aligned} P_{c,\phi}^\pm &= c_\pm \left( (D_n + i\tau\phi')^2 + |\xi'|^2 - \frac{1}{c_\pm} |\xi_t|^2 \right) \\ &= c_\pm \left( (D_n + i\tau\phi')^2 + \sqrt{|\xi'|^2 - \frac{1}{c_\pm} |\xi_t|^2}^2 \right) \\ &= c_\pm \left( D_n + i\tau\phi' + i\sqrt{|\xi'|^2 - \frac{1}{c_\pm} |\xi_t|^2} \right) \left( D_n + i\tau\phi' - i\sqrt{|\xi'|^2 - \frac{1}{c_\pm} |\xi_t|^2} \right) \end{aligned}$$

Nous obtenons alors dans ces régions une factorisation comme dans [LRL13] et des estimées du même type.

2.  $\mathcal{GH}_\epsilon^\pm : \left[ |\xi'|^2 - c_\pm^{-1} |\xi_t|^2 \leq 2\epsilon (|\xi'|^2 + |\xi_t|^2) \right]$ . Ce sont les régions glancing et hyperboliques (voir par exemple [Hör85, Chapter 23.2] ou [GL20]). L'observation clef est que pour  $\epsilon$  suffisamment petit dans ces régions nous avons  $|\xi_t| \gtrsim |\xi'|$ . Grâce au poids  $e^{-\delta \frac{D_t^2}{2\tau}}$ , nous pouvons considérer les termes  $D_t v \sim \xi_t v$  comme erreurs admissibles. Le fait que  $\xi_t$  est grand par rapport à  $\xi'$  permet essentiellement de contrôler les termes tangentiels. Il nous reste alors à contrôler les termes avec la dérivée normale  $\|D_n v\|$  et  $|D_n v|$ . Pour faire cela on s'inspire en particulier de la méthode du commutateur utilisée par Lebeau-Robbiano dans [LR95, Section 3.A].

Ici nous avons décrit la stratégie pour le cas modèle. Pour traiter le cas général nous avons besoin d'outils d'analyse microlocale, ce qui rajoute certaines difficultés techniques. Aussi, il faut vérifier qu'on peut perturber le poids  $\phi$  avec des termes qui ne dépendent pas uniquement de  $x_n$  : ceci correspond à la convexification géométrique.

### L'utilisation du Théorème 1.2.7

Une fois que nous avons obtenu l'inégalité (1.2.6) nous sommes en mesure d'exploiter la machinerie de [LL19]. En effet, la première étape est de la rendre quantitative, à savoir obtenir une estimée du type (1.2.3).

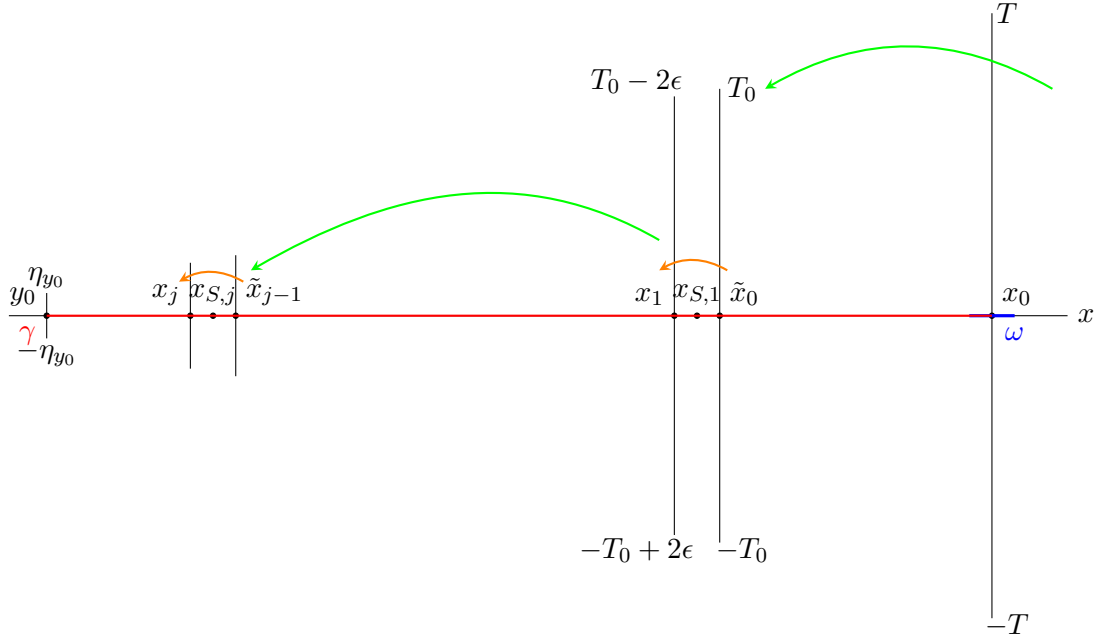


FIGURE 1.2 : Le processus d'itération dans l'espace-temps. Nous transportons l'information d'un point  $x_0 \in \omega$  à  $y_0 \in \mathcal{M}$  en suivant le chemin  $\gamma$ . Les points  $x_{S,j}$  sont les points d'intersection de  $\gamma$  avec  $S$ . Les flèches vertes correspondent à la propagation de l'information dans le contexte lisse où nous utilisons les résultats de [LL19]. Les flèches oranges propagent l'information à travers l'interface en utilisant notre nouvelle estimée locale et ce, en perdant un temps arbitrairement petit.

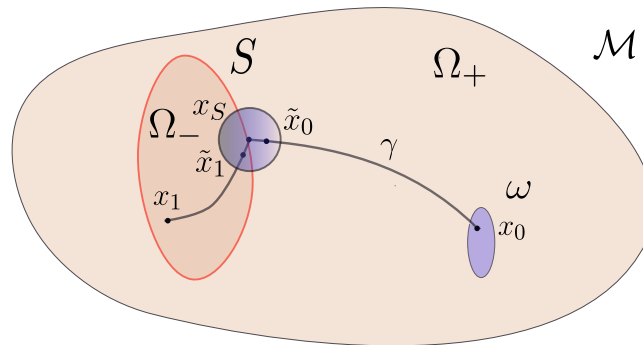


FIGURE 1.3 : Première étape du processus d'itération. L'information est propagée de  $x_0$  à  $x_1$ . Le processus est ensuite répété à partir de  $x_1$ .

Avec l'estimation quantitative locale près de l'interface à notre disposition, nous sommes capables d'obtenir une estimation globale. Nous commençons par propager uniquement les

estimations à basse fréquence. Pour ce faire, nous utilisons certaines propriétés d'itération abstraites introduites dans [LL19]. Nous considérons un point arbitraire  $y_0 \in \mathcal{M}$  et suivons un chemin allant d'un point  $x_0 \in \omega$  à  $y_0 \in \mathcal{M}$ . Tant que le chemin reste dans  $\Omega_+$  ou dans  $\Omega_-$ , la propagation est garantie par le résultat de [LL19]. Si le chemin rencontre l'interface  $S$ , nous utilisons notre nouvelle estimée locale quantitative (qui est un analogue de (1.2.3) déduit de (1.2.6)) qui permet de propager l'unicité d'un côté de  $S$  à l'autre. Nous poursuivons ensuite la propagation en utilisant à nouveau [LL19]. Ceci est illustré dans les figures 1.3 et 1.2.

A la fin de ce processus d'itération on considère aussi les hautes fréquences afin de contrôler toute la solution et on utilise des estimées d'énergie pour l'équation des ondes pour faire intervenir les données initiales. Ceci conclut la preuve du Théorème 1.2.4. Les Théorèmes 1.2.5 et 1.2.6 sont conséquences de celui-ci.

### 1.2.4 Perspectives

Une question naturelle serait de savoir si on peut généraliser le résultat obtenu pour des métriques qui sont complètement différentes de chaque côté de l'interface. En effet, nous avons travaillé ici avec l'opérateur  $\partial_t^2 - \operatorname{div}_g(c(x)\nabla_g)$  ce qui implique que les deux métriques sont conformes. Dans le cas anisotrope on considère une matrice  $A(x) = (a_{jk}(x))_{1 \leq j, k \leq n}$  définie positive dont les coefficients présentent un saut à travers l'interface  $S$  et on travaille avec l'opérateur  $\partial_t^2 - \operatorname{div}(A(x) \cdot \nabla)$ . Le fait d'avoir deux métriques conformes a permis de se placer en coordonnées géodésiques normales des deux côtés à la fois, ce qui est crucial pour la preuve de notre inégalité de Carleman locale. Toutefois, dans [LRL13] les auteurs travaillent avec l'opérateur  $\operatorname{div}(A(x) \cdot \nabla)$  et contournent cet obstacle en construisant "à la main" des coordonnées dans lesquelles  $S = \{x_n = 0\}$  et  $\operatorname{div}(A(x) \cdot \nabla)$  a une forme simple. Cela suggère que, au moins pour l'estimée de Carleman locale, notre preuve pourrait probablement fonctionner pour le cas anisotrope aussi, en espérant pouvoir absorber les termes supplémentaires venant de la forme plus compliquée de  $\partial_t^2 - \operatorname{div}(A(x) \cdot \nabla)$  dans ces nouvelles coordonnées.

Une autre question concerne la régularité optimale de  $c$ . Ici nous avons travaillé avec des coefficients qui sont  $C^\infty$  par morceaux. Comme déjà remarqué c'est le comportement de l'opérateur des ondes dans la région elliptique qui semble être le plus important. Or, nous savons que la régularité limite pour avoir de l'unicité pour les opérateurs elliptiques est Lipschitz. En effet, en dessous de cette régularité nous avons des exemples d'opérateurs elliptiques pour lesquels la propriété d'unicité locale n'est pas satisfaite (voir [Pli63], [Mil74]). Des estimées de Carleman pour des opérateurs elliptiques avec une métrique Lipschitz ont été, par exemple, démontrés dans [Hör85, Chapter 17], [LL21b]. Il semble alors naturel de conjecturer que notre résultat peut se généraliser à des coefficients qui sont Lipschitz par morceaux. Pour prouver cela il faut généraliser les résultats de [LL19] pour des métriques Lipschitz (cf [LL23a]) ainsi que notre estimée de Carleman locale près de l'interface. En ce qui concerne ce deuxième point remarquons qu'on a utilisé des arguments d'analyse microlocale afin de se localiser dans des régions appropriées et exploiter la technique elliptique de [LRL13]. Une telle approche ne pourrait fonctionner en régularité Lipschitz. Toutefois, dans le cas des coefficients constants on peut travailler à l'aide de la transformée de Fourier uniquement dans l'espace des fréquences et prouver notre estimée sans faire appel au calcul pseudodifférentiel. Ensuite on pourrait traiter le cas des coefficients variables comme une perturbation du cas constant en "gelant" les coefficients. C'est la stratégie employée



dans [DCFL<sup>+</sup>17] dans le cas elliptique.

Aussi, il serait intéressant d'appliquer notre résultat à des questions venant des problèmes inverses. Cela est à l'origine de nos motivations puisque une métrique avec des sauts peut être utile pour modéliser des ondes sismiques. La question que l'on se pose est de savoir si on est capable de déterminer la métrique (ou un potentiel) en "observant" l'application de Dirichlet-vers-Neumann pendant un certain temps. La "technique de contrôle frontière" introduite dans [Bel87] repose de façon essentielle sur des résultats de prolongement unique/contrôle approché. Un résultat de prolongement unique quantitatif pourrait donner une estimée de stabilité dans le résultat d'injectivité entre la métrique/potentiel et l'application de Dirichlet-vers-Neumann correspondante.

Finalement, une autre piste dans la direction des coefficients singuliers consiste à diminuer la régularité du potentiel  $V$  dans l'opérateur des ondes  $\partial_t^2 - \Delta_g + V$ . Nous savons depuis les travaux de Jerison et Kenig [JK85] que le prolongement unique est satisfait pour des opérateurs du type  $-\Delta + V(x)$  si  $V \in L_{\text{loc}}^{n/2}$  seulement. On peut alors se poser la question du prolongement unique (qualitative et quantitative) pour les solutions de l'équation des ondes associées. Notons qu'il s'agit d'un problème difficile qui nécessite de nouveaux outils. En effet, on doit passer par des estimées de Carleman  $L^p$  (voir par exemple [KRS87, Sog92, Tat97a, DSF05]) qui sortent du cadre habituel  $L^2$ .

### 1.3 Construction de modes propres pour un problème de transmission

Le point de départ pour le travail présenté dans cette partie est la question de l'optimalité de l'estimée quantitative du Théorème 1.2.5 présentée dans la partie précédente. Notre but est de donner des exemples de situations où l'estimation tunnel du Théorème 1.2.5 est saturée, et la saturation est provoquée par la présence de la discontinuité à travers l'hypersurface  $S$ .

On note dans cette partie  $\mathcal{M}$  un sous-ensemble compact et connexe de  $\mathbb{R}^2$  avec un bord lisse  $\partial\mathcal{M}$  et  $S \subset \mathcal{M}$  une hypersurface compacte lisse telle que  $\text{Int}(\mathcal{M}) \setminus S = \Omega_- \cup \Omega_+$  avec  $\Omega_- \cap \Omega_+ = \emptyset$ . Nous considérons l'opérateur  $-\Delta_c = -\text{div}(c\nabla \cdot)$  agissant sur  $\mathcal{M}$  avec  $c$  étant strictement positif et lisse par morceaux mais ayant un saut à travers l'interface  $S$ .

Afin d'exhiber des solutions de l'équation des ondes (1.2.4) qui saturent l'estimée du Théorème 1.2.5 on cherche à construire des fonctions propres de  $-\Delta_c$  dont la masse se concentrent exponentiellement près de l'interface  $S$ . Ces modes propres sont souvent appelés "Whispering Gallery Modes" (modes de "chuchotement de galerie", WGM dans la suite).

#### 1.3.1 Résultats antérieurs et motivations

Les données  $\mathcal{M}, S$  et  $c$  que nous allons considérer sont à symétrie radiale. L'idée de construire des exemples explicites de fonctions propres sur des surfaces de révolution remonte à [Leb96] et [All98].

Même si notre motivation principale pour la construction de WGM vient de la théorie du contrôle, il est intéressant de noter que ces objets ont des applications physiques importantes. Une application vient de l'optoélectronique. En effet, le cas où  $\mathcal{M}$  est un disque et  $S$  un cercle concentrique plus petit contenu dans son intérieur peut être consi-

déré comme un modèle jouet pour étudier la section orthogonale d'une fibre optique. En effet, dans ce cas,  $\Omega_-$  peut être considéré comme le cœur de la fibre optique entouré d'une gaine  $\Omega_+$ . Si l'on suppose que  $c_- < c_+$ , l'indice de réfraction dans  $\Omega_-$  est plus élevé que dans  $\Omega_+$ , et par conséquent, la lumière reste localisée dans le cœur par réflexion interne totale à la frontière entre le cœur et la gaine. En ce qui concerne les applications à l'optoélectronique, les WGM ont été étudiés numériquement et d'un point de vue théorique dans [BDDM20, BDM21]. Dans ces références, le problème des valeurs propres de  $-\Delta_c$  est étudié dans un domaine non borné, devenant ainsi un problème de résonance. Dans [PV99] un problème similaire a été étudié et des propriétés de concentration polynomiale ont été obtenues mais dans des géométries beaucoup plus générales.

Dans un travail récent [BBAD22], les auteurs obtiennent des propriétés de concentration et de non-concentration pour les fonctions propres de  $-\Delta_c$  en fonction de la régularité du coefficient  $c$ . La discontinuité du coefficient  $c$  correspond au cas de *milieux stratifiés*.

### 1.3.2 Principaux résultats obtenus

Le coefficient  $c$  étant discontinu, on doit être attentif avec le domaine de  $-\Delta_c$ . On définit l'espace

$$\mathcal{A} = \{u \in H_0^1(\mathcal{M}) \mid u|_{\Omega_-} \in H^2(\Omega_-), u|_{\Omega_+} \in H^2(\Omega_+), c_+ \partial_\nu(u|_{\Omega_-})|_S = c_- \partial_\nu(u|_{\Omega_+})|_S\}. \quad (1.3.1)$$

En utilisant des arguments standards de théorie spectrale nous pouvons voir que  $-\Delta_c$ , avec domaine  $\mathcal{A}$ , est positif, auto-adjoint et à résolvante compacte. Nous en déduisons que son spectre contient uniquement des valeurs propres positives  $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$  de multiplicité finie avec  $\lambda_j \rightarrow +\infty$ .

Nous avons le résultat suivant, qui nous donne l'existence des WGM dans certains contextes géométriques :

**Théorème 1.3.1.** *Sous certaines hypothèses sur  $\mathcal{M}$ ,  $S$  et  $c = c_- \mathbf{1}_{\Omega_-} + c_+ \mathbf{1}_{\Omega_+}$  il existe des suites  $(\lambda_n)_{n \in \mathbb{N}}$  et  $(u_n)_{n \in \mathbb{N}}$  et  $\lambda_n \rightarrow +\infty$  et  $u_n \in \mathcal{A}$  tels que pour tout  $\omega \subset \mathcal{M}$  avec  $\text{dist}(\bar{\omega}, S) > 0$  il existe  $C, d > 0$  avec :*

$$\|u_n\|_{L^2(\omega)} \leq C e^{-d\sqrt{\lambda_n}}, \quad -\Delta_c u_n = \lambda_n u_n, \quad u_n|_{\partial\mathcal{M}} = 0, \quad \|u_n\|_{L^2(\mathcal{M})} = 1.$$

Les données  $\mathcal{M}, S, c$  que nous considérons sont à symétrie radiale. Avant de donner plus de précisions sur  $\mathcal{M}, S, c$  on énonce une conséquence immédiate du Théorème 1.3.1.

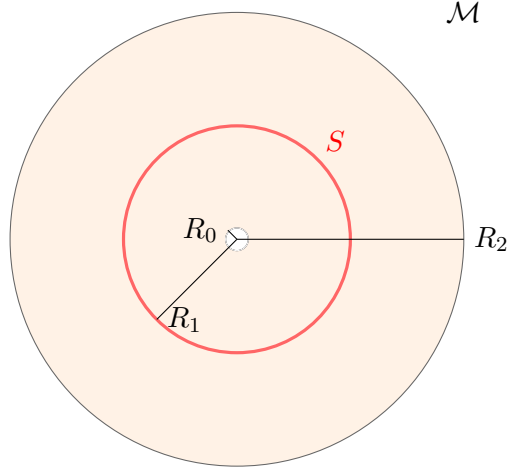
On note  $\Lambda(w) := \frac{\|(w(0), \partial_t w(0))\|_{H^1 \times L^2}}{\|(w(0), \partial_t w(0))\|_{L^2 \times H^{-1}}}$ .

**Théorème 1.3.2.** *Dans le cadre du Théorème 1.3.1, il existe des solutions  $(w_n)_{n \in \mathbb{N}}$  de (1.2.4) avec  $\|w_n(0)\|_{L^2} = 1$ ,  $\partial_t w_n(0) = 0$  et telles que pour tout  $\omega \subset \mathcal{M}$  avec  $\text{dist}(\bar{\omega}, S) > 0$  et pour tout  $T > 0$  il existe  $C, d > 0$  avec :*

$$\|w_n\|_{L^2((0,T) \times \omega)} \leq C e^{-d\Lambda(w_n)}.$$

Le théorème 1.3.2 est une conséquence immédiate du théorème 1.3.1. En effet, si l'on prend  $u_n$  comme dans le Théorème 1.3.1, alors

$$w_n(t, x) := \cos(\sqrt{\lambda_n} t) u_n(x).$$

FIGURE 1.4 : La couronne  $\mathcal{M}$  avec l'interface  $S$ .

satisfait (1.2.4) avec  $(w_n, \partial_t w_n)|_{t=0} = (u_n, 0)$ ,  $\Lambda(w_n) = \sqrt{\lambda_n + 1}$  et l'inégalité

$$\|u_n\|_{L^2(\omega)} \leq C e^{-d\sqrt{\lambda_n}}$$

implique que, pour une autre constante  $C > 0$

$$\|w_n\|_{L^2((0,T)\times\omega)} \leq C e^{-d\sqrt{\lambda_n}}.$$

Les solutions  $(w_n)$  que nous avons obtenues montrent alors que l'estimée du Théorème 1.2.5 est optimale par rapport à l'exponentielle.

Dans le même contexte géométrique, l'inégalité spectrale suivante pour les fonctions propres de  $-\Delta_c$  est prouvée dans [LR10, Théorème 1.2] (et généralisée dans [LRL13]) :

**Théorème 1.3.3** (Théorème 1.2 dans [LR10]). *Soit  $(u_j)_{j \in \mathbb{N}}$  une base hilbertienne des fonctions propres de l'opérateur  $-\Delta_c$  avec des conditions au bord de Dirichlet, satisfaisant les conditions de transmission usuelles. Notons par  $\lambda_j$  les valeurs propres associées dans l'ordre croissant. Alors pour tout  $\omega \subset \text{Int}(\mathcal{M})$  avec  $\omega \neq \emptyset$  il existe  $C > 0$  tel que pour tout  $(a_j)_{j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$  :*

$$\sum_{\lambda_j \leq \lambda} |a_j|^2 \leq C e^{C\sqrt{\lambda}} \int_{\omega} \left| \sum_{\lambda_j \leq \lambda} a_j u_j(x) \right|^2 dx.$$

Les fonctions propres construites dans le Théorème 1.3.1 prouvent que l'estimation spectrale du Théorème 1.3.3 ci-dessus est optimale en général, même pour une seule fonction propre.

Le modèle de base avec lequel on travaille est le cas où  $\mathcal{M}$  est une couronne et  $S$  est un cercle concentrique comme dans la Figure 1.4. Plus précisément nous désignons par  $B_R$  la boule ouverte dans  $\mathbb{R}^2$  de rayon  $R$  centrée sur 0 et étudions le problème des fonctions propres pour  $(\mathcal{A}, -\Delta_c)$  dans le contexte géométrique suivant : Soit  $0 < R_0 < R_1 < R_2$  et  $\mathcal{M} = \overline{B_{R_2}} \setminus B_{R_0} \subset \mathbb{R}^2$ ,  $\Omega_- = B_{R_1} \setminus \overline{B_{R_0}}$ ,  $\Omega_+ = B_{R_2} \setminus \overline{B_{R_1}}$ ,  $S = \overline{B_{R_1}} \setminus B_{R_1}$ . On prend le coefficient  $c = c_- \mathbf{1}_{\Omega_-} + c_+ \mathbf{1}_{\Omega_+}$  constant par morceaux et la condition que l'on impose sur  $c$  est

$$\frac{c_-}{R_1^2} < \frac{c_+}{R_2^2}.$$

Cette condition sera expliquée dans les paragraphes suivants.

Notre motivation initiale est la construction de solutions comme dans le Théorème 1.3.1 afin de saturer certaines estimées d'observabilité comme expliqué ci-dessus. Toutefois, nous obtenons également des résultats plus précis sur le comportement des fonctions propres que nous construisons. Pour plus de détails nous référons au Théorème 1.3.6 ci-dessous.

Le fait de travailler dans une couronne permet de se libérer de certaines difficultés techniques liées au fait que, quand on travaille en coordonnées polaires, nous avons une singularité en  $r = 0$ . Nous expliquons toutefois comment en se basant sur [LL21b] on peut inclure d'autres exemples de surfaces  $\mathcal{M}$  comme le cas où  $\mathcal{M}$  est un disque ou une surface de révolution diffeomorphe au disque.

Etant donné que nous sommes intéressés par le comportement de fonctions propres près de l'interface, une question naturelle est de savoir si on peut restreindre les conditions géométriques sur  $\mathcal{M}$ ,  $S$  et  $c$  uniquement dans un voisinage de l'interface. En ce qui concerne les fonctions propres elles-mêmes cette question semble être difficile. Nous obtenons toutefois une généralisation du Théorème 1.3.2 en se basant uniquement sur une hypothèse de symétrie uniquement au voisinage de l'interface.

### 1.3.3 Outils et idée de preuve

#### Reduction à un problème 1d

Nous commençons par le cas où  $\mathcal{M}$  est une couronne et  $S$  un cercle comme dans la figure 1.4. Nous suivons l'approche de [LL21b] afin de nous ramener à l'étude d'un opérateur de Schrödinger 1D. On travaille en coordonnées polaires et cherche des solutions de  $-\Delta_c u = \lambda u$  sous la forme  $u(r, \theta) = e^{in\theta} f(r)$ . Cela nous amène à l'étude du problème 1D sur  $(R_0, R_2)$   $P_h f = E_h f$  avec

$$P_h := \sum_{\pm} \mathbf{1}_{\Omega_{\pm}} \left( -h^2 \frac{1}{r} \partial_r (c_{\pm} r \partial_r) \right) + V_c(r), \quad V_c(r) := \frac{c}{r^2}, \quad E_h = h^2 \lambda, \quad h = 1/n,$$

ayant comme domaine

$$\mathcal{A}_r = \{f \in H_0^1((R_0, R_2)) \mid f|_{(R_0, R_1)} \in H^2((R_0, R_1)), f|_{(R_1, R_2)} \in H^2((R_1, R_2)), \\ c_+ \partial_r (f|_{(R_0, R_1)})(R_1) = c_- \partial_r (f|_{(R_1, R_2)})(R_1)\}. \quad (1.3.2)$$

En considérant des oscillations fortes dans la direction angulaire, c'est-à-dire  $n \rightarrow \infty$ , on peut interpréter  $h$  comme un petit paramètre semiclassique. Nous avons ainsi réduit le problème à l'étude d'un opérateur de Schrödinger semiclassique avec un potentiel  $V_c$  ayant un saut de discontinuité en  $r = R_1$ .

L'intérêt de cette formulation est que les propriétés spectrales des opérateurs de Schrödinger dans la limite semiclassique ont déjà été étudiés en profondeur (voir par exemple les ouvrages de Helffer [Hel88] et Dimassi-Sjöstrand [DS99]) et on dispose d'un outil relativement élémentaire et assez robuste pour montrer la décroissance exponentielle des fonctions propres : les estimées d'Agmon.

#### Estimées d'Agmon

Nous partons de l'idée intuitive suivante : Une particule classique se déplaçant dans  $]R_0, R_2[$  soumise à un potentiel  $V$  a une énergie constante égale à  $E = |\xi|^2 + V(x)$  où  $x$  est la variable

de position et  $\xi$  la variable de quantité de mouvement. Puisque  $|\xi|^2 \geq 0$  la particule doit toujours se trouver dans la région  $\{x \in ]R_0, R_2[ \mid V(x) \leq E\}$  : c'est la région *classiquement autorisée* et son complémentaire la région *classiquement interdite*. Considérons maintenant une particule quantique d'énergie  $E$ . Sa fonction d'onde  $\psi_h$  satisfait à l'équation

$$(-h^2\Delta + V)\psi_h = E\psi_h.$$

Si l'on interprète la limite  $h \rightarrow 0$  comme la limite "du quantique au classique" alors on s'attend à ce que la masse de  $\psi_h$  dans la région classiquement interdite tende vers 0 dans la limite  $h \rightarrow 0$ . Les estimées d'Agmon permettent de démontrer cela mathématiquement, en donnant aussi une décroissance exponentielle en  $h$ . Afin d'étudier la décroissance des fonctions propres de  $-\Delta + V$  en géométrie non bornée, Agmon [Agm82] a introduit une distance qui a été ensuite adaptée dans le contexte semiclassique par Simon [Sim83] et Helffer-Sjöstrand [HS84].

Pour les définitions et stratégie de preuve nous suivons [Hel88, Chapitre 3]. Dans notre contexte la région classiquement autorisée est donnée par

$$K_E = \{r \in [R_0, R_2] \mid V_c(r) \leq E\},$$

et nous définissons la distance d'Agmon au niveau d'énergie  $E$  par

$$d_{A,E}(r) = \inf_{y \in K_E} \left| \int_y^r \sqrt{\frac{(V_c(s) - E)_+}{c(s)}} ds \right|,$$

où  $a_+ = \max(a, 0)$  et on suppose que  $E \geq \inf V_c =: E_0$ . On voit alors que la distance d'Agmon quantifie le fait d'être dans la région classiquement interdite. L'estimée clef que nous obtenons est la suivante :

**Théorème 1.3.4.** *Il existe  $\eta > 0$  tel que pour tout  $E \in [E_0, E_0 + \eta]$  et pour tout  $\epsilon(h)$  avec  $\epsilon(h) \xrightarrow{h \rightarrow 0} 0$  pour tout  $\delta > 0$ , il existe  $C, h_0$  tels que pour  $\psi_h$  satisfaisant*

$$\psi_h \in \mathcal{A}_r, \quad P_h \psi_h = (E + \epsilon(h))\psi_h, \quad \|\psi_h\|_{L^2} = 1,$$

nous avons pour  $0 < h \leq h_0$  :

$$\left\| h \partial_r \left( e^{\frac{d_{A,E}}{h}} \psi_h \right) \right\|_{L^2((R_0, R_2))} + \left\| e^{\frac{d_{A,E}}{h}} \psi_h \right\|_{L^2((R_0, R_2))} \leq C e^{\delta/h}.$$

Nous voyons alors aussitôt que dès que  $d_{A,E} \geq d > 0$  la masse de  $\psi_h$  est exponentiellement petite dans cette région. Ceci arrive précisément lorsque la particule se trouve dans la région classiquement interdite. La preuve de cette inégalité repose de façon essentielle sur l'identité suivante :

**Lemme 1.3.5.** *Soit  $\phi$  une fonction Lipschitz sur  $[R_0, R_2]$  et  $f \in \mathcal{A}_r$ . Alors nous avons :*

$$\begin{aligned} \int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} f) \right|^2 r dr - \int_{R_0}^{R_2} c |\partial_r \phi|^2 e^{2\phi/h} |f|^2 r dr \\ = -\operatorname{Re} \int_{(R_0, R_1) \cup (R_1, R_2)} e^{2\phi/h} ch^2 \left( \partial_r^2 + \frac{1}{r} \partial_r \right) f \cdot \bar{f} r dr. \end{aligned}$$

Nous rappelons la définition de l'espace  $\mathcal{A}_r$  dans (1.3.2).

Pour la preuve de l'inégalité d'Agmon on doit appliquer l'identité ci-dessus à  $\phi = (1 - \delta)d_{A,E}$  (qui est Lipschitz) et  $f$  notre fonction propre. Un point très important est aussi le fait que la distance d'Agmon satisfait à l'équation eikonale

$$c |\partial_r d_{A,E}|^2 = (V_c(r) - E)_+, \quad \text{dans } \mathcal{D}'((R_0, R_2)).$$

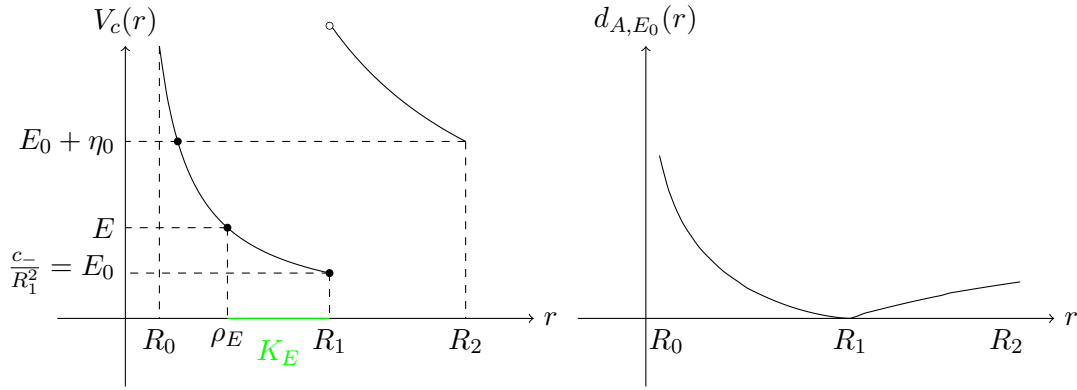


FIGURE 1.5 : A gauche : Le potentiel  $V_c$ . La région classiquement autorisée  $K_E$  en vert. A droite : Le graphe de la distance d'Agmon  $d_{A,E_0}$  associée au niveau d'énergie minimal  $E_0$ .

### Constructions de fonctions propres

L'inégalité d'Agmon du Théorème 1.3.4 montre que toute fonction propre d'énergie proche de  $E_0$  a une masse exponentiellement petite dans la région classiquement interdite associée à  $E_0$ . Il faut alors s'assurer que, dès qu'on est à une distance strictement positive de  $R_1$  on est forcément dans la région classiquement interdite, voir la Figure 1.5. Ceci est le cas précisément lorsque le minimum du potentiel  $V_c$  est atteint en  $r = R_1$  et c'est la raison pour laquelle on impose que  $\frac{c_-}{R_1^2} < \frac{c_+}{R_2^2}$ . Cette hypothèse implique que  $\min V_c = \frac{c_-}{R_1^2} = E_0$  et  $K_{E_0} = \{R_1\}$ .

Il reste finalement à montrer qu'il est possible de trouver des fonctions propres de  $P_h$  associées à l'énergie  $E_0$  (ou au moins suffisamment proche de  $E_0$ ). Pour faire cela on construit des quasimodes de précision  $\epsilon(h) = h^{2/3}$  (c'est le  $\epsilon$  du Théorème 1.3.4) qui donnent l'existence des fonctions propres cherchées à l'aide du théorème spectral pour les opérateurs auto-adjoints.

Cela donne l'existence de fonctions propres  $\psi_h \in \mathcal{A}_r$  de  $P_h$  qui se concentrent exponentiellement près de  $R_1$ . Puis pour construire les fonctions propres de  $\Delta_c$  sur la couronne on n'a qu'à poser  $u_n(r, \theta) = e^{in\theta} \psi_h(r)$  (on rappelle que  $h = 1/n$ ,  $\lambda_n = n^2 E$ ). Les fonctions propres ( $u_n$ ) ainsi définies ont une concentration exponentielle près de  $r = R_1$ , ce qui correspond à l'interface  $S$ . Ceci conclut la preuve du Théorème 1.3.1.

### Une estimée d'Agmon précisée

L'estimée d'Agmon du Théorème 1.3.4 permet de montrer une estimée sur un voisinage *symétrique* de l'interface et ceci est suffisant pour la preuve du Théorème 1.3.1. Toutefois une question intéressante est de savoir si l'on peut obtenir des informations plus fines sur la concentration de fonctions propres autour de l'interface. En particulier, l'hypothèse faite sur le coefficient  $c$  est que  $c_-$  est petit devant  $c_+$ . Sous cette hypothèse, au moins pour le cas d'une onde, on s'attend à ce qu'une partie de l'onde reste piégée à l'intérieur de  $\Omega_-$  en suivant les lois d'optique géométrique comme expliqué dans la Section 1.2.2. On s'attend alors à ce qu'il y ait plus de masse à gauche de  $R_1$  (c'est-à-dire dans  $\Omega_-$  quand on revient en deux dimensions). L'inégalité du Théorème 1.3.4 n'est pas suffisante pour

capter le comportement des fonction propres arbitrairement proche de l'interface. Cela est dû à la perte  $\delta$  dans le membre droite de l'inégalité du Théorème 1.3.4 qui nous oblige à fixer une distance strictement positive de  $R_1$  avant d'appliquer le Théorème 1.3.4.

Afin de contourner ce problème nous montrons une inégalité d'Agmon raffinée en suivant une technique due à Hellfer-Sjöstrand [HS84] qui a été employée dans le cas où  $E_0$  est le minimum d'un puits non dégénéré. Cette technique a aussi été utilisée dans [LL21b, Preuve du Théorème 3.9]. Le point de départ est toujours l'identité-clef du Lemme 1.3.5 mais avec un choix de  $\phi$  différent. Nous montrons l'estimée suivante :

**Théorème 1.3.6.** *Soit  $\psi_h$  satisfaisant*

$$\psi_h \in \mathcal{A}_r, \quad P_h \psi_h = (E_0 + \epsilon(h)) \psi_h, \quad \|\psi_h\|_{L^2} = 1,$$

avec  $|\epsilon(h)| \leq C_1 h^{2/3}$  pour un  $C_1 > 0$ . Alors il existe  $C, h_0 > 0$  tels que pour  $0 < h \leq h_0$  on a

$$\|\psi_h\|_{L^2((R_0, r))} \leq C e^{-\frac{C \log h}{h^{1/3}}} e^{-\frac{d_{A, E_0}(r)}{h}}, \quad \text{pour tout } r \in R_0 \leq r \leq R_1,$$

$$\|\psi_h\|_{L^2((r, R_2))} \leq C e^{-\frac{C \log h}{h^{1/3}}} e^{-\frac{d_{A, E_0}(r)}{h}}, \quad \text{pour tout } r \in R_1 \leq r \leq R_2.$$

Remarquons maintenant que :

$$d_{A, E_0}(r) \underset{r \rightarrow R_1^+}{\sim} \alpha_+(r - R_1), \quad d_{A, E_0}(r) \underset{r \rightarrow R_1^-}{\sim} \alpha_-(R_1 - r)^{\frac{3}{2}},$$

avec  $\alpha_+ = \sqrt{1 - \frac{c_-}{c_+} \frac{1}{R_1}}$  et  $\alpha_- = \frac{2\sqrt{2}}{3R_1\sqrt{R_1}}$ .

En combinant alors le Théorème 1.3.6 avec les formules ci-dessus décrivant le comportement de  $d_{A, E_0}$  près de  $R_1$  nous obtenons le corollaire suivant.

**Corollaire 1.3.7.** *Soit  $\psi_h$  comme dans le Théorème 1.3.6. Alors pour tout  $\delta \in (1/3, 1)$  il existe  $c_\delta > 0$  tel que :*

$$\|\psi_h\|_{L^2((R_1 - h^{\frac{2}{3}(1-\delta)}, R_1 + h^{1-\delta}))} = 1 + O(e^{-\frac{c_\delta}{h^\delta}}).$$

En d'autres termes, la masse de la fonction propre se concentre exponentiellement pour  $h \rightarrow 0$  à l'intérieur de  $(R_1 - h^{\frac{2}{3}(1-\delta)}, R_1 + h^{1-\delta})$ . Cela suggère qu'il pourrait y avoir plus de masse à gauche de l'interface, et ce de façon quantitative.

## D'autres surfaces de révolution

Remarquons que dans le cas où  $c$  est constant par morceaux, on pourrait probablement faire des calculs explicites en utilisant les fonctions de Bessel. En effet, leur comportement asymptotique est bien compris (voir par exemple [Olv74]). Cependant, notre approche a l'avantage de s'appuyer uniquement sur des estimations d'énergie robustes. En particulier, elle s'applique avec des modifications mineures à des géométries radiales et à des coefficients avec de saut plus généraux.

En effet, cette méthode permet de traiter toutes les surfaces de révolution qui sont traités dans [LL21b, Section 3]. Une difficulté technique qui surgit quand on travaille avec des surfaces de révolution plus générales est que la distance d'Agmon explose près du

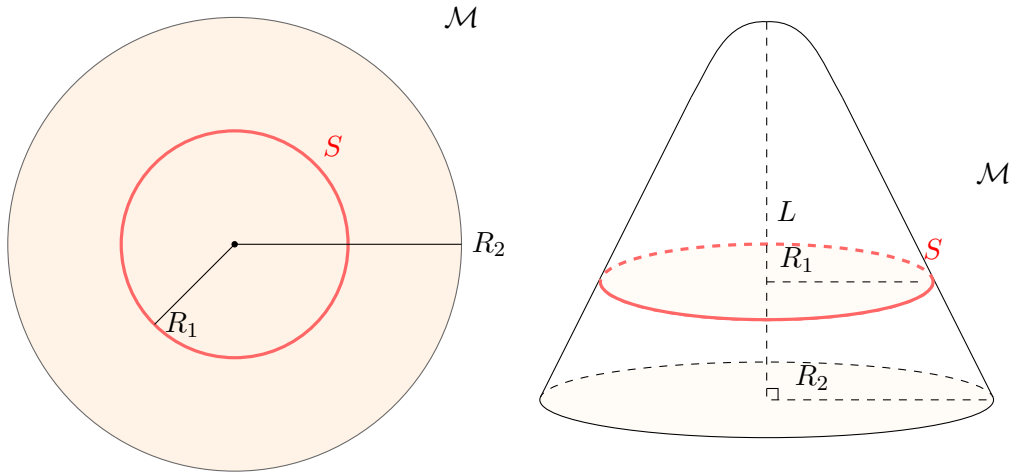


FIGURE 1.6 : D'autres exemples de situations géométriques pour lesquelles le Théorème 1.3.1 s'applique.

pôle nord (c'est-à-dire  $s = 0$ ). Nous pouvons déjà constater ceci dans le graphe de la distance d'Agmon (1.5) où l'on voit qu'elle explose près de  $r = 0$ . Dans [LL21b, Preuve du Théorème 3.9] il est expliqué comment on peut contourner ce problème.

Finalement la preuve du résultat local que nous obtenons repose sur l'observation suivante : on considère une fonction propre  $f$  de  $-\Delta_c$  sur un voisinage  $\mathcal{V}$  de  $S$  qui vérifie les conditions du Théorème 1.3.1. Alors  $\chi f$  avec  $\chi$  un cut-off qui localise dans un voisinage de  $\mathcal{V}$  n'a aucune raison d'être une fonction propre de  $-\Delta_c$  sur  $\mathcal{M}$ . Toutefois l'estimée d'Agmon du Théorème 1.3.4 permet de montrer que  $\chi f$  est un quasimode exponentiellement précis, à savoir  $(-\Delta_c - \lambda)\chi f = O(e^{-c\sqrt{\lambda}}) \|\chi f\|_{L^2}$ . Grâce à la linéarité du problème (1.2.4) il se trouve que cela est suffisant pour prouver l'existence des solutions avec les propriétés du Théorème 1.3.2.

### 1.3.4 Perspectives

Une question naturelle serait de chercher à obtenir des informations plus précises sur le comportement et la concentration des fonctions propres que nous construisons. Le Théorème 1.3.6 et son Corollaire 1.3.7 donnent déjà un résultat dans cette direction. Le résultat obtenu indique que la masse de fonctions propres se concentre à une distance  $h^{\frac{2}{3}(1-\delta)}$  à gauche de l'interface et  $h^{(1-\delta)}$  avec  $\delta \in (1/3, 1)$ . Cela suggère que la vitesse de concentration serait de  $h^{2/3}$  à gauche et  $h$  à droite ce qui se traduit par le fait de pouvoir prendre  $\delta \in (0, 1)$ . La raison pour laquelle on impose que  $\delta > 1/3$  vient de la perte  $e^{-\frac{C \log h}{h^{1/3}}}$  dans l'estimée du Théorème ci-dessus. Cette perte est due à la précision  $\epsilon(h) = O(h^{2/3})$ . Une précision en  $O(h)$  permettrait d'obtenir une estimée sans le terme  $e^{-\frac{C \log h}{h^{1/3}}}$  et prendre par conséquent  $\delta \in (0, 1)$  dans le corollaire.

Outre la motivation que nous venons de donner la question sur la localisation du spectre de  $P_h$  est intéressante en soi. Nous avons construit un quasimode peu sophistiqué de précision  $O(h^{2/3})$  pour  $E_0$  qui nous a donné l'existence d'une fonction propre associée à la valeur propre  $E_h$  avec  $|E - E_h| = O(h^{2/3})$ . Le quasimode que nous avons construit a un support qui localise à gauche de l'interface de sorte que les conditions de transmission soient



automatiquement satisfaites. Nous pourrions alors essayer de construire des quasimodes plus raffinés qui ont un support sur un voisinage de l'interface. Le modèle local approprié semble être une fonction de type Airy à gauche de l'interface et une exponentielle à droite. Toutefois le raccord de ces fonctions engendre certains problèmes techniques. Cela pourrait indiquer que le saut du potentiel fait que la distance entre la première valeur propre et  $E_0$  est d'ordre plus grand que  $O(h)$ .

Enfin, une question intéressante (et probablement difficile) serait d'obtenir l'existence de fonctions propres qui se concentrent autour de l'interface dans des géométries plus générales. D'un point de vue heuristique on s'intéresse uniquement aux propriétés des fonctions propres sur un voisinage arbitrairement petit de l'interface. Par conséquent nous aurions envie de faire l'hypothèse de révolution uniquement près de l'interface (ou pas du tout) et non pas sur toute la variété. Néanmoins il n'est pas clair comment traiter la région loin de l'interface afin d'obtenir une vraie fonction propre définie sur toute la variété avec les propriétés cherchées.

## 1.4 Prolongement unique pour des opérateurs de Schrödinger avec des coefficients partiellement Gevrey

### 1.4.1 Résultats antérieurs et motivations

Dans la première partie nous nous sommes intéressés à des propriétés d'unicité satisfaites par des solutions de l'équation des ondes. Pour la partie qualitative du problème nous avons vu que le théorème le plus général est le théorème de Tataru-Hörmander-Robbiano-Zuily 1.1.6, quantifié dans [LL19]. Motivés par plusieurs applications venant de la théorie du contrôle, on s'intéresse ici au cas des opérateurs de Schrödinger

$$P = i\partial_t + \sum_{j,k=1}^d g^{jk}(x) \partial_{x_j} \partial_{x_k} + \sum_{j=1}^d b^j(t, x) \partial_{x_j} + \mathbf{q}(t, x), \quad (1.4.1)$$

avec  $g^{jk}$  une matrice elliptique en  $x_0 \in V \subset \mathbb{R}^d$ , c'est à dire satisfaisant

$$\sum_{j,k=1}^d g^{jk}(x_0) \xi_j \xi_k \geq C |\xi|^2, \text{ pour tout } \xi \in \mathbb{R}^d.$$

Rappelons que le théorème de Tataru-Hörmander-Robbiano-Zuily 1.1.6 dans toute sa généralité porte sur des opérateurs à coefficients partiellement analytiques et s'applique aussi à  $P$ . La condition de pseudoconvexité est une condition qui porte sur le symbole principal de l'opérateur. Pour  $P$ , la partie principale "classique" est  $\sum_{j,k=1}^d g^{jk} \partial_{x_j} \partial_{x_k}$ . Toutefois, négliger la dérivée temporelle change complètement la structure de l'opérateur et on peut se demander s'il est possible de modifier ces résultats afin de mieux respecter son anisotropie. Pour l'opérateur de Schrödinger il faudrait un scaling qui fait qu'une dérivée en temps a le même poids que deux dérivées en espace.

Isakov montre dans [Isa93] que le théorème de pseudoconvexité de Hörmander 1.1.4 reste valide dans le contexte anisotrope en modifiant de façon appropriée la définition du crochet de Poisson. Dans un scaling comme celui de Schrödinger on définit

$$\{p(t, x, \xi_t, \xi_x), q(t, x, \xi_t, \xi_x)\} := \partial_{\xi_t} p(t, x, \xi_t, \xi_x) \cdot \partial_x q(t, x, \xi_t, \xi_x) - \partial_x p(t, x, \xi_t, \xi_x) \cdot \partial_{\xi_t} q(t, x, \xi_t, \xi_x).$$

La différence par rapport au crochet de Poisson classique est qu'ici on ne dérive pas par rapport à  $t$  et sa variable duale  $\xi_t$ , puisque la différentiation par rapport à  $t$  réduit l'ordre par deux dans le scaling anisotrope. Dans le théorème classique de Hörmander 1.1.4 les conditions de pseudoconvexité ne sont jamais satisfaites pour l'opérateur de Schrödinger. En utilisant le crochet de Poisson anisotrope et le résultat d'Isakov on obtient un résultat d'unicité non-trivial sous une condition analogue à la pseudoconvexité classique (donc n'incluant pas toute hypersurface non-caractéristique). Par conséquent, le résultat d'unicité global obtenu en itérant le résultat local d'Isakov fournit une condition géométrique très restrictive sur l'ensemble d'observation. Pour les applications aux problèmes de contrôle ou aux problèmes inverses, des estimations globales de Carleman pour des opérateurs de Schrödinger ont été prouvées par exemple dans [BP02] (coefficients d'ordre supérieur constants) et dans [TX07, Lau10] (coefficients variables). Une condition de pseudoconvexité faible a également été prouvée suffisante dans [MOR08] dans le cas d'une métrique plate et dans [Lau10] pour une métrique variable. Cependant, dans tous ces exemples, une condition de convexité similaire à celle d'Isakov est requise et aboutit à des hypothèses géométriques fortes.

Dans le contexte des coefficients partiellement analytiques Tjøn généralise dans [Tj00] le théorème de Tataru-Hörmander-Robbiano-Zuily dans un context anisotrope en obtenant une famille plus large d'hypersurfaces. Toutefois, dans le cas particulier d'un opérateur de Schrödinger le théorème de Tataru-Hörmander-Robbiano-Zuily donne déjà un résultat géométrique optimal. Nous avons :

**Théorème 1.4.1** (Tataru-Hörmander-Robbiano-Zuily pour des opérateurs de Schrödinger). *Soit  $\Omega = I \times V$  où  $I \subset \mathbb{R}$  est un intervalle ouvert,  $V \subset \mathbb{R}^d$  un ouvert et  $(t_0, x_0) \in \Omega$ . Supposons que  $g^{jk} \in C^1(V)$  est une matrice elliptique à valeurs réelles en  $x_0$  et que les coefficients  $b^j, q$  sont analytiques par rapport à  $t$  et bornés par rapport à  $x$ . Soit  $S = \{\Psi = 0\}$  une hypersurface non-caractéristique par rapport à  $P$  en  $(t_0, x_0)$ , c'est-à-dire,*

$$\sum_{j,k=1}^d g^{jk}(x_0) \partial_{x_j} \Psi(t_0, x_0) \partial_{x_k} \Psi(t_0, x_0) \neq 0. \quad (1.4.2)$$

Alors il existe un voisinage  $\mathcal{V}$  de  $(t_0, x_0)$  tel que, pour  $P$  définie dans (1.4.1), nous ayons

$$\begin{cases} Pu = 0 \text{ sur } \Omega \\ u = 0 \text{ sur } \Omega \cap \{\Psi > 0\} \\ u \in H^1(\Omega) \end{cases} \implies u = 0 \text{ sur } \mathcal{V}.$$

Du point de vue de la régularité, l'analyticité en temps est bien sûr très exigeante. Toutefois, du point de vue géométrique, l'hypothèse de non-caractéristicité est optimale : elle exclut seulement les surfaces tangentes à  $\{t = t_0\}$ , pour lesquelles nous savons que la propriété du prolongement unique locale n'est, en général, pas vérifiée (cela impliquerait sinon une vitesse de propagation finie pour les solutions de Schrödinger). En appliquant ce Théorème itérativement à une famille d'hypersurfaces bien choisies (voir par exemple [LL19, Section 6.2]), nous obtenons un résultat de prolongement unique global sous une condition géométrique optimale, toujours sous l'hypothèse d'analyticité en temps pour tous les coefficients.

Le point de départ pour le travail que nous présentons dans ce paragraphe vient de la question : Peut-on améliorer (au moins partiellement) le théorème de Tataru-Hörmander-Robbiano-Zuily dans le cas spécifique de l'opérateur de Schrödinger, en tenant compte de

son caractère anisotrope? Puisque sa condition géométrique est déjà optimale c'est la régularité exigée pour les coefficients que nous espérons relaxer. Outre son intérêt théorique un tel résultat aurait des conséquences à la théorie du contrôle pour l'équation de Schrödinger, qui est un sujet de recherche très actif avec beaucoup de questions ouvertes (voir par exemple le survey [Lau14]). En effet, nous avons déjà remarqué qu'un résultat d'unicité équivaut à un résultat de contrôle approché. Toutefois, un résultat de prolongement unique peut être aussi une étape cruciale pour l'obtention d'un résultat de contrôle/observabilité exact/exacte. Nous avons en particulier le résultat suivant qui porte sur l'observabilité pour l'équation de Schrödinger dans le disque :

**Théorème 1.4.2** (Observabilité pour l'équation de Schrödinger dans le disque, Théorème 1.2 dans [ALM16]). *Soit  $\mathbb{D}$  le disque unité dans  $\mathbb{R}^2$ . On note  $u(t, z)$  l'unique solution de*

$$(i\partial_t + \Delta + V(t, z)) u(t, z) = 0, \quad t \in \mathbb{R}, \quad z \in \mathbb{D}, \quad u(0, z) = u_0 \in L^2(\mathbb{D}).$$

*Soit  $T > 0$  et  $\omega \subset \mathbb{D}$  un ouvert avec  $\omega \cap \partial\mathbb{D} \neq \emptyset$ . Nous supposons que le potentiel  $V \in C^\infty([0, T] \times \mathbb{D}; \mathbb{R})$  satisfait :*

$$u(t, z) = 0 \text{ dans } T \times \omega \implies u = 0 \text{ partout .}$$

*Alors il existe  $C = C(V, \omega, T) > 0$  telle que*

$$\|u_0\|_{L^2(\mathbb{D})}^2 \leq C \int_0^T \|u(t, z)\|_{L^2(\omega)}^2 dt, \quad (1.4.3)$$

*pour toute donnée initiale  $u_0 \in L^2(\mathbb{D})$ .*

La famille la plus générale de potentiels satisfaisant la propriété du Théorème 1.4.2 est, jusqu'à maintenant, celle des potentiels analytiques par rapport à  $t$ .

### 1.4.2 Principaux résultats obtenus

*Les résultats présentes dans cette partie ont été obtenus en collaboration avec Camille Laurent et Matthieu Léautaud.*

Notre résultat principal fournit une généralisation partielle du théorème de Tataru-Hörmander-Robbiano-Zuily dans le cas particulier de l'opérateur de Schrödinger en relaxant l'hypothèse d'analyticité pour les coefficients d'ordre inférieur. Un cadre naturel pour étudier des problèmes en haute régularité est donné par les espaces de Gevrey. Ces espaces ont été introduits par Gevrey dans [Gev18] afin d'étudier les propriétés de régularité pour des solutions de l'équation de la chaleur.

**Définition** Pour  $d \in \mathbb{N}$ ,  $U \subset \mathbb{R}^d$  un ouvert et  $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$  un espace de Banach on dit que  $f : U \rightarrow \mathcal{B}$  est une fonction Gevrey  $s$ , et l'on note  $f \in \mathcal{G}^s(U; \mathcal{B})$ , si  $f \in C^\infty(U; \mathcal{B})$  est telle que pour tout compact  $K \subset U$ , il existe  $C, R > 0$  tels que pour tout  $\alpha \in \mathbb{N}^d$

$$\max_{t \in K} \|\partial^\alpha f(t)\|_{\mathcal{B}} \leq CR^{|\alpha|} \alpha!^s.$$

Observons que pour  $s = 1$ , cet espace coïncide avec les fonctions réelles analytiques mais dès que  $s > 1$ ,  $\mathcal{G}^s(U; \mathbb{C})$  contient des fonctions non triviales à support compact.

On suppose que la métrique  $g$  est Lipschitz et que les coefficients à valeurs complexes  $b^j$  et  $V$  sont Gevrey 2 par rapport à  $t$  et bornés par rapport à  $x$ . Notre théorème stipule alors que la propriété d'unicité locale est satisfaite par  $P$  à travers toute hypersurface non caractéristique. Plus précisément nous avons le théorème :

**Théorème 1.4.3** (Prolongement unique local pour des opérateurs de Schrödinger). *Soit  $U = I \times V$  où  $I \subset \mathbb{R}$  est un intervalle ouvert,  $V \subset \mathbb{R}^d$  un ouvert et  $(t_0, x_0) \in \Omega$ . Supposons que  $g^{jk} \in W^{1,\infty}(V)$  est une matrice elliptique à valeurs réelles en  $x_0$  et que les coefficients  $b^j, \mathfrak{q} \in \mathcal{G}^2(I; L^\infty(V; \mathbb{C}))$ . Soit  $S = \{\Psi = 0\}$  une hypersurface non-caractéristique par rapport à  $P$  au sens de (1.4.2). Alors il existe un voisinage  $\mathcal{V}$  de  $(t_0, x_0)$  tel que pour,  $P$  définie dans (1.4.1), nous ayons*

$$\begin{cases} Pu = 0 \text{ sur } U \\ u = 0 \text{ sur } U \cap \{\Psi > 0\} \\ u \in L^2(I; H^1(V)) \end{cases} \implies u = 0 \text{ sur } \mathcal{V}.$$

La principale nouveauté de notre résultat est que l'hypothèse d'analyticité a été relaxée pour les termes d'ordre inférieur, ce qui apporte une amélioration significative par rapport au théorème général de Tataru-Hörmander-Robbiano-Zuily dans le cas d'un opérateur de Schrödinger. Nous soulignons également la faible régularité Lipschitz supposée pour la partie principale de l'opérateur (voir [LL23a] pour les ondes). On note que c'est la régularité limite pour les opérateurs elliptiques. En effet, en dimension  $n \geq 3$ , nous avons des contre-exemples montrant que la propriété d'unicité locale n'est pas vérifiée pour des opérateurs elliptiques généraux (même sous forme de divergence) avec des coefficients  $C^{0,\alpha}$  pour tout  $\alpha < 1$ , voir [Pli63] et [Mil74].

Une question qui vient naturellement est de savoir s'il est possible de réduire la condition sur la régularité de la solution. Ceci est principalement motivé par le fait que les résultats d'observabilité exacte qui portent souvent sur des solutions  $L^2$  (voir par exemple le Théorème 1.4.2). Il se trouve que les propriétés de microlocalisation du poids  $e^{-|D_t|^2}$  ainsi que l'ellipticité partielle de l'opérateur de Schrödinger permettent de résoudre ce problème. Nous obtenons en particulier le résultat suivant :

**Théorème 1.4.4.** *Dans le contexte du Théorème 1.4.3 nous supposons en plus que  $b^j = 0$ . Alors il existe un voisinage  $\mathcal{V}$  de  $(t_0, x_0)$  tel que nous ayons*

$$\begin{cases} Pu = 0 \text{ sur } \Omega \\ u = 0 \text{ sur } \Omega \cap \{\Psi > 0\} \\ u \in L^2(\Omega) \end{cases} \implies u = 0 \text{ sur } \mathcal{V}.$$

Puisque nous pouvons propager l'unicité à travers toute hypersurface non caractéristique, nous sommes capables d'appliquer successivement ce résultat local pour obtenir le résultat global suivant :

**Théorème 1.4.5** (Prolongement unique global pour des opérateurs de Schrödinger). *Soit  $(\mathcal{M}, g)$  une variété riemannienne connexe lisse avec ou sans bord  $\partial\mathcal{M}$ . Considérons l'opérateur*

$$P := P = i\partial_t + \Delta_g + b \cdot \nabla_g + \mathfrak{q}(t, x),$$

où  $\Delta_g$  est l'opérateur de Laplace-Beltrami sur  $\text{Int}(\mathcal{M})$ ,  $\nabla_g$  le gradient riemannien et  $b$  une 1-forme. Supposons que  $g \in W_{\text{loc}}^{1,\infty}(\text{Int}(\mathcal{M}))$ ,  $\mathbf{q} \in \mathcal{G}^2(I; L_{\text{loc}}^\infty(\text{Int}(\mathcal{M}); \mathbb{C}))$  et que toutes les composantes de  $b$  sont dans  $\mathcal{G}^2(I; L_{\text{loc}}^\infty(\text{Int}(\mathcal{M}); \mathbb{C}))$ . Soit  $T > 0$  et  $\omega$  un ouvert non vide de  $\mathcal{M}$ . On a alors :

$$\begin{cases} Pu = 0 \text{ sur } (0, T) \times \text{Int}(\mathcal{M}) \\ u = 0 \text{ sur } (0, T) \times \omega \\ u \in L_{\text{loc}}^2((0, T); H_{\text{loc}}^1(\text{Int}(\mathcal{M}))) \end{cases} \implies u = 0 \text{ sur } (0, T) \times \text{Int}(\mathcal{M}).$$

Si nous supposons en plus que  $b = 0$  alors on a

$$\begin{cases} Pu = 0 \text{ sur } (0, T) \times \text{Int}(\mathcal{M}) \\ u = 0 \text{ sur } (0, T) \times \omega \\ u \in L_{\text{loc}}^2((0, T) \times \text{Int}(\mathcal{M})) \end{cases} \implies u = 0 \text{ sur } (0, T) \times \text{Int}(\mathcal{M}).$$

### Applications à la théorie du contrôle

Comme nous l'avons déjà souligné un résultat de prolongement unique équivaut à un résultat de contrôlabilité approchée. Précisons alors la conséquence principale du Théorème 1.4.5 obtenue à l'aide de la dualité présentée dans le Paragraphe 1.1.2. Soit  $T > 0$ ,  $(\mathcal{M}, g)$  une variété riemannienne et  $\omega \subset \mathcal{M}$  un ouvert. On considère le système de contrôle suivant :

$$\begin{cases} i\partial_t + \Delta_g v + \mathbf{q}v = \mathbb{1}_\omega f \text{ sur } (0, T) \times \text{Int}(\mathcal{M}), \\ v = 0 \text{ sur } (0, T) \times \partial\mathcal{M} \text{ si } \partial\mathcal{M} \neq \emptyset, \\ v(0, \cdot) = v_0, \text{ sur } \text{Int}(\mathcal{M}). \end{cases} \quad (1.4.4)$$

Nous supposons que  $(\mathcal{M}, g)$  est complète de sorte que  $\Delta_g$  soit autoadjoint sur  $L^2(\mathcal{M})$  avec  $H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})$  comme domaine. Si l'on suppose que  $\mathbf{q} \in L^\infty((0, T) \times \mathcal{M}; \mathbb{C})$ ,  $f \in L^2((0, T) \times \mathcal{M})$  et  $v_0 \in L^2(\mathcal{M}; \mathbb{C})$  le problème (1.4.4) est bien posé avec une unique solution  $v \in C^0([0, T]; L^2(\mathcal{M}; \mathbb{C}))$ . La seconde partie du Théorème 1.4.5 nous donne alors le corolaire suivant :

**Corollaire 1.4.6.** *Soit  $T > 0$ ,  $(\mathcal{M}, g)$  est une variété riemannienne complète connexe avec ou sans bord et  $\mathbf{q} \in L^\infty((0, T) \times \mathcal{M}; \mathbb{C}) \cap \mathcal{G}^2((0, T); L^\infty(\mathcal{M}; \mathbb{C}))$ . Alors pour tout ouvert non vide  $\omega \subset \mathcal{M}$  le système (1.4.4) est approximativement contrôlable au sens de la Définition 2 pour  $H = L^2(\mathcal{M})$ .*

Outre le contrôle approché, le prolongement unique peut jouer un rôle crucial pour l'obtention d'un résultat de contrôle exact/observabilité exacte. En effet, pour des équations de type ondes ou Schrödinger la preuve d'un tel résultat se base souvent en une analyse qui distingue les basses et les hautes fréquences. De façon générale la partie basses fréquences est étroitement liée à des propriétés de prolongement unique. Pour le cas des ondes ceci est expliqué en détails dans [LL16].

Nous avons déjà présenté dans le Théorème 1.4.2 un résultat de contrôle exact pour l'équation de Schrödinger où la condition de prolongement unique figure clairement dans l'énoncé. Nous obtenons alors comme conséquence immédiate des Théorèmes 1.4.2 et 1.4.5 le théorème suivant.

**Théorème 1.4.7** (Application à l'observabilité pour Schrödinger). *Considérons le système d'observation suivant :*

$$\begin{cases} i\partial_t u + \Delta_g u + \mathbf{q}u = 0 & \text{sur } (0, T) \times \text{Int}(\mathcal{M}), \\ u = 0, & \text{sur } (0, T) \times \partial\mathcal{M} \quad \text{si } \partial\mathcal{M} \neq \emptyset, \\ u(0, \cdot) = u_0, & \text{sur } \text{Int}(\mathcal{M}). \end{cases} \quad (1.4.5)$$

Supposons que  $(\mathcal{M}, g) = (\mathbb{D}, \text{Eucl})$  est le disque euclidien,  $\mathbf{q} \in C^\infty([0, T] \times \mathbb{D}; \mathbb{R}) \cap \mathcal{G}^2((0, T); L^\infty(\mathbb{D}; \mathbb{R}))$  est un potentiel à valeurs réelles et  $\omega$  est un ouvert non vide de  $\mathbb{D}$  avec  $\omega \cap \partial\mathbb{D} \neq \emptyset$ . Soit  $T > 0$ , alors le système (1.4.5) est observable depuis  $\omega$  pour toute donnée initiale  $u_0 \in L^2(\mathbb{D})$  au sens de (1.4.3).

Notre contribution dans le théorème ci-dessus est le fait d'avoir relaxé la régularité exigée pour le potentiel  $\mathbf{q}$ .

Nous avons choisi d'énoncer comme théorème uniquement le cas du disque parce que le résultat de [ALM16] fait directement appel à la propriété du prolongement unique, qui vient de l'analyse basses fréquences. Cependant nous pouvons appliquer notre résultat de prolongement unique global du Théorème 1.4.5 dans d'autres situations où des résultats d'observabilité exacte/ contrôle exact ont été obtenus, en rentrant un peu plus dans leurs preuves. Mentionnons les deux situations suivantes :

1.  $(\mathcal{M}, g)$  est une variété riemannienne compacte lisse sans bord,  $\mathbf{q} \in L^\infty((0, T) \times \mathcal{M}; \mathbb{C}) \cap \mathcal{G}^2((0, T); L^\infty(\mathcal{M}; \mathbb{C}))$  et  $\omega$  satisfait la condition du contrôle géométrique de [RT74, BLR92] en temps fini ;
2.  $(\mathcal{M}, g) = (\mathbb{T}^d; \text{Eucl})$  est le tore plat, le potentiel  $\mathbf{q} \in C^0([0, T] \times \mathbb{T}^d; \mathbb{R}) \cap \mathcal{G}^2((0, T); L^\infty(\mathbb{T}^d; \mathbb{R}))$  est à valeur réelles et  $\omega \subset \mathbb{T}^d$  est un ouvert non vide ;

Dans le cas 1 l'analyse donnant l'estimée d'observabilité dans le régime "hautes fréquences" vient de [Leb92b] et dans le cas 2 elle est faite dans [AM14].

### 1.4.3 Outils et idée de preuve

La preuve du Théorème 1.4.3 repose sur deux points clefs. Le premier est une inégalité de Carleman adaptée à l'anisotropie de l'opérateur de Schrödinger. Le deuxième est un résultat de conjugaison.

#### L'estimée de Carleman

Comme nous l'avons vu, depuis le travail fondateur de [Car39] les inégalités de Carleman sont l'outil principal pour la preuve des résultats de prolongement unique. L'idée clef introduite par Tataru dans [Tat95] est que l'on peut encore montrer des résultats d'unicité en utilisant le poids microlocal  $e^{-\frac{\varepsilon|D_t|^2}{2\tau}} e^{\tau\phi}$  au lieu du poids classique. L'avantage de ce poids est qu'il localise dans un régime basse fréquence et que l'on peut par conséquent restreindre l'hypothèse de pseudoconvexité classique à un sous-ensemble plus petit de l'espace des phases. Si l'on suppose que  $D_t, D_x$  et  $\tau$  ont le même poids alors  $\frac{|D_t|^2}{2\tau}$  est d'ordre 1. Ce poids est bien adapté à l'opérateur des ondes. Dans le scaling de Schrödinger, si on voit  $D_t$  comme un opérateur "d'ordre 2", l'opérateur  $\frac{|D_t|^2}{2\tau^3}$  est "d'ordre 1". Le poids  $e^{-\frac{|D_t|^2}{2\tau^3}}$  semble

être alors un candidat naturel. Notre estimée de Carleman est précisément une estimée comme celle obtenue dans [Tat95, Hör97, RZ98, Tat99b] mais avec ce nouveau poids. Nous avons le théorème suivant (pour un énoncé plus précis, voir le Théorème 4.2.1) :

**Théorème 1.4.8.** *Soit  $\mathbf{x}_0 = (t_0, x_0) \in \Omega \subset \mathbb{R}^{1+n}$  et  $\mu > 0$ . Alors sous certaines hypothèses sur  $\phi$  nous avons que pour tout  $k \in \mathbb{N}$  il existe  $d, C, \tau_0 > 0$  tels que pour tout  $\tau \geq \tau_0$  et  $w \in C_c^\infty(B(\mathbf{x}_0, r))$ , on a*

$$C \left\| e^{-\frac{\mu|D_t|^2}{2\tau^3}} e^{\tau\phi} (i\partial_t + \Delta_g) w \right\|_{L^2}^2 + C e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{H_t^{-k} H_x^1}^2 \geq \tau \left\| e^{-\frac{\mu|D_t|^2}{2\tau^3}} e^{\tau\phi} w \right\|_{H_t^1}^2,$$

avec la définition

$$\|v\|_{H_t^1}^2 := \tau^2 \|v\|_{L^2}^2 + \|D_x v\|_{L^2}^2 + \tau^{-2} \|D_t v\|_{L^2}^2.$$

Avant de présenter les aspects plus techniques de ce théorème la première question qu'on peut se poser est : qu'avons-nous gagné avec ce nouveau poids ? D'un point de vue heuristique, une réponse partielle peut être donnée par les résultats de [LL19]. En effet, dans [LL19] le théorème général de Tataru-Hörmander-Robbiano-Zuily est quantifié de manière optimale. L'étape clef est le résultat local du Théorème 3.1 dans [LL19] qui dit que l'estimation de Carleman avec un poids  $e^{-\frac{\epsilon|D_t|^2}{2\tau}} e^{\tau\phi}$  permet de propager l'information jusqu'à une fréquence  $D_t \sim \tau$  donnant un reste d'ordre  $e^{-\frac{\epsilon\tau}{2}}$ . Dans le scaling anisotrope de l'opérateur de Schrödinger, nous avons que  $D_t \sim \tau^2$  et on peut donc espérer que le poids  $e^{-\frac{\mu|D_t|^2}{2\tau^3}} e^{\tau\phi}$  permet de propager l'information jusqu'à des fréquences inférieures à  $D_t \sim \tau^2$  donnant un reste d'ordre  $e^{-\frac{\mu\tau}{2}}$ . Le terme d'erreur anisotrope  $\tau^{-2} \|D_t v\|_{L^2}^2$  qu'on s'autorise dans l'estimée de Carleman est étroitement lié au choix du nouveau poids  $e^{-\frac{\mu|D_t|^2}{2\tau^3}}$ .

La preuve du Théorème 1.4.8 est basée sur l'inégalité de Carleman elliptique démontrée dans [LL21b, Appendix A] en régularité Lipschitz (voir [LL23a] pour le cas des ondes) et se fait en deux étapes. La première consiste à faire la preuve pour  $\mu = 0$  ce qui permet de se libérer du terme non local  $e^{-\frac{\epsilon|D_t|^2}{2\tau}}$ . L'idée est alors de traiter le terme  $i\partial_t$  comme une perturbation de l'opérateur elliptique  $\Delta_g$ , ce qui permet de suivre la stratégie de preuve employée dans [LL21b, Appendix A]. En particulier, les conditions sur le poids  $\phi$  sont les mêmes que dans [LL21b, Theorem A.5]. Ensuite, on fait la preuve pour  $\mu > 0$  en traitant les termes additionnels comme une perturbation du cas  $\mu = 0$ . Il est intéressant de remarquer que dans le cadre des ondes nous avons le poids  $e^{-\frac{\epsilon|D_t|^2}{2\tau}}$  et il faut prendre  $\epsilon$  petit. Ici,  $\mu > 0$  est un paramètre libre que l'on peut choisir comme on veut. Ceci pourrait être une manifestation de la vitesse infinie de propagation satisfaite par les solutions de l'équation de Schrödinger.

Remarquons enfin le terme d'erreur  $e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{H_t^{-k} H_x^1}^2$  dans le membre de gauche. Le fait de pouvoir prendre des normes très faibles en  $t$  est une conséquence de la localisation du poids  $e^{-\frac{\mu|D_t|^2}{2\tau^3}}$ . Il nous sera utile pour réduire la régularité de la solution dans le théorème de l'unicité.

### Le résultat de conjugaison

Dans les travaux de [Tat95, Hör97, RZ98, Tat99b] une des difficultés techniques majeures est celle de l'opérateur conjugué. Dans une inégalité de Carleman classique nous avons un



opérateur  $P$  et un poids  $e^{\tau\phi}$ . L'inégalité

$$\|e^{\tau\phi}Pu\| \geq C \|e^{\tau\phi}u\|, \quad u \in C_0^\infty(B((t_0, x_0), r))$$

est équivalente à

$$\|e^{\tau\phi}Pe^{-\tau\phi}v\| \geq C \|v\|, \quad v \in C_0^\infty(B((t_0, x_0), r))$$

Ceci est une étape cruciale car elle permet en quelque sorte de se débarrasser du poids  $e^{\tau\phi}$  quitte à étudier le nouvel opérateur  $P_\phi := e^{\tau\phi}Pe^{-\tau\phi}$  : c'est l'opérateur conjugué. Lorsqu'on travaille avec le poids  $e^{-\frac{|D_t|^2}{2\tau}}e^{\tau\phi}$  nous affrontons une difficulté : l'inverse de  $e^{-\frac{|D_t|^2}{2\tau}}$  n'est même pas défini sur l'espace de Schwartz. Nous partons alors du calcul suivant :

**Lemme 1.4.9** (Lemma 3.3.1 dans [LL20]). *Pour  $u \in \mathcal{S}(\mathbb{R}^{n+1})$ , on a*

$$e^{-\varepsilon\frac{|D_t|^2}{2\varsigma}}(tu) = \left(t + i\varepsilon\frac{D_t}{\varsigma}\right) e^{-\varepsilon\frac{|D_t|^2}{2\varsigma}}u.$$

On peut itérer ce calcul pour trouver

$$e^{-\varepsilon\frac{|D_t|^2}{2\varsigma}}(t^k u) = \left(t + i\varepsilon\frac{D_t}{\varsigma}\right)^k e^{-\varepsilon\frac{|D_t|^2}{2\varsigma}}u.$$

Cela implique qu'au moins lorsque nous avons une fonction  $f = f(t, x)$  qui est un polynôme en  $t$  on peut définir un opérateur conjugué. Toutefois, même si  $f$  est analytique en  $t$  il n'est pas facile de définir un opérateur conjugué. En effet, le candidat naturel serait une somme infinie de termes  $\left(t + i\varepsilon\frac{D_t}{\varsigma}\right)^k$  mais cette somme n'est pas définie telle quelle. L'approche de [Hör97] consiste à ajouter des troncatures en  $t$  et  $\xi/\tau$  de sorte que cette série d'opérateurs devienne convergente. Cela définit un opérateur conjugué approché : c'est-à-dire un opérateur modulo une erreur exponentiellement petite, qui est admissible au vu de notre estimée de Carleman du Théorème 1.4.8. Dans [Tat99b] Tataru définit l'opérateur conjugué comme suit : Si  $f$  est analytique en  $t_0$ , alors elle admet une extension holomorphe dans un petit voisinage complexe de  $t_0$ . On note cette extension par  $\tilde{f}$ . Il définit alors l'opérateur

$$F_\tau := \text{op}^w \left( \chi(t)\eta \left( \frac{\varepsilon\xi}{\tau} \right) \tilde{f} \left( t + i\frac{\varepsilon\xi}{\tau} \right) \right),$$

qui est un opérateur conjugué approché au sens suivant :

$$e^{-\varepsilon\frac{|D_t|^2}{2\tau}}f = F_\tau e^{-\varepsilon\frac{|D_t|^2}{2\tau}} + O(e^{-\tau/\varepsilon}).$$

Ici,  $\text{op}^w$  désigne la quantification de Weyl (voir (4.3.5) pour une définition) et  $\chi, \eta$  sont des troncatures appropriées. Bien sûr, l'approche de Hörmander et celle de Tataru sont étroitement liées.

Nous suivons alors l'approche de Tataru en remplaçant  $\tilde{f}$  par une *extension quasi-analytique* de  $f$ . Une extension quasi-analytique  $\tilde{f}$  d'une fonction lisse  $f$  est une extension complexe de  $f$  telle que  $\partial_{\bar{z}}\tilde{f}$  soit nulle à tout ordre pour  $z \in \mathbb{R}$ . Le fait crucial est qu'une telle extension se comporte particulièrement bien dans le cas d'une fonction Gevrey. Pour un intervalle compact  $K \subset \mathbb{R}$  et  $\rho > 0$ , on écrit

$$K_\rho := \{x \in \mathbb{R}, \text{dist}(x, K) \leq \rho\} \subset \mathbb{R}, \quad \text{et} \quad K'_\rho := K_\rho + i[-\rho, \rho] \subset \mathbb{C}.$$

Nous avons le lemme suivant (voir [BJ20, Lemma 1.2, Remark 1.7]) :



**Lemme 1.4.10.** *Soit  $s > 1$ . Pour tout intervalle compact  $K \subset \mathbb{R}$  et  $\rho > 0$ , il existe  $C_0, A > 0$  tels que pour tout  $R > 0$ , il existe  $C > 0$  et une application linéaire continue*

$$\begin{aligned} \mathcal{G}^{s,R}(K; \mathbb{C}) &\rightarrow \mathcal{G}^{s,AR}(K'_\rho; \mathbb{C}) \\ f &\mapsto \tilde{f}, \end{aligned}$$

tels que pour tout  $f \in \mathcal{G}^{s,R}(K; \mathbb{C})$ ,

$$\begin{aligned} \tilde{f}(t) &= f(t) \text{ pour } t \in K, \quad \|\tilde{f}\|_{s,AR,K'_\rho} \leq C \|f\|_{s,R,K}, \\ |\partial_{\bar{z}} \tilde{f}(z)| &\leq C \exp\left(-\frac{1}{C_0(R \operatorname{Im}(z))^{\frac{1}{s-1}}}\right), \quad \text{pour } z \in K'_\rho. \end{aligned}$$

L'inégalité ci-dessus traduit le fait que  $\tilde{f}$  est une extension quasi-analytique de  $f$ , bien adaptée à la régularité Gevrey  $\mathcal{G}^s$ .

On considère alors un intervalle compact  $K$ ,  $f \in \mathcal{G}^{2,R}(K; \mathbb{C})$  et  $\rho > 0$ . On note par  $\tilde{f}$  l'extension quasi-analytique de  $f$  dans  $K'_\rho$  donnée par le Lemme 1.4.10. Nous définissons

$$F_h := \operatorname{op}^w \left( \chi(t) \eta(h^{2/3} \xi) \tilde{f}(t + ih\xi) \right),$$

avec  $\chi$  et  $\theta$  des cut-off appropriés. Une version simplifiée de notre résultat de conjugaison est alors :

**Théorème 1.4.11.** *Considérer l'opérateur  $F_h$  comme défini ci-dessus et  $\theta \in C_c^\infty(J)$  avec  $J \Subset K$ . Alors il existe  $C$  and  $c$  tels que pour tout  $u \in \mathcal{S}(\mathbb{R})$  nous avons :*

$$\left\| F_h e^{-\frac{h}{2}|D_t|^2} \theta u - e^{-\frac{h}{2}|D_t|^2} f \theta u \right\|_{L^2} \leq C e^{-ch^{-1/3}} \|u\|_{L^2},$$

pour tout  $0 < h \leq h_0$ .

Dans la version la plus générale (voir la Proposition 4.3.2) de ce résultat nous considérons une fonction  $f$  à valeurs dans les applications linéaires entre deux espace de Hilbert. Cela permet de traiter le cas où  $f$  est à valeurs dans  $L^\infty$ .

Nous avons ici choisi une formulation semiclassique. Le lien entre le petit paramètre  $h$  et le grand paramètre  $\tau$  dans l'inégalité de Carleman est donné par  $h = 1/\tau^3$ . Nous voyons alors que le cut-off  $\eta$  localise en des fréquences plus petites que  $h^{-2/3} \sim \tau^2$ , ce qui est cohérent avec l'intuition que nous avons grâce aux résultats quantitatifs de [LL19].

Nous pouvons maintenant voir l'intérêt d'avoir introduit le poids  $e^{-\frac{|D_t|^2}{2\tau^3}}$  : le résultat de conjugaison fournit une erreur en  $e^{-d(\tau^3)^{1/3}} \|u\|_{L^2} = e^{-d\tau} \|u\|_{L^2}$  qui est admissible.

La preuve du Théorème 1.4.11 utilise de façon cruciale l'estimée satisfaite par  $\partial_{\bar{z}} \tilde{f}$  donnée par le Lemme 1.4.10. L'idée essentielle est que dans le cas où  $\tilde{f}$  est holomorphe les termes correspondants à  $\partial_{\bar{z}} \tilde{f}$  sont nuls. Dans notre situation nous avons des termes additionnels que nous arrivons à contrôler grâce à la petitesse exponentielle de  $\partial_{\bar{z}} \tilde{f}$ .

#### Fin de la preuve des Théorèmes 1.4.3 et 1.4.4

Avec l'inégalité de Carleman du Théorème 1.4.8 et le résultat de conjugaison du Théorème 1.4.11 nous sommes en mesure de conclure la preuve du Théorème 1.4.4. En combinant ces deux résultats clefs on perturbe notre inégalité de Carleman avec des termes

d'ordre inférieur qui sont Gevrey 2 par rapport à  $t$ . Afin d'exploiter cette nouvelle estimée de Carleman on doit vérifier qu'on peut trouver un poids  $\phi$ , qui à la fois, satisfait les conditions du théorème et dont les lignes de niveau sont étroitement liées à celles de l'hypersurface non caractéristique  $\{\Psi = 0\}$ . Cela conclut la preuve du Théorème 1.4.3.

Pour réduire la régularité de la solution et prouver ainsi le Théorème 1.4.4 on se base sur une remarque de Hörmander [Hör97, Remark p.205] et les deux observations suivantes :

- Le poids  $e^{-\frac{|D_t|^2}{2\tau^3}}$  permet de réduire la régularité nécessaire en temps ;
- La régularité elliptique de  $\Delta_g$  dans  $(i\partial_t + \Delta_g + V)u = 0$  permet alors de réduire la régularité nécessaire en espace.

#### 1.4.4 Perspectives

La première question qui vient tout naturellement est de savoir s'il est possible d'avoir aussi une métrique (coefficients principaux) dépendant du temps  $t$ , avec la même hypothèse de régularité que pour les autres termes : partiellement Gevrey 2. Le résultat clef de conjugaison nous dit que nous pouvons définir un opérateur conjugué raisonnable. Quand nous avons affaire à des termes inférieurs nous ne sommes pas obligés de démontrer une estimée de Carleman avec ce nouvel opérateur conjugué. En effet, nous exploitons le fait que notre estimée est stable par perturbations d'ordre inférieur. Par conséquent, si l'on veut inclure des termes principaux qui dépendent du temps il faudra redémontrer l'estimée de Carleman. A priori, il n'y a pas de raison pour qu'une telle estimée ne soit pas vraie. Le problème est purement de nature technique : notre preuve est basée sur des calculs et intégration par parties explicites. Inclure l'opérateur conjugué tel qu'on l'a défini semble être trop compliqué à ce stade. Il faudrait sans doute une preuve plus compacte qui puisse inclure l'opérateur conjugué comme une perturbation admissible.

Aussi, nous avons déjà vu que l'estimée de Carleman de [Tat95, Hör97, RZ98, Tat99b] a été quantifiée dans [LL19] en donnant le résultat d'observabilité approchée du Théorème 1.2.2. Dans le cas de l'opérateur de Schrödinger le résultat de [LL19] donne une inégalité similaire mais le terme  $\Lambda = \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\|(u_0, u_1)\|_{L^2 \times H^{-1}}}$  que nous avons pour les ondes devient  $\frac{\|u_0\|_{H^2}}{\|u_0\|_{L^2}}$ . La quantité  $\Lambda$  peut être interprétée comme la fréquence typique des données initiales. On s'attend alors à ce que la fréquence typique de la donnée apparaisse aussi pour Schrödinger. Il est normal qu'on ne puisse pas atteindre le résultat optimal dans [LL19] puisqu'une partie de l'analyse n'est pas adaptée à l'anisotropie de l'opérateur de Schrödinger. On espère alors utiliser notre nouvelle estimée de Carleman avec les techniques de [LL19] pour pouvoir démontrer l'estimée quantitative optimale pour Schrödinger. Une fois que nous avons obtenu l'estimée d'observabilité approchée optimale on aimerait obtenir des informations sur le comportement de la constante associée lorsque  $T \rightarrow 0^+$ . Ceci pourrait être combiné avec des résultats de contrôle/observabilité exact/exacte et donner des informations quantitatives explicites sur le coût du contrôle/constante d'observabilité.

La vitesse infinie de propagation satisfaite par les solutions de l'équation de Schrödinger motive une autre notion de prolongement unique : si nous avons une information sur une partie d'une tranche temporelle  $V \subset \{t = t_0\}$  peut-on la propager à une autre partie  $V' \subset \{t = t_0\}$ ? Bien entendu, ceci n'est pas possible puisque nous pouvons prescrire la donnée initiale en  $t = 0$  comme nous le souhaitons. Il se trouve néanmoins qu'il est possible de montrer un résultat dans cet esprit si l'on suppose que la solution décroît

exponentiellement en temps dans un voisinage uniforme de l'espace. Plus précisément, nous supposons que  $S \subset \{t = t_0\}$  est une hypersurface,  $V$  un voisinage de  $x_0$ ,  $u$  une solution de  $Pu = 0$  et il existe  $c_1 > 0$  avec

$$\lim_{t \rightarrow t_0} u(t, x) e^{\frac{c_1}{|t-t_0|}} = 0,$$

uniformément en  $x \in S^+ \cap V$ . On veut alors savoir s'il existe  $c_2 > 0$  tel que

$$\lim_{t \rightarrow t_0} u(t, x) e^{\frac{c_2}{|t-t_0|}} = 0,$$

pour  $x$  dans un *voisinage* de  $x_0$ . De tels résultats de prolongement unique dit *anisotrope* ont été démontrés dans [Tat97a] pour des surfaces satisfaisant une condition de pseudo-convexité. Il pourrait alors être intéressant d'utiliser notre résultat de conjugaison dans le but d'inclure des poids Gevrey dans une estimée de Carleman anisotrope contenant aussi le poids microlocal  $e^{-\frac{|D_t|^2}{2\tau^3}}$ . Ceci pourrait généraliser les résultats de [Tat97a] en incluant une famille plus large de surfaces.

Première partie

**On unique continuation for waves  
in singular media**



## Chapitre 2

# Quantitative unique continuation for waves in singular media

*Ce chapitre est la reproduction de la prépublication [Fil22] soumise pour publication.*

In this chapter we prove quantitative unique continuation results for wave operators of the form  $\partial_t^2 - \operatorname{div}(c(x)\nabla\cdot)$  where the scalar coefficient  $c$  is discontinuous across an interface of codimension one in a bounded domain or on a compact Riemannian manifold. We do not make any assumptions on the geometry of the interface or on the sign of the jumps of the coefficient  $c$ . The key ingredient is a local Carleman estimate for a wave operator with discontinuous coefficients. We then combine this estimate with the recent techniques of Laurent-Léautaud [LL19] to propagate local unique continuation estimates and obtain a global stability inequality. As a consequence, we deduce the cost of the approximate controllability for waves propagating in this geometry.

### 2.1 Introduction

For a wave operator  $P$  the question of unique continuation consists in asking whether a partial observation of a wave on a small set  $\omega \subset \Omega$  is sufficient to determine *the whole* wave. If this property holds, then the next natural question is if we can *quantify* it. This is expressed via a stability estimate of the form

$$\|u\|_{\Omega} \lesssim \phi(\|u\|_{\omega}, \|Pu\|_{\Omega}, \|u\|_{\Omega}), \quad (2.1.1)$$

with  $\phi$  satisfying

$$\phi(a, b, c) \xrightarrow{a, b \rightarrow 0} 0, \quad \text{with } c \text{ bounded.}$$

Such estimates have numerous applications in control theory, spectral geometry and inverse problems. Concerning the wave operator a seminal unique continuation result was obtained by Robbiano in [Rob91] and refined by Hörmander in [Hör92]. The optimal version of this *qualitative* result was finally attained in the so called Tataru, Hörmander, Robbiano-Zuily Theorem [Tat95, Tat99b, Hör97, RZ98]. This theorem deals in fact with the more general case of operators with partially analytic coefficients and, in the particular case of a wave operator with coefficients independent of time, gives uniqueness across any non characteristic hypersurface. Recently, in [LL19] the authors proved a quantitative

version of the latter theorem which, for the wave equation, is optimal with respect to the observation time and the stability estimate obtained. Note that, a *qualitative* uniqueness result is equivalent to an approximate controllability result, and a *quantified* version of it gives an estimate of the control cost. The quantitative unique continuation result of [LL19] applies to (variants of) the operator  $\partial_t^2 - \Delta_g$  where  $\Delta_g$  is an elliptic operator with  $C^\infty$  coefficients. See also [BKL16] for a related set of estimates concerning the wave operator.

However, in many contexts, waves propagate through singular media and therefore in the presence of non smooth coefficients. E.g. in the case of seismic waves [Sym83] or acoustic waves [YDdH<sup>+</sup>17, AdHG17, CdHKU19] propagating through the Earth's crust. Models proposing to describe such phenomena use discontinuous metrics and more precisely metrics which are piece-wise regular but presenting jumps along some hypersurfaces. See for instance the Mohorovičić discontinuity between the Earth's crust and the mantle. Another example arises in medical imaging. The human brain [FdLKBH01, MCCP17] has two main components: white and grey matter. These two have very different electric conductivities and models describing the situation are very similar to the preceding example.

The question of quantitative unique continuation across a jump discontinuity seems to be well understood in the elliptic/parabolic context. One of the first results (in the parabolic case) is [DOP02] where the operator  $\partial_t - \operatorname{div}(c\nabla \cdot)$  is studied with a monotonicity assumption imposed on the scalar coefficient  $c = c(x)$ : the observation should take place in the region where the coefficient  $c$  is smaller. In this article a global Carleman estimate was proved. Later, in the elliptic case in [LR10] a similar result was obtained but without any restriction on the sign of the jump of the coefficient. These techniques were extended to the parabolic context in [LR11]. The most recent (and general) result to the best of our knowledge is proved by Le Rousseau and Lerner [LRL13] where the anisotropic case ( $-\operatorname{div}(A(x)\nabla \cdot)$ , with  $A$  a matrix jumping across an interface) is treated. Other interesting works in the elliptic context include [DCFL<sup>+</sup>17, FLVW16]. There, the problem of reducing the regularity of the leading coefficients away from the interface has been addressed.

The question of *exact* control for waves with jumps at an interface has already been addressed in the book [Lio88]. A controllability result is proved for the operator  $\partial_t^2 - \operatorname{div}(c\nabla \cdot)$  with  $c$  a piece-wise constant coefficient under a geometric assumption on the jump hypersurface and a sign condition on the jump. One of the first Carleman estimates was proved in the discontinuous setting in [BMO07]. With the same assumption on the coefficient and assuming that the interface is convex the authors prove linear quantitative stability estimates. Recently, in [BGM21] quantitative results were proved as well for interfaces that interpolate between star-shaped and convex. Other related works are [Gag18] and [BCDCG20].

However, to our knowledge the question of stability estimates without any particular geometric assumption on the interface has not been studied yet. This is the main object of this article.

### 2.1.1 Setting and statement of main results

Let  $(\mathcal{M}, g)$  be a smooth connected compact  $n$ -dimensional Riemannian manifold with or without boundary. We consider  $S$  an  $(n - 1)$ -dimensional submanifold of  $\mathcal{M}$  without boundary. We assume that  $\mathcal{M} \setminus S = \Omega_- \cup \Omega_+$  with  $\Omega_- \cap \Omega_+ = \emptyset$ .

We consider a scalar coefficient  $c(x) = \mathbb{1}_{\Omega_-} c_-(x) + \mathbb{1}_{\Omega_+} c_+(x)$  with  $c_\pm \in C^\infty(\overline{\Omega}_\pm)$  satisfying  $0 < c_{\min} < c(x) < c_{\max}$  uniformly on  $\Omega_- \cup \Omega_+$  to ensure ellipticity. We shall

work with the wave operator  $P$  defined as

$$P = \partial_t^2 - \operatorname{div}_g(c(x)\nabla_g), \quad \text{on } \mathbb{R}_t \times \Omega_- \cup \Omega_+. \quad (2.1.2)$$

We consider for  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  the following evolution problem:

$$\begin{cases} Pu = 0 & \text{in } (0, T) \times \Omega_- \cup \Omega_+ \\ u|_{S_-} = u|_{S_+} & \text{in } (0, T) \times S \\ (c\partial_\nu u)|_{S_-} = (c\partial_\nu u)|_{S_+} & \text{in } (0, T) \times S \\ u = 0 & \text{in } (0, T) \times \partial\mathcal{M} \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{in } \mathcal{M}, \end{cases} \quad (2.1.3)$$

where we denote by  $\partial_\nu$  a nonvanishing vector field defined in a neighborhood of  $S$ , normal to  $S$  (for the metric  $g$ ), pointing into  $\Omega_+$  and normalized for  $g$ . We denote as well by  $u|_{S_\pm}$  the traces of  $u|_{\Omega_\pm}$  on the hypersurface  $S$ .

Notice that there are two extra equations in our system. These are some natural transmission conditions that we impose in the interface. These conditions imply that the underlying elliptic operator is self-adjoint on its domain and one can show using classical methods (for instance with the Hille-Yosida Theorem) that the system (2.1.3) is well posed. For more details on this we refer to Section 2.2.

Our first result provides a quantitative unique continuation result from an observation region  $\omega$  for the discontinuous wave operator  $P$ .

In Section 2.1.3 we introduce  $\mathcal{L}(\mathcal{M}, \omega) = \sup_{x \in \mathcal{M}} \operatorname{dist}(x, \omega)$ , the ‘‘largest distance’’ of the subset  $\omega$  to a point of  $\mathcal{M}$ , where  $\operatorname{dist}$  is a distance function adapted to  $(\mathcal{M}, g, c)$ .

**Theorem 2.1.1.** *Consider  $(\mathcal{M}, g), S, \Omega_\pm$  and  $P$  as defined in (2.1.2). Then for any nonempty open subset  $\omega$  of  $\mathcal{M}$  and any  $T > 2\mathcal{L}(\mathcal{M}, \omega)$ , there exist  $C, \kappa, \mu_0$  such that for any  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  and  $u$  solving (2.1.3) one has, for any  $\mu \geq \mu_0$ ,*

$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq Ce^{\kappa\mu} \|u\|_{L^2((0, T) \times \omega)} + \frac{C}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}.$$

*If moreover  $\partial\mathcal{M} \neq \emptyset$  and  $\Gamma$  is a non empty open subset of  $\partial\mathcal{M}$ , for any  $T > 2\mathcal{L}(\mathcal{M}, \Gamma)$ , there exist  $C, \kappa, \mu_0 > 0$  such that for any  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  and  $u$  solving (2.1.3), we have*

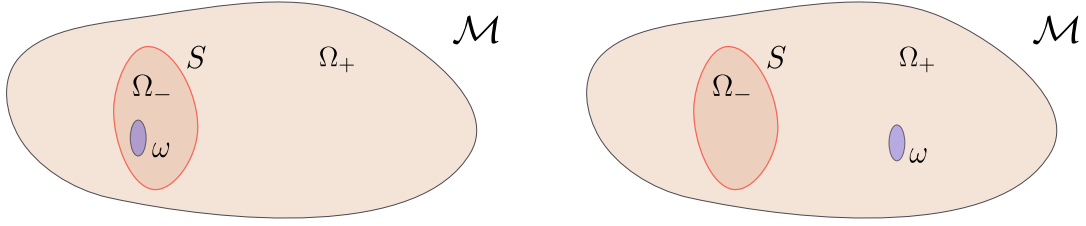
$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq Ce^{\kappa\mu} \|\partial_{\nu_\Gamma} u\|_{L^2((0, T) \times \Gamma)} + \frac{C}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}.$$

**Remark 2.1.2.** In fact one can take  $\mu > 0$  in the statement of the above theorem. However we preferred to state it in this way in order to stress out the fact that this estimate is interesting only when  $\mu$  is large.

With the above one can recover the following qualitative result: ‘‘If we do not see anything from  $\omega$  during a time  $T$  strictly larger than  $2\mathcal{L}(\mathcal{M}, \omega)$ , then there is no wave at all.’’ Indeed, if  $\|u\|_{L^2((0, T) \times \omega)} = 0$ , then letting  $\mu \rightarrow +\infty$  in the above inequality implies that  $(u_0, u_1) = 0$ .

An important aspect of this theorem is that there is no assumption on the sign of the jump of the coefficient  $c$  and consequently the observation region  $\omega$  can be chosen indifferently on  $\Omega_-$  or  $\Omega_+$ . Let us explain why this is quite surprising. Suppose, to fix





(a) The observation takes place inside  $\Omega_-$ . (b) The observation takes place inside  $\Omega_+$ . If  $c_- < c_+$  then a part of the wave may be trapped inside  $\Omega_-$ . Nevertheless, the quantitative unique continuation and its consequences still hold.

ideas, that  $c_- < c_+$  are two constants. We can then interpret  $c_-$  and  $c_+$  as the speed of propagation of a wave travelling through two isotropic media  $\Omega_-$  and  $\Omega_+$  with different refractive indices,  $n_-$  and  $n_+$  respectively (recall that  $n_{\pm} = 1/c_{\pm}$ ). Imagine that a wave starts travelling from a region that is inside  $\Omega_-$ . One has  $\frac{c_-}{c_+} = \frac{n_+}{n_-}$  and therefore the assumption  $c_- < c_+$  translates to  $n_- > n_+$ . Then Snell-Descartes law states that when a wave travels from a medium with a higher refractive index to one with a lower refractive index there is a critical angle from which there is *total internal reflection*, that is no refraction at all. At the level of geometric optics, that is to say, in the high frequency regime such a wave stays trapped inside  $\Omega_-$ . Therefore one expects that, at least at high frequency, no information propagates from  $\Omega_-$  to  $\Omega_+$ , following the laws of geometric optics. Our result (see Theorem 2.1.3) states that the intensity of waves in  $\Omega_+$  is at least exponentially small in terms of the typical frequency  $\Lambda$  of the wave.

We can reformulate Theorem 2.1.1 in a way closer to quantitative estimates such as (2.1.1). Indeed, optimizing the inequalities of Theorem 2.1.1 with respect to  $\mu$  yields the following result (which we state only in the interior observation case):

**Theorem 2.1.3.** *Under the assumptions of Theorem 2.1.1 there exists  $C > 0$  such that for all  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  with  $(u_0, u_1) \neq (0, 0)$  one has:*

$$\begin{aligned} \|(u_0, u_1)\|_{H^1 \times L^2} &\leq C e^{C\Lambda} \|u\|_{L^2((0,T) \times \omega)}, \\ \|(u_0, u_1)\|_{L^2 \times H^{-1}} &\leq C \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\log\left(1 + \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\|u\|_{L^2((0,T) \times \omega)}}\right)}. \end{aligned} \quad (2.1.4)$$

where  $\Lambda = \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\|(u_0, u_1)\|_{L^2 \times H^{-1}}}$ .

Note that  $\Lambda$  represents the typical frequency of the initial data. Theorem 2.1.3 is a direct consequence of Theorem 2.1.1 and Lemma A.3 in [LL19]. Notice that the function

$$x \mapsto \frac{1}{\log(1 + 1/x)},$$

appearing in the right hand side of (2.1.4) has been tacitly extended by continuity by 0 when  $x = 0$ .

In [Rob95, proof of Theorem 2, Section 3] it is shown that such a quantitative information can lead to an estimate for the cost of the approximate controllability. We state the

case of internal control, a similar result holds for approximate boundary controllability as well.

**Theorem 2.1.4** (Cost of approximate interior control). *Consider  $\mathcal{M}$ ,  $S$  and  $\omega \subset \mathcal{M}$  as before. Then for any  $T > 2\mathcal{L}(\mathcal{M}, \omega)$  there exist  $C, c > 0$  such that for any  $\epsilon > 0$  and any  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ , there exists  $f \in L^2((0, T) \times \omega)$  with*

$$\|f\|_{L^2((0, T) \times \omega)} \leq C e^{c/\epsilon} \|(u_0, u_1)\|_{H_0^1(\mathcal{M}) \times L^2(\mathcal{M})}$$

such that the solution of

$$\begin{cases} Pu = \mathbf{1}_\omega f & \text{in } (0, T) \times \Omega_- \cup \Omega_+ \\ u|_{S_-} = u|_{S_+} & \text{in } (0, T) \times S \\ (c\partial_\nu u)|_{S_-} = (c\partial_\nu u)|_{S_+} & \text{in } (0, T) \times S \\ u = 0 & \text{in } (0, T) \times \partial\mathcal{M} \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{in } \mathcal{M}, \end{cases}$$

satisfies

$$\|(u, \partial_t u)|_{t=T}\|_{L^2 \times H^{-1}} \leq \epsilon \|(u_0, u_1)\|_{H_0^1 \times L^2}.$$

In other words, if we act on the region  $\omega$  during a time  $T > 2\mathcal{L}(\mathcal{M}, \omega)$  we can drive our solution from energy 1 (in  $H^1 \times L^2$ ) to  $\epsilon$  close to 0 (in  $L^2 \times H^{-1}$ ). Additionally, this comes with an estimate of the energy of the control which is of the order of  $e^{c/\epsilon}$ . In the analytic context and without the presence of an interface it was shown in [Leb92a] that this form of exponential cost is optimal in the absence of Geometric Control Condition [BLR92].

In the more general hypoelliptic context of [LL21a] the result of Theorem 2.1.4 is stated as *approximate observability* for the wave equation. It is shown by the authors of this article that such a property implies some resolvent estimates (Proposition 1.11 in [LL21a]) which in turn give a logarithmic energy decay estimate for the damped wave equation (see Theorem 1.5 in [LL21a]). Consequently, Theorem 2.1.4 combined with the results of [LL21a] provides a different proof for theorems that were already obtained using Carleman estimates for elliptic/parabolic operators (see [LR10] or [LRL13]).

**Remark 2.1.5.** We have assumed that the interface  $S$  decomposes  $\mathcal{M}$  in two disjoint parts  $\Omega_+$  and  $\Omega_-$ . However the same results can be obtained for other more general geometric situations as well. This comes from the fact that the key ingredient is a *local* quantitative estimate (see Theorem 2.5.1). See also the figure in [LRL13, Section 1.3.2]

## 2.1.2 Strategy of the proof and organization of the paper

### The Carleman estimate

One of the main tools for dealing with problems of local unique continuation across a hypersurface  $\{\phi = 0\}$  is *Carleman estimates*. The idea, introduced by Carleman in [Car39], is to prove an inequality involving a weight function  $\psi$  and a large parameter  $\tau$ , of the form

$$\|e^{\tau\psi} Pu\|_{L^2} \gtrsim \|e^{\tau\psi} u\|_{L^2}, \quad \tau \geq \tau_0,$$

uniform in  $\tau$ . The weight function  $\psi$  is closely related to the level sets of the function  $\phi$  which defines implicitly the hypersurface. In an heuristic way, the chosen weight reinforces the sets where  $u$  is zero and propagates smallness from sets where  $\psi$  is big to sets where  $\psi$  is small. Since Carleman estimates are already quantitative in nature they provide a good starting point for results of the form (2.1.1). We point out the fact that this is a *local* problem. In order to obtain a *global* result one needs in general to propagate the local one by passing through an appropriate family of hypersurfaces.

The core of this article is to prove a local Carleman estimate *in a neighborhood of the interface*, containing a microlocal weight in the spirit of [Tat95, Hör97, RZ98]. The presence of discontinuous coefficients complicates significantly this task. In general, for a Carleman estimate to hold a condition involving the principal symbol of the operator and the hypersurface needs to hold, the so-called pseudoconvexity condition (see for instance [Hör67]). These results are based on microlocal analysis arguments and some regularity is necessary for the estimate to hold. In our case we explicitly construct an appropriate weight function and show our estimate for this particular weight. Our proof is inspired by that of Lerner-Le Rousseau in the elliptic case [LRL13] and relies in a factorization argument. Even though the behavior of our (hyperbolic) operator may be very different we consider in our context the wave operator as a “perturbation” of a Laplacian, in the spirit of [Tat95, Hör97, RZ98, LL19]. Let us explain why. For the sake of exposition, consider that  $\mathcal{M} = \mathbb{R}^n$ ,  $g$  is the Euclidean metric and  $c$  is piecewise constant with a jump across the hypersurface  $S = \{x_n = 0\}$ . That is  $c = c_+ \mathbf{1}_{\{x_n > 0\}} + c_- \mathbf{1}_{\{x_n < 0\}}$ . Then the principal symbol of the wave operator  $\square = \partial_t^2 - c\Delta$  is

$$\sigma(\square) = -\xi_t^2 + c|\xi|^2.$$

The inequality we want to prove contains the weight  $e^{-\epsilon \frac{D_t^2}{2\tau}}$  which heuristically localizes close to  $\{\xi_t = 0\}$ , in other words localizes in a microlocal region where our operator is *close to being elliptic*.

In order to obtain a Carleman estimate with a weight  $\psi$  one obtains an inequality for the conjugated operator  $P_\psi$  defined by  $P_\psi := e^{\tau\psi} P e^{-\tau\psi}$ . In [LRL13] the authors use an idea which can be traced back to Calderón [Cal58]. They factorize the conjugated operator  $P_\psi$  as a product of two first-order operators and prove estimates for each first order factor. In the elliptic case  $P = -c\Delta$  and for a weight  $\psi$  depending only on  $x_n$  one has the following factorization:

$$\begin{aligned} P_\psi &= c_+ (D_n + i(\tau\psi' + |\xi'|)) (D_n + i(\tau\psi' - |\xi'|)) \\ &\quad + c_- (D_n + i(\tau\psi' - |\xi'|)) (D_n + i(\tau\psi' + |\xi'|)) \\ &= c_+ (D_n + ie_+) (D_n + if_+) + c_- (D_n + if_-) (D_n + ie_-). \end{aligned}$$

Here the operator has been identified with its symbol in the tangential variables. This allows to work on the normal variable  $x_n$  and treat the other ones as parameters. In more technical terms, we can use the tools of tangential symbolic calculus in all the variables but  $x_n$  and try to obtain good one-dimensional estimates for the first order operators  $D_n + ie_\pm$  and  $D_n + if_\pm$ . The general principle is that the sign of  $e_\pm, f_\pm$  determines the quality of the one dimensional estimates. The choice of the weight function  $\psi$  is made so that the following key property is satisfied: we can cover the tangential dual space by  $\Gamma_1, \Gamma_2$  such that  $(x', \tau, \xi') \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1} = \Gamma_1 \cup \Gamma_2$  with  $f_+ \geq 0$  on  $\Gamma_1$  and  $f_- \leq 0$  on  $\Gamma_2$ .

In the proof of the present article, in which  $P$  is a wave operator, we work essentially in two microlocal zones. In the first zone, the operator  $P$  is not microlocally elliptic. In this zone, we consider terms involving  $D_t$  as admissible errors. Using the operator  $e^{-\epsilon \frac{D_t^2}{2\tau}}$  allows us then to obtain the desired estimate. In the second zone, the operator  $P$  is microlocally elliptic one can follow the proof provided in the elliptic case in [LRL13].

In Section 2.2 we give the precise statement of the local Carleman estimate that we prove and we describe adapted local coordinates in a small neighborhood of the interface to prepare the proof. In Section 2.4 we prove the Carleman estimate. We considered helpful to give first a proof for a toy model (constant coefficients case) in Section 2.3. In this case one can simply work on the Fourier domain without having to work with pseudodifferential operators (see [Ler19, Chapter 10]). At the same time it allows to understand the core of the arguments which will be used in the general case too.

### Using the Carleman estimate

The next step is to use the Carleman estimate. To do this, one needs to obtain a local quantitative version which can be iterated. This has been done in the smooth case by Laurent and Léautaud in [LL19]. There, the estimate has the same form as that obtained in [LL19] in a (arbitrarily small) neighborhood of the interface  $S$ . Thus, we are able to use it “once” to pass on the other side of  $S$  and then combine it with the results of [LL19]. In Section 2.5.1 we show how one can use the techniques of [LL19] to obtain such a local quantitative estimate in a neighborhood of the interface where the coefficients jump. Finally in Section 2.5.3 we prove that indeed, one can propagate the quantitative estimate by combining our local estimate with the analogous results in the smooth case.

#### 2.1.3 Some notations and definitions

We recall in this Section some elementary geometric facts and we give the precise definition of the distance which is used in the statement of Theorem 2.1.1 and Theorem 2.1.4.

Let us recall that the geometric setting is given at the beginning of Section 2.1.1. The interface  $S$  has a natural metric given by the restriction of  $g$  to  $TS$ . In local coordinates we have:

$$\partial_\nu = \sum_j \nu^j \partial_{x_j}, \quad \nu^j = \lambda \sum_k n_k g^{jk}, \quad |\nu_g| = 1,$$

with  $g^{ij}g_{jk} = \delta_{ik}$ ,  $\lambda^2 = \left(\sum_{i,j} g^{ij}n_i n_j\right)^{-1}$  and  $n$  is the normal to  $S$  for the Euclidean metric in the chosen local coordinates pointing into  $\Omega_+$ . We denote by  $(\cdot, \cdot)_g = g(\cdot, \cdot)$  the inner product on  $T\mathcal{M}$ . The Riemannian gradient of a function  $f$  is defined in an intrinsic manner by

$$(\nabla_g f, X)_g = df(X), \quad \text{for any smooth vector field } X.$$

The integral of a function  $f$  is defined by  $\int f := \int_{\mathcal{M}} f(x) d\text{Vol}_g(x)$ , where  $d\text{Vol}_g(x)$  is the Riemannian volume element. The divergence operator, acting on a vector field  $X$  is defined by the relation

$$\int u \operatorname{div}_g X = - \int (\nabla_g u, X)_g, \quad \text{for all } u \in C_0^\infty(\text{Int}\mathcal{M}).$$

Let us recall the expression of these objects in local coordinates. We consider  $f$  a smooth function on  $\mathcal{M}$  and  $X = \sum_i X^i \partial_{x_i}$ ,  $Y = \sum_i Y^i \partial_{x_i}$  two smooth vector fields on  $\mathcal{M}$ . We have:

$$(X, Y)_g = \sum_{i,j} g_{ij} X^i Y^j, \quad \nabla_g f = \sum_{i,j} g^{ij} (\partial_j f) \partial_{x_i}, \quad \operatorname{div}_g(X) = \frac{1}{\sqrt{\det g}} \sum_i \partial_i (\sqrt{\det g} X_i).$$

We finally have:

$$\operatorname{div}_g(c(x) \nabla_g f) = \frac{1}{\sqrt{\det g}} \sum_{i,j} \partial_{x_i} (c g^{ij} \sqrt{\det g} \partial_{x_j} f), \quad x \in \Omega_- \cup \Omega_+.$$

We want to define the natural distance associated to the operator  $P$  appearing in Theorem 2.1.1. Consider the piecewise smooth metric  $g_c$  by  $g_c := \mathbf{1}_{\Omega_-} c_-^{-1} g + \mathbf{1}_{\Omega_+} c_+^{-1} g$ .

**Definition 1.** An admissible path  $C^\infty([0, 1]; \mathcal{M}) \ni \gamma : [0, 1] \rightarrow \mathcal{M}$  is a path satisfying the conditions:

- $\gamma$  does not have self-intersections
- $\gamma$  intersects the interface  $S$  a finite number of times
- $\gamma(t) \in S \implies \gamma'(t) \perp T_{\gamma(t)} S$

In particular, if  $\gamma$  is an admissible path then the map  $t \mapsto |\gamma'(t)|_{g_c(\gamma(t))} \in L^\infty([0, 1]; \mathcal{M})$  is bounded and consequently we can define its length by the usual formula:

$$\operatorname{length}(\gamma) = \int_0^1 |\gamma'(t)|_{g_c(\gamma(t))} dt.$$

We now define the distance we will be working with:

**Definition 2.** The distance of two points  $x_0, x_1 \in \mathcal{M}$  is defined as:

$$\operatorname{dist}(x_0, x_1) = \inf \{ \operatorname{length}(\gamma) \mid \gamma \text{ admissible path, } \gamma(0) = x_0, \gamma(1) = x_1 \}.$$

We can now as well define the *largest distance of a subset  $E \subset \mathcal{M}$  to  $\mathcal{M}$*  by

$$\mathcal{L}(\mathcal{M}, E) := \sup_{x \in \mathcal{M}} \operatorname{dist}(x, E), \tag{2.1.5}$$

where

$$\operatorname{dist}(x, E) = \inf_{y \in E} \operatorname{dist}(x, y).$$

**Remark 2.1.6.** Notice that the conditions imposed on the family of admissible paths do not pose any important restriction since any Lipschitz path can be replaced by an admissible one up to increasing its length by an  $\epsilon > 0$  arbitrarily small. Since we take the infimum of these lengths the distance remains the same.

## 2.2 The Carleman estimate

The key ingredient for the proof of Theorem 2.1.1 is a local Carleman estimate. Since we will work in space-time it is convenient to consider  $\Sigma := \mathbb{R}_t \times S$  with  $S$  the smooth interface of the manifold  $\mathcal{M}$  defined in Section 2.1.1. Therefore,  $\Sigma$  is a smooth hypersurface of  $\mathbb{R}_t \times \mathcal{M}$ . We define as well  $\Omega_{t,\pm} := \mathbb{R}_t \times \Omega_\pm$ .

### 2.2.1 General transmission conditions

We want to derive a Carleman estimate for the wave operator

$$\square = P := \partial_t^2 - \operatorname{div}_g(c(x)\nabla_g),$$

where the scalar coefficient  $c$  satisfies  $0 < c_{\min} < c(x) < c_{\max} < +\infty$  uniformly for  $x \in \Omega_- \cup \Omega_+$  to make sure that the ellipticity property is satisfied. We recall as well that  $c|_{\Omega_{t,\pm}} \in C^\infty(\overline{\Omega}_{t,\pm})$  but it jumps across the interface  $S$ . Since the operator  $P$  has discontinuous coefficients one needs to be careful with its domain. Indeed, given a function  $u = \mathbb{1}_{\Omega_{t,-}}u_- + \mathbb{1}_{\Omega_{t,+}}u_+$  with  $u_\pm \in C^\infty(\mathbb{R}_t \times \mathcal{M})$  one has in the distributional sense

$$\nabla_g u = \mathbb{1}_{\Omega_{t,-}}\nabla_g u_- + \mathbb{1}_{\Omega_{t,+}}\nabla_g u_+ + (u_- - u_+)\delta_\Sigma\nu,$$

where  $\delta_\Sigma$  is the surface measure on  $\Sigma$  and  $\nu$  is the unit normal vector field pointing into  $\Omega_{t,+}$ . We impose then that

$$u_-|_\Sigma = u_+|_\Sigma, \quad (2.2.1)$$

and the singular term is removed. Similarly, calculating

$$\operatorname{div}(c(x)\nabla_g u),$$

we see that the condition

$$c_+\partial_\nu u_+|_\Sigma = c_-\partial_\nu u_-|_\Sigma \quad (2.2.2)$$

combined with (2.2.1) gives the equality

$$\operatorname{div}_g(c(x)\nabla_g u) = \mathbb{1}_{\Omega_{t,-}}\operatorname{div}(c_-\nabla_g u_-) + \mathbb{1}_{\Omega_{t,+}}\operatorname{div}(c_+\nabla_g u_+).$$

We define then  $\mathcal{W}$  as the space containing functions of the form

$$u = \mathbb{1}_{\Omega_{t,-}}u_- + \mathbb{1}_{\Omega_{t,+}}u_+, \quad (2.2.3)$$

with  $u_\pm \in C_0^\infty(\mathbb{R}_t \times \mathcal{M})$  and such that (2.2.1) and (2.2.2) hold. These conditions are called *transmission conditions* and for  $u \in \mathcal{W}$  one has  $Pu \in L^2$ . The above calculations show as well that  $P$  is formally self adjoint on this domain and therefore by classical methods (energy estimates or semi-group theory) one has that the evolution problem (2.1.3) is well posed.

**Remark 2.2.1.** The first transmission condition expresses the continuity across the interface and the second one the continuity for the normal flux. Notice that the second condition excludes many smooth functions from the space  $\mathcal{W}$ . On the other hand elements of  $\mathcal{W}$  are Lipschitz continuous and in particular one has  $\mathcal{W} \subset H^1$ .

For technical reasons it will be useful to work with non-homogeneous transmission conditions as well. More precisely, we shall denote by  $\mathcal{W}^{\theta,\Theta}$  the space of functions of the form (2.2.3) satisfying additionally the following *non homogeneous transmission conditions*:

$$u_-|_\Sigma = u_+|_\Sigma + \theta \quad (2.2.4)$$

$$c_+\partial_\nu u_+|_\Sigma = c_-\partial_\nu u_-|_\Sigma + \Theta, \quad (2.2.5)$$

where  $\theta$  and  $\Theta$  are smooth functions of the interface  $\Sigma$ .

### 2.2.2 Local setting in a neighborhood of the interface

Since we show a local Carleman estimate we can state it directly in adapted local coordinates. In a sufficiently small neighborhood  $V$  of a point  $x_0$  of  $S$  one can use normal geodesic coordinates with respect to the spatial variables  $x$  (see [Hör85], Appendix C.5, [LRLR22] Section 9.4). In such a coordinate system the interface  $S$  is given by  $S = \{x_n = 0\}$  and the *principal part* of the operator  $P$  denoted by  $P_2$  takes the form (on both sides of the interface)

$$P_2 = \partial_t^2 - c(x)(\partial_{x_n}^2 - r(x, \partial_{x'}/i)),$$

with  $r(x, \xi')$  an  $x$ -family of second order polynomials in  $\xi'$  that satisfy

$$r(x, \xi') \in \mathbb{R}, \quad C_1 |\xi'|^2 \leq r(x, \xi') \leq C_2 |\xi'|^2,$$

for  $x \in V$ ,  $\xi' \in \mathbb{R}^{n-1}$  and  $0 < C_1 < C_2 < \infty$ . The transmission conditions become after this change of variables:

$$u_{-|x_n=0} = u_{+|x_n=0} + \theta \tag{2.2.6}$$

$$c_+ \partial_{x_n} u_{+|x_n=0} = c_- \partial_{x_n} u_{-|x_n=0} + \Theta, \tag{2.2.7}$$

In this setting, the two sides of the interface become  $\Omega_{t,+} = \{x_n > 0\}$  and  $\Omega_{t,-} = \{x_n < 0\}$ . We shall use the notation

$$H_{\pm} = 1_{\Omega_{t,\pm}}.$$

and we have for the coefficient  $c$ :

$$c(x) = H_- c_-(x) + H_+ c_+(x).$$

The space  $\mathcal{W}^{\theta, \Theta}$  consists in this local context of functions  $u$  of the form

$$u = H_- u_- + H_+ u_+, \quad u_{\pm} \in C_0^{\infty}(\mathbb{R}_t \times V),$$

satisfying also (2.2.6) and (2.2.7). We define as well

$$P^{\pm} := \partial_t^2 - \operatorname{div}(c_{\pm} \nabla \cdot).$$

Following [Tat95, Hör97, RZ98] we are seeking to prove a Carleman estimate containing the microlocal weight

$$Q_{\delta, \tau}^{\phi} = e^{-\frac{\delta |D_t|^2}{2\tau}} e^{\tau \phi}. \tag{2.2.8}$$

We shall take  $\phi$  in the following form:

$$\phi = \phi(x_n) = \left( \alpha_- x_n + \frac{\beta x_n^2}{2} \right) H_- + \left( \alpha_+ x_n + \frac{\beta x_n^2}{2} \right) H_+, \quad \alpha_{\pm}, \beta > 0. \tag{2.2.9}$$

The parameters  $\alpha_{\pm}$  and  $\beta$  will be chosen in the sequel. The parameter  $\beta$  will be taken large and is related to the sub-ellipticity property (see for instance [Hör63] or [Hör94]) which is necessary for a Carleman estimate to hold. The choice of  $\alpha_{\pm}$  comes from the construction in the microlocally elliptic case (see Lemma 2.4.11). It is a geometric condition on the interface that requires the jump of  $\partial_{x_n} \phi$  (i.e.  $\alpha_+ - \alpha_- > 0$ ) to be sufficiently large (see 2.4.32).

The following is our main Carleman estimate and its proof will occupy a large part of this article:

**Theorem 2.2.2.** *In the geometric situation presented just above let  $(t_0, x_0) \in \mathbb{R}_t \times V$ . Then there exists an appropriate weight  $\phi$  and some positive constants  $C, \tau_0, \delta, d, r_0$  such that*

$$\begin{aligned} C \left\| H_- Q_{\delta, \tau}^\phi P^- u_- \right\|_{L^2}^2 + C \left\| H_+ Q_{\delta, \tau}^\phi P^+ u_+ \right\|_{L^2}^2 \\ + C e^{-d\tau} \left( \tau^3 \left\| e^{\tau\phi} u \right\|_{L^2}^2 + \tau \left\| H_+ \nabla e^{\tau\phi} u_+ \right\|_{L^2}^2 + \tau \left\| H_- \nabla e^{\tau\phi} u_- \right\|_{L^2}^2 \right) + CT_{\theta, \Theta} \\ \geq \tau^3 \left\| Q_{\delta, \tau}^\phi u \right\|_{L^2}^2 + \tau \left\| H_+ \nabla Q_{\delta, \tau}^\phi u_+ \right\|_{L^2}^2 + \tau \left\| H_- \nabla Q_{\delta, \tau}^\phi u_- \right\|_{L^2}^2, \end{aligned}$$

for  $u \in \mathcal{W}^{\theta, \Theta}$  such that  $\text{supp } u \subset B((t_0, x_0), r_0)$  and  $\tau \geq \tau_0$ , where

$$T_{\theta, \Theta} = \tau^3 \left| Q_{\delta, \tau}^\phi \theta \right|_{L^2(\Sigma)}^2 + \tau \left| Q_{\delta, \tau}^\phi \nabla \theta \right|_{L^2(\Sigma)}^2 + \tau \left| Q_{\delta, \tau}^\phi \Theta \right|_{L^2(\Sigma)}^2. \quad (2.2.10)$$

Several remarks are in order:

**Remark 2.2.3.** Observe that the choice of  $\phi$  is well adapted to the geometric situation: for small  $|x_n|$  we have that  $\Sigma = \{x_n = 0\} = \{\phi = 0\}$  and the two sides of the interface are given by  $\{\phi > 0\}$  and  $\{\phi < 0\}$ . We also have (for  $|x_n|$  sufficiently small) that  $\phi' > 0$ , and consequently  $\phi$  is larger in  $\Omega^+$ :  $\Omega^+$  is the observation region and the Carleman estimate of Theorem 2.2.2 will propagate uniqueness from  $\Omega^+$  to  $\Omega^-$ . In particular, the weight function  $\phi$  is *suitable for observation* in the sense of [LRL13, Property (1-12)].

**Remark 2.2.4.** Notice that in the case of homogeneous transmission conditions  $\theta = \Theta = 0$  one has  $T_{\theta, \Theta} = 0$ . The calculations carried out in Section 2.2.1 imply that  $Pu \in L^2$  and therefore

$$\left\| H_- P^- u_- \right\|_{L^2} + \left\| H_+ P^+ u_+ \right\|_{L^2} = \|Pu\|_{L^2}.$$

Using moreover the fact that  $u \in H^1$  we can write the estimate of Theorem 2.2.2 in a more concise way as:

$$C \left( \left\| Q_{\delta, \tau}^\phi Pu \right\|_{L^2}^2 + e^{-d\tau} \left\| e^{\tau\phi} u \right\|_{H_\tau^1}^2 \right) \geq \tau \left\| Q_{\delta, \tau}^\phi u \right\|_{H_\tau^1}^2,$$

where the  $\|\cdot\|_{H_\tau^1}$  norm is defined as

$$\|w\|_{H_\tau^1} := \tau \|w\|_{L^2} + \|\nabla w\|_{L^2}.$$

**Remark 2.2.5.** As in [LRL13] we explicitly construct an appropriate weight function  $\phi$ . It will initially depend only in the variable  $x_n$  with  $\Sigma = \{x_n = 0\}$ . In Section 2.4.5 we shall use a perturbation argument to allow dependence upon the other variables as well.

As usual, to prove Theorem 2.2.2 we work with the conjugated operator  $P_{\phi, \delta}$  defined in our case by the relation:

$$e^{-\frac{\delta |D_t|^2}{2\tau}} e^{\tau\phi} P^\pm = P_{\phi, \delta}^\pm e^{-\frac{\delta |D_t|^2}{2\tau}} e^{\tau\phi}. \quad (2.2.11)$$



For a general weight  $\phi$  the operators  $P_{\phi,\delta}^\pm$  do not exist, however in the case where  $\phi$  is quadratic in  $t$  one can show that we have the following expression for  $P_{\phi,\delta}^\pm$  (see for instance [Hör97, Chapter 2]):

$$\begin{aligned} P_{\phi,\delta}^\pm &= c_\pm(x)(D_n + i\tau\phi' - \delta\kappa_{t,t}'')^2 \\ &\quad + c_\pm(x) \sum_{1 \leq j,k \leq n-1} b_{jk}(x)(D_j + i\tau\partial_j\phi - \delta\phi_{t,x_j}'')(D_k + i\tau\partial_k\phi - \delta\phi_{t,x_k}'') \\ &\quad - (D_t + i\tau\phi_t' - \delta\phi_{t,t}'')^2 \end{aligned} \quad (2.2.12)$$

This will be used at a later stage where we will *convexify* our initial weight  $\phi$  as we know that this is in general a necessary procedure for our Carleman estimate to be used if one wishes to obtain qualitative or quantitative uniqueness results. We shall however initially consider a weight  $\phi$  depending solely on the variable  $x_n$ , and a perturbation argument will be used to allow some convexification. In the case where  $\phi$  is independent of  $t$  the conjugated operator  $e^{\tau\phi}Pe^{-\tau\phi}$  commutes with the Fourier multiplier  $e^{-\frac{\delta D_t^2}{2\tau}}$  and it takes the following particularly simple form:

$$P_\phi^\pm = c_\pm(x)(D_n + i\tau\phi')^2 + c_\pm(x) \sum_{1 \leq j,k \leq n-1} b_{jk}(x)D_jD_k - D_t^2$$

As for the smooth case shown by Tataru we will prove a sub elliptic estimate concerning the conjugated operator, which will act on functions of the form  $w = Q_{\delta,\tau}^\phi u$  for  $u \in \mathcal{W}^{\theta,\Theta}$ . We therefore have to understand the action of the conjugated operator on the transmission conditions. We use the following expressions:

$$\begin{aligned} w_\pm(t, x) &= \left(\frac{\tau}{2\pi\delta}\right)^{\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{\tau}{2\delta}(t-s)^2} e^{\tau\phi} u_\pm(s, x) ds \\ \partial_n w_\pm(t, x) &= \left(\frac{\tau}{2\pi\delta}\right)^{\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{\tau}{2\delta}(t-s)^2} e^{\tau\phi} (\tau\phi_\pm' u_\pm + \partial_n u_\pm)(s, x) ds. \end{aligned}$$

Let us define

$$V_t := \mathbb{R}_t \times V. \quad (2.2.13)$$

One has that  $u \in \mathcal{W}^{\theta,\Theta}$  is equivalent to  $w \in \mathcal{W}_\phi^{\theta,\Theta}$  with the space  $\mathcal{W}_\phi^{\theta,\Theta}$  being defined as the space containing functions  $w$  such that

$$w = H_- w_- + H_+ w_+, \quad w_\pm \in C_0^\infty(V_t),$$

satisfying additionally the following modified *transmission conditions*:

$$w_{-|\Sigma} = w_{+|\Sigma} + \theta_\phi \quad (2.2.14)$$

$$c_+(D_n w_+ + i\tau\alpha_+ w_+)_{|\Sigma} = c_-(D_n w_- + i\tau\alpha_- w_-)_{|\Sigma} + \Theta_\phi, \quad (2.2.15)$$

where  $\theta_\phi = Q_{\delta,\tau}^\phi \theta$  and  $\Theta_\phi = Q_{\delta,\tau}^\phi \Theta$ . The following proposition is the main step in the proof of Theorem 2.2.2:

**Proposition 2.2.6.** *Let  $(t_0, x_0) \in \Sigma$ . There exist a suitable weight  $\phi$  and  $C, \tau_0, r_0 > 0$  such that*

$$\begin{aligned} C \left( & \left\| P_\phi^- v_- \right\|_{L^2(\mathbb{R}^{n+1})}^2 + \left\| P_\phi^+ v_+ \right\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \|H_+ D_t v_+\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \|H_- D_t v_-\|_{L^2(\mathbb{R}^{n+1})}^2 \right. \\ & \left. + \tau \left| (D_t v_+) |_\Sigma \right|_{L^2(\Sigma)}^2 + \tau \left| (D_t v_-) |_\Sigma \right|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \right) \\ & \geq \tau^3 \|v\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \|H_+ \nabla v_+\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \|H_- \nabla v_-\|_{L^2(\mathbb{R}^{n+1})}^2 \\ & \quad + \tau^3 |v_+|_{L^2(\Sigma)}^2 + \tau^3 |v_-|_{L^2(\Sigma)}^2 + \tau \left| (\nabla v_+) |_\Sigma \right|_{L^2(\Sigma)}^2 + \tau \left| (\nabla v_-) |_\Sigma \right|_{L^2(\Sigma)}^2, \end{aligned}$$

for  $v \in \mathcal{W}_\phi^{\theta, \Theta}$  such that  $\text{supp } v \subset B((t_0, x_0), r_0)$  and  $\tau \geq \tau_0$ .

Proposition 2.2.6 provides a sub elliptic estimate for the conjugated operator which contains an admissible error (compared to a standard Carleman estimate) in its left hand side, which we will call  $E_t$  for convenience in the sequel:

$$E_t(v) := \tau \|H_+ D_t v_+\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \|H_- D_t v_-\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \left| (D_t v_+) |_\Sigma \right|_{L^2(\Sigma)}^2 + \tau \left| (D_t v_-) |_\Sigma \right|_{L^2(\Sigma)}^2. \quad (2.2.16)$$

### 2.2.3 Proof of Theorem 2.2.2 from Proposition 2.2.6

One should notice that Theorem 2.2.2 is not a straightforward consequence of Proposition 2.2.6. Indeed when one considers  $w = Q_{\delta, \tau}^\phi u$  the function  $w$  is not necessarily compactly supported even though this is the case for  $u$  and consequently Proposition 2.2.6 cannot be applied directly. In particular when we pass from Proposition 2.2.6 to Theorem 2.2.2 the Gaussian weight localizes close to  $\{D_t = 0\}$  and that is why  $E_t$  is an admissible remainder term. Nevertheless the passage from Proposition 2.2.6 to Theorem 2.2.2 is quite classical ([Tat95, Hör97]). In our context one has additionally to deal with the terms coming from the interface. Let us present a proof here:

*Proof that Proposition 2.2.6 implies Theorem 2.2.2.* We consider an element  $u \in \mathcal{W}^{\theta, \Theta}$  satisfying additionally

$$\text{supp } u \subset B(0, r/4),$$

with  $r := \frac{r_0}{2}$  and  $r_0$  fixed by Proposition 2.2.6. We define as before  $w = Q_{\delta, \tau}^\phi u = e^{-\frac{\delta |D_t|^2}{2\tau}} e^{\tau \phi} u$  which is *not* compactly supported on the time variable  $t$ . We take then  $\zeta \in C_0^\infty((-r, r); [0, 1])$  with  $\zeta = 1$  on  $(-r/2, r/2)$ . This implies that the function  $v := \zeta(t)w(t, x) \in \mathcal{W}_\phi^{\theta, \Theta}$  satisfies additionally  $\text{supp } v \subset B(0, r_0)$  which means that we can apply Proposition 2.2.6 to it. We neglect the last two surface terms and write the estimate of Proposition 2.2.6 in a more compact and slightly weaker form to obtain:

$$\sum_{\pm} \left( \left\| H_\pm P_\phi^\pm v \right\|_{L^2}^2 + \tau \|H_\pm D_t v_\pm\|_{L^2}^2 + \tau |D_t v_\pm|_{L^2(\Sigma)}^2 + T_{\zeta\theta, \zeta\Theta} \right) \gtrsim \tau \|v\|_{H_t^1}^2 + \tau^3 |v_\pm|_{L^2(\Sigma)}^2.$$

Using the fact that  $\zeta(t) \leq 1$ , the definition of  $T_{\theta, \Theta}$  in (2.2.10) and the property (2.B.1) of  $Q_{\delta, \tau}^\phi$  we see that  $T_{\zeta\theta, \zeta\Theta} \leq CT_{\theta, \Theta}$ . This yields:

$$\sum_{\pm} \left( \|H_{\pm} P_{\phi}^{\pm} v\|_{L^2}^2 + \tau \|H_{\pm} D_t v_{\pm}\|_{L^2}^2 + \tau \|D_t v_{\pm}\|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \right) \gtrsim \tau \|v\|_{H_{\tau}^1}^2 + \tau^3 |v_{\pm}|_{L^2(\Sigma)}^2. \quad (2.2.17)$$

Now we estimate for  $w$ :

$$\begin{aligned} \tau \|v\|_{H_{\tau}^1}^2 + \tau^3 |w_{\pm}|_{L^2(\Sigma)}^2 &\lesssim \tau \|\zeta w\|_{H_{\tau}^1}^2 + \tau \|(1 - \zeta)w\|_{H_{\tau}^1}^2 + \tau^3 |\zeta w_{\pm}|_{L^2(\Sigma)}^2 + \tau^3 |(1 - \zeta)w_{\pm}|_{L^2(\Sigma)}^2 \\ &\lesssim \sum_{\pm} \left( \|H_{\pm} P_{\phi}^{\pm} \zeta w\|_{L^2}^2 + \tau \|H_{\pm} D_t \zeta w_{\pm}\|_{L^2}^2 + \tau \|D_t \zeta w_{\pm}\|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \right) \\ &\quad + \tau \|(1 - \zeta)w\|_{H_{\tau}^1}^2 + \tau^3 |(1 - \zeta)w_{\pm}|_{L^2(\Sigma)}^2, \end{aligned} \quad (2.2.18)$$

thanks to (2.2.17). For the last two terms we remark that

$$(1 - \zeta)w = (1 - \zeta)e^{-\frac{\delta|D_t|^2}{2\tau}} e^{\tau\phi} u = (1 - \zeta)e^{-\frac{\delta|D_t|^2}{2\tau}} (\check{\chi} e^{\tau\phi} u),$$

with  $\check{\chi} = \check{\chi}(t) \in C_0^\infty((-r/3, r/3))$  with  $\check{\chi} = 1$  in a neighborhood of  $[-r/4, r/4]$  which implies that  $\check{\chi}u = u$ . Since  $1 - \zeta$  is supported away from  $(-r/2, r/2)$  one can apply Lemma 2.B.2 which gives:

$$\tau \|(1 - \zeta)w\|_{H_{\tau}^1}^2 + \tau^3 |(1 - \zeta)w_{\pm}|_{L^2(\Sigma)}^2 \leq C\tau e^{-c\frac{\tau}{\delta}} \|e^{\tau\phi} u\|_{H_{\tau}^1}^2 + C\tau^3 e^{-c\frac{\tau}{\delta}} |e^{\tau\phi} u_{\pm}|_{L^2(\Sigma)}^2.$$

Therefore it remains to estimate the following three terms, appearing in the right hand side of (2.2.18):

- We take a function  $\tilde{\chi} = \tilde{\chi}(t)$  such that  $\tilde{\chi} = 1$  on  $\text{supp}(\zeta')$  and  $\tilde{\chi} = 0$  on  $(-r/3, r/3)$  and find for the first one:

$$\begin{aligned} \sum_{\pm} \|H_{\pm} P_{\phi}^{\pm} \zeta w\|_{L^2}^2 &\lesssim \sum_{\pm} \left( \|H_{\pm} \zeta P_{\phi}^{\pm} w\|_{L^2}^2 + \|H_{\pm} [P_{\phi}^{\pm}, \zeta] w\|_{L^2}^2 \right) \\ &\lesssim \sum_{\pm} \left( \|H_{\pm} P_{\phi}^{\pm} w\|_{L^2}^2 + \|H_{\pm} [P_{\phi}^{\pm}, \zeta] w\|_{L^2}^2 \right) \\ &= \sum_{\pm} \left( \|H_{\pm} P_{\phi}^{\pm} w\|_{L^2}^2 + \|H_{\pm} [P_{\phi}^{\pm}, \zeta] \tilde{\chi} w\|_{L^2}^2 \right) \\ &\lesssim \sum_{\pm} \|H_{\pm} P_{\phi}^{\pm} w\|_{L^2}^2 + \|\tilde{\chi} w\|_{H_{\tau}^1}^2 \\ &\lesssim \sum_{\pm} \|H_{\pm} P_{\phi}^{\pm} w\|_{L^2}^2 + e^{-c\frac{\tau}{\delta}} \|e^{\tau\phi} u\|_{H_{\tau}^1}^2 \end{aligned} \quad (2.2.19)$$

where we used the properties of the support of  $\tilde{\chi}$  and  $u$  combined with Lemma 2.B.2.

- For the second term we have, using the support of  $\zeta'$ , as well (to alleviate notation we drop the  $\pm$ ):

$$\begin{aligned} \tau \|D_t \zeta w\|_{L^2}^2 &\lesssim \tau \|\zeta' w\|_{L^2}^2 + \tau \|D_t w\|_{L^2}^2 \\ &\lesssim \tau e^{-c\frac{\tau}{\delta}} \|e^{\tau\phi} u\|_{L^2}^2 + \tau \|D_t w\|_{L^2}^2. \end{aligned}$$

To estimate  $\tau \|D_t w\|_{L^2}^2$  we work on the Fourier domain (with respect to the time variable  $t$ ) and distinguish between the frequencies smaller or bigger than  $\sigma\tau$  for an arbitrary  $\sigma > 0$  (as in [LL19, Section 5.2]). One has:

$$\begin{aligned} \|D_t w\|_{L^2} &\leq \left\| 1_{|D_t| \leq \sigma\tau} D_t w \right\|_{L^2} + \left\| 1_{|D_t| \geq \sigma\tau} D_t w \right\|_{L^2} \\ &= \left\| 1_{|D_t| \leq \sigma\tau} D_t w \right\|_{L^2} + \left\| 1_{|D_t| \geq \sigma\tau} D_t e^{-\frac{\delta|D_t|^2}{2\tau}} e^{\tau\phi} u \right\|_{L^2} \\ &\leq \left\| 1_{|D_t| \leq \sigma\tau} D_t w \right\|_{L^2} + \max_{\xi_t \geq \sigma\tau} \left( \xi_t e^{-\frac{\delta\xi_t^2}{2\tau}} \right) \left\| e^{\tau\phi} u \right\|_{L^2}. \end{aligned}$$

We see that if  $\tau \geq \frac{1}{\sigma^2\delta}$  the function  $s \mapsto s e^{-\frac{\delta s^2}{2\tau}}$  is decreasing on the interval  $[\sigma\tau, +\infty)$ . We obtain therefore for  $\tau \geq \max(\tau_1, \frac{1}{\sigma^2\delta})$  the estimate:

$$\tau \|D_t w\|_{L^2}^2 \lesssim \sigma^2 \tau^3 \|w\|_{L^2}^2 + \sigma^2 \tau^3 e^{-\tau\sigma^2\delta} \left\| e^{\tau\phi} u \right\|_{L^2}^2. \quad (2.2.20)$$

Above and in the sequel the hidden constant will be independent of  $\sigma$ .

- For the third term  $\tau |D_t \zeta w_{\pm}|_{L^2(\Sigma)}^2$  in (2.2.18) one can proceed exactly as above to find:

$$\tau |D_t \zeta w|_{L^2(\Sigma)}^2 \lesssim \tau e^{-c\frac{\tau}{\delta}} \left| e^{\tau\phi} u \right|_{L^2(\Sigma)}^2 + \sigma^2 \tau^3 |w|_{L^2(\Sigma)}^2 + \sigma^2 \tau^3 e^{-\tau\sigma^2\delta} \left| e^{\tau\phi} u \right|_{L^2(\Sigma)}^2. \quad (2.2.21)$$

We inject estimates (2.2.19), (2.2.20) and (2.2.21) in the right hand side of (2.2.18) to obtain:

$$\begin{aligned} \tau \|w\|_{H^1_\tau}^2 + \tau^3 |w|_{L^2(\Sigma)}^2 &\lesssim \|P_\phi w\|_{L^2}^2 + \left( \tau e^{-c\frac{\tau}{\delta}} + \sigma^2 \tau^3 e^{-\tau\sigma^2\delta} \right) \left\| e^{\tau\phi} u \right\|_{H^1_\tau}^2 \\ &\quad + \left( \tau^3 e^{-c\frac{\tau}{\delta}} + \sigma^2 \tau^3 e^{-\tau\sigma^2\delta} \right) \left| e^{\tau\phi} u \right|_{L^2(\Sigma)}^2 \\ &\quad + \sigma^2 \tau^3 \|w\|_{L^2}^2 + \sigma^2 \tau^3 |w|_{L^2(\Sigma)}^2 + T_{\theta, \Theta}. \end{aligned}$$

We now choose  $\sigma$  sufficiently small to absorb the last two terms above in the left hand side of our estimate. Then there exists  $d > 0$  such that for  $\tau \geq \max(\tau_0, \frac{1}{\sigma^2\delta})$ :

$$\begin{aligned} \sum_{\pm} \left( \tau \|w_{\pm}\|_{H^1_\tau}^2 + \tau^3 |w_{\pm}|_{L^2(\Sigma)}^2 \right) &\lesssim \sum_{\pm} \left( \|H_{\pm} P_\phi^\pm w_{\pm}\|_{L^2}^2 + e^{-d\tau} \left( \left\| e^{\tau\phi} u_{\pm} \right\|_{H^1_\tau}^2 + \left| e^{\tau\phi} u_{\pm} \right|_{L^2(\Sigma)}^2 \right) + T_{\theta, \Theta} \right) \\ &\lesssim \sum_{\pm} \left( \|H_{\pm} P_\phi^\pm w_{\pm}\|_{L^2}^2 + e^{-d\tau} \left\| e^{\tau\phi} u_{\pm} \right\|_{H^1_\tau}^2 + T_{\theta, \Theta} \right) \\ &\lesssim \sum_{\pm} \left( \|H_{\pm} Q_{\delta, \tau}^\phi P^\pm u_{\pm}\|_{L^2}^2 + e^{-d\tau} \left\| e^{\tau\phi} u_{\pm} \right\|_{H^1_\tau}^2 + T_{\theta, \Theta} \right), \end{aligned}$$

where we have used the trace estimate  $\left| e^{\tau\phi} u \right|_{L^2(\Sigma)}^2 \lesssim \left\| e^{\tau\phi} u \right\|_{H^1}^2$  as well as the definition of the conjugated operator  $P_\phi$ . This concludes the proof of Theorem 2.2.2 from Proposition 2.2.6.  $\square$

## 2.3 Proof of Proposition 2.2.6 for a toy model

The goal of this section is to prove the subelliptic estimate of Proposition 2.2.6 in the particular situation where the coefficient  $c$  is piecewise constant. This case works as a sketch of proof since it is technically simpler but at the same time it allows to understand the core of the arguments that will be used for the proof of the general case in Section 2.4.

**Notations:** Before going further let us fix some useful notations that will be used in the sequel. We write  $\mathbb{R}_+^{n+1} = \{x_n > 0\} \times \mathbb{R}^n$  with the analogous definition for  $\mathbb{R}_-^{n+1}$ . In particular  $\mathbb{R}_+$  will refer to  $\{x_n > 0\}$  and  $\mathbb{R}_-$  to  $\{x_n < 0\}$ . We note  $(a, b)$  the inner product in  $L^2(\mathbb{R}^{n+1})$  and  $x' = (x_1, \dots, x_{n-1})$ . The inner product on  $L^2(\mathbb{R}_+^{n+1}), L^2(\mathbb{R}_-^{n+1}), \Sigma$  will be denoted by  $(\cdot, \cdot)_+, (\cdot, \cdot)_-, (\cdot, \cdot)_\Sigma$  respectively. When we consider norms on  $\Sigma$  the argument will automatically be considered to be restricted in  $\Sigma$ , even though we shall not always write  $|\Sigma$  to simplify our notation. We will simply write  $\hat{v}$  for the *partial* Fourier transform of  $v$  in the variables  $(t, x')$ , whose dual variables are  $(\xi_t, \xi')$ . We recall that the space of functions  $\mathcal{W}_\phi^{\theta, \Theta}$  and the small neighborhood  $V$  have been defined in Section 2.2.2. To alleviate notation we denote

$$\mathcal{W}_\phi := \mathcal{W}_\phi^{0,0},$$

for the case of homogeneous transmission conditions.

We recall that we work in the setting introduced in Section 2.2.2. We suppose additionally only in this section that the coefficient  $c$  is piecewise constant. We write then  $c = H_- c_- + H_+ c_+$  with  $c_\pm > 0$  constants and we consider homogeneous transmission conditions  $\theta = \Theta = 0$ . This allows to write  $v_-(0) = v_+(0) = v(0)$  for  $v \in \mathcal{W}_\phi$  and it implies as well that  $P_\phi v \in L^2(V_i)$ .

### 2.3.1 Factorization and first estimates

*In the sequel when we use the notation  $\lesssim$  or  $\gtrsim$  the implicit constant will depend on the coefficients of  $P_\phi$  (here  $c_\pm$ ) and on the coefficients  $\alpha_\pm, \beta$  of our weight function  $\phi$ . The constants denoted by  $C$  will depend on the same variables and they can be different from one line to another.*

In the elliptic case of [LRL13] the authors use a factorization argument which takes advantage of the fact that  $-\Delta$  is positive and therefore one can define its square root. In our proof of Proposition 2.2.6 we use and extend this idea of factorization, however as we no longer have the positivity property this factorization is not always possible. Before giving the details let us observe that we can identify the operator  $P_\phi$  with its symbol in the tangential variables  $(t, x')$ . Indeed, using Plancherel's theorem and denoting by  $\mathcal{F}_{t, x'}$  the partial Fourier transform in the variables  $(t, x')$  we have:

$$\begin{aligned} (2\pi)^{n/2} \|P_\phi v\|_{L^2(\mathbb{R}^{n+1})} &= \|\mathcal{F}_{t, x'} P_\phi v\|_{L^2(\mathbb{R}^{n+1})} \\ &= \left\| H_+ \left( c_+ (D_n + i\tau\phi')^2 + c_+ |\xi'|^2 - \xi_t^2 \right) \mathcal{F}_{t \rightarrow \xi_t, x' \rightarrow \xi'} v_+ \right\|_{L^2(\mathbb{R}^{n+1})} \\ &\quad + \left\| H_- \left( c_- (D_n + i\tau\phi')^2 + c_- |\xi'|^2 - \xi_t^2 \right) \mathcal{F}_{t \rightarrow \xi_t, x' \rightarrow \xi'} v_- \right\|_{L^2(\mathbb{R}^{n+1})}. \end{aligned}$$

Here we used the fact that the coefficients of  $P$  (and thus of  $P_\phi$  since  $\phi = \phi(x_n)$ ) are constant. In the general case, although this identification is no longer valid, symbolic calculus will allow to exploit the core of the arguments carried out below. We write:

$$P_\phi^\pm = c_\pm \left( (D_n + i\tau\phi')^2 + |\xi'|^2 - \frac{1}{c_\pm} |\xi_t|^2 \right).$$

For  $\epsilon > 0$  small to be chosen, we distinguish the following regions of the tangential frequency space  $\mathbb{R}^n \ni (\xi', \xi_t)$ :

1.  $\mathcal{E}_\epsilon^\pm : \boxed{|\xi'|^2 - c_\pm^{-1} |\xi_t|^2 \geq \epsilon(|\xi'|^2 + |\xi_t|^2)}$ . This is the elliptic region. Here we can factorize in precisely the same way as in the elliptic case [LRL13] with the same estimates for the first order factors. For  $(\xi_t, \xi') \in \mathcal{E}_\epsilon^- \cap \mathcal{E}_\epsilon^+$  we can proceed exactly as in the elliptic case.
2.  $\mathcal{GH}_\epsilon^\pm : \boxed{|\xi'|^2 - c_\pm^{-1} |\xi_t|^2 < \epsilon(|\xi'|^2 + |\xi_t|^2)}$ . This is the union of the hyperbolic and glancing regions (see for instance [Hör85, Chapter 23.2] or [GL20]). The important fact is that here we have for sufficiently small  $\epsilon$  that  $|\xi_t| \gtrsim |\xi'|$ . Since  $\|D_t v\| = \|\xi_t v\|$  and  $|D_t v| = |\xi_t v|$  are admissible remainder terms (in view of the statement of Proposition 2.2.6) we obtain directly a useful estimate on all tangential derivatives. It thus only remains to obtain an estimate on  $\|D_n v\|$  and  $|D_n v|$ .

We have thus considered two regions adapted to the geometry in each side of the interface (that is  $\{x_n < 0\}$  and  $\{x_n > 0\}$ ). This gives us four regions which cover all of the tangent dual space. Note that making an assumption on the *sign* of the jump could reduce the number of regions one has to deal with but we do not make such an assumption.

The following simple Lemma will be used in multiple occasions in the sequel:

**Lemma 2.3.1.** *One has for  $v \in \mathcal{W}_\phi$  and all  $\tau > 0$ :*

$$\tau |D_n \hat{v}_\pm(0)|^2 \lesssim \tau |D_n \hat{v}_\mp(0)|^2 + \tau^3 |\hat{v}(0)|^2$$

This means that if we have one of the two terms  $\tau |D_n \hat{v}_+|^2_{L^2(\Sigma)}$  or  $\tau |D_n \hat{v}_-|^2_{L^2(\Sigma)}$  we automatically have the other one too, modulo the surface term  $\tau^3 |\hat{v}|^2$ .

*Proof.* This is a result of the transmission conditions (2.2.14) and (2.2.15) and of the fact that partial Fourier transform commutes with restriction on  $\Sigma = \{x_n = 0\}$ . The first transmission condition allows to write  $\hat{v}_-(0) = \hat{v}_+(0) = \hat{v}(0)$  and (2.2.15) reads

$$D_n \hat{v}_-(0) = \frac{c_+}{c_-} (D_n \hat{v}_+ + i\tau \alpha_+ \hat{v}_+(0)) - i\tau \alpha_- \hat{v}_-(0),$$

and therefore

$$|c_- D_n \hat{v}_-(0) - c_+ D_n \hat{v}_+(0)| \lesssim \tau |\hat{v}(0)|,$$

from which the Lemma follows. □

Consequently, if for instance one has  $\|P_\phi v\|_{L^2(\mathbb{R}^{n+1})}^2 + E_t(v) \gtrsim \tau |D_n v_-|^2_{L^2(\Sigma)} + \tau^3 |v(0)|^2_{L^2(\Sigma)}$  then thanks to Lemma 2.3.1 we have:

$$\tau |D_n v_+(0)|^2_{L^2(\Sigma)} \lesssim \tau |D_n v_-(0)|^2_{L^2(\Sigma)} + \tau^3 |v(0)|^2_{L^2(\Sigma)} \lesssim \|P_\phi v\|_{L^2(\mathbb{R}^{n+1})}^2 + E_t(v).$$

As we aim at proving a Carleman estimate (which involves a large parameter  $\tau$ ) we expect that regions depending on  $\tau$  arise. The following lemma furnishes such a region which is rather favorable:

**Lemma 2.3.2.** *For all  $\tau, c_0 > 0$ ,  $v \in \mathcal{W}_\phi$  and  $Y \subset \{|\xi_t| \geq c_0(\tau + |\xi'|)\}$  one has*

$$\begin{aligned} & \tau \left\| \widehat{D_t v_+} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau \left\| \widehat{D_t v_-} \right\|_{L^2(\mathbb{R}_- \times Y)}^2 + \tau \left| \widehat{D_t v_+}(0) \right|_{L^2(\Sigma \cap Y)}^2 + \tau \left| \widehat{D_t v_-}(0) \right|_{L^2(\Sigma \cap Y)}^2 \\ & \quad \gtrsim \tau^3 \|\widehat{v}\|_{L^2(\mathbb{R} \times Y)}^2 + \tau^3 |v(0)|_{L^2(\Sigma \cap Y)}^2 + \tau \left| \widehat{\nabla_{x'} v}(0) \right|_{L^2(\Sigma \cap Y)}^2 \\ & \quad \quad \quad + \tau \left\| \widehat{\nabla_{x'} v_-} \right\|_{L^2(\mathbb{R}_- \times Y)}^2 + \tau \left\| \widehat{\nabla_{x'} v_+} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2. \end{aligned}$$

*Proof.* We simply write

$$\begin{aligned} \tau \left\| \widehat{D_t v_\pm} \right\|_{L^2(\mathbb{R} \times Y)}^2 &= \tau \|\xi_t \widehat{v}_\pm\|_{L^2(\mathbb{R} \times Y)}^2 \gtrsim \tau \|(\tau + |\xi'|) \widehat{v}_\pm\|_{L^2(\mathbb{R} \times Y)}^2 \\ &\gtrsim \tau^3 \|\widehat{v}_\pm\|_{L^2(\mathbb{R} \times Y)}^2 + \tau \left\| \widehat{\nabla_{x'} v} \right\|_{L^2(\mathbb{R} \times Y)}^2 \end{aligned}$$

And:

$$\begin{aligned} \tau |D_t v|_{L^2(\Sigma \cap Y)}^2 &= \tau |\xi_t \widehat{v}|_{L^2(\Sigma \cap Y)}^2 \gtrsim \tau |(\tau + |\xi'|) \widehat{v}|_{L^2(\Sigma \cap Y)}^2 \\ &\gtrsim \tau^3 |\widehat{v}|_{L^2(\Sigma \cap Y)}^2 + \tau \left| \widehat{\nabla_{x'} v} \right|_{L^2(\Sigma \cap Y)}^2. \end{aligned}$$

□

The following lemma allows to obtain the volume derivatives modulo the reminder of the terms of the right hand-side of Proposition 2.2.6.

**Lemma 2.3.3.** *One has for  $v \in \mathcal{W}_\phi$  and  $\tau \geq \tau_0$ :*

$$\begin{aligned} & \|P_\phi v\|_{L^2(\mathbb{R}^{n+1})}^2 + E_t(v) + \left( \tau^3 \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^2 |v(0)|_{L^2(\Sigma)}^2 + |D_n v_+(0)|_{L^2(\Sigma)}^2 + |D_n v_-(0)|_{L^2(\Sigma)}^2 \right) \\ & \quad \quad \quad \gtrsim \tau \|\nabla v_-\|_{L^2(\mathbb{R}_-^{n+1})}^2 + \tau \|\nabla v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2. \end{aligned}$$

**Remark 2.3.4.** Lemma 2.3.3 allows to forget the volume norm of the derivatives in the proof of our proposition. From now on we will try to obtain all the other terms of Proposition 2.2.6. Notice also that there is no restriction on the frequency support of  $v$ .

*Proof.* We start with the positive half-line. One has the following elementary inequality:

$$\left\| P_\phi^+ v_+ \right\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^2 \|v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 \geq 2\tau \operatorname{Re}(P_\phi^+ v_+, H_+ v_+).$$

Now we can integrate by parts. Indeed, recall that:

$$P_\phi^+ = c^+(D_n^2 + \tau\beta + 2i\tau\phi'D_n - \tau^2|\phi'|^2) + c^+|D'|^2 - D_t^2$$

and that the definition of the weight function  $\phi$  gives  $\phi'(x_n) = \alpha_{\pm} + \beta x_n$ , therefore:

$$\begin{aligned} 2\tau \operatorname{Re}(P_{\phi}^+ v, H_+ v) &= c_+ \left( 2\tau \|D_n v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + 2\tau \int_{\mathbb{R}^n} \operatorname{Re}(\partial_{x_n} v_+(0) \bar{v}_+(0)) dx' dt \right. \\ &\quad + 2\tau^2 \beta \|v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 - 2\tau^2 \alpha_+ |v_+(0)|^2 - 2\tau^2 \beta \|v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 \\ &\quad \left. - 2\tau^3 \|\phi' v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + 2\tau \|D' v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 \right) - 2\tau \|D_t v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2. \end{aligned} \quad (2.3.1)$$

As one has

$$\left| 2\tau \int_{\mathbb{R}^n} \operatorname{Re}(\partial_{x_n} v_+(0) \bar{v}_+(0)) dx' dt \right| \lesssim \|D_n v_+(0)\|_{L^2(\Sigma)}^2 + \tau^2 \|v_+(0)\|_{L^2(\Sigma)}^2,$$

(2.3.1) can be rewritten as

$$2\tau \operatorname{Re}(P_{\phi}^+ v, H_+ v)_{L^2(\mathbb{R}_+^{n+1})} = C\tau \|\nabla v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + R,$$

with

$$|R| \lesssim \tau^3 \|v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|D_t v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^2 \|v_+(0)\|_{L^2(\Sigma)}^2 + \|D_n v_+(0)\|_{L^2(\Sigma)}^2.$$

which implies the desired inequality for  $v_+$ . Since the proof above is insensitive to the sign of the boundary terms coming from the integration by parts in (2.3.1) we also obtain the desired inequality for the negative half-line.  $\square$

The following simple calculation will be at the core of the one dimensional estimates that will be used in the sequel, with  $s = x_n$ :

**Lemma 2.3.5.** *One has for  $\gamma > 0$ ,  $\lambda \in \mathbb{C}$  and  $v \in C_c^1(\mathbb{R})$ :*

$$\begin{aligned} \|(D_s + i\gamma s + \lambda)v\|_{L^2(\mathbb{R}_+)}^2 &= \|D_s v + \operatorname{Re} \lambda v\|_{L^2(\mathbb{R}_+)}^2 + \|(\gamma s + \operatorname{Im} \lambda)v\|_{L^2(\mathbb{R}_+)}^2 \\ &\quad + \gamma \|v\|_{L^2(\mathbb{R}_+)}^2 + \operatorname{Im} \lambda |v(0)|^2, \end{aligned} \quad (2.3.2)$$

and

$$\begin{aligned} \|(D_s + i\gamma s + \lambda)v\|_{L^2(\mathbb{R}_-)}^2 &= \|D_s v + \operatorname{Re} \lambda v\|_{L^2(\mathbb{R}_-)}^2 + \|(\gamma s + \operatorname{Im} \lambda)v\|_{L^2(\mathbb{R}_-)}^2 \\ &\quad + \gamma \|v\|_{L^2(\mathbb{R}_-)}^2 - \operatorname{Im} \lambda |v(0)|^2. \end{aligned} \quad (2.3.3)$$

In particular one can deduce the following downgraded estimates:

$$\|(D_s + i\gamma s + \lambda)v\|_{L^2(\mathbb{R}_+)}^2 \geq \|(\gamma s + \operatorname{Im} \lambda)v\|_{L^2(\mathbb{R}_+)}^2 + \operatorname{Im} \lambda |v(0)|^2, \quad (2.3.4)$$

$$\|(D_s + i\gamma s + \lambda)v\|_{L^2(\mathbb{R}_-)}^2 \geq \gamma \|v\|_{L^2(\mathbb{R}_-)}^2 + \|(\gamma s + \operatorname{Im} \lambda)v\|_{L^2(\mathbb{R}_-)}^2 - \operatorname{Im} \lambda |v(0)|^2. \quad (2.3.5)$$

*Proof of Lemma 2.3.5.* We develop:

$$\begin{aligned} \|(D_s + \operatorname{Re} \lambda + i\gamma s + i \operatorname{Im} \lambda)v\|_{L^2(\mathbb{R}_+)}^2 &= \|(D_s + \operatorname{Re} \lambda)v\|_{L^2(\mathbb{R}_+)}^2 + \|(\gamma s + \operatorname{Im} \lambda)v\|_{L^2(\mathbb{R}_+)}^2 \\ &\quad + 2 \operatorname{Re}(H_+(D_s + \operatorname{Re} \lambda)v, i\gamma s v + i \operatorname{Im} \lambda v). \end{aligned}$$



We have  $\operatorname{Re}(H_+ \operatorname{Re} \lambda v, i\gamma sv + i \operatorname{Im} \lambda v) = 0$  and then integrate by parts:

$$(H_+ D_s v, i \operatorname{Im} \lambda v) = \operatorname{Im} \lambda |v(0)|^2 - \overline{(H_+ D_s v, i \operatorname{Im} \lambda v)},$$

and

$$(H_+ D_s v, i\gamma sv) = \gamma \|v\|_{L^2(\mathbb{R}_+)}^2 - \overline{(H_+ D_s v, i\gamma sv)},$$

which allows to explicitly obtain the real parts and get the stated equality. For the proof of (2.3.3) observe that the boundary term comes out with a negative sign.  $\square$

We first deal with the case  $\tau \gg |\xi'| + |\xi_t|$ . Here we can use the ellipticity of

$$(D_n + i\tau\phi')^2$$

as an 1D operator, and then a perturbation argument. Recall that  $V_t$  has been defined in (2.2.13).

**Lemma 2.3.6.** *There exists  $\sigma_0, \tau_0 > 0$  and  $\mathcal{V} \Subset V_t$  such that for all  $v \in \mathcal{W}_\phi$  with  $\operatorname{supp} v \subset \mathcal{V}$  and*

$$Y \subset \{\tau \geq \frac{1}{\sigma} (|\xi_t| + |\xi'|)\} \quad (2.3.6)$$

one has for all  $\sigma \leq \sigma_0$  and  $\tau \geq \tau_0$  :

$$\begin{aligned} & \|\widehat{P_\phi v}\|_{L^2(\mathbb{R} \times Y)}^2 + \tau \|\widehat{D_t v_+}\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau \|\widehat{D_t v_-}\|_{L^2(\mathbb{R}_- \times Y)}^2 \\ & \quad + \tau \left| \widehat{D_t v_+}(0) \right|_{L^2(\Sigma \cap Y)}^2 + \tau \left| \widehat{D_t v_-}(0) \right|_{L^2(\Sigma \cap Y)}^2 \\ & \gtrsim \tau^3 \|\widehat{v}\|_{L^2(\mathbb{R} \times Y)}^2 + \tau^3 \left| \widehat{v}(0) \right|_{L^2(\Sigma \cap Y)}^2 + \tau \left| \widehat{(\nabla v_+)}(0) \right|_{L^2(\Sigma \cap Y)}^2 + \tau \left| \widehat{(\nabla v_-)}(0) \right|_{L^2(\Sigma \cap Y)}^2. \end{aligned}$$

*Proof.* We identify, with a slight abuse of notation, the operator with its symbol. One has:

$$c_+^{-1} P_\phi^+ = (D_n + i\tau\phi')^2 + |\xi'|^2 - c_+^{-1} \xi_t^2 := A_+ + R_+$$

where  $A_+ := (D_n + i\tau\phi')^2$  is elliptic as a 1D operator and  $R_+ := |D'|^2 - c_+^{-1} D_t^2$ . One has with  $w = (D_n + i\tau\phi')\widehat{v}_+ = (D_n + i\tau\alpha_+ + i\tau\beta x_n)\widehat{v}_+$ , using (2.3.4):

$$\begin{aligned} \|A_+ \widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 &= \|(D_n + i\tau\alpha_+ + i\tau\beta x_n)w\|_{L^2(\mathbb{R}_+ \times Y)}^2 \\ &\geq \|(\tau\beta x_n + \tau\alpha_+)w\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \alpha_+ \tau |w|_{L^2(\Sigma \cap Y)}^2 \gtrsim \tau^2 \|w\|_{L^2(\mathbb{R}_+ \times Y)}^2 + |w|_{L^2(\Sigma \cap Y)}^2 \\ &\gtrsim \tau^2 \left( \|D_n \widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \|(\tau\alpha_+ + \tau\beta x_n)\widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau\alpha_+ |\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 \right) \\ &\quad + \alpha_+ \tau |(D_n + i\tau\alpha)\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2. \end{aligned}$$

In particular we find that

$$\|A_+ \widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 \gtrsim \tau^4 \|\widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^3 |\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 + \tau |(D_n + i\tau\alpha)\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2,$$

and then using  $\tau^3 |D_n \widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 \lesssim \tau^3 |\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 + \tau |(D_n + i\tau\alpha)\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2$  we get

$$\|A_+ \widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 \gtrsim \tau^4 \|\widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^3 |\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 + \tau |D_n \widehat{v}_+|_{L^2(\Sigma \cap Y)}^2.$$

Lemma 2.3.1 then implies:

$$\|A_+ \widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 \gtrsim \tau^4 \|\widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^3 |\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 + \tau |D_n \widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 + \tau |D_n \widehat{v}_-|_{L^2(\Sigma \cap Y)}^2. \quad (2.3.7)$$

Recalling that  $A_+ = P_\phi - R_+$  we now explain how  $\|R_+ \widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}$  can be absorbed as an error term. We estimate as follows:

$$\begin{aligned} \|P_\phi^+ \widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 &\geq C \|A_+ \widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 - \|R_+ \widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 \\ &\geq C \left( \tau^4 \|\widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^3 |\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 + \tau |D_n \widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 \right) - \|R_+ \widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 \\ &= \left( (C\tau^4 - R^*R) \widehat{v}_+, \widehat{v}_+ \right)_+ + C\tau^3 |\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 + C\tau |D_n \widehat{v}_+|_{L^2(\Sigma \cap Y)}^2. \end{aligned}$$

Using (2.3.6) we have

$$C\tau^4 - R^*R = C\tau^4 - (|\xi'|^2 - c_+ \xi_t^2)^2 \geq \left( C - \sigma^4 (1 + c_+^{-1})^2 \right) \tau^4.$$

Then, taking  $\sigma \leq \left( \frac{C}{2(1 + c_+^{-1})^2} \right)^{1/4}$  we deduce

$$\|P_\phi^+ \widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 \gtrsim \tau^4 \|\widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^3 |\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 + \tau |D_n \widehat{v}_+|_{L^2(\Sigma \cap Y)}^2.$$

Using the fact that  $\tau \gtrsim |\xi'|$  together with the transmission condition (2.2.14) we obtain the tangential terms :

$$\tau^3 |\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 \gtrsim \tau \left( |\xi'|^2 + |\xi_t|^2 \right)^{1/2} \widehat{v} \Big|_{L^2(\Sigma \cap Y)}^2 = \tau \left| \widehat{\nabla v} \right|_{L^2(\Sigma \cap Y)}^2.$$

This yields

$$\|P_\phi^+ \widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 \gtrsim \tau^4 \|\widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^3 |\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 + \tau \left| \widehat{\nabla_{x'} v} \right|_{L^2(\Sigma \cap Y)}^2 + \tau |D_n \widehat{v}_+|_{L^2(\Sigma \cap Y)}^2.$$

and finally thanks to Lemma 2.3.1:

$$\begin{aligned} \|P_\phi^+ \widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \|\widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 \\ \gtrsim \tau^4 \|\widehat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^3 |\widehat{v}_+|_{L^2(\Sigma \cap Y)}^2 + \tau \left| \widehat{\nabla v_-} \right|_{L^2(\Sigma \cap Y)}^2 + |\nabla \widehat{v}_+|_{L^2(\Sigma \cap Y)}^2. \end{aligned} \quad (2.3.8)$$

The only missing term is the volume norm on the negative half-line. Here one needs to use  $P_\phi^-$ . We write

$$c_-^{-1} P_\phi^- = (D_n + i\tau\phi')^2 + |\xi'|^2 - c_-^{-1} \xi_t^2 = A_- + R_-, \quad A_- := (D_n + i\tau\phi')^2$$

In the region under consideration  $\|R_- \widehat{v}_-\|_{L^2(\mathbb{R}_- \times Y)}$  is a perturbation of  $\|A_- \widehat{v}_-\|_{L^2(\mathbb{R}_- \times Y)}$ . After repeating the same steps as for the positive half-line one finds:

$$\|P_\phi^- \widehat{v}_-\|_{L^2(\mathbb{R}_- \times Y)}^2 + \tau^3 |\widehat{v}_-|_{L^2(\Sigma \cap Y)}^2 + \tau \left| \widehat{\nabla v_-} \right|_{L^2(\Sigma \cap Y)}^2 \gtrsim \tau^4 \|\widehat{v}_-\|_{L^2(\mathbb{R}_- \times Y)}^2. \quad (2.3.9)$$

To finish the proof of the Lemma one can simply multiply (2.3.8) by a sufficiently large constant and add it to (2.3.9).  $\square$

We now give a Lemma for the region  $\tau \lesssim |\xi'| + |\xi_t|$ .

**Lemma 2.3.7.** *For all  $c_1, c_2 > 0$  there exists  $\mathcal{V} \Subset V_t$  and  $\tau_0 > 0$  such that for all  $v \in \mathcal{W}_\phi$  with  $\text{supp } v \subset \mathcal{V}$  and*

$$Y \subset \{\tau \leq c_1(|\xi_t| + |\xi'|)\} \cap \{|\xi_t| \geq c_2|\xi'|\}$$

one has:

$$\begin{aligned} & \left\| \widehat{P_\phi v} \right\|_{L^2(\mathbb{R} \times Y)}^2 + \tau \left\| \widehat{D_t v_+} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau \left\| \widehat{D_t v_-} \right\|_{L^2(\mathbb{R}_- \times Y)}^2 \\ & \quad + \tau \left| \widehat{D_t v_+}(0) \right|_{L^2(\Sigma \cap Y)}^2 + \tau \left| \widehat{D_t v_-}(0) \right|_{L^2(\Sigma \cap Y)}^2 \\ & \gtrsim \tau^3 \|\widehat{v}\|_{L^2(\mathbb{R} \times Y)}^2 + \tau^3 \left| \widehat{v(0)} \right|_{L^2(\Sigma \cap Y)}^2 + \tau \left| \widehat{(\nabla v_+)}(0) \right|_{L^2(\Sigma \cap Y)}^2 + \tau \left| \widehat{(\nabla v_-)}(0) \right|_{L^2(\Sigma \cap Y)}^2. \end{aligned}$$

for  $\tau \geq \tau_0$ .

*Proof.* Observe that in such a region one has in particular

$$|\xi_t| \gtrsim |\xi'| + \tau,$$

and consequently we are in the regime of Lemma 2.3.2. This implies:

$$\begin{aligned} & \left\| \widehat{P_\phi^+ v_+} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau \left\| \widehat{D_t v_+} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau \left| \widehat{D_t v_+}(0) \right|_{L^2(\Sigma \cap Y)}^2 \\ & \gtrsim \tau^3 \|\widehat{v_+}\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau \left\| \widehat{\nabla_{x'} v_+} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^3 \left| \widehat{v(0)} \right|_{L^2(\Sigma \cap Y)}^2 \\ & \quad + \tau \left| \widehat{(\nabla v_+)}(0) \right|_{L^2(\Sigma \cap Y)}^2 + \tau \left| \widehat{(\nabla v_-)}(0) \right|_{L^2(\Sigma \cap Y)}^2. \end{aligned} \quad (2.3.10)$$

The only remaining term is then  $\tau |D_n \widehat{v_+}|_{L^2(\Sigma \cap Y)}^2$ . To obtain it we use the commutator technique (see [LR95, Section 3A]). we write

$$c_+^{-1} P_\phi^+ = Q_2 + i\tau Q_1,$$

with  $Q_2 = D_n^2 - \tau^2 |\phi'|^2 + |\xi'|^2 - c_+^{-1} |\xi_t|^2$  and  $Q_1 = \phi' D_n + D_n \phi'$ . We integrate by parts taking into account the boundary terms to find:

$$\left\| c_+^{-1} P_\phi^+ \widehat{v_+} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2 = \|Q_2 \widehat{v_+}\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^2 \|Q_1 \widehat{v_+}\|_{L^2(\mathbb{R}_+ \times Y)}^2 + i\tau ([Q_2, Q_1] \widehat{v_+}, \widehat{v_+})_+ + \tau \mathcal{B}(\widehat{v}). \quad (2.3.11)$$

With  $\mathcal{B}(\widehat{v})$  the boundary term which can be written as (see for instance [LR95] or [LRLR22, Proposition 3.24]):

$$\mathcal{B}(\widehat{v}) = 2(\phi' D_n \widehat{v}, D_n \widehat{v})_{\Sigma \cap Y} + (M_1 \widehat{v}, D_n \widehat{v})_{\Sigma \cap Y} + (M_1' D_n \widehat{v}, \widehat{v})_{\Sigma \cap Y} + (M_2 \widehat{v}, \widehat{v})_{\Sigma \cap Y},$$

where  $M_j$  is a Fourier multiplier by a polynomial of degree  $j$  in  $(\xi', \xi_t, \tau)$ . Using this as well as the Young inequality yields, for arbitrary  $\delta > 0$ :

$$|\mathcal{B}(\widehat{v}) - 2(\phi' D_n \widehat{v}, D_n \widehat{v})_{\Sigma \cap Y}| \lesssim \tau^2 |\widehat{v}|_{L^2(\Sigma \cap Y)}^2 + (1 + \delta^{-1}) |\nabla_{x'} \widehat{v}|_{L^2(\Sigma \cap Y)}^2 + \delta |D_n \widehat{v_+}|_{L^2(\Sigma \cap Y)}^2.$$

We combine this last inequality with (2.3.11), recalling that  $\phi' > 0$  in the support of  $\hat{v}$  and by choosing  $\delta$  sufficiently small to find, for  $\tau$  sufficiently large:

$$\begin{aligned} & \left\| P_\phi^+ \hat{v}_+ \right\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^3 \|\hat{v}_+\|_{L^2(\Sigma \cap Y)}^2 + \tau \|\nabla_{x'} \hat{v}_+\|_{L^2(\Sigma \cap Y)}^2 \\ & \gtrsim \|Q_2 \hat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^2 \|Q_1 \hat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + i\tau ([Q_2, Q_1] \hat{v}_+, \hat{v}_+)_+ + \tau \|D_n \hat{v}_+\|_{L^2(\Sigma \cap Y)}^2. \end{aligned} \quad (2.3.12)$$

This almost gives us the desired term: we need to take care of the commutator. In our case, this can be done in a very simple way. Indeed, one can write:

$$i[Q_2, Q_1] = B_0 Q_2 + B_1 Q_1 + B_2,$$

with  $B_j$  a Fourier multiplier by a polynomial of degree  $j$  in  $(\xi', \xi_t, \tau)$ . This implies that:

$$|i\tau (B_0 Q_2 \hat{v}_+, \hat{v}_+)| \lesssim \tau^{-\frac{1}{2}} \|Q_2 \hat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^{\frac{5}{2}} \|\hat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2,$$

as well as

$$\begin{aligned} & |i\tau (B_1 Q_1 \hat{v}_+, \hat{v}_+)| \\ & \lesssim \tau \|Q_1 \hat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^3 \|\hat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^{\frac{1}{2}} \left\| \widehat{\nabla_{x'} v_+} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau \left\| \widehat{D_t v} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2 \end{aligned}$$

and

$$|i\tau (B_2 \hat{v}_+, \hat{v}_+)| \lesssim \tau^3 \|\hat{v}_+\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau \left\| \widehat{\nabla_{x'} v_+} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau \left\| \widehat{D_t v} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2.$$

We can now inject these three estimates in the commutator term in (2.3.12) taking  $\tau \geq \tau_0$ ,  $\tau_0$  large to absorb the terms  $\|Q_1\|^2$ ,  $\|Q_1\|^2$  to finally find:

$$\begin{aligned} & \left\| \widehat{P_\phi^+ v_+} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau \left\| \widehat{D_t v_+} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau \left\| \widehat{D_t v_+(0)} \right\|_{L^2(\Sigma \cap Y)}^2 + \tau^3 \|\widehat{v_+}\|_{L^2(\mathbb{R}_+ \times Y)}^2 \\ & + \tau \left\| \widehat{\nabla_{x'} v_+} \right\|_{L^2(\mathbb{R}_+ \times Y)}^2 + \tau^3 \left\| \widehat{v(0)} \right\|_{L^2(\Sigma \cap Y)}^2 \gtrsim \tau \|D_n \widehat{v_+(0)}\|_{L^2(\Sigma \cap Y)}^2. \end{aligned} \quad (2.3.13)$$

We can now multiply (2.3.10) by a large constant and add it to (2.3.13) to conclude the proof of Lemma 2.3.7.  $\square$

### 2.3.2 End of the proof for the toy model

We finish here the proof of Proposition 2.2.6 for the constant coefficient case. One has to deal with all the possible cases and put together the estimates Section 2.3.1. We have the following *partition* of  $\mathbb{R}^n \ni (\xi', \xi_t)$ :

$$\begin{aligned} \mathbb{R}^n &= \left\{ \tau \geq \frac{1}{\sigma} (|\xi'| + |\xi_t|) \right\} \sqcup \left( \left\{ \tau < \frac{1}{\sigma} (|\xi'| + |\xi_t|) \right\} \cap \mathcal{E}_\epsilon^- \cap \mathcal{E}_\epsilon^+ \right) \sqcup \left( \left\{ \tau < \frac{1}{\sigma} (|\xi'| + |\xi_t|) \right\} \cap \mathcal{G}_\epsilon \mathcal{H}_\epsilon \right) \\ &= Y_1^\sigma \sqcup Y_2^{\sigma, \epsilon} \sqcup Y_3^{\sigma, \epsilon} \end{aligned}$$

with

$$\begin{aligned} Y_1^\sigma &:= \{\tau \geq \frac{1}{\sigma}(|\xi'| + |\xi_t|)\}, & Y_2^{\sigma,\epsilon} &:= \{\tau < \frac{1}{\sigma}(|\xi'| + |\xi_t|)\} \cap \mathcal{E}_\epsilon^- \cap \mathcal{E}_\epsilon^+, \\ Y_3^{\sigma,\epsilon} &:= \{\tau < \frac{1}{\sigma}(|\xi'| + |\xi_t|)\} \cap (\mathcal{G}_\epsilon \mathcal{H}_\epsilon^- \cup \mathcal{G}_\epsilon \mathcal{H}_\epsilon^+). \end{aligned}$$

We recall the notations/definitions of Section 2.3.1. The crucial remark is that in all of the above regions with the exception of  $\mathcal{E}^- \cap \mathcal{E}^+$  we have  $|\xi_t| \gtrsim |\xi'|$ .

In this particular toy model we are dealing with here, we work on the Fourier domain and we can simply restrict ourselves in each of those regions and prove the sought estimate. In the general case treated in Section 2.4 symbolic calculus will be used and as a result we will have to use *overlapping* regions and an associated *smooth* partition of unity. This being said we deal with the three regions above to conclude:

$\boxed{Y_1^\sigma}$  Here we just apply the result of Lemma 2.3.6 which gives:

$$\begin{aligned} & \left\| \widehat{P_\phi v} \right\|_{L^2(\mathbb{R} \times Y_1^\sigma)}^2 + \tau \left\| \widehat{D_t v_+} \right\|_{L^2(\mathbb{R}_+ \times Y_1^\sigma)}^2 + \tau \left\| \widehat{D_t v_-} \right\|_{L^2(\mathbb{R}_- \times Y_1^\sigma)}^2 \\ & \quad + \tau \left| \widehat{D_t v_+}(0) \right|_{L^2(\Sigma \cap Y_1^\sigma)}^2 + \tau \left| \widehat{D_t v_-}(0) \right|_{L^2(\Sigma \cap Y_1^\sigma)}^2 \\ & \gtrsim \tau^3 \left\| \widehat{v_+} \right\|_{L^2(\mathbb{R}_+ \times Y_1^\sigma)}^2 + \tau^3 \left| \widehat{v}(0) \right|_{L^2(\Sigma \cap Y_1^\sigma)}^2 + \tau \left| \widehat{(\nabla v_+)}(0) \right|_{L^2(\Sigma \cap Y_1^\sigma)}^2 + \tau \left| \widehat{(\nabla v_-)}(0) \right|_{L^2(\Sigma \cap Y_1^\sigma)}^2. \end{aligned} \tag{2.3.14}$$

This region fixes the choice of  $\sigma \leq \sigma_0$ .

$\boxed{Y_2^{\sigma,\epsilon}}$  In this region our operator is elliptic and one can follow the proof of [LRL13]. Indeed here one has an elliptic factorization on both sides of the interface. That is

$$c_+^{-1} P_\phi^+ v_+ = (D_n + i\tau\phi' + im_+)(D_n + i\tau\phi' - im_+)v_+, \quad \phi' = \alpha_+ + \beta x_n,$$

and

$$c_-^{-1} P_\phi^- v_- = (D_n + i\tau\phi' + im_-)(D_n + i\tau\phi' - im_-)v_-, \quad \phi' = \alpha_- + \beta x_n.$$

with

$$m_\pm = \sqrt{|\xi'|^2 - c_\pm^{-1} \xi_t^2} \gtrsim |\xi'| + |\xi_t|.$$

Consequently the arguments used in [LRL13] can be used in this microlocal region as well. We refer to Lemma 2.4.11 for a rigorous proof in the general case. Notice that this region forces:

$$\frac{\alpha_+}{\alpha_-} > \sup_{x_n=0} \frac{m_+}{m_-}.$$

We obtain the following estimate:

$$\begin{aligned} & \left\| \widehat{P_\phi v} \right\|_{L^2(\mathbb{R} \times Y_2^{\sigma,\epsilon})}^2 + \tau \left\| \widehat{D_t v_+} \right\|_{L^2(\mathbb{R}_+ \times Y_2^{\sigma,\epsilon})}^2 + \tau \left\| \widehat{D_t v_-} \right\|_{L^2(\mathbb{R}_- \times Y_2^{\sigma,\epsilon})}^2 \\ & \quad + \tau \left| \widehat{D_t v_+}(0) \right|_{L^2(\Sigma \cap Y_2^{\sigma,\epsilon})}^2 + \tau \left| \widehat{D_t v_-}(0) \right|_{L^2(\Sigma \cap Y_2^{\sigma,\epsilon})}^2 \\ & \gtrsim \tau^3 \left\| \widehat{v_+} \right\|_{L^2(\mathbb{R}_+ \times Y_2^{\sigma,\epsilon})}^2 + \tau^3 \left| \widehat{v}(0) \right|_{L^2(\Sigma \cap Y_2^{\sigma,\epsilon})}^2 + \tau \left| \widehat{(\nabla v_+)}(0) \right|_{L^2(\Sigma \cap Y_2^{\sigma,\epsilon})}^2 + \tau \left| \widehat{(\nabla v_-)}(0) \right|_{L^2(\Sigma \cap Y_2^{\sigma,\epsilon})}^2. \end{aligned} \tag{2.3.15}$$

$\boxed{Y_3^{\sigma,\epsilon}}$  Here we are in the situation of Lemma 2.3.7. We obtain:

$$\begin{aligned} & \left\| \widehat{P_\phi v} \right\|_{L^2(\mathbb{R} \times Y_3^{\sigma,\epsilon})}^2 + \tau \left\| \widehat{D_t v_+} \right\|_{L^2(\mathbb{R}_+ \times Y_3^{\sigma,\epsilon})}^2 + \tau \left\| \widehat{D_t v_-} \right\|_{L^2(\mathbb{R}_- \times Y_3^{\sigma,\epsilon})}^2 \\ & \quad + \tau \left| \widehat{D_t v_+}(0) \right|_{L^2(\Sigma \cap Y_3^{\sigma,\epsilon})}^2 + \tau \left| \widehat{D_t v_-}(0) \right|_{L^2(\Sigma \cap Y_3^{\sigma,\epsilon})}^2 \\ & \gtrsim \tau^3 \left\| \widehat{v_+} \right\|_{L^2(\mathbb{R}_+ \times Y_3^{\sigma,\epsilon})}^2 + \tau^3 \left| \widehat{v(0)} \right|_{L^2(\Sigma \cap Y_3^{\sigma,\epsilon})}^2 + \tau \left| \widehat{(\nabla v_+)}(0) \right|_{L^2(\Sigma \cap Y_3^{\sigma,\epsilon})}^2 + \tau \left| \widehat{(\nabla v_-)}(0) \right|_{L^2(\Sigma \cap Y_3^{\sigma,\epsilon})}^2. \end{aligned} \quad (2.3.16)$$

*End of the proof of proposition 2.2.6 for the toy model.* We can now finish our proof in this particular setting by simply putting together the results of the three above regions. Indeed, adding (2.3.14), (2.3.15) and (2.3.16) yield:

$$\begin{aligned} & \left\| \widehat{P_\phi v} \right\|_{L^2(\mathbb{R} \times \mathbb{R}^n)}^2 + \tau \left\| \widehat{D_t v_+} \right\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^n)}^2 + \tau \left\| \widehat{D_t v_-} \right\|_{L^2(\mathbb{R}_- \times \mathbb{R}^n)}^2 \\ & \quad + \tau \left| \widehat{D_t v_+}(0) \right|_{L^2(\Sigma)}^2 + \tau \left| \widehat{D_t v_-}(0) \right|_{L^2(\Sigma)}^2 \\ & \gtrsim \tau^3 \left\| \widehat{v} \right\|_{L^2(\mathbb{R} \times \mathbb{R}^n)}^2 + \tau^3 \left| \widehat{v(0)} \right|_{L^2(\Sigma)}^2 + \tau \left| \widehat{(\nabla v_+)}(0) \right|_{L^2(\Sigma)}^2 + \tau \left| \widehat{(\nabla v_-)}(0) \right|_{L^2(\Sigma)}^2, \end{aligned}$$

which using the Plancherel theorem translates to,

$$\begin{aligned} & \left\| P_\phi v \right\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \left\| H_+ D_t v_+ \right\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \left\| H_- D_t v_- \right\|_{L^2(\mathbb{R}^{n+1})}^2 \\ & \quad + \tau \left| (D_t v_+) \right|_{L^2(\Sigma)}^2 + \tau \left| (D_t v_-) \right|_{L^2(\Sigma)}^2 \\ & \gtrsim \tau^3 \left\| v \right\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau^3 \left| v \right|_{L^2(\Sigma)}^2 + \tau \left| (\nabla v_+) \right|_{L^2(\Sigma)}^2 + \tau \left| (\nabla v_-) \right|_{L^2(\Sigma)}^2. \end{aligned} \quad (2.3.17)$$

Notice that the only missing term from (2.3.17) is the volume norm of the gradients. Proposition 2.2.6 is then a result of (2.3.17) and Lemma 2.3.3.  $\square$

## 2.4 Proof of Proposition 2.2.6 for the general case

### 2.4.1 Notation, microlocal regions and first estimates

The proof for the general case uses essentially the ideas introduced in Section 2.3. The main difference is that one needs to consider this time sub-regions of the whole phase space and not only of the frequency space. To do so one needs to use some microlocal analysis tools. We refer to Appendix 2.A for some basic properties that will be used in the sequel.

We first define, for  $m \in \mathbb{R}$  the class of standard *tangential* smooth symbols  $\mathcal{S}^m$ . These are the functions  $a \in C^\infty(t, x, \xi_t, \xi')$  satisfying for all  $(\alpha, \beta) \in \mathbb{N}^{n+1} \times \mathbb{N}^n$ :

$$\sup_{(t,x,\xi_t,\xi')} (1 + |\xi'|^2 + |\xi_t|^2)^{\frac{-m+|\beta|}{2}} \left| (\partial_{t,x}^\alpha \partial_{\xi_t,\xi'}^\beta) a(t, x, \xi_t, \xi') \right| < \infty.$$

We will also work in the class of smooth tangential symbols depending on a large parameter. This class will be denoted by  $\mathcal{S}_\tau^m$  and contains the functions  $a \in C^\infty(t, x, \xi_t, \xi', \tau)$  satisfying for all  $(\alpha, \beta) \in \mathbb{N}^{n+1} \times \mathbb{N}^n$ :

$$\sup_{\substack{(t,x,\xi_t,\xi') \\ \tau \geq 1}} (\tau^2 + |\xi'|^2 + |\xi_t|^2)^{\frac{-m+|\beta|}{2}} \left| (\partial_{t,x}^\alpha \partial_{\xi_t,\xi'}^\beta) a(t, x, \xi_t, \xi', \tau) \right| < \infty.$$

To an element  $a$  of  $\mathcal{S}^m$  we associate an operator  $\text{op}^w(a) \in \Psi^m$ <sup>1</sup>, which is an element of the class of *tangential* pseudodifferential operators. Notice that to alleviate our notation we do not use the tangential notation for the symbols, since all of the symbols we will consider will be tangential. The same remark applies to the notation  $\Psi^m$  which refers to tangential operators even though it is not explicit in the notation. We have the analogous notation for

$$\Psi_\tau^m = \text{op}^w(\mathcal{S}_\tau^m).$$

Let us denote  $\lambda^2 := 1 + |\xi'|^2 + |\xi_t|^2$  and  $\lambda_\tau^2 = \tau^2 + |\xi'|^2 + |\xi_t|^2$ . We introduce the following Sobolev norms, defined in the tangential variables:

$$|u(x_n, \cdot)|_{H^s} = |\text{op}^w(\lambda^s)u(x_n, \cdot)|_{L^2(\mathbb{R}^n)}, \quad |u(x_n, \cdot)|_{H_\tau^s} = |\text{op}^w(\lambda_\tau^s)u(x_n, \cdot)|_{L^2(\mathbb{R}^n)}.$$

**Remark 2.4.1.** In the sequel we will have to consider symbols  $a \in \mathcal{S}^m$  independent of  $\tau$ . However the natural symbol class for our Carleman estimate is  $\mathcal{S}_\tau^m$ . Lemma 2.A.2 provides with a sufficient condition for making use of pseudodifferential calculus mixing operators in  $\Psi^m$  and  $\Psi_\tau^m$ .

**Remark 2.4.2.** Of crucial importance is the fact that the Carleman estimate we are seeking to prove (in fact most Carleman estimates in general) is insensitive to perturbations with respect to elements of the class

$$\Psi^1 + \tau\Psi^0 + \Psi^0 D_n,$$

up to taking even larger values for our parameter  $\tau$ . Let us briefly recall why. Suppose that Proposition 2.2.6 is proved for  $P_\phi$  and consider  $T \in \Psi^1 + \tau\Psi^0 + \Psi^0 D_n$ , that is  $T = S_1 + \tau S_0 + \tilde{S}_0 D_n$  for  $S_j$  tangential operators of order  $j$ . Since  $\|(P_\phi + T)v\|_{L^2(\mathbb{R}^{n+1})} \leq \|P_\phi v\|_{L^2(\mathbb{R}^{n+1})} + \|Tv\|_{L^2(\mathbb{R}^{n+1})}$  one simply needs to show that  $\|Tv\|_{L^2(\mathbb{R}^{n+1})}$  can be absorbed in the right hand side of our estimate. By Sobolev regularity of the pseudo differential calculus one has:

$$\|S_1 v\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|v\|_{L^2(\mathbb{R}; H^1)},$$

which yields

$$\tau^{\frac{3}{2}} \|v\|_{L^2(\mathbb{R}^{n+1})} + \tau \|\nabla_{x'} v\|_{L^2(\mathbb{R}^{n+1})} - \|S_1 v\|_{L^2(\mathbb{R}^{n+1})} \gtrsim \tau^{\frac{3}{2}} \|v\|_{L^2(\mathbb{R}^{n+1})} + \tau \|\nabla_{x'} v\|_{L^2(\mathbb{R}^{n+1})}, \quad \forall \tau \geq \tau_0$$

for  $\tau$  sufficiently large. Similarly, using

$$\|\tau S_0 v\|_{L^2(\mathbb{R}^{n+1})} \lesssim \tau \|v\|_{L^2(\mathbb{R}^{n+1})}, \quad \|\tilde{S}_0 D_n v\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|D_n v\|_{L^2(\mathbb{R}^{n+1})},$$

we see that the perturbation is absorbed in our estimate. We can therefore from now on replace without any loss of generality our operator  $P_\phi$  by an element of  $P_\phi + \Psi^1 + \tau\Psi^0 + \Psi^0 D_n$ . Remark as well that if  $L \in \mathcal{D}^1$  is a differential operator of order one then  $L_\phi = e^{\tau\phi} L e^{-\tau\phi} \in \Psi^1 + \tau\Psi^0 + \Psi^0 D_n$ .

<sup>1</sup>Notice that in our definition we use the Weyl quantization and not the standard one.

Recall that we are working in the local setting of Section 2.2.2 with the operator:

$$P_2^\pm = -D_t^2 + c_\pm(x)D_n^2 + c_\pm(x)Q(x, D_{x'}),$$

with

$$Q(x, \xi') := \sum_{1 \leq j, k \leq n-1} b_{jk}(x) \xi_j \xi_k \quad (2.4.1)$$

and  $b_{jk}$  satisfying

$$b_1 |\xi'|^2 \leq Q(x, \xi') \leq b_2 |\xi'|^2, \quad b_1, b_2 > 0.$$

The conjugated operator is given then by

$$P_\phi = H_- P_\phi^- + H_+ P_\phi^+,$$

where

$$c_\pm^{-1}(x) P_\phi^\pm = (D_n + i\tau\phi')^2 + Q(x, D_{x'}) - c_\pm^{-1}(x) D_t^2.$$

We now consider the analog of the microlocal regions used in the toy model, for  $\epsilon > 0$  small to be chosen:

$$\mathcal{E}_\epsilon^\bullet := \{(t, x, \xi', \xi_t) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R} \text{ such that } Q(x, \xi') - c_\bullet^{-1}(x) \xi_t^2 \geq \epsilon(|\xi'|^2 + |\xi_t|^2)\}, \quad (2.4.2)$$

$$\mathcal{GH}_\epsilon^\bullet := \{(t, x, \xi', \xi_t) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R} \text{ such that } Q(x, \xi') - c_\bullet(x) \xi_t^2 \leq 2\epsilon(|\xi'|^2 + |\xi_t|^2)\}. \quad (2.4.3)$$

Notice that for  $\bullet = +$  or  $\bullet = -$ ,  $\mathcal{E}_\epsilon^\bullet$  and  $\mathcal{GH}_\epsilon^\bullet$  overlap and are conic in  $(\xi_t, \xi')$  which will allow to construct an associated partition of unity. Another important remark is that whenever we are in the regions  $\mathcal{GH}_\epsilon^\bullet$  we have (for  $\epsilon$  sufficiently small) thanks to the ellipticity of  $b_{jk}$  that  $|\xi_t| \gtrsim |\xi'|$ .

The Carleman estimate we want to show (see Proposition 2.2.6) concerns functions  $v$  supported in a compact set  $K \subset V$ . However the space of compactly supported functions is not stable by pseudo differential operators. Let us denote by  $\pi_{t,x}$  the projection in the physical space of an element of  $\mathbb{R}_t \times \mathbb{R}_x \times \mathbb{R}_{\xi_t} \times \mathbb{R}_{\xi'}$ . The natural space in which we will be working is the Schwartz space  $\mathcal{S}$  and we will use cut-off functions  $\chi$  satisfying  $\pi_{t,x}(\text{supp } \chi) \subset K$ . That is the projection of their support on the physical space will be contained on a compact set  $K$ . This will allow us to suppose that  $(t, x)$  lies on a compact set. Indeed, if we consider an additional cut-off function  $\tilde{\chi} = \tilde{\chi}(t, x)$  to the left of our operator, with  $\text{supp } \tilde{\chi} \subset K$  and  $\tilde{\chi} = 1$  on  $\pi_{t,x}(\text{supp } \chi)$  then we have for  $u \in \mathcal{S}(\mathbb{R}^{n+1})$ :

$$\|P_\phi \text{op}^w(\chi)u\|_{L^2(\mathbb{R}^{n+1})} \geq \|\tilde{\chi} P_\phi \text{op}^w(\chi)u\|_{L^2(\mathbb{R}^{n+1})} - \|(1 - \tilde{\chi}) P_\phi \text{op}^w(\chi)u\|_{L^2(\mathbb{R}^{n+1})},$$

since  $(1 - \tilde{\chi}) P_\phi \text{op}^w(\chi) \in \Psi_\tau^{-\infty}$  for  $\chi \in \mathcal{S}_\tau^0$ , the last term above yields an error term which can be absorbed in our estimate. Up to replacing  $P_\phi$  by  $\tilde{\chi} P_\phi$  we can indeed suppose that  $(t, x)$  lies on a compact set of  $\mathbb{R}^{n+1}$ .

We shall work in the space

$$\mathcal{S}_c = \{u \in \mathcal{S}(\mathbb{R}^{n+1}); \text{ there exists } \eta > 0, \text{ sup } u \subset \mathbb{R}_t \times \mathbb{R}_{x'}^{n-1} \times (-\eta, \eta)_{x_n}\}.$$



Since all of the pseudo differential operators we consider are tangential, the support of a function in the  $x_n$  direction is preserved and the above space is therefore stable by application of pseudodifferential operators in  $\Psi^m$  or  $\Psi_\tau^m$ .

In the sequel, the implicit constants may depend on the coefficients  $b_{jk}, c_\pm$  of  $P_\phi$ , on the coefficients of  $\phi$  ( $\alpha_\pm, \beta$ ) and they may also depend on  $\epsilon$ . However the value of  $\epsilon$  will be a small fixed value depending on  $\alpha_\pm, b_{jk}$  and  $c_\pm$ . More precisely,  $\epsilon$  is fixed by the above remark guaranteeing that being in the regions  $\mathcal{GH}_\epsilon^\pm$  implies that  $|\xi_t| \gtrsim |\xi'|$ . Once the value of  $\epsilon$  has been fixed in this way, the implicit constants depend on  $b_{jk}, c_\pm, \alpha_\pm, \beta$ . The choice of the coefficients  $\alpha_\pm$  and  $\beta$  is done in Lemma 2.4.11. We take  $\alpha_\pm$  such that the geometric condition (2.4.32) is satisfied and  $\beta$  large such that the sub-ellipticity condition (2.4.31) is satisfied.

Let us recover some of the basic estimates of Section 2.3. Recall the definition of the space  $\mathcal{W}_\phi^{\theta, \Theta}$  as well as the setting given in Section 2.2.2. In particular, elements of  $\mathcal{W}_\phi^{\theta, \Theta}$  have small support contained in  $V_t$ . We start with the lemma giving the trace of the normal derivatives modulo the surface norm:

**Lemma 2.4.3.** *Let  $\tilde{v} = H_- \tilde{v}_- + H_+ \tilde{v}_+ \in \mathcal{W}_\phi^{\theta, \Theta}$  and suppose that  $\chi \in \mathcal{S}^0$  or that  $\chi \in \mathcal{S}_\tau^0$ . Consider  $v_\pm = \text{op}^w(\chi) \tilde{v}_\pm$ . Then one has:*

$$\tau |D_n v_\pm|_{L^2(\Sigma)}^2 \lesssim \tau |D_n v_\mp|_{L^2(\Sigma)}^2 + \tau^3 |v_+|_{L^2(\Sigma)}^2 + \tau^3 |v_-|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} + |\tilde{v}_\mp|_{L^2(\Sigma)}^2.$$

*Proof.* This is a result of the transmission conditions and of the fact that  $\text{op}^w(\chi)$  is a tangential operator. This implies that it commutes with restriction on  $\Sigma$  (here restriction on  $x_n = 0$ ). We use then (2.2.14) and (2.2.15). The first one allows to write  $v_-(0) = v_+(0) + \text{op}^w(\chi) \theta_\phi$  and (2.2.15) reads

$$D_n v_-(0) = \frac{c_+(x)}{c_-(x)} (D_n v_+(0) + i\tau \alpha_+ v_+(0)) - i\tau \alpha_- v_-(0) + \frac{1}{c_-(x)} \text{op}^w(\chi) \Theta_\phi + [D_n, \text{op}^w(\chi)] \tilde{v}_+.$$

Since  $|[D_n, \text{op}^w(\chi)] \tilde{v}_\pm| = |\text{op}^w(\partial_{x_n} \chi) \tilde{v}_\pm|$ , using the fact that  $0 < c_{\min} < c(x) < c_{\max}$  we find that there exist constants  $C_1, C_2$  and  $C_3$  depending on  $c_\pm$  and  $\alpha_\pm$  such that

$$\tau |D_n v_-|_{L^2(\Sigma)}^2 \leq C_1 \tau |D_n v_+|_{L^2(\Sigma)}^2 + C_2 \tau^3 |v_\pm|_{L^2(\Sigma)}^2 + C_3 T_{\theta, \Theta} + |\tilde{v}_\pm|_{L^2(\Sigma)}^2,$$

where we have used the fact that since  $\chi \in \mathcal{S}^0$  one has  $\|\text{op}^w(\chi)\|_{L^2 \rightarrow L^2} \lesssim 1$ . This gives one of the two desired inequalities and we can get the second one using again (2.2.15).  $\square$

Recall that  $E_t(v)$  has been defined in (2.2.16). The following lemma allows to obtain the volume norms of the derivatives modulo the remainder of the terms. This corresponds to Lemma 2.3.3 in the toy model of Section 2.3.

**Lemma 2.4.4.** *There exists  $\tau_0 > 0$  such that for  $v \in \mathcal{W}_\phi^{\theta, \Theta}$  and  $\tau \geq \tau_0$  we have:*

$$\begin{aligned} & \|H_- P_\phi v_-\|_{L^2(\mathbb{R}^{n+1})}^2 + \|H_+ P_\phi v_+\|_{L^2(\mathbb{R}^{n+1})}^2 + E_t(v) + \tau^3 \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 \\ & + \tau^2 |v_-(0)|_{L^2(\Sigma)}^2 + \tau^2 |v_+(0)|_{L^2(\Sigma)}^2 + |D_n v_+(0)|_{L^2(\Sigma)}^2 + |D_n v_-(0)|_{L^2(\Sigma)}^2 \\ & \gtrsim \tau \|\nabla v_-\|_{L^2(\mathbb{R}_-^{n+1})}^2 + \tau \|\nabla v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2. \end{aligned}$$

*Proof.* We start with the positive half-space  $\mathbb{R}_+^{n+1}$ . One has the following elementary inequality:

$$\left\| P_\phi^+ v_+ \right\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^2 \|v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 \geq 2\tau \operatorname{Re}(P_\phi^+ v_+, H_+ v_+).$$

Now we can integrate by parts. Indeed, recall the form of our operator:

$$c_+^{-1}(x)P_\phi^+ = (D_n^2 + \tau\beta + 2i\tau\phi'D_n - \tau^2|\phi'|^2) + Q(x, D_{x'}) - c_+^{-1}(x)D_t^2$$

and that the definition of the weight function  $\phi$  gives  $\phi'(x_n) = \alpha_\pm + \beta x_n$ , therefore:

$$\begin{aligned} 2\tau \operatorname{Re}(P_\phi^+ v, H_+ v) &\gtrsim 2\tau \|D_n v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + 2\tau \int_{\mathbb{R}^n} \operatorname{Re}(\partial_{x_n} v_+(0) \bar{v}_+(0)) dx' dt \\ &\quad + 2\tau^2 \beta \|v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 - 2\tau^2 \alpha_+ |v_+|_{L^2(\Sigma)}^2 - 2\tau^2 \beta \|v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 \\ &\quad - 2\tau^3 \|\phi' v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + 2C\tau \|D' v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 - 2C\tau \|D_t v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2. \end{aligned}$$

Here we used the fact that  $c$  is bounded as well as the ellipticity of  $b_{jk}$ . Indeed, up to adding an element of  $\Psi^1$  (which does not have any influence on the estimate we are seeking to prove, see Remark 2.4.2) we can replace  $Q(x, D_{x'})$  by  $-\operatorname{div}(B \cdot \nabla_{x'})$  with  $B = (b_{jk})_{1 \leq j, k \leq n-1}$  which satisfies

$$(-\operatorname{div}(B \cdot \nabla_{x'}) v, v)_+ \gtrsim \|\nabla_{x'} v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2.$$

As one has

$$\left| 2\tau \int_{\mathbb{R}^n} \operatorname{Re}(\partial_{x_n} v_+(0) \bar{v}_+(0)) dx' dt \right| \lesssim |D_n v_+|_{L^2(\Sigma)}^2 + \tau^2 |v_+|_{L^2(\Sigma)}^2,$$

the above inequality can be written as

$$2\tau \operatorname{Re}(P_\phi^+ v, H_+ v)_{L^2(\mathbb{R}_+^{n+1})} \gtrsim \tau \|\nabla v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + R,$$

with

$$|R| \lesssim \tau^3 \|v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|D_t v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^2 |v_+|_{L^2(\Sigma)}^2 + |D_n v_+|_{L^2(\Sigma)}^2.$$

which gives the sought result. Since the proof above is insensitive with respect to the sign of the boundary terms coming from the integration by parts we also obtain the desired inequality on the negative half-space  $\mathbb{R}_-^{n+1}$ .  $\square$

When we are in a microlocal region where  $|\xi_t|$  is large compared to  $\tau$  and  $|\xi'|$  we automatically have a very good estimate:

**Lemma 2.4.5.** *Let  $c_0 > 0$  and  $\chi \in \mathcal{S}_\tau^0$  with  $\operatorname{supp}(\chi) \subset V \times \{|\xi_t| \geq c_0(\tau + |\xi'|)\}$ . Then there exists  $\tau_0$  such that:*

$$\begin{aligned} \tau \|D_t v\|_{L^2(\mathbb{R}_\pm^{n+1})}^2 + \|u\|_{L^2(\mathbb{R}_\pm^{n+1})}^2 + |u|_{L^2(\Sigma)}^2 + \tau |D_t v|_{L^2(\Sigma)}^2 \\ \gtrsim \tau \|v\|_{L^2(\mathbb{R}_\pm; H_\pm^1)}^2 + \tau^3 |v|_{L^2(\Sigma)}^2 + \tau \|\nabla_{x'} v\|_{L^2(\mathbb{R}_\pm^{n+1})}^2 + \tau |\nabla_{x'} v|_{L^2(\Sigma)}^2, \end{aligned}$$

for  $\tau \geq \tau_0$ ,  $u \in \mathcal{S}_c(\mathbb{R}^{n+1})$  and  $v = \operatorname{op}^w(\chi)u$ .

**Remark 2.4.6.** Notice that thanks to Lemma 2.A.2 the support assumption on  $\chi$  implies that if  $\chi_0$  is an element of  $\mathcal{S}^0$  independent of  $\tau$  one has in fact that  $\text{op}^w(\chi_0)\text{op}^w(\chi) \in \Psi_\tau^0$ .

*Proof.* Let  $0 \leq \tilde{\chi} \leq 1$  satisfy the same properties as  $\chi$  with moreover  $\tilde{\chi} = 1$  on  $\text{supp } \chi$ . Defining

$$\mathcal{S}_\tau^2 \ni \tilde{a} := \xi_t^2 \tilde{\chi} + (1 - \tilde{\chi})\lambda_\tau^2,$$

we notice that  $\tilde{a} \gtrsim \lambda_\tau^2$ . Let  $a := \xi_t^2$  and remark that  $\text{op}^w(a)\text{op}^w(\chi) = \text{op}^w(\tilde{\chi}a)\text{op}^w(\chi) + R$  with  $R = \text{op}^w((1 - \tilde{\chi})a)\text{op}^w(\chi) \in \Psi_\tau^{-\infty}$ , since  $\tilde{\chi} = 1$  on  $\text{supp } \chi$ . We then obtain:

$$\begin{aligned} & (\text{op}^w(a)\text{op}^w(\chi)u, \text{op}^w(\chi)u)_+ \\ &= (\text{op}^w(\tilde{a})\text{op}^w(\chi)u, \text{op}^w(\chi)u)_+ - (\text{op}^w((1 - \tilde{\chi})\lambda_\tau^2)\text{op}^w(\chi)u, \text{op}^w(\chi)u)_+ + (Ru, \text{op}^w(\chi)u)_+ \end{aligned} \quad (2.4.4)$$

The last two terms yield an operator in  $\Psi_\tau^{-\infty}$  which implies in particular

$$| - (\text{op}^w((1 - \tilde{\chi})\lambda_\tau^2)\text{op}^w(\chi)u, \text{op}^w(\chi)u)_+ + (Ru, \text{op}^w(\chi)u)_+ | \lesssim \tau^{-1} \|u\|_{L^2(\mathbb{R}_+^{n+1})}^2.$$

We now can apply Gårding's inequality in the context of tangential pseudodifferential calculus with a large parameter (Lemma 2.A.1) to the first term to find (recall that  $v = \text{op}^w(\chi)u$ ):

$$\text{Re}(\text{op}^w(\tilde{a})v, v)_+ \gtrsim \|v\|_{L^2(\mathbb{R}_+; H_\tau^1)}^2.$$

The estimates above combined with the equality (2.4.4) yield

$$\|D_t v\|_{L^2(\mathbb{R}_+^{n+1})}^2 = (\text{op}^w(a)v, v)_+ \gtrsim \|v\|_{L^2(\mathbb{R}_+; H_\tau^1)}^2 - \tau^{-1} \|u\|_{L^2(\mathbb{R}_+^{n+1})}^2,$$

We multiply the above estimate by  $\tau$  and write the  $H_\tau^1$  norm as

$$\|\cdot\|_{H_\tau^1}^2 \sim \tau^2 \|\cdot\|_{L^2}^2 + \|\nabla \cdot\|_{L^2}^2,$$

to deduce

$$\tau \|D_t v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \|u\|_{L^2(\mathbb{R}_+^{n+1})}^2 \gtrsim \tau^3 \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|\nabla_{x'} v\|_{L^2(\mathbb{R}_+^{n+1})}^2.$$

In a similar fashion, using the same notation we apply Gårding's inequality on  $\Sigma$  (which here is  $\mathbb{R}^n$ ). Notice that since the operators considered above are tangential they commute with the restriction on  $\Sigma$  and we find

$$\text{Re}(\text{op}^w(\tilde{a})v, v)_\Sigma \gtrsim |v|_{H_\tau^1}^2 - \tau^{-1} |u|_{L^2(\Sigma)}^2,$$

which implies in particular

$$\tau |D_t v|_{L^2(\Sigma)}^2 + |u|_{L^2(\Sigma)}^2 \gtrsim \tau^3 |v|_{L^2(\Sigma)}^2 + \tau |\nabla_{x'} v|_{L^2(\Sigma)}^2.$$

We obtain the same estimates when integrating in the negative half-line.  $\square$

We shall now show the desired estimate when micro-localized in a region where

$$\tau \gg |\xi_t| + |\xi'|.$$

From an heuristic point of view, in a such a region  $(D_n + i\tau\phi')^2$  is the most important term of  $P_\phi$ . Since  $D_n + i\tau\phi'$  is elliptic as an 1D operator we expect that this will give a good estimate. Let us define (as in [LRL13])  $\psi \in C^\infty(\mathbb{R})$  nonnegative with  $\psi = 0$  in  $[0, 1]$  and  $\psi = 1$  in  $[2, +\infty)$  and then

$$\psi_\sigma(\tau, \xi_t, \xi') := \psi \left( \frac{\sigma\tau}{(1 + |\xi_t|^2 + |\xi'|^2)^{\frac{1}{2}}} \right) \in \mathcal{S}_\tau^0, \quad (2.4.5)$$

with  $\sigma$  small to be chosen. The choice of  $\psi$  implies that  $\tau \geq (|\xi_t|^2 + |\xi'|^2)^{1/2}/\sigma$  on the support of  $\psi_\sigma$ . We have the following lemma:

**Lemma 2.4.7.** *There exists  $\sigma_0, \tau_0$  and  $\mathcal{V} \Subset V_t$  such that for  $0 < \sigma < \sigma_0$  we have:*

$$\begin{aligned} & \|H_- P_\phi v_-\|_{L^2(\mathbb{R}^{n+1})}^2 + \|H_+ P_\phi v_+\|_{L^2(\mathbb{R}^{n+1})}^2 + \|u\|_{L^2(\mathbb{R}^{n+1})}^2 + |u_+|_{L^2(\Sigma)}^2 + |u_-|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \\ & \gtrsim \tau \|v\|_{L^2(\mathbb{R}; H_\tau^\perp)}^2 + \tau^3 |v_+|_{L^2(\Sigma)}^2 + \tau^3 |v_-|_{L^2(\Sigma)}^2 + \tau |\nabla v_+|_{L^2(\Sigma)}^2 + \tau |\nabla v_-|_{L^2(\Sigma)}^2, \end{aligned}$$

for all  $\tau \geq \tau_0$ ,  $u \in \mathcal{W}_\phi^{\theta, \Theta}$  with  $\text{supp } u \subset \mathcal{V}$  and  $v = \text{op}^w(\psi_\sigma)u$ .

**Remark 2.4.8.** In fact here we obtain a slightly better estimate (without the loss of a half-derivative in the volume norm). But we prefer to state it like this since that is how we use it in view of the final estimate, when all regions are put together.

*Proof.* Recall that we write  $u = H_- u_- + H_+ u_+$  and  $v_\pm = \text{op}^w(\psi_\sigma)u_\pm$ . We start by working in the positive half-space  $\mathbb{R}_+^{n+1}$ . As one could expect this is where we obtain most terms of the sought estimate, since it is the observation region.

We write:

$$c_+^{-1}(x)P_\phi^+ = (D_n + i\tau\phi')^2 + Q(x, D_{x'}) - c_+^{-1}D_t^2 := A_+ + R_+,$$

where we have defined  $A_+ := (D_n + i\tau\phi')^2$  and  $R_+ := Q(x, D_{x'}) - c_+^{-1}D_t^2$ . One has with  $w = (D_n + i\tau\phi')v_+ = (D_n + i\tau\alpha_+ + i\tau\beta x_n)v_+$ , using (2.3.4) twice:

$$\begin{aligned} \|A_+ v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 &= \|(D_n + i\tau\alpha_+ + i\tau\beta x_n)w\|_{L^2(\mathbb{R}_+^{n+1})}^2 \\ &\geq \|(\tau\beta x_n + \tau\alpha_+)w\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \alpha_+ \tau |w|_{L^2(\Sigma)}^2 \\ &\gtrsim \tau^2 \left( \|D_n v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \|(\tau\alpha_+ + \tau\beta x_n)v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau\alpha_+ |v_+|_{L^2(\Sigma)}^2 \right) \\ &\quad + \alpha_+ \tau |(D_n + i\tau\alpha)v_+|_{L^2(\Sigma)}^2. \end{aligned}$$

In particular using the fact that  $\alpha_+ > 0$  and taking  $|x_n|$  sufficiently small we find that for  $u$  supported in a small neighborhood  $\mathcal{V} \Subset V_t$  one has

$$\|A_+ v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 \gtrsim \tau^4 \|v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^3 |v_+|_{L^2(\Sigma)}^2 + \tau |(D_n + i\tau\alpha)v_+|_{L^2(\Sigma)}^2.$$

We deduce then

$$\|A_+ v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 \gtrsim \tau^4 \|v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^3 |v_+|_{L^2(\Sigma)}^2 + \tau |D_n v_+|_{L^2(\Sigma)}^2.$$

Recalling that  $A_+ = c^{-1}P_\phi^+ - R_+$  we obtain

$$\begin{aligned} \left\| P_\phi^+ v_+ \right\|_{L^2(\mathbb{R}_+^{n+1})}^2 &\geq C \|A_+ v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 - \|R_+ v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 \\ &\geq C \left( \tau^4 \|v_+\|_{L^2(\mathbb{R}^+)}^2 + \tau^3 |v_+|_{L^2(\Sigma)}^2 + \tau |D_n v_+|_{L^2(\Sigma)}^2 \right) - \|R_+ v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 \\ &= \left( (C\tau^4 - R_+^* R_+) v_+, v_+ \right)_+ + C\tau^3 |v_+|_{L^2(\Sigma)}^2 + C\tau |D_n v_+|_{L^2(\Sigma)}^2, \end{aligned} \quad (2.4.6)$$

with  $C$  positive constant depending on the coefficients of  $P_\phi$  and of  $\phi$ . Observe now that the principal symbol  $r_+^2 = \left( Q(x, \xi') - c_+^{-1}(x) |\xi_t|^2 \right)^2 = \left( \sum_{1 \leq j, k \leq n-1} b_{jk}(x) \xi_j \xi_k - c_+^{-1}(x) |\xi_t|^2 \right)^2$  of  $R_+^* R_+ \in \Psi^4$  satisfies

$$r_+^2 \leq \max(b_2, c_{\min}^{-1}) (|\xi_t|^2 + |\xi'|^2)^2.$$

Using the fact that

$$\tau \geq \frac{1}{\sigma} (1 + |\xi_t|^2 + |\xi'|^2)^{\frac{1}{2}},$$

on the support of  $\psi_\sigma$  one obtains the existence of  $\sigma_0$  sufficiently small depending on the coefficients of  $P_\phi$  and of  $\phi$  such that for all  $0 < \sigma \leq \sigma_0$ :

$$C\tau^4 - r_+^2 \geq \lambda_\tau^4, \quad (2.4.7)$$

on the support of  $\psi_\sigma$ . We consider now  $\tilde{\psi} \in C^\infty(\mathbb{R}^+)$  with  $\tilde{\psi} = 1$  in  $[1/3, \infty)$  and  $\tilde{\psi} = 0$  in  $[0, 1/4]$  and then define  $\tilde{\psi}_\sigma$  similarly to  $\psi_\sigma$ :

$$\tilde{\psi}_\sigma(\tau, \xi_t) := \tilde{\psi} \left( \frac{\sigma\tau}{(1 + |\xi_t|^2 + |\xi'|^2)^{\frac{1}{2}}} \right) \in \mathcal{S}_\tau^0.$$

We write:

$$\begin{aligned} \left( (C\tau^4 - R_+^* R_+) v_+, v_+ \right) &= \left( (C\tau^4 - R_+^* R_+) \text{op}^w(\tilde{\psi}_\sigma) v_+, v_+ \right)_+ \\ &\quad + \left( (C\tau^4 - R_+^* R_+) (\text{op}^w(1 - \tilde{\psi}_\sigma) v_+, v_+ \right)_+. \end{aligned} \quad (2.4.8)$$

Observe that  $\tilde{\psi}_\sigma = 1$  on the support of  $\psi_\sigma$  and that  $\tau \lesssim |\xi'| + |\xi_t|$  on the support of  $1 - \tilde{\psi}_\sigma$  and this gives thanks to Lemma 2.A.2 that in fact  $\text{op}^w(1 - \tilde{\psi}_\sigma) \text{op}^w(\psi_\sigma) \in \Psi_\tau^{-\infty}$ .

As a consequence, we have in particular:

$$\left| \left( (C\tau^4 - R_+^* R_+) (\text{op}^w(1 - \tilde{\psi}_\sigma) v_+, v_+ \right)_+ \right| \lesssim \|u_+\|_{L^2(\mathbb{R}_+^{n+1})}^2.$$

We consider now

$$\tilde{a}(t, x, \xi_t, \xi', \tau) = (C\tau^4 - r_+^2) \tilde{\psi}_\sigma + (1 - \tilde{\psi}_\sigma) \lambda_\tau^4 \gtrsim \lambda_\tau^4,$$

by construction of  $\tilde{\psi}_\sigma$  and (2.4.7). We obtain therefore the following relation:

$$\begin{aligned} \left( (C\tau^4 - R_+^* R_+) \text{op}^w(\tilde{\psi}_\sigma) v_+, v_+ \right)_+ &= (\text{op}^w(\tilde{a}) v_+, v_+)_+ - \left( \text{op}^w((1 - \tilde{\psi}_\sigma) \lambda_\tau^4) v_+, v_+ \right)_+ \\ &\quad + (S v_+, v_+)_+, \end{aligned}$$

where  $S \in \Psi_\tau^3$  is a subprincipal term coming from the pseudodifferential calculus. Indeed, remark that in fact  $C\tau^4 - R_+^*R_+ \in \mathcal{D}_\tau^4 \subset \Psi_\tau^4$ . In particular one can control this term by

$$\left| (Sv_+, v_+) \right| \lesssim \|v_+\|_{L^2(\mathbb{R}_+; H_\tau^{3/2})}^2$$

As before, using that  $\text{op}^w(1 - \tilde{\psi}_\sigma)\text{op}^w(\psi_\sigma) \in \Psi_\tau^{-\infty}$  one has

$$\left| \left( \text{op}^w((1 - \tilde{\psi}_\sigma)\lambda_\tau^4)v_+, v_+ \right)_+ \right| \lesssim \|u_+\|_{L^2(\mathbb{R}_+^{n+1})}^2.$$

We use then the fact that  $\tilde{a} \gtrsim \lambda_\tau^4$  which thanks to Gårding's inequality with a large parameter (Lemma 2.A.1) yields:

$$(\text{op}^w(\tilde{a})v_+, v_+) \gtrsim \|v_+\|_{L^2(\mathbb{R}_+; H_\tau^2)}^2. \quad (2.4.9)$$

Putting (2.4.6), (2.4.8), (2.4.9) together we find, taking  $\tau$  large enough

$$\|P_\phi^+ v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \|u_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 \gtrsim \tau^4 \|v_+\|_{L^2(\mathbb{R}_+)}^2 + \tau^3 |v_+|_{L^2(\Sigma)}^2 + \tau |D_n v_+|_{L^2(\Sigma)}^2.$$

Since  $v_+$  localizes in a region where  $\tau \gtrsim |\xi'|$  one can simply control the trace of the tangential derivatives (thanks also to the transmission condition (2.2.14)):

$$\tau^3 |v_+|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \gtrsim \tau^3 |v_\pm|_{L^2(\Sigma)}^2 = \tau |\tau \hat{v}_\pm|_{L^2(\Sigma)}^2 \gtrsim \tau |\lambda \hat{v}_\pm|_{L^2(\Sigma)}^2 = \tau |\nabla_{x'} v_\pm|_{L^2(\Sigma)}^2.$$

This yields:

$$\begin{aligned} & \|P_\phi^+ v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \|u_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + T_{\theta, \Theta} \\ & \gtrsim \tau^4 \|v_+\|_{L^2(\mathbb{R}_+)}^2 + \tau^3 |v_+|_{L^2(\Sigma)}^2 + \tau |\nabla_{x'} v_-|_{L^2(\Sigma)}^2 + \tau |\nabla_{x'} v_+|_{L^2(\Sigma)}^2 + \tau |D_n v_+|_{L^2(\Sigma)}^2. \end{aligned}$$

and finally thanks to Lemma 2.4.3:

$$\begin{aligned} & \|P_\phi^+ v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \|u_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + |u_-|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \\ & \gtrsim \tau^4 \|v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^3 |v_-|_{L^2(\Sigma)}^2 + \tau^3 |v_+|_{L^2(\Sigma)}^2 + \tau |\nabla v_-|_{L^2(\Sigma)}^2 + |\nabla v_+|_{L^2(\Sigma)}^2. \end{aligned} \quad (2.4.10)$$

The only missing term is the volume norm on the negative half-line. Here one needs to use  $P_\phi^-$ . We write

$$c_-^{-1}(x)P_\phi^- = (D_n + i\tau\phi')^2 + Q(x, D_{x'}) - c_-^{-1}D_t^2 = A_- + R_-.$$

In the region under consideration  $\|R_-v\|_{L^2(\mathbb{R}_-^{n+1})}$  is a perturbation of  $\|A_-v\|_{L^2(\mathbb{R}_-^{n+1})}$ . The difference is that here we integrate on the negative half-line and the boundary terms come with the opposite sign when one calculates  $\|A_-v\|_{L^2(\mathbb{R}_-^{n+1})}^2$ . More precisely, after repeating the same steps as above one finds

$$\begin{aligned} & \|P_\phi^+ v_-\|_{L^2(\mathbb{R}_-^{n+1})}^2 + \|u_-\|_{L^2(\mathbb{R}_-^{n+1})}^2 + \tau^3 |v_-|_{L^2(\Sigma)}^2 \\ & + \tau^3 |v_+|_{L^2(\Sigma)}^2 + \tau |\nabla v_-|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \gtrsim \tau^4 \|v_-\|_{L^2(\mathbb{R}_-^{n+1})}^2. \end{aligned} \quad (2.4.11)$$

To finish the proof of the Lemma one can simply multiply (2.4.10) by a sufficiently large constant and add it to (2.4.11).  $\square$

### 2.4.2 Microlocal estimates in the non-elliptic regions

In (2.4.2) and (2.4.3) we have defined two microlocal regions on each side of the interface. In this section we prove microlocal estimates inside the non elliptic regions. As expected most surface terms are estimated by the positive half-space where the observation takes place.

Lemma 2.4.7 proves the appropriate estimate in the sub-region  $\tau \gg |\xi_t| + |\xi'|$ . We can consequently localize in its complementary region in the sequel. The following lemma deals with the microlocal sub-regions where the operator  $P_\phi$  is **not** elliptic microlocally. In this case the error terms in  $\xi_t$  become very useful.

**Lemma 2.4.9 (Non-elliptic positive half-space).** *Let  $K \subset \mathbb{R}^{n+1}$  be a compact set,  $c_0 > 0$ ,  $Y \subset \{|\xi_t| \geq c_0|\xi'|\}$ . Consider  $\chi \in \mathcal{S}^0$  with  $\text{supp}(\chi) \subset Y$  and  $\pi_{t,x}(\text{supp} \chi) \subset K$ . Then for all  $\sigma > 0$  there exists  $\tau_0 > 0$  such that one has:*

$$\begin{aligned} & \left\| P_\phi^+ v \right\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|D_t v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau |D_t v|_{L^2(\Sigma)}^2 + \tau \|u\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau |u|_{L^2(\Sigma)}^2 \\ & \qquad \qquad \qquad \gtrsim \tau \|v\|_{L^2(\mathbb{R}_+; H_x^1)}^2 + \tau^3 |v|_{L^2(\Sigma)}^2 + \tau |\nabla v|_{L^2(\Sigma)}^2, \end{aligned}$$

for all  $\tau \geq \tau_0$ ,  $u \in \mathcal{S}_c(\mathbb{R}^{n+1})$  and  $v = \text{op}^w(\chi)\text{op}^w(1 - \psi_\sigma)u$ .

*Proof.* Here we are microlocally in a region where  $\xi_t$  is large. We consider as before an auxiliary function  $\check{\psi} \in C^\infty(\mathbb{R}^+)$  with  $\check{\psi} = 1$  in  $[4, \infty)$  and  $\check{\psi} = 0$  in  $[0, 3]$  and then define  $\check{\psi}_\sigma$  similarly to  $\psi_\sigma$ :

$$\check{\psi}_\sigma(\tau, \xi_t) := \check{\psi} \left( \frac{\sigma\tau}{(1 + |\xi_t|^2 + |\xi'|^2)^{\frac{1}{2}}} \right) \in \mathcal{S}_\tau^0.$$

To simplify notation we consider additionally  $\theta_\sigma := 1 - \psi_\sigma$  and  $\check{\theta}_\sigma := 1 - \check{\psi}_\sigma$ . Remark then  $\theta_\sigma$  and  $\check{\theta}_\sigma$  localize in a region where  $|\xi_t| + |\xi'| \gtrsim \tau$  with moreover  $\check{\theta}_\sigma = 1$  on  $\text{supp} \theta_\sigma$ . We introduce also  $\tilde{\chi}$  satisfying the same properties as  $\chi$  with  $\tilde{\chi} = 1$  on  $\text{supp} \chi$ .

Observe now that on the one hand one has  $|\xi_t| + |\xi'| \gtrsim \tau$  on the support of  $\check{\theta}_\sigma$ , and on the other hand  $|\xi_t| \gtrsim |\xi'|$  on the support of  $\tilde{\chi}$ . Consequently  $\check{\theta}_\sigma \tilde{\chi}$  localizes in the regime of Lemma 2.4.5 and belongs to  $\mathcal{S}_\tau^0$  thanks to Lemma 2.A.2. This yields the estimate:

$$\begin{aligned} & \tau \left\| D_t \text{op}^w(\check{\theta}_\sigma \tilde{\chi}) v \right\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + |v|_{L^2(\Sigma)}^2 + \tau \left| D_t \text{op}^w(\check{\theta}_\sigma \tilde{\chi}) v \right|_{L^2(\Sigma)}^2 \\ & \qquad \qquad \qquad \gtrsim \tau^3 \left\| \text{op}^w(\check{\theta}_\sigma \tilde{\chi}) v \right\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^3 \left| \text{op}^w(\check{\theta}_\sigma \tilde{\chi}) v \right|_{L^2(\Sigma)}^2 \\ & \qquad \qquad \qquad + \tau \left\| \nabla_{x'} \text{op}^w(\check{\theta}_\sigma \tilde{\chi}) v \right\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \left| \nabla_{x'} \text{op}^w(\check{\theta}_\sigma \tilde{\chi}) v \right|_{L^2(\Sigma)}^2. \quad (2.4.12) \end{aligned}$$

One then has:

$$\begin{aligned} \tau^{\frac{3}{2}} \left\| \text{op}^w(\check{\theta}_\sigma \tilde{\chi}) v \right\|_{L^2(\mathbb{R}_+^{n+1})} &= \tau^{\frac{3}{2}} \left\| v - \text{op}^w(1 - \check{\theta}_\sigma \tilde{\chi}) v \right\|_{L^2(\mathbb{R}_+^{n+1})} \\ &\geq \tau^{\frac{3}{2}} \|v\|_{L^2(\mathbb{R}_+^{n+1})} - \tau^{\frac{3}{2}} \left\| \text{op}^w(1 - \check{\theta}_\sigma \tilde{\chi}) v \right\|_{L^2(\mathbb{R}_+^{n+1})} \\ &= \tau^{\frac{3}{2}} \|v\|_{L^2(\mathbb{R}_+^{n+1})} - \tau^{\frac{3}{2}} \|Ru\|_{L^2(\mathbb{R}_+^{n+1})}, \end{aligned}$$

with  $R = \text{op}^w(1 - \check{\theta}_\sigma \check{\chi}) \text{op}^w(\chi) \text{op}^w(\theta_\sigma)$ . Since

$$\text{supp}(1 - \check{\theta}_\sigma \check{\chi}) \cap \text{supp} \theta_\sigma \cap \text{supp} \chi = \emptyset,$$

one has immediately that  $R \in \Psi^{-\infty}$ . Since  $\theta_\sigma \in \mathcal{S}_\tau^0$  has support in a region where  $\tau \lesssim |\xi_t| + |\xi'|$  Lemma 2.A.2 implies  $R \in \Psi_\tau^{-\infty}$  and therefore

$$\tau^{\frac{3}{2}} \|Ru\|_{L^2(\mathbb{R}_+^{n+1})} \lesssim \|u\|_{L^2(\mathbb{R}_+^{n+1})}.$$

We control in a similar fashion all the terms in (2.4.12) in which  $\text{op}^w(\check{\theta}_\sigma \check{\chi})$  appears. Taking then  $\tau \geq \tau_0$ ,  $\tau_0$  large yields

$$\begin{aligned} \tau \|D_t v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau |D_t v|_{L^2(\Sigma)}^2 + \|u\|_{L^2(\mathbb{R}_+^{n+1})}^2 + |u|_{L^2(\Sigma)}^2 &\gtrsim \tau^3 \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^3 |v|_{L^2(\Sigma)}^2 \\ &+ \tau \|\nabla_{x'} v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau |\nabla_{x'} v|_{L^2(\Sigma)}^2. \end{aligned} \quad (2.4.13)$$

The only term we need now is the trace of the normal derivative. Here we have no factorization and we will use the commutator technique (see [LR95, Section 3A], [LRLR22, Chapter 3.4]). In our case we already control almost all of the terms thanks to (2.4.13).

We write

$$c_+^{-1}(x)P_\phi^+ = Q_2 + i\tau Q_1,$$

with

$$Q_2 = D_n^2 - \tau^2 |\phi'|^2 + Q(x, D_{x'}) - c_+^{-1}(x)D_t^2 \quad (2.4.14)$$

and

$$Q_1 = \phi' D_n + D_n \phi'. \quad (2.4.15)$$

We integrate by parts taking into account the boundary terms to find:

$$\left\| c_+^{-1} P_\phi^+ v \right\|_{L^2(\mathbb{R}_+^{n+1})}^2 = \|Q_2 v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^2 \|Q_1 v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + i\tau ([Q_2, Q_1]v, v)_+ + \tau \mathcal{B}(v). \quad (2.4.16)$$

After some calculations (which are the same as in [LRLR22, Chapter 3.4]) we see that the boundary term  $\mathcal{B}(v)$  can be written as:

$$\mathcal{B}(v) = 2(\phi' D_n v, D_n v)_\Sigma + (M_1 v, D_n v)_\Sigma + (M_1' D_n v, v)_\Sigma + (M_2 v, v)_\Sigma,$$

where  $M_j \in \mathcal{D}_{\top, \tau}^j$ , that is a tangential differential operator of order  $j$  depending on  $\tau$  (see Section 2.A.1 for a definition). Using this as well as the Young inequality yields, for arbitrary  $\delta_2 > 0$ :

$$|\mathcal{B}(v) - 2(\phi' D_n v, D_n v)_\Sigma| \lesssim (1 + \delta_2^{-1}) \left( \tau^2 |v|_{L^2(\Sigma)}^2 + |\nabla_{x'} v|_{L^2(\Sigma)}^2 \right) + \delta_2 |D_n v|_{L^2(\Sigma)}^2.$$

We want to combine this last inequality with (2.4.16). To do this we recall that  $\phi' > 0$  close to  $\Sigma$  (that is for  $|x_n|$  small enough) and we choose  $\delta_2$  sufficiently small. One finds then, for  $\tau$  sufficiently large:

$$\begin{aligned} \left\| P_\phi^+ v \right\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^3 |v|_{L^2(\Sigma)}^2 + \tau |\nabla_{x'} v|_{L^2(\Sigma)}^2 \\ \gtrsim \|Q_2 v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^2 \|Q_1 v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + i\tau ([Q_2, Q_1]v, v)_+ + \tau |D_n v|_{L^2(\Sigma)}^2. \end{aligned} \quad (2.4.17)$$



We need to take care of the commutator to deduce an estimate on  $\tau |D_n v|_{L^2(\Sigma)}^2$ . We simply write:

$$i[Q_2, Q_1] = B_0 D_n^2 + C_1 D_n + C_2,$$

with  $C_j$  tangential operators of order  $j$  depending on  $\tau$ . Since  $\phi' > 0$  close to  $\Sigma$  we can use (2.4.14) and (2.4.15) to express  $D_n^2$  and  $D_n$  in terms of  $Q_2, Q_1$ . This yields the following expression:

$$i[Q_2, Q_1] = B_0 Q_2 + B_1 Q_1 + B_2,$$

with  $B_j$  tangential operators of order  $j$  depending on  $\tau$ . This implies that:

$$|i\tau (B_0 Q_2 v, v)| \lesssim \tau^{-\frac{1}{2}} \|Q_2 v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^{\frac{5}{2}} \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2,$$

as well as

$$|i\tau (B_1 Q_1 v, v)| \lesssim \tau \|Q_1 v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^3 \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|\nabla_{x'} v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|D_t v\|_{L^2(\mathbb{R}_+^{n+1})}^2$$

and

$$|i\tau (B_2 v, v)| \lesssim \tau^3 \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|\nabla_{x'} v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|D_t v\|_{L^2(\mathbb{R}_+^{n+1})}^2.$$

We inject now these last three estimates in the commutator term in (2.4.17) and take  $\tau \geq \tau_0$ ,  $\tau_0$  large to absorb the terms with  $\|Q_2\|$  and  $\|Q_1\|$  to find:

$$\|P_\phi^+ v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^3 \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|\nabla_{x'} v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^3 |v|_{L^2(\Sigma)}^2 + \tau |\nabla_{x'} v|_{L^2(\Sigma)}^2 \gtrsim \tau |D_n v|_{L^2(\Sigma)}^2 \quad (2.4.18)$$

We multiply finally (2.4.13) by a large constant and add it to (2.4.18) to obtain:

$$\begin{aligned} & \|P_\phi^+ v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|D_t v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau |D_t v|_{L^2(\Sigma)}^2 + \|u\|_{L^2(\mathbb{R}_+^{n+1})}^2 + |u|_{L^2(\Sigma)}^2 \\ & \gtrsim \tau^3 \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^3 |v|_{L^2(\Sigma)}^2 + \tau |D_n v|_{L^2(\Sigma)}^2 + \tau \|\nabla_{x'} v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau |\nabla_{x'} v|_{L^2(\Sigma)}^2, \end{aligned}$$

which gives immediately the sought estimate.  $\square$

**Lemma 2.4.10 (Non elliptic negative half-space).** *Let  $K \subset \mathbb{R}^{n+1}$  be a compact set,  $c_0 > 0$ ,  $Y \subset \{|\xi_t| \geq c_0 |\xi'|\}$ . Consider  $\chi \in \mathcal{S}^0$  with  $\text{supp}(\chi) \subset Y$  and  $\pi_{t,x}(\text{supp} \chi) \subset K$ . Then for all  $\sigma > 0$ , there exists  $\tau_0 > 0$  such that:*

$$\tau \|D_t v\|_{L^2(\mathbb{R}_-^{n+1})}^2 + \tau |D_t v|_{L^2(\Sigma)}^2 + \|u\|_{L^2(\mathbb{R}_-^{n+1})}^2 + |u|_{L^2(\Sigma)}^2 \gtrsim \tau \|v\|_{L^2(\mathbb{R}_-; H_\sigma^1)},$$

for all  $\tau \geq \tau_0$ ,  $u \in \mathcal{S}_c(\mathbb{R}^{n+1})$  and  $v = \text{op}^w(\chi) \text{op}^w(1 - \psi_\sigma) u$ .

*Proof.* We introduce, as in the proof of Lemma 2.4.9 the function  $\check{\theta}_\sigma$  and  $\check{\chi}$  such that  $\check{\theta}_\sigma \check{\chi}$  has a support contained in the region of Lemma 2.4.5, that is  $\{\tau \gtrsim |\xi_t| + |\xi'|\}$ . We obtain as above the sought estimate.  $\square$

### 2.4.3 Microlocal estimate in the elliptic region

We now state the desired estimate in a microlocal region where the operator  $P_\phi$  is **elliptic**. In this case we adapt step by step the proof of [LRL13]. Indeed, here we can factorize our operator as in [LRL13, Eq. (2-17)] and obtain the same estimates for the first order factors. Notice that in this regime the error terms in  $D_t$  in the left hand side are useless.

We recall that  $\psi_\sigma$  has been defined in (2.4.5). The aim of this subsection is to prove the following lemma:

**Lemma 2.4.11 (Elliptic region).** *Let  $K \subset \mathbb{R}^{n+1}$  be a compact set,  $\chi \in \mathcal{S}^0$  with  $\text{supp}(\chi) \subset \mathcal{E}_\epsilon^- \cap \mathcal{E}_\epsilon^+$  and  $\pi_{t,x}(\text{supp} \chi) \subset K$ . Then for all  $\sigma > 0$  there exist  $\mathcal{V} \Subset V_t$  and  $\tau_0 > 0$  such that:*

$$\begin{aligned} & \|H_- P_\phi v_-\|_{L^2(\mathbb{R}^{n+1})}^2 + \|H_+ P_\phi v_+\|_{L^2(\mathbb{R}^{n+1})}^2 + \|u\|_{L^2(\mathbb{R}^{n+1})}^2 + |u_+|_{L^2(\Sigma)}^2 + |u_-|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \\ & \gtrsim \tau \|v\|_{L^2(\mathbb{R}; H_\tau^\pm)}^2 + \tau^3 |v_+|_{L^2(\Sigma)}^2 + \tau^3 |v_-|_{L^2(\Sigma)}^2 \\ & \quad + \tau |\nabla v_+|_{L^2(\Sigma)}^2 + \tau |\nabla v_-|_{L^2(\Sigma)}^2 + \|\nabla v_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \|\nabla v_-\|_{L^2(\mathbb{R}_-^{n+1})}^2, \end{aligned}$$

for all  $\tau \geq \tau_0$  and  $v_\pm = \text{op}^w(1 - \psi_\sigma) \text{op}^w(\chi) u_\pm$  with  $u \in \mathcal{W}_\phi^{\theta, \Theta}$  satisfying  $\text{supp} u \subset \mathcal{V}$ .

To simplify we remove the  $\pm$  notation from  $v_\pm$ . Let  $\tilde{\chi}$  satisfy the same properties as  $\chi$ , satisfying additionally  $\tilde{\chi} = 1$  on  $\text{supp} \chi$  and define

$$s(x, \xi_t, \xi') := Q(x, \xi') - c_+^{-1}(x) \xi_t^2,$$

which means that  $c_+^{-1}(x) P_\phi^+ = (D_n + i\tau\phi')^2 + \text{op}^w(s) + \tilde{R}^+$ , where  $\tilde{R}^+ \in \Psi^1$ . We write:

$$c_+^{-1}(x) P_\phi^+ v = \left( (D_n + i\tau\phi')^2 + \text{op}^w(s\tilde{\chi}) \right) v + \text{op}^w((1 - \tilde{\chi})s) \text{op}^w(1 - \psi_\sigma) \text{op}^w(\chi) u + \tilde{R}^+ u. \quad (2.4.19)$$

The support condition on  $\tilde{\chi}$  guarantees that  $\text{op}^w((1 - \tilde{\chi})s) \text{op}^w(1 - \psi_\sigma) \text{op}^w(\chi) \in \Psi_\tau^{-\infty}$ . We define now the symbol

$$m_+^2 := s\tilde{\chi} + (1 - \tilde{\chi})\lambda^2. \quad (2.4.20)$$

To justify the slightly abusive notation of  $m_+^2$  we notice that by definition of the region  $\mathcal{E}_\epsilon^+$  (see (2.4.2)) one has  $s \gtrsim \epsilon\lambda^2$ , on the support of  $\tilde{\chi}$  and consequently  $m_+^2 \gtrsim \epsilon\lambda^2$ . This means that  $m_+^2 \in \mathcal{S}^2$  is elliptic positive and therefore it is indeed a square of another symbol in  $\mathcal{S}^1$ . We now write:

$$\text{op}^w(s\tilde{\chi})v = \text{op}^w(m_+^2)v - \text{op}^w((1 - \tilde{\chi})\lambda^2) \text{op}^w(1 - \psi_\sigma) \text{op}^w(\chi)u,$$

with  $\text{op}^w((1 - \tilde{\chi})\lambda^2) \text{op}^w(1 - \psi_\sigma) \text{op}^w(\chi) \in \Psi_\tau^{-\infty}$ . Coming back to (2.4.19) we have obtained

$$c_+^{-1}(x) P_\phi^+ v = \left( (D_n + i\tau\phi')^2 + \text{op}^w(m_+^2) \right) v + \tilde{R}^+ u + R^+ u,$$

with  $R^+ \in \Psi_\tau^{-\infty}$ . In particular  $\|R^+ u\|_{L^2(\mathbb{R}_+^{n+1})}^2 \lesssim \|u\|_{L^2(\mathbb{R}_+^{n+1})}^2$ , which can be absorbed in the left hand side of the sought estimate. Using the positive ellipticity of  $m_+^2$  we define its square root which we denote by  $m_+ \in \mathcal{S}^1$ . Using symbolic calculus we obtain

$$\begin{aligned} c_+^{-1}(x) P_\phi^+ v &= \left( (D_n + i\tau\phi')^2 + M_+^2 \right) v + R_1^+ u \\ &= (D_n + i\tau\phi' - iM_+)(D_n + i\tau\phi' + iM_+)v + R_2^+ u \\ &= (D_n + i\tau\phi' + iM_+)(D_n + i\tau\phi' - iM_+)v + R_3^+ u, \end{aligned} \quad (2.4.21)$$

where  $M_+ = \text{op}^w(m_+) \in \Psi^1$  and the operators  $R_j^+ \in \Psi^1 + \tau\Psi^0 + \Psi^0 D_n$  which is an admissible perturbation (see Remark 2.4.2). Similarly, using that  $\text{supp}(\chi) \subset \mathcal{E}_\epsilon^- \cap \mathcal{E}_\epsilon^+$  we may write:

$$\begin{aligned} c_-^{-1}(x)P_\phi^- v &= \left( (D_n + i\tau\phi')^2 + M_-^2 \right) v + R_1^- u \\ &= (D_n + i\tau\phi' - iM_-)(D_n + i\tau\phi' + iM_-)v + R_2^- u \\ &= (D_n + i\tau\phi' + iM_-)(D_n + i\tau\phi' - iM_-)v + R_3^- u, \end{aligned} \quad (2.4.22)$$

where  $M_- = \text{op}^w(m_-)$  and  $m_- \in \mathcal{S}^1$  is (similar to  $m_+$ ) *positive elliptic* and *homogeneous* of degree one. Recall as well that  $\text{op}^w(1 - \psi_\sigma)$  localizes in a region where  $\tau \lesssim |\xi_t| + |\xi'|$ . We can therefore suppose without loss of generality, up to introducing an admissible error in our estimate, that  $m_\pm \in \mathcal{S}_\tau^1$ .

So far we have obtained microlocally a factorization as in [LRL13] with the same weight  $\phi$  and with operators  $M_\pm$  having real, positive elliptic, homogeneous symbols of degree one. This is sufficient for the proof of the elliptic case to hold in our context too. Let us present the arguments of [LRL13] in the present context.

We have two first-order factors on each side of the interface. Using the same notation as in [LRL13] we write:

$$\begin{aligned} D_n + i(\tau\phi' + M_\pm) &:= D_n + iE_\pm, & e_\pm &= \tau\phi' + m_\pm, & E_\pm &= \text{op}^w(e_\pm) \\ D_n + i(\tau\phi' - M_\pm) &:= D_n + iF_\pm, & f_\pm &= \tau\phi' - m_\pm, & F_\pm &= \text{op}^w(f_\pm). \end{aligned}$$

The crucial remark here is that one has always

$$\mathcal{S}_\tau^1 \ni e_\pm \gtrsim \lambda_\tau,$$

thanks to the positive ellipticity of  $m_\pm$ , whereas the sign of  $f_\pm$  may change. The quality of the estimate obtained depends on the ellipticity/positivity of the factors above, therefore one needs to distinguish cases concerning their sign.

We collect now a series of lemmas giving some first order estimates. These correspond in the elliptic case to the estimates obtained in [LRL13, Sections from 3.2 to 3.7] and the proofs are similar.

For the elliptic factors  $e_\pm$  one has:

**Lemma 2.4.12 (Positive factor on the positive half-space).** *Let  $e_+ \in \mathcal{S}_\tau^1$  such that  $e_+ \gtrsim \lambda_\tau$  and  $E_+ = \text{op}^w(e_+)$ . Then for all  $l \in \mathbb{R}$  there exists  $\tau_0 > 0$  such that*

$$\|(D_n + iE_+)v\|_{L^2(\mathbb{R}_+; H_\tau^l)} \gtrsim |v|_{H_\tau^{l+1/2}} + \|v\|_{L^2(\mathbb{R}_+; H_\tau^{l+1})} + \|D_n v\|_{L^2(\mathbb{R}_+; H_\tau^l)}, \quad (2.4.23)$$

for  $\tau \geq \tau_0$  and  $v \in \mathcal{S}_c(\mathbb{R}^{n+1})$ .

*Proof.* We sketch the proof for  $l = 0$ . An integration by parts in the  $x_n$  variable yields:

$$2 \text{Re}((D_n + iE_+)v, i\text{op}^w(\lambda_\tau)v)_+ = |v|_{H_\tau^{1/2}}^2 + 2 \text{Re}(E_+ v, \text{op}^w(\lambda_\tau)v)_+,$$

and then we simply use the fact that  $e_+ \gtrsim \lambda_\tau$  as well as Gårding's inequality with a large parameter which implies:

$$(E_+ v, \text{op}^w(\lambda_\tau)v)_+ = (\text{op}^w(\lambda_\tau)E_+ v, v)_+ \gtrsim \|v\|_{L^2(\mathbb{R}_+; H_\tau^1)}^2,$$

and (2.4.23) without the term  $\|D_n v\|$  follows for  $l = 0$  from Young's inequality. To introduce  $\|D_n v\|$  in our estimate we simply use the expression

$$\|(D_n + iE_+)v\|_{L^2(\mathbb{R}_+^{n+1})}^2 = \|D_n v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + 2 \operatorname{Re}(D_n v, iE_+ v)_+ + \|E_+ v\|_{L^2(\mathbb{R}_+^{n+1})}^2,$$

and we bound from above the last two terms by  $\|v\|_{L^2(\mathbb{R}_+; H_\tau^1)}^2$ . For general  $l$  we just consider

$$2 \operatorname{Re} \left( (D_n + iE_+)v, i \operatorname{op}^w(\lambda_\tau^{2l+1})v \right)_+$$

instead of  $2 \operatorname{Re} \left( (D_n + iE_+)v, i \operatorname{op}^w(\lambda_\tau)v \right)_+$ .  $\square$

Estimate 2.4.23 is of great quality as one could expect since we have an elliptic factor in the observation region. In the sequel, to alleviate notation, we sketch the proofs for  $l = 0$ .

**Lemma 2.4.13 (Positive factor on the negative half-space).** *Let  $e_- \in \mathcal{S}_\tau^1$  such that  $e_- \gtrsim \lambda_\tau$  and  $E_- = \operatorname{op}^w(e_-)$ . Then for all  $l \in \mathbb{R}$  we have that there exists  $\tau_0 > 0$  such that*

$$\|(D_n + iE_-)v\|_{L^2(\mathbb{R}_-; H_\tau^l)} + |v|_{H_\tau^{l+1/2}} \gtrsim \|v\|_{L^2(\mathbb{R}_-; H_\tau^{l+1})} + \|D_n v\|_{L^2(\mathbb{R}_-; H_\tau^l)}, \quad (2.4.24)$$

for  $\tau \geq \tau_0$  and  $v \in \mathcal{S}_c(\mathbb{R}^{n+1})$ .

*Proof.* The proof is the same as for (2.4.23) but the boundary term comes with different sign.  $\square$

Since the factors  $f_\pm$  do not have a constant sign we need to consider appropriate microlocal regions.

**Lemma 2.4.14 (Positive  $f_+$  on the positive half-space).** *Let  $f_+ \in \mathcal{S}_\tau^1$  and  $F_+ = \operatorname{op}^w(f_+)$ . We consider  $c_0 > 0$  and  $\chi = \chi(t, x, \xi_t, \xi', \tau) \in \mathcal{S}_\tau^0$  such that  $f_+ \geq c_0 \lambda_\tau$  on  $\operatorname{supp} \chi$ . Then for all  $l \in \mathbb{R}$  there exists  $\tau_0 > 0$  such that*

$$\begin{aligned} \|(D_n + iF_+) \operatorname{op}^w(\chi)v\|_{L^2(\mathbb{R}_+; H_\tau^l)} + \|v\|_{L^2(\mathbb{R}_+^{n+1})} & \quad (2.4.25) \\ & \gtrsim |\operatorname{op}^w(\chi)v|_{H_\tau^{l+1/2}} + \|\operatorname{op}^w(\chi)v\|_{L^2(\mathbb{R}_+; H_\tau^{l+1})} \\ & \quad + \|D_n \operatorname{op}^w(\chi)v\|_{L^2(\mathbb{R}_+; H_\tau^l)}, \end{aligned}$$

for  $\tau \geq \tau_0$  and  $v \in \mathcal{S}_c(\mathbb{R}^{n+1})$ .

*Proof.* The proof only uses the positive ellipticity of  $f_+$  on the support of  $\chi$  and is exactly the same as the proof of Lemma 3.3 in [LRL13]. The (harmless) error term on the left hand side comes from the microlocalization.  $\square$

**Lemma 2.4.15 (Positive  $f_-$  on the negative half-space).** *Let  $e_- \in \mathcal{S}_\tau^1$  such that  $e_- \gtrsim \lambda_\tau$  and  $E_- = \operatorname{op}^w(e_-)$ . Let  $f_- \in \mathcal{S}_\tau^1$  and  $F_- = \operatorname{op}^w(f_-)$ . We consider  $c_0 > 0$  and  $\chi = \chi(t, x, \xi_t, \xi', \tau) \in \mathcal{S}_\tau^0$  such that  $f_- \geq c_0 \lambda_\tau$  on  $\operatorname{supp} \chi$ . Then for all  $l \in \mathbb{R}$  there exists  $\tau_0 > 0$  such that*

$$\begin{aligned} \|(D_n + iF_-)(D_n + iE_-) \operatorname{op}^w(\chi)v\|_{L^2(\mathbb{R}_-; H_\tau^l)} + \|v\|_{L^2(\mathbb{R}_-^{n+1})} + \|D_n v\|_{L^2(\mathbb{R}_-^{n+1})} \\ + |(D_n + iE_-) \operatorname{op}^w(\chi)v|_{H_\tau^{l+1/2}} \gtrsim \|(D_n + iE_-) \operatorname{op}^w(\chi)v\|_{L^2(\mathbb{R}_-; H_\tau^{l+1})}, \end{aligned} \quad (2.4.26)$$

for  $\tau \geq \tau_0$  and  $v \in \mathcal{S}_c(\mathbb{R}^{n+1})$ .

*Proof.* The proof is as [LRL13, Lemma 3.4]. We set  $u = (D_n + iE_-)\text{op}^w(\chi)v$  and write

$$2 \operatorname{Re}((D_n + iF_-)u, \text{op}^w(\lambda_\tau)u)_- = -|u|_{H_\tau^{1/2}} + 2 \operatorname{Re}(F_-u, \text{op}^w(\lambda_\tau)u)_-.$$

We then use the fact that  $f_- \geq c_0\lambda_t$  on the support of  $\chi$  combined with Gårding's inequality with a large parameter.  $\square$

Even though one would expect that the crucial surface terms should come from the observation region  $\mathbb{R}^{n+1}$  there is a situation in which the “good estimate” arises from the negative side  $\mathbb{R}_-^{n+1}$ . This is the content of the next estimate.

**Lemma 2.4.16 (Negative  $f_-$  on the negative half-space).** *Let  $e_- \in \mathcal{S}_\tau^1$  such that  $e_- \gtrsim \lambda_\tau$  and  $E_- = \text{op}^w(e_-)$ . Let  $f_- \in \mathcal{S}_\tau^1$  and  $F_- = \text{op}^w(f_-)$ . We consider  $c_0$  and  $\chi = \chi(t, x, \xi_t, \xi', \tau) \in \mathcal{S}_\tau^0$  such that  $f_- \leq -c_0\lambda_\tau$  on  $\operatorname{supp} \chi$ . Then there exists  $\tau_0 > 0$  such that*

$$\begin{aligned} & \|(D_n + iF_-)(D_n + iE_-)\text{op}^w(\chi)v\|_{L^2(\mathbb{R}_-^{n+1})} + \|v\|_{L^2(\mathbb{R}_-^{n+1})} \\ & + \|D_nv\|_{L^2(\mathbb{R}_-^{n+1})} \gtrsim |(D_n + iE_-)\text{op}^w(\chi)v|_{H_\tau^{1/2}} + \|(D_n + iE_-)\text{op}^w(\chi)v\|_{L^2(\mathbb{R}_-; H_\tau^1)}, \end{aligned} \quad (2.4.27)$$

for  $\tau \geq \tau_0$  and  $v \in \mathcal{S}_c(\mathbb{R}^{n+1})$ .

*Proof.* This is the analogue of Lemma 3.5 in [LRL13]. The proof is in the previous Lemma, here however the fact that  $f_-$  is negative elliptic on the support of  $\chi$  will allow us to control the crucial surface term.  $\square$

We finally give the estimate when  $f_\pm$  has no constant sign. Here we exhibit an estimate with loss of a half derivative. Notice that in the estimates above, no assumption on the coefficients of the weight function  $\phi$  was made. The sub-ellipticity (pseudoconvexity) assumption below (2.4.31) is used in the next region. As in the classic case (with no interface) (see for example [Hör94, Chapter 28]) this translates into taking  $\beta$  large enough. We sketch the proof. We recall that the weight  $\phi$  is defined in (2.2.9).

**Lemma 2.4.17 (Changing signs for  $f_\pm$ ).** *Let  $m_\pm \in \mathcal{S}_\tau^1$  be real positive elliptic homogeneous symbols. Define  $f_\pm = \tau\phi' - m_\pm$  and  $F_\pm = \text{op}^w(f_\pm)$ . Then for all  $l \in \mathbb{R}$  there exist  $\mathcal{V} \Subset V_t$  and  $\tau_0 > 0$  such that*

$$\|(D_n + iF_\pm)v_\pm\|_{L^2(\mathbb{R}_\pm; H_\tau^l)} + |v_\pm|_{H_\tau^{l+1/2}} \gtrsim \tau^{-\frac{1}{2}} \left( \|v_\pm\|_{L^2(\mathbb{R}_\pm^{n+1}; H_\tau^{l+1})} + \|D_nv_\pm\|_{L^2(\mathbb{R}_\pm^{n+1}; H_\tau^l)} \right), \quad (2.4.28)$$

for  $\tau \geq \tau_0$  and  $v \in \mathcal{S}_c(\mathbb{R}^{n+1})$  with  $\operatorname{supp} v \subset \mathcal{V}$ .

*Proof.* We do the proof for  $l = 0$ . Since the symbol of  $F_+ \in \Psi_\tau^1$  is real we have  $F_+^* = F_+$ , according to the Weyl quantization. We compute then

$$\begin{aligned} \|(D_n + iF_+)v\|_{L^2(\mathbb{R}_+^{n+1})}^2 &= \|D_nv\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \|F_+v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + (D_nv, iF_+v)_+ + (iF_+v, D_nv)_+ \\ &\geq \|F_+v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + (D_nv, iF_+v)_+ + (iF_+v, D_nv)_+ \\ &= \operatorname{Re} \left( (F_+^2v, v)_+ + i([D_n, F_+]v, v)_+ + (F_+v, v)_\Sigma \right) \\ &\geq \operatorname{Re} \left( \tau^{-1}\mu(F_+^2v, v)_+ + i([D_n, F_+]v, v)_+ + (F_+v, v)_\Sigma \right) \\ &= \tau^{-1} \operatorname{Re}(\mu F_+^2 + i\tau[D_n, F_+]v, v)_+ + \operatorname{Re}(F_+v, v)_\Sigma, \end{aligned} \quad (2.4.29)$$

for all  $\tau \geq \mu$ . Now since  $F_+ \in \Psi_\tau^1$  one has

$$|\operatorname{Re} i(F_+ v, v)_\Sigma| \lesssim |v|_{H_\tau^{1/2}}^2 \quad (2.4.30)$$

which goes to the left hand side of (2.4.28). We need therefore to estimate the term

$$\operatorname{Re}(\mu F_+^2 + i\tau[D_n, F_+]v, v)_+.$$

The principal symbol of  $\mu F_+^2 + i\tau([D_n, F_+]v, v)_+$  in the class  $\Psi_\tau^2$  is

$$\mu f_+^2 + \tau\{\xi_n, f_+\} = \mu f_+^2 + \tau \partial_{x_n} f_+ \in \mathcal{S}_\tau^2.$$

Notice in particular that the commutator kills the non tangential part. This is the point where the sub-ellipticity condition appears. Indeed, for  $\beta$  sufficiently large one has that for  $\mu$  sufficiently large,

$$\mu f_+^2 + \tau\{\xi_n, f_+\} \gtrsim \lambda_\tau^2. \quad (2.4.31)$$

To prove (2.4.31) we start by noticing that there exists  $\epsilon_1 > 0$  such that the following implication holds:

$$|f_+| \leq \epsilon_1 \lambda_\tau \implies \tau \sim \lambda = (|\xi'|^2 + |\xi_t|^2)^{\frac{1}{2}},$$

where the notation  $\tau \sim \lambda$  means that there exists  $C > 0$  such that  $1/C\tau \leq \lambda \leq C\tau$ . The above implication is a consequence of the facts  $f_+ = \tau\phi' - m_+$ ,  $\phi' \gtrsim 1$  and  $m_+ \lesssim \lambda$ . We distinguish two cases:

1. Suppose that  $|f_+| \leq \epsilon_1 \lambda_\tau$ . Then the above implies  $\tau \sim \lambda$  and we can compute as follows

$$\{\xi_n, f_+\} = \tau\beta - \partial_{x_n} m_+ \gtrsim \beta\lambda - \partial_{x_n} m_+.$$

Using homogeneity and compactness we obtain that for  $\beta$  large enough one has

$$\beta\lambda - \partial_{x_n} m_+ \gtrsim \lambda,$$

and since  $\tau \sim \lambda$  this implies  $\tau\{\xi_n, f_+\} \gtrsim \lambda_\tau^2$ , which implies (2.4.31).

2. If  $|f_+| \geq \epsilon_1 \lambda_\tau$  then again by homogeneity and compactness one has for  $\mu$  sufficiently large and all  $\tau \geq \mu$  the sought estimate (2.4.31).

One can then apply Gårding's inequality with a large parameter which gives

$$\operatorname{Re}(\mu F_+^2 + i\tau[D_n, F_+]v, v)_+ \gtrsim \|v\|_{L^2(\mathbb{R}_+; H_\tau^1)}^2.$$

Combined with (2.4.29) and (2.4.30), this proves inequality (2.4.28) in the positive half-space. The proof for the negative half-space is exactly the same.  $\square$

We have presented the first order estimates that we will iterate. In order for the iteration to work, we need to impose a condition on the coefficients of the weight function  $\phi$ . The crucial assumption that we use is the following **geometric hypothesis** of [LRL13]. We choose the coefficients  $\alpha_-$ ,  $\alpha_+$  such that:

$$\frac{\alpha_+}{\alpha_-} > \sup_{\substack{t, x', \xi_t, \xi' \\ |(\xi_t, \xi')| \geq 1}} \frac{m_+(t, 0, x', \xi_t, \xi')}{m_-(t, 0, x', \xi_t, \xi')},$$

where the symbols  $m_{\pm}$  defined above (see (2.4.7)) are real positive elliptic and homogeneous of degree one. This geometric hypothesis can be stated as well by saying that:

$$\exists \mu > 1, \quad \frac{\alpha_+}{\alpha_-} = \mu^2 \sup_{\substack{t, x', \xi_t, \xi' \\ |(\xi_t, \xi')| \geq 1}} \frac{m_+(t, 0, x', \xi_t, \xi')}{m_-(t, 0, x', \xi_t, \xi')}. \quad (2.4.32)$$

We now recall how the geometric assumption on the weight (2.4.32) allows to effectively combine all of the above estimates. This is expressed through the following lemma.

**Lemma 2.4.18.** *Let  $\mu_0 > \mu > 1$  and  $\alpha_{\pm}$ , be positive numbers such that (2.4.32) holds. For  $s > 0$  define the following subsets of  $\mathbb{R}_{x', t}^n \times \mathbb{R}_{\xi_t, \xi'}^n \times \mathbb{R}_+^*$  by:*

$$\begin{aligned} \Gamma_s &= \{(t, x', \xi_t, \xi', \tau); |(\xi', \xi_t)| < 2 \text{ or } \tau \alpha_+ > sm_+(t, 0, x', \xi_t, \xi'), \\ \tilde{\Gamma}_s &= \{(t, x', \xi_t, \xi', \tau); |(\xi', \xi_t)| > 1 \text{ and } \tau \alpha_+ < sm_+(t, 0, x', \xi_t, \xi')\}. \end{aligned}$$

Then there exists  $\eta, \tau_0 > 0$  such that for  $|x_n| \leq \eta$  and  $\tau \geq \tau_0$  we have  $\mathbb{R}_{x', t}^n \times \mathbb{R}_{\xi_t, \xi'}^n \times \mathbb{R}_+^* = \Gamma_{\mu_0} \cup \tilde{\Gamma}_{\mu}$  and

$$\begin{aligned} \Gamma_{\mu_0} &\subset \{(t, x', \xi_t, \xi', \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+^*; f_+(t, x, \xi_t, \xi') \geq C\lambda_{\tau} \text{ for } 0 \leq x_n \leq \eta\}, \\ \tilde{\Gamma}_{\mu} &\subset \{(t, x', \xi_t, \xi', \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+^*; f_-(t, x, \xi_t, \xi') \leq -C\lambda_{\tau} \text{ for } -\eta \leq x_n \leq 0\}. \end{aligned}$$

The proof of this Lemma uses the ellipticity and homogeneity of  $m_{\pm}$  and is exactly the same as in Section 4A of [LRL13].

The end of the proof is also similar to [LRL13]. Let us sketch it here for the sake of completeness. The crucial property of Lemma 2.4.18 is that we have covered the tangential dual space by two regions such that  $f_+$  is positive elliptic on the one region and  $f_-$  is negative elliptic on the other. We consider a homogeneous partition of unity

$$1 = \chi_{\Gamma, 0}(t, x', \xi_t, \xi', \tau) + \chi_{\Gamma, 1}(t, x', \xi_t, \xi', \tau), \quad \text{supp } \chi_{\Gamma, 0} \subset \Gamma_{\mu_0}, \quad \text{supp } \chi_{\Gamma, 1} \subset \tilde{\Gamma}_{\mu}.$$

Remark that the derivatives of  $\chi_{\Gamma, j}$  are supported in a region where  $\tau \lesssim |\xi'| + \xi_t$  and consequently one has  $\chi_{\Gamma, j} \in \mathcal{S}_{\tau}^0$ .

We now prove the estimate of Lemma 2.4.11 in the interior of each of the microlocal sub-regions given by Lemma 2.4.18. After this, the last step will be to put together these two microlocal estimates.

#### Estimate in $\Gamma_{\mu_0}$

We consider  $\Xi_0 = \text{op}^w(\chi_{\Gamma, 0}) \in \Psi_{\tau}^0$ . We start by applying (2.4.23) to get:

$$\|(D_n + iE_+)(D_n + iF_+)\Xi_0 v_+\|_{L^2(\mathbb{R}_+^{n+1})} \gtrsim |(D_n + iF_+)\Xi_0 v_+|_{H_{\tau}^{1/2}} + \|(D_n + iF_+)\Xi_0 v_+\|_{L^2(\mathbb{R}_+; H_{\tau}^1)}.$$

The localization of  $\chi_{\Gamma, 0}$  allows to use Lemma (2.4.25) which gives

$$\|(D_n + iF_+)\Xi_0 v_+\|_{L^2(\mathbb{R}_+; H_{\tau}^1)} + \|v_+\|_{L^2} \gtrsim |\Xi_0 v_+|_{H_{\tau}^{3/2}} + \|\Xi_0 v_+\|_{L^2(\mathbb{R}_+; H_{\tau}^2)} + \|D_n \Xi_0 v_+\|_{L^2(\mathbb{R}_+; H_{\tau}^1)}.$$

We obtain therefore:

$$\begin{aligned} &\|(D_n + iE_+)(D_n + iF_+)\Xi_0 v_+\|_{L^2(\mathbb{R}_+^{n+1})} + \|v_+\|_{L^2} \\ &\gtrsim |(D_n + iF_+)\Xi_0 v_+|_{H_{\tau}^{1/2}} + |\Xi_0 v_+|_{H_{\tau}^{3/2}} + \|\Xi_0 v_+\|_{L^2(\mathbb{R}_+; H_{\tau}^2)} + \|D_n \Xi_0 v_+\|_{L^2(\mathbb{R}_+; H_{\tau}^1)}. \end{aligned}$$

Using the fact that

$$|(D_n + iF_+)\Xi_0 v_+|_{H_\tau^{1/2}} \geq |D_n \Xi_0 v_+|_{H_\tau^{1/2}} - |F_+ \Xi_0 v_+|_{H_\tau^{1/2}} \geq |D_n \Xi_0 v_+|_{H_\tau^{1/2}} - C |\Xi_0 v_+|_{H_\tau^{3/2}},$$

we finally get

$$\begin{aligned} & \|(D_n + iE_+)(D_n + iF_+)\Xi_0 v_+\|_{L^2(\mathbb{R}_+^{n+1})} + \|v_+\|_{L^2} \\ & \gtrsim |D_n \Xi_0 v_+|_{H_\tau^{1/2}} + |\Xi_0 v_+|_{H_\tau^{3/2}} + \|\Xi_0 v_+\|_{L^2(\mathbb{R}_+; H_\tau^2)} + \|D_n \Xi_0 v_+\|_{L^2(\mathbb{R}_+; H_\tau^1)}. \end{aligned} \quad (2.4.33)$$

For the negative half-space we start by using (2.4.28) for  $l = 1/2$  which yields:

$$\|(D_n + iF_-)(D_n + iE_-)\Xi_0 v_-\|_{L^2(\mathbb{R}_-^{n+1})} + |(D_n + iE_-)\Xi_0 v_-|_{L^2} \gtrsim \|(D_n + iE_-)\Xi_0 v_-\|_{L^2(\mathbb{R}_-; H_\tau^{1/2})}.$$

Estimate (2.4.24) gives:

$$\|(D_n + iE_-)\Xi_0 v_-\|_{L^2(\mathbb{R}_-; H_\tau^{1/2})} + |\Xi_0 v_-|_{H_\tau^1} \gtrsim \|\Xi_0 v_-\|_{L^2(\mathbb{R}_-; H_\tau^{3/2})} + \|\Xi_0 D_n v_-\|_{L^2(\mathbb{R}_-; H_\tau^{1/2})},$$

and consequently

$$\begin{aligned} & \|(D_n + iF_-)(D_n + iE_-)\Xi_0 v_-\|_{L^2(\mathbb{R}_-^{n+1})} + |D_n \Xi_0 v_-|_{L^2} + |\Xi_0 v_-|_{H_\tau^1} \\ & \gtrsim \|\Xi_0 v_-\|_{L^2(\mathbb{R}_-; H_\tau^{3/2})} + \|\Xi_0 D_n v_-\|_{L^2(\mathbb{R}_-; H_\tau^{1/2})}. \end{aligned} \quad (2.4.34)$$

We can now multiply (2.4.33) by a large constant and add it to (2.4.34), take  $\tau$  large and use the transmission conditions to find:

$$\begin{aligned} & \|P_\phi^+ \Xi_0 v_+\|_{L^2(\mathbb{R}_+^{n+1})} + \|P_\phi^- \Xi_0 v_-\|_{L^2(\mathbb{R}_-^{n+1})} + T_{\theta, \Theta}^{1/2} + \tau^{1/2} |v_+|_{L^2(\Sigma)} + \|v_+\|_{L^2(\mathbb{R}_+^{n+1})} \\ & \gtrsim \tau^{1/2} \left( |\Xi_0 D_n v_-|_{L^2(\Sigma)} + |\Xi_0 D_n v_+|_{L^2(\Sigma)} + |\Xi_0 v_-|_{H_\tau^1} + |\Xi_0 v_+|_{H_\tau^1} \right) \\ & \quad + \|\Xi_0 v\|_{L^2(\mathbb{R}; H_\tau^{3/2})} + \|\Xi_0 D_n v_-\|_{L^2(\mathbb{R}; H_\tau^{1/2})} + \|\Xi_0 D_n v_+\|_{L^2(\mathbb{R}; H_\tau^{3/2})}. \end{aligned} \quad (2.4.35)$$

This gives the desired estimate in the microlocal region  $\Gamma_{\mu_0}$ .

### Estimate in $\tilde{\Gamma}_\mu$

We consider  $\Xi_1 = \text{op}^w(\chi_{\Gamma,1}) \in \Psi_\tau^0$ . We apply estimate (2.4.23) which gives:

$$\|(D_n + iE_+)(D_n + iF_+)\Xi_1 v_+\|_{L^2(\mathbb{R}_+^{n+1})} \gtrsim |(D_n + iF_+)\Xi_1 v_+|_{H_\tau^{1/2}} + \|(D_n + iF_+)\Xi_1 v_+\|_{L^2(\mathbb{R}_+; H_\tau^1)}. \quad (2.4.36)$$

Thanks to the localization of  $\chi_{\Gamma,1}$  we can use the estimate (2.4.27) for the negative half-space:

$$\begin{aligned} & \|(D_n + iF_-)(D_n + iE_-)\Xi_1 v_-\|_{L^2(\mathbb{R}_-^{n+1})} + \|v_-\|_{L^2(\mathbb{R}_-^{n+1})} + \|D_n v_-\|_{L^2(\mathbb{R}_-^{n+1})} \\ & \gtrsim |(D_n + iE_-)\Xi_1 v_-|_{H_\tau^{1/2}} + \|(D_n + iE_-)\Xi_1 v_-\|_{L^2(\mathbb{R}_-; H_\tau^1)}. \end{aligned} \quad (2.4.37)$$

Estimates (2.4.36) and (2.4.37) imply in particular that we control:

$$\begin{aligned} & c_+ \|P_\phi \Xi_1 v_+\|_{L^2(\mathbb{R}_+^{n+1})} + c_- \|P_\phi \Xi_1 v_+\|_{L^2(\mathbb{R}_-^{n+1})} + \|v_-\|_{L^2(\mathbb{R}_-^{n+1})} + \|D_n v_-\|_{L^2(\mathbb{R}_-^{n+1})} \\ & \gtrsim \tau^{1/2} \left( |c_+ (D_n + iF_+)\Xi_1 v_+|_{L^2(\Sigma)} + |c_- (D_n + iE_-)\Xi_1 v_-|_{L^2(\Sigma)} \right). \end{aligned}$$



Using the transmission conditions (2.2.14), (2.2.15) as well as the triangle inequality we find:

$$\begin{aligned} & \tau^{1/2} \left( |c_+(D_n + iF_+) \Xi_1 v_+|_{L^2(\Sigma)} + |c_-(D_n + iE_-) \Xi_1 v_-|_{L^2(\Sigma)} \right) \\ & \geq \tau^{1/2} |(c_- M_- + c_+ M_+) \Xi_1 v_+|_{L^2(\Sigma)} - C \left( T_{\theta, \Theta}^{1/2} + \tau^{1/2} |v_+|_{L^2(\Sigma)} \right). \end{aligned}$$

We now use the positive ellipticity of  $M_{\pm}$  combined with Lemmata 2.A.1 and 2.A.2 to obtain:

$$\begin{aligned} & \|(D_n + iE_-) \Xi_1 v_-|_{H_\tau^1} + \|(D_n + iE_-) \Xi_1 v_- \|_{L^2(\mathbb{R}_-; H_\tau^1)} + T_{\theta, \Theta}^{1/2} + \tau^{1/2} |v_+|_{L^2(\Sigma)} \\ & \gtrsim |\Xi_1 v_-|_{H_\tau^1} + |\Xi_1 v_+|_{H_\tau^1}. \end{aligned}$$

Hence we control

$$\begin{aligned} & c_+ \|P_\phi \Xi_1 v_+\|_{L^2(\mathbb{R}_+^{n+1})} + c_- \|P_\phi \Xi_1 v_+\|_{L^2(\mathbb{R}_-^{n+1})} + \|v_-\|_{L^2(\mathbb{R}_-^{n+1})} + \|D_n v_-\|_{L^2(\mathbb{R}_-^{n+1})} \\ & + T_{\theta, \Theta}^{1/2} + \tau^{1/2} |v_+|_{L^2(\Sigma)} \gtrsim \tau^{1/2} \left( |\Xi_1 v_-|_{H_\tau^1} + |\Xi_1 v_+|_{H_\tau^1} + |\Xi_1 D_n v_-|_{L^2(\Sigma)} + |\Xi_1 D_n v_+|_{L^2(\Sigma)} \right. \\ & \left. + \|(D_n + iF_+) \Xi_1 v_+\|_{L^2(\mathbb{R}_+; H_\tau^{1/2})} + \|(D_n + iE_-) \Xi_1 v_+\|_{L^2(\mathbb{R}_+; H_\tau^{1/2})} \right). \end{aligned}$$

From here one can proceed as for the region  $\Gamma_{\mu_0}$  by using estimate (2.4.24) for the term

$$\|(D_n + iE_-) \Xi_1 v_+\|_{L^2(\mathbb{R}_+; H_\tau^{1/2})}$$

and estimate (2.4.28) for

$$\|(D_n + iF_+) \Xi_1 v_+\|_{L^2(\mathbb{R}_+; H_\tau^{1/2})}.$$

We finally obtain the desired estimate microlocalized in  $\tilde{\Gamma}_\mu$ :

$$\begin{aligned} & \|P_\phi^+ \Xi_1 v_+\|_{L^2(\mathbb{R}_+^{n+1})} + \|P_\phi^- \Xi_1 v_-\|_{L^2(\mathbb{R}_-^{n+1})} + T_{\theta, \Theta}^{1/2} + \tau^{1/2} |v_+|_{L^2(\Sigma)} + \|v_+\|_{L^2(\mathbb{R}_+^{n+1})} \quad (2.4.38) \\ & + \|D_n v_-\|_{L^2(\mathbb{R}_-^{n+1})} \\ & \gtrsim \tau^{1/2} \left( |\Xi_1 D_n v_-|_{L^2(\Sigma)} + |\Xi_1 D_n v_+|_{L^2(\Sigma)} + |\Xi_1 v_-|_{H_\tau^1} + |\Xi_1 v_+|_{H_\tau^1} \right) \\ & + \|\Xi_1 v\|_{L^2(\mathbb{R}; H_\tau^{3/2})} + \|\Xi_1 D_n v_-\|_{L^2(\mathbb{R}; H_\tau^{1/2})} + \|\Xi_1 D_n v_+\|_{L^2(\mathbb{R}; H_\tau^{3/2})}. \end{aligned}$$

*End of the proof of Lemma 2.4.11.* To finish the proof of Lemma 2.4.11 one has to add estimates (2.4.35) and (2.4.38). This yields the estimate:

$$\begin{aligned} & \|P_\phi \Xi_0 v\|_{L^2(\mathbb{R}^{n+1})} + \|P_\phi \Xi_1 v\|_{L^2(\mathbb{R}^{n+1})} + T_{\theta, \Theta}^{1/2} + \tau^{1/2} |v_+|_{L^2(\Sigma)} + \|v\|_{L^2(\mathbb{R}^{n+1})} + \|D_n v_-\|_{L^2(\mathbb{R}_-^{n+1})} \\ & \gtrsim \sum_{j=0,1} \tau^{1/2} \left( |\Xi_j D_n v_-|_{L^2(\Sigma)} + |\Xi_j D_n v_+|_{L^2(\Sigma)} + |\Xi_j v_-|_{H_\tau^1} + |\Xi_j v_+|_{H_\tau^1} \right) \\ & + \|\Xi_j v\|_{L^2(\mathbb{R}; H_\tau^{3/2})} + \|\Xi_j D_n v_-\|_{L^2(\mathbb{R}; H_\tau^{1/2})} + \|\Xi_j D_n v_+\|_{L^2(\mathbb{R}; H_\tau^{3/2})}. \end{aligned}$$

The right hand side can be estimated from below by simply using the triangle inequality as well as the fact that  $\Xi_0 + \Xi_1 = \text{Id}$ . To bound from above the left hand side we argue

with commutators noticing that  $P_\phi^\pm \in \Psi_\tau^2$ ,  $\Xi_0, \Xi_1 \in \Psi_\tau^0$  and therefore  $[P_\phi^\pm, \Xi_j] \in \Psi_\tau^1$ . This, combined with Sobolev regularity of the pseudo differential calculus allows to estimate as follows:

$$\begin{aligned} \left\| P_\phi^\pm \Xi_j v_\pm \right\|_{L^2(\mathbb{R}_\pm^{n+1})} &\leq \left\| \Xi_j P_\phi^\pm v_\pm \right\|_{L^2(\mathbb{R}_\pm^{n+1})} + \left\| [P_\phi^\pm, \Xi_j] v_\pm \right\|_{L^2(\mathbb{R}_\pm^{n+1})} \\ &\lesssim \left\| P_\phi^\pm v_\pm \right\|_{L^2(\mathbb{R}_\pm^{n+1})} + \|v_\pm\|_{L^2(\mathbb{R}_\pm; H_\tau^1)}. \end{aligned}$$

We take  $\tau \geq \tau_0$ ,  $\tau_0$  large to absorb the error term  $\|v_\pm\|_{L^2(\mathbb{R}_\pm; H_\tau^1)}$ . This concludes the proof of Lemma 2.4.11.  $\square$

#### 2.4.4 Patching together microlocal estimates: End of the proof of Proposition 2.2.6

*End of the proof of Proposition 2.2.6.* We are now ready to finish the proof of the sub-elliptic estimate for  $P$ . We have considered above two regions on each side of the interface (given in our local coordinates by  $\Sigma = \{x_n = 0\}$ ). This yields the following covering of  $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n-1} \ni (t, x, \xi_t, \xi')$ :

$$\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n-1} = (\mathcal{E}_\epsilon^- \cap \mathcal{E}_\epsilon^+) \cup Y.$$

where we have defined

$$Y := (\mathcal{E}_\epsilon^- \cap \mathcal{GH}_\epsilon^+) \cup (\mathcal{E}_\epsilon^+ \cap \mathcal{GH}_\epsilon^-) \cup (\mathcal{GH}_\epsilon^- \cap \mathcal{GH}_\epsilon^+).$$

Let us recall that the regions above have been defined in (2.4.2) and (2.4.3). The crucial remark here is that the definition of our microlocal regions implies that  $Y \subset \mathcal{GH}_\epsilon^- \cup \mathcal{GH}_\epsilon^+$  so that

$$|\xi_t| \gtrsim |\xi'| \quad \text{on } Y.$$

Notice that the definition of the conic regions above imply in particular that  $Y$  and  $\mathcal{E}_\epsilon^- \cap \mathcal{E}_\epsilon^+$  overlap due to the factor  $2\epsilon$  in the definition of  $\mathcal{GH}_\epsilon^\pm$ . That means that we can consider an associated homogeneous partition of unity. More precisely, given a compact set  $K$  of  $\mathbb{R}^{n+1}$  we can introduce  $\chi_j \in \mathcal{S}^0$  homogeneous,  $\text{supp } \chi_1 \cap \{|\xi'| + |\xi_t| \geq 1\} \subset \mathcal{E}_\epsilon^- \cap \mathcal{E}_\epsilon^+$ ,  $\text{supp } \chi_2 \cap \{|\xi'| + |\xi_t| \geq 1\} \subset Y$ <sup>2</sup>,  $\pi_{t,x}(\text{supp } \chi_j) \subset K$  and

$$\chi_1 + \chi_2 = 1.$$

We pick now an element  $u$  of  $\mathcal{W}_\phi^{\theta, \Theta}$  which is compactly supported and write

$$\text{op}^w(1 - \psi_\sigma)u = v_1 + v_2, \quad v_j := \text{op}^w(\chi_j)\text{op}^w(1 - \psi_\sigma)u, j = 1, 2.$$

One needs now to simply put together the estimates already obtained according to the microlocalization of  $v_1$  and  $v_2$ :

<sup>2</sup>In fact one has to construct the partition of unity on the cosphere bundle. That is the reason why we excluded a compact set which includes the zero section. However this does not pose any problem since a function localized on a compact subset of the phase space yields a residual operator and does not have any impact on our estimates.

- We apply Lemma 2.4.11, with  $v_1 = \text{op}^w(\chi_1)\text{op}^w(1 - \psi_\sigma)u$  and we obtain the desired estimate microlocalized in  $\mathcal{E}_\epsilon^- \cap \mathcal{E}_\epsilon^+$ :

$$\begin{aligned} & \|P_\phi v_1\|_{L^2(\mathbb{R}^{n+1})}^2 + \|u\|_{L^2(\mathbb{R}^{n+1})}^2 + |u|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \\ & \gtrsim \tau \|v_1\|_{L^2(\mathbb{R}; H_\tau^1)}^2 + \tau^3 |v_1|_{L^2(\Sigma)}^2 + \tau \left| \nabla v_1^+ \right|_{L^2(\Sigma)}^2 + \tau \left| \nabla v_1^- \right|_{L^2(\Sigma)}^2, \end{aligned} \quad (2.4.39)$$

where we write  $v_1 = H_- v_1^- + H_+ v_1^+$ .

- We consider  $v_2 = \text{op}^w(\chi_2)\text{op}^w(1 - \psi_\sigma)u = H_- v_2^- + H_+ v_2^+$ . Since  $\chi_2$  localizes in particular in a region where  $|\xi_t| \gtrsim |\xi'|$  we can apply Lemma 2.4.10 to  $v_2^-$  to find:

$$\begin{aligned} & \|P_\phi^- v_2^-\|_{L^2(\mathbb{R}_-^{n+1})}^2 + \tau \|D_t v_2^-\|_{L^2(\mathbb{R}_-^{n+1})}^2 + \tau \left| D_t v_2^- \right|_{L^2(\Sigma)}^2 + \|u\|_{L^2(\mathbb{R}_-^{n+1})}^2 + |u|_{L^2(\Sigma)}^2 \\ & \gtrsim \tau \|v_2^-\|_{L^2(\mathbb{R}_-; H_\tau^1)}^2. \end{aligned} \quad (2.4.40)$$

Thanks to the localization of  $\chi_2$  we can apply Lemma 2.4.9 as well to  $v_2^+$

$$\begin{aligned} & \|P_\phi^+ v_2^+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|D_t v_2^+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \left| D_t v_2^+ \right|_{L^2(\Sigma)}^2 + \|u\|_{L^2(\mathbb{R}_+^{n+1})}^2 + |u|_{L^2(\Sigma)}^2 \\ & \gtrsim \tau \|v_2^+\|_{L^2(\mathbb{R}_+; H_\tau^1)}^2 + \tau^3 |v_2^+|_{L^2(\Sigma)}^2 + \tau \left| \nabla_x v_2^+ \right|_{L^2(\Sigma)}^2. \end{aligned}$$

We use then Lemma 2.4.3 and (2.2.15) to control the trace of the derivative of  $v_2^-$  to deduce

$$\begin{aligned} & \|P_\phi^+ v_2^+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|D_t v_2^+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \left| D_t v_2^+ \right|_{L^2(\Sigma)}^2 + \|u\|_{L^2(\mathbb{R}_+^{n+1})}^2 + |u|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \\ & \gtrsim \tau \|v_2^+\|_{L^2(\mathbb{R}_+; H_\tau^1)}^2 + \tau |v_2^+|_{L^2(\Sigma)}^2 + \tau \left| \nabla_x v_2^+ \right|_{L^2(\Sigma)}^2 + \tau \left| \nabla_x v_2^- \right|_{L^2(\Sigma)}^2. \end{aligned} \quad (2.4.41)$$

We can then multiply as usual the above estimate (2.4.41) by a large constant and add it to (2.4.40) to obtain:

$$\begin{aligned} & \|P_\phi v_2\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \|D_t v_2^\pm\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \left| D_t v_2^\pm \right|_{L^2(\Sigma)}^2 + \|u\|_{L^2(\mathbb{R}^{n+1})}^2 + |u|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \\ & \gtrsim \tau \|v_2\|_{L^2(\mathbb{R}; H_\tau^1)}^2 + \tau^3 |v_2|_{L^2(\Sigma)}^2 + \tau \left| \nabla v_2^+ \right|_{L^2(\Sigma)}^2 + \tau \left| \nabla v_2^- \right|_{L^2(\Sigma)}^2. \end{aligned} \quad (2.4.42)$$

Summarising, we have thus shown, for  $j \in \{1, 2\}$ :

$$\begin{aligned} & \|P_\phi \text{op}^w(\chi_j)\text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \|D_t \text{op}^w(\chi_j)\text{op}^w(1 - \psi_\sigma)u_\pm\|_{L^2(\mathbb{R}_\pm^{n+1})}^2 \\ & + \tau \left| D_t \text{op}^w(\chi_j)\text{op}^w(1 - \psi_\sigma)u_\pm \right|_{L^2(\Sigma)}^2 + \|u\|_{L^2(\mathbb{R}^{n+1})}^2 + |u|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \\ & \gtrsim \tau \|\text{op}^w(\chi_j)\text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}; H_\tau^1)}^2 + \tau^3 |\text{op}^w(\chi_j)\text{op}^w(1 - \psi_\sigma)u|_{L^2(\Sigma)}^2 \\ & + \tau \left| \nabla \text{op}^w(\chi_j)\text{op}^w(1 - \psi_\sigma)u_\pm \right|_{L^2(\Sigma)}^2. \end{aligned}$$

We add the two estimates above. We control the right hand side from below by using the triangle inequality as well as the fact

$$\text{op}^w(1 - \psi_\sigma)u = \sum_j \text{op}^w(\chi_j)\text{op}^w(1 - \psi_\sigma)u.$$

For the left hand side we argue with commutators. For instance we control:

$$\begin{aligned} \|P_\phi \text{op}^w(\chi_j)\text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}^{n+1})} &\leq \|\text{op}^w(\chi_j)P_\phi \text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}^{n+1})} \\ &\quad + \|[P_\phi, \text{op}^w(\chi_j)]\text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}^{n+1})}. \end{aligned}$$

Recall the notations  $\check{\theta}_\sigma = 1 - \check{\psi}_\sigma$  and  $\theta_\sigma = 1 - \psi_\sigma$  with  $\check{\psi}_\sigma$  as defined in the proof of Lemma 2.4.9. We write

$$[P_\phi, \text{op}^w(\chi_j)] = [P_\phi, \text{op}^w(\chi_j)\text{op}^w(\check{\theta}_\sigma)] + [P_\phi, \text{op}^w(\chi_j)\text{op}^w(1 - \check{\theta}_\sigma)].$$

Now on the one hand  $\text{supp}(1 - \check{\theta}_\sigma) \cap \text{supp} \theta_\sigma = \emptyset$  and on the other hand  $\theta_\sigma$  localizes in a region where  $\tau \lesssim |\xi'| + |\xi_t|$ . This implies that

$$[P_\phi, \text{op}^w(\chi_j)\text{op}^w(1 - \check{\theta}_\sigma)]\text{op}^w(\theta_\sigma) \in \Psi_\tau^{-\infty},$$

therefore

$$\|[P_\phi, \text{op}^w(\chi_j)\text{op}^w(1 - \check{\theta}_\sigma)]\text{op}^w(\theta_\sigma)u\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|u\|_{L^2(\mathbb{R}^{n+1})}.$$

We have  $[P_\phi, \text{op}^w(\chi_j)\text{op}^w(\check{\theta}_\sigma)] \in \Psi_\tau^1$ . The above considerations as well as the fact that  $\text{op}^w(\chi_j) \in \Psi^0$  finally yield:

$$\begin{aligned} \|P_\phi \text{op}^w(\chi_j)\text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}^{n+1})} &\leq \|\text{op}^w(\chi_j)P_\phi \text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}^{n+1})} \\ &\quad + \|[P_\phi, \text{op}^w(\chi_j)]\text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}^{n+1})} \\ &\lesssim \|\text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}; H_\tau^1)} + \|u\|_{L^2(\mathbb{R}^{n+1})} \\ &\quad + \|P_\phi \text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}^{n+1})}. \end{aligned}$$

We use the same argument for the other terms of the right hand side (noticing that  $\xi_t \in \mathcal{S}_\tau^1$ ). We finally obtain, taking also  $\tau$  large to absorb the error terms (as for instance  $\|\text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}; H_\tau^1)}$ )

$$\begin{aligned} &\|P_\phi \text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \|D_t \text{op}^w(1 - \psi_\sigma)u_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|D_t \text{op}^w(1 - \psi_\sigma)u_-\|_{L^2(\mathbb{R}_-^{n+1})}^2 \\ &+ \tau \|D_t \text{op}^w(1 - \psi_\sigma)u_+\|_{L^2(\Sigma)}^2 + \tau \|D_t \text{op}^w(1 - \psi_\sigma)u_-\|_{L^2(\Sigma)}^2 + \|u\|_{L^2(\mathbb{R}^{n+1})}^2 + |u|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \\ &\gtrsim \tau \|\text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}; H_\tau^1)}^2 + \tau^3 |\text{op}^w(1 - \psi_\sigma)u|_{L^2(\Sigma)}^2 \\ &\quad + \tau |\nabla \text{op}^w(1 - \psi_\sigma)u_+|_{L^2(\Sigma)}^2 + \tau |\nabla \text{op}^w(1 - \psi_\sigma)u_-|_{L^2(\Sigma)}^2. \end{aligned} \quad (2.4.43)$$

Lemma 2.4.7 furnishes an estimate in the complementary sub-region:

$$\begin{aligned} &\|P_\phi \text{op}^w(\psi_\sigma)u\|_{L^2(\mathbb{R}^{n+1})}^2 + \|u\|_{L^2(\mathbb{R}^{n+1})}^2 + T_{\theta, \Theta} + |u_-|_{L^2(\Sigma)}^2 + |u_+|_{L^2(\Sigma)}^2 \\ &\gtrsim \tau \|\text{op}^w(\psi_\sigma)u\|_{L^2(\mathbb{R}; H_\tau^1)}^2 + \tau^3 |\text{op}^w(\psi_\sigma)u|_{L^2(\Sigma)}^2 \\ &\quad + \tau |\nabla \text{op}^w(\psi_\sigma)u_+|_{L^2(\Sigma)}^2 + \tau |\nabla \text{op}^w(\psi_\sigma)u_-|_{L^2(\Sigma)}^2. \end{aligned} \quad (2.4.44)$$

To finish the proof we add (2.4.43) and (2.4.44). For the right hand side we simply use the triangle inequality. For example:

$$\|\text{op}^w(1 - \psi_\sigma)u\|_{L^2(\mathbb{R}^{n+1})} + \|\text{op}^w(\psi_\sigma)u\|_{L^2(\mathbb{R}^{n+1})} \geq \|u\|_{L^2(\mathbb{R}^{n+1})},$$

since  $\text{op}^w(1 - \psi_\sigma) + \text{op}^w(\psi_\sigma) = \text{Id}$ . For the left hand side we argue with commutators as before. We notice that:

$$P_\phi \in \mathcal{D}_\tau^2, \text{op}^w(\psi_\sigma) \in \Psi_\tau^0, 1 - \text{op}^w(\psi_\sigma) \in \Psi_\tau^0 \text{ and } D_t \in \Psi_\tau^1.$$

Therefore in particular

$$[P_\phi, \text{op}^w(\psi_\sigma)] \in \Psi_\tau^1, [P_\phi, 1 - \text{op}^w(\psi_\sigma)] \in \Psi_\tau^1 \text{ and } [D_t, 1 - \text{op}^w(\psi_\sigma)] \in \Psi_\tau^0.$$

We take as usual  $\tau$  sufficiently big to absorb the errors terms and we have thus proven:

$$\begin{aligned} & \|P_\phi u\|_{L^2(\mathbb{R}^{n+1})}^2 + \tau \|D_t u_+\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|D_t u_-\|_{L^2(\mathbb{R}_-^{n+1})}^2 \\ & + \tau |D_t u_+|_{L^2(\Sigma)}^2 + \tau |D_t u_-|_{L^2(\Sigma)}^2 + \|u\|_{L^2(\mathbb{R}; H_\tau^1)}^2 + |u|_{L^2(\Sigma)}^2 + T_{\theta, \Theta} \\ & \gtrsim \tau \|u\|_{L^2(\mathbb{R}; H_\tau^1)}^2 + \tau^3 |u|_{L^2(\Sigma)}^2 + \tau |\nabla u_+|_{L^2(\Sigma)}^2 + \tau |\nabla u_-|_{L^2(\Sigma)}^2. \end{aligned} \quad (2.4.45)$$

Observe that the only term that is missing in the above estimate compared to the statement of Proposition 2.2.6 is the volume norm of the derivatives. The latter are estimated thanks to Lemma 2.4.4. Indeed, we multiply (2.4.45) by a large constant and add it to the estimate of Lemma 2.4.4. This finishes the proof of Proposition 2.2.6.  $\square$

### 2.4.5 Convexification: A perturbation argument

For the sequel it is important to notice that the quality of the estimates obtained for the first order factors (such as (2.4.23) etc) depend on the *imaginary* part of the operator only. Indeed, consider  $L, M \in \Psi^1$  with *real* symbols. One has:

$$\|(D_n + L + iM)v\|^2 = \|(D_n + L)v\|^2 + \|Mv\|^2 + 2 \text{Re}(D_n v, iMv) + 2 \text{Re}(Lv, iMv).$$

Now the fact that  $L$  and  $M$  have real principal symbols implies (since we are working with the Weyl quantization) that

$$2 \text{Re}(Lv, iMv) = i([L, M]v, v).$$

Consequently one has  $[L, M] \in \Psi^1$  and

$$|([L, M]v, v)| \lesssim \|v\|_{L^2(\mathbb{R}; H^{1/2})}^2,$$

which can be absorbed in first-order factor estimates such as (2.4.23).

The Carleman estimate we have obtained so far involves a weight function  $\phi$  depending only on the variable  $x_n$  which in our local coordinates describes the interface  $\Sigma = \{x_n = 0\}$ . However it is important for applications to allow dependence in the other variables too.

We show here that indeed this is possible if one changes “slightly” the weight function  $\phi$ . Recall that we have  $\phi = \alpha_{\pm}x_n + \frac{\beta x_n^2}{2}$ , and consider the new weight

$$\psi = \phi + \kappa(t, x_n, x'), \quad \kappa \text{ real valued quadratic polynomial.}$$

We have taken  $\kappa$  quadratic polynomial for technical reasons related to the action of  $e^{-\delta \frac{D_t^2}{2\tau}}$  (see [Tat95, Hör97, RZ98, LL20]). However this should be sufficient for the applications. We shall now verify that if  $\|\kappa'\|_{L^\infty}$  is sufficiently small then the steps carried out in the preceding sections remain valid.

**Proposition 2.4.19 (Geometric convexification).** *Consider the new weight  $\psi = \phi + \kappa(t, x_n, x')$  with  $\kappa$  a real valued quadratic polynomial. Then there exist  $\eta, \delta_0 > 0$  depending on the coefficients of  $\phi$  and  $P$  such that if  $\|\kappa'\|_{L^\infty} \leq \eta$  and  $\delta \leq \delta_0$  then the estimates of Proposition 2.2.6 and consequently of Theorem 2.2.2 remain valid with the weight  $\psi$ .*

We shall show that the crucial sub-elliptic estimate Proposition 2.2.6 remains valid with the new weight. To do so, we shall revisit the key arguments and show that up to taking  $\|\kappa'\|_{L^\infty}$  small we can see the new conjugated operator as a perturbation of the conjugated operator with the weight depending only on  $x_n$ .

Recall the microlocal weight  $Q_{\delta, \tau}^\psi$  and the conjugated operator  $P_\psi^\pm$  have been defined in (2.2.8) and (2.2.11).

Recall as well that we have the following formula for  $P_\psi$ : (see [LL20, Chapter 3.3.1] or [Hör97])

$$\begin{aligned} c_\pm^{-1}(x)P_\psi^\pm &= (D_n + i\tau\phi' + i\tau\kappa'_{x_n} - \delta\kappa''_{t,t}D_t)^2 \\ &\quad + \sum_{1 \leq j, k \leq n-1} b_{jk}(x)(D_j + i\tau\partial_j\kappa - \delta\kappa''_{t,x_j}D_t)(D_k + i\tau\kappa'_{x_k} - \delta\kappa''_{t,x_k}D_t) \\ &\quad - c_\pm^{-1}(x)((1 - \delta\kappa''_{t,t})D_t + i\tau\partial_t\kappa)^2. \end{aligned}$$

In the sequel we shall denote by  $\tilde{P}_\psi$  the principal part of the operator  $P_\psi$ .

Notice that since  $\kappa$  is supposed to be quadratic  $\kappa''_{t,x_j}, \kappa''_{t,t}$  are actually constants. We shall write  $\psi'_{x_n} = \partial_{x_n}\psi$ . Let us start by showing that the result of Lemma 2.4.4 remains valid, up to taking small values for  $\delta$  and  $\|\kappa'\|_{L^\infty}$ . We write:

$$\begin{aligned} c_+^{-1}(x)\tilde{P}_\psi &= (D_n + i\tau\psi'_{x_n})^2 - 2\delta\kappa''_{t,t}D_t(D_n + \tau\psi'_{x_n}) \\ &\quad + Q(x, D_{x'}) - c_+^{-1}(x)(1 - \delta\kappa''_{t,t})D_t^2 + \tilde{R}_+, \end{aligned}$$

with  $\tilde{R}_+$  a tangential differential operator (see Section 2.A.1) satisfying

$$(\tau\tilde{R}_+v, v)_+ \lesssim \|\kappa'\|_{L^\infty} \left( \tau \|\nabla_{x'}v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^3 \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 \right) + \tau\delta \|\nabla_{t,x'}\|_{L^2(\mathbb{R}_+^{n+1})}^2. \quad (2.4.46)$$

We notice that  $\psi'_{x_n}(0) = \alpha_{\pm} + \kappa'_{x_n}(0, t, x')$ , therefore for  $|x_n|$  sufficiently small and for  $\alpha_{\pm}/\|\kappa'\|_{L^\infty}$  sufficiently large the estimates involving  $\phi'$  remain valid for  $\psi'$  as well. One

obtains then:

$$\begin{aligned}
\|\tilde{P}_\psi v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^2 \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 &\geq 2\tau \operatorname{Re}(\tilde{P}_\psi v, v)_+ \\
&\gtrsim \tau \operatorname{Re}(P_\phi^+ v, v)_+ + (\tau \tilde{R}_+ v, v)_+ - 2\tau \delta \kappa''_{t,t}(D_n v + i\tau \psi'_{x_n} v, D_t v)_+ \\
&\gtrsim \tau \operatorname{Re}(P_\phi^+ v, v)_+ + (\tau \tilde{R}_+ v, v)_+ \\
&\quad - (\tau \delta \|D_n v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \delta \tau^3 \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \delta \tau \|D_t v\|_{L^2(\mathbb{R}_+^{n+1})}^2).
\end{aligned}$$

Using (2.4.46) combined with the estimate obtained in Lemma 2.4.4 for  $\tau \operatorname{Re}(P_\phi^+ v, v)_+$  we obtain, up to taking  $\delta \leq \delta_0$  the same result as in Lemma 2.4.4 but for the convexified weight.

We now investigate what happens with respect to the microlocal regions considered in Section 2.4 and show that in fact we have the same estimates. We recall that there are three main regions. The first is the one where  $\tau$  is large compared to  $|\xi'| + |\xi_t|$ , the second is the non-elliptic region and the third is the elliptic one.

- We localize with  $\operatorname{op}^w(\psi_\sigma)$  in a region where  $\tau \geq \frac{1}{\sigma}(|\xi'| + |\xi_t|)$ . (recall that  $\psi_\sigma$  has been defined in (2.4.5)). This is the region covered in Lemma 2.4.7 and we check that the change of weight function from  $\phi$  to  $\psi$  only adds acceptable error terms. Indeed we write:

$$c_+^{-1}(x)\tilde{P}_\psi = (D_n - \delta \kappa''_{t,t} D_t + i\tau \psi'_{x_n})^2 = \tilde{A}_+ + \tilde{R}_+,$$

where  $\tilde{A}_+ = (D_n - \delta \kappa''_{t,t} D_t + i\tau \psi'_{x_n})^2$  and  $\tilde{R}_+$  has a principal symbol  $\tilde{r}_+$  satisfying

$$\tilde{r}_+ \lesssim |\xi'|^2 + |\xi_t|^2 + \|\kappa'\|_{L^\infty}^2 \tau^2. \quad (2.4.47)$$

As in the proof of Lemma 2.4.7 we have a very good estimate for  $\|\tilde{A}_+ v\|_{L^2(\mathbb{R}_+^{n+1})}^2$ .

We calculate for the first order factor:

$$\begin{aligned}
\|(D_n - \delta \kappa''_{t,t} D_t + i\tau \psi'_{x_n})^2 v\|_{L^2(\mathbb{R}_+^{n+1})}^2 &= \|(D_n - \delta \kappa''_{t,t} D_t)^2\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \|\tau \psi'_{x_n} v\|_{L^2(\mathbb{R}_+^{n+1})}^2 \\
&\quad + 2 \operatorname{Re}(D_n v, i\tau \psi'_{x_n} v)_+ - 2\delta \operatorname{Re}(D_t v, i\tau \psi'_{x_n} v)_+.
\end{aligned}$$

Since an integration by parts in the  $t$  variable yields

$$2\delta \operatorname{Re}(D_t v, i\tau \psi'_{x_n} v)_+ = 0,$$

we obtain the usual estimate:

$$\|(D_n - \delta \kappa''_{t,t} D_t + i\tau \psi'_{x_n})^2 v\|_{L^2(\mathbb{R}_+^{n+1})}^2 \gtrsim \tau^2 \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau |v|_{L^2(\Sigma)}^2.$$

We iterate twice to get, exactly as in the proof of Lemma 2.4.7:

$$\|A_+ v\|_{L^2(\mathbb{R}_+^{n+1})}^2 \gtrsim \tau^4 \|v\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau^3 |v|_{L^2(\Sigma)}^2 + \tau |D_n v|_{L^2(\Sigma)}^2.$$

And we estimate in the same way:

$$\begin{aligned} \|\tilde{P}_{\tilde{\phi}}v\|_{L^2(\mathbb{R}_+^{n+1})}^2 &\geq C \|\tilde{A}_+v\|_{L^2(\mathbb{R}_+^{n+1})}^2 - \|\tilde{R}_+v\|_{L^2(\mathbb{R}_+^{n+1})}^2 \\ &\geq C \left( \tau^4 \|v\|_{L^2(\mathbb{R}_+)}^2 + \tau^3 |v|_{L^2(\Sigma)}^2 + \tau |D_n v|_{L^2(\Sigma)}^2 \right) - \|\tilde{R}_+v\|_{L^2(\mathbb{R}_+^{n+1})}^2 \\ &= \left( (C\tau^4 - \tilde{R}_+^2)v, v \right)_+ + C\tau^3 |v|_{L^2(\Sigma)}^2 + C\tau |D_n v|_{L^2(\Sigma)}^2, \end{aligned}$$

with  $C$  positive constant depending on the the coefficients of  $P_{\tilde{\phi}}$  and of  $\tilde{\phi}$ . According to (2.4.47) we now have

$$C\tau^4 - \tilde{r}_+^2 \geq C\tau^4 - \tilde{C} \left( |\xi'|^2 + |\xi_t|^2 - \|\kappa'\|_{L^\infty}^2 \tau^4 \right).$$

Choosing  $\|\kappa'\|_{L^\infty}^2 \leq \eta^2$  with  $\eta$  sufficiently small depending on  $C$  and  $\tilde{C}$  we have

$$C\tau^4 - \tilde{C} \left( |\xi'|^2 + |\xi_t|^2 - \|\kappa'\|_{L^\infty}^2 \tau^4 \right) \geq \frac{C}{2} \tau^4 - \tilde{C} (|\xi'|^2 + |\xi_t|^2).$$

Then one can choose  $\sigma \leq \sigma_0$  small such that

$$C\tau^4 - \tilde{r}_+^2 \geq \lambda_\tau^4,$$

on the support of  $\psi_\sigma$ . This fixes the choice of  $\sigma_0$ . Then we obtain the same estimate as in Lemma 2.4.7 with the weight  $\psi$ .

- We localize now with  $\text{op}^w(1 - \psi_\sigma)$  in the sub-region where  $\tau \lesssim \frac{1}{\sigma} (|\xi'| + |\xi_t|)$  with  $\sigma \leq \sigma_0$ . In this region one has for  $\|\kappa'\|_{L^\infty} \leq \eta$ , with  $\eta$  sufficiently small that

$$1/C (|\xi'|^2 + |\xi_t|^2) \leq |\xi'|^2 + |\xi_t|^2 - \tau^2 \|\kappa'\|_{L^\infty}^2 \leq C (|\xi'|^2 + |\xi_t|^2),$$

for some  $C > 0$ .

We now investigate the non-elliptic and elliptic regions.

### Non elliptic region

We can treat the non elliptic region as in the proof of Lemmata 2.4.9 and 2.4.10. In this region the localization of  $\text{op}^w(1 - \psi_\sigma)$  implies that  $\tau \lesssim |\xi'| + |\xi_t|$  and being outside the elliptic region implies that  $|\xi_t| \gtrsim |\xi'|$  and thus  $|\xi_t| \gtrsim \tau + |\xi'|$ . Therefore by the same arguments as in the proofs of Lemmata 2.4.9 and 2.4.10 one needs to obtain only one of the trace of the normal derivatives (for the other one we use as again Lemma 2.4.3). The commutator technique works here exactly as before. Indeed, we consider

$$\tilde{Q}_2 = c(x)^{-1} \frac{\tilde{P}_\psi + \tilde{P}_\psi^*}{2}, \quad \tilde{Q}_1 = c(x)^{-1} \frac{\tilde{P}_\psi - \tilde{P}_\psi^*}{2i},$$

and we decompose

$$c(x)^{-1} \tilde{P}_\psi = \tilde{Q}_2 + i\tau \tilde{Q}_1.$$

We observe that

$$\tilde{Q}_2 = (D_n - \delta \kappa_{i,t}'' D_t)^2 - \tau^2 \left( |\psi'_{x_n}|^2 + Q(\nabla_{x'} \kappa) \right) + T_2,$$



and

$$\tilde{Q}_1 = \psi'_{x_n} D_n + D_n \psi'_{x_n} + T_1,$$

where  $T_j$  are *tangential* operators of order  $j$ . We can then proceed exactly as in the proof of Lemma 2.4.9. What is crucial in the proof of this lemma is the sign of  $\phi'(0)$  (which is positive close to  $\Sigma$ ). If  $\phi$  is replaced by  $\psi$  one can also have  $\psi'_{x_n}(0) > 0$ , if we choose  $\alpha_+ / \|\kappa'\|_{L^\infty} = \phi'(0^+) / \|\kappa'\|_{L^\infty}$  is sufficiently large.

### Elliptic region

This is the region  $\mathcal{E}_\epsilon^- \cap \mathcal{E}_\epsilon^+$  with the definition of (2.4.2). Here we are in the situation of Lemma 2.4.11 and we follow [LRL13, Section 4E]. We revisit the factorization argument. To do this we check that in this microlocal region one can define a square root for the operator

$$\begin{aligned} S := & \sum_{1 \leq j, k \leq n-1} b_{jk}(x) (D_j + i\tau\kappa'_{x_j} - \delta\kappa''_{t,x_j} D_t) (D_k + i\tau\kappa'_{x_k} - \delta\kappa''_{t,x_k} D_t) \\ & - c^{-1}(x) ((1 - \delta\kappa''_{t,t}) D_t + i\tau\partial_t \kappa)^2. \end{aligned} \quad (2.4.48)$$

Since its principal symbol  $s$  is no longer real, we study its real part. A sufficient condition for defining a square root is that its real part is positive elliptic. We thus compute:

$$\begin{aligned} \operatorname{Re}(s) = & \sum_{1 \leq j, k \leq n-1} b_{jk}(x) \left( \xi_j \xi_k - \delta\kappa''_{t,x_j} \xi_j \xi_t - \tau^2 \kappa'_{x_j} \kappa'_{x_k} - \delta\kappa''_{t,x_k} \xi_t \xi_k + \delta^2 \kappa''_{t,x_j} \kappa''_{t,x_k} \xi_t^2 \right) \\ & - c^{-1}(x) (1 - \delta\kappa''_{t,t} \xi_t^2 + c^{-1}(x) \tau^2 (\kappa'_t)^2) \\ = & Q(x, \xi') - c^{-1}(x) \xi_t^2 + r, \end{aligned}$$

with

$$|r| \lesssim \delta \|\kappa''\|_{L^\infty} (|\xi'|^2 + |\xi_t|^2) + \tau^2 \|\kappa'\|_{L^\infty}^2. \quad (2.4.49)$$

When microlocalized in  $\mathcal{E}_\epsilon^- \cap \mathcal{E}_\epsilon^+$  one has

$$Q(x, \xi') - c^{-1}(x) \xi_t^2 \geq \epsilon (|\xi'|^2 + |\xi_t|^2).$$

Combining this with estimate (2.4.49) we see that for  $\delta \leq \delta_0$  we have

$$\operatorname{Re}(s) \geq C\epsilon (|\xi'|^2 + |\xi_t|^2) - \tau^2 \|\kappa'\|_{L^\infty}^2.$$

Recalling that  $\tau \leq \frac{1}{\sigma} (|\xi'|^2 + |\xi_t|^2)$  in the support of  $\operatorname{op}^w(1 - \psi_\sigma)$  we obtain that up to taking  $\eta$  small enough we have for  $\|\kappa'\|_{L^\infty} \leq \eta$  in the elliptic region:

$$\operatorname{Re}(s) \gtrsim |\xi'|^2 + |\xi_t|^2.$$

Using a cut-off  $\tilde{\chi}$  which localizes in the elliptic region we define then (as for the definition of  $m_+$  in 2.4.7):

$$\tilde{s} := \tilde{\chi} s + (1 - \tilde{\chi}) \lambda^2.$$

We use then the principal value of the square root for complex numbers to define

$$\tilde{m} = \tilde{s}^{\frac{1}{2}} \in \mathcal{S}^1, \quad \operatorname{Re} \tilde{m} \gtrsim (|\xi'| + |\xi_t|).$$

Consequently we obtain the following almost-factorization:

$$\begin{aligned} & c^{-1}(x)\tilde{P}_\psi v \\ &= \left( D_n - \delta\kappa''_{t,t}D_t + i\tau\psi'_{x_n} - i\text{op}^w(\tilde{m}) \right) \left( D_n - \delta\kappa''_{t,t}D_t + i\tau\psi'_{x_n} + i\text{op}^w(\tilde{m}) \right) v + \tilde{R}v, \end{aligned}$$

where  $v = \text{op}^w(1 - \psi_\sigma)\text{op}^w(\chi)u$  with  $u \in \mathcal{S}_c(\mathbb{R}^{n+1})$ ,  $\chi \in \mathcal{S}^0$  with  $\text{supp}(\chi) \subset \mathcal{E}_\epsilon^- \cap \mathcal{E}_\epsilon^+$  and  $\tilde{R} \in \Psi^1 + \tau\Psi^0 + \Psi^0D_n$ . We have already seen in the beginning of Section 2.4.5 that the imaginary part of the first-order factors determines the quality of the estimate we obtain. We write then:

$$\begin{aligned} c^{-1}(x)\tilde{P}_\psi &= \left( D_n - \delta\kappa''_{t,t}D_t + \text{op}^w(\text{Im } \tilde{m}) + i(\tau\psi'_{x_n} - \text{op}^w(\text{Re } \tilde{m})) \right) \\ &\quad \cdot \left( D_n - \delta\kappa''_{t,t}D_t - \text{op}^w(\text{Im } \tilde{m}) + i(\tau\psi'_{x_n} + \text{op}^w(\text{Re } \tilde{m})) \right) v + \tilde{R}v, \end{aligned}$$

and we focus on the imaginary part of the first order factors above. Since  $\text{op}^w(\text{Re } \tilde{m})$  satisfies the same estimates (elliptic positive) as  $m$  (as defined in Section 2.4.3) the proof remains valid, up to taking  $\alpha_\pm / \|\kappa'\|_{L^\infty}$  sufficiently large and under the same geometric hypothesis similarly to [LRL13, Section 4.5].

Taking everything into account we have obtained the same estimates as in Section 2.4 with  $\psi$  in place of  $\phi$  and in the same microlocal regions. One can then patch these estimates together and obtain the desired result with the convexified weight, exactly as in Section 2.4. This finishes the perturbation argument and therefore the proof of Proposition 2.4.19.  $\square$

## 2.5 The quantitative estimates

With Theorem 2.2.2 at hand we are now ready to obtain the desired quantitative estimates following [LL19]. Firstly we obtain a local quantitative estimate (the analogue of Theorem 3.1 in [LL19]). This estimate allows to propagate the information quantitatively from a small neighborhood of one point belonging to one side of the interface to some other neighborhood of the other side. For this estimate one needs to make sure that the methods used in [LL19] can also be adapted to our context.

We can then use this new local quantitative estimate to cross the interface and then continue the propagation process by directly using the results of [LL19] which are valid as soon as the coefficients of our operator are smooth with respect to the space variable.

### 2.5.1 Some definitions and statement of the local estimate

Before stating the Theorem we need to introduce some notation from [LL19]. We only propagate *low frequency* information with respect to time. Let  $m(t)$  be a smooth radial function, compactly supported in  $|t| < 1$  such that  $m(t) = 1$  for  $|t| < 3/4$ . We shall denote by  $M^\mu$  the Fourier multiplier defined  $M^\mu = m(\frac{D_t}{\mu})$ , that is

$$M^\mu u(t, x) = \mathcal{F}_t^{-1} \left( m \left( \frac{\xi_t}{\mu} \right) \mathcal{F}_t(u)(\xi_t, x) \right) (t).$$

Therefore the upper index  $\mu$  translates to an operator that localizes to times frequencies smaller than  $\mu$ . We shall also use a regularization operator. Given a function

$f \in L^\infty(\mathbb{R}^{n+1})$  we set

$$f_\lambda(t, x) := e^{-\frac{|D_t|^2}{\lambda}} f = \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} f(s, x) e^{-\frac{\lambda}{4}|t-s|^2} ds.$$

That is, the lower index  $\lambda$  produces an analytic function with respect to the time variable. We will need also the combination of the two procedures above. Given  $\lambda, \mu > 0$ , we write  $M_\lambda^\mu$  for the Fourier multiplier defined by  $M_\lambda^\mu = m_\lambda\left(\frac{D_t}{\mu}\right)$  or more precisely:

$$(M_\lambda^\mu u)(t, x) = \mathcal{F}_t^{-1} \left( m_\lambda \left( \frac{\xi_t}{\mu} \right) \mathcal{F}_t(u)(\xi_t, x) \right) (t).$$

That is, we first regularize and then localize.

Let us consider as well a smooth function  $\sigma \in C^\infty(\mathbb{R})$  such that  $\sigma = 1$  in a neighborhood of  $(-\infty, 1]$ , and  $\sigma = 0$  in a neighborhood of  $[2, +\infty)$ . Given a point  $(t_0, x_0) \in \Sigma$  we write

$$\sigma_R(t, x) := \sigma \left( \frac{|(t, x) - (t_0, x_0)|}{R} \right). \quad (2.5.1)$$

We can now state the local quantitative estimate. We recall that  $\Sigma$  is defined as  $\mathbb{R}_t \times S$  and that we are in the geometric situation presented in Section 2.1.1.

**Theorem 2.5.1.** *Let  $(t_0, x_0) \in \Sigma$  given locally by  $\Sigma = \{\phi = 0\}$ . Then there exists  $R_0 > 0$  such that for any  $R \in (0, R_0)$  there exist  $r, \rho, \tilde{\tau}_0 > 0$  such that for any  $\theta \in C_0^\infty(\mathbb{R}_t \times \mathcal{M})$  with  $\theta(x) = 1$  on a neighborhood of  $\{\phi \geq 2\rho\} \cap B((t_0, x_0), 3R)$ , for all  $c_1, \kappa > 0$  there exist  $C, \kappa', \beta_0 > 0$  such that for all  $\beta \leq \beta_0$ , we have*

$$C e^{\kappa\mu} \left( \|M_{c_1\mu}^\mu \theta_{c_1\mu} u\|_{H^1} + \|Pu\|_{L^2(B((t_0, x_0), 4R))} \right) + C e^{-\kappa'\mu} \|u\|_{H^1} \geq \|M_{c_1\mu}^{\beta\mu} \sigma_{r, c_1\mu} u\|_{H^1},$$

for all  $\mu \geq \tilde{\tau}_0/\beta$  and  $u \in \mathcal{W}$  compactly supported.

**Remark 2.5.2.** In the statement of Theorem 2.5.1 uniqueness is propagated quantitatively from  $\Omega_+$  to  $\Omega_-$ . However, we have the same result in the other direction as well. Indeed, this comes from the fact that since there is no assumption on the jump of the coefficient  $c$ , the geometric situation as presented in Section 2.1.1 is completely symmetrical with respect to  $\Omega_+$  and  $\Omega_-$  up to changing the sign of  $c_+ - c_-$ . This will be important in the proof of the semi-global estimate (proof of Theorem 2.5.14) where the local quantitative estimate will be applied successively in chosen points of the interface.

## 2.5.2 Proof of Theorem 2.5.1

We work as usual in geodesic normal coordinates as explained in the local setting of Section 2.2.2. This does not pose any problem since the estimate we are seeking to prove is invariant by change of coordinates in the  $x$  variable. In our context, the first and most important step for the proof of Theorem 2.5.1 will be to state a Carleman estimate with a *geometrically* convexified weight. That is the purpose of Proposition 2.5.3. This estimate will provide the analogue of Corollary 3.6 in [LL19] and will be the starting point of the quantified version of Theorem 2.5.1.

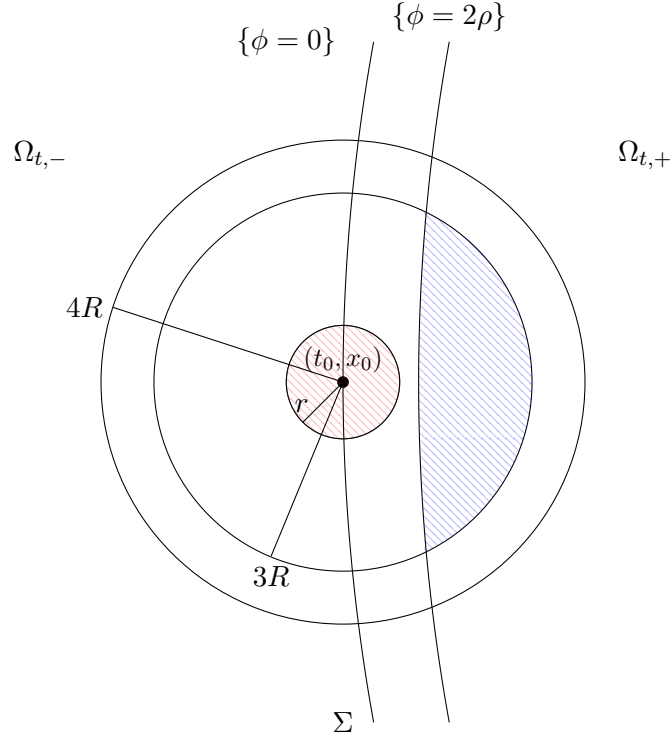


Figure 2.2: Geometry of the local quantitative estimate. The function  $\theta$  localizes in the blue region and  $\sigma$  in the red one. This allows to propagate information from the blue to the red region.

**Proposition 2.5.3.** *Let  $(t_0, x_0) \in \Sigma$  given locally by  $\Sigma = \{\phi = 0\} = \{x_n = 0\}$ . Then there exist  $\Omega$  a neighborhood of  $(t_0, x_0)$ , a function  $\psi : \Omega \rightarrow \mathbb{R}$  which is a quadratic polynomial in  $t$  and  $R_0 > 0$  such that  $B((t_0, x_0), 4R_0) \subset \Omega$  and for any  $R \in (0, R_0]$ , there exist  $\epsilon, \delta, \rho, r, d, \tau_0, C > 0$  such that  $\delta \leq \frac{d}{8}$  and*

1. *The Carleman estimate*

$$C \left( \|Q_{\epsilon, \tau}^{\psi} P u\|_{L^2(\Omega_{t,-} \cup \Omega_{t,+})}^2 + e^{-d\tau} \|e^{\tau\psi} u\|_{H_1^1}^2 + T_{\theta, \Theta} \right) \geq \tau \|Q_{\epsilon, \tau}^{\psi} u\|_{H_1^1}^2,$$

holds for all  $\tau \geq \tau_0$  and all  $u \in \mathcal{W}^{\theta, \Theta}$  with  $\text{supp } u \subset B((t_0, x_0), 4R)$ ;

2. *One has*

$$(B((t_0, x_0), 5R/2) \setminus B((t_0, x_0), R/2) \cap \{-9\delta \leq \psi \leq 2\delta\}) \Subset \{\phi > 2\rho\} \cap B((t_0, x_0), 3R), \quad (2.5.2)$$

$$\{\delta/4 \leq \psi \leq 2\delta\} \cap B((t_0, x_0), 5R/2) \Subset \{\phi > 2\rho\} \cap B((t_0, x_0), 3R), \quad (2.5.3)$$

$$B((t_0, x_0), 2r) \Subset \{-\delta/2 \leq \psi \leq \delta/2\} \cap B((t_0, x_0), R). \quad (2.5.4)$$

**Remark 2.5.4.** The first item is the Carleman estimate we have already obtained and the second one says that we can have this estimate with a weight function whose level sets are

appropriately curved with respect to the interface  $\Sigma$ . This is the geometric convexification part.

*Proof.* We suppose to simplify that  $(t_0, x_0) = 0$ . Theorem 2.2.2 gives us the desired estimate with a weight function  $\phi$  defined in (2.2.9). Proposition 2.4.19 gives the existence of  $\tilde{\delta}$  sufficiently small such that the same estimate is valid with the weight  $\psi$  defined as

$$\psi = \phi - \tilde{\delta}|(t, x)|^2. \quad (2.5.5)$$

More precisely one has the existence of  $R_0, \epsilon, d, \tau_0$  and  $C$  such that

$$C \left( \|Q_{\epsilon, \tau}^\psi P u\|_{L^2(\Omega_{t, -} \cup \Omega_{t, +})}^2 + e^{-d\tau} \|e^{\tau\psi} u\|_{H_\tau^1}^2 + T_{\theta, \Theta} \right) \geq \tau \|Q_{\epsilon, \tau}^\psi u\|_{H_\tau^1}^2,$$

for all  $\tau \geq \tau_0$  and all  $u \in \mathcal{W}^{\theta, \Theta}$  with  $\text{supp } u \subset B(0, 4R)$  and  $R \leq R_0$ . Consider now  $\delta > 0$  such that

$$\delta \leq \frac{\tilde{\delta}R^2}{4 \cdot 10} \Leftrightarrow \frac{\tilde{\delta}R^2}{4} \geq 10\delta.$$

This implies that for  $z = (t, x) \in B(0, 5R/2) \setminus B(0, R/2) \cap \{-9\delta \leq \psi \leq 2\delta\}$  one has

$$\psi \geq -9\delta \Rightarrow \phi \geq \tilde{\delta}|z|^2 - 9\delta \geq \tilde{\delta}\frac{R^2}{4} - 9\delta \geq \delta.$$

We choose then  $\rho = \frac{\delta}{10}$  and (2.5.2) is satisfied. For the second condition we consider again  $z = (t, x) \in \{\delta/4 \leq \psi \leq 2\delta\}$  and we have

$$\psi \geq \frac{\delta}{4} \Rightarrow \phi \geq \frac{\delta}{4} > 2\rho = \frac{\delta}{5},$$

which shows that (2.5.3) is satisfied as well. The last property is simply a continuity statement. Indeed, since  $\psi(0) = \phi(0) = 0$  and  $\psi$  is continuous there exists  $0 < r < R/2$  sufficiently small such that

$$B(0, 2r) \Subset \{-\delta/2 \leq \psi \leq \delta/2\} \cap B(0, R).$$

We choose  $\delta \leq \min(\frac{\tilde{\delta}R^2}{4 \cdot 10}, \frac{d}{8})$  and, with  $\rho = \delta/10$  and  $r$  as above, the proposition is proved.  $\square$

From this point on, one would like to follow the proof of Theorem 3.1 in [LL19] from Step 2: Using the Carleman estimate (in the present setting Proposition 2.5.3). The major difference is that in our context the coefficients of  $P$  are no longer smooth, neither is the weight  $\psi$ . We show however that one can overcome this difficulty with a few modifications. Since this is a rather long and technical proof we only sketch the key arguments and explain, where necessary, what changes in our situation.

**Remark 2.5.5.** Recall that the weight  $\phi$  constructed in Section 2.2 is Lipschitz continuous and in particular one has  $\phi, \psi \in W^{1, \infty}(B(0, 4R))$ .

**Remark 2.5.6.** With the notation introduced in Section 2.2 one has

$$\|Q_{\epsilon, \tau}^\psi P u\|_{L^2(\Omega_{t, -} \cup \Omega_{t, +})}^2 = \|H_- Q_{\epsilon, \tau}^\psi P^- u_-\|_{L^2}^2 + \|H_+ Q_{\epsilon, \tau}^\psi P^+ u_+\|_{L^2}^2.$$

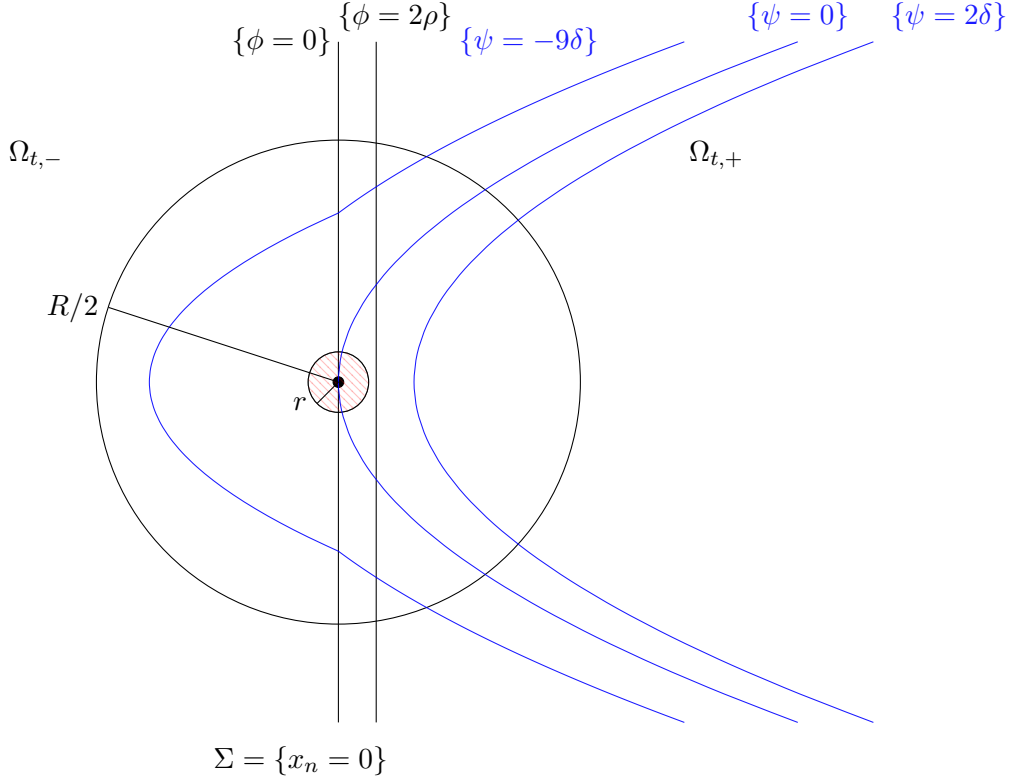


Figure 2.3: We have appropriately curved the level sets of  $\phi$  with the help of the convexified weight  $\psi$ , so that properties (2.5.2) to (2.5.4) of Proposition 2.5.3 are satisfied. Notice that the level sets of  $\psi$  present a singularity when crossing the interface  $\Sigma$ .

We suppose to simplify that  $(t_0, x_0) = 0 \in \Sigma$ . We consider then  $\phi, \psi, R, d, \tau_0, C$  as given by Proposition 2.5.3. We shall use the localization and regularization parameters  $\lambda, \mu > 0$  and we will suppose that  $\lambda \sim \mu$ , that is

$$1/\tilde{C}\mu \leq \lambda \leq \tilde{C}\mu,$$

for some  $\tilde{C} > 0$ .

We introduce now some cut-off functions that will allow us to localize and apply our Carleman estimate. We define  $\chi(s)$  as a smooth function supported in  $(-8, 1)$  such that  $\chi(s) = 1$  for  $s \in [-7, 1/2]$  and set

$$\chi_\delta(s) := \chi(s/\delta). \quad (2.5.6)$$

We define as well  $\tilde{\chi}$  with  $\tilde{\chi} = 1$  on  $(-\infty, 3/2)$  and supported in  $s \leq 2$ , then  $\tilde{\chi}_\delta(s) := \tilde{\chi}(s/\delta)$ .

Following [LL19], for  $u \in \mathcal{W}$  compactly supported we wish to apply our Carleman estimate of Proposition 2.5.3 to

$$\sigma_{2R} \sigma_{R, \lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta, \lambda}(\psi) u,$$

where we recall that  $\sigma_R$  has been defined in (2.5.1). Note that, even though  $u \in \mathcal{W}$  the function  $\sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)u$  does not satisfy the homogeneous transmission conditions. This is why we need to consider non homogeneous transmission conditions.

One should notice that the fact that the operator  $e^{-\frac{|D_t|^2}{\lambda}}$  is tangential with respect to the variable  $x_n$  implies that

$$f \in \mathcal{W}_\phi^{\theta,\Theta} \implies f_\lambda \in \mathcal{W}_\phi^{\theta_\lambda,\Theta_\lambda}.$$

Now the definition of  $\sigma_R$  in (2.5.1) gives  $(\partial_{x_n}\sigma_R)|_\Sigma = 0$  and  $(\partial_{x_n}\sigma_{R,\lambda})|_\Sigma = 0$ . This is true for  $\tilde{\chi}_\delta(\psi)$  also. To see this, we observe that by definition the derivative  $\partial_{x_n}\tilde{\chi}_\delta(\psi)$  is supported in  $\{\psi \geq 3/2\delta\}$  which according to the definition of  $\psi$  in (2.5.5) is away from the interface  $\Sigma$ . However, the term  $\chi_\delta(\psi)$  may not be constant on  $\Sigma$ . More precisely, we have that:

$$\sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)u \in \mathcal{W}^{\theta,\Theta},$$

with  $\theta = 0$  (since  $\sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)u$  remains continuous) and

$$\Theta = \left( (c_+ - c_-)\sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)u\partial_{x_n}(\chi_{\delta,\lambda})(\psi) \right) \Big|_\Sigma.$$

Notice that the definition of  $\chi_\delta$  implies that

$$\text{supp } \partial_{x_n}\chi_\delta|_\Sigma \subset \Sigma \cap \{-8\delta \leq \psi \leq -7\delta\}.$$

This support property combined with Lemma 2.B.7 allow to estimate the term  $T_{\theta,\Theta}$  appearing in the left hand side of the estimate of Proposition 2.5.3 in the following way:

$$T_{\theta,\Theta} = \tau \left| e^{-\epsilon\frac{|D_t|^2}{2\tau}} e^{\tau\psi} \Theta \right|_{L^2(\Sigma)}^2 \lesssim \tau \left| e^{\tau\psi} \Theta \right|_{L^2(\Sigma)}^2 \quad (2.5.7)$$

$$\begin{aligned} &= \tau \left| e^{\tau\psi} \left( (c_+ - c_-)\sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)u\partial_{x_n}(\chi_{\delta,\lambda})(\psi) \right) \right|_{|\Sigma|}^2 \Big|_{L^2(\Sigma)}^2 \\ &\lesssim \tau \mu e^{-14\delta\tau} e^{\frac{\tau^2}{\mu}} |u|_{L^2(\Sigma)}^2 \lesssim \tau \mu e^{-14\delta\tau} e^{\frac{\tau^2}{\mu}} \|u\|_{H^1}^2. \end{aligned} \quad (2.5.8)$$

The other term in the left hand side of Proposition 2.5.3 that we need to estimate is

$$\left\| Q_{\epsilon,\tau}^\psi P\sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)u \right\|_{L^2(\Omega_{t,-} \cup \Omega_{t,+})}.$$

We use again  $\text{supp } \chi_\delta \subset (-\infty, \delta)$  with Lemma 2.B.7 to obtain

$$\begin{aligned} &\left\| Q_{\epsilon,\tau}^\psi P\sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)u \right\|_{L^2(\Omega_{t,-} \cup \Omega_{t,+})} \\ &\leq \left\| Q_{\epsilon,\tau}^\psi \sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)Pu \right\|_{L^2(\Omega_{t,-} \cup \Omega_{t,+})} \\ &\quad + \left\| Q_{\epsilon,\tau}^\psi [\sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi), P]u \right\|_{L^2(\Omega_{t,-} \cup \Omega_{t,+})} \\ &\lesssim \mu^{1/2} e^{C\frac{\tau^2}{\mu}} e^{\delta\tau} \|Pu\|_{L^2(B(0,4R) \cap (\Omega_{t,-} \cup \Omega_{t,+}))} \\ &\quad + \left\| Q_{\epsilon,\tau}^\psi [\sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi), P]u \right\|_{L^2(\Omega_{t,-} \cup \Omega_{t,+})}. \end{aligned} \quad (2.5.9)$$

We need therefore to estimate the commutator appearing in (2.5.9). This is the purpose of the following Lemma.

**Lemma 2.5.7.** *There exists  $R_0 > 0$  such that for any  $R \in (0, R_0)$  there exist  $r, \rho > 0$  such that for any  $\theta \in C_0^\infty(\mathbb{R}^{n+1})$  such that  $\theta(x) = 1$  on a neighborhood of  $\{\phi \geq 2\rho\} \cap B(0, 3R)$ , there exist  $C > 0$ ,  $c > 0$  and  $N > 0$  such that*

$$Ce^{2\delta\tau} \left\| M_\lambda^{2\mu} \theta_\lambda u \right\|_{H^1} + C\mu^{1/2}\tau \left( e^{-8\delta} + e^{-\frac{c\mu^2}{8\tau}} + e^{-c\mu} e^{\delta\tau} \right) e^{C\frac{\tau^2}{\mu}} e^{\delta\tau} \|u\|_{H^1} \quad (2.5.10)$$

$$\geq \left\| Q_{\epsilon, \tau}^\psi [\sigma_{2R} \sigma_{R, \lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta, \lambda}(\psi), P] u \right\|_{L^2(\Omega_{t, -} \cup \Omega_{t, +})}, \quad (2.5.11)$$

for any  $u \in \mathcal{W}$  compactly supported,  $\mu \geq 1$ ,  $\lambda \sim \mu$  and  $\tau \geq 1$ .

This is the difference with respect to the situation of Lemma 3.7 in [LL19]: the coefficients  $p_\alpha$  are no longer smooth. They remain bounded however and the regularization-localization operators are tangential to the interface where the coefficients may jump. This allows to use the same techniques. Moreover in our situation we can exploit the fact that the coefficients  $p_\alpha$  are independent of  $t$  and therefore they commute with  $Q_{\epsilon, \tau}^\psi$ .

*Proof.* The preceding remark allows us to write our operator as

$$P = \sum_{|\alpha| \leq 2} p_\alpha(x) \partial^\alpha,$$

with  $p_\alpha = H_- p_\alpha^- + H_+ p_\alpha^+$  and  $p_\alpha^\pm = p_\alpha^\pm(x)$  smooth functions independent of  $t$ .

This implies:

$$\begin{aligned} \sum_{\pm} \left\| H_\pm Q_{\epsilon, \tau}^\psi [\sigma_{2R} \sigma_{R, \lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta, \lambda}(\psi), P_\pm] u_\pm \right\|_{L^2} &= \sum_{|\alpha| \leq 2} \left\| H_\pm Q_{\epsilon, \tau}^\psi [\sigma_{2R} \sigma_{R, \lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta, \lambda}(\psi), p_\alpha(x) \partial^\alpha] u \right\|_{L^2} \\ &= \sum_{|\alpha| \leq 2} \left\| Q_{\epsilon, \tau}^\psi p_\alpha(x) [\sigma_{2R} \sigma_{R, \lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta, \lambda}(\psi), \partial^\alpha] u \right\|_{L^2} \\ &= \sum_{|\alpha| \leq 2} \left\| p_\alpha(x) Q_{\epsilon, \tau}^\psi [\sigma_{2R} \sigma_{R, \lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta, \lambda}(\psi), \partial^\alpha] u \right\|_{L^2} \\ &\leq C \sum_{|\alpha| \leq 2} \left\| Q_{\epsilon, \tau}^\psi [\sigma_{2R} \sigma_{R, \lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta, \lambda}(\psi), \partial^\alpha] u \right\|_{L^2}. \end{aligned} \quad (2.5.12)$$

By the Leibniz rule we can write

$$\begin{aligned} &\partial^\alpha (\sigma_{2R} \sigma_{R, \lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta, \lambda}(\psi) u) \\ &= \sum_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = \alpha} C_{(\alpha_i)} \partial^{\alpha_1} (\chi_{\delta, \lambda}(\psi)) \partial^{\alpha_2} (\sigma_{2R}) \partial^{\alpha_3} (\sigma_{R, \lambda}) \partial^{\alpha_4} (\tilde{\chi}_\delta(\psi)) \partial^{\alpha_5} u. \end{aligned}$$

We split then the commutators in (2.5.12) in a sum of differential operators of order one as follows:

$$\sum_{|\alpha| \leq 2} [\partial^\alpha, \sigma_{2R} \sigma_{R, \lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta, \lambda}(\psi)] = B_1 + B_2 + B_3 + B_4,$$

where:

1.  $B_1$  contains the terms with  $\alpha_1 \neq 0$  and  $\alpha_2 = \alpha_4 = 0$ ;



2.  $B_2$  contains some terms with  $\alpha_2 \neq 0$ ;
3.  $B_3$  contains the terms with  $\alpha_3 \neq 0$  and  $\alpha_1 = \alpha_2 = \alpha_4 = 0$ ;
4.  $B_4$  contains some terms with  $\alpha_4 \neq 0$ .

Now we provide estimates for each of the terms above. Before that we further decompose  $B_1$  in two terms by observing

$$(\chi_{\delta,\lambda})' = \frac{1}{\delta}(\chi')_{\delta,\lambda} = \frac{1}{\delta}(\mathbb{1}_{[\delta/2,\delta]}\chi')_{\delta,\lambda} + \frac{1}{\delta}(\mathbb{1}_{[-8\delta,-7\delta]}\chi')_{\delta,\lambda} = \chi_{\delta,\lambda}^+ + \chi_{\delta,\lambda}^-,$$

with  $\chi_{\delta,\lambda}^+ := \frac{1}{\delta}(\mathbb{1}_{[\delta/2,\delta]}\chi')_{\delta,\lambda}$  and  $\chi_{\delta,\lambda}^- := \frac{1}{\delta}(\mathbb{1}_{[-8\delta,-7\delta]}\chi')_{\delta,\lambda}$ , and we have used the properties of  $\text{supp } \chi'$ .

1. This allows to decompose  $B_1$  as a sum of generic terms of the form

$$B_{\pm} = b_{\pm}\partial^{\gamma} = f\sigma_{2R}\partial^{\beta}(\sigma_{R,\lambda})\chi_{\delta,\lambda}^{\pm}\tilde{\chi}_{\delta}(\psi)\partial^{\gamma}, \quad (2.5.13)$$

where  $|\beta|, |\gamma| \leq 1$ ,  $f \in L^{\infty}(\mathbb{R}^{n+1})$ , compactly supported and analytic in  $t$ . Notice that in the absence of the regularization parameter  $\lambda$  the terms of  $B_+$  would be supported in

$$\left(\{\delta/2 \leq \psi \leq \delta\} \cap B(0, 2R)\right) \subset \left(\{\phi > 2\rho\} \cap \{\psi \leq \delta\} \cap B(0, 2R)\right)$$

and those of  $B_-$  in  $\{-8\delta \leq \psi \leq -7\delta\} \cap B(0, 2R)$ .

2.  $B_2$  consists of terms where there is at least one derivative on  $\sigma_{2R}$  and contains terms of the form

$$\tilde{b}\partial^{\beta}(\sigma_{R,\lambda})(\chi_{\delta,\lambda})^{(k)}(\psi)\tilde{\chi}_{\delta}(\psi)\partial^{\gamma},$$

where  $k, |\beta|, |\gamma| \leq 1$  with  $\tilde{b}$  bounded and supported in  $B(0, 4R) \setminus B(0, 2R)$ .

3.  $B_3$  consists of terms where there is at least one derivative on  $\sigma_{R,\lambda}$  and none on  $\chi_{\delta,\lambda}(\psi)$ ,  $\tilde{\chi}_{\delta}(\psi)$  and  $\sigma_R$ . These are terms of the form

$$f\sigma_{2R}\partial^{\beta}(\sigma_{R,\lambda})\chi_{\delta,\lambda}(\psi)\tilde{\chi}_{\delta}(\psi)\partial^{\gamma},$$

where  $f$  is bounded and independent of  $t$ ,  $|\beta| = 1$  and  $|\gamma| \leq 1$ . Notice that in the absence of the regularization parameter  $\lambda$  these terms would be supported in

$$\left(\{-8\delta \leq \psi \leq -7\delta\} \cap B(0, 2R) \setminus B(0, R)\right) \subset \left(\{\phi > 2\rho\} \cap \{\psi \leq \delta\} \cap B(0, 2R)\right).$$

4.  $B_4$  consists of terms where there is at least one derivative on  $\tilde{\chi}_{\delta}(\psi)$  and contains terms of the form

$$\tilde{b}\partial^{\beta}(\sigma_{R,\lambda})(\chi_{\delta,\lambda})^{(k)}(\psi)\partial^{\gamma},$$

where  $k, |\beta|, |\gamma| \leq 1$  and the function  $\tilde{b}$  is bounded and supported in  $B(0, 4R) \cap \{3\delta/2 \leq \psi \leq 2\delta\}$ .

To prove Lemma 2.5.7 one needs to provide appropriate estimates for a generic term from each of the four groups defined above. What happens is that terms containing derivatives of a non regularized function (that is without the subscript  $\lambda$ ) are easier to handle since they localize exactly. To deal with derivatives of regularized functions requires more work since they produce additional errors coming from the non exact localization properties.

**Estimating  $B_-$**  (defined in 2.5.13). We use Lemma 2.B.7 applied to  $\chi_\delta^-$  to find:

$$\left\| Q_{\epsilon, \tau}^\psi B_- u \right\|_{L^2} \leq \left\| e^{\tau\psi} B_- \right\|_{L^2} \leq C_\delta \lambda^{1/2} e^{-7\delta} e^{\frac{\tau^2}{\lambda}} \|u\|_{H^1} \leq C \mu^{1/2} e^{-7\delta} e^{C \frac{\tau^2}{\mu}} \|u\|_{H^1}.$$

**Estimating  $B_2$ .** We use Lemma 2.B.1 applied to  $\tilde{b}$  and  $\partial^\beta(\sigma_{R, \lambda})$  which have supports away from each other. We apply then Lemma 2.B.7 to  $\chi_\delta^{(k)}$  using its support properties to find:

$$\begin{aligned} \left\| Q_{\epsilon, \tau}^\psi B_2 u \right\|_{L^2} &\leq \left\| e^{\tau\psi} B_2 u \right\|_{L^2} \leq \left\| \tilde{b} \partial^\beta(\sigma_{R, \lambda}) \right\|_{L^\infty} \left\| \chi_\delta^{(k)} u \right\|_{L^2} \leq C \lambda^{1/2} e^{\delta\tau} e^{\frac{\tau^2}{\lambda}} e^{-c\lambda} \|u\|_{H^1} \\ &\leq C \mu^{1/2} e^{\delta\tau} e^{C \frac{\tau^2}{\mu}} e^{-c\mu} \|u\|_{H^1}. \end{aligned}$$

**Estimating  $B_4$ .** We use Lemma 2.B.1 applied to  $(\chi_{\delta, \lambda})^{(k)}(\psi)$  and  $\mathbb{1}_{[3\delta/2, 2\delta]}$  to find thanks to the localization of  $\tilde{\chi}'_\delta(\psi)$ :

$$\left\| Q_{\epsilon, \tau}^\psi B_4 u \right\|_{L^2} \leq \left\| e^{\tau\psi} B_4 u \right\|_{L^2} \leq C e^{2\delta\tau} e^{-c\mu} \|u\|_{H^1}.$$

**First estimates on  $B_+$**  (defined in (2.5.13)) **and  $B_3$ .** These are the most difficult terms since here the derivative does not localize *exactly*. We have for  $B_*$  with  $* = +$  or  $* = 3$ :

$$\begin{aligned} \left\| Q_{\epsilon, \tau}^\psi B_* u \right\|_{L^2} &= \left\| e^{-\epsilon \frac{|D_t|^2}{2\tau}} e^{\tau\psi} B_* u \right\|_{L^2} \leq \left\| e^{-\epsilon \frac{|D_t|^2}{2\tau}} M_\lambda^\mu e^{\tau\psi} B_* u \right\|_{L^2} + \left\| e^{-\epsilon \frac{|D_t|^2}{2\tau}} (1 - M_\lambda^\mu) e^{\tau\psi} B_* u \right\|_{L^2} \\ &\leq \left\| M_\lambda^\mu e^{\tau\psi} B_* u \right\|_{L^2} + C \lambda^{1/2} \left( e^{-\frac{\epsilon\mu^2}{8\tau}} + e^{-c\mu} \right) e^{C \frac{\tau^2}{\mu}} e^{\delta\tau} \|u\|_{H^1}, \end{aligned}$$

where we have applied successively Lemma 2.B.8 and Lemma 2.B.7. We estimate now the first term in the above inequality, with  $B_* = b_* \partial^\gamma$ :

$$\left\| M_\lambda^\mu e^{\tau\psi} B_* u \right\|_{L^2} \leq \left\| M_\lambda^\mu e^{\tau\psi} b_* (1 - M_\lambda^{2\mu}) \partial^\gamma u \right\|_{L^2} + \left\| M_\lambda^\mu e^{\tau\psi} b_* M_\lambda^{2\mu} \partial^\gamma u \right\|_{L^2}.$$

We apply then Lemma 2.B.9 which gives

$$\left\| M_\lambda^\mu e^{\tau\psi} b_* (1 - M_\lambda^{2\mu}) \partial^\gamma u \right\|_{L^2} \leq C \tau^N e^{C \frac{\tau^2}{\mu}} e^{2\delta\tau - c\mu} \|u\|_{H^1}.$$

Using the fact that

$$\left\| M_\lambda^\mu e^{\tau\psi} b_* M_\lambda^{2\mu} \partial^\gamma u \right\|_{L^2} \leq \left\| e^{\tau\psi} b_* M_\lambda^{2\mu} \partial^\gamma u \right\|_{L^2}$$

we have thus obtained

$$\left\| Q_{\epsilon, \tau}^\psi B_* u \right\|_{L^2} \leq \left\| e^{\tau\psi} b_* M_\lambda^{2\mu} \partial^\gamma u \right\|_{L^2} + C \mu^{1/2} \tau^N \left( e^{-\frac{\epsilon\mu^2}{8\tau}} + e^{\delta\tau} e^{-c\mu} \right) e^{C\frac{\tau^2}{\mu}} e^{\delta\tau} \|u\|_{H^1}.$$

That is we “almost commuted”  $M_\lambda^\mu$  with  $e^{\tau\psi} B_*$ . To finish the proof of Lemma 2.5.7 we need therefore to estimate  $\left\| e^{\tau\psi} b_* M_\lambda^{2\mu} \partial^\gamma u \right\|_{L^2}$ , for  $b_* = b_+$  and  $b_* = b_3$ .

This is done exactly as in [LL19]. As we have already seen in the course of this proof, in our case  $b_*$  is less regular. However,  $u \in H^1$ , and  $b_* \in L^\infty$  satisfies the same localization properties as in [LL19]. For the sake of completeness we sketch the estimate for  $b_+$ .

**Estimating  $B_+$ .** A generic term of  $B_+$  has the form

$$b_+ \partial^\gamma = f \tilde{b}_\lambda \chi_{\delta, \lambda}^+(\psi) \tilde{\chi}(\psi) \partial^\gamma,$$

where  $\tilde{b} = \partial^\beta(\sigma_R)$ ,  $|\beta| \leq 1$ , is supported in  $B(0, 2R)$  and  $f$  is bounded. We decompose  $\mathbb{R}^{n+1}$  as

$$\begin{aligned} \mathbb{R}^{n+1} &= O_1 \cup O_2 \cup O_3, \quad \text{with} \\ O_1 &= \{\psi \notin [\delta/4, 2\delta]\} \cap B(0, 5R/2), \\ O_2 &= B(0, 5R/2)^c, \\ O_3 &= \{\psi \in [\delta/4, 2\delta]\} \cap B(0, 5R/2). \end{aligned}$$

For the region  $O_1$  we use the fact that  $\chi_\delta^+$  is supported in  $[\delta/2, \delta]$  and then Lemma 2.B.1 with  $f_2 = \mathbb{1}_{[\delta/4, 2\delta]^c}$ . For the region  $O_2$  we exploit as well the almost localization by using Lemma 2.B.1 and Lemma 2.B.7.

For the region  $O_3$  we start by noticing that thanks to the geometric convexification property (2.5.3) one can find a smooth  $\tilde{\theta}$  with  $\tilde{\theta} = 1$  on a neighborhood of  $O_3$  and supported in  $\{\phi > 2\rho\} \cap B(0, 3R)$ . We estimate then

$$\left\| e^{\tau\psi} b_+ M_\lambda^{2\mu} \partial^\gamma u \right\|_{L^2(O_3)} \leq C e^{\delta\tau} \left\| M_\lambda^{2\mu} \partial^\gamma u \right\|_{L^2(O_3)} \leq C e^{\delta\tau} \left\| \tilde{\theta}_\lambda M_\lambda^{2\mu} \partial^\gamma u \right\|_{L^2}.$$

The final step is to commute  $\theta_\lambda$  with  $M_\lambda^{2\mu}$ . Let  $\tilde{\tilde{\theta}} \in C_0^\infty$  be such that  $\tilde{\tilde{\theta}} = 1$  on a neighborhood of  $\text{supp } \tilde{\theta}$  and supported in  $\{\phi > 2\rho\} \cap B(0, 3R)$ . Now recall that from the assumption of Theorem 2.5.1 we are given  $\theta \in C_0^\infty$  with  $\theta = 1$  on  $\{\phi > 2\rho\} \cap B(0, 3R)$  and consequently one has that  $\theta = 1$  in a neighborhood of  $\text{supp } \tilde{\tilde{\theta}}$ . We use then Lemma 2.B.4 which gives

$$\left\| \tilde{\theta}_\lambda M_\lambda^{2\mu} \partial^\gamma u \right\|_{L^2} \leq C \left\| \tilde{\tilde{\theta}} M_\lambda^{2\mu} u \right\|_{H^1} + C e^{-c\lambda} \|u\|_{H^1},$$

and then Lemma 2.B.6:

$$\left\| \tilde{\tilde{\theta}} M_\lambda^{2\mu} u \right\|_{H^1} \leq \left\| M_\lambda^{2\mu} \theta_\lambda u \right\|_{H^1} + C e^{-c\mu} \|u\|_{H^1}.$$

Consequently, we have obtained in each region  $O_*$  an estimate of the same type as is (2.5.10). This gives the estimate for  $B_+$ .

**Estimating  $B_3$ .** This is done in a similar manner to  $B_+$ .

We have finally obtained estimates of the type of (2.5.10) for all of the generic terms of the commutator. This proves Lemma 2.5.7.  $\square$

With the commutator estimate at hand we can now show:

**Lemma 2.5.8.** *There exists  $R_0 > 0$  such that for any  $R \in (0, R_0)$  there exist  $r, \rho > 0$  such that for any  $\theta \in C_0^\infty(\mathbb{R}^{n+1})$  such that  $\theta(x) = 1$  on a neighborhood of  $\{\phi > 2\rho\} \cap B(0, 3R)$ , there exist  $\mu_0, C, c, N > 0$  such that:*

$$C\mu^{1/2}e^{C\frac{\tau^2}{\lambda}}e^{\delta\tau}\|Pu\|_{L^2(B(0,4R))} + C\mu^{1/2}\tau^N\left(e^{-8\delta\tau} + e^{-\frac{\epsilon\mu^2}{8\tau}} + e^{\delta\tau - c\mu}\right)e^{C\frac{\tau^2}{\mu}}e^{\delta\tau}\|u\|_{H^1} \\ + Ce^{2\delta\tau}\|M_\lambda^{2\mu}\theta_\lambda u\|_{H^1} \geq \tau\|Q_{\epsilon,\tau}^\psi\sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)u\|_{H_\tau^1},$$

for  $u \in \mathcal{W}$  compactly supported,  $\mu \geq \mu_0, \lambda \sim \mu$  and  $\tau \geq \tau_0$ .

*Proof.* We apply the estimate of Proposition 2.5.3 to  $w := \sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)u$ . This gives:

$$C\left(\|Q_{\epsilon,\tau}^\psi Pw\|_{L^2(\Omega_{t,-} \cup \Omega_{t,+})}^2 + e^{-d\tau}\|e^{\tau\psi}w\|_{H_\tau^1}^2 + T_{\theta,\Theta}\right) \geq \tau\|Q_{\epsilon,\tau}^\psi w\|_{H_\tau^1}^2,$$

We need therefore to estimate the three terms appearing in the left hand side of the above inequality.

- For the term  $\|Q_{\epsilon,\tau}^\psi Pw\|_{L^2(\Omega_{t,-} \cup \Omega_{t,+})}^2$  we simply combine estimate (2.5.9) with Lemma 2.5.7.
- For the term  $T_{\theta,\Theta}$  we use the estimate (2.5.8).
- It remains to deal with  $e^{-d\tau}\|e^{\tau\psi}w\|_{H_\tau^1}^2$ . Recall that  $\chi_\delta$  is supported in  $(-8\delta, \delta)$ . This implies, using Lemma 2.B.7:

$$e^{-d\tau}\|e^{\tau\psi}w\|_{H_\tau^1} = e^{-d\tau}\|e^{\tau\psi}\sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)u\|_{H_\tau^1} \\ \leq Ce^{-d\tau}\lambda^{1/2}\tau e^{\delta\tau}e^{\frac{\tau^2}{\lambda}}\|u\|_{H^1} \\ \leq \mu^{1/2}\tau e^{(\delta-d)\tau}e^{C\frac{\tau^2}{\mu}}\|u\|_{H^1} \\ \leq \mu^{1/2}\tau e^{-7\delta\tau}e^{C\frac{\tau^2}{\mu}}\|u\|_{H^1},$$

where for the last inequality we used the fact  $\delta \leq \frac{d}{8}$ , thanks to Proposition 2.5.3.

This finishes the proof of Lemma 2.5.8.  $\square$

### The complex analysis argument

The final step of the proof of Theorem 2.5.1 consists in transferring the estimate provided by Lemma 2.5.8 from

$$\|Q_{\epsilon,\tau}^\psi\sigma_{2R}\sigma_{R,\lambda}\tilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)u\|_{H_\tau^1}$$

to  $\|M_{c_1\mu}^{\beta\mu}\sigma_{\tau,c_1\mu}u\|_{H^1}$ . The presence of the microlocal weight  $Q_{\epsilon,\tau}^\psi$  makes this part highly non-trivial and one has to work by duality. In [Hör97, Tat95] the authors prove a qualitative unique continuation result. They use the following strategy:

1. For any test function  $f$  define the distribution  $h_f = \psi_*(fu)$  by

$$\langle h_f, w \rangle_{\mathcal{E}'(\mathbb{R}), C^\infty(\mathbb{R})} = \langle fu, w(\psi) \rangle_{\mathcal{E}'(\mathbb{R}^{n+1}), C^\infty(\mathbb{R}^{n+1})}.$$

That measures  $fu$  along the level sets of  $\psi$ .

2. Consider the Fourier transform of  $h_f$  and use

- the Carleman estimate to bound the quantity  $\widehat{h}_f(i\tau)$  for  $\tau$  large
- an *a priori* estimate on  $\widehat{h}_f(\zeta)$  for  $\zeta \in \mathbb{C}$  which gives sub-exponential growth

Thanks to a Phragmén–Lindelöf theorem transfer the estimate provided by the Carleman estimate from the upper imaginary axis to the whole upper plane.

3. Use a Paley-Wiener theorem to deduce from the bound obtained for the Fourier transform of  $h_f$  an information about its support.

Now in [LL19] both the Phragmén–Lindelöf and the Paley-Wiener theorem are replaced by some precise estimates for  $\widehat{h}_f$ . In our case the proof works in exactly the same way. The only difference is that here  $\psi$  and  $u$  are no longer smooth but they are Lipschitz continuous. However this does not affect the proof.

**Lemma 2.5.9.** *Under the above assumptions, there exists  $\tilde{\tau}_0$  such that for any  $\kappa, c_1 > 0$ , there exist  $\beta_0, C, c > 0$ , such that for any  $0 < \beta < \beta_0$ , for all  $\mu \geq \frac{\tilde{\tau}_0}{\beta}$  and  $u \in \mathcal{W}$  compactly supported, one has:*

$$\begin{aligned} & C e^{-c\mu} \left( e^{\kappa\mu} \left( \|M_\lambda^{2\mu} \theta_\lambda u\|_{H^1} + \|Pu\|_{L^2(B(0,4R))} \right) + \|u\|_{H^1} \right) \\ & \geq \|M^{\beta\mu} \sigma_{2R} \sigma_{R,\lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta,\lambda}(\psi) \eta_{\delta,\lambda}(\psi) u\|_{H^1}, \end{aligned}$$

with  $\lambda = 2c_1\mu$ , and

$$\eta \in C_0^\infty((-4, 1)), \quad \eta = 1 \text{ in } [-1/2, 1/2] \quad \text{and} \quad \eta_\delta(s) := \eta(s/\delta).$$

*Proof.* Recall that  $\psi \in H^1$  (see (2.5.5)). For any test function  $f \in \mathcal{S}(\mathbb{R}^{n+1})$  we define then the distribution

$$\langle h_f, w \rangle_{\mathcal{E}'(\mathbb{R}), C^\infty(\mathbb{R})} := \langle (M^{\beta\mu} f \sigma_{2R} \sigma_{R,\lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta,\lambda}(\psi), w(\psi))_{H^{-1}(\mathbb{R}^{n+1}), H_0^1(\mathbb{R}^{n+1})}.$$

We work with  $w = \eta_{\delta,\lambda}$  and estimate the quantity  $\langle h_f, \eta_{\delta,\lambda} \rangle_{\mathcal{E}'(\mathbb{R}), C^\infty(\mathbb{R})}$ . This done exactly as in [LL19]. We use the formula for the Fourier transform of a compactly supported distribution to obtain an a priori estimate on  $\widehat{h}_f(\xi)$  for  $\xi \in \mathbb{R}$ . Then we use Lemma 2.5.8 to obtain an estimate for  $\widehat{h}_f(\zeta)$  for  $\zeta = i\tau$ . All the complex analysis arguments that follow remain valid in our context. Indeed, these arguments do not involve the  $t, x$  space but they are carried out in the complexification of our Carleman large parameter  $\tau$ .  $\square$

*End of the proof of Theorem 2.5.1.* We are ready to finish the proof the local quantitative estimate. The last thing we need to do is to estimate the term

$$\left\| M^{\beta\mu} \sigma_{2R} \sigma_{R,\lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta,\lambda}(\psi) \eta_{\delta,\lambda}(\psi) u \right\|_{H^1}$$

appearing in the right hand side of Lemma 2.5.9. This is done as in [LL19]. Indeed, all the operations are tangential and thanks to our Proposition 2.5.3 the geometric context is the same as in [LL19]. We sketch the end of the proof in a concise way.

We have:

$$\begin{aligned} \left\| M_\lambda^{\beta\mu} \sigma_{2R} \sigma_{R,\lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta,\lambda}(\psi) \eta_{\delta,\lambda}(\psi) u \right\|_{H^1} &\leq \left\| M_\lambda^{\frac{\beta\mu}{2}} (1 - M^{\beta\mu}) \sigma_{R,\lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta,\lambda}(\psi) \eta_{\delta,\lambda}(\psi) u \right\|_{H^1} \\ &\quad + \left\| M_\lambda^{\frac{\beta\mu}{2}} M^{\beta\mu} \sigma_{R,\lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta,\lambda}(\psi) \eta_{\delta,\lambda}(\psi) u \right\|_{H^1}. \end{aligned}$$

To control the first term we use Lemma 2.B.1. For the second one we use Lemma 2.5.9. We find for  $0 < \beta < \beta_0$ , for all  $\mu \geq \frac{\tilde{r}_0}{\beta}$  and  $\lambda = 2c_1\mu$ :

$$\left\| M_\lambda^{\beta\mu} \sigma_{2R} \sigma_{R,\lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta,\lambda}(\psi) \eta_{\delta,\lambda}(\psi) u \right\|_{H^1} \leq C e^{-c\mu} \left( e^{\kappa\mu} \left( \left\| M_\lambda^{2\mu} \theta_\lambda u \right\|_{H^1} + \|Pu\|_{L^2(B(0,4R))} \right) + \|u\|_{H^1} \right).$$

Next we combine the above estimate with Lemma 2.B.6:

$$\begin{aligned} \left\| M^{\frac{\beta\lambda}{4}} \sigma_{r,\lambda} u \right\|_{H^1} &\leq C e^{-c\mu} \left( e^{\kappa\mu} \left( \left\| M_\lambda^{2\mu} \theta_\lambda u \right\|_{H^1} + \|Pu\|_{L^2(B(0,4R))} \right) + \|u\|_{H^1} \right) \\ &\quad + \left\| \sigma_{r,\lambda} M_\lambda^{\frac{\beta\mu}{2}} (1 - \sigma_{2R} \sigma_{R,\lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta,\lambda}(\psi) \eta_{\delta,\lambda}(\psi)) u \right\|_{H^1}, \end{aligned}$$

where  $r$  is given by Proposition 2.5.3. In particular Proposition 2.5.3 implies thanks to the property (2.5.4) that  $\sigma_R = \chi_\delta(\psi) = \tilde{\chi}_\delta(\psi) = \eta_\delta(\psi) = 1$  on a neighborhood of  $\text{supp } \sigma_r$ . We can then finish the proof exactly as in [LL19]. Indeed, we take  $\Pi \in C_0^\infty$  with  $\Pi = 1$  on a neighborhood of  $\text{supp } \sigma_r$  and such that  $\sigma_{2R} = \sigma_R = \chi_\delta(\psi) = \tilde{\chi}_\delta(\psi) = \eta_\delta(\psi) = 1$  on a neighborhood of  $\text{supp } \Pi$ . Then

$$\begin{aligned} &\left\| \sigma_{r,\lambda} M_\lambda^{\frac{\beta\mu}{2}} (1 - \sigma_{2R} \sigma_{R,\lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta,\lambda}(\psi) \eta_{\delta,\lambda}(\psi)) u \right\|_{H^1} \\ &\leq \left\| \sigma_{r,\lambda} M_\lambda^{\frac{\beta\mu}{2}} (1 - \sigma_{2R} \sigma_{R,\lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta,\lambda}(\psi) \eta_{\delta,\lambda}(\psi)) (1 - \Pi) u \right\|_{H^1} \\ &\quad + \left\| \sigma_{r,\lambda} M_\lambda^{\frac{\beta\mu}{2}} (1 - \sigma_{2R} \sigma_{R,\lambda} \tilde{\chi}_\delta(\psi) \chi_{\delta,\lambda}(\psi) \eta_{\delta,\lambda}(\psi)) \Pi u \right\|_{H^1}. \end{aligned}$$

We control from above the first term by using Lemma 2.B.5 and for the second one we combine Lemmata 2.B.1 and 2.B.3.

This finishes the proof of Theorem 2.5.1 up to renaming the constants appearing in the statement of the theorem.  $\square$

### 2.5.3 Propagation of information and applications

In this section we make use of  $\mathcal{L}(\mathcal{M}, E)$ . This is the ‘‘largest distance’’ of a subset  $E \subset \mathcal{M}$  to a point of  $\mathcal{M}$  and has been defined in (2.1.5).

We introduce now the tools of [LL19, Section 4] in order to explain how one can propagate information by applying successively the local quantitative estimate.

**Definition 3.** Fix an open subset  $\Omega$  of  $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}_x^n$  and two *finite* collections  $(V_j)_{j \in J}$  and  $(U_i)_{i \in I}$  of bounded open sets in  $\mathbb{R}^{n+1}$ . We say that  $(V_j)_{j \in J}$  is *under the dependence* of  $(U_i)_{i \in I}$ , denoted

$$(V_j)_{j \in J} \trianglelefteq (U_i)_{i \in I},$$

if for any  $\theta_i \in C_0^\infty(\mathbb{R}^{n+1})$  such that  $\theta_i(t, x) = 1$  on a neighborhood of  $\overline{U_i}$ , for any  $\tilde{\theta}_j \in C_0^\infty(V_j)$  and for all  $\kappa, \alpha > 0$ , there exist  $C, \kappa', \beta, \mu_0$  such that for all  $\mu \geq \mu_0$  and  $u \in \mathcal{W}$  compactly supported, one has:

$$C \left( \sum_{i \in I} \|M_\mu^{\alpha\mu} \theta_{i,\mu} u\|_{H^1} + \|Pu\|_{L^2(\Omega_{t,+} \cup \Omega_{t,-})} \right) + C e^{-\kappa'\mu} \|u\|_{H^1} \geq \sum_{j \in J} \|M_\mu^{\beta\mu} \tilde{\theta}_{j,\mu} u\|_{H^1},$$

where  $P$  is as defined in Section 2.2.2.

The motivation behind this definition becomes apparent when one looks at the local quantitative estimate of Theorem 2.5.1. If one forgets about the localization and regularization indexes then the definition says simply that  $V$  depends on  $U$  if information on  $U$  controls information on  $V$ , and this comes with a precise estimate.

The Carleman estimate we have obtained (Theorem 2.2.2) as well as Carleman estimates in general provide already some sort of quantitative estimate which propagates information. However, what is of fundamental importance here is that the relation of dependence as defined above (in fact in [LL19]) can be **propagated** (in an optimal way). Indeed, the crucial property that one needs in order to iterate such a dependence relation is *transitivity*. That is if a set  $A$  depends on a set  $B$  in the sense of the definition above, and  $B$  depends on a set  $C$  then one would like to say that  $A$  depends on  $C$ . That is why the slightly stronger notion of *strong dependence* is introduced:

**Definition 4.** Fix an open subset  $\Omega$  of  $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}_x^n$  and two *finite* collections  $(V_j)_{j \in J}$  and  $(U_i)_{i \in I}$  of bounded open sets in  $\mathbb{R}^{n+1}$ . We say that  $(V_j)_{j \in J}$  is *under the strong dependence* of  $(U_i)_{i \in I}$ , denoted by

$$(V_j)_{j \in J} \triangleleft (U_i)_{i \in I},$$

if there exist  $\tilde{U}_i \Subset U_i$  such that  $(V_j)_{j \in J} \trianglelefteq (\tilde{U}_i)_{i \in I}$ .

To facilitate the lecture we re-write here the basic properties of the relation  $\triangleleft$  summarised in Proposition 4.5 of [LL19]. We shall use these properties for the proof of Theorem 2.5.14 in order to iterate in an abstract way the local estimate of Theorem 2.5.1.

**Proposition 2.5.10** (Proposition 4.5 in [LL19]). *One has:*

1.  $(V_j)_{j \in J} \triangleleft (U_i)_{i \in I}$  implies  $(V_j)_{j \in J} \trianglelefteq (U_i)_{i \in I}$ .
2. If  $(V_j)_{j \in J} \triangleleft (U_i)_{i \in I}$  with  $U_i = U$  for all  $i \in I$ , then  $(V_j)_{j \in J} \triangleleft U$ .
3. If  $V_i \Subset U_i$  for any  $i \in I$ , then  $(V_i)_{i \in I} \triangleleft (U_i)_{i \in I}$ .
4. If  $V_i \Subset U_i$  for any  $i \in I$ , then  $\cup_{i \in I} V_i \triangleleft (U_i)_{i \in I}$ .

5. If  $V_i \triangleleft U_i$  for any  $i \in I$ , then  $(V_i)_{i \in I} \triangleleft (U_i)_{i \in I}$ . In particular, if  $U_i \triangleleft U$  for any  $i \in I$ , then  $(U_i)_{i \in I} \triangleleft U$ .

6. The relation is transitive, that is:

$$[(V_j)_{j \in J} \triangleleft (U_i)_{i \in I} \text{ and } (U_i)_{i \in I} \triangleleft (W_k)_{k \in K}] \Rightarrow (V_j)_{j \in J} \triangleleft (W_k)_{k \in K}.$$

**Remark 2.5.11.** Notice that the relation  $\triangleleft$  depends on the operator  $P$ , therefore technically speaking Proposition 2.5.10 is not exactly the same as Proposition 4.5 in [LL19] since in our case the coefficients of  $P$  present a jump discontinuity. However this does not have any impact in the proof.

We can now formulate the result of Theorem 2.5.1 in terms of relations of dependence. Indeed, one has:

**Corollary 2.5.12.** *In the geometric situation of Theorem 2.5.1 let  $(t_0, x_0) \in \Sigma$  given locally by  $\Sigma = \{\phi = 0\}$ . Then there exists  $R_0$  such that for any  $R \in (0, R_0)$ , there exist  $r, \rho > 0$  such that*

$$B((t_0, x_0), r) \triangleleft \{\phi > \rho\} \cap B((t_0, x_0), 4R).$$

**Remark 2.5.13.** In fact the constants  $R, \rho, r$  depend only on  $x_0$  and not on  $t_0$ . This comes from the fact that the coefficients of  $P$  and the interface  $\Sigma$  are independent of  $t$ .

*Proof.* The proof is as in Corollary 4.6 in [LL19].  $\square$

With the local quantitative estimate of Theorem 2.5.1 at our disposal we are now ready to propagate these estimates to obtain a global one. We will start by propagating the low frequency estimates only (such as the estimate provided by Theorem 2.5.1). This will be done by using some abstract iteration properties of the relation  $\triangleleft$  as defined in 4. We will then follow a path from a point  $x_0 \in \mathcal{M}$  to another one  $x_1 \in \mathcal{M}$ . As long as the path stays either in  $\Omega_+$  or in  $\Omega_-$  the propagation is guaranteed by the result of [LL19]. If the path crosses the interface  $S$  then we use Theorem 2.5.1 which allows to propagate uniqueness from one side of  $S$  to the other losing only an  $\epsilon$  of time. Then we continue the propagation using again [LL19]. We state the main theorem:

**Theorem 2.5.14.** *Let  $(\mathcal{M}, g)$  be a smooth compact connected  $n$ -dimensional Riemannian manifold with (or without) boundary and  $S$  a  $(n-1)$ -dimensional smooth submanifold of  $\mathcal{M}$ . We suppose that  $\mathcal{M} \setminus S = \Omega_- \cup \Omega_+$  with  $\Omega_- \cap \Omega_+ = \emptyset$ . Consider  $P$  as defined in (2.1.2). For any nonempty open subset  $\omega$  of  $\mathcal{M} \setminus S$  and any  $T > \mathcal{L}(\mathcal{M}, \omega)$ , there exist  $\eta, C, \kappa, \mu_0$  such that for any  $u \in H^1((-T, T) \times \mathcal{M})$  and  $f \in L^2((-T, T) \times \mathcal{M})$  solving*

$$\begin{cases} Pu = f & \text{in } (-T, T) \times \Omega_- \cup \Omega_+ \\ u|_{S_-} = u|_{S_+} & \text{in } (-T, T) \times S \\ (c\partial_\nu u)|_{S_-} = (c\partial_\nu u)|_{S_+} & \text{in } (-T, T) \times S \\ u = 0 & \text{in } (-T, T) \times \partial\mathcal{M}, \end{cases} \quad (2.5.14)$$

one has, for any  $\mu \geq \mu_0$ ,

$$\|u\|_{L^2((-\eta, \eta) \times \mathcal{M})} \leq Ce^{\kappa\mu} \left( \|u\|_{L^2((-T, T) \times \omega)} + \|f\|_{L^2((-T, T) \times \mathcal{M})} \right) + \frac{C}{\mu} \|u\|_{H^1((-T, T) \times \mathcal{M})}.$$



The following lemma will be used for the proof of Theorem 2.5.14 in order to transport the information locally from one side of the interface to the other.

**Lemma 2.5.15.** *Consider a point  $x \in S = \{\phi = 0\}$ . Let  $R_0, R, r, \rho$  be the associated constants given by Corollary 2.5.12. Then for any  $T > 0$ , for any  $\epsilon > 0$  and any subset  $U \subset \mathbb{R}^{n+1}$  such that*

$$[-T, T] \times \left( \{\phi > \frac{\rho}{2}\} \cap B(x, r), 4R \right) \Subset U.$$

there exists  $r_\epsilon > 0$  with

$$[-T + 2\epsilon, T - 2\epsilon] \times B(x, r_\epsilon) \triangleleft U.$$

*Proof.* Let us consider a finite covering of

$$[-T + \epsilon, T - \epsilon] \times B(x, r/2) \subset \bigcup_{i \in I} B((t_i, x), r), \quad I \text{ finite.}$$

Now recall that from Corollary 2.5.12 and definitions of the above quantities we have as well:

$$B((t_i, x), r) \triangleleft \{\phi > \rho\} \cap B((t_i, x), 4R), \quad i \in I. \quad (2.5.15)$$

We also have that

$$\bigcup_{i \in I} \left( \{\phi > \rho\} \cap B((t_i, x), 4R) \right) \Subset [-T, T] \times \left( \{\phi > \frac{\rho}{2}\} \cap B(x, r), 4R \right)$$

and consequently thanks to the assumption made on  $U$  we obtain that

$$\{\phi > \rho\} \cap B((t_i, x), 4R) \Subset U, \quad \text{for } i \in I.$$

We apply then item 3 of Proposition 2.5.10 which gives

$$\left( \{\phi > \rho\} \cap B((t_i, x), 4R) \right)_{i \in I} \triangleleft U.$$

By transitivity of  $\triangleleft$  by (2.5.15) we get

$$\left( B((t_i, x), r) \right)_{i \in I} \triangleleft U. \quad (2.5.16)$$

We now use the following geometric fact, which is a consequence of the triangle inequality: For a given  $\epsilon > 0$  there exists  $r_\epsilon$  such that we have

$$[-T + 2\epsilon, T - 2\epsilon] \times B(x, r_\epsilon) \Subset \bigcup_{i \in I} B\left((t_i, x), \frac{r}{2}\right).$$

Consequently, item 3 of Proposition 2.5.10 gives:

$$[-T + 2\epsilon, T - 2\epsilon] \times B(x, r_\epsilon) \triangleleft \bigcup_{i \in I} B\left((t_i, x), \frac{r}{2}\right). \quad (2.5.17)$$

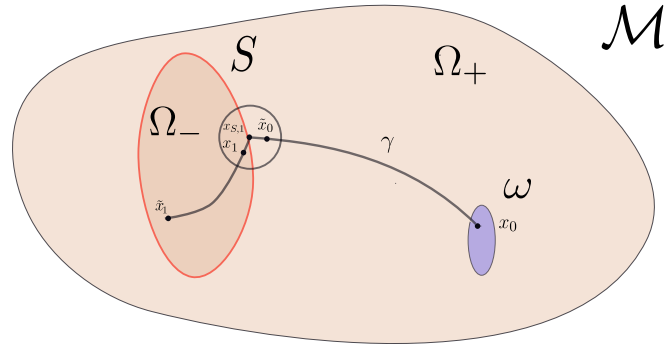


Figure 2.4: First step of the iteration process for the proof of Theorem 2.5.14. Information is propagated from  $x_0$  to  $x_1$ . The process is then repeated starting from  $x_1$ .

The compact inclusion  $B\left((t_i, x), \frac{r}{2}\right) \Subset B((t_i, x), r)$  implies according to item 4 of Proposition 2.5.10:

$$\bigcup_{i \in I} B\left((t_i, x), \frac{r}{2}\right) \Subset B((t_i, x), r) \Big|_{i \in I}. \quad (2.5.18)$$

Finally, we combine (2.5.16), (2.5.17), (2.5.18) using as well the transitivity (property 6 in Proposition 2.5.10)) to find

$$[-T + 2\epsilon, T - 2\epsilon] \times B(x, r_\epsilon) \Subset U,$$

which finishes the proof of the Lemma.  $\square$

We are now ready to give the proof of Theorem 2.5.14.

*Proof of Theorem 2.5.14.* The proof is divided in two steps:

### Step 1: Abstract propagation of low frequency information

For technical reasons related to the second step of the proof we consider an open set  $\tilde{\omega}$  with  $\tilde{\omega} \Subset \omega$ .

We want to propagate uniqueness from  $\tilde{\omega}$  to a neighborhood of an arbitrary point of  $\mathcal{M}$ , say  $y_0 \in \mathcal{M}$ . From the definition of  $\mathcal{L}(\mathcal{M}, \tilde{\omega})$  we can find a point  $x_0 \in \tilde{\omega}$  and an admissible path  $\gamma : [0, 1] \rightarrow \mathcal{M}$  of length  $l$  such that  $T > \mathcal{L}(\mathcal{M}, \tilde{\omega}) > l$ ,  $\gamma(0) = x_0$  and  $\gamma(1) = y_0$ .

By assumption (see Section 2.1.3), the path  $\gamma$  intersects the interface  $S$  a finite number of times  $N$ . We call  $x_{S,j}$ ,  $j \in \{1, 2, \dots, N\}$  the intersection points. Moreover, the conditions made on the family of admissible paths imply that we can use the same coordinates as in [LL19, proof of Theorem 6.3], [Leb92a, pp 21-22]) in a neighborhood of this path. In particular, in these coordinates the path is straightened out, that is  $\gamma(s) = (0, sl)$ . We now apply Corollary 2.5.12 to  $x_{S,1} \in S$  which gives us some constants  $r_{S,1}, \rho_{S,1}$ . We then choose a point  $\tilde{x}_0$  with

$$\tilde{x}_0 \in \{0 < \phi_{S,1} < \rho/4\} \cap B(x_{S,1}, r_{S,1}) \cap \gamma\left((\gamma^{-1}(x_0), \gamma^{-1}(x_{S,1}))\right).$$

Let us now look at the path joining  $x_0$  to  $\tilde{x}_0$ . This path is entirely either in  $\Omega_+$  or  $\Omega_-$  and in particular one can apply the results of [LL19]. More precisely one can construct an

appropriate foliation such that one can apply Theorem 4.7 of [LL19]. The construction is exactly as in the proof of Theorem 6.3 in [LL19]. The difference is that here we only use the foliation to apply Theorem 4.7 which solely concerns *low frequency* information. That is because in our case, this is only the first step of the propagation. Indeed, since we will need to continue the iteration we can not consider high frequencies yet.

Applying Theorem 4.7 of [LL19] gives us a set  $U \subset \mathbb{R}_t \times \Omega_+$  such that

$$U \triangleleft [-T, T] \times \tilde{\omega},$$

and the precise construction of [LL19, Proof of theorem 6.3] implies that  $U$  contains a set of the form

$$[-T_0, T_0] \times \left( \left\{ \phi_{S,1} > \frac{\rho}{2} \right\} \cap B(x_{S,1}, r_{S,1}), 4R \right) \Subset U. \quad (2.5.19)$$

The time  $T_0$  is equal to  $T_0 = T - \tilde{T}_0$  where  $\tilde{T}_0$  is the time given by [LL19] to propagate the information from  $x_0$  to  $\tilde{x}_0$ . This is any time greater than the length of the path joining those two points.

We can now apply Lemma 2.5.15 which ensures that for arbitrary  $\epsilon > 0$  there exists  $r_\epsilon$  with

$$[-T_0 + 2\epsilon, T_0 - 2\epsilon] \times B(x_{S,1}, r_\epsilon) \triangleleft U.$$

Since by construction of  $U$  we have  $U \triangleleft [-T, T] \times \tilde{\omega}$ , using once again the transitivity of  $\triangleleft$  we have finally shown that

$$[-T_0 + 2\epsilon, T_0 - 2\epsilon] \times B(x_{S,1}, r_\epsilon) \triangleleft [-T, T] \times \tilde{\omega}.$$

Recalling that  $x_{S,1} \in S$  the above property says that we managed to pass on (a possibly very small) neighborhood of the other side of the interface, and this by losing an arbitrarily small time. In particular we have shown that we can find a  $x_1 \in \gamma(\gamma^{-1}(x_{S,1}), 1]$  such that

$$[-T_0 + 2\epsilon, T_0 - 2\epsilon] \times V(x_1) \triangleleft [-T, T] \times \tilde{\omega}, \quad (2.5.20)$$

where  $V(x_1)$  is a (small) neighborhood of  $x_1$ . We can now repeat this propagation procedure starting from the point  $x_1$ . We consider a point  $\tilde{x}_1$  which is on the same side of the interface as  $x_1$ , that is  $\tilde{x}_1 \in \gamma((\gamma^{-1}(x_{S,1}), \gamma^{-1}(x_{S,2})))$  and sufficiently close to the interface and we propagate the information from  $x_1$  to  $\tilde{x}_1$  using Theorem 4.7 of [LL19]. Then we pass on the other side of the interface using Lemma 2.5.15. Iterating this process  $N$  times and using the transitivity of  $\triangleleft$  gives

$$[-T_N + 2N\epsilon, T_N - 2N\epsilon] \times V(y_0) \triangleleft [-T, T] \times \tilde{\omega},$$

with  $T_N = T - \sum_{0 \leq j \leq N} \tilde{T}_j$  and  $\tilde{T}_j$  any number strictly larger than the length of the path joining  $x_j$  to  $\tilde{x}_j$ . Notice that the definition of  $\mathcal{L}(\mathcal{M}, \tilde{\omega})$  (see (2.1.5)) implies thanks to our assumption  $T > \mathcal{L}(\mathcal{M}, \tilde{\omega})$  that  $T_N > 0$  and since  $\epsilon$  can be chosen arbitrarily small we can have

$$\eta_{y_0} := T_N - 2N\epsilon > 0.$$

To sum up, for an arbitrary  $y_0 \in \mathcal{M}$  we were able to find an  $\eta_{y_0} > 0$  and a neighborhood  $V(y_0)$  of  $y_0$  such that

$$[-\eta_{y_0}, \eta_{y_0}] \times V(y_0) \triangleleft [-T, T] \times \tilde{\omega}. \quad (2.5.21)$$

We have thus completed the iteration procedure.

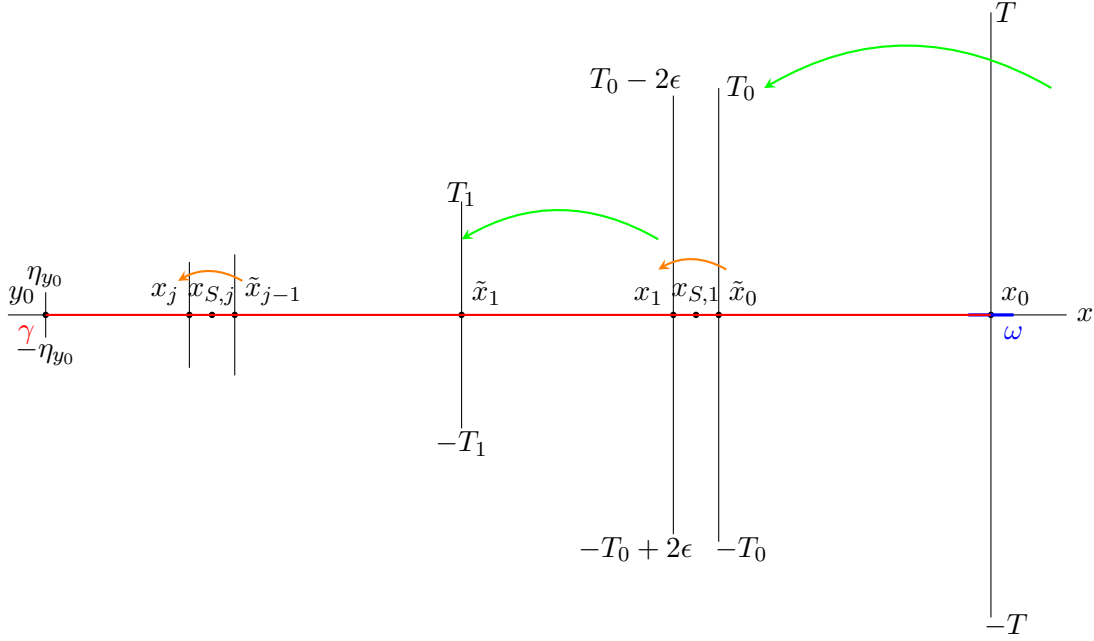


Figure 2.5: The iteration process of the proof of Theorem 2.5.14 in space-time. We transport information from a point  $x_0 \in \omega$  to  $y_0 \in \mathcal{M}$  by following the path  $\gamma$ . The points  $x_{S,j}$  are the intersection points of  $\gamma$  with  $S$ . The green arrows correspond to propagation of information in the smooth context where we use the results of [LL19]. The orange arrows propagate the information through the interface using Theorem 2.5.1 and this, by losing an arbitrarily small time.

**Step 2: Unfolding the definition of  $\triangleleft$**

The last part of the proof will be to write down the definition of  $\triangleleft$  and obtain the desired estimate using a compactness argument for  $\mathcal{M}$ .

Define  $U_{y_0}$  as

$$U_{y_0} := [-\eta_{y_0}, \eta_{y_0}] \times V(y_0).$$

Consider a sufficiently small neighborhood  $\tilde{V}(y_0)$  of  $y_0$  such that  $K \Subset U_{y_0}$ , where we define  $K$  as

$$K := \left[-\frac{\eta_{y_0}}{2}, \frac{\eta_{y_0}}{2}\right] \times \tilde{V}(y_0).$$

Pick  $\chi \in C_0^\infty(U_{y_0})$  such that  $\chi = 1$  on a neighborhood  $U_\chi$  of  $K$  and  $\phi \in C_0^\infty((-T, T) \times \omega)$  with  $\phi = 1$  on a neighborhood of  $\tilde{\omega}$ . The definition of  $\triangleleft$  and (2.5.21) imply that for any  $\kappa > 0$ , there exist  $C, \beta, \kappa', \mu_0 > 0$  such that for  $\mu \geq \mu_0$ <sup>3</sup>,

$$Ce^{\kappa\mu} \left( \|M_\mu^\mu \phi_\mu u\|_{H^1} + \|Pu\|_{L^2(-T, T) \times \mathcal{M}} \right) + Ce^{-\kappa'\mu} \|u\|_{H^1} \geq \|M_\mu^{\beta\mu} \chi_\mu u\|_{H^1}. \quad (2.5.22)$$

<sup>3</sup>Here and all along the proof we write directly, with a slight abuse of notation, the estimates in an invariant way. In fact one writes down the estimates given by the definition of  $\triangleleft$  in some appropriate coordinates and then passes into the global ones. This does not pose any problem since the estimates we consider are invariant by changes of coordinates.

We want to use the almost localization of  $\phi_\mu$  in order to control the left hand side of the above quantity. To do so we use Lemma 2.B.10 which gives us

$$\left\| M_\mu^\mu \phi_\mu u \right\|_{H^1} \leq C\mu \|u\|_{L^2((-T,T) \times \omega)} + Ce^{-c\mu} \|u\|_{H^1}.$$

We want to inject the estimate above in (2.5.22). We need to be sure that the term  $\|u\|_{H^1}$  will be multiplied by a negative exponential. We recall that  $\kappa > 0$  can be chosen arbitrarily small and we impose  $\kappa < c/2$ . Since  $\mu \leq Ce^{\kappa\mu}$  we obtain with a new constant  $c' > 0$ :

$$\left\| M_\mu^{\beta\mu} \chi_\mu u \right\|_{H^1} \leq Ce^{2\kappa\mu} \left( \|u\|_{L^2((-T,T) \times \omega)} + \|Pu\|_{L^2(-T,T) \times \mathcal{M}} \right) + Ce^{-c'\mu} \|u\|_{H^1}. \quad (2.5.23)$$

We need to exploit the almost localization of  $\chi_\mu$  to control from below the left hand side of (2.5.23). We take  $\tilde{\chi} \in C_0^\infty(U_\chi)$  with  $\tilde{\chi} = 1$  in a neighborhood of  $K$ . We use Lemma 2.B.1 to control:

$$\begin{aligned} \|\tilde{\chi}u\|_{L^2} &\leq \|\tilde{\chi}\chi_\mu u\|_{L^2} + \|\tilde{\chi}(1 - \chi_\mu)u\|_{L^2} \leq C\|\chi_\mu u\|_{L^2} + Ce^{-c\mu} \|u\|_{H^1} \\ &\lesssim \left\| M_\mu^{\beta\mu} \chi_\mu u \right\|_{L^2} + \left\| (1 - M_\mu^{\beta\mu}) \chi_\mu u \right\|_{L^2} + e^{-c\mu} \|u\|_{H^1}. \end{aligned} \quad (2.5.24)$$

We control the second term as follows:

$$\begin{aligned} \left\| (1 - M_\mu^{\beta\mu}) \chi_\mu u \right\|_{L^2} &= \left\| \left( 1 - m_\mu \left( \frac{\xi_t}{\beta\mu} \right) \right) \widehat{\chi_\mu u} \right\|_{L^2} = \left\| \frac{\left( 1 - m_\mu \left( \frac{\xi_t}{\beta\mu} \right) \right)}{|\xi_t| + |\xi_x|} (|\xi_t| + |\xi_x|) \widehat{\chi_\mu u} \right\|_{L^2} \\ &\lesssim \left\| \frac{\left( 1 - m_\mu \left( \frac{\xi_t}{\beta\mu} \right) \right)}{|\xi_t| + |\xi_x|} \right\|_{L^\infty} \|u\|_{H^1}. \end{aligned}$$

In the region  $|\xi_t| \geq \frac{\beta\mu}{2}$  we simply have

$$\frac{\left( 1 - m_\mu \left( \frac{\xi_t}{\beta\mu} \right) \right)}{|\xi_t| + |\xi_x|} \leq \frac{C}{\mu}.$$

In the region  $|\xi_t| \leq \frac{\beta\mu}{2}$  we use the support of  $m$  and Lemma 2.B.1 to find

$$\frac{\left( 1 - m_\mu \left( \frac{\xi_t}{\beta\mu} \right) \right)}{|\xi_t| + |\xi_x|} \leq Ce^{-c\mu},$$

hence in particular

$$\left\| \frac{\left( 1 - m_\mu \left( \frac{\xi_t}{\beta\mu} \right) \right)}{|\xi_t| + |\xi_x|} \right\|_{L^\infty} \leq \frac{C}{\mu}.$$

Combining estimates (2.5.24) and (2.5.23) yields:

$$\|\tilde{\chi}u\|_{L^2} \leq Ce^{2\kappa\mu} \left( \|u\|_{L^2((-T,T) \times \omega)} + \|Pu\|_{L^2(-T,T) \times \mathcal{M}} \right) + \frac{C}{\mu} \|u\|_{H^1}.$$

Since  $\tilde{\chi} = 1$  on  $K$  we have:

$$\|u\|_{L^2(K)} = \|\tilde{\chi}u\|_{L^2(K)} \leq \|\tilde{\chi}u\|_{L^2}.$$

This yields, thanks to the previous estimate and the explicit definition of  $K$  the final estimate:

$$\|u\|_{L^2\left(\left[-\frac{\eta_{y_0}}{2}, \frac{\eta_{y_0}}{2}\right] \times \tilde{V}(y_0)\right)} \leq C e^{2\kappa\mu} \left( \|u\|_{L^2((-T, T) \times \omega)} + \|Pu\|_{L^2(-T, T) \times \mathcal{M}} \right) + \frac{C}{\mu} \|u\|_{H^1},$$

where  $\tilde{V}_{y_0}$  is a (small) neighborhood of  $y_0$ ,  $y_0$  is an arbitrary point of  $\mathcal{M}$  and  $\eta_{y_0} > 0$  is an associated strictly positive time. One can cover the manifold  $\mathcal{M}$  by such neighborhoods and by compactness we can extract a finite covering

$$\mathcal{M} \subset \bigcup_{j \in J \text{ finite}} \tilde{V}(y_j),$$

such that

$$\|u\|_{L^2\left(\left[-\frac{\eta_{y_j}}{2}, \frac{\eta_{y_j}}{2}\right] \times \tilde{V}(y_j)\right)} \leq C_j e^{2\kappa_j\mu} \left( \|u\|_{L^2((-T, T) \times \omega)} + \|Pu\|_{L^2(-T, T) \times \mathcal{M}} \right) + \frac{C_j}{\mu} \|u\|_{H^1},$$

for  $\mu \geq \mu_0$ . Let  $\eta := \min \eta_{y_j}/2$ ,  $C := \max C_j$ ,  $c := \min c'_j$ . We have:

$$\begin{aligned} \|u\|_{L^2([-\eta, \eta] \times \mathcal{M})} &\leq \sum_{j \in J} \|u\|_{L^2\left(\left[-\frac{\eta_{y_j}}{2}, \frac{\eta_{y_j}}{2}\right] \times \tilde{V}(y_j)\right)} \\ &\leq |J| \left( C e^{2\kappa\mu} \left( \|u\|_{L^2((-T, T) \times \omega)} + \|Pu\|_{L^2(-T, T) \times \mathcal{M}} \right) + \frac{C}{\mu} \|u\|_{H^1} \right). \end{aligned}$$

The proof of Theorem 2.5.14 is now complete.  $\square$

In the preceding theorem we chose to present a proof in the case where our observation domain  $\omega$  is an open subset of  $\mathcal{M}$ . The point is that the difficulty of our problem comes from the interface inside  $\mathcal{M}$  where the metric may jump. The Theorem and its proof show how we can propagate the information when crossing the interface in a way that is compatible with the quantitative results of [LL19]. Since in [LL19] the boundary case has been treated too, we can as well formulate the analogous result in the case of boundary observation. More precisely one has:

**Theorem 2.5.16.** *Under the assumptions of Theorem 2.5.14 assume additionally that  $\partial\mathcal{M}$  is non empty and consider  $\Gamma$  a non empty open subset of  $\partial\Gamma$ . Then for any  $T > \mathcal{L}(\mathcal{M}, \Gamma)$ , there exist  $\eta, C, \kappa, \mu_0 > 0$  such that for any  $u \in H^1((-T, T) \times \mathcal{M})$  and  $f \in L^2((-T, T) \times \mathcal{M})$  solving 2.5.14, we have for any  $\mu \geq \mu_0$*

$$\|u\|_{L^2([-\eta, \eta] \times \mathcal{M})} \leq C e^{\kappa\mu} \left( \|\partial_{\nu_\Gamma} u\|_{L^2((-T, T) \times \Gamma)} + \|f\|_{L^2(-T, T) \times \mathcal{M}} \right) + \frac{C}{\mu} \|u\|_{H^1((-T, T) \times \mathcal{M})}.$$

One can now combine the two preceding theorems with classical energy estimates for solutions of the wave equation that allow to relate  $\|u\|_{H^1}$  and  $\|u\|_{L^2}$  with the energy of its initial data  $(u, \partial_t u)|_{t=0}$  to obtain the following slightly more general version of Theorem 2.1.1:

**Theorem 2.5.17.** *Let  $(\mathcal{M}, g)$  be a smooth compact connected  $n$ -dimensional Riemannian manifold with (or without) boundary and  $S$  an  $(n - 1)$ -dimensional smooth submanifold of  $\mathcal{M}$ . We write  $\mathcal{M} \setminus S = \Omega_- \cup \Omega_+$ . Consider  $P$  as defined in (2.1.2). For any nonempty open subset  $\omega$  of  $\mathcal{M} \setminus S$  and any  $T > \mathcal{L}(\mathcal{M}, \omega)$ , there exist  $C, \kappa, \mu_0$  such that for any  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ ,  $f \in L^2((-T, T) \times \mathcal{M})$  and  $u$  solving*

$$\begin{cases} Pu = f & \text{in } (-T, T) \times \Omega_- \cup \Omega_+ \\ u|_{S_-} = u|_{S_+} & \text{in } (-T, T) \times S \\ (c\partial_\nu u)|_{S_-} = (c\partial_\nu u)|_{S_+} & \text{in } (-T, T) \times S \\ u = 0 & \text{in } (-T, T) \times \partial\mathcal{M} \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{in } \mathcal{M}, \end{cases} \quad (2.5.25)$$

one has, for any  $\mu \geq \mu_0$ ,

$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq Ce^{\kappa\mu} \left( \|u\|_{L^2((-T, T) \times \omega)} + \|f\|_{L^2((-T, T) \times \mathcal{M})} \right) + \frac{C}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}.$$

If moreover  $\mathcal{M} \neq \emptyset$  and  $\Gamma$  is a non empty open subset of  $\partial\mathcal{M}$ , for any  $T > \mathcal{L}(\mathcal{M}, \Gamma)$ , there exist  $C, \kappa, \mu_0 > 0$  such that for any  $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ ,  $f \in L^2((-T, T) \times \mathcal{M})$  and  $u$  solving (2.5.25), we have

$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq Ce^{\kappa\mu} \left( \|\partial_{\nu_\Gamma} u\|_{L^2((-T, T) \times \Gamma)} + \|f\|_{L^2((-T, T) \times \mathcal{M})} \right) + \frac{C}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}.$$

## 2.A A few facts on pseudodifferential calculus

We collect here some facts and notations concerning the symbolic calculus which is an essential ingredient for the proof of the Carleman estimate. We follow here the exposition of [LRL13] and [LRLR22, Chapter 2].

### 2.A.1 Differential operators

For  $(t, x) \in \mathbb{R}_t \times \mathbb{R}_x^n$  we write  $(t, x) = (t, x', x_n)$ . The variable  $x_n$  is normal to the interface. We use as well the notation  $D_{x_j} := \frac{1}{i} \partial_{x_j}$ .

We denote by  $\mathcal{D}_\tau^m$  the set of differential operators depending on  $\tau$ , that is operators of the form

$$P(t, x, D_t, D_x, \tau) = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(t, x) \tau^j D_{t,x}^\alpha.$$

Their principal symbols are defined as

$$\sigma(P) = \sum_{j+|\alpha|=m} a_{j,\alpha}(t, x) \tau^j (\xi_t \xi)^\alpha.$$

The set of tangential operators depending on the large parameter  $\tau$  is denoted by  $\mathcal{D}_{\top, \tau}^m$  and contains operators of the form

$$P(t, x, D_t, D_{x'}, \tau) = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(t, x) \tau^j (D_t D_{x'})^\alpha$$

with principal symbols defined as

$$\sigma(P) = \sum_{j+|\alpha|=m} a_{j,\alpha}(t,x) \tau^j(\xi_t \xi')^\alpha.$$

### 2.A.2 Standard tangential classes

For  $m \in \mathbb{R}$  we define the class *tangential symbols*  $\mathcal{S}^m$  as the smooth functions on  $\mathbb{R}^{n+1} \times \mathbb{R}^n$  such that for all  $(\alpha, \beta) \in \mathbb{N}^{n+1} \times \mathbb{N}^n$ ,

$$\sup_{(t,x,\xi_t,\xi')} (1 + |\xi'|^2 + |\xi_t|^2)^{\frac{-m+|\beta|}{2}} |(\partial_{t,x}^\alpha \partial_{\xi_t,\xi'}^\beta) a(t,x,\xi_t,\xi')| < \infty.$$

We shall mainly work with the *Weyl quantization* which associates to  $a \in \mathcal{S}^m$  an operator denoted by  $\text{op}^w(a)$  defined by

$$(\text{op}^w(a)u)(t,x',x_n) =$$

$$(2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i((t,x')-(s,y')) \cdot (\xi_t,\xi')} a\left(\frac{t+s}{2}, \frac{x'+y'}{2}, x_n, \xi'\right) u(y',s,x_n) ds dy' d\xi' d\xi_t.$$

These integrals may not be defined in the classical sense (Lebesgue integration). They are however well defined as *oscillatory integrals*.

We denote by  $\Psi^m$  the set of these pseudodifferential operators. We define as well

$$\mathcal{S}^{-\infty} := \bigcap_{m \in \mathbb{R}} \mathcal{S}^m, \quad \Psi^{-\infty} := \bigcap_{m \in \mathbb{R}} \Psi^m.$$

Notice that even though the operators above are tangential we do not use any special notation since all pseudodifferential operators we consider are tangential.

A basic feature of the Weyl quantization is that we have the exact equality:

$$(\text{op}^w(a))^* = \text{op}^w(\bar{a}),$$

where we denote by  $*$  the adjoint operator on  $L^2$ . In particular, operators associated to real valued symbols are (formally) self-adjoint.

For  $a \in \mathcal{S}^m$  we call *principal symbol*,  $\sigma(a)$ , the equivalence class of  $a$  in  $\mathcal{S}^m/\mathcal{S}^{m-1}$ .

Many times we refer to the *Sobolev regularity* property of pseudodifferential calculus, namely:

$$\text{op}^w(a) : L^2(\mathbb{R}_{x_n}; H^{s+m}(\mathbb{R}_{t,x'}^n)) \rightarrow L^2(\mathbb{R}_{x_n}; H^s(\mathbb{R}_{t,x'}^n)) \text{ continuously,}$$

where  $a \in \mathcal{S}^m$ .

Consider now  $a_1 \in \mathcal{S}^{m_1}$  and  $a_2 \in \mathcal{S}^{m_2}$ . Then there exists a  $c \in \mathcal{S}^{m_1+m_2}$  such that we have

$$\text{op}^w(a_1)\text{op}^w(a_2) = \text{op}^w(c),$$

and we denote  $c := a_1 \sharp a_2$  where  $\sharp$  is called the *Moyal product*. One has, for any  $N \in \mathbb{N}$  the following asymptotic formula:

$$(a_1 \sharp a_2)(t,x,\xi_t,\xi) - \sum_{j < N} (i\omega(D_{t,x'}, D_{\xi_t,\xi'}; D_{y'}, D_{\eta'}))^j a_1(t,x,\xi_t,\xi) a_2(y,\eta)|_{y=(t,x), \eta=(\xi_t,\xi)} \in \mathcal{S}^{m_1+m_2-N}, \quad (2.A.1)$$

with  $\omega(a,b;c,d) = c \cdot b - a \cdot d$ . This formula implies:

1.  $\text{op}^w(a_1)\text{op}^w(a_2) = \text{op}^w(a_1 a_2) + \text{op}^w(r_1)$ ,  $r_1 \in \mathcal{S}^{m_1+m_2-1}$ .
2.  $[\text{op}^w(a_1), \text{op}^w(a_2)] = \text{op}^w\left(\frac{1}{i}\{a_1, a_2\}\right) + \text{op}^w(r_3)$ ,  $r_3 \in \mathcal{S}^{m_1+m_2-2}$ .



### 2.A.3 Tangential classes with a large parameter

Since we want to show a Carleman estimate which involves a large parameter, the natural class in our context is that of *pseudodifferential operators with a large parameter*. For  $\tau \geq 1$  we define

$$\lambda_\tau^2 = \tau^2 + |\xi'|^2 + |\xi_t|^2,$$

The class denoted by  $\mathcal{S}_\tau^m$  contains the functions  $a \in C^\infty(t, x, \xi_t, \xi', \tau)$  satisfying for all  $(\alpha, \beta) \in \mathbb{N}^{n+1} \times \mathbb{N}^n$ :

$$\sup_{\substack{(t,x,\xi_t,\xi') \\ \tau \geq 1}} \lambda_\tau^{-m+|\beta|} |\partial_{t,x}^\alpha \partial_{\xi_t,\xi'}^\beta a(t, x, \xi_t, \xi', \tau)| < \infty.$$

We set  $\Psi_\tau^m := \{\text{op}^w(a), a \in \mathcal{S}_\tau^m\}$  and

$$\mathcal{S}_\tau^{-\infty} := \bigcap_{m \in \mathbb{R}} \mathcal{S}_\tau^m, \quad \Psi_\tau^{-\infty} := \bigcap_{m \in \mathbb{R}} \Psi_\tau^m.$$

We denote by  $(\cdot, \cdot)$  the inner product on  $L^2(\mathbb{R}^{n+1})$  defined by  $(f, g) = \int_{\mathbb{R}^{n+1}} f \bar{g}$  and by  $(\cdot, \cdot)_\pm$  its restriction on  $L^2(\mathbb{R}_\pm^{n+1})$ .

We introduce the following Sobolev norms, defined in the tangential variables:

$$|u(x_n, \cdot)|_{H^s} = |\text{op}^w(\lambda^s)u(x_n, \cdot)|_{L^2(\mathbb{R}^n)}, \quad |u(x_n, \cdot)|_{H_\tau^s} = |\text{op}^w(\lambda_\tau^s)u(x_n, \cdot)|_{L^2(\mathbb{R}^n)}.$$

The above norms define the (usual) Sobolev space  $H^s$  and the Sobolev space including a large parameter  $H_\tau^s$ . We use many times that for  $s = 1$  one has the equivalence

$$\|\cdot\|_{H_\tau^1}^2 \sim \tau^2 \|\cdot\|_{L^2}^2 + \|\nabla \cdot\|_{L^2}^2.$$

All the properties listed in Section 2.A.2 in the classical case remain valid in the context of the large parameter. In particular we have the *Sobolev regularity* property, for  $a \in \mathcal{S}_\tau^m$ :

$$\text{op}^w(a) : L^2(\mathbb{R}_{x_n}; H_\tau^{s+m}(\mathbb{R}_{t,x'}^n)) \rightarrow L^2(\mathbb{R}_{x_n}; H_\tau^s(\mathbb{R}_{t,x'}^n)),$$

continuously and uniformly in  $\tau \geq 1$ . This yields that for  $a \in \mathcal{S}_\tau^m$  we have

$$|(\text{op}^w(a)u, u)| \lesssim \|u\|_{L^2(\mathbb{R}; H_\tau^{m/2})}^2.$$

Many times in the article we absorb error terms by taking  $\tau$  *sufficiently large*. By that we invoke the following property: If  $m' > m$  then

$$\|\cdot\|_{H_\tau^m} \leq C\tau^{-(m'-m)} \|\cdot\|_{H_\tau^{m'}},$$

for  $\tau \geq 1$ .

The main tool to transfer some positivity properties of the symbol to some estimate for the corresponding operator is *Gårding's inequality*. We shall use it in the context of operators involving a large parameter. For a proof we refer to [LRLR22, Chapter 2]. There the standard quantization is used, the same proof works however for the Weyl quantization.

**Lemma 2.A.1 (Gårding's inequality with a large parameter).** Consider  $a \in \mathcal{S}_\tau^m$  with principal symbol  $a_m$ . Suppose that there exist  $C > 0$  and  $R > 0$  such that

$$\operatorname{Re} a_m(t, x, \xi_t, \xi', \tau) \geq C\lambda_\tau^m, \quad x \in \mathbb{R}^{n+1}, (\xi_t, \xi') \in \mathbb{R}^n, \tau \geq 1, |(\xi_t, \xi', \tau)| \geq R,$$

then there exist  $C'$  and  $\tau_0$  such that

$$\operatorname{Re}(\operatorname{op}^w(a)u, u) \geq C' \|u\|_{L^2(\mathbb{R}; H_\tau^{m/2})}^2,$$

for  $u \in \mathcal{S}(\mathbb{R}^{n+1})$  and  $\tau \geq \tau_0$ .

Before stating the last lemma of this section let us recall a definition.

**Definition 5.** The **essential support** of a symbol  $\mathcal{S}_\tau^m$ , denoted by  $\operatorname{essupp}(a)$  is the complement of the largest open set of  $\mathbb{R}_{t,x}^{n+1} \times \mathbb{R}_{\xi_t, \xi'}^n \times \{\tau \geq 1\}$  where the estimates for  $\mathcal{S}_\tau^{-\infty}$  hold. More precisely, a point  $(t_0, x_0, s_0, \xi'_0) \in \mathbb{R}^{n+1} \times (\mathbb{R}^n \setminus \{0\})$  **does not** lie in the essential support of  $a$  if there exists a neighborhood  $\mathcal{U}$  of  $(t_0, x_0)$  and a conic neighborhood  $\mathcal{V}$  of  $(s_0, \xi'_0)$  such that for all  $m \in \mathbb{R}$  and all  $(\alpha, \beta) \in \mathbb{N}^{n+1} \times \mathbb{N}^n$  one has

$$\sup_{\substack{(t,x) \in \mathcal{U}, (\xi_t, \xi') \in \mathcal{V} \\ \tau \geq 1}} \lambda_\tau^{-m+|\beta|} |\partial_{t,x}^\alpha \partial_{\xi_t, \xi'}^\beta a(t, x, \xi_t, \xi', \tau)| < \infty.$$

Although the natural classes for us to work with are  $\mathcal{S}_\tau^m$  involving the large parameter  $\tau$ , for technical reasons we also have to deal with symbols in  $\mathcal{S}^m$ . Since  $\mathcal{S}^m \not\subset \mathcal{S}_\tau^m$  one has to make sure that the chosen objects belong to the appropriate spaces. The following Lemma [LRL13, Lemma A.4] will be then very useful:

**Lemma 2.A.2.** Let  $m, m' \in \mathbb{R}$ ,  $a_1(t, x, \xi_t, \xi') \in \mathcal{S}^m$  and  $a_2(t, x, \xi_t, \xi', \tau) \in \mathcal{S}_\tau^{m'}$  such that the essential support of  $a_2$  is contained in a region where  $|(\xi_t, \xi')| \gtrsim \tau$ . Then

$$\operatorname{op}^w(a_1)\operatorname{op}^w(a_2) = \operatorname{op}^w(b_1), \quad \operatorname{op}^w(a_2)\operatorname{op}^w(a_1) = \operatorname{op}^w(b_2),$$

with  $b_1, b_2 \in \mathcal{S}_\tau^{m+m'}$ . Moreover one has the same asymptotic formula as (2.A.1) with the remainder in  $\mathcal{S}_\tau^{m+m'-N}$ .

## 2.B Some lemmata used in the quantitative estimates

In this Section we collect some estimates coming essentially from [LL19] concerning the regularization and localization operators that we introduced for the quantitative estimates.

We define for a function  $f \in L^\infty(\mathbb{R}^{n+1})$  and  $\lambda > 0$ :

$$f_\lambda(t, x) := e^{-\frac{|D_t|^2}{\lambda}} f = \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} f(s, x) e^{-\frac{\lambda}{4}|t-s|^2} ds.$$

We use many times the fact that

$$\|f_\lambda\|_{L^2} \leq \left\| e^{-\frac{|\cdot|^2}{\lambda}} \right\|_{L^\infty(\mathbb{R}_t)} \|\mathcal{F}_t(f)(\xi_t, x)\|_{L^2} = \|f\|_{L^2}.$$

Notice also that we have

$$f \geq 0 \implies f_\lambda \geq 0,$$

and consequently

$$f \geq g \implies f_\lambda \geq g_\lambda. \quad (2.B.1)$$

We now recall several Lemmas from [LL19] that we use in the main part of this article.

**Lemma 2.B.1** (Lemma 2.3 in [LL19]). *For any  $d > 0$ , there exist  $C, c > 0$  such that for any  $f_1, f_2 \in L^\infty(\mathbb{R}^{n+1})$  such that  $\text{dist}(\text{supp } f_1, \text{supp } f_2) \geq d$  and all  $\lambda \geq 0$ , we have*

$$\|f_{1,\lambda} f_2\|_{L^\infty} \leq C e^{-c\lambda} \|f_1\|_{L^\infty} \|f_2\|_{L^\infty}, \quad \|f_{1,\lambda} f_{2,\lambda}\|_{L^\infty} \leq C e^{-c\lambda} \|f_1\|_{L^\infty} \|f_2\|_{L^\infty}.$$

**Lemma 2.B.2** (Lemma 2.4 in [LL19]). *Let  $f_2 \in C^\infty(\mathbb{R}^{n+1})$  with all derivatives bounded, and  $d > 0$ . Then for every  $k \in \mathbb{N}$ , there exist  $C, c > 0$  such that for all  $f_1 \in H^k(\mathbb{R}^{n+1})$  such that  $\text{dist}(\text{supp } f_1, \text{supp } f_2) \geq d$  and all  $\lambda \geq 0$  we have*

$$\|f_{1,\lambda} f_2\|_{H^k} \leq C e^{-c\lambda} \|f_1\|_{H^k}.$$

**Lemma 2.B.3.** *Let  $\psi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be a Lipschitz continuous function,  $f_1 \in C^\infty(\mathbb{R}^{n+1})$  with bounded derivatives and  $f_2 \in C_0^\infty(\mathbb{R}^{n+1})$  such that  $\text{dist}(\text{supp } f_1(\psi), \text{supp } f_2) > 0$ . Then, for  $k \in \{0, 1\}$  there exist  $C, c > 0$  such that for all  $\lambda > 0$ , we have*

$$\|f_{1,\lambda}(\psi) f_2\|_{H^k \rightarrow H^k} \leq C e^{-c\lambda}.$$

Lemma 2.B.3 is essentially Lemma 2.5 from [LL19]. In its statement the Lemma requires for  $\psi$  to be smooth. However, since we only need to control derivatives of order at most one Lipschitz regularity is sufficient.

**Lemma 2.B.4** (Lemma 2.6 in [LL19]). *Let  $f_1, f_2 \in C_0^\infty(\mathbb{R}^{n+1})$  such that  $f_1 = 1$  in a neighborhood of  $\text{supp } f_2$ . Then there exist  $C, c > 0$  such that for all  $\lambda > 0$ , and all  $u \in H^1(\mathbb{R}^{n+1})$ , we have*

$$\|f_{2,\lambda} \partial^\alpha u\|_{L^2} \leq C \|f_{1,\lambda} u\|_{H^1} + C e^{-c\lambda} \|u\|_{H^1}, \quad \text{for all } |\alpha| \leq 1.$$

We recall that the operators  $M_\lambda^\mu$  have been defined in Section 2.5.1.

**Lemma 2.B.5** (Lemma 2.10 in [LL19]). *Let  $f_1$  and  $f_2$  be in  $C^\infty$  bounded as well as their derivatives with  $\text{dist}(\text{supp } f_1, \text{supp } f_2) \geq d > 0$ . Then for any  $k \in \mathbb{N}$ , there exist  $C, c > 0$  such that for all  $\mu > 0$  and  $\lambda > 0$ , we have*

$$\|f_{1,\lambda} M_\lambda^\mu f_{2,\lambda}\|_{H^k \rightarrow H^k} \leq C e^{-c\frac{\mu^2}{\lambda}} + C e^{-c\lambda}, \quad \|f_{1,\lambda} M_\lambda^\mu f_2\|_{H^k \rightarrow H^k} \leq C e^{-c\frac{\mu^2}{\lambda}} + C e^{-c\lambda}$$

**Lemma 2.B.6** (Lemma 2.11 in [LL19]). *Let  $f \in C_0^\infty(\mathbb{R}^{n+1})$ . Then there exist  $C, c > 0$  such that for all  $\mu > 0$ ,  $\lambda > 0$  and  $u \in H^1(\mathbb{R}^{n+1})$ , one has*

$$\|f_\lambda M_\lambda^\mu u\|_{H^1} \leq \|f_\lambda M_\lambda^{2\mu} u\|_{H^1} + C \left( e^{-c\frac{\mu^2}{\lambda}} + e^{-c\lambda} \right) \|u\|_{H^1}.$$

**Lemma 2.B.7** (Lemma 2.13 in [LL19]). *There exists  $C > 0$  such that for all  $D \in \mathbb{R}$ ,  $\tilde{\chi} \in L^\infty(\mathbb{R})$  such that  $\text{supp } \tilde{\chi} \subset (-\infty, D]$ , for all  $\lambda, \tau > 0$ , we have*

$$\left\| e^{\tau\psi} \tilde{\chi}_\lambda(\psi) \right\|_{L^\infty} \leq C \|\tilde{\chi}\|_{L^\infty} \langle \lambda \rangle^{1/2} e^{D\tau} e^{\frac{\tau^2}{\lambda}},$$

for all  $\psi \in C^0(\mathbb{R}^{n+1}; \mathbb{R})$ .

**Lemma 2.B.8** (Lemma 2.14 in [LL19]). *There exist  $C, c$  such that, for any  $\epsilon, \tau, \lambda, \mu > 0$ , for any  $k \in \mathbb{N}$ , we have:*

$$\left\| e^{-\frac{\epsilon|D_t|^2}{2\tau}} (1 - M_\mu^\lambda) \right\|_{H^k \rightarrow H^k} \leq e^{-\frac{\epsilon\mu^2}{8\tau}} + C e^{-c\lambda}.$$

**Lemma 2.B.9.** *Let  $\psi$  be a locally Lipschitz continuous real valued function on  $\mathbb{R}_t \times \mathbb{R}_x^n$ , which is a quadratic polynomial in  $t$ , let  $R_\sigma > 0$ , and  $\sigma \in C_0^\infty(B(0, R_\sigma))$ . Let  $\chi \in C_0^\infty(\mathbb{R})$  with  $\text{supp}(\chi) \subset (-\infty, 1)$ , and  $\tilde{\chi} \in C_0^\infty(\mathbb{R})$  such that  $\tilde{\chi} = 1$  on a neighborhood of  $(-\infty, 3/2)$ ,  $\text{supp } \tilde{\chi} \subset (-\infty, 2)$ , and set  $\chi_\delta(s) := \chi(s/\delta)$ ,  $\tilde{\chi}_\delta(s) := \tilde{\chi}(s/\delta)$ . Let  $f$  be bounded, compactly supported and real analytic in the variable  $t$  in a neighborhood of  $\bar{B}_{\mathbb{R}_t}(0, R_\sigma)$  and define*

$$g := e^{\tau\psi} \chi_{\delta,\lambda}(\psi) \tilde{\chi}_\delta(\psi) f \sigma_\lambda.$$

Then one has the following estimate: for all  $c, \delta > 0$  there exist  $c_0, C, N > 0$  such that for any  $\tau, \mu \geq 1$  and  $\frac{\mu}{c} \leq \lambda \leq c\mu$ , we have

$$\left\| M_\lambda^{\mu/2} g (1 - M_\lambda^\mu) \right\|_{L^2 \rightarrow L^2} \leq C \tau^N e^{\frac{\tau^2}{\lambda}} e^{2\delta\tau} e^{-c_0\mu}.$$

*Proof.* This is essentially Lemma 2.17 in [LL19]. Its proof is based upon Lemma 2.15 from [LL19] and there the assumptions on  $\psi$  and  $f$  are more restrictive. However, since we only need to use a version of [LL19, Lemma 2.17] for the case  $k = 0$  we see that the proof works as well in the less restrictive case of Lipschitz regularity for  $\psi$  (our function  $\psi$  remains a quadratic polynomial in  $t$ ) and boundedness for  $f$ .  $\square$

Lemma 2.B.9 is used Section 2.5.2 for the first estimates on the terms  $B_*$ . In our case  $f$  will be equal to functions which will be either independent of  $t$  or simply a polynomial in  $t$ , therefore the real analyticity property is preserved as well.

The following lemma is taken from [LL22] and allows to replace in the quantitative estimates the observation term  $\|u\|_{H^1(\omega)}$  by the weaker  $\|u\|_{L^2(\omega)}$ . It is used in the proof of the semi-global estimate of Theorem 2.5.14.

**Lemma 2.B.10** (Lemma 5.2 in [LL22]). *Let  $\Omega$  be a bounded set of  $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}_x^n$ . Let  $P$  be a differential operator of order 2, defined in a neighborhood of  $\Omega$ , with real principal symbol and coefficients independent of the variable  $t$ . Suppose as well that  $P$  is elliptic in  $\{\xi_t = 0\}$ . Let  $\omega \Subset \Omega$  and  $\theta \in C_0^\infty(\omega)$ . Then there exists  $C > 0$  such that for all  $u \in C_0^\infty(\mathbb{R}^{n+1})$  and  $\mu \geq 1$ , we have*

$$\left\| M_\mu^\mu u \right\|_{H^1} \leq C \mu \|u\|_{L^2(\omega)} + C \|Pu\|_{L^2(\Omega)} + C e^{-c\mu} \|u\|_{H^1}.$$



## Chapitre 3

# Interlude : Whispering gallery modes for a transmission problem

*Ce chapitre est la reproduction de l'article [Fil23].*

In this chapter we construct a specific family of eigenfunctions for a Laplace operator with coefficients having a jump across an interface. These eigenfunctions have an exponential concentration arbitrarily close to the interface, and therefore could be considered as whispering gallery modes. The proof is based on an appropriate Agmon estimate. We deduce as a corollary that the quantitative unique continuation result for waves propagating in singular media proved by the author in [Fil22] is optimal.

### 3.1 Introduction

Let  $\mathcal{M}$  be a compact, connected subset of  $\mathbb{R}^n$  with smooth boundary  $\partial\mathcal{M}$  and  $S$  a smooth hypersurface (that is a  $C^\infty$  compact submanifold of  $\mathcal{M}$  of codimension 1) contained in the interior of  $\mathcal{M}$  such that we have the following partition:  $\text{Int}(\mathcal{M}) \setminus S = \Omega_- \cup \Omega_+$  with  $\Omega_- \cap \Omega_+ = \emptyset$ . Notice that in particular each connected component of  $\text{Int}(\mathcal{M}) \setminus S$  is contained either in  $\Omega_-$  or in  $\Omega_+$ . We consider the operator  $-\Delta_c := -\text{div}(c\nabla \cdot)$  acting on  $\mathcal{M}$  with  $c$  being strictly positive and piecewise smooth but having a jump across the interface  $S$ . In this note we construct, for specific choices of  $\Omega_-$ ,  $\Omega_+$ ,  $S$  and  $c$  eigenfunctions of  $-\Delta_c$  which concentrate exponentially near the interface  $S$ . These maximally vanishing eigenfunctions are sometimes called *whispering gallery modes* (WGM). We show that (see Theorem 3.2.8 or Theorem 3.3.4 for a more precise statement):

**Theorem 3.1.1.** *There are sets  $\mathcal{M}$ ,  $S$  and coefficients  $c = c_- \mathbf{1}_{\Omega_-} + c_+ \mathbf{1}_{\Omega_+}$  such that there exist sequences  $(\lambda_n)_{n \in \mathbb{N}}$  and  $(u_n)_{n \in \mathbb{N}}$  with  $\lambda_n \rightarrow +\infty$  and  $u_n$  satisfying the transmission conditions*

$$(u_n|_{\Omega_-})|_S = (u_n|_{\Omega_+})|_S \text{ and } c_+ \partial_\nu(u_n|_{\Omega_-})|_S = c_- \partial_\nu(u_n|_{\Omega_+})|_S, \quad (3.1.1)$$

*such that for all  $\omega \subset \mathcal{M}$  with  $\text{dist}(\bar{\omega}, S) > 0$  there exist  $C, d > 0$  with:*

$$\|u_n\|_{L^2(\omega)} \leq C e^{-d\sqrt{\lambda_n}}, \quad -\Delta_c u_n = \lambda_n u_n, \quad u_n|_{\partial\mathcal{M}} = 0, \quad \|u_n\|_{L^2(\mathcal{M})} = 1. \quad (3.1.2)$$

The data  $\mathcal{M}, S, c$  considered in the proof of Theorem 3.1.1 have a rotational symmetry and possible sets for  $\mathcal{M}$  include a disk or an annulus. The proof of this is based on an *Agmon estimate* (see Sections 3.2.2, 3.2.3). Its main advantage is that it is relatively elementary and it allows to handle with minor modifications different geometries  $\mathcal{M}, S$ . Our basic toy model will be the case where  $\mathcal{M}$  is an annulus (see Figure 3.1 below) but in Section 3.3 we explain how one can deal with more general surfaces of revolution.

Our main motivation comes from tunneling estimates in control theory (see for instance [LL21b, Section 1.2]). Constructing such eigenfunctions allows to saturate certain observability estimates and therefore deduce their optimality. The idea of exhibiting such examples on surfaces of revolution can be traced back to [Leb96] and [All98].

Another motivation for studying WGM comes from optoelectronics. Indeed, the case where  $\mathcal{M}$  is a disk and  $S$  a smaller concentric circle in its interior can be seen as a toy model for the orthogonal section of an optical fiber. Indeed, in this case  $\Omega_-$  can be considered as the core of the optical fiber surrounded by a cladding  $\Omega_+$ . For the proof of Theorem 3.1.1 we shall assume that  $c_- < c_+$  which means that the refractive index in  $\Omega_-$  is higher than in  $\Omega_+$  and consequently light stays localized in the core by total internal reflection in the boundary between the core and the cladding. See as well the remarks after Theorem 3.1.3. Concerning the applications to optoelectronics, WGM have been studied numerically as well as from a theoretical point of view in [BDDM20, BDM21]. In these references the eigenvalue problem for  $-\Delta_c$  is studied in an unbounded domain, thus becoming a resonance problem. In [PV99] a related problem was studied and polynomial concentration properties were obtained but in more general geometries.

In the recent work [BBAD22] the authors obtain concentration and non-concentration properties for the eigenfunctions of  $-\Delta_c$  depending on the regularity of the coefficient  $c$ . Discontinuity of the coefficient  $c$  corresponds to the case of *layered media*. The methods employed and the geometric context are however different from ours since the proofs are not based on Agmon estimates but rather on the explicit form of the Green kernel of the solutions.

### 3.1.1 Optimality of unique continuation results for operators with jumps

In the geometric context described in the beginning of the introduction, the following spectral inequality for eigenfunctions of  $-\Delta_c$  is proved in [LR10, Theorem 1.2] (and generalized in [LRL13]):

**Theorem 3.1.2** (Theorem 1.2 in [LR10]). *Let  $(u_j)_{j \in \mathbb{N}}$  be a Hilbert basis of eigenfunctions of the operator  $-\Delta_c$  with Dirichlet boundary conditions, satisfying the transmission conditions (3.1.1). Denote by  $\lambda_j$  the associated eigenvalues, sorted in an increasing sequence. Then for any  $\omega \subset \text{Int}(\mathcal{M})$  with  $\omega \neq \emptyset$  there exists  $C > 0$  such that one has, for any  $a_j \in \mathbb{C}$ :*

$$\sum_{\lambda_j \leq \lambda} |a_j|^2 \leq C e^{C\sqrt{\lambda}} \int_{\omega} \left| \sum_{\lambda_j \leq \lambda} a_j u_j(x) \right|^2 dx.$$

The eigenfunctions exhibited in Theorem 3.1.1 prove that the spectral estimate of Theorem 3.1.2 above is sharp in general, even for a single eigenfunction.

The discontinuities of the operator  $-\Delta_c$  can be used to describe waves propagating in non-homogeneous media. The coefficient  $c$  can be interpreted as the square of the speed

of propagation. We consider the following system which describes the evolution of such a wave,

$$\begin{cases} (\partial_t^2 - \Delta_c)w = 0 & \text{in } (0, T) \times \Omega_- \cup \Omega_+ \\ w|_{S_-} = w|_{S_+} & \text{in } (0, T) \times S \\ (c\partial_\nu w)|_{S_-} = (c\partial_\nu w)|_{S_+} & \text{in } (0, T) \times S \\ w = 0 & \text{in } (0, T) \times \partial\mathcal{M} \\ (w, \partial_t w)|_{t=0} = (w_0, w_1) & \text{in } \mathcal{M}, \end{cases} \quad (3.1.3)$$

where we denote by  $\partial_\nu$  the outward unit normal vector to  $S$  pointing into  $\Omega_+$  and by  $w|_{S_\pm}$  the traces of  $w|_{\Omega_\pm}$  on  $S$ . Notice that if the initial data are in  $H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  the solution  $w$  belongs to  $C^0((0, T); H_0^1(\mathcal{M}))$  and the traces  $w|_{S_\pm}$  are well defined. The condition on the normal derivatives is then expressed in the weak formulation of the problem. See for instance [Lio88, Chapter 6]. In unique continuation problems one tries to recover the whole wave from a partial observation. In [Fil22] the following quantitative unique continuation result is proved.

**Theorem 3.1.3** (Theorem 1.3 in [Fil22]). *For any non empty subset  $\omega \subset \mathcal{M}$  there exist  $C, T > 0$  such that for all  $(w_0, w_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  with  $(w_0, w_1) \neq (0, 0)$  and  $w$  solution of (3.1.3) one has:*

$$\|(w_0, w_1)\|_{H^1 \times L^2} \leq C e^{C\Lambda} \|w\|_{L^2((0, T) \times \omega)},$$

$$\text{where } \Lambda = \frac{\|(w_0, w_1)\|_{H^1 \times L^2}}{\|(w_0, w_1)\|_{L^2 \times H^{-1}}}.$$

Theorem 3.1.3 above generalizes Theorem 1.1 in [LL19] where smooth coefficients are considered. An important aspect of Theorem 3.1.3 is that there is no assumption on the sign of the jump of the coefficient  $c$ . Suppose, to fix ideas, that  $c_- < c_+$  are two constants. We interpret  $c_-$  and  $c_+$  as the square of the speed of propagation of a wave travelling through two isotropic media  $\Omega_-$  and  $\Omega_+$  with different refractive indices,  $n_-$  and  $n_+$  respectively (recall that  $n_\pm = 1/\sqrt{c_\pm}$ ). Imagine that a wave starts travelling from a region that is inside  $\Omega_-$ . One has  $\sqrt{\frac{c_-}{c_+}} = \frac{n_+}{n_-}$  and therefore the assumption  $c_- < c_+$  translates to  $n_- > n_+$ . Then Snell-Descartes law states that when a wave travels from a medium with a higher refractive index to one with a lower refractive index there is a critical angle for which there is *total internal reflection*, that is no refraction at all. At the level of geometric optics, that is to say, in the high frequency regime such a wave stays trapped inside  $\Omega_-$ . Therefore one expects that, at least at high frequency, no information propagates from  $\Omega_-$  to  $\Omega_+$ , following the laws of geometric optics. However, Theorem 3.1.3 states that a part of the wave can always be observed from  $\Omega_+$  with an intensity at least exponentially small in terms of the typical frequency  $\Lambda$  of the wave.

In this article we show that indeed, in situations where  $\frac{c_-}{c_+}$  is sufficiently small depending on the geometric context one can find waves that are exponentially localized, in the high frequency limit, arbitrarily close to the interface  $S$ . As a consequence, we deduce that the estimate of Theorem 3.1.3 is, in general, optimal. For a solution  $w$  of (3.1.3) we define  $\Lambda(w) := \frac{\|(w(0), \partial_t w(0))\|_{H^1 \times L^2}}{\|(w(0), \partial_t w(0))\|_{L^2 \times H^{-1}}}$ . We have the following corollary of Theorem 3.1.1:



**Theorem 3.1.4** (Whispering gallery waves). *In the geometric setting of Theorem 3.1.1 there exist solutions  $(w_n)_{n \in \mathbb{N}}$  of (3.1.3) with  $\|w_n(0)\|_{L^2} = 1$ ,  $\partial_t w_n(0) = 0$  and such that for all  $\omega \subset \mathcal{M}$  with  $\text{dist}(\bar{\omega}, S) > 0$  and  $T > 0$  there exist  $C, d > 0$  with:*

$$\|w_n\|_{L^2((0,T) \times \omega)} \leq C e^{-d\Lambda(w_n)}. \quad (3.1.4)$$

Theorem 3.1.4 is an immediate consequence of Theorem 3.1.1. Indeed, take  $u_n$  as in Theorem 3.1.1, then

$$w_n(t, x) := \cos(\sqrt{\lambda_n t}) u_n(x).$$

satisfies (3.1.3) with  $(w_n, \partial_t w_n)|_{t=0} = (u_n, 0)$ ,  $\Lambda(w_n) = \sqrt{\lambda_n} + 1$  and (3.1.2) implies (3.1.4), up to changing the constant  $C$ .

The plan of the article is as follows. In Section 3.1.2 we describe the domain of the operator  $-\Delta_c$ . Then in Section 3.2 we give a detailed proof of Theorem 3.1.1 in the case of an annulus based on an Agmon estimate for a 1D semiclassical Schrödinger operator. In 3.2.3 we prove a refined Agmon estimate (Proposition 3.2.27) which gives a better description of the behavior of the eigenfunctions constructed in Section 3.2. In Section 3.3 we explain how our arguments can be used to include more general surfaces of revolution. Finally in subsection 3.3.2 we prove in Theorem 3.3.7 a result using only some local geometric assumptions in the neighborhood of the interface.

### 3.1.2 Domain and self-adjointness of the operator

Let us recall first some basic facts concerning the operator  $-\Delta_c$ , its domain and some general spectral properties. Given a function  $u = \mathbb{1}_{\Omega_-} u_- + \mathbb{1}_{\Omega_+} u_+$  with  $u_{\pm} \in C^\infty(\mathcal{M})$  one has in the distributional sense

$$\nabla u = \mathbb{1}_{\Omega_-} \nabla u_- + \mathbb{1}_{\Omega_+} \nabla u_+ + (u_- - u_+) \delta_S \nu,$$

where  $\delta_S$  is the surface measure on  $S$  and  $\nu$  is the unit normal vector field pointing into  $\Omega_+$ . We impose then that

$$u_-|_S = u_+|_S, \quad (3.1.5)$$

and the singular term is removed. Similarly, calculating

$$\text{div}(c(x) \nabla u),$$

we see that the condition

$$c_+ \partial_\nu u_+|_S = c_- \partial_\nu u_-|_S \quad (3.1.6)$$

combined with (3.1.5) gives the equality

$$\text{div}(c(x) \nabla u) = \mathbb{1}_{\Omega_-} \text{div}(c_- \nabla u_-) + \mathbb{1}_{\Omega_+} \text{div}(c_+ \nabla u_+).$$

We define then  $\mathcal{W}$  as the space of functions of the form

$$u = \mathbb{1}_{\Omega_-} u_- + \mathbb{1}_{\Omega_+} u_+,$$

with  $u_{\pm} \in C_0^\infty(\mathcal{M})$  and such that (3.1.5) and (3.1.6) hold. These conditions are called *transmission conditions* and for  $u \in \mathcal{W}$  one has  $-\Delta_c u \in L^2$ . With  $W$  as initial domain

$-\Delta_c$  is symmetric and bounded from below. Indeed, writing  $(\cdot, \cdot)$  for the inner product in  $L^2$  one has for  $u \in \mathcal{W}$

$$(-\Delta_c u, u) = - \int_{\Omega_-} \operatorname{div}(c_- \nabla u_-) \bar{u} - \int_{\Omega_+} \operatorname{div}(c_+ \nabla u_+) \bar{u} = \int_{\Omega_-} c_- |\nabla u_-|^2 dx + \int_{\Omega_+} c_+ |\nabla u_+|^2 dx,$$

where we have used the transmission conditions and an integration by parts.

One can then consider the Friedrichs extension (see for instance [Lew18, Chapter 3.2.4]) of  $-\Delta_c$ , which is a self-adjoint extension of  $-\Delta_c$  whose domain is given by:

$$\mathcal{A} = \{u \in H_0^1(\mathcal{M}) \mid -\Delta_c u \in L^2(\mathcal{M})\}.$$

Using elliptic regularity arguments (see eg [LRLR13, Appendix C.2]) one can see that in fact

$$\mathcal{A} = \{u \in H_0^1(\mathcal{M}) \mid u|_{\Omega_-} \in H^2(\Omega_-), u|_{\Omega_+} \in H^2(\Omega_+), c_+ \partial_\nu(u|_{\Omega_-})|_S = c_- \partial_\nu(u|_{\Omega_+})|_S\}. \quad (3.1.7)$$

In the sequel we shall denote by  $u_\pm$  the restriction  $u|_{\Omega_\pm}$  of  $u \in \mathcal{A}$  on  $\Omega_-$  and  $\Omega_+$ .

The operator  $-\Delta_c$  with domain  $\mathcal{A}$  is then positive, self-adjoint and has a compact resolvent. We deduce that its spectrum solely consists of positive eigenvalues  $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$  of finite multiplicity with  $\lambda_j \rightarrow +\infty$ .

**Acknowledgements** The author would like to thank C. Laurent and M. Léautaud for discussions, encouragements, and patient guidance. The author thanks as well the anonymous referees for valuable remarks and suggestions that helped to improve the quality of this paper and Jeffrey Galkowski for pointing out the reference [PV99].

## 3.2 The case of an annulus

We denote by  $B_R$  the open ball in  $\mathbb{R}^2$  of radius  $R$  centered at 0 and study the eigenfunction problem for  $(\mathcal{A}, -\Delta_c)$  in the following geometric context: Let  $0 < R_0 < R_1 < R_2$  and set  $\mathcal{M} = \overline{B_{R_2}} \setminus B_{R_0} \subset \mathbb{R}^2$ ,  $\Omega_- = B_{R_1} \setminus \overline{B_{R_0}}$ ,  $\Omega_+ = B_{R_2} \setminus \overline{B_{R_1}}$ ,  $S = \overline{B_{R_1}} \setminus B_{R_1}$ . For simplicity we assume  $c$  piecewise constant, that is  $c = \mathbf{1}_{\Omega_-} c_- + \mathbf{1}_{\Omega_+} c_+$ , with  $0 < c_- < c_+$  to be chosen later on.

### 3.2.1 Reduction to a semiclassical Schrödinger operator

Following [LL21c, LL21b] we want to reduce the problem to the study of a semiclassical Schrödinger operator. We work in polar coordinates, in which the operator  $-\Delta_c$  takes the form

$$-\Delta_c = -\frac{1}{r} \partial_r (c r \partial_r) + \frac{c}{r^2} \partial_\theta^2.$$

For a function  $u \in \mathcal{A}$  we have (see Section 3.1.2):

$$-\Delta_c u(r, \theta) = - \sum_{\pm} \mathbf{1}_{\Omega_\pm} \left( c_\pm \partial_r^2 u_\pm + \frac{c_\pm}{r} \partial_r u_\pm + \frac{c_\pm}{r^2} \partial_\theta^2 u_\pm \right), \quad u \in \mathcal{A},$$

where we recall that we write  $u = \mathbf{1}_{\Omega_-} u_- + \mathbf{1}_{\Omega_+} u_+$ . We look for solutions of the eigenvalue problem  $-\Delta_c u = \lambda u$  under the form  $u(r, \theta) = e^{in\theta} f(r)$  which yields

$$\sum_{\pm} \mathbf{1}_{\Omega_\pm} \left( -\frac{c_\pm}{n^2} \left( \partial_r^2 f_\pm(r) + \frac{1}{r} \partial_r f_\pm(r) \right) + \left( \frac{c_\pm}{r^2} - \frac{\lambda}{n^2} \right) f_\pm(r) \right) = 0. \quad (3.2.1)$$

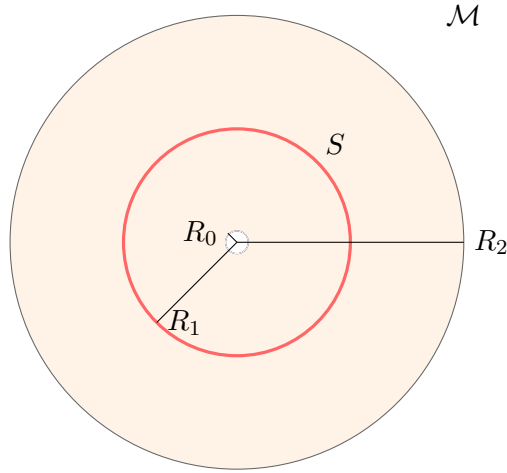


Figure 3.1: The annulus  $\mathcal{M}$  with the interface  $S$  where the coefficient  $c$  jumps.

We consider large angular momenta,  $n \rightarrow +\infty$  and interpret  $h := 1/n$  as a semiclassical parameter. Equation (3.2.1) writes then as  $P_h f = E_h f$ , where

$$P_h := \sum_{\pm} \mathbb{1}_{\Omega_{\pm}} \left( -h^2 \frac{1}{r} \partial_r (c_{\pm} r \partial_r) \right) + V_c(r), \quad V_c(r) := \frac{c}{r^2}, \quad E_h = h^2 \lambda, \quad (3.2.2)$$

and  $V_c(r)$  can be seen as the effective potential with a jump discontinuity at  $r = R_1$ . We define  $V_-$  as:

$$V_-(r) = \frac{c_-}{r^2} \quad (3.2.3)$$

and we write  $E_0 := \frac{c_-}{R_1^2}$ .

We now study the eigenvalue problem for the one dimensional semiclassical Schrödinger operator  $P_h$ . We define the space

$$\mathcal{A}_r = \{f \in H_0^1((R_0, R_2)) \mid f|_{(R_0, R_1)} \in H^2((R_0, R_1)), f|_{(R_1, R_2)} \in H^2((R_1, R_2)), \\ c_+ \partial_r (f|_{(R_0, R_1)})(R_1) = c_- \partial_r (f|_{(R_1, R_2)})(R_1)\}. \quad (3.2.4)$$

We write as well  $f_- := f|_{(R_0, R_1)}$  and  $f_+ := f|_{(R_1, R_2)}$ . The space  $\mathcal{A}_r$  satisfies:

$$f \in \mathcal{A}_r \Leftrightarrow f(r) e^{in\theta} \in \mathcal{A}, \quad n \in \mathbb{Z},$$

and  $P_h$  is self-adjoint with domain  $\mathcal{A}_r$ , for the same reason as in Section 3.1.2.

In the following proposition we show the existence of eigenfunctions for  $P_h$  close to some energy levels. For the proof we start by constructing a rough quasimode and then use the self-adjointness of the operator (see for instance [LL21b, Lemma 3.6], [Zwo12, Chapter 12.5]).

We also point out the references [Chr15] in which the question of concentration of quasimodes is considered in a related setting and the recent preprint [HM22] where 1d semiclassical Schrödinger operators with singular potential are studied.

**Proposition 3.2.1** (Existence of eigenfunctions for the 1D operator). *There exists  $h_0$  such that for all  $E \in [E_0, V_c(R_0))$  there exists  $C > 0$  such that for all  $h \in (0, h_0]$ , there exist  $E_h$  and  $\psi_h \in \mathcal{A}_r$  with*

$$P_h \psi_h = E_h \psi_h, \quad |E_h - E| \leq Ch^{2/3}.$$

*Proof.* The operator  $P_h$  with domain  $\mathcal{A}_r$  is self-adjoint in  $L^2((R_0, R_2), r dr)$ . Note that we can estimate indifferently with norms in  $L^2((R_0, R_2), r dr)$  or  $L^2((R_0, R_2), dr)$  since they are equivalent.

We write  $E = V_-(\rho_E)$  for  $\rho_E = \sqrt{\frac{c_-}{E}}$  where we recall that  $V_-$  is defined in (3.2.3). Consider  $\chi \in C_0^\infty((-1, 0))$  such that  $\chi = 1$  in a neighborhood of  $-1/2$  and define  $f_h(r) = h^{-1/3} \chi(h^{-2/3}(r - \rho_E))$ . One then has that for  $h \leq h_0$  with  $h_0$  sufficiently small  $f_h(r) \in C_0^\infty((R_0, R_1))$ . This implies in particular that  $f_h \in \mathcal{A}_r$ . We estimate now:

$$\begin{aligned} & \left\| -ch^2 \left( \partial_r^2 f_h(r) + \frac{1}{r} \partial_r f_h(r) \right) \right\|_{L^2}^2 \\ &= \int ch^4 \left( h^{-5/3} \chi''(h^{-2/3}(r - \rho_E)) + \frac{1}{r} h^{-1} \chi'(h^{-2/3}(r - \rho_E)) \right)^2 dr \\ &\leq Ch^{4/3}. \end{aligned} \quad (3.2.5)$$

To estimate the next term we use that  $\text{supp}(f_h) \subset (R_0, R_1)$ . We use now the fact that  $V_-$  is uniformly Lipschitz in  $[R_0, R_1]$  which implies that, in the support of  $f_h$ , one has

$$|V_c(r) - E| = |V_-(r) - E| \leq C|r - \rho_E|,$$

and hence

$$\|(V_c(r) - E) f_h(r)\|_{L^2}^2 \leq C \int (r - \rho_E)^2 h^{-2/3} \chi^2(h^{-2/3}(r - \rho_E)) dr \leq Ch^{4/3}. \quad (3.2.6)$$

Putting together (3.2.5) and (3.2.6) yields

$$\|(P_h - E) f_h\|_{L^2} \leq Ch^{2/3}. \quad (3.2.7)$$

Noticing finally that

$$\|f_h\|_{L^2}^2 = \int h^{-2/3} \chi^2(h^{-2/3}(r - \rho_E)) dr = c_0,$$

we can write (3.2.7) as  $\|(P_h - E) f_h\|_{L^2} \leq Ch^{2/3} \|f_h\|_{L^2}$ . If now  $E \notin \text{Sp}(P_h)$  one can write

$$\begin{aligned} \|(P_h - E) f_h\|_{L^2} &\leq Ch^{2/3} \|f_h\|_{L^2} = Ch^{2/3} \left\| (P_h - E)^{-1} (P_h - E) f_h \right\| \\ &\leq Ch^{2/3} \left\| (P_h - E)^{-1} \right\|_{L^2 \rightarrow L^2} \|(P_h - E) f_h\|_{L^2}, \end{aligned}$$

which gives

$$\left\| (P_h - E)^{-1} \right\|_{L^2 \rightarrow L^2} \geq C^{-1} h^{-2/3}.$$

Finally, since  $P_h$  is self-adjoint one has  $\left\| (P_h - E)^{-1} \right\|_{L^2 \rightarrow L^2} = \frac{1}{\text{dist}(\text{Sp}(P_h), E)}$  and consequently

$$\text{dist}(\text{Sp}(P_h), E) \leq Ch^{2/3}, \quad (3.2.8)$$

which holds in the case  $E \in \text{Sp}(P_h)$  as well. The existence of  $E_h, \psi_h$  is then a result of (3.2.8) and of the fact that the spectrum of  $P_h$  consists solely of eigenvalues.  $\square$

### 3.2.2 The Agmon estimate

We follow [Hel88, Chapter 3] (see as well [DS99, Chapter 6.B], [LL23c]). We start by defining the appropriate Agmon distance, which corresponds to a distance to the *classically allowed region* for the potential  $V_c$  at the energy level  $E$ . The classically allowed region  $K_E$  is defined as

$$K_E = \{r \in [R_0, R_2] \mid V_c(r) \leq E\}, \quad (3.2.9)$$

where we recall that  $V_c(r) = \frac{c}{r^2}$  and  $c = \mathbf{1}_{(R_0, R_1)}c_- + \mathbf{1}_{(R_1, R_2)}c_+$ . The condition we impose on the coefficient  $c$  is then (see Figure 3.2):

$$0 < c_- < c_+, \quad V_-(R_1) < V_c(R_2), \quad (3.2.10)$$

and as a consequence the function  $\tilde{V}$  defined as  $\tilde{V} = V_c(r)$  for  $r \in [R_0, R_2] \setminus \{R_1\}$  and  $\tilde{V}(R_1) = E_0$  attains its minimum at  $r = R_1$ . We define the appropriate Agmon distance for  $E \geq E_0$  as

$$d_{A,E}(r) = \inf_{y \in K_E} \left| \int_y^r \sqrt{\frac{(V_c(s) - E)_+}{c(s)}} ds \right|, \quad (3.2.11)$$

where  $a_+ = \max(a, 0)$ .

In the sequel we shall focus on energy levels situated close to the minimum  $E_0$ . We remark that assumption (3.2.10) and continuity of  $V_-$  in  $[R_0, R_1]$  imply that there exists  $\eta_0 > 0$  such that

$$K_{E_0+\eta} \subset [R_0, R_1], \quad \forall 0 < \eta < \eta_0. \quad (3.2.12)$$

For  $E \in [E_0, E_0 + \eta]$ ,  $\eta \leq \eta_0$  we can define  $\rho_E := V_c^{-1}(E)$  (which is invertible in  $[R_0, R_1]$ ) and obtain that  $K_E = [\rho_E, R_1]$  as well as the following explicit expressions:

$$\begin{aligned} d_{A,E}(r) &= \int_r^{\rho_E} \sqrt{\frac{(V_c(s) - E)}{c_-(s)}} ds, \quad \text{for } r \in [R_0, \rho_E], \\ d_{A,E}(r) &= 0, \quad \text{for } r \in [\rho_E, R_1], \\ d_{A,E}(r) &= \int_{R_1}^r \sqrt{\frac{(V_c(s) - E)}{c_+(s)}} ds, \quad \text{for } r \in [R_1, R_2]. \end{aligned} \quad (3.2.13)$$

One has in particular that  $d_{A,E}$  is  $C$ -Lipschitz with  $C = \left( \frac{\max\{V_c(s) - E\}}{c_-} \right)^{1/2}$ .

The following identity is the key ingredient of the Agmon estimate. Notice that all quantities appearing in the following lemma are well defined. Indeed, since  $f \in \mathcal{A}_r$  one has  $f \in H^1$  and the left hand side is well defined.

**Lemma 3.2.2.** *Let  $\phi$  be real valued Lipschitz continuous on  $[R_0, R_2]$  and  $f \in \mathcal{A}_r$ . Then one has:*

$$\begin{aligned} \int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} f) \right|^2 r dr - \int_{R_0}^{R_2} c |\partial_r \phi|^2 e^{2\phi/h} |f|^2 r dr \\ = - \operatorname{Re} \int_{(R_0, R_1) \cup (R_1, R_2)} e^{2\phi/h} ch^2 \left( \partial_r^2 + \frac{1}{r} \partial_r \right) f \cdot \bar{f} r dr. \end{aligned}$$

*Proof.* We write  $f = \mathbb{1}_{(R_0, R_1)} f_- + \mathbb{1}_{(R_1, R_2)} f_+$ , split the integrals and integrate by parts. We have:

$$\begin{aligned} - \int_{R_0}^{R_1} c_- e^{2\phi/h} \left( \partial_r^2 + \frac{1}{r} \partial_r \right) f_- \cdot \bar{f}_- r dr &= - \int_{R_0}^{R_1} c_- \partial_r (r \partial_r f_-) e^{2\phi/h} \cdot \bar{f}_- dr \\ &= \int_{R_0}^{R_1} c_- \partial_r f_- \partial_r (e^{2\phi/h} \bar{f}_-) r dr - c_- R_1 \partial_r f_- (R_1) e^{2\phi/h} \bar{f}_- (R_1), \end{aligned} \quad (3.2.14)$$

and similarly

$$\begin{aligned} - \int_{R_1}^{R_2} c_+ e^{2\phi/h} \left( \partial_r^2 + \frac{1}{r} \partial_r \right) f_+ \cdot \bar{f}_+ r dr &= - \int_{R_1}^{R_2} c_+ \partial_r (r \partial_r f_+) e^{2\phi/h} \cdot \bar{f}_+ dr \\ &= \int_{R_1}^{R_2} c_+ \partial_r f_+ \partial_r (e^{2\phi/h} \bar{f}_+) r dr + c_+ R_1 \partial_r f_+ (R_1) e^{2\phi/h} \bar{f}_+ (R_1). \end{aligned} \quad (3.2.15)$$

Now by definition of the space  $\mathcal{A}_r$  in (3.2.4) we find that

$$c_- R_1 \partial_r f_- (R_1) e^{2\phi/h} \bar{f}_- (R_1) = c_+ R_1 \partial_r f_+ (R_1) e^{2\phi/h} \bar{f}_+ (R_1).$$

Adding (3.2.14) and (3.2.15) the boundary terms cancel out and we obtain

$$\begin{aligned} - \int_{(R_0, R_1) \cup (R_1, R_2)} c e^{2\phi/h} h^2 \left( \partial_r^2 + \frac{1}{r} \partial_r \right) f \cdot \bar{f} r dr \\ = \int_{R_0}^{R_2} c 2h \partial_r \phi e^{2\phi/h} \bar{f} \partial_r f r dr + \int_{R_0}^{R_2} c h^2 e^{2\phi/h} |\partial_r f|^2 r dr, \end{aligned}$$

which gives the sought identity after taking real parts.  $\square$

We can now show the following Agmon estimate. We recall that  $\eta_0$  is defined by (3.2.12).

**Proposition 3.2.3.** *Let  $E \in [E_0, E_0 + \frac{\eta_0}{2}]$  and  $\epsilon(h)$  with  $\epsilon(h) \xrightarrow{h \rightarrow 0} 0$ . Then for all  $\delta > 0$ , there exist  $C, h_0$  such that for all  $\psi_h$  satisfying*

$$P_h \psi_h = (E + \epsilon(h)) \psi_h, \quad \|\psi_h\|_{L^2} = 1,$$

one has, for  $h \leq h_0$ :

$$\left\| h \partial_r \left( e^{\frac{d_{A,E}}{h}} \psi_h \right) \right\|_{L^2((R_0, R_2))} + \left\| e^{\frac{d_{A,E}}{h}} \psi_h \right\|_{L^2((R_0, R_2))} \leq C e^{\delta/h}.$$

*Proof.* We recall that  $d_{A,E}$  is Lipschitz continuous and consider the weight  $\phi = (1 - \delta) d_{A,E}$  with  $0 < \delta < 1$ . Let us write  $E_h := E + \epsilon(h)$ . We can then apply the identity of Lemma 3.2.2 with  $\psi_h$  which solves  $P_h \psi_h = E_h \psi_h$ , or equivalently

$$- \mathbb{1}_{(R_0, R_1)} c h^2 \left( \partial_r^2 + \frac{1}{r} \partial_r \right) \psi_{h,-} = \mathbb{1}_{(R_0, R_1)} (E_h - V_c(r)) \psi_{h,-},$$

and

$$- \mathbb{1}_{(R_1, R_2)} c h^2 \left( \partial_r^2 + \frac{1}{r} \partial_r \right) \psi_{h,+} = \mathbb{1}_{(R_1, R_2)} (E_h - V_c(r)) \psi_{h,+}.$$

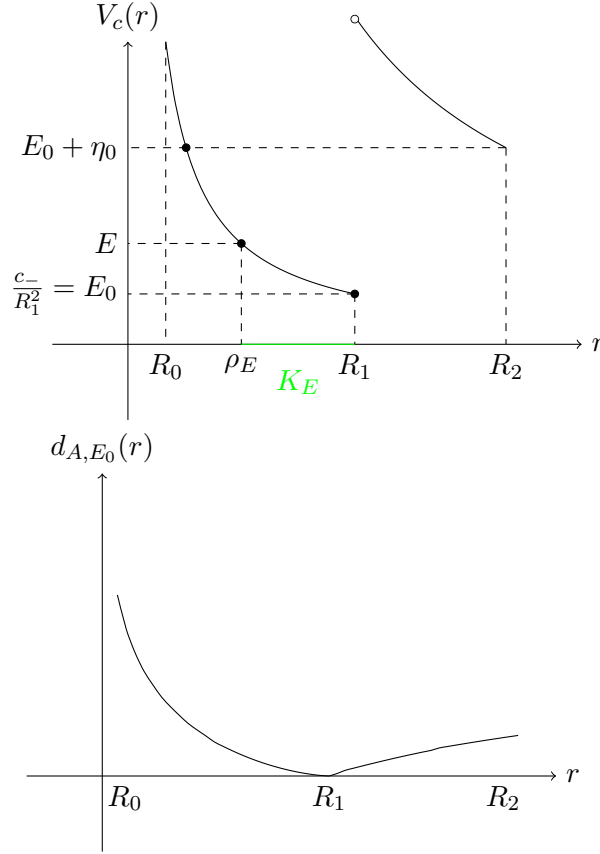


Figure 3.2: Up: The potential  $V_c$  with the points of interest. For  $E < E_0 + \eta_0 = \frac{c_+}{R_2^2}$  the potential is continuous and injective in the classically allowed region  $K_E$  which is in green. Down: The graph of the Agmon distance related to the minimal energy level  $E_0$ . Notice the  $C^1$  singularity at  $r = R_1$  due to the jump of the coefficient  $c$ .

We find:

$$\int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 r dr - \int_{R_0}^{R_2} c |\partial_r \phi|^2 e^{2\phi/h} |\psi_h|^2 r dr = \int_{R_0}^{R_2} e^{2\phi/h} (E_h - V_c(r)) |\psi_h|^2 r dr.$$

Let us define  $I_\alpha^+ := \{V - E > \alpha^2\}$ ,  $I_\alpha^- = \{V - E \leq \alpha^2\}$  with  $\alpha > 0$  small to be chosen. We split the integrals according to  $(R_0, R_2) = I_\alpha^+ \cup I_\alpha^-$  and write the above equality as

$$\begin{aligned} \int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 r dr + \int_{I_\alpha^+} e^{2\phi/h} (V_c(r) - E_h - c |\partial_r \phi|^2) |\psi_h|^2 r dr \\ = - \int_{I_\alpha^-} e^{2\phi/h} (V_c(r) - E_h - c |\partial_r \phi|^2) |\psi_h|^2 r dr. \end{aligned} \quad (3.2.16)$$

To control the integral on  $I_\alpha^+$  from below we notice that the Agmon distance satisfies the eikonal equation:

$$c |\partial_r d_{A,E}|^2 = (V_c(r) - E)_+, \quad \text{in } \mathcal{D}'((R_0, R_2)). \quad (3.2.17)$$

Hence, taking  $h \leq h_0 = h_0(\alpha, \delta)$

$$\begin{aligned} \int_{I_\alpha^+} e^{2\phi/h} (V_c(r) - E_h - c|\partial_r \phi|^2) |\psi_h|^2 r dr &= \int_{I_\alpha^+} e^{2\phi/h} \left( (V_c(r) - E)(1 - (1 - \delta)^2) - \epsilon(h) \right) |\psi_h|^2 r dr \\ &\geq \frac{\alpha^2 \delta}{2} \int_{I_\alpha^+} e^{2\phi/h} |\psi_h|^2 r dr, \end{aligned} \quad (3.2.18)$$

where  $h_0$  is such that  $\epsilon(h) \leq \frac{\alpha^2 \delta}{2}$  for all  $h \leq h_0$ . For the integral on  $I_\alpha^-$  we have

$$\left| \int_{I_\alpha^-} e^{2\phi/h} (V_c(r) - E_h - c|\partial_r \phi|^2) |\psi_h|^2 r dr \right| \leq C \int_{I_\alpha^-} e^{2\phi/h} |\psi_h|^2 r dr, \quad (3.2.19)$$

where  $C$  depends on  $\max V_c$ . Putting (3.2.16), (3.2.18), (3.2.19) together we find

$$\int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 r dr + \alpha^2 \delta \int_{I_\alpha^+} e^{2\phi/h} |\psi_h|^2 r dr \leq C \int_{I_\alpha^-} e^{2\phi/h} |\psi_h|^2 r dr,$$

which gives

$$\int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 r dr + \alpha^2 \delta \int_{R_0}^{R_2} e^{2\phi/h} |\psi_h|^2 r dr \leq C \int_{I_\alpha^-} e^{2\phi/h} |\psi_h|^2 r dr. \quad (3.2.20)$$

We estimate now  $\phi$  in  $I_\alpha^-$ . Taking  $\alpha^2 \leq \frac{\eta_0}{2}$  we have, with  $\rho_E = V_c^{-1}(E)$

$$d_{A,E}(r) = \int_r^{\rho_E} \sqrt{\frac{(V_c(s) - E)}{c(s)}} ds, \quad \text{for } r \in I_\alpha^-.$$

Using that  $V_c^{-1}$  is Lipschitz on  $[R_0, \rho_E]$  implies that for  $r \in I_\alpha^-$  one has

$$|r - \rho_E| \leq C |V_c(r) - V_c(\rho_E)| \leq C \alpha^2.$$

Choosing then  $\alpha_0 = \alpha_0(\delta)$  sufficiently small we obtain for  $r \in I_\alpha^-$  and  $\alpha \leq \alpha_0$ :

$$\phi(r) = (1 - \delta) d_{A,E} = (1 - \delta) \int_r^{\rho_E} \sqrt{\frac{(V_c(s) - E)}{c(s)}} ds \leq C(1 - \delta) |r - \rho_E| \leq \tilde{C} \alpha^2 \leq \delta.$$

which combined with (3.2.20) yields, using  $R_0 > 0$ ,

$$\int_{R_0}^{R_2} h^2 \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 dr + \alpha^2 \delta \int_{R_0}^{R_2} e^{2\phi/h} |\psi_h|^2 dr \leq C e^{2\delta/h}. \quad (3.2.21)$$

One needs finally to replace  $\phi$  by  $d_{A,E}$  in the above estimate. To do this we write, with  $d_{A,E} \leq M$ :

$$\begin{aligned} \int_{R_0}^{R_2} h^2 \left| \partial_r \left( e^{\frac{d_{A,E}}{h}} \psi_h \right) \right|^2 dr &= \int_{R_0}^{R_2} h^2 \left| \partial_r \left( e^{\frac{\delta d_{A,E}}{h}} e^{\frac{\phi}{h}} \psi_h \right) \right|^2 dr \\ &\leq C \int_{R_0}^{R_2} h^2 e^{\frac{2\delta d_{A,E}}{h}} \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 dr + C \delta^2 \int_{R_0}^{R_2} |d'_{A,E}|^2 e^{\frac{2\delta d_{A,E}}{h}} e^{2\phi/h} |\psi_h|^2 dr \\ &\leq C h^2 e^{\frac{2\delta M}{h}} \int_{R_0}^{R_2} \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 dr + C \delta^2 \|V_c - E\|_{L^\infty} e^{\frac{2\delta M}{h}} \int_{R_0}^{R_2} e^{2\phi/h} |\psi_h|^2 dr. \end{aligned}$$

Combining this together with (3.2.21) concludes the proof of Proposition 3.2.3.  $\square$



**Remark 3.2.4** (On the regularity of the potential  $V_c$ ). In order to control from above the integral on  $I_\alpha^-$  we used that  $V_-^{-1}$  is Lipschitz on  $[R_0, \rho_E]$  in order to establish the following property:

$$|V_c(r) - V_c(\rho_E)| \leq \alpha^2 \implies |r - \rho_E| \leq C\alpha^2.$$

However one can replace this step by using the uniform continuity of  $V_-^{-1}$ . This is done in [LL23c] where the authors prove Agmon estimates for continuous potentials. In particular the Agmon estimate of Proposition 3.2.3 and its consequences hold for potentials  $V$  having the following local behavior:

$$V(r) = (a_- + b_-(R_1 - r)^{\alpha_-}(1 + \epsilon_-(r))\mathbb{1}_{r < R_1} + (a_+ + b_+(-R_1 + r)^{\alpha_+}(1 + \epsilon_+(r))\mathbb{1}_{r > R_1}$$

where  $\epsilon_\pm$  are smooth functions that vanish at  $r = R_1$ ,  $R_1$  is a global strict minimum for  $V$  and  $\alpha_\pm \geq 0$ . This type of potential appears in [FS09] where a different problem is studied.

It follows from the Agmon estimate of Proposition 3.2.3 that the mass of eigenfunctions close to the minimum energy level  $E_0$  should be concentrated close to the point  $R_1$  where the coefficient  $c$  exhibits a jump. This is the following corollary. We recall that  $E_0 = V_-(R_1)$  is the infimum of the potential  $V_c$

**Corollary 3.2.5.** *Let  $\epsilon(h)$  with  $\epsilon(h) \xrightarrow{h \rightarrow 0} 0$ . There exists  $h_0 > 0$  such that for all  $\psi_h$  satisfying*

$$P_h \psi_h = (E_0 + \epsilon(h))\psi_h, \quad \|\psi_h\|_{L^2} = 1, \quad \psi_h \in \mathcal{A}_r, \quad (3.2.22)$$

and all  $\varepsilon > 0$  there exist  $C, d > 0$  such that for  $0 < h \leq h_0$

$$\|\psi_h\|_{L^2((R_0, R_2) \setminus [R_1 - \varepsilon, R_1 + \varepsilon])} \leq C e^{-\frac{d}{h}}.$$

*Proof.* Consider a solution of (3.2.22). The associated Agmon distance to the energy level  $E_0$  (see (3.2.13)) satisfies

$$d_{A, E_0}(r) \geq m > 0, \quad \text{for } r \in (R_0, R_2) \setminus [R_1 - \varepsilon, R_1 + \varepsilon].$$

The result then follows from the estimate of Proposition 3.2.3 by taking  $\delta \leq m/2$ .  $\square$

### 3.2.3 Finer information on the concentration of the eigenfunctions

We proved so far, using the Agmon estimate 3.2.3 and its corollary a *symmetric* estimate with respect to the interface which gives exponential concentration of the eigenfunctions. This is sufficient in order to prove the optimality of the unique continuation results stated in the introduction.

However, a natural question is to know whether eigenfunctions are more localized on one side of the interface than on the other. This is partly motivated by the geometric optics example given in the introduction where one expects that eigenfunctions have a bigger mass in the left of  $R_1$ . Heuristically, this can already be seen in the behavior of the Agmon distance since it grows faster on the right of  $R_1$  than on the left. Indeed, one has that  $d'_{A, E}(r) \rightarrow 0$  when  $r \rightarrow R_1^-$  and  $|d'_{A, E}(r)| \rightarrow \sqrt{1 - \frac{c_-}{c_+} \frac{1}{R_1}} > 0$  when  $r \rightarrow R_1^+$ . See as well its graph in Figure 3.2. Unfortunately, the Agmon estimate of Proposition 3.2.3 is not sufficient to capture the behavior of the eigenfunctions arbitrarily close to the interface

with uniform constants. This is due to the fact that there is a loss of  $\delta > 0$  and therefore one has to fix a distance from the interface *and then* use the Agmon estimate. We explain in this section how can obtain a refined Agmon estimate without this loss. For this improved version we follow [Hel88, Chapter 3] where it is used in the case of a non-degenerate well. See as well [LL21b, Theorem 3.9].

**Proposition 3.2.6** (Refined concentration estimate). *Consider  $\psi_h$  satisfying (3.2.22) with  $|\epsilon(h)| \leq C_1 h^{2/3}$  for some  $C_1 > 0$ . Then there exist  $C, h_0 > 0$  such that for  $0 < h \leq h_0$  one has*

$$\|\psi_h\|_{L^2((R_0, r))} \leq C e^{-\frac{C \log h}{h^{1/3}}} e^{-\frac{d_{A, E_0}(r)}{h}}, \quad \text{for all } r \in R_0 \leq r \leq R_1,$$

and

$$\|\psi_h\|_{L^2((R_1, r))} \leq C e^{-\frac{C \log h}{h^{1/3}}} e^{-\frac{d_{A, E_0}(r)}{h}}, \quad \text{for all } r \in R_1 \leq r \leq R_2.$$

*Proof.* Let us use the notation of Proposition 3.2.3 and its proof. The proof is similar to the proof of Proposition 3.2.3 but now we define  $I_{C_0}^- := \{d_{A, E_0} \leq C_0 h^{2/3}\}$ ,  $I_{C_0}^+ := \{d_{A, E_0} > C_0 h^{2/3}\}$  with  $C_0 > 0$  to be chosen. We define as well  $\phi$  by

$$\begin{aligned} \phi(r) &= d_{A, E_0}(r) - C_0 h^{2/3} \log(C_0), \quad r \in I_{C_0}^- \\ &= d_{A, E_0}(r) - C_0 h^{2/3} \log\left(\frac{d_{A, E_0}(r)}{h^{2/3}}\right), \quad r \in I_{C_0}^+. \end{aligned}$$

We find then as in (3.2.16):

$$\begin{aligned} \int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 r dr + \int_{I_{C_0}^+} e^{2\phi/h} (V_c(r) - E_h - c|\partial_r \phi|^2) |\psi_h|^2 r dr \\ = - \int_{I_{C_0}^-} e^{2\phi/h} (V_c(r) - E_h - c|\partial_r \phi|^2) |\psi_h|^2 r dr, \end{aligned} \quad (3.2.23)$$

where  $E_h = E_0 + \epsilon(h)$ . To control the integral on  $I_{C_0}^+$  we need to bound from below  $V_c(r) - E_h - c|\partial_r \phi|^2$  in this region. We have  $c|\partial_r \phi|^2 = c|\partial_r d_{A, E_0}|^2 (1 - \frac{C_0 h^{2/3}}{d_{A, E_0}})^2$ . Using (3.2.17) we find, for  $r \in I_{C_0}^+$ :

$$\begin{aligned} V_c(r) - E_h - c|\partial_r \phi|^2 &= (V_c - E_0) - (V_c - E_0) \left(1 - \frac{C_0 h^{2/3}}{d_{A, E_0}}\right)^2 - \epsilon(h) \\ &= (V_c - E_0) \left(1 - \left(1 - \frac{C_0 h^{2/3}}{d_{A, E_0}}\right)^2 - \frac{\epsilon(h)}{V_c - E_0}\right) \\ &= (V_c - E_0) \left(\frac{2C_0 h^{2/3}}{d_{A, E_0}} - \left(\frac{C_0 h^{2/3}}{d_{A, E_0}}\right)^2 - \frac{\epsilon(h)}{V_c - E_0}\right) \\ &\geq (V_c - E_0) \left(\frac{C_0 h^{2/3}}{d_{A, E_0}} - \frac{\epsilon(h)}{V_c - E_0}\right), \end{aligned} \quad (3.2.24)$$

where for the last inequality we used that  $d_{A, E_0}(r) \geq C_0 h^{2/3}$ . At this point let us observe that there exists  $C_2 > 0$  such that

$$d_{A, E_0}(r) \leq C_2 (V_c(r) - E_0).$$

Let us verify this for  $r$  in a small neighborhood of  $R_1$ . Remark that for  $r \rightarrow R_1^+$  one has  $d_{A,E_0}(r) \rightarrow 0$  and  $V_c(r) - E_0 \rightarrow \frac{c_+}{R_1^2} - \frac{c_-}{R_1^2} > 0$ . Therefore for  $\varepsilon > 0$  small one has for  $r \in (R_1, R_1 + \varepsilon)$  that  $d_{A,E_0}(r) \leq C(V_c(r) - E_0)$ . For  $r \in (R_1 - \varepsilon, R_1)$  with  $\varepsilon$  small we have  $V_c(r) - E_0 \underset{r \rightarrow R_1^-}{\sim} (r - R_1)V'_-(R_1)$  with  $V_-(r) = \frac{c_-}{r^2}$  and hence  $V_c(r) - E_0 \geq \tilde{c}|r - R_1|$ .

Since  $d_{A,E_0}$  is Lipschitz we have as well  $d_{A,E_0}(r) = d_{A,E_0}(r) - d_{A,E_0}(R_1) \leq C|r - R_1|$ . Consequently for  $r \in (R_1 - \varepsilon, R_1 + \varepsilon)$  we have  $d_{A,E_0}(r) \leq C(V_c(r) - E_0)$ . For  $r \in [R_0, R_2] \setminus (R_1 - \varepsilon, R_1 + \varepsilon)$  we have that  $V_c(r) - E_0 \geq \tilde{C} > 0$  and since  $d_{A,E_0}$  is bounded we obtain  $d_{A,E_0}(r) \leq C(V_c(r) - E_0)$  in this region as well. This gives the existence of  $C_2$  with  $d_{A,E_0}(r) \leq C_2(V_c(r) - E_0)$  for  $r \in [R_0, R_2]$ .

Now recall that  $\epsilon(h) \leq C_1 h^{2/3}$ . Choosing then  $C_0$  sufficiently large with respect to  $C_1$  and  $C_2$  we find using (3.2.24):

$$V_c(r) - E_h - c|\partial_r \phi|^2 \geq Ch^{2/3}, \quad r \in I_{C_0}^+,$$

which combined with (3.2.23) yields:

$$\begin{aligned} \int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 r dr + h^{2/3} \int_{I_{C_0}^+} e^{2\phi/h} |\psi_h|^2 r dr \\ \leq C \int_{I_{C_0}^-} e^{2\phi/h} (V_c(r) - E_h - c|\partial_r \phi|^2) |\psi_h|^2 r dr \\ \leq C \int_{I_{C_0}^-} e^{2\phi/h} |\psi_h|^2 r dr \end{aligned}$$

and hence

$$\int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 r dr + h^{2/3} \int_{R_0}^{R_2} e^{2\phi/h} |\psi_h|^2 r dr \leq C \int_{I_{C_0}^-} e^{2\phi/h} |\psi_h|^2 r dr.$$

Since  $\phi(r) \leq d_{A,E_0}(r) \leq C_0 h^{2/3}$  on  $I_{C_0}^-$  the last integral can be controlled from above, using that  $\psi_h$  is normalized:

$$\int_{I_{C_0}^-} e^{2\phi/h} |\psi_h|^2 r dr \leq Ce^{Ch^{-1/3}}.$$

The last two lines imply

$$h^{2/3} \int_{R_0}^{R_2} e^{2\phi/h} |\psi_h|^2 r dr \leq Ce^{Ch^{-1/3}}. \quad (3.2.25)$$

As in the proof of Proposition 3.2.3 we need to replace  $\phi$  by  $d_{A,E_0}$  in the expression above. To do so, remark that  $e^{\frac{2d_{A,E_0}}{h}} = e^{2\phi/h} e^{2C_0 h^{-1/3} \log C_0}$  for  $r \in I_{C_0}^-$  and  $e^{\frac{2d_{A,E_0}}{h}} = e^{2\phi/h} e^{2C_0 h^{-1/3} \log \left( \frac{d_{A,E_0}(r)}{h^{2/3}} \right)}$  for  $I_{C_0}^+$ . In all cases we can estimate, using that  $d_{A,E_0}(r)$  is bounded

$$e^{\frac{2d_{A,E_0}}{h}} \leq e^{2\phi/h} e^{-Ch^{-1/3} \log h}. \quad (3.2.26)$$

Combining (3.2.25) and (3.2.26) yields the existence of  $C > 0$  with

$$\int_{R_0}^{R_2} e^{2\frac{d_{A,E_0}}{h}} |\psi_h|^2 r dr \leq Ce^{-Ch^{-1/3} \log h}. \quad (3.2.27)$$

We remark finally that  $d_{A,E_0}$  is decreasing in  $[R_0, R_1]$  and increasing in  $[R_1, R_2]$  and we obtain the proposition as a consequence of (3.2.27).  $\square$

We note now that:

$$d_{A,E_0}(r) \underset{r \rightarrow R_1^+}{\sim} \alpha_+(r - R_1),$$

with  $\alpha_+ = \sqrt{1 - \frac{c_-}{c_+} \frac{1}{R_1}}$  and using for  $r \leq R_1$  the formula  $d_{A,E_0}(r) = \int_r^{R_1} \sqrt{\frac{1}{s^2} - \frac{1}{R_1^2}} ds$  we find

$$d_{A,E_0}(r) \underset{r \rightarrow R_1^-}{\sim} \alpha_-(R_1 - r)^{\frac{3}{2}},$$

with  $\alpha_- = \frac{2\sqrt{2}}{3R_1\sqrt{R_1}}$ . These asymptotics, combined with Proposition 3.2.6 give a precise result on the concentration of such eigenfunctions  $\psi_h$  on an arbitrarily small neighborhood of the interface. In particular we can now study the concentration of the eigenfunctions on a  $O(h^\beta)$  neighborhood of the interface. Using the first estimate of Proposition 3.2.6 with the asymptotic behavior of  $d_{A,E_0}$  on the left of the interface gives the existence of  $C, c > 0$  such that for  $r > 0$  one has:

$$\|\psi_h\|_{L^2(R_0, R_1-r)} \leq C e^{-Ch^{-1/3} \log h} e^{-\frac{cr^{3/2}}{h}}.$$

And we find as well:

$$\|\psi_h\|_{L^2(R_1+r, R_2)} \leq C e^{-Ch^{-1/3} \log h} e^{-\frac{cr}{h}}.$$

The two above estimates imply the following corollary:

**Corollary 3.2.7.** *Let  $\psi_h$  satisfy the assumptions of Proposition 3.2.6. Then for all  $\delta \in (1/3, 1)$  there exist  $c_\delta > 0$  such that:*

$$\|\psi_h\|_{L^2((R_1 - h^{\frac{2}{3}(1-\delta)}, R_1 + h^{1-\delta}))} = 1 + O(e^{-\frac{c_\delta}{h^\delta}}).$$

In other words the mass of the eigenfunction concentrates exponentially for  $h \rightarrow 0$  inside  $(R_1 - h^{\frac{2}{3}(1-\delta)}, R_1 + h^{1-\delta})$ . Once again this suggests that there might be more mass in the left of the interface and in a quantitative way.

### 3.2.4 Back to the two dimensional annulus

We now put all the pieces together to state our result for the initial operator  $-\Delta_c = -\operatorname{div}(c\nabla \cdot)$  defined on the annulus. Recall that the space  $\mathcal{A}$  has been defined in (3.1.7).

**Theorem 3.2.8.** *Consider  $\mathcal{M}, S$  as defined in the beginning of Section 3.2. Suppose that*

$$\frac{c_-}{R_1^2} < \frac{c_+}{R_2^2}.$$

*Then there exist sequences  $(\lambda_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, (u_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$  and  $E_0 > 0$  with  $\lambda_n \underset{n \rightarrow +\infty}{\sim} E_0 n^2$  such that for all  $\omega \subset \mathcal{M}$  with  $\operatorname{dist}(\bar{\omega}, S) > 0$  there exist  $C, d > 0$  satisfying:*

$$-\Delta_c u_n = \lambda_n u_n, \quad \|u_n\|_{L^2(\mathcal{M})} = 1, \quad \|u_n\|_{L^2(\omega)} \leq C e^{-dn},$$

*for all  $n \in \mathbb{N}$ .*

*Proof.* Let  $h = 1/n$  and consider  $\psi_h \in \mathcal{A}_r$  satisfying

$$P_h \psi_h = \left( E_0 + O(h^{2/3}) \right) \psi_h, \quad \|\psi_h\|_{L^2((R_0, R_2), r dr)} = \frac{1}{\sqrt{2\pi}}.$$

The existence of such a family  $(\psi_h)_h$  is given by Proposition 3.2.1. We define then

$$u_n(r, \theta) := e^{in\theta} \psi_h(r).$$

It follows (see the remark after the definition of  $\mathcal{A}_r$  in (3.2.4)) that  $u_n \in \mathcal{A}$ ,  $\|u_n\|_{L^2(\mathcal{M})} = 1$  and that (see (3.2.1))

$$-\Delta_c u_n = n^2 \left( E_0 + O(n^{-2/3}) \right) u_n.$$

Let finally  $\omega \subset \mathcal{M}$  satisfy  $\text{dist}(\bar{\omega}, S) > 0$ . That means that there exists  $\varepsilon > 0$  with

$$\begin{aligned} \|u_n\|_{L^2(\omega)} &\leq \|u_n\|_{L^2(\mathcal{M} \setminus (B(0, R_1 + \varepsilon) \setminus B(0, R_1 - \varepsilon)))} = \|\psi_h\|_{L^2((R_0, R_2) \setminus [R_1 - \varepsilon, R_1 + \varepsilon])}, \\ &\leq C e^{-dn}, \end{aligned}$$

thanks to Corollary 3.2.5. □

**Remark 3.2.9.** Let us remark that in the case of  $c$  being piecewise constant one could probably do explicit calculations using Bessel functions. Indeed, their asymptotic behavior is well understood (see for instance [Olv74]). However, our approach has the advantage of relying only on robust energy estimates. In particular, it applies with minor modifications to more general rotationally invariant geometries and jump coefficients (see Section 3.3).

### 3.3 The disk and other surfaces of revolution

We considered in Section 3.2 the case where  $\mathcal{M}$  is an annulus. The reason we presented the proof for an annulus is that the singularity coming from the change of variables to polar coordinates at 0 disappears, allowing us to work exclusively in one dimension. Of course, from a heuristic point of view this should not be a problem since we are interested in the behaviour of the eigenfunctions away from zero. However, in the case of the disk it is sometimes simpler to work with the initial operator  $-\Delta_c$  and not with its 1-dimensional analogue (3.2.2) since in this case it becomes more intricate to describe its domain  $\mathcal{A}_r$  of self-adjointness when  $r \in (0, R_2)$ .

#### 3.3.1 Maximally vanishing eigenfunctions

In this section we briefly explain how the method presented in Section 3.2 can be used to exhibit maximally vanishing eigenfunctions in the case of the disk or even for some surfaces of revolution embedded in  $\mathbb{R}^3$  diffeomorphic to a disk. For the geometric description of such manifolds we follow [LL21b, Section 3] and [LL21c, Section 4].

Let  $(\mathcal{M}, g)$  be an embedded 2D submanifold of  $\mathbb{R}^3$  having  $\mathbb{S}^1$  as an effective isometry group. We denote by  $\mathbb{S}^1 \times \mathcal{M} \ni (\theta, m) \rightarrow \mathcal{R}_\theta m$  the action of  $\mathbb{S}^1$  on  $\mathcal{M}$  which satisfies  $\mathcal{R}_\theta \mathcal{M} = \mathcal{M}$  and we suppose that it has exactly one fixed point denoted by  $N \in \mathcal{M}$  (which we call the north pole). Notice that  $\partial \mathcal{M}$  has a single connected component (called ‘‘south boundary’’) which is also invariant by  $\mathcal{R}_\theta$ . We define  $L = \text{dist}_g(N, \partial \mathcal{M})$ . Then one can find coordinates

$$\mathcal{M} \ni m \rightarrow \zeta(m) = (s, \theta) \in (0, L] \times \mathbb{S}^1,$$

such that the metric becomes

$$(\zeta^{-1})^*g = ds^2 + R(s)d\theta^2.$$

Given a geodesic  $\gamma_0$  that connects the north pole to the south boundary of  $\mathcal{M}$  (hence in particular  $\text{length}(\gamma_0) = L$ ) we define  $\theta$  as the unique angle such that the new geodesic  $\mathcal{R}_\theta(\gamma_0)$  passes through  $m$ . We can then parametrize the geodesic  $\mathcal{R}_\theta(\gamma_0)$  by arclength and the coordinate  $s$  is defined as the distance of  $m$  to the north pole, via the parametrization of  $\mathcal{R}_\theta(\gamma_0)$ . See [LL21c, Section 4.1] for a more precise definition of these coordinates. The smooth function  $R : (0, L] \rightarrow \mathbb{R}_*^+$  can be interpreted as the Euclidean distance in  $\mathbb{R}^3$  of a point of  $\mathcal{M}$  to the symmetry axis. The disk of radius  $L$  centered at 0 corresponds to the case  $R(s) = s$ .

In the  $(s, \theta)$  coordinates, the Riemannian volume form is  $R(s)dsd\theta$  and the Laplace-Beltrami operator is given by

$$\Delta_{s,\theta} = \frac{1}{R(s)}\partial_s(R(s)\partial_s) + \frac{1}{R^2(s)}\partial_\theta^2.$$

Choose now a point  $s_0 \in (0, L)$  and define  $\Omega_- = \{N\} \cup \zeta^{-1}((0, s_0) \times \mathbb{S}^1)$ ,  $\Omega_+ = \zeta^{-1}((s_0, L) \times \mathbb{S}^1)$ ,  $S = \zeta^{-1}(\{s_0\} \times \mathbb{S}^1)$ . We consider as well the coefficient  $c = c(s) = \mathbf{1}_{\Omega_-}c_- + \mathbf{1}_{\Omega_+}c_+$ , and

$$\Delta_c := \frac{1}{R(s)}\partial_s(c(s)R(s)\partial_s) + \frac{c(s)}{R^2(s)}\partial_\theta^2,$$

which is well defined and self adjoint in the space  $\mathcal{A}$ , defined in (3.1.7) as explained in Section 3.1.2. More precisely, for  $u \in \mathcal{A}$  one has

$$\partial_s(c(s)R(s)\partial_s u) = \mathbf{1}_{\Omega_-}c_- \partial_s(R(s)\partial_s u_-) + \mathbf{1}_{\Omega_+}c_+ \partial_s(R(s)\partial_s u_+), \quad \text{in } \mathcal{D}'(\mathcal{M}),$$

where  $u_\pm = u|_{\Omega_\pm}$ . The effective potential becomes now  $V_c(s) = \frac{c(s)}{R^2(s)}$ , and we define  $V_c(s_0) = \frac{c_-}{R^2(s_0)}$  and  $E_0 = \min V_c$ . Consider now the operator  $\tilde{P}_h$  given by

$$\tilde{P}_h = -h^2 \left( \frac{1}{R(s)}\partial_s(c(s)R(s)\partial_s) \right) + \frac{c(s)}{R^2(s)}\partial_\theta^2.$$

It can be as shown as in [LL21b, Section 3.2] that if  $\lambda \in \mathbb{R}$  is an eigenvalue of  $\tilde{P}_h$  then there is an eigenfunction of the form  $e^{in\theta}\psi(s)$  with  $n \in \mathbb{Z}$  and  $\psi \in L^2((0, R_2), R(s)ds)$ . This and the arguments used in the proof of Proposition 3.2.1 give us the following:

**Proposition 3.3.1.** *There exists  $C > 0$  such that for all  $n \in \mathbb{N}$  there exists  $u_n \in \mathcal{A}$  and  $E_n > 0$  with:*

$$-\Delta_c u_n = n^2 E_n u_n, \quad |E_n - E| \leq \frac{C}{n^{2/3}}, \quad \|u_n\|_{L^2(\mathcal{M})} = 1, \quad u_n(s, \theta) = e^{in\theta}\psi_n(s).$$

We can then repeat the same steps as in Section 3.2. We define as in (3.2.11) the Agmon distance to the energy level  $E$  by

$$d_{A,E}(s) = \inf_{y \in K_E} \left| \int_y^r \sqrt{\frac{(V_c(x) - E)_+}{c(x)}} dx \right|,$$

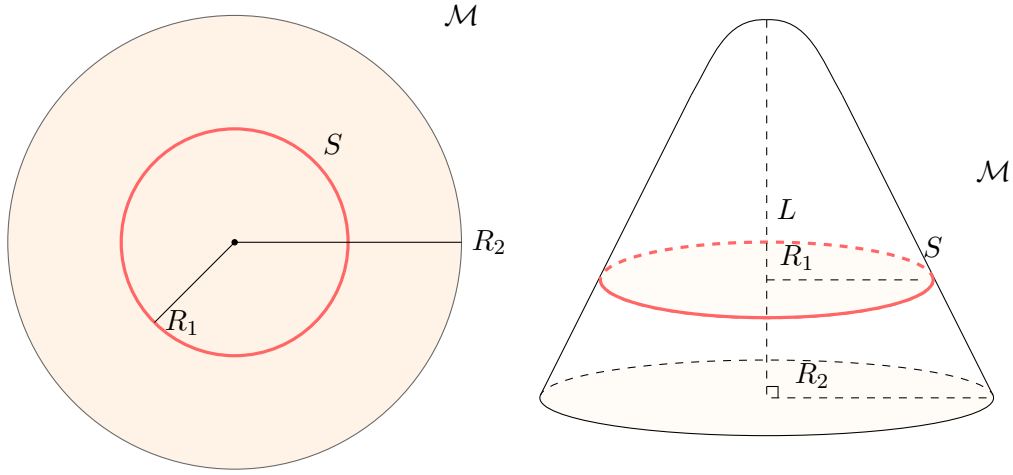


Figure 3.3: A disk and a surface of revolution diffeomorphic to the disk for which Theorem 3.3.4

applies under an assumption for the coefficient  $c$  similar to the one considered for the annulus.

which satisfies the equation  $c|\partial_s d_{A,E}|^2 = (V_c(s) - E)_+$ , in  $\mathcal{D}'((0, L))$ . Working directly on  $\mathcal{M}$  and using the fact that  $u_n \in \mathcal{A}$  allows us to obtain the key identity of Lemma 3.2.2. We define the Agmon distance on  $\mathcal{M}$  as the pullback by  $\zeta$  of the Agmon distance defined for  $s \in (0, L]$ .

We now assume that  $V_c^{-1}(E_0) = \{s_0\}$ . That is to say

$$\frac{1}{R^2(s)} > \frac{1}{R^2(s_0)}, \quad s \in (0, s_0) \quad \text{and} \quad \frac{c_-}{R^2(s_0)} < \min_{s \in [s_0, L]} \frac{c_+}{R^2(s)}.$$

With the key identity at hand we obtain then the Agmon estimate of Proposition 3.3.2. The proof is very similar to the proof of Proposition 3.2.3, but needs some care with respect to the degeneracy at the pole  $N$  where the Agmon distance tends to infinity (see Figure 3.2). This issue is treated in [LL23c, Theorem 3.9] and we omit it.

**Proposition 3.3.2.** *For all  $\delta > 0$  and  $\epsilon(n)$  with  $\epsilon(n) \xrightarrow{n \rightarrow \infty} 0$ , there exist  $C, n_0$  such that for all  $u_n$  satisfying*

$$-\Delta_c u_n = n^2(E_0 + \epsilon(n))u_n, \quad \|u_n\|_{L^2} = 1, \quad u_n \in \mathcal{A},$$

one has for  $n \geq n_0$ :

$$\left\| e^{n(1-\delta)d_{A,E_0}} u_n \right\|_{L^2(\mathcal{M})} \leq C e^{n\delta}.$$

**Remark 3.3.3.** Consider a neighborhood  $\mathcal{V}$  of the interface which is away from the pole  $N$ . Using that in  $\mathcal{V}$  the Agmon distance  $d_{A,E_0}$  and its derivative are bounded we can then as in the proof of Proposition 3.2.3 control as well the radial derivative of the associated eigenfunction  $u_{n,\mathcal{V}}$  of  $-\Delta_c$  on  $\mathcal{V}$ :

$$\frac{1}{n} \left\| e^{nd_{A,E_0}} \partial_s u_{n,\mathcal{V}} \right\|_{L^2(\mathcal{V})} + \left\| e^{nd_{A,E_0}} u_{n,\mathcal{V}} \right\|_{L^2(\mathcal{V})} \leq C e^{n\delta}.$$

This will be helpful in order to obtain a result using separation of variables only locally (see Section 3.3.2).

Proposition 3.3.2 implies finally the following result:

**Theorem 3.3.4.** *Consider  $\mathcal{M}, S, c$  as defined in Section 3.3. Suppose that  $V_c^{-1}(E_0) = \{s_0\}$ .*

*Then there exist sequences  $(\lambda_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, (u_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$  with  $\lambda_n \underset{n \rightarrow +\infty}{\sim} E_0 n^2$  such that for all  $\omega \subset \mathcal{M}$  with  $\text{dist}_g(\bar{\omega}, S) > 0$  there exist  $C, d > 0$  satisfying:*

$$-\Delta_c u_n = \lambda_n u_n, \quad \|u_n\|_{L^2(\mathcal{M})} = 1, \quad \|u_n\|_{L^2(\omega)} \leq C e^{-dn},$$

for all  $n \in \mathbb{N}$ .

**Remark 3.3.5.** If one supposes that the function  $s \mapsto R(s)$  is increasing (this is the case if for instance  $\mathcal{M}$  is a disk or a cone-like surface) then the assumption  $V_c^{-1}(E_0) = \{s_0\}$  is equivalent to

$$\frac{c_-}{R_1^2} < \frac{c_+}{R_2^2},$$

where  $R_1 := R(s_0)$  and  $R_2 := R(L)$ . This is the same assumption as (3.2.10) for the annulus.

**Remark 3.3.6.** Theorem 3.3.4 and its proof are equally valid in the case where the coefficient  $c$  is rotationally invariant and piecewise smooth (not necessarily constant) satisfying  $0 < c_{\min} \leq c \leq c_{\max}$ . Notice that in the case of a disk or an annulus, in order to satisfy  $V_c^{-1}(E_0) = \{s_0\}$  with  $s_0 \in (0, L)$  the coefficient  $c(s)$  has indeed to present a jump discontinuity at  $s = s_0$  if one supposes that  $c$  is constant in  $\Omega_+$  (in terms of physical applications that means that the refractive index of the outer cladding of the optical fiber is constant).

### 3.3.2 A result assuming separation of variables only locally

Since we are interested in the local behavior of the eigenfunctions in a neighborhood of the interface a natural question is whether rotational invariance can be assumed only locally. Notice that this assumption is crucial for our proof since it allows to work in polar coordinates, separate the variables and reduce the problem to a 1-dimensional semiclassical Schrödinger equation. As far as *eigenfunctions* are considered this question seems to be difficult and beyond the scope of this article. However, localising close to the interface yields exponentially precise quasimodes. It turns out that thanks to the linearity of the wave equation 3.1.3 this is sufficient to obtain solutions of 3.1.3 that concentrate close to the interface. We recall the definition  $\Lambda(w_n) = \frac{\|(w_n(0), \partial_t w_n(0))\|_{H^1 \times L^2}}{\|(w_n(0), \partial_t w_n(0))\|_{L^2 \times H^{-1}}}$ . We have the following result:

**Theorem 3.3.7.** *Let  $\mathcal{M}$  be compact surface and  $S$  a smooth hypersurface as in Section 3.3. Assume that there exists a neighborhood  $\mathcal{V}$  of  $S$  such that the assumptions of Theorem 3.3.4 are satisfied on  $\mathcal{V}$ . Then for all  $\omega \subset \mathcal{M}$  with  $\text{dist}(\bar{\omega}, S) > 0$  there exist a sequence  $(\lambda_n)_{n \in \mathbb{N}}$  with  $\lambda_n \rightarrow \infty$  such that for all  $T > 0$  there exist  $c > 0$  and solutions  $(w_n)_{n \in \mathbb{N}}$  of (3.1.3) with  $w_n \in C^0((0, T); H_0^1(\mathcal{M}))$ ,  $\|w_n(0)\|_{L^2} \approx 1$ ,  $\partial_t w_n(0) = i\sqrt{\lambda_n} w_n$ ,  $\Lambda(w_n) \underset{n \rightarrow +\infty}{\sim} c\sqrt{\lambda_n}$  such that there exist  $C, d > 0$  with:*

$$\|w_n\|_{L^2((0, T) \times \omega)} \leq C e^{-d\sqrt{\lambda_n}}.$$



**Remark 3.3.8.** If one supposes that the function  $s \mapsto R(s)$  is increasing (this is the case if for instance  $\mathcal{M}$  is a disk or a cone-like surface) then in order to satisfy the assumption  $V_c^{-1}(E_0) = \{s_0\}$  locally, that is in small neighborhood of the interface, it suffices to have  $c_- < c_+$ .

*Proof of Theorem 3.3.7.* Given  $\omega \subset \mathcal{M}$  with  $\text{dist}(\bar{\omega}, S) > 0$  we can suppose without loss of generality that the neighborhood  $\mathcal{V}$  is such that  $\omega \subset \mathcal{M} \setminus \mathcal{V}$ . Let  $\chi \in C_0^\infty(\mathcal{V})$  be a cut-off with  $\chi = 1$  on  $S_\mu$  where we write  $S_\mu = \{x | \text{dist}(x, S) \leq \mu\}$  for  $\mu > 0$  small. The assumption implies that on  $\mathcal{V}$  we are in the geometric context of Section 3.3.1. In particular we can take  $\chi$  that depends only on the  $s$  variable. Proposition 3.3.1 gives the existence of eigenfunctions  $f_n$  of  $-\Delta_c$  on  $\mathcal{V}$  satisfying

$$-\Delta_c f_n = \lambda_n f_n, \quad \text{on } \mathcal{V}, \quad \|f_n\|_{L^2} = 1, \quad f_n \in H_0^1(\mathcal{V}),$$

with  $\lambda_n \sim E_0 n^2$ . We define then  $g_n$  by  $g_n := \chi f_n \in H_0^1(\mathcal{M})$ . The key observation is that

$$-\Delta_c g_n = \chi(-\Delta_c f_n) + [-\Delta_c, \chi]f_n.$$

In the support of  $\chi$  one has  $-\Delta_c f_n = \lambda_n f_n$  and  $[-\Delta_c, \chi]$  is an order one differential operator with derivatives only in  $s$  and coefficients supported on the support of  $\chi'$  which is away from the interface  $S$ . Using Remark 3.3.3 we estimate then:

$$\|[-\Delta_c, \chi]f_n\|_{L^2} \leq C \|f_n\|_{L^2(\mathcal{V} \setminus S_\mu)} + C \|\partial_s f_n\|_{L^2(\mathcal{V} \setminus S_\mu)} \leq C e^{-dn}. \quad (3.3.1)$$

For  $T > 0$  define now  $w_n$  as the solution of system 3.1.3 with initial data

$$(w_n, \partial_t w_n)|_{t=0} = (g_n, i\sqrt{\lambda_n} g_n).$$

In particular we have  $\|w_n\|_{L^2(\mathcal{M})} = \|\chi f_n\|_{L^2(\mathcal{M})} \approx 1$ ,  $\|w_n\|_{H^1(\mathcal{M})} \approx 1 + \sqrt{\lambda_n}$  and  $\Lambda(w_n) \underset{n \rightarrow +\infty}{\sim} c\sqrt{\lambda_n}$ . We define as well  $v_n$  to be the approximate solution

$$v_n := \chi e^{i\sqrt{\lambda_n} t} f_n = e^{i\sqrt{\lambda_n} t} g_n.$$

Define finally by  $r_n$  the difference  $r_n := w_n - v_n$ . Denoting  $\partial_t^2 - \Delta_c := \square$  we have:

$$\begin{aligned} -\square r_n &= \square v_n = \chi \square e^{i\sqrt{\lambda_n} t} f_n + [\square, \chi] e^{i\sqrt{\lambda_n} t} f_n = [\square, \chi] e^{i\sqrt{\lambda_n} t} f_n \\ &= [-\Delta_c, \chi] e^{i\sqrt{\lambda_n} t} f_n = e^{i\sqrt{\lambda_n} t} [-\Delta_c, \chi] f_n, \end{aligned}$$

and using 3.3.1 we finally obtain

$$\|\square r_n\|_{L^2((0,T) \times \mathcal{M})} \leq \|[-\Delta_c, \chi] f_n\|_{L^2((0,T) \times \mathcal{M})} \leq C e^{-dn}.$$

Since we have as well  $(r_n, \partial_t r_n)|_{t=0} = (0, 0)$  standard energy estimates for the wave equation yield

$$\|r_n\|_{L^2((0,T) \times \mathcal{M})} \leq C e^{-dn}.$$

We can now conclude the proof of the theorem. Indeed we observe that

$$\begin{aligned} \|r_n\|_{L^2((0,T) \times \mathcal{M})} &= \|w_n - v_n\|_{L^2((0,T) \times \mathcal{M})} \geq \|w_n - v_n\|_{L^2((0,T) \times \mathcal{M} \setminus \mathcal{V})} = \|w_n\|_{L^2((0,T) \times \mathcal{M} \setminus \mathcal{V})} \\ &\geq \|w_n\|_{L^2((0,T) \times \omega)}, \end{aligned}$$

since by definition of  $v_n$  and  $\chi$ ,  $v_n$  vanishes outside  $\mathcal{V}$ . This finishes the proof of Theorem 3.3.7.  $\square$

Deuxième partie

**On unique continuation for  
Schrödinger operators**



# Chapitre 4

## Schrödinger operators with partially Gevrey coefficients

*Ce chapitre est la reproduction de [FLL23].*

In this chapter we prove a local unique continuation result for Schrödinger operators with time independent Lipschitz metric and lower order terms which are Gevrey 2 in time and bounded in space. This implies global unique continuation from any open set in a complete connected Riemannian manifold. These results relax in the same geometric setting the analyticity assumption in time of the Tataru-Robbiano-Zuily-Hörmander theorem for these operators. The proof is based on a Tataru-Robbiano-Zuily-Hörmander type Carleman estimate with a nonlocal weight adapted to the anisotropy of the Schrödinger operator and a conjugation result with Gevrey coefficients.

### 4.1 Introduction and main results

#### 4.1.1 Background and results

In this article we are interested in the *unique continuation* problem for a large family of time-dependent Schrödinger operators. For a general differential operator

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \quad \text{where } D_{x_j} = \frac{\partial_{x_j}}{i}, \quad m \in \mathbb{N}, \quad (4.1.1)$$

on an open set  $\Omega \subset \mathbb{R}^n$  the problem of local unique continuation is the question: given  $x_0 \in \Omega \subset \mathbb{R}^n$  and  $S \ni x_0$  a smooth oriented hypersurface, do we have:

$$(Pu = 0 \text{ in } \Omega, \quad u = 0 \text{ in } S^- \cap \Omega) \implies x_0 \notin \text{supp}(u),$$

where we denote by  $S^-$  one side of the oriented hypersurface  $S$ ? If the local unique continuation property holds for a sufficiently large family of hypersurfaces one can propagate it and obtain a *global* result. For  $\omega$  a small subset of  $\Omega$  such a result takes the following form:

$$(Pu = 0 \text{ in } \Omega, \quad u = 0 \text{ in } \omega) \implies u = 0 \text{ in } \Omega.$$

Unique continuation properties are sometimes motivated by control theory, where a global unique continuation from a set  $\omega$  is equivalent to an approximate controllability result from

$\omega$  for an appropriate adjoint problem, see e.g. [LL23b]. On the one hand, the Holmgren-John theorem [Hör63, Theorem 5.3.1] yields unique continuation assuming all coefficients of  $P$  (i.e. all  $a_\alpha$ 's for all  $|\alpha| \leq m$ ) are real-analytic and the hypersurface  $S$  is non-characteristic, that is to say

$$p_m(x_0, d\Psi(x_0)) \neq 0, \quad \text{where } S = \{\Psi = 0\}, \quad (4.1.2)$$

and

$$p_m(x, \xi) := \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha \quad (4.1.3)$$

is the so-called principal symbol of the operator  $P$ . On the other hand, if one is interested in  $C^\infty$  (or  $C^k$ ) regularity, Hörmander's theorem [Hör94, Theorem 28.3.4] yields unique continuation under a (rather strong) so called pseudoconvexity condition (that is to be checked on the whole cotangent space over  $x_0$ ). The seminal result of Robbiano [Rob91] for hyperbolic operators, subsequently improved in [Hör92], paved the way to a more general theorem that would bridge the gap between the  $C^\infty$  and the analytic case. This was achieved by Tataru, Robbiano-Zuily and Hörmander in the series of papers [Tat95, RZ98, Hör97, Tat99b] proving a general unique continuation result for operators having partially analytic coefficients, containing as a particular cases both the Holmgren-John and the Hörmander theorems. We refer to [LL19, LL20, LL23b] for further discussions and comments on these results.

In this article, motivated by applications to control theory (see Section 4.1.2 below), we are interested in the particular case of Schrödinger operators

$$P = i\partial_t + \sum_{j,k=1}^d \partial_{x_j} g^{jk}(x) \partial_{x_k} + \sum_{j=1}^d b^j(t, x) \partial_{x_j} + \mathbf{q}(t, x). \quad (4.1.4)$$

Compared to the general situation in (4.1.1)–(4.1.3), we have here  $n = 1 + d$ ,  $x = (t, x)$ ,  $m = 2$ , and the principal symbol of  $P$  is  $p_2(x, \xi) = p_2(t, x, \xi_t, \xi_x) = -\sum_{j,k} g^{jk}(x) \xi_{x_j} \xi_{x_k}$ . The latter does not depend on  $\xi_t$  (and, in particular, is the same as for the heat operator (4.1.4) in which  $i\partial_t$  is replaced by  $-\partial_t$ ). The classical theorem of Hörmander is empty in this case. Taking advantage of the anisotropic (or quasi-homogeneous) nature of the Schrödinger operator, Isakov proved in [Isa93] that the results of Hörmander [Hör94, Chapter 28] can be generalized to the anisotropic case with an appropriate modification of the symbol classes and Poisson bracket (see also [Tat97b]). In the context of (4.1.4), this theorem applies for coefficients  $g^{jk}$ ,  $b^j$ ,  $\mathbf{q}$  that have finite regularity, under a pseudoconvexity condition on the hypersurface. The latter is a very strong local geometric assumption on the surface, which necessarily leads to a very strong geometric assumption of the observation set in an associated global unique continuation statement. For applications to control or inverse problems, related global Carleman estimates for Schrödinger operators have been proved for instance in [BP02] (constant leading order coefficients) and in [TX07, Lau10] (Riemannian manifolds or varying coefficients). A weak pseudoconvexity condition has also been proved sufficient in [MOR08] for a flat metric and in [Lau10] with varying metrics. Yet, in all of these examples, a kind convexity related to the one of Isakov is required and ends up to strong geometrical assumptions.

The Tataru-Hörmander-Robbiano-Zuily, applied to the Schrödinger operator (4.1.4), yields the following local unique continuation statement (here  $C^\omega$  means real-analytic, see Definition 6 below and the remark thereafter).

**Theorem 4.1.1** (Tataru-Hörmander-Robbiano-Zuily for Schrödinger operators). *Assume  $\Omega = I \times V$  where  $I \subset \mathbb{R}$  is an open interval and  $V \subset \mathbb{R}^d$  an open set, and let  $(t_0, x_0) \in \Omega$ . Assume  $g^{jk} \in C^1(V)$  is a real-valued elliptic matrix at  $x_0$ , that  $b^j, V \in C^\omega(I; L^\infty(V; \mathbb{C}))$ . Let  $\Psi \in C^1(\Omega; \mathbb{R})$  such that  $\{\Psi = 0\}$  is non characteristic for  $P$  at  $(t_0, x_0)$ , that is to say,*

$$\sum_{j,k=1}^d g^{jk}(x_0) \partial_{x_j} \Psi(t_0, x_0) \partial_{x_k} \Psi(t_0, x_0) \neq 0. \quad (4.1.5)$$

Then, there is  $W$  a neighborhood of  $(t_0, x_0)$  such that, for  $P$  defined in (4.1.4),

$$\left( Pu = 0 \quad \text{in } \Omega, \quad u \in H^1(\Omega), \quad u = 0 \text{ in } \{\Psi > 0\} \right) \implies u = 0 \text{ in } W.$$

From the point of view of regularity requirements, analyticity in time is of course very demanding. From the geometric point of view however, the non-characteristicity assumption is optimal: it excludes only surfaces tangent to  $\{t = t_0\}$ , for which we know that local unique continuation may fail (this would otherwise imply finite speed of propagation for Schrödinger equations). Applied iteratively to appropriate families of hypersurfaces (see e.g. [LL19, Section 6.2]), this result thus leads to global unique continuation under optimal geometric condition, still assuming analyticity in time of the coefficients. Note finally that Tjoen [Tj00] proved a quasi-homogeneous variant of the Tataru-Hörmander-Robbiano-Zuily theorem in a general setting and Masuda [Mas67] proved a global uniqueness result in the case of  $C^2$  principal coefficients and time independent coefficients. A challenging problem is to understand to which extent the time-analyticity condition can be relaxed under optimal geometric conditions. For the wave operator, we refer to the discussion in [LL23b] and the counterexamples of Alinhac-Baouendi [AB79, Ali83, AB95]. In this direction, our results relaxe the time analyticity assumption of Theorem 4.1.1 down to Gevrey regularity.

**Definition 6.** Given  $d \in \mathbb{N}^*$ ,  $U \subset \mathbb{R}^d$  an open set and  $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$  a Banach space, we say that  $f$  is a  $s$ -Gevrey function valued in  $\mathcal{B}$ , denoted  $f \in \mathcal{G}^s(U; \mathcal{B})$ , if  $f \in C^\infty(U; \mathcal{B})$  is such that for every compact set  $K \subset U$ , there are constants  $C, R > 0$  such that for all  $\alpha \in \mathbb{N}^d$

$$\max_{t \in K} \|\partial^\alpha f(t)\|_{\mathcal{B}} \leq CR^{|\alpha|} \alpha!^s, \quad (4.1.6)$$

These spaces were introduced by Gevrey [Gev18] to investigate regularity properties for solutions of the heat equation between real-analyticity and  $C^\infty$  regularity. Recall that for  $s = 1$ ,  $\mathcal{G}^1(U; \mathcal{B}) = C^\omega(U; \mathcal{B})$  is the space of real-analytic  $\mathcal{B}$ -valued functions. However, for  $s > 1$ ,  $\mathcal{G}^s(U; \mathcal{B})$  contains nontrivial compactly supported functions. In what follows, we mostly consider the case  $d = 1$ ,  $t$  being the time variable (but also consider  $d = 2$  in Section 4.3.1). Our main result may be summarized as follows.

**Theorem 4.1.2** (Local unique continuation for Schrödinger operators). *Assume  $\Omega = I \times V$  where  $I \subset \mathbb{R}$  is an open interval and  $V \subset \mathbb{R}^d$  an open set, and let  $(t_0, x_0) \in \Omega$ . Assume  $g^{jk} \in W^{1,\infty}(V)$  is a real-valued elliptic matrix at  $x_0$ , that  $b^j, V \in \mathcal{G}^2(I; L^\infty(V; \mathbb{C}))$ . Let*

$\Psi \in C^1(\Omega; \mathbb{R})$  such that  $\{\Psi = 0\}$  is non characteristic for  $P$  at  $(t_0, x_0)$ , in the sense of (4.1.5). Then, there is  $W$  a neighborhood of  $(t_0, x_0)$  such that, for  $P$  defined in (4.1.4),

$$\left( Pu = 0 \quad \text{in } \Omega, \quad u \in L^2(I; H^1(V)), \quad u = 0 \text{ in } \{\Psi > 0\} \right) \implies u = 0 \text{ in } W.$$

If in addition  $b^j = 0$  for all  $j \in \{1, \dots, d\}$ , then

$$\left( Pu = 0 \quad \text{in } \Omega, \quad u \in L^2(\Omega), \quad u = 0 \text{ in } \{\Psi > 0\} \right) \implies u = 0 \text{ in } W.$$

With respect to Theorem 4.1.1 we relax the analyticity-in-time assumption for the lower order terms to a Gevrey 2 condition. We also relax the  $C^1$  regularity of the main coefficients, replaced here by Lipschitz regularity; in the elliptic context (and therefore in our context as well) this is essentially the minimal regularity in dimension  $d \geq 3$  for local uniqueness to hold (see [Pli63] and [Mil74] for  $C^{0,\alpha}$  counterexamples for all  $\alpha < 1$ , for operators in divergence forms or not). Finally if  $b^j = 0$ , we also relax the regularity of the solution required for unique continuation to hold. Successive applications of Theorem 4.1.2 through a family of well-chosen non-characteristic hypersurfaces yield the following global result (see [LL19, Section 6.2]).

**Theorem 4.1.3.** *Let  $(\mathcal{M}, g)$  be a connected smooth Riemannian manifold with or without boundary  $\partial\mathcal{M}$  and consider the differential operator*

$$P := P = i\partial_t + \Delta_g + b \cdot \nabla_g + \mathfrak{q}(t, x),$$

where  $\Delta_g$  is the Laplace-Beltrami operator on  $\text{Int}(\mathcal{M})$ ,  $\nabla_g$  the Riemannian gradient, and  $b$  a one form. Suppose that  $g \in W_{\text{loc}}^{1,\infty}(\text{Int}(\mathcal{M}))$ , that  $V \in \mathcal{G}^2(I; L_{\text{loc}}^\infty(\text{Int}(\mathcal{M}); \mathbb{C}))$  and that all components of  $b$  belong to  $\mathcal{G}^2(I; L_{\text{loc}}^\infty(\text{Int}(\mathcal{M}); \mathbb{C}))$ . Let  $T > 0$  and  $\omega$  be a nonempty open set of  $\mathcal{M}$ . Then we have

$$\begin{cases} Pu = 0 \text{ in } (0, T) \times \text{Int}(\mathcal{M}) \\ u \in L^2(0, T; H_{\text{loc}}^1(\text{Int}(\mathcal{M}))) \\ u = 0 \text{ in } (0, T) \times \omega \end{cases} \implies u = 0 \text{ in } (0, T) \times \text{Int}(\mathcal{M}).$$

If in addition  $b = 0$ , then

$$\begin{cases} Pu = 0 \text{ in } (0, T) \times \text{Int}(\mathcal{M}) \\ u \in L_{\text{loc}}^2((0, T) \times \text{Int}(\mathcal{M})) \\ u = 0 \text{ in } (0, T) \times \omega \end{cases} \implies u = 0 \text{ in } (0, T) \times \text{Int}(\mathcal{M}).$$

We finally mention that other notions of uniqueness have been extensively investigated for solutions of Schrödinger equations during the last years. One such notion is the following: Assume that a solution  $u = u(t, x)$  of the Schrödinger equation on  $\mathbb{R}_t \times \mathbb{R}_x$  vanishes in  $|x| > R$  for some  $R > 0$  at two different times  $t_0$  and  $t_1$ . Can we then conclude that  $u$  vanishes everywhere? This question has been addressed in [EKPV06, IK06, DS07]. All of these results use as a key tool Carleman inequalities.

### 4.1.2 Application to controllability and observability

#### Approximate controllability

As already alluded, unique continuation properties for evolution equations are often equivalent to approximate controllability results for an appropriate dual problem, see e.g. the introduction of [LL23b] for the wave equation. In particular, Theorem 4.1.3 has an “approximate controllability” counterpart. For simplicity of the exposition, we only treat the internal control of  $L^2$  solutions with  $b = 0$ . Given  $T > 0$ ,  $(\mathcal{M}, g)$  a Riemannian manifold, and  $\omega \subset \mathcal{M}$  an open set, we consider the control problem

$$\begin{cases} i\partial_t v + \Delta_g v + \mathbf{q}v = \mathbb{1}_\omega f, & \text{in } (0, T) \times \text{Int}(\mathcal{M}), \\ v = 0, & \text{on } (0, T) \times \partial\mathcal{M} \quad \text{if } \partial\mathcal{M} \neq \emptyset, \\ v(0, \cdot) = v_0, & \text{in } \text{Int}(\mathcal{M}). \end{cases} \quad (4.1.7)$$

Here,  $f$  is a control force acting on the system on the small open set  $\omega$  and one would like to control the state  $v$  of the equation. We assume that  $(\mathcal{M}, g)$  is complete, so that  $\Delta_g$  is selfadjoint on  $L^2(\mathcal{M})$  with domain  $H^2 \cap H_0^1(\mathcal{M})$ . Assuming for instance that  $\mathbf{q} \in L^\infty((0, T) \times \mathcal{M}; \mathbb{C})$ ,  $f \in L^2((0, T) \times \mathcal{M}; \mathbb{C})$  and  $v_0 \in L^2(\mathcal{M}; \mathbb{C})$ , (4.1.7) admits a unique solution  $v \in C^0([0, T]; L^2(\mathcal{M}; \mathbb{C}))$ . The (second) unique continuation result of Theorem 4.1.3 combined with classical duality results (see [DR77, Lio88]) yields the following corollary.

**Corollary 4.1.1.** *Assume  $(\mathcal{M}, g)$  is a complete connected Riemannian manifold with or without boundary and  $\mathbf{q} \in L^\infty((0, T) \times \mathcal{M}; \mathbb{C}) \cap \mathcal{G}^2((0, T); L^\infty(\mathcal{M}; \mathbb{C}))$ . For any nonempty open set  $\omega \subset \mathcal{M}$ , for all  $v_0, v_1 \in L^2(\mathcal{M}; \mathbb{C})$  and for all precision  $\varepsilon > 0$ , there is  $f \in L^2((0, T) \times \omega)$  such that the solution to (4.1.7) satisfies  $\|v(T, \cdot) - v_1\|_{L^2(\mathcal{M})} \leq \varepsilon$ .*

Note that we actually only need to assume  $\mathbf{q} \in \mathcal{G}^2(I; L_{\text{loc}}^\infty(\mathcal{M}; \mathbb{C}))$  for some nonempty open set  $I \subset (0, T)$ .

#### Observability, exact controllability

Unique continuation also plays a key role in proofs of exact controllability results, or equivalently, observability estimates. For wave-type and Schrödinger equations, the proof of the latter often decompose on a high frequency and a low frequency analyses. We refer to the introduction of [LL16] for a detailed account in the case of the wave equation. The low frequency part of the analysis amounts to a unique continuation like Theorem 4.1.3. The observation system is the following free Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta_g u + qu = 0 & \text{in } (0, T) \times \text{Int}(\mathcal{M}), \\ u = 0, & \text{on } (0, T) \times \partial\mathcal{M} \quad \text{if } \partial\mathcal{M} \neq \emptyset, \\ u(0, \cdot) = u_0, & \text{in } \text{Int}(\mathcal{M}), \end{cases} \quad (4.1.8)$$

dual to the control problem (4.1.7) if  $\mathbf{q} = \bar{q}$ . As in the preceding section, for simplicity of the exposition, we only discuss the internal observability/control of  $L^2$  solutions with  $b = 0$  to illustrate some applications of our results.

**Theorem 4.1.4.** *Assume that*

1.  $(\mathcal{M}, g) = (\mathbb{D}, \text{Eucl})$  is the Euclidean disk,  $q \in C^\infty([0, T] \times \mathbb{D}; \mathbb{R}) \cap \mathcal{G}^2((0, T); L^\infty(\mathbb{D}; \mathbb{R}))$  is real valued and  $\omega$  is any nonempty open set such that  $\omega \cap \partial\mathbb{D} \neq \emptyset$ ;



Then for any  $T > 0$ , there is  $C > 0$  such that for all  $u_0 \in L^2(\mathcal{M})$ , the solution  $u$  to (4.1.8) satisfies

$$\|u_0\|_{L^2(\mathcal{M})}^2 \leq C \int_0^T \int_{\omega} |u(t, x)|^2 dx dt.$$

Our contribution in Theorem 4.1.4 is to include more general time-dependent potentials  $q$ , using Theorem 4.1.3 for the “low frequency” part of the proof. See [AM12] for an overview of the situation in high-frequency.

In case 1, Theorem 4.1.4 is a direct combination of [ALM16, Theorem 1.2] and Theorem 4.1.3. Note that the  $C^\infty$  regularity can be relaxed, see [ALM16, Remark 1.6]. With some work, one may also generalize Theorem 4.1.4 to the following situations:

1. either  $(\mathcal{M}, g)$  is any smooth compact Riemannian manifold,  $q \in L^\infty((0, T) \times \mathcal{M}; \mathbb{C}) \cap \mathcal{G}^2((0, T); L^\infty(\mathcal{M}; \mathbb{C}))$  and  $\omega$  satisfies the Geometric Control Condition of [RT74, BLR92] in finite time;
2. or  $(\mathcal{M}, g) = (\mathbb{T}^d, \text{Eucl})$  is the flat square torus,  $q \in C^0([0, T] \times \mathbb{T}^d; \mathbb{R}) \cap \mathcal{G}^2((0, T); L^\infty(\mathbb{T}^d; \mathbb{R}))$  is real valued and  $\omega$  is any nonempty open set;

In case 1, the high frequency result is essentially proved in [Leb92b] (in the  $H^1$  context though, see also [DGL06]). In the  $L^2$  setting, one can check that the high frequency observability inequality is insensitive to addition of  $L^\infty$  potentials. Recall that the Geometric Control Condition of [RT74, BLR92] states in this context that every maximal geodesic curve intersects  $\omega$ .

In case 2, the “high-frequency” result is taken from [AM14] (see also [AFKM15] for irrational tori), where potentials can even be taken continuous outside of a set of measure zero. Note actually that the proof of Theorem 4.1.4 in this case actually relies on a finite (but possibly large) number of applications of Theorem 4.1.3 to sub-problems. See also [BBZ13] for an observability inequality on  $\mathbb{T}^2$  with a time independent  $L^2$  potential.

In the context of manifolds of negative curvature, it is likely that the high-frequency result of [AR12] and that the spectral results of [Ana08, DJ18, DJN22] adapt to prove, in the geometric settings of these articles, an observability inequality for (4.1.8) with  $q \in L^\infty((0, T) \times \mathcal{M}; \mathbb{C}) \cap \mathcal{G}^2((0, T); L^\infty(\mathcal{M}; \mathbb{C}))$ .

As a direct corollary of the observability statement of Theorem 4.1.4, we deduce an exact controllability statement for System (4.1.7) (see [DR77, Lio88]).

**Corollary 4.1.2.** *Assume that one of the three assumptions of Theorem 4.1.4 is satisfied. Then, for all  $v_0, v_1 \in L^2(\mathcal{M}; \mathbb{C})$ , there is  $f \in L^2((0, T) \times \omega)$  such that the solution to (4.1.7) satisfies  $v(T, \cdot) = v_1$ .*

### 4.1.3 Idea of the proof, structure of the paper

Since the pioneering work of [Car39], Carleman inequalities are one of the main tools for proving unique continuation results. Carleman estimates are weighted inequalities of the form

$$\|e^{\tau\phi} Pu\|_{L^2} \geq C \|e^{\tau\phi} u\|_{L^2}, \quad \tau \geq \tau_0,$$

which are uniform in the large parameter  $\tau$  and are applied to compactly supported functions  $u$ . The weight  $e^{\tau\phi}$  enforces the set where the solution is zero and allows to propagate uniqueness by sending  $\tau \rightarrow \infty$ . The key idea introduced by Tataru in [Tat95] is that one

can still show uniqueness results using the microlocal weight  $e^{-\frac{\varepsilon|D_t|^2}{2\tau}} e^{\tau\phi}$  instead of the classical one. The advantage of this weight is that it localizes in a low frequency regime and one can consequently restrict the classical pseudoconvexity assumption in a smaller subset of the phase space, namely where  $\xi_t = 0$ .

The first crucial step for the proof of our main result Theorem 4.1.2 is to show a Carleman estimate adapted to the anisotropy of the Schrödinger equation. If one supposes that  $\tau, D_t$  and  $D_x$  have the same weight then  $\frac{|D_t|^2}{\tau}$  is of “order one”. This is the natural scaling for the wave equation. However in the Schrödinger context one should rather consider that  $D_t$  is homogeneous to  $D_x^2 \sim \tau^2$ . Therefore with new definition of homogeneity/order, the operator  $\frac{|D_t|^2}{\tau^3}$  becomes of order one. The Carleman estimate of Theorem 4.2.1 is precisely a Carleman estimate as in [Tat95, Hör97, RZ98, Tat99b] but with this new weight. Its proof is based on [LL21b, Appendix A] where the last two authors showed a Carleman estimate for elliptic operators with Lipschitz coefficients. Indeed, the key step for the proof of Theorem 4.2.1 is the sub-elliptic estimate of Proposition 4.2.6. In order to prove the key Proposition 4.2.6 we proceed in two steps: the first step corresponds to the case where the weight function is the classical one, that is  $e^{\tau\phi}$ . The second step is then to consider the weight  $e^{-\frac{\mu|D_t|^2}{2\tau^3}} e^{\tau\phi}$  as a perturbation of the previous one yielding some admissible errors in our estimate. A similar perturbation argument is also used in the proofs of [Tat95, Hör97, RZ98, Tat99b]. In our situation there is however a remarkable difference:  $\mu$  need not be small and can be chosen as large as one desires.

This being said one can naturally ask: what did we gain by having a Carleman estimate with weight  $e^{-\frac{|D_t|^2}{2\tau^3}} e^{\tau\phi}$  instead of  $e^{-\frac{|D_t|^2}{2\tau}} e^{\tau\phi}$ ? Heuristically, a partial answer can be given by the results of [LL19]. Indeed, in [LL19] the general theorem of Tataru-Hörmander-Robbiano-Zuily is quantified in an optimal way. The key step is the local result of Theorem 3.1 in [LL19] which says that the Carleman estimate with weight  $e^{-\frac{|D_t|^2}{2\tau}} e^{\tau\phi}$  allows to propagate information up to a frequency  $D_t \sim \tau$ . In the anisotropic scaling of the Schrödinger operator we have  $D_t \sim \tau^2$  and therefore one can hope that the weight  $e^{-\frac{|D_t|^2}{2\tau^3}} e^{\tau\phi}$  allows to propagate information up to frequencies smaller than  $D_t \sim \tau^2$ . Even though this is not directly related to our qualitative problem, it is an indication that having this new weight allows to “propagate more information”.

In our qualitative context the advantage of having this new weight is seen via the conjugation results of Section 4.3. The issue of defining a conjugate for  $e^{-\frac{|D_t|^2}{2\tau}}$  with respect to  $P$  is one the main difficulties in [Tat95, RZ98, Hör97, Tat99b]. Even in the case where the coefficients of  $P$  are analytic with respect to  $t$  a conjugate operator for  $e^{-\frac{|D_t|^2}{2\tau}}$  does not necessarily exist. However, one can define an approximate conjugate up to an exponentially small admissible error which is given by  $e^{-d\tau} \|u\|$ . Our conjugation result roughly says that if the coefficients of  $P$  are 2-Gevrey instead of analytic then we can define a conjugated operator of  $P$  with  $e^{-\frac{|D_t|^2}{2\lambda}}$  up to an error given by  $e^{-d\lambda^{1/3}} \|u\|$ , which is an admissible error term in view of the new weight  $e^{-\frac{|D_t|^2}{2\tau^3}}$ . For the proof of the conjugation result we follow the strategy of Tataru [Tat99b]. Working with non-analytic functions yields however certain non trivial technical difficulties. A crucial object for our proof is the so called *quasi analytic extensions* of a smooth function. Such an extension is particularly well behaved in the case of a Gevrey function.

The plan of this article is as follows: In Section 4.2 we recall some notation from Riemannian geometry and then prove the key Carleman estimate of Theorem 4.2.1. The proof relies on the sub-elliptic estimate of Proposition 4.2.6 that we show in two steps. The first step corresponds to the case  $\mu = 0$  and then we use a perturbation argument for the case  $\mu > 0$ .

In Section 4.3 we prove the conjugation result. The content of this section is independent of the rest of this article and we prove the result in a slightly abstract Banach space setting. This allows to rigorously define the notion of being "partially Gevrey". After recalling some basic definitions and properties of semiclassical analysis we prove the main result.

Finally, in Section 4.4 we combine the results of the two previous sections. We show a Carleman estimate with Gevrey lower order terms, construct an appropriate weight function and prove the uniqueness result. In sub-section 4.4.3 we explain how one can exploit the localization properties of the weight  $e^{-|D_t|^2}$  in order to reduce the regularity of the solution in the uniqueness result.

**Acknowledgements.** The authors would like to thank Nicolas Burq for having pointed to them the reference [Mas67]. The third author is partially supported by the Institut Universitaire de France and the Agence Nationale de la Recherche under grants SALVE (ANR-19-CE40-0004) and ADYCT (ANR-20-CE40-0017).

## 4.2 The Carleman estimate

### 4.2.1 Toolbox of Riemannian geometry

The proof of the Carleman estimate below (as many proofs of Carleman inequalities for operators with low-regularity coefficients) relies on an integration by parts. Although we work here in a local setting, it is still convenient to formulate our integration by parts formula in a Riemannian geometric framework following [LL21b, Appendix A], which we recall now (see [GHL90]).

We work in a relatively compact open set  $V \subset \mathbb{R}^d$ . We denote by  $g = (g_{jk})_{1 \leq j, k \leq d}$  a Lipschitz metric on  $V$ , (that is,  $x \mapsto g_x(\cdot, \cdot)$  is a Lipschitz family of symmetric bilinear forms on  $TV$  that is uniformly bounded from below by a positive constant). We denote by  $\langle \cdot, \cdot \rangle_g = g(\cdot, \cdot)$  the inner product in  $TV = V \times \mathbb{R}^d$ . Remark that this notation omits to mention the point  $x \in V$  at which the inner products takes place: this allows to write  $\langle X, Y \rangle_g$  as a function on  $V$  (the dependence on  $x$  is omitted here as well) when  $X$  and  $Y$  are two vector fields on  $V$ . We also denote for a vector field  $X$ ,  $|X|_g^2 = \langle X, X \rangle_g$ . In  $V$ , for  $f$  a smooth function and  $X = \sum_i X^i \frac{\partial}{\partial x_i}$ ,  $Y = \sum_i Y^i \frac{\partial}{\partial x_i}$  smooth vector fields on  $V$ , we

write

$$\begin{aligned}\langle X, Y \rangle_g &= \sum_{i=1}^d g_{ij} X^i Y^j, \quad \nabla_g f = \sum_{i,j=1}^d g^{ij} (\partial_j f) \frac{\partial}{\partial x_i}, \\ \operatorname{div}_g(X) &= \sum_{i=1}^d \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} X_i), \quad \Delta_g f = \operatorname{div}_g \nabla_g f = \sum_{i,j=1}^d \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij} \partial_j f), \\ D_X Y &= \sum_{i=1}^d \left( \sum_{j=1}^d X^j \frac{\partial Y^i}{\partial x_j} + \sum_{j,k=1}^d \Gamma_{j,k}^i X^j Y^k \right) \frac{\partial}{\partial x_i},\end{aligned}$$

where  $(g^{-1})_{ij} = g^{ij}$  and the Christoffel symbols are defined by

$$\Gamma_{j,k}^i = \frac{1}{2} \sum_{l=1}^d g^{il} (\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}),$$

(see for instance [GHL90, p71]). Note in particular that the Lipschitz regularity of  $g$  writes  $g_{ij} \in W^{1,\infty}(V)$ , and implies  $g^{ij} \in W^{1,\infty}(V)$ . This entails, if  $f, X, Y$  are smooth, that  $\langle X, Y \rangle_g \in W^{1,\infty}(V)$ ,  $\nabla_g f$  is a Lipschitz vector field,  $\Delta_g f \in L^\infty(V)$  and  $D_X Y$  is an  $L^\infty$  vector field on  $V$ , since the definitions of  $\Delta_g$  and  $D_X$  involve one derivative of the coefficients of  $g$ . Let us now collect some properties of these objects, that we shall use below. For  $f, g$  two smooth functions on  $V$  and  $X = \sum_i X^i \frac{\partial}{\partial x_i}$ ,  $Y = \sum_i Y^i \frac{\partial}{\partial x_i}$  two smooth vector fields on  $V$ , we have

$$\begin{aligned}\operatorname{div}_g(fX) &= \langle \nabla_g f, X \rangle_g + f \operatorname{div}_g(X), \\ D_X(fY) &= (Xf)Y + f D_X Y, \\ D_X(\langle Y, Z \rangle_g) &= \langle D_X Y, Z \rangle_g + \langle Y, D_X Z \rangle_g.\end{aligned}$$

We define (see [GHL90, Exercice 2.65] or [LL21b] for more on the the Hessian)

$$\operatorname{Hess}_g(f)(X, Y) = (D_X df)(Y) = \sum_{i,j} X^i Y^j \left[ \partial_{ij}^2 f - \Gamma_{ij}^k \partial_k f \right],$$

which again is in  $L^\infty(V)$  for a Lipschitz metric  $g$  and  $L^\infty$  vector fields  $X, Y$ . Note also that the Hessian of  $f$  is symmetric, that is  $\operatorname{Hess}_g(f)(X, Y) = \operatorname{Hess}_g(f)(Y, X)$ . For any function  $f$  and any vector field  $X$  and  $Y$ , we have (see e.g. [LL21b, Lemma A.1])

$$\operatorname{Hess}_g(f)(X, Y) = \langle D_X \nabla_g f, Y \rangle_g.$$

Concerning integrals, we write in this section

$$\int f = \int_V f(x) \sqrt{\det g(x)} dx,$$

where  $\sqrt{\det g(x)} dx$  is the Riemannian density. With this notation, a useful integration by parts formula writes as follows: For all  $f \in H^2(V)$  and  $h \in H^1(V)$  one of which having compact support in  $V$ , we have

$$\int (\Delta_g f) h = - \int \langle \nabla_g f, \nabla_g h \rangle_g.$$

As we are interested in complex-valued functions, we set  $(f, g)_{L^2(V)} = \int f \bar{h}$  for the  $L^2$  hermitian product. We are moreover interested in time-dependent functions, and in the context of spacetime integration, we will denote  $\iint f = \int_{\mathbb{R}^t} \int_V f(t, x) \sqrt{\det g(x)} dx dt$  and  $(f, g) = \int f \bar{h}$ .

### 4.2.2 The Carleman weight

We denote by  $U = I \times V$  where  $I$  is a bounded open interval of  $\mathbb{R}$  and  $V$  is a relatively compact open subset of  $\mathbb{R}^d$  equipped with a Lipschitz metric  $g$ . In this section, we set  $P := i\partial_t + \Delta_g$  where  $\Delta_g$  is defined in Section 4.2.1.

For a smooth real-valued weight function  $\phi$  (later on, we will assume that it is polynomial of order 2), the Carleman estimate below will make use of the operator

$$Q_{\mu,\tau}^\phi u = e^{-\mu \frac{|D_t|^2}{2\tau^3}} e^{\tau\phi} u.$$

Note the difference with the usual  $e^{-\mu \frac{|D_t|^2}{2\tau}} e^{\tau\phi}$ . This is coherent since for waves,  $\frac{|D_t|^2}{2\tau}$  is of order one and now  $\frac{|D_t|^2}{2\tau^3}$  is of order one since  $\frac{D_t}{\tau}$  is of order one. In all the rest of the proof,  $\mu$  does not have any role and could be any constant. We have chosen to keep it along the proof since we believe it helps to follow the perturbation of the pseudodifferential weight. We now describe the conjugation by  $e^{-\mu \frac{|D_t|^2}{2\tau^3}}$ .

**Lemma 4.2.1** (Lemma 3.12 in [LL23b]). *Let  $u \in \mathcal{S}(\mathbb{R}^{n+1})$  and  $\varsigma > 0$ , then*

$$e^{-\frac{|D_t|^2}{2\varsigma}} (tu) = \left( t + i \frac{D_t}{\varsigma} \right) e^{-\frac{|D_t|^2}{2\varsigma}} u.$$

This implies the following conjugation of monomials.

**Lemma 4.2.2** (Lemma 3.14 in [LL23b]). *Assume  $\phi$  is a real polynomial of degree two in the variable  $t$ . For all  $k \in \{0, \dots, d\}$  (with the convention  $t = x_0$ ,  $D_0 = D_t$ ) we have*

$$Q_{\mu,\tau}^\phi D_k = (D_k)_{\phi,\mu} Q_{\mu,\tau}^\phi,$$

where (denoting  $\phi''_{t,x_k} = \partial_t \partial_{x_k} \phi$ )

$$(D_k)_{\phi,\mu} = D_k + i\tau \partial_{x_k} \phi(z) - \mu \phi''_{t,x_k} \frac{D_t}{\tau^2}.$$

**Corollary 4.2.3** (The “conjugated operator”). *Let*

$$P = i\partial_t + \Delta_g = -D_t - \sum_{i,j=1}^d \frac{1}{\sqrt{\det g}} D_i \sqrt{\det g} g^{ij} D_j,$$

with all coefficients independent of  $t$ . Let  $\phi$  be a real-valued function being quadratic in  $t$ . Then, for any  $\mu > 0$ ,

$$Q_{\mu,\tau}^\phi P = P_{\phi,\mu} Q_{\mu,\tau}^\phi, \quad \text{with}$$

$$P_{\phi,\mu} = - \left( D_t + i\tau \partial_t \phi(z) - \mu \phi''_{t,t} \frac{D_t}{\tau^2} \right) - \sum_{j,k=1}^d \frac{1}{\sqrt{\det g}} \left( D_j + i\tau \partial_j \phi(z) - \mu \phi''_{t,j} \frac{D_t}{\tau^2} \right) \sqrt{\det g} g^{jk} \left( D_k + i\tau \partial_k \phi(z) - \mu \phi''_{t,k} \frac{D_t}{\tau^2} \right).$$

We define the anisotropic norm

$$\|v\|_{H_t^1}^2 := \tau^2 \|v\|_{L^2}^2 + \|D_x v\|_{L^2}^2 + \tau^{-2} \|D_t v\|_{L^2}^2, \quad (4.2.1)$$

and the spatial part

$$\|v\|_{H_{\tau,x}^1}^2 := \tau^2 \|v\|_{L^2}^2 + \|D_x v\|_{L^2}^2. \quad (4.2.2)$$

Before stating our main Carleman estimate we need to define the following two important quantities, see [LL21b, Theorem A.5]. Given  $\phi \in W^{2,\infty}(U; \mathbb{R})$ ,  $f \in W^{1,\infty}(U; \mathbb{R})$ ,  $X$  a smooth *complex valued* vector field on  $V$  we set

$$\begin{aligned} \mathcal{B}_{g,\phi,f}(X) &:= 2 \operatorname{Re} \left( \operatorname{Hess}_g(\phi)(X, \bar{X}) \right) - (\Delta_g \phi) |X|_g^2 + f |X|_g^2, \\ \mathcal{E}_{g,\phi,f} &:= 2 \operatorname{Hess}_g(\phi)(\nabla_g \phi, \nabla_g \phi) + (\Delta_g \phi) |\nabla_g \phi|_g^2 - f |\nabla_g \phi|_g^2, \end{aligned}$$

where the Hessian is with respect to the  $x$  variable only, see Section 4.2.1 and where we have written  $|X|_g^2 = \langle X, \bar{X} \rangle_g$ . Note the difference with [LL21b, Theorem A.5] where functions/vectorfields were all assumed real-valued. Note that for a Lipschitz metric  $g$  on  $V$ , we have  $\mathcal{E}_{g,\phi,f} \in L^\infty(U)$  and  $\mathcal{B}_{g,\phi,f}(X) \in L^\infty(U)$  for any bounded vector field  $X$  on  $V$  and we stress the fact that these two quantities are time-dependent (they are defined on  $U = I \times V$ ).

**Remark 4.2.4.** In what we follows we use the notation  $C$  for a constant whose value may change from one line to another. It may depend on the metric  $g$ , the norms  $\|\phi\|_{W^{2,\infty}}$  and  $\|f\|_{W^{1,\infty}}$  where  $f \in W^{1,\infty}$  is an auxiliary function. It is important to notice that all these quantities consume at most one derivative of  $g$  and therefore are compatible with the Lipschitz regularity of  $g$ .

Let us now state the main result of this section, which is a Carleman estimate in the spirit of [Tat95, Hör97, RZ98, Tat99b] but with two main differences:

1. The Fourier multiplier is now  $e^{-\frac{\mu|D_t|^2}{2\tau^3}}$  instead of  $e^{-\frac{\mu|D_t|^2}{2\tau}}$
2. We use the anisotropic norm defined in (4.2.1).

In Section 4.4.1 we shall show that this estimate remains valid for more general operators that  $P$ .

**Theorem 4.2.1** (Carleman estimate). *Let  $\mathbf{x}_0 = (t_0, x_0) \in \Omega \subset \mathbb{R}^{1+d}$ . Assume that  $\phi$  and  $f$  satisfy the following:  $\phi$  is a quadratic real-valued polynomial,  $f \in W^{1,\infty}(\Omega; \mathbb{R})$ , there exist  $r > 0$  such that  $|\nabla_g \phi|_g^2 > 0$  on  $\bar{B}(\mathbf{x}_0, r)$ , and  $C_0 > 0$  such that for any vector field  $X$ , we have almost everywhere on  $B(\mathbf{x}_0, r)$ :*

$$\begin{aligned} \mathcal{B}_{g,\phi,f}(X) &\geq 2C_0 |X|_g^2, \\ \mathcal{E}_{g,\phi,f} &\geq 2C_0 |\nabla_g \phi|_g^2. \end{aligned}$$

*Then, for all  $k \in \mathbb{N}$  there exist  $d, C, \tau_0 > 0$  such that for all  $\tau \geq \tau_0$  and  $w \in C_c^\infty(B(\mathbf{x}_0, \frac{r}{8}))$ , we have*

$$C \left\| Q_{\mu,\tau}^\phi P w \right\|_{L^2}^2 + C e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{H_t^{-k} H_x^1}^2 \geq \tau \left\| Q_{\mu,\tau}^\phi w \right\|_{H_\tau^1}^2. \quad (4.2.3)$$

**Remark 4.2.5.** The error term  $e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{H_t^{-k} H_x^1}^2$  in the left hand side of the estimate is obtained thanks to the localization properties of the gaussian weight. For the uniqueness result for  $H^1$  solutions one only needs to apply this theorem for  $k = 0$ . The advantage however of having a better control with respect to the time variable is that one can use the partial ellipticity of  $P$  and reduce the regularity of the solution. For more details on this see Section 4.4.3.

The main step for the proof of Theorem 4.2.1 is the following subelliptic estimate:

**Proposition 4.2.6** (The subelliptic estimate). *Let  $\mathbf{x}_0 = (t_0, x_0) \in \Omega \subset \mathbb{R}^{1+d}$ . Assume that  $\phi$  and  $f$  satisfy the assumptions of Theorem 4.2.1. Then, there exist  $C, \tau_0 > 0$  such that for all  $\tau \geq \tau_0$  and  $v \in C_c^\infty(B(\mathbf{x}_0, r))$ , we have*

$$C \|P_{\phi, \mu} v\|_{L^2}^2 + C\tau^{-1} \|D_t v\|_{L^2}^2 \geq \tau \|v\|_{H^1}^2. \quad (4.2.4)$$

**Remark 4.2.7.** Consider  $R$  an order 1 differential operator of the form  $R = R_1 + \tau a + \frac{b}{\tau^2} D_t + \frac{c}{\tau} D_t$ , where  $R_1$  is of order one with no time derivatives and bounded coefficients and the coefficients  $a, b, c$  are bounded as well. Suppose that estimate (4.2.4) holds with  $P_{\phi, \mu} + R$  instead of  $P_{\phi, \mu}$ . We can write then:

$$\begin{aligned} \|P_{\phi, \mu} v\|_{L^2}^2 + \tau^{-1} \|D_t v\|_{L^2}^2 &\gtrsim \|(P_{\phi, \mu} + R)v\|_{L^2}^2 + \tau^{-1} \|D_t v\|_{L^2}^2 - \|Rv\|_{L^2}^2 \\ &\gtrsim \tau \|v\|_{H^1}^2 + \tau^{-1} \|D_t v\|_{L^2}^2 - \|Rv\|_{L^2}^2 \\ &\gtrsim \tau \|v\|_{H^1}^2 + \tau^{-1} \|D_t v\|_{L^2}^2 \\ &\quad - \left( \tau^2 \|v\|_{L^2}^2 + \|\nabla_x u\|_{L^2}^2 + \frac{\|D_t v\|_{L^2}^2}{\tau^4} + \frac{\|D_t v\|_{L^2}^2}{\tau^2} \right), \end{aligned}$$

which implies (4.2.4) up to taking  $\tau_0$  sufficiently large. Let us now define

$$P := \sum_{jk} g^{jk}(x) \partial_j \partial_k = - \sum_{jk} g^{jk}(x) D_j D_k. \quad (4.2.5)$$

We denote similarly by  $P_{\phi, \mu}$  the conjugated operator  $P$  with respect to  $Q_{\mu, t}^\phi$  given by

$$\begin{aligned} P_{\phi, \mu} &= - \left( D_t + i\tau \partial_t \phi(z) - \mu \phi''_{t,t} \frac{D_t}{\tau^2} \right) \\ &\quad - \sum_{j,k=1}^d g^{jk}(x) \left( D_j + i\tau \partial_j \phi(z) - \mu \phi''_{t,j} \frac{D_t}{\tau^2} \right) \left( D_k + i\tau \partial_k \phi(z) - \mu \phi''_{t,k} \frac{D_t}{\tau^2} \right). \end{aligned}$$

Remark now that since the metric  $g$  is Lipschitz and time independent, the commutator

$$\left[ \left( D_j + i\tau \partial_j \phi(z) - \mu \phi''_{t,j} \frac{D_t}{\tau^2} \right), \sqrt{\det gg^{jk}} \right] = \left[ D_j, \sqrt{\det gg^{jk}} \right],$$

is a 0 order differential operator with bounded coefficients. It follows then that estimate (4.2.4) for  $P_{\phi, \mu}$  implies the same estimate for  $P_{\phi, \mu}$ . This remark will be crucial in the sequel, as it will allow us to pass from one operator to another.

Let us now show how the subelliptic estimate implies our main Carleman estimate.

*Proof of Theorem 4.2.1 from Proposition 4.2.6.* Suppose for simplicity that  $t_0 = 0$  and let  $r_0 := r/2$  with  $r$  given by the assumptions. Consider  $w \in C_c^\infty(B(\mathbf{x}_0, r_0/4))$  and  $\chi \in C_c^\infty((-r_0, r_0))$  with  $\chi = 1$  on  $(-r_0/2, r_0/2)$ . We have:

$$\tau \left\| Q_{\mu, \tau}^\phi w \right\|_{H_\tau^1}^2 \leq C\tau \left\| \chi Q_{\mu, \tau}^\phi w \right\|_{H_\tau^1}^2 + C\tau \left\| (1 - \chi) Q_{\mu, \tau}^\phi w \right\|_{H_\tau^1}^2. \quad (4.2.6)$$

Consider  $\tilde{\chi} \in C_c^\infty((-r_0/3, r_0/3))$  with  $\tilde{\chi} = 1$  in a neighborhood of  $[-r_0/4, r_0/4]$ , so that  $w = \tilde{\chi}w$ . Using the support properties of  $\chi, \tilde{\chi}$  and  $w$  together with Lemma 4.A.1 we estimate the second term in (4.2.6) as follows:

$$\begin{aligned} \tau \left\| (1 - \chi) Q_{\mu, \tau}^\phi w \right\|_{H_\tau^1}^2 &\leq C\tau \left\| (1 - \chi) Q_{\mu, \tau}^\phi w \right\|_{H_{\tau, x}^1}^2 + C\tau^{-1} \left\| D_t (1 - \chi) Q_{\mu, \tau}^\phi w \right\|_{L^2}^2 \\ &= C\tau \left\| (1 - \chi) e^{-\mu \frac{|D_t|^2}{2\tau^3}} e^{\tau\phi} \tilde{\chi} w \right\|_{H_{\tau, x}^1}^2 + C\tau^{-1} \left\| D_t (1 - \chi) Q_{\mu, \tau}^\phi w \right\|_{L^2}^2 \\ &\leq C\tau e^{-2c \frac{\tau^3}{\mu}} \left\| e^{\tau\phi} w \right\|_{H_t^{-k} H_x^1}^2 + C\tau^{-1} \left\| [D_t, (1 - \chi)] e^{-\mu \frac{|D_t|^2}{2\tau^3}} e^{\tau\phi} \tilde{\chi} w \right\|_{L^2}^2 \\ &\quad + C\tau^{-1} \left\| (1 - \chi) D_t Q_{\mu, \tau}^\phi w \right\|_{L^2}^2 \\ &\leq C e^{-c \frac{\tau^3}{\mu}} \left\| e^{\tau\phi} w \right\|_{H_t^{-k} H_x^1}^2 + C\tau^{-1} \left\| D_t Q_{\mu, \tau}^\phi w \right\|_{L^2}^2. \end{aligned} \quad (4.2.7)$$

We estimate now the second term in (4.2.7). To do so we consider  $\sigma > 0$  a small constant to be chosen later on and we distinguish between frequencies smaller or larger than  $\sigma\tau^2$ . We also assume  $\sigma\tau^2 \geq 1$  and get:

$$\begin{aligned} \left\| D_t Q_{\mu, \tau}^\phi w \right\|_{L^2} &\leq \left\| D_t \mathbf{1}_{|D_t| \leq \sigma\tau^2} Q_{\mu, \tau}^\phi w \right\|_{L^2} + \left\| D_t^k \mathbf{1}_{|D_t| \geq \sigma\tau^2} D_t^{-k} e^{-\mu \frac{|D_t|^2}{2\tau^3}} e^{\tau\phi} w \right\|_{L^2} \\ &\leq \sigma\tau^2 \left\| Q_{\mu, \tau}^\phi w \right\|_{L^2} + \max_{\xi_t \geq \sigma\tau^2} (\xi_t^k e^{-\mu \frac{|\xi_t|^2}{2\tau^3}}) \left\| e^{\tau\phi} w \right\|_{H_t^{-k} L_x^2}. \end{aligned}$$

Now the function  $\mathbb{R}^+ \ni s \mapsto s^k e^{-\mu \frac{|s|^2}{2\tau^3}}$  reaches its maximum at  $s = \sqrt{\frac{k\tau^3}{\mu}}$  and is decreasing on  $[\sqrt{\frac{k\tau^3}{\mu}}, \infty)$ . As a consequence, if  $\sigma\tau^2 \geq \sqrt{\frac{k\tau^3}{\mu}}$  which translates to  $\tau \geq \frac{k}{\sigma^2\mu}$ , one has that  $\max_{\xi_t \geq \sigma\tau^2} (\xi_t^k e^{-\mu \frac{|\xi_t|^2}{2\tau^3}}) = \sigma^k \tau^{2k} e^{-\mu \frac{\sigma^2\tau^4}{2\tau^3}} = \sigma^k \tau^{2k} e^{-\mu \frac{\sigma^2\tau}{2}}$ . We obtain therefore, for  $\tau \geq \tau_0$  that

$$\left\| D_t Q_{\mu, \tau}^\phi w \right\|_{L^2} \leq \sigma\tau^2 \left\| Q_{\mu, \tau}^\phi w \right\|_{L^2} + \sigma^k \tau^{2k} e^{-\mu \frac{\sigma^2\tau}{2}} \left\| e^{\tau\phi} w \right\|_{H_t^{-k} L_x^2}. \quad (4.2.8)$$

We now estimate the term  $\tau \left\| \chi Q_{\mu, \tau}^\phi w \right\|_{H_\tau^1}^2$  appearing in (4.2.6). Thanks to the supports of  $\chi$  and  $w$  we can apply the subelliptic estimate of Proposition 4.2.6 to  $\chi Q_{\mu, \tau}^\phi w$ , this gives:

$$\begin{aligned} \tau \left\| \chi Q_{\mu, \tau}^\phi w \right\|_{H_\tau^1}^2 &\leq C \left\| P_{\phi, \mu} \chi Q_{\mu, \tau}^\phi w \right\|_{L^2}^2 + C\tau^{-1} \left\| D_t \chi Q_{\mu, \tau}^\phi w \right\|_{L^2}^2 \\ &\leq C \left\| P_{\phi, \mu} \chi Q_{\mu, \tau}^\phi w \right\|_{L^2}^2 + C\tau^{-1} \left\| Q_{\mu, \tau}^\phi w \right\|_{L^2}^2 + C\tau^{-1} \left\| D_t Q_{\mu, \tau}^\phi w \right\|_{L^2}^2 \\ &\leq C \left\| P_{\phi, \mu} \chi Q_{\mu, \tau}^\phi w \right\|_{L^2}^2 + C\sigma^2\tau^3 \left\| Q_{\mu, \tau}^\phi w \right\|_{L^2}^2 + C\sigma^{2k}\tau^{4k-1} e^{-\mu\sigma^2\tau} \left\| e^{\tau\phi} w \right\|_{H_t^{-k} L_x^2}^2, \end{aligned} \quad (4.2.9)$$



where for the last inequality, we used  $\sigma\tau^2 \geq 1$  and (4.2.8). We need finally to estimate the following term:

$$\begin{aligned} \left\| P_{\phi,\mu}\chi Q_{\mu,\tau}^\phi w \right\|_{L^2} &\leq \left\| \chi P_{\phi,\mu} Q_{\mu,\tau}^\phi w \right\|_{L^2} + \left\| [P_{\phi,\mu}, \chi] Q_{\mu,\tau}^\phi w \right\|_{L^2} \\ &\leq C \left\| P_{\phi,\mu} Q_{\mu,\tau}^\phi w \right\|_{L^2} + C \left\| [P_{\phi,\mu}, \chi] Q_{\mu,\tau}^\phi w \right\|_{L^2} \\ &= C \left\| Q_{\mu,\tau}^\phi P w \right\|_{L^2} + C \left\| [P_{\phi,\mu}, \chi] e^{-\mu \frac{|D_t|^2}{2\tau^3}} e^{\tau\phi} \tilde{\chi} w \right\|_{L^2} \end{aligned}$$

Now, we decompose  $[P_{\phi,\mu}, \chi]$  as  $[P_{\phi,\mu}, \chi] = R_1 + R_2 + R_3$  where  $R_1$  is an order one differential operator in  $\tau$  with no derivatives and with coefficients supported in  $\text{supp}(\chi')$ ,  $R_2$  can be written as  $R_2 = \frac{1}{\tau^4} b(x) D_t$  with  $b$  bounded and  $R_3$  can be written  $R_3 = \frac{\tilde{R}_3}{\tau^2}$  with  $\tilde{R}_3$  a differential operator of order 1 in  $x$  with bounded coefficients supported in  $\text{supp}(\chi')$ . Using Lemma 4.A.1 we obtain

$$\left\| R_1 e^{-\mu \frac{|D_t|^2}{2\tau^3}} e^{\tau\phi} \tilde{\chi} w \right\|_{L^2} + \left\| R_2 e^{-\mu \frac{|D_t|^2}{2\tau^3}} e^{\tau\phi} \tilde{\chi} w \right\|_{L^2} + \left\| R_3 e^{-\mu \frac{|D_t|^2}{2\tau^3}} e^{\tau\phi} \tilde{\chi} w \right\|_{L^2} \leq C e^{-c \frac{\tau^3}{\mu}} \left\| e^{\tau\phi} w \right\|_{H_t^{-k} H_x^1}.$$

Putting the two last inequalities together we get

$$\left\| P_{\phi,\mu}\chi Q_{\mu,\tau}^\phi w \right\|_{L^2} \leq C \left\| Q_{\mu,\tau}^\phi P w \right\|_{L^2} + C e^{-c \frac{\tau^3}{\mu}} \left\| e^{\tau\phi} w \right\|_{H_t^{-k} H_x^1}. \quad (4.2.10)$$

Combining (4.2.6), (4.2.7), (4.2.8), (4.2.9) and (4.2.10) we find that there are constants  $C, c > 0$  such that for any  $\sigma > 0$  and  $\tau \geq \tau_0$  we have:

$$\begin{aligned} &\tau \left\| Q_{\mu,\tau}^\phi w \right\|_{H_t^1}^2 \\ &\leq C \left\| Q_{\mu,\tau}^\phi P w \right\|_{L^2}^2 + C \sigma^2 \tau^3 \left\| Q_{\mu,\tau}^\phi w \right\|_{L^2}^2 + C \left( e^{-c \frac{\tau^3}{\mu}} + \sigma^{2k} \tau^{4k-1} e^{-\mu \sigma^2 \tau} \right) \left\| e^{\tau\phi} w \right\|_{H_t^{-k} H_x^1}^2. \end{aligned}$$

Choosing then  $\sigma > 0$  sufficiently small to absorb the term  $\sigma^2 \tau^3 \left\| Q_{\mu,\tau}^\phi w \right\|_{L^2}^2$  and taking  $\tau \geq \tau_0$  with  $\tau_0$  sufficiently large finishes the proof of Theorem 4.2.3 from Proposition 4.2.6.  $\square$

### 4.2.3 Proof of the subelliptic estimate

Let us consider the "classical" conjugated operator given by

$$P_\phi := e^{\tau\phi} (i\partial_t + \Delta_g) e^{-\tau\phi}.$$

Remark that  $P_\phi$  corresponds to  $P_{\phi,\mu}$  for  $\mu = 0$ . We start by showing the desired subelliptic estimate in this particular case. We then prove that the additional terms coming from the difference  $\|(P_{\phi,\mu} - P_\phi)u\|_{L^2}^2$  can be absorbed in our estimate.

The case  $\mu = 0$

**Proposition 4.2.8.** *Let  $\Omega \subset \mathbb{R}^{1+d}$ . Assume that  $\phi \in W^{2,\infty}(\Omega; \mathbb{R})$  and  $f \in W^{1,\infty}(\Omega; \mathbb{R})$ . Then, for any  $u \in C_0^\infty(\Omega)$  and  $\tau \geq 0$ , we have for any  $\delta > 0$*

$$\begin{aligned} & \iint |P_\phi u|^2 + 2 \frac{\|\Delta\phi\|_{L^\infty}^2 + \|f\|_{L^\infty}}{\delta} \frac{1}{\tau} \|u_t\|_{L^2}^2 + R(u) \\ & \geq \tau^3 \iint \left[ 2\text{Hess}_g(\phi)(\nabla_g\phi, \nabla_g\phi) + (\Delta_g\phi) |\nabla_g\phi|_g^2 - f |\nabla_g\phi|_g^2 - \delta \right] |u|^2 \\ & \quad + \tau \iint 2 \text{Re}(\text{Hess}_g(\phi)(\nabla_g u, \nabla_g \bar{u})) - (\Delta_g\phi) |\nabla_g u|_g^2 + f |\nabla_g u|_g^2, \end{aligned}$$

with a remainder  $R(u)$  satisfying

$$|R(u)| \leq C\tau^2 \|u\|_{L^2}^2 + C \|\nabla_g u\|_{L^2}^2. \quad (4.2.11)$$

Our computations are inspired by [Lau10] for Schrödinger and [LL21b] for the elliptic case. They rely on Riemannian tools presented in Section 4.2.1. In [Lau10], it led, for Schrödinger operators, to require a positive (space) Hessian for the weight function. Yet, the possibility of having a term  $\frac{1}{\tau} \|u_t\|_{L^2}^2$  in the left hand side and the flexibility of the function  $f$  gives more freedom. This will allow later to relax this convexity condition and to obtain a condition similar to the elliptic case as presented in [LL21b].

*Proof of Proposition 4.2.8.* We start by computing

$$\begin{aligned} P_\phi u & := e^{\tau\phi}(i\partial_t + \Delta_g)(e^{-\tau\phi}u) \\ & = iu_t - i\tau\phi_t u + \Delta_g u - 2\tau \langle \nabla_g\phi, \nabla_g u \rangle_g - \tau(\Delta_g\phi)u + \tau^2 |\nabla_g\phi|_g^2 u, \end{aligned}$$

where we denote  $v_t := \partial_t v$ . We then decompose the conjugated operator  $P_\phi$  as

$$P_\phi = i\partial_t - i\tau\phi_t + Q_2 + Q_1$$

with

$$\begin{aligned} Q_1 u & := -2\tau \langle \nabla_g\phi, \nabla_g u \rangle_g - \tau f u \\ Q_2 u & := \Delta_g u + \tau^2 |\nabla_g\phi|_g^2 u - \tau(\Delta_g\phi)u + \tau f u = \tilde{Q}_2 u + R_2 u \end{aligned}$$

where  $\tilde{Q}_2$  is the principal part of  $Q_2$ , that is

$$\tilde{Q}_2 u = \Delta_g u + \tau^2 |\nabla_g\phi|_g^2 u, \quad \text{and} \quad R_2 u = \tau(-\Delta_g\phi + f)u.$$

Now, we write ( $\|\cdot\|$  denotes the  $L^2$  norm for short)

$$\begin{aligned} 2\|P_\phi u\|^2 + 2\|R_2 u\|^2 + 2\|\tau\phi_t u\|^2 & \geq \|P_\phi u - R_2 u + i\tau\phi_t u\|^2 \\ & = \|iu_t + Q_1 u + \tilde{Q}_2 u\|^2, \end{aligned} \quad (4.2.12)$$

where we estimate the remainder as

$$\|R_2 u\|^2 \leq \tau^2 \|f - \Delta_g\phi\|_{L^\infty}^2 \|u\|_{L^2}^2 \quad (4.2.13)$$

$$\|\tau\phi_t u\|^2 \leq \tau^2 \|\phi_t\|_{L^\infty}^2 \|u\|_{L^2}^2. \quad (4.2.14)$$

Hence, we are left to produce a lower bound for

$$\|iu_t + Q_1u + \tilde{Q}_2u\|^2 = \|Q_1u\|^2 + \|iu_t + \tilde{Q}_2u\|^2 + 2\operatorname{Re}(iu_t, Q_1u) + 2\operatorname{Re}(Q_1u, \tilde{Q}_2u).$$

We decompose

$$2\operatorname{Re}(iu_t, Q_1u) = -4\tau\operatorname{Re}(iu_t, \langle \nabla_g\phi, \nabla_gu \rangle_g) - 2\tau\operatorname{Re}(iu_t, fu) = 2I_1 + I_2.$$

Expanding  $2\operatorname{Re}a = a + \bar{a}$  for  $I_1$  and performing an integration by parts in  $t$  for the first term, we get

$$\begin{aligned} I_1 &= \tau \iint i \langle \nabla_g\phi, \nabla_gu \rangle_g \bar{u}_t - i\tau \iint \langle \nabla_g\phi, \nabla_g\bar{u} \rangle_g u_t \\ &= \tau \iint -i \left[ \langle \nabla_g\phi_t, \nabla_gu \rangle_g + \langle \nabla_g\phi, \nabla_gu_t \rangle_g \right] \bar{u} - i\tau \iint \langle \nabla_g\phi, \nabla_g\bar{u} \rangle_g u_t \end{aligned}$$

Integration by parts in  $x$  yields

$$-i \iint \langle \nabla_g\phi, \nabla_g\bar{u} \rangle_g u_t = i \iint (\Delta\phi)\bar{u}u_t + i \iint \langle \nabla_g\phi, \nabla_gu_t \rangle_g \bar{u}.$$

As a consequence

$$I_1 = \tau \iint -i \langle \nabla_g\phi_t, \nabla_gu \rangle_g \bar{u} + i\tau \iint (\Delta\phi)\bar{u}u_t.$$

By Cauchy-Schwarz

$$2 \left| \tau \iint -i \langle \nabla_g\phi_t, \nabla_gu \rangle_g \bar{u} \right| \leq \|\nabla_g\phi_t\|_{L^\infty} \tau^2 \|u\|_{L^2}^2 + \|\nabla_gu\|_{L^2}^2$$

By Cauchy-Schwarz, we get

$$2 \left| \tau \iint (\Delta\phi)\bar{u}u_t \right| \leq \frac{1}{2}\delta\tau^3 \|u\|_{L^2}^2 + 2\frac{\|\Delta\phi\|_{L^\infty}^2}{\delta} \frac{1}{\tau} \|u_t\|_{L^2}^2$$

By Cauchy-Schwarz again, we get  $|I_2| \leq \frac{1}{2}\delta\tau^3 \|u\|_{L^2}^2 + \frac{\|f\|_{L^\infty}^2}{\delta} \frac{1}{\tau} \|u_t\|_{L^2}^2$ . In particular, we get the intermediate result

$$2\operatorname{Re}(iu_t, Q_1u) \leq \delta\tau^3 \|u\|_{L^2}^2 + 2\frac{\|\Delta\phi\|_{L^\infty}^2 + \|f\|_{L^\infty}}{\delta} \frac{1}{\tau} \|u_t\|_{L^2}^2 + C\tau^2 \|u\|_{L^2}^2 + C\|\nabla_gu\|_{L^2}^2 \quad (4.2.15)$$

The following Lemma is the analogue of Lemma A.7 in [LL21b] but for complex valued functions.

**Lemma 4.2.9.** *One has:*

$$\begin{aligned} (Q_1u, \tilde{Q}_2u) &= \tau^3 \iint \left( 2\operatorname{Hess}_g(\phi)(\nabla_g\phi, \nabla_g\phi) + (\Delta_g\phi) |\nabla_g\phi|_g^2 - f |\nabla_g\phi|_g^2 \right) |u|^2 \\ &\quad + 2\tau \iint \operatorname{Re}(\operatorname{Hess}_g(\phi)(\nabla_gu, \nabla_g\bar{u})) - (\Delta_g\phi) |\nabla_gu|_g^2 + f |\nabla_gu|_g^2 \\ &\quad + \tau \operatorname{Re} \iint u \langle \nabla_gf, \nabla_g\bar{u} \rangle_g. \end{aligned}$$

In the estimates of Lemma 4.2.9, the remainder term is

$$R_3(u) = -\operatorname{Re} \tau \iint u \langle \nabla_g f, \nabla_g \bar{u} \rangle_g, \quad |R_3(u)| \leq \frac{\|\nabla_g f\|_{L^\infty}}{2} \left( \|\nabla_g u\|_{L^2}^2 + \tau^2 \|u\|_{L^2}^2 \right),$$

This combined with (4.2.12), (4.2.13) (4.2.15)(4.2.14) and Lemma 4.2.9, concludes the proof of the proposition with

$$R(u) = \|R_2 u\|^2 + \|\tau \phi_t u\|^2 + |R_3(u)| + C\tau^2 \|u\|_{L^2}^2 + C \|\nabla_g u\|_{L^2}^2.$$

□

### The case $\mu > 0$ : End of the proof of Proposition 4.2.6

Thanks to Remark 4.2.7 it suffices to consider  $P_{\phi, \mu}$  as defined in the remark. We start by expressing it in terms of  $P_\phi$ . Recall that by assumption  $\phi$  is a quadratic polynomial and therefore  $\partial_{t,j}^2 \phi$  are actually constants. We have:

$$\begin{aligned} P_{\phi, \mu} &= P_\phi - \sum_{jk} g^{jk}(x) (D_j + i\tau \partial_j \phi) \mu \partial_{t,k}^2 \phi \frac{D_t}{\tau^2} \\ &\quad - \sum_{jk} g^{jk}(x) \mu \partial_{t,j}^2 \phi \frac{D_t}{\tau^2} (D_k + i\tau \partial_k \phi) + \mu^2 \sum_{jk} \partial_{t,k}^2 \phi \cdot \partial_{t,j}^2 \phi g^{jk}(x) \frac{D_t^2}{\tau^4} + \tilde{R}_1 \\ &= P_\phi - 2\mu \sum_{jk} \partial_{t,k}^2 \phi g^{jk}(x) \frac{D_j D_t}{\tau^2} + \mu^2 \sum_{jk} \partial_{t,k}^2 \phi \cdot \partial_{t,j}^2 \phi g^{jk}(x) \frac{D_t^2}{\tau^4} + \tilde{R}_2 \\ &= P_\phi + 2\mu \sum_{jk} \partial_{t,k}^2 \phi g^{jk}(x) \frac{\partial_i \partial_t}{\tau^2} - \mu^2 \sum_{jk} \partial_{t,k}^2 \phi \cdot \partial_{t,j}^2 \phi g^{jk}(x) \frac{\partial_t^2}{\tau^4} + \tilde{R}_2 \\ &:= \tilde{P}_{\phi, \mu} + \tilde{R}_2. \end{aligned}$$

where the operators  $\tilde{R}_j$  belong to the class of admissible perturbations considered in Remark 4.2.7. It suffices then to show the estimate of Proposition 4.2.6 for the operator  $\tilde{P}_{\phi, \mu}$ . We decompose

$$\tilde{P}_{\phi, \mu} = i\partial_t - i\tau \partial_t \phi + \tilde{Q}_{1, \mu} + \tilde{Q}_{2, \mu},$$

where

$$\tilde{Q}_{1, \mu} = Q_1$$

and

$$\begin{aligned} \tilde{Q}_{2, \mu} &= Q_2 + 2\mu \sum_{jk} \partial_{t,k}^2 \phi g^{jk}(x) \frac{\partial_i \partial_t}{\tau^2} - \mu^2 \sum_{jk} \partial_{t,k}^2 \phi \cdot \partial_{t,j}^2 \phi g^{jk}(x) \frac{\partial_t^2}{\tau^4} \\ &= \tilde{Q}_2 + R_2 + 2\mu \sum_{jk} \partial_{t,k}^2 \phi g^{ij}(x) \frac{\partial_j \partial_t}{\tau^2} - \mu^2 \sum_{jk} \partial_{t,k}^2 \phi \cdot \partial_{t,j}^2 \phi g^{jk}(x) \frac{\partial_t^2}{\tau^4} \\ &= \tilde{Q}_{2, \mu} + R_2 \end{aligned}$$

where

$$\tilde{Q}_{2, \mu} := \tilde{Q}_2 + 2\mu \sum_{jk} \partial_{t,k}^2 \phi g^{jk}(x) \frac{\partial_i \partial_t}{\tau^2} - \mu^2 \sum_{ij} \partial_{t,k}^2 \phi \cdot \partial_{t,j}^2 \phi g^{jk}(x) \frac{\partial_t^2}{\tau^4}, \quad (4.2.16)$$

and the operators  $Q_1, Q_2, \tilde{Q}_2$  are as defined in Section 4.2.3. We follow the same steps to absorb the term  $R_2$  and  $i\tau\partial_t\phi$ . We need as before to control

$$\begin{aligned} \|i\partial_t u + \tilde{Q}_{1,\mu}u + \tilde{Q}_{2,\mu}u\|_{L^2}^2 &= \|Q_1u\|^2 + \|i\partial_t u + \tilde{Q}_{2,\mu}u\|^2 + 2\operatorname{Re}(iu_t, Q_1u) + 2\operatorname{Re}(Q_1u, \tilde{Q}_{2,\mu}u) \\ &= \|Q_1u\|^2 + \|i\partial_t u + \tilde{Q}_{2,\mu}u\|^2 + 2\operatorname{Re}(iu_t, Q_1u) + 2\operatorname{Re}(Q_1u, \tilde{Q}_2u) \\ &\quad + 2\operatorname{Re}(Q_1u, (\tilde{Q}_{2,\mu} - \tilde{Q}_2)u). \end{aligned} \quad (4.2.17)$$

It follows that in order to finish the proof of Proposition 4.2.6 it suffices to show that the term

$$2\operatorname{Re}(Q_1u, (\tilde{Q}_{2,\mu} - \tilde{Q}_2)u)$$

yields an admissible error in view of the estimate (4.2.4). This is the content of the following Lemma.

**Lemma 4.2.10.** *There exist  $C, \tau_0 > 0$  such that for  $u \in C_0^\infty(\Omega)$  one has:*

$$|2\operatorname{Re}(Q_1u, (\tilde{Q}_{2,\mu} - \tilde{Q}_2)u)| \leq C \|u\|_{H_x^1}^2, \quad \text{for all } \tau \geq \tau_0.$$

*Proof.* We recall that  $Q_1u = -2\tau \langle \nabla_g \phi, \nabla_g u \rangle_g - \tau fu$  and we write  $\tilde{Q}_{2,\mu} - \tilde{Q}_2 = L_1 + L_2$  with

$$L_1 := 2\mu \sum_{jk} \partial_{t,k}^2 \phi g^{jk}(x) \frac{\partial_j \partial_t}{\tau^2},$$

and

$$L_2 := -\mu^2 \sum_{jk} \partial_{t,k}^2 \phi \cdot \partial_{t,j}^2 \phi g^{jk}(x) \frac{\partial_t^2}{\tau^4}.$$

One needs to control then:

$$\begin{aligned} -4\operatorname{Re}(\tau \langle \nabla_g \phi, \nabla_g u \rangle_g, L_1u) - 4\operatorname{Re}(\tau \langle \nabla_g \phi, \nabla_g u \rangle_g, L_2u) - 2\operatorname{Re}(\tau fu, L_1u) - 2\operatorname{Re}(\tau fu, L_2u) \\ := 2(A_1 + A_2 + A_3 + A_4). \end{aligned} \quad (4.2.18)$$

We start with the terms  $A_3$  and  $A_4$  which are the easiest ones. We integrate by parts in  $t$  and find for  $A_3$ :

$$\begin{aligned} A_3 &= -2\mu \operatorname{Re} \left( \frac{1}{\tau} fu, \partial_t \sum_{jk} \partial_{t,k}^2 \phi g^{jk}(x) \partial_j u \right) \\ &= 2\mu \operatorname{Re} \left( \frac{1}{\tau} (\partial_t f)u, \sum_{jk} \partial_{t,k}^2 \phi g^{jk}(x) \partial_j u \right) + 2\mu \operatorname{Re} \left( \frac{1}{\tau} f \partial_t u, \sum_{jk} \partial_{t,k}^2 \phi g^{jk}(x) \partial_j u \right) \\ &:= A_{31} + A_{32}. \end{aligned}$$

By Cauchy-Schwarz we get for a constant  $C$  depending on  $f, \phi$  and  $g$

$$|A_{31}| \leq C \left( \frac{\|u\|_{L^2}^2}{\tau} + \frac{\|\nabla_x u\|_{L^2}^2}{\tau} \right) \quad \text{and} \quad |A_{32}| \leq C \left( \frac{\|D_t u\|_{L^2}^2}{\tau^2} + \|\nabla_x u\|_{L^2}^2 \right),$$

which give

$$|A_3| \leq C \left( \|u\|_{L^2}^2 + \|\nabla_x u\|_{L^2}^2 + \frac{\|D_t u\|_{L^2}^2}{\tau^2} \right). \quad (4.2.19)$$

Similarly, integrating by parts in time gives

$$|A_4| \leq \frac{C}{\tau^3} \left( \|u\|_{L^2}^2 + \|D_t u\|_{L^2}^2 \right). \quad (4.2.20)$$

We now turn our attention to  $A_1$ . Here one needs to use the real part in order to decrease the number of derivatives. We write  $\langle \nabla_g \phi, \nabla_g u \rangle_g = \sum_{jk} g^{jk}(x) \partial_j \phi \partial_k u$ ,  $2 \operatorname{Re} a = a + \bar{a}$  and calculate as follows:

$$\begin{aligned} -A_1 &= 2 \operatorname{Re} \left( \tau \sum_{jk} g^{jk}(x) \partial_j \phi \partial_k u, 2\mu \sum_{lm} \partial_{t,m}^2 \phi g^{lm}(x) \frac{\partial_l \partial_t}{\tau^2} u \right) \\ &= \frac{2\mu}{\tau} \left( \sum_{jk} g^{jk}(x) \partial_j \phi \partial_k u, \sum_{lm} \partial_{t,m}^2 \phi g^{lm}(x) \partial_l \partial_t u \right) \end{aligned} \quad (4.2.21)$$

$$+ \frac{2\mu}{\tau} \left( \sum_{lm} \partial_{t,m}^2 \phi g^{lm}(x) \partial_l \partial_t u, \sum_{jk} g^{jk}(x) \partial_j \phi \partial_k u \right). \quad (4.2.22)$$

Integrating by parts in  $t$  in the first term in (4.2.21) yields

$$\begin{aligned} &\left( \sum_{jk} g^{jk}(x) \partial_j \phi \partial_k u, \sum_{lm} \partial_{t,m}^2 \phi g^{lm}(x) \partial_l \partial_t u \right) \\ &= - \left( \sum_{jk} g^{jk}(x) \partial_{j,t}^2 \phi \partial_k u, \sum_{lm} \partial_{t,m}^2 \phi g^{lm}(x) \partial_l u \right) - \left( \sum_{jk} g^{jk}(x) \partial_j \phi \partial_k \partial_t u, \sum_{lm} \partial_{t,m}^2 \phi g^{lm}(x) \partial_l u \right) \\ &= - \left( \sum_{jk} g^{jk}(x) \partial_{j,t}^2 \phi \partial_k u, \sum_{lm} \partial_{t,m}^2 \phi g^{lm}(x) \partial_l u \right) - \left( \sum_{lm} \partial_{t,m}^2 \phi g^{lm}(x) \partial_l \partial_t u, \sum_{jk} g^{jk}(x) \partial_j \phi \partial_k u \right), \end{aligned}$$

which together with (4.2.21) implies that

$$A_1 = \frac{2\mu}{\tau} \left( \sum_{jk} g^{jk}(x) \partial_{j,t}^2 \phi \partial_k u, \sum_{lm} \partial_{t,m}^2 \phi g^{lm}(x) \partial_l u \right).$$

This immediately implies, thanks to Cauchy-Schwarz:

$$|A_1| \leq \frac{C}{\tau} \|\nabla_x u\|_{L^2}^2. \quad (4.2.23)$$

Finally, to estimate  $A_2$  we proceed similarly by writing:

$$\begin{aligned} A_2 &= 2 \operatorname{Re} \left( \tau \sum_{jk} g^{jk}(x) \partial_j \phi \partial_k u, \mu^2 \sum_{lm} \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \frac{\partial_t^2}{\tau^4} u \right) \\ &= \frac{\mu^2}{\tau^3} \left( \sum_{jk} g^{jk}(x) \partial_j \phi \partial_k u, \sum_{lm} \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_t^2 u \right) \end{aligned} \quad (4.2.24)$$

$$\begin{aligned} &+ \frac{\mu^2}{\tau^3} \left( \sum_{lm} \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_t^2 u, \sum_{jk} g^{jk}(x) \partial_j \phi \partial_k u \right) \\ &= \frac{\mu^2}{\tau^3} \left( \sum_{jk} g^{jk}(x) \partial_j \phi \partial_k u, \sum_{lm} \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_t^2 u \right) \end{aligned} \quad (4.2.25)$$

$$+ \frac{\mu^2}{\tau^3} \sum_{jklm} \left( \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_t^2 u, g^{jk}(x) \partial_j \phi \partial_k u \right). \quad (4.2.26)$$

To facilitate our notation we write in what follows  $S_j$  for differential operators of order 0 with bounded coefficients that depend on  $g$  and  $\phi$ . We integrate by parts in  $t$  in the first term in (4.2.24) to find:

$$\begin{aligned} &\left( \sum_{jk} g^{jk}(x) \partial_j \phi \partial_k u, \sum_{lm} \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_t^2 u \right) \\ &= - \left( \sum_{jk} g^{jk}(x) \partial_{t,t}^2 \phi \partial_k u, \sum_{lm} \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_t u \right) \\ &\quad - \left( \sum_{jk} g^{jk}(x) \partial_j \phi \partial_{t,k}^2 u, \sum_{lm} \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_t u \right) \\ &= A_{21} + A_{22} \end{aligned} \quad (4.2.27)$$

We integrate by parts in  $x$  and then in  $t$  to find for  $A_{22}$ :

$$\begin{aligned} A_{22} &= - \left( \sum_{jk} g^{jk}(x) \partial_j \phi \partial_{t,k}^2 u, \sum_{lm} \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_t u \right) \\ &= - \left( \sum_{jk} \partial_k \left( g^{jk}(x) \partial_j \phi \partial_t u \right), \sum_{lm} \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_t u \right) \\ &\quad + \left( S_1 \partial_t u, \sum_{lm} \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_t u \right) \\ &= \sum_{jklm} \left( g^{jk}(x) \partial_j \phi \partial_t u, \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_{t,k}^2 u \right) + (S_2 \partial_t u, \partial_t u) \\ &= - \sum_{jklm} \left( g^{jk}(x) \partial_j \phi \partial_t^2 u, \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_k u \right) + (S_2 \partial_t u, \partial_t u) + \sum_j (S_{3,j} \partial_t u, \partial_j u) \\ &= - \sum_{jklm} \left( \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_t^2 u, g^{jk}(x) \partial_j \phi \partial_k u \right) + (S_2 \partial_t u, \partial_t u) + \sum_j (S_{3,j} \partial_t u, \partial_j u). \end{aligned} \quad (4.2.28)$$

Now putting together (4.2.24), (4.2.27) and (4.2.28) gives the equality:

$$A_2 = \frac{\mu^2}{\tau^3} \left( A_{21} + (S_2 \partial_t u, \partial_t u) + \sum_j (S_{3,j} \partial_t u, \partial_j u) \right),$$

where

$$A_{21} = - \left( \sum_{jk} g^{jk}(x) \partial_{j,t}^2 \phi \partial_k u, \sum_{lm} \partial_{t,m}^2 \phi \cdot \partial_{t,l}^2 \phi g^{lm}(x) \partial_t u \right).$$

We obtain therefore

$$|A_2| \leq \frac{C}{\tau^3} \left( \|\nabla_x u\|_{L^2}^2 + \|D_t u\|_{L^2}^2 \right). \quad (4.2.29)$$

Putting (4.2.23), (4.2.29), (4.2.19) and (4.2.20) in (4.2.18) finishes the proof of the Lemma.  $\square$

*End of the proof of the subelliptic estimate.* We can now conclude the proof of Proposition 4.2.6. Indeed, thanks to the estimates in Section 4.2.3, decomposition (4.2.17) and Lemma 4.2.10 one has that there exists  $C, \tau_0 > 0$  such that for all  $\delta > 0, \tau \geq \tau_0$  and  $u \in C_0^\infty(\Omega)$  one has

$$\begin{aligned} & \int |\mathbb{P}_{\phi,\mu} u|^2 + \frac{C}{\delta\tau} \|u_t\|_{L^2}^2 + R'(u) \\ & \geq \tau^3 \int \left[ 2\text{Hess}_g(\phi)(\nabla_g \phi, \nabla_g \phi) + (\Delta_g \phi) |\nabla_g \phi|_g^2 - f |\nabla_g \phi|_g^2 - \delta \right] |u|^2 \\ & \quad + \tau \int 2 \text{Re} (\text{Hess}_g(\phi)(\nabla_g u, \nabla_g \bar{u})) - (\Delta_g \phi) |\nabla_g u|_g^2 + f |\nabla_g u|_g^2, \end{aligned}$$

with a remainder  $R'(u)$  satisfying

$$|R'(u)| \leq C\tau^2 \|u\|_{L^2}^2 + C \|\nabla_g u\|_{L^2}^2.$$

The assumptions in the statement of Proposition 4.2.6 guarantee that

$$2\text{Hess}_g(\phi)(\nabla_g \phi, \nabla_g \phi) + (\Delta_g \phi) |\nabla_g \phi|_g^2 - f |\nabla_g \phi|_g^2 - \delta \geq C_\phi - \delta, \quad C_\phi > 0$$

and

$$\tau \int 2 \text{Re} (\text{Hess}_g(\phi)(\nabla_g u, \nabla_g \bar{u})) - (\Delta_g \phi) |\nabla_g u|_g^2 + f |\nabla_g u|_g^2 \geq 2C_0\tau |\nabla_g u|_g^2.$$

We then choose  $\delta = \frac{C_\phi}{2}$ , take  $\tau \geq \tau_0$  with  $\tau_0$  sufficiently large to absorb the term  $R'$  in the right hand side of the estimate and the proof of Proposition 4.2.6 is now complete.  $\square$

#### 4.2.4 Choice of weight function via convexification

In this section, we explain how to construct weight functions  $(\check{\phi}, f)$  that *almost* satisfy the assumptions of Theorem 4.2.1, via the usual convexification procedure. In the present context (as opposed to the usual situation), this also requires a smart choice of the function  $f$ .

The main difference with respect to the assumptions of Theorem 4.2.1 is that the function  $\check{\phi}$  that we construct here is not a quadratic polynomial. In Section 4.4.2 we shall see however that since the positivity of the quantities  $\mathcal{B}$  and  $\mathcal{E}$  is a condition that only involves derivatives up to order 2 one can replace  $\check{\phi}$  by its Taylor expansion at order 2.



**Lemma 4.2.11** (Explicit convexification). *Let  $\Psi \in W^{2,\infty}(\Omega; \mathbb{R})$  and  $G \in W^{2,\infty}(\mathbb{R})$ , and choose*

$$\check{\phi} = G(\Psi) \quad \text{and} \quad f = 2G''(\Psi) |\nabla_g \Psi|_g^2. \quad (4.2.30)$$

Then we have

$$\begin{aligned} \mathcal{B}_{g,\check{\phi},f}(X) &= 2G'(\Psi) \operatorname{Re}(\operatorname{Hess}_g(\Psi)(X, \bar{X})) + 2G''(\Psi) \left| \langle \nabla_g \Psi, X \rangle_g \right|^2 \\ &\quad + \left( G''(\Psi) |\nabla_g \Psi|_g^2 - G'(\Psi) \Delta_g \Psi \right) |X|_g^2, \\ \mathcal{E}_{g,\check{\phi},f} &= G'(\Psi)^2 \left[ 2G'(\Psi) \operatorname{Hess}_g(\Psi)(\nabla_g \Psi, \nabla_g \Psi) + G''(\Psi) |\nabla_g \Psi|_g^4 + G'(\Psi) \Delta_g \Psi |\nabla_g \Psi|_g^2 \right]. \end{aligned}$$

To state the next corollary, for  $B$  an  $L_{\text{loc}}^\infty$  section of bilinear forms on  $TV$ , we define  $|B|_g(x) = \sup_{X \in T_x V \setminus 0} \frac{|B(x, X, \bar{X})|}{|X|_g^2}$  which yields a  $L^\infty$  function on  $V$ .

**Corollary 4.2.12.** *Let  $\Psi \in W^{2,\infty}(\Omega; \mathbb{R})$ ,  $\lambda > 0$  and define  $\check{\phi}, f$  as in (4.2.30) with  $G(t) = e^{\lambda t} - 1$ . Then, for any  $\lambda > 0$  and any vector field  $X$ , we have almost everywhere on  $U$*

$$\begin{aligned} \mathcal{B}_{g,\check{\phi},f}(X) &\geq \lambda e^{\lambda \Psi} |X|_g^2 \left( \lambda |\nabla_g \Psi|_g^2 - 2|\operatorname{Hess}_g(\Psi)|_g - \Delta_g \Psi \right), \\ \mathcal{E}_{g,\check{\phi},f} &\geq \lambda e^{\lambda \Psi} |\nabla_g \check{\phi}|_g^2 \left( \lambda |\nabla_g \Psi|_g^2 - 2|\operatorname{Hess}_g(\Psi)|_g + \Delta_g \Psi \right). \end{aligned}$$

*Proof of Corollary 4.2.12.* With this choice of  $G$ , Lemma 4.2.11 gives

$$\mathcal{B}_{g,\check{\phi},f}(X) = \lambda e^{\lambda \Psi} \left[ 2 \operatorname{Re}(\operatorname{Hess}_g(\Psi)(X, \bar{X})) + 2\lambda \left| \langle \nabla_g \Psi, X \rangle_g \right|^2 - \Delta_g \Psi |X|_g^2 + \lambda |\nabla_g \Psi|_g^2 |X|_g^2 \right],$$

together with

$$\mathcal{E}_{g,\check{\phi},f} = \lambda^3 e^{3\lambda \Psi} \left[ 2 \operatorname{Hess}_g(\Psi)(\nabla_g \Psi, \nabla_g \Psi) + \lambda |\nabla_g \Psi|_g^4 + \Delta_g \Psi |\nabla_g \Psi|_g^2 \right],$$

which yields the sought result.  $\square$

*Proof of Lemma 4.2.11.* We first have  $d\check{\phi} = G'(\Psi)d\Psi$  and  $\nabla_g \check{\phi} = G'(\Psi)\nabla_g \Psi$ . We then compute the Hessian and the Laplacian as

$$\begin{aligned} \operatorname{Hess}_g(\check{\phi})(X, Y) &= \left\langle D_X \nabla_g \check{\phi}, Y \right\rangle_g = \left\langle D_X (G'(\Psi) \nabla_g \Psi), Y \right\rangle_g \\ &= G'(\Psi) \left\langle D_X \nabla_g \Psi, Y \right\rangle_g + G''(\Psi) d\Psi(X) \left\langle \nabla_g \Psi, Y \right\rangle_g \\ &= G'(\Psi) \operatorname{Hess}_g(\Psi)(X, Y) + G''(\Psi) \left\langle \nabla_g \Psi, X \right\rangle_g \left\langle \nabla_g \Psi, Y \right\rangle_g, \end{aligned}$$

and

$$\Delta_g \check{\phi} = \operatorname{div}_g(G'(\Psi)\nabla_g \Psi) = G'(\Psi)\Delta_g \Psi + G''(\Psi) |\nabla_g \Psi|_g^2.$$

In particular, we have

$$\begin{aligned} \operatorname{Hess}_g(\check{\phi})(\nabla_g \check{\phi}, \nabla_g \check{\phi}) &= G'(\Psi) \operatorname{Hess}_g(\Psi)(\nabla_g \check{\phi}, \nabla_g \check{\phi}) + G''(\Psi) \left| \left\langle \nabla_g \Psi, \nabla_g \check{\phi} \right\rangle_g \right|^2 \\ &= G'(\Psi)^2 \left[ G'(\Psi) \operatorname{Hess}_g(\Psi)(\nabla_g \Psi, \nabla_g \Psi) + G''(\Psi) \left| |\nabla_g \Psi|_g^2 \right|^2 \right], \end{aligned}$$

together with

$$\Delta_g \check{\phi} \left| \nabla_g \check{\phi} \right|_g^2 = G'(\Psi)^2 |\nabla_g \Psi|_g^2 \left( G'(\Psi) \Delta_g \Psi + G''(\Psi) |\nabla_g \Psi|_g^2 \right)$$

As a consequence, we obtain

$$\begin{aligned} \mathcal{B}_{g,\check{\phi},f}(X) &= 2G'(\Psi) \operatorname{Re}(\operatorname{Hess}_g(\Psi)(X, \bar{X})) + 2G''(\Psi) \left| \langle \nabla_g \Psi, X \rangle_g \right|^2 \\ &\quad + \left( -G'(\Psi) \Delta_g \Psi - G''(\Psi) |\nabla_g \Psi|_g^2 + f \right) |X|_g^2, \end{aligned}$$

as well as

$$\begin{aligned} \mathcal{E}_{g,\check{\phi},f}(x) &= G'(\Psi)^2 \left[ 2G'(\Psi) \operatorname{Hess}_g(\Psi)(\nabla_g \Psi, \nabla_g \Psi) + 2G''(\Psi) |\nabla_g \Psi|_g^4 \right. \\ &\quad \left. + \left( G'(\Psi) \Delta_g \Psi + G''(\Psi) |\nabla_g \Psi|_g^2 - f \right) |\nabla_g \Psi|_g^2 \right]. \end{aligned}$$

Now, recalling the choice  $f = 2G''(\Psi) |\nabla_g \Psi|_g^2$  concludes the proof of the lemma.  $\square$

### 4.3 Conjugation with a partially Gevrey function

In [Tat95, RZ98, Hör97, Tat99b] part of the difficulty consists in defining an appropriate conjugated operator even in the case where the coefficients of  $P$  depend analytically on the time variable. Here, we exploit the anisotropic nature of  $P$  to allow conjugation with  $s$ -Gevrey in time functions, for an appropriate  $s > 1$  adapted to the scaling of the Schrödinger operator. Our strategy is based on the proof of Proposition 4.1 in [Tat99b].

#### 4.3.1 Gevrey functions and Banach valued symbols

For notations, definitions and basic properties of Gevrey functions we follow [BJ20]. We recall Definition 6 where the space  $\mathcal{G}^s(\Omega; \mathcal{B})$  of Gevrey  $s$  Banach valued is defined. For  $f \in \mathcal{G}^s(\Omega; \mathcal{B})$  and  $K \subset \Omega \subset \mathbb{R}^d$  a compact set, we write

$$\|f\|_{s,R,K} := \sup_{\alpha \in \mathbb{N}^d} \max_{t \in K} \frac{\|\partial^\alpha f(t)\|_{\mathcal{B}}}{R^{|\alpha|} \alpha!^s}. \quad (4.3.1)$$

In what follows, we only consider the case  $d = 1$  ( $t$  being the time variable) and  $d = 2$  for extensions to  $\mathbb{C} \simeq \mathbb{R}^2$  of such Gevrey functions. Note that if the compact set  $K$  is given and  $R > 0$  is fixed, the set of functions  $f \in C^\infty(K; \mathcal{B})$  satisfying (4.1.6), which we denote  $\mathcal{G}^{s,R}(K; \mathcal{B})$ , is a Banach space normed by (4.3.1).

Given a compact interval  $K \subset \mathbb{R}$  and  $\rho > 0$ , we write

$$K_\rho := \{x \in \mathbb{R}, \operatorname{dist}(x, K) \leq \rho\} \subset \mathbb{R}, \quad \text{and} \quad K'_\rho := K_\rho + i[-\rho, \rho] \subset \mathbb{C}. \quad (4.3.2)$$

The following lemma is (a particular case of) a straightforward adaptation of Lemma 1.2 and Remark 1.7 in [BJ20] to the case of Banach valued functions.

**Lemma 4.3.1.** Fix  $s > 1$ . For any compact interval  $K \subset \mathbb{R}$  and  $\rho > 0$ , there exist  $C_0, A > 0$  such that for any  $R > 0$ , there exist  $C > 0$  and a continuous linear map

$$\begin{aligned} \mathcal{G}^{s,R}(K; \mathcal{B}) &\rightarrow \mathcal{G}^{s,AR}(K'_\rho; \mathcal{B}) \\ f &\mapsto \tilde{f}, \end{aligned}$$

such that for all  $f \in \mathcal{G}^{s,R}(K; \mathcal{B})$ ,

$$\tilde{f}(t) = f(t) \text{ for } t \in K, \quad \|\tilde{f}\|_{s,AR,K'_\rho} \leq C \|f\|_{s,R,K}, \quad (4.3.3)$$

$$\|\partial_{\bar{z}} \tilde{f}(z)\|_{\mathcal{B}} \leq C \|f\|_{s,R,K} \exp\left(-\frac{1}{C_0(R|\operatorname{Im}(z)|)^{\frac{1}{s-1}}}\right), \quad \text{for } z \in K'_\rho. \quad (4.3.4)$$

Estimate (4.3.4) translates the fact that  $\tilde{f}$  is an *almost analytic extension* of  $f$  well-adapted to the Gevrey regularity  $\mathcal{G}^s$ .

Consider now  $\mathcal{X}, \mathcal{Y}$  two separable Hilbert spaces and denote by  $\mathcal{L}(\mathcal{X}, \mathcal{Y})$  the space of bounded operators from  $\mathcal{X}$  to  $\mathcal{Y}$ , which is a Banach space as well for  $\|\cdot\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})}$ . We recall some facts of semiclassical analysis in dimension 1 with values in  $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ . We consider a family of symbols depending on a (small) parameter  $h \in (0, 1)$ . We say that  $a \in S^m(\mathbb{R} \times \mathbb{R}; \mathcal{L}(\mathcal{X}, \mathcal{Y}))$  if  $a \in C^\infty(\mathbb{R} \times \mathbb{R}; \mathcal{L}(\mathcal{X}, \mathcal{Y}))$  depends implicitly on  $h \in (0, 1)$  and satisfies: for all  $\alpha, \beta \in \mathbb{N}^n$  there is  $C_{\alpha\beta} > 0$  such that

$$\left\| \partial_x^\alpha \partial_\xi^\beta a(t, \xi, h) \right\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})} \leq C_{\alpha\beta} \langle \xi \rangle^{m-\beta}, \quad \text{for all } (t, \xi, h) \in \mathbb{R} \times \mathbb{R} \times (0, 1).$$

We then quantify (using the Weyl quantization) such a symbol as

$$(\operatorname{op}^w(a)u)(t) := \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} e^{i(t-s)\xi} a\left(\frac{t+s}{2}, \xi\right) u(s) ds d\xi. \quad (4.3.5)$$

According to [Hör85, Paragraph 18.1 Remark 2 p 117],

- for all  $a \in S^m(\mathbb{R} \times \mathbb{R}; \mathcal{L}(\mathcal{X}, \mathcal{Y}))$ ,  $\operatorname{op}^w(a)$  maps continuously  $\mathcal{S}(\mathbb{R}; \mathcal{X})$  into  $\mathcal{S}(\mathbb{R}; \mathcal{Y})$  uniformly in  $h \in (0, 1)$ ;
- for all  $a \in S^0(\mathbb{R} \times \mathbb{R}; \mathcal{L}(\mathcal{X}, \mathcal{Y}))$ ,  $\operatorname{op}^w(a)$  maps continuously  $L^2(\mathbb{R}; \mathcal{X})$  into  $L^2(\mathbb{R}; \mathcal{Y})$  uniformly in  $h \in (0, 1)$ .

If  $a \in S^0(\mathbb{R} \times \mathbb{R}; \mathcal{L}(\mathcal{X}, \mathcal{Y}))$  has compact support in  $\mathbb{R} \times \mathbb{R}$  (with support possibly depending on the parameter  $h \in (0, 1)$ ), then

$$(\operatorname{op}^w(a)u)(t) = \int_{\mathbb{R}} \mathcal{K}(t, s) u(s) ds, \quad \mathcal{K}(t, s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(t-s)\xi} a\left(\frac{t+s}{2}, \xi\right) d\xi,$$

where the Schwartz kernel  $\mathcal{K}$  of the operator  $\operatorname{op}^w(a)$  satisfies  $\mathcal{K} \in C^\infty(\mathbb{R} \times \mathbb{R}; \mathcal{L}(\mathcal{X}, \mathcal{Y}))$ .

Note that in the application we have in mind, for a domain  $\Omega \subset \mathbb{R}^n$ , we choose  $\mathcal{X} = \mathcal{Y} = L^2(\Omega)$  and  $\mathcal{B} = L^\infty(\Omega)$  and observe the embedding  $L^\infty(\Omega) = \mathcal{B} \hookrightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y}) = \mathcal{L}(L^2(\Omega))$  (via the application that maps to a bounded function  $f$  the multiplication operator by  $f$ ) with  $\|\cdot\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})} \leq \|\cdot\|_{\mathcal{B}}$ .

### 4.3.2 The conjugated operator

In this section we define for  $t_0 \in \mathbb{R}$  and  $r_0 > 0$  the compact interval  $K := [t_0 - r_0, t_0 + r_0]$ . The compact  $K$  and also the radius  $R$ , used in definition (4.3.1) will be fixed for the rest of this section.

Given  $f \in \mathcal{G}^{s,R}(K; \mathcal{L}(\mathcal{X}, \mathcal{Y}))$  and  $\rho > 0$  we denote by  $\tilde{f}(z)$  the almost analytic extension of  $f$  in  $K'_\rho$  given by Lemma 4.3.1 (which depends implicitly on the parameter  $\rho$ ).

Along the proof, we will need some cut-off functions satisfying the following properties:  $\chi^0 \in C_c^\infty((-4, 4); [0, 1])$  with  $\chi = 1$  in a neighborhood of  $[-3, 3]$ ,  $\theta^0 \in C_c^\infty((-1, 1); [0, 1])$  and  $\eta^0 \in C_c^\infty((-3, 3); [0, 1])$  with  $\eta = 1$  in a neighborhood of  $[-2, 2]$ .

Take now  $r$  with  $0 < r < \min(\frac{r_0}{4}, \frac{\rho}{3})$ . We will define  $\chi(t) = \chi^0((t - t_0)/r)$ ,  $\theta(t) = \theta^0((t - t_0)/r)$  and  $\eta(\xi) = \eta^0(\xi/r)$ . In particular, they satisfy

- $\chi \in C_c^\infty((t_0 - 4r, t_0 + 4r); [0, 1])$  with  $\chi = 1$  in a neighborhood of  $[t_0 - 3r, t_0 + 3r]$
- $\theta \in C_c^\infty((t_0 - r, t_0 + r); [0, 1])$
- $\eta \in C_c^\infty((-3r, 3r); [0, 1])$  with  $\eta = 1$  in a neighborhood of  $[-2r, 2r]$ .

The functions  $\chi$ ,  $\theta$  and  $\eta$  depend implicitly on  $r$  and  $t_0$ , but we will not write anymore this dependence for better readability.

With  $h \in (0, 1)$ , we set

$$\begin{aligned} \tilde{f}^r(z) &:= \chi(\operatorname{Re} z) \eta(h^{-1/3} \operatorname{Im} z) \tilde{f}(z), \quad z \in \mathbb{C} \quad \text{and hence} \quad (4.3.6) \\ \tilde{f}^r(t + ih\xi) &= \chi(t) \eta(h^{2/3} \xi) \tilde{f}(t + ih\xi), \quad (t, \xi) \in \mathbb{R} \times \mathbb{R}. \end{aligned}$$

Observe that the function  $(t, \xi) \mapsto \tilde{f}^r(t + ih\xi)$  is smooth, compactly supported in  $\mathbb{R} \times \mathbb{R}$ , and belongs to  $S^0(\mathbb{R} \times \mathbb{R}; \mathcal{L}(\mathcal{X}, \mathcal{Y}))$ . According to the above discussion, we define the operator

$$F_h := \operatorname{op}^w(\tilde{f}^r(t + ih\xi)). \quad (4.3.7)$$

It maps continuously  $\mathcal{S}(\mathbb{R}; \mathcal{X})$  into  $\mathcal{S}(\mathbb{R}; \mathcal{Y})$  uniformly in  $h \in (0, 1)$  and  $L^2(\mathbb{R}; \mathcal{X})$  into  $L^2(\mathbb{R}; \mathcal{Y})$  uniformly in  $h \in (0, 1)$ .

We are now ready to state the following result, which guarantees that we have a reasonable conjugate for the operator  $e^{-\frac{h}{2}|D_t|^2} f$ .

**Proposition 4.3.2.** *Let  $\rho > 0$  and  $0 < r < \min(\frac{r_0}{4}, \frac{\rho}{3})$ . For any  $f \in \mathcal{G}^{2,R}(K; \mathcal{L}(\mathcal{X}, \mathcal{Y}))$ , we define the operator  $F_h$  by (4.3.7). Then there exist  $C = C(K, \rho, r)$  and  $c = c(K, \rho, r)$  such that there exist  $C > 0$  and  $h_0 > 0$  such that for all  $u \in \mathcal{S}(\mathbb{R}; \mathcal{X})$  one has:*

$$\left\| \chi F_h e^{-\frac{h}{2}|D_t|^2} \theta u - e^{-\frac{h}{2}|D_t|^2} f \theta u \right\|_{L^2(\mathbb{R}; \mathcal{Y})} \leq C \|f\|_{2,R,K} e^{-\frac{ch^{-1/3}}{R}} \|u\|_{L^2(\mathbb{R}; \mathcal{X})},$$

for all  $0 < h \leq h_0$ .

**Remark 4.3.3.** Taking  $h = 1/\tau^3$  one sees that  $F_h$  is a conjugated operator for  $f$  by  $e^{-\frac{h}{2}|D_t|^2}$  up to an exponentially small remainder controlled by  $e^{-\tau} \|u\|_{L^2}$ , which is an admissible error in the Carleman estimate. Notice that with this scaling, the frequency cut-off  $\eta$  localizes in frequencies smaller than  $h^{-2/3} \sim \tau^2$ . This is coherent with the proof of the Carleman estimate as well as with its quantitative counterpart, where we see that the anisotropy of  $P$  allows to advance further in the low frequency regime than for a wave-type operator.

**Remark 4.3.4.** Proposition 4.3.2 provides with a substitute of Lemma 4.2.1 in the case where  $t$  is replaced by a Gevrey function.

**Lemma 4.3.5.** *Setting*

$$R_h := \chi F_h e^{-\frac{h}{2}|D_t|^2} \theta - \chi e^{-\frac{h}{2}|D_t|^2} f \theta \in \mathcal{L}(L^2(\mathbb{R}, \mathcal{X}), L^2(\mathbb{R}, \mathcal{Y})), \quad (4.3.8)$$

we have

$$(R_h u)(t) = \int_{\mathbb{R}} \mathcal{K}_h(t, s) u(s) ds, \quad u \in \mathcal{S}(\mathbb{R}, \mathcal{X}) \text{ with} \quad (4.3.9)$$

$$\mathcal{K}_h(t, s) = -\frac{1}{2\pi} \mathcal{I}_{1,h} + C_h \mathcal{I}_{2,h}(t, s) \quad C_h := \frac{1}{2\pi} \left( \frac{1}{2\pi h} \right)^{1/2}, \quad \text{and}$$

$$\mathcal{I}_{1,h}(t, s) := \chi(t) \theta(s) f(s) \int_{\mathbb{R}} e^{-i(s-t)\xi} (1 - \eta(h^{2/3}\xi)) e^{-\frac{h|\xi|^2}{2}} d\xi, \quad (4.3.10)$$

$$\mathcal{I}_{2,h}(t, s) := \chi(t) \theta(s) \int_{\mathbb{R} \times \mathbb{R}} \left( \tilde{f}^r \left( \frac{t+w}{2} + ih\xi \right) - \eta(h^{2/3}\xi) f(s) \right) e^{i(t-w)\xi} e^{-\frac{|w-s|^2}{2h}} dw d\xi. \quad (4.3.11)$$

*Proof of Lemma 4.3.5.* Recalling the definition of the Weyl quantization in (4.3.5) and that of  $F_h$  in (4.3.7), we have

$$(F_h u)(t) = \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} e^{i(t-w)\xi} \tilde{f}^r \left( \frac{t+w}{2} + ih\xi \right) u(w) dw d\xi.$$

Combined with formula (4.A.2), this implies

$$\begin{aligned} & (\chi F_h e^{-\frac{h}{2}|D_t|^2} \theta u)(t) \\ &= \frac{1}{2\pi} \cdot \left( \frac{1}{2\pi h} \right)^{1/2} \chi(t) \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}} e^{i(t-w)\xi} \tilde{f}^r \left( \frac{t+w}{2} + ih\xi \right) \theta(s) u(s) e^{-\frac{|w-s|^2}{2h}} dw d\xi ds. \end{aligned} \quad (4.3.12)$$

Using again formula (4.A.2) as well as the formula for the Fourier transform of a Gaussian (4.A.1) we find

$$\begin{aligned} (\chi e^{-\frac{h}{2}|D_t|^2} f \theta u)(t) &= \left( \frac{1}{2\pi h} \right)^{1/2} \chi(t) \int_{\mathbb{R}} f(s) \theta(s) u(s) e^{-\frac{|t-s|^2}{2h}} ds \\ &= \left( \frac{1}{2\pi h} \right)^{1/2} \chi(t) \int_{\mathbb{R}} f(s) \theta(s) u(s) \left( \frac{1}{2\pi} (2\pi h)^{1/2} \int_{\mathbb{R}} e^{-i(s-t)\xi} e^{-\frac{h|\xi|^2}{2}} d\xi \right) ds \\ &= \left( \frac{1}{2\pi h} \right)^{1/2} \chi(t) \int_{\mathbb{R}} f(s) \theta(s) u(s) \left( \frac{1}{2\pi} (2\pi h)^{1/2} \int_{\mathbb{R}} e^{-i(s-t)\xi} \eta(h^{2/3}\xi) e^{-\frac{h|\xi|^2}{2}} d\xi \right) ds \\ &\quad + \left( \frac{1}{2\pi h} \right)^{1/2} \chi(t) \int_{\mathbb{R}} f(s) \theta(s) u(s) \left( \frac{1}{2\pi} (2\pi h)^{1/2} \int_{\mathbb{R}} e^{-i(s-t)\xi} (1 - \eta(h^{2/3}\xi)) e^{-\frac{h|\xi|^2}{2}} d\xi \right) ds. \end{aligned}$$

We now use once more (4.A.1) in order to replace  $e^{-\frac{h|\xi|^2}{2}}$  by  $(\frac{1}{2\pi h})^{1/2} \int_{\mathbb{R}} e^{-iw\xi} e^{-\frac{|w|^2}{2h}} dw$  in the first term of the sum above. We find then:

$$\begin{aligned} & (\chi e^{-\frac{h}{2}|D_t|^2} f \theta u)(t) \\ &= \frac{1}{2\pi} \left( \frac{1}{2\pi h} \right)^{1/2} \chi(t) \int_{\mathbb{R}} f(s) \theta(s) u(s) \left( \int_{\mathbb{R}} e^{-i(s-t)\xi} \eta(h^{2/3}\xi) \int_{\mathbb{R}} e^{-iw\xi} e^{-\frac{|w|^2}{2h}} dw d\xi \right) ds \\ &\quad + \frac{1}{2\pi} \chi(t) \int_{\mathbb{R}} f(s) \theta(s) u(s) \left( \int_{\mathbb{R}} e^{-i(s-t)\xi} (1 - \eta(h^{2/3}\xi)) e^{-\frac{h|\xi|^2}{2}} d\xi \right) ds. \end{aligned} \quad (4.3.13)$$

We finally perform the change of variable  $w \rightarrow w - s$  in the integral with respect to  $w$  to express the first term in (4.3.13) in the following way:

$$\begin{aligned} & \frac{1}{2\pi} \left( \frac{1}{2\pi h} \right)^{1/2} \chi(t) \int_{\mathbb{R}} f(s) \theta(s) u(s) \left( \int_{\mathbb{R}} e^{-i(s-t)\xi} \eta(h^{2/3}\xi) \int_{\mathbb{R}} e^{-iw\xi} e^{-\frac{|w|^2}{2h}} dw d\xi \right) ds \\ &= \frac{1}{2\pi} \left( \frac{1}{2\pi h} \right)^{1/2} \chi(t) \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}} e^{i(t-w)\xi} \eta(h^{2/3}\xi) f(s) \theta(s) u(s) e^{-\frac{|w-s|^2}{2h}} dw d\xi ds. \end{aligned} \quad (4.3.14)$$

The result is then a consequence of (4.3.12), (4.3.13) and (4.3.14).  $\square$

As a preparation for the proof of Proposition 4.3.2, we first estimates the terms  $\mathcal{I}_{j,h}$  in (4.3.10)–(4.3.11).

**Lemma 4.3.6.** *Let  $\rho > 0$  and  $r$  as in Proposition 4.3.2. For any  $f \in \mathcal{G}^{2,R}(K; \mathcal{L}(\mathcal{X}, \mathcal{Y}))$ , define  $\mathcal{I}_{1,h}, \mathcal{I}_{2,h}$  as in (4.3.10) and (4.3.11). Then, there exist  $C, c, h_0 > 0$  depending on  $K, \rho, r$  such that*

$$\|\mathcal{I}_{j,h}\|_{L^\infty(\mathbb{R} \times \mathbb{R}; \mathcal{L}(\mathcal{X}; \mathcal{Y}))} \leq C \|f\|_{2,R,K} e^{-\frac{ch^{-1/3}}{R}}, \quad j = 1, 2,$$

for all  $0 < h \leq h_0$ .

*Proof of Lemma 4.3.6. Proof for  $j = 1$*

We start with the proof for  $j = 1$ . We remark that in the support of  $1 - \eta(h^{2/3}\xi)$  one has  $h^{2/3}|\xi| \geq 2r$  which implies that  $h|\xi|^2 \geq ch^{-1/3}$  in the support of  $1 - \eta(h^{2/3}\xi)$ . We estimate then, for  $h \leq h_0$  with  $h_0$  sufficiently small:

$$\begin{aligned} \|\mathcal{I}_{1,h}(t, s)\|_{\mathcal{L}(\mathcal{X}; \mathcal{Y})} &\leq \|\mathbb{1}_{\text{supp } \theta} f(s)\|_{\mathcal{L}(\mathcal{X}; \mathcal{Y})} \int_{\mathbb{R}} \left| (1 - \eta(h^{2/3}\xi)) e^{-\frac{h|\xi|^2}{2}} \right| d\xi \\ &\leq C \|f\|_{L^\infty(\text{supp}(\theta); \mathcal{L}(\mathcal{X}; \mathcal{Y}))} \int_{\mathbb{R}} \left| (1 - \eta(h^{2/3}\xi)) e^{-\frac{h|\xi|^2}{4}} e^{-\frac{h|\xi|^2}{4}} \right| d\xi \\ &\leq C e^{-ch^{-1/3}} \int_{\mathbb{R}} e^{-\frac{h|\xi|^2}{4}} d\xi \|f\|_{L^\infty(\text{supp}(\theta); \mathcal{L}(\mathcal{X}; \mathcal{Y}))} \\ &\leq C e^{-ch^{-1/3}} \|f\|_{L^\infty(\text{supp}(\theta); \mathcal{L}(\mathcal{X}; \mathcal{Y}))} \leq C e^{-ch^{-1/3}} \|f\|_{2,R,K}, \end{aligned} \quad (4.3.15)$$

where we used the fact that  $f$  is Gevrey (and hence continuous) and  $\theta$  is compactly supported in  $K$ .

**Proof for  $j = 2$**

We now turn our attention to  $\mathcal{I}_{2,h}(t, s)$ . In the definition of  $\mathcal{I}_{2,h}(t, s)$  we change variable by writing  $(w, \xi) \in \mathbb{R}^2 \rightarrow z \in \mathbb{C}$  with

$$z = \frac{t+w}{2} + ih\xi, \quad \text{whence} \quad w = 2\text{Re}(z) - t, \quad h\xi = \text{Im}(z), \quad \text{and} \quad dw \wedge d\xi = \frac{i}{h} dz \wedge d\bar{z}. \quad (4.3.16)$$

The factor  $e^{i(t-w)\xi} e^{-\frac{|w-s|^2}{2h}}$  rewrites as  $e^{i(t-w)\xi} e^{-\frac{|w-s|^2}{2h}} = e^{\frac{1}{h}\Phi(t,s,z)}$ , with

$$\begin{aligned}\Phi(t, s, z) &= i(t-w)h\xi - \frac{(w-s)^2}{2} = 2i(t - \operatorname{Re}(z)) \operatorname{Im}(z) - \frac{(2 \operatorname{Re}(z) - t - s)^2}{2} \\ &= 2it \operatorname{Im}(z) - 2i \operatorname{Re}(z) \operatorname{Im}(z) - 2 \operatorname{Re}(z)^2 - \frac{t^2 + s^2}{2} + 2t \operatorname{Re}(z) + 2s \operatorname{Re}(z) - ts \\ &= 2tz + s(z + \bar{z} - t) - (z + \bar{z})z - \frac{t^2 + s^2}{2} \\ &= -\frac{(t-s)^2}{2} + (z-s)(2t - z - \bar{z}).\end{aligned}\quad (4.3.17)$$

Then, recalling the definition of  $\tilde{f}^r(z)$  in (4.3.6), we can write  $\mathcal{I}_{2,h}$  as

$$\mathcal{I}_{2,h}(t, s) = \frac{i}{h} \chi(t) \theta(s) \int \eta(h^{-1/3} \operatorname{Im} z) \left( \chi(\operatorname{Re} z) \tilde{f}(z) - f(s) \right) e^{-\frac{|t-s|^2}{2h}} e^{\frac{1}{h}(z-s)(2t-z-\bar{z})} dz \wedge d\bar{z}.\quad (4.3.18)$$

Defining

$$\check{b}_s(z) = \theta(s) \frac{\chi(\operatorname{Re} z) \tilde{f}(z) - f(s)}{z-s},\quad (4.3.19)$$

we may rewrite

$$\mathcal{I}_{2,h}(t, s) = -i\chi(t) \int_{\mathbb{C}} \eta(h^{-1/3} \operatorname{Im} z) \check{b}_s(z) \partial_{\bar{z}} \left( e^{-\frac{|t-s|^2}{2h}} e^{\frac{1}{h}(z-s)(2t-z-\bar{z})} \right) dz \wedge d\bar{z}.\quad (4.3.20)$$

We will now check that we are in position to integrate by parts using Lemma 4.A.4.

First, we prove that  $\check{b}_s \in C^1(\mathbb{C})$ . It is smooth away from  $s$ , so we only need to check the regularity close to  $z = s$ . We decompose  $\check{b}_s(z) = \theta(s) \chi(\operatorname{Re} z) \frac{\tilde{f}(z) - f(s)}{z-s} - \theta(s) (1 - \chi)(\operatorname{Re} z) \frac{f(s)}{z-s}$ . The first term is  $C^1(\mathbb{C})$  thanks to Lemma 4.A.3 applied to  $\tilde{f}(\cdot - s)$ . For the second term, we observe that for  $s \in (t_0 - r, t_0 + r)$  in the support of  $\theta$  and for  $\operatorname{Re}(z) \notin (t_0 - 3r, t_0 + 3r)$  in the support of  $1 - \chi$ , we have  $|z - s| \geq |\operatorname{Re}(z) - s| \geq 2r$ . This gives the regularity of the second term.

Now, we will need a few estimates to get the integrability of the terms considered in the integration by parts. We consider first the exponential term. We compute its phase.

$$-\frac{(t-s)^2}{2} + (z-s)(2t-z-\bar{z}) = 2i(t - \operatorname{Re}(z)) \operatorname{Im}(z) - \frac{(2 \operatorname{Re}(z) - t - s)^2}{2}.\quad (4.3.21)$$

Since  $(2 \operatorname{Re}(z) - t - s)^2 \geq \operatorname{Re}(z)^2 - C_{t,s}$  for some well chosen constants, we get

$$\left| e^{-\frac{|t-s|^2}{2h}} e^{\frac{1}{h}(z-s)(2t-z-\bar{z})} \right| \leq e^{\frac{C_{t,s}}{h}} e^{-\frac{\operatorname{Re}(z)^2}{2h}}.\quad (4.3.22)$$

We get also

$$\begin{aligned}\left| \partial_{\bar{z}} \left( e^{-\frac{|t-s|^2}{2h}} e^{\frac{1}{h}(z-s)(2t-z-\bar{z})} \right) \right| &= |z-s| \left| e^{-\frac{|t-s|^2}{2h}} e^{\frac{1}{h}(z-s)(2t-z-\bar{z})} \right| \\ &\leq e^{\frac{C_{t,s}}{h}} (|\operatorname{Im}(z)| + |\operatorname{Re}(z) - s|) e^{-\frac{\operatorname{Re}(z)^2}{2h}}.\end{aligned}\quad (4.3.23)$$

Since  $\eta$  localizes the imaginary part in a compact set and now (4.3.22) and (4.3.23) are obtained, we are left to prove  $L^\infty$  estimates on  $\check{b}_s(z)$  and  $\partial_{\bar{z}}\check{b}_s$ .

We have

$$\left\| \check{b}_s(z) \right\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} \leq \left\| \tilde{f} \right\|_{W^{1,\infty}(K'_\rho;\mathcal{L}(\mathcal{X};\mathcal{Y}))}, \quad \text{for } \operatorname{Re}(z) \in (t_0 - 3r, t_0 + 3r),$$

since  $\chi(\operatorname{Re} z) = 1$  for such  $z$ . For  $\operatorname{Re}(z) \notin (t_0 - 3r, t_0 + 3r)$  and  $s \in \operatorname{supp} \theta$ , we have  $|z - s| \geq 2r$ , which implies

$$\left\| \check{b}_s(z) \right\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} \leq C \left\| \tilde{f} \right\|_{L^\infty(K'_\rho;\mathcal{L}(\mathcal{X};\mathcal{Y}))}, \quad \text{for } \operatorname{Re}(z) \notin (t_0 - 3r, t_0 + 3r),$$

with a constant  $C$  depending only on  $r$ . Putting the two estimates above together we obtain that  $\check{b}_s \in C_b^0(\mathbb{C})$  and there is  $C = C(r) > 0$  such that

$$\left\| \check{b}_s(z) \right\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} \leq C \left\| \tilde{f} \right\|_{W^{1,\infty}(K'_\rho;\mathcal{L}(\mathcal{X};\mathcal{Y}))}, \quad z \in \mathbb{C}. \quad (4.3.24)$$

Secondly, we compute

$$\partial_{\bar{z}}\check{b}_s(z) = \theta(s) \frac{\chi'(\operatorname{Re} z)}{2(z-s)} \tilde{f}(z) + \theta(s) \frac{\chi(\operatorname{Re} z) \partial_{\bar{z}} \tilde{f}(z)}{z-s}, \quad (4.3.25)$$

and notice that the first term is smooth and bounded given the relative support properties of  $\theta$  and  $\chi'$ . For the second term, using (4.3.4) for Gevrey 2 functions and the fact that  $s \in \mathbb{R}$ , we obtain, for  $z \in K'_\rho$  (the value of the constant  $C$  may change from one line to another):

$$\begin{aligned} \left\| \frac{\chi(\operatorname{Re} z) \partial_{\bar{z}} \tilde{f}(z)}{z-s} \right\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} &\leq \frac{1}{|z-s|} C \|f\|_{2,R,K} \exp\left(-\frac{1}{C_0 R |\operatorname{Im} z|}\right) \\ &\leq \frac{1}{|\operatorname{Im} z|} C \|f\|_{2,R,K} \exp\left(-\frac{1}{C_0 R |\operatorname{Im} z|}\right) \\ &\leq C \|f\|_{2,R,K} \exp\left(-\frac{1}{2C_0 R |\operatorname{Im} z|}\right). \end{aligned} \quad (4.3.26)$$

Combining the previous estimate and (4.3.25), we get

$$\left\| \partial_{\bar{z}}\check{b}_s(z) \right\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} \leq C \left\| \tilde{f} \right\|_{L^\infty(K'_\rho;\mathcal{L}(\mathcal{X};\mathcal{Y}))} + C \|f\|_{2,R,K}, \quad z \in \mathbb{C}.$$

As announced before, the  $L^\infty$  bounds on  $\partial_{\bar{z}}\check{b}_s$  and  $\check{b}_s$ , combined with the localization of  $\eta$ , (4.3.22) and (4.3.23) give the integrability of all the terms involved in the integration by parts. All assumptions of Lemma 4.A.7 are therefore satisfied and we may now integrate by parts in (4.3.20), yielding

$$\mathcal{I}_{2,h}(t, s) = i\chi(t) \int_{\mathbb{C}} \partial_{\bar{z}} \left( \eta(h^{-1/3} \operatorname{Im} z) \check{b}_s(z) \right) e^{-\frac{|t-s|^2}{2h}} e^{\frac{1}{h}(z-s)(2t-z-\bar{z})} dz \wedge d\bar{z}. \quad (4.3.27)$$



Recalling (4.3.25), we now decompose (4.3.27) as

$$\begin{aligned} \mathcal{I}_{2,h} &= \mathcal{I}_{21,h} + \mathcal{I}_{22,h} + \mathcal{I}_{23,h}, \quad \text{with} \\ \mathcal{I}_{21,h}(t, s) &:= i\chi(t)\theta(s) \int_{\mathbb{C}} \eta(h^{-1/3} \operatorname{Im} z) \frac{\chi'(\operatorname{Re} z)}{2(z-s)} \tilde{f}(z) e^{-\frac{|t-s|^2}{2h}} e^{\frac{1}{h}(z-s)(2t-z-\bar{z})} dz \wedge d\bar{z}, \end{aligned} \quad (4.3.28)$$

$$\mathcal{I}_{22,h}(t, s) := i\chi(t)\theta(s) \int_{\mathbb{C}} \eta(h^{-1/3} \operatorname{Im} z) \frac{\chi(\operatorname{Re} z) \partial_{\bar{z}} \tilde{f}(z)}{z-s} e^{-\frac{|t-s|^2}{2h}} e^{\frac{1}{h}(z-s)(2t-z-\bar{z})} dz \wedge d\bar{z} \quad (4.3.29)$$

$$\mathcal{I}_{23,h}(t, s) := -\frac{1}{2} h^{-1/3} \chi(t) \int_{\mathbb{C}} \eta'(h^{-1/3} \operatorname{Im} z) \check{b}_s(z) e^{-\frac{|t-s|^2}{2h}} e^{\frac{1}{h}(z-s)(2t-z-\bar{z})} dz \wedge d\bar{z}. \quad (4.3.30)$$

We now estimate each term separately. We start with  $\mathcal{I}_{21,h}$  and rewrite the integral in the original variables (4.3.16) as

$$\mathcal{I}_{21,h}(t, s) = ih\chi(t)\theta(s) \int_{\mathbb{R} \times \mathbb{R}} \eta(h^{2/3}\xi) \chi' \left( \frac{t+w}{2} \right) \frac{\tilde{f} \left( \frac{t+w}{2} + ih\xi \right)}{2 \left( \frac{t+w}{2} + ih\xi - s \right)} e^{i(t-w)\xi} e^{-\frac{|w-s|^2}{2h}} dw d\xi.$$

Observe now that  $\operatorname{supp}(\chi') \subset (t_0 - 4r, t_0 - 3r) \cup (t_0 + 3r, t_0 + 4r)$ . Therefore the integrand above is supported in  $|\frac{t+w}{2} - t_0| \geq 3r$  (thanks to the support of  $\chi'$ ) and  $|t - t_0| < 4r$  (thanks to the support of  $\chi$ ). This implies that  $|w - t_0| \geq 2r$  for otherwise one would have

$$\left| \frac{t+w}{2} - t_0 \right| \leq \left| \frac{t-t_0}{2} \right| + \left| \frac{w-t_0}{2} \right| < 2r + r = 3r.$$

Since in the support of  $\theta$  we have  $|s - t_0| < r$  we find finally that  $|w - s| \geq r$  in the support of the integral. Notice finally that, if  $\chi' \left( \frac{t+w}{2} \right) \neq 0$  and  $\theta(s) \neq 0$  one has

$$\left| \frac{t+w}{2} + ih\xi - s \right| \geq \left| \frac{t+w}{2} - s \right| \geq \left| \frac{t+w}{2} - t_0 \right| - |t_0 - s| \geq 2r.$$

We can then estimate as follows:

$$\begin{aligned} \|\mathcal{I}_{21,h}\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} &\leq \frac{h}{4r} \|\tilde{f}\|_{L^\infty(K'_r; \mathcal{L}(\mathcal{X};\mathcal{Y}))} \int_{\mathbb{R} \times \mathbb{R}} |\eta(h^{2/3}\xi)| e^{-\frac{|w-s|^2}{4h}} e^{\frac{r^2}{4h}} dw d\xi \\ &\leq \frac{h}{4r} \|\tilde{f}\|_{L^\infty(K'_r; \mathcal{L}(\mathcal{X};\mathcal{Y}))} e^{-\frac{r^2}{4}h} \int_{[-3rh^{-2/3}, 3rh^{-2/3}]} d\xi \int_{\mathbb{R}} e^{-\frac{|w-s|^2}{4h}} dw. \end{aligned}$$

This implies the stronger bound

$$\|\mathcal{I}_{21,h}(t, s)\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} \leq C \|\tilde{f}\|_{L^\infty(K'_r; \mathcal{L}(\mathcal{X};\mathcal{Y}))} e^{-ch^{-1}} \leq C e^{-ch^{-1}} \|f\|_{2,R,K}, \quad (4.3.31)$$

where the last inequality follows from (4.3.3).

We now study the integral  $\mathcal{I}_{22,h}$  defined in (4.3.29). Recall that  $\operatorname{supp} \eta \subset [-3r, 3r]$ , so that the domain of integration is contained in  $|\operatorname{Im} z| \leq 3rh^{1/3}$ . Using (4.3.26), we can then estimate the corresponding integral as follows:

$$\begin{aligned} \|\mathcal{I}_{22,h}(t, s)\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} &\leq \int_{\mathbb{C}} \left\| \eta(h^{-1/3} \operatorname{Im} z) \frac{\chi(\operatorname{Re} z) \partial_{\bar{z}} \tilde{f}(z)}{z-s} e^{-\frac{|t-s|^2}{2h}} e^{\frac{1}{h}(z-s)(2t-z-\bar{z})} \right\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} |dz \wedge d\bar{z}| \\ &\leq C \|f\|_{2,R,K} \exp \left( -\frac{h^{-1/3}}{6rC_0R} \right) \int_{K'_r} \left| e^{-\frac{|t-s|^2}{2h}} e^{\frac{1}{h}(z-s)(2t-z-\bar{z})} \right| |dz \wedge d\bar{z}| \\ &\leq C \|f\|_{2,R,K} e^{-\frac{ch^{-1/3}}{R}}. \end{aligned} \quad (4.3.32)$$

In this last inequality of (4.3.32), we used the fact that

$$\int_{K'_\rho} \left| e^{-\frac{|t-s|^2}{2h}} e^{\frac{1}{h}(z-s)(2t-z-\bar{z})} \right| |dz \wedge d\bar{z}| \leq \int_{K'_\rho} |dz \wedge d\bar{z}| \leq C.$$

which follows from (4.3.21).

The last term we need to control is the integral  $\mathcal{I}_{23,h}$  in (4.3.30). In the original coordinates (4.3.16), we have

$$\mathcal{I}_{23,h}(t, s) = i \frac{h^{2/3}}{2} \chi(t) \int_{\mathbb{R} \times \mathbb{R}} \eta'(h^{2/3}\xi) \check{b}_s \left( \frac{t+w}{2} + ih\xi \right) e^{i(t-w)\xi} e^{-\frac{|w-s|^2}{2h}} dw d\xi.$$

We look at the integral in  $w$  and treat  $\xi$  as a parameter satisfying  $2rh^{-2/3} \leq |\xi| \leq 3rh^{-2/3}$  thanks to the support of  $\eta'$ . The change of variable  $w \rightarrow w + s$  allows to rewrite the integral as follows:

$$\int_{\mathbb{R}} \check{b}_s \left( \frac{t+w}{2} + ih\xi \right) e^{i(t-w)\xi} e^{-\frac{|w-s|^2}{2h}} dw = e^{-i(s-t)\xi} \int_{\mathbb{R}} g_{h\xi,t,s}(w) e^{-iw\xi} e^{-\frac{|w|^2}{2h}} dw, \quad (4.3.33)$$

$$\text{with } \tilde{g}_{\xi,t,s}(w) := \check{b}_s \left( \frac{t+s+w}{2} + i\tilde{\xi} \right). \quad (4.3.34)$$

Using (4.3.33), we obtain

$$\begin{aligned} \|\mathcal{I}_{23,h}(t, s)\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} &= \left\| \frac{h^{2/3}}{2} \chi(t) \int_{\mathbb{R} \times \mathbb{R}} \eta'(h^{2/3}\xi) \check{b}_s \left( \frac{t+w}{2} + ih\xi \right) e^{i(t-w)\xi} e^{-\frac{|w-s|^2}{2h}} dw d\xi \right\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} \\ &\leq \frac{1}{2} \chi(t) h^{2/3} \left\| \int_{\mathbb{R}} \eta'(h^{2/3}\xi) e^{-i(s-t)\xi} \left( \int_{\mathbb{R}} g_{h\xi,t,s}(w) e^{-iw\xi} e^{-\frac{|w|^2}{2h}} dw \right) d\xi \right\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} \\ &\leq \frac{1}{2} \chi(t) h^{2/3} \int_{\mathbb{R}} |\eta'(h^{2/3}\xi)| \left\| \int_{\mathbb{R}} g_{h\xi,t,s}(w) e^{-iw\xi} e^{-\frac{|w|^2}{2h}} dw \right\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} d\xi. \end{aligned}$$

Recalling that  $\text{supp } \chi \subset (t_0 - 4r, t_0 + 4r)$  together with the definition of  $g_{h\xi,t,s}$  in (4.3.34), of  $\check{b}_s$  in (4.3.19) and  $\text{supp } \chi \subset (t_0 - 4r, t_0 + 4r)$ , Lemma 4.3.7 (below) now implies

$$\begin{aligned} &\chi(t) \int_{\mathbb{R}} |\eta'(h^{2/3}\xi)| \left\| \int_{\mathbb{R}} g_{h\xi,t,s}(w) e^{-iw\xi} e^{-\frac{|w|^2}{2h}} dw \right\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} d\xi \\ &\leq C \int_{\mathbb{R}} |\eta'(h^{2/3}\xi)| d\xi e^{-\frac{ch^{-1/3}}{R}} \|f\|_{2,R,K}. \end{aligned}$$

Combining the two estimates above and recalling the support of  $\eta$  yields

$$\|\mathcal{I}_{23,h}(t, s)\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} \leq h^{2/3} \int_{-3rh^{-2/3}}^{3rh^{-2/3}} d\xi e^{-\frac{ch^{-1/3}}{R}} \|f\|_{2,R,K} \leq C e^{-\frac{ch^{-1/3}}{R}} \|f\|_{2,R,K}, \quad (4.3.35)$$

for  $h \leq h_0$ .

Putting together (4.3.31), (4.3.32) and (4.3.35) yields for some constants  $C$  and  $c$  depending only on  $K, \rho, r$ :

$$\|\mathcal{I}_{2,h}(t, s)\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} \leq C e^{-\frac{ch^{-1/3}}{R}} \|f\|_{2,R,K},$$

which concludes the proof of Lemma 4.3.6.  $\square$

We can now turn to the proof of Proposition 4.3.2.

*Proof of Proposition 4.3.2.* For  $u \in \mathcal{S}(\mathbb{R}; \mathcal{X})$ , we start by writing

$$\begin{aligned} \chi F_h e^{-\frac{h}{2}|D_t|^2} \theta u - e^{-\frac{h}{2}|D_t|^2} f \theta u &= \left( \chi F_h e^{-\frac{h}{2}|D_t|^2} \theta u - \chi e^{-\frac{h}{2}|D_t|^2} f \theta u \right) - (1 - \chi) e^{-\frac{h}{2}|D_t|^2} (f \theta u) \\ &= R_h u - (1 - \chi) e^{-\frac{h}{2}|D_t|^2} (f \theta u), \end{aligned} \quad (4.3.36)$$

where  $R_h$  is defined in (4.3.8). The second term in (4.3.36) is bounded using Lemma 4.A.1 by

$$\begin{aligned} \left\| (1 - \chi) e^{-\frac{h}{2}|D_t|^2} (f \theta u) \right\|_{L^2(\mathbb{R}; \mathcal{Y})} &\leq C e^{-c/h} \|f \theta u\|_{H^{-k}(\mathbb{R}; \mathcal{Y})} \\ &\leq C e^{-c/h} \|f\|_{W^{k, \infty}(\text{supp}(\theta); \mathcal{L}(\mathcal{X}, \mathcal{Y}))} \|u\|_{H^{-k}(\mathbb{R}; \mathcal{X})} \end{aligned} \quad (4.3.37)$$

thanks to the supports of  $(1 - \chi)$  and  $\theta$ . Concerning the first term in (4.3.36), the kernel of  $R_h$  is given  $\mathcal{K}_h(t, s)$  given by (4.3.9) according to Lemma 4.3.5. Since  $\mathcal{K}(t, s) = -\frac{1}{2\pi} \mathcal{I}_{1,h}(t, s) + C_h \mathcal{I}_{2,h}(t, s)$ , Lemmata 4.3.5 and 4.3.6 yield

$$\|\mathcal{K}(\cdot, \cdot)\|_{L^\infty(\mathbb{R} \times \mathbb{R}; \mathcal{L}(\mathcal{X}, \mathcal{Y}))} \leq C e^{-\frac{ch^{-1/3}}{R}} \|f\|_{2,R,K}. \quad (4.3.38)$$

Combining Lemmata 4.3.5 and 4.3.6 and recalling  $\text{supp } \mathcal{K}_h \subset (t_0 - 4r, t_0 + 4r) \times (t_0 - r, t_0 + r)$ , the Cauchy-Schwartz inequality yields

$$\|R_h u\|_{L^2(\mathbb{R}; \mathcal{Y})} = \left\| \int \mathcal{K}_h(\cdot, s) u(s) ds \right\|_{L^2(\mathbb{R}; \mathcal{Y})} \leq C e^{-\frac{ch^{-1/3}}{R}} \|f\|_{2,R,K} \|u\|_{L^2((t_0-r, t_0+r); \mathcal{X})}.$$

This, together with (4.3.36) and (4.3.37), implies

$$\left\| \chi F_h e^{-\frac{h}{2}|D_t|^2} \theta u - e^{-\frac{h}{2}|D_t|^2} f \theta u \right\|_{L^2} \leq C e^{-\frac{ch^{-1/3}}{R}} \|f\|_{2,R,K} \|u\|_{L^2},$$

and concludes the proof of Proposition 4.3.2.  $\square$

In the course of Lemma 4.3.6, we have used the following Lemma.

**Lemma 4.3.7.** *Let  $g_{h\xi,t,s}$  be as in (4.3.34) and fix  $c_2 > c_1 > 0$ . Then there exist  $C > 0$ ,  $c > 0$  and  $h_0$  depending on  $K, \rho, r, c_1, c_2$  such that for  $t \in (t_0 - 4r, t_0 + 4r)$ ,  $s \in \mathbb{R}$ ,  $h \in (0, h_0)$  and  $c_1 h^{-2/3} \leq |\xi| \leq c_2 h^{-2/3}$  one has:*

$$\left\| \int_{\mathbb{R}} g_{h\xi,t,s}(w) e^{-iw\xi} e^{-\frac{|w|^2}{2h}} dw \right\|_{\mathcal{L}(\mathcal{X}; \mathcal{Y})} \leq C e^{-\frac{ch^{-1/3}}{R}} \|f\|_{2,R,K}.$$

*Proof.* First, thanks to the definition of  $\check{b}_s$  and the support of  $\theta$ , we can assume without loss of generality that  $s \in (t_0 - r, t_0 + r)$ , for otherwise the integral is zero. We start by separating the integral in two terms:

$$\int_{\mathbb{R}} g_{h\xi,t,s}(w) e^{-iw\xi} e^{-\frac{|w|^2}{2h}} dw = \int_{|w| \geq r} g_{h\xi,t,s}(w) e^{-iw\xi} e^{-\frac{|w|^2}{2h}} dw + \int_{|w| \leq r} g_{h\xi,t,s}(w) e^{-iw\xi} e^{-\frac{|w|^2}{2h}} dw.$$

For the integral in  $|w| \geq r$  we can proceed as in (4.3.15) to obtain the stronger bound

$$\left\| \int_{|w| \geq r} g_{h\xi,t,s}(w) e^{-iw\xi} e^{-\frac{|w|^2}{2h}} dw \right\|_{\mathcal{L}(\mathcal{X}; \mathcal{Y})} \leq C e^{-ch^{-1}} \|\tilde{f}\|_{W^{1, \infty}(K'_\rho; \mathcal{L}(\mathcal{X}, \mathcal{Y}))} \leq C e^{-ch^{-1}} \|f\|_{2,R,K}.$$

thanks to (4.3.24).

We now work in the region  $|w| \leq r$  and remark that for  $t \in (t_0 - 4r, t_0 + 4r)$ ,  $s \in (t_0 - r, t_0 + r)$  and  $|w| \leq r$  one has for  $z = \frac{t+s+w}{2} + ih\xi$  that

$$|\operatorname{Re}(z) - t_0| \leq \left| \frac{t - t_0}{2} \right| + \left| \frac{s - t_0}{2} \right| + \left| \frac{w}{2} \right| \leq 3r,$$

and  $|\operatorname{Im}(z)| = h|\xi|$ . Therefore in this region we have  $\chi(\operatorname{Re} z) = 1$  and consequently

$$\check{b}_s(z) = \theta(s) \frac{\chi(\operatorname{Re} z) \tilde{f}(z) - f(s)}{z - s} = \theta(s) \frac{\tilde{f}(z) - f(s)}{z - s}.$$

This implies as in (4.3.26) that, for  $\operatorname{Im}(z) \leq h_0$ :

$$\left\| \partial_{\bar{z}} \check{b}_s(z) \right\|_{\mathcal{L}(\mathcal{X}; \mathcal{Y})} \leq C \|f\|_{2,R,K} \exp\left(-\frac{1}{2C_0 R |\operatorname{Im} z|}\right). \quad (4.3.39)$$

To alleviate the notation we write  $g$  for  $g_{h\xi, t, s}$ . We know thanks to (4.3.34) that  $g$  admits a complex extension in  $[-r, r] + i[-\rho/2, \rho/2]$  for  $h \leq h_0$  given by

$$g(w + iv) := \check{b}_s\left(\frac{t + s + w}{2} + ih\xi + \frac{iv}{2}\right),$$

that is

$$g(z) = \check{b}_s\left(\frac{z}{2} + \frac{t + s}{2} + ih\xi\right),$$

which implies

$$\partial_{\bar{z}} g(z) = \frac{1}{2} \partial_{\bar{z}} \check{b}_s\left(\frac{z}{2} + \frac{t + s}{2} + ih\xi\right). \quad (4.3.40)$$

We now write the integral we want to control as

$$\int_{-r}^r g(z) e^{-iz\xi} e^{-\frac{z^2}{2h}} dz = \int_{-r}^r g(z) e^{-\frac{h\xi^2}{2}} e^{-\frac{(z+ih\xi)^2}{2h}} dz.$$

We consider now  $\sigma \in (0, \frac{1}{2})$  to be chosen later on. We let  $\Omega = [-r, r] \times [-\sigma h\xi, 0]$  in case  $\xi \in [c_1 h^{-2/3}, c_2 h^{-2/3}]$  (see Figure 4.1), resp.  $\Omega = [-r, r] \times [0, -\sigma h\xi]$  in case  $\xi \in [-c_2 h^{-2/3}, -c_1 h^{-2/3}]$ . Stoke's theorem applies, see (4.A.5), and yields:

$$\begin{aligned} \int_{-r}^r g(z) e^{-\frac{h\xi^2}{2}} e^{-\frac{(z+ih\xi)^2}{2h}} dz &= \int_{\Gamma_1} g(z) e^{-\frac{h\xi^2}{2}} e^{-\frac{(z+ih\xi)^2}{2h}} dz + \int_{\Gamma_2} g(z) e^{-\frac{h\xi^2}{2}} e^{-\frac{(z+ih\xi)^2}{2h}} dz \\ &\quad + \int_{\Gamma_3} g(z) e^{-\frac{h\xi^2}{2}} e^{-\frac{(z+ih\xi)^2}{2h}} dz + \int_{\Omega} \partial_{\bar{z}}(g(z)) e^{-iz\xi} e^{-\frac{z^2}{2h}} dz \wedge d\bar{z}, \end{aligned} \quad (4.3.41)$$

where the contours (oriented counterclockwise, see Figure 4.1 in the case  $\xi > 0$ ) are defined by

$$\begin{aligned} \Gamma_1 &= \{z \in \mathbb{C}, \operatorname{Re} z = -r, \quad -\sigma h\xi \leq \operatorname{Im} z \leq 0\}, \\ \Gamma_2 &= \{z \in \mathbb{C}, -r \leq \operatorname{Re} z \leq r, \quad \operatorname{Im} z = -\sigma h\xi\}, \\ \Gamma_3 &= \{z \in \mathbb{C}, \operatorname{Re} z = r, \quad -\sigma h\xi \leq \operatorname{Im} z \leq 0\}, \end{aligned}$$

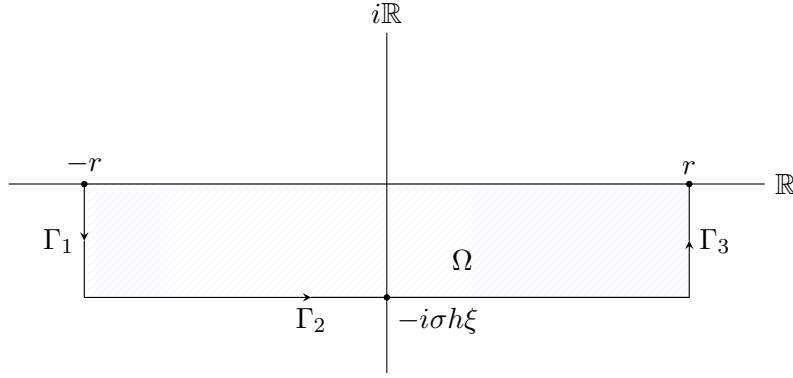


Figure 4.1: The domain  $\Omega$  where we apply Stokes' theorem in case  $\xi > 0$  (the picture in case  $\xi < 0$  is the symmetric about the real axis). Notice that  $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup [-r, r]$ . Recall as well that in this regime we have  $\xi \sim h^{-2/3}$  and therefore  $h\xi \sim h^{1/3}$ . As  $h$  goes to 0 the domain  $\Omega$  collapses to the segment  $[-r, r]$ .

if  $\xi > 0$  and

$$\begin{aligned}\Gamma_1 &= \{z \in \mathbb{C}, \operatorname{Re} z = -r, \quad 0 \leq \operatorname{Im} z \leq \sigma h \xi\}, \\ \Gamma_2 &= \{z \in \mathbb{C}, -r \leq \operatorname{Re} z \leq r, \quad \operatorname{Im} z = \sigma h \xi\}, \\ \Gamma_3 &= \{z \in \mathbb{C}, \operatorname{Re} z = r, \quad 0 \leq \operatorname{Im} z \leq \sigma h \xi\},\end{aligned}$$

if  $\xi < 0$ . We now estimate all terms in the right hand-side of (4.3.41).

We start with the last term in the right hand-side of (4.3.41). Using (4.3.40) and (4.3.39) together with the fact that  $z \in \Omega$  in particular  $|\operatorname{Im} z| \leq \sigma h |\xi| \leq \frac{1}{2} h |\xi|$  (since  $\sigma \leq \frac{1}{2}$ ), we obtain

$$\begin{aligned}\|\partial_{\bar{z}}(g(z))\|_{\mathcal{L}(\mathcal{X}; \mathcal{Y})} &\leq C \|f\|_{2,R,K} \exp\left(-\frac{1}{C_0 R |\operatorname{Im}(z/2) + h\xi|}\right) \\ &\leq C \|f\|_{2,R,K} \exp\left(-\frac{1}{C_0 R (|\operatorname{Im}(z/2)| + h|\xi|)}\right) \\ &\leq C \|f\|_{2,R,K} \exp\left(-\frac{1}{2C_0 R |h\xi|}\right) \\ &\leq C \|f\|_{2,R,K} \exp\left(-\frac{1}{2C_0 c_2 R h^{1/3}}\right) = C \|f\|_{2,R,K} e^{-\tilde{c} h^{-1/3}},\end{aligned}\quad (4.3.42)$$

where  $c_2$  is given by  $|\xi| \leq c_2 h^{-2/3}$  and  $\tilde{c} := \frac{1}{2C_0 c_2 R}$ . We write  $z = \alpha + i\beta$  with  $\alpha, \beta \in \mathbb{R}$  and notice that for  $z \in \Omega$  we have  $|\beta| \leq \sigma h |\xi|$  and  $\beta \xi < 0$  (in both cases). As a consequence, we deduce

$$\left| e^{-iz\xi} e^{-\frac{z^2}{2h}} \right| = e^{\beta\xi} e^{-\frac{\alpha^2 - \beta^2}{2h}} \leq e^{\frac{\sigma^2 h^2 |\xi|^2}{2h}} \leq e^{\frac{\sigma^2 h |\xi|^2}{2}} \leq e^{\frac{\sigma^2 c_2^2}{2h^{1/3}}}.$$

Together with (4.3.42) this yields

$$\begin{aligned}\int_{\Omega} \left\| \partial_{\bar{z}}(g(z)) e^{-iz\xi} e^{-\frac{z^2}{2h}} \right\|_{\mathcal{L}(\mathcal{X}; \mathcal{Y})} |dz \wedge d\bar{z}| &\leq C \|f\|_{2,R,K} e^{-(\tilde{c} - \sigma^2 c_2^2/2) h^{-1/3}} \\ &\leq C \|f\|_{2,R,K} e^{-\tilde{c}/2 h^{-1/3}},\end{aligned}$$

after having chosen  $\sigma := \min(\frac{\tilde{c}^{1/2}}{c_2}, \frac{1}{2})$ . With  $\sigma$  now fixed we control the other three terms in (4.3.41).

- For  $\alpha + i\beta = z \in \Gamma_1$  we have  $\alpha^2 = r^2$  and estimate the real part of the second exponential, using  $(\beta + h\xi)^2 \leq (h\xi)^2$  (in both cases  $-\sigma h\xi \leq \beta \leq 0$  if  $\xi \geq 0$  and  $0 \leq \beta \leq -\sigma h\xi$  if  $\xi < 0$ ), as

$$\operatorname{Re} \left( \frac{(z + ih\xi)^2}{2h} \right) = \frac{r^2 - (\beta + h\xi)^2}{2h} \geq \frac{r^2 - h^2\xi^2}{2h} \geq \frac{r^2 - c_2^2 h^{2/3}}{2h} \geq \frac{r^2}{4h} \geq 0,$$

for  $h$  sufficiently small. This implies

$$\begin{aligned} \int_{\Gamma_1} \left\| g(z) e^{-\frac{h\xi^2}{2}} e^{-\frac{(z+ih\xi)^2}{2h}} \right\|_{\mathcal{L}(\mathcal{X}; \mathcal{Y})} dz &\leq C \left\| \tilde{f} \right\|_{W^{1,\infty}(K'_\rho; \mathcal{L}(\mathcal{X}; \mathcal{Y}))} e^{-\frac{h\xi^2}{2}} \\ &\leq C \left\| \tilde{f} \right\|_{W^{1,\infty}(K'_\rho; \mathcal{L}(\mathcal{X}; \mathcal{Y}))} e^{-ch^{-1/3}}, \end{aligned} \quad (4.3.43)$$

thanks to (4.3.24).

- For the integral in  $\Gamma_3$  we proceed exactly as for  $\Gamma_1$ .
- For  $\alpha + i\beta = z \in \Gamma_2$  we have  $\beta = -\sigma h\xi$  and  $\alpha \in [-r, r]$ , and we obtain

$$\begin{aligned} \operatorname{Re} \left( \frac{h|\xi|^2}{2} + \frac{(z + ih\xi)^2}{2h} \right) &= \frac{h\xi^2}{2} + \frac{\alpha^2 - (\beta + h\xi)^2}{2h} \\ &\geq \frac{h\xi^2}{2} - \frac{(\beta + h\xi)^2}{2h} = \frac{h\xi^2}{2} (1 - (1 - \sigma)^2) \\ &\geq \frac{\sigma h\xi^2}{2} \geq \frac{\sigma c_1^2}{2} h^{-1/3}, \end{aligned}$$

for  $|\xi| \geq c_1 h^{-2/3}$ . The estimate of  $\int_{\Gamma_3}$  in (4.3.41) then proceeds as that of  $\int_{\Gamma_1}$  in (4.3.43).

This concludes the proof of Lemma 4.3.7.  $\square$

## 4.4 The uniqueness theorem

### 4.4.1 Adding partially Gevrey lower order terms

With the results of Section 4.3 at our disposition we can now add in the Carleman estimate of Theorem 4.2.1 lower order terms with coefficients which are Gevrey 2 with respect to  $t$  and bounded with respect to  $x$ . Let  $I \subset \mathbb{R}$  and  $V \subset \mathbb{R}^n$  be open sets and define  $\Omega := I \times V$ . Consider the operator

$$P_{\mathcal{G}} := i\partial_t + \Delta_g + \sum_j b_j(t, x) \partial_j + \mathbf{q}(t, x). \quad (4.4.1)$$

We suppose that the metric  $g$  is Lipschitz with coefficients independent of  $t$  and  $b_j, \mathbf{q} \in \mathcal{G}^2(I; L^\infty(V; \mathbb{C}))$ . We then have:

**Theorem 4.4.1** (The Carleman estimate with Gevrey terms). *Let  $\mathbf{x}_0 = (t_0, x_0) \in \Omega \subset \mathbb{R}^{1+d}$ . Assume that  $\phi$  and  $f$  satisfy the assumptions of Theorem 4.2.1. Then there exist  $r, d, C, \tau_0 > 0$  such that for all  $\tau \geq \tau_0$  and  $w \in C_c^\infty(B(\mathbf{x}_0, r))$ , we have*

$$C \left\| Q_{\mu, \tau}^\phi P_G w \right\|_{L^2}^2 + C e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{L_t^2 H_x^1}^2 \geq \tau \left\| Q_{\mu, \tau}^\phi w \right\|_{H_t^1}^2. \quad (4.4.2)$$

Moreover, if  $b_j = 0$ ,  $j \in \{1, \dots, d\}$  we have:

$$C \left\| Q_{\mu, \tau}^\phi P_G w \right\|_{L^2}^2 + C e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{H_t^{-1} H_x^1}^2 + C e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{L^2}^2 \geq \tau \left\| Q_{\mu, \tau}^\phi w \right\|_{H_t^1}^2. \quad (4.4.3)$$

*Proof.* We define  $R := \sum_j b_j \partial_j + \mathfrak{q}$  and estimate

$$\left\| Q_{\mu, \tau}^\phi P_G w \right\|_{L^2}^2 \gtrsim \left\| Q_{\mu, \tau}^\phi P w \right\|_{L^2}^2 - \left\| Q_{\mu, \tau}^\phi R w \right\|_{L^2}^2.$$

We then use (4.2.3) of Theorem 4.2.1 for  $k = 0$  to find:

$$\begin{aligned} \left\| Q_{\mu, \tau}^\phi P_G w \right\|_{L^2}^2 + e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{L_t^2 H_x^1}^2 &\gtrsim \left\| Q_{\mu, \tau}^\phi P w \right\|_{L^2}^2 + e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{L_t^2 H_x^1}^2 - \left\| Q_{\mu, \tau}^\phi R w \right\|_{L^2}^2 \\ &\gtrsim \tau \left\| Q_{\mu, \tau}^\phi w \right\|_{H_t^1}^2 - \left\| Q_{\mu, \tau}^\phi R w \right\|_{L^2}^2. \end{aligned} \quad (4.4.4)$$

We now estimate the last term with the help of Proposition 4.3.2, up to reducing  $r$ . In order for all the setting of Section 4.3 to apply, we pick  $r_0$  small enough so that  $K = (t_0 - r_0, t_0 + r_0) \subset I$ . As for  $\rho > 0$  is arbitrary. If  $r$  is the one given by Theorem 4.2.1, we reduce it again in order to ensure the assumption  $0 < r < \min(\frac{r_0}{4}, \frac{\rho}{3})$ . We select  $\chi, \theta, \eta$  with the additional assumption that  $\theta = 1$  on  $[t_0 - r/2, t_0 + r/2]$ . We denote by  $B_{j,h}$  the approximate conjugated operator associated to  $b_j$  as defined in Section 4.3, that is  $B_{j,h} = F_h$  as defined in (4.3.7), in the case  $f = b_j$  and  $h$  is linked to  $\tau$  via  $h = \mu/\tau^3$ . We will keep however the  $h$  notation for the conjugated operator. The function  $b_j \in \mathcal{G}^2(K; L^\infty(V; \mathbb{C}))$  is identified with the multiplication operator (noted with the same name)  $b_j \in \mathcal{G}^{s,R}(K; \mathcal{L}(\mathcal{X}, \mathcal{Y}))$  with an appropriate  $R > 0$ . That is, we make the choice  $\mathcal{X} = \mathcal{Y} = L^2(V)$  and  $\mathcal{B} = L^\infty(V)$ .

We now assume  $w \in C_c^\infty(B(\mathbf{x}_0, r/2))$  so that  $\theta w = w$ . Applying Proposition 4.3.2 with  $u = e^{\tau\phi} \partial_j w = \theta u$  gives

$$\left\| \chi B_{j,h} e^{-\frac{\mu|D_t|^2}{2\tau^3}} u - e^{-\frac{\mu|D_t|^2}{2\tau^3}} b_j u \right\|_{L^2(\mathbb{R}; L^2(V))} \leq C e^{-c\tau} \|u\|_{L^2(\mathbb{R}; L^2(V))}.$$

Combining this estimate with the fact that, as noticed in Section 4.3,  $\|B_{j,h}\|_{L^2 \rightarrow L^2}$  is bounded uniformly in  $h$ , we obtain:

$$\begin{aligned} \left\| Q_{\mu, \tau}^\phi b_j \partial_j w \right\|_{L^2} &= \left\| e^{-\frac{\mu|D_t|^2}{2\tau^3}} b_j u \right\|_{L^2} \lesssim \left\| B_{j,h} e^{-\frac{\mu|D_t|^2}{2\tau^3}} u \right\|_{L^2} + e^{-c\tau} \|u\|_{H_t^{-k} L_x^2} \\ &\lesssim \left\| e^{-\frac{\mu|D_t|^2}{2\tau^3}} u \right\|_{L^2} + e^{-c\tau} \|u\|_{L^2} \\ &= \left\| Q_{\mu, \tau}^\phi \partial_j w \right\|_{L^2} + e^{-c\tau} \left\| e^{\tau\phi} \partial_j w \right\|_{L^2} \\ &\lesssim \tau \left\| Q_{\mu, \tau}^\phi w \right\|_{L^2} + \left\| Q_{\mu, \tau}^\phi w \right\|_{H_x^1} + e^{-c\tau} \left\| e^{\tau\phi} w \right\|_{L_t^2 H_x^1}. \end{aligned}$$

We proceed similarly for the potential  $\mathfrak{q}$  to find

$$\left\| Q_{\mu,\tau}^{\phi} \mathfrak{q} w \right\|_{L^2} \lesssim \left\| Q_{\mu,\tau}^{\phi} w \right\|_{L^2} + e^{-c\tau} \left\| e^{\tau\phi} w \right\|_{L^2}. \quad (4.4.5)$$

Adding these estimates and using the triangular inequality yields

$$\left\| Q_{\mu,\tau}^{\phi} R w \right\|_{L^2} \lesssim \tau \left\| Q_{\mu,\tau}^{\phi} w \right\|_{L^2} + \left\| Q_{\mu,\tau}^{\phi} w \right\|_{H_x^1} + e^{-c\tau} \left\| e^{\tau\phi} w \right\|_{H_t^{-k} H_x^1}. \quad (4.4.6)$$

Estimate (4.4.6) allows to absorb the last term in (4.4.4) up to taking  $\tau \geq \tau_0$  with  $\tau$  sufficiently large. This concludes the proof of (4.4.2) up to renaming the constants  $r, C, c, d$  and  $\tau_0$ . The proof (4.4.3) is the same after applying (4.2.3) of Theorem 4.2.1 for  $k = 1$  and using only the estimate (4.4.5) for the potential  $\mathfrak{q}$ .  $\square$

#### 4.4.2 Using the Carleman estimate

We state now a first version of the uniqueness theorem for smooth solutions. In the next section we shall explain how one can significantly reduce the required regularity of the solution.

**Theorem 4.4.2.** *Assume  $\Omega = I \times V$  where  $I \subset \mathbb{R}$  is an open interval and  $V \subset \mathbb{R}^d$  an open set. Consider the operator  $P_G$  defined in (4.4.1). Let  $\mathbf{x}_0 := (t_0, x_0) \in \Omega$  and  $\Psi \in C^2(\Omega)$  with  $\Psi(\mathbf{x}_0) = 0$  and  $|\nabla_g \Psi(\mathbf{x}_0)|_g > 0$ . Then there exists a neighborhood  $W$  of  $\mathbf{x}_0$  such that for any  $u \in L^2(I; H^1(V))$  one has*

$$\begin{cases} P_G u = 0 \text{ in } \Omega \\ u = 0 \text{ in } \Omega \cap \{\Psi > 0\} \end{cases} \implies u = 0 \text{ in } W.$$

Theorem 4.4.2 is a result of the Carleman estimate of Theorem 4.4.1. Nevertheless, in order to use it one needs to construct a weight function that:

- Satisfies the assumptions of Theorem 4.4.1, that is the assumptions of Theorem 4.2.1,
- Has level sets that are appropriately curved with respect to the level sets of  $\Psi$ . This is the geometric convexification part.

This is the content of the following Lemma:

**Lemma 4.4.1.** *Let  $\Psi \in C^2(\Omega; \mathbb{R})$  with  $|\nabla_g \Psi|_g^2 > 0$ . Suppose that  $\Psi(\mathbf{x}_0) = 0$ . Then there exists a quadratic polynomial  $\phi$  and a function  $f$  satisfying the assumptions of Theorem 4.2.1 together with the following properties:  $\phi(\mathbf{x}_0) = 0$  and there exists  $r_0$  such that for any  $0 < r < r_0$  there exists  $\eta > 0$  so that  $\phi(\mathbf{x}) \leq -\eta$  for  $\mathbf{x} \in \{\Psi \leq 0\} \cap \{r/2 \leq |\mathbf{x} - \mathbf{x}_0| \leq r\}$  where we write  $\mathbf{x} = (t, x)$ .*

*Proof.* Without loss of generality, we assume  $\bar{\Omega}$  compact. Given  $\Psi \in C^2(\Omega; \mathbb{R})$  define  $\check{\phi} = G(\Psi)$  and  $f$  as in (4.2.30) with  $G(s) = e^{\lambda s} - 1$ . Note in particular that  $\check{\phi}$  and  $\Psi$  have the same level sets. Then using Corollary 4.2.12, one has, for  $\lambda$  large enough,

$$\begin{aligned} \mathcal{B}_{g,\check{\phi},f}(X) &\geq 2C_0 |X|_g^2, \\ \mathcal{E}_{g,\check{\phi},f} &\geq 2C_0 \left| \nabla_g \check{\phi} \right|_g^2 > 0. \end{aligned}$$



Now define  $\check{\phi}_T$  by

$$\check{\phi}_T := \sum_{|\alpha| \leq 2} \frac{1}{\alpha!} (\partial^\alpha \check{\phi})(\mathbf{x}_0) (\mathbf{x} - \mathbf{x}_0)^\alpha.$$

Observe that both quantities  $\mathcal{B}_{g, \check{\phi}, f}$  and  $\mathcal{E}_{g, \check{\phi}, f}$  involve derivatives of order at most 2 of  $\check{\phi}$ . Since  $\Psi$  is  $C^2$  and  $G$  is smooth one has that  $\check{\phi} = G(\Psi)$  is  $C^2$  as well. Since  $(\partial^\alpha \check{\phi}_T)(\mathbf{x}_0) = (\partial^\alpha \check{\phi})(\mathbf{x}_0)$  for  $\alpha \leq 2$  we obtain by continuity that there exists  $r_1 > 0$  such that

$$\begin{aligned} \mathcal{B}_{g, \check{\phi}_T, f}(\mathbf{x})(X) &\geq C_0 |X|_g^2, \\ \mathcal{E}_{g, \check{\phi}_T, f}(\mathbf{x}) &\geq C_0 \left| \nabla_g \check{\phi}_T \right|_g^2(\mathbf{x}) > 0, \end{aligned}$$

for  $\mathbf{x} \in \overline{B}(\mathbf{x}_0, r_1)$ . Define finally  $\phi$  by

$$\phi := \check{\phi}_T - \delta |\mathbf{x} - \mathbf{x}_0|^2.$$

Then by the same stability argument with respect to  $C^2$  perturbations we have that for  $\delta > 0$  small enough:

$$\begin{aligned} \mathcal{B}_{g, \phi, f}(\mathbf{x})(X) &\geq \frac{C_0}{2} |X|_g^2, \\ \mathcal{E}_{g, \phi, f}(\mathbf{x}) &\geq \frac{C_0}{2} |\nabla_g \phi|_g^2(\mathbf{x}) > 0, \end{aligned}$$

for  $\mathbf{x} \in \overline{B}(\mathbf{x}_0, r_1)$ . That means that  $\phi$  satisfies indeed the assumptions of Theorem 4.2.1. The last property is an immediate consequence of the fact that  $\check{\phi}$  and  $\Psi$  have the same level sets and  $\check{\phi}_T$  is the order 2 Taylor expansion of  $\check{\phi}$ .  $\square$

We have now all the necessary ingredients for the proof of Theorem 4.4.2.

*Proof of Theorem 4.4.2.* Consider  $u$  a solution of  $P_{\mathcal{G}}u = 0$  such that  $u = 0$  in  $\Omega \cap \{\Psi > 0\}$ . Let  $\phi$  be as in Lemma 4.4.1. Theorem 4.4.1 implies that there exist  $r, \mathbf{d}, C, \tau_0 > 0$  such that for all  $\tau \geq \tau_0$  and  $w \in C_c^\infty(B(\mathbf{x}_0, r))$ , we have

$$C \left\| Q_{\mu, \tau}^\phi P_{\mathcal{G}} w \right\|_{L^2}^2 + C e^{-\mathbf{d}\tau} \left\| e^{\tau\phi} w \right\|_{L_t^2 H_x^1}^2 \geq \tau \left\| Q_{\mu, \tau}^\phi w \right\|_{H^1}^2.$$

We have moreover:

1.  $\phi(\mathbf{x}_0) = 0$  and there exists  $\eta > 0$  so that  $\phi(\mathbf{x}) \leq -\eta$  for  $\mathbf{x} \in \{\Psi \leq 0\} \cap \{r \geq |\mathbf{x} - \mathbf{x}_0| \geq r/2\}$
2.  $\phi(\mathbf{x}) \leq \mathbf{d}/4$  for  $|\mathbf{x} - \mathbf{x}_0| \leq r$ .

Property 1 comes from Lemma 4.4.1 and Property 2 is just the continuity of  $\phi$ , up to reducing  $r$ . Notice that the Carleman estimate of Theorem 4.4.2 remains valid by standard density arguments for functions  $w \in L^2(I; H^1(V))$  such that  $P_{\mathcal{G}}w \in L^2$  and  $\text{supp } w \subset B(\mathbf{x}_0, r)$ . Let  $\chi \in C_0^\infty(B(\mathbf{x}_0, r))$  with  $\chi = 1$  in  $B(\mathbf{x}_0, r/2)$ . We apply the Carleman estimate above to  $w = \chi u$ . We control

$$\begin{aligned} \left\| Q_{\mu, \tau}^\phi P_{\mathcal{G}} \chi u \right\|_{L^2} &\leq \left\| Q_{\mu, \tau}^\phi \chi P_{\mathcal{G}} u \right\|_{L^2} + \left\| Q_{\mu, \tau}^\phi [P_{\mathcal{G}}, \chi] u \right\|_{L^2} = \left\| Q_{\mu, \tau}^\phi [P_{\mathcal{G}}, \chi] u \right\|_{L^2} \\ &\leq \left\| e^{\tau\phi} [P_{\mathcal{G}}, \chi] u \right\|_{L^2} \leq e^{-\eta\tau} \|u\|_{L_t^2 H_x^1}, \end{aligned}$$

thanks to the supports of  $\chi'$  and  $u$ , Property 1 and the fact that  $[P_G, \chi]$  has no time derivatives. We have as well

$$e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{L_t^2 H_x^1} \leq e^{-d\tau/4} \|u\|_{L_t^2 H_x^1},$$

thanks to Property 2. We finally obtain that there exists a  $\delta > 0$  such that

$$\left\| Q_{\mu, \tau}^\phi \chi u \right\|_{L^2} \leq \|Q_{\mu, \tau}^\phi \chi u\|_{H_t^1}^2 \leq C e^{-\delta\tau},$$

which implies that  $\left\| Q_{\mu, \tau}^{\phi+\delta} \chi u \right\|_{L^2} \leq C$  uniformly in  $\tau \geq \tau_0$ . Lemma 4.A.2 gives  $\text{supp}(\chi u) \subset \{\phi \leq -\delta\}$ . Since  $\phi(\mathbf{x}_0) = 0$  and  $\chi = 1$  in  $B(\mathbf{x}_0, r/2)$  one has that  $W = B(\mathbf{x}_0, r/2) \cap \{\phi > -\delta/2\}$  is a neighborhood of  $\mathbf{x}_0$  in which  $\chi u = u = 0$  and the proof of Theorem 4.4.2 is complete  $\square$

**Remark 4.4.2.** Observe that, in fact, we only used the weaker estimate

$$C \left\| Q_{\mu, \tau}^\phi P_G w \right\|_{L^2}^2 + C e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{H^1}^2 \geq \|Q_{\mu, \tau}^\phi w\|_{L^2}^2.$$

#### 4.4.3 Reducing the regularity of the solution

Theorem 4.4.2 concerns solutions  $u$  of the Schrödinger equation  $P_G = 0$  which are in  $L_t^2 H_x^1$ . In view of the regularity of the coefficients this is almost optimal in the sense that if  $u$  is less regular then  $Pu$  may be not even defined as a distribution. In this section we are interested with solutions  $u \in L^2$  of  $P_G u = 0$  in the particular case where  $b_j = 0$  for all  $j$ . Reducing the regularity of the solution to  $L^2$  is interesting, especially with respect to applications in control theory.

The idea is to use the Carleman estimate of (4.4.3). This allows to exploit the ellipticity of  $\Delta_g$  and gain some regularity. Applying (4.4.3) (which is still valid for a compactly supported function  $w \in L^2$  such that  $P_G w \in L^2$  by standard density arguments) yields:

$$C \left\| Q_{\mu, \tau}^\phi P_G w \right\|_{L^2}^2 + C e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{H_t^{-1} H_x^1}^2 + C e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{L^2}^2 \geq \tau \|Q_{\mu, \tau}^\phi w\|_{H_t^1}^2. \quad (4.4.7)$$

One would like then to proceed exactly as in the proof of Theorem 4.4.2, with  $w = \chi u$ . The only problem is that the terms  $\|[P_G, \chi]u\|_{L^2}$  and  $\left\| e^{\tau\phi} \chi u \right\|_{H_t^{-1} H_x^1}$  can no longer be bounded by  $\|u\|_{H^1}$  since the solution  $u$  is only in  $L^2$ .

We denote by  $\langle D_t \rangle$  the Fourier multiplier  $(1 + |D_t|^2)^{1/2}$ . We then bound from above

the term  $\|Q_{\mu,\tau}^\phi P_{\mathcal{G}}w\|_{L^2}$  appearing in the left hand side of (4.4.7) as follows:

$$\begin{aligned}
\|Q_{\mu,\tau}^\phi P_{\mathcal{G}}w\|_{L^2} &= \left\| \langle D_t \rangle e^{-\frac{\mu|D_t|^2}{2\tau^3}} \langle D_t \rangle^{-1} e^{\tau\phi} P_{\mathcal{G}}w \right\|_{L^2} \\
&= \left\| \mathbf{1}_{|D_t| \leq \tau^2} \langle D_t \rangle e^{-\frac{\mu|D_t|^2}{2\tau^3}} \langle D_t \rangle^{-1} e^{\tau\phi} P_{\mathcal{G}}w \right\|_{L^2} \\
&\quad + \left\| \mathbf{1}_{|D_t| \geq \tau^2} \langle D_t \rangle e^{-\frac{\mu|D_t|^2}{2\tau^3}} \langle D_t \rangle^{-1} e^{\tau\phi} P_{\mathcal{G}}w \right\|_{L^2} \\
&\leq C\tau^2 \left\| \langle D_t \rangle^{-1} e^{\tau\phi} P_{\mathcal{G}}w \right\|_{L^2} + C \max_{\xi_t \geq \tau^2} (\xi_t e^{-\mu \frac{\xi_t^2}{2\tau^3}}) \left\| \langle D_t \rangle^{-1} e^{\tau\phi} P_{\mathcal{G}}w \right\|_{L^2} \\
&\leq C\tau^2 \left\| e^{\tau\phi} P_{\mathcal{G}}w \right\|_{H_t^{-1}L_x^2} + C e^{-c\tau} \left\| e^{\tau\phi} P_{\mathcal{G}}w \right\|_{H_t^{-1}L_x^2} \\
&\leq C\tau^2 \left\| e^{\tau\phi} P_{\mathcal{G}}w \right\|_{H_t^{-1}L_x^2}
\end{aligned}$$

for  $\tau \geq \tau_0$ , where we proceeded as in (4.2.8). Combining the above with (4.4.7) gives:

$$C\tau^4 \left\| e^{\tau\phi} P_{\mathcal{G}}w \right\|_{H_t^{-1}L_x^2}^2 + C e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{H_t^{-1}H_x^1}^2 + C e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{L^2}^2 \geq \tau \|Q_{\mu,\tau}^\phi w\|_{H_t^1}^2. \quad (4.4.8)$$

We apply it to  $w = \chi u$  with  $\chi$  as in the proof of Theorem 4.4.2 and we need to control from above

$$\tau^2 \left\| e^{\tau\phi} P_{\mathcal{G}}(\chi u) \right\|_{H_t^{-1}L_x^2} = \tau^2 \left\| e^{\tau\phi} [P_{\mathcal{G}}, \chi]u \right\|_{H_t^{-1}L_x^2}.$$

Observe that  $[P_{\mathcal{G}}, \chi]$  is a differential operator of order 1 with no time derivatives and coefficients supported in the support of  $\chi'$ . Therefore the same holds for its adjoint operator  $[P_{\mathcal{G}}, \chi]^*$ . Consider now  $\theta \in C^\infty(\Omega)$  with  $\theta = 1$  in  $\{\Psi < 0\}$  and  $\theta = 0$  in  $\{\Psi > \varepsilon\}$  with  $\varepsilon$  sufficiently small such that  $\phi(\mathbf{x}) \leq -\eta/2$  for  $\mathbf{x} \in \{\Psi \leq \varepsilon\} \cap \{r/2 \leq |\mathbf{x} - \mathbf{x}_0| \leq r\}$ . Thanks to the support of  $u$  we have  $\theta u = u$ . As in the proof of Theorem 4.4.2 one has that  $e^{\tau\phi} \leq e^{-\eta\tau/2}$  in the support of  $\chi' \cap \text{supp}(\theta)$ . This gives

$$\left\| [P_{\mathcal{G}}, \chi]^* e^{\tau\phi} u \right\|_{H_t^1 L_x^2} = \left\| [P_{\mathcal{G}}, \chi]^* e^{\tau\phi} \theta u \right\|_{H_t^1 L_x^2} \leq C\tau e^{-\eta\tau/2} \|u\|_{H_t^1 H_x^1}.$$

That means that if we define the operator  $S : v \mapsto [P_{\mathcal{G}}, \chi]^* e^{\tau\phi} \theta v$  for  $v \in \mathcal{S}$  we have

$$\|S\|_{H_t^1 H_x^1 \rightarrow H_t^1 L_x^2} \leq C\tau e^{-\eta\tau/2}$$

By duality this yields

$$\|S^*\|_{H_t^{-1}L_x^2 \rightarrow H_t^{-1}H_x^{-1}} = \|S\|_{H_t^1 H_x^1 \rightarrow H_t^1 L_x^2} \leq C\tau e^{-\eta\tau/2}$$

which implies finally  $\|S^*\|_{H_t^{-1}H_x^1 \rightarrow H_t^{-1}L_x^2} \leq C\tau e^{-\eta\tau/2}$ . Coming back to the quantity we need to control we have then

$$\begin{aligned}
\tau^2 \left\| e^{\tau\phi} [P_{\mathcal{G}}, \chi]u \right\|_{H_t^{-1}L_x^2} &= \tau^2 \|S^*u\|_{H_t^{-1}L_x^2} \leq C\tau^3 e^{-\eta\tau/2} \|u\|_{H_t^{-1}H_x^1} \\
&\leq C e^{-\eta\tau/4} \|u\|_{H_t^{-1}H_x^1}.
\end{aligned}$$

Similarly, one has:

$$e^{-d\tau} \left\| e^{\tau\phi} w \right\|_{H_t^{-1}H_x^1} \leq e^{-d\tau/4} \|u\|_{H_t^{-1}H_x^1}.$$

As before, combining the last two estimates with (4.4.8) and using the support of  $\chi$  gives the existence of some  $\delta > 0$  and a small open set  $U \subset \mathbb{R}^d$  with

$$\|Q_{\mu,\tau}^\phi w\|_{L^2} \leq C e^{-\delta\tau} \left( \|u\|_{H_t^{-1}H_x^1(U)} + \|u\|_{L_t^2L_x^2(U)} \right).$$

Therefore, the last step in order to finish the proof and conclude as in the proof of Theorem 4.4.2 is to prove that  $\|u\|_{H_t^{-1}H_x^1}$  is indeed finite. This comes from the fact that  $u$  solves  $P_{\mathcal{G}}u = (i\partial_t + \Delta_g + \mathbf{q}(t, x))u = 0$  and from the ellipticity of  $\Delta_g$ . Indeed, by elliptic regularity we have, for  $U \Subset \tilde{U} \subset \mathbb{R}^d$ :

$$\begin{aligned} \|u\|_{H_t^{-1}H_x^1(U)} &\leq C \left( \|\Delta_g u\|_{H_t^{-1}L_x^2(\tilde{U})} + \|u\|_{H_t^{-1}L_x^2(\tilde{U})} \right) \\ &\leq C \left( \|\langle D_t \rangle^{-1} \Delta_g u\|_{L^2} + \|\langle D_t \rangle^{-1} u\|_{L^2} \right) \\ &\leq C \left( \|\langle D_t \rangle^{-1} D_t u\|_{L^2} + \|\langle D_t \rangle^{-1} V u\|_{L^2} + \|\langle D_t \rangle^{-1} u\|_{L^2} \right), \end{aligned} \quad (4.4.9)$$

where we used the equality  $\Delta_g u = D_t u - V u$ . Now observe that  $\|\langle D_t \rangle^{-1} D_t\|_{L^2 \rightarrow L^2} = \left\| \frac{\xi_t}{\langle \xi_t \rangle} \right\|_{L^\infty} \leq 1$  and similarly  $\|\langle D_t \rangle^{-1}\|_{L^2 \rightarrow L^2} \leq 1$ . Using this in (4.4.9) yields

$$\|u\|_{H_t^{-1}H_x^1(U)} \leq C \|u\|_{L^2}.$$

We can then conclude exactly as in the proof of Theorem 4.4.2. This proves the second part of Theorem 4.1.2 stated in the introduction.

## 4.A Tools

### 4.A.1 A technical lemma

For further reference, we first collect the formula

$$\mathcal{F}(e^{-\frac{|\cdot|^2}{\lambda}})(\xi) = (\pi\lambda)^{n/2} e^{-\lambda \frac{|\xi|^2}{4}}, \quad (4.A.1)$$

and hence

$$e^{-\frac{h}{2}|D_t|^2} f(t) = \left( \frac{1}{2\pi h} \right)^{1/2} \int f(s) e^{-\frac{|t-s|^2}{2h}} ds. \quad (4.A.2)$$

**Lemma 4.A.1.** *Let  $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$  be a normed vector space,  $\chi_1, \chi_2 \in C^\infty(\mathbb{R})$  with all derivatives bounded and such that  $\text{dist}(\text{supp } f_1, \text{supp } f_2) \geq d > 0$ . Then for every  $k, m \in \mathbb{N}$ , there exist  $C, c > 0$  such that for all  $u \in \mathcal{S}(\mathbb{R}; \mathcal{X})$  and all  $\lambda > 0$  we have*

$$\left\| \chi_1 e^{-\frac{|D_t|^2}{\lambda}} (\chi_2 u) \right\|_{H^k(\mathbb{R}; \mathcal{X})} \leq C e^{-c\lambda} \|u\|_{H^{-m}(\mathbb{R}; \mathcal{X})}.$$

See e.g. [LL19, Lemma 2.4] in case  $m = k = 0$ .

*Proof.* We start with  $k = m = 0$  and recall (4.A.2). Using the support properties of  $\chi_1, \chi_2$ , this implies

$$\begin{aligned} \chi_1 e^{-\frac{|D_t|^2}{\lambda}} (\chi_2 u)(t) &= \left( \frac{\lambda}{4\pi} \right)^{1/2} \chi_1(t) \int_{|t-s| \geq d} e^{-\frac{\lambda}{4}(s-t)^2} \chi_2(s) u(s) ds \\ &= \left( \frac{\lambda}{4\pi} \right)^{1/2} \chi_1(t) \mathbf{1}_{|\cdot| \geq d} e^{-\frac{\lambda}{4}(\cdot)^2} * \chi_2(\cdot)(t) \end{aligned}$$

The Young inequality thus yields

$$\begin{aligned} \left\| \chi_1 e^{-\frac{|D_t|^2}{\lambda}} (\chi_2 u) \right\|_{L^2(\mathbb{R}; \mathcal{X})} &\leq \left( \frac{\lambda}{4\pi} \right)^{1/2} \|\chi_1\|_{L^\infty} \left\| \mathbf{1}_{|\cdot| \geq d} e^{-\frac{\lambda}{4}(\cdot)^2} \right\|_{L^1(\mathbb{R})} \|\chi_2 u\|_{L^2(\mathbb{R}; \mathcal{X})} \\ &\leq \left( \frac{\lambda}{4\pi} \right)^{1/2} \|\chi_1\|_{L^\infty} \|\chi_2\|_{L^\infty} \left\| \mathbf{1}_{|\cdot| \geq d} e^{-\frac{\lambda}{4}(\cdot)^2} \right\|_{L^1(\mathbb{R})} \|u\|_{L^2(\mathbb{R}; \mathcal{X})}. \end{aligned}$$

The result for  $k = m = 0$  then follows from the fact that

$$\begin{aligned} \frac{1}{2} \left\| \mathbf{1}_{|\cdot| \geq d} e^{-\frac{\lambda}{4}(\cdot)^2} \right\|_{L^1(\mathbb{R})} &= \int_d^\infty e^{-\frac{\lambda}{8}s^2} e^{-\frac{\lambda}{8}s^2} ds \leq e^{-\frac{\lambda}{8}d^2} \int_0^\infty e^{-\frac{\lambda}{8}s^2} ds \\ &\leq \frac{C e^{-\frac{\lambda}{8}d^2}}{\sqrt{\lambda}} \int_0^\infty e^{-s^2} ds \leq C e^{-c\lambda}. \end{aligned}$$

As a preparation for the general case, we prove a similar estimate if  $e^{-\frac{|D_t|^2}{\lambda}}$  is replaced by  $D_t^k e^{-\frac{|D_t|^2}{\lambda}}$  for  $k \in \mathbb{N}$ . Notice that from (4.A.2), we have

$$\begin{aligned} D_t^k e^{-\frac{|D_t|^2}{\lambda}} f &= \left( \frac{\lambda}{4\pi} \right)^{1/2} \int_{\mathbb{R}} D_t^k e^{-\frac{\lambda}{4}(s-t)^2} f(s) ds \\ &= \left( \frac{\lambda}{4\pi} \right)^{1/2} \sum_{0 \leq k_1, k_2 \leq k} \alpha_{k_1, k_2} \int_{\mathbb{R}} \lambda^{k_1} (s-t)^{k_2} e^{-\frac{\lambda}{4}(s-t)^2} f(s) ds, \end{aligned}$$

where  $\alpha_{k_1, k_2} \in \mathbb{C}$  do not depend on  $\lambda$ . As a consequence, proceeding as above with the Young inequality, we obtain

$$\begin{aligned} \left\| \chi_1 D_t^k e^{-\frac{|D_t|^2}{\lambda}} (\chi_2 u) \right\|_{L^2(\mathbb{R}; \mathcal{X})} &\leq C_k \lambda^{k+1/2} \sum_{0 \leq k_2 \leq k} \left\| \chi_1(t) \mathbf{1}_{|\cdot| \geq d} e^{-\frac{\lambda}{4}(\cdot)^2} (\cdot)^{k_2} * (\chi_2 u)(t) \right\|_{L^2(\mathbb{R}; \mathcal{X})} \\ &\leq C_k \lambda^{k+1/2} \|\chi_1\|_{L^\infty} \|\chi_2\|_{L^\infty} \sum_{0 \leq k_2 \leq k} \left\| \mathbf{1}_{|\cdot| \geq d} e^{-\frac{\lambda}{4}(\cdot)^2} (\cdot)^{k_2} \right\|_{L^1(\mathbb{R})} \|u\|_{L^2(\mathbb{R}; \mathcal{X})}. \end{aligned}$$

Using now

$$\begin{aligned} \left\| \mathbf{1}_{|\cdot| \geq d} e^{-\frac{\lambda}{4}(\cdot)^2} (\cdot)^{k_2} \right\|_{L^1(\mathbb{R})} &= 2 \int_d^\infty e^{-\frac{\lambda}{8}s^2} e^{-\frac{\lambda}{8}s^2} s^{k_2} ds \leq 2 e^{-\frac{\lambda}{8}d^2} \int_0^\infty s^{k_2} e^{-\frac{\lambda}{8}s^2} ds \\ &= e^{-\frac{\lambda}{8}d^2} \left( \frac{8}{\lambda} \right)^{\frac{k_2+1}{2}} \Gamma\left(\frac{k_2+1}{2}\right) \leq C_{k_2} e^{-c_{k_2}\lambda}, \end{aligned}$$

Combining these two lines, we finally deduce that for any  $k \in \mathbb{N}$  and any  $\chi_1, \chi_2 \in L^\infty(\mathbb{R})$  such that  $\text{dist}(\text{supp } f_1, \text{supp } f_2) \geq d > 0$ , there are  $C_k, c_k > 0$  such that for all  $u \in \mathcal{S}(\mathbb{R}; \mathcal{X})$ ,

$$\left\| \chi_1 D_t^k e^{-\frac{|D_t|^2}{\lambda}} (\chi_2 u) \right\|_{L^2(\mathbb{R}; \mathcal{X})} \leq C_k e^{-c_k \lambda} \|u\|_{L^2(\mathbb{R}; \mathcal{X})}. \quad (4.A.3)$$

Now, we prove the following statement: for all  $k, \ell, m \in \mathbb{N}$ , for all  $\chi_1, \chi_2 \in C_b^\infty(\mathbb{R})$  such that  $\text{dist}(\text{supp } f_1, \text{supp } f_2) \geq d > 0$  there are  $C, c > 0$  such that for all  $u \in \mathcal{S}(\mathbb{R}; \mathcal{X})$ ,

$$\left\| D_t^\ell \chi_1 D_t^k e^{-\frac{|D_t|^2}{\lambda}} (\chi_2 D_t^m u) \right\|_{L^2(\mathbb{R}; \mathcal{X})} \leq C e^{-c\lambda} \|u\|_{L^2(\mathbb{R}; \mathcal{X})}. \quad (4.A.4)$$

To this aim, given  $\ell, m \in \mathbb{N}$ , we consider the induction assumption

$$(4.A.4) \text{ is satisfied for all } k \in \mathbb{N}. \quad (A(\ell, m))$$

We notice first that  $(A(0, 0))$  is (4.A.3). Then, we assume  $(A(\ell, m))$  and prove  $(A(\ell + 1, m + 1))$ . For this, we decompose and expand

$$\begin{aligned} D_t^{\ell+1} \chi_1 D_t^k e^{-\frac{|D_t|^2}{\lambda}} \chi_2 D_t^{m+1} &= D_t^\ell (\chi_1 D_t + [D_t, \chi_1]) D_t^k e^{-\frac{|D_t|^2}{\lambda}} (D_t \chi_2 + [\chi_2, D_t]) D_t^m \\ &= D_t^\ell \chi_1 D_t^{k+2} e^{-\frac{|D_t|^2}{\lambda}} \chi_2 D_t^m + i D_t^\ell \chi_1 D_t^{k+1} e^{-\frac{|D_t|^2}{\lambda}} \chi_2' D_t^m \\ &\quad - i D_t^\ell \chi_1' D_t^{k+1} e^{-\frac{|D_t|^2}{\lambda}} \chi_2 D_t^m + D_t^\ell \chi_1' D_t^k e^{-\frac{|D_t|^2}{\lambda}} \chi_2' D_t^m, \end{aligned}$$

and notice that the induction assumption  $(A(\ell, m))$  applies to all of these four terms since  $\text{supp } \chi_j' \subset \text{supp } \chi_j$ ,  $j = 1, 2$ . This concludes the proof of (4.A.4).

To conclude the proof of the lemma, we deduce from (4.A.4) (for  $k = 0$ ) that for  $\ell, m \in \mathbb{N}$ , and all  $v \in \mathcal{S}(\mathbb{R}; \mathcal{X})$ ,

$$\left\| (1 + D_t^2)^\ell \chi_1 e^{-\frac{|D_t|^2}{\lambda}} (\chi_2 (1 + D_t^2)^m v) \right\|_{L^2(\mathbb{R}; \mathcal{X})} \leq C e^{-c\lambda} \|v\|_{L^2(\mathbb{R}; \mathcal{X})}.$$

Letting  $v := (1 + D_t^2)^{-m} u$  in this expression, we deduce that for all  $u \in \mathcal{S}(\mathbb{R}; \mathcal{X})$ ,

$$\begin{aligned} \left\| \chi_1 e^{-\frac{|D_t|^2}{\lambda}} (\chi_2 u) \right\|_{H^{2\ell}(\mathbb{R}; \mathcal{X})} &= \left\| (1 + D_t^2)^\ell \chi_1 e^{-\frac{|D_t|^2}{\lambda}} (\chi_2 (1 + D_t^2)^m v) \right\|_{L^2(\mathbb{R}; \mathcal{X})} \\ &\leq C e^{-c\lambda} \|v\|_{L^2(\mathbb{R}; \mathcal{X})} = C e^{-c\lambda} \|u\|_{H^{-2m}(\mathbb{R}; \mathcal{X})}. \end{aligned}$$

This concludes the proof of the lemma (for even integers, and thus for all integers).  $\square$

The following is Proposition 2.1 in [Hör97] that we state here for the reader's convenience.

**Lemma 4.A.2.** *Let  $u \in L^2(\mathbb{R}^n)$  and let  $\phi$  be a smooth real valued function. For  $\tau > 0$  let  $A_\tau$  be continuous bounded functions in  $\mathbb{R}^n$  such that the Fourier multipliers  $A_\tau(\xi) \rightarrow 1$  uniformly on every compact set when  $\tau \rightarrow \infty$ . If*

$$\left\| A_\tau(D) e^{\tau\phi} u \right\|_{L^2} \leq C, \quad \tau \geq \tau_0$$

*then it follows that  $\text{supp } u \subset \{\phi \leq 0\}$ .*

The following regularity Lemma was useful in the conjugation argument.

**Lemma 4.A.3.** *Let  $h \in C^3(U)$  with  $U \subset \mathbb{C}$  open set containing 0 so that  $|\partial_{\bar{z}} h| \leq C|z|^2$ . Then,  $w(z) = \frac{h(z) - h(0)}{z} \in C^1(U)$ .*

*Proof.* The only problem is close to  $z = 0$ . We write  $h(z) = h(0) + z \int_0^1 \partial_z h(sz) ds + \bar{z} \int_0^1 \partial_{\bar{z}} h(sz) ds$  and we have

$$w(z) = \int_0^1 \partial_z h(sz) ds + \frac{\bar{z}}{z} \int_0^1 \partial_{\bar{z}} h(sz) ds$$

The only term left to consider is  $u(z) = \frac{\bar{z}}{z} \int_0^1 \partial_{\bar{z}} h(sz) ds$ . The estimate on  $\partial_{\bar{z}} h$  implies that  $u(z)$  can be continuously extended by 0 at 0 so, we are left to consider the derivatives of  $u$ .

$$\nabla u = \nabla \left( \frac{\bar{z}}{z} \right) \int_0^1 \partial_{\bar{z}} h(sz) ds + \frac{\bar{z}}{z} \int_0^1 s \nabla \partial_{\bar{z}} h(sz) ds.$$

By Taylor expansion, the assumption  $\partial_{\bar{z}} h \in C^2$  and  $|\partial_{\bar{z}} h| \leq C|z|^2$  implies  $\partial_{\bar{z}} h(0) = 0$ ,  $\nabla \partial_{\bar{z}} h(0) = 0$  and therefore,  $|\nabla \partial_{\bar{z}} h(z)| \leq C|z|$ . Since  $|\nabla \left( \frac{\bar{z}}{z} \right)| \leq C|z|^{-1}$ , we obtain that  $\nabla u$  can be continuously extended by 0 at  $z = 0$ . This gives the expected result.  $\square$

#### 4.A.2 Integration by parts formulæ

Given a bounded  $C^1$  (or piecewise  $C^1$ ) domain  $\Omega \subset \mathbb{C}$  and a  $C^1$  one form  $\omega$  defined in a neighborhood of  $\Omega$ , we recall the Stokes formula

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega.$$

Here,  $\partial\Omega$  is given the orientation coming from the canonical orientation  $\mathbb{C}$ .

Now given a Banach space  $\mathcal{B}$  and a function  $f_0 : \mathbb{R}^2 \simeq \mathbb{C} \rightarrow \mathcal{B}$ , and under the identification  $f_0(x, y) = f(z, \bar{z})$ , we apply the above formula with the one Banach-valued form  $\omega(x, y) = f(z, \bar{z}) dz$  to obtain

$$\int_{\partial\Omega} f(z, \bar{z}) dz = \int_{\Omega} d(f(z, \bar{z}) dz) = \int_{\Omega} \partial_z f(z, \bar{z}) dz \wedge dz + \partial_{\bar{z}} f(z, \bar{z}) d\bar{z} \wedge dz = \int_{\Omega} \partial_{\bar{z}} f(z, \bar{z}) d\bar{z} \wedge dz,$$

that is

$$\int_{\partial\Omega} f(z, \bar{z}) dz = \int_{\Omega} \partial_{\bar{z}} f(z, \bar{z}) d\bar{z} \wedge dz. \quad (4.A.5)$$

Here  $\partial\Omega$  is oriented so that  $\Omega$  lies to the left of  $\partial\Omega$ , see for instance [Hör90, Chapter 1, Section 1.2]. Applying this to  $f = gh$  with  $g \in C^1(\mathbb{C}; \mathbb{C})$ ,  $h \in C^1(\mathbb{C}; \mathcal{B})$ , we deduce

$$\int_{\Omega} g \partial_{\bar{z}} h d\bar{z} \wedge dz = \int_{\Omega} g h dz - \int_{\Omega} h \partial_{\bar{z}} g d\bar{z} \wedge dz. \quad (4.A.6)$$

If now  $g \in C^1(\mathbb{C})$  and  $h \in C^1(\mathbb{C}; \mathcal{B})$  satisfy  $hg \rightarrow 0$  at infinity and  $g \partial_{\bar{z}} h \in L^1(\mathbb{C}; \mathcal{B})$ ,  $h \partial_{\bar{z}} g \in L^1(\mathbb{C}; \mathcal{B})$ , then we may choose  $\Omega = B(0, R)$  and let  $R \rightarrow +\infty$ , yielding the following statement.

**Lemma 4.A.4.** *Assume  $\mathcal{B}$  is a Banach space,  $g \in C^1(\mathbb{C}; \mathbb{C})$  and  $h \in C^1(\mathbb{C}; \mathcal{B})$  satisfy  $g \partial_{\bar{z}} h \in L^1(\mathbb{C}; \mathcal{B})$ ,  $h \partial_{\bar{z}} g \in L^1(\mathbb{C}; \mathcal{B})$  and  $\int_{\partial B(0, R)} \|hg\|_{\mathcal{B}}(z) dz \rightarrow 0$  as  $R \rightarrow +\infty$ . Then*

$$\int_{\mathbb{C}} g \partial_{\bar{z}} h d\bar{z} \wedge dz = - \int_{\mathbb{C}} h \partial_{\bar{z}} g d\bar{z} \wedge dz. \quad (4.A.7)$$

Note finally that  $z = x + iy$  and  $\bar{z} = x - iy$  so that

$$d\bar{z} \wedge dz = d(x - iy) \wedge d(x + iy) = 2i dx \wedge dy,$$

where  $dx \wedge dy$  is the usual Lebesgue measure on  $\mathbb{R}^2$  (oriented).

Note also that if  $f$  is holomorphic, we recover the usual deformation of contour principle  $\int_{\partial\Omega} f(z) dz = 0$ .

# Bibliographie

- [AB79] Serge Alinhac and Mohamed Salah Baouendi. Construction de solutions nulles et singulières pour des opérateurs de type principal. In *Séminaire Goulaouic-Schwartz (1978/1979)*, pages Exp. No. 22, 6. École Polytech., Palaiseau, 1979.
- [AB95] Serge Alinhac and Mohamed Salah Baouendi. A nonuniqueness result for operators of principal type. *Math. Z.*, 220(4) :561–568, 1995.
- [AdHG17] Giovanni Alessandrini, Maarten V. de Hoop, and Romina Gaburro. Uniqueness for the electrostatic inverse boundary value problem with piecewise constant anisotropic conductivities. *Inverse Problems*, 33(12) :125013, 24, 2017.
- [AFKM15] Nalini Anantharaman, Clotilde Fermanian Kammerer, and Fabricio Macià. Semiclassical completely integrable systems : long-time dynamics and observability via two-microlocal Wigner measures. *Amer. J. Math.*, 137(3) :577–638, 2015.
- [Agm82] Shmuel Agmon. *Lectures on exponential decay of solutions of second-order elliptic equations : bounds on eigenfunctions of N-body Schrödinger operators*, volume 29 of *Mathematical*. Princeton University Press, Princeton, NJ ; University of Tokyo Press, Tokyo, 1982.
- [Ali83] Serge Alinhac. Non-unicité du problème de Cauchy. *Ann. of Math. (2)*, 117(1) :77–108, 1983.
- [All98] Brice Allibert. Contrôle analytique de l'équation des ondes et de l'équation de Schrödinger sur des surfaces de révolution. *Comm. Partial Differential Equations*, 23(9-10) :1493–1556, 1998.
- [ALM16] Nalini Anantharaman, Matthieu Léautaud, and Fabricio Macià. Wigner measures and observability for the Schrödinger equation on the disk. *Invent. Math.*, 206(2) :485–599, 2016.
- [AM12] Nalini Anantharaman and Fabricio Macià. The dynamics of the Schrödinger flow from the point of view of semiclassical measures. In *Spectral geometry*, volume 84 of *Proc. Sympos. Pure Math.*, pages 93–116. Amer. Math. Soc., Providence, RI, 2012.



- [AM14] Nalini Anantharaman and Fabricio Macià. Semiclassical measures for the Schrödinger equation on the torus. *J. Eur. Math. Soc. (JEMS)*, 16(6) :1253–1288, 2014.
- [Ana08] Nalini Anantharaman. Entropy and the localization of eigenfunctions. *Ann. of Math. (2)*, 168(2) :435–475, 2008.
- [AR12] Nalini Anantharaman and Gabriel Rivière. Dispersion and controllability for the Schrödinger equation on negatively curved manifolds. *Anal. PDE*, 5(2) :313–338, 2012.
- [Bah87] Hajer Bahouri. Dépendance non linéaire des données de Cauchy pour les solutions des équations aux dérivées partielles. *J. Math. Pures Appl. (9)*, 66(2) :127–138, 1987.
- [BBAD22] Assia Benabdallah, Matania Ben-Artzi, and Yves Dermenjian. Concentration and non-concentration of eigenfunctions of second-order elliptic operators in layered media. *arXiv :2212.05872*, 2022.
- [BBZ13] Jean Bourgain, Nicolas Burq, and Maciej Zworski. Control for Schrödinger operators on 2-tori : rough potentials. *J. Eur. Math. Soc. (JEMS)*, 15(5) :1597–1628, 2013.
- [BCDCG20] Remi Buffe, Marcelo M. Cavalcanti, Valéria N. Domingos Cavalcanti, and Ludovick Gagnon. Control and exponential stability for a transmission problem of a viscoelastic wave equation. working paper or preprint, February 2020.
- [BDDM20] Stéphane Balac, Monique Dauge, Yannick Dumeige, and Zoïs Moitier. Mathematical analysis of whispering gallery modes in graded index optical micro-disk resonators. <https://hal.science/hal-02157635/document>, 2020.
- [BDM21] Stéphane Balac, Monique Dauge, and Zoïs Moitier. Asymptotics for 2d whispering gallery modes in optical micro-disks with radially varying index. *hal-02528150*, version 2, 2021.
- [Bel87] Mikhail Belishev. An approach to multidimensional inverse problems for the wave equation. *Dokl. Akad. Nauk SSSR*, 297(3) :524–527, 1987.
- [BGM21] Lucie Baudouin, Pamela Godoy, and Alberto Mercado. Carleman estimates for the wave equation in heterogeneous media with non-convex interface. working paper or preprint, April 2021.
- [BJ20] Yannick Guedes Bonthonneau and Malo Jézéquel. FBI transform in Gevrey classes and Anosov flows. *preprint, arXiv :2001.03610*, 2020.
- [BKL16] Roberta Bosi, Yaroslav Kurylev, and Matti Lassas. Stability of the unique continuation for the wave operator via Tataru inequality and applications. *J. Differential Equations*, 260(8) :6451–6492, 2016.

- [BLR92] Claude Bardos, Gilles Lebeau, and Jeffrey Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30 :1024–1065, 1992.
- [BMO07] Lucie Baudouin, Alberto Mercado, and Axel Osses. A global carleman estimate in a transmission wave equation and application to a one-measurement inverse problem. *Inverse Problems*, 23(1) :257, 2007.
- [BP02] Lucie Baudouin and Jean-Pierre Puel. Uniqueness and stability in an inverse problem for the Schrödinger equation. *Inverse Problems*, 18(6) :1537–1554, 2002.
- [Cal58] Alberto Pedro Calderón. Uniqueness in the Cauchy problem for partial differential equations. *Amer. J. Math.*, 80 :16–36, 1958.
- [Car39] Torsten Carleman. Sur un problème d’unicité pour les systèmes d’équations aux dérivées partielles à deux variables indépendantes. *Ark. Mat. Astr. Fys.*, 26B(17) :1–9, 1939.
- [CdHKU19] Peter Caday, Maarten V. de Hoop, Vitaly Katsnelson, and Gunther Uhlmann. Reconstruction of piecewise smooth wave speeds using multiple scattering. *Trans. Amer. Math. Soc.*, 372(2) :1213–1235, 2019.
- [Chr15] Hans Christianson. Unique continuation for quasimodes on surfaces of revolution : Rotationally invariant neighbourhoods. In *Annales de l’Institut Fourier*, volume 65, pages 1617–1645, 2015.
- [Cor07] Jean-Michel Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.
- [DCFL<sup>+</sup>17] Michele Di Cristo, Elisa Francini, C-L Lin, S Vessella, and J-N Wang. Carleman estimate for second order elliptic equations with lipschitz leading coefficients and jumps at an interface. *J. Math. Pures Appl.*, 108(2) :163–206, 2017.
- [DGL06] Belhassen Dehman, Patrick Gérard, and Gilles Lebeau. Stabilization and control for the nonlinear Schrödinger equation on a compact surface. *Math. Z.*, 254(4) :729–749, 2006.
- [DJ18] Semyon Dyatlov and Long Jin. Semiclassical measures on hyperbolic surfaces have full support. *Acta Math.*, 220(2) :297–339, 2018.
- [DJN22] Semyon Dyatlov, Long Jin, and Stéphane Nonnenmacher. Control of eigenfunctions on surfaces of variable curvature. *J. Amer. Math. Soc.*, 35(2) :361–465, 2022.
- [DOP02] Anna Doubova, Axel Osses, and Jean-Pierre Puel. Exact controllability to trajectories for semilinear heat equations with discontinuous diffusion coefficients. *ESAIM Control Optim. Calc. Var.*, 8 :621–661, 2002.

- [DR77] Szymon Dolecki and David L. Russell. A general theory of observation and control. *SIAM J. Control Optim.*, 15(2) :185–220, 1977.
- [DS99] Mouez Dimassi and Johannes Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1999.
- [DS07] Hongjie Dong and Wolfgang Staubach. Unique continuation for the Schrödinger equation with gradient vector potentials. *Proceedings of the American Mathematical Society*, 135(7) :2141–2149, 2007.
- [DSF05] David Dos Santos Ferreira. Sharp  $L^p$  Carleman estimates and unique continuation. *Duke Mathematical Journal*, 129(3) :503 – 550, 2005.
- [EKPV06] Luis Escauriaza, Carlos E. Kenig, Gustavo Ponce, and Luis Vega. On uniqueness properties of solutions of Schrödinger equations. *Communications in Partial Differential Equations*, 31(12) :1811–1823, 2006.
- [Eva98] Lawrence C. Evans. *Partial differential equations*. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1998.
- [FdLKBH01] Vera M. Fernandez de Lima, Joao E. Kogler, Jocelyn Bennaton, and Wolfgang Hanke. Wave onset in central gray matter - its intrinsic optical signal and phase transitions in extracellular polymers. *An. Acad. Bras. Ciênc.*, 73(3) :1678–2690, 2001.
- [Fil22] Spyridon Filippas. Quantitative unique continuation for wave operators with a jump discontinuity across an interface and applications to approximate control. *arXiv preprint arXiv :2210.04634*, 2022.
- [Fil23] Spyridon Filippas. Whispering gallery modes for a transmission problem. *ESAIM : COCV*, 29 :75, 2023.
- [FLL23] Spyridon Filippas, Camille Laurent, and Matthieu Léautaud. Unique continuation for Schrödinger operators with partially Gevrey coefficients. *work in progress*, 2023.
- [FLVW16] Elisa Francini, C-L Lin, Sergio Vessella, and J-N Wang. Three-region inequalities for the second order elliptic equation with discontinuous coefficients and size estimate. *Journal of Differential Equations*, 261(10) :5306–5323, 2016.
- [FS09] Leonid Friedlander and Michael Solomyak. On the spectrum of the Dirichlet Laplacian in a narrow strip. *Israel journal of mathematics*, 170 :337–354, 2009.
- [Gag18] Ludovick Gagnon. Sufficient Conditions for the Controllability of Wave Equations with a Transmission Condition at the Interface. 28 pages, 30 figures, December 2018.
- [Gev18] Maurice Gevrey. Sur la nature analytique des solutions des équations aux dérivées partielles. premier mémoire. In *Annales Scientifiques de l'École Normale Supérieure*, volume 35, pages 129–190, 1918.

- [GHL90] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry*. Universitext. Springer-Verlag, Berlin, second edition, 1990.
- [GL20] Jeffrey Galkowski and Matthieu Léautaud. Control from an interior hypersurface. *Transactions of the American Mathematical Society*, 373(5) :3177–3233, 2020.
- [Hel88] Bernard Helffer. *Semi-classical analysis for the Schrödinger operator and applications*, volume 1336 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.
- [HM22] Luc Hillairet and Jeremy L. Marzuola. Eigenvalue spacing for 1d singular Schrödinger operators, 2022.
- [Hör63] Lars Hörmander. *Linear Partial Differential Operators*. Springer-Verlag, Berlin, 1963.
- [Hör67] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119 :147–171, 1967.
- [Hör85] Lars Hörmander. *The Analysis of Linear Partial Differential Operators*, volume III. Springer-Verlag, 1985. Second printing 1994.
- [Hör90] Lars Hörmander. *An introduction to complex analysis in several variables*, volume 7 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, third edition, 1990.
- [Hör92] Lars Hörmander. A uniqueness theorem for second order hyperbolic differential equations. *Comm. Partial Differential Equations*, 17(5-6) :699–714, 1992.
- [Hör94] Lars Hörmander. *The analysis of linear partial differential operators. IV*, volume 275 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1994. Fourier integral operators, Corrected reprint of the 1985 original.
- [Hör97] Lars Hörmander. On the uniqueness of the Cauchy problem under partial analyticity assumptions. In *Geometrical optics and related topics (Cortona, 1996)*, volume 32 of *Progr. Nonlinear Differential Equations Appl.*, pages 179–219. Birkhäuser Boston, Boston, MA, 1997.
- [HS84] Bernard Helffer and Johannes Sjöstrand. Multiple wells in the semiclassical limit. I. *Comm. Partial Differential Equations*, 9(4) :337–408, 1984.
- [IK06] Alexandru D. Ionescu and Carlos E. Kenig. Uniqueness properties of solutions of Schrödinger equations. *Journal of Functional Analysis*, 232(1) :90–136, 2006.
- [Isa93] Victor Isakov. Carleman type estimates in an anisotropic case and applications. *Journal of Differential Equations*, 105(2) :217–238, 1993.

- [JK85] David Jerison and Carlos E. Kenig. Unique continuation and absence of positive eigenvalues for Schrodinger operators. *Annals of Mathematics*, 121(3) :463–488, 1985.
- [Joh60] Fritz John. Continuous dependence on data for solutions of partial differential equations with a prescribed bound. *Comm. Pure Appl. Math.*, 13 :551–585, 1960.
- [KRS87] Carlos E. Kenig, Alberto Ruiz, and Christopher D. Sogge. Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. *Duke Math. J.*, 54(1) :329–347, 1987.
- [Lau10] Camille Laurent. Global controllability and stabilization for the nonlinear Schrödinger equation on some compact manifolds of dimension 3. *SIAM J. Math. Anal.*, 42(2) :785–832, 2010.
- [Lau14] Camille Laurent. Internal control of the Schrödinger equation. *Mathematical Control and Related Fields*, 4(2) :161–186, 2014.
- [Leb92a] Gilles Lebeau. Contrôle analytique. I. Estimations a priori. *Duke Math. J.*, 68(1) :1–30, 1992.
- [Leb92b] Gilles Lebeau. Contrôle de l'équation de Schrödinger. *J. Math. Pures Appl. (9)*, 71(3) :267–291, 1992.
- [Leb96] Gilles Lebeau. Équation des ondes amorties. In *Algebraic and geometric methods in mathematical physics (Kaciveli, 1993)*, volume 19 of *Math. Phys. Stud.*, pages 73–109. Kluwer Acad. Publ., Dordrecht, 1996.
- [Ler19] Nicolas Lerner. *Carleman Inequalities : An Introduction and More*, volume 353. Springer, 2019.
- [Lew18] Mathieu Lewin. *Théorie spectrale et mécanique quantique*. Lecture notes. <https://hal.archives-ouvertes.fr/cel-01935749v4>, 2018.
- [Lio88] Jacques-Louis Lions. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, volume 8 of *Recherches en Mathématiques Appliquées*. Masson, Paris, 1988.
- [LL16] Camille Laurent and Matthieu Léautaud. Uniform observability estimates for linear waves. *ESAIM Control Optim. Calc. Var.*, 22(4) :1097–1136, 2016.
- [LL19] Camille Laurent and Matthieu Léautaud. Quantitative unique continuation for operators with partially analytic coefficients. Application to approximate control for waves. *J. Eur. Math. Soc. (JEMS)*, 21(4) :957–1069, 2019.
- [LL20] Camille Laurent and Matthieu Léautaud. *Unique continuation and applications*. Lecture notes. unpublished, <http://leautaud.perso.math.cnrs.fr/files/UCPApplications.pdf>, 2020.
- [LL21a] Camille Laurent and Matthieu Léautaud. Logarithmic decay for linear damped hypoelliptic wave and Schrödinger equations. *SIAM J. Control Optim.*, 59(3) :1881–1902, 2021.

- [LL21b] Camille Laurent and Matthieu Léautaud. Observability of the heat equation, geometric constants in control theory, and a conjecture of Luc Miller. *Anal. PDE*, 14(2) :355–423, 2021.
- [LL21c] Camille Laurent and Matthieu Léautaud. On uniform observability of gradient flows in the vanishing viscosity limit. *J. Éc. polytech. Math.*, 8 :439–506, 2021.
- [LL22] Camille Laurent and Matthieu Léautaud. Tunneling estimates and approximate controllability for hypoelliptic equations. *Memoirs of the American Mathematical Society*, 276(1357), 2022.
- [LL23a] Camille Laurent and Matthieu Léautaud. Carleman estimates for waves with Lipschitz metrics and applications. *work in progress*, 2023.
- [LL23b] Camille Laurent and Matthieu Léautaud. Lectures on unique continuation for waves. *submitted, arxiv.org/pdf/2307.02155*, 2023.
- [LL23c] Camille Laurent and Matthieu Léautaud. Uniform observation of semiclassical Schrödinger eigenfunctions on an interval. *Tunisian J. Math.*, 5 :125–170, 2023.
- [LR95] Gilles Lebeau and Luc Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20 :335–356, 1995.
- [LR10] Jérôme Le Rousseau and Luc Robbiano. Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations. *Arch. Rational Mech. Anal.*, 105 :953–990, 2010.
- [LR11] Jérôme Le Rousseau and Luc Robbiano. Local and global Carleman estimates for parabolic operators with coefficients with jumps at interfaces. *Invent. Math.*, 183 :245–336, 2011.
- [LRL13] Jérôme Le Rousseau and Nicolas Lerner. Carleman estimates for anisotropic elliptic operators with jumps at an interface. *Analysis & PDE*, 6(7) :1601–1648, 2013.
- [LRLR13] Jérôme Le Rousseau, Matthieu Léautaud, and Luc Robbiano. Controllability of a parabolic system with a diffuse interface. *Journal of the European Mathematical Society*, 15(4) :1485–1574, 2013.
- [LRLR22] Jérôme Le Rousseau, Gilles Lebeau, and Luc Robbiano. *Elliptic Carleman Estimates and Applications to Stabilization and Controllability, Volume I : Dirichlet Boundary Conditions on Euclidean Space*. Progress in Nonlinear Differential Equations and Their Applications. Springer International Publishing, 2022.
- [Mas67] Kyûya Masuda. A unique continuation theorem for solutions of the Schrödinger equations. *Proceedings of the Japan Academy*, 43(5) :361–364, 1967.

- [MCCP17] Fanny Morin, Matthieu Chabanas, Hadrien Courtecuisse, and Yohan Payan. Biomechanical modeling of brain soft tissues for medical applications. In *Biomechanics of Living Organs. Hyperelastic Constitutive Laws for Finite Element Modeling*, pages 127–146. Academic Press, 2017.
- [Mil74] Keith Miller. Nonunique continuation for uniformly parabolic and elliptic equations in self-adjoint divergence form with Hölder continuous coefficients. *Arch. Rational Mech. Anal.*, 54 :105–117, 1974.
- [MOR08] Alberto Mercado, Axel Osses, and Lionel Rosier. Inverse problems for the Schrödinger equation via Carleman inequalities with degenerate weights. *Inverse Problems*, 24(1) :015017, 18, 2008.
- [Olv74] Frank W. J. Olver. *Asymptotics and special functions*. Computer Science and Applied Mathematics. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1974.
- [Phu10] Kim Dang Phung. Waves, damped wave and observation. In *Some problems on nonlinear hyperbolic equations and applications*, volume 15 of *Ser. Contemp. Appl. Math. CAM*, pages 386–412. Higher Ed. Press, 2010.
- [Pli63] Andrzej Pliś. On non-uniqueness in Cauchy problem for an elliptic second order differential equation. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 11 :95–100, 1963.
- [PV99] Georgi Popov and Georgi Vodev. Resonances near the real axis for transparent obstacles. *Communications in mathematical physics*, 207(2) :411–438, 1999.
- [Rob91] Luc Robbiano. Théorème d’unicité adapté au contrôle des solutions des problèmes hyperboliques. *Comm. Partial Differential Equations*, 16(4-5) :789–800, 1991.
- [Rob95] Luc Robbiano. Fonction de coût et contrôle des solutions des équations hyperboliques. *Asymptotic Anal.*, 10 :95–115, 1995.
- [RT74] Jeffrey Rauch and Michael Taylor. Exponential decay of solutions to hyperbolic equations in bounded domains. *Indiana Univ. Math. J.*, 24 :79–86, 1974.
- [Rus71a] David L. Russell. Boundary value control of the higher-dimensional wave equation. *SIAM Journal on Control*, 9(1) :29–42, 1971.
- [Rus71b] David L. Russell. Boundary value control theory of the higher-dimensional wave equation, part ii. *SIAM Journal on Control*, 9(3) :401–419, 1971.
- [RZ98] Luc Robbiano and Claude Zuily. Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients. *Invent. Math.*, 131(3) :493–539, 1998.
- [Sim83] Barry Simon. Instantons, double wells and large deviations. *Bull. Amer. Math. Soc. (N.S.)*, 8(2) :323–326, 1983.

- [Sog92] Christopher D. Sogge. Uniqueness in cauchy problems for hyperbolic differential operators. *Transactions of the American Mathematical Society*, 333(2) :821–833, 1992.
- [Sym83] William W. Symes. A trace theorem for solutions of the wave equation, and the remote determination of acoustic sources. *Math. Methods Appl. Sci.*, 5(2) :131–152, 1983.
- [Tat95] Daniel Tataru. Unique continuation for solutions to PDE’s; between Hörmander’s theorem and Holmgren’s theorem. *Comm. Partial Differential Equations*, 20(5-6) :855–884, 1995.
- [Tat97a] Daniel Tataru. Carleman estimates, unique continuation and controllability for anizotropic pde’s. *Contemporary Mathematics*, 209 :267–280, 1997.
- [Tat97b] Daniel Tataru. Carleman estimates, unique continuation and controllability for anizotropic PDEs. In *Optimization methods in partial differential equations (South Hadley, MA, 1996)*, volume 209 of *Contemp. Math.*, pages 267–279. Amer. Math. Soc., Providence, RI, 1997.
- [Tat99a] Daniel Tataru. *Carleman estimates, unique continuation and applications*. Lecture notes. unpublished, <https://math.berkeley.edu/~tataru/papers/ucpnotes.ps>, 1999.
- [Tat99b] Daniel Tataru. Unique continuation for operators with partially analytic coefficients. *J. Math. Pures Appl. (9)*, 78(5) :505–521, 1999.
- [T’j00] Laurent T’joën. Uniqueness in the Cauchy problem for quasi-homogeneous operators with partially holomorphic coefficients. *Osaka J. Math.*, 37(4) :925–951, 2000.
- [TW09] Marius Tucsnak and Georges Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts : Basel Textbooks. Birkhäuser Verlag, Basel, 2009.
- [TX07] Roberto Triggiani and Xiangjin Xu. Pointwise Carleman estimates, global uniqueness, observability, and stabilization for Schrödinger equations on Riemannian manifolds at the  $H^1(\Omega)$ -level. In *Control methods in PDE-dynamical systems*, volume 426 of *Contemp. Math.*, pages 339–404. Amer. Math. Soc., Providence, RI, 2007.
- [YDdH<sup>+</sup>17] Chunquan Yu, Elizabeth A. Day, Maarten V. de Hoop, Michel Campillo, and van der Hilst Robert D. Mapping mantle transition zone discontinuities beneath the central pacific with array processing of ss precursors. *J. Geophys. Res.*, 122(12) :10364–10378, 2017.
- [Zwo12] Maciej Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.