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Julien Trevisan

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Julien Trevisan. Convergence en loi de l'erreur dans le problème des points d'un réseau. Systèmes dynamiques [math.DS]. Université Paris Cité, 2022. Français. NNT : 2022UNIP7246 . tel-04421115

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Université Paris Cité
École doctorale n°386 : Sciences Mathématiques de Paris Centre

Institut de Mathématiques de Jussieu-Paris Rive Gauche

Convergence en loi de l'erreur dans le problème des points d'un réseau

Par Julien TREVISAN

Thèse de doctorat de Mathématiques fondamentales

Dirigée par Bassam Fayad

Présentée et soutenue publiquement le 29/09/2022

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Convergence en loi de l'erreur dans le problème des points d'un réseau

Julien Trevisan

20 juillet 2022

Remerciements

2019-2022. Un extrait condensé de ces trois années laborieuses et tumultueuses que contient ce manuscrit de thèse. Bien sûr, je suis l'homme derrière ces lignes. Mais derrière moi se trouvent une foule de personnes et d'institutions qui me sont précieuses, qui font partis de moi et en qui je me reconnais. Est-ce que ce manuscrit aurait pu voir le jour sans elles, c'est-à-dire sans moi ? De manière assurée, je ne le crois pas. Je tiens donc à les remercier avec ces quelques lignes, nécessairement trop courtes.
La première personne que je veux remercier est la personne qui a bien voulu encadrer ce travail : M. Bassam Fayad. Merci à lui pour la liberté qu'il m'a accordé, pour le développement de mon esprit auquel il a bien voulu contribuer avec son énergie, son intuition géniale, sa rigueur, ses relectures attentives. Plus d'une fois j'aurais pu me perdre en chemin s'il ne m'avait pas rattrapé.
Merci aussi à Marie-Claude Arnaud, ma tutrice, pour les rendez-vous réguliers que j'ai pu avoir avec elle et des conseils qu'elle a pu me donner.
Je suis aussi très reconnaissant envers Yann Bugeaud et Dmitry Kleinbock qui ont bien voulu rapporter cette thèse. Merci aussi à Valérie Berthé, Dmitry Dolgopyat, Sébastien Gouëzel, Jean-François Quint qui ont accepté de faire partie de mon jury. Merci à eux, et aux rapporteurs, pour l'intérêt qu'ils ont porté à ma thèse.
Je remercie également l'Institut de Mathématiques de Jussieu-Paris Rive Gauche pour m'avoir accueilli chaleureusement au sein du laboratoire, pour m'avoir assisté dans mes démarches et pour m'avoir permis de participer à des séminaires à l'étranger. Je remercie également l'université Paris Cité pour m'avoir permis de donner des travaux dirigés et des travaux pratiques à des étudiants en licence. Une respiration au sein de mon activité de recherche. Merci aux étudiants pour leur accueil, j'espère leur avoir dispensé un enseignement qui leur sera utile pour le reste de leur vie, qui leur a permis une ouverture de leur esprit et peut-être donner l'envie de continuer sur cette belle voie que sont les mathématiques.
Merci aussi aux amis du Vent Se Lève pour m'avoir accueilli parmi eux. De riches et passionnantes discussions, une ouverture et une formation à d'autres sujets que ceux directement liés à mon travail, de la fraternité, du sens ... Voilà en quelques mots comment je résumerai mon expérience au sein de ce média qui essaie, contre vents et marées, d'apporter sa pierre à l'édifice.
Cette ouverture et ce sens, j'ai aussi pu les trouver dans les belles rencontres que j'ai pu faire dans le 13ème arrondissement. Alexis, Anne-Laure, Antoine, Kahina, Maxime, Mathilde, Omar, Sofiane, merci à vous.
Merci à mes amis et anciens camarades de l'ENS Paris-Saclay, Jules, Guilhem, Vianney, Florent, Cheikh. Je suis heureux d'avoir pu intégrer cette école car cela m'a donné la possibilité de vous rencontrer, de belles personnes, très agréables, toujours prêts à aider.

Que de douces années passés avec vous à l'école, mais qui se sont écoulées trop vite ... Je sais néanmoins que nous maintiendrons des liens forts et ce quelque soient les vents sur lesquels la vie nous portera.
Merci aussi à tous mes amis de la maison du Mexique (ou en lien avec celles-ci). Jeffrey, Mario, Timo, Sonia, Mary Carmen, Paolina, nos soirées entre amis constituent d'agréables souvenirs qui valsent toujours dans un coin de mon esprit et qui sont amenées à perdurer, je l'espère.
Merci aux amis de « Bueno si breve », Pauline, Rodolphe, Hector pour les discussions de philosophie antique intenses (et particulièrement ardue pour ma part, il faut bien le reconnaître), toujours suivies d'un agréable moment convivial. J'espère que ces moments, sous une forme ou une autre, seront amenées à perdurer.
Merci à David pour son amitié et nos discussions - parfois ardentes - assurément pléthoriques comme relevé précédemment par tes soins. Je sens qu'elles nous font progresser tous les deux et j'imagine que c'est aussi ton cas. Merci à toi aussi et à Julie, ta compagne, pour m'avoir fait découvrir votre région d'origine, même si la conduite de la voiture a été quelque peu décoiffant ...
Merci à ma famille, mes parents Flavio et Patricia, mon frère Mathieu, ma belle-sœur Anne, ma nièce Louise, mes oncles et tantes Jean-François, Jean-Luc, Giuliana, Michèle, Serge, Simone, mes cousins et mes cousines, Alexandra, Jonathan, Jérémie, Julie, Léa, Ludovic, Tristan, mon grand-père Dino. Vous êtes mes racines qui me permettent de me maintenir debout et vous revoir me fait toujours beaucoup de bien. Merci aussi à mon oncle défunt, Léonard, pour avoir là toutes ces années durant. J'aurais aimé que tu sois présent à ma soutenance ... Je sais qu'elle t'aurait, à toi aussi, rendu fier ...
Enfin comment terminer ces remerciements sans remercier celle qui m'accompagne chaque jour ? Alexandra, merci de ta présence au quotidien et d'avoir choisi d'être avec moi, de participer à mes joies. Le soutien mutuel que nous avons l'un pour l'autre m'est fondamental. Merci aussi de m'avoir fait rencontrer ta très belle famille. Merci à eux de m'avoir accueilli comme l'un des leurs. Leur rencontre a été une de mes grandes leçons d'humanisme et m'a permis d'apprendre tellement ...
Je ne peux m'empêcher de penser à ces mots qu'a tenu Jaurès dans son discours du 31 Juillet 1892 au « Grand lycée » de Toulouse : « c'est que, plus les conditions accoutumées de l'activité individuelle sont parallèles à la marche des faits et menacées peut-être par l'approche d'une crise [...] plus il importe aussi, jeunes gens, que vous sachiez créer en vous-mêmes [...] des individualités énergiques et résistantes. Il faut que vous appreniez à dire « moi », non par les témérités de l'indiscipline ou de l'orgueil, mais par la force de la vie intérieure. ». J'espère que ce modeste manuscrit contribuera à la croissance de la vie intérieur de ses lecteurs, tout comme son écriture l'a faite pour moi. Et qu'elle permettra, comme le reste de mon action future, le « [jaillissement] dans la vieille forêt humaine [de] l'immortelle fraîcheur des sources. ».

Table des matières

1 Introduction **7**

1.1 Réseaux de $\mathbb{R}^d$ 8

1.2 L'espace des réseaux unimodulaires $\mathscr{S}_d$ 12

1.3 Le problème des points d'un réseau 17

1.4 Nouveaux résultats 23

1.5 A brief introduction for non-french speakers 33

2 The case of boxes **36**

2.1 Introduction 37

2.2 Probability preliminaries 39

2.3 Preliminaries on the lattice space 43

2.4 Study of $S(\omega, L, T)$ 50

2.5 Asymptotical study of $\mathcal{R}$ 63

3 The case of ovals **84**

3.1 Introduction 85

3.2 Heuristic explanation and plan of the proof 89

3.3 Reduction to the study of the Siegel transform 93

3.4 Study of $S_{A,prime}(L, t)$ when $t \to \infty$ 103

3.5 Asymptotic study of $\tilde{S}_A(\omega, L)$ 111

4 The case of analytic and stricly convex sets **115**

4.1 Introduction 116

4.2 Heuristic explanation and plan of the proof 120

4.3 Reduction to the study of the Siegel transform 122

4.4 Study of $S_{A,prime}(L,t)$ when $t \to \infty$ 133

4.5 Asymptotic study of $\tilde{S}_A(\omega, L)$ and of $\tilde{S}_A(\omega_1, \omega_2, L)$ 144

4.6 Elements of proof of Theorem 12 145

5 Return to the case of boxes 146

5.1 Introduction 147

5.2 Calculation of a Fourier transform, heuristic and plan of the rest of the paper 151

5.3 Proof of Theorem 14 155

5.4 Proof of Theorem 13 157

5.5 Proof of Theorem 15 178

5.6 Proof of Theorem 16 182

6 The particular case of $\mathbb{Z}^d$ and boxes 189

6.1 Introduction 190

6.2 A bit of heuristic and plan of the paper 192

6.3 Simplification of the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ 193

6.4 Proof of Theorem 23 195

6.5 Proof of Theorem 24 198

Index of notations

The next list describes several notations that are being used within the body of this thesis

δ_t Geodesic flow over $\mathscr{S}_d$

$\hat{f}$ Fourier transform of the function f

$\mathbb{E}$ Expectation

$\mathbb{P}$ Probability

$\mathbb{V}$ Variance

$\mathcal{R}(tS+X,L)$ Difference between $N(tS+X,L)$ and $t^d \frac{\mathrm{Vol}(S)}{\mathrm{Covol}(L)}$

$\mathcal{S}(f)$ Siegel transform of the function f

$\mathscr{S}_d$ Space of unimodular lattices of $\mathbb{R}^d$

μ_d The unique Haar probability measure over $\mathbb{R}^d$

$\mathrm{Covol}(L)$ Covolume of the lattice L of $\mathbb{R}^d$

$\mathrm{Diag}(a_1,\cdots,a_d)$ Diagonal matrix that contains on the diagonal, in this order, $a_1,\ldots,a_d$

$f_*\mu$ Push forward of a measure μ by a function f

$N(tS+X,L)$ Cardinal number of the set $(tS+X)\cap L$ with L being a lattice and S being a set of $\mathbb{R}^d$

Chapitre 1

Introduction

Dans ce chapitre d'introduction, nous exposons tout d'abord le cadre mathématique minimal, nécessaire pour comprendre les différents travaux présentés dans ce manuscrit.

Dans la Section 1.1, nous rappelons la définition de réseaux de $\mathbb{R}^d$ et motivons leur introduction.

Dans la Section 1.2, nous nous intéressons à l'espace des réseaux dits unimodulaires $\mathscr{S}_d$ et donnons quelques propriétés géométriques, métriques et dynamiques de celui-ci. Nous parlons aussi du lien entre dynamiques sur $\mathscr{S}_d$ et approximations diophantiennes.

Dans la Section 1.3, nous énonçons le problème des points d'un réseau sous une forme abstraite et générale, expliquons son lien avec différents domaines des mathématiques et faisons un état de l'art à ce sujet.

Dans la Section 1.4, nous présentons les différents résultats originaux obtenus lors de la rédaction de cette thèse. *Ces résultats seront encadrés.*

In Section 1.5, the non-french speaker will find a brief presentation of the mathematical problems addressed in this dissertation.

Sommaire

1.1 Réseaux de $\mathbb{R}^d$ 8

1.2 L'espace des réseaux unimodulaires $\mathscr{S}_d$ 12

1.3 Le problème des points d'un réseau 17

1.3.1 Formulation du problème 17

1.3.2 Résultats dans le cas déterministe 18

1.3.3 Approche avec de l'aléatoire 20

1.4 Nouveaux résultats 23

1.4.1 Estimation de l'erreur $\mathcal{R}$ lorsque le réseau L est aléatoire et lorsque S est un parallélogramme P. 23
1.4.2 Estimation de l'erreur $\mathcal{R}$ lorsque le réseau L est aléatoire et lorsque S est une ellipse $\mathcal{E}$ 25
1.4.3 Estimation de l'erreur $\mathcal{R}$ lorsque le réseau L est aléatoire et lorsque S est un corps strictement convexe analytique 28
1.4.4 Estimation de l'erreur $\mathcal{R}$ lorsque le réseau L est fixé, lorsque t est aléatoire et lorsque S est un rectangle 30
1.5 A brief introduction for non-french speakers 33

1.1 Réseaux de $\mathbb{R}^d$

Définition d'un réseau de $\mathbb{R}^d$

Commençons par introduire la notion fondamentale au sein de ce manuscrit : celle des réseaux de $\mathbb{R}^d$ avec $d \in \mathbb{N} - \{0\}$. Une première manière de définir un réseau de $\mathbb{R}^d$ est de dire que c'est un ensemble de points obtenus en pavant $\mathbb{R}^d$ à partir d'un parallélotope dont l'un des sommets est 0.

Une autre manière de définir un réseau de $\mathbb{R}^d$ est de dire que c'est un sous-groupe discret de $\mathbb{R}^d$ qui engendre, en tant que $\mathbb{R}$-espace vectoriel, $\mathbb{R}^d$.

Enfin, une troisième manière équivalente de définir un réseau de $\mathbb{R}^d$ est de dire que c'est un sous-groupe discret L de $\mathbb{R}^d$ tel que $\mathbb{R}^d/L$ soit compacte.

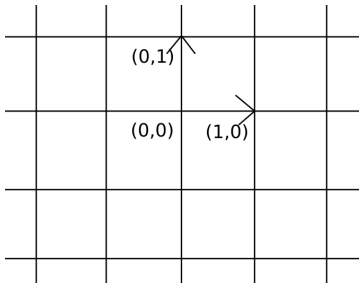
En fait, tout réseau de $\mathbb{R}^d$ est de la forme $M\mathbb{Z}^d$, avec $M \in GL_d(\mathbb{R})$ et l'espace des réseaux de $\mathbb{R}^d$ s'identifie à l'espace quotient $GL_d(\mathbb{R})/SL_d(\mathbb{Z})$.

Définition du réseau dual

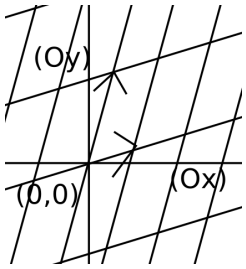
Avant d'illustrer la notion de réseau $\mathbb{R}^d$, nous rappelons qu'à partir d'un réseau L de réseau $\mathbb{R}^d$, un autre réseau, noté habituellement $L^\perp$, peut être construit et est appelé le réseau dual de L. Celui-ci est défini par l'égalité suivante :

$$L^\perp = \Big\{ v \in \mathbb{R}^2 \mid \forall l \in L, \ < l, v > \in \mathbb{Z} \Big\}.$$

Par ailleurs, si la matrice $M \in GL_d(\mathbb{R})$ représente le réseau L, $(M^{-1})^T$ représente le réseau dual $L^\perp$. Le réseau dual $L^\perp$ est un objet qui intervient de manière cruciale lorsque l'on fait de l'analyse harmonique avec des fonctions L-périodiques.



(a) Le réseau $\mathbb{Z}^2$



(b) Un réseau L différent de $\mathbb{Z}^2$

FIGURE 1.1 : Deux exemples de réseau de $\mathbb{R}^2$

Illustration et définition d'un domaine fondamental

Sur la Figure 1.1, nous représentons deux réseaux de $\mathbb{R}^2$: à gauche le réseau $\mathbb{Z}^2$, à droite un réseau L différent de $\mathbb{Z}^2$ qui n'est pas à maille carrée. Des vecteurs qui engendrent le réseau, en tant que groupe additif, ont été dessinés sur chacune des deux images. Sur la figure de droite on a aussi dessiné les axes de coordonnées.

On voit sur ces deux images qu'il y a une maille (carrée dans le cas de $\mathbb{Z}^2$ et, dans le second cas, c'est un parallélogramme) qui permet de reconstituer le réseau. Cette observation colle avec la première définition de réseau. Une telle maille est ce qui s'appelle un domaine fondamentale du réseau associé.

Plus formellement, soit L un réseau de $\mathbb{R}^d$. On dit que D est un domaine fondamental mesurable de L si D est mesurable et si on a :

$$\dot{\bigcup_{l \in L}} l + D = \mathbb{R}^d.$$

Tous les domaines fondamentaux mesurables de L sont de même mesure de Lebesgue finie. Cette mesure commune est appelée covolume du réseau L et est notée $\mathrm{Covol}(L)$. Par ailleurs, les réseaux de $\mathbb{R}^d$ qui sont de covolume 1 sont dits unimodulaires et l'espace des réseaux unimodulaires s'identifie à $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$. Il sera noté par la suite $\mathscr{S}_d$.

Sur la Figure 1.2, on a dessiné un réseau L et colorié en rouge l'aire correspondant à la coaire (on utilise le terme coaire à la place du terme covolume dans le cas de la dimension 2).

Applications de la théorie des réseaux

Une première utilisation des réseaux en mathématiques a été faite en 1801 par Gauss dans [33] avec l'utilisation du réseau $\mathbb{Z}[i]$. Puis, dans le livre [65] de Minkowski publié pour la première fois en 1896, les réseaux ont été étudiés de manière systématique. Il y a énoncé et prouvé son fameux théorème, dit théorème du corps convexe de Minkowski :

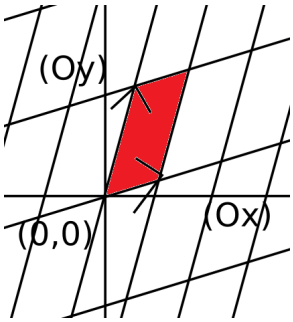


FIGURE 1.2 : Un réseau L en dimension 2 et sa coaire

Théorème ([65])**.** *Soit $C \subset \mathbb{R}^d$ une partie mesurable, symétrique et convexe et soit L un réseau de $\mathbb{R}^d$. Supposons que $2^d Covol(L) < Vol(C)$. Alors il existe $l \in L - \{0\}$ tel que $l \in C$.*

La conclusion reste valable si l'on suppose que $2^d Covol(L) \leqslant Vol(C)$ et que le convexe C est compact.

Signalons au passage que la seconde partie du théorème a été généralisée dans ce qu'on appelle le second théorème de Minkowski, lui aussi énoncé et prouvé dans [65], et qu'une autre généralisation se trouve dans [91].

Le théorème du corps convexe de Minkowski a eu des applications importantes en théorie des nombres. Il permet en effet de prouver les deux théorèmes suivants :

Théorème (Fermat-Euler)**.** *Soit p un nombre premier. Si $p = 1 \pmod 4$ alors il existe $a, b \in \mathbb{Z}$ tel que $p = a^2 + b^2$.*

Théorème (Théorème des quatre carrés - Lagrange)**.** *Tout entier $n \in \mathbb{N}$ est somme de quatre carrés.*

Il faut aussi noter que les réseaux de $\mathbb{R}^d$ ont de nombreuses applications dans d'autres domaines. Ils sont par exemple utilisés en cryptographie (voir à ce sujet [17]).

Désignons par $\|\cdot\|$ la norme euclidienne usuelle de $\mathbb{R}^d$. Le problème du plus court vecteur, c'est-à-dire le fait de trouver de manière algorithmique, pour un réseau L donné, un vecteur $l \in L$ non nul qui réalise $\min_{l \in L - \{0\}} \|l\|$, est un problème qui se trouve au cœur de la solution de différents problèmes informatiques : calculer des facteurs irréductibles de polynômes (voir à ce sujet [45] et [30]), calculer des polynômes minimaux de nombres algébriques (voir [45]) etc.

Enfin le problème de réduction des réseaux, c'est-à-dire le fait, pour un réseau donné, de trouver une base « sympathique » qui engendre un réseau en tant que groupe additif, est au cœur de nombreux problèmes informatiques (voir à ce sujet la thèse de Carine Jaber [44]).

Généralisations de la notion de réseau de $\mathbb{R}^d$

Pour terminer cette section, signalons deux généralisations de la notion de réseau de $\mathbb{R}^d$. La première est la notion de cristal topologique (voir à ce sujet [94]). La seconde est la généralisation suivante qui nous sera utile pour la suite et que l'on tire de [67]. Soit G un groupe de Lie. Soit Γ un sous-groupe de G. Appelons μ une mesure de Haar (invariante par multiplication à gauche) du groupe G et rappelons que les mesures de Haar invariante à gauche sont proportionnelles entre elles. Un domaine fondamental (au sens métrique) $\mathcal{D}$ de Γ relativement à G est un sous-ensemble mesurable de G tel que :

- $\mathcal{D}\Gamma = G$
- pour tout $\gamma \in \Gamma - \{e\}$, où e est l'élément neutre de G, $\mu(\mathcal{D}\gamma \cap \mathcal{D}) = 0$

Signalons au passage que les domaines fondamentaux définis précédemment sont des domaines fondamentaux au sens métrique.

On dit que Γ est un réseau de G si :

- Γ est discret
- il existe (et donc tous) un domaine fondamental (au sens métrique) Γ relativement à G qui est de mesure μ finie

Lorsque Γ est un réseau de G, il existe une unique mesure sur G/Γ qui soit de probabilité et qui soit G-invariante à gauche et à droite.

Un réseau L de $\mathbb{R}^d$ est aussi un réseau en ce sens. Plus intéressant, l'espace $SL_d(\mathbb{Z})$ est un réseau de $SL_d(\mathbb{R})$. Pour la preuve de ce résultat, on pourra consulter [5], [68] ou encore [13]. La clé pour le prouver, et le résultat est intéressant en soi pour comprendre ce qu'est $\mathscr{S}_d$, est le fait suivant. Soient $s, t > 0$. Appelons :

$$\mathbf{A} = \Big\{a = \mathrm{Diag}(a_{11}, \cdots, a_{dd}) \mid \det(a) = 1, \forall i,\, a_{ii} > 0\Big\},$$

$$A_s = \Big\{a \in \mathbf{A} \mid \frac{a_{ii}}{a_{i+1i+1}} \leqslant s,\, \forall 1 \leqslant i < d\Big\},$$

$$\mathbf{N} = \Big\{T \in \mathbb{M}_d(\mathbb{R}) \mid T \textit{ est triangulaire supérieure et les coefficients diagonaux valent 1}\Big\} \textit{ et}$$

$$N_t = \Big\{u \in \mathbf{N} \mid |u_{ij}| \leqslant t,\, \forall 1 \leqslant i < j \leqslant d\Big\}.$$

Théorème. *Si $s \geqslant \frac{2}{\sqrt{3}}$ et $t \geqslant \frac{1}{2}$ alors $G = SO_d(\mathbb{R})A_sN_tSL_d(\mathbb{Z})$.*

Un ensemble de la forme $SO_d(\mathbb{R})A_sN_t$ est appelé un domaine de Siegel.

En conséquence de ce théorème, $SL_d(\mathbb{Z})$ est un réseau de $SL_d(\mathbb{R})$ et il existe donc une unique mesure de probabilité qui soit $SL_d(\mathbb{R})$ invariante sur $\mathscr{S}_d$ et on la désignera par la suite par μ_d.

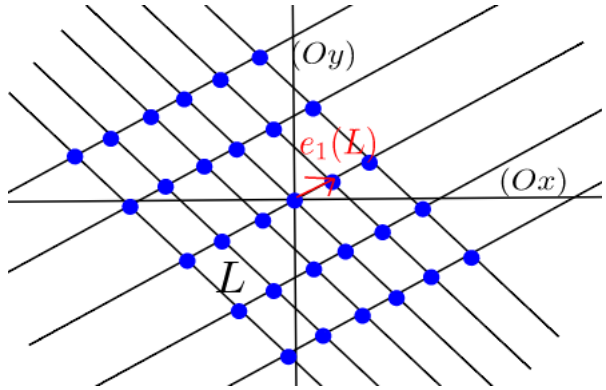


FIGURE 1.3 : Illustration de plus court vecteur

1.2 L'espace des réseaux unimodulaires $\mathscr{S}_d$

Topologie et géométrie à l'infini

Disons tout d'abord quelques mots quant à l'aspect topologique de $\mathscr{S}_d$. Une suite de réseaux unimodulaires $(L_n)_{n\in\mathbb{N}}$ converge, au sens de la topologie quotient, vers un réseau unimodulaire L si, et seulement si, on peut trouver une suite de d vecteurs $(l_1^{(n)}, \cdots, l_d^n)_{n\in\mathbb{N}}$ et d vecteurs $(l_1, \cdots, l_d)$ tels que pour tout $n \in \mathbb{N}$, $(l_1^{(n)}, \cdots, l_d^n)$ soit une $\mathbb{Z}$-base de L_n et tel que $(l_1, \cdots, l_d)$ soit une $\mathbb{Z}$-base de L et tel que pour tout $i \in \{1, \cdots, d\}$, $(l_i^{(n)})$ converge vers l_i.

À l'aide de cette remarque, on démontre le critère de compacité, dit de Mahler (voir à ce sujet [61], [14] et [59]) suivant : pour tout sous-ensemble M de $\mathscr{S}_d$, M est relativement compact si, et seulement si, il existe un voisinage U de 0 dans $\mathbb{R}^d$ tel que

$$\forall L \in M,\ L \cap U = \{0\}.$$

Formulé autrement, cela signifie que M est relativement compacte si, et seulement si, il existe une constante $C > 0$ tel que pour $L \in M$, $\|L\| \geqslant C$ où pour tout réseau unimodulaire $L \in \mathscr{S}_d$, $\|L\|$ est définie par

$$\|L\| = \min_{l\in L-\{0\}} \|l\|.$$

On a ainsi une caractérisation des compacts de $\mathscr{S}_d$.

Dans la Figure 1.3, le vecteur $e_1(L)$ est un des plus courts vecteurs du réseau L.

Par ailleurs, il existe une distance d sur $SL_d(\mathbb{R})$, qui induit sa topologie naturelle, qui est invariante à gauche par $SO_d(\mathbb{R})$, qui est invariante à droite par $SL_d(\mathbb{R})$ et telle que pour $a = (a_{ij}) \in \mathbf{A}$,

$$d(a, e) = (\sum_{i=1}^{d} (\log(a_{ii}))^2)^{\frac{1}{2}}$$

(voir Corollaire 1.3 dans [68]).

(a) C_2 (b) C_3

FIGURE 1.4 : Représentations de C_2 et de C_3

Comme d est $SL_d(\mathbb{R})$ invariante à droite et que $SL_d(\mathbb{Z})$ est discret, on peut munir $\mathscr{S}_d$ de la distance quotient, $\overline{d}$, qui est définie par : pour tout $x, x' \in SL_d(\mathbb{R})$,

$$\overline{d}(xSL_d(\mathbb{Z}), x'SL_d(\mathbb{Z})) = \inf_{\gamma,\gamma' \in SL_d(\mathbb{Z})} d(x\gamma, x'\gamma').$$

Cette distance induit la même topologie sur $\mathscr{S}_d$ que la topologie quotient.

Appelons C_d le cône convexe des $t = (t_1, \cdots, t_d)$ de $\mathbb{R}^d$ tels que $\sum_{i=1}^d t_i = 0$ et $t_i \leqslant t_{i+1}$ pour tout $1 \leqslant i < d$. On considère C_d muni de la distance d_{C_d} induite par la distance usuelle sur $\mathbb{R}^d$.

Alors, on dispose du résultat suivant, qui nous renseigne sur la géométrie à l'infini de $\mathscr{S}_d$: il existe une application f de C_d dans $\mathscr{S}_d$ telle qu'il existe $\lambda \geqslant 1$, $\epsilon > 0$ tels que pour tout $x, y \in C_d$,

$$-\epsilon + \frac{1}{\lambda} d_{C_d}(x, y) \leqslant \overline{d}(f(x), f(y)) \leqslant \lambda d_{C_d}(x, y) + \epsilon$$

et il existe $c \geqslant 0$ tel que pour tout $x' \in \mathscr{S}_d$, $\overline{d}(x', f(C_d)) \leqslant c$. On dit ainsi que C_d et $\mathscr{S}_d$ sont quasi-isométriques (pour en savoir plus sur cette relation, voir [34]). Ce résultat nous renseigne sur la géométrie « à l'infini » de $\mathscr{S}_d$.

Flot géodésique et théorie ergodique

Sur l'espace des réseaux unimodulaires $\mathscr{S}_d$ muni de μ_d, le flot géodésique préserve la mesure μ_d et est définie par :

$$L \in \mathscr{S}_d \longmapsto \delta_t L \in \mathscr{S}_d$$

où $t = (t_1, \cdots, t_d) \in H$ avec $H = \left\{(x_1, \cdots, x_d) \in \mathbb{R}^d \mid t_1 + \cdots + t_d = 0\right\}$ et

$$\delta_t = \mathrm{Diag}(e^{t_1}, \cdots, e^{t_d}).$$

Définissons le sous-groupe discret de $GL_d(\mathbb{R})$ suivant :

$$\Delta = \left\{\delta_t \mid t \in \mathbb{Z}^d \cap H\right\}$$

et rappelons qu'un sous-groupe G de $GL_d(\mathbb{R})$ est dit ergodique sur $\mathscr{S}_d$ (par rapport à l'action naturelle) si pour tout ensemble A mesurable de $\mathscr{S}_d$, si A est G-invariant alors, par rapport à μ_d, A est de mesure pleine ou bien est négligeable.

On dispose alors du théorème de Moore (voir à ce sujet [66] et [96]) qui énonce qu'un sous-groupe G de $GL_d(\mathbb{R})$ est ergodique sur $\mathscr{S}_d$ si, et seulement si, G n'est inclus dans aucun sous-groupe compact de $GL_d(\mathbb{R})$.

En particulier, Δ est ergodique et le flot (discret) associé l'est aussi. Définissons, pour tout $r > 0$,

$$\Delta_r = \left\{\delta_t \mid t \in H \cap \mathbb{Z}^d,\ \|t\| < r\right\}$$

et on appelle n_r le cardinal de Δ_r.

En utilisant le théorème 2.8 qui se trouve dans [56], on obtient le théorème ergodique individuel :

Théorème. *Soit ψ une fonction intégrable sur $\mathscr{S}_d$ relativement à la mesure μ_d. Alors, pour presque tout $L \in \mathscr{S}_d$, on dispose de l'égalité suivante :*

$$\lim_{r\to\infty} \frac{1}{n_r} \sum_{\delta\in\Delta_r} \psi(\delta L) = \int_{\mathscr{S}_d} \psi(L) d\mu_d(L).$$

Ce théorème s'étend aux sous-groupes discrets de $SL_d(\mathbb{R})$ qui sont ergodiques sur $\mathscr{S}_d$.

On peut montrer que $L \in \mathscr{S}_d \longmapsto \log(\|L\|)$ est intégrable sur $\mathscr{S}_d$. En conséquence de ce résultat et du théorème précédent, on obtient le résultat de Géométrie des nombres suivant. Soit g une matrice de $SL_d(\mathbb{R})$ tel que la suite $(g^j)_{j\in\mathbb{Z}}$ ne soit pas bornée. Alors, pour presque tout réseau $L \in \mathscr{S}_d$, on a :

$$\lim_{n\to\infty} \Big(\prod_{j=0}^{m-1} \|g^j L\|\Big)^m = \exp(J_d)$$

où $J_d = \int_{\mathscr{S}_d} \log(\|L\|) d\mu_d(L)$.

Ce théorème peut être considéré comme un analogue multidimensionnel du théorème de Khintchin concernant la convergence de la moyenne géométrique des quotients partiels pour presque tout nombre réel x (voir à ce sujet le chapitre 7 de [19]).

Avant de passer à un aspect plus géométrique du flot géodésique $(\delta_t)_{t\in H}$, intéressons-nous à la nature topologique des orbites $\mathcal{O}(L) = \{\delta_t L \mid t \in H\}$ avec $L \in \mathscr{S}_d$. Le critère de Malher donne que : $\mathcal{O}(L)$ est compact si, et seulement si,

$$\inf_{\delta\in H} \|\delta L\| > 0.$$

Suivant [82], on peut reformuler ce critère d'une autre manière. Appelons, pour tout $L \in \mathscr{S}_d$, pour tout $r > 0$:

$$\nu(L, r) = \min \Big\{ |l_1 \cdots l_d| \mid l = (l_1, \cdots, l_d) \in L \ et \ 0 < \| < r \Big\}$$

(on convient que si l'ensemble est vide, son minimum est nul) et

$$\mathrm{Num}(L) = \lim_{r \to \infty} \nu(L, r) = \inf\{|l_1 \cdots l_d| \mid l = (l_1, \cdots, l_d) \in L - \{0\}\}.$$

Alors, dans [85], il a été démontré que :

$$\mathrm{Num}(L) = d^{-\frac{d}{2}} \inf\{\|\delta L\| \mid \delta \in \Delta\}$$

et le critère de Mahler dans le cas de $\mathcal{O}(L)$ se reformule de la manière suivante : $\mathcal{O}(L)$ est compacte si, et seulement si, $\mathrm{Num}(L) > 0$. Un tel réseau sera dit admissible. L'ensemble des réseaux admissibles est μ_d-négligeable et est dense dans $\mathscr{S}_d$ (voir [82]).

Lorsque L vérifiera que $\nu(L, r) > 0$ pour tout r assez grand, on dira que le réseau est faiblement admissible. On notera qu'un réseau faiblement admissible n'est pas nécessairement admissible, la réciproque étant quant à elle vraie.
L'égalité précédente souligne qu'il y a un lien entre avoir un vecteur « proche » des axes de coordonnées et avoir un point de l'orbite $\mathcal{O}(L)$ qui soit « proche » de l'infini.

Hyperbolicité du flot géodésique

Passons à un aspect plus géométrique du flot géodésique $(\delta_t)_{t \in H}$. Celui-ci constitue sur $\mathscr{S}_d$ un flot partiellement hyperbolique au sens suivant. Appelons $\mathcal{T}\mathscr{S}_d$ l'espace tangent de $\mathscr{S}_d$. Alors, on a la décomposition suivante :

$$\mathcal{T}\mathscr{S}_d = E_0 \oplus \sum_{1 \leqslant q < p \leqslant d} E^+_{qp} \oplus E^-_{qp}$$

où E_0 est tangent à l'orbite de $(\delta_t)_{t \in H}$ et $E^\pm_{qp}$ sont des distributions invariantes de dimension 1. Les exposants de Lyapunov, pour $1 \leqslant q < p \leqslant d-1$ sont $\pm\lambda_{qp}$ où $\lambda_{qp} = t_q - t_p$ et, pour $p = d$, ce sont $\pm\lambda_q$ où $\lambda_q = \sum_{i=1}^{d-1} t_i + t_q$.

$E^\pm_{qp}$ sont tangent aux feuilletages $W^\pm_{qp}$ qui sont des feuilletages d'orbites pour les groupes $h^\pm_{qp}$ où $h^+_{qp}(u)$ est la matrice avec des 1 sur la diagonale, u dans la $p$ième colonne de la $q$ième ligne tandis que $h^-_{qp}(u)$ est sa transposée. C'est cette propriété essentielle qui est utilisée dans [28] et qui permet de prouver des théorèmes centraux limites dans le cadre d'approximations diophantiennes. Rappelons que des théorèmes limites sont obtenus dans un cadre plus général, le cadre hyperbolique, dans [25]. La méthode utilisée par Dolgopyat est d'ailleurs proche de celle utilisée par Le Borgne dans [58].

Cette propriété géométrique, cette propriété d'hyperbolicité, se traduit en terme

de mélange. Appelons H^s l'espace de Sobolev d'indice $s \in \mathbb{N}$ muni de sa norme usuelle relativement à l'espace $\mathscr{S}_d$ muni de μ_d. Posons pour toute fonction A intégrable sur $(\mathscr{S}_d, \mu_d)$:

$$\mu_d(A) = \int_{x \in \mathscr{S}_d} A(x) d\mu_d(x).$$

On appelle $A(M\cdot)$ la fonction qui à $x \in \mathscr{S}_d \longmapsto A(Mx) \in \mathbb{R}$.

On a alors, d'après [55] et d'après [71], qu'il existe $C > 0$, $\kappa > 0$, $s \in \mathbb{N} - \{0\}$ tel que pour tout $A, B \in H^s$:

$$|\mu\left(A(\cdot)B(\delta_t\cdot)\right) - \mu(A(\cdot))\mu(B(\cdot))| \leqslant C\|A\|_s\|B\|_s e^{-\kappa \max |t_i|}.$$

On dit aussi que le flot géodésique (δ_t) mélange (à l'ordre 1) à vitesse exponentielle.

Au passage, cela implique (voir [26]) qu'il existe $C > 0$ (indépendant de A et de B) tel que :

$$|\mu\left(A(\cdot)B(h_{qp}^{\pm}(u)\cdot)\right) - \mu(A(\cdot))\mu(B(\cdot))| \leqslant C\|A\|_s\|B\|_s|u|^{-\kappa}.$$

Ces propriétés de mélange ont été utilisées de manière clé, par exemple, dans [26]. La propriété de mélange exponentielle du flot géodésique a aussi été utilisée de manière clé dans une partie de nos travaux.

Flot géodésique et approximations diophantiennes

Le flot géodésique $(\delta_t)_{t \in H}$ est aussi très utile en théorie des approximations diophantiennes. En effet, soit $A \in \mathbb{M}_{m,n}(\mathbb{R})$. Soit $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. Supposons ψ décroissante. Une question centrale en théorie des approximations diophantiennes est la suivante : existe-t-il une infinité de $q \in \mathbb{Z}^n$ tel qu'il existe $p \in \mathbb{Z}^m$ tel que

$$\|Aq + p\| \leqslant \psi(\|q\|^n) \text{ ?}$$

Si la réponse à cette question est positive, on dit que A est ψ-approximable.

En 1842, Dirichlet dans [23] et [24] prouve, via le principe des tiroirs, que toute matrice A est ψ_0-approximable avec $\psi_0 = \frac{1}{x}$. À partir de là, on peut demander ce qu'il en est pour d'autres fonctions ψ, par exemple celles qui décroissent plus vite que ψ_0. Le théorème de Khintchine-Groshev donne des éléments de réponse. Il énonce que :

- Si $\int_1^\infty \psi(x)dx$ converge alors presque aucune (au sens de la mesure de Lebesgue) matrice A n'est ψ-approximable
- Si $\int_1^\infty \psi(x)dx$ diverge alors presque toute matrice A est ψ-approximable.

Ce théorème peut être redémontrer via, ce qu'on appelle, le *principe de Dani*. Ce principe énonce qu'il y a un lien entre, d'une part, l'approximation diophantienne d'une

matrice A et, d'autre part, la trajectoire, sous l'action du flot géodésique $(\delta_t)_{t\in H}$, du réseau $\Lambda_A\mathbb{Z}^{m+n}$ où

$$\Lambda_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}$$

avec I_m désignant la matrice identité d'ordre m et I_n celle d'ordre n.

Ce principe s'illustre par exemple dans le théorème suivant. Appelons c l'infinimum des $d > 0$ tel que toute matrice $A \in \mathbb{M}_{m,n}$ soit $d\psi_0$-approximable. Perron en 1921 a démontré dans [70] que c était strictement positif (c dépend a priori de m et de n). En d'autres termes, pour tout $m, n \geqslant 1$, il existe une matrice $A \in \mathbb{M}_{m,n}$, il existe $d > 0$ tel que A n'est pas $d\psi_0$-approximable. Une telle matrice sera dite mal-approximable. Le principe de Dani, démontré dans [21], s'illustre ainsi de la manière suivante : A est mal-approximable si, et seulement si, la trajectoire du flot géodésique de Λ_A, $\{\delta_t\Lambda_A\}_{t\in H}$, est bornée.

A ensuite été montré dans [54] que ce principe pouvait s'appliquer dans un cadre plus large. Grâce à cela on peut, *in fine*, retrouver le théorème de Khintchine-Groshev. Pour plus de détails à ce sujet, on pourra aussi consulter [53].

Signalons au passage que le principe de Dani a été appliquée dans [27] ce qui a permis d'obtenir de nouveaux résultats en théorie probabiliste des approximations diophantiennes. On le retrouve aussi dans [28], [8] et dans [63]. Dans ce dernier, ce principe est couplé à l'utilisation du théorème de Ratner (qui donne une description géométrique des mesures ergodiques et invariantes sous un flot unipotent, voir à ce sujet [67] ou [72]).

Enfin, notons que Beck dans [3] a aussi donné des résultats de nature probabiliste sur les approximations diophantiennes simultanées. Ces derniers peuvent aussi être interprétés comme des résultats de distribution de points de réseaux appartenant à un certain ensemble de réseaux. Ces réseaux qui appartiennent à cet ensemble sont d'ailleurs du type Λ_A. Toutefois, l'auteur dans cet article n'a pas eu recours au principe de Dani.

1.3 Le problème des points d'un réseau

1.3.1 Formulation du problème

Le problème des points d'un réseau se formule de manière générale de la manière suivante. Soit L un réseau de $\mathbb{R}^d$. Soit $t > 0$. Soit $X \in \mathbb{R}^d$. Soit S un ensemble mesurable de $\mathbb{R}^d$ de volume finie. Quel est le nombre de points $N(tS + X, L)$ du réseau L qui appartiennent à l'ensemble S d'abord dilaté de t puis translaté par le vecteur X ?

Lorsque l'ensemble S est suffisamment régulier (voir [9]), on peut montrer que l'on

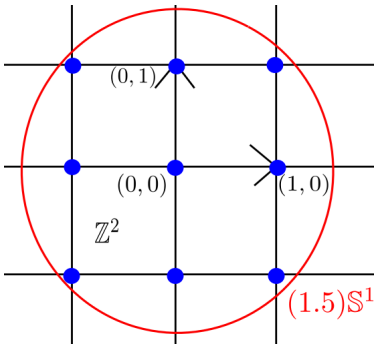


FIGURE 1.5 : Illustration du problème de Gauss avec $t = 1,5$

a :

$$N(tS + X, L) = t^d \frac{\mathrm{Vol}(S)}{\mathrm{Covol}(L)} + o(t^d).$$

Au passage, signalons que ce résultat a été étendu dans un cadre plus général par Gorodnik et Nevo dans [35].

Partant de là, il est naturel de s'intéresser à l'erreur $\mathcal{R}(tS + X, L)$ définie par :

$$\mathcal{R}(tS + X, L) = N(tS + X, L) - t^d \frac{\mathrm{Vol}(S)}{\mathrm{Covol}(L)}.$$

Estimer l'erreur $\mathcal{R}$ dans différentes situations permet notamment d'obtenir des informations sur les fonctions ζ de Epstein (voir à ce sujet [86], [77]), sur la distribution des valeurs du laplacien sur les tores $\mathbb{R}^2/L$ où L est un réseau (voir à ce sujet [62]) ou encore sur les approximations diophantiennes en utilisant le réseau Λ_A (voir à ce sujet [95]).

En pratique, via une dilatation, on se ramène toujours à l'étude de l'erreur $\mathcal{R}$ en supposant le réseau L unimodulaire.

1.3.2 Résultats dans le cas déterministe

Gauss, dans [32], est l'un des premiers à s'être intéressé à ce problème dans le cas particulier où $S = \mathbb{D}^2$, dans le cas où $X = 0$ et dans le cas où $L = \mathbb{Z}^2$. Il a montré, par un argument géométrique simple, que dans ce cas-là, on a l'estimation suivante sur $\mathcal{R}$:

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t)$$

où $O(t)$ désigne une quantité qui, lorsqu'elle est divisée par t, est bornée quand $t \to \infty$. Dans la Figure 1.5, on a représenté le cas abordé par Gauss avec $t = 1,5$. On voit dans ce cas-là avec la figure que $N((1,5)\mathbb{D}^2, \mathbb{Z}^2) = 9$.

Hardy, dans un article publié en 1917, à savoir [36], a fait la conjecture suivante : pour tout $\epsilon > 0$,

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^{\frac{1}{2}+\epsilon}).$$

Divers résultats se sont succédés depuis Gauss pour essayer d'avoir la meilleure estimation possible de $\mathcal{R}$ dans ce cas. Citons par exemple l'avancée de Hlawka [41] en 1950, où il a obtenu, via de l'analyse de Fourier, le résultat suivant valable en dimension $d \geqslant 2$:

$$\mathcal{R}(t\mathbb{D}^d, \mathbb{Z}^d) = O(t^{\frac{d(d-1)}{d+1}}).$$

En dimension 2, l'exposant 2/3 avait déjà été obtenu par Van der Corput dans sa thèse de 1919 [20].

En 1988, Iwaniec et Mozzochi ont prouvé dans [43] en se basant sur le travail [12] datant de 1986, qu'on avait l'estimation suivante : pour tout $\epsilon > 0$,

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^{\frac{7}{11}+\epsilon}).$$

(ce qui constitue une estimation meilleure que celle de Hlawka et Van der Corput puisque $2/3 \approx 0.667$ et $7/11 \approx 0.636$).
Huxley, dans le prolongement de ce qui a été fait par Iwaniec et Mozzochi, a obtenu en 2003 (voir [42]) la meilleure estimation suivante :

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^K \log(t)^\Lambda)$$

où $K = \frac{131}{208}$ et $\Lambda = \frac{18627}{8320}$. À notre connaissance, c'est le meilleur résultat connu à ce jour. Pour un historique un peu plus complet sur cette question, on pourra consulter l'introduction de l'article [6].

En dimension 3, le meilleur résultat a été obtenu pour l'heure par Heath-Brown dans [39], à savoir :

$$\mathcal{R}(t\mathbb{D}^3, \mathbb{Z}^3) = O(t^{\frac{21}{16}+\epsilon})$$

(la conjecture dans ce cas est que le meilleur exposant possible soit égale à 1 et l'article de Tsang [90] va dans ce sens). Ce résultat a par ailleurs été étendu en 2009 dans [15] à une classe de convexe réguliers auquel appartient le disque de dimension 3, $\mathbb{D}^3$.

Enfin, pour des dimensions supérieures, on pourra consulter l'article de Chamizo [16] publié en 1998.

Le problème des points d'un réseau est étudié avec d'autres types de formes, moins régulières. Par exemple, Skriganov a étudié dans divers articles le cas des polytopes convexes de dimension d. Rappelons qu'un polytope convexe P de dimension d est défini comme l'enveloppe convexe d'un nombre fini de points de $\mathbb{R}^d$. Par exemple, le carré ou encore les parallélépipèdes rectangles sont des polytopes convexes (de dimensions respectives 2 et 3). Dans la Figure 1.6, on a illustré le problème des points d'un réseau dans le cas où l'ensemble S est pris comme étant égal à un pentagone P (on a juste dessiné le contour du pentagone pour pouvoir visualiser les points de L qui sont à l'intérieur du pentagone).

Dans le cas d'un réseau admissible $L \in \mathscr{S}_d$ et d'un hyperparallélépipède P dont les côtés sont parallèles aux axes, Skriganov a prouvé à la fin du XXème siècle, dans [84] et dans [85] que :

$$\max_{X \in \mathbb{R}^d} |\mathcal{R}(tP + X, L)| = O(t^{d-1}).$$

Il a d'ailleurs reprouvé ce résultat via une autre approche, basée sur de la dynamique, dans [82].

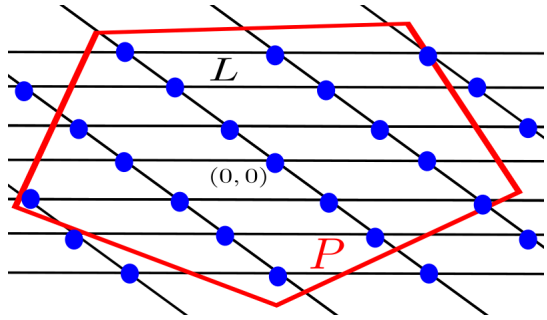


FIGURE 1.6 : Illustration du problème des points d'un réseau dans le cas d'un pentagone P

1.3.3 Approche avec de l'aléatoire

Le résultat de Skriganov

Dans [82] Skriganov a aussi prouvé le résultat suivant, ce qui constitue le résultat principal de son article : pour presque tout réseau $L \in \mathscr{S}_d$ unimodulaire, pour tout polytope P, pour tout $\epsilon > 0$,

$$\max_{X \in \mathbb{R}^d} |\mathcal{R}(tP + X, L)| = O(t^{d-1+\epsilon}).$$

Pour montrer ce résultat, Skriganov montre un premier résultat qui permet, *grosso modo*, d'estimer $\max_{X \in \mathbb{R}^d} |\mathcal{R}(tP + X, L)|$ via la somme ergodique :

$$S_1(L', r) = \sum_{\delta \in \Delta_r} \frac{1}{\|\delta L'\|^d}$$

avec $L' \in \mathscr{S}_d$.

Cette somme ergodique peut ensuite être estimée via le théorème ergodique que l'on a rappelé précédemment (toutefois, il faut prendre garde au fait que $L \longmapsto \|\delta L\|^d$ n'est pas intégrable sur $\mathscr{S}_d$).

Il faut noter qu'au début du XXème siècle, le cas de la dimension 2 avait déjà été étudié par Khintchine dans [50] et par Hardy et Littlewood dans [37]. Il avait été aussi étudié spécifiquement par Skriganov dans [83].

Skriganov dans [82] a donc traité le cas de la pire erreur lorsque l'ensemble considéré est un polytope convexe de dimension d et lorsque le réseau $L \in \mathscr{S}_d$ est tiré de manière aléatoire.

Suivant la même approche que Skriganov, mais de manière antérieure à celui-ci, Schmidt a prouvé en 1960 dans [79] que pour tout $d \geqslant 3$, pour tout ensemble mesurable S de $\mathbb{R}^d$, pour tout $\epsilon > 0$, pour presque tout réseau $L \in \mathscr{S}_d$,

$$\mathcal{R}(tS, L) = O(t^{\frac{d}{2}} \log(t)^{1+\epsilon}).$$

Il a aussi prouvé que pour $d = 2$, on avait le résultat suivant : pour tout ensemble mesurable S de $\mathbb{R}^2$, pour tout $\epsilon > 0$, pour presque tout réseau $L \in \mathscr{S}_2$,

$$\mathcal{R}(tS, L) = O(t \log(t)^{2+\epsilon}).$$

Dans [1], il a été prouvé qu'en se restreignant aux vecteurs premiers (c'est-à-dire les vecteurs d'un réseau qui ne sont pas des multiples non triviaux d'autres vecteurs du réseau), il n'était pas nécessaire de rajouter un facteur $\log(t)$ en dimension 2 par rapport aux dimensions supérieures.

Cette différence de traitement entre la dimension $d = 2$ et $d \geqslant 3$ tient au fait que Schmidt pour sa preuve a eu recours à la formule de Rogers d'ordre 2. Or celle-ci ne s'applique que lorsque $d \geqslant 3$ comme on le rappelle ci-après.

Formule de Siegel et de Rogers

Soit f une fonction mesurable de $\mathbb{R}^d$ dans $\mathbb{R}$ à support compact. On définit la transformée de Siegel de f, $\mathcal{S}(f)$ de la manière suivante : pour tout $L \in \mathscr{S}_d$

$$\mathcal{S}(f)(L) = \sum_{l \text{ premier} \in L} f(l).$$

Alors la formule de Siegel démontré dans [81] (qui est valable aussi en dimension $d = 2$) donne que :

$$\int_{L \in \mathscr{S}_d} \mathcal{S}(f)(L) d\mu_d(L) = \zeta(d)^{-1} \int_{x \in \mathbb{R}^d} f(x) dx$$

où $\zeta(d) = \sum_{n \geqslant 1} \frac{1}{n^d}$ tandis que la formule de Rogers d'ordre 2 donne que pour $d \geqslant 3$,

$$\int_{L \in \mathscr{S}_d} (\mathcal{S}(f)(L))^2 d\mu_d(L) = \zeta(d)^{-2} (\int_{x \in \mathbb{R}^d} f(x) dx)^2$$

$$+\zeta(d)^{-1} \int_{x \in \mathbb{R}^d} f(x)^2 dx + \zeta(d)^{-1} \int_{x \in \mathbb{R}^d} f(-x) f(x) dx.$$

Rogers l'a démontré dans [74] (article qui date de 1955). Siegel correspond, en quelque sorte, à une formule de Rogers d'ordre 1.

En réalité, en dimension 2, la formule de Rogers est encore valable ... en un certain sens. On a (voir [46] ou [47]) : il existe une isométrie $\iota : L^2(\mathbb{R}^2)_{paire} \longrightarrow L^2(\mathbb{R}^2)_{paire}$ tel que pour toute fonction f mesurable à support compact paire

$$\int_{L \in \mathscr{S}_2} (\mathcal{S}(f)(L))^2 d\mu_2(L) = \zeta(2)^{-2} (\int_{x \in \mathbb{R}^2} f(x) dx)^2 + 2\zeta(2)^{-1} \int_{x \in \mathbb{R}^2} f(x)^2 + f(x) \iota(f)(x) dx.$$

La formule de Rogers d'ordre 2 et la formule de Siegel ont joué un rôle clé dans notre travail, ainsi que dans les travaux [52], [51], [80], [78] ou encore dans [64]. Des analogues de la formule de Siegel peuvent aussi être trouvés dans d'autres cadres : voir, par exemple, [31].

Le cas d'un oval régulier avec dilatation aléatoire

Autour de l'année 1992, on s'est aussi intéressé à un autre type de problème. Que se passe-t-il, en dimension 2, quand l'ensemble S est choisi, que le réseau L est choisi

comme étant égal à $\mathbb{Z}^2$, que le paramètre de translation $X \in \mathbb{R}^2$ est fixé mais que le paramètre de dilatation t est pris aléatoire et grand ? Dans [10], Bleher a prouvé le résultat suivant.

Théorème. *Soit γ une courbe de classe C^7 simple, régulière, fermée, convexe et de courbure positive. Appelons Ω_γ le domaine délimité par la courbe γ. Supposons que $(0,0) \in \Omega_\gamma$. Soit $\alpha \in \mathbb{R}^2$. Soit ρ une densité de probabilité sur $[0,1]$. Supposons que t soit distribuée sur $[0,T]$ selon la mesure de probabilité $\frac{1}{T}\rho(\frac{t}{T})dt$.*

Alors, sous ces hypothèses, $\frac{\mathcal{R}(t\Omega_\gamma+\alpha,\mathbb{Z}^2)}{\sqrt{t}}$ converge en loi quand $T \to \infty$. Par ailleurs, la loi limite ne dépend pas de ρ, admet un moment d'ordre 2 et est d'espérance nulle.

Ce résultat constitue une généralisation de [38] et de [11]. Dans [38], Heath-Brown a obtenu ce résultat de convergence en loi dans le cas où $\Omega_\gamma = \mathbb{D}^2$ et où $\alpha = 0$ (il a aussi obtenu que le moment d'ordre 9 de la loi limite est finie). Dans [11], ce résultat avait été généralisé au cas où α est quelconque et le comportement de la densité de la loi limite en l'infini a, par ailleurs, été précisé (les auteurs obtiennent, en particulier, que la loi limite admet des moments de tous les ordres). La démarche suivie pour obtenir ces résultats a été précieuse pour nos recherches.

Un autre axe d'approche, empilement de sphères

Terminons cette sous-section en disant un bref mot sur les résultats obtenus par Kim dans [51] et dans [52] et par Schmidt dans [78] et dans [80]. Dans ces articles, les auteurs respectifs s'intéressent au problème de comptage que l'on a décrit dans cette section mais suivant un angle différent du notre. La question que ces auteurs se posent est la suivante : étant donné un ensemble mesurable S de $\mathbb{R}^d$, quelle est la probabilité qu'un réseau $L \in \mathscr{S}_d$ ait $k \in \mathbb{N}$ points dans S ? Autrement dit que vaut $\mu_d\left(N(S,L) = k\right)$? Et les auteurs s'intéressent à cette question en particulier quand d devient grand.

Kim, dans [51] obtient ainsi le résultat suivant. Soit S un ensemble mesurable de $\mathbb{R}^d$ symétrique par rapport à l'origine de volume V. Soit $\epsilon > 0$. Supposons que $k \in \mathbb{N}$ soit tel que $k \leqslant (\frac{d}{2})^{\frac{1}{2}-\epsilon}$ et supposons que $V \leqslant \frac{\sqrt{\frac{d}{2}}-k}{8}$. Appelons $F_{\frac{V}{2}}$ la fonction de répartition associée à une loi de Poisson de moyenne $\frac{V}{2}$. Alors, on a :

$$\lim_{d\to\infty} \mu_d\left(N(S, L - \{0\}) \leqslant k\right) - F_{\frac{V}{2}}(k) = 0.$$

De plus, la vitesse de convergence vers 0 dépend de ϵ mais pas de k. Ce résultat améliore, en un certain sens, un résultat précédent de [73] (k et V pouvant varier avec d). Il a ensuite été amélioré pour des ensembles S ouverts dans [52].

Schmidt a, quant à lui, prouvé le résultat suivant. Il existe des constants $c, C > 0$ telles que pour tout $V < Cd$, pour tout $d \geqslant 13$,

$$\mu_d\left(N(B(V), L) = 1\right) = e^{-\frac{V}{2}} + O(e^{-cd})$$

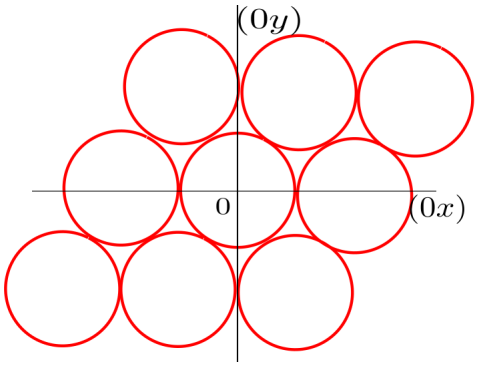


FIGURE 1.7 : Un empilement périodique de cercles

où $B(V)$ est la boule ouverte de centre 0 et de volume V. Rappelons par ailleurs que la probabilité qu'une variable aléatoire suivant une loi de Poisson de paramètre $\frac{V}{2}$ prenne la valeur 0 est de $e^{-\frac{V}{2}}$. Ce résultat recoupe donc le résultat de Kim dont on a parlé précédemment.

Ces résultats sont liés au problème de l'*empilement des sphères*. Ce problème s'énonce de la manière suivante. Existe-t-il un empilement périodique optimal des hypersphères dans l'espace euclidien $\mathbb{R}^d$? Si oui, quel est-il ? Optimal doit s'entendre au sens suivant : un tel empilement sera dit optimal si sa densité, c'est-à-dire si le volume d'une hypersphère divisé par le covolume du réseau sous-jacent, est maximale. Dans la Figure 1.7, on a illustré, en dimension 2, un empilement d'hypersphères (dans ce cas, des cercles) périodique.
La maximisation de la densité d'un empilement périodique d'hypersphères revient en fait à maximiser la quantité suivante :

$$\mathrm{Vol}(\mathcal{E})$$

où $\mathcal{E}$ est un ellipsoïde centré en 0 et tel que $\mathcal{E} \cap \mathbb{Z}^d = \{0\}$. Ce qui revient, *in fine*, à maximiser $V > 0$ tel qu'il existe $L \in \mathscr{S}_d$ tel que $N(B(V), L) = 1$. Pour en savoir plus à ce sujet, on pourra consulter [2], [22], [57], [75], [92], [76] ou encore [18].

1.4 Nouveaux résultats

1.4.1 Estimation de l'erreur $\mathcal{R}$ lorsque le réseau L est aléatoire et lorsque S est un parallélogramme P.

Dans le résultat principal de [82] obtenu par Skriganov, l'accent est mis sur le pire scénario pour le terme d'erreur lorsque le paramètre d'échelle t varie dans $\mathbb{R}_+$ *via* une borne supérieure presque sûre. De plus, dans ce cas et pour un réseau fixe, le terme d'erreur va osciller lorsque t va croître. Une autre approche consiste à étudier le comportement moyen du terme d'erreur lorsqu'il y a une certaine incertitude sur le réseau. En ajoutant de l'incertitude sur le réseau, et avec la bonne normalisation, on peut espérer se débarrasser du phénomène oscillatoire et obtenir une convergence en loi lorsque t tend vers l'infini. Ce type d'approche est inspiré des travaux de Kesten [48, 49] traitant des

propriétés diophantiennes statistiques des nombres réels : dans ces articles, il a étudié la distribution des écarts associés à des rotations circulaires dont les deux paramètres, α et x, sont pris aléatoirement sur le cercle. Il a montré qu'après normalisation par un facteur logarithmique, la distribution des écarts converge en loi vers une loi de Cauchy.

Dans le Chapitre 2, nous avons étudié le problème de Skriganov dans le cas où la dimension est $d = 2$ et où P est un parallélogramme et où le réseau L est distribué selon $\tilde{\mu}_2$, une mesure de probabilité absolument continue par rapport à μ_2 et telle que sa densité est régulière et bornée sur $\mathscr{S}_2$. On notera qu'une telle densité peut très bien être centrée dans un petit voisinage autour d'un réseau donné.

Appelons $\tilde{\lambda}_2$ une mesure de probabilité absolument continue par rapport à λ_2, la mesure de Haar normalisée sur $\mathbb{R}^2/L$ et la densité de $\tilde{\lambda}_2$ est choisie comme étant régulière. Le résultat principal du Chapitre 2 est alors le suivant :

Théorème 1. *Supposons que L soit distribué selon $\tilde{\mu}_2$ et X selon $\tilde{\lambda}_2$ alors, lorsque t tend vers l'infini, $\frac{\mathcal{R}(tP+X,L)}{\log(t)}$ converge en loi vers une loi de Cauchy centrée.*

Disons quelques mots sur ce résultat. Il dit qu'en moyenne autour d'un réseau $L \in \mathscr{S}_2$ et autour d'un vecteur de translation $X \in \mathbb{R}^2$, l'erreur $\mathcal{R}(tP + X, L)$ est de la taille de $\log(t)$ lorsque t devient grand.

Pour prouver ce résultat, le fait que la bonne normalisation de l'erreur $\mathcal{R}$ était $\log(t)$ a été suggéré par le résultat de Skriganov obtenu dans [82]. De plus, nous avons suivi une approche similaire à celle suivie dans [26], dans lequel les auteurs étendent les résultats de Kesten obtenus dans [48], [49] en dimension supérieure. Pourtant, même si l'approche est similaire, le contexte était différent et il y a eu de nouveaux ingrédients à introduire. Donnons maintenant quelques idées clés de la preuve.

La première idée est que l'étude de $\frac{\mathcal{R}(tP+X,L)}{\log(t)}$, lorsque $t \to \infty$, se ramène à l'étude, lorsque $T \to \infty$, de la somme géodésique à poids aléatoires normalisée suivante :

$$S_2(L,T) = \frac{1}{T}\sum_{t=0}^{T-1} \frac{\theta_t(L)}{\|\delta_{(t,-t)}L\|^2}$$

où $(\theta_t)_{t\in\mathbb{N}}$ est une famille de variables aléatoires réelles symétriques, identiquement distribuées, à support compact et non nulles. Pour obtenir ce résultat, nous utilisons des outils de l'analyse harmonique.

Dans un deuxième temps, en utilisant des outils de probabilité usuels, nous montrons que nous pouvons supposer que les termes qui interviennent dans S_2 correspondent aux temps t tels que $\|\delta_t L\|$ est petit (par rapport à $\frac{1}{\log(t)}$) et est un minimum local. Ainsi, de cette sorte, on peut obtenir de l'indépendance quand $t \to \infty$.

Troisièmement, en utilisant la théorie ergodique, nous montrons que nous pouvons supposer les θ_t indépendants entre eux et de L.

En partant de là, nous pouvons finalement prouver la convergence en loi vers une

loi de Cauchy centrée lorsque $T \to \infty$. Le fait que le flux géodésique sur $\mathscr{S}_2$ mélange à une vitesse exponentielle joue un rôle clé dans l'application du critère de convergence vers une loi de Cauchy centrée exposé dans [26].

Appelons

$$S(\omega, L, T) = \frac{1}{T} \sum_{t=0}^{T-1} \frac{\theta_t(\omega)}{\|\delta_{(t,-t)} L\|^2}$$

où $(\theta_t)_{t\in\mathbb{N}}$ sont des variables aléatoires partant de l'espace probabilisé $(\Omega, \mathbb{P}_1)$, réelles, symétriques, non nulles et à support compact et notons, qu'au passage, on prouve le résultat suivant :

Théorème 2. *Lorsque L est distribué selon la loi de probabilités $\tilde{\mu}_2$ et ω selon $\mathbb{P}_1$,*

$$S(\omega, L, T)$$

converge, quand $T \to \infty$, vers une loi de Cauchy centrée.

1.4.2 Estimation de l'erreur $\mathcal{R}$ lorsque le réseau L est aléatoire et lorsque S est une ellipse $\mathcal{E}$

Dans le Chapitre 3, toujours en suivant l'approche de Kesten, nous avons étudié le cas où la dimension d vaut 2, où $S = \mathcal{E}$ est une ellipse de centre 0 et où le réseau $L \in \mathscr{S}_2$ est aléatoire. Pour énoncer le résultat principal de ce chapitre, posons $\Pi = \{(k_1, k_2) \in \mathbb{Z}^2 \mid k_1 \wedge k_2 = 1,\ k_1 \geqslant 0\}$ où nous convenons que si $k_1 = 0$, $k_1 \wedge k_2 = 1$ implique que $k_2 = 1$ et rappelons la définition suivante :

Definition 1. *Pour tout $i \in \{1, 2\}$, nous appelons*

$$\|L\|_i = \min\{r > 0 \mid B_f(0, r) \text{ contient } i \text{ vecteurs de } L \text{ linéairement indépendants}\}$$

où $B_f(0, r)$ est le disque fermée de centre 0 poour la norme euclidienne usuelle $\|\cdot\|$ de rayon r. Ces quantités sont les minima successifs du réseau L.

En réalité, pour μ_2-presque tout réseau $L \in \mathscr{S}_2$, $\|L\|_2 > \|L\|_1$ et il existe un unique couple de vecteurs $(e_1(L), e_2(L))$ tel que $(e_1(L))_1 > 0$, $\|e_1(L)\| = \|L\|_1$, $(e_2(L))_1 > 0$ et $\|e_2(L)\| = \|L\|_2$.

Pour un réseau $L \in \mathscr{S}_2$ typique (au sens de la mesure μ_2), on appelle $\mathcal{P}_+(L)$ l'ensemble des vecteurs e de L tels que $e = k_1 e_1(L) + k_2 e_2(L)$ avec $(k_1, k_2) \in \Pi$.

Dans la Figure 1.8, on a dessiné un réseau L de $\mathbb{R}^2$ ainsi que le couple associé $(e_1(L), e_2(L))$. Avant de passer à la suite, rappelons le théorème suivant qui énonce les relations dites de Mahler en dimension 2 :

Théorème. *Pour tout entier k tels que $1 \leqslant k \leqslant 2$, on a*

$$1 \leqslant \|L\|_k \|L^\perp\|_{3-k}.$$

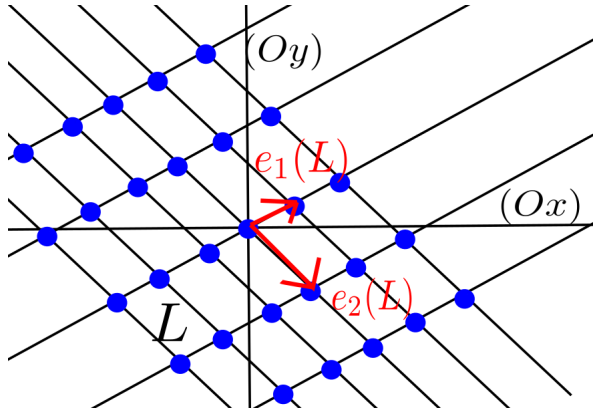


FIGURE 1.8 : Un réseau L et le couple correspondant $(e_1(L), e_2(L))$

Ce théorème nous informe donc, en particulier, qu'en dimension $d = 2$, si un réseau L admet un court vecteur petit (ou, avec une vision plus topologique, est proche de l'infini), il en est de même pour son réseau dual L. Pour en savoir plus à ce sujet, on consultera [14] et [59].

Rappelons aussi le fait qu'une variable aléatoire réelle Z est symétrique si $\mathbb{P}_Z = \mathbb{P}_{-Z}$ où, pour toute variable aléatoire X, $\mathbb{P}_X$ désigne la loi de la variable aléatoire X.

Soit $\tilde{\mu}_2$ une mesure de probabilité absolument continue par rapport à μ_2, de densité σ, supposée bornée et régulière sur $\mathscr{S}_2$, relativement à μ_2. Il y a deux cas différents qui sont abordés par le résultat principal du Chapitre 3. Le premier est lorsque $\tilde{\mu}_2$ est à support compact, *id est* lorsqu'il existe $\alpha > 0$ tel que :

$$\tilde{\mu}_2(\{L \in \mathscr{S}_2 \mid \|L\|_1 < \alpha\}) = 0. \tag{1.1}$$

Le second est lorsque $\tilde{\mu}_2$ n'est pas à support compact et vérifie la condition (plus forte) suivante : il existe $m > 0$, tel que pour tout $\alpha > 0$ tel que pour tout L qui appartient à l'évènement $(\|L\|_1 < \alpha)$,

$$\sigma(L) \geqslant m. \tag{1.2}$$

Un exemple de telle mesure $\tilde{\mu}_2$ est donnée par la mesure de Haar normalisée μ_2.

Soit $\mathbb{T}^\infty = (\mathbb{T}^1)^\Pi$ et appelons λ_∞ la mesure produit de Lebesgue normalisée sur $\mathbb{T}^\infty$. Soit M l'unique matrice, modulo l'action de $SL_2(\mathbb{Z})$, de $SL_2(\mathbb{R})$ transformant l'ellipse $\mathcal{E}$ de centre 0 en un disque de centre 0. Dans le Chapitre 3, le résultat principal qui y est exposé s'énonce ainsi de la manière suivante :

Théorème 3. *Pour tous nombres réels $a < b$,*

$$\lim_{t\to\infty} \tilde{\mu}_2\left(L \in \mathscr{S}_2 \mid \frac{\mathcal{R}(t\mathcal{E}, L)}{\sqrt{t}} \in [a,b]\right)$$

$$= (\lambda_\infty \times (M^{-1})_*\tilde{\mu}_2)\left((\theta, L) \in \mathbb{T}^\infty \times \mathscr{S}_2 \mid S(\theta, L^\perp) \in [a,b]\right)$$

où $(M^{-1})_*\tilde{\mu}_2$ désigne la mesure poussée en avant à partir de $\tilde{\mu}_2$ par $L \in \mathscr{S}_2 \longmapsto$

$M^{-1}L \in \mathscr{S}_2$ et où pour $\theta = (\theta_e) \in \mathbb{T}^\infty$,

$$S(\theta, L) = \frac{2}{\pi} \sum_{e \in \mathcal{P}_+(L)} \frac{\phi(\theta_e)}{\|e\|^{\frac{3}{2}}}$$

avec

$$\phi(\theta_e) = \sum_{m \geqslant 1} \frac{\cos(2\pi m \theta_e - \frac{3\pi}{4})}{m^{\frac{3}{2}}}. \tag{1.3}$$

De plus, $S(\theta, L^\perp)$ (ou, de manière équivalente, $S(\theta, L)$) :

- *converge presque partout*
- *est symétrique et est d'espérance égale à 0*
- *admet un moment d'ordre $1 + \kappa$ et ce pour tout κ tel que $0 \leqslant \kappa < \frac{1}{3}$*
- *admet des moments à tous les ordres p tels que $1 \leqslant p < \infty$ quand $\tilde{\mu}_2$ est à support compact*
- *n'admet pas de moment d'ordre $\frac{4}{3}$ quand $\tilde{\mu}_2$ n'est pas à support compact sous la condition (1.2).*

Ce résultat, essentiellement, nous dit qu'en moyenne autour d'un réseau $L \in \mathscr{S}_2$ l'erreur $\mathcal{R}(t\mathcal{E}, L)$ est de la taille de $\sqrt{t}$ lorsque t devient grand. C'est un résultat complémentaire à celui obtenu dans [10]. On en déduit d'ailleurs le corollaire suivant :

Corollaire 1. *Pour μ_2 presque tout réseau $L \in \mathscr{S}_2$, pour toute suite $(\tilde{\mu}_{2,n})$ de mesures de probabilités à support compact absolument continues par rapport à μ_2, de densités régulières telle que la suite associé des supports converge vers $\{L\}$, il existe une sous-suite $(n_k)_{k \in \mathbb{N}}$, il existe des réels $t_{n_k} \to \infty$ tel que $(\frac{\mathcal{R}(t_{n_k}\mathbb{D}^2, L')}{\sqrt{t_{n_k}}})$ converge en loi vers la loi de la variable aléatoire réelle $\theta \in (\mathbb{T}^\infty, \lambda_\infty) \longmapsto S(\theta, L^\perp)$.*

Donnons quelques éléments clés de la preuve du Théorème 3.

Tout d'abord, nous ramenons l'étude au cas où $\mathcal{E} = \mathbb{D}^2$ et nous utilisons des outils de l'analyse harmonique afin de ramener l'étude de la convergence en loi de $\mathcal{R}$ à l'étude de la convergence en loi de la somme partielle de la série de Fourier, évaluée en 0, associée à $X \longmapsto \mathcal{R}(t\mathbb{D}^2 + X, L)$. De plus, nous connaissons une approximation des coefficients de Fourier de cette fonction (voir [27] ou [10]). Nous pouvons alors nous ramener à étudier la somme partielle de cette série de Fourier approchée. Nous observons que cette dernière quantité peut être interprétée comme *une transformée de Siegel* (qui dépend ici de t, le paramètre de dilatation).
À ce stade de la preuve, la quantité que nous devons étudier est du même type que celle de [11]. Pourtant, dans [11], c'est le paramètre t qui est aléatoire ce qui permet aux auteurs d'utiliser la théorie des fonctions quasi-périodiques. Nous ne pouvons pas utiliser cette théorie dans notre cas car l'aléatoire porte sur le réseau L, ce qui nécessite une nouvelle approche.

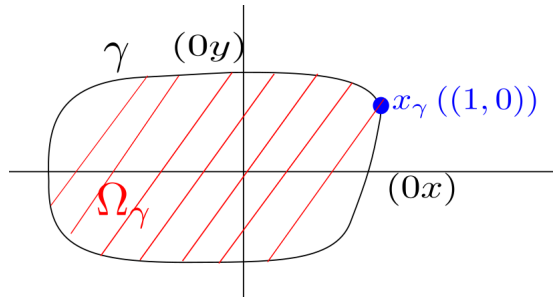


FIGURE 1.9 : Une courbe γ satisfaisant les hypothèses, ainsi que Ω_γ et $x_\gamma\,((1,0))$

L'étude peut finalement être réduite à ce que nous avons appelé dans le Chapitre 3 une transformée de Siegel modifiée avec des poids aléatoires. Cette réduction constitue le passage clé de la preuve et nécessite de faire de la dynamique sur $\mathscr{S}_2$ (voir Section 4 du Chapitre 3).
Nous prouvons finalement un résultat général de convergence presque sûre (et nous en profitons pour étudier l'existence des moments de la limite).

Il est intéressant de noter, par ailleurs, qu'un certain nombre de problèmes de comptage en moyenne se ramènent à l'étude de transformées de Siegel modifiées avec des poids aléatoires.

1.4.3 Estimation de l'erreur $\mathcal{R}$ lorsque le réseau L est aléatoire et lorsque S est un corps strictement convexe analytique

Dans le Chapitre 4, on énonce et on prouve deux résultats plus généraux que le Théorème 3.

Plus précisément, supposons que $S = \Omega_\gamma$ où γ est une courbe analytique simple, fermée et strictement convexe, où Ω_γ désigne le domaine délimité par γ et où $0 \in \Omega_\gamma$.

Appelons $x_\gamma(\xi)$ le point de γ où la normale unitaire extérieure à γ coïncide avec $\frac{\gamma}{\|\gamma\|}$ et appelons $\rho_\gamma(\xi)$ le rayon de courbure de γ au point $x_\gamma(\xi)$. Sur la Figure 1.9, on a dessiné une telle courbe γ, ainsi que son domaine Ω_γ ainsi que le point $x_\gamma\,((1,0))$.

Appelons $\mathbb{T}^{\infty,2} = (\mathbb{T}^1)^\Pi \times (\mathbb{T}^1)^\Pi$ et désignons par $\lambda_{\infty,2}$ la mesure de Lebesgue normalisée sur le produit $\mathbb{T}^{\infty,2}$.
Alors, avec les notations de la sous-section précédente, le premier résultat que l'on prouve dans le Chapitre 4 est le suivant :

Théorème 4. *Il existe une fonction de répartition $\mathcal{D}_\gamma(z)$ telle que pour tout réel z, on a :*

$$\lim_{t\to\infty} \tilde{\mu}_2\left(\frac{\mathcal{R}(t\Omega_\gamma, L)}{\sqrt{t}} \in]-\infty, z]\right) = \mathcal{D}_\gamma(z).$$

Dans le cas où Ω_γ est symétrique, $\mathcal{D}_\gamma(z)$ est la fonction de répartition de la variable

aléatoire réelle

$$S_\gamma(\theta, L^\perp) = \frac{2}{\pi} \sum_{e \in \mathcal{P}_+(L^\perp)} \frac{\rho_\gamma(e)\phi(\theta_e)}{\|e\|^{\frac{3}{2}}}$$

où $\theta = (\theta_e) \in \mathbb{T}^\infty$ *est tiré selon la loi de probabilité* λ_∞ *et* L *selon* $\tilde{\mu}_2$.

Dans le cas non symétrique, $\mathcal{D}_\gamma(z)$ *est la fonction de répartition de la variable aléatoire réelle*

$$S_\gamma(\theta, L^\perp) = \frac{1}{\pi} \sum_{e \in \mathcal{P}_+(L^\perp)} \frac{\phi_{\gamma,2}(\theta_e, e)}{\|e\|^{\frac{3}{2}}}$$

où $\theta = (\theta_{1,e}, \theta_{2,e}) \in \mathbb{T}^{\infty,2}$ *est tiré selon la loi de probabilité* $\lambda_{\infty,2}$ *et* L *selon* $\tilde{\mu}_2$ *et où*

$$\phi_{\gamma,2}(\theta_e, e) = \sum_{m \geqslant 1} \frac{\rho_\gamma(e)\cos(2\pi m\theta_{1,e} - \frac{3\pi}{4}) + \rho_\gamma(-e)\cos(2\pi m\theta_{2,e} - \frac{3\pi}{4})}{m^{\frac{3}{2}}}. \tag{1.4}$$

De plus, dans les deux cas, S_γ *admets les mêmes propriétés que* $S(\theta, L)$ *listées dans le Théorème* 3.

En réalité, et c'est le second résultat du Chapitre 4, le Théorème 4 peut être généralisé de la manière suivante :

Théorème 5. *Pour tout* $\alpha \in \mathbb{R}^2$, *il existe une fonction de répartition* $\mathcal{D}_{\gamma,\alpha}(z)$ *telle que pour tout réel* z, *on a :*

$$\lim_{t \to \infty} \tilde{\mu}_2 \left(\frac{\mathcal{R}(t\Omega_\gamma + \alpha, L)}{\sqrt{t}} \in]-\infty, z] \right) = \mathcal{D}_{\gamma,\alpha}(z).$$

Dans le cas où Ω_γ *est symétrique,* $\mathcal{D}_{\gamma,\alpha}(z)$ *est la fonction de répartition de la variable aléatoire réelle*

$$S_{\gamma,\alpha}(\theta, L^\perp) = \frac{2}{\pi} \sum_{e \in \mathcal{P}_+(L^\perp)} \frac{\rho_\gamma(e)\phi_\alpha(\theta_e, e)}{\|e\|^{\frac{3}{2}}}$$

où $\theta = (\theta_e) \in \mathbb{T}^\infty$ *est tiré selon* λ_∞ *et* L *selon* $\tilde{\mu}_2$ *et où*

$$\phi_\alpha(\theta_e, e) = \sum_{m \geqslant 1} \frac{\cos(2\pi m\theta_e + 2\pi m < \alpha, e > -\frac{3\pi}{4})}{m^{\frac{3}{2}}}.$$

Dans le cas non symétrique, $\mathcal{D}_{\gamma,\alpha}(z)$ *est la fonction de répartition de la variable aléatoire réelle*

$$S_\gamma(\theta, L^\perp) = \frac{1}{\pi} \sum_{e \in \mathcal{P}_+(L^\perp)} \frac{\phi_{\alpha,\gamma,2}(\theta_e, e)}{\|e\|^{\frac{3}{2}}}$$

où $\theta = (\theta_{1,e}, \theta_{2,e}) \in \mathbb{T}^{\infty,2}$ est tiré selon $\lambda_{\infty,2}$ et L selon $\tilde{\mu}_2$ et où

$$\phi_{\alpha,\gamma,2}(\theta_e, e) = \sum_{m \geqslant 1} \frac{\rho_\gamma(e)\cos(2\pi m\theta_{1,e} + 2\pi m < \alpha, e > -\frac{3\pi}{4})}{m^{\frac{3}{2}}}$$

$$+ \sum_{m \geqslant 1} \frac{\rho_\gamma(-e)\cos(2\pi m\theta_{2,e} - 2\pi m < \alpha, e > -\frac{3\pi}{4})}{m^{\frac{3}{2}}}.$$

De plus, dans les deux cas, S_γ admets les mêmes propriétés que $S(\theta, L)$ listés dans le Théorème 3.

Ainsi, dans ce cadre, même après translation par un vecteur $\alpha \in \mathbb{R}^2$, l'erreur $\mathcal{R}$ (pour $S = \Omega_\gamma$) en moyenne autour d'un réseau $L \in \mathscr{S}_2$ est de l'ordre de $\sqrt{t}$. C'est en partant de la lecture de [26] et de [10] que l'on a prouvé ces deux généralisations du Théorème 3.

Par ailleurs, pour prouver le Théorème 4, nous suivons la même approche que celle suivie pour prouver le Théorème 3. Le Théorème 5 est, quant à lui, une simple généralisation du Théorème 4.

1.4.4 Estimation de l'erreur $\mathcal{R}$ lorsque le réseau L est fixé, lorsque t est aléatoire et lorsque S est un rectangle

Dans le Chapitre 5, nous étudions le cas où $L \in \mathscr{S}_2$ est un réseau fixé et où le paramètre aléatoire est le paramètre de dilatation t qui est tiré selon la loi de probabilité $\rho(\frac{t}{T})\frac{1}{T}dt$ où ρ est une densité de probabilité sur $[0, 1]$. On choisit S comme étant égal à

$$\mathrm{Rect}(a, b) = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \leqslant a,\ |x_2| \leqslant b\}$$

où $a, b > 0$.

En d'autres termes, on suit l'approche déployée dans [11] et dans [10] et on l'applique dans le cas des rectangles (centrées en 0).

On pose, pour f une fonction localement intégrable sur $\mathbb{R}^2$:

$$\mathbb{E}_{X \in \mathbb{R}^2/L}(f) = \frac{1}{\mathrm{Covol}(L)} \int_{X \in \mathbb{R}^2/L} f(X)dX.$$

Posons aussi pour tout réseau $L \in \mathscr{S}_2$ faiblement admissible :

$$V(L, t) = \sum_{\substack{l \in L \\ 0 < \|l\| \leqslant t}} \frac{1}{l_1^2 l_2^2}.$$

Alors, on prouve dans le Chapitre 5, le résultat suivant :

Théorème 6. *Soit L un réseau unimodulaire admissible de $\mathbb{R}^2$.*

Alors, tout d'abord, il existe $C > 0$ tel que pour tout t assez grand :

$$\frac{1}{C}\log(t) \leqslant V(L,t) \leqslant C\log(t)$$

ce qui sera dénoté par $V(L,t) = \Theta(\log(t))$.

Ensuite, pour $S = Rect(a,b)$,

$$\mathbb{E}_{X\in L}\left(\left(\frac{\mathcal{R}(tS+X,L)}{\sqrt{V(L^\perp,t)}}\right)^2\right) \text{ converge en loi et en probabilité vers } \frac{1}{4\pi^4}$$

lorsque $t \in [0,T]$ est tiré selon $\frac{1}{T}\rho(\frac{t}{T})dt$ et lorsque $T \to \infty$.

Ce théorème signifie qu'en un certain sens l'erreur $\mathcal{R}$ dans ce cas est de l'ordre de $\sqrt{\log(t)}$. Nous avons, par ailleurs, dû d'abord moyenner en X car il semblerait que, même en laissant X aléatoire et en regardant si on obtient dans ce cas une convergence en loi, on aurait un phénomène d'oscillation qui nous empêcherait d'obtenir la convergence désirée.

La preuve de ce théorème réside sur l'usage d'outils d'analyse harmonique et sur des estimations de sommes discrètes par des intégrales.

À partir de cette approche et dans le même chapitre on démontre le résultat suivant :

Théorème 7. *Pour tout $\epsilon > 0$, pour μ_2- presque tout L, on a que*

$$V(L,t) = O(\log(t)^{2+\epsilon})$$

et

$$\frac{V(L,t)}{\log(t)^2} \underset{t\to\infty}{\to} \infty.$$

De plus, on a aussi que :

$$\mathbb{E}_{X\in L}\left(\left(\frac{\mathcal{R}(tP+X,L)}{\sqrt{V(L^\perp,t)}}\right)^2\right) \text{ converge en loi et en probabilité vers } \frac{1}{4\pi^4}$$

où $t \in [0,T]$ est tiré selon la loi de probabilité $\frac{1}{T}\rho(\frac{t}{T})dt$ et où $T \to \infty$.

Enfin, dans le cas où $a = b$ et où $L = \mathbb{Z}^2$ (qui n'a pas été abordé dans les cas précédents), on prouve aussi le résultat suivant dans le Chapitre 5 :

Théorème 8. *Pour tout $x \in \mathbb{R}$, lorsque $t \in [0,T]$ est tiré selon la loi de probabilité $\frac{1}{T}\rho(\frac{\cdot}{T})dt$ sur $[0,T]$ alors, lorsque $T \to \infty$, $\frac{\mathcal{R}(tRect(a,a)+(x,x),\mathbb{Z}^2)}{t}$ converge en loi.*

De plus, la loi limite β a un support compact inclus dans $[-4,4]$ et pour tout $k \in \mathbb{N}$, on a que

$$\int_{x\in\mathbb{R}} x^k d\beta(x) = a_k$$

où

$$a_k = \frac{4^k(1+(-1)^k)(y^{k+1}+(1-y)^{k+1})}{2(k+1)}$$

avec $y = |t_{2,0} - t_{1,0}|$ et où $t_{2,0}$ est le premier temps $t \geqslant 0$ tel que $-t+x \in \mathbb{Z}$ et $t_{1,0}$ est le premier temps $t \geqslant 0$ tel que $t+x \in \mathbb{Z}$.

En particulier, on voit que la normalisation est beaucoup plus importante qu'avant et dans ce cas l'erreur $\mathcal{R}$ est de l'ordre de t.

La preuve réside sur des calculs élémentaires et sur l'application du critère des moments (utilisé pour démontrer la convergence en loi). Dans le Chapitre 6, on généralise le résultat précédent à toute dimension $d \geqslant 2$.

Plus formellement, soit ρ une densité de probabilité sur $[0,1]$ et soient $a > 0$ et $\mathcal{C}(a)$ l'ensemble défini par

$$\mathcal{C}(a) = \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d \ |\forall i \in [1,d],\ |x_i| \leqslant a\}. \tag{1.5}$$

Avec ces notations, on prouve les résultats suivants dans le Chapitre 6 :

Théorème 9. *Pour tout $x \in \mathbb{R}$, quand $t \in [0,T]$ est tiré selon la loi de probabilité $\frac{1}{T}\rho(\frac{\cdot}{T})dt$ sur $[0,T]$ alors, quand $T \to \infty$, $\frac{\mathcal{R}(t\mathcal{C}(a)+X,\mathbb{Z}^d)}{t^{d-1}}$ où $X = (x, \cdots, x)$ converge en loi. De plus, la loi limite admet comme fonction caractéristique la fonction suivante*

$$\varphi(u) = \frac{\sin(d2^{d-1}uy) + \sin(d2^{d-1}u(1-y))}{d2^{d-1}u}$$

où $y = |t_{2,0} - t_{1,0}|$ avec $t_{2,0}$ le premier temps $t \geqslant 0$ tel que $-t+x \in \mathbb{Z}$ et $t_{1,0}$ le premier temps $t \geqslant 0$ tel que $t+x \in \mathbb{Z}$. D'ailleurs, on a $y = |1-2\{x\}|$ où $\{x\}$ désigne la partie fractionnaire de x.

Théorème 10. *Pour $x_1, \cdots, x_d$ des variables aléatoires réelles indépendantes identiquement distribuées selon la loi uniforme sur $[-\frac{1}{2}, \frac{1}{2}]$, quand $t \in [0,T]$ est tiré selon la loi de probabilité $\frac{1}{T}\rho(\frac{\cdot}{T})dt$ sur $[0,T]$ et quand t et $x_1, \cdots, x_d$ sont indépendants entre eux alors, quand $T \to \infty$, $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ avec $X = (x_1, \cdots, x_d)$ converge en loi.*

De plus, la loi limite admet comme fonction caractéristique la fonction suivante

$$\varphi(u) = (2\frac{1-\cos(2^{d-1}u)}{(2^{d-1}u)^2})^d.$$

On remarque que dans les deux cas abordés, la normalisation de l'erreur $\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)$ est en t^{d-1}. De plus, les deux cas étudiés correspondent à deux cas extrêmes : le cas du Théorème 9 est un cas où les x_i sont liés (en fait, ils sont tous égaux) tandis que le cas

du Théorème 10 est un cas où les x_i sont indépendants entre eux.

La preuve de ces deux théorèmes repose sur des calculs élémentaires et sur l'usage d'outils de probabilités de base.

1.5 A brief introduction for non-french speakers

Lattices of $\mathbb{R}^d$ We define a lattice of $\mathbb{R}^d$ as a discrete subgroup of $(\mathbb{R}^d, +)$ that generates, as a $\mathbb{R}$-vector space, $\mathbb{R}^d$. For every lattice L of $\mathbb{R}^d$ there exists F measurable such that $\mathbb{R}^d = \dot{\cup}_{l\in L} F + l$. Such F is call a fundamental domain of L. Its volume does not depend on F and is called the covolume of the lattice L. It is denoted by $\mathrm{Covol}(L)$. We say that a lattice L is unimodular if its covolume is equal to 1. The space of unimodular lattices $\mathscr{S}_d$ can be identified to the homogeneous space $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$. Moreover, the existence of the Siegel domains gives us that there exists μ_d a probability measure over $\mathscr{S}_d$ that is $SL_d(\mathbb{R})$ invariant. As a consequence, it defines the notion of a random unimodular lattice.

The space of unimodular lattices $\mathscr{S}_d$ On the space $\mathscr{S}_d$, the geodesic flow is identified to $(\delta_t)_{t\in H}$ where $H = \{(x_1, \cdots, x_d) \in \mathbb{R}^d \mid x_1 + \cdots + x_d = 0\}$. This flow conserves the Haar measure μ_d and the discrete underlying geodesic flow is ergodic. Furthermore, the geodesic flow on $\mathscr{S}_d$ is partially hyperbolic and mixes at exponential speed and its stable and unstable directions are tangent to orbit foliations given by unipotent groups. Moreover, there is a link, indicated by the Dani principle (*cf* [21]), between Diophantine approximation of a matrix $A \in \mathbb{M}_{m,n}(\mathbb{R})$ and the orbit of the unimodular lattice $\Lambda_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \mathbb{Z}^{m+n}$ under the action of the geodesic flow.

The lattice counting problem The lattice counting problem, which is the main subject of this dissertation, is the following. Let L be a lattice of $\mathbb{R}^d$. Let S be a measurable set of $\mathbb{R}^d$ with a finite volume. What is the number of points $N(tS + X, L)$ that belong to L and to the set $tS + X$ where $t > 0$ and $X \in \mathbb{R}^d$? In fact, when S is regular enough, one has $N(tS + X, L) = \frac{\mathrm{Vol}(S)}{\mathrm{Covol}(L)} t^d + o(t^d)$. The next step is to study the error term $\mathcal{R}(tS + X, L) = N(tS + X, L) - \frac{\mathrm{Vol}(S)}{\mathrm{Covol}(L)} t^d$. The historical Gauss circle problem is a particular case of this problem : it is the case when $S = \mathbb{D}^2$, $X = 0$ and $L = \mathbb{Z}^2$. Gauss in [32] proved that, in that case, the error was dominated by t. The still open Hardy's conjecture (see [36]) says that the error $\mathcal{R}$ is dominated by $t^{\frac{1}{2}+\epsilon}$ for every $\epsilon > 0$. There has been several improvements in this direction (see the introduction of the article [6] for more details).

Instead of considering a pure deterministic problem, we can add some randomness. For example, in [11], they proved that, when t is random and becomes large, for a fixed $\alpha \in \mathbb{R}^2$, when $S = \Omega_\gamma$ is a regular oval such that $0 \in \Omega_\gamma$, $\mathcal{R}(t\Omega_\gamma + \alpha, \mathbb{Z}^2)$ converges in distribution. In this dissertation, we study the behavior $\mathcal{R}$ when there is also some randomness.

Study of $\mathcal{R}$ when $S = P$ is a parallelogram and when the lattice $L \in \mathscr{S}_2$ is random In Chapter 2, we study the error $\mathcal{R}$ when the lattice L is random, when $X \in \mathbb{R}^2/L$ is random, when $S = P$ is a parallelogram and when t goes to infinity. The main result is Theorem 1 and it says that, when the error is normalized by $\log(t)$, there is a convergence in distribution towards a Cauchy centred distribution.

Study of $\mathcal{R}$ in the case of ellipses and when the lattice $L \in \mathscr{S}_2$ is random In Chapter 3, we study the error $\mathcal{R}$ when the lattice L is random, when $X = 0$ and when $S = \mathcal{E}$ is an ellipse and when t goes to infinity. The main result is Theorem 3 and it says that, when the error is normalized by $\sqrt{t}$, as suggested by [82] and by the Hardy's conjecture, there is a convergence in distribution towards an explicit and non trivial distribution. In Theorem 3 we also study the existence of the moments of the limit distribution.

Generalisation In Chapter 4, we extend the previous convergence in distribution result (with Theorem 4) to the case where $S = \Omega_\gamma$ is an oval such that the curve, given by its boundary, is analytical. We also show that such a result holds even if, instead of considering the set $t\Omega_\gamma$, one considers the set $t\Omega_\gamma + \alpha$ with $\alpha \in \mathbb{R}^2$ being fixed (see Theorem 5). Furthermore, the limit distribution is made explicit.

Study of $\mathcal{R}$ when $S =$ Rect is a rectangle, when the lattice $L \in \mathscr{S}_2$ is random and when t is random and goes to infinity In Chapter 5, we study several different cases. The first one is when $S =$ Rect is a rectangle centred on 0 with edges parallel to the coordinate axis, when L is an admissible lattice (in the sense of Skriganov, see [82]) and when t is random and becomes large. In that case, we show that, after having normalized the error by a quantity estimated by $\sqrt{\log(t)}$ and after having averaged on $X \in \mathbb{R}^2/L$, $\mathcal{R}^2$ converges towards a constant (depending on the covolume of L) (see Theorem 6).

The second one is when $S =$ Rect is a rectangle centred on 0 with edges parallel to the coordinate axis and when L is a typical lattice (*id est* when it is weakly admissible in the sense of Skriganov). In that case, we show that, after having normalized the error by a quantity estimated by $\log(t)$ and after having averaged on $X \in \mathbb{R}^2/L$, $\mathcal{R}^2$ converges towards a constant (depending on the covolume of L) (see Theorem 7).

The third one is when $S =$ Sq is a square centred on 0 with edges parallel to the coordinate axis and when $L = \mathbb{Z}^2$. In that case, we show that, with $x \in \mathbb{R}$, there is a non trivial convergence in distribution of the error $\frac{\mathcal{R}(t\mathrm{Sq}+(x,x),\mathbb{Z}^2)}{t}$ when t is random and becomes large (see Theorem 8).

Generalisation In Chapter 6, we generalise the last result in dimension $d \geqslant 3$. We set S to be an hypercube $\mathcal{C}$ centred on 0 with edges parallel to coordinate axis. We take $L = \mathbb{Z}^d$. We study the case when t is random and becomes large. Then, when $X = (x, \cdots, x) \in \mathbb{R}^d$, we show that $\frac{\mathcal{R}(t\mathcal{C}+X,\mathbb{Z}^d)}{t^{d-1}}$ converges in distribution towards a

non trivial distribution (see Theorem 9). Also, when the X_i are random, distributed according to the uniform distribution on $[-\frac{1}{2}, \frac{1}{2}]$, independent between them and from t, we show that $\frac{\mathcal{R}(t\mathcal{C}+X,\mathbb{Z}^d)}{t^{d-1}}$ converges in distribution towards a non trivial distribution (see Theorem 10).

Chapter 2

The case of boxes

(Résumé en français) On étudie l'erreur commise lorsque l'on estime le nombre de points d'un réseau unimodulaire qui se trouvent dans un parallélogramme dilaté et translaté par son aire. À l'aide d'un des travaux de Skriganov, on voit que cette erreur peut être vue comme une somme ergodique portant sur le flot géodésique discret sur l'espace des réseaux unimodulaires. En normalisant correctement et en utilisant des outils d'un travail de Fayad et Dolgopyat, on montre qu'un certain processus converge en loi vers un processus de Poisson. On en déduit que la somme ergodique converge en loi vers une loi de Cauchy centrée lorsque le réseau est distribué selon la mesure de Haar normalisée. Fort de cette expérience, on applique le même type d'approche, avec plus de difficultés, pour étudier le comportement asymptotique de l'erreur et on montre que l'erreur, normalisée par $\log(t)$ avec t le paramètre de dilatation du parallélogramme, converge en loi vers une loi de Cauchy centrée quand le paramètre de dilatation tend vers l'infini et lorsque le réseau est distribué selon la mesure de Haar normalisée et lorsque le vecteur de translation est aléatoire.

(English abstract) We study the error of the number of unimodular lattice points that fall into a dilated and translated parallelogram. By using an article from Skriganov, we see that this error can be compared to an ergodic sum that involves the discrete geodesic flow over the space of unimodular lattices. With the right normalization, we show, by using tools from a previous work of Fayad and Dolgopyat, that a certain point process converges in distribution towards a Poisson process and deduce that the ergodic sum converges towards a Cauchy centred distribution when the unimodular lattice is distributed according to the normalized Haar measure. Strong from this experience, we apply the same kind of approach, with more difficulties, to the study of the asymptotic behaviour of the error and show that this error, normalized by $\log(t)$ with t the factor of dilatation of the parallelogram, also converges in distribution towards a Cauchy centred distribution when the dilatation parameter tends to infinity and when the lattice and the vector of translation are random.

Contents

2.1 Introduction **37**
2.2 Probability preliminaries **39**
2.3 Preliminaries on the lattice space **43**
2.4 Study of $S(\omega, L, T)$ **50**
2.4.1 Introduction to the problem 50
2.4.2 Elimination of terms with too large a denominator 51
2.4.3 Existence of and centring on local minima 52
2.4.4 Study of the asymptotic behaviour of S_2 55
2.5 Asymptotical study of $\mathcal{R}$ **63**
2.5.1 First considerations on the error $\mathcal{R}$ 63
2.5.2 Single term study 64
2.5.3 Transfer of the problem to the checking of the hypothesis of Theorem 4 76
2.5.4 Verification of hypothesis of Theorem 4 77
2.5.5 Return to $\tilde{S}$ and proof of Theorem 1 81

2.1 Introduction

In this paper, we are going to take interest in a particular case of estimating the number of points of a lattice that belong to a given set. This type of problem has an ancient origin insofar as the Gauss circle problem is a problem of this type.
Let us give $d \in \mathbb{N} - \{0, 1\}$. Let us give $X \in \mathbb{R}^d$ and L a lattice of $\mathbb{R}^d$ and P a measurable set of $\mathbb{R}^d$ such that its volume is strictly positive and finite. We would like to evaluate the following cardinal number when $t \to \infty$:

$$N(tP + X, L) = |(tP + X) \cap L|.$$

Under a certain assumption of regularity on the set P, it can be shown that :

$$N(tP + X, L) = t^d \frac{\mathrm{Vol}(P)}{\mathrm{Covol}(L)} + o(t^d).$$

It is then natural to look at the error defined by :

$$\mathcal{R}(tP + X, L) = N(tP + X, L) - t^d \frac{\mathrm{Vol}(P)}{\mathrm{Covol}(L)}.$$

In the case where P is the unit disk $\mathbb{D}^2$, the Hardy's conjecture made in [36] states that we should have, for every $\epsilon > 0$,

$$\mathcal{R}(tD^2, \mathbb{Z}^2) = O(t^{\frac{1}{2}+\epsilon}).$$

One of the result in this direction was established by Iwaniec and Mozzchi in [43]. They have shown that for every $\epsilon > 0$,

$$\mathcal{R}(tD^2, \mathbb{Z}^2) = O(t^{\frac{7}{11}+\epsilon}).$$

In the case where P is a convex polytope and where $L \in \mathscr{S}_d$ (which is the space of unimodular lattices of $\mathbb{R}^d$, *id est* the space of lattices that have a covolume equal to 1) is typical, Skriganov has managed to link $\mathcal{R}$ to an ergodic sum which has given him, in particular, the following estimate :

Theorem ([82])**.** *For almost all $L \in \mathscr{S}_d$, for all $\epsilon > 0$,*

$$\max_{X \in \mathbb{R}^d} |\mathcal{R}(tP + X, L)| = O(\log(t)^{d-1+\epsilon})$$

where $\mathscr{S}_d$ has the unique Haar probability measure μ_d.

Note that this theorem does not hold for any lattice $L \in \mathscr{S}_d$. It is sufficient to consider the lattice $L = \mathbb{Z}^d$ and for P a hypercube centred around 0 whose sides are parallel to the axes. The maximum error in this case is of order t^{d-1}.
Following Kesten's approach in [48] and in [49], we are interested in the asymptotic behaviour of $\mathcal{R}$ when $L \in \mathscr{S}_2$ and $X \in \mathbb{R}^2/L$ are random.
More formally, let us give $\tilde{\mu}_2$ a probability measure absolutely continuous with respect to μ_2, of regular and bounded density on $\mathscr{S}_2$ and where μ_2 refers to the normalised Haar measure on $\mathscr{S}_2$. By calling λ_2 the normalized Lebesgue measure on $\mathbb{R}^2/L$, we also give ourselves $\tilde{\lambda}_2$ a probability measure absolutely continuous with respect to λ_2, of regular density on $\mathbb{R}^2/L$.
We will then show the following result assuming that P is a parallelogram :

Theorem 1. *When $L \in \mathscr{S}_2$ is distributed according to the measure $\tilde{\mu}_2$ and X is distributed according to $\tilde{\lambda}_2$ then we have :*

$$\frac{\mathcal{R}(tP + X, L)}{\log(t)} \underset{t \to \infty}{\overset{\mathcal{L}}{\to}} \mathcal{C}_c$$

where $\overset{\mathcal{L}}{\to}$ means that the convergence takes place in distribution and where $\mathcal{C}_c$ refers to a centred Cauchy distribution.

Following Skriganov, it seemed to us judicious, in order to study this normalised error $\frac{\mathcal{R}}{\log(t)}$, to take interest in the study of the distribution of quantities - and that was fruitful : $(\|\delta_t L\|^{-2})_{t \in [0, T-1]}$ when $T \to \infty$, where $\delta_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ and where $\|L\|$, for any lattice L, is defined by :

$$\|L\| = \inf\{\|l\| | l \in L - \{0\}\}$$

where $\|X\|$ refers to the usual euclidean norm of $X \in \mathbb{R}^2$.
More precisely, when we start to study the asymptotic convergence of $\frac{\mathcal{R}}{\log(t)}$, we see that we are reduced to the study of a finite sum on $l \in L$ whose terms are fractions. Heuristically, moreover, the numerator behaves like a function which oscillates faster and faster in a segment centred around 0, while the denominator contains a term of the form $Num(l)\log(t) = l_1 l_2 \log(t)$. The terms will therefore not be negligible only when $Num(l)$ is small. Now, in this case, $Num(l)$ can be seen as a $\|\delta_t L\|^2$.
Thus, a good indicator to know if $\frac{\mathcal{R}}{\log(t)}$ converges in distribution is to know if :

$$S(\omega, L, T) = \frac{1}{T} \sum_{t \in [0, T-1]} \frac{\theta_t(\omega)}{\|\delta_t L\|^2}$$

converges in distribution when $T \to \infty$ (T plays here the role of $\log(t)$) with $(\omega, L) \in \Omega \times \mathscr{S}_2$ where $(\Omega, \mathbb{P}_1)$ is a probability space and the θ_t are real random variables with compact support, symmetrical, identically distributed, independent between them. We recall that a real random variable X is said to be symmetric if $\mathbb{P}_X = \mathbb{P}_{-X}$.
We will thus first show the following theorem before showing Theorem 1 :

Theorem 2. *When L is distributed according to the probability distribution μ_2 and ω according to $\mathbb{P}_1$, $S(\omega, L, T)$ converges, when $T \to \infty$, to a centred Cauchy distribution.*

Plan of the paper. In the following, we will present the proofs of Theorem 1 and of Theorem 2. To do so, in the next section, we will expose some preliminaries of probabilities. Then, we will recall some results on the space of lattices. We will then prove Theorem 2 which is relevant in view of what we have stated previously. It will give us an idea of the approach we will have to follow to prove Theorem 1. Then we will treat a simplified case resulting from the study of $\mathcal{R}$ (but which will however be more complicated than the study of $S(\omega, L, T)$ as we shall see). We will finally finish the study of $\mathcal{R}$.

2.2 Probability preliminaries

The point process which are Poisson processes will play an essential role in our convergence studies. We will therefore give some basic reminders on this subject. We will essentially follow the exposition made in [26].
A random variable N has a Poisson distribution with parameter $\lambda > 0$ if for any integer $k \geqslant 0$, $\mathbb{P}(N = k) = e^{-\lambda}\frac{\lambda^k}{k!}$. This distribution will be noted $\mathcal{P}(\lambda)$. We then have the following results :

- If $N_1, \cdots, N_m$ are independent random variables which follow Poisson distributions of respective parameters λ_j then $N = \sum_{j=1}^m N_j$ admits a Poisson distribution of parameter $\sum_{j=1}^m \lambda_j$.

- Conversely, if we give ourselves N points distributed according to a Poisson distribution of parameter λ, that we colour each point independently with a colour $j \in \{1, \cdots, m\}$ chosen with a probability p_j (and thus $\sum_{j=1}^m p_j = 1$) and that we call N_j the number of points of colour j obtained then the N_j are independent and follow Poisson distributions of respective parameters $\lambda_j = p_j\lambda$.

Let $(\mathbf{X}, \mathbf{m})$ be a measured space. We call a Poisson process on this space a point process on $\mathbf{X}$ such that if $\mathbf{X}_1, \cdots, \mathbf{X}_m$ are disjoint (measurable) sets and if N_j is the number of points which fall in $\mathbf{X}_j$ then the N_j are random variables which admit for respective distributions $\mathcal{P}(\mathbf{m}(\mathbf{X}_j))$. This definition is consistent thanks to the first previous stated fact. We will note $\{x_j\} \sim \mathfrak{P}(\mathbf{X}, \mathbf{m})$ to indicate that $\{x_j\}$ is a Poisson process of parameter $(\mathbf{X}, \mathbf{m})$. If $\mathbf{X} \subset \mathbb{R}^d$ and if $\mathbf{m} \ll \lambda_d$ (the Lebesgue measure on $\mathbb{R}^d$) then $\mathbf{m}$

has a density with respect to λ_d and this density will be refered to as the intensity of the Poisson process.

We thus have the following lemma, which will be useful to us later:

Lemma 1. *(a) If $\{\theta_j\} \sim \mathfrak{P}(\mathfrak{X}, \mathbf{m})$ and $\{\theta'_j\} \sim \mathfrak{P}(\mathfrak{X}, \mathbf{m}')$ are independent then*

$$\{\theta_j\} \cup \{\theta'_j\} \sim \mathfrak{P}(\mathfrak{X}, \mathbf{m} + \mathbf{m}').$$

(b) If $\{\theta_j\} \sim \mathfrak{P}(\mathfrak{X}, \mathbf{m})$ and $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is a measurable application then

$$\{f(\theta_j)\} \sim \mathfrak{P}(\mathfrak{Y}, f_*\mathbf{m}).$$

(c) Let $\mathfrak{X} = \mathfrak{Y} \times Z$, $\mathbf{m} = \nu \times \lambda$ where λ is a probability measure over Z. Then $\{(\theta_j, \Gamma_j)\} \sim \mathfrak{P}(\mathfrak{X}, \mathbf{m})$ if, and only if, $\{\theta_j\} \sim \mathfrak{P}(\mathfrak{Y}, \nu)$ and Γ_j are random variables independent from $\{\theta_j\}$ and between them and are all distributed according to λ.

(d) If in (c) $\mathfrak{Y} = Z = \mathbb{R}$ then $\tilde{\theta} = \{\Gamma_j \theta_j\}$ is a Poisson process. If $\{\theta_j\}$ is of parameter $f(\theta)d\theta$ then $\tilde{\theta}$ is of parameter $\tilde{f}(\theta)d\theta$ where

$$\tilde{f}(\theta) = \mathbb{E}_\Gamma(f(\frac{\theta}{\Gamma})\frac{1}{|\Gamma|}).$$

Another lemma that will be useful is the following :

Lemma 2. *If $\{\Xi_i\}$ follows a Poisson process over $\mathbb{R}^+$ of constant intensity $C > 0$ and if the $\{\Gamma_i\}$ are real random variables that are independent, between them and with the Ξ_i, identically distributed, with probability distributions that are symmetric and with a compact support then*

$$\lim_{\epsilon \to 0} \frac{1}{\rho} \sum_{\Xi_i < \frac{1}{\epsilon}} \frac{\Gamma_i}{\Xi_i}$$

is distributed according to the standard Cauchy distribution $\mathcal{C}(0, 1)$ where $\rho = \frac{C}{2}\mathbb{E}(|\Gamma|)\pi$.

A variant of this, which will also be useful, is the following :

Lemma 3. *If $\{\Xi_i\}$ follows a Poisson process over $\mathbb{R}$ of constant intensity $C > 0$ and if the $\{\Gamma_i\}$ are real random variables that are independent, between them and with the Ξ_i, identically distributed, with probability distributions that are symmetric and with a compact support then*

$$\lim_{\epsilon \to 0} \frac{1}{\rho} \sum_{|\Xi_i| < \frac{1}{\epsilon}} \frac{\Gamma_i}{\Xi_i}$$

is distributed according to the standard Cauchy distribution $\mathcal{C}(0, 1)$ where $\rho = C\mathbb{E}(|\Gamma|)\pi$.

A proof of these last two lemmas can be found in [29]. We deduce from the definition of convergence in distribution the following two lemmas :

Lemma 4. *If $\{\Xi_i^T\}$ converge,when $T \to \infty$, towards a Poisson process over $\mathbb{R}^+$ of constant intensity $C > 0$ and if the $\{\Gamma_i^T\}$ converge towards real random variables that are independent between them, identically distributed, of probability distributions that are symmetric and with a compact support and independent from the limit process of $\{\Xi_i^T\}$ then*

$$\lim_{\epsilon \to 0} \lim_{T \to \infty} \frac{1}{\rho} \sum_{\Xi_{i,T} < \frac{1}{\epsilon}} \frac{\Gamma_{i,T}}{\Xi_{i,T}}$$

is distributed according to the standard Cauchy distribution $\mathcal{C}(0,1)$ where $\rho = \frac{C}{2}\mathbb{E}(|\Gamma|)\pi$.

Lemma 5. *If $\{\Xi_i^T\}$ converge,when $T \to \infty$, towards a Poisson process over $\mathbb{R}$ of constant intensity $C > 0$ and if the $\{\Gamma_i^T\}$ converge towards real random variables that are independent between them, identically distributed, of probability distributions that are symmetric and with a compact support and independent from the limit process of $\{\Xi_i^T\}$ then*

$$\lim_{\epsilon \to 0} \lim_{T \to \infty} \frac{1}{\rho} \sum_{|\Xi_{i,T}| < \frac{1}{\epsilon}} \frac{\Gamma_{i,T}}{\Xi_{i,T}}$$

is distributed according to the standard Cauchy distribution $\mathcal{C}(0,1)$ where $\rho = C\mathbb{E}(|\Gamma|)\pi$.

In order to apply these two lemmas, we will have to prove that, asymptotically, suitable point process behave like Poisson processes. To do this, a tool that will be particularly useful to us is Theorem 6.1 proved in [26] which is particularly adapted when we are dealing with flows that mix sufficiently quickly (exponential speed, for example). This will be our case, as we will see a little further on. We recall now the statement of this theorem.

Let $(\Omega, \mathbb{P})$ be a probability space. We denote by $\mathbb{E}$ the expectation relatively to $\mathbb{P}$.

We give us $\lambda_1(x) = ax$ a non-zero linear form on $\mathbb{R}$.

Let $(\mathbf{X}, \mathbf{m})$ be a measurable space. Let $\mathbf{Q}$ be a countable collection of finite partitions of $\mathbf{X}$ such that $\mathbf{Q}$ converges to the point partition.

For all M, consider a sequence $\{\xi_t^M\}_{t\in[0,M]}$ of random variables taking positive integer values and a sequence $\{\nu_t^M\}_{t\in[0,M]}$ of random variables of Ω valued in $\mathbf{X}$. We can imagine, for example, that ξ_t^M is a function which indicates if a mixing flow, at time t, enters a ball of radius $\frac{1}{\sqrt{M}}$ (located at the infinity of the space) and ν_t^M indicates up to which point the flow has entered at this time.

For any partition $Q = (K_1, \cdots, K_P) \in \mathbf{Q}$, we suppose that ξ_t^M can be written $\xi_t^M = \sum_{p=1}^P \xi_{t,p}^M$ where $\xi_{t,p}^M$ takes values in the natural numbers and on the set $\{\xi_t^M = 1\}$, $\xi_{t,p}^M$ satisfies :

$$\xi_{t,p}^M = \mathbf{1}_{\nu_t^M \in K_p}.$$

Let $\eta_t^M = \mathbf{1}_{\xi_t^M = 1}$ and $\eta_{t,p}^M = \mathbf{1}_{\xi_{t,p}^M = \xi_t^M = 1}$. All the variables depend on M and in the following we will omit the letter M for simplicity.

In all this part, when we use the notation $Y = O(X)$ it means that $|Y| \leqslant C|X|$ where C depends possibly on $\mathbf{Q}$ but not on M, t or on $\overline{\delta}$ (introduced a little further on).

We suppose that for all M fixed, a sequence of partitions F_t, $t \in \Pi$ of $(\Omega, \mathbb{P})$ is given. For $\omega \in \Omega$, we call $F_t(\omega)$ the element of F_t such that $\omega \in F_t$. We call $\mathcal{F}_t$ the σ-algebra generated by F_t.

We suppose that the following hypotheses are verified : there exists $R > 0$ (which does not depend on M) and a measurable set E such that

$$\mathbb{P}(E^c) = O(M^{-100})$$

and

- (h1) For all $t \in [0, M]$,
$$\mathbb{E}(\xi_t) = O(M^{-1});$$
- (h2) For all $t \in [0, M]$,
$$\mathbb{P}(\xi_t > 1) = O(M^{-2});$$
- (h3) For all $t \neq t' \in [0, M]$,
$$\mathbb{P}(\xi_t \geqslant 1, \xi_{t'} \geqslant 1) = O(M^{-2});$$
- (h4) For $t, t' \in [0, M]$, with $\lambda_1(t) > \lambda_1(t') + R\log(M)$, for all $p \in [1, P]$ and for all $\omega \in E$:
$$\text{(h4a) } \mathbb{E}(\xi_t \,|\mathcal{F}_{t'})(\omega) = \tfrac{\mathbf{cm}(\mathbf{X})}{M} + O(M^{-2})$$
$$\text{(h4b) } \mathbb{E}(\xi_{t,p} \,|\mathcal{F}_{t'})(\omega) = \tfrac{\mathbf{cm}(K_p)}{M} + O(M^{-2})$$
$$\text{(h4c) } \mathbb{E}(\eta_{t,p} \,|\mathcal{F}_{t'})(\omega) = \tfrac{\mathbf{cm}(K_p)}{M} + O(M^{-2})$$
- (h5) For all $t, \bar{t} \in [0, M]$ with $\lambda_1(\bar{t}) > \lambda_1(t) + R\log(M)$, for all $p \in [1, P]$, for all $\omega \in E$,
$$\xi_{t,p} \textit{ is constant over } F_{\bar{t}}(\omega);$$
- (h6) The algebras $\{\mathcal{F}_t\}$ have a filtration-type property in the sense that for $t, \bar{t} \in [0, M]$, with $\lambda_1(\bar{t}) > \lambda_1(t) + R\log(M)$, for all $\omega \in E$,
$$F_{\bar{t}}(\omega) \subset F_t(\omega).$$

Theorem 3 ([26])**.** *When $(\lambda_1, \{\xi_t\}, \{\nu_t\})$ verify the hypothesis (h1) up to (h6), the sequence of point process*
$$\{\nu_t^M, \frac{t}{M}\}_{\xi_t^M=1,\ t\in[0,M]}$$
converges when $M \to \infty$ towards a Poisson process of constant intensity $\mathbf{c}$ on the space $(\mathbf{X} \times J, \mathbf{m} \times Leb)$.

By adding hypotheses, we can have a stronger conclusion which will be useful to obtain, among other things, identically distributed real random variables independent between them, and with respect to an *ad hoc* Poisson process. This will be useful because of Lemma 1. Let us specify these hypotheses and the conclusion.
With the notations of the theorem 3, let us suppose that we have $\lambda_2(t)$ an affine form such that $\lambda_2(t) > \lambda_1(t)$ (where λ_1 can now be a non-constant affine form instead of a linear and non-zero one) on $[0, M]$.
Suppose that for all M we have a sequence of ζ_t^M with values in $(\tilde{\mathbf{X}}, \tilde{\mathbf{m}})$, a probability space. This space is moreover supposed to have a countable collection of finite partitions of $\tilde{\mathbf{X}}$ which converges to the point partition. Suppose finally that for each element

$\tilde{Q} = (\tilde{K}_1, \cdots, \tilde{K}_P) \in \tilde{\mathbf{Q}}$, we have $\tilde{E}$ such that $\mathbb{P}(\tilde{E}^c) = O(M^{-100})$ and such that the following hypotheses are satisfied :

(h7) There exists a sequence ν_M such that $\nu_M \to 0$ when $M \to \infty$ and $R > 0$ such that for all $t, t' \in \Pi_M$ that satisfy $\lambda_2(t) \geqslant \lambda_1(t') + R\log(M) \geqslant \lambda_1(t) + 2R\log(M)$, we have for all $\omega \in \tilde{E}$ such that $\xi_t(\omega) = 1$,

$$|\mathbb{P}(\zeta_t^M \in \tilde{K}_p | \mathcal{F}_{t'})(\omega) - \tilde{\mathbf{m}}(\tilde{K}_p)| \leqslant \nu_M.$$

(h8) For all $t, \bar{t} \in \Pi_M$ with $\lambda_1(\bar{t}) > \lambda_2(t) + R\log(M)$, for all $p \in [1, P]$, for all $\omega \in \tilde{E}$ such that $\xi_t(\omega) = 1$,

$$\mathbf{1}_{\zeta_t \in \tilde{K}_p} \textit{ is constant over } F_{\bar{t}}(\omega).$$

We then have the following result, which is stronger than that of Theorem 3 :

Theorem 4 ([26])**.** *When $(\lambda_1, \lambda_2, \{\xi_t\}, \{\nu_t\}, \{\zeta_t\})$ verify the hypothesis (h1) up to (h8), the sequence of point process*

$$\{\nu_t^M,\ \zeta_t^M,\ \frac{t}{M}\}_{\xi_t^M = 1,\ t \in \Pi_M}$$

converges when $M \to \infty$ towards a Poisson process of constant intensity $\mathbf{c}$ over the space

$$(\mathbf{X} \times \tilde{\mathbf{X}} \times J,\ \mathbf{m} \times \tilde{\mathbf{m}} \times Leb).$$

Note that in these two theorems, $[0, M]$ can be replaced by $[-M, 0]$, it does not change their validity. We will use this remark in Section 4.

2.3 Preliminaries on the lattice space

In this section we recall some results on the space of unimodular lattices and the action of the geodesic flow on it. They come essentially from [82] and [26].

We have previously said that the space of unimodular lattices in dimension d is denoted by $\mathscr{S}_d$. In other words, we have the equality : $\mathscr{S}_d = SL_d(\mathbb{R})/SL_d(\mathbb{Z})$.

Definition 2. *Let $L \in \mathscr{S}_d$. We define $L^\perp \in \mathscr{S}_d$ the dual lattice of L as follows :*

$$L^\perp = \{l \in \mathbb{R}^d \mid \forall k \in L,\ < k, l > \in \mathbb{Z}\}$$

where $< \cdot, \cdot >$ denotes the usual scalar product.

This definition comes into play in particular when one starts to manipulate Fourier series.
We have $\mathscr{S}_d = SL_d(\mathbb{R})/SL_d(\mathbb{Z})$ and so $\mathscr{S}_d$ is therefore provided with a Lie group structure. Yet, for $d \geqslant 2$, this space is not compact. But he still admits a Haar measure that is also a probability measure and we denote it by μ_d. As a consequence, this measure is also unimodular and we can do measured dynamic over this space.

Definition 3. *A subgroup H of $SL_d(\mathbb{R})$ is said to be ergodic (for the natural action on $\mathscr{S}_d$) if any measurable H-invariant set of $\mathscr{S}_d$ has measure zero or full measure.*

We have then the following theorem (Moore's theorem) :

Theorem 5. *A subgroup H is ergodic if, and only if, H is not included in any compact subgroup of $SL_d(\mathbb{R})$.*

In particular, by setting $\Gamma = \{(t_1, \ldots, t_d) \in \mathbb{R}^d \mid t_1 + \cdots + t_d = 0\}$, we deduce from this theorem that the subgroup $\Delta = \{\mathrm{Diag}(e^{t_1}, \cdots, e^{t_d}) \mid (t_1, \cdots, t_d) \in \mathbb{Z}^d \cap \Gamma\} \subset SL_d(\mathbb{R})$ is ergodic. As a consequence, the action of the (discretised) geodesic flow $(\delta, L) \in \Delta \times \mathscr{S}_d \longmapsto \delta L$ is ergodic.
Let us set :

$$\Delta_r = \{\mathrm{Diag}(e^{t_1}, \cdots, e^{t_d}) \mid (t_1, \cdots, t_d) \in \mathbb{Z}^d \cap \Gamma \text{ tel que } \|(t_1, \cdots, t_d)\| < r\}.$$

We also call $n_r = |\Delta_r|$ and we observe that the sequence of sets (Δ_r) is increasing and converge towards Δ. From the ergodicity of Δ, we obtain the following theorem (known as the individual ergodic theorem) :

Theorem 6. *Let $\psi \in L^1(\mathscr{S}_d, \mu_d)$. Then, for almost every $L \in \mathscr{S}_d$ (in the sense of the probabilty measure μ_d), one has that :*

$$\lim_{r \to \infty} \frac{1}{n_r} \sum_{\delta \in \Delta_r} \psi(\delta L) = \int_{\mathscr{S}_d} \psi(L) d\mu_d(L).$$

In [82], the following lemma, which gives examples of integrable functions on $\mathscr{S}_d$ and which will be useful to us, is proved :

Lemma 6. *For every $\epsilon > 0$, $L \longmapsto \|L\|^{-d+\epsilon}$ is integrable over $\mathscr{S}_d$.*

We still need to define some characteristic numbers of a lattice $L \in \mathscr{S}_d$.

Definition 4. *Let $k \in \{1, \cdots, d\}$. We set :*

$$\|L\|_k = \inf\{r > 0 \mid B_f(0, r) \text{ contains a free family of } k \text{ vectors of } L\}$$

where $B_f(0, r)$ stands for the closed ball of center 0 and with radius r relatively to $\|\|$. In particular, one has : $\|L\|_1 = \|L\|$.

Definition 5. *A vector l of a lattice L will be said to be prime if for all $l' \in L$, the equality $l = kl'$ with $k \in \mathbb{Z}$ implies that $k = \pm 1$.*

Any vector l such that $\|l\| = \|L\|_k$, for any $k \in \{1, \cdots, d\}$, is a prime vector and for any $\delta \in \Delta$, if l is a prime vector then δl is also a prime vector (of the lattice δL).

Let f, f_1, f_2 be piecewise C^∞-functions with compact support from $\mathbb{R}^d$ into $\mathbb{R}$. For all $L \in \mathscr{S}_d$, we set :

$$\mathcal{S}(f)(L) = \sum_{\substack{l \in L \\ l \text{ prime}}} f(l) \text{ and } \overline{\mathcal{S}}(f)(L) = \sum_{\substack{l_1 \neq \pm l_2 \in L \\ l_1,\, l_2 \text{ prime}}} f_1(l_1) f_2(l_2).$$

Definition 6. *$\mathcal{S}(f)$ is called the Siegel transform of f.*

We call ζ the Riemann zeta function.
Then we have the following identities (called Siegel and Rogers identities respectively) :

Lemma 7 (voir [64], [93],[47])**.**

$$\int_{\mathscr{S}_d} \mathcal{S}(f) d\mu_d = \zeta(d)^{-1} \int_{\mathbb{R}^d} f d\lambda_d \ ,$$

if $d \geqslant 3$,

$$\int_{\mathscr{S}_d} \overline{\mathcal{S}}(f) d\mu_d = \zeta(d)^{-2} \int_{\mathbb{R}^d} f_1 d\lambda_d \int_{\mathbb{R}^d} f_2 d\lambda_d \text{ and we deduce that}$$

$$\int_{\mathscr{S}_d} \mathcal{S}(f)^2 d\mu_d = \zeta(d)^{-1} \int_{\mathbb{R}^d} f^2 d\lambda_d + \zeta(d)^{-1} \int_{\mathbb{R}^d} f(x) f(-x) dx + \zeta(d)^{-2} (\int_{\mathbb{R}^d} f d\lambda_d)^2.$$

If $d = 2$, when f is even, one has that

$$(b) \int_{\mathscr{S}_2} \mathcal{S}(f)^2 d\mu_2 \leqslant C \int_{\mathbb{R}^2} f^2 d\lambda + c_2 (\int_{\mathbb{R}^2} f d\lambda)^2$$

where $C > 0$.

Rogers' formula can of course be generalised to any finite product of functions.
Let us suppose for the moment that $d = 2$. From Rogers' and Siegel's formulas, we deduce the following lemma which is one of the bases of our reflection :

Lemma 8. *For every $a > 0$ large enough,*

$$\mu_2\Big(\{L \in \mathscr{S}_2 | \|L\|^{-2} > a\}\Big) = Ca^{-1}$$

where we have set $C = \frac{1}{2} c_1 \pi = \frac{3}{\pi}$.

To prove this lemma we need two preliminary results.

Lemma 9. *Let l_1, l_2 be two vectors of L prime. Suppose that their respective first coordinates are positive. Suppose that one of them has a norm 2 equal to $\|L\|$. Then, if they belong to the same straight line passing through 0, they are equal.*

Proof. Let us suppose that $l_1 \neq l_2$ and that, for example, $\|l_1\| = \|L\|$. Then, we have $l_2 = kl_1$ with $k \geqslant 0$, $k \notin \mathbb{N}$ because l_2 is prime. By the way, as $\|L\| = \|l_1\|$, necessarily $k > 1$. Let us set $v = l_2 - \lfloor k \rfloor l_1$. Then, one has :

$$\|v\| < \|l_1\|$$

which is absurd. □

From this proof, we deduce that we do not need to assume $d = 2$.
This leads to the following lemma :

Lemma 10. *If $\|L\| < 1$ then there exists only two vectors of L, l and $-l$, such that $\|l\| = \|-l\| = \|L\|$.*

Proof. Let suppose that there exist l_1,l_2 such that $\|l_1\| = \|l_2\| = \|L\|$. They are necessarily prime and even if it means taking their opposites, we can suppose that their first coordinate is positive or null. The lemma 9 then gives that either $l_1 = l_2$, either (l_1, l_2) is a free family of $\mathbb{R}^2$. In this last case one has that $\|L\|_2 = \|L\|_1$.
Yet, we have : $\mathrm{covol}(L) = \|L\|_1\|L\|_2 \sin(\alpha)$ where α is the non-oriented angle between two vectors h, k of L such that h, k have their first coordinate positive or null and $\|h\| = \|L\|_1$ and $\|k\| = \|L\|_2$. Thus, we obtain : $\mathrm{covol}(L) < 1$, which is excluded and therefore $l_1 = l_2$. □

Let us now prove the Lemma 8 :

Proof of Lemma 8. Let $a > 0$ be given. For all $x \in \mathbb{R}^2$, $f(x) = \mathbf{1}_{B_f(0,a^{-1/2})}(x)$. We call $\mathcal{S}(f)$ the Siegel transform associated with f. For any $a > 0$ large enough, for any $L \in \mathscr{S}_2$, $\mathcal{S}(f) \in \{0, 2\}$. Indeed, for $a > 0$ large enough, $a^{-\frac{1}{2}} < 1$ and so the Lemma 10 applies. Thus, we have :

$$\frac{\mathbb{E}(\mathcal{S}(f))}{2} = \mu_2\Big(\{L \in \mathscr{S}_2 | \|L\|^{-2} > a\}\Big).$$

Lemma 7 then gives the desired result. □

Note by the way that the reasoning in the Lemma 10 gives the following lemma :

Lemma 11. *Let $L \in \mathscr{S}_2$. We suppose that $\|L\|_2 \leqslant 1$. Then, two cases arise :*

- *Either $\|L\| = \|L\|_2$. In this case, $\|L\| = 1$ and L was obtained from $\mathbb{Z}^2$ with a rotation. Furthermore, there are exactly four distinct vectors of L such that their norms is equal to $\|L\|$.*
- *Either $\|L\| < \|L\|_2$. In this case, there are exactly two distinct vectors of L such that their norms is equal to $\|L\|$.*

In greater dimension, that is, for $d \geqslant 3$, the Lemma 8 is generalized as follows :

Lemma 12.

$$\mu_d\Big(\{L \in \mathscr{S}_d | \|L\|^{-d} > a\}\Big) = Da^{-1} + O(\frac{1}{a^2})$$

where one has set $D = \frac{\zeta(d)^{-1} Vol(B_f(0,1))}{2}$.

The problem, which must be dealt with very often in dimension $d \geqslant 3$, is the following : even if $\|L\|$ is small, this does not mean that $\|L\|_2$ is large, contrary to the dimension 2. Let us see the proof.

Proof. We set, like before, for all $x \in \mathbb{R}^d$, $f(x) = \mathbf{1}_{B_f(0,a^{-\frac{1}{d}})}(x)$. We call $\mathcal{S}(f)$ the Siegel transform of f and $\overline{\mathcal{S}}(f)$ the function defined as before.
On the one hand, one has, according to Lemma 7 :

$$\mathbb{E}(\mathcal{S}(f)) = \zeta(d)^{-1} \int_{\mathbb{R}^d} f d\lambda_d. \tag{2.1}$$

From this, we derive, by homogeneity of the volume of a ball :

$$\mathbb{E}(\mathcal{S}(f)) = \frac{\zeta(d)^{-1}\mathrm{Vol}(B_f(0,1))}{a}.$$

On the other hand, one has : $\mathbb{E}(\mathcal{S}(f)) = 2\mathbb{P}(\mathcal{S}(f) = 2) + \mathbb{E}(\mathcal{S}(f)\mathbf{1}_{\mathcal{S}(f)\geqslant 3})$.
Yet, one has also : $\mathcal{S}(f)\mathbf{1}_{\mathcal{S}(f)\geqslant 3} \leqslant \overline{\mathcal{S}}(f)\mathbf{1}_{F\geqslant 3} \leqslant \overline{\mathcal{S}}(f)$.
So, one gets that : $\mathbb{E}(\mathcal{S}(f)\mathbf{1}_{\mathcal{S}(f)\geqslant 3}) \leqslant \mathbb{E}(\overline{\mathcal{S}}(f)) = O(\frac{1}{a^2})$ according to Lemma 7.
Thus, one gets that :

$$\mathbb{E}(\mathcal{S}(f)) = 2\mathbb{P}(\mathcal{S}(f) = 2) + O(\frac{1}{a^2}). \tag{2.2}$$

With Equation (2.1) and Equation (2.2), one obtains that :

$$\mathbb{P}(\mathcal{S}(f) = 2) = \frac{\zeta(d)^{-1}\mathrm{Vol}(B_f(0,1))}{2}\frac{1}{a} + O(\frac{1}{a^2}). \tag{2.3}$$

Finally, let us note that

$$\mu_d(\{L \in \mathscr{S}_d | \|L\|^{-d} > a\}) = \mathbb{P}(F > 0) = \mathbb{P}(F = 2) + \mathbb{P}(F \geqslant 3)$$

and that

$$\mathbb{P}(F \geqslant 3) \leqslant \mathbb{E}(F\mathbf{1}_{F\geqslant 3}) = O(\frac{1}{a^2}).$$

So, one has that :

$$\mu_d\Big(\{L \in \mathscr{S}_d \mid \|L\|^{-d} > a\}) = \mathbb{P}(\mathcal{S}(f) = 2) + O(\frac{1}{a^2}).$$

Hence the desired result according to Equation (2.3). □

Recall that for all $k = (k_1, \cdots, k_d)$ such that $k_1 + \cdots + k_d = 0$, we have set in the introduction $\delta_k = \mathrm{Diag}(e^{k_1}, \cdots, e^{k_d})$. Let us now end this subsection with two small lemmas that will allow us to calculate smaller lattice lengths.

Lemma 13. *For all $L \in \mathscr{S}_d$, for all $k \in \Gamma$,*

$$e^{-\|k\|_\infty}\|L\| \leqslant \|\delta_k L\| \leqslant e^{\|k\|_\infty}\|L\|. \tag{2.4}$$

Proof. This is immediate via the definitions of δ_k and $\|\cdot\|$. □

The Lemma 13 Le lemme 13 will be essentially useful to us in dimension 2. We have the following lemma, in any dimension :

Lemma 14. *Let $L \in \mathscr{S}_d$. Let us suppose that $\|L\|_2 > \|L\|_1$. Let l be a vector of L such that $\|l\| = \|L\| > 0$. Then for all $k \in \mathbb{Z}^d \cap \Gamma$ such that $\|L\|_2 > e^{2\|k\|_\infty}\|L\|_1$, one has*

$$\|\delta_k L\|_1 = \|\delta_k l\|.$$

Proof. Let k, l as in the statement of the lemma. As $\|L\|_2 > \|L\|_1$, for every $h \in L - \{l, -l\}$, h prime, $\|h\| \geqslant \|L\|_2$ according to Lemma 9.
Thus, one has :

$$\|\delta_k h\| \geqslant e^{-\|k\|_\infty}\|h\| \geqslant e^{-\|k\|_\infty}\|L\|_2.$$

Yet, by hypothesis, one has

$$e^{-\|k\|_\infty}\|L\|_2 > e^{\|k\|_\infty}\|L\|_1.$$

Hence, one gets that :

$$\|\delta_k h\| > e^{\|k\|_\infty}\|L\|_1 \geqslant \|\delta_k l\|.$$

So, one gets the wanted result. □

In dimension 2, we have a variant of this lemma based on the Lemma 13 :

Lemma 15. *Let $L \in \mathscr{S}_2$. Let us suppose that $\|L\| < 1$. One calls l a vector of L such that $\|l\| = \|L\|$. Then, for every $k \in \mathbb{Z}$ such that $e^{|k|}\|L\| < 1$,*

$$\|\delta_k L\| = \|\delta_k l\|.$$

Let us define now a quantity that will appear in our problem.

Definition 7. *Let $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$. The quantity $Num(x)$ is defined by :*

$$Num(x) = \prod_{i=1}^{d} x_i.$$

For $L \in \mathscr{S}_d$, for $r > \|L\|$, one defines

$$\nu(L, r) = \inf\Big\{Num(l) \mid 0 < \|l\| < r\Big\}.$$

One says that a lattice L is weakly admissible if for all $r > \|L\|$, $\nu(L, r) > 0$.

We can note by the way that $\nu(L, \cdot)$ is a decreasing function. In [82] (see Lemma 4.5), the following proposition is proved:

Proposition 1 ([82])**.** *For all $\beta > 0$, for almost all $L \in \mathscr{S}_d$, there exists $C > 0$ such that for all $l \in L - \{0\}$,*

$$|Num(l)| \geqslant C|\log(\|l\|)|^{1-d-\beta}.$$

In particular, almost all lattice $L \in \mathscr{S}_d$ is weakly admissible.

This proposition means that typically, if a vector $l \in L$ becomes large, it will not be able to approach the coordinate axes too quickly. Equivalently, it means that typically the speed of divergence to infinity of $(\delta_t L)_{t\in\Gamma_r}$ is not too large. Skriganov proves this proposition using the Borel-Cantelli lemma.

To finish this section, let us look at the geodesic flow on $\mathscr{S}_d$ and at a more "differential" aspect of it (whereas before we were interested in more "metric" things). The results that will be stated will be in two dimension and generalize to higher dimension. Let us first keep in mind that the action of δ_t (for $t \in \mathbb{R}$) on $\mathscr{S}_2$ is partially hyperbolic in the sense that :

$$T\mathscr{S}_2 = E_0 + E_1^+ + E_1^-$$

where E_0 is tangent to the orbits of the geodesic flow and $E_1^\pm$ are invariant distributions of dimension 1.

Definition 8. *The corresponding Lyapunov exponents are $\pm\lambda_1$ where*

$$\lambda_1 = 2t.$$

Moreover, $E_1^\pm$ are tangent to the foliations $W_1^\pm$ for the foliations of orbits of the groups $h_1^\pm$ where

$$h_1^+ = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \text{ and}$$

$h_1^-(u)$ is the transpose of $h_1^+(u)$.
In the following we will denote E_1^+ by E_1 and W_1^+ by W_1 and h_1^+ by h_1 and the results stated will still be valid for E_1^-, W_1^- and h_1^-. Moreover $\|\cdot\|_{H^s}$ will denote the Sobolev norm of index s.

Definition 9. *Let $s, r \in \mathbb{N}$. One says that $A : \mathscr{S}_2 \to \mathbb{R}$ belongs to $H^{s,r}$ and satisfies $\|A\|_{H^{s,r}} = K$ if for all $1 \geqslant \epsilon > 0$, there exist H^s-functions $A^- \leqslant A \leqslant A^+$ such that*

$$\|A^+ - A^-\|_{L^1(\mu_{\mathscr{S}_2})} \leqslant \epsilon \text{ et } \|A^\pm\|_{H^s} \leqslant K\epsilon^{-r}.$$

Definition 10. *One says that γ is a W_1-curve of length L if there exists $y \in \mathscr{S}_2$ such that*

$$\gamma = \{h_1(\tau)y \mid \tau \in [0, L]\}.$$

In that case, for a function $A : \mathscr{S}_2 \to \mathbb{R}$, we are going to use the notation

$$\int_\gamma A = \frac{1}{L}\int_0^L A(h_1(s)y)ds.$$

Definition 11. *Let $\kappa_0 > 0$.Let $l > 0$ and $\mathcal{P}$ a partition of $\mathscr{S}_2$ into W_1-curves of length L. Let us call $\gamma(x)$ the element of $\mathcal{P}$ that owns x. Given a sequence, finite ou infinite, of integers (k_n) and a function $A \in H^{s,r}$, one says that $\mathcal{P}$ is a κ_0-representative relatively to $((k_n), A)$ if for all n,*

$$\mu_2\Big(x \in \mathscr{S}_2 \mid |\int_{g^{k_n}\gamma(x)} A - \mu_{\mathscr{S}_2}(A)| \geqslant \mathcal{K}_A L_n^{-\kappa_0}\Big) \leqslant L_n^{-\kappa_0} \tag{2.5}$$

where $\mathcal{K}_A = \|A\|_{s,r}+1$, $L_n = Le^{\lambda_1(k_n)}$ is the length of $g^{k_n}\gamma(x)$ and $\mu_2(A) = \int_{\mathscr{S}_2} A(x)d\mu_2(x)$. The point x such that for all n,

$$|\int_{g^{k_n}\gamma(x)} A - \mu_2(A)| \leqslant \mathcal{K}_A L_n^{-\kappa_0}$$

are said to be representative relatively to $(\mathcal{P}, (k_n), A)$.
Let us notice by the way that $\mathcal{P}$ *is* κ_0*-representative relatively to* $((k_n), A)$ *and if*

$$\sum_n (L_n)^{-\kappa_0} \leqslant \epsilon$$

then the set of representative points is of measure at least $1 - \epsilon$.

In [26] (see Proposition 7.3), the following proposition is proved :

Proposition 2 ([26])**.** *(a) There exists* $s, \kappa_0, \epsilon_0 > 0$ *such that for all* $0 \leqslant r \leqslant s$, $0 < \epsilon \leqslant \epsilon_0$, *for every finite family of functions* $\mathfrak{F}$ *of* $H^{s,r}$, *for every* $l > 0$, *for every sequence* (k_n) *such that*

$$\sum_n (Le^{\lambda_1(k_n)})^{-\kappa_0} \leqslant \epsilon,$$

there exists a partition $\mathcal{P}$ *of* $\mathscr{S}_2$ *into* W_1*-curves of length* l *which is* κ_0*-representative relatively to* $((k_n), F)$ *and this holds for every* $F \in \mathfrak{F}$.
(b) If $L \in \mathscr{S}_2$ *is distributed according to a probability measure* $\tilde{\mu}_2$ *which has a bounded density relatively to* μ_2 *then the result (a) is still valid provided that in the definition of representative partition, Equation* (2.5) *is replaced by*

$$\tilde{\mu}_2\Big(x \in \mathscr{S}_2 \mid \mid \int_{g^{k_n}\gamma(x)} A - \mu_{\mathscr{S}_2}(A)| \geqslant \mathcal{K}_A L_n^{-\kappa_0}\Big) \leqslant \tilde{C} L_n^{-\kappa_0}$$

where $\tilde{C}$ *is the upper bound of the density* f *of* $\tilde{\mu}_2$ *relatively to* μ_2.

This proposition is based on the fact that the geodesic flow mixes at exponential speed. Still in [26](see Lemme A.2), one can find the following lemma, which will enable us to estimate some quantities of the type $\|A\|_{H^{s,r}}$.

Lemma 16 ([26])**.** *For every non-negative integer* s, *for every* $R > 0$, *there exists a constant* $C(R, s)$ *such that : for* $f \in C^{s,r}(\mathbb{R}^d)$ *with compact support included in the Euclidean ball of centre* 0 *and of radius* R *then* $\mathcal{S}(f) \in H^{s,r}$ *and*

$$\|\mathcal{S}(f)\|_{H^{s,r}} \leqslant C(R, s)\|f\|_{C^{s,r}}.$$

It should be noted that $C^{s,r}$ is defined in a similar way to $H^{s,r}$ and that the space C^s is the space of regular functions, with compact support, with norm $\|\cdot\|_{\infty,s}$ (s indicates the maximum order of derivation taken into account).

2.4 Study of $S(\omega, L, T)$

2.4.1 Introduction to the problem

In this section, we will specify the asymptotic behaviour of $S(\omega, L, r)$. Let $(\theta_t)_{t\in\mathbb{N}}$ be a sequence of real random variables whose *alea* ω belongs to $(\Omega, \mathbb{P}_1)$ a probabilised

space and such that these real random variables are independent, identically distributed and symmetrical, i.e. such that $\mathbb{P}_{\theta_t} = \mathbb{P}_{-\theta_t}$.

Let us note that the construction of such an object, where $\Omega = \mathscr{S}_2$ and with independence with respect to L, can be done in the following way: for all $t \geqslant 1$, for all $n \geqslant 1$, we set $\theta_{t,n} = f(\delta_t^{n^2} \cdot)$ where f is a regular measurable function of $\mathscr{S}_2$ with values in a segment $[-a, a]$ where $a > 0$ such that $\mu_2\Big(f(L) \in [0, b]\Big) = \mu_2\Big(f(L) \in [-b, 0]\Big)$ and this for all $0 \leqslant b \leqslant a$. The sequence of variables (θ_t) can be thought as the limit, when $n \to \infty$, of $(\theta_{t,n})_{t \geqslant 1}$ (modulo a shift of index).
However, in general, it is not certain that this limit exists. On the other hand, as each of these sequences (in n) of random variables is tense, one will be able to find an extractor ϕ such that $(\theta_{t,\phi(n)})_{t \in \mathbb{N}}$ converges towards a sequence of real random variables identically distributed (as μ_2 is a Haar measure), symmetrical, with compact support and independent between them and from L (as the geodesic flow mixes at exponential speed).
Note that this construction can be generalized to the dimension $d \geqslant 2$.

We are now interested in the behaviour when $T \to \infty$ of $S(\omega, L, T)$. The main result that we are going to prove on this subject is Theorem 2. In the following subsections, we prove this theorem.

2.4.2 Elimination of terms with too large a denominator

Let $\alpha > 0$. For every $\epsilon > 0$, let us set :

$$A_1(\epsilon, T, L) = \Big\{ i \in [0, T-1] \mid \|\delta_i L\|^2 T \leqslant \frac{1}{\epsilon} \Big\}. \tag{2.6}$$

Let us set also

$$S_1(\omega, L, \epsilon, T) = \sum_{t \in A_1(\epsilon, T, L)} \frac{\theta_t(\omega)}{T \|\delta_t L\|^2}. \tag{2.7}$$

Proposition 3. *For every $\epsilon > 0$ small enough, for every T large enough, one has :*

$$\mathbb{P}\Big(|S(\omega, L, T) - S_1(\omega, L, \epsilon, T)| \geqslant \alpha \Big) \leqslant \alpha.$$

This proposition basically tells us that terms whose denominator is too large can be neglected.

Proof. One has, for $L \in \mathscr{S}_2$, for $T \in \mathbb{N} - \{0\}$:

$$\sum_{i=0}^{T-1} \frac{\theta_i(\omega)}{T \|\delta_i L\|^2} - \sum_{\substack{0 \leqslant i \leqslant T-1 \\ T\|\delta_i L\|^2 \leqslant \frac{1}{\epsilon}}} \frac{\theta_i(\omega)}{T \|\delta_i L\|^2} = \sum_{\substack{0 \leqslant i \leqslant T-1 \\ T\|\delta_i L\|^2 > \frac{1}{\epsilon}}} \frac{\theta_i(\omega)}{T \|\delta_i L\|^2}. \tag{2.8}$$

Now, as the θ_i are identically distributed random variables, with expectations equal to zero and as the $L \longmapsto \delta_j L$ are identically distributed and as we work on the probability

product space $\Omega \times \mathscr{S}_2$, we have :

$$\mathbb{E}(\sum_{\substack{0\leqslant t\leqslant T-1\\ T\|\delta_i t\|^2>\frac{1}{\epsilon}}} \frac{\theta_i(\omega)}{T\|\delta_i L\|^2}) = 0 \tag{2.9}$$

and

$$\begin{aligned}\mathbb{V}(\frac{1}{T}\sum_{\substack{0\leqslant t\leqslant T-1\\ T\|\delta_i L\|^2>\frac{1}{\epsilon}}} \frac{\theta_i(\omega)}{\|\delta_i L\|^2}) &= T\frac{1}{T^2}\mathbb{V}(\frac{\theta_0(\omega)}{\|\delta_0 L\|^2}\mathbf{1}_{T\|\delta_0 L\|^2>\frac{1}{\epsilon}})\\ &= \frac{1}{T}\mathbb{E}((\frac{\theta_0(\omega)}{\|\delta_0 L\|^2}\mathbf{1}_{T\|\delta_0 L\|^2>\frac{1}{\epsilon}})^2)\\ &\leqslant \frac{M^2}{T}E(\frac{1}{\|L\|^4}\mathbf{1}_{T\|L\|^2>\frac{1}{\epsilon}}))\end{aligned} \tag{2.10}$$

where $M = \|\theta_0\|_\infty \geqslant 0$.
Now Lemma 8 gives that for all $\epsilon > 0$, for all T large enough, $\mathbb{E}(\frac{1}{\|L\|^4}\mathbf{1}_{T\|L\|^2>\frac{1}{\epsilon}})) \sim \epsilon T$. Hence the desired result via the Bienaymé-Chebyshev inequality. □

Proposition 3 thus brings us back to the study of the quantity $S_1(\omega, L, T, \epsilon)$ where ϵ can be assumed, and will be assumed in the following, to be strictly between 0 and 1.

2.4.3 Existence of and centring on local minima

Let us set

$$A_2(\epsilon, T, L) = \{i \in [0, T-1] \cap A_1(\epsilon, T, L) \mid \|\delta_{i-1}L\| > \|\delta_i L\| < \|\delta_{i+1}L\|\}. \tag{2.11}$$

The following proposition assures us that we can "centre" the terms of S_1 on these local minima:

Proposition 4. *There exists $k_d(i) \leqslant 0$ and $k_m(i) \geqslant 0$ for all $i \in A_2(\epsilon, T, L)$ such that for all $\alpha > 0$, for every $\epsilon > 0$ small enough, for every T large enough, one has :*

$$\mathbb{P}(|S_1(\omega, L, \epsilon, T) - S_2(\omega, L, \epsilon, T)| \geqslant \alpha) \leqslant \alpha$$

where, for every $L \in \mathscr{S}_2$ and $\omega \in \Omega$,

$$S_2(\omega, L, \epsilon, T) = \sum_{i\in A_2(\epsilon,T,L)} \sum_{k_d(i)\leqslant k\leqslant k_m(i)} \frac{\theta_{i+k}(\omega)}{T\|\delta_{i+k}L\|^2} \tag{2.12}$$

and where the following properties are verified :

- $|k_d(i)|, |k_m(i)| \geqslant \frac{\log(T)}{2} + C_\epsilon$ *where* $C_\epsilon = \frac{\log(\epsilon)}{2} - 3$ *;*
- *all the sets* $\{i+k\}_{k_d(i)\leqslant k\leqslant k_m(i)}$ *are pairwise disjoint ;*
- *for every* $k \in [k_d(i), k_m(i)]$, $1 > \|\delta_{i+k}L\| > \|\delta_i L\|$.

It should be noted that $k_d(i)$ and $k_m(i)$ possibly depend on $\|\delta_i L\|$. We will see a little further on that we can eliminate this possible dependence.
Before continuing, we need two preliminary lemmas.
Let l_1, l_2 be two non zero reals and let $l = (l_1, l_2)$. For every $a > 0$, one has :

$$f_{(l_1,l_2)}(a) = a l_1^2 + \frac{1}{a} l_2^2.$$

Lemma 17. *$f_{(l_1,l_2)}$ is strictly decreasing until $a_0 = |\frac{l_1}{l_2}|$, where it reaches its minimum, and then is strictly increasing.*

Proof. A study of the variations of $f_{(l_1,l_2)}$ via a derivative calculation gives the result. □

Lemma 18. *Let $\beta > 0$, $C > 0$ and $L \in \mathscr{S}_2$ such that for every $l \in L - \{0\}$,*

$$|Num(l)| \geqslant C |\log(\|l\|)|^{-1-\beta}.$$

Then there exists $D > 0$ such that for every T large enough, for every $i \in [-\log(T), T + \log(T)]$

$$\frac{\|\delta_i L\|^2}{2} \geqslant D T^{-1-\beta} \tag{2.13}$$

if we assume that $\|L\| \geqslant \sqrt{\epsilon}$.

The first hypothesis of this lemma is realised after discarding a small number of lattices, according to Proposition 1. The same is true for the second hypothesis according to Lemma 8. We can thus suppose these two hypotheses verified and we will do it thereafter.

Proof. Let $i \in [-\log(T), T + \log(T)]$. Let us call $l = (l_1, l_2) \in L - \{0\}$ a vector such that $\|\delta_i l\| = \|\delta_i L\|$.
Furthermore, by assuming that $\|L\| > 1$, the arithmetic-geometric inequality gives, on the one hand, that :

$$\frac{\|\delta_i L\|^2}{2} \geqslant |l_1 l_2| \geqslant C \log(\|l\|)^{-1-\beta}.$$

On the other hand, according to Minkowski's theorem:

$$e^{2i} l_1^2 + e^{-2i} l_2^2 = \|\delta_i L\|^2 \leqslant \frac{4}{\pi}$$

and thus, as $i \in [-\log(T), T + \log(T)]$,

$$\|l\|^2 \leqslant \frac{4(e^{2(T+\log(T))})}{\pi}.$$

So, there exists $\tilde{C} > 0$ (independent from i and L) such that for all T large enough,

$$\log(\|l\|)^{-1-\beta} \geqslant \tilde{C} T^{-1-\beta}$$

and in particular

$$\frac{\|\delta_i L\|^2}{2} \geqslant |l_1 l_2| \geqslant C \tilde{C} T^{-1-\beta}. \tag{2.14}$$

So one gets the wanted result in the case $\|L\|_2 > 1$. The other case is straightforward because one has still in that case that $\|l\|_2 \geqslant \|L\| \geqslant \sqrt{\epsilon}$ and we can always take T larger or D smaller if necessary. □

We can now tackle the proof of Proposition 4. The main idea is that if i is not a local minimum then we have a direction to follow which will lead us to a local minimum according to Lemma 17.

Proof of Proposition 4. • In this first part of the proof, we explain how, from a term with a small denominator, we can reduce to a term with a small denominator which is a local minimum.

We take T large enough so that $\log(\epsilon) + \log(T) - 2 > 0$. In particular, one has $\frac{1}{\epsilon T} < 1$. By definition, one has $A_2(\epsilon, T, L) \subset A_1(\epsilon, T, L)$.
Conversely, let $i \in A_1(\epsilon, T, L)$. Either one has $i \in A_2(\epsilon, T, L)$, or one has $\|\delta_{i-1}L\| \leqslant \|\delta_i L\|$ or $\|\delta_i L\| \geqslant \|\delta_{i+1}L\|$.
Let us place ourselves in the case where we only have $\|\delta_{i-1}L\| \leqslant \|\delta_i L\|$ with $T-1 \geqslant i \geqslant 0$.
Let us call $k \in \delta_i L$ such that $\|\delta_i L\| = \|k\|_2$. Because of the fact that $\|\delta_i L\| \leqslant \sqrt{\frac{1}{\epsilon T}} < 1$, one has :

$$\|\delta_i L\|^2 = k_1^2 + k_2^2 \ \textit{and} \ \|\delta_{i-1}L\|^2 = e^{-2}k_1^2 + e^2 k_2^2.$$

If you eliminate the non-weakly admissible lattices (see Definition 7), which constitute a negligible set for μ_2, one can suppose that $k_1, k_2 \neq 0$. Then one has :

$$f_{(k_1,k_2)}(e^{-2}) = \|\delta_{i-1}L\|^2 \leqslant f_{(k_1,k_2)}(1) = \|\delta_i L\|^2$$

and, in particular $f_{(k_1,k_2)}$ can be used to calculate the shortest vectors around time i.
According to Lemma 17, there exists a unique $m_i \in \mathbb{Z}$ such that $m_i < i$ and such that for all $m_i \leqslant k \leqslant i-1$, $\|\delta_k L\| < \|\delta_{k+1}L\|$ and $\|\delta_{m_i-1}L\| > \|\delta_{m_i}L\|$. In particular, $m_i \in A_2(\epsilon, T, L)$.

Let us suppose that we now place ourselves in the case where we only have $\|\delta_i L\| \geqslant \|\delta_{i+1}L\|$ with $0 \leqslant i \leqslant T-1$. Like before, one obtains a unique m_i such that $m_i \in \mathbb{Z}$, $m_i > i$ and for every $k \in [i, m_i - 1]$, $\|\delta_k L\| > \|\delta_{k+1}L\|$ et $\|\delta_{m_i}L\| < \|\delta_{m_i+1}L\|$. In particular, $m_i \in A_2(\epsilon, T, L)$.

Let us assume now that one has $0 \leqslant i \leqslant T-1$ and $\|\delta_{i-1}L\| \leqslant \|\delta_i L\| \geqslant \|\delta_{i+1}L\|$. Lemma 17 makes it impossible.

So, for every $i \in A_1(\epsilon, T, L)$, one has defined $m_i \in A_2(\epsilon, T, L)$ that can be interpreted as "the first local minimum that can be encountered by following the direction indicated"(in the case $i \in A_2(\epsilon, T, L)$, we simply set $m_i = i$).

• In a second time, one constructs the $k_d(i)$ and the $k_m(i)$ and we show that the study of S_1 can be reduced to the study of S_2 thanks to what was done before.

We set, for every $i \in A_2(\epsilon, T, L) - \{T-1\}$, $k_m(i)$ the largest non-negative integer such that for every $k \in [0, k_m(i) - 1]$, $\|\delta_{i+k}L\| \leqslant \|\delta_{i+k+1}L\| \leqslant \sqrt{\frac{1}{\epsilon T}}$. We define for the other direction in an analogous manner $k_d(i)$ for all $i \in A_2(\epsilon, T, L) - \{0\}$. Let us remark that $k_d(i) \leqslant 0$ and $k_m(i) \geqslant 0$.
Let us set, in the case where $0 \in A_2(\epsilon, T, L)$, $k_d(0) = -\left\lfloor \frac{\log(\epsilon)}{2} + \frac{\log(T)}{2} - 2 \right\rfloor$. We do the same thing in the case where $T-1 \in A_2(\epsilon, T, L)$ with $k_m(T-1)$. We did not put the

dependence on L, on ϵ or on T of k_d and k_m for the sake of simplicity.

The reasoning that allows us to construct m_i gives us in addition that to study the convergence in distribution of S_1 is to study the convergence in distribution of

$$S_2(\omega, L, \epsilon, T) = \sum_{i \in A_2(\epsilon,T,L)} \sum_{k_d(i) \leqslant k \leqslant k_m(i)} \frac{\theta_{i+k}(\omega)}{T\|\delta_{i+k}L\|^2} \tag{2.15}$$

(S_2 may contain terms before 0 and after $T-1$).
Indeed, Lemma 18 applies here and so gives us that, for all T large enough :

$$P(|S_1(\omega, L, \epsilon, T) - S_2(\omega, L, \epsilon, T)| \neq 0) \leqslant D \log(T) T^{\beta} \|\theta_0\|_\infty \mathbb{P}(T\|L\|^2 \leqslant \frac{1}{\epsilon}), \tag{2.16}$$

by using Equation (2.13), the compacity of the common distribution of the θ_i and where $D > 0$. By choosing $\beta > 0$ such that $\beta < 1$, we obtain the wanted result thanks to Lemma 8. So, now, the study is reduced to the study of S_2.

Let $i \in A_1(\epsilon, T, L)$. We observe that $\|\delta_i L\|^2 \leqslant \frac{1}{\epsilon T}$ and so, for $k \in \mathbb{Z}$ such that $|k| < \frac{\log(\epsilon)}{2} + \frac{\log(T)}{2}$, one has $\|\delta_{i+k}L\|^2 < 1$ according to Lemma 13.
By using Lemma 15 and Lemma 17, one has that for every $i \in A_1(\epsilon, T, L)$, the sets

$$[-\lfloor \frac{\log(\epsilon)}{2} + \frac{\log(T)}{2} - 2 \rfloor, \lfloor \frac{\log(\epsilon)}{2} + \frac{\log(T)}{2} - 2 \rfloor] + \{i\}$$

(with $0 \leqslant i \leqslant T-1$) are pairwise disjoint.
Even if it means reducing the $k_d(i)$ and increasing $k_m(i)$ in order to have $k_d(i) \leqslant -\lfloor \frac{\log(\epsilon)}{2} + \frac{\log(T)}{2} - 2 \rfloor$ and $k_m(i) \geqslant \lfloor \frac{\log(\epsilon)}{2} + \frac{\log(T)}{2} - 2 \rfloor$ and this for any i, we can effectively assume that these two inequalities are satisfied.
Indeed, this amounts to adding instants $t \in [-2\log(T), T + 2\log(T)]$ in S_2 such that

$$T\|\delta_t L\|^2 > \frac{1}{\epsilon}$$

and Equation (2.9) and Equation (2.10) show that this addition is negligible. □

We can now focus our attention on the asymptotic study of S_2.

2.4.4 Study of the asymptotic behaviour of S_2

Let us set

$$\Gamma_i(\omega, L, \epsilon, T) = \sum_{k_d(i) \leqslant k \leqslant k_m(i)} \frac{\theta_{i+k}(\omega)\|\delta_i L\|^2}{\|\delta_{i+k}L\|^2}, \tag{2.17}$$

$$\Xi_i(L, T) = T\|\delta_i L\|^2 \tag{2.18}$$

so that one has

$$S_2(\omega, L, \epsilon, T) = \sum_{i \in A_2(\epsilon,T,L)} \frac{\Gamma_i(\omega, L, T, \epsilon)}{\Xi_i(L, T)}. \tag{2.19}$$

To apply Lemma 4 and thus obtain Theorem 2, we will now show that, in our case: $\{\Gamma_i(\omega, L, \epsilon, T)\}_{i\in A_2(\epsilon,T,L)}$ are asymptotically independent real random variables identically distributed, symmetrical and whose support is compact and independent of $\{\Xi_i(L,T)\}_{i\in A_2(\epsilon,T,L)}$ and that $\{\Xi_i(L,T)\}_{i\in A_2(\epsilon,T,L)}$ converges to a Poisson process on $[0, \frac{1}{\epsilon}]$ of constant intensity independent of ϵ. Let us first focus on the asymptotic behaviour of Γ_i.

Asymptotic behaviour of Γ_i

In this subsubsection, we will prove the following proposition which determines the asymptotic behaviour of Γ_i :

Proposition 5. *$\{\Gamma_i(\omega, L, \epsilon, T)\}_{i\in A_2(\epsilon,T,L)}$ are asymptotically independent real random variables with the same distribution, which is symmetric and has a compact support, and are independent of $\{\Xi_i(T,L)\}_{i\in A_2(\epsilon,T,L)}$.*

To prove this proposition, we need some lemmas :

Lemma 19. *For every T large enough, for every $i \in A_2(\epsilon, T, L)$, for every $k \in [k_d(i), k_m(i)]$, there exists $K > 0$ such that*

$$\frac{\|\delta_i L\|^2}{\|\delta_{i+k} L\|^2} \leqslant \frac{K}{\cosh(2|k|)}. \tag{2.20}$$

Proof. Let i and k as in the statement of the lemma. By calling $h \in (\delta_i L) - \{0\}$ the vector such that $\|\delta_i L\| = \|h\|$ and such that $h_1 \geqslant 0$, one has according to Lemma 15 :

$$\frac{\|\delta_i L\|^2}{\|\delta_{i+k} L\|^2} = \frac{h_1^2 + h_2^2}{e^{2k} h_1^2 + e^{-2k} h_2^2}. \tag{2.21}$$

By calling (r, α) the polar coordinates of h (and so $r > 0$ and $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$), one has :

$$\frac{\|\delta_i L\|^2}{\|\delta_{i+k} L\|^2} = \frac{1}{\cos^2(\alpha) e^{2k} + \sin^2(\alpha) e^{-2k}}.$$

However, we have : $\frac{\|\delta_i L\|^2}{\|\delta_{i+k} L\|^2} \leqslant 1$ for $k = -1$ and $k = 1$. So, there exists $\frac{\pi}{2} > \kappa > 0$ such that: $\frac{\pi}{2} - \kappa > |\alpha| > \kappa$. Hence the lemma. □

Let us note by the way that according to the hypotheses made just after Lemma 18 with $\beta = \frac{1}{2}$ and according to the theorem of Minkowski which gives that for any lattice L', $\|L'\| \leqslant \frac{4}{\pi}$, we have therefore according to Lemma 19 and Proposition 4 that for any i, $|k_d(i)|, |k_m(i)| \leqslant \log(T)$ as soon as T is large enough.
Thanks to the previous lemma, we can establish that $k_d(i)$ and $k_m(i)$ can be chosen so that they do not depend on $\|\delta_i L\|$ for $i \in A_2(\epsilon, T, L)$. This is the purpose of the following lemma:

Lemma 20. *Let us set :*

$$S_3(\omega, L, \epsilon, T) = \sum_{i \in A_2(\epsilon,T,L)} \frac{1}{\Xi_i(L,T)} \sum_{-(\frac{\log(T)}{2}+C_\epsilon) \leqslant k \leqslant \frac{\log(T)}{2}+C_\epsilon} \frac{\theta_{i+k}(\omega)\|\delta_i L\|^2}{\|\delta_{i+k} L\|^2}.$$

Then, when $T \to \infty$, $S_3(\omega, L, \epsilon, T) - S_2(\omega, L, \epsilon, T)$ converges almost surely towards 0.

Thus, it can be considered hereafter that: $k_d(i) = -(\frac{\log(T)}{2} + C_\epsilon)$ and $k_m(i) = \frac{\log(T)}{2} + C_\epsilon$.

Proof. One has :

$$|S_3(\omega, L, \epsilon, T) - S_2(\omega, L, \epsilon, T)| \leqslant \sum_{i \in A_2(\epsilon,T,L)} \Big(\sum_{|k| \geqslant \frac{\log(T)}{2}+C_\epsilon} \frac{2K\|\theta_0\|_\infty}{\cosh(2|k|)} \Big) \frac{1}{T\|\delta_i L\|^2} \tag{2.22}$$

because $|k_d(i)|, |k_m(i)| \geqslant \frac{\log(T)}{2} + C_\epsilon$ and because $\Xi_i(L,T) = T\|\delta_i L\|^2$.
Yet, one has:

$$\sum_{|k| \geqslant \frac{\log(T)}{2}+C_\epsilon} \frac{1}{\cosh(2|k|)} \sim \frac{H_\epsilon}{T}$$

when $T \to \infty$ and where $H_\epsilon > 0$.
Hence, it follows from Equation (2.22) that :

$$|S_3(\omega, L, \epsilon, T) - S_2(\omega, L, \epsilon, T)| \leqslant \sum_{i \in A_2(\epsilon,T,L)} \frac{M_\epsilon}{T^2\|\delta_i L\|^2} \tag{2.23}$$

where $M_\epsilon > 0$.
Now the quantity on the right-hand side of Inequation (2.23) converges almost surely towards 0 when $T \to \infty$ according to Lemma 3.2 from [82] (which is based on the individual ergodic theorem). Hence the desired result. □

Lemma 21. *Let us assume that $\|L\| < 1$. Let $\alpha(L)$ the angle between the line generated by a vector l of L such that $\|l\| = \|L\|$ and the abscissa axis. Conditionally to the event $(\|L\| < 1)$, $\alpha(L)$ is independent from $\|L\|$.*

Proof. Let $-\frac{\pi}{2} \leqslant a \leqslant b \leqslant \frac{\pi}{2}$ and $0 < r_1 \leqslant r_2 < 1$. Let us set

$$f_1(x) = \mathbf{1}_{r_1 \leqslant \|x\| \leqslant r_2}$$

and

$$f_2(x) = \mathbf{1}_{0<\|x\|<1} \mathbf{1}_{\alpha(x) \in [a,b]}$$

where $\alpha(x)$ is the angle between $(1,0)$ and x.
Let us set :

$$\overline{\mathcal{S}}(f)(L) = \sum_{l_1 \neq \pm l_2 \in L} f_1(l_1) f_2(l_2).$$

Then Lemma 7 can be applied and it gives us that:

$$\begin{aligned} \mu_2\Big(\alpha(L) \in [a,b], \|L\| \in [r_1, r_2]\Big) &= \frac{1}{2}\mathbb{E}(\overline{\mathcal{S}}(f)) \\ &= \frac{1}{2}\zeta(2)^{-2} \int_{\mathbb{R}^2} f_1(x)dx \int_{\mathbb{R}^2} f_2(x)dx \end{aligned}$$

Switching to polar coordinates, we have:

$$\int_{\mathbb{R}^2} f_1(x)dx = \int_{\substack{r_1<r<r_2 \\ 0\leqslant\alpha\leqslant 2\pi}} rdrd\alpha = \pi(r_2^2 - r_1^2)$$

and

$$\int_{\mathbb{R}^2} f_2(x)dx = \int_{\substack{0<r<1 \\ a\leqslant\alpha\leqslant b}} rdrd\alpha = \frac{b-a}{2}.$$

From which we derive:

$$\mu_2\Big(\alpha(L) \in [a,b], \|L\| \in [r_1,r_2] \mid \|L\| < 1\Big) = \mu_2\Big(\alpha(L) \in [a,b] \mid \|L\| < 1\Big)\mu_2\Big(\|L\| \in [r_1,r_2] \mid \|L\|$$

□

We can now prove Proposition 5.

Proof of Proposition 5. The sum $\sum_{k\in\mathbb{Z}} \frac{1}{\cosh(2|k|)}$ converges.
Thus, thanks to Lemma 19, thanks to the fact that $\|\theta_0\|_\infty < \infty$ and by using Equation (2.17), we obtain that the Γ_i, for $i \in A_1(T,\epsilon,L)$, converge towards real random variables with compact support. These variables are symmetrical because the θ_i are symmetrical and independent of each other.
Furthermore, as the θ_i are mutually independent real random variables, as μ_2 is $SL_2(\mathbb{R})$-invariant and as $|k_d(i)|, |k_m(i)| \to \infty$, it is clear that asymptotically the Γ_i are identically distributed.

Let us now see that asymptotically the Γ_i are independent of the Ξ_i.
Let us first look at what happens between Γ_i and Ξ_i. Recall that $\Xi_i = T\|\delta_i L\|^2$ and that Γ_i is written :

$$\Gamma_i(\omega, L, T, \epsilon) = \sum_{k_d(i)\leqslant k\leqslant k_m(i)} \frac{\theta_{i+k}(\omega)\|\delta_i L\|^2}{\|\delta_{i+k} L\|^2},$$

where $k_m(i) = \frac{\log(T)}{2} + C_\epsilon$ and $k_d(i) = -k_m(i)$.
Equation (2.21) shows that $\frac{\theta_{i+k}(\omega)\|\delta_i L\|^2}{\|\delta_{i+k} L\|^2}$ does not depend on $\|\delta_i L\|$ but only on $\alpha(\delta_i L)$.
However, $\alpha(\delta_i L)$ is independent from $\|\delta_i L\|$, as μ_2 is a Haar measure and according to Lemma 21.
So, asymptotically, Γ_i is independent from $\Xi_i = T\|\delta_i L\|^2$.

Furthermore, let us observe that for $i \neq j \in A_2(\epsilon, L, T)$, a gap can be created between the blocks $\{i+k\}$ and $\{j+k\}$ of size, Let us say, at least $\log(\log(T))$, i.e. instead of considering $k_m(h)$, we can consider $\tilde{k}_m(h) = k_m(h) - \frac{\log(\log(T))}{2}$ and instead of considering $k_d(h)$, we can consider $\tilde{k}_d(h) = k_d(h) + \frac{\log(\log(T))}{2}$, without this making any significant difference to the asymptotic study of the Γ_h, of the Ξ_h or of S_2 (see Equation (2.22) and Equation (2.23)).

Finally, using the exponential mixing speed of the geodesic flow on $\mathscr{S}_2$ and using the independence of θ_k, we obtain that asymptotically :

1) The Γ_i are asymptotically independent of each other

2) The Γ_i are asymptotically independent of $\{\Xi_i\}$

□

Let us now study the asymptotic behaviour of $\{\Xi_i\}_{i\in A_2(\epsilon,T,L)}$.

Convergence of $\{\Xi_i\}_{i\in A_1(\epsilon,T,L)}$ towards a Poisson process on $[0, \frac{1}{\epsilon}]$ and proof of Theorem 2

The purpose of this section is to establish the following proposition:

Proposition 6. *$\{\Xi_i(L,T)\}_{i\in A_2(\epsilon,T,L)}$ converges towards a Poisson process on $[0, \frac{1}{\epsilon}]$ of constant intensity $D = \zeta(2)^{-1}\int_0^{\frac{\pi}{2}} \mathbf{1}_{e^{-2}\cos^2(\theta)+e^2\sin^2(\theta)\leqslant 1}\mathbf{1}_{e^2\cos^2(\theta)+e^{-2}\sin^2(\theta)\leqslant 1}d\theta$.*

This proposal gives us a better idea of the distribution at infinity of the points of the geodesic trajectory in the space $\mathscr{S}_2$. It is also this kind of thing that we are interested in [69].
To establish the desired convergence, we will rely on Theorem 3. Here, even if it means looking at the sum which interests us as going from 0 to T (instead of $T-1$), T plays the role of M.
Moreover, we pose

$$\nu_0^T(L) = (\mathbf{1}_{\|\delta_{-1}L\|\geqslant\|L\|\leqslant\|\delta_1 L\|})T\|L\|^2,$$
$$\nu_t^T(L) = \nu_0^T(\delta_t L),$$
$$\xi_t^T(L) = \mathbf{1}_{\nu_t^T(L)\in]0,\frac{1}{\epsilon}]}$$

and the sequence of partitions considered $\mathbf{Q}$ is given by $(]\frac{k}{p}\frac{1}{\epsilon}, \frac{k+1}{p}\frac{1}{\epsilon}])_{p\in\mathbb{N}-\{0\},\ k\in[0,p-1]}$.
Note that we have, for T large enough (depending on $\epsilon > 0$):

$$\xi_t^T(L) = \frac{1}{2}\mathcal{S}(f_0^T)(\delta_t L) \tag{2.24}$$

where $F_0^T(L)$ is the Siegel transform of the function f_0^T defined by :

$$f_0^T(x) = \mathbf{1}_{\|\delta_{-1}x\|\geqslant\|x\|\leqslant\|\delta_1 x\|}\mathbf{1}_{0<T\|x\|^2\leqslant\frac{1}{\epsilon}}.$$

Finally, recall that λ_1 was defined in Definition 8.
Now that we have made these choices, we can state the following proposition in two parts:

Proposition 7. *One has:*

- *$\{\Xi_t(L,T)\}_{t\in A_2(\epsilon,T,L)}$ is exactly the same process as $\{\nu_t^M\}_{\xi_t^M=1,\ t\in[0,M]}$*
- *$(\lambda_1, \{\xi_t\}, \{\nu_t\})$ satisfy the hypothesis (h1) to (h6) of Theorem 3.*

The first part of the proposition is obvious from the choices made. It remains to demonstrate the second part of the proposition on which we will concentrate now.
To see that the hypotheses (h1) up to (h6) are verified, we will follow an analogous approach to that followed in [26]. Only analogous, because the zones at infinity in $\mathscr{S}_2$ which interest us are not the same.
Before showing Proposition 7, we need some preliminary lemmas.

Lemma 22. *For every $t \neq t' \in \mathbb{Z}$, for every $b > 0$, for every $T > 0$ large enough (the size depending only on b), one has :*

$$\mathbb{P}\Big(T\|\delta_t L\|^2 \leqslant b, T||\delta_{t'} L||^2 \leqslant b\Big) = \frac{D_1 b^2}{T^2} + \frac{D_2 b}{T} \arccos(\frac{1-e^{-2|t-t'|}}{1+e^{-2|t-t'|}})$$

where $D_1, D_2 > 0$.

Proof. Let us set, for all $x \in \mathbb{R}$, $f_t(x) = \mathbf{1}_{\|\delta_t x\|^2 \leqslant \frac{b}{T}}$ et $f_{t'}(x) = \mathbf{1}_{||\delta_{t'} x||^2 \leqslant \frac{b}{T}}$. Then, by considering $\mathcal{S}(f_t)$ and $\mathcal{S}(f_{t'})$ by following the approach of Lemma 8, by applying the formulas of Lemma 7 and by remarking that for all $r > 0$

$$\int_{\mathbb{R}^2} \mathbf{1}_{B_f(0,r)}(\delta_t \cdot)\mathbf{1}_{B_f(0,r)}(\delta_{t'} \cdot) = 2r^2 \arccos(\frac{1-e^{-2|t-t'|}}{1+e^{-2|t-t'|}}),$$

one gets the wanted result. □

To prove that (h4) and (h5) are verified, we are going to follow a similar approach that was followed in [26]. So, let us set $K =]0, \frac{1}{\epsilon}]$ and $K_i =]\frac{i}{p}\frac{1}{\epsilon}, \frac{i+1}{p}\frac{1}{\epsilon}]$ for $0 \leqslant i \leqslant p-1$ with $p \geqslant 1$ (p is implied in the notations from before). With these notations, one has : $\xi_t = \mathbf{1}_{\nu_t \in K}$ and $\xi_{t,i} = \mathbf{1}_{\nu_t \in K_i}$.
Let us also set $\hat{K} = \Big\{x \in]0, \infty[\mid d(x, \partial K) \leqslant T^{-1000}\Big\}$ and also $\hat{K}_i = \Big\{x \in]0, \infty[\mid d(x, \partial K_i) \leqslant T^{-1000}\Big\}$.
Let us define $\hat{\xi}_t = \mathbf{1}_{\nu_t \in \hat{K}}$ and also $\hat{\xi_{t,i}} = \mathbf{1}_{\nu_t \in \hat{K}_i}$. We have then the following lemmas :

Lemma 23. *One has:*

$$\|\xi_0^T\|_{H^{s,s}} = O(1). \tag{2.25}$$

For all $i \in \{1, \ldots, p\}$, one has also :

$$\|\xi_{0,i}^T\|_{H^{s,s}} = O(1). \tag{2.26}$$

Proof. Lemma 16, applied to F_0^T, gives that $F_0^T \in H^{s,s}$ and that $\|F_0^T\|_{H^{s,s}} = O(1)$.
To show this last point, we proceed by approximating *via* functions that are piecewisely affines the indicator function $\mathbf{1}_{B_f(0, \frac{1}{\sqrt{\epsilon T}})}$ and by regularizing these functions with approximation of the unity.
So one gets the first wanted result according to Equation (2.24). We do the same thing to get the second result. □

Lemma 24. *One has:*

$$\mu_{\mathscr{S}_2}(\hat{\xi_{0,i}}) = O(T^{-1000}) \text{ and} \tag{2.27}$$

$$\mu_{\mathscr{S}_2}(\hat{\xi_0}) = O(T^{-1000}) \tag{2.28}$$

Proof. It is an application of Lemma 7. □

Let us prove now Proposition 7.

Proof of Proposition 7. • Verification of (h1)
Let us recall that for all T large enough (relatively to a quantity dependent on $\epsilon > 0$)

$$\xi_t^T(L) = \frac{1}{2}\mathcal{S}(f_t^T)(L)$$

where the function f_t^T is defined by:

$$f_t^T(x) = \mathbf{1}_{\|\delta_{-1}\delta_t x\| \geqslant \|\delta_t x\| \leqslant \|\delta_1 \delta_t x\|} \mathbf{1}_{0 < T\|\delta_t x\|^2 \leqslant \frac{1}{\epsilon}}. \tag{2.29}$$

By using Lemma 7, by making a change of variable $u = \delta_t x$ and by passing into polar coordinates, one has:

$$\mathbb{E}(\xi_t^T) = \frac{D}{T}\frac{1}{\epsilon} \tag{2.30}$$

where

$$D = \zeta(2)^{-1} \int_0^{\frac{\pi}{2}} \mathbf{1}_{e^{-2}\cos^2(\theta) + e^2 \sin^2(\theta) \leqslant 1} \mathbf{1}_{e^2 \cos^2(\theta) + e^{-2}\sin^2(\theta) \leqslant 1} d\theta.$$

Before continuing, let us note that the variations of the function $\theta \longmapsto \frac{1}{a^2}\cos^2(\theta) + a^2 \sin^2(\theta)$, with $a > 0$, enables us to see that the value of D is in fact :

$$D = \zeta(2)^{-1}(\arccos(\frac{1}{\sqrt{e^2+1}}) - \arccos(\frac{e}{\sqrt{e^2+1}})).$$

(which can be simplified more by using a formula about differences of arccos)

• Verification of (h2)
(h2) is verified because $\xi_t \in \{0, 1\}$.

• Verification of (h3)
If we have $t, t' \in \Pi$, different such that $\xi_t, \xi_{t'} = 1$ then $t, t' \in A_2(\epsilon, T, L)$ and are so separated of $\log(T) + 2C_\epsilon$ according to what was seen previously.
By using Lemma 22, one sees that :

$$P(\xi_t \geqslant 1, \xi_{t'} \geqslant 1) \leqslant O(\frac{1}{T}\arccos(\frac{1 - e^{-2(\log(T)+2C_\epsilon)}}{1 + e^{-2(\log(T)+2C_\epsilon)}})) \tag{2.31}$$

and the term in arccos is a $O(\frac{1}{T})$.

• Verification of (h4a), (h4b) et (h4c)
In our case, let us remark that $\eta_{t,p} = \xi_{t,p}$. So, it is enough to have (h4a) and (h4b) verified to get that the three hypothesis are verified.
We fix $T > 0$. We fix $t \in \Pi$ (the exponent T is implied). Let us call $\Pi^+(t)$ the set of elements $t' \in [0, T]$ such that $\lambda_1(t') > \lambda_1(t) + R\log(T)$ where $R > 0$ will be fixed later on.
One considers also $\mathfrak{F} = \{\xi_0, \xi_{0,1}, \cdots, \xi_{0,p}\}$. One calls F_t the partition into W_1-curves of

size $L_t = (e^{\lambda_1(t)}T^{1000})^{-1}$ given by Proposition 2 which is κ_0-representative relatively to $(\Pi^+(t), \mathfrak{F})$. It is possible insofar as :

$$\sum_{t\in\Pi}\ \sum_{t'\in\Pi^+(t)} (L_t e^{\lambda_1(t')})^{-\kappa_0} \leqslant T^2 T^{1000\kappa_0} T^{-R\kappa_0} = O(T^{-1000}) \tag{2.32}$$

for R chosen large enough.
In particular, if one calls $\tilde{E}_1$ the set of $L \in \mathscr{S}_2$ such that for all $t \in \Pi$, L is a representative point of $(F_t, \Pi^+(t), \mathfrak{F})$, one sees that

$$\mu_2(\tilde{E}_1^c) = O(T^{-1000}).$$

For $L \in \tilde{E}_1$, according to the definition of representative point, according to the calculus made to get Equation (2.30), according to Lemma 23, one sees that (h4a) and (h4b) are verified by choosing $E = \tilde{E}_1$ and by taking this partition F_t.

• Verification of (h5)
Let us give $t \in [0, T]$, $\bar{t} \in \Pi^+(t)$ and $\gamma_{\bar{t}} \in F_{\bar{t}}$. Even if it means taking R larger, one can asssume that the length of $\delta_t\gamma_t$, which is $e^{-\lambda_1(\bar{t})}T^{-1000}e^{\lambda_1(t)}$, is smaller than T^{-10^9}.
This being said, if ξ_t is null over γ_t, the result is clearly acquired on $\gamma_{\bar{t}}$. Otherwise, there exists $\overline{L} \in \gamma_{\bar{t}}$ and p such that $\xi_{t,p}(\overline{L}) = 1$ and so there must exist $u \in \overline{L}$ such that $\nu_t \in K_p$.
Moreover, let us note that $\delta_t\gamma_{\bar{t}} = \{h(\tau)\delta_t\overline{L}\}_{\tau\in\mathscr{I}}$ where $\mathscr{I}$ is a closed interval that owns 0 of length smaller than T^{-10^9}. Then we observe that $h(\tau)\delta_t\overline{L}$ admits $h(\tau)\delta_t u$ as smallest non-zero vector and

$$T\|h(\tau)\delta_t u\|^{-2} = T\|\delta_t u\|^{-2} + O(T^{-10^9+1}).$$

Even if it means replacing the condition $t \in A_2(\epsilon, T, L)$ by

$$t \in \tilde{A}_2(\epsilon, L, T) = \{i \in [0, T-1] \cap A(\epsilon, T, L) \mid \|\delta_{i-1}L\| > \|\delta_i L\| + \frac{1}{T^{2001}} < \|\delta_{i+1}L\|\},$$

if $\delta_t\gamma_{\bar{t}}$ does not intersect (or more exactly its image)

$$\{x \in]0, \infty[\mid d(x, \partial K_p) \leqslant T^{-10^3}\},$$

it is completely included in K_p and so $\xi_{t,p} = 1$ along this curve.
The measure of the set of L such that $\delta_t\gamma_{\bar{t}}$ intersects $\{x \in]0, \infty[\mid d(x, \partial K_p) \leqslant T^{-10^3}\}$ is so $O(T^{-1000})$ because this set is included in the set of L that "belong" to $\hat{K}_p$ and which has a measure of $O(T^{-1000})$ according to Lemma 24. And when, before, we have replaced the set, we have a neglected a set and the measure of this set is $O(T^{-1000})$.
By taking the complement of the union of all these exceptional events, for $t \in [0, T]$, for $\bar{t} \in \Pi^+(t)$ (and by considering the intersection with E_1), one obtains a measurable set E_2 such that $\mu_2(E_2^c) = O(T^{-998})$ on which (h5) is valid (as well as (h4)).

• Verification of (h6)
The size of the F_t is $L_t = (e^{\lambda_1(t)}T^{1000})^{-1}$ and that of $F_{\bar{t}}$ is $L_{\bar{t}} = (e^{\lambda_1(\bar{t})}T^{1000})^{-1}$, if one sets

$$E_3 = \Big\{L \in \mathscr{S}_2 \mid F_{\bar{t}}(L) \subset F_t(L) \text{ pour tout } t \in [0, T],\ \bar{t} \in \Pi^+(t)\Big\},$$

one sees that the measure of the complement is $O(T^{-998})$ even if it means taking R larger. Indeed, at $t, \bar{t}$ fixed, for every curve of F_t, the total length of the sets that do

not meet the desired condition is at most $2L_{\bar{t}}$.

On the set $E = E_1 \cap E_2 \cap E_3$, one sees that (h4) to (h6) are verified. Furthermore, $\mu_2(E^c) = O(T^{-100})$. More, (h1) to (h3) are verified. Hence the wanted result. □

Proof of Theorem 2. By using Lemma 5 and Proposition 4, Proposition 5 and Proposition 7 which enables us to apply Theorem 3, one gets finally the validity of Theorem 2. □

Note : the result remains valid if we assume L distributed according to $\tilde{\mu}_2$ because the geodesic flow mixes at exponential speed.

2.5 Asymptotical study of $\mathcal{R}$

2.5.1 First considerations on the error $\mathcal{R}$

The problem that interests us now is that of convergence in distribution of $\frac{\mathcal{R}(tP+X,L)}{\log(t)}$ where $X \in \mathbb{R}^2$, $t > 0$, P is a parallelogram of non-empty interior and $L \in \mathscr{S}_2$. For simplicity, we can assume, as we are working with L distributed according to $\tilde{\mu}_2$ and X according to $\tilde{\lambda}_2$, that the parallelogram P is a rectangle whose sides are parallel to the coordinate axes (this can be done by deforming P *via* the rotation matrices and *via* the matrices of the form $\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$) and, even if it means deforming P *via* the matrices of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$, we can assume that P is, in fact, a square whose sides are parallel to the coordinate axes and finally, even if means translating P, we can assume that the centre of P is $(0,0)$. Note that we have used all the degrees of freedom of the problem that we had *a priori*.
This simplification made, let us call A_1 the lower right-hand corner of P of coordinates $(a,-a)$, A_2 the upper right-hand corner of P of coordinates (a,a), A_3 the upper left corner of P with coordinates $(-a,a)$ and A_4 the lower left-hand corner of P of coordinates $(-a,-a)$ with $a > 0$.
We now need to introduce some notations to recall a result of [82] which is necessary to establish the desired convergence in distribution. Let us call :

- $\tau = \frac{\log(t)^{\frac{1}{4}}}{t}$
- $t^{\pm} = t \pm \beta\tau$ with $\beta \in \mathbb{R}$ well-chosen
- $\rho > 0$ such that $\tau = \rho^{-\theta}$ where $\theta \in]0,1[$ with θ as near as 1 that one wants
- $\lambda(l) = \frac{l_1^2}{l_1^2+l_2^2}$
- ω_1 is a function of compact support $\subset B_f(0,\frac{1}{4})$, of class C^∞, non-negative, smaller than 1 and such that $\omega_1 = 1$ over $B_f(0,\frac{1}{8})$, with spherical symmetry

- ω_2 is the Fourier transform of $\frac{\omega_1}{\int \omega_1}$ (and is therefore a rapidly decreasing function with spherical symmetry)

With these notations, we can now pose :

$$\tilde{S}_1^{\pm}(\tau, t, P^0, R_1, X, L) = \frac{1}{\log(t)} \frac{1}{(2\pi i)^2} \sum_{l \in L^{\perp} - \{0\}} \frac{1}{(R_1 l)_1 (R_1 l)_2} \lambda(R_1 l) \omega_2(\tau l) \omega_1(\rho^{-1} l) e^{2i\pi < l, t^{\pm} P^0 + X >}$$

where $R_1 \in SO_2(\mathbb{R})$ and where $R_1 L^{\perp}$ is assumed to be weakly admissible (see Definition 7).
Finally, let us call $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$\tilde{S}^{\pm}(\tau, t, X, L) = \sum_{i=1}^{4} (-1)^i (\tilde{S}_1^{\pm}(\tau, t, A_i, I_2, X, L) - \tilde{S}_1^{\pm}(\tau, t, A_i, R, X, L)).$$

The two sums $\tilde{S}^{\pm}$ are in fact two sums over what is called the set of flags of the square P, which in this particular case is composed of 8 elements (see [82] for more details).
The following result is given by [82]:

Proposition 8. *For every $\alpha > 0$, for every t large enough,*

$$\mathbb{P}\left(\alpha + \tilde{S}^{+}(\tau, t, X, L) \geqslant \frac{\mathcal{R}(tP + X, L)}{\log(t)} \geqslant \tilde{S}^{-}(\tau, t, X, L) - \alpha\right) \geqslant 1 - \alpha.$$

Thus, to show Theorem 1, it is sufficient to show that $\tilde{S}^{-}$ and $\tilde{S}^{+}$ have a common asymptotic distribution which is a centred Cauchy distribution.
Moreover, the study of the asymptotic convergence of $\tilde{S}^{-}$ can be conducted in the same way as that of $\tilde{S}^{+}$ and the limit distribution will be independent of the sign $\pm$. As a consequence, we have to and will only deal with $\tilde{S}^{+}$ and, from now on, we rename it $\tilde{S}$. All along the proof, it will be pointed out what would have happened if we had dealt with $\tilde{S}^{-}$ instead of $\tilde{S}^{-}$.

2.5.2 Single term study

For pedagogical purposes, we will simplify the situation by dealing with only a single term of $\tilde{S}$ namely :

$$\tilde{S}_2(\tau, t, X, L) = \tilde{S}_1^{+}(\tau, t, A_2, I_2, X, L). \tag{2.33}$$

The purpose of this subsection is to show the following result:

Theorem 7. *When $L \in \mathscr{S}_2$ is distributed according to $\tilde{\mu}_2$, when X is distributed according to $\tilde{\lambda}_2$, $\tilde{S}_2$ converges in distribution towards a Cauchy centred distribution.*

One clarification, however: as we have seen in the previous section, we can and will assume that there is $D > 0$ such that for all $l \in L - \{0\}$,

$$|l_1 l_2| \geqslant D |\log(\|l\|_2)|^{-1-\beta} \tag{2.34}$$

(with $\beta > 0$) and that

$$\|L\|_1 \geqslant \sqrt{\epsilon}. \tag{2.35}$$

Moreover, we will replace $L^\perp$ by L in the study of $\tilde{S}_1$ (which is not a problem because $L \longmapsto L^\perp$ is a regular involution).

Heuristically, the term in the denominator $|Num(l)|$ is in fact related to a certain $\|\delta_k L\|$, while in the numerator we have a term that will behave at infinity as a θ_k. Since we have managed to prove Theorem 2, we should succeed in proving this last theorem.

Let us introduce:

$$\tilde{S}_9(\tau, t, \epsilon, X, L) = \frac{2}{(2\pi i)^2} \sum_{h \in \tilde{I}_-(t)} \frac{\Gamma_{h,t}}{\Xi_{h,t}} \tag{2.36}$$

where

$$\Xi_{h,t} = \mathrm{Num}(l(L,h)) \log(t) \ , \tag{2.37}$$

$$\Gamma_{h,t} = \sum_{k \in \mathbb{N} - \{0\}} \frac{1}{k^2} \cos(2k\pi < l(L,h), t^+ A_2 + X >) \ \textit{and} \tag{2.38}$$

$$\tilde{I}_-(t) = \{h \in [-\lceil \log(t) \rceil, 0] \mid \|\delta_h L\|^2 < 2\cosh(1) |\mathrm{Num}(e(\delta_h L))| \text{ and } |\mathrm{Num}(e(\delta_h L))| \leqslant \frac{1}{\epsilon \log(t)}\} \tag{2.39}$$

with $e(L)$ the unique vector of L such that its first coordinate is positive and such that $\|e(L)\| = \|L\|$ ($e(L)$ is defined for almost any lattice L), $\epsilon > 0$ and with $l(L,h) = \delta_h^{-1} e(\delta_h L)$.
With these notations, we are going to proceed the following way to show Theorem 7. First, in the second subsection, we are going to show the following proposition which allows us to move from the asymptotic study of $\tilde{S}_2$ to that of $\tilde{S}_9$:

Proposition 9. *For every $\alpha > 0$, for every $\epsilon > 0$ small enough, for every t large enough,*

$$\mathbb{P}(|\tilde{S}_2 - \tilde{S}_9| \geqslant \alpha) \leqslant \alpha.$$

This proposal will be the subject of the following subsubsection and is obtained after a number of successive reductions, which require estimates of integrals, the use of tools from Fourier analysis and classical analysis, the geometry of lattices and the use of dynamic tools on the space of lattices with the geodesic flow (the ergodic theorem for example).
A third subsection will be devoted to the development of a framework to reduce the asymptotic study of the $\{\Xi_{h,t}\}$ and of the $\{\Gamma_{h,t}\}$ to the verification of hypothesis (h1) to (h8) of Theorem 4. The following subsection will be dedicated to the verification of these assumptions which will ensure that the $\{\Xi_{h,t}\}$ asymptotically form a Poisson process and the $\{\Gamma_{h,t}\}$ are asymptotically independent, identically distributed, symmetric, compactly supported real random variables independent of $\{\Xi_{h,t}\}$. It will enable us, *in fine*, *via* Lemma 4 to get the validity of Theorem 7.
In the fifth and final subsection of this section, building on the results of the previous subsections, we will return to the study of $\tilde{S}_1$ and prove Theorem 1.

Proof of Proposition 9 - "Forgetting" of the function ω_1

Let us introduce the following quantity:

$$\tilde{S}_3(\tau,t,X,L)=\frac{1}{\log(t)}\frac{1}{(2\pi i)^2}\sum_{l\in J_3(L,\alpha,t)}\frac{1}{l_1l_2}\lambda(l)\omega_2(\tau l)e^{2i\pi<l,t^+A_2+X>}$$

where

$$J_3(L,\alpha,t)=\left\{l\in L-\{0\}\mid \|l\|\leqslant\frac{1}{8}t^{\alpha}\right\}$$

with $1<\alpha<\frac{1}{\theta}$ (later, we are going to make it go towards 1).
The purpose of this subsection is then to prove the following proposition:

Proposition 10. *$|\tilde{S}_2-\tilde{S}_3|$ converges in probability towards 0.*

Heuristically, this proposition means that we replace ω_1 by its value in 0, which is 1, and that we cut the terms in l whose norm is too large, which is what the function ω_1 does when l in norm is greater than $\frac{1}{4}t^{\frac{1}{\theta}}$. It is therefore quite natural to prove this. To prove this proposition, we need the following lemma, stated in dimension $d\geqslant 2$.

Lemma 25. *For every $\epsilon_1>0$, for all $\beta>0$, for every $\gamma>0$,*

$$\int_{\substack{l\in\mathbb{R}^d\\ 1<\|l\|\leqslant t^{1+\epsilon_1}\\ |l_1\cdots l_d|\geqslant D|\log(\|l\|)|^{1-d-\beta}}}\frac{1}{|l_1\cdots l_d|}dl_1\cdots dl_d=O(\log(t)^d).$$

Proof. By continuity and symmetry, this is equivalent to showing that:

$$\int_{\substack{l\in\mathbb{R}^d\\ 1<\|l\|\leqslant t^{1+\epsilon_1}\\ l_1\cdots l_d\geqslant D\log(\|l\|)^{1-d-\beta}\\ 0<l_1\leqslant l_2\leqslant\cdots\leqslant l_d}}\frac{1}{l_1\cdots l_d}dl_1\cdots dl_d=O(\log(t)^d).$$

Let us call I this last integral.
The system of inequalities on l implies that :
for every $i\in\{1,\cdots,d\}$, $l_i\leqslant t^{1+\epsilon_1}$ and $l_i^i(t^{1+\epsilon_1})^{d-i}\geqslant D\log(\|l\|)^{1-d-\beta}\geqslant C\log(t^{1+\epsilon_1})^{1-d-\beta}$.
Thus, for any i :

$$t^{1+\epsilon_1}\geqslant l_i\geqslant C_i\frac{\log(t)^{\frac{1-d-\beta}{i}}}{t^{(1+\epsilon_1)\frac{d-i}{i}}}.$$

As a consequence, for every t large enough, by using Fubini's theorem, one gets that :

$$I\leqslant\prod_{i=1}^{d}\int_{t^{1+\epsilon_1}\geqslant l_i\geqslant C_i\frac{\log(t)^{\frac{1-d-\beta}{i}}}{t^{(1+\epsilon_1)\frac{d-i}{i}}}}\frac{1}{l_i}dl_i.$$

However, for any i, $\int_{t^{1+\epsilon_1}\geqslant l_i\geqslant C_i\frac{\log(t)^{\frac{1-d-\beta}{i}}}{t^{(1+\epsilon_1)\frac{d-i}{i}}}}\frac{1}{l_i}dl_i=O(\log(t))$. So one gets the wanted result.

□

Now let us prove Proposition 10.

Proof of Proposition 10. Given the properties on ω_1 and the definition of ρ, we have (by implying the various variables) :

$$|\tilde{S}_2 - \tilde{S}_3| \leqslant \frac{M}{\log(t)} \sum_{\substack{l \in L - \{0\} \\ \frac{t^{\frac{1}{\theta}}}{4} \geqslant \|l\| \geqslant \frac{1}{8} t^{\alpha}}} \frac{1}{|\mathrm{Num}(l)|} |\omega_2(\tau l)|. \tag{2.40}$$

Yet, ω_2 is a rapidly decreasing function and for l like in the sum, one gets $\|\tau l\| \geqslant \tau t^{\alpha} \geqslant t^{\alpha-1}$ and $\alpha - 1 > 0$.
So, from Equation (2.40) one gets that :

$$|\tilde{S}_2 - \tilde{S}_3| \leqslant \frac{M}{\log(t) t^{17(\alpha-1)}} \sum_{\substack{l \in L - \{0\} \\ \frac{t^{\frac{1}{\theta}}}{4} \geqslant \|l\| \geqslant \frac{1}{8} t^{\alpha}}} \frac{1}{|\mathrm{Num}(l)|}. \tag{2.41}$$

As a consequence, one gets the wanted result thanks to Lemma 7 and thanks to Equation (2.34) which enables us to use Lemma 25. □

As a result, the study is reduced to the study of $\tilde{S}_3$ [1].

Proof of Proposition 9 **- Centring on prime vectors**

Let us introduce:

$$\tilde{S}_4(\tau, t, X, L) = \frac{2}{\log(t)} \frac{1}{(2\pi i)^2} \sum_{l \in J_4(L,\alpha,t)} \frac{1}{l_1 l_2} \lambda(l) \sum_{k \in \mathbb{N} - \{0\}} \frac{\omega_2(k\tau l)}{k^2} \cos(2k\pi < l, t^+ A_2 + X >) \tag{2.42}$$

where

$$J_4(L, \alpha, t) = \Big\{ l \in L - \{0\} \mid \|l\| \leqslant \frac{1}{8} t^{\alpha},\ l \text{ prime},\ l_1 > 0 \Big\}.$$

The purpose of this subsection is to prove the following proposition:

Proposition 11. $\tilde{S}_4 - \tilde{S}_3$ *converges in probability towards* 0.

This proposition tells us, on the one hand, that we can group together all the terms which are multiples of a prime vector l, and, on the other hand, that we can just consider the prime vectors l whose first coordinate is strictly positive (via a parity argument and even if we have to eliminate a negligible set of lattices).

Proof. First, by using the invariance by the transformation $l \longmapsto -l$ of $\omega_2(l)$, of $\mathrm{Num}(l)$ (in general it depends on the parity of the dimension d) and of $\lambda(l)$, it comes that :

$$\tilde{S}_3(\tau, t, X, L) = \frac{2}{\log(t)} \frac{1}{(2\pi i)^2} \sum_{\substack{l \in L \\ \|l\| \leqslant \frac{1}{8} t^{\alpha} \\ l_1 > 0}} \frac{1}{l_1 l_2} \lambda(l) \omega_2(\tau l) \cos(2\pi < l, t^+ A_2 + X >). \tag{2.43}$$

[1] We see in the proof of Proposition 10 that we have upper bounded the modulus of the terms $e^{2i\pi < l, t^+ A_2 + X >}$ by 1. So this proof is also valid by having t^- instead of t^+.

From Equation (2.42) and from Equation (2.43), one gets that :

$$|\tilde{S}_3 - \tilde{S}_4| \leqslant \frac{M}{\log(t)} \sum_{\substack{l\in L \\ \|l\|\geqslant \frac{1}{8}t^\alpha \\ l_1>0}} \frac{|\omega_2(\tau l)|}{|\mathrm{Num}(l)|}.$$

The conclusion is the same as for Proposition 10. □

The study is so reduced to the study of $\tilde{S}_4$ [2].

Proof of Proposition 9 - Reduction of the sum to terms l such that $Num(l)$ is small

Let $\epsilon > 0$. Let us introduce:

$$\tilde{S}_5(\tau, t, \epsilon, X, L) = \frac{2}{\log(t)} \frac{1}{(2\pi i)^2} \sum_{l\in J_5(L,\alpha,\epsilon,t)} \frac{1}{l_1 l_2} \lambda(l) \sum_{k\in\mathbb{N}-\{0\}} \frac{\omega_2(k\tau l)}{k^2} \cos(2k\pi < l, t^+ A_2 + X >) \tag{2.44}$$

where

$$J_5(L, \alpha, \epsilon, t) = \Big\{ l \in L - \{0\} \mid \|l\| \leqslant \frac{1}{8} t^\alpha,\ l \text{ prime},\ l_1 > 0,\ |Num(l)| \leqslant \frac{1}{\log(t)\epsilon} \Big\}.$$

The purpose of this subsection is then to prove the following proposition:

Proposition 12. *For every $\kappa > 0$, for every $\epsilon > 0$ small enough, for every t large enough, one has*

$$\mathbb{P}\Big(|\tilde{S}_4(\tau, t, X, L) - \tilde{S}_5(\tau, t, \epsilon, X, L)| \geqslant \kappa \Big) \leqslant \kappa.$$

This proposition essentially tells us that we can reduce ourselves to the terms $l \in J_4(L, \alpha, t)$ who see their *Num* being small (which is natural since, as the inverse quantity is involved in the sum, it corresponds to terms that contribute a lot to the sum).
The proof is based on the following lemma stated in dimension $d \geqslant 2$:

Lemma 26. *There $M > 0$ such that for all $\epsilon_1 > 0$,*

$$\int_{\substack{l\in\mathbb{R}^d \\ 1\leqslant\|l\|\leqslant t^{1+\epsilon_1} \\ |Num(l)|\geqslant \frac{1}{\epsilon\log(t)^{d-1}}}} \frac{1}{l_1^2 \cdots l_d^2} dl_1 \cdots dl_d \leqslant M \log(t)^{2d-2} \epsilon.$$

Proof. Because of a symmetry argument, we can assume that the l_i are strictly positive. We set $\phi : (l_1, \cdots, l_d) \longmapsto (l_1, l_1 l_2, \cdots, l_1 \cdots l_d)$. Then ϕ is a C^∞-diffeomorphism from $(\mathbb{R}_+ - \{0\})^d$ on itself and the Jacobian matrix in $(l_1, \cdots, l_d)$, denoted by $\mathrm{Jac}(l_1, \cdots, l_d)$, satisfies that :

$$\mathrm{Jac}(l_1, \cdots, l_d) = \prod_{i=1}^{d-1} \phi_i(l_1, \cdots, l_d). \tag{2.45}$$

[2] One sees that, in the proof of Proposition 11, essentially used a parity argument and upper bounded the terms $|\cos(2\pi < l, t^+ A_2 + X >)|$ by 1. This proof is so still valid if one considers t^- instead of t^+.

Furthermore, as l belongs to the domain of the integral referred to in Lemma 26, one sees that for every $i \in \{1, \cdots, d-1\}$, $b_i = \frac{1}{\epsilon \log(t)^{d-1} t^{(1+\epsilon_1)(n-i)}} \leqslant \phi_i(l) \leqslant t^{(1+\epsilon_1)i} = h_i$ et $\phi_d(l) \geqslant \frac{1}{\epsilon \log(t)^{d-1}} = b_d$.
From Equation (2.45), one gets that :

$$\int_{\substack{\forall 1 \leqslant i \leqslant d,\ l_i > 0 \\ 1 \leqslant \|l\|_2 \leqslant t^{1+\epsilon_1} \\ \mathrm{Num}(l) \geqslant \frac{1}{\epsilon \log(t)^{d-1}}}} \frac{1}{l_1^2 \cdots l_d^2} dl_1 \cdots dl_d \leqslant \int_{\substack{\forall 1 \leqslant i \leqslant d-1,\ b_i \leqslant u_i \leqslant h_i \\ b_d \leqslant u_d}} \frac{1}{u_d^2 \prod_{i=1}^{d-1} u_i} du. \tag{2.46}$$

The right-hand member can be calculated and we obtain the wanted result. □

Now, let us prove Proposition 12.

Proof of Proposition 12. By calling Δ the norm 2 squared of the difference between $\tilde{S}_4$ and $\tilde{S}_5$ relatively to X, the Parseval's formula gives us that :

$$\Delta = \frac{M_1}{\log(t)^2} \sum_{\substack{l \in J_4(L,\alpha,t) \\ |Num(l)| > \frac{1}{\epsilon \log(t)}}} \frac{1}{(l_1 l_2)^2} (\lambda(l))^2 \sum_{k \in \mathbb{N} - \{0\}} \frac{(\omega_2(k\tau l))^2}{k^4} \tag{2.47}$$

where $M_1 > 0$.
By using the fact that ω_2 and λ are bounded and by integrating relatively to $L \in \mathscr{S}_2$ and by using Lemma 7 and then Lemma 26 (in the case where $d = 2$), one gets that for every t large enough :

$$\mathbb{E}(\Delta) \leqslant \frac{M_2}{\log(t)^2} \epsilon \log(t)^2$$

where $M_2 > 0$. So, one gets the wanted resuilt. □

As a consequence, the study is reduced to, for $\epsilon > 0$, the study of $\tilde{S}_5$ [3].

Proof of Proposition 9 - Transition to a geodesic sum

The aim of this subsubsection is to pass from a sum on $l \in L - \{0\}$ to a sum on $\delta_h L$ with h integer. To speak more precisely, let us introduce some notations. For $L \in \mathscr{S}_2$ weakly admissible, let $e(L)$ be the unique vector of L such that $e(L)_1 > 0$ and $\|e(L)\| = \|L\|_1$ ($e(L)$ is necessarily prime) and let us give $\eta > \alpha - 1$, as close as one wants of $\alpha - 1$.
Let us introduce now the set

$$I(t,\eta) = \Big\{ h \in [-\lceil (1+\eta)\log(t) \rceil, \lceil (1+\eta)\log(t) \rceil] \mid \|\delta_h L\|^2 < 2\cosh(1) |\mathrm{Num}(e(\delta_h L))| \text{ and } |\mathrm{Num}(e(\delta_h L))| \leqslant \frac{1}{\epsilon \log(t)} \Big\}$$

[3] We see that in the proof, if we replace t^+ by t^-, it still works: the term $\cos(2k\pi < l, t^+ A_2 + X >)$ occurs only at the beginning of the proof and we eliminate it via Parseval's formula, which still works by replacing t^+ by t^-.

the vectors L being of the form

$$l(L,h) = \delta_h^{-1} e(\delta_h L) \tag{2.48}$$

where h belongs to $\mathbb{Z}$ and the sum

$$\tilde{S}_6(\tau,t,\epsilon,X,L) = \frac{2}{(2\pi i)^2} \sum_{h\in I(t,\eta)} \frac{1}{\Xi_{h,t}} \lambda(l(L,h)) \tag{2.49}$$

$$\Big(\sum_{k\in\mathbb{N}-\{0\}} \frac{\omega_2(k\tau l(L,h))}{k^2} \cos(2k\pi < l(L,h), t^+A_2 + X >) \Big).$$

Let us recall that $\Xi_{h,t}$ was defined by Equation (2.37).
The goal of this subsubsection is to prove the following proposition:

Proposition 13. *$\tilde{S}_5 - \tilde{S}_6$ converges in probability towards* 0.

This proposition enables us to return to the study of a geodesic sum, namely $\tilde{S}_6$. The important fact is that the points l of the sum $\tilde{S}_5$ are in fact of the form $l(L,h)$ with $h \in I(t,\eta)$ (see Lemma 28 and Lemma 30) and that the terms of $\tilde{S}_6$ which are in addition relatively to the sum $\tilde{S}_5$ are negligible (which is given by Lemma 28, Lemma 29 and Lemma 30).
To show this proposition, we first need to go back and clarify the beginning of section 7 of [82].
Let us set $\mathcal{P} =]0,\infty[\times(\mathbb{R}-\{0\})$.
In this section, Skriganov introduces the group action defined by $(\delta,x) \in \Delta\times\mathcal{P} \longmapsto \delta x$. This one admits as fundamental domain:

$$\mathcal{F}_\Delta = \{\ m(e^y, e^{-y}) \mid m>0,\ y\in[-\frac{1}{2},\frac{1}{2}[\ \}\dot{\cup}\{\ m(e^y,-e^{-y}) \mid m>0,\ y\in[-\frac{1}{2},\frac{1}{2}[\ \}.$$

Lemma 27. *For every $x=(x_1,x_2)\in\mathcal{P}$, the unique $h(x)\in\mathbb{Z}$ such that $\delta_{h(x)}x \in \mathcal{F}_\Delta$ is given by :*
$h(x) = \frac{1}{2}\lceil \log(\frac{|x_2|}{x_1})\rceil$ *if* $\lceil \log(\frac{|x_2|}{x_1})\rceil$ *is even and otherwise* $h(x) = \frac{1}{2}\lfloor \log(\frac{|x_2|}{x_1})\rfloor$.
Moreover, one has: $\delta_{h(x)}x = m(e^y, sgn(x_2)e^{-y})$ *where* $m = \sqrt{x_1|x_2|}$ *and* $y = h(x) - \frac{1}{2}\log(\frac{|x_2|}{x_1})$.

Proof. The proof is elementary. □

We notice that, even if we have to remove from $\mathcal{P}$ a countable set of portions of lines, we can suppose that $y \in]-\frac{1}{2},\frac{1}{2}[$. This symmetrizes the situation and because of this fact we will consider lattices L which never touch this set of lines. This set constitutes a measurable set of full measure.
We then have the following lemma:

Lemma 28. *Let $l\in L$ as in the area of the sum S_5. Then $h(l)\in I(t,\eta)$.*

Proof. Let l as in the statement of the lemma. Then one has $\|\delta_{h(l)}l\|^2 = 2\cosh(2y)|l_1l_2|$ and so, as $y\in]-\frac{1}{2},\frac{1}{2}[$,

$$2|l_1l_2| \leqslant \|\delta_{h(l)}l\|^2 < 2\cosh(1)|l_1l_2|. \tag{2.50}$$

Thus, since $|l_1 l_2|$ is sufficiently small and l is prime, provided that t is large enough, $\delta_{t(l)} l$ is a shortest vector of the lattice $\delta_{t(l)} L$ according to the reasoning that led to Lemma 10 and since $(\delta_{h(l)} l)_1 > 0$, we have

$$e(\delta_{h(l)} L) = \delta_{h(l)} l. \tag{2.51}$$

Furthermore, as for every x, for every $k \in \mathbb{Z}$, $\mathrm{Num}(\delta_k x) = \mathrm{Num}(x)$, one has, according to Equation (2.51) :

$$Num(e(\delta_{h(l)} L)) \leqslant \frac{1}{\epsilon \log(t)}. \tag{2.52}$$

Finally, one has: $h(l) = \lceil \frac{1}{2} \log(\frac{|l_2|}{l_1}) \rceil$ ou $h(l) = \lfloor \frac{1}{2} \log(\frac{|l_2|}{l_1}) \rfloor$ et $\frac{|l_2|}{l_1} = \frac{l_1 |l_2|}{l_1^2} = \frac{l_2^2}{l_1 |l_2|}$.
Yet, according to Equation (2.34) :

$$D\alpha^{-1-\beta} \log(t)^{-1-\beta} \leqslant D \log(\|l\|)^{-1-\beta} \leqslant l_1 |l_2| \leqslant \frac{1}{\epsilon \log(t)}.$$

So, as $\|l\| \leqslant t^\alpha$, it comes that for t large enough (the largeness depending on D, α, β et ϵ),

$$-\lceil (1+\eta) \log(t) \rceil \leqslant t(l) \leqslant \lceil (1+\eta) \log(t) \rceil$$

where $\eta > \alpha - 1$ and is as close as one wants of $\alpha - 1$. So one gets the wanted result. □

The following lemma provides a form of reciprocal:

Lemma 29. *For $h \in I(t, \eta)$,*

$$l(L,h) \in V(L,t) = \{l \in L | l_1 > 0,\ l \text{ is prime},\ \|l\| \leqslant C t^{1+\eta},\ l_1 |l_2| \leqslant \frac{1}{\epsilon \log(t)}\}$$

where $C > 0$ is a constant.

Proof. One sets $l = l(L,h) = \delta_h^{-1} e(\delta_h L)$. It is clear that $l_1 > 0$ and that l is prime according to the definition of e.
Moreover, Minkowski's theorem gives us that there is $C > 0$ such that:

$$\|e(\delta_h L)\| \leqslant C.$$

So, one gets that there exists $C > 0$ such that

$$\|L\|_2 \leqslant C t^{1+\eta}.$$

Finally, one notes that $\mathrm{Num}(l) = \mathrm{Num}(e(\delta_h L))$ and so $l(L,h) \in V(L,t)$. □

One sees that, with $l(L, \cdot)$, we do not fall exactly in the domain of S_5 *a priori* (but we do not fall too far from it). But $l(L, \cdot)$ still plays the role of reciprocal of $t(l)$:

Lemma 30. *Let $k \in L$ prime such that $|Num(k)| \leqslant \frac{1}{\epsilon \log(t)}$. Provided that t is large enough, one has $l(L, t(k)) = k$.*
Conversely, let $h \in \mathbb{Z}$ such that $|Num(e(\delta_h L))| \leqslant \frac{1}{\epsilon \log(t)}$ and $\|\delta_h L\|^2 < 2\cosh(1) |Num(e(\delta_h L))|$.
If t is large enough, $t(l(L,h)) = h$.

Note that this lemma implies that $h \in I(\eta, t)$ are local minima (in the sense defined in the previous section).

Proof. Let us deal with the first part of the lemma and take $t > 2\cosh(1)$ and k as in the statement. Then according to Equation (2.50), $e(\delta_{t(k)}L) = \delta_{t(k)}k$. Thus, we have :

$$l(L, t(k)) = \delta_{t(k)}^{-1} e(\delta_{t(k)}L) = k.$$

Let us now deal with the second part of the lemma and take again $t > 2\cosh(1)$ and h as in the statement. Recall that $t(l(L,h))$ is the unique $p \in \mathbb{Z}$ such that

$$\delta_p l(L,h) \in \mathcal{F}_\Delta.$$

However, one has $l(L,h) = \delta_h^{-1} e(\delta_h L)$ and so $\delta_h l(L,h) = e(\delta_h L)$.
One has also :

$$||\delta_h L||^2 < 2\cosh(1)|\mathrm{Num}(e(\delta_h L))|.$$

As $e(\delta_h L)$ can be written : $e(\delta_h L) = \sqrt{|\mathrm{Num}(e(\delta_h L))|}(e^y, \pm e^{-y})|$ where $y \in \mathbb{R}$, the previous inequality implies that:

$$2\cosh(2y) < 2\cosh(1).$$

So $y \in]-\frac{1}{2}, \frac{1}{2}[$ and $e(\delta_h L) \in \mathcal{F}_\Delta$. □

We can now prove Proposition 13.

Proof of Proposition 13. One has, according to Lemma 27, Lemma 28, Lemma 29 and Lemma 30, for every t large enough:

$$|(\tilde{S}_5 - \tilde{S}_6)| \leqslant \frac{M}{\log(t)} \sum_{l \in H(L,\alpha,\epsilon,\eta,t)} |\lambda(l)| \frac{1}{|l_1 l_2|} \sum_{k \in \mathbb{N}-\{0\}} \frac{|\omega_2(k\tau l)|}{k^2} |\cos(2k\pi < l, t^+ A_2 + X >)|$$

where $(\tilde{S}_5 - \tilde{S}_6)$ is evaluated at $(\tau, t, \epsilon, X, L)$ and

$$H(L,\alpha,\epsilon,\eta,t) = \{l \in L - \{0\} \mid \frac{1}{8}t^\alpha < \|l\| \leqslant Ct^{1+\eta},\ l \text{ prime},\ l_1 > 0,\ |Num(l)| \leqslant \frac{1}{\log(t)\epsilon}\}.$$

We conclude as we concluded the proof of Proposition 11 [4]. □

Proof of Proposition 9 - Reduction of the number of terms in the sum

We set $\tilde{I}(t) = I(t,\eta) \cap \{h \in \mathbb{Z} \mid |h| \leqslant \lceil \log(t) \rceil\}$ and

$$\tilde{S}_7(\tau, t, \epsilon, X, L) = \frac{2}{(2\pi i)^2} \sum_{h \in \tilde{I}(t)} \frac{1}{\Xi_{h,t}} \lambda(l(L,h)) \tag{2.53}$$

$$\left(\sum_{k \in \mathbb{N}-\{0\}} \frac{\omega_2(k\tau l(L,h))}{k^2} \cos(2k\pi < l(L,h), t^+ A_2 + X >) \right).$$

[4] At the end of the proof, the term in $|\cos|$ was upper bounded by 1. The proof is therefore still valid by replacing t^+ by t^-.

Proposition 14. $\mathbb{P}(\tilde{S}_7 - \tilde{S}_6 \neq 0) \leqslant M\frac{\eta}{\epsilon}$

This proposition says, *grosso modo*, that the factor η does not play any role in the problem we are studying (which is understandable, it is a quantity which is not intrinsic to the problem) and that we can reduce between what passes between $-\lceil \log(t) \rceil$ and $\lceil \log(t) \rceil$.

Proof. We have, for ever $h \in I(t,\eta)$, $\|\delta_h L\|^2 < 2\cosh(1)|\mathrm{Num}(e(\delta_h L))|$ and

$$|\mathrm{Num}(e(\delta_h L))| \leqslant \frac{1}{\epsilon \log(t)}.$$

By using this observation and Lemma 8, one has :

$$\mathbb{P}(\tilde{S}_7 - \tilde{S}_6 \neq 0) \leqslant \frac{M\eta \log(t)}{\epsilon \log(t)}.$$

So, one gets the wanted result. □

As one can take the parameter η as small as one wants (by taking α as close of 1 as one wants), the study is reduced to the study of $\tilde{S}_7$.

Proof of Proposition 9 - Elimination of "close" terms from the y-axis

We introduce $\tilde{I}_-(t) = \tilde{I}(t) \cap \mathbb{R}_-$ and the two following quantities :

$$\tilde{S}_8(\tau, t, \epsilon, X, L) = \frac{2}{(2\pi i)^2} \sum_{h \in \tilde{I}_-(t)} \frac{1}{\Xi_{h,t}} \tag{2.54}$$

$$\Big(\sum_{k \in \mathbb{N} - \{0\}} \frac{\omega_2(k\tau l(L,h))}{k^2} \cos(2k\pi < l(L,h), t^+ A_2 + X >) \Big).$$

The purpose of this subsection is to prove the following proposition

Proposition 15. $\tilde{S}_7 - \tilde{S}_8$ *converges towards* 0 *in probability.*

The purpose of this proposition is to remove the terms $h > 0$ and to simplify the term in λ. Actually, the terms h in question will become large as t grows and this will correspond to l terms such that $l_1 l_2$ will be small and l_2 will be large (in absolute value), i.e. "close" to the y-axis and the term in λ will therefore be close to 0. As for the terms corresponding to $h < 0$, these will become smaller and smaller as t grows and this will correspond to l terms such that $l_1 l_2$ will be small and l_1 will be large (in absolute value), i.e. "close" to the x-axis and the λ term will thus be close to 1.

Proof. Let us give $h \in \tilde{I}(t)$ such that $h \geqslant 0$. We omit L and h in $l(L,h)$ for the sake of simplification, and we pose $l(L,h) = (l_1, l_2)$. Let us assume $l_1 \geqslant l_2$. As $h \in \tilde{I}(t)$, one has

$$e^{2h} l_1^2 + e^{-2h} l_2^2 < \frac{2\cosh(1)}{\epsilon \log(t)}.$$

So, one gets:

$$l_1^2 < \frac{2\cosh(1)e^{-2h}}{\epsilon\log(t)}$$

and

$$l_2^2 < \frac{\cosh(1)}{\epsilon\log(t)\cosh(2h)}.$$

But, according to Equation (2.35), $l_1^2 + l_2^2 \geqslant \epsilon$, which is excluded for t large enough. Thus, one has $l_1 \leqslant l_2$. As $h \in \tilde{I}(t)$, one gets

$$|l_1| \leqslant \frac{1}{\sqrt{\epsilon\log(t)}} \ \textit{and} \tag{2.55}$$

$$|l_2| \geqslant \sqrt{\frac{\epsilon}{2}}. \tag{2.56}$$

From the expression of h, from the coordinates of l given by the lemma 27, we note in passing that, for t large enough,

$$h \geqslant H\log(\log(t))$$

where H is a constant > 0.
Thanks to Equation (2.55) and to Equation (2.56), one has also :

$$0 \leqslant \lambda(l) \leqslant \frac{1}{\epsilon^2\log(t)}.$$

In the case where $h \leqslant 0$, one gets that : $l_1 \geqslant l_2$,

$$|l_2| \leqslant \frac{1}{\sqrt{\epsilon\log(t)}},$$

$$l_1 \geqslant \sqrt{\frac{\epsilon}{2}},$$

$$h \leqslant -H\log(\log(t)) \ \textit{and}$$

$$|\lambda(l) - 1| \leqslant \frac{1}{\epsilon^2\log(t)}.$$

In view of the preceding remarks, we see that there exists a constant $F > 0$ such that :

$$|(\tilde{S}_8 - \tilde{S}_7)(\tau, t, P^0, X, L)| \leqslant \frac{F}{\epsilon^2\log(t)^2}\sum_{h\in\tilde{I}(t)}\frac{1}{||\delta_h L||^2} \tag{2.57}$$

and, thanks to Lemma 3.2 of [82], the last quantity, for almost every $L \in \mathscr{S}_2$, is $O(\frac{1}{\log(t)^{\frac{1}{2}}})$ (this lemma from [82] is a consequence of Theorem 6 and of Lemma 6). So, one gets the wanted result [5]. □

[5] We can see that here too the absolute value of cos is upper bounded by 1. This proof is therefore also valid when we replace t^+ by t^-.

Proof of Proposition 9 - Replacement of ω_2 by $\omega_2(0)$ and conclusion

As ω_2 is the Fourier transform of $\frac{\omega_1}{\int \omega_1}$, $\omega_2(0) = 1$ and so one has:

$$\tilde{S}_9(\tau, t, \epsilon, X, L) = \frac{2}{(2\pi i)^2} \sum_{h \in \tilde{I}_-(t)} \frac{1}{\Xi_{h,t}} \tag{2.58}$$
$$(\sum_{k \in \mathbb{N}-\{0\}} \frac{\omega_2(0)}{k^2} \cos(2k\pi < l(L,h), t^+ A_2 + X >))$$

according to Equation (2.36).
The purpose of this subsubsection is to prove the following proposition which allows to replace the ω_2 term by its single value in 0 :

Proposition 16. *$\tilde{S}_8 - \tilde{S}_9$ converges in probability towards 0.*

Proof. Let us note that, for $k \in \mathbb{N} - \{0\}$, for $h \in \tilde{I}(t)$:

$$\|k\tau l(L,h)\| \leqslant k\tau e^{\log(t)} \|\delta_h L\| \leqslant k\tau e^{\log(t)} \sqrt{\frac{2\cosh(1)}{\epsilon \log(t)}}.$$

As $\tau = \frac{\log(t)^{\frac{1}{4}}}{t}$, one has with $l = l(L,h)$:

$$\|k\tau l\| \leqslant k \frac{1}{\log(t)^{\frac{1}{4}}} \sqrt{\frac{2\cosh(1)}{\epsilon}}.$$

So, according to the mean value theorem, and as ω is a rapidly decreasing function :

$$|\frac{\omega(k\tau l)\cos(2\pi k < l, t^+ A_2 + X >)}{k^2} - \frac{\omega(0)\cos(2\pi k < l, t^+ A_2 + X >)}{k^2}| \leqslant \min(\frac{F}{k^2}, \frac{F}{k\log(t)^{\frac{1}{4}}}) \tag{2.59}$$

where F is a constant > 0 which depends on ϵ. With Equation (2.59), one finds that :

$$|\sum_{k \in \mathbb{N}-\{0\}} \frac{\omega_2(k\tau l)}{k^2} \cos(2k\pi < l(L,h), t^+ A_2 + X > - \sum_{k \in \mathbb{N}-\{0\}} \frac{\omega_2(0)}{k^2} \cos(2k\pi < l(L,h), t^+ A_2 + X > |$$
$$\leqslant \sum_{k=1}^{\infty} \min(\frac{F}{k^2}, \frac{F}{k\log(t)^{\frac{1}{4}}}) = O(\frac{1}{\log(t)^{\frac{1}{8}}}) \tag{2.60}$$

where the O depends only on ϵ. So, one finds that:

$$|\tilde{S}_8 - \tilde{S}_9| \leqslant F \frac{\sum_{h \in \tilde{I}_-(t)} ||\delta_h L||^{-2}}{\log(t)^{1+\frac{1}{8}}} \tag{2.61}$$

where $F > 0$ is a constant depending on ϵ.
Thanks again to Lemma 3.2 of [82], one finds that this term is $O(\frac{1}{\log(t)^{\frac{1}{16}}})$. So, one gets the wanted result [6]. □

[6] Yet again, we have upper bounded the absolute value of cos by 1. The proof is still valid if we replace t^+ by t^-.

Proof of Proposition 9. It is a consequence of Proposition 10, Proposition 11, Proposition 12, Proposition 13, Proposition 14, Proposition 15, Proposition 16 [7]. □

2.5.3 Transfer of the problem to the checking of the hypothesis of Theorem 4

Now, with $\tilde{S}_9$, we are going to conclude like in the previous section, *id est* by showing the following proposition:

Proposition 17. *One has:*

- *1)* $\{\Xi_{h,t}\}_{h\in\tilde{I}_-(t)}$ *converges towards a Poisson process over* $[-\frac{1}{\epsilon},\frac{1}{\epsilon}]-\{0\}$ *of constant intensity (independent from* ϵ*).*

- *2) The real random variables* $(\Gamma_{h,t})_{h\in\tilde{I}_-(t)}$ *converge asymptotically towards real random variables that are symmetric, independent, identically distributed, with compact support and independent from* $\{\Xi_{h,t}\}_{h\in\tilde{I}_-(t)}$.

Concerning item 2), as $t^+ = t + b\tau$, as $\tau = \frac{\log(t)^{\frac{1}{4}}}{t}$, it is enough to show the result by replacing t^+ by t (even if it means adapting the reasoning leading to Equation (2.61) [8]. Moreover, one can replace $\log(t)$ by $\lceil\log(t)\rceil$.

Let us set some new notations:

- $M = \lceil\log(t)\rceil$
- $\nu_0^M(L) = (M\mathrm{Num}(e(\cdot))\mathbf{1}_{2\cosh(1)|\mathrm{Num}(e(\cdot))|>||\cdot||^2})(L)$
- $\nu_h^M(L) = \nu_0^M(\delta_h L)$,
- $\xi_h^M = \mathbf{1}_{\nu_h^M\in[-\frac{1}{\epsilon},\frac{1}{\epsilon}]-\{0\}}$
- $\zeta_h^M(L,X) =< e(\delta_h L), \delta_h^{-1}(e^M A_2 + X) > (\mathrm{mod}\ 1)$

If we show the next proposition, the validity of Proposition 17 will follow :

Proposition 18. *One has:*

- *1)* $(\nu_h^M)_{\xi_h^M=1,\ h\in[-M,0]}$ *converges towards a Poisson process over* $[-\frac{1}{\epsilon},\frac{1}{\epsilon}]-\{0\}$ *of constant intensity (independent from* ϵ*).*

[7] Since all the propositions in question are still valid if we replace t^+ by t^-, the reduction from $tildeS_2$ to $tildeS_9$ is still valid with t^- instead of t^+.

[8] Let us remark that this reasoning makes only intervene $|t-t^+|$. However, $|t-t^+| = |t-t^-|$. As a consequence, Proposition 17 will still be valid if we replace t^+ by t^- and, as a result, $\tilde{S}_2^+$ and $\tilde{S}_2^-$ will converge towards the same centred Cauchy distribution.

- *2)* $(\zeta_h^M)_{\xi_h^M=1,\ h\in[-M,0]}$ *converge asymptotically towards random variables which are valued in* $\mathbb{R}/\mathbb{Z}$*, independent, identically distributed according to the normalized Haar measure over* $\mathbb{R}/\mathbb{Z}$*, independent from* $(\nu_h^M)_{\xi_h^M=1,\ h\in[-M,0]}$.

Proof that it is enough to show Proposition 18. It is enough to observe that:

- $h \in [-M, 0]$ signifies exactly that $h \in \tilde{I}_-(t)$.

- $(\xi_h^M)(L) = 1$ signifies exactly that $|\mathrm{Num}(l(L,h))\lceil \log(t)\rceil| \leqslant \frac{1}{\epsilon}$, and, in that case, $(\nu_h^M)(L) = \mathrm{Num}(l(L,h))\lceil \log(t)\rceil$

- $(\zeta_h^M)(L, X) = (< l(L,k), e^M A_2 + X >)$

$\square$

Moreover, the dominating term in $< e(\delta_h L), \delta_h^{-1}(e^M A_2 + X) >$ is $< e(\delta_h L), \delta_h^{-1} e^M A_2 >$. So, we can assume, and we will do it in the following, that:

$$\zeta_h^M(L) =< e(\delta_h L), \delta_h^{-1} e^M A_2 > \pmod 1$$

(in particular, one can forget the dependence on X).
Let us focus now on the proof of Proposition 18.

2.5.4 Verification of hypothesis of Theorem 4

Theorem 4, the remark that follows it and the proposition that follows after enable us to show Proposition 18. In order to state this proposition, let us keep in mind the notations of the previous subsection and let us set :

- $\lambda_1(h) = -2h$
- $\mathbf{X} = [-\frac{1}{\epsilon}, \frac{1}{\epsilon}] - \{0\}$
- $\mathbf{Q} = (\mathbf{Q}_P)_{P\in\mathbb{N}-\{0\}}$ where $\forall P \in \mathbb{N} - \{0\}$, $\mathbf{Q}_P = (\mathbf{X}_{k,P})_{k\in\{0,\cdots,P-1\}}$ with

  for all $k \in [0; P-1]$, $\mathbf{X}_{k,P} = [-\frac{k+1}{P}\frac{1}{\epsilon}, -\frac{k}{P}\frac{1}{\epsilon}[\cup]\frac{k}{P}\frac{1}{\epsilon}, \frac{k+1}{P}\frac{1}{\epsilon}]$
- $\tilde{\mathbf{X}} = \mathbb{R}/\mathbb{Z}$ and $\tilde{\mathbf{Q}} = ((\tilde{\mathbf{X}}_{k,P})_{k\in\{0,\cdots,P-1\}})_{P\in\mathbb{N}-\{0\}}$ a collection of partitions of intervals of length $\frac{1}{P}$ with $P \in \mathbb{N} - \{0\}$
- $\lambda_2(h) = M - h$ so that we have $\lambda_2 > \lambda_1$ sur $\mathrm{Int}(\Pi_M)$

The proposition that we are going to show is the following:

Proposition 19. $(\lambda_1, \lambda_2, \{\xi_t\}, \{\nu_t\}, \{\zeta_t\})$ *satisfy hypothesis from (h1) to (h8) of Thereom* 4.
Proposition 18*, as a consequence, is valid.*

Proof of the second part of Proposition 19. It is implied by Lemma 1 and Theorem 4. □

We will now show that (h1) to (h8) are verified here.

Verification of hypothesis (h1) to (h8) and conclusion

The purpose of this section is to verify the following proposition which is the first part of Proposition 19 :

Proposition 20. $(\lambda_1, \lambda_2, \{\xi_t\}, \{\nu_t\}, \{\zeta_t\})$ *satisfy (h1) to (h8) of Theorem* 4.

We need some preliminary lemmas, the equivalents of which can be found in the previous section and the proofs are more or less the same (Green-Riemann's theorem is useful for the second lemma).

Lemma 31. $\xi_{0,P}$, ξ_0 *belong to* $H^{s,s}$ *and*

$$\|\xi_{0,P}, \xi_0\|_{H^{s,s}} = O(1).$$

Let us set $\hat{\mathbf{X}} = \{x \in]0, \infty[\mid d(x, \partial\mathbf{X}) \leqslant M^{-1000}\}$ and $\hat{\mathbf{X}}_i = \{x \in]0, \infty[\mid d(x, \partial\mathbf{X}_i) \leqslant M^{-1000}\}$ (the index P being implied).
Let us also define $\hat{\xi_m} = \mathbf{1}_{\nu_m \in \hat{\mathbf{X}}}$ et $\hat{\xi_{m,i}} = \mathbf{1}_{\nu_m \in \hat{\mathbf{X}}_i}$ (the exponent M being implied). Then one has:

Lemma 32.

$$\tilde{\mu}_{\mathscr{S}_2}(\hat{\xi_{0,i}}) = O(T^{-1000}) \tag{2.62}$$

$$\tilde{\mu}_{\mathscr{S}_2}(\hat{\xi_0}) = O(T^{-1000}) \tag{2.63}$$

Before passing to the proof of Proposition 20, let's say a few words about it. (h1) to (h6) are checked in a analogous way to what was done in the previous section. The same is true for (h8). Only the verification of (h7) stands out and the penultimate paragraph is dedicated to it.

Proof of Proposition 20. Let us assume that $\mu_2 = \tilde{\mu}_2$, the general case is treated in an analogous way and with the help of the results of the simplified case that we are now treating.
• Verification of (h1)

By the same reasoning as in the previous section, using the lemma 7, we find, by posing $D = \int_{u>0,\ v>0} \mathbf{1}_{uv\leqslant 1}\mathbf{1}_{2\cosh(1)uv>\|(u,v)\|^2}$, that

$$\mathbb{E}(\xi_k^M) = \frac{2D}{\epsilon M}. \tag{2.64}$$

• Verification of (h2)
Given the definition of ξ_k^M, it is clear that (h2) is verified.
• Verification of (h3)
We have seen that if we had before a k such that $\xi_k^M = 1$ then necessarily this k was a local minimum of $p \in \mathbb{Z} \longmapsto \|\delta_p L\|^2$ and the k such that $xi_k^M = 1$ are k such that

$$\|\delta_p L\|^2 \leqslant \frac{2\cosh(1)e}{\epsilon M}.$$

Thus, by the same reasoning that led to the Inequation $(refeq120)$, we see that $(h3)$ is verified: the k that fit are sufficiently distant.
• Verification of (h4a), (h4b) et (h4c)
Let $p \in \mathbb{N} - \{0\}$. Here too, we will have $\eta_{k,p} = \xi_{k,p}$. Therefore, it is only necessary to check (h4a) and (h4b), (h4c) being then checked automatically.
Following the same reasoning as in the previous section, and using the W_1^--foliation (which corresponds to the matrix $h_1(\tau)$ transposed, whereas the W_1^+-foliation corresponds to $h_1(\tau)$), one can see that there exists $\tilde{E}_1$ and F_t such that for all $t \in \Pi$, such that (h4) is verified. We have this because of Lemma 31.
• Verification of (h5)
Here again, we follow the same reasoning as in the previous section. Here again, we must replace the set we are working on, $\tilde{I}_-(t)$, by the set $\tilde{\tilde{I}}_-(t)$ where :

$$\tilde{\tilde{I}}_-(t) = \{h \in [-\lceil\log(t)\rceil, 0] \mid \|\delta_h L\|^2 + M^{-2001} \leqslant 2\cosh(1)|\mathrm{Num}(e(\delta_h L))| \text{ and } |\mathrm{Num}(e(\delta_h L))| \leqslant \frac{1}{\epsilon_4 \log(t)} - M^{-2001}\},$$

Doing so costs only a $O(M^{-1000})$.
To complete the same reasoning, we must use the equality shown in the previous section, namely

$$M\|h(\tau)\delta_k u\|^{-2} = M\|\delta_k u\|^{-2} + O(M^{-10^9+1})$$

(valid as long as the foliation used is sufficiently thin), use the fact that

$$|\mathrm{Num}(e(h(\tau)\delta_k L))| = |\mathrm{Num}(e(\delta_k L))|(1 + O(\tau))$$

when $\tau \to 0$ and use Lemma 32.
• Verification of (h6)
Verification of (h6) is done in a very similar way to what was done in the previous section.
• Verification of (h7)
We now have a set of E on which from (h1) to (h6) are verified. Let give us k, k' in $[-M, 0]$ such that $\lambda_2(k) \geqslant \lambda_1(k') + R\log(M) \geqslant \lambda_1(k) + 2R\log(M)$. Let $L \in E$ such that $\xi_k(L) = \xi_{k,p}(L) = 1$ where $p \in \{0, \cdots, P-1\}$. One wants to show, in order to prove (h7), that :

$$\mu_2(< e(\delta_k L), \delta_k^{-1}(e^M A_2) > (\mathrm{mod}\ 1) \in \tilde{\mathbf{X}}_{j,P}|\mathcal{F}_{k'})(L) = \mathrm{Leb}(\tilde{\mathbf{X}})_{j,P}(1 + o(1)).$$

Let us recall that

$$h_\tau = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$$

and set $\gamma_{k'} = F_{k'}(L)$. Then, one has:

$$\gamma_{k'} = \{h_\tau^T \overline{L}\}_{0 \leqslant \tau \leqslant (e^{\lambda_1(k')} M^{1000})^{-1}}$$

where $\overline{L} \in \mathscr{S}_2$.
Even if it means increasing R, one can assume that, thanks to (h5), that $\xi_{k,p} = 1$ over $\gamma_{k'}$, and so, in particular, by setting

$$e = (e_1, e_2) = e(\delta_k \overline{L}),$$

e satisfies :

$$2\cosh(1)|\mathrm{Num}(e)| > \|e\|^2 \ \textit{and}\ M|\mathrm{Num}(e)| \in \mathbf{X}_p.$$

Yet, one has $\delta_k h_\tau^T = h^T_{e^{\lambda_1(k)}\tau} \delta_k$ and, even if it means taking R larger, one can assume that $e^{\lambda_1(k)}\tau \leqslant 1$ and so one has:

$$e(\delta_k h_\tau \overline{L}) = h^T_{e^{\lambda_1(k)}\tau} e = \begin{pmatrix} e_1 \\ \tau e^{\lambda_1(k)} e_1 + e_2 \end{pmatrix}.$$

So, one gets:

$$< e(\delta_k h_\tau^T L), \delta_k^{-1} e^M A_2 >= e_1 e^{-k} e^M (A_2)_1 + e_2 e^k e^M (A_2)_2 + e^{\lambda_1(k)} \tau e_1 e^k e^M (A_2)_2 (\mathrm{mod}\ 1).$$

Yet, given the conditions verified by e, one has $|e_1| > \beta$ where β is a positive constant. Moreover, one has $e^M e^k e^{\lambda_1(k)} = e^{\lambda_2(k)}$ (see the beginning of the section 5.9) and τ varies within an interval of length $(e^{\lambda_1(k')} M^{1000})^{-1}$. So, as $\lambda_2(k) \geqslant \lambda_1(k') + R\log(M)$, one gets the wanted result as $(A_2)_2 \neq 0$.
• Verification of (h8)
One knows that: $F_{\overline{k}}(L) = \{h_\tau \overline{L}\}_{0 \leqslant \tau \leqslant (e^{\lambda_1(\overline{k})} M^{1000})^{-1}}$. According to (h5), the value of ξ_k is 1 over $F_{\overline{k}}(L)$. The calculation made previously gives that :

$$< e(\delta_k h_\tau^T L), \delta_k^{-1} e^M A_2 >= e_1 e^{-k} e^M (A_2)_1 + e_2 e^k e^M (A_2)_2 + \tau e_1 e^{\lambda_2(k)} (A_2)_2 + (\mathrm{mod}\ 1).$$

So, if $\mathbf{1}_{\zeta_k \in \tilde{K}_p}$ is not constant over $F_{\overline{k}}(L)$, ζ_k is in $O(M^{-1000})$-neighborhood of $\partial \tilde{\mathbf{X}}_p$. If we set

$$\begin{aligned} \tilde{E} = \{L \in E | \forall k,\ \overline{k} \in [-M, 0] \text{ with } \lambda_1(\overline{k}) > \lambda_2(k) + R\log(M) \text{ one has} \\ \forall p \in [0, P-1],\ \mathbf{1}_{\zeta_k \in \tilde{\mathbf{X}}_p} \text{ is constant over } F_{\overline{k}}(L)\} \end{aligned}$$

then (h8) is verified over $\tilde{E}$. Finally Lemma 7 enables us to show that $\mu_2(E - \tilde{E}) = O(M^{-998})$. □

We can now prove Theorem 7 :

Proof of Theorem 7. Proposition 20 and proposition 19 give then that Proposition 18 is true. Proposition 17 is so also valid. Hence the wanted result thanks to Lemma 1, Lemma 3 and Proposition 9. □

2.5.5 Return to $\tilde{S}$ and proof of Theorem 1

We will now prove Theorem 1. Recall that the sum to be studied is as follows:

$$\tilde{S}(\tau,t,X,L)=\sum_{i=1}^{4}(-1)^i(\tilde{S}(\tau,t,A_i,I_2,X,L)-\tilde{S}(\tau,t,A_i,R,X,L)) \tag{2.65}$$

and the various sums relate to terms $l\in L$ and not on terms $l\in L^{\perp}$.
The way to prove Theorem 1 in that case is the following. For every term in $\tilde{S}^{\pm}$, we reduce the corresponding sum thanks to what was done in the previous section, the results extending naturally. Then we group these 8 new sums by small calculations. Finally we conclude by following the same approach as before (application of Theorem 4).
More precisely, after some reductions and small calculations, we see that the asymptotic study of $\tilde{S}$ is reduced to the study of :

$$\Sigma(t,P,\epsilon,X,L)=\frac{2}{\pi^2\log(t)}\sum_{h\in\tilde{I}(L,t,\epsilon)}\frac{1}{\mathrm{Num}(l(L,h))} \tag{2.66}$$
$$\sum_{m\in\mathbb{N}-\{0\}}\frac{\sin(2m\pi(l(L,h))_1t^+a)\sin(2m\pi(l(L,h))_2t^+a)\cos(2m\pi<l(L,h),X>)}{m^2}$$

where

$$\tilde{I}(L,t,\epsilon)=\Big\{h\in[-\lceil\log(t)\rceil,\lceil\log(t)\rceil]\mid\|\delta_hL\|^2<2\cosh(1)|\mathrm{Num}(e(\delta_hL))| \text{ and } |\mathrm{Num}(e(\delta_hL))|\leqslant\frac{1}{\epsilon\log(t)}\Big\}. \tag{2.67}$$

Starting from there, a proof of Theorem 1 is the following:

Proof of Theorem 1. By following the same method as before, we show that:

- $\{\mathrm{Num}(l(L,h))\}_{h\in\tilde{I}(L,t,\epsilon)}$ converges towards a Poisson process over $[-\frac{1}{\epsilon},\frac{1}{\epsilon}]$ of constant intensity, independent from ϵ
- The sequence of random variables $((l(L,h))_1ta)_{h\in\tilde{I}(L,t,\epsilon)}$ valued in $\mathbb{R}/\mathbb{Z}$ converge towards independent identically distributed random variables independent, whose common distribution is the normalized Haar measure over $\mathbb{R}/\mathbb{Z}$. Furthermore, they become independent, at infinity, of $\{\mathrm{Num}(l(L,h))\}_{h\in\tilde{I}(L,t,\epsilon)}$.
- The sequence of random variables $((l(L,h))_2ta)_{h\in\tilde{I}(L,t,\epsilon)}$ valued in $\mathbb{R}/\mathbb{Z}$ converge towards independent identically distributed random variables independent, whose common distribution is the normalized Haar measure over $\mathbb{R}/\mathbb{Z}$. Furthermore, they become independent, at infinity, of $\{\mathrm{Num}(l(L,h))\}_{h\in\tilde{I}(L,t,\epsilon)}$.

One also shows that, modulo 1, the sequence of random variables $((l(L,h))_1ta)_{h\in\tilde{I}(L,t,\epsilon)}$ becomes independent from $((l(L,h))_2ta)_{h\in\tilde{I}(L,t,\epsilon)}$ when $t\to\infty$.

Furthermore, all these properties also hold for $(< l(L,h), X >)$ modulo 1.
Thanks to that, one obtains that:

$$\Big(\sum_{m\in\mathbb{N}-\{0\}} \frac{\sin(2m\pi(l(L,h))_1 t^+ a)\sin(2m\pi(l(L,h))_2 t^+ a)\cos(2m\pi < l(L,h), X >)}{m^2}\Big)_{h\in\tilde{I}(L,t,\epsilon)}$$

converge to mutually independent non-zero real random variables with a common distribution that is symmetric and whose support is compact, and that are asymptotically independent of $(\mathrm{Num}(l(L,h)))_{h\in\tilde{I}(L,t,\epsilon)}$. We can then apply Lemma 5. Since this reasoning also works by replacing t^+ by t^-, we get the desired result. □

Hereafter, we first expose the reduction that the previous section allows us to do on the 8 terms of the sum $\tilde{S}$ and then some small calculations that allow us to reduce the study to the sum Σ in the form exposed here [9].

Reduction of the 8 terms $\tilde{S}_1$ of $\tilde{S}$

Following what was done throughout the previous section and posing (for A a point of $\mathbb{R}^2$ and $R_1 \in SO_2(\mathbb{R})$):

$$I(R_1, L, t, \epsilon) = \{h \in [-\lceil\log(t)\rceil, 0] \mid \|\delta_h R_1 L\|^2 < 2\cosh(1)|\mathrm{Num}(e(\delta_h R_1 L))| \text{ and } |\mathrm{Num}(e(\delta_h R_1 L))| \leqslant \frac{1}{\epsilon\log(t)}\} \tag{2.68}$$

$$G_1(\tau, t, A, R_1, X, l) = \sum_{m\in\mathbb{N}-\{0\}} \frac{\cos(2m\pi < l, t^+ R_1 A + R_1 X >)}{m^2} \tag{2.69}$$

$$\Sigma_1(\tau, t, A, R_1, X, L) = \frac{2}{\log(t)}\frac{1}{(2\pi i)^2}\sum_{h\in I(R_1,L,t,\epsilon)} \frac{1}{\mathrm{Num}(l(R_1 L, h))} G_1(\tau, t, A, R_1, X, l(R_1 L, h)) \tag{2.70}$$

$$\Sigma(\tau, t, X, L) = \sum_{i=1}^{4}(-1)^i(\tilde{S}_r(\tau, t, A_i, I_2, X, L) - \tilde{S}_r(\tau, t, A_i, R, X, L)) \tag{2.71}$$

(where we recall that $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$) we get the following proposition:

Proposition 21. *For every $\alpha > 0$, for every $\epsilon > 0$ small enough, for t large enough, one has*

$$\mathbb{P}\Big(|\tilde{S}_1 - \Sigma| \leqslant \alpha\Big) \leqslant \alpha.$$

The study is reduced to the study of Σ [10].

[9] These two parts work independently of whether we consider t^+ or t^-.

[10] Given what we have noted throughout the passage from $\tilde{S}_2$ to $\tilde{S}_9$, this reduction also applies if we replace t^+ by t^-.

A last regrouping of terms

The purpose of this subsection is to see that $\Sigma(\tau, t, X, L)$ can also be written :

Proposition 22.

$$\Sigma(t, P, \epsilon, X, L) = \frac{-1}{2\pi^2 \log(t)} \sum_{h \in \tilde{I}(L,t,\epsilon)} \frac{1}{Num(l(L,h))} \sum_{m \in \mathbb{N}-\{0\}} \sum_{i=1}^{4} (-1)^i \frac{\cos(2m\pi < l(L,h), t^+ A_i + X >)}{m^2}. \tag{2.72}$$

The key to this proposition is that when we have an h that is in $I(R, L, t, \epsilon)$, $-h$ belongs to $I(I_2, L, t, \epsilon)$ and vice versa. This is just the translation of the fact that if a vector $l \in L$ is large in norm and "close" to the x-axis, it is "far" from the y-axis and vice versa [11].

Proof. For t large enough, if one has h such that $h \in I(I_2, L, t, \epsilon)$ then

$$e(\delta_h L) = e(R\delta_{-h} RL) = \pm Re(\delta_{-h} RL)$$

and so $-h \in I(R, L, t, \epsilon)$ and $\mathrm{Num}(e(\delta_h L)) = -\mathrm{Num}(e(\delta_{-h} RL))$.
Reversing the role of R and I_2, we have the converse.
By using Equation (2.68), Equation (2.69) and Equation (2.71) and the previous remarks in this proof, the desired result is obtained by grouping the terms corresponding to the different flags. □

A last rewriting

In order to arrive at the final Σ formula given at the beginning of this subsection, we need to perform a final rewriting of the $\tilde{S}_{r,2}(t, P, \epsilon, X, L)$ terms.
Let us recall that A_1 is of coordinates $(a, -a)$, A_2 is of coordinates (a, a), A_3 is of coordinates $(-a, a)$ and A_4 is of coordinates $(-a, -a)$. An application of the usual trigonometric formulas then gives :

Proposition 23.

$$\tilde{S}_{r,2}(t, P, \epsilon, X, L) = \frac{2}{\pi^2 \log(t)} \sum_{h \in \tilde{I}(L,t,\epsilon)} \frac{1}{Num(l(L,h))} \tag{2.73}$$
$$\sum_{m \in \mathbb{N}-\{0\}} \frac{\sin(2m\pi (l(L,h))_1 t^+ a) \sin(2m\pi (l(L,h))_2 t^+ a) \cos(2m\pi < l(L,h), X >)}{m^2}.$$

This proposition is still valid if we replace t^+ by t^-.

[11] This proposition is independent of the fact that $t = t^+$. It is therefore still valid by replacing t^- by t^+.

Chapter 3

The case of ovals

(Résumé en français) Nous étudions l'erreur du nombre de points de réseaux unimodulaires qui tombent dans une ellipse dilatée et centrée autour de 0. Le résultat principal est que l'erreur, lorsqu'elle est normalisée par $\sqrt{t}$ avec t le paramètre de dilatation et lorsque le réseau est aléatoire, converge en loi vers une loi explicite. Pour cela, nous utilisons d'abord de l'analyse harmonique pour réduire l'étude de l'erreur normalisée à l'étude d'une transformée de Siegel $\mathcal{S}(f_t)(L)$ qui dépend de t. Ensuite, lorsque $t \to \infty$, nous montrons que $\mathcal{S}(f_t)$ se comporte en loi comme une transformée de Siegel modifiée avec des poids aléatoires $\mathcal{S}(F)(\theta, L)$ où θ est une suite de variables aléatoires indépendantes et identiquement distribuées sur le tore. Enfin, nous montrons que cette dernière quantité converge presque sûrement et nous étudions l'existence des moments de sa loi.

(English abstract) We study the error of the number of unimodular lattice points that fall into a dilated and centred ellipse around 0. The main result is that the error, when normalized by $\sqrt{t}$ with t the parameter of dilatation and when the lattice is random, converges in distribution towards an explicit distribution. For this, we first use harmonic analysis to reduce the study of the normalized error to the study of a Siegel transform $\mathcal{S}(f_t)(L)$ that depends on t. Then, when $t \to \infty$, we show that $\mathcal{S}(f_t)$ behaves in distribution as a modified Siegel transform with random weights $\mathcal{S}(F)(\theta, L)$ where θ is a sequence of independent and identically distributed random variables on the torus. Finally, we show that this last quantity converges almost surely and we study the existence of the moments of its distribution.

Contents

3.1 Introduction **85**

3.2 Heuristic explanation and plan of the proof **89**

3.3 Reduction to the study of the Siegel transform **93**

3.3.1 Proof of Lemma 33 94

3.3.2 Proof of Lemma 34 101

3.4 Study of $S_{A,prime}(L,t)$ when $t \to \infty$ 103
3.4.1 Reductions for the study of $S_{A,prime}(L,t)$ and proof of Theorem 8 103
3.4.2 Foliation and local estimates 106
3.4.3 Proof of Proposition 29 107
3.4.4 Proof of Proposition 28 109
3.5 Asymptotic study of $\tilde{S}_A(\omega, L)$ 111

3.1 Introduction

The lattice point problem is an open problem in the Geometry of Numbers, at least since Carl Friedrich Gauss took interest in which became the famous Gauss circle problem. The general problem states as follows.
Let d be an integer greater than 1. We recall the following definition :

Definition 12. *A subset L of $\mathbb{R}^d$ is a lattice if it is a subgroup of $\mathbb{R}^d$ such that L is discrete and span$(L) = \mathbb{R}^d$.*

Let P be a measurable subset of $\mathbb{R}^d$ of non-zero finite Lebesgue measure. We want to evaluate the following cardinal number when $t \to \infty$:

$$N(tP + X, L) = |(tP + X) \cap L|$$

where $X \in \mathbb{R}^d$, L is a lattice of $\mathbb{R}^d$ and $tP + X$ denotes the set P dilated by a factor t relatively to 0 and then translated by the vector X.
Under mild regularity conditions on the set P, one can show that :

$$N(tP + X, L) = t^d \frac{\mathrm{Vol}(P)}{\mathrm{Covol}(L)} + o(t^d)$$

where $o(f(t))$ denotes a quantity such that, when divided by $f(t)$, it goes to 0 when $t \to \infty$ and where Covol(L) is defined as :

Definition 13. *The covolume of a lattice L of $\mathbb{R}^d$, Covol(L), is the Lebesgue measure of a measurable fundamental set of L. Furthermore, a lattice is said to be unimodular if its covolume is equal to 1.*
When $d = 2$, instead of using the term covolume, we use the term coarea.

We are interested in the error term

$$\mathcal{R}(tP + X, L) = N(tP + X, L) - t^d \frac{\mathrm{Vol}(P)}{\mathrm{Covol}(L)}.$$

In the case where $d = 2$ and where P is the unit disk $\mathbb{D}^2$, Hardy's conjecture in [36] stipulates that we should have for all $\epsilon > 0$,

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^{\frac{1}{2}+\epsilon})$$

where $Y = O(f(t))$ is a quantity such that

$$|Y| \leqslant C|f(t)|$$

with $C \geqslant 0$.
One of the result in this direction has been established by Iwaniec and Mozzochi in [43]. They have proven that for all $\epsilon > 0$,

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^{\frac{7}{11}+\epsilon}).$$

This result has been recently improved by Huxley in [42]. Indeed, he has proven that :

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^K \log(t)^\Lambda)$$

where $K = \frac{131}{208}$ and $\Lambda = \frac{18627}{8320}$.
This last result is based on estimating what are called *exponential sums*. In this case, the error is considered in a deterministic way.
Another approach was followed first by Heath-Brown in [38] and then by Bleher, Cheng, Dyson and Lebowitz in [11]. They took interest in the case where the dilatation parameter t is random. More precisely, they assumed that t was being distributed according to the measure $\rho(\frac{t}{T})dt$ (that is absolutely continuous relatively to Lebesgue measure) and where ρ is a probability density on $[0, 1]$ and T is parameter that goes to infinity. In that case, Bleher, Cheng, Dyson and Lebowitz showed the following result (which generalizes the previous result of Heath-Brown) :

Theorem ([11])**.** *Let $\alpha \in [0, 1[^2$. There exists a probability density p_α on $\mathbb{R}$ such that for every piecewise continuous and bounded function $g : \mathbb{R} \longrightarrow \mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(\frac{\mathcal{R}(t\mathbb{D}^2 + \alpha, \mathbb{Z}^2)}{\sqrt{t}})\rho(\frac{t}{T})dt = \int_{\mathbb{R}} g(x)p_\alpha(x)dx.$$

Furthermore p_α can be extended as an analytic function over $\mathbb{C}$ and satisfies that for every $\epsilon > 0$,

$$p_\alpha(x) = O(e^{-|x|^{4-\epsilon}})$$

when $x \in \mathbb{R}$ and when $|x| \to \infty$.

In our case, we keep t deterministic as in the original Gauss problem but we let the lattice L be a random unimodular lattice and we study $\mathcal{R}$. This approach was first initiated by Kesten in [48] and in [49]. It should be noted that several counting problems have followed this approach : we can cite, for example, [9], [27], [64], [3], [60] and [93].
We denote by $\mathscr{S}_2$ the space of unimodular lattices and it can be seen as the quotient space $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$. We denote by μ_2 the unique Haar probability measure on it. Let us set :

$$\Pi = \{(k_1, k_2) \in \mathbb{Z}^2 \mid k_1 \wedge k_2 = 1,\ k_1 \geqslant 0\} \tag{3.1}$$

where we agree that if $k_1 = 0$, $k_1 \wedge k_2 = 1$ means that $k_2 = 1$.
We denote by $\|\cdot\|$ the usual euclidean norm over $\mathbb{R}^2$.
We need to define some additional objects.

Definition 14. *For every $i \in \{1,2\}$, we call*

$$\|L\|_i = \min\{r > 0 \mid B_f(0,r) \text{ contains } i \text{ vectors of } L \text{ lineary independent}\}$$

where $B_f(0,r)$ is the closed centred ball on 0 for the norm $\|\cdot\|$ of radius r. These two quantities are the successive minima of the lattice L.
In fact, for almost all $L \in \mathscr{S}_2$, $\|L\|_2 > \|L\|_1$ and there exists only one couple of vectors $(e_1(L), e_2(L))$ such that $(e_1(L))_1 > 0$, $\|e_1(L)\| = \|L\|_1$, $(e_2(L))_1 > 0$ and $\|e_2(L)\| = \|L\|_2$.
In the rest of the article, for the sake of simplicity, for L a lattice, we will use the notation $\|L\|$ instead of $\|L\|_1$.
For a lattice $L \in \mathscr{S}_2$, we also say that a vector of L is prime if it is not a non-trivial integer multiple of another vector of L.
In fact, for every $M \in SL_2(\mathbb{R})$, a vector $l \in L$ is prime if, and only if, $Ml \in ML$ is prime and a vector $(k_1, k_2) \in \mathbb{Z}^2$ is prime if, and only if, $k_1 \wedge k_2 = 1$. With these notations, one has that, for a generic lattice $L \in \mathscr{S}_2$, $e \in L$ is prime if, and only if, e can be written as $e = k_1 e_1(L) + k_2 e_2(L)$ with $k_1 \wedge k_2 = 1$.
Finally, for a generic lattice $L \in \mathscr{S}_2$, we call $\mathcal{P}_+(L)$ the set of vectors e of L such that $e = k_1 e_1(L) + k_2 e_2(L)$ with $(k_1, k_2) \in \Pi$. All the vectors of $\mathcal{P}_+(L)$ are prime vectors according to the previous remark.

We recall also the fact that we say that a real random variable Z is symmetrical if $\mathbb{P}_Z = \mathbb{P}_{-Z}$ where, for every random variable X, $\mathbb{P}_X$ stands for the distribution of the random variable X.
Let $\tilde{\mu}_2$ be a probability measure that has a smooth bounded density σ with respect to μ_2. There are two different cases that are addressed in our main result, which is Theorem 8. The first one is when $\tilde{\mu}_2$ is compactly supported, *id est* when there exists $\alpha > 0$ such that :

$$\tilde{\mu}_2(\{L \in \mathscr{S}_2 \mid \|L\|_1 < \alpha\}) = 0. \tag{3.2}$$

The second one is when $\tilde{\mu}_2$ is non-compactly supported under the following condition : there exists $m > 0$, there exists $\alpha > 0$ such that for all L that belongs to the event $(\|L\| < \alpha)$,

$$\sigma(L) \geqslant m. \tag{3.3}$$

An example of such a measure $\tilde{\mu}_2$ is given by the normalized Haar measure μ_2.
Let $\mathcal{E}$ be an ellipse centred around 0. Let us call M a matrix that transforms $\mathcal{E}$ into a disk and that belongs to $SL_2(\mathbb{R})$. M is unique modulo the natural action of $SL_2(\mathbb{Z})$.
Let us set : $\mathbb{T}^\infty = (\mathbb{T}^1)^\Pi$ where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and let us call λ_∞ the normalized Lebesgue measure product over $\mathbb{T}^\infty$.
The main result of this article is the following theorem :

Theorem 8. *For every real numbers $a < b$,*

$$\lim_{t\to\infty} \tilde{\mu}_2\left(L \in \mathscr{S}_2 \mid \frac{\mathcal{R}(t\mathcal{E}, L)}{\sqrt{t}} \in [a,b]\right) = (\lambda_\infty \times (M^{-1})_* \tilde{\mu}_2)\left((\theta, L) \in \mathbb{T}^\infty \times \mathscr{S}_2 \mid S(\theta, L^\perp) \in [a,b]\right)$$

where $(M^{-1})_*\tilde{\mu}_2$ is the push-forward of $\tilde{\mu}_2$ by M^{-1} and where $\theta = (\theta_e) \in \mathbb{T}^\infty$,

$$S(\theta, L) = \frac{2}{\pi} \sum_{e \in \mathcal{P}_+(L)} \frac{\phi(\theta_e)}{\|e\|^{\frac{3}{2}}}$$

with

$$\phi(\theta_e) = \sum_{m\geqslant 1} \frac{\cos(2\pi m\theta_e - \frac{3\pi}{4})}{m^{\frac{3}{2}}}. \tag{3.4}$$

Furthermore, $S(\theta, L)$ (and $S(\theta, L^{\perp})$) :

- *converges almost surely*
- *is symmetrical and its expectation is equal to* 0
- *admits a moment of order $1+\kappa$ for any $0 \leqslant \kappa < \frac{1}{3}$*
- *$S(\theta, L)$ admits moments of all order $1 \leqslant p < \infty$ when $\tilde{\mu}_2$ is compactly supported*
- *does not admit a moment of order $\frac{4}{3}$ when $\tilde{\mu}_2$ is non-compactly supported under the condition (3.3).*

Basically, what our theorem shows is that Hardy's conjecture is true on average on $L \in \mathscr{S}_2$ whereas, in [11], it was shown that Hardy's conjecture is true on average on t. Furthermore, as a consequence of Theorem 8, one can see that for almost every $L \in \mathscr{S}_2$, when one takes $L' \in \mathscr{S}_2$ distributed according to $\tilde{\mu}_{2,n}$ that are compactly supported such that the sequence of their supports converge towards $\{L\}$, there exists a sequence $t_n \to \infty$ such that $\frac{\mathcal{R}(t_n\mathbb{D}^2, L')}{\sqrt{t_n}}$ converge in distribution towards the distribution of the real random variable $\theta \in (\mathbb{T}^\infty, \lambda_\infty) \longmapsto S(\theta, L)$.
Moreover, we notice that, when $\tilde{\mu}_2$ is non-compactly supported under condition (3.3), the tail of the distribution is by far larger than the tail of the limit distribution of [11] or than the tail of the limit distribution in the compact case. The main reason is that the magnitude of $S(\theta, L)$ is of order $\frac{1}{\|L\|^{\frac{3}{2}}}$ and one has

$$\mu_2\Big(\|L\|_1^{\frac{-3}{2}} > \beta\Big) = \frac{C}{\beta^{\frac{4}{3}}}$$

with C a positive constant (see Section 5 for more details).

The proof of Theorem 8 can be reduced to the case $\mathcal{E} = \mathbb{D}^2$. Indeed, since the parameter of dilatation goes to infinity, we can replace it by $\frac{t}{\text{Area}(\mathcal{E})}$ and so the study is reduced to the case where $\mathcal{E}$ is of area 1. Furthermore, we note that in that case there exists $M \in SL_2(\mathbb{R})$ such that $M\mathcal{E} = \mathbb{D}^2$ and so the study is reduced to the case where $\mathcal{E} = \mathbb{D}^2$ by considering that L is distributed according to the probability measure $(M^{-1})_*\tilde{\mu}_2$. As a consequence, for the rest of the article, we are going to suppose

$$\mathcal{E} = \mathbb{D}^2.$$

The next section is a preliminary dedicated to a summation formula. It will enable us to give a heuristic explanation of the approach that we will follow. At the end of the section, we will give the plan of the paper.

3.2 Heuristic explanation and plan of the proof

We are first going to recall a summation formula (about this subject, see also Section 2 of [10]). This summation formula is based on the Poisson formula applied to the indicator function of an appropriate convex set. Yet it is not quite rigorous to use it because there is a convergence problem due to the lack of regularity of the indicator function. In fact, it can be used once the problem is regularised with a smooth cutoff function. We are not going to treat the regularization issue in this section, it is treated in Section 3. In this section, we will pretend the summation formula is valid as it is, since it will give a good insight of how the estimation of the limit distribution can be tackled, and we leave the technical issue of regularization for later.
After having given heuristic explanations about the proof of Theorem 8, we will expose a detailed plan of the proof.
Let γ be a simple, smooth, closed, convex and with a positive curvature curve in $\mathbb{R}^2$ such that $(0,0)$ lies inside γ. For Ω_γ the domain enclosed by γ, we call $\mathbf{1}_{t\Omega_\gamma}$ the characteristic function of $t\Omega_\gamma$.
Then one has that :

$$N(t\Omega_\gamma, \mathbb{Z}^2) = \sum_{n\in\mathbb{Z}^2} \mathbf{1}_{t\Omega_\gamma}(n).$$

By the Poisson summation formula (yet, as we said earlier, it is not quite rigorous to use it because $\mathbf{1}_{t\Omega_\gamma}$ is not smooth enough and so there is a convergence problem), one has that :

$$N(t\Omega_\gamma, \mathbb{Z}^2) = \sum_{n\in\mathbb{Z}^2} \widehat{\mathbf{1}_{t\Omega_\gamma}}(2\pi n)$$

with $\widetilde{\mathbf{1}_{t\Omega_\gamma}}$ the Fourier transform of $\mathbf{1}_{t\Omega_\gamma}$ defined by

$$\widehat{\mathbf{1}_{t\Omega_\gamma}}(\xi) = \int_{\mathbb{R}^2} e^{ix\xi}\mathbf{1}_{t\Omega_\gamma}(x)dx.$$

Note that $\widehat{\mathbf{1}_{t\Omega_\gamma}}(0) = \text{Area}(t\Omega_\gamma)$. So we obtain that :

$$\mathcal{R}(t\Omega_\gamma, \mathbb{Z}^2) = \sum_{n\in\mathbb{Z}^2-\{0\}} \widehat{\mathbf{1}_{t\Omega_\gamma}}(2\pi n). \tag{3.5}$$

We need some notations before moving forward. For $\xi \in \mathbb{R}^2 - \{0\}$, let us call $x_\gamma(\xi)$ the point on γ where the outer normal to γ coincides with $\frac{\xi}{\|\xi\|}$ where $\|\xi\|$ refers to the usual euclidean norm of $\mathbb{R}^2$. Let us call also $\rho_\gamma(\xi)$ the curvature radius of γ at $x_\gamma(\xi)$ and

$$Y_\gamma(\xi) =< \xi, x_\gamma(\xi) >$$

where $< \cdot, \cdot >$ denotes the usual scalar product over $\mathbb{R}^2$.
With these notations, the authors in [10] obtain by giving an asymptotic expression of $\widetilde{\mathbf{1}_{t\Omega_\gamma}}$ and by using (3.5) :

$$\frac{\mathcal{R}(t\Omega_\gamma, \mathbb{Z}^2)}{\sqrt{t}} = \frac{1}{\pi} \sum_{n\in\mathbb{Z}^2-\{0\}} \frac{\sqrt{\rho_\gamma(n)}\cos(2\pi t Y_\gamma(n) - \frac{3\pi}{4})}{\|n\|^{\frac{3}{2}}} + O(\frac{1}{t}). \tag{3.6}$$

Before saying what this summation formula gives in our case we need to recall the definition of a dual lattice :

Definition 15. *For $L \in \mathscr{S}_2$, its dual lattice $L^\perp$ is defined in the following way*

$$L^\perp = \{x \in \mathbb{R}^2 \mid \forall l \in L, \ < l, x > \in \mathbb{Z}\}.$$

Furthermore, if M is a matrix in $SL_2(\mathbb{R})$ that represents L then $(M^{-1})^T$ is a matrix that represents $L^\perp$ where $(M^{-1})^T$ is the transpose matrix of M^{-1}.

Let $L \in \mathscr{S}_2$ and $M \in SL_2(\mathbb{R})$ being a matrix that represents the lattice L. Then, one has that :

$$\mathcal{R}(t\mathbb{D}^2, L) = \mathcal{R}(tM^{-1}\mathbb{D}^2, \mathbb{Z}^2).$$

By using this last fact, by the previously exposed summation formula and after some calculations (that we will expose later), one gets that :

$$\frac{\mathcal{R}(t\mathbb{D}^2, L)}{\sqrt{t}} = \frac{1}{\pi} \sum_{l \in L^\perp - \{0\}} \frac{\cos(2\pi t \|l\| - \frac{3\pi}{4})}{\|l\|^{\frac{3}{2}}} + O(\frac{1}{t}).$$

Let us recall the following definition :

Definition 16. *Let f be a function on $\mathbb{R}^2$.*
Let us define formally :

$$\mathcal{S}(f)(L) = \sum_{\substack{e \in L \\ e \text{ prime}}} f(e).$$

$\mathcal{S}(f)$ is called the Siegel transform of f.
If f has a compact support, $\mathcal{S}(f)$ is well-defined.

Let us set $f_t(x) = \frac{1}{\pi} \frac{\cos(2\pi t \|x\| - \frac{3\pi}{4})}{\|x\|^{\frac{3}{2}}} \mathbf{1}_{\mathbb{R}^2 - \{0\}}(x)$. As a consequence of the summation formula, heuristically the asymptotic behaviour of $\frac{\mathcal{R}(t\mathbb{D}^2, L)}{\sqrt{t}}$ is the same as the asymptotic behaviour of the following Siegel transform

$$\mathcal{S}(f_t)(L^\perp) = \frac{1}{\pi} \sum_{l \in L^\perp - \{0\}} \frac{\cos(2\pi t \|l\| - \frac{3\pi}{4})}{\|l\|^{\frac{3}{2}}}. \tag{3.7}$$

We can, and we will, study it when $t \to \infty$ by replacing $L^\perp$ by L (because L is supposed to be random).
By the way, here we see that a natural approach would be to replace the randomness of t by randomness of $\|l\|$ with L random and distributed according to $\tilde{\mu}_2$. However, in [11] and in [10], this randomness on t enables the authors to use results about what is called *almost periodic* function (developed in [7] and in [38]). More precisely, the authors use the following theorem : for every $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$, if for every $\epsilon > 0$, there exists a trigonometric polynomial P_ϵ such that

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T \min(1, |F(t) - P_\epsilon(t)|) \leqslant \epsilon$$

then $F(t)$ admits an asymptotic distribution.
Here, in our case, we can not work in this framework (because of the difference of nature of the randomness). So, we have to take another approach and the approach we are going to follow is inspired from [27].

We now give the main steps of the proof of Theorem 8.

First step. By regularizing the problem, we are going to show that the quantity $S_{A,prime}(L,t)$, for $A>0$ a fixed parameter that is taken large enough, when $t\to\infty$, is close, in probability, to $\frac{\mathcal{R}(t\mathbb{D}^2,L)}{\sqrt{t}}$ where $S_{A,prime}(L,t)$ is defined by :

$$S_{A,prime}(L,t)=\frac{1}{\pi}\sum_{\substack{l\in L^{\perp}\ prime\\ 0<\|l\|\leqslant A}}\frac{1}{\|l\|^{\frac{3}{2}}}\sum_{m\in\mathbb{N}-\{0\}}\frac{\cos(2\pi tm\|l\|-\frac{3\pi}{4})}{m^{\frac{3}{2}}}. \tag{3.8}$$

It is different from the right-hand side of Equation (3.7) because, first, the sum has been cut and it has to be cut because of a problem of convergence of the right-hand side of Equation (3.7) ; second, we take into account a phenomenon of multiplicity (if l appears in the sum, $2l$ is also going to appear). In [11], and in [26], such a phenomenon was also taken into account. In both cases and in our case, it is done to get independence at infinity.
By using a simple parity argument, by using the remark that follows Definition 14 and by replacing L by $L^{\perp}$ (which is done only for a matter of convenience), one has that :

$$S_{A,prime}(L^{\perp},t)=\frac{2}{\pi}\sum_{\substack{k_1\wedge k_2=1\\ k_1\geqslant 0\\ \|k_1e_1(L)+k_2e_2(L)\|\leqslant A}}\frac{\phi(t\|k_1e_1(L)+k_2e_2(L)\|)}{\|k_1e_1(L)+k_2e_2(L)\|^{\frac{3}{2}}} \tag{3.9}$$

where the function ϕ was defined by the Equation (3.4). We have done that so that from this stage onwards we consider vectors of $\mathcal{P}_+(L)$ with an indexation (k_1,k_2) in the fixed set Π (that does not depend on L). Moreover, in probability, we can reduce to the case where $\|(k_1,k_2)\|\leqslant A'$, with $A'>0$ so that we deal with a fixed indexation in (k_1,k_2). So, one can think that the dependence on L in the inequality $\|k_1e_1(L)+k_2e_2(L)\|\leqslant A$ is secondary. Finally, let us add that this indexation will be very useful for the second step (for more details, see Section 4).

Second step. We will show that the family of variables $(t\|k_1e_1(L)+k_2e_2(L)\|)$, whose values are in $\mathbb{R}/\mathbb{Z}$, become, when $t\to\infty$, independent from one another and indeed converge towards independent and identically distributed random variables whose common distribution is given by the normalized Haar measure on $\mathbb{R}/\mathbb{Z}$. The idea here is basically the same as in [11] and in [38] where the respective authors used the fact that the square roots of square free integers are $\mathbb{Z}$-free. In our case, to prove the result, we will decompose the space of unimodular lattices into small geodesic segments, calculate the Taylor series of $\|k_1e_1(L)+k_2e_2(L)\|$ at order 1 on such a segment and show that the coefficients of order 1 are $\mathbb{Z}$-free. These coefficients are going to depend on L, they will not be fixed numbers like in [11] and in [38].
We will also prove that the variables $(t\|k_1e_1(L)+k_2e_2(L)\|)$, modulo 1, become independent, when $t\to\infty$, from the variable L due to the presence of the factor t.

Third step. Thanks to the first and second step, we will see that the asymptotic distribution of $\frac{\mathcal{R}(t\mathbb{D}^2,L)}{\sqrt{t}}$ is the distribution of

$$S(\theta,L)=\sum_{\substack{k=(k_1,k_2)\\ k_1\wedge k_2=1\\ k_1\geqslant 0}}\frac{\phi(\theta_k)}{\|k_1e_1(L)+k_2e_2(L)\|^{\frac{3}{2}}} \tag{3.10}$$

under the assumption that the quantity $S(\theta, L)$ is well-defined and where $\theta = (\theta_k) \in \mathbb{T}^\infty$ is distributed according to λ_∞, $L \in \mathscr{S}_2$ is distributed according to $\tilde{\mu}_2$. It should be noted that the $(\phi(\theta_k))$ is a family of independent and identically distributed non-zero real random variables. Furthermore, all of them are symmetrical and have a compact support. The third step of the proof will consist in showing the following general result. Let the Z_k be real non-zero independent identically distributed real random variables from a probability space $\Omega \ni \omega$ that are symmetrical and have a compact support. Then we are going to show that

$$\tilde{S}_A(\omega, L) = \sum_{\substack{k=(k_1,k_2) \\ k_1 \wedge k_2 = 1 \\ k_1 \geqslant 0 \\ \|k_1 e_1(L) + k_2 e_2(L)\| \leqslant A}} \frac{Z_k(\omega)}{\|k_1 e_1(L) + k_2 e_2(L)\|^{\frac{3}{2}}} \tag{3.11}$$

converges almost surely when $A \to \infty$. $\tilde{S}_A$ looks like a Siegel transform, but it is not so, because of the numerator of the considered terms. We are going to call it a modified Siegel transform with random weights. An object of this type was already studied in [27].

As a consequence of this general result of almost sure convergence, $S(\theta, L)$, as a particular case, will be well-defined as an almost sure limit.

We will also study the existence of moments of the almost sure limit. In particular, we are going to see that the optimal κ is $\frac{1}{3}$ when $\tilde{\mu}_2$ is not compactly supported under condition 3.3 (see statement of Theorem 8). Heuristically, it is because the magnitude of $\tilde{S}$ is given by the term

$$\frac{Z_{1,0}}{\|L\|_1^{\frac{3}{2}}}$$

with $Z_{1,0} \neq 0$. Thus, if $\tilde{\mu}_2$ satisfies condition (3.3), any of its moment is finite if, and only if, the corresponding moment of $\|L\|^{-\frac{3}{2}}$ is finite. The study of the moments of $\|L\|^{-\frac{3}{2}}$ will give us the wanted result.

After doing all of that, we will finally get the validity of Theorem 8.

Plan of the paper. The next section will be dedicated to deal with the first step of the proof, namely it will show that $\frac{\mathcal{R}(t\mathbb{D}^2, L)}{\sqrt{t}}$ is close in probability to $S_{A,prime}(L^\perp, t)$ when A is a fixed parameter taken large enough and, then, t goes to infinity (see Proposition 24). We have to "cut" the sum because of the problem of convergence of the Fourier series of $X \longmapsto \frac{\mathcal{R}(t\mathbb{D}^2 + X, L)}{\sqrt{t}}$ which is due to the lack of regularity of the indicator function $\mathbf{1}_{t\mathbb{D}^2}$. To do so, we are going to proceed by *regularization* which means here that we are going to smooth the indicator function $\mathbf{1}_{t\mathbb{D}^2}$ via a Gaussian kernel.

In Section 4, we tackle the second step of the proof, that is to prove that $(t\|k_1 e_1(L) + k_2 e_2(L)\|)$ become independent when $t \to \infty$. We also show that they converge towards random variables that are identically distributed according to the normalized Haar measure over $\mathbb{R}/\mathbb{Z}$ and that they become independent, when $t \to \infty$, from L and so that $\frac{\mathcal{R}(t\mathbb{D}^2, L)}{\sqrt{t}}$ has the same distribution as $S(\theta, L)$ defined in Equation (3.10).

In Section 5, which is the last section, we are going to tackle the third step of the proof, namely study the convergence of $\tilde{S}_A(\omega, L)$ when $A \to \infty$ and the existence of moments of its limit.

In the rest of the article, all the calculus of expectation $\mathbb{E}$, of variance $\mathbb{V}$ and of probability $\mathbb{P}$ will be made according to the measure $\tilde{\mu}_2$. Furthermore, the expression typical

is going to signify $\tilde{\mu}_2 -$ *almost surely*. In fact, we are going to suppose that $\tilde{\mu}_2 = \mu_2$ in Section 3 because we prove a result in probability and we have the following inequality :

$$\tilde{\mu}_2 \leqslant C\mu_2$$

where $C > 0$ because $\tilde{\mu}_2$ admits a bounded density relatively to μ_2.

3.3 Reduction to the study of the Siegel transform

The main object of this section is to show the following proposition :

Proposition 24. *For every $\alpha > 0$, for every $A > 0$ large enough, for every t large enough, one has that :*

$$\mathbb{P}(\Delta_{A,prime}(L,t) \geqslant \alpha) \leqslant \alpha$$

where

$$\Delta_{A,prime}(L,t) = |\frac{\mathcal{R}(t\mathbb{D}^2, L)}{\sqrt{t}} - S_{A,prime}(L,t)| \ . \tag{3.12}$$

This proposition basically says that we can reduce the asymptotical study of $\frac{\mathcal{R}(t\mathbb{D}^2,L)}{\sqrt{t}}$ to the study of its Fourier transform, taking into account a phenomenon of multiplicity. In fact, due to the triangle inequality, we only have to prove the following two lemmas :

Lemma 33. *For every $\alpha > 0$, for every $A > 0$ large enough, for every t large enough, one has that :*

$$\mathbb{P}(\Delta_A(L,t) \geqslant \alpha) \leqslant \alpha$$

where

$$\Delta_A(L,t) = |\frac{\mathcal{R}(t\mathbb{D}^2, L)}{\sqrt{t}} - H_A(L,t)| \ \textit{with} \tag{3.13}$$

$$H_A(L,t) = \frac{1}{\pi} \sum_{\substack{l \in L^\perp \\ 0 < \|l\| \leqslant A}} \frac{\cos(2\pi t\|l\| - \frac{3\pi}{4})}{\|l\|^{\frac{3}{2}}}. \tag{3.14}$$

Lemma 34. *For every $\alpha > 0$, for every $A > 0$ large enough, for every t large enough, one has that :*

$$\mathbb{P}(|S_{A,prime}(L,t) - H_A(L,t)| \geqslant \alpha) \leqslant \alpha.$$

Proof of Proposition 24. One has that :

$$\Delta_{A,prime}(L,t) \leqslant \Delta_A(L,t) + |S_{A,prime}(L,t) - H_A(L,t)|. \tag{3.15}$$

Lemma 33 and Lemma 34 imply then the wanted result. □

Let us say a few words about Lemmas 33 and 34 before following with their respective proofs. Lemma 33 says that the study of $\frac{\mathcal{R}(t\mathbb{D}^2,L)}{\sqrt{t}}$ can be reduced to the study of its Fourier transform. Lemma 34 says that the phenomenon of multiplicity (the fact

that for a prime vector l, $2l$, $3l$ etc. appear in the sum $H_A(L,t)$ when $A \to \infty$) is not so important. We only have to gather all the multiples of a prime vector (which corresponds to the infinite sum over m, see Equation (3.8)), so that we focus on prime vectors.

3.3.1 Proof of Lemma 33

First, we are going to prove Lemma 33. To do so, we are following closely the approach of [10], yet with some differences because in our case it is not the radius of dilatation that is random but the lattice (or, equivalently and in a certain sense, the oval).
For $x \in \mathbb{R}^2$ and $t > 0$, let us define

$$\lambda(x;t) = \frac{t^2}{4\pi} e^{-\frac{t^2}{4\pi}\|x\|^2}$$

and, for $M \in SL_2(\mathbb{R})$,

$$\lambda_M(x;t) = \lambda(Mx;t). \tag{3.16}$$

We recall that :

$$\int_{\mathbb{R}^2} \lambda_M(x;t)dx = 1 \tag{3.17}$$

and that the Fourier transform of $\lambda_M(\cdot;t)$ can be expressed as

$$\widehat{\lambda_M}(\xi;t) = e^{-\frac{\|(M^{-1})^T\xi\|^2}{t^2}}. \tag{3.18}$$

We introduce the following function :

$$\chi_{\mathbb{S}^1,M}(x;t) = (\mathbf{1}_{t\Omega_{M^{-1}\mathbb{S}^1}} * \lambda_M(\cdot;t))(x) = \int_{\mathbb{R}^2} \mathbf{1}_{t\Omega_{M^{-1}\mathbb{S}^1}}(y)\lambda_M(x-y;t)dy \tag{3.19}$$

(it is a regularization of the function $\mathbf{1}_{t\Omega_{M^{-1}\mathbb{S}^1}}$).
Let us also set :

$$N_{reg}(t\mathbb{D}^2, M) = \sum_{n\in\mathbb{Z}^2} \chi_{\mathbb{S}^1,M}(n;t) \ \textit{and} \tag{3.20}$$

(the index "reg" stands for regularized)

$$F(M,t) = \frac{N_{reg}(t\mathbb{D}^2, M) - \text{Area}(t\mathbb{D}^2)}{\sqrt{t}}. \tag{3.21}$$

Let L be a unimodular lattice such that $e_1(L)$ and $e_2(L)$ are well-defined and let

$$M = [e_1(L), e_2(L)] \ \textit{if} \det([e_1(L), e_2(L)]) > 0 \tag{3.22}$$

and

$$M = [e_2(L), e_1(L)] \ \textit{if} \det([e_2(L), e_1(L)]) > 0. \tag{3.23}$$

Then M is a matrix that represents L.
Now, let us call :

$$\Delta_1(L,t) = |\frac{\mathcal{R}(t\mathbb{D}^2, L)}{\sqrt{t}} - F(M,t)| \ \textit{and} \ (\Delta_2)_A(L,t) = |F(M,t) - H_A(L,t)| \tag{3.24}$$

so one has that :

$$\Delta_A(L,t) \leqslant \Delta_1(L,t) + (\Delta_2)_A(L,t). \tag{3.25}$$

The proof of Lemma 33 lies on the two following lemmas :

Lemma 35. *$\Delta_1(L,t)$ converges almost surely to 0 when $t \to \infty$.*

Lemma 36. *For all $\alpha > 0$, for all A large enough, for all t large enough,*

$$\mathbb{P}((\Delta_2)_A(L,t) \geqslant \alpha) \leqslant \alpha.$$

Proof of Lemma 33. It is the direct consequence of Equation (3.25) and of Lemmas 35 and 36. □

Lemma 35 basically tells us that the study of $\frac{\mathcal{R}(t\mathbb{D}^2,L)}{\sqrt{t}}$ can be reduced to the study of one of its regularized Fourier series, whereas Lemma 36 means that the asymptotical study of this regularized Fourier series can be brought back to the study of the non-regularized Fourier series.
The next subsubsection is dedicated to the proof of Lemma 35 and the subsubsection after it is dedicated to the proof of Lemma 36.

Proof of Lemma 35

The proof of Lemma 35 is based on two sublemmas. The first one is the following :

Sublemma 1. *For all $x \in \mathbb{R}^2$, for all $t > 0$,*

$$|\chi_{\mathbb{S}^1,M}(x;t) - \mathbf{1}_{t\Omega_{M^{-1}\mathbb{S}^1}}(x)| \leqslant e^{-\frac{t^2}{4}dist(Mx,t\mathbb{S}^1)^2}$$

where for all $z \in \mathbb{R}^2$,

$$dist(z,t\mathbb{S}^1) = \inf_{y\in t\mathbb{S}^1} |z-y|.$$

Proof. One has that :

$$|\chi_{\mathbb{S}^1,M}(x;t) - \mathbf{1}_{t\Omega_{M^{-1}\mathbb{S}^1}}(x)| = |\int_{y\notin t\mathbb{D}^2} \frac{t^2}{4\pi} e^{-\frac{\|Mx-y\|^2}{4}} dy| \text{ if } Mx \in t\mathbb{D}^2$$

and

$$|\chi_{\mathbb{S}^1,M}(x;t) - \mathbf{1}_{t\Omega_{M^{-1}\mathbb{S}^1}}(x)| = |\int_{y\in t\mathbb{D}^2} \frac{t^2}{4\pi} e^{-\frac{\|Mx-y\|^2}{4}} dy| \text{ if } Mx \notin t\mathbb{D}^2$$

because of Equation (3.17) and by making the change of variable $y = Mu$.
The proof of Lemma 3.2 from [10] gives the wanted result. □

The second sublemma gives an estimate of $\mathrm{dist}(Mn, t\mathbb{S}^1)$. To state it, we need a notation. For a vector x of $\mathbb{R}^2$, let us set $r = \|x\|$. Then, one has that there exists $C > 0$ small enough so that for every $x \in \mathbb{R}^2$,

$$\mathrm{dist}(x,t\mathbb{S}^1) \geqslant C|r-t|. \tag{3.26}$$

We deduce the following sublemma :

Sublemma 2. *For all $L \in \mathscr{S}_2$, for all $t > 0$, for all $n \in \mathbb{Z}^2$, we have that :*

$$dist(Mn,t\mathbb{S}^1) \geqslant C|\|Mn\| - t|.$$

Now we can prove Lemma 35.

Proof of Lemma 35. By using the Equation (3.24), we have that :

$$\Delta_1(L,t) \leqslant \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}^2} |\chi_{\mathbb{S}^1,M}(n;t) - \mathbf{1}_{t\Omega_{M^{-1}\mathbb{S}^1}}(n)|$$

because, also, $\mathcal{R}(t\Omega_{\mathbb{S}^1}, L) = \mathcal{R}(t\Omega_{M^{-1}\mathbb{S}^1}, \mathbb{Z}^2)$. So, Sublemma 1 and Sublemma 2 imply that :

$$\Delta_1(L,t) \leqslant \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}^2} e^{-C^2\frac{t^2}{4}|\|Mn\|-t|^2}.$$

The essential part of the right-hand side are the terms such that $n \in \mathbb{Z}^2$ verify that

$$|\|Mn\| - t| \leqslant \frac{1}{t^{3/4}}. \tag{3.27}$$

Yet the number of $n \in \mathbb{Z}^2$ that belong to such an annulus is of order $t^{\frac{1}{4}}$.
So, one has finally :

$$\Delta_1(L,t) = O(\frac{1}{t^{\frac{1}{4}}}).$$

□

Proof of Lemma 36

To prove Lemma 36, we first need to give another expression of $F(M,t)$, obtained via the Poisson formula. It is the object of the following lemma :

Lemma 37.

$$F(M,t) = \frac{1}{\pi} \sum_{n\in\mathbb{Z}^2-\{0\}} \frac{1}{\|(M^{-1})^T n\|^{\frac{3}{2}}} \cos(2\pi t\|(M^{-1})^T n\| - \frac{3\pi}{4})\widetilde{\lambda_M}(2\pi n;t) + O_M(t^{-1})$$

where the M in index of O_M indicates that it depends on M (or, equivalently, on the lattice L).

To prove it, we first need a calculatory sublemma :

Sublemma 3. *Let $D \in SL_2(\mathbb{R})$ and $\gamma = D\mathbb{S}^1$. Then one has for every $\xi \in \mathbb{R}^2 - \{0\}$:*

$$x_\gamma(\xi) = D\frac{D^T\xi}{\|D^T\xi\|},$$

$$\rho_\gamma(\xi) = \frac{\|\xi\|^3}{\|D^T\xi\|^3} \text{ and}$$

$$Y_\gamma(\xi) = \|D^T\xi\|$$

where D^T is the transpose of the matrix D.

Proof. Let us set $y = D^{-1}x_\gamma(\xi) \in \mathbb{S}^1$. One knows that :

$$k = y \tag{3.28}$$

where k is the outer normal to $\mathbb{S}^1$ at y.
Let us call T the unit tangent vector of $\mathbb{S}^1$ such that (k, T) is an orthonormal and direct basis of $\mathbb{R}^2$. Let us call R the rotation matrix of angle $\frac{\pi}{2}$, so that

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and so one has :

$$Rk = T. \tag{3.29}$$

By definition of $x_\gamma(\xi)$ and because D is a linear map such that $\det(D) > 0$, one knows that DT is a tangent vector of γ at the point $x_\gamma(\xi)$ such that $(\frac{\xi}{\|\xi\|}, DT)$ is an orthogonal and direct basis of $\mathbb{R}^2$. By using (3.29), we now know that : (ξ, DRk) is an orthogonal and direct basis of $\mathbb{R}^2$.
By using the fundamental property of the adjoint of an operator, we get that :

$$(R^{-1}D^T\xi, k) \text{ is an orthogonal basis of } \mathbb{R}^2. \tag{3.30}$$

So one gets that : $R^{-1}D^T\xi = aT$ where a is a non-zero real.
Finally, we have

$$D^T\xi = -ak \tag{3.31}$$

because of (3.29).
Equation (3.28) and the fact that k is unitary give us that :

$$x_\gamma(\xi) = \pm D\frac{D^T\xi}{\|D^T\xi\|}.$$

Yet, because (ξ, DRk) is a direct basis of $\mathbb{R}^2$ we know that :

$$(\xi, \pm DRD^T\xi) \textit{ is a direct basis.}$$

Yet, one can see that

$$RDR = -(D^{-1})^T$$

and so one gets that :

$$(R\xi, \mp\xi) \textit{ is a direct basis.}$$

So, one gets finally that :

$$x_\gamma(\xi) = D\frac{D^T\xi}{\|D^T\xi\|}. \tag{3.32}$$

As a consequence, we get immediately that :

$$Y_\gamma(\xi) = \|D^T\xi\|.$$

Then by using the parametrization $t \longmapsto (\cos(t), \sin(t))$ of the circle and the expression of a curvature radius when using a parametrization, one gets that at a point X of $\gamma = D\mathbb{S}^1$,

$$\rho(X) = \|(D^{-1})^T D^{-1}X\|^3.$$

So, with Equation (3.32), finally one gets that :

$$\rho_\gamma(\xi) = \frac{\|\xi\|^3}{\|D^T\xi\|^3}.$$

□

We can now tackle the proof of Lemma 37.

Proof of Lemma 37. According to Equation (3.21), the Poisson summation formula and because of the fact that $\widehat{\mathbf{1}_{t\Omega_{M^{-1}\mathbb{S}^1}}}(0) = \text{Area}(t\Omega_{M^{-1}\mathbb{S}^1})$ one has that :

$$F(M,t) = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}^2-\{0\}} \widehat{\mathbf{1}_{t\Omega_{M^{-1}\mathbb{S}^1}}}(2\pi n)\widehat{\lambda_M}(2\pi n;t). \tag{3.33}$$

Yet, according to Lemma 2.1 from [10], one has that :

$$\widehat{\mathbf{1}_{t\Omega_{M^{-1}\mathbb{S}^1}}}(\xi) = \sqrt{t}\|\xi\|^{-\frac{3}{2}} \sum_{\pm} \sqrt{2\pi\rho_{M^{-1}\mathbb{S}^1}(\pm\xi)}\exp(\pm i(tY_{M^{-1}\mathbb{S}^1}(\pm\xi)-\frac{3\pi}{4}))+O_M(t^{-\frac{1}{2}}\|\xi\|^{-\frac{5}{2}}) \tag{3.34}$$

By using Sublemma 3, we get with Equation (3.33) :

$$F(M,t) = \frac{1}{\pi} \sum_{n\in\mathbb{Z}^2-\{0\}} \frac{1}{\|(M^{-1})^T n\|^{\frac{3}{2}}} \cos(2\pi t\|(M^{-1})^T n\| - \frac{3\pi}{4})\widehat{\lambda_M}(2\pi n;t) + O_M(t^{-1}). \tag{3.35}$$

□

Using Equation (3.18), Equation (3.24), the fact that if M represents a lattice L, $(M^{-1})^T$ represents the dual lattice $L^\perp$, and the previous lemma, that is Lemma 37, one gets that :

$$(\Delta_2)_A(L,t) \leqslant \Delta_{2,1}(L,t) + (\Delta_{2,2})_A(L,t) + (\Delta_{2,3})_A(L,t) \tag{3.36}$$

where

$$\Delta_{2,1}(L,t) = O_M(t^{-1}) \tag{3.37}$$

$$(\Delta_{2,2})_A(L,t) = |\sum_{\substack{l\in L^\perp \\ 0<\|l\|\leqslant A}} \frac{1}{\|l\|^{\frac{3}{2}}} \cos(2\pi t\|l\| - \frac{3\pi}{4})(1-e^{-(2\pi)^2\frac{\|l\|^2}{t^2}})| \tag{3.38}$$

$$(\Delta_{2,3})_A(L,t) = |\sum_{\substack{l\in L^\perp \\ A<\|l\|}} \frac{1}{\|l\|^{\frac{3}{2}}} \cos(2\pi t\|l\| - \frac{3\pi}{4})e^{-(2\pi)^2\frac{\|l\|^2}{t^2}}|. \tag{3.39}$$

So, if we prove the following lemmas, we will get Lemma 36 and, *in fine*, get Lemma 33 :

Lemma 38. $\Delta_{2,1}(L,t)$ *converges almost surely to* 0 *when* $t\to\infty$.

Let us remark, by the way, that this last lemma is immediate according to Equation (3.37).

Lemma 39. *For all* $A>0$, $(\Delta_{2,2})_A(L,t)$ *converges to* 0 *when* $t\to\infty$.

Lemma 40. *For all* $\alpha>0$, *for all* A *large enough, for all* t *large enough,*

$$\mathbb{P}((\Delta_{2,3})_A(L,t)\geqslant\alpha)\leqslant\alpha.$$

Proof of Lemma 36. Let $\alpha > 0$. Let us take A large enough so that for all t large enough,

$$\mathbb{P}((\Delta_{2,3})_A(L,t) \geqslant \alpha) \leqslant \alpha.$$

It is possible according to Lemma 40.
According to Lemma 39, according to Lemma 38 and because the almost-sure convergence imply the convergence in probability, even if it means taking t larger, one can suppose that :

$$\mathbb{P}((\Delta_{2,1})(L,t) \geqslant \alpha) \leqslant \alpha \text{ and}$$
$$\mathbb{P}((\Delta_{2,2})_A(L,t) \geqslant \alpha) \leqslant \alpha.$$

By using Equation 3.36, one gets the wanted result. □

Before following with the proof of Lemma 33, let us say a few words about Lemma 39 and Lemma 40. The first tells us that the non-regularized Fourier series is "close" enough to the regularized Fourier series whereas the second one tells us that the large terms of the regularized Fourier series do not matter, in a certain sense, for our study. It remains only to prove Lemma 39 and Lemma 40. We will do just that in the next two subsubsections.

Proof of Lemma 39

Proof of Lemma 39. Let $l \in L^{\perp}$. Then one has :

$$|1 - e^{-(2\pi)^2 \frac{\|l\|^2}{t^2}}| \leqslant \frac{(2\pi)^2 \|l\|^2}{t^2}. \tag{3.40}$$

With this equation and with Equation (3.38), one gets that :

$$(\Delta_{2,2})_A(L,t) \leqslant \sum_{\substack{l \in L^{\perp} \\ 0 < \|l\| \leqslant A}} \frac{(2\pi)^2}{t^2} \|l\|^{\frac{1}{2}}. \tag{3.41}$$

It follows that there exists $C(L) > 0$ such that :

$$(\Delta_{2,2})_A(L,t) \leqslant C(L) \frac{A^{\frac{5}{2}}}{t^2}. \tag{3.42}$$

□

Proof of Lemma 40

To prove Lemma 40 we need to use what are called *Siegel* and *Rogers formulas*. Theses formulas will also be useful later in this paper.
By setting $c_k = \zeta(2)^{-k}$ for k an integer larger than 1 and where ζ denotes the ζ function of Riemann, one has the following formulas :

Lemma 41 ([64],[93],[47])**.** *For f a piecewise smooth function with compact support on $\mathbb{R}^2$, one has :*

- $$\int_{\mathscr{S}_2} \mathcal{S}(f) d\mu_2 = c_1 \int_{\mathbb{R}^2} f d\lambda$$
- *When f is even,*
$$(b) \int_{\mathscr{S}_2} \mathcal{S}(f)^2 d\mu_2 \leqslant C \int_{\mathbb{R}^2} f^2 d\lambda + c_2 (\int_{\mathbb{R}^2} f d\lambda)^2$$
where $C > 0$.

With this lemma, we are going to prove two lemmas that will enable us to prove Lemma 40 by using Chebyshev's inequality : the first one is intended to estimate the expectation of $(\Delta_{2,3})_A(L,t)$ to see that it goes to 0 when $t \to \infty$ (uniformly in A), the second one is intended to estimate its variance to see that it can be as uniformly small in t as one wants if A is chosen large enough. Until the end of this section, we are going to suppose $A > 1$.

Lemma 42.
$$\mathbb{E}((\Delta_{2,3})_A(L,t)) = O(\frac{1}{t}).$$

Proof. One has :
$$(\Delta_{2,3})_A(L,t) = \sum_{\substack{l \in L^\perp \text{ prime} \\ A < \|l\|}} f(l) \tag{3.43}$$
where
$$f(l) = \frac{1}{\|l\|^{\frac{3}{2}}} \sum_{k \in \mathbb{N} - \{0\}} \frac{\cos(2k\pi t \|l\| - \frac{3\pi}{4})}{k^{\frac{3}{2}}} e^{-(2k\pi)^2 \frac{\|l\|^2}{t^2}}. \tag{3.44}$$
Lemma 41 gives us then that :
$$\mathbb{E}((\Delta_{2,3})_A(L,t)) = C \int_{\mathbb{R}^2} f(x) \mathbf{1}_{\|x\| > A} dx \tag{3.45}$$
where $C > 0$ because the density of $\tilde{\mu}_2$ is supposed to be bounded.
By passing into polar coordinates, one gets that :
$$\mathbb{E}((\Delta_{2,3})_A(L,t)) = C \int_{r=A}^{\infty} \frac{\sum_{k \in \mathbb{N} - \{0\}} \frac{\cos(2k\pi t r - \frac{3\pi}{4})}{k^{\frac{3}{2}}} e^{-(2k\pi)^2 \frac{r^2}{t^2}}}{r^{\frac{1}{2}}} dr \tag{3.46}$$
(the constant C has changed but it does not matter). Lebesgue's dominated convergence theorem gives us that :
$$\mathbb{E}((\Delta_{2,3})_A(L,t)) = C \sum_{k \in \mathbb{N} - \{0\}} \frac{1}{k^{\frac{3}{2}}} \int_{r=A}^{\infty} \frac{\cos(2k\pi t r - \frac{3\pi}{4})}{r^{\frac{1}{2}}} e^{-(2k\pi)^2 \frac{r^2}{t^2}} dr. \tag{3.47}$$
Finally, an integration by part gives us that :
$$\begin{aligned} \mathbb{E}((\Delta_{2,3})_A(L,t)) = C \sum_{k \in \mathbb{N} - \{0\}} \frac{1}{k^{\frac{3}{2}}} (& - \frac{\sin(2k\pi t A - \frac{3\pi}{4}) e^{-(2k\pi)^2 \frac{A^2}{t^2}}}{2k\pi t A^{\frac{1}{2}}} \\ & + \int_A^\infty \frac{\sin(2k\pi t A - \frac{3\pi}{4}) e^{-(2k\pi)^2 \frac{A^2}{t^2}}}{4k\pi t r^{\frac{3}{2}}} dr \\ & + \int_A^\infty \frac{\sin(2k\pi t r - \frac{3\pi}{4})}{2k\pi t} \frac{(2\pi k)^2}{t^2} 2 r^{\frac{1}{2}} e^{-(2k\pi)^2 \frac{r^2}{t^2}} dr). \end{aligned} \tag{3.48}$$

By using that $r^{\frac{1}{2}} \leqslant r$ (because $A > 1$), one has finally, by estimating the three terms of the right member :

$$\mathbb{E}((\Delta_{2,3})_A(L,t)) = O(\frac{1}{t}).$$

□

Lemma 43.

$$\mathbb{V}\Big((\Delta_{2,3})_A(L,t)\Big) = O(\frac{1}{A})$$

where the O can be chosen independent from t.

Proof. By using the same notation as before, by using again Lemma 41 and by using Lemma 42, one gets that :

$$\mathbb{V}\Big((\Delta_{2,3})_A(L,t)\Big) \leqslant C \int_{\mathbb{R}^2} f^2(x) \mathbf{1}_{\|x\|>A} dx. \tag{3.49}$$

So, by passing into polar coordinates, one gets that :

$$\mathbb{V}\Big((\Delta_{2,3})_A(L,t)\Big) \leqslant C2\pi \int_{r=A}^{\infty} \frac{1}{r^2} (\sum_{k\in\mathbb{N}-\{0\}} \frac{\cos(2k\pi tr - \frac{3\pi}{4})}{k^{\frac{3}{2}}} e^{-(2k\pi)^2 \frac{r^2}{t^2}})^2 dr. \tag{3.50}$$

Because $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} < \infty$, we get the wanted result. □

We can now prove Lemma 40.

Proof of Lemma 40. Chebyshev's inequality gives the wanted result if, first, we choose A large enough and, second, we choose t large enough so that $\mathbb{E}((\Delta_{2,3})_A(L,t))$ and $\mathbb{V}\Big((\Delta_{2,3})_A(L,t)\Big)$ are small enough. These choices are possible according to Lemmas 42 and 43. □

So, now the proof of Lemma 33 is complete and we will conclude this section by proving Lemma 34 so that the proof of Proposition 24 will be complete.

3.3.2 Proof of Lemma 34

To prove Lemma 34, we are going to take the same kind of approach as before : estimate the expectation and the variance of the quantity $S_A - S_{A,prime}$ and get the result via Chebyshev's inequality.
We have that :

$$H_A(L,t) - S_{A,prime}(L,t) = \sum_{l\in L^{\perp}prime} f(l) \tag{3.51}$$

where

$$f(l) = \frac{1}{\|l\|^{\frac{3}{2}}} \mathbf{1}_{0<\|l\|\leqslant A} \sum_{k\geqslant \lfloor \frac{A}{\|l\|} \rfloor + 1} \frac{\cos(2k\pi t\|l\| - \frac{3\pi}{4})}{k^{\frac{3}{2}}}. \tag{3.52}$$

With this expression, we see that we are going to have a little problem of integrability at 0 when using Lemma 41. That's why, we have to exclude 0 and we will suppose that L is chosen so that $\|L^{\perp}\|_1 \geqslant \epsilon$ where $0 < \epsilon < 1$. Only a small number of lattices is excluded according to this lemma that we recall :

Lemma 44. *For every $0 < \epsilon < 1$, one has that*

$$\mathbb{P}(\|L\|_1 < \epsilon) = O(\epsilon^2).$$

Proof. It is a consequence of Lemma 41 by taking

$$\mathcal{S}(f)(L) = \sum_{\substack{l \in L \\ l \ prime}} \mathbf{1}_{B_f(0,\epsilon)}(l)$$

where $\mathbf{1}_{B_f(0,\epsilon)}(l)$ is the indicator function of the closed ball for the norm $\|\|$ centred on 0 of radius ϵ. □

Thus, for the chosen lattices, we have :

$$H_A(L,t) - S_{A,prime}(L,t) = \sum_{l \in L^{\perp} prime} f(l)\mathbf{1}_{\|l\| \geqslant \epsilon} = \Delta_{3,\epsilon,A,t}(L). \tag{3.53}$$

Lemma 45.

$$\mathbb{E}(\Delta_{3,\epsilon,A,t}(L)) = O_{\epsilon,A}(\frac{1}{t}).$$

Proof. By using Lemma 41, one gets that :

$$\mathbb{E}(\Delta_{3,\epsilon,A,t}(L)) = C \int_{\epsilon}^{A} \frac{1}{r^{\frac{1}{2}}} \sum_{k \geqslant \frac{A}{r}} \frac{\cos(2k\pi tr - \frac{3\pi}{4})}{k^{\frac{3}{2}}} dr. \tag{3.54}$$

Lebesgue's dominated convergence theorem gives us that :

$$\int_{\epsilon}^{A} \frac{1}{r^{\frac{1}{2}}} \sum_{k \geqslant \frac{A}{r}} \frac{\cos(2k\pi tr - \frac{3\pi}{4})}{k^{\frac{3}{2}}} dr = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} \int_{\max(\frac{A}{k},\epsilon)}^{A} \frac{\cos(2k\pi tr - \frac{3\pi}{4})}{r^{\frac{1}{2}}} dr. \tag{3.55}$$

An integration by part as in the proof of Lemma 42 and Equation (3.54) give us finally that :

$$\mathbb{E}(\Delta_{3,\epsilon,A,t}(L)) = O_{\epsilon,A}(\frac{1}{t}). \tag{3.56}$$

□

Lemma 46. *There exists $K > 0$ such that :*

$$\mathbb{V}\Big(\Delta_{3,\epsilon,A,t}(L)\Big) \leqslant K(-\frac{\log(\epsilon)}{A} + \frac{\log(A)}{A}).$$

Proof. Lemma 41 gives us that :

$$\mathbb{V}\Big(\Delta_{3,\epsilon,A,t}(L)\Big) \leqslant C2\pi \int_{\epsilon}^{A} \frac{1}{r^2} (\sum_{k \geqslant \frac{A}{r}} \frac{\cos(2\pi ktr - \frac{3\pi}{4})}{k^{\frac{3}{2}}})^2 dr. \tag{3.57}$$

Yet one also has that for all $x > 0$:

$$\sum_{k \geqslant x} \frac{1}{k^{\frac{3}{2}}} \leqslant \frac{D}{x^{\frac{1}{2}}} \tag{3.58}$$

where $D > 0$.
Thus, Equation (3.57) and Equation (3.58) imply that :

$$\mathbb{V}\Big(\Delta_{3,\epsilon,A,t}(L)\Big) \leqslant \frac{2\pi CD}{A} \int_{\epsilon}^{A} \frac{1}{r} dr. \tag{3.59}$$

□

We can now give the proof of Lemma 34.

Proof of Lemma 34. First we take $1 > \epsilon > 0$ small enough so that the measure of the neglected lattices, *id est* the lattices such that $\|L\|_1 < \epsilon$, is small enough. It is possible according to Lemma 44.
Then we take A large enough so that $\mathbb{V}\Big(\Delta_{3,\epsilon,A,t}(L)\Big)$ is small enough. It is possible according to Lemma 46.
Finally, we take t large enough so that $\mathbb{E}(\Delta_{3,\epsilon,A,t}(L))$ is small enough, which is possible according to Lemma 45, and conclude by using Chebyshev's inequality. □

So, we are now brought back to the study of $S_{A,prime}(L,t)$ when $t \to \infty$ and the next section is dedicated to it.
We are going to replace $L^{\perp}$ by L (it changes nothing because we are studying the asymptotic convergence in distribution with $L \in \mathscr{S}_2$ distributed according to $\tilde{\mu}_2$).

3.4 Study of $S_{A,prime}(L,t)$ when $t \to \infty$

3.4.1 Reductions for the study of $S_{A,prime}(L,t)$ and proof of Theorem 8

Before entering in the main object of this section, we need to do a small rewriting of $S_{A,prime}(L,t)$.
We recall that a vector $l \in L$ is prime if, and only if, $Kl \in KL$ is prime where $K \in SL_2(\mathbb{R})$. Furthermore, a vector $(l_1, l_2) \in \mathbb{Z}^2$ is prime if, and only if, $l_1 \wedge l_2 = 1$.
By using the symmetry $l \longmapsto -l$, we deduce that $S_{A,prime}(L,t)$ can be rewritten as follows :

$$S_{A,prime}(L,t) = \frac{2}{\pi} \sum_{k \in \Pi_A(L)} \frac{Z_k(L,t)}{Y_k(L)} \tag{3.60}$$

where, for $k = (k_1, k_2) \in \mathbb{Z}^2$,

$$Y_k(L) = \|k_1 e_1(L) + k_2 e_2(L)\|^{\frac{3}{2}}, \tag{3.61}$$

$$Z_k(L,t) = \sum_{m\in\mathbb{N}-\{0\}} \frac{\cos(2\pi t m\|k_1e_1(L)+k_2e_2(L)\| - \frac{3\pi}{4})}{m^{\frac{3}{2}}}, \tag{3.62}$$

and where

$$\Pi_A(L) = \{(k_1,k_2)\in\mathbb{Z}^2 \mid k_1\wedge k_2 = 1,\ k_1 \geqslant 0\ \|k_1e_1(L)+k_2e_2(L)\| \leqslant A\} \tag{3.63}$$

(for the definition of $e_1(L)$ and $e_2(L)$ see Definition 14).

The goal of this section is to prove the following proposition :

Proposition 25. *$\{Z_k(L,t)\}_{k\in\Pi}$ converge, when $t\to\infty$, towards independent identically distributed real random variables that have a compact support, are symmetrical and are independent of L.*

In the next section we are going to consider the sums of the type

$$\tilde{S}_A(\omega,L) = \sum_{k\in\Pi_A(L)} \frac{Z_k(\omega)}{\|k_1e_1(L)+k_2e_2(L)\|^{\frac{3}{2}}}$$

where Z_k are non-zero real independent identically distributed random variables from $\Omega\ni\omega$ that are symmetrical and have a compact support. We will show the following proposition :

Proposition 26. *$\tilde{S}_A(\omega,L)$ converges almost surely when $A\to\infty$. Furthermore, the almost sure limit $\lim_{A\to\infty}\tilde{S}_A(\omega,L)$:*

- *is symmetrical and its expectation is equal to 0*
- *admits moment of order $1+\kappa$ for any $0\leqslant\kappa<\frac{1}{3}$*
- *does not admit a moment of order $\frac{4}{3}$ when $\sigma(L)\geqslant m$ where $m>0$ and where L belongs to an event of the form $(\|L\|<\alpha)$ with $\alpha>0$.*
- *when there exists $\alpha>0$ such that $\tilde{\mu}_2\Big(\{L\in\mathscr{S}_2 \mid \|L\|_1<\alpha\}\Big)=0$ then it admits moments of all order $1\leqslant p<\infty$.*

We are going to see now that it is enough to prove Proposition 25 and Proposition 26 to establish Theorem 8, with the exception of the exact form of the limit distribution (yet it is given by Proposition 27).

Proof of Theorem 8. Let $\psi\in C_c^\infty(\mathbb{R})$. Let $\epsilon>0$. According to Proposition 24, we can take A as large as we want and then t as large as we want so that :

$$|\mathbb{E}\left(\psi(\frac{\mathcal{R}(t\mathbb{D}^2,L)}{\sqrt{t}})\right) - \mathbb{E}\left(\psi(S_{A,prime}(L,t))\right)| \leqslant \epsilon. \tag{3.64}$$

Furthermore, thanks to Proposition 25, one has that :

$$|\mathbb{E}\left(\psi(S_{A,prime}(L,t))\right) - \mathbb{E}\left(\psi(\tilde{S}_A(\omega,L))\right)| \leqslant \epsilon \tag{3.65}$$

where $Z_k(\omega)$ in $\tilde{S}_A(\omega, L)$ are given by Proposition 25.
Moreover, Proposition 26 gives us that :

$$|\mathbb{E}\left(\psi(\tilde{S}_A(\omega,L))\right) - \mathbb{E}\left(\psi(\lim_{A\to\infty}\tilde{S}_A(\omega,L))\right)| \leqslant \epsilon \tag{3.66}$$

with $\lim_{A\to\infty}\tilde{S}_A(\omega,L))$ that verify all the listed properties.
So, Equation (3.64), Equation (3.65) and Equation (3.66) give the wanted result. □

The main reason why Z_k are going to become independent from L is the presence of the factor t.
The main reasons why the rest of Proposition 25 will be true are the presence of the factor t in Z_k and the fact that the coefficients of order 1 of the Taylor series of $(\|k_1e_1(L)+k_2e_2(L)\|)_{k\in\Pi_A(L)}$ on a small geodesic segment are $\mathbb{Z}$-free.

In order to prove Proposition 25, it is actually enough to prove the following proposition :

Proposition 27. *For $k=(k_1,k_2)$, let*

$$\theta_k(L,t) = t\|k_1e_1(L)+k_2e_2(L)\| \mod 1. \tag{3.67}$$

Then, we have that $\{\theta_k(L,t)\}_{k\in\Pi}$ converge, when $t\to\infty$, towards random variable that are independent identically distributed, are distributed according to the Lebesgue measure λ over $\mathbb{R}/\mathbb{Z}$ and are independent from L.

Thanks to this proposition, we now understand why the limit distribution of $\frac{\mathcal{R}(t\mathbb{D}^2,L)}{\sqrt{t}}$ is given by $S(\theta, L^\perp)$.
To prove this last proposition, it is sufficient to prove the following proposition where $e(\theta)$ stands for $\exp(i2\pi\theta)$:

Proposition 28. *For every $l\in\mathbb{N}-\{0\}$, for every $\psi\in C_c^\infty(\mathscr{S}_2)$, for every $(p_1,\cdots,p_l)\in\mathbb{Z}^l-\{0\}$ and for every $k_h\in\Pi$ that are all distinct with $h\in[1,l]$, one has :*

$$\mathbb{E}\left(\psi(L)e(\sum_{h=1}^{l}p_h\theta_{k_h})\right) \underset{t\to\infty}{\to} 0. \tag{3.68}$$

Before passing to the proof of Proposition 28, let us give some heuristic about it.
Basically, by working with a foliation of the space $\mathscr{S}_2$ given by small enough geodesic segments, we are first going to have :

$$\mathbb{E}\left(\psi(L)e(\sum_{h=1}^{l}p_h\theta_{k_h})\right) \approx \mathbb{E}(\psi)\mathbb{E}\left(e(\sum_{h=1}^{l}p_h\theta_{k_h})\right)$$

because ψ is almost constant on such a segment.
The right member will go to 0 when t goes to infinity because a Riemann-Lebesgue lemma will apply because quantities "close" to the variables θ_k are typically $\mathbb{Z}$-free (see the heuristic explanation of the second step in Section 2).
The rest of this section is now dedicated to the proof of Proposition 28.

3.4.2 Foliation and local estimates

We recall that a foliation of the space $\mathscr{S}_2$ is given by the orbits of the group δ where

$$\delta(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}.$$

To prove Proposition 28, we are going to look at what it is happening on a small "segment" of the form

$$J_\epsilon(L) = \{\delta(\lambda)L \mid \lambda \in [\frac{1}{1+\epsilon}, 1+\epsilon]\} \tag{3.69}$$

where $L \in \mathscr{S}_2$ and $\epsilon > 0$ can be taken as small as possible. More precisely, we are going to show, when $t \to \infty$, the independence of the (θ_k) and of L over smalls segments of the form $J_\epsilon(L)$, as well as the fact that the (θ_k) are identically distributed and distributed according to the normalized Haar measure over $\mathbb{R}/\mathbb{Z}$.
Let us call $s : (x, y) \in \mathbb{R}^2 \longmapsto (x, -y)$ and let us set, for $k = (k_1, k_2) \in \Pi$:

$$W_k(L) = \frac{< k_1e_1(L) + k_2e_2(L), s(k_1e_1(L) + k_2e_2(L)) >}{\|k_1e_1(L) + k_2e_2(L)\|}. \tag{3.70}$$

On a segment of the form $J_\epsilon(L)$, the following lemma basically tells us how we can estimate the quantities Z_k :

Lemma 47. *For a typical $L \in \mathscr{S}_2$, there exists $\epsilon > 0$ small enough such that for every $\lambda \in [\frac{1}{1+\epsilon}, 1+\epsilon]$,*

$$e_1(\delta(\lambda)L) = \delta(\lambda)e_1(L) \text{ and } e_2(\delta(\lambda)L) = \delta(\lambda)e_2(L).$$

Furthermore, for such a lattice L, for such λ, for $k = (k_1, k_2) \in \Pi$, we have for $h = \lambda - 1$,

$$\begin{aligned} &\|k_1e_1(\delta(\lambda)L) + k_2e_2((\delta(\lambda)L)\| = \|k_1e_1(L) + k_2e_2(L)\| \\ &+ W_k(L)h + O_{k_1,k_2,L}(h^2). \end{aligned} \tag{3.71}$$

Proof. First let us prove that for a typical L, for $\epsilon > 0$ small enough, for every $\lambda \in [\frac{1}{1+\epsilon}, 1+\epsilon]$, $e_1(\delta(\lambda)L) = \delta(\lambda)e_1(L)$ and $e_2(\delta(\lambda)L) = \delta(\lambda)e_2(L)$.
One has that for every $k \in \mathbb{R}^2$, for every $\lambda \in [\frac{1}{1+\epsilon}, 1+\epsilon]$

$$\|k\| \geqslant \frac{1}{1+\epsilon}\|\delta(\lambda)k\| \tag{3.72}$$

and

$$\|\delta(\lambda)k\| \geqslant \frac{1}{1+\epsilon}\|k\|. \tag{3.73}$$

We also note that for a typical lattice L, we have that

$$\min_{l \in L - \{\pm e_1(L), \pm e_2(L), 0\}} \|l\| \geqslant \|e_2(L)\| > \|e_1(L)\|. \tag{3.74}$$

So there exists $\kappa > 0$ such that for every $k \in L - \{\pm e_1(L), 0\}$,

$$\|k\| \geqslant \|e_1(L)\| + \kappa. \tag{3.75}$$

This last equation, Equation (3.72) and Equation (3.73) give us that for every $k \in L - \{\pm e_1(L), 0\}$,

$$\|\delta(\lambda)k\| \geqslant \frac{1}{1+\epsilon}\|k\| \geqslant \frac{1}{(1+\epsilon)^2}\|\delta(\lambda)e_1(L)\| + \frac{\kappa}{1+\epsilon}. \tag{3.76}$$

This last equation and the fact that $(\delta(\lambda)e_1(L))_1 > 0$ give us that, for a typical lattice L, for every ϵ small enough, $\delta(\lambda)e_1(L) = e_1(\delta(\lambda)L)$.
Let us prove that, for a typical lattice L, even if it means reducing $\epsilon > 0$, for every $\lambda \in [\frac{1}{1+\epsilon}, 1+\epsilon]$, $e_2(\delta(\lambda)L) = \delta_\lambda e_2(L)$.
We have the wanted result when the first inequality in Equation (3.74) is strict (by reasoning the same way as before).
So let us suppose that this last inequality is an equality. Let us call l such that $l_1 \geqslant 0$ and such that $\|e_2(L)\| = \|l\|$. If $l_1 > 0$ it means that the couple $(e_1(L), e_2(L))$ is not well-defined. If $l_1 = 0$ then L belongs to a negligible set according to Lemma 4.5 from the article [82].
Now, let us show that we have Equation (3.71). It is in fact a basic calculation of Taylor series. Indeed, one has, thanks to the facts that $e_1(\delta(\lambda)L) = \delta(\lambda)e_1(L)$ and $e_2(\delta(\lambda)L) = \delta(\lambda)e_2(L)$:

$$\|k_1e_1(\delta(\lambda)L) + k_2e_2(\delta(\lambda)L)\|^2 \tag{3.77}$$
$$= \|k_1e_1(L) + k_2e_2(L) + hs(k_1e_1(L) + k_2e_2(L)) + O_{k_1,k_2,L}(h^2)\|^2$$
$$= \|k_1e_1(L) + k_2e_2(L)\|^2 + 2h < k_1e_1(L) + k_2e_2(L), s(k_1e_1(L) + k_2e_2(L)) > + O_{k_1,k_2,L}(h^2). \tag{3.78}$$

By applying the square root on Equation (3.77), one gets the Equation (3.71). □

To prove Proposition 28 we see, in light of Lemma 47, that it would be convenient to prove the following proposition :

Proposition 29. *For a typical $L \in \mathscr{S}_2$, for every $m \in \mathbb{N} - \{0\}$, for every family $(p_1, \cdots, p_m) \in \mathbb{Z}^m$, for every $k_1, \cdots, k_m \in \Pi$ all distinct if*

$$\sum_{i=1}^{m} p_i W_{k_i}(L) = 0 \tag{3.79}$$

then $p_1 = \cdots = p_m = 0$.
In other words, for a typical $L \in \mathscr{S}_2$,

$$(W_{k_i}(L))_{i \geqslant 1}$$

is a $\mathbb{Z}$-free family.

The next subsection is dedicated to prove this proposition.

3.4.3 Proof of Proposition 29

To prove Proposition 29, we are following closely what was done in the Section 5 of [27] and we need two preliminary lemmas.

To state the first one, we call, for every $k = (k_1, k_2) \in \Pi$,

$$f_k : (A = (X_1, Y_1), B = (X_2, Y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \longmapsto \frac{< k_1 X + k_2 Y, s(k_1 X + k_2 Y) >}{\|k_1 X + k_2 Y\|} \quad (3.80)$$

where $X = (X_1, X_2)$ and $Y = (Y_1, Y_2) \in \mathbb{R}^2$.

Lemma 48. *For every $m \in \mathbb{N} - \{0\}$, for every family $(p_1, \cdots, p_m) \in \mathbb{Z}^m$, for every $k_1, \cdots, k_m \in \Pi$ all distinct, one has the following implication : if $\sum_{i=1}^m p_i f_{k_i}$ is a polynomial function then all the p_i must be equal to 0.*

Proof. Let us set :

$$g : \delta \in \mathbb{R} \longmapsto \frac{< (1, \delta), s((1, \delta)) >}{\|(1, \delta)\|} = \frac{1 - \delta^2}{\sqrt{1 + \delta^2}}. \quad (3.81)$$

Let us suppose that $\sum_{i=1}^m p_i f_{k_i}$ is a polynomial. Let us give $j \in [1, m]$. We are going to show that $p_j = 0$.
Let $\beta \in \mathbb{R}^2$ such that $< k_j, \beta > \neq 0$. Let $\alpha \in \mathbb{R}^2$ and let $X = \alpha$ and $Y = \delta\alpha + \theta\beta$. Then, one has :

$$f_k(A, B) = | < k, \alpha > | g(\delta + \theta \frac{< k, \beta >}{| < k, \alpha > |}).$$

As $\sum_{i=1}^m p_i f_{k_i}$ is a polynomial then $\sum_{i=1}^m p_i | < k_i, \alpha > | g(\delta + \theta \frac{<k_i, \beta>}{|<k_i, \alpha>|})$ is a polynomial in β whose degree is bounded by a number that does not depend on α (nor in β).
So there exists $K \geqslant 2$ such that the K-th derivative of $\sum_{i=1}^m p_i | < k_i, \alpha > | g(\delta + \theta \frac{<k_i, \beta>}{|<k_i, \alpha>|}$ relatively to θ is equal to 0. It means that the terms in front of θ^K is equal to 0. Hence the following equation :

$$\sum_{i=1}^m h_i \frac{< k_i, \beta >^K}{| < k_i, \alpha > |^{K-1}} = 0 \quad (3.82)$$

where $h_i = p_i g^{(K)}(\delta)$.
Now, since the k_i belong to Π and are all distinct, it is possible to choose α so that $< k_j, \alpha >> 0$ is arbitrary small while $< k_i, \alpha >$ remain bounded away from zero for every $i \neq j$. Thus, we must have $h_j = 0$. Yet, g is not a polynomial so there exists $\delta \in \mathbb{R}$ such that $g^{(K)}(\delta) \neq 0$ and it gives us that $p_j = 0$. □

Lemma 48 enables us to prove the following lemma.

Lemma 49. *For a typical $L \in \mathscr{S}_2$, for every $m \in \mathbb{N} - \{0\}$, for every family $(p_1, \cdots, p_m) \in \mathbb{Z}^m$, for every $k_1, \cdots, k_m \in \Pi$ all distinct if*

$$\sum_{i=1}^m p_i W_{k_i}(L) = 0 \quad (3.83)$$

then

$$\sum_{i=1}^m p_i f_{k_i} = 0.$$

Proof. First, we note that, for every $k \in \Pi$, $W_k(L) = f_k((e_1(L))_1, (e_2(L))_1, (e_1(L))_2, (e_2(L))_2)$. Thanks to this remark, we see that Lemma 49 is a direct consequence of the facts that $\mathscr{S}_2$ is a variety of dimension 3 that can be seen, locally, as an open set of $\{M \in GL_2(\mathbb{R}) \mid det(M) = 1\}$, that det is a polynomial function whereas $\sum_{i=1}^m p_i f_{k_i}$ is a polynomial function only in the trivial case according to Lemma 48. □

We can now give the proof of Proposition 29.

Proof of Proposition 29. It is a direct consequence of Lemma 48 and Lemma 49. □

3.4.4 Proof of Proposition 28

Before starting the proof of Proposition 28, we only need a simple lemma :

Lemma 50. *For every $m \in \mathbb{N} - \{0\}$, for every family $(p_1, \cdots, p_m) \in \mathbb{Z}^m - \{0\}$, for every $k_1, \cdots, k_m \in \Pi$, all distinct, for every $0 < \epsilon < 1$, there exists $a > 0$ and K_ϵ a measurable set of $\mathscr{S}_2$ such that $\tilde{\mu}_2(\mathscr{S}_2 - K_\epsilon) \leqslant \epsilon$ and such that for all $L \in K_\epsilon$,*

$$|\sum_{i=1}^m p_i W_{k_i}(L)| \geqslant a \ .$$

Proof. It is a direct consequence of Proposition 29. □

Now, by using the foliation given by $\delta(\lambda)$ and previous results, we can now prove Proposition 28.

Proof of Proposition 28. The proof in all its generality can be made as in the case where $\tilde{\mu}_2 = \mu_2$ because $\tilde{\mu}_2$ admits a bounded and regular density relatively to μ_2. So, we will suppose for simplicity that $\tilde{\mu}_2 = \mu_2$.
Let $l \geqslant 1$. Let $\psi \in C_c^\infty(\mathscr{S}_2)$. Let $(p_1, \cdots, p_l) \in \mathbb{Z}^l - \{0\}$.
For all $\epsilon > 0$, we call $\mathcal{F}_\epsilon$ the tribe on $\mathscr{S}_2$ generated by the $J_\epsilon(L)$. Let $1 > \epsilon_1 \geqslant \epsilon_2 > 0$.
According to Lemma 47 and Lemma 50, there exists a measurable part K_{ϵ_1} such that $\mu_2(K_{\epsilon_1}) \geqslant 1 - \epsilon_1$, a real $M > 0$ and a real $a > 0$, such that

- for every $L \in K_{\epsilon_1}$, for every $\lambda \in [\frac{1}{1+\epsilon_1}, 1 + \epsilon_1]$:

$$|\psi(\delta(h)L) - \psi(L)| \leqslant M|h| \tag{3.84}$$

  where $h = \lambda - 1$

- for every $L \in K_{\epsilon_1}$, for every $\lambda \in [\frac{1}{1+\epsilon_1}, 1 + \epsilon_1]$, Equation (3.71) is verified.

- for every $L \in K_{\epsilon_1}$,

$$|\sum_{i=1}^l p_i W_{k_i}(L)| \geqslant a \ . \tag{3.85}$$

Furthermore, we are going to suppose, even if it means making ϵ_2 goes to 0, that for every $L \in K_{\epsilon_1}$,

$$J_{\epsilon_2}(L) \subset K_{\epsilon_1}. \tag{3.86}$$

Claim. With these notations, we have, for all $L \in K_{\epsilon_1}$, that :

$$\mathbb{E}\left(\psi e(\sum_{i=1}^{l} p_i \theta_{k_i})|\mathcal{F}_{\epsilon_2}\right)(L) = O(\epsilon_2) + O(\frac{1}{at\epsilon_2}) + \frac{1}{a}O(\epsilon_2). \tag{3.87}$$

To this end, let us set, for all $\epsilon > 0$,

$$\delta(\epsilon) = 1 + \epsilon - \frac{1}{1+\epsilon}.$$

Then one has for every $L \in K_{\epsilon_1}$ according to Lemma 47 :

$$\begin{aligned}
&\mathbb{E}(\psi e(\sum_{j=1}^{l} p_j \theta_{k_j})|\mathcal{F}_{\epsilon_2})(L) \\
&= \frac{1}{\delta(\epsilon_2)} \int_{\frac{1}{1+\epsilon_2}-1}^{\epsilon_2} \left(\psi e(\sum_{j=1}^{l} p_j \theta_{k_j})\right)(\delta(h)L)dh \\
&= \psi(L)\frac{1}{\delta(\epsilon_2)} \int_{\frac{1}{1+\epsilon_2}-1}^{\epsilon_2} \left(e(\sum_{j=1}^{l} p_j \theta_{k_j})\right)(\delta(h)L)dh + O(\epsilon_2) \\
&= \left(\psi e(\sum_{j=1}^{l} p_j \theta_{k_j})\right)(L)\frac{1}{\delta(\epsilon_2)} \int_{\frac{1}{1+\epsilon_2}-1}^{\epsilon_2} e^{itD_1(L)h+itD_2(L,h)}dh + O(\epsilon_2)
\end{aligned} \tag{3.88}$$

where

$$D_1(L) = \sum_{j=1}^{l} p_j W_{k_j}(L), \tag{3.89}$$

$$D_2(L,h) = \sum_{j=1}^{l} p_j \theta_{k_j}(\delta(\lambda)L) - \sum_{j=1}^{l} p_j \theta_{k_j}(L) - D_1(L)h \textit{ such that} \tag{3.90}$$

$$D_2(L,h) = O(h^2) \textit{ and} \tag{3.91}$$

$D_2(L,\cdot)$ is smooth around 0.
Thus, by integrating by part and by using Equation (3.85), one gets that for all $L \in K_{\epsilon_1}$:

$$\begin{aligned}
&\frac{1}{\delta(\epsilon_2)} \int_{\frac{1}{1+\epsilon_2}-1}^{\epsilon_2} e^{itD_1(L)h+itD_2(L,h)}dh \\
&= \frac{1}{\delta(\epsilon_2)} \left(\left[\frac{e^{itD_1(L)h+itD_2(L,h)}}{itD_1(L)}\right]_{\frac{1}{1+\epsilon_2}-1}^{\epsilon_2} + \frac{1}{D_1(L)} \int_{\frac{1}{1+\epsilon_2}-1}^{\epsilon_2} (D_2(L,\cdot))'(h) e^{itD_1(L)h+itD_2(L,h)}dh\right) \\
&= O(\frac{1}{at\epsilon_2}) + \frac{1}{a}O(\epsilon_2).
\end{aligned} \tag{3.92}$$

Finally, Equation (3.88) and Equation (3.92) give the wanted claim.

Thanks to Equation (3.87), the fact that $\mu_2(K_{\epsilon_1}) \geqslant 1 - \epsilon_1$ and because of Equation (3.86), we have that :

$$\begin{aligned}
&|\mathbb{E}\left(\psi e(\sum_{j=1}^{l} p_j\theta_{k_j})\right)| \\
&\leqslant |\mathbb{E}\left(\psi e(\sum_{j=1}^{l} p_j\theta_{k_j})\mathbf{1}_{K_{\epsilon_1}^c}\right)| + |\mathbb{E}\left(\psi e(\sum_{j=1}^{l} p_j\theta_{k_j})\mathbf{1}_{K_{\epsilon_1}}\right)| \\
&\leqslant \|\psi\|_\infty \epsilon_1 + O(\epsilon_2) + O(\frac{1}{at\epsilon_2}) + \frac{1}{a}O(\epsilon_2). \qquad (3.93)
\end{aligned}$$

By first choosing $\epsilon_1 > 0$ small enough (note that a depends on ϵ_1), then choosing $\epsilon_2 > 0$ and finally choosing t large enough, we obtain the wanted result. □

The study is now reduced to the study of the convergence, when $A \to \infty$, of a sum of the type

$$\tilde{S}_A(\omega, L) = \sum_{k=(k_1,k_2)\in\Pi_A(L)} \frac{Z_k(\omega)}{\|k_1e_1(L) + k_2e_2(L)\|^{\frac{3}{2}}}$$

where Z_k are real non-zero independent and identically distributed random variables from $\Omega \ni \omega$ that are symmetrical and have a compact support. In the next section, we are going to study the sums of this type and prove Proposition 26, which will conclude the proof of Theorem 8.

3.5 Asymptotic study of $\tilde{S}_A(\omega, L)$

The goal of this section is to prove Proposition 26. To prove this proposition, we first need some lemmas.
The following lemma basically gives us that the magnitude of $\lim_{A\to\infty} \tilde{S}_A(\omega, L)$ is given by $\frac{X_{1,0}}{\|L\|_1}$.

Lemma 51. *Let us set, for* $0 < \epsilon < 1 < A$,

$$B_{k,1,A}(\omega, L, \epsilon) = \frac{Z_k(\omega)}{\|k_1e_1(L) + k_2e_2(L)\|^{\frac{3}{2}}}\mathbf{1}_{A\geqslant\|k_1e_1(L)+k_2e_2(L)\|\geqslant\epsilon}, \qquad (3.94)$$

$$B_{k,2}(\omega, L, \epsilon) = \frac{Z_k(\omega)}{\|k_1e_1(L) + k_2e_2(L)\|^{\frac{3}{2}}}\mathbf{1}_{\|k_1e_1(L)+k_2e_2(L)\|<\epsilon}, \qquad (3.95)$$

$$\tilde{S}_{\epsilon,1,A}(\omega, L) = \sum_{k\in\Pi} H_{k,1,A}(\omega, L, \epsilon) \text{ and} \qquad (3.96)$$

$$\tilde{S}_{\epsilon,2}(\omega, L) = \sum_{k\in\Pi} H_{k,2}(\omega, L, \epsilon) \qquad (3.97)$$

so that

$$\tilde{S}_A(\omega, L) = \tilde{S}_{\epsilon,1,A}(\omega, L) + \tilde{S}_{\epsilon,2}(\omega, L).$$

Then there exists $(M_s)_{s\in\mathbb{N}-\{0,1\}}$, a sequence of positive real numbers (that does not depend on A), such that for all $s \geqslant 2$:

$$\mathbb{E}(\sum_{k\in\Pi} |H_{k,1}|^s(\omega, L, A, \epsilon)) \leqslant M_s. \tag{3.98}$$

Furthermore, one has :

$$\tilde{S}_{\epsilon,2}(\omega, L) = \frac{Z_{1,0}}{\|L\|_1^{\frac{3}{2}}} \mathbf{1}_{\|L\|_1<\epsilon} \tag{3.99}$$

when $\epsilon > 0$ is chosen small enough.

To prove this lemma, we need to recall the second theorem of Minkowski.

Theorem 9 ([65])**.** *There exist $K_1, K_2 > 0$ such that for all $L \in \mathscr{S}_2$,*

$$K_1 \geqslant \|L\|_1 \|L\|_2 \geqslant K_2.$$

We can now prove Lemma 51.

Proof of Lemma 51. The first fact is a direct consequence of Lemma 41 and of the fact that the Z_k are of compact support.
Indeed, first let us say that it is enough to deal with the case $\tilde{\mu}_2 = \mu_2$. With the notations of Lemma 41, let us set

$$f(x) = \frac{1}{\|x\|^3} \mathbf{1}_{\epsilon\leqslant\|x\|\leqslant A} \tag{3.100}$$

and

$$\mathcal{S}(f)(L) = \sum_{l \text{ prime } \in L} f(l). \tag{3.101}$$

From the facts that the X_k are independent, identically distributed, symmetrical with a compact support, one gets that :

$$\mathbb{E}(\sum_{k\in\Pi} (B_{k,1,A})^2(\omega, L, \epsilon)) = \mathbb{E}(Z_{1,0}^2)\mathbb{E}(F). \tag{3.102}$$

Lemma 41 gives then that :

$$\mathbb{E}(\sum_{k\in\Pi} (B_{k,1,A})^2(\omega, L, \epsilon)) = \mathbb{E}(Z_{1,0}^2) \int_{\mathbb{R}^2} f(x)dx. \tag{3.103}$$

By passing into polar coordinates, one gets finally that :

$$\mathbb{E}(\sum_{k\in\Pi} (B_{k,1,A})^2(\omega, L, \epsilon)) = 2\pi\mathbb{E}(Z_{1,0}^2)(\frac{1}{\epsilon} - \frac{1}{A}). \tag{3.104}$$

We do the same for the other $k \geqslant 3$.
Concerning the second fact, if there exists $(k_1, k_2) \in \Pi$ such that $\|k_1e_1(L)+k_2e_2(L)\| < \epsilon$ then $\|L\|_1 < \epsilon$ and the reverse is also true.
In this case, according to Theorem 9, one has

$$\|k_1e_1(L) + k_2e_2(L)\| \geqslant \|L\|_2 > \frac{K_2}{\epsilon} \tag{3.105}$$

where $(k_1, k_2) \in \Pi - \{(1,0)\}$.
Thus, by choosing $\epsilon > 0$ so that $\frac{K_2}{\epsilon} > \epsilon$, we get that :

$$\tilde{S}_{\epsilon,2}(\omega, L) = \frac{Z_{1,0}}{\|L\|_1^{\frac{3}{2}}} \tag{3.106}$$

when $\|L\|_1 < \epsilon$.
Otherwise, $\tilde{S}_{\epsilon,2}(\omega, L) = 0$ and $\frac{Z_{1,0}}{\|L\|^{\frac{3}{2}}} \mathbf{1}_{\|L\|_1 < \epsilon} = 0$. And so the second result is also true. □

The following lemma precise Lemma 44 and it gives us a better understanding of the distribution of $\frac{X_{1,0}}{\|L\|_1^{\frac{3}{2}}}$. The proof of it is basically the same as the proof of Lemma 44.

Lemma 52. *When $\sigma(L) \geqslant m$ where $m > 0$ and where L belongs to an event of the form $(\|L\| < \alpha)$ with $\alpha > 0$, there exists $C > 0$ such that*

$$\mathbb{P}(\|L\|_1^{-\frac{3}{2}} > \beta) \geqslant \frac{C}{\beta^{\frac{4}{3}}}$$

and this inequality is true for every β large enough.

Proof. The proof is a direct application of Lemma 41. □

We recall a classic tool from probability theory :

Lemma 53. *For X a real random variable, one has for every $k \geqslant 1$*

$$\mathbb{E}(|X|^k) = \int_0^\infty t^{k-1} P(|X| > t) dt.$$

Thus, by using the fact that $Z_{1,0}$ is bounded and is different from 0 and by using Lemma 52 and Lemma 53, we get the following fact about the behaviour $\frac{Z_{1,0}}{\|L\|_1^{\frac{3}{2}}}$:

Lemma 54. *For every $0 \leqslant \kappa < \frac{1}{3}$, $\frac{Z_{1,0}(\omega)}{\|L\|_1^{\frac{3}{2}}}$ admits a moment of order $1 + \kappa$.*
Furthermore, $\frac{Z_{1,0}(\omega)}{\|L\|_1^{\frac{3}{2}}}$ does not admit a moment of order $\frac{4}{3}$ when $\sigma(L) \geqslant m$ where $m > 0$ and where L belongs to an event of the form $(\|L\| < \alpha)$ with $\alpha > 0$.

We finally remind the reader of the following obvious lemma :

Lemma 55. *If X is a real random variable integrable then if it is symmetrical, its expectation is equal to zero.*

We can now prove Proposition 26.

Proof of Proposition 26. Because of Lemma 55, we only need first to prove that $\tilde{S}_A(\omega, L)$ converges almost surely when $A \to \infty$ and that the limit random variable is symmetrical and admits moment of order $1+\alpha$ and that for all $0 \leqslant \alpha < \frac{1}{3}$: the fact that the expectation of the limit is null will be a consequence of Lemma 55.

• Let us prove that $\tilde{S}_A(\omega, L)$ converges almost surely when $A \to \infty$. It is enough to prove that it is the case on all the events $\{\|L\|_1 \geqslant \epsilon\}$ where $\epsilon > 0$. So, let us give $\epsilon > 0$.
On such an event, one has :

$$\tilde{S}_A(\omega, L) = \tilde{S}_{\epsilon,1,A}(\omega, L). \tag{3.107}$$

Yet, one has, because the X_k are symmetrical and integrable,

$$\mathbb{E}(B_{k,1,A}(\omega, L, \epsilon)) = 0. \tag{3.108}$$

By using Equation (3.98) (see Lemma 51) and Equation (3.108) and because the Z_k are independent between them and their expectations are equal to zero, one gets that :

$$\tilde{S}_{\epsilon,1,A}(\omega, L) \textit{ converges almost surely when } A \to \infty.$$

Because of Equation (3.107), one gets that $\tilde{S}_A(\omega, L)$ converges almost surely when $A \to \infty$ on all the events $\{\|L\|_1 \geqslant \epsilon\}$ where $\epsilon > 0$.

• The fact that $\lim_{A\to\infty} \tilde{S}_A(\omega, L)$ is symmetrical follows directly from the fact that the Z_k are symmetrical.

• It only remains to prove that $\lim_{A\to\infty} \tilde{S}_A(\omega, L)$ admits a moment of order $1+\kappa$ and that for all $0 \leqslant \kappa < \frac{1}{3}$; that $\lim_{A\to\infty} \tilde{S}_A(\omega, L)$ does not admit a moment of order $\frac{4}{3}$ when $\sigma(L) \geqslant m$ where $m > 0$ and where L belongs to an event of the form $(\|L\| < \alpha)$ with $\alpha > 0$. ; and that if $\tilde{\mu}_2$ is such that there exists $\alpha > 0$ such that $\tilde{\mu}_2(\{L \in \mathscr{S}_2 \mid \|L\|_1 < \alpha\}) = 0$ then the limit $\lim_{A\to\infty} \tilde{S}_A(\omega, L)$ admits moments for all finite orders $1 \leqslant p < \infty$. These facts are a direct consequence of Lemma 51, of Lemma 54 and of Fatou's lemma. □

Chapter 4

The case of analytic and stricly convex sets

(Résumé en français)Nous étudions l'erreur du nombre de points d'un réseau unimodulaire qui tombent dans un ensemble strictement convexe et analytique possédant l'origine et qui est dilaté d'un facteur t. Le but est de généraliser le résultat de [89]. On montre d'abord que l'étude de l'erreur, lorsqu'elle est normalisée par $\sqrt{t}$, lorsque ce paramètre tend vers l'infini et lorsque le réseau considéré est aléatoire, se ramène à l'étude d'une transformée de Siegel $\mathcal{S}(f_t)(L)$ qui dépend de t. Ensuite, on se ramène à l'étude du comportement asymptotic d'une transformée de Siegel avec poids aléatoires, $\mathcal{S}(F)(\theta, L)$ où θ est un second paramètre aléatoire. Puis, on montre que cette dernière quantité converge presque sûrement et on étudie l'existence des moments de sa loi. Enfin, on montre que ce résultat est encore valable si l'on translate, après dilatation, l'ensemble strictement convexe d'un vecteur $\alpha \in \mathbb{R}^2$ fixé.

(English abstract)We study the error of the number of points of a unimodular lattice that fall in a strictly convex and analytic set having the origin and that is dilated by a factor t. The aim is to generalize the result of [89]. We first show that the study of the error, when it is normalized by $\sqrt{t}$, when this parameter tends to infinity and when the considered lattice is random, is reduced to the study of a Siegel transform $\mathcal{S}(f_t)(L)$ which depends on t. Then, we come back to the study of the asymptotic behaviour of a Siegel transform with random weights, $\mathcal{S}(F)(\theta, L)$ where θ is a second random parameter. Then, we show that this last quantity converges almost surely and we study the existence of moments of its distribution. Finally, we show that this result is still valid if we translate, after dilation, the strictly convex set of a fixed vector $\alpha \in \mathbb{R}^2$. study the existence of the moments of its distribution.

Contents

4.1 Introduction **116**

4.2 Heuristic explanation and plan of the proof **120**

4.3 Reduction to the study of the Siegel transform **122**

4.3.1 Proof of Lemma 56 123
4.3.2 Proof of Lemma 57 131
4.4 Study of $S_{A,prime}(L,t)$ when $t \to \infty$ 133
4.4.1 Reductions for the study of $S_{A,prime}(L,t)$ and proof of Theorem 10 133
4.4.2 Foliation and local estimates 136
4.4.3 Proof of Proposition 35 138
4.4.4 Proof of Proposition 34 142
4.5 Asymptotic study of $\tilde{S}_A(\omega, L)$ and of $\tilde{S}_A(\omega_1, \omega_2, L)$ 144
4.6 Elements of proof of Theorem 12 145

4.1 Introduction

The lattice point problem is an open problem in the Geometry of Numbers, at least since Carl Friedrich Gauss took interest in which became the famous Gauss circle problem. The general problem states as followed.
Let d be an integer greater than 1. We recall the following definition :

Definition 17. *A subset L of $\mathbb{R}^d$ is a lattice if it is a subgroup of $\mathbb{R}^d$ such that L is discrete and span$(L) = \mathbb{R}^d$.*

Let P be a measurable subset of $\mathbb{R}^d$ of non-zero finite Lebesgue measure. We want to evaluate the following cardinal number when $t \to \infty$:

$$N(tP + X, L) = |(tP + X) \cap L|$$

where $X \in \mathbb{R}^d$, L is a lattice of $\mathbb{R}^d$ and $tP + X$ denotes the set P dilated by a factor t relatively to 0 and then translated by the vector X.
Under mild regularity conditions on the set P, one can show that :

$$N(tP + X, L) = t^d \frac{\mathrm{Vol}(P)}{\mathrm{Covol}(L)} + o(t^d)$$

where $o(f(t))$ denotes a quantity such that, when divided by $f(t)$, it goes to 0 when $t \to \infty$ and where Covol(L) is defined in the following definition :

Definition 18. *The covolume of a lattice L of $\mathbb{R}^d$, covol(L), is the Lebesgue measure of a measurable fundamental set of L. Furthermore, a lattice is said to be unimodular if its covolume is equal to 1.*
When $d = 2$, instead of using the term covolume, we use the term coarea.

We are interested in the error term

$$\mathcal{R}(tP + X, L) = N(tP + X, L) - t^d \frac{\mathrm{Vol}(P)}{\mathrm{Covol}(L)}.$$

In the case where $d = 2$ and where P is the unit disk $\mathbb{D}^2$, Hardy's conjecture in [36] stipulates that we should have for all $\epsilon > 0$,

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^{\frac{1}{2}+\epsilon}).$$

One of the result in this direction has been established by Iwaniec and Mozzochi in [43]. They have proven that for all $\epsilon > 0$,

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^{\frac{7}{11}+\epsilon}).$$

This result has been recently improved by Huxley in [42]. Indeed, he has proven that :

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^K \log(t)^\Lambda)$$

where $K = \frac{131}{208}$ and $\Lambda = \frac{18627}{8320}$.
In dimension 3, Heath-Brown has proven in [40] that :

$$\mathcal{R}(t\mathbb{D}^3, \mathbb{Z}^3) = O(t^{\frac{21}{16}+\epsilon}).$$

These last two results are all based on estimating what are called *exponential sums*.
Another approach was followed first by Heath-Brown in [38] and then by Bleher, Cheng, Dyson and Lebowitz in [11]. They took interest in the case where the dilatation parameter t is random. More precisely, they assumed that t was being distributed according to the measure $\rho(\frac{t}{T})dt$ (that is absolutely continuous relatively to Lebesgue measure) and where ρ is a probability density on $[0, 1]$ and T is parameter that goes to infinity. In that case, Bleher, Cheng, Dyson and Lebowitz showed the following result (which generalizes the result of Heath-Brown) :

Theorem ([11])**.** *Let $\alpha \in [0, 1[^2$. There exists a probability density p_α on $\mathbb{R}$ such that for every piecewise continuous and bounded function $g : \mathbb{R} \longrightarrow \mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g\Big(\frac{\mathcal{R}(t\mathbb{D}^2 + \alpha, \mathbb{Z}^2}{\sqrt{t}}\Big)\rho(\frac{t}{T})dt = \int_{\mathbb{R}} g(x)p_\alpha(x)dx.$$

Furthermore p_α can be extended as an analytic function over $\mathbb{C}$ and satisfies that for every $\epsilon > 0$,

$$p_\alpha(x) = O(e^{-|x|^{4-\epsilon}})$$

when $x \in \mathbb{R}$ and when $|x| \to \infty$.

In our case, we keep t deterministic as in the original Gauss problem but we let the lattice L be a random unimodular lattice and we study $\mathcal{R}$. This approach was first initiated by Kesten in [48] and in [49]. It should be noted that several counting problems have followed this approach : we can cite, for example, [9], [27], [64], [3], [60] and [93].
We denote by $\mathscr{S}_2$ the space of unimodular lattices and it can be seen as the quotient space $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$. We denote by μ_2 the unique Haar probability measure on it. Let us set :

$$\Pi = \{(k_1, k_2) \in \mathbb{Z}^2 \mid k_1 \wedge k_2 = 1,\ k_1 \geqslant 0\} \tag{4.1}$$

where we agree that if $k_1 = 0$, $k_1 \wedge k_2 = 1$ means that $k_2 = 1$.
We denote by $\|\cdot\|$ the usual euclidean norm over $\mathbb{R}^2$.
We need to define some additional objects.

Definition 19. *For every $i \in \{1,2\}$, we call*

$$\|L\|_i = \min\{r > 0 \mid B_f(0,r) \text{ contains } i \text{ vectors of } L \text{ lineary independent}\}$$

where $B_f(0,r)$ is the closed centred ball on 0 for the norm $\|\cdot\|$ of radius r. These two quantities are the successive minima of the lattice L.
In fact, for almost all $L \in \mathscr{S}_2$, $\|L\|_2 > \|L\|_1$ and there exists only one couple of vectors $(e_1(L), e_2(L))$ such that $(e_1(L))_1 > 0$, $\|e_1(L)\| = \|L\|_1$, $(e_2(L))_1 > 0$ and $\|e_2(L)\| = \|L\|_2$.
In the rest of the article, for the sake of simplicity, for L a lattice, we will use the notation $\|L\|$ instead of $\|L\|_1$.
For a lattice $L \in \mathscr{S}_2$, we also say that a vector of L is prime if it is not a non-trivial integer multiple of another vector of L.
In fact, for every $M \in SL_2(\mathbb{R})$, a vector $l \in L$ is prime if, and only if, $Ml \in ML$ is prime and a vector $(k_1, k_2) \in \mathbb{Z}^2$ is prime if, and only if, $k_1 \wedge k_2 = 1$. With these notations, one has that, for a generic lattice $L \in \mathscr{S}_2$, $e \in L$ is prime if, and only if, e can be written as $e = k_1 e_1(L) + k_2 e_2(L)$ with $k_1 \wedge k_2 = 1$.
Finally, for a generic lattice $L \in \mathscr{S}_2$, we call $\mathcal{P}_+(L)$ the set of vectors e of L such that $e = k_1 e_1(L) + k_2 e_2(L)$ with $(k_1, k_2) \in \Pi$. All the vectors of $\mathcal{P}_+(L)$ are prime vectors according to the previous remark.

We recall also the fact that we say that a real random variable Z is symmetrical if $\mathbb{P}_Z = \mathbb{P}_{-Z}$ where, for every random variable X, $\mathbb{P}_X$ stands for the distribution of the random variable X.
Let $\tilde{\mu}_2$ be a probability measure that has a smooth bounded density σ with respect to μ_2. There are two different cases that are addressed in our main result, which is Theorem 10. The first one is when $\tilde{\mu}_2$ is compactly supported, *id est* when there exists $\alpha > 0$ such that :

$$\tilde{\mu}_2\Big(\{L \in \mathscr{S}_2 \mid \|L\|_1 < \alpha\}\Big) = 0. \tag{4.2}$$

The second one is when $\tilde{\mu}_2$ is non-compactly supported under the following condition : there exists $m > 0$, there exists $\alpha > 0$ such that for all L that belongs to the event $(\|L\| < \alpha)$,

$$\sigma(L) \geqslant m. \tag{4.3}$$

An example of such a measure $\tilde{\mu}_2$ is given by the normalized Haar measure μ_2.
Let $\mathcal{E}$ be an ellipse centred around 0. Let us call M a matrix that transforms $\mathcal{E}$ into a disk and that belongs to $SL_2(\mathbb{R})$. M is unique modulo the natural action of $SL_2(\mathbb{Z})$.
Let us set : $\mathbb{T}^\infty = (\mathbb{T}^1)^\Pi$ where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and let us call λ_∞ the normalized Lebesgue measure product over $\mathbb{T}^\infty$.
The main result of the previous article [89] is the following theorem :

Theorem 10. *For every real numbers $a < b$,*

$$\lim_{t\to\infty} \tilde{\mu}_2\left(L \in \mathscr{S}_2 \mid \frac{\mathcal{R}(t\mathcal{E}, L)}{\sqrt{t}} \in [a,b]\right) = (\lambda_\infty \times (M^{-1})_* \tilde{\mu}_2)\Big((\theta, L) \in \mathbb{T}^\infty \times \mathscr{S}_2 \mid S(\theta, L^\perp) \in [a,b]\Big)$$

where $(M^{-1})_*\tilde{\mu}_2$ is the push-forward of $\tilde{\mu}_2$ by M^{-1} and where $\theta = (\theta_e) \in \mathbb{T}^\infty$,

$$S(\theta, L) = \frac{2}{\pi} \sum_{e \in \mathcal{P}_+(L)} \frac{\phi(\theta_e)}{\|e\|^{\frac{3}{2}}}$$

with

$$\phi(\theta_e) = \sum_{m\geqslant 1} \frac{\cos(2\pi m\theta_e - \frac{3\pi}{4})}{m^{\frac{3}{2}}}. \tag{4.4}$$

Furthermore, $S(\theta, L)$ (and $S(\theta, L^{\perp})$:

- *converges almost surely*
- *is symmetrical and its expectation is equal to* 0
- *admits a moment of order $1+\kappa$ for any $0 \leqslant \kappa < \frac{1}{3}$*
- *$S(\theta, L)$ admits moments of all order $1 \leqslant p < \infty$ when $\tilde{\mu}_2$ is compactly supported*
- *does not admit a moment of order $\frac{4}{3}$ when $\tilde{\mu}_2$ is non-compactly supported under the condition (*4.3*).*

In this article, we want to extend this last result. More precisely, let us suppose that $\mathcal{E} = \Omega_\gamma$ where γ is analytic curve that is simple, closed and strictly convex and where $0 \in \Omega_\gamma$. Let us also call $x_\gamma(\xi)$ the point on γ where the outer normal to γ coincides with $\frac{\gamma}{\|\gamma\|}$ and $\rho_\gamma(\xi)$ the curvature radius of γ at $x_\gamma(\xi)$. Let us set finally $\mathbb{T}^{\infty,2} = (\mathbb{T}^1)^{\Pi} \times (\mathbb{T}^1)^{\Pi}$ and let us call $\lambda_{\infty,2}$ the normalized Lebesgue measure product over $\mathbb{T}^{\infty,2}$.
Then, we want to prove the following theorem :

Theorem 11. *There exists a distribution function $\mathcal{D}_\gamma(z)$ such that for every real z we have :*

$$\lim_{t\to\infty} \tilde{\mu}_2 \left(\frac{\mathcal{R}(t\Omega_\gamma, L)}{\sqrt{t}} \in]-\infty, z] \right) = \mathcal{D}_\gamma(z).$$

In the case where Ω_γ is symmetric, $\mathcal{D}_\gamma(z)$ is the distribution function of

$$S_\gamma(\theta, L) = \frac{2}{\pi} \sum_{e\in\mathcal{P}_+(L)} \frac{\rho_\gamma(e)\phi(\theta_e)}{\|e\|^{\frac{3}{2}}}$$

with $\theta = (\theta_e) \in \mathbb{T}^\infty$ being distributed according to λ_∞ and L being distributed according to $\tilde{\mu}_2$.
In the non symmetric case, $\mathcal{D}_\gamma(z)$ is the distribution function of

$$S_\gamma(\theta, L) = \frac{1}{\pi} \sum_{e\in\mathcal{P}_+(L)} \frac{\phi_{\gamma,2}(\theta_e, e)}{\|e\|^{\frac{3}{2}}}$$

where $\theta = (\theta_{1,e}, \theta_{2,e}) \in \mathbb{T}^{\infty,2}$ being distributed according to $\lambda_{\infty,2}$ and L being distributed according to $\tilde{\mu}_2$ and with

$$\phi_{\gamma,2}(\theta_e, e) = \sum_{m\geqslant 1} \frac{\rho_\gamma(e)\cos(2\pi m\theta_{1,e} - \frac{3\pi}{4}) + \rho_\gamma(-e)\cos(2\pi m\theta_{2,e} - \frac{3\pi}{4})}{m^{\frac{3}{2}}}. \tag{4.5}$$

Furthermore, in both cases, S_γ admits the same properties of $S(\theta, L)$ listed in Theorem 10.

In fact, this theorem can be generalized as followed :

Theorem 12. *For every $\alpha \in \mathbb{R}^2$, there exists a distribution function $\mathcal{D}_{\gamma,\alpha}(z)$ such that for every real z we have :*

$$\lim_{t\to\infty} \tilde{\mu}_2 \left(\frac{\mathcal{R}(t\Omega_\gamma + \alpha, L)}{\sqrt{t}} \in]-\infty, z] \right) = \mathcal{D}_{\gamma,\alpha}(z).$$

In the case where Ω_γ is symmetric, $\mathcal{D}_\gamma(z)$ is the distribution function of

$$S_\gamma(\theta, L) = \frac{2}{\pi} \sum_{e\in\mathcal{P}_+(L)} \frac{\rho_\gamma(e)\phi_\alpha(\theta_e, e)}{\|e\|^{\frac{3}{2}}}$$

with $\theta = (\theta_e) \in \mathbb{T}^\infty$ being distributed according to λ_∞ and L being distributed according to $\tilde{\mu}_2$ and where

$$\phi_\alpha(\theta_e, e) = \sum_{m\geqslant 1} \frac{\cos(2\pi m\theta_e + 2\pi m < \alpha, e > -\frac{3\pi}{4})}{m^{\frac{3}{2}}}.$$

In the non symmetric case, $\mathcal{D}_{\gamma,\alpha}(z)$ is the distribution function of

$$S_\gamma(\theta, L) = \frac{1}{\pi} \sum_{e\in\mathcal{P}_+(L)} \frac{\phi_{\alpha,\gamma,2}(\theta_e, e)}{\|e\|^{\frac{3}{2}}}$$

where $\theta = (\theta_{1,e}, \theta_{2,e}) \in \mathbb{T}^{\infty,2}$ being distributed according to $\lambda_{\infty,2}$ and L being distributed according to $\tilde{\mu}_2$ and with

$$\phi_{\alpha,\gamma,2}(\theta_e, e) = \sum_{m\geqslant 1} \frac{1}{m^{\frac{3}{2}}}$$
$$\Big(\rho_\gamma(e)\cos(2\pi m\theta_{1,e} + 2\pi m < \alpha, e > -\frac{3\pi}{4}) + \rho_\gamma(-e)\cos(2\pi m\theta_{2,e} - 2\pi m < \alpha, e > -\frac{3\pi}{4})\Big).$$

Furthermore, in both cases, S_γ admits the same properties of $S(\theta, L)$ listed in Theorem 10.

This last theorem is a generalization of Theorem 10 on two planes. The first one is the shape of the sets : it treats the more general case of analytic curves, not only the case of the ellipses centred on 0. The second one is the presence of a translation parameter α whereas Theorem 10 can be deduced by assuming that $\alpha = 0$ (no translation of $t\mathcal{E}$). In the next section, we give a brief heuristic explanation of the approach that we will follow. It is basically the same as in [89]. At the end of the section, we will give the plan of the paper.

4.2 Heuristic explanation and plan of the proof

First, let us explain the different steps of the proof of Theorem 11, Theorem 12 being a simple generalization of Theorem 11.
First step. By regularizing the problem and using the Poisson summation formula, we are going to show that the quantity $S_{A,prime}(L, t)$, for $A > 0$ a fixed parameter that is

taken large enough, when $t \to \infty$, is close, in probability, to $\frac{\mathcal{R}(t\Omega_\gamma, L)}{\sqrt{t}}$ where $S_{A,prime}(L,t)$ is defined by :

$$S_{A,prime}(L,t) = \frac{1}{\pi} \sum_{\substack{l \in L^\perp \ prime \\ 0 < \|l\| \leqslant A}} \frac{1}{\|l\|^{\frac{3}{2}}} \tag{4.6}$$

$$\sum_{m \in \mathbb{N} - \{0\}} \frac{\rho_\gamma(l)\cos(2\pi t m Y_\gamma(l) - \frac{3\pi}{4}) + \rho_\gamma(-l)\cos(2\pi t m Y_\gamma(-l) - \frac{3\pi}{4})}{m^{\frac{3}{2}}}$$

where $Y_\gamma(\xi)$, for all $\xi \in \mathbb{R}^2 - \{0\}$, is defined by

$$Y_\gamma(\xi) = < \xi, x_\gamma(\xi) >$$

(it is a positive homogeneous function of order 1).
Let us say a few remarks. First, the vectors l prime considered are such that $\|l\| \leqslant A$. We have to limit the norm of the considered vectors because of a convergence problem. Second, we take into account a phenomenon of multiplicity (if l appears in the sum, $2l$ is also going to appear). In [11] such a phenomenon was also taken into account. This was done in order to get independence at infinity, as it was also done in [26], and we do that for the same goal.
By using the remark that is in Definition 19 and by replacing L by $L^\perp$ (which is done only for a matter of convenience), one has that :

$$S_{A,prime}(L^\perp,t) = \frac{1}{\pi} \sum_{\substack{k_1 \wedge k_2 = 1 \\ k_1 \geqslant 0 \\ \|k_1 e_1(L) + k_2 e_2(L)\| \leqslant A}} \frac{1}{\|k_1 e_1(L) + k_2 e_2(L)\|^{\frac{3}{2}}} \tag{4.7}$$

$$\phi_{\gamma,2}\left((tY_\gamma(k_1 e_1(L) + k_2 e_2(L)), tY_\gamma(-(k_1 e_1(L) + k_2 e_2(L)))), k_1 e_1(L) + k_2 e_2(L)\right)$$

where the function $\phi_{\gamma,2}$ was defined by the Equation (4.5). We have done that so from this stage onwards we consider vectors of $\mathcal{P}_+(L)$ with a fixed indexation (that does not depend on L). Furthermore, this indexation will be very useful for the second step (for more details, see Section 4).

Second step. In the non symmetric case, we will show that the family of variables $\Big(tY_\gamma(k_1 e_1(L) + k_2 e_2(L)), tY_\gamma(-(k_1 e_1(L) + k_2 e_2(L))\Big)$, whose values are in $(\mathbb{R}/\mathbb{Z})^2$, become, when $t \to \infty$, independent from one another and indeed converge towards independent and identically distributed random variables whose common distribution is given by the normalized Haar measure on $(\mathbb{R}/\mathbb{Z})^2$. The idea here is basically the same as in [11] and in [38] where the respective authors used the fact that the square roots of square free integers are $\mathbb{Z}$-free. It is a generalization of what was done in [89]. In our case, to prove the result, we will decompose the space of unimodular lattices into small geodesic segments, calculate the Taylor series of $Y_\gamma(k_1 e_1(L) + k_2 e_2(L))$ and of $Y_\gamma(-(k_1 e_1(L) + k_2 e_2(L)))$ at order 1 on such a segment and show that the coefficients of order 1 are $\mathbb{Z}$-free.
We will also prove that these variables become independent, when $t \to \infty$, from the variable L due to the presence of the factor t.
In the symmetric case, instead of considering $\Big(tY_\gamma(k_1 e_1(L) + k_2 e_2(L)), tY_\gamma(-(k_1 e_1(L) + k_2 e_2(L))\Big)$, we consider the family of variables $\Big(tY_\gamma(k_1 e_1(L) + k_2 e_2(L))\Big)$, whose values are in $(\mathbb{R}/\mathbb{Z})$, and show the same results.

Third step. Thanks to the first and second step, we will see that the asymptotic distribution of $\frac{\mathcal{R}(t\Omega_\gamma,L)}{\sqrt{t}}$ is the distribution of $S_\gamma(\theta,L)$ (see Theorem 11) under the assumption that the quantity $S(\theta,L)$ is well-defined. This last fact will be quasi immediate because what was done in the last section of [89] can be generalized directly in our case. Furthermore, all the listed properties of $S_\gamma(\theta,L)$ are also going to be obtained immediately. After doing all of that, we will finally get the validity of Theorem 11.

Plan of the paper. The next section will be dedicated to deal with the first step of the proof, namely it will show that $\frac{\mathcal{R}(t\Omega_\gamma,L)}{\sqrt{t}}$ is close in probability with $S_{A,prime}(L^\perp,t)$ when A is a fixed parameter taken large enough and t goes to infinity (see Proposition 30). We have to "cut" the sum because of the problem of convergence of the Fourier series of $X \longmapsto \frac{\mathcal{R}(t\Omega_\gamma+X,L)}{\sqrt{t}}$ which is due to the lack of regularity of the indicator function $\mathbf{1}_{t\Omega_\gamma}$. To prove this, we are going to proceed by *regularization* which means here that we are going to smooth the indicator function $\mathbf{1}_{t\Omega_\gamma}$ via a Gaussian kernel.
In Section 4, we tackle the second step of the proof, that is the fact, in the non symmetric case, that the $(tY_\gamma(k_1e_1(L)+k_2e_2(L), tY_\gamma(-(k_1e_1(L)+k_2e_2(L)))$ become independent when $t\to\infty$. We also show that they converge towards random variables that are identically distributed according to the normalized Haar measure over $(\mathbb{R}/\mathbb{Z})^2$ and that they become independent, when $t\to\infty$, from L and so that $\frac{\mathcal{R}(t\Omega_\gamma,L)}{\sqrt{t}}$ has the same distribution of $S_\gamma(\theta,L)$. We also deal with the symmetric case, which is simpler.
In Section 5 we are going to tackle the third step of the proof, namely study the convergence of $\tilde{S}_A(\omega,L)$ when $A\to\infty$ and the existence of moments of its limit.
In Section 6, which is the last section, we give the approach, based on the approach to prove Theorem 11, to prove Theorem 12.
In the rest of the article, all the calculus of expectation $\mathbb{E}$, of variance $\mathbb{V}$ and of probability $\mathbb{P}$ will be made according to the measure $\tilde{\mu}_2$. Furthermore, the expression typical is going to signify $\tilde{\mu}_2-$ *almost surely.* In fact, like we have said in Section 1, we are going to suppose that $\tilde{\mu}_2=\mu_2$ in Section 3 and in Section 4 because all the results extend to the general case.

4.3 Reduction to the study of the Siegel transform

The main object of this section is to show the following proposition :

Proposition 30. *For every $\alpha>0$, for every $A>0$ large enough, for every t large enough, one has that :*

$$\mathbb{P}(\Delta_{A,prime}(L,t)\geqslant\alpha)\leqslant\alpha$$

where

$$\Delta_{A,prime}(L,t)=|\frac{\mathcal{R}(t\Omega_\gamma,L)}{\sqrt{t}}-S_{A,prime}(L,t)|\ . \tag{4.8}$$

This proposition basically says that we can reduce the asymptotical study of $\frac{\mathcal{R}(t\Omega_\gamma,L)}{\sqrt{t}}$ to the study of its Fourier transform, taking into account a phenomenon of multiplicity. In fact, due to the triangle inequality, we only have to prove the following two lemmas :

Lemma 56. *For every $\alpha > 0$, for every $A > 0$ large enough, for every t large enough, one has that :*

$$\mathbb{P}(\Delta_A(L,t) \geqslant \alpha) \leqslant \alpha$$

where

$$\Delta_A(L,t) = |\frac{\mathcal{R}(t\Omega_\gamma, L)}{\sqrt{t}} - H_A(L,t)| \text{ with} \tag{4.9}$$

$$H_A(L,t) = \frac{1}{2\pi} \sum_{\substack{l \in L^\perp \\ 0 < \|l\| \leqslant A}} \frac{1}{\|l\|^{\frac{3}{2}}} \left(\rho_\gamma(l) \cos(2\pi t Y_\gamma(l) - \frac{3\pi}{4}) + \rho_\gamma(-l) \cos(2\pi t Y_\gamma(-l) - \frac{3\pi}{4}) \right) \tag{4.10}$$

which becomes when Ω_γ is symmetric

$$H_A(L,t) = \frac{1}{\pi} \sum_{\substack{l \in L^\perp \\ 0 < \|l\| \leqslant A}} \frac{\rho_\gamma(l) \cos(2\pi t Y_\gamma(l) - \frac{3\pi}{4}))}{\|l\|^{\frac{3}{2}}}. \tag{4.11}$$

Lemma 57. *For every $\alpha > 0$, for every $A > 0$ large enough, for every t large enough, one has that :*

$$\mathbb{P}(|S_{A,prime}(L,t) - H_A(L,t)| \geqslant \alpha) \leqslant \alpha.$$

Proof of Proposition 30. One has that :

$$\Delta_{A,prime}(L,t) \leqslant \Delta_A(L,t) + |S_{A,prime}(L,t) - H_A(L,t)|. \tag{4.12}$$

The Lemma 56 and the Lemma 57 imply then the wanted result. □

Let us say a few words about Lemmas 56 and 57 before following with their respective proofs. The Lemma 56 says that the study of $\frac{\mathcal{R}(t\Omega_\gamma, L)}{\sqrt{t}}$ can be reduced to the study of its Fourier transform. The Lemma 57 says that the phenomenon of multiplicity (the fact that for a prime vector l, $2l$, $3l$ etc. appear in the sum $H_A(L,t)$ when $A \to \infty$) is not so important. We only have to gather all the multiples of a prime vector (which corresponds to the infinite sum over m, see equation (4.6)), so that we focus on prime vectors.

4.3.1 Proof of Lemma 56

First, we are going to prove the Lemma 56. To do so, we are following closely the approach of [10], yet with some differences because in our case it is not the radius of dilatation that is random but the lattice (or, equivalently and in a certain sense, the oval).
For $x \in \mathbb{R}^2$ and $t > 0$, let us define

$$\lambda(x;t) = \frac{t^2}{4\pi} e^{-\frac{t^2}{4\pi}\|x\|^2}$$

and, for $M \in SL_2(\mathbb{R})$,

$$\lambda_M(x;t) = \lambda(Mx;t). \tag{4.13}$$

We recall that :

$$\int_{\mathbb{R}^2} \lambda_M(x;t)dx = 1 \tag{4.14}$$

and that the Fourier transform of $\lambda_M(\cdot;t)$ can be expressed as

$$\widehat{\lambda_M}(\xi;t) = e^{-\frac{\|(M^{-1})^T\xi\|^2}{t^2}}. \tag{4.15}$$

We introduce the following function :

$$\chi_{\gamma,M}(x;t) = (\mathbf{1}_{t\Omega_{M^{-1}\gamma}} * \lambda_M(\cdot;t))(x) = \int_{\mathbb{R}^2} \mathbf{1}_{t\Omega_{M^{-1}\gamma}}(y)\lambda_M(x-y;t)dy \tag{4.16}$$

(it is a regularization of the function $\mathbf{1}_{t\Omega_{M^{-1}\gamma}}$).
Let us also set :

$$N_{reg}(t\Omega_\gamma, M) = \sum_{n\in\mathbb{Z}^2} \chi_{\gamma,M}(n;t) \ \textit{and} \tag{4.17}$$

(the index "reg" stands for regularized)

$$F(M,t) = \frac{N_{reg}(t\Omega_\gamma, M) - \mathrm{Area}(t\Omega_\gamma)}{\sqrt{t}}. \tag{4.18}$$

Let L be a unimodular lattice such that $e_1(L)$ and $e_2(L)$ are well-defined and let

$$M = [e_1(L), e_2(L)] \ \textit{if} \ \det([e_1(L), e_2(L)]) > 0 \tag{4.19}$$

and

$$M = [e_2(L), e_1(L)] \ \textit{if} \ \det([e_2(L), e_1(L)]) > 0. \tag{4.20}$$

Then M is a matrix that represents L and one has immediately that :

$$\mathcal{R}(t\Omega_\gamma, L) = \mathcal{R}(t\Omega_{M^{-1}\gamma}, \mathbb{Z}^2).$$

Now, let us call :

$$\Delta_1(L,t) = |\frac{\mathcal{R}(t\Omega_\gamma, L)}{\sqrt{t}} - F(M,t)| \ \textit{and} \ (\Delta_2)_A(L,t) = |F(M,t) - H_A(L,t)| \tag{4.21}$$

so one has that :

$$\Delta_A(L,t) \leqslant \Delta_1(L,t) + (\Delta_2)_A(L,t). \tag{4.22}$$

The proof of Lemma 56 lies on the two following lemmas :

Lemma 58. *The quantity $\Delta_1(L,t)$ converges to 0 when $t \to \infty$.*

Lemma 59. *For all $\alpha > 0$, for all A large enough, for all t large enough,*

$$\mathbb{P}((\Delta_2)_A(L,t) \geqslant \alpha) \leqslant \alpha.$$

Proof of Lemma 56. It is the direct consequence of Equation (4.22) and of Lemma 58 and Lemma 59. □

Lemma 58 basically tells us that the study of $\frac{\mathcal{R}(t\Omega_\gamma,L)}{\sqrt{t}}$ can be reduced to the study of one of its regularized Fourier series, whereas Lemma 59 means that the asymptotical study of this regularized Fourier series can be brought back to the study of the non-regularized Fourier series.
The next subsubsection is dedicated to the proof of Lemma 58 and the subsubsection after it is dedicated to the proof of Lemma 59.

Proof of Lemma 58

The proof of Lemma 58 is based on two sublemmas. The first one is the following :

Sublemma 4. *For all $x \in \mathbb{R}^2$, for all $t > 0$,*

$$|\chi_{\gamma,M}(x;t) - \mathbf{1}_{t\Omega_{M^{-1}\gamma}}(x)| \leqslant e^{-\frac{t^2}{4} dist(Mx,t\gamma)^2}$$

where for all $z \in \mathbb{R}^2$,

$$dist(z, t\gamma) = \inf_{y \in t\gamma} |z - y|.$$

Proof. One has that :

$$|\chi_{\gamma,M}(x;t) - \mathbf{1}_{t\Omega_{M^{-1}\gamma}}(x)| = |\int_{y \notin t\Omega_\gamma} \frac{t^2}{4\pi} e^{-\frac{\|Mx-y\|^2}{4}} dy| \text{ if } Mx \in t\Omega_\gamma$$

and

$$|\chi_{\gamma,M}(x;t) - \mathbf{1}_{t\Omega_{M^{-1}\gamma}}(x)| = |\int_{y \in t\Omega_\gamma} \frac{t^2}{4\pi} e^{-\frac{\|Mx-y\|^2}{4}} dy| \text{ if } Mx \notin t\Omega_\gamma$$

because of Equation (4.14) and by making the change of variable $y = Mu$.
The proof of Lemma 3.2 from [10] gives the wanted result. □

The second sublemma gives an estimate of $\mathrm{dist}(Mn, t\gamma)$. To state it, we need some notations. Like in [10], let the curve γ be defined in the polar coordinates (r, φ) by the equation

$$r = \Gamma(\varphi). \tag{4.23}$$

Let us define :

$$r_\gamma(x) = \frac{\|x\|}{\Gamma(\varphi(x))} \tag{4.24}$$

where $\varphi(x)$ is the angular coordinate of x. Then, one has that there exists $C > 0$ small enough so that for every $x \in \mathbb{R}^2$,

$$\mathrm{dist}(x, t\gamma) \geqslant C|r_\gamma(x) - t|. \tag{4.25}$$

We deduce the following sublemma :

Sublemma 5. *For all $L \in \mathscr{S}_2$, for all $t > 0$, for all $n \in \mathbb{Z}^2$, we have that :*

$$dist(Mn, t\gamma) \geqslant C|r_\gamma(Mn) - t|.$$

Now we can prove Lemma 58.

Proof of Lemma 58. By using the Equation (4.21), we have that :

$$\Delta_1(L,t) \leqslant \frac{1}{\sqrt{t}} \sum_{n \in \mathbb{Z}^2} |\chi_{\gamma,M}(n;t) - \mathbf{1}_{t\Omega_{M^{-1}\gamma}}(n)|$$

because, also, $\mathcal{R}(t\Omega_\gamma, L) = \mathcal{R}(t\Omega_{M^{-1}\gamma}, \mathbb{Z}^2)$. So, Sublemma 4 and Sublemma 5 imply that :

$$\Delta_1(L,t) \leqslant \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}^2} e^{-C^2\frac{t^2}{4}|r_\gamma(Mn)-t|^2}.$$

The essential part of the right-hand side are the terms such that $n \in \mathbb{Z}^2$ verify that

$$|r_\gamma(Mn) - t| \leqslant \frac{1}{t^{3/4}}. \tag{4.26}$$

Yet the number of $n \in \mathbb{Z}^2$ that belong to such an annulus is of order $t^{\frac{1}{4}}$.
So, one has finally :

$$\Delta_1(L,t) = O(\frac{1}{t^{\frac{1}{4}}}).$$

□

Proof of Lemma 59

To prove Lemma 59, we first need to give another expression of $F(M,t)$, obtained via the Poisson formula. It is the object of the following lemma :

Lemma 60.

$$F(M,t) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}^2-\{0\}} \frac{\widehat{\lambda_M}(2\pi n;t)}{\|(M^{-1})^T n\|^{\frac{3}{2}}} \tag{4.27}$$
$$\Big(\rho_\gamma((M^{-1})^T n)\cos(2\pi t Y_\gamma((M^{-1})^T n) - \frac{3\pi}{4}) + \rho_\gamma(-(M^{-1})^T n)\cos(2\pi t Y_\gamma(-(M^{-1})^T n) - \frac{3\pi}{4})\Big)$$
$$+ O_M(t^{-1})$$

which becomes in the symmetric case

$$F(M,t) = \frac{1}{\pi} \sum_{n\in\mathbb{Z}^2-\{0\}} \frac{\widehat{\lambda_M}(2\pi n;t)}{\|(M^{-1})^T n\|^{\frac{3}{2}}} \rho_\gamma((M^{-1})^T n)\cos(2\pi t Y_\gamma((M^{-1})^T n) - \frac{3\pi}{4}) + O_M(t^{-1})$$

where the M in index of O_M is to signal that it depends on M (or, equivalently, on the lattice L).

To prove it, we first need a calculatory sublemma :

Sublemma 6. *Let γ be a simple, closed, analytic, strictly convex curve such that $0 \in \Omega_\gamma$. Let $D \in SL_2(\mathbb{R})$. Let $\tilde{\gamma} = D\gamma$. Then one has for every $\xi \in \mathbb{R}^2 - \{0\}$:*

$$x_{\tilde{\gamma}}(\xi) = Dx_\gamma(D^T\xi),$$

$$\rho_{\tilde{\gamma}}(\xi) = \frac{\|\xi\|^3}{\|D^T\xi\|^3}\rho_\gamma(D^T\xi) \text{ and}$$

$$Y_{\tilde{\gamma}}(\xi) = Y_\gamma(D^T\xi)$$

where D^T is the transpose of the matrix D.

Proof. Let us call T the unit tangent vector at $x_\gamma(D^T\xi)$ such that $(D^T\xi, T)$ is a orthogonal and direct basis of $\mathbb{R}^2$. Then, DT is a unit tangent vector at $Dx_\gamma(D^T\xi)$.
Let us call k the unit normal exterior vector of $\tilde{\gamma}$ at $Dx_\gamma(D^T\xi)$. Then, one knows that k is orthogonal to DT and (k, DT) is direct because $\det(D) = 1$. By property of the adjoint operator, one knows that $D^T k$ is orthogonal to T and thus there exists $\alpha \in \mathbb{R} - \{0\}$ such that

$$D^T k = \alpha D^T \xi. \tag{4.28}$$

So, one gets that :

$$k = \alpha \xi. \tag{4.29}$$

Yet, one has also that (k, DT) and $(D^T\xi, T)$ are direct and orthogonal basis of $\mathbb{R}^2$. So, one must have $\alpha > 0$ and it gives us the first wanted result.
Now, concerning the third equality, one has that, by definition :

$$Y_{\tilde{\gamma}}(\xi) =< \xi, x_{\tilde{\gamma}}(\xi) > \tag{4.30}$$

So, the first equality of Sublemma 6 gives us that :

$$Y_{\tilde{\gamma}}(\xi) =< \xi, Dx_\gamma(D^T\xi) > . \tag{4.31}$$

By using the adjoint property, one finds the wanted result :

$$Y_{\tilde{\gamma}}(\xi) =< D^T\xi, x_\gamma(D^T\xi) >= Y_\gamma(D^T\xi). \tag{4.32}$$

So, one gets the third equality.
Concerning the second equality, one knows that $t \longmapsto \gamma(t)$ is a parametrization of the curve γ and that $t \longmapsto D\gamma(t)$ is a parametrization of the curve $D\gamma$. So, one can use these parametrizations to compute ρ_γ and $\rho_{\tilde{\gamma}}$.
By using the fact that $D \in SL_2(\mathbb{R})$, one has that :

$$\rho_{\tilde{\gamma}}(D\gamma(t)) = (\frac{\|D\gamma'(t)\|}{\|\gamma'(t)\|})^{\frac{3}{2}} \rho_\gamma(\gamma(t)). \tag{4.33}$$

Let us call t_0 the instant such that

$$\gamma(t_0) = x_\gamma(D^T\xi)$$

and so, according to the first result of the Sublemma 6, one has that

$$D\gamma(t_0) = x_{\tilde{\gamma}}(\xi). \tag{4.34}$$

Let us set $\gamma(t) = (x(t), y(t))$ and $D\gamma(t) = (\phi_1(t), \phi_2(t))$.
Then one has at the instant $t = t_0$:

$$\alpha R D^T \xi = (x'(t_0), y'(t_0)) \tag{4.35}$$

and

$$\beta R \xi = (\phi_1'(t_0), \phi_2'(t_0)) \tag{4.36}$$

with $\alpha, \beta > 0$ and $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SO_2(\mathbb{R})$.
By using the fact that $DRD^T = R$, one finds that :

$$\alpha = \beta. \tag{4.37}$$

By using Equation (4.34), Equation (4.35), Equation (4.36), Equation (4.37) and by using Equation (4.33) at the instant $t = t_0$, one has the third wanted equality. □

We can now tackle the proof of Lemma 60.

Proof of Lemma 60. According to the Equation (4.18), the Poisson summation formula and because of the fact that $\widehat{\mathbf{1}_{t\Omega_{M^{-1}\gamma}}}(0) = \text{Area}(t\Omega_{M^{-1}\gamma})$ one has that :

$$F(M,t) = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}^2-\{0\}} \widehat{\mathbf{1}_{t\Omega_{M^{-1}\gamma}}}(2\pi n)\widehat{\lambda_M}(2\pi n;t). \tag{4.38}$$

Yet, according to Lemma 2.1 from [10], one has that :

$$\widehat{\mathbf{1}_{t\Omega_{M^{-1}\gamma}}}(\xi) = \sqrt{t}\|\xi\|^{-\frac{3}{2}} \sum_{\pm} \sqrt{2\pi\rho_{M^{-1}\gamma}(\pm\xi)} \exp\Big(\pm i(tY_{M^{-1}\gamma}(\pm\xi) - \frac{3\pi}{4})\Big) + O_M(t^{-\frac{1}{2}}\|\xi\|^{-\frac{5}{2}}) \tag{4.39}$$

By using Sublemma 6 with $M^{-1} = D$, we get with Equation (4.38) and by grouping the n and $-n$ terms in the Fourier series, we get the wanted result. □

Let us set

$$\nu(l,t) = \rho_\gamma(l)\cos(2\pi t Y_\gamma(l) - \frac{3\pi}{4}). \tag{4.40}$$

Using the Equation (4.15), the Equation (4.21), the fact that if M represents a lattice L, $(M^{-1})^T$ represents the dual lattice $L^\perp$, and the previous lemma, that is Lemma 60, one gets that :

$$(\Delta_2)_A(L,t) \leqslant \Delta_{2,1}(L,t) + (\Delta_{2,2})_A(L,t) + (\Delta_{2,3})_A(L,t) \tag{4.41}$$

where

$$\Delta_{2,1}(L,t) = O_M(t^{-1}) \tag{4.42}$$

$$(\Delta_{2,2})_A(L,t) = \frac{1}{2\pi}\Big|\sum_{\substack{l\in L^\perp \\ 0<\|l\|\leqslant A}} \frac{1}{\|l\|^{\frac{3}{2}}}(\nu(l,t)+\nu(-l,t))(1-e^{-(2\pi)^2\frac{\|l\|^2}{t^2}})\Big| \tag{4.43}$$

$$(\Delta_{2,3})_A(L,t) = \frac{1}{2\pi}\Big|\sum_{\substack{l\in L^\perp \\ A<\|l\|}} \frac{1}{\|l\|^{\frac{3}{2}}}(\nu(l,t)+\nu(-l,t))e^{-(2\pi)^2\frac{\|l\|^2}{t^2}}\Big|. \tag{4.44}$$

So, if we prove the following lemmas, we will get Lemma 59 and, *in fine*, get Lemma 56 :

Lemma 61. *$\Delta_{2,1}(L,t)$ converges almost surely to 0 when $t\to\infty$.*

Let us remark, by the way, that this last lemma is immediate according to equation (4.42).

Lemma 62. *For all $A>0$, $(\Delta_{2,2})_A(L,t)$ converges to 0 when $t\to\infty$.*

Lemma 63. *For all $\alpha>0$, for all A large enough, for all t large enough,*

$$\mathbb{P}((\Delta_{2,3})_A(L,t) \geqslant \alpha) \leqslant \alpha.$$

Proof of Lemma 59. Let $\alpha > 0$. Let us take A large enough so that for all t large enough,

$$\mathbb{P}((\Delta_{2,3})_A(L,t) \geqslant \alpha) \leqslant \alpha.$$

It is possible according to Lemma 63.
According to Lemma 62, according to Lemma 61 and because the almost-sure convergence imply the convergence in probability, even if it means taking t larger, one can suppose that :

$$\mathbb{P}((\Delta_{2,1})(L,t) \geqslant \alpha) \leqslant \alpha \text{ and}$$

$$\mathbb{P}((\Delta_{2,2})_A(L,t) \geqslant \alpha) \leqslant \alpha.$$

By using equation 4.41, one gets the wanted result. □

Before following with the proof of Lemma 56, let us say a few words about Lemma 62 and Lemma 63. The first tells us that the non-regularized Fourier series is "close" enough to the regularized Fourier series whereas the second one tells us that the large terms of the regularized Fourier series do not matter, in a certain sense, for our study. It remains only to prove Lemma 62 and Lemma 63. Because the density of $\tilde{\mu}_2$ is bounded, we only need to prove these lemmas for $\tilde{\mu}_2 = \mu_2$ and we will make this assumption for the rest of the section. We are now going to prove Lemma 62 and Lemma 63.

Proof of Lemma 62

Proof of Lemma 62. Let $l \in L^{\perp}$. Then one has :

$$|1 - e^{-(2\pi)^2 \frac{\|l\|^2}{t^2}}| \leqslant \frac{(2\pi)^2 \|l\|^2}{t^2}. \tag{4.45}$$

With this equation and with Equation (4.43), one gets that :

$$(\Delta_{2,2})_A(L,t) \leqslant \sum_{\substack{l \in L^{\perp} \\ 0 < \|l\| \leqslant A}} \frac{M}{t^2} \|l\|^{\frac{1}{2}} \tag{4.46}$$

with $M > 0$ because ν is a bounded function. It follows that there exists $C(L) > 0$ such that :

$$(\Delta_{2,2})_A(L,t) \leqslant C(L) \frac{A^{\frac{5}{2}}}{t^2}. \tag{4.47}$$

□

Proof of Lemma 63

To prove Lemma 63 we need to use what are called *Siegel* and *Rogers formulas*. Theses formulas will also be useful later in this paper.
By setting $c_k = \zeta(2)^{-k}$ for k an integer larger than 1 and where ζ denotes the ζ function of Riemann, one has the following formulas :

Lemma 64 ([64],[93],[47]). *For f a piecewise smooth function with compact support on $\mathbb{R}^2$, one has :*

- $$\int_{\mathscr{S}_2} \mathcal{S}(f) d\mu_2 = c_1 \int_{\mathbb{R}^2} f d\lambda$$
- *When f is even,*
$$(b) \int_{\mathscr{S}_2} \mathcal{S}(f)^2 d\mu_2 \leqslant C \int_{\mathbb{R}^2} f^2 d\lambda + c_2 (\int_{\mathbb{R}^2} f d\lambda)^2$$
where $C > 0$.

With this lemma, we are going to prove two lemmas that will enable us to prove Lemma 63 by using Chebyshev's inequality : the first one is intended to estimate the expectation of $(\Delta_{2,3})_A(L,t)$ to see that it goes to 0 when $t \to \infty$ (uniformly in A), the second one is intended to estimate its variance to see that it can be as uniformly small in t as one wants if A is chosen large enough. Until the end of this section, we are going to suppose $A > 1$.

Lemma 65.
$$\mathbb{E}((\Delta_{2,3})_A(L,t)) = O(\frac{1}{t}).$$

Proof. One has :
$$(\Delta_{2,3})_A(L,t) = \sum_{\substack{l \in L^\perp \\ A < \|l\|}} f(l) \tag{4.48}$$
where
$$f(l) = \frac{1}{2\pi} \frac{1}{\|l\|^{\frac{3}{2}}} (\nu(l,t) + \nu(-l,t)) e^{-(2\pi)^2 \frac{\|l\|^2}{t^2}}. \tag{4.49}$$
The Lemma 64 gives us then that :
$$|\mathbb{E}((\Delta_{2,3})_A(L,t))| = C| \int_{\mathbb{R}^2} f(x) \mathbf{1}_{\|x\|>A} dx|. \tag{4.50}$$

By passing into polar coordinates (r,θ), one gets that :
$$|\mathbb{E}((\Delta_{2,3})_A(L,t))| = 2C| \int_{\theta=0}^{2\pi} \int_{r \geqslant A} \frac{\cos(2\pi t r h(\theta) - \frac{3\pi}{4}) e^{-(2\pi)^2 \frac{r}{t^2}}}{r^{\frac{1}{2}}} dr d\theta| \tag{4.51}$$
by setting $h(\theta) = Y_\gamma((\cos(\theta), \sin(\theta))$ and by using the fact that Y_γ is positively homogeneous. Furthermore, an integration by part gives us that :
$$\begin{aligned} &\int_{r \geqslant A} \frac{\cos(2\pi t r h(\theta) - \frac{3\pi}{4}) e^{-(2\pi)^2 \frac{r}{t^2}}}{r^{\frac{1}{2}}} dr \\ &= -\frac{e^{-\frac{(2\pi)^2 r^2}{t^2}} \sin(2\pi t A h(\theta) - \frac{3\pi}{4})}{2\pi t h(\theta) A^{\frac{1}{2}}} \\ &+ \frac{1}{4\pi t h(\theta)} \int_{r \geqslant A} \sin(2\pi t r h(\theta) - \frac{3\pi}{4}) \frac{e^{-\frac{(2\pi)^2 r^2}{t^2}}}{r^{\frac{3}{2}}} dr \\ &+ \frac{4\pi}{t^3 h(\theta)} \int_{r \geqslant A} \sin(2\pi t r h(\theta) - \frac{3\pi}{4}) e^{-\frac{(2\pi)^2 r^2}{t^2}} r^{\frac{1}{2}} dr. \end{aligned} \tag{4.52}$$

By using that $r^{\frac{1}{2}} \leqslant r$ (because $A > 1$) and the fact that $h(\theta)$ admits a positive lower bound (because $0 \in \Omega_\gamma$), by estimating the three terms of the right member, one gets that :

$$|\int_{r \geqslant A} \frac{\cos(2\pi t r h(\theta) - \frac{3\pi}{4}) e^{-(2\pi)^2 \frac{r}{t^2}}}{r^{\frac{1}{2}}} dr| \leqslant \frac{C}{t} \tag{4.53}$$

with $C > 0$ that does not depend on θ.
By using Equation (4.51) and Equation (4.53), one gets that :

$$\mathbb{E}((\Delta_{2,3})_A(L,t)) = O(\frac{1}{t}).$$

□

Lemma 66.

$$\mathbb{V}((\Delta_{2,3})_A(L,t)) = O(\frac{1}{A})$$

where the O can be chosen independent from t.

Proof. By using the same notation as before, by using again the Lemma 64 and by using the Lemma 65, one gets that :

$$\mathrm{Var}((\Delta_{2,3})_A(L,t)) \leqslant C \int_{\mathbb{R}^2} f^2(x) \mathbf{1}_{\|x\|>A} dx. \tag{4.54}$$

So, by passing into polar coordinates and by using the fact that ν is bounded, one gets that :

$$\mathrm{Var}((\Delta_{2,3})_A(L,t)) \leqslant C \int_{r=A}^{\infty} \frac{1}{r^2} dr. \tag{4.55}$$

By integrating, we get the wanted result. □

We can now prove the Lemma 63.

Proof of Lemma 63. The Chebyshev's inequality gives the wanted result if, first, we choose A large enough and, second, we choose t large enough so that $\mathbb{E}((\Delta_{2,3})_A(L,t))$ and $\mathrm{Var}((\Delta_{2,3})_A(L,t))$ are small enough. These choices are possible according to Lemmas 65 and 66. □

So, now the proof of Lemma 56 is complete and we will conclude this section by proving the Lemma 57 so that the proof of Proposition 30 will be complete.

4.3.2 Proof of Lemma 57

To prove the Lemma 57, we are going to take the same kind of approach as before : estimate the expectation and the variance of the quantity $S_A - S_{A,prime}$ and get the result via the Chebyshev's inequality.
We have that :

$$H_A(L,t) - S_{A,prime}(L,t) = \sum_{l \in L^\perp prime} f(l) \tag{4.56}$$

where

$$f(l) = \frac{1}{2\pi\|l\|^{\frac{3}{2}}} \mathbf{1}_{0<\|l\|\leqslant A} \sum_{k\geqslant \lfloor \frac{A}{\|l\|}\rfloor+1} \frac{(\nu(kl,t)+\nu(-kl,t))}{k^{\frac{3}{2}}} \tag{4.57}$$

with ν being defined by Equation 4.40.
With this expression, we see that we are going to have a little problem of integrability at 0 if we use the Lemma 64. That's why, we have to exclude 0 and we will suppose that L is chosen so that $\|L^{\perp}\|_1 \geqslant \epsilon$ where $0 < \epsilon < 1$. Only a small number of lattices is excluded according to this lemma :

Lemma 67. *For every $0 < \epsilon < 1$, one has that*

$$\mathbb{P}(\|L\|_1 < \epsilon) = O(\epsilon^2).$$

Proof. It is a consequence of Lemma 64 by taking

$$\mathcal{S}(f)(L) = \sum_{l\in L} \mathbf{1}_{B_f(0,\epsilon)}(l)$$

where $\mathbf{1}_{B_f(0,\epsilon)}(l)$ is the indicator function of the closed ball for the norm $\|\cdot\|$ centred on 0 of radius ϵ. □

Thus, for the chosen lattices, we have :

$$H_A(L,t) - S_{A,prime}(L,t) = \sum_{l\in L^{\perp} prime} f(l)\mathbf{1}_{\|l\|\geqslant\epsilon} = \Delta_{3,\epsilon,A,t}(L) \tag{4.58}$$

(this equation defines $\Delta_{3,\epsilon,A,t}(L)$).

Lemma 68.

$$\mathbb{E}(\Delta_{3,\epsilon,A,t}(L)) = O_{\epsilon,A}(\frac{1}{t}).$$

Proof. By using the Lemma 64, one gets that :

$$\mathbb{E}(\Delta_{3,\epsilon,A,t}(L)) \leqslant C \int_{r=\epsilon}^{A} \int_{\theta=0}^{2\pi} \frac{1}{r^{\frac{1}{2}}} \sum_{k\geqslant\frac{A}{r}} \frac{\nu(kr(\cos(\theta),\sin(\theta)),t)}{k^{\frac{3}{2}}} dr. \tag{4.59}$$

Lebesgue's dominated convergence theorem gives us that :

$$\int_{\epsilon}^{A} \frac{1}{r^{\frac{1}{2}}} \sum_{k\geqslant\frac{A}{r}} \frac{\nu(kr(\cos(\theta),\sin(\theta)),t)}{k^{\frac{3}{2}}} dr = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} \int_{\max(\frac{A}{k},\epsilon)}^{A} \frac{\nu(kr(\cos(\theta),\sin(\theta)),t)}{r^{\frac{1}{2}}} dr. \tag{4.60}$$

An integration by part (on the variable r) as in the proof of Lemma 65 and the Equation (4.59) give us finally that :

$$\mathbb{E}(\Delta_{3,\epsilon,A,t}(L)) = O_{\epsilon,A}(\frac{1}{t}). \tag{4.61}$$

□

Lemma 69. *There exists $K > 0$ such that :*

$$Var(\Delta_{3,\epsilon,A,t}(L)) \leqslant K(-\frac{\log(\epsilon)}{A} + \frac{\log(A)}{A}).$$

Proof. Lemma 64 gives us that :

$$\mathrm{Var}(\Delta_{3,\epsilon,A,t}(L)) \leqslant C \int_{r=\epsilon}^{A} \int_{\theta=0}^{2\pi} \frac{1}{r^2} \Big(\sum_{k \geqslant \frac{A}{r}} \frac{\nu(kr(\cos(\theta), \sin(\theta)), t)}{k^{\frac{3}{2}}} \Big)^2 dr. \tag{4.62}$$

Yet one also has that for all $x > 0$:

$$\sum_{k \geqslant x} \frac{1}{k^{\frac{3}{2}}} \leqslant \frac{D}{x^{\frac{1}{2}}} \tag{4.63}$$

where $D > 0$.
Thus, Equation (4.62), Equation (4.63) and the fact that ν is bounded imply that :

$$\mathbb{V}(\Delta_{3,\epsilon,A,t}(L)) \leqslant \frac{2\pi CD}{A} \int_{\epsilon}^{A} \frac{1}{r} dr. \tag{4.64}$$

□

We can now give the proof of Lemma 57.

Proof of Lemma 57. First we take $1 > \epsilon > 0$ small enough so that the measure of the neglected lattices, *id est* the lattices such that $\|L\|_1 < \epsilon$, is small enough. It is possible according to Lemma 67.
Then we take A large enough so that $\mathbb{V}(\Delta_{3,\epsilon,A,t}(L))$ is small enough. It is possible according to Lemma 69.
Finally, we take t large enough so that $\mathbb{E}(\Delta_{3,\epsilon,A,t}(L))$ is small enough, which is possible according to Lemma 68, and conclude by using Chebyshev's inequality. □

So, we are now brought back to the study of $S_{A,prime}(L,t)$ when $t \to \infty$ and the next section is dedicated to it.
We are going to replace $L^{\perp}$ by L (it changes nothing because we are studying the asymptotic convergence in distribution with $L \in \mathscr{S}_2$ distributed according to $\tilde{\mu}_2$).

4.4 Study of $S_{A,prime}(L,t)$ when $t \to \infty$

4.4.1 Reductions for the study of $S_{A,prime}(L,t)$ and proof of Theorem 10

Before entering in the main object of this section, we need to do a small rewriting of $S_{A,prime}(L,t)$.
We recall that a vector $l \in L$ is prime if, and only if, $Kl \in KL$ is prime where

$K \in SL_2(\mathbb{R})$. Furthermore, a vector $(l_1, l_2) \in \mathbb{Z}^2$ is prime if, and only if, $l_1 \wedge l_2 = 1$. By using the symmetry $l \longmapsto -l$, we deduce that $S_{A,prime}(L,t)$ can be rewritten as followed :

$$S_{A,prime}(L,t) = \frac{1}{\pi} \sum_{k \in \Pi_A(L)} \frac{Z_k(L,t)}{W_k(L)} \tag{4.65}$$

where, for $k = (k_1, k_2) \in \mathbb{Z}^2$,

$$W_k(L) = \|k_1 e_1(L) + k_2 e_2(L)\|^{\frac{3}{2}}, \tag{4.66}$$

$$Z_k(L,t) = \sum_{m \in \mathbb{N} - \{0\}} \frac{\nu(m(k_1 e_1(L) + k_2 e_2(L)), t) + \nu(-m(k_1 e_1(L) + k_2 e_2(L)), t)}{m^{\frac{3}{2}}}, \tag{4.67}$$

and where

$$\Pi_A(L) = \{(k_1, k_2) \in \mathbb{Z}^2 \mid k_1 \wedge k_2 = 1,\ k_1 \geqslant 0\ \|k_1 e_1(L) + k_2 e_2(L)\| \leqslant A\} \tag{4.68}$$

and here we agree that if $(k_1, k_2) \in \Pi_A(L)$ then $k_1 = 0$ implies that $k_2 = 1$ (for the definition of $e_1(L)$ and $e_2(L)$ see Definition 19).

Let us recall that :

$$\Pi = \{(k_1, k_2) \in \mathbb{Z}^2 \mid k_1 \wedge k_2 = 1,\ k_1 \geqslant 0\}. \tag{4.69}$$

Our goal now is to prove the following proposition :

Proposition 31. *In the symmetric case, $\{Z_k(L,t)\}_{k \in \Pi}$ converge, when $t \to \infty$, in distribution towards $\{\rho_\gamma(k_1 e_1(L) + k_2 e_2(L))\tilde{Z}_k(\omega)\}_{k \in \Pi}$ where $\tilde{Z}_k(\omega)$, with $\omega \in \Omega$, are independent identically distributed real random variables that have a compact support, are symmetrical and are non-zero.*
In the non symmetric case, $\{Z_k(L,t)\}_{k \in \Pi}$ converge, when $t \to \infty$, in distribution towards $\{\rho_\gamma(k_1 e_1(L) + k_2 e_2(L))\tilde{Z}_k(\omega_1) + \rho_\gamma(-(k_1 e_1(L) + k_2 e_2(L)))\tilde{Z}_k(\omega_2)\}_{k \in \Pi}$ where $(\omega_1, \omega_2) \in \Omega \times \Omega$, where $\tilde{Z}_k(\omega)$ are independent identically distributed real random variables that have a compact support, are symmetrical and are non-zero.

In the next section we are going to consider, in the symmetric case, the sums of the type

$$\tilde{S}_A(\omega, L) = \sum_{k \in \Pi_A(L)} \frac{\rho_\gamma(k_1 e_1(L) + k_2 e_2(L))\tilde{Z}_k(\omega)}{\|k_1 e_1(L) + k_2 e_2(L)\|^{\frac{3}{2}}}$$

and the sums of the type, in the non symmetric case,

$$\tilde{S}_A(\omega_1, \omega_2, L) = \sum_{k \in \Pi_A(L)} \frac{\rho_\gamma(k_1 e_1(L) + k_2 e_2(L))\tilde{Z}_k(\omega_1) + \rho_\gamma(-(k_1 e_1(L) + k_2 e_2(L)))\tilde{Z}_k(\omega_2)}{\|k_1 e_1(L) + k_2 e_2(L)\|^{\frac{3}{2}}}$$

where Z_k are non-zero real independent identically distributed random variables from $\Omega \ni \omega$ that are symmetrical and have a compact support and where $(\omega_1, \omega_2) \in \Omega \times \Omega$. Proposition 32 tells us that :

Proposition 32. *The sums of these types :*

- *converge almost surely*
- *their respective limits are symmetrical and their expectations are equal to* 0
- *their respective limits admit moment of order* $1+\kappa$ *for any* $0 \leqslant \kappa < \frac{1}{3}$
- *their respective limits do not admit a moment of order* $\frac{4}{3}$ *when* $\sigma(L) \geqslant m$ *where* $m > 0$ *and where* L *belongs to an event of the form* $(\|L\| < \alpha)$ *with* $\alpha > 0$.
- *when there exists* $\alpha > 0$ *such that* $\tilde{\mu}_2(\{L \in \mathscr{S}_2 \mid \|L\|_1 < \alpha\}) = 0$ *then their respective limits admit moments of all order* $1 \leqslant p < \infty$.

We are going to see now that it is enough to prove Proposition 31 and Proposition 32 to establish Theorem 10, with the exception of the exact form of the limiting distribution (yet it is given by Proposition 33).

Proof of Theorem 10. Let $\psi \in C_c^\infty(\mathbb{R})$. Let $\epsilon > 0$. According to Proposition 30, we can take A as large as we want and then t as large as we want so that :

$$|\mathbb{E}\left(\psi(\frac{\mathcal{R}(t\Omega_\gamma, L)}{\sqrt{t}})\right) - \mathbb{E}\left(\psi(S_{A,prime}(L,t))\right)| \leqslant \epsilon. \tag{4.70}$$

Thanks to Proposition 31, one has also that :

$$|\mathbb{E}\left(\psi(S_{A,prime}(L,t))\right) - \mathbb{E}\left(\psi(\tilde{S}_A(\omega, L))\right)| \leqslant \epsilon \tag{4.71}$$

where the $Z_k(\omega)$ in $\tilde{S}_A(\omega, L)$ are given by Proposition 31.
Furthermore, Proposition 32 gives us that :

$$|\mathbb{E}\left(\psi(\tilde{S}_A(\omega, L))\right) - \mathbb{E}\left(\psi(\lim_{A\to\infty} \tilde{S}_A(\omega, L))\right)| \leqslant \epsilon \tag{4.72}$$

with $\lim_{A\to\infty} \tilde{S}_A(\omega, L)$ that verify all the listed properties.
So, Equation (4.70), Equation (4.71) and Equation (4.72) give the wanted result. □

The main reason why the Z_k are going to be independent from L is the presence of the factor t.
The main reasons why the rest of Proposition 31 will be true are the presence of the factor t in Z_k and the fact that the coefficients of order 1 of the Taylor series of $(Y(k_1e_1(L) + k_2e_2(L)))_{k\in\Pi_A(L)}$ on a small geodesic segment are $\mathbb{Z}$-free and, in the non symmetric case, $(Y(k_1e_1(L) + k_2e_2(L)), Y(-(k_1e_1(L) + k_2e_2(L)))_{k\in\Pi_A(L)}$ on a small geodesic segment are $\mathbb{Z}$-free.

In order to prove Proposition 31, it is actually enough to prove the following proposition by using the definition of ν (see Equation (4.40)) :

Proposition 33. *For* $k = (k_1, k_2)$, *let*

$$\theta_k(L,t) = tY_\gamma(k_1e_1(L) + k_2e_2(L))) \mod 1. \tag{4.73}$$

Then, we have that $\{\theta_k(L,t)\}_{k\in\Pi}$ *converge, when* $t \to \infty$, *towards random variable that are independent identically distributed, are distributed according to the Lebesgue*

measure λ over $\mathbb{R}/\mathbb{Z}$ and are independent from L.
In the non symmetric case, we have that $\{\theta_k(L,t)\}_{k\in\Pi}$ and $\{\theta_{-k}(L,t)\}_{k\in\Pi}$ converge, when $t\to\infty$, towards random variable that are independent identically distributed, are distributed according to the Lebesgue measure λ over $\mathbb{R}/\mathbb{Z}$ and are independent from L.

Thanks to this proposition, we now understand why the limit distribution of $\frac{\mathcal{R}(t\mathbb{D}^2,L)}{\sqrt{t}}$ is given by $S(\theta,L)$.
To prove this last proposition, it is sufficient to prove the following proposition where $e(\theta)$ stands for $\exp(i2\pi\theta)$:

Proposition 34. *For every $l\in\mathbb{N}-\{0\}$, for every $\psi\in C_c^\infty(\mathscr{S}_2)$, for every $(p_1,\cdots,p_l)\in\mathbb{Z}^l-\{0\}$, one has :*

$$\mathbb{E}\left(\psi(L)e(\sum_{h=1}^{l}p_h\theta_{k_h})\right)\underset{t\to\infty}{\to}0 \tag{4.74}$$

where the $k_h\in\Pi$ are all distinct.
In the non symmetric case, one has that for every $l\in\mathbb{N}-\{0\}$, for every $\psi\in C_c^\infty(\mathscr{S}_2)$, for every $(p_{-l},\cdots,p_{-1},p_1,\cdots,p_l)\in\mathbb{Z}^{2l}-\{0\}$, one has :

$$\mathbb{E}\left(\psi(L)e(\sum_{h=1}^{l}p_h\theta_{k_h}+\sum_{h=1}^{l}p_{-h}\theta_{-k_h})\right)\underset{t\to\infty}{\to}0 \tag{4.75}$$

where the $k_h\in\Pi$ are all distinct.

Before passing to the proof of Proposition 34, let us give some heuristic about it in the symmetric case, the non symmetric case being similar here.
Basically, by working with a foliation of the space $\mathscr{S}_2$ given by small enough geodesic segments, we are first going to have :

$$\mathbb{E}\left(\psi(L)e(\sum_{h=1}^{l}p_h\theta_{k_h})\right)\approx\mathbb{E}(\psi)\mathbb{E}\left(e(\sum_{h=1}^{l}p_h\theta_{k_h})\right)$$

due to the presence of the factor t in θ.
The right member will go to 0 when t goes to infinity because a Riemann-Lebesgue lemma will apply because quantities "close" to the variables θ_k are typically $\mathbb{Z}$-free (see the heuristic explanation of the second step).
The rest of this section is now dedicated to the proof of the Proposition 34.

4.4.2 Foliation and local estimates

We recall that a foliation of the space $\mathscr{S}_2$ is given by the orbits of the group δ where

$$\delta(\lambda)=\begin{pmatrix}\lambda & 0\\ 0 & \frac{1}{\lambda}\end{pmatrix}.$$

To prove Proposition 34, we are going to look at what it is happening on a small "segment" of the form

$$J_\epsilon(L)=\{\delta(\lambda)L\mid\lambda\in[\frac{1}{1+\epsilon},1+\epsilon]\} \tag{4.76}$$

where $L \in \mathscr{S}_2$ and $\epsilon > 0$ can be taken as small as possible. More precisely, we are going to show, when $t \to \infty$, the independence of the (θ_k) and of L over smalls segments of the form $J_\epsilon(L)$, as well as the fact that the (θ_k) are identically distributed and distributed according to the normalized Haar measure over $\mathbb{R}/\mathbb{Z}$.
Let us call $s : (x,y) \in \mathbb{R}^2 \longmapsto (x,-y)$ and let us set for $k = (k_1,k_2) \in \Pi$ or $k \in -\Pi$:

$$W_k(L) = dY_\gamma(k_1e_1(L) + k_2e_2(L))\,(s(k_1e_1(L) + k_2e_2(L))) \tag{4.77}$$

where $dY_\gamma(k_1e_1(L) + k_2e_2(L))(\cdot)$ stands for the differential of Y_γ at the point $k_1e_1(L) + k_2e_2(L)$.
On a segment of the form $J_\epsilon(L)$, the following lemma basically tells us how we can estimate the quantities $Y_\gamma(k_1e_1(L) + k_2e_2(L))$:

Lemma 70. *For a typical $L \in \mathscr{S}_2$, there exists $\epsilon > 0$ small enough such that for every $\lambda \in [\frac{1}{1+\epsilon}, 1+\epsilon]$,*

$$e_1(\delta(\lambda)L) = \delta(\lambda)e_1(L) \text{ and } e_2(\delta(\lambda)L) = \delta(\lambda)e_2(L).$$

Furthermore, for such a lattice L, for such λ, for $k = (k_1,k_2) \in \Pi$ or for $k \in -\Pi$, we have for $h = \lambda - 1$,

$$\begin{aligned} &Y_\gamma(k_1e_1(\delta(\lambda)L) + k_2e_2(\delta(\lambda)L)) = Y_\gamma(k_1e_1(L) + k_2e_2(L)) \\ &+ hW_k(L) + O_{k_1,k_2,L}(h^2). \end{aligned} \tag{4.78}$$

Proof. The first fact was proven in [89] (see Lemma 15).
Let us note that, as γ is analytical, Y_γ is regular. As a consequence, the second fact is obtained from the first fact of Lemma 70 and by a simple calculus of Taylor series. □

To prove Proposition 34 we see, in light of Lemma 70, that it would be convenient to prove the following proposition :

Proposition 35. *For a typical $L \in \mathscr{S}_2$, for every $m \in \mathbb{N} - \{0\}$, for every family $(p_1, \cdots, p_m) \in \mathbb{Z}^m$, for every $k_1, \cdots, k_m \in \Pi$ all distinct if*

$$\sum_{i=1}^{m} p_i W_{k_i}(L) = 0 \tag{4.79}$$

then $p_1 = \cdots = p_m = 0$.
In other words, for a typical $L \in \mathscr{S}_2$,

$$(W_k(L))_{k\in\Pi}$$

is a $\mathbb{Z}$-free family.
In the non symmetric case, for a typical $L \in \mathscr{S}_2$, for every $m \in \mathbb{N} - \{0\}$, for every family $(p_{-m}, \cdots, p_{-1}, p_1, \cdots, p_m) \in \mathbb{Z}^{2m}$, for every $k_1, \cdots, k_m \in \Pi$ all distinct if

$$\sum_{i=1}^{m} p_i W_{k_i}(L) + \sum_{i=1}^{m} p_{-i} W_{-k_i}(L) = 0 \tag{4.80}$$

then $p_{-m} = \cdots = p_{-1} = p_1 = \cdots = p_m = 0$.
In other words, in the non symmetric case, for a typical $L \in \mathscr{S}_2$,

$$(W_k(L))_{k\in-\Pi\cup\Pi}$$

is a $\mathbb{Z}$-free family.

The next subsection is dedicated to prove this proposition.

4.4.3 Proof of Proposition 35

To prove Proposition 35, we are following closely what was done in the Section 5 of [27] and we need four preliminary lemmas.
To state these lemmas we need to put in place some notations. Let us call

$$P : X \in \mathbb{R}^2 \longmapsto dY_\gamma(X)(s(X)). \tag{4.81}$$

Let us also set for every $k = (k_1, k_2) \in -\Pi \cup \Pi$,

$$f_k : (A = (X_1, Y_1), B = (X_2, Y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \longmapsto P(k_1 X + k_2 Y) \tag{4.82}$$

where $X = (X_1, X_2)$ and $Y = (Y_1, Y_2) \in \mathbb{R}^2$.
Let us set for every $L \in GL_2(\mathbb{R})$,

$$g_L : \delta \in \mathbb{R} \longmapsto P(L(1, \delta)). \tag{4.83}$$

Let us also set for every $L \in GL_2(\mathbb{R})$,

$$\tilde{g}_L : \delta \in \mathbb{R} \longmapsto P(L(-1, -\delta)). \tag{4.84}$$

Lemma 71. *If γ is analytic, we have that for any $L \in GL_2(\mathbb{R})$, g_L is (real) analytic and not equal to a polynomial.*

Proof. First, let us note that we $X \in \mathbb{R}^2 \longmapsto dY_\gamma(X)(s(X))$ is positively homogeneous because $X \longmapsto Y_\gamma(X)$ also is positively homogeneous.
So, one has that for every $\delta \in \mathbb{R}$,

$$g_L(\delta) = \sqrt{1+\delta^2} P(L(\frac{1}{\sqrt{1+\delta^2}}, \frac{\delta}{\sqrt{1+\delta^2}})). \tag{4.85}$$

We know that g_L is analytic because γ is analytic. So, let us suppose that f_L is equal to a polynomial. Let us observe that $P(L(\frac{1}{\sqrt{1+\delta^2}}, \frac{\delta}{\sqrt{1+\delta^2}}))$ is bounded so that g_L can only be of degree at most one.
From this fact, one has that, for every $\lambda, \delta \in \mathbb{R}$, $(P \circ L)(\lambda, \delta)$ can be written as :

$$(P \circ L)(\lambda, \delta) = a_0 \lambda + a_1 \delta \tag{4.86}$$

where $a_0, a_1 \in \mathbb{R}$.
By making the change of variable $u = L(\lambda, \delta)$, one gets finally that there exists $b_0, b_1 \in \mathbb{R}$ such that for all $\lambda, \delta \in \mathbb{R}$:

$$P(\lambda, \delta) = b_0 \lambda + b_1 \delta. \tag{4.87}$$

Yet, by using Equation (4.81), one gets that :

$$x \partial_x Y_\gamma - y \partial_y Y_\gamma = b_0 x + b_1 y. \tag{4.88}$$

Yet, Y_γ is positively homogeneous and so for $x > 0$, one has $Y_\gamma(x, y) = x Y_\gamma(1, \frac{y}{x})$. Using differentiation, one gets that :

$$\partial_y Y_\gamma(x, y) = \partial_y Y_\gamma(1, \frac{y}{x}). \tag{4.89}$$

Using Equation (4.88) and Equation (4.89), by diving by x and making x goes to infinity one gets that :

$$\partial_x Y_\gamma(x,y) \to b_0 \tag{4.90}$$

when $x \to \infty$.
So, one gets that (because Y_γ is analytic) :

$$\partial_x Y_\gamma(x,y) = K(y) \tag{4.91}$$

where $K(y)$ is a regular function.
By interchanging the role of x and y, one gets that :

$$Y_\gamma(x,y) = A_0 xy + A_1 \tag{4.92}$$

where A_0 and A_1 belong to $\mathbb{R}$. Yet by setting $x = 1$ and $y = \delta$ and by using Lemma 5.2 from [27], we obtain that it is impossible. □

In the non symmetric case, we will need the following lemma.

Lemma 72. *The following alternative holds. Let $k \geqslant 2$, $k \in 2\mathbb{N}$. Either*
(i) There exists $L \in GL_2(\mathbb{R})$ and $\delta, \delta' \in \mathbb{R}$ such that

$$\frac{g_L^{(k)}(\delta)}{g_L^{(k)}(\delta')} \neq \frac{\tilde{g}_L^{(k)}(\delta)}{\tilde{g}_L^{(k)}(\delta')} \tag{4.93}$$

or
(ii) Ω_γ has a center of symmetry.

Proof. Let us suppose that (i) does not hold. Let $L = Id$. We have that $g_{Id}^{(k)}(\cdot) = c\tilde{g}_{Id}^{(k)}(\cdot)$ for some constant c. In other words

$$(\frac{\partial}{\partial\delta})^k P(1,\delta) = c(\frac{\partial}{\partial\delta})^k P(-1,-\delta). \tag{4.94}$$

Since for $x > 0$ we have $P(x,y) = xP(1,\frac{y}{x})$, it follows that

$$\partial_y^k P(x,y) = c\partial_y^k P(-x,-y). \tag{4.95}$$

Since γ is analytic, this equality in fact holds identically. In particular :

$$\partial_y^k P(-x,-y) = c\partial_y^k P(x,y). \tag{4.96}$$

As a consequence, and because of Lemma 71, one must have $c = \pm 1$.
Furthermore, from Equation (4.95), one has necessarily that :

$$P(x,y) - cP(-x,-y) = \sum_{l=0}^{k-1} a_l(x) y^l. \tag{4.97}$$

Let us set now $H(x,y) = Y_\gamma(x,y) - cY_\gamma(-x,-y)$. Equation (4.97) and Equation (4.97) give us that :

$$x\partial_x H(x,y) - y\partial_y H(x,y) = \sum_{l=0}^{k-1} a_l(x) y^l. \tag{4.98}$$

Furthermore, one has, by positive homogeneity of $H(x,y)$, that $\partial_x H(x,y) = \partial_x H(x/y,1)$. Equation (4.98) gives us then that the growth of $\partial_y H(x,y)$, at x fixed, is of order y^{d-1}. As $H(x,y)$ is analytic, we must have :

$$\partial_y H(x,y) = \sum_{l=0}^{k-1} \tilde{b}_l(x) y^l \tag{4.99}$$

and so

$$H(x,y) = \sum_{l=0}^{k} b_l(x) y^l \tag{4.100}$$

where the $b_l(x)$ are analytic functions.
Yet, $H(x,y) = yH(\frac{x}{y},1)$. So, one must have : $H(x,y) = b_0(x) + b_1(x)y$. So, we see that we have reached the same point of the demonstration of Lemma 5.3 of [27]. So, by reasoning the same way, one gets finally that :

$$Y_\gamma(X) - Y_\gamma(-X) =< X, v > \tag{4.101}$$

with v being a vector of $\mathbb{R}^2$. By shifting the origin to x_0, $Y_\gamma(X)$ is replaced by $Y_\gamma(X)+ < X, x_0 >$ and $Y_\gamma(-X)$ by $Y_\gamma(-X)- < X, x_0 >$. So, after shifting the origin to $\frac{v}{2}$, we get $Y_\gamma(X) = Y_\gamma(-X)$ so that Ω_γ is symmetric. □

Let us assume WLOG that Equation (4.93) holds for $L = Id$.

Lemma 73. *For every $m \in \mathbb{N} - \{0\}$, for every family $(p_1, \cdots, p_m) \in \mathbb{Z}^m$, for every $k_1, \cdots, k_m \in \Pi$ all distinct, one has the following implication : if $\sum_{i=1}^m p_i f_{k_i}$ is a polynomial function then all the p_i must be equal to 0.*
In the non symmetric case, for every $m \in \mathbb{N}-\{0\}$, for every family $(p_{-m}, \cdots, p_{-1}, p_1, \cdots, p_m) \in \mathbb{Z}^{2m}$, for every $k_1, \cdots, k_m \in \Pi$ all distinct if $\sum_{i=1}^m p_i f_{k_i} + \sum_{i=1}^m p_{-i} f_{-k_i}$ is a polynomial function then all the p_i must be equal to 0.

Proof. Let us suppose that $\sum_{i=1}^m p_i f_{k_i}$ is a polynomial. Let us give $j \in [1,m]$. We are going to show that $p_j = 0$.
Let $\beta \in \mathbb{R}^2$ such that $< k_j, \beta > \neq 0$. Let $\alpha \in \mathbb{R}^2$ and let $X = \alpha$ and $Y = \delta\alpha + \theta\beta$. Then, one has :

$$f_k(A,B) = | < k, \alpha > | g_{Id}(\delta + \theta \frac{< k, \beta >}{| < k, \alpha > |})$$

if $< k, \alpha >> 0$ or

$$f_k(A,B) = | < k, \alpha > | \tilde{g}_{Id}(\delta + \theta \frac{< k, \beta >}{| < k, \alpha > |})$$

if $< k, \alpha >< 0$.
As $\sum_{i=1}^m p_i f_{k_i}$ is a polynomial then $\sum_{i=1}^m p_i f_k(A,B)$ is a polynomial in β whose degree is bounded by a number that does not depend on α (nor in β).
So there exists $K \geqslant 2$ such that the K-th derivative of $\sum_{i=1}^m f_k(A,B)$ relatively to θ is equal to 0. It means that the terms in front of θ^K is equal to 0. Hence the following equation :

$$\sum_{i=1}^m h_i \frac{< k_i, \beta >^K}{| < k_i, \alpha > |^{K-1}} = 0 \tag{4.102}$$

where $h_i = p_i g_{Id}^{(K)}(\delta)$ if $< k_i, \alpha >> 0$ and $h_i = p_i g_{Id}^{(K)}(\delta)$ if $< k_i, \alpha >< 0$. Now, since the k_i belong to Π and are all distinct, it is possible to choose α so that $< k_j, \alpha >> 0$ is arbitrary small while $< k_i, \alpha >$ remain bounded away from zero for every $i \neq j$. Thus, we must have $h_j = 0$. Yet, g is not a polynomial so there exists $\delta \in \mathbb{R}$ such that $g_{Id}^{(K)}(\delta) \neq 0$ and it gives us that $p_j = 0$. In the non symmetric case, Equation (4.102) becomes

$$\sum_{i=1}^{m} h_i \frac{< k_i, \beta >^K}{| < k_i, \alpha > |^{K-1}} = 0. \tag{4.103}$$

where $h_i = p_i g_{Id}^{(K)}(\delta) + p_{-i}\tilde{g}_{Id}^{(K)}(\delta)$ if $< k_i, \alpha >> 0$ and $h_i = p_i \tilde{g}_{Id}^{(K)}(\delta) + p_{-i} g_{Id}^{(K)}(\delta)$ if $< k_i, \alpha >< 0$.
Let us consider for example the case where the first alternative holds. As before, we must have $p_j g_{Id}^{(K)}(\delta) + p_{-j}\tilde{g}_{Id}^{(K)}(\delta) = h_j = 0$ for any choice of δ. Since we assume that Equation (4.93), this implies that $p_j = p_{-j} = 0$. □

Lemma 73 enables us to prove the following lemma.

Lemma 74. *For a typical $L \in \mathscr{S}_2$, for every $m \in \mathbb{N}-\{0\}$, for every family $(p_1, \cdots, p_m) \in \mathbb{Z}^m$, for every $k_1, \cdots, k_m \in \Pi$ all distinct if*

$$\sum_{i=1}^{m} p_i W_{k_i}(L) = 0 \tag{4.104}$$

then

$$\sum_{i=1}^{m} p_i f_{k_i} = 0.$$

In the non symmetric case, for a typical $L \in \mathscr{S}_2$, for every $m \in \mathbb{N} - \{0\}$, for every family $(p_{-m}, \cdots, p_{-1}, p_1, \cdots, p_m) \in \mathbb{Z}^{2m}$, for every $k_1, \cdots, k_m \in \Pi$ all distinct if

$$\sum_{i=1}^{m} p_i W_{k_i}(L) + \sum_{i=1}^{m} p_{-i} W_{-k_i}(L) = 0 \tag{4.105}$$

then

$$\sum_{i=1}^{m} p_i f_{k_i} + \sum_{i=1}^{m} p_{-i} f_{-k_i} = 0.$$

Proof. First, we note that, for every $k \in -\Pi \cup \Pi$,

$$W_k(L) = f_k\Big((e_1(L))_1, (e_2(L))_1, (e_1(L))_2, (e_2(L))_2\Big).$$

Thanks to this remark, we see that Lemma 74 is a direct consequence of the facts that $\mathscr{S}_2 = SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, that det is a polynomial function whereas $\sum_{i=1}^m p_i f_{k_i}$ and $\sum_{i=1}^m p_i f_{k_i} + \sum_{i=1}^m p_{-i} f_{-k_i}$ are polynomial functions, for a typical L, only in the trivial case according to Lemma 73. □

We can now give the proof of Proposition 35.

Proof of Proposition 35. It is a direct consequence of Lemma 73 and Lemma 74. □

4.4.4 Proof of Proposition 34

As we have at our disposal Proposition 35, the proof of Proposition 34 is the same as the proof of Proposition 5 in [89]. We are going to present it again for completeness. Before starting the proof of Proposition 34, we only need a simple lemma :

Lemma 75. *For every $m \in \mathbb{N} - \{0\}$, for every family $(p_1, \cdots, p_m) \in \mathbb{Z}^m - \{0\}$, for every $k_1, \cdots, k_m \in \Pi$, all distinct, for every $0 < \epsilon < 1$, there exists $a > 0$ and K_ϵ a measurable set of $\mathscr{S}_2$ such that $\tilde{\mu}_2(\mathscr{S}_2 - K_\epsilon) \leqslant \epsilon$ and such that for all $L \in K_\epsilon$,*

$$|\sum_{i=1}^{m} p_i W_{k_i}(L)| \geqslant a \ .$$

In the non symmetric case, for every $m \in \mathbb{N} - \{0\}$, for every family $(p_{-m}, \cdots, p_{-1}, p_1, \cdots, p_m) \in \mathbb{Z}^{2m} - \{0\}$, for every $k_1, \cdots, k_m \in \Pi$, all distinct, for every $0 < \epsilon < 1$, there exists $a > 0$ and K_ϵ a measurable set of $\mathscr{S}_2$ such that $\tilde{\mu}_2(\mathscr{S}_2 - K_\epsilon) \leqslant \epsilon$ and such that for all $L \in K_\epsilon$,

$$|\sum_{i=1}^{m} p_i W_{k_i}(L) + \sum_{i=1}^{m} p_{-i} W_{-k_i}(L)| \geqslant a \ .$$

Proof. It is a direct consequence of Proposition 35. □

Now, by using the foliation given by $\delta(\lambda)$ and previous results, we can now prove Proposition 34.

Proof of Proposition 34. The proof in all its generality can be made as in the case where $\tilde{\mu}_2 = \mu_2$. So, we will suppose for simplicity that $\tilde{\mu}_2 = \mu_2$.
We consider the case where Ω_γ is symmetric. The non symmetric case is similar.
Let $l \geqslant 1$. Let $\psi \in C_c^\infty(\mathscr{S}_2)$. Let $(p_1, \cdots, p_l) \in \mathbb{Z}^l - \{0\}$.
For all $\epsilon > 0$, we call $\mathcal{F}_\epsilon$ the tribe on $\mathscr{S}_2$ generated by the $J_\epsilon(L)$. Let $1 > \epsilon_1 \geqslant \epsilon_2 > 0$.
According to Lemma 70 and Lemma 75, there exists a measurable part K_{ϵ_1} such that $\mu_2(K_{\epsilon_1}) \geqslant 1 - \epsilon_1$, a real $M > 0$ and a real $a > 0$, such that

- for every $L \in K_{\epsilon_1}$, for every $\lambda \in [\frac{1}{1+\epsilon_1}, 1 + \epsilon_1]$:
$$|\psi(\delta(h)L) - \psi(L)| \leqslant M|h| \tag{4.106}$$
where $h = \lambda - 1$
- for every $L \in K_{\epsilon_1}$, for every $\lambda \in [\frac{1}{1+\epsilon_1}, 1 + \epsilon_1]$, Equation (4.78) is verified.
- for every $L \in K_{\epsilon_1}$,
$$|\sum_{i=1}^{l} p_i W_{k_i}(L)| \geqslant a \ . \tag{4.107}$$

Furthermore, we are going to suppose, even if it means making ϵ_2 goes to 0, that for every $L \in K_{\epsilon_1}$,

$$J_{\epsilon_2}(L) \subset K_{\epsilon_1}. \tag{4.108}$$

Claim. With these notations, we have, for all $L \in K_{\epsilon_1}$, that :

$$\mathbb{E}\left(\psi e(\sum_{i=1}^{l} p_i\theta_{k_i})|\mathcal{F}_{\epsilon_2}\right)(L) = O(\epsilon_2) + O(\frac{1}{at\epsilon_2}) + \frac{1}{a}O(\epsilon_2). \tag{4.109}$$

To this end, let us set, for all $\epsilon > 0$,

$$\delta(\epsilon) = 1 + \epsilon - \frac{1}{1+\epsilon}.$$

Then one has for every $L \in K_{\epsilon_1}$ according to Lemma 70 :

$$\begin{aligned}
&\mathbb{E}(\psi e(\sum_{j=1}^{l} p_j\theta_{k_j})|\mathcal{F}_{\epsilon_2})(L)\\
&= \frac{1}{\delta(\epsilon_2)}\int_{\frac{1}{1+\epsilon_2}-1}^{\epsilon_2}\left(\psi e(\sum_{j=1}^{l} p_j\theta_{k_j})\right)(\delta(h)L)dh\\
&= \psi(L)\frac{1}{\delta(\epsilon_2)}\int_{\frac{1}{1+\epsilon_2}-1}^{\epsilon_2}\left(e(\sum_{j=1}^{l} p_j\theta_{k_j})\right)(\delta(h)L)dh + O(\epsilon_2)\\
&= \left(\psi e(\sum_{j=1}^{l} p_j\theta_{k_j})\right)(L)\frac{1}{\delta(\epsilon_2)}\int_{\frac{1}{1+\epsilon_2}-1}^{\epsilon_2} e^{itD_1(L)h+itD_2(L,h)}dh + O(\epsilon_2)
\end{aligned} \tag{4.110}$$

where

$$D_1(L) = \sum_{j=1}^{l} p_j W_{k_j}(L), \tag{4.111}$$

$$D_2(L,h) = \sum_{j=1}^{l} p_j\theta_{k_j}(\delta(\lambda)L) - \sum_{j=1}^{l} p_j\theta_{k_j}(L) - D_1(L)h \text{ such that} \tag{4.112}$$

$$D_2(L,h) = O(h^2) \text{ and} \tag{4.113}$$

$D_2(L,\cdot)$ is smooth around 0.
Thus, by integrating by part and by using Equation (4.107), one gets that for all $L \in K_{\epsilon_1}$:

$$\begin{aligned}
&\frac{1}{\delta(\epsilon_2)}\int_{\frac{1}{1+\epsilon_2}-1}^{\epsilon_2} e^{itD_1(L)h+itD_2(L,h)}dh\\
&= \frac{1}{\delta(\epsilon_2)}\left(\left[\frac{e^{itD_1(L)h+itD_2(L,h)}}{itD_1(L)}\right]_{\frac{1}{1+\epsilon_2}-1}^{\epsilon_2} + \frac{1}{D_1(L)}\int_{\frac{1}{1+\epsilon_2}-1}^{\epsilon_2}(D_2(L,\cdot))'(h)e^{itD_1(L)h+itD_2(L,h)}dh\right)\\
&= O(\frac{1}{at\epsilon_2}) + \frac{1}{a}O(\epsilon_2).
\end{aligned} \tag{4.114}$$

Finally, Equation (4.110) and Equation (4.114) give the wanted claim.

Thanks to Equation (4.109), the fact that $\mu_2(K_{\epsilon_1}) \geqslant 1 - \epsilon_1$ and because of Equation

(4.108), we have that :

$$\begin{aligned}
&|\mathbb{E}\left(\psi e(\sum_{j=1}^{l} p_j\theta_{k_j})\right)| \\
&\leqslant |\mathbb{E}\left(\psi e(\sum_{j=1}^{l} p_j\theta_{k_j})\mathbf{1}_{K^c_{\epsilon_1}}\right)| + |\mathbb{E}\left(\psi e(\sum_{j=1}^{l} p_j\theta_{k_j})\mathbf{1}_{K_{\epsilon_1}}\right)| \\
&\leqslant \|\psi\|_\infty\epsilon_1 + O(\epsilon_2) + O(\frac{1}{at\epsilon_2}) + \frac{1}{a}O(\epsilon_2). \qquad (4.115)
\end{aligned}$$

By first choosing $\epsilon_1 > 0$ small enough (note that a depends on ϵ_1), then choosing $\epsilon_2 > 0$ and finally choosing t large enough, we obtain the wanted result. □

We are now brought back to the study of the convergence, in the symmetric case, of the sums of the type

$$\tilde{S}_A(\omega, L) = \sum_{k\in\Pi_A(L)} \frac{\rho_\gamma(k_1e_1(L) + k_2e_2(L))\tilde{Z}_k(\omega)}{\|k_1e_1(L) + k_2e_2(L)\|^{\frac{3}{2}}}$$

and of the sums of the type, in the non symmetric case,

$$\tilde{S}_A(\omega_1, \omega_2, L) = \sum_{k\in\Pi_A(L)} \frac{\rho_\gamma(k_1e_1(L) + k_2e_2(L))\tilde{Z}_k(\omega_1) + \rho_\gamma(-(k_1e_1(L) + k_2e_2(L)))\tilde{Z}_k(\omega_2)}{\|k_1e_1(L) + k_2e_2(L)\|^{\frac{3}{2}}}$$

where Z_k are non-zero real independent identically distributed random variables from $\Omega \ni \omega$ are symmetrical and have a compact support and where $(\omega_1, \omega_2) \in \Omega \times \Omega$. In the next section, we are going to study the sums of this type and prove Proposition 32, which will conclude the proof of Theorem 10.

4.5 Asymptotic study of $\tilde{S}_A(\omega, L)$ and of $\tilde{S}_A(\omega_1, \omega_2, L)$

The study is in fact essentially done in Section 5 of [89]. We only need to notice the following fact to apply the same method :

Lemma 76. *There exists $m, M > 0$ such that for every $\xi \in \mathbb{R}^2 - \{0\}$,*

$$M \geqslant \rho_\gamma(\xi) \geqslant m.$$

Proof. As γ is strictly convex, from the definition of ρ_γ, $\rho_\gamma > 0$.
Furthermore, for every $\xi \in \mathbb{R}^2 - \{0\}$,

$$\rho_\gamma(\xi) = \rho_\gamma(\frac{\xi}{\|\xi\|}).$$

S^1 being compact, we obtain immediately the wanted result. □

Proof of Proposition 32. We use the same method of the last section of [89] and use Lemma 76. □

4.6 Elements of proof of Theorem 12

To prove Theorem 12 we follow the same approach that was followed to prove Theorem 11.
Namely, as a first step, we prove the equivalent of Proposition 30. Namely, let us set

$$\Delta_{A,prime}(L,\alpha,t) = |\frac{\mathcal{R}(t\Omega_\gamma+\alpha,L)}{\sqrt{t}} - S_{A,prime}(L,\alpha,t)| \tag{4.116}$$

with

$$S_{A,prime}(L,\alpha,t) = \frac{1}{\pi}\sum_{\substack{l\in L^\perp\ prime\\ 0<\|l\|\leqslant A}} \frac{1}{\|l\|^{\frac{3}{2}}}\sum_{m\in\mathbb{N}-\{0\}}\frac{1}{m^{\frac{3}{2}}} \tag{4.117}$$

$$\Big(\rho_\gamma(l)\cos(2\pi t m Y_\gamma(l)+2\pi m<\alpha,l>-\frac{3\pi}{4})+\rho_\gamma(-l)\cos(2\pi t m Y_\gamma(-l)-2\pi m<\alpha,l>-\frac{3\pi}{4})\Big).$$

Then, one can prove, as Proposition 30 was proven :

Proposition 36. *For every $\beta>0$, for every $A>0$ large enough, for every t large enough, one has that :*

$$\mathbb{P}(\Delta_{A,prime}(L,\alpha,t)\geqslant\beta)\leqslant\beta.$$

Second, as Proposition 33 was already proven, the limit distribution of $S_{A,prime}(L,\alpha,t)$ when $t\to\infty$ and with $\alpha\in\mathbb{R}^2$ fixed and with L being distributed according to $\tilde{\mu}_2$ is, in the symmetric case, is the distribution of the almost-sure limit of

$$S_{\gamma,A}(\theta,L) = \frac{2}{\pi}\sum_{\substack{(k_1,k_2)\in\Pi\\ \|e(k_1,k_2,L)\|\leqslant A}}\frac{\rho_\gamma(e(k_1,k_2,L))\phi_\alpha(\theta_{(k_1,k_2)},e(k_1,k_2,L))}{\|e(k_1,k_2,L)\|^{\frac{3}{2}}} \tag{4.118}$$

when $A\to\infty$ and where $\theta=(\theta_{(k_1,k_2)})\in\mathbb{T}^\infty$ being distributed according to λ_∞ and L being distributed according to $\tilde{\mu}_2$ and where $e(k_1,k_2,L)=k_1e_1(L)+k_2e_2(L)$.
In the non symmetric case, the limit distribution of $S_{A,prime}(L,\alpha,t)$ when $t\to\infty$ and with $\alpha\in\mathbb{R}^2$ fixed and with L being distributed according to $\tilde{\mu}_2$ is the distribution of the almost-sure limit of

$$S_{\gamma,A}(\theta,L) = \frac{1}{\pi}\sum_{\substack{(k_1,k_2)\in\Pi\\ \|e(k_1,k_2,L)\|\leqslant A}}\frac{\phi_{\alpha,\gamma,2}(\theta_{(k_1,k_2)},e(k_1,k_2,L))}{\|e(k_1,k_2,L)\|^{\frac{3}{2}}} \tag{4.119}$$

when $A\to\infty$ and where $\theta=(\theta_{(k_1,k_2)})=(\theta_{1,(k_1,k_2)},\theta_{2,(k_1,k_2)})\in\mathbb{T}^{\infty,2}$ being distributed according to $\lambda_{\infty,2}$ and L being distributed according to $\tilde{\mu}_2$.
Finally, we conclude the proof by proving the equivalent of Proposition 32. To do so, we follow the exact same approach that was used to prove Proposition 32.

Chapter 5

Return to the case of boxes

(Résumé en français)Nous étudions l'erreur du nombre de points d'un réseau L qui appartiennent à un rectangle, centré en 0, dont les axes sont parallèles aux axes de coordonnées, dilaté d'un facteur t puis translaté d'un vecteur $X \in \mathbb{R}^2$. Quand nous considérons le moment d'ordre 2 de l'erreur relativement à $X \in \mathbb{R}^2/L$, on montre que, quand t est aléatoire et devient grand et quand l'erreur est normalisée par une quantité qui se comporte, dans le cas admissible, comme $\sqrt{\log(t)}$, elle converge en loi vers une constante positive explicite. Dans le cas d'un réseau L typique, on montre que ce résultat tient toujours mais la normalisation est toutefois plus importante, autour de $\log(t)$. On montre aussi que quand $L = \mathbb{Z}^2$, l'erreur, quand elle est normalisée par t, converge en loi quand t est aléatoire et devient grand et on calcule les moments de la loi limite.

(Engish abstract)We study the error of the number of points of a lattice L that belong to a rectangle, centred at 0, whose axes are parallel to the coordinate axes, dilated by a factor t and then translated by a vector $X \in \mathbb{R}^2$. When we consider the second order moment of the error relatively to $X \in \mathbb{R}^2/L$, one shows that, when t is random and becomes large and when the error is normalized by a quantity which behaves, in the admissible case, as $\sqrt{\log(t)}$, it converges in distribution to an explicit positive constant. In the case of a typical lattice L, we show that this result still holds but the normalisation is more important, around $\log(t)$. We also show that when $L = \mathbb{Z}^2$, the error, when normalized by t, converges in distribution when t is random and becomes large and we compute the moments of the limit distribution. study the existence of the moments of its distribution.

Contents

5.1 Introduction **147**

5.2 Calculation of a Fourier transform, heuristic and plan of the rest of the paper **151**

5.2.1 Calculation of the Fourier transform of $\mathbf{1}_{tP+X}$ 152

5.2.2 Elements of heuristic and plan of the rest of the paper 152

5.3 Proof of Theorem 14 . . . . . **155**
5.4 Proof of Theorem 13 . . . . . **157**
5.4.1 Estimates of $V(L,t)$ when L is admissible 157
5.4.2 Smoothing and reduction to the study of a Fourier series . . 159
5.4.3 Asymptotic behaviour of S 168
5.4.4 Conclusion 177
5.5 Proof of Theorem 15 . . . . . **178**
5.5.1 Estimation of $\tilde{V}(L,r)$ in the typical case 178
5.5.2 Result about $V(L,t)$ in the typical case 180
5.5.3 Conclusion 181
5.6 Proof of Theorem 16 . . . . . **182**
5.6.1 An expression of $\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}$ 182
5.6.2 Reduction of the study of $\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}$ 183
5.6.3 Convergence in distribution of $\Delta(t,x)$ 184
5.6.4 Conclusion 188

5.1 Introduction

Let P be a measurable subset of $\mathbb{R}^d$ of non-zero finite Lebesgue measure. We want to evaluate the following cardinal number when $t \to \infty$:

$$N(tP+X,L) = |(tP+X)\cap L|$$

where $X \in \mathbb{R}^d$, L is a lattice of $\mathbb{R}^d$ and $tP+X$ denotes the set P dilated by a factor t relatively to 0 and then translated by the vector X.
Under mild regularity conditions on the set P, one can show that :

$$N(tP+X,L) = t^d \frac{\mathrm{Vol}(P)}{\mathrm{Covol}(L)} + o(t^d)$$

where $o(f(t))$ denotes a quantity such that, when divided by $f(t)$, it goes to 0 when $t \to \infty$ and where $\mathrm{Covol}(L)$ is the volume of a fundamental set of the lattice L.
We are interested in the error term

$$\mathcal{R}(tP+X,L) = N(tP+X,L) - t^d \frac{\mathrm{Vol}(P)}{\mathrm{Covol}(L)}.$$

In the case where $d=2$ and where P is the unit disk $\mathbb{D}^2$, Hardy's conjecture in [36] stipulates that we should have for all $\epsilon > 0$,

$$\mathcal{R}(t\mathbb{D}^2,\mathbb{Z}^2) = O(t^{\frac{1}{2}+\epsilon})$$

where $Y = O(X)$ means that there exists $D > 0$ such that $|Y| \leqslant D|X|$.
One of the result in this direction has been established by Iwaniec and Mozzochi in [43]. They have proven that for all $\epsilon > 0$,

$$\mathcal{R}(t\mathbb{D}^2,\mathbb{Z}^2) = O(t^{\frac{7}{11}+\epsilon}).$$

This result has been recently improved by Huxley in [42]. Indeed, he has proven that :

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^K \log(t)^\Lambda)$$

where $K = \frac{131}{208}$ and $\Lambda = \frac{18627}{8320}$.
In dimension 3, Heath-Brown has proven in [40] that :

$$\mathcal{R}(t\mathbb{D}^3, \mathbb{Z}^3) = O(t^{\frac{21}{16}+\epsilon}).$$

These last two results are all based on estimating what are called *exponential sums*. Furthermore, in both cases, the error is considered in a deterministic way.
Another approach was followed first by Heath-Brown in [38] and then by Bleher, Cheng, Dyson and Lebowitz in [11]. They took interest in the case where the dilatation parameter t is random. More precisely, they assumed that t was being distributed according to the measure $\rho(\frac{t}{T})dt$ (that is absolutely continuous relatively to Lebesgue measure) and where ρ is a probability density on $[0,1]$ and T is parameter that goes to infinity. In that case, Bleher, Cheng, Dyson and Lebowitz showed the following result (which generalizes the result of Heath-Brown) :

Theorem ([11])**.** *Let $\alpha \in [0,1[^2$. There exists a probability density p_α on $\mathbb{R}$ such that for every piecewise continuous and bounded function $g : \mathbb{R} \longrightarrow \mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g\Big(\frac{\mathcal{R}(t\mathbb{D}^2+\alpha, \mathbb{Z}^2)}{\sqrt{t}}\Big)\rho(\frac{t}{T})dt = \int_{\mathbb{R}} g(x)p_\alpha(x)dx.$$

Furthermore p_α can be extended as an analytic function over $\mathbb{C}$ and satisfies that for every $\epsilon > 0$,

$$p_\alpha(x) = O(e^{-|x|^{4-\epsilon}})$$

when $x \in \mathbb{R}$ and when $|x| \to \infty$.

We want to follow this approach on another problem. Namely, let us give $a > 0$ and $b > 0$ and let us define $\mathrm{Rect}(a,b)$ the rectangle centred around $(0,0)$ whose summits are (a,b), $(-a,b)$, $(-a,-b)$ and $(a,-b)$.
Let us recall the following definitions :

Definition 20. *For a lattice L of $\mathbb{R}^d$, its dual lattice $L^\perp$ is defined by*

$$L^\perp = \{x \in \mathbb{R}^d \mid \forall l \in L,\ < l, x > \in \mathbb{Z}\}$$

where $<,>$ is the usual euclidean scalar product over $\mathbb{R}^d$.

Definition 21. *A lattice L of $\mathbb{R}^d$ is called admissible if there exists $C > 0$ such that for all $l = (l_1, \cdots, l_d) \in L - \{0\}$,*

$$|Num(l)| \geqslant C$$

where $Num(l) = l_1 \cdots l_d$.
During the rest of this article, we are going to use the following notation :

$$Num(L) = \inf\{|Num(l)| \mid l \in L - \{0\}\}.$$

With this notation, saying that L is admissible is equivalent to saying that $Num(L) > 0$.

We recall that if L is an admissible lattice, $L^\perp$ is also admissible (see, for example, [85]).
Let us also define the following quantity :

Definition 22. *For every lattice L of $\mathbb{R}^2$, for every $t > 0$, one sets :*

$$V(L,t) = \sum_{\substack{l\in L\\ 0<\|l\|\leqslant t}} \frac{1}{l_1^2 l_2^2}$$

when for all $l \in L$ such that $0 < \|l\| \leqslant t$, $Num(l) \neq 0$ and where $\|\cdot\|$ is the usual euclidean norm over $\mathbb{R}^2$.

We are going to use the following notation : for $f(t,X)$ a function of $\mathbb{R}^+ \times \mathbb{R}^2$ such that $f(t,\cdot)$ is L-periodic,

$$\mathbb{E}_{X\in\mathbb{R}^2/L}(f)(t) = \frac{1}{covol(L)} \int_{X\in\mathbb{R}^2/L} f(t,X)dX.$$

Let us give a probability density ρ over $[0,1]$.
One of the goal of this article is to prove the following theorem :

Theorem 13. *Let L be an admissible lattice of $\mathbb{R}^2$.*
Then, first, there exists $C > 0$ such that for all t large enough :

$$\frac{1}{C}\log(t) \leqslant V(L,t) \leqslant C\log(t)$$

and we are going to write it $V(L,t) = \Theta(\log(t))$.
Second, if $P = Rect(a,b)$,

$$\mathbb{E}_{X\in\mathbb{R}^2/L}\left((\frac{\mathcal{R}(tP+X,L)}{\sqrt{V(L^\perp,t)}})^2 \right) \text{ converges in distribution and in probability towards } \frac{1}{4\pi^4 Covol(L)^2}$$

when $t \in [0,T]$ is distributed according to $\frac{1}{T}\rho(\frac{t}{T})dt$ and when $T \to \infty$.

Theorem 13 can be interpreted as followed : after averaging on X, when t is large and random, the error committed by making the approximation

$$N(t\mathrm{Rect}(a,b) + X, \Gamma_{\alpha,\alpha'}) \approx t^2 \frac{\mathrm{Area}(\mathrm{Rect}(a,b))}{\mathrm{Covol}(\Gamma_{\alpha,\alpha'})}$$

is of order $\sqrt{\log(t)}$. This normalization is in fact suggested by [88].
This theorem is also in fact suggested by Theorem 14 that we will prove (see the next section for some heuristic explanations). We are going to state it.
Let us designate the space of unimodular lattices of $\mathbb{R}^d$ by the notation $\mathscr{S}_d$.
We are calling Δ_r the following set :

$$\Delta_r = \{\mathrm{Diag}(e^{t_1},\cdots,e^{t_d}) \mid (t_1,\cdots,t_d) \in \mathbb{Z}^d,\ t_1+\cdots+t_d = 0 \text{ and } \|(t_1,\cdots,t_d)\| \leqslant r\}.$$

We have that $\Delta_r \subset \mathscr{S}_d$ and $|\Delta_r| = n_r r^{d-1} + o(r^{d-1})$ when $r \to \infty$.
We also call

$$\Delta = \{\mathrm{Diag}(e^{t_1},\cdots,e^{t_d}) \mid (t_1,\cdots,t_d) \in \mathbb{Z}^d \text{ and } t_1+\cdots+t_d = 0\} \text{ and} \tag{5.1}$$

for every lattice L of $\mathbb{R}^d$, we call

$$\|L\| = \min\{\|l\| \mid l \in L - \{0\}\}.$$

With these notations, the Theorem 14 states as the following :

Theorem 14. *Let $L \in \mathscr{S}_d$ be an admissible lattice. Let us give $(\Theta_\delta)_{\delta\in\Delta}$ a sequence of independent identically distributed real random variables that are symmetrical and admit moment of order 3 and whose alea ω belongs to a probability space Ω.*
Let us set :

$$\tilde{V}(L,r) = \sum_{\delta\in\Delta_r} \frac{1}{\|\delta L\|^{2d}}$$

and

$$\tilde{S}(L,\omega,r) = \sum_{\delta\in\Delta_r} \frac{\theta_\delta(\omega)}{\|\delta L\|^{d}}.$$

Then one has :

$$\tilde{V}(L,r) = \Theta(r^{d-1})$$

and $\frac{\tilde{S}(L,\omega,r)}{\sqrt{\tilde{V}(L,r)}}$ converges in distribution towards the standard normal distribution when $r \to \infty$.

Theorem 14 says that for an admissible lattice $\tilde{S}(L,\omega,r)$, normalized by a quantity of order $r^{\frac{d-1}{2}}$, converges, in distribution, towards a normal centred distribution. When L is typical, this must not be true : the regularization must be stronger because the orbit δL goes repeatedly into the cusp of the space $\mathscr{S}_d$, *id est* the zone where $\|\delta L\|$ is small.
In the typical case (typical in the sense of the unique Haar probability measure μ_d over $\mathscr{S}_d$), we are going to prove the following result:

Theorem 15. *For every $\epsilon > 0$, for a typical $L \in \mathscr{S}_d$, one has that*

$$\tilde{V}(L,r) = O(r^{2d-2+\epsilon}) \tag{5.2}$$

and

$$\frac{\tilde{V}(L,r)}{r^{d-1}} \underset{r\to\infty}{\to} \infty. \tag{5.3}$$

In the case where $d = 2$, one has also that for every $\epsilon > 0$, for L a typical lattice, one has that

$$V(L,t) = O(\log(t)^{2+\epsilon})$$

and

$$\frac{V(L,t)}{\log(t)^2} \underset{t\to\infty}{\to} \infty.$$

Furthermore, in this case, if $P = Rect(a,b)$,

$$\mathbb{E}_{X\in\mathbb{R}^2/L}\left((\frac{\mathcal{R}(tP+X,L)}{\sqrt{V(L^\perp,t)}})^2 \right) \text{ converges in distribution and in probability towards } \frac{1}{4\pi^4 Covol(L}$$

when $t \in [0,T]$ is distributed according to $\frac{1}{T}\rho(\frac{t}{T})dt$ and when $T \to \infty$.

In particular, in the typical case, the convergence in distribution and in probability of Theorem 13 still holds. Yet, the normalization is larger in this case : "around" $\log(t)$ whereas before it was in $\sqrt{\log(t)}$.
Finally, we are also going to tackle another extreme case : the case where $L = \mathbb{Z}^2$. $\mathbb{Z}^2$ is a unimodular lattice such that there exists $l \in L - \{0\}$ such that $Num(l) = 0$ (for example $(1,0)$). Typically, for $L \in \mathscr{S}_d$, $\mathrm{Num}(L)$ is null but there does not exists a non-zero $l \in L$ such that $Num(l) = 0$.
In that case, with ρ being a probability density over $[0,1]$, we are going to prove the following theorem :

Theorem 16. *For all $x \in \mathbb{R}$, when $t \in [0,T]$ is distributed according to the probability measure $\frac{1}{T}\rho(\frac{t}{T})dt$ on $[0,T]$ then, when $T \to \infty$, $\frac{\mathcal{R}(tRect(a,a)+(x,x),\mathbb{Z}^2)}{t}$ converges in distribution. Furthermore, the limit distribution β has a compact support included in $[-4,4]$ and for every $k \in \mathbb{N}$, one has that*

$$\int_{x\in\mathbb{R}} x^k d\beta(x) = a_k$$

where

$$a_k = \frac{4^k(1+(-1)^k)(y^{k+1}+(1-y)^{k+1})}{2(k+1)}$$

with $y = |t_{2,0} - t_{1,0}|$ where $t_{2,0}$ is the first $t \geqslant 0$ such that $-t + x \in \mathbb{Z}$ and $t_{1,0}$ is the first $t \geqslant 0$ such that $t + x \in \mathbb{Z}$.

In particular, we see that the normalization in this case is much more important than before and that the error $\mathcal{R}(t\mathrm{Rect}(a,a)+(x,x),\mathbb{Z}^2)$ in this case is of order t.
In the next section, we are going to give some heuristic ideas about all these results and then give the plan of the rest of the paper.

5.2 Calculation of a Fourier transform, heuristic and plan of the rest of the paper

To give some heuristic explanations, we will apply the Poisson formula which states that for a smooth and compact supported function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and for L a lattice of $\mathbb{R}^2$, one has that for every $X \in \mathbb{R}^2$

$$\sum_{l\in L} f(l+X) = \frac{1}{\mathrm{Covol}(L)} \sum_{l\in L^\perp} \hat{f}(l) e^{-2i\pi<l,X>} \tag{5.4}$$

where the Fourier transform $\hat{f}$ is defined by : for every $\xi \in \mathbb{R}^2$

$$\hat{f}(\xi) = \int_{x\in\mathbb{R}^2} f(x) e^{2i\pi<\xi,x>} dx. \tag{5.5}$$

In our case, we are interested into the following quantity :

$$N(tP+X,L) = \sum_{l\in L} \mathbf{1}_{tP+X}(l) \tag{5.6}$$

where $\mathbf{1}_{tP+X}$ is the indicator function of the set $tP+X$ with $P = \mathrm{Rect}(a,b)$.
Yet, the function $\mathbf{1}_{tP+X}$ is not smooth enough. But the Poisson formula gives us a good idea of the phenomena that are at play and that is because the Poisson formula applies after having realized a *smoothing* of the studied problem (see Section 4). So, we are going to calculate the Fourier transform of $\mathbf{1}_{tP+X}$ in the next subsection. Then we will give the heuristic and announce the plan of the rest of the paper.

5.2.1 Calculation of the Fourier transform of $\mathbf{1}_{tP+X}$

The main object of this subsection is to prove the following proposition :

Proposition 37. *For $P = Rect(a,b)$, for $l \in \mathbb{R}^2$, for $t > 0$, one has :*

$$\widehat{\mathbf{1}_{tP+X}}(l) = \frac{1}{\pi^2}\frac{\sin(2\pi t l_1 a)}{l_1}\frac{\sin(2\pi t l_2 b)}{l_2}e^{2i\pi<l,X>}$$

where we convey that $\frac{\sin(0)}{0} = 1$.

Proof. Let $l \in \mathbb{R}^2$ and $t > 0$. By making the change of variable $x = ty + X$, one has that :

$$\widehat{\mathbf{1}_{tP+X}}(l) = e^{2i\pi<l,X>}t^2\int_{y\in P} e^{2i\pi t<l,y>}dy. \tag{5.7}$$

Yet, we recall that $P = \mathrm{Rect}(a,b)$ and so one has :

$$\int_{y\in P} e^{2i\pi t<l,y>} = \frac{\sin(2\pi t l_1 a)}{\pi t l_1}\frac{\sin(2\pi t l_2 b)}{\pi t l_2}. \tag{5.8}$$

So, with Equation (5.7) and Equation (5.8), one gets that :

$$\widehat{\mathbf{1}_{tP+X}}(l) = \frac{e^{2i\pi<l,X>}}{\pi^2}\frac{\sin(2\pi t l_1 a)}{l_1}\frac{\sin(2\pi t l_2 b)}{l_2}. \tag{5.9}$$

□

Now, we are going to give some heuristic explanations about the main results of this paper and the plan of the rest of the paper.

5.2.2 Elements of heuristic and plan of the rest of the paper

Heuristically, the Poisson formula (see Equation (5.4)) gives us that :

$$N(tP+X,L) = \sum_{l\in L}\mathbf{1}_{tP+X}(l) = \frac{1}{\mathrm{Covol}(L)}\sum_{l\in L^\perp}\frac{1}{\pi^2}\frac{\sin(2\pi t l_1 a)}{l_1}\frac{\sin(2\pi t l_2 b)}{l_2}e^{2i\pi<l,X>} \tag{5.10}$$

with $P = \mathrm{Rect}(a,b)$.
Yet, and this link was used a lot in [88], in the typical case, the smallest $|l_1 l_2|$ can be seen as a $\|\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix} L\|$. This link can also be seen in the following proposition (that will be useful for us later) :

Proposition 38 ([82]).

$$Num(L) = d^{-\frac{d}{2}} \inf\{\|\delta L\|^d \mid \delta \in \Delta\}$$

where Δ *was defined by Equation* 5.1.

So it suggests that the term $\frac{\sin(2\pi t l_1 a)}{l_1}\frac{\sin(2\pi t l_2 b)}{l_2}e^{2i\pi<l,X>}$ can be as seen as $\frac{\theta_\delta(\omega)}{\|\delta L\|^d}$ and so there exists a link between Theorem 13 and Theorem 14.
Yet, to prove Theorem 13 from Theorem 14, there are two difficulties. The first one is that Theorem 13 and Theorem 14 are about admissible unimodular lattices, which form a negligible set with respect to μ_2 and the relation about $|l_1 l_2|$ and $\|\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} L\|$ is not certain in this case.
The second one is the fact that the $\sin(2\pi t l_1 a)$ and $\sin(2\pi t l_2 a)$ do not behave like independent random variable when $T \to \infty$ and with t being distributed according to the probability measure $\frac{1}{T}\rho(\frac{t}{T})dt$. Sure we can, like we have done in [88] and in [89], reduce the study to the $l \in L^\perp$ that are prime, which is a notion defined by:

Definition 23. *For a lattice* L*, we say that a vector of* L *is prime if it not a non-trivial integer multiple of another vector of* L*.*

Yet, even in that case, it is not true that the $\sin(2\pi t l_1 a)$ and the $\sin(2\pi t l_2 b)$ behave asymptotically like independent random variables.
Indeed, let $\alpha \neq \alpha'$ be real irrational numbers with bounded partial quotients in their continued fractions and let us define :

$$\Gamma_{\alpha,\alpha'} = \{(n + m\alpha, n + m\alpha') \mid n, m \in \mathbb{Z}^2\}. \tag{5.11}$$

Then, we know from [85] that $\Gamma_{\alpha,\alpha'}$ is admissible in the sense of Definition 20. So, it is also the case of

$$\Gamma^\perp_{\alpha,\alpha'} = \frac{1}{\alpha' - \alpha}\{(n + m\alpha, n + m\alpha') \mid n, m \in \mathbb{Z}^2\}. \tag{5.12}$$

Then, if we consider, for $k \geqslant 2$,

$$v_1(k) = \frac{1}{\alpha' - \alpha}(k + (k+1)\alpha, k + (k+1)\alpha'), \tag{5.13}$$

$$v_2(k) = \frac{1}{\alpha' - \alpha}(k + 1 + (k+2)\alpha, k + 1 + (k+2)\alpha'), \tag{5.14}$$

$$v_3(k) = \frac{1}{\alpha' - \alpha}(k + 2 + (k+3)\alpha, k + 2 + (k+3)\alpha'), \tag{5.15}$$

we have that : $v_1(k), v_2(k)$ and $v_3(k)$ are prime vectors of $\Gamma^\perp_{\alpha,\alpha'}$ and

$$-v_1(k) + 2v_2(k) = v_3(k) \tag{5.16}$$

which prevents from getting the wanting asymptotic independence.
That is why, to get rid of this problem, we consider $\mathbb{E}_{X\in\mathbb{R}^2/L}\left((\frac{\mathcal{R}(tP+X,L)}{\sqrt{V(L^\perp,t)}})^2\right)$ instead of

considering $\frac{\mathcal{R}(tP+X,L)}{\sqrt{V(L^\perp,t)}}$.
By doing so, the Parseval formula gives us that, if we want to prove Theorem 13, heuristically we have to study the convergence in distribution of

$$\frac{G(L^\perp,t)}{V(L^\perp,t)} = \frac{2}{\pi^4 \mathrm{Covol}(L)^2 V(L^\perp,t)} \sum_{l\in J_2(L^\perp,t)} \frac{1}{(l_1 l_2)^2} \sum_{k=1}^{\infty} \frac{(\sin(2\pi k t l_1)\sin(2\pi k t l_2))^2}{k^4} \tag{5.17}$$

where

$$J_2(L^\perp,t) = \{l \in L^\perp \mid 0 < \|l\| \leqslant t \ , \ l \text{ prime and } l_1 > 0\} \tag{5.18}$$

and where $t \in [0,T]$ is being distributed according to the probability measure $\frac{1}{T}\rho(\frac{\cdot}{T})dt$ and where $T \to \infty$. We have implicitly used Equation (5.10) and centred the sum of the right-hand side on the prime vectors whose norm are smaller than t. We have to cut the sum because of a problem of convergence.
The final ideas that we use to prove Theorem 13 are, first, the idea to use the well-known formula $\sin^2(\cdot) = \frac{1-\cos(2\cdot)}{2}$. Then there are two different types of quantities that must be dealt with.
The first one is of the type :

$$\frac{1}{V(L^\perp,t)} \sum_{\substack{l\in L^\perp-\{0\} \\ \|l\|\leqslant t}} \frac{1}{l_1^2 l_2^2} \tag{5.19}$$

and we show quickly that such a quantity converges almost surely when $T \to \infty$.
The other type of quantities have a term of the form $\cos(tf(l))$ or a product of two $\cos(tf(l))$ (with f being a function of l) in the numerator of the terms. In that case, we show that the moment of order 2 of the quantities of this type converge to 0 when $T \to \infty$.
We use the moment of order 2 because it is quite convenient since we are dealing with numerators that have a term of the form $\cos(tf(l))$. Yet, this last part is a calculatory one. Furthermore, we have to underline the fact that these calculations still work for the typical L considered in Theorem 15, not only for admissible lattice L. They are, in that sense, intrinsic. The estimate of $V(L,t)$ contained in Theorem 13 is basically derived from the estimate of the integral

$$\int_{\substack{l\in\mathbb{R}^2 \\ A\leqslant\|l\|\leqslant t \\ |\mathrm{Num}(l)|\geqslant C}} \frac{1}{(l_1 l_2)^2}$$

where $A > 0$, $C > 0$ and $t \to \infty$. It concludes the heuristic explanations that we wanted to give about Theorem 13.

The proof of Theorem 14 is quicker and it is basically an application of the central limit theorem with error term (see Theorem 17).
We already have said a few words about the last part of Theorem 15. We will now give some explanations about the estimates V and $\tilde{V}$ presented in this theorem.
The estimates of $\tilde{V}$ are applications of the ergodic theorem whereas the estimates of V is deduced from an upper estimate of

$$\int_{\substack{l\in\mathbb{R}^2 \\ A\leqslant\|l\|\leqslant t \\ |\mathrm{Num}(l)|\geqslant C|\log(\|l\|)|^{-1-\alpha}}} \frac{1}{l_1^2 l_2^2} dl_1 dl_2$$

where $C > 0$, $A > 0$ and $\alpha > 0$ with $t \to \infty$ and from results of [87] and of [82] (see Theorem 20).
Finally, concerning the Theorem 16, the appropriate normalization of the error term $\mathcal{R}$ is t because, to within a multiplicative factor, it is the perimeter of $t\mathrm{Rect}(a,a)+X$. Indeed, in this case, we can easily compute $\mathcal{R}$. The wanted convergence of Theorem 16 is then deduced from it and from the application of the method of moments.

Plan of the paper. In Section 3, we prove Theorem 14.
In Section 4, we prove Theorem 13. The first subsection is dedicated to obtain the upper and lower estimate on $V(L,t)$ in the case where L is admissible.
Let us set :

$$S(L,X,t) = \frac{2}{\pi^2 covol(L)} \sum_{l \in J_2(L^\perp,t)} \frac{1}{l_1 l_2} \sum_{k=1}^{\infty} \frac{\sin(2\pi k t l_1)\sin(2\pi t l_2)\cos(2\pi k < l, X >)}{k^2}$$

where we recall that $J_2(L^\perp,t) = \{l \in L^\perp \mid 0 < \|l\| \leqslant t \text{ , } l \text{ prime and } l_1 > 0\}$.
The second subsection is dedicated to show that we can reduce the study of the convergence in distribution of $\mathbb{E}_{X\in\mathbb{R}^2/L}\left((\frac{\mathcal{R}(tP+X,L)}{\sqrt{V(L^\perp,t)}})^2\right)$ to the study of the convergence in distribution of $\mathbb{E}_{X\in\mathbb{R}^2/L}\left(\frac{S(L,X,t)}{\sqrt{V(L^\perp,t)}}^2\right)$ when t is being distributed according to $\frac{1}{T}\rho(\frac{t}{T})dt$ and when $T \to \infty$ (see Proposition 40). The third subsection is dedicated to show that $\mathbb{E}_{X\in\mathbb{R}^2/L}\left(\frac{S(L,X,t)}{\sqrt{V(L^\perp,t)}}^2\right)$ (see Proposition 44) converges in distribution when t is being distributed according to $\frac{1}{T}\rho(\frac{t}{T})dt$ and when $T \to \infty$. The fourth subsection concludes the proof of Theorem 13.
In Section 5, we give the proof of Theorem 15. The first subsection is dedicated to the estimates of $\tilde{V}(L,r)$. The second subsection is dedicated to the estimates of $V(L,t)$. The third subsection concludes the proof of Theorem 15, using, in particular, the third subsection of Section 4.
In Section 6, we give the proof of Theorem 16. In the first subsection, we give a simple expression of $\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}$ (and before we see that the a, in the statement of Theorem 16, can be chosen equal to 1). In the second subsection, we can reduce the study of the convergence in distribution of $\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}$, when $T \to \infty$, to a simpler quantity. In the third subsection, we are going to apply the method of moments to get the convergence of this simpler quantity. In the fourth subsection, we conclude the proof of Theorem 16.

5.3 Proof of Theorem 14

To prove Theorem 14, we need to recall two important theorems. The first one is the central limit theorem with error term (see for example [4]) :

Theorem 17. *Let $(Z_i)_{1\leqslant i\leqslant n}$ be a sequence of real random variables independent such that for all $1 \leqslant i \leqslant n$, $\mathbb{E}(Z_i) = 0$ and Z_i admits a moment of order 3. Let us call :*

$$T = \sum_{i=1}^{n} \mathbb{E}(|Z_i|^3) \text{ and } V = \sum_{i=1}^{n} \mathbb{E}(Z_i^2).$$

Then for every real λ, one has

$$|\mathbb{P}(\frac{\sum_{i=1}^{n} Z_i}{\sqrt{V}} \geqslant \lambda) - \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{\infty} e^{-\frac{u^2}{2}} du| < \frac{40T}{V^{\frac{3}{2}}}.$$

The second one is a theorem due to Minkowski :

Theorem 18. *There exists $K > 0$ such that for all $L \in \mathscr{S}_d$,*

$$\|L\| \leqslant K.$$

We can now prove Theorem 14.

Proof of Theorem 14. Let L be an admissible lattice. So, according to Proposition 38 and Equation 5.1, there exists $C > 0$ such that for every $\delta \in \Delta$, one has

$$\|\delta L\| \geqslant C. \tag{5.20}$$

Theorem 18 gives us that :

$$\|\delta L\| \leqslant K. \tag{5.21}$$

Because of Equation (5.20) and Equation (5.21), one has

$$|\Delta_r| K^{-2d} \leqslant \tilde{V}(L, r) \leqslant |\Delta_r| C^{-2d}. \tag{5.22}$$

Yet, the cardinal number of Δ_r satisfies that

$$|\Delta_r| = n_r r^{d-1} + o_r(r^{d-1}). \tag{5.23}$$

So we get the first wanted result.
According to Theorem 18, one has :

$$|\mathbb{P}(\frac{\tilde{S}(L, \omega, r)}{\sqrt{\tilde{V}(L, r)}} \geqslant \lambda) - \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{\infty} e^{-\frac{u^2}{2}} du| < \frac{40T_1(r)}{V_1(r)^{\frac{3}{2}}} \tag{5.24}$$

where

$$V_1(r) = \sum_{\delta \in \Delta_r} \mathbb{E}((\frac{\theta_\delta}{\|\delta L\|^d})^2) \tag{5.25}$$

and

$$T_1(r) = \sum_{\delta \in \Delta_r} \mathbb{E}(|\frac{\theta_\delta}{\|\delta L\|^d}|^3). \tag{5.26}$$

Yet, one has :

$$V_1(r) = \Theta(r^{d-1}) \tag{5.27}$$

and

$$T_1(r) = \Theta(r^{d-1}) \tag{5.28}$$

because the θ_δ are independent, symmetrical, identically distributed and their common distribution admit a moment of order 3 and because of Equation (5.20) and of Equation (5.21).
Thus, one gets that :

$$|\mathbb{P}(\frac{\tilde{S}(L, \omega, r)}{\sqrt{\tilde{V}(L, r)}} \geqslant \lambda) - \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{\infty} e^{-\frac{u^2}{2}} du| = O(\frac{1}{r^{\frac{d-1}{2}}}). \tag{5.29}$$

From this last Equation and by making $r \to \infty$, one gets the wanted result. □

5.4 Proof of Theorem 13

5.4.1 Estimates of $V(L,t)$ when L is admissible

In this subsection, we are going to prove the following proposition, which is the first assertion of Theorem 13

Proposition 39. *Let L be an admissible lattice of $\mathbb{R}^2$. One has that there exists $C>0$ (big enough) such that for all t large enough :*

$$\frac{1}{C}\log(t) \leqslant V(L,t) \leqslant C\log(t).$$

The main idea is that $V(L,t)$ can be compared to

$$\int_{\substack{l\in\mathbb{R}^2\\ A\leqslant\|l\|\leqslant t+B\\ |\mathrm{Num}(l)|\geqslant C}} \frac{1}{l_1^2 l_2^2} dl_1 dl_2$$

when L is admissible and when C, A and B are positive constants that are well-chosen. And this last integral behaves like $\log(t)$ to within one multiplicative constant.
This last fact is the object of the following lemma. In this lemma, we use the notation

$$f(t) \sim_{t\to\infty} g(t)$$

to express the facts that, for t large enough, $g(t)\neq 0$ and that $\frac{f(t)}{g(t)}\to 1$ when $t\to\infty$.

Lemma 77. *For all $C>0$, for all $A>0$, for all t large enough, one has :*

$$\int_{\substack{l\in\mathbb{R}^2\\ A\leqslant\|l\|\leqslant t\\ |Num(l)|\geqslant C}} \frac{1}{l_1^2 l_2^2} dl_1 dl_2 \sim_{t\to\infty} \frac{8\log(t)}{C}.$$

Proof. Let us set :

$$J(t) = \int_{\substack{l\in\mathbb{R}^2\\ A\leqslant\|l\|\leqslant t\\ |\mathrm{Num}(l)|\geqslant C}} \frac{1}{l_1^2 l_2^2} dl_1 dl_2 \tag{5.30}$$

and let us remark by the way that it is enough to prove the result for A large enough. By passing into polar coordinates (r,θ) and by using the symmetries, one has :

$$J(t) = 8\int_{\substack{A\leqslant r\leqslant t\\ 0\leqslant\theta\leqslant\frac{\pi}{4}\\ \sin(2\theta)\geqslant\frac{2C}{r^2}}} \frac{1}{r^3}\frac{4}{\sin(2\theta)^2} dr d\theta. \tag{5.31}$$

By making the changes of variable $\theta' = 2\theta$ and, then, $u=\tan(\theta')$, one gets from Equation (5.31) and by taking A large enough :

$$J(t) = 16\int_{A\leqslant r\leqslant t} \frac{1}{r^3}(-1+\frac{r^2}{2C}\sqrt{1-(\frac{2C}{r^2})^2})dr. \tag{5.32}$$

From Equation (5.32) and because of the fact that $\frac{1}{r^3}(-1+\frac{r^2}{2C}\sqrt{1-(\frac{2C}{r^2})^2}) = \frac{1}{2rC}+O(\frac{1}{r^2})$ (when $r \to \infty$), we finally get that :

$$J(t) \sim_{t\to\infty} \frac{8}{C}\log(t). \tag{5.33}$$

So we get the wanted result. □

We can now prove Proposition 39 :

Proof of Proposition 39. First, as L is admissible, there exists $C > 0$ such that for all $l \in L - \{0\}$,

$$|l_1 l_2| \geqslant C.$$

So, according to Definition 22 and by integration by parts, there exists $A > 0$, $B > 0$ and $D > 0$ such that :

$$\frac{1}{D}\int_{\substack{l\in\mathbb{R}^2\\ A\leqslant\|l\|\leqslant t+B\\ |\mathrm{Num}(l)|\geqslant C}} \frac{1}{l_1^2 l_2^2} dl_1 dl_2 \leqslant V(L,t) \leqslant D\int_{A\leqslant\|l\|\leqslant t+B} \min(\frac{1}{C^2}, \frac{1}{(l_1 l_2)^2}) dl_1 dl_2. \tag{5.34}$$

Lemma 77 gives us in particular that :

$$\int_{\substack{l\in\mathbb{R}^2\\ A\leqslant\|l\|\leqslant t+B\\ |\mathrm{Num}(l)|\geqslant C}} \frac{1}{l_1^2 l_2^2} dl_1 dl_2 = \Theta(\log(t)). \tag{5.35}$$

and

$$\int_{A\leqslant\|l\|\leqslant t+B} \min(\frac{1}{C^2}, \frac{1}{(l_1 l_2)^2}) dl_1 dl_2 = \Theta(\log(t)) + \frac{1}{C^2}\int_{\substack{A\leqslant\|l\|\leqslant t+B\\ |Num(l)|\leqslant C}} dl_1 dl_2. \tag{5.36}$$

Yet, one has that :

$$\int_{\substack{A\leqslant\|l\|\leqslant t+B\\ |Num(l)|\leqslant C}} dl_1 dl_2 = \Theta(\log(t)). \tag{5.37}$$

Equation (5.34), Equation (5.35), Equation (5.36) and Equation (5.37) give us then the wanted result. □

Remark. We think that in dimension $d \geqslant 3$, we must have that for all $C > 0$, for all $A > 0$, one has :

$$\int_{\substack{l\in\mathbb{R}^d\\ A\leqslant\|l\|\leqslant t\\ |\mathrm{Num}(l)|\geqslant C}} \frac{1}{l_1^2\cdots l_d^2} dl_1\cdots dl_d = \Theta(\log(t)^{d-1}). \tag{5.38}$$

In fact, the upper part of Equation (5.38) can be proven like that :

Proof. By symmetry, we are brought back to the case where $l_i > 0$ for all i.
We set

$$\phi : (l_1, \cdots, l_d) \longmapsto (l_1, l_1 l_2, \cdots, l_1\cdots l_d).$$

Then ϕ is a C^∞-diffeomorphism from $(\mathbb{R}_+ - \{0\})^d$ to itself and the jacobian matrix evaluated on $(l_1, \cdots, l_d)$, denoted by $\mathrm{Jac}(l_1, \cdots, l_d)$, satisfies that :

$$\mathrm{Jac}(l_1, \cdots, l_d) = \prod_{i=1}^{d-1} \phi_i(l_1, \cdots, l_d). \tag{5.39}$$

Furthermore, because l belongs to the domain of the integral, we see that for all $i \in \{1, \cdots, d-1\}$,

$$b_i = \frac{1}{Ct^{(1+\epsilon_1)(d-i)}} \leqslant \phi_i(l) \leqslant t^{(1+\epsilon_1)i} = h_i \text{ and}$$
$$\phi_d(l) \geqslant C = b_d.$$

Hence, from (5.39), we get that :

$$\int_{\substack{\forall 1 \leqslant i \leqslant d,\ l_i > 0 \\ A \leqslant \|l\|_2 \leqslant t \\ \mathrm{Num}(l) \geqslant C}} \frac{1}{l_1^2 \cdots l_d^2} dl_1 \cdots dl_d \leqslant \int_{\substack{\forall 1 \leqslant i \leqslant d-1,\ b_i \leqslant u_i \leqslant h_i \\ b_d \leqslant u_d}} \frac{1}{u_d^2 \prod_{i=1}^{d-1} u_i} du. \tag{5.40}$$

The right hand side can be easily calculated and gives the wanted result. □

To prove the lower part of the Equation (5.38), we think that one way is to use hyperspherical coordinates.

We are now going to tackle the proof of the second part of Theorem 13. It should be noted that, by density, it is enough to treat the case where the support of ρ is included in $[\alpha, 1]$ with $0 < \alpha < 1$.

5.4.2 Smoothing and reduction to the study of a Fourier series

In this subsection, we are going to prove the following proposition :

Proposition 40. *Let L be an admissible lattice of $\mathbb{R}^2$.*
Then, for all T large enough, we have that :

$$\mathbb{E}_{X \in \mathbb{R}^2/L} \left(\left(\frac{\mathcal{R}(t\mathbb{D}^2 + X, L)}{\sqrt{V(L^\perp, t))}} - \frac{S(L^\perp, X, t)}{\sqrt{V(L^\perp, t)}} \right)^2 \right) \to 0$$

where the convergence towards 0 is uniform in $\alpha T \leqslant t \leqslant T$ when $T \to \infty$, where

$$S(L, X, t) = \frac{2}{\pi^2 covol(L)} \sum_{l \in J_2(L,t)} \frac{1}{l_1 l_2} \sum_{k=1}^{\infty} \frac{\sin(2\pi k t l_1) \sin(2\pi k t l_2) \cos(2\pi k < l, X >)}{k^2} \tag{5.41}$$

and where

$$J_2(L, t) = \{l \in L \mid 0 < \|l\| \leqslant t,\ l \text{ prime and } l_1 > 0\}. \tag{5.42}$$

Proposition 40 basically tells us that the asymptotical study of $\mathbb{E}_{X \in \mathbb{R}^2/L} \left(\left(\frac{\mathcal{R}(tP+X,L)}{\sqrt{V(L^\perp,t)}} \right)^2 \right)$, when $T \to \infty$, can be reduced to the study of $\mathbb{E}_{X \in \mathbb{R}^2/L} \left(\frac{S(L^\perp,X,t,T)}{\sqrt{V(L^\perp,t)}} \right)$ (it is in fact

stronger). It is suggested by the Poisson formula. Yet, we can not use it directly to prove this fact because the indicator function $\mathbf{1}_{tP+X}$, with

$$P = Rect(a, b)$$

is not regular enough.
So, the first thing we are going to do to prove Proposition 40 is to *smooth* the studied problem. It is the object of the next subsubsection. The study of $\frac{\mathcal{R}(tP+X,L)}{\sqrt{V(L^\perp,t)}}$ is going to be reduced to the study of two Fourier series. Then, we will do some calculations to simplify this study and finally the study of $\frac{\mathcal{R}(tP+X,L)}{\sqrt{V(L^\perp,t)}}$ is going to be reduced to the study of $\frac{S(L^\perp,X,t)}{\sqrt{V(L^\perp,t)}}$.

Reduction to the study of two Fourier series

This subsubsection is dedicated to the *smoothing* of the problem. We are going to do it like in [82].
Let us give a C^∞-function, of compact support included in $B_f(0,1)$, such that

$$\int_{\mathbb{R}^2} \omega(x)dx = 1,$$

$\omega(x) \geqslant 0$ and such that ω is spherically symmetric (so will be its Fourier transform).
Let us recall the following definition :

Definition 24. *Let $\mathbf{O} \subset \mathbb{R}^2$ be a connect compact set of $\mathbb{R}^2$ with a frontier that is regularly piecewise. Let $1 > \tau > 0$. A couple of compact region $(\mathbf{O}_\tau^+, \mathbf{O}_\tau^-)$ is called a τ-co-approximation if $\mathbf{O}_\tau^- \subset \mathbf{O} \subset \mathbf{O}_\tau^+$ and if the points at the frontiers $\partial\mathbf{O}_\tau^\pm$ are at least distant from τ of the frontier $\partial\mathbf{O}$.*

For example, we can think that $\mathbf{O} = P$ and, in that case, a τ-co-approximation is given by $(1 \pm \beta\tau)P$, with $\beta > 0$ well-chosen.
Let us define :

$$\mathfrak{R}_\tau^\pm(\mathbf{O}, X) = \frac{1}{covol(L)} \sum_{l \in L^\perp - \{0\}} \hat{\mathbf{1}}_{\mathbf{O}_\tau^\pm}(l)\hat{\omega}(\tau l)e^{2i\pi<l,X>}. \tag{5.43}$$

For $\mathbf{O} = tP = t\mathrm{Rect}(a, b)$, there exists $\beta > 0$ (independent from t) such that $\mathbf{O}_\tau^\pm = (t \pm \beta\tau)P$ is a τ-co-approximation.
Let us take :

$$\tau = \tau(t) = \frac{\log(t)^\gamma}{t} \tag{5.44}$$

where $\frac{1}{2} > \gamma > 0$.
The main object of this subsubsection is to prove the following proposition :

Proposition 41. *For this choice of τ and of τ-co-approximation $(t \pm \beta\tau)P$, there exists $B > 0$ (independent from X) such that for all T large enough*

$$\frac{\mathfrak{R}_\tau^-(tP, X)}{\sqrt{V(L^\perp, t)}} - B\log(T)^{\gamma-\frac{1}{2}} \leqslant \frac{\mathcal{R}(tP + X, L)}{\sqrt{V(L^\perp, t)}} \leqslant \frac{\mathfrak{R}_\tau^+(tP, X)}{\sqrt{V(L^\perp, t)}} + B\log(T)^{\gamma-\frac{1}{2}}.$$

Basically, it says that one can brought back the study of $\frac{\mathcal{R}(tP+X,L)}{\sqrt{V(L^\perp,t)}}$ to the study of the two quantities $\frac{\mathfrak{R}_\tau^\pm(tP,X)}{\sqrt{V(L^\perp,t)}}$, which represent the *smoothed* Fourier series of $\frac{\mathcal{R}(tP+X,L)}{\sqrt{V(L^\perp,t)}}$. To prove this proposition, the following lemma is going to be useful :

Lemma 78. *Let us take* $0<\tau<1$. *Then, one has :*

$$\mathfrak{R}_\tau^-(\mathbf{O},X)+\frac{Area(\mathbf{O}_\tau^-)-Area(\mathbf{O})}{covol(L)}\leqslant\mathcal{R}(\mathbf{O}+X,L)\leqslant\mathfrak{R}_\tau^+(\mathbf{O},X)+\frac{Area(\mathbf{O}_\tau^+)-Area(\mathbf{O})}{covol(L)}.$$

Proof. Let us set $\omega_\tau(x)=\tau^{-2}\omega(\tau^{-1}x)$ for all $x\in\mathbb{R}^2$. Then, one has :

$$\hat{\omega}_\tau(\xi)=\hat{\omega}(\tau\xi) \tag{5.45}$$

for all $\xi\in\mathbb{R}^2$.
Let us consider the following convolution products :

$$h^\pm(x)=(\omega_\tau*\mathbf{1}_{\mathbf{O}_\tau^\pm})(x)=\int_{\mathbb{R}^2}\omega_\tau(x-y)\mathbf{1}_{\mathbf{O}_\tau^\pm}(y)dy. \tag{5.46}$$

The functions $h^\pm$ are C^∞ over $\mathbb{R}^2$ and have a compact support.
By using Definition 24, one has, for all $x\in\mathbb{R}^2$:

$$h^-(x)\leqslant\mathbf{1}_{\mathbf{O}}(x)\leqslant h^+(x). \tag{5.47}$$

By replacing $x\in\mathbb{R}^2$ by $l-x$ with $l\in L$ and by summing over $l\in L$, one gets with Equation (5.47) :

$$\sum_{l\in L}h^-(l-X)\leqslant N(\mathbf{O}+X,L)\leqslant\sum_{l\in L}h^+(l-X). \tag{5.48}$$

By using the Poisson formula and the fact that Fourier transform a convolution product into an usual product, one gets the wanted result thanks to Equation (5.45). □

We can now tackle the proof of Proposition 41 :

Proof of Proposition 41. According to Lemma 78, one has that, for every $t>0$ and X,

$$\mathfrak{R}_\tau^-(tP,X)+\frac{\mathrm{Area}((t-\beta\tau)P)-\mathrm{Area}(tP)}{covol(L)}\leqslant\mathcal{R}(tP+X,L)\leqslant\mathfrak{R}_\tau^+(tP,X)+\frac{\mathrm{Area}((t+\beta\tau)P)-\mathrm{Area}(tP)}{covol(L)}. \tag{5.49}$$

Yet, one has the two following equations :

$$\mathrm{Area}((t-\beta\tau)P)-\mathrm{Area}(tP)=\mathrm{Area}(P)(-2\beta\tau t+\beta^2\tau^2) \tag{5.50}$$

and

$$\mathrm{Area}((t+\beta\tau)P)-\mathrm{Area}(tP)=\mathrm{Area}(P)(2\beta\tau t+\beta^2\tau^2). \tag{5.51}$$

Yet here $\tau=\frac{\log(t)^\gamma}{t}$ and so one gets that for every T large enough, for every $T\geqslant t\geqslant\alpha T$,

$$|\frac{\pm2\beta\tau t+\beta^2\tau^2}{\sqrt{V(L^\perp,t)}}|\leqslant M\log(T)^{\gamma-\frac{1}{2}} \tag{5.52}$$

where $M > 0$. To obtain this last equation we have used the fact that $\sqrt{V(L^\perp, t)} = \Theta(\sqrt{\log(t)})$ that is given by proposition 39.
By using Equation 5.49, Equation 5.50, Equation 5.51 and Equation 5.52, one gets the wanted result. □

The study is now reduced to the study, when $T \to \infty$, of the two quantities $\frac{\mathfrak{R}_\tau^\pm(tP,X)}{\sqrt{V(L^\perp,t)}}$. In the next section, we are going to reduce the study of $\frac{\mathfrak{R}_\tau^+(tP,X)}{\sqrt{V(L^\perp,t)}}$ to the study of the Fourier series $\frac{S(L^\perp,X,t,T)}{\sqrt{V(L^\perp,t)}}$ and the results are also going to hold for $\frac{\mathfrak{R}_\tau^-(tP,X)}{\sqrt{V(L^\perp,t)}}$.

Simplification of the Fourier series $\frac{\mathfrak{R}_\tau^+(tP,X)}{\sqrt{V(L^\perp,t)}}$

The main object of this subsubsection is to prove the following proposition :

Proposition 42. *Let us suppose that L is admissible. Then, one has, for all $\gamma > 0$,*

$$\mathbb{E}_{X\in\mathbb{R}^2/L}\left(\left(\frac{\mathfrak{R}_\tau^+(tP,X) - S(L^\perp,t,X)}{\sqrt{V(t,L^\perp)}}\right)^2\right) \to 0$$

where $\tau = \frac{\log(t)^\gamma}{t}$ and where the convergence towards 0 is uniform in t such that $\alpha T \leqslant t \leqslant T$ and when $T \to \infty$.

To do this, we will need several intermediate lemmas. We will use tools from Fourier Analysis, integration calculus and Geometry of numbers.
In the rest of this section, we are going to suppose that L is admissible.
Under this hypothesis, the dual lattice of L is also going to be admissible.

Reduction of $\mathfrak{R}_\tau^+$ to a finite sum. Let us introduce the following sum :

$$S_1(L^\perp,t,X) = \frac{1}{\pi^2 covol(L)} \sum_{\substack{l\in L^\perp \\ 0<\|l\|\leqslant t}} \frac{\hat{\omega}(\tau l)\sin(2\pi l_1 t^+)\sin(2\pi l_2 t^+)e^{2i\pi<l,X>}}{l_1 l_2} \tag{5.53}$$

where $t^+ = t + \beta\tau$. Then the main object of this paragraph is to prove the following lemma :

Lemma 79. *Then, one has for all T large enough,*

$$\left|\frac{|\mathfrak{R}_\tau^+(tP,X) - S_1(L^\perp,t,X)|}{\sqrt{V(t,L^\perp)}}\right| = O(\log(T)^{\gamma-\frac{1}{2}}).$$

where the O is uniform in $X \in \mathbb{R}^2$ and in t when $\alpha T \leqslant t \leqslant T$.

Recall that γ is a fixed parameter such that $0 < \gamma < \frac{1}{2}$.
To prove this lemma, we are going to need a lemma that gives an upper estimate of certain integrals :

Lemma 80. *Let us set for $N \geqslant 2$ and for $C > 0$,*

$$D_{N,C} = \{x = (x_1, x_2) \in \mathbb{R}^+ - \{0\} \mid \|x\| \geqslant N,\ |Num(x)| \geqslant C\}. \tag{5.54}$$

Then one has, for all $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ measurable, there exists $M > 0$ such that for every N large enough,

$$\int_{x \in D_{N,C}} \frac{g(\|x\|)}{|x_1 x_2|} dx_1 dx_2 \leqslant M \int_{r=N}^{\infty} g(r) \frac{\log(r)}{r} dr.$$

Proof. Let us call :

$$J(N, C) = \int_{x \in D_{N,C}} \frac{g(\|x\|)}{|x_1 x_2|} dx_1 dx_2. \tag{5.55}$$

By passing in polar coordinates, we get that for every N large enough :

$$J(N, C) \leqslant K_1 \int_{r \geqslant N} \frac{g(r)}{r} \Big(\int_{\arcsin(2\frac{C}{r^2})}^{\frac{\pi}{2}} \frac{d\theta}{\sin(\theta)}\Big) dr \tag{5.56}$$

where $K_1 > 0$.
Yet one calculates that:

$$\Big(\int_{\arcsin(2\frac{C}{r^2})}^{\frac{\pi}{2}} \frac{d\theta}{\sin(\theta)}\Big) = \frac{1}{2} \log\Big(\frac{(1 + \sqrt{1 - \frac{2C}{r^2}})^2}{4\frac{C}{r^2}}\Big). \tag{5.57}$$

With the Inequation (5.56), we get that :

$$J(N, C) \leqslant K_2 \int_{r \geqslant N} \frac{g(r) \log(r)}{r} dr \tag{5.58}$$

where $K_2 > 0$ and so we get the wanted result. □

We can now prove Lemma 79 :

Proof of Lemma 79. L is supposed to be admissible and so $L^\perp$ must also be admissible. So there exists $C > 0$ such that for all $x \in L^\perp - \{0\}$,

$$|\mathrm{Num}(x)| \geqslant C. \tag{5.59}$$

Furthermore, ω is supposed to be regular and of compact support. In consequence, $\hat{\omega}$ belongs to the Schwartz space and so for all $A > 2$, there exists $M > 0$ such that for all $x \in \mathbb{R}^2$:

$$|\tilde{\omega}(x)| \leqslant \frac{M}{1 + \|x\|^A}. \tag{5.60}$$

Proposition 37 and the fact that $\mathbf{O}_\tau^+ = (t + \beta\tau)P$ with $P = \mathrm{Rect}(a, b)$ give us that :

$$|\mathfrak{R}_\tau^+(tP, X) - S_1(L^\perp, t, X)| \leqslant K_1 \sum_{\substack{l \in L^\perp \\ \|l\| > t}} \frac{1}{|l_1 l_2|(1 + \|\tau l\|^A)} \tag{5.61}$$

where $K_1 > 0$. We have also used implicitly Equation (5.43).
Yet, one has also :

$$\sum_{\substack{l\in L^\perp\\ \|l\|>t}} \frac{1}{|l_1 l_2|(1+\|\tau l\|^A)} \leqslant K_2 \left(\int_{x\in D_{t,C}} \frac{1}{|x_1x_2|(1+\|\tau x\|^A)} dx_1dx_2 + \int_{\substack{\|x\|>t\\ |Num(x)|<C}} \frac{1}{C(1+\|\tau x\|^A)} dx \right) \tag{5.62}$$

where $K_2 > 0$.
Lemma 80 and a quick calculation give us then that for all T large enough (we have to keep in mind that $\alpha T \leqslant t \leqslant T$) :

$$\sum_{\substack{l\in L^\perp\\ \|l\|\leqslant t}} \frac{1}{|l_1 l_2|(1+\|\tau l\|^A)} \leqslant K_3 \left(\int_{r=t}^{\infty} \frac{\log(r)}{r(1+(\tau r)^A)} dr + \int_{r=t}^{\infty} \frac{1}{r^2(1+(\tau r)^A)} dr \right) \tag{5.63}$$

where $K_3 > 0$.
However, one has also that :

$$\int_{r=t}^{\infty} \frac{\log(r)}{r(1+(\tau r)^A)} dr \leqslant \frac{K_4 \log(t)}{\tau^A t^A} \tag{5.64}$$

where $K_4 > 0$ (and depends on A).
Yet one has that : $\tau = \frac{\log(t)^\gamma}{t}$ with $\frac{1}{2} > \gamma > 0$.
By using Equation (5.61), Equation (5.63) and Equation (5.64) and by using the fact that $V(L^\perp, t) = \Theta(\log(t))$, one gets that :

$$\frac{|\mathfrak{R}_\tau^+(tP, X) - S_1(L^\perp, t, X)|}{\sqrt{V(t, L^\perp)}} \leqslant K_5 \left(\frac{\log(t)^{\frac{1}{2}}}{\log(t)^{A\gamma}} + \frac{1}{\sqrt{\log(t)}} \right) \tag{5.65}$$

where $K_5 > 0$ and depends on A.
Then, we can take A large enough so one has that :

$$\frac{\log(t)^{\frac{1}{2}}}{\log(t)^{A\gamma}} \leqslant \frac{K_6}{\log(t)} \tag{5.66}$$

with $K_6 > 0$.
By using Equation (5.65) and Equation (5.66), one has that :

$$\frac{|\mathfrak{R}_\tau^+(tP, X) - S_1(L^\perp, t, X)|}{\sqrt{V(t, L^\perp)}} \leqslant \frac{K_7}{\sqrt{\log(T)}} \tag{5.67}$$

with $K_7 > 0$ and for T large enough. Thus the wanted result. □

We are now brought back to the study of $S_1(L^\perp, t, X)$ and now we are going, mainly, to "center" it on the prime vectors of $L^\perp$.

Centring of $S_1(L^\perp, t, X)$ over prime vectors. Before stating the main lemma of this paragraph, let us make a small remark. Because $L^\perp$ is admissible and because $\tilde{\omega}$ is spherically symmetric, by parity, one has that :

$$S_1(L^\perp, t, X) = \frac{2}{\pi^2 covol(L)} \sum_{l\in J_1(L^\perp,t)} \frac{\tilde{\omega}(\tau l)\sin(2\pi l_1 t^+)\sin(2\pi l_2 t^+)\cos(2\pi < l, X >)}{l_1 l_2} \tag{5.68}$$

where

$$J_1(L^\perp, t) = \{l \in L^\perp \mid 0 < \|l\| \leqslant t \text{ and } l_1 > 0\}. \tag{5.69}$$

Let us define :

$$S_2(L^\perp, t, X) = \frac{2}{\pi^2 covol(L)} \sum_{l \in J_2(L^\perp, t)} \frac{Z_l(X, t)}{l_1 l_2} \tag{5.70}$$

where

$$Z_l(X, t) = \sum_{k=1}^{\infty} \frac{\tilde{\omega}(k\tau l) \sin(2k\pi l_1 t^+) \sin(2k\pi l_2 t^+) \cos(2k\pi < l, X >)}{k^2} \tag{5.71}$$

and we recall that

$$J_2(L^\perp, t) = \{l \in L^\perp \mid 0 < \|l\| \leqslant t \ , \ l \text{ prime and } l_1 > 0\}.$$

Then the main statement of this subsection is the following lemma :

Lemma 81. *For every T large enough, one has :*

$$|\frac{|S_2(L^\perp, t, X) - S_1(L^\perp, t, X)|}{\sqrt{V(L^\perp, t)}}| = O(\log(T)^{\gamma - \frac{1}{2}})$$

and the O is uniform in $X \in \mathbb{R}^2$ and in t such that $\alpha T \leqslant t \leqslant T$.

It basically says that the essential information of S_1 is contained in its prime terms.

Proof. One has :

$$|S_2(L^\perp, t, X) - S_1(L^\perp, t, X)| \leqslant K_1 \sum_{\substack{l \in L^\perp \\ \|l\| \geqslant t}} \frac{|\tilde{\omega}(\tau l)|}{|l_1 l_2|} \tag{5.72}$$

where $K_1 > 0$.
So, we can apply the same argument as before in the proof of Lemma 79 and one gets that :

$$|\frac{|S_2(L^\perp, t, X) - S_1(L^\perp, t, X)|}{\sqrt{V(L^\perp, t)}}| = O(\log(T)^{\gamma - \frac{1}{2}}). \tag{5.73}$$

□

So, the study of S_1, when $T \to \infty$, is now reduced to the study of S_2, when $T \to \infty$. In the next paragraph, we are going to simplify the sum S_2.

Replacing $\tilde{\omega}$ by 1 in S_2. Let us define :

$$S_3(L^\perp, t, X) = \frac{2}{\pi^2 covol(L)} \sum_{l \in J_2(L^\perp, t)} \frac{\tilde{Z}_l(X, t)}{l_1 l_2} \tag{5.74}$$

where

$$\tilde{Z}_l(X, t) = \sum_{k=1}^{\infty} \frac{\sin(2k\pi l_1 t^+) \sin(2k\pi l_2 t^+) \cos(2k\pi < l, X >)}{k^2} \tag{5.75}$$

(we have replaced $\tilde{\omega}$ by its value at 0, which is 1 because $\int_{\mathbb{R}^2} \omega = 1$).
The main statement of this subsection is the following lemma :

Lemma 82. *Let us set :*

$$\Delta(L^{\perp},t)=\int_{X\in\mathbb{R}^2/L^{\perp}}(\frac{|S_2(L^{\perp},t,X)-S_3(L^{\perp},t,X)|}{\sqrt{V(L^{\perp},t)}})^2d\tilde{\lambda}_2(X). \tag{5.76}$$

Then $\Delta(L^{\perp},t)$ goes to 0 uniformly in t that are such that $\alpha T\leqslant t\leqslant T$ with $T\to\infty$.

Proof. The Parseval formula applies here and gives us that :

$$\Delta(L^{\perp},t)\leqslant\frac{K_1}{\log(t)}\sum_{l\in J_1(L^{\perp},t)}(\frac{\tilde{\omega}(\tau l)-1}{l_1l_2})^2 \tag{5.77}$$

where $K_1>0$ and we have used the fact that $V(L^{\perp},t)=\Theta(\log(t))$.
Let us take $\tilde{\gamma}$ such that $\gamma<\tilde{\gamma}<\frac{1}{2}$ and let us set $t_1=\frac{t}{\log(t)^{\tilde{\gamma}}}$.
Then, with this notation, one has, thanks to Equation (5.77) :

$$\Delta(L^{\perp},t)\leqslant K_2(\Delta_1+\Delta_2) \tag{5.78}$$

where $K_2>0$,

$$\Delta_1=\frac{1}{\log(t)}\sum_{l\in J_1(L^{\perp},t_1)}(\frac{\tau\|l\|}{l_1l_2})^2 \text{ and} \tag{5.79}$$

$$\Delta_2=\frac{1}{\log(t)}\sum_{l\in J_1(L^{\perp},t)-J_1(L^{\perp},t_1)}\frac{1}{l_1^2l_2^2}. \tag{5.80}$$

Then Lemma 77, the fact $\tau=\frac{\log(t)^{\gamma}}{t}$ and the definition of t_1 give us that for T large enough (and for t such that $\alpha T\leqslant t\leqslant T$) :

$$\Delta_1\leqslant K_3\log(t)^{2(\gamma-\tilde{\gamma})} \tag{5.81}$$

with $K_3>0$ and

$$\Delta_2\leqslant\frac{K_4}{\log(t)}\int_{t\log(t)^{-\tilde{\gamma}}}^{t}\frac{1}{r}dr=O(\frac{1}{\sqrt{\log(t)}}). \tag{5.82}$$

with $K_4>0$.
So, one gets finally that, when $T\to\infty$,

$$\Delta_1\to 0 \text{ and } \Delta_2\to 0$$

uniformly in t for $\alpha T\leqslant t\leqslant T$.

□

So, now, we are brought back to the study of S_3 when $T\to\infty$.
In the next paragraph, we are going to simply S_3 and its study will be reduced to the study of S when $T\to\infty$ as wanted.

Replacing t^+ by t and proof of proposition 42. We recall that we have defined S by the Equation (5.41). The main object of this paragraph is to prove the following lemma :

Lemma 83. *Let us call (here) Δ the following quantity :*

$$\Delta(L^\perp, t) = \sqrt{\int_{X\in\mathbb{R}^2/L^\perp} (\frac{|S_3(L^\perp, t, X) - S(L^\perp, t, X)|}{\sqrt{V(L^\perp, t)}})^2 d\tilde{\lambda}_2(X)}. \tag{5.83}$$

Then $\Delta(L^\perp, t)$ goes to 0 uniformly in t that are such that $\alpha T \leqslant t \leqslant T$ with $T \to \infty$.

Mainly, this lemma says that we can replace t^+ in the two sin terms by t. It is reasonable insofar as t^+ is very close to t.

Proof. Let us set :

$$S_4(L^\perp, t, X) = \frac{2}{\pi^2 covol(L)} \sum_{l\in J_2(L^\perp,t)} \frac{W_l(X,t)}{l_1 l_2} \tag{5.84}$$

where

$$W_l(X,t) = \sum_{k=1}^{\infty} \frac{\sin(2k\pi l_1 t)\sin(2k\pi l_2 t^+)\cos(2k\pi < l, X >)}{k^2}. \tag{5.85}$$

Then the triangle inequality gives us that :

$$\Delta \leqslant \sqrt{\Delta_1} + \sqrt{\Delta_2} \tag{5.86}$$

where

$$\Delta_1 = \int_{X\in\mathbb{R}^2/L^\perp} (\frac{|S_3(L^\perp, t, X) - S_4(L^\perp, t, X)|}{\sqrt{V(L^\perp, t)}})^2 d\tilde{\lambda}_2(X) \tag{5.87}$$

and

$$\Delta_2 = \int_{X\in\mathbb{R}^2/L^\perp} (\frac{|S_4(L^\perp, t, X) - S(L^\perp, t, X)|}{\sqrt{V(L^\perp, t)}})^2 d\tilde{\lambda}_2(X). \tag{5.88}$$

Let us take $\tilde{\gamma}$ such that $\gamma < \tilde{\gamma} < \frac{1}{2}$ and let us set $t_1 = \frac{t}{\log(t)^{\tilde{\gamma}}}$.
The Parseval formula and the mean value theorem apply here and give us that :

$$\Delta_1 \leqslant K_2(\Delta_{1,1} + \Delta_{1,2}) \tag{5.89}$$

where $K_2 > 0$,

$$\Delta_{1,1} = \frac{1}{\log(t)} \sum_{l\in J_1(L^\perp,t_1)} (\frac{\tau\|l\|}{l_1 l_2})^2, \tag{5.90}$$

$$\Delta_{1,2} = \frac{1}{\log(t)} \sum_{l\in J_1(L^\perp,t)-J_1(L^\perp,t_1)} \frac{1}{l_1^2 l_2^2} \tag{5.91}$$

and

$$\Delta_2 \leqslant K_2(\Delta_{1,1} + \Delta_{1,2}). \tag{5.92}$$

We have used the fact that $V(L^\perp,) = \Theta(\log(t))$.
Then Equation (5.81), Equation (5.82), Equation (5.86), Equation (5.92) and Equation (5.89) give that Δ goes uniformly to 0 in t for t such that $\alpha T \leqslant t \leqslant T$ □

We can now prove Proposition 42.

Proof of proposition 42. Proposition 42 is a direct consequence of the lemmas 79, 81, 82 and 83. □

Case of t^- and proof of Proposition 40

By following exactly the same approach that has been used to prove proposition 42, we prove the following proposition :

Proposition 43. *Let us suppose that L is admissible. Then, one has, for all $\gamma > 0$,*

$$\mathbb{E}_L\left(\left(\frac{|\mathfrak{R}_\tau^-(tP,X) - S(L^\perp,t,X)|}{\sqrt{V(t,L^\perp)}}\right)^2\right) \to 0.$$

where $\tau = \frac{\log(t)^\gamma}{t}$ and where the convergence towards 0 is uniform in t such that $\alpha T \leqslant t \leqslant T$ and when $T \to \infty$.

We are now capable to prove Proposition 40 :

Proof of Proposition 40. It is a direct consequence of Proposition 41, Proposition 42 and Proposition 43 and of the triangle inequality. □

So, finally, it is enough to study the behaviour, when $T \to \infty$, of S. It is the object of the next subsection.

5.4.3 Asymptotic behaviour of S

According to the proof of proposition 40, by replacing $L^\perp$ by L (if one is admissible, the other also is and conversely), to show the second assertion of Theorem 13, one only needs to prove that $\mathbb{E}_{L^\perp}\left(\left(\frac{S(L,X,t)}{\sqrt{V(L,t)}}\right)^2\right)$ converges in distribution and in probability towards $\frac{1}{4\pi^4 \mathrm{Covol}(L^\perp)^2}$. Furthermore, this limit constant will be the same for $\mathbb{E}_{L^\perp}\left(\left(\frac{S(L,X,t)}{\sqrt{V(L,t)}}\right)^2\right)$ and $\mathbb{E}_{L^\perp}\left(\left(\frac{S(L,X,t)}{\sqrt{V(L,t)}}\right)^2\right)$.
With the Parseval formula, we see that we only need to prove the following proposition :

Proposition 44.

$$G(L,t) = \frac{2}{\pi^4 Covol(L^\perp)^2} \sum_{l \in J_2(L,t)} \frac{1}{(l_1 l_2)^2} \sum_{k=1}^{\infty} \frac{(\sin(2\pi k t l_1)\sin(2\pi k t l_2))^2}{k^4}$$

normalized by $V(L,t)$ converges in distribution and in probability towards $\frac{1}{4\pi^4 Covol(L^\perp)^2}$.

The rest of this subsection is dedicated to prove this last proposition.

A small remark and approach

By using the fact that one has for every $x \in \mathbb{R}$,

$$\sin(x)^2 = \frac{1-\cos(2x)}{2},$$

one gets that :

$$G(L,t) = G_1(L,t) - G_2(L,t) - G_3(L,t) + G_4(L,t) \tag{5.93}$$

where

$$G_1(L,t) = \frac{1}{2\pi^4 \mathrm{Covol}(L^\perp)^2} \sum_{l \in J_2(L,t)} \frac{1}{(l_1 l_2)^2} \sum_{k=1}^{\infty} \frac{1}{k^4}, \tag{5.94}$$

$$G_2(L,t) = \frac{1}{2\pi^4 \mathrm{Covol}(L^\perp)^2} \sum_{l \in J_2(L,t)} \frac{1}{(l_1 l_2)^2} \sum_{k=1}^{\infty} \frac{\cos(4\pi k t l_1)}{k^4}, \tag{5.95}$$

$$G_3(L,t) = \frac{1}{2\pi^4 \mathrm{Covol}(L^\perp)^2} \sum_{l \in J_2(L,t)} \frac{1}{(l_1 l_2)^2} \sum_{k=1}^{\infty} \frac{\cos(4\pi k t l_2)}{k^4} \quad \textit{and} \tag{5.96}$$

$$G_4(L,t) = \frac{1}{2\pi^4 \mathrm{Covol}(L^\perp)^2} \sum_{l \in J_2(L,t)} \frac{1}{(l_1 l_2)^2} \sum_{k=1}^{\infty} \frac{\cos(4\pi k t l_1)\cos(4\pi k t l_2)}{k^4}. \tag{5.97}$$

To get the validity of Proposition 44, we only need to show the three following propositions :

Proposition 45.

$$\frac{G_1(L,t)}{V(L,t)} \to \frac{1}{4\pi^4 \mathit{Covol}(L^\perp)^2}$$

when $t \to \infty$.

Proposition 46. $\frac{G_2(L,t)}{V(L,t)}$ *and* $\frac{G_3(L,t)}{V(L,t)}$ *converge in distribution and in probability towards* 0 *when* t *is distributed according to the probability measure* $\frac{1}{T}\rho(\frac{t}{T})dt$ *and when* $T \to \infty$.

Proposition 47. $\frac{G_4(L,t)}{V(t,T)}$ *converge in distribution and in probability towards* 0 *when* t *is distributed according to the probability measure* $\frac{1}{T}\rho(\frac{t}{T})dt$ *and when* $T \to \infty$.

Proposition 45 is just a use of different definitions whereas Proposition 46 and Proposition 47 work because, basically, there are, relatively to G_1, additional oscillatory terms that make the normalized sum go to 0.
Now we are going to prove in this order these three propositions.

Proof of Proposition 45

The proof of this proposition is straightforward :

Proof of Proposition 45. One has :

$$G_1(L,t) = \frac{1}{2\pi^4 \mathrm{Covol}(L^\perp)^2} \sum_{l \in J_2(L,t)} \frac{1}{(l_1 l_2)^2} \sum_{k=1}^{\infty} \frac{1}{k^4}$$

where

$$J_2(L,t) = \{l \in L \mid 0 < \|l\| \leqslant t \ , \ l \text{ prime and } l_1 > 0\}$$

and

$$V(L,t) = \sum_{\substack{l \in L \\ 0 < \|l\| \leqslant t}} \frac{1}{l_1^2 l_2^2}.$$

By making t goes forward infinity, one gets the wanted result. □

Next, we are going to prove Proposition 46.

Proof of Proposition 46

Before giving the proof of Proposition 46, we need several lemmas.
Let us introduce :

$$\tilde{G}_2(L,t,T) = \frac{1}{2\pi^4 \mathrm{Covol}(L^\perp)^2} \sum_{l \in J_2(L,T)} \frac{1}{(l_1 l_2)^2} \sum_{k=1}^{\infty} \frac{\cos(4\pi k t l_1)}{k^4}$$

and

$$\tilde{G}_3(L,t,T) = \frac{1}{2\pi^4 \mathrm{Covol}(L^\perp)^2} \sum_{l \in J_2(L,T)} \frac{1}{(l_1 l_2)^2} \sum_{k=1}^{\infty} \frac{\cos(4\pi k t l_2)}{k^4}.$$

The first lemma basically says that we can study $\frac{\tilde{G}_2(L,t,T)}{V(L,t)}$ and $\frac{\tilde{G}_3(L,t,T)}{V(L,t)}$ instead of studying, respectively, $\frac{G_2(L,t)}{V(L,t)}$ and $\frac{G_3(L,t)}{V(L,t)}$:

Lemma 84. *One has, when $T \to \infty$,*

$$\mathbb{E}\left(\left| \frac{\tilde{G}_2(L,t,T)}{V(L,t)} - \frac{G_2(L,t)}{V(L,t)} \right| \right) \to 0$$

and

$$\mathbb{E}\left(\left| \frac{\tilde{G}_3(L,t,T)}{V(L,t)} - \frac{G_3(L,t)}{V(L,t)} \right| \right) \to 0$$

(Note that the expectation are calculated relatively to t with t being distributed according to the probability measure $\frac{1}{T}\rho(\frac{t}{T})dt$).

Proof. We are only going to prove the first fact because the proof is going to be symmetrical in (l_1, l_2).
One has that :

$$\frac{\tilde{G}_2(L,t,T)}{V(L,t)} - \frac{G_2(L,t)}{V(L,t)} = \frac{D}{V(L,t)} \sum_{l \in J_2(L,T) - J_2(L,t)} \frac{1}{(l_1 l_2)^2} \sum_{k=1}^{\infty} \frac{\cos(4\pi k t l_1)}{k^4} \tag{5.98}$$

where D is a positive constant.
By integrating and because t belongs to $[\alpha T, T]$, one gets that :

$$\mathbb{E}\left(|\frac{\tilde{G}_2(L,t,T)}{V(L,t)} - \frac{G_2(L,t)}{V(L,t)}|\right) \leqslant \frac{D}{\log(T)}\mathbb{E}\left(\sum_{l\in J_2(L,T)-J_2(L,t)} \frac{1}{(l_1l_2)^2}\sum_{k=1}^{\infty}\frac{1}{k^4}\right) \tag{5.99}$$

where D is a positive constant, possibly different from the previous one. We have also used Proposition 39.
Yet, one has that (see Lemma 77) :

$$\sum_{l\in J_2(L,T)-J_2(L,t)} \frac{1}{(l_1l_2)^2}\sum_{k=1}^{\infty}\frac{1}{k^4} \leqslant D\log(\frac{T}{t}). \tag{5.100}$$

So, with Equation (5.99), one has that :

$$\mathbb{E}\left(|\frac{\tilde{G}_2(L,t,T)}{V(L,t)} - \frac{G_2(L,t)}{V(L,t)}|\right) \leqslant \frac{D}{\log(T)}\mathbb{E}(\log(\frac{T}{t})). \tag{5.101}$$

Yet, a quick calculation gives us that :

$$\mathbb{E}(\log(\frac{T}{t})) = O(1) \tag{5.102}$$

when $T \to \infty$.
So, thanks to Equation (5.101) and thanks to Equation (5.102), one gets the first wanted result. □

The second lemma is an estimating one.

Lemma 85. *For every $A > 0$, for every $C > 0$, one has :*

$$\int_{\substack{A\leqslant\|l\|\leqslant T\\ l_1>0,\ l_2>0\\ l_1l_2\geqslant C}} \frac{1}{l_1^3l_2^2}dl_1dl_2 = O(T) \text{ and}$$

$$\int_{\substack{A\leqslant\|l\|\leqslant T\\ l_1>0,\ l_2>0\\ l_1l_2\geqslant C}} \frac{1}{l_1^2l_2^3}dl_1dl_2 = O(T).$$

Remark. Thanks to this lemma, we can show very quickly that the expectation of $\frac{\tilde{G}_2(L,t,T)}{V(L,t)}$ and of $\frac{\tilde{G}_2(L,t,T)}{V(L,t)}$ tends to 0 when $T \to \infty$.

Proof. By symmetry, one has :

$$\int_{\substack{A\leqslant\|l\|\leqslant T\\ l_1>0,\ l_2>0\\ l_1l_2\geqslant C}} \frac{1}{l_1^3l_2^2}dl_1dl_2 = \int_{\substack{A\leqslant\|l\|\leqslant T\\ l_1>0,\ l_2>0\\ l_1l_2\geqslant C}} \frac{1}{l_1^2l_2^3}dl_1dl_2 \tag{5.103}$$

So, we only have to prove the first equality of lemma 85.
By passing into polar coordinates (r,θ), one has that :

$$\int_{\substack{A\leqslant\|l\|\leqslant T\\ l_1>0,\ l_2>0\\ l_1l_2\geqslant C}} \frac{1}{l_1^3l_2^2}dl_1dl_2 = \int_{\substack{A\leqslant r\leqslant T\\ \frac{\pi}{2}>\theta>0\\ \sin(2\theta)\geqslant\frac{2C}{r^2}}} \frac{1}{r^4}\frac{1}{\cos^3(\theta)\sin^2(\theta)}drd\theta. \tag{5.104}$$

We see that, when $t \to \infty$, there are *a priori* two essential parts of this last integral : the first one is when r is large and θ is closed to $\frac{1}{2}\arcsin(\frac{2C}{r^2})$; the second is when r is large and θ is closed to $\frac{\pi}{2} - \frac{1}{2}\arcsin(\frac{2C}{r^2})$.
By using the facts that when $\theta \to 0$, $\sin(\theta) \sim \theta$ and when $\theta \to \frac{\pi}{2}$, $\cos(\theta) \sim \frac{\pi}{2} - \theta$ and by calculating, one gets that the first essential part, let us call it $A_1(T)$, is estimated as followed

$$A_1(T) = O(1) \tag{5.105}$$

whereas the second essential part, let us call it $A_2(T)$, is estimated as followed

$$A_2(T) = O(T). \tag{5.106}$$

By using Equation (5.104), one gets that :

$$\int_{\substack{A \leqslant \|l\| \leqslant T \\ l_1>0,\ l_2>0 \\ l_1 l_2 \geqslant C}} \frac{1}{l_1^3 l_2^2} dl_1 dl_2 = O(T). \tag{5.107}$$

□

We can now tackle the proof of Proposition 46.

Proof of Proposition 46. The proof will be symmetrical relatively to the transformation $l_1 \leftarrow l_2$. So, we only need to give the proof of the result that concerns $G_2(L,t)$.
According to Lemma 84 and Markov's inequality, we only need to see that $\frac{\tilde{G}_2(L,t,T)}{V(L,t)}$ converges in distribution and in probability towards 0.
To obtain the fact that $\frac{\tilde{G}_2(L,t,T)}{V(L,t)} \to 0$ in probability, we are going to show that its second moment goes to 0.
One has, according to the definition of $\frac{\tilde{G}_2(L,t,T)}{V(L,t)}$, that :

$$\mathbb{E}\Big((\frac{\tilde{G}_2(L,t,T)}{V(L,t)})^2\Big) \leqslant D\mathbb{E}\left(\frac{1}{V(L,t)^2} \sum_{l,l' \in J_2(L,T)} \frac{1}{(l_1 l_2)^2} \frac{1}{(l_1' l_2')^2} \sum_{k,k' \geqslant 1} \frac{\cos(4\pi k t l_1)}{k^4} \frac{\cos(4\pi k' t l_1')}{k'^4}\right) \tag{5.108}$$

where $D > 0$.
By integrating, by using a usual trigonometric formula and by using Proposition 39, one gets that :

$$\mathbb{E}\Big((\frac{\tilde{G}_2(L,t,T)}{V(L,t))})^2\Big) \leqslant O(A_1(T) + A_2(T)). \tag{5.109}$$

where

$$A_1(T) = \sum_{l,l' \in J_2(L,T)} \frac{1}{(l_1 l_2)^2} \frac{1}{(l_1' l_2')^2} \sum_{k,k' \geqslant 1} \frac{1}{(kk')^4} \max(\frac{1}{\log(T)^2}, \frac{1}{\log(T)^2 T |k l_1 + k' l_1'|}) \tag{5.110}$$

and

$$A_2(T) = \sum_{l,l' \in J_2(L,T)} \frac{1}{(l_1 l_2)^2} \frac{1}{(l_1' l_2')^2} \sum_{k,k' \geqslant 1} \frac{1}{(kk')^4} \max(\frac{1}{\log(T)^2}, \frac{1}{\log(T)^2 T |k l_1 - k' l_1'|}). \tag{5.111}$$

To get the wanted result, it is enough to show that $A_1(T)$ and $A_2(T)$ tend to 0 when $T \to \infty$. Furthermore, we are only going to show that $A_2(T)$ tend to 0 when $T \to \infty$, the proof for $A_1(T)$ being symmetrical.
Let us remark that one has :

$$A_2(T) \leqslant A_{2,1}(T) + A_{2,2}(T) \tag{5.112}$$

where

$$A_{2,1}(T) = \sum_{l,l' \in J_2(L,T)} \frac{1}{(l_1 l_2)^2} \frac{1}{(l_1' l_2')^2} \sum_{\substack{k,k' \geqslant 1 \\ \min(k,k') \geqslant \lceil \log(T) \rceil}} \frac{1}{(kk')^4} \max\left(\frac{1}{\log(T)^2}, \frac{1}{\log(T)^2 T |kl_1 - k'l_1'|}\right) \tag{5.113}$$

and

$$A_{2,2}(T) = \sum_{l,l' \in J_2(L,T)} \frac{1}{(l_1 l_2)^2} \frac{1}{(l_1' l_2')^2} \sum_{\lceil \log(T) \rceil \geqslant k,k' \geqslant 1} \frac{1}{(kk')^4} \max\left(\frac{1}{\log(T)^2}, \frac{1}{\log(T)^2 T |kl_1 - k'l_1'|}\right). \tag{5.114}$$

From Lemma 77 and by using a usual equivalent, one gets that :

$$A_{2,1}(T) \to 0 \tag{5.115}$$

when $T \to \infty$. As a consequence, we only need to look at the behaviour $A_{2,2}(T)$.
There are two types of terms in $A_{2,2}(T)$: those such that kl_1 is close to $k'l_1'$, for example at a distance less than $\frac{1}{T}$ and the others.
In the first case, it forms a sum that is estimated by

$$\frac{D}{\log(T)^2} \int_{\substack{l_1 > 0,\ l_2 > 0 \\ l_1 l_2 \geqslant C \\ D \leqslant \|l\| \leqslant T \log(T)}} \frac{1}{l_1^2 l_2^2} \frac{1}{T l_1} dl_1 dl_2$$

with $D > 0$ and this last quantity goes to zero according to Lemma 85 (we have first integrated over l_2' and then used the fact kl_1 is close to $k'l_1'$).
So, finally, we only need to show that the following quantity goes to 0 when T goes to infinity :

$$J(T) = \sum_{\substack{\lceil \log(T) \rceil \geqslant k,k' \geqslant 1 \\ |kl_1 - k'l_1'| \geqslant \frac{1}{T}}} \frac{1}{(l_1 l_2)^2} \frac{1}{(l_1' l_2')^2} \frac{1}{(kk')^4} \max\left(\frac{1}{\log(T)^2}, \frac{1}{\log(T)^2 T |kl_1 - k'l_1'|}\right). \tag{5.116}$$

Yet, one has that :

$$J(T) \leqslant J_1(T) + J_2(T) \tag{5.117}$$

where

$$J_1(T) = \sum_{\substack{\lceil \log(T) \rceil \geqslant k,k' \geqslant 1 \\ |kl_1 - k'l_1'| \geqslant 1}} \frac{1}{(l_1 l_2)^2} \frac{1}{(l_1' l_2')^2} \frac{1}{(kk')^4} \max\left(\frac{1}{\log(T)^2}, \frac{1}{\log(T)^2 T |kl_1 - k'l_1'|}\right) \tag{5.118}$$

and

$$J_2(T) = \sum_{\substack{\lceil \log(T) \rceil \geqslant k,k' \geqslant 1 \\ 1 \geqslant |kl_1 - k'l_1'| \geqslant \frac{1}{T}}} \frac{1}{(l_1 l_2)^2} \frac{1}{(l_1' l_2')^2} \frac{1}{(kk')^4} \max\left(\frac{1}{\log(T)^2}, \frac{1}{\log(T)^2 T |kl_1 - k'l_1'|}\right). \tag{5.119}$$

Yet, according to Lemma 77, one has that :

$$J_1(T) \leqslant \frac{K}{T\log(T)^2} O(\log(T)^2) \tag{5.120}$$

and so $J_1(T) \to 0$ when $T \to \infty$.
Furthermore, with P being a constant that can be chosen as large as one wants, one has, for T large :

$$J_2(T) \leqslant \frac{K}{T\log(T)^2} \int_{(l_1,l_2,l_1',l_2')\in Dom(T)} \frac{1}{(l_1 l_2 l_1' l_2')^2} \frac{1}{l_1 - l_1'} dl_1 dl_1' dl_2 dl_2' + o(1) \tag{5.121}$$

where

$$\begin{aligned} Dom(T) = \{(l_1, l_2, l_1', l_2') \in \mathbb{R}^4 \mid T\log(T) \geqslant l_1 \geqslant P, \\ T\log(T) \geqslant l_1' \geqslant P-1,\ T\log(T) \geqslant l_2', l_2 > 0, \\ l_1' + 1 \geqslant l_1 \geqslant l_1' + \frac{1}{T} \text{ and } l_1' l_2',\ l_1 l_2 \geqslant C\} \end{aligned}$$

and $o(1)$ is a quantity that goes to 0 when $T \to \infty$.
By integrating in l_2 and in l_2', one gets from Equation (5.121) that :

$$J_2(T) \leqslant \frac{D}{T\log(T)^2} \int_{\substack{T\log(T)\geqslant l_1 \geqslant P \\ T\log(T)\geqslant l_1' \geqslant P-1 \\ l_1'+1\geqslant l_1 \geqslant l_1'+\frac{1}{T}}} \frac{1}{l_1 l_1'} \frac{1}{l_1 - l_1'} dl_1 dl_1' + o(1). \tag{5.122}$$

By using the fact that $0 < \frac{1}{l_1(l_1-l_1')} \leqslant \frac{1}{l_1'(l_1-l_1')}$ and by integrating on l_1, one gets with Equation (5.122) that :

$$J_2(T) \leqslant \frac{K}{T\log(T)^2} \int_{T\geqslant l_1' \geqslant P-1} \frac{\log(T)}{(l_1')^2} dl_1' = O(\frac{1}{T\log(T)}) + o(1). \tag{5.123}$$

So, we have finally that $J_2(T) \to 0$, when $T \to \infty$, and so does $J(T)$, $A_{2,2}(T)$ and $A_2(T)$.
By exchanging l_1 and l_2, we obtain the fact that $A_1(T) \to 0$ when $T \to \infty$.
Finally, with Equation (5.109), one gets the wanted result. □

Proof of Proposition 47

To prove Proposition 47, we are going to follow the same approach as was used just before.
Before entering into the proof of Proposition 47, we need the following preparatory lemma. Let us introduce :

$$\tilde{G}_4(L,t,T) = \frac{1}{2\pi^4 \mathrm{Covol}(L^\perp)^2} \sum_{l\in J_2(L,T)} \frac{1}{(l_1 l_2)^2} \sum_{k=1}^{\infty} \frac{\cos(4\pi k t l_1)\cos(4\pi k t l_2)}{k^4}. \tag{5.124}$$

Then we have the following lemma that in particular says that $\frac{G_4(L,t)}{V(L,t)} - \frac{\tilde{G}_4(L,t,T)}{V(L,t)}$ tends, in probability, towards 0.

Lemma 86. *One has, when $T \to \infty$,*

$$\mathbb{E}\left(|\frac{\tilde{G}_4(L,t,T)}{V(L,t)} - \frac{G_4(L,t)}{V(L,t)}|\right) \to 0.$$

Proof. We use the same estimates as in the proof of Lemma 84. □

We can now tackle the proof of Proposition 47.

Proof of Proposition 47. According to Lemma 86 and the Markov's inequality, we only need to prove that $\frac{\tilde{G}_4(L,t,T)}{V(L,t)}$ tends in probability towards 0.
To do so, by using a usual trigonometric formula, we have that :

$$\frac{\tilde{G}_4(L,t,T)}{V(L,t)} = D(U_1(L,t,T) + U_2(L,t,T)) \tag{5.125}$$

with $D > 0$ and where

$$U_1(L,t,T) = \frac{1}{V(L,t)} \sum_{l \in J_2(L,T)} \frac{1}{(l_1 l_2)^2} \sum_{k=1}^{\infty} \frac{\cos(4\pi k t(l_1 - l_2))}{k^4} \tag{5.126}$$

and

$$U_2(L,t,T) = \frac{1}{V(L,t)} \sum_{l \in J_2(L,T)} \frac{1}{(l_1 l_2)^2} \sum_{k=1}^{\infty} \frac{\cos(4\pi k t(l_1 + l_2))}{k^4}. \tag{5.127}$$

So, to prove that $\frac{\tilde{G}_4(L,t,T)}{V(L,t)}$ tends to 0 in probability, we only need to show that the moments of order 2 of $U_1(L,t,T)$ and $U_2(L,t,T)$ tend to 0 in probability. We are going to prove this fact for $U_1(L,t,T)$ and the proof will be valid by symmetry for U_2 and we will so get the wanted result.
We have that :

$$\mathbb{E}(U_1(L,t,T)^2) \leqslant D \sum_{l,l' \in J_2(L,T)} \frac{1}{(l_1 l_2 l_1' l_2')^2} \sum_{k,k' \geqslant 1} \frac{1}{(kk')^4} \min\left(\frac{1}{\log(T)^2}, \frac{h(kl,kl') + h(kl,-kl')}{T\log(T)^2}\right) \tag{5.128}$$

where $D > 0$ is a constant and

$$h(l,l') = \frac{1}{|l_1 - l_2 - (l_1' - l_2')|}. \tag{5.129}$$

We are going to show that :

$$Z(L,T) = \sum_{l,l' \in J_2(L,T)} \frac{1}{(l_1 l_2 l_1' l_2')^2} \sum_{k,k' \geqslant 1} \frac{1}{(kk')^4} \min\left(\frac{1}{\log(T)^2}, \frac{h(kl,kl')}{T\log(T)^2}\right) \to 0 \tag{5.130}$$

when $T \to \infty$. Furthermore, the proof will still be valid if we exchange l' and $-l'$. So we will get the wanted result due to Equation (5.128).
One has that :

$$Z(L,T) = \Delta_1(L,T) + \Delta_2(L,T) \tag{5.131}$$

where

$$\Delta_1(L,T) = \sum_{l,l' \in J_2(L,T)} \frac{1}{(l_1 l_2 l_1' l_2')^2} \sum_{\substack{k,k' \geqslant 1 \\ k \text{ or } k' > \log(T)}} \frac{1}{(kk')^4} \min\left(\frac{1}{\log(T)^2}, \frac{h(kl,kl')}{T\log(T)^2}\right) \tag{5.132}$$

and

$$\Delta_2(L,T) = \sum_{l,l' \in J_2(L,T)} \frac{1}{(l_1 l_2 l_1' l_2')^2} \sum_{\substack{k,k' \geqslant 1 \\ k \text{ and } k' \leqslant \log(T)}} \frac{1}{(kk')^4} \min\left(\frac{1}{\log(T)^2}, \frac{h(kl,kl')}{T\log(T)^2}\right). \tag{5.133}$$

Yet, according to Lemma 77, one has that :

$$\Delta_1(L,T) \leqslant \frac{D}{\log(T)^8} \Big(\int_{\substack{A \leqslant \|l\| \leqslant T \\ |l_1 l_2| \geqslant C}} \frac{1}{l_1^2 l_2^2} dl_1 dl_2\Big)^2 + o(1) \underset{T\to\infty}{\to} 0 \tag{5.134}$$

where $D > 0$ is a constant.
So, we only need to prove that $\Delta_2(L,T) \to 0$ when $T \to \infty$.
Yet, one has also :

$$\Delta_2(L,T) = \Delta_{2,1}(L,T) + \Delta_{2,2}(L,T) \tag{5.135}$$

where

$$\Delta_{2,1}(L,T) = \sum_{l,l' \in J_2(L,T)} \frac{1}{(l_1 l_2 l_1' l_2')^2} \sum_{\substack{k,k' \geqslant 1 \\ k \text{ and } k' \leqslant \log(T) \\ h(kl,kl') \leqslant 1}} \frac{1}{(kk')^4} \frac{h(kl,kl')}{T\log(T)^2} \text{ and} \tag{5.136}$$

$$\Delta_{2,2}(L,T) = \sum_{l,l' \in J_2(L,T)} \frac{1}{(l_1 l_2 l_1' l_2')^2} \sum_{\substack{k,k' \geqslant 1 \\ k \text{ and } k' \leqslant \log(T) \\ h(kl,kl') > 1}} \frac{1}{(kk')^4} \frac{h(kl,kl')}{T\log(T)^2}. \tag{5.137}$$

Yet, there exists $A > 0$ such that :

$$\Delta_{2,1}(L,T) \leqslant \frac{K}{T\log(T)^2} \left(\int_{A \leqslant \|l\| \leqslant T\log(T)} \frac{1}{l_1^2 l_2^2}\right)^2 \to 0 \tag{5.138}$$

when $T \to \infty$ and where K is a positive constant. The right-hand side converges towards 0 because of Lemma 77.
Because of Equation (5.135), it only remains to prove that $\Delta_{2,2}(L,T)$ converges towards 0 when $T \to \infty$.
Yet, one has that :

$$\Delta_{2,2}(L,T) \leqslant \frac{K}{\log(T)^2} \int_{\substack{A \leqslant \|l\|, \|l'\| \leqslant T\log(T) \\ |l_1 l_2|, |l_1' l_2'| \geqslant C \\ |l_1 - l_2 - (l_1' - l_2')| < 1}} f(l,l') dl dl' \tag{5.139}$$

where $K > 0$ and $f(l,l') = \left(\frac{1}{l_1 l_2 l_1' l_2'}\right)^2$.
Because of Equation (5.139) and for symmetry reasons, it is enough to show that the following quantity converges towards 0 when $T \to \infty$:

$$J(T) = \frac{1}{\log(T)^2} \int_{(l,l') \in I(T)} f(l,l') dl dl' \tag{5.140}$$

where

$$I(T) = \{(l,l') \in \mathbb{R}^4 \mid l_1, l_2, l_1', l_2' > 0 \ , \\ l_1 > l_2 \ , \ l_1' > l_2' \ , \ 1 > l_1 - l_2 - (l_1' - l_2') > 0 \ , \\ A \leqslant \|l\|, \|l'\| \leqslant T\log(T) \ \textit{and} \ l_1 l_2, l_1' l_2' \geqslant C\}. \tag{5.141}$$

Let us call $I_1(T)$ the set of all (l,l') that belong to $I(T)$ and that verify $l_1 - l_2 \leqslant 2$. Let us call also $I_2(T) = I(T) - I_1(T)$.
Let us note that if $(l,l') \in I_1(T)$ then $l_1' - l_2' \leqslant 2$.
So, one has :

$$J_1(T) = \frac{1}{\log(T)^2} \int_{(l,l') \in I_1(T)} f(l,l') dl dl' \underset{T\to\infty}{\to} 0 \tag{5.142}$$

because $\int_{(l,l')\in I_1(T)} f(l,l')dldl'$ is bounded (l and l' are close to the axis $y = x$).
Let us set :

$$J_2(T) = \frac{1}{\log(T)^2} \int_{(l,l') \in I_2(T)} f(l,l') dl dl' \tag{5.143}$$

so that

$$J(T) = J_1(T) + J_2(T). \tag{5.144}$$

As a consequence from Equation (5.142), it is enough to prove that $J_2(T) \to 0$ when $T \to \infty$ in order to prove that $\Delta_{2,2}(L,T)$ converges towards 0 when $T \to \infty$.
It is easy to see that an *a priori* important part of the integral $\int_{(l,l')\in I_2(T)} f(l,l')dldl'$ is the l and l' such that l_1 and l_1' are large, for example larger than $\log(\log(T))$, and l_2 and l_2' are small, for example smaller than $\frac{1}{\log(\log(T))}$. The rest of the integral, when divided by $\log(T)^2$, goes to 0 when $T \to \infty$.
But we have also that $1 > l_1 - l_2 - (l_1' - l_2') > 0$ because $(l,l') \in I(T)$. So, it implies in particular that, for T large enough,

$$\frac{3}{2} > l_1 - l_1' > -\frac{1}{2}.$$

Hence, by integrating first in l_2 and in l_2' and by using the fact that the lattice is admissible, we have that $J_2(T)$ is estimated as followed :

$$J_2(T) \leqslant \frac{K_1}{\log(T)^2} \int_{\substack{T\log(T) \geqslant l_1, l_1' \geqslant \log(\log(T)) \\ \frac{3}{2} > l_1 - l_1' > -\frac{1}{2}}} \frac{1}{l_1 l_1'} dl_1 dl_1' + o(1) \tag{5.145}$$

where K_1 is a positive constant (that depends on C).
From Equation (5.145), by integrating, first, in l_1, second, in l_1', one gets that :

$$J_2(T) \leqslant \frac{K_2}{\log(T)} + o_T(1) \tag{5.146}$$

where $K_2 > 0$, which concludes the proof.

□

5.4.4 Conclusion

We can now give the full proof of Theorem 13.

Proof of Theorem 13. Proposition 39 gives us the first part of Theorem 13. The second part of Theorem 13 is the consequence of Proposition 40 and of Proposition 44 (with L being replaced by $L^{\perp}$; the validity of this last Proposition is a consequence of Proposition 45, Proposition 46 and Proposition 47).

□

5.5 Proof of Theorem 15

The goal of this section is to establish Theorem 15. It is a natural extension of Theorem 13 and of a part of 14. We see that, in that case, the normalization is larger than before : at least $\log(t)$ whereas the normalization before was in $\sqrt{\log(t)}$.
To show this last theorem, we are going to proceed in three times : first, we will show that the lower and upper estimates about $\tilde{V}(L,r)$ hold, second we will show that the lower and upper estimates about $\tilde{V}(L,r)$ hold and third we will conclude the proof of Theorem 15 by using the third subsection of Section 4.

5.5.1 Estimation of $\tilde{V}(L,r)$ in the typical case

The goal of this subsection is to show the following proposition :

Proposition 48. *For every $\epsilon > 0$, for a typical $L \in \mathscr{S}_d$, one has that*

$$\tilde{V}(L,r) = O(r^{2d-2+\epsilon}) \tag{5.147}$$

and

$$\frac{\tilde{V}(L,r)}{r^{d-1}} \underset{r\to\infty}{\to} \infty. \tag{5.148}$$

The proof of this proposition relies heavily on [82] and we need to recall a fundamental theorem.

Definition 25. *A subgroup $G \subset SL_d(\mathbb{R})$ is called ergodic on the homogeneous space $\mathscr{S}_d$ if for every G-invariant measurable subset $A \subset \mathscr{S}_d$, $\mu_d(A) = 0$ or $\mu_d(A) = 1$ where μ_d is the unique Haar and probability measure over $\mathscr{S}_d$.*

The Moore's ergodic theorem gives us in fact that G is ergodic if, and only if, G is not contained in any compact subgroup of $SL_d(\mathbb{R})$. As a consequence, Δ is ergodic and thus we have the following fundamental theorem

Theorem 19. *Let ψ a function integrable over $(\mathscr{S}_d, \mu_d)$. Then, for almost all $L \in \mathscr{S}_d$ (in the sense of the measure μ_d), one has that*

$$\lim_{r\to\infty} \frac{1}{|\Delta_r|} \sum_{\delta\in\Delta_r} \psi(\delta L) = \int_{\mathscr{S}_d} \psi(L) d\mu_d(L).$$

We can now give the proof of Proposition 48.

Proposition 48. First, one has that :

$$\tilde{V}(L,r) \leqslant \left(\sum_{\delta \in \Delta_r} \frac{1}{\|\delta L\|^d} \right)^2 . \tag{5.149}$$

Yet, Lemma 3.2 from [82] gives us that for every $\epsilon > 0$, for a typical $L \in \mathscr{S}_d$,

$$\left(\sum_{\delta \in \Delta_r} \frac{1}{\|\delta L\|^d} \right) = O(r^{d-1+\epsilon}). \tag{5.150}$$

So, Equation (5.149) and Equation (5.150) give us the wanted first result of Proposition 48.
Second, one has that :

$$\sqrt{\frac{\tilde{V}(r,L)}{r^{d-1}}} \geqslant \frac{K_1}{r^{d-1}} \sum_{\delta \in \Delta_r} \frac{1}{\|\delta L\|^d} \tag{5.151}$$

where $K_1 > 0$. We have obtained this last equation by using the concavity of the square root and because the cardinal number of Δ_r is of order r^{d-1} (see Equation (5.23)).
Let then $m \geqslant 1$.
One has that :

$$\frac{1}{r^{d-1}} \sum_{\delta \in \Delta_r} \frac{1}{\|\delta L\|^d} \geqslant \frac{1}{r^{d-1}} \sum_{\delta \in \Delta_r} \min \left(m, \frac{1}{\|\delta L\|^d} \right) . \tag{5.152}$$

Yet, $L \longmapsto \min \left(m, \frac{1}{\|L\|^d} \right)$ is integrable over $\mathscr{S}_d$. So, one has for a typical $L \in \mathscr{S}_d$:

$$\frac{1}{r^{d-1}} \sum_{\delta \in \Delta_r} \min \left(m, \frac{1}{\|\delta L\|^d} \right) \underset{r \to \infty}{\to} K_2 \int_{\mathscr{S}_d} \min(m, \frac{1}{\|L\|^d}) d\mu_d \tag{5.153}$$

where $K_2 > 0$ does not depend on m.
By using Equation (5.151), Equation (5.152) and Equation (5.153), one gets that for every $m \geqslant 1$, for a typical $L \in \mathscr{S}_d$:

$$\liminf_{r \to \infty} \sqrt{\frac{\tilde{V}(r,L)}{r^{d-1}}} \geqslant K \int_{\mathscr{S}_d} \min(m, \frac{1}{\|L\|^d}) d\mu_d \tag{5.154}$$

where $K > 0$ and $m > 0$.
By making $m \to \infty$, by using Fatou's lemma and by using the fact that

$$\int_{\mathscr{S}_d} \frac{1}{\|L\|^d} d\mu_d = \infty,$$

one gets the wanted result. □

5.5.2 Result about $V(L,t)$ in the typical case

The goal of this subsection is to prove the following proposition :

Proposition 49. *For every $\epsilon > 0$, for L a typical lattice, one has that*

$$V(L,t) = O(\log(t)^{2+\epsilon})$$

and

$$\frac{V(L,t)}{\log(t)^2} \underset{t\to\infty}{\to} \infty.$$

In fact, the most important part of $V(L,t)$ are the terms $l_1^2 l_2^2$ that are the smallest possible. So, we need to know more about how small can be $|l_1 l_2|$. In fact, we have the following result :

Theorem 20 ([87], [82])**.** *For a typical $L \in \mathscr{S}_2$, there exists a sequence $(l_n)_{n\in\mathbb{N}}$ such that*

$$\|l_n\| \underset{n\to\infty}{\to} \infty \text{ and } \log(\|l_n\|)|(l_n)_1(l_n)_2| \underset{n\to\infty}{\to} 0.$$

Furthermore, for all $\alpha > 0$, for a typical $L \in \mathscr{S}_2$, there exists $C > 0$ such that for all $l \in L - \{0\}$,

$$|l_1 l_2| \geqslant C|\log(\|l\|)|^{-1-\alpha}.$$

As a consequence of Theorem 20, we see that, to establish Proposition 49, the following lemma will be convenient :

Lemma 87. *For all $C > 0$, for all $A > 0$, for all $\alpha > 0$, one has that :*

$$\int_{\substack{l\in\mathbb{R}^2 \\ A\leqslant\|l\|\leqslant t \\ |Num(l)|\geqslant C|\log(\|l\|)|^{-1-\alpha}}} \frac{1}{l_1^2 l_2^2} dl_1 dl_2 = O(\log(t)^{2+\alpha}).$$

The proof of this lemma is basically the same as the proof of Lemma 77.

Proof. First, let us say that it is enough to prove the result for A fixed and large enough. Then, let us set :

$$J(t) = \int_{\substack{l\in\mathbb{R}^2 \\ A\leqslant\|l\|\leqslant t \\ |\mathrm{Num}(l)|\geqslant C|\log(\|l\|)|^{-1-\alpha}}} \frac{1}{l_1^2 l_2^2} dl_1 dl_2. \tag{5.155}$$

By passing into polar coordinates (r,θ) and by using the symmetries, one has :

$$J(t) = 8\int_{\substack{A\leqslant r\leqslant t \\ 0\leqslant\theta\leqslant\frac{\pi}{4} \\ \sin(2\theta)\geqslant\frac{2C\log(r)^{-1-\alpha}}{r^2}}} \frac{1}{r^3}\frac{4}{\sin(2\theta)^2} dr d\theta. \tag{5.156}$$

By making the changes of variable $\theta' = 2\theta$ and, then, $u = \tan(\theta')$, one gets from Equation (5.31) and by taking A large enough :

$$J(t) = 16\int_{A\leqslant r\leqslant t^{1+\epsilon_1}} \frac{1}{r^3}(-1 + \frac{r^2\log(r)^{1+\alpha}}{2C}\sqrt{1 - (\frac{2C}{\log(r)^{1+\alpha}r^2})^2})dr. \tag{5.157}$$

From Equation (5.157), for all A large enough, one has that :

$$J(t) = O(\log(t)^{2+\alpha}) \tag{5.158}$$

when $t \to \infty$. So we get the wanted result. □

We can now prove Proposition 49.

Proof of Proposition 49. Let $\epsilon > 0$. For a typical $L \in \mathscr{S}_2$, there exists $C > 0$ such that for all $l \in L - \{0\}$

$$|l_1 l_2| \geqslant C|\log(\|l\|)|^{-1-\epsilon} \tag{5.159}$$

and there exists a sequence $(l_n)_{n\in\mathbb{N}} \in L^{\mathbb{N}}$ such that

$$\|l_n\| \underset{n\to\infty}{\to} \infty \text{ and} \tag{5.160}$$

$$\log(\|l_n\|)|(l_n)_1(l_n)_2| \underset{n\to\infty}{\to} 0. \tag{5.161}$$

So, first, one has that there exist $A, D > 0$ such that :

$$V(L,t) \leqslant D \int_{\substack{l\in\mathbb{R}^2 \\ A\leqslant\|l\|\leqslant t \\ |\mathrm{Num}(l)|\geqslant C|\log(\|l\|)|^{-1-\epsilon}}} \frac{1}{l_1^2 l_2^2} dl_1 dl_2 = O(\log(t)^{2+\epsilon}) \tag{5.162}$$

according to the definition of $V(L,t)$ and because of Equation (5.159).
So, because of Lemma 87, one gets that :

$$V(L,t) = O(\log(t)^{2+\epsilon}). \tag{5.163}$$

Furthermore, a consequence of Equation (5.160) and of Equation (5.161) is the fact that :

$$\liminf_{t\to\infty} \frac{V(L,t)}{\log(t)^2} = \infty \tag{5.164}$$

also due to the definition of $V(L,t)$. □

We can now conclude the proof of Theorem 15.

5.5.3 Conclusion

Proof of Theorem 15. The first assertion of Theorem 15 is proven by Proposition 48.
The second assertion of Theorem 15 is proven by Proposition 49.
The third part of Theorem 15, that concerns the convergence in distribution, is shown as the second part of Theorem 13 (see 4.2 and 4.3). □

5.6 Proof of Theorem 16

In this section, we are going to show Theorem 16. Let $x \in \mathbb{R}$ and $a > 0$.
Instead of considering t, we can consider $\frac{t}{a}$. So, we can suppose, and we are going to make this assumption in the rest of this section, that $a = 1$ in the study of $\frac{\mathcal{R}(t\mathrm{Rect}(a,a)+(x,x),\mathbb{Z}^2)}{t}$.
Now we are going to give a simple expression of $\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}$.

5.6.1 An expression of $\frac{\mathcal{R}(t\mathbf{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}$

The main object of this subsection is to prove the following proposition :

Proposition 50. *We have for every $x \in \mathbb{R}^2$, for every $t > 0$, that*

$$\frac{\mathcal{R}(tRect(1,1)+(x,x),\mathbb{Z}^2)}{t} = \frac{(\lfloor t+x \rfloor - \lceil -t+x \rceil + 1)^2 - 4t^2}{t}. \tag{5.165}$$

The proof is quite straightforward.

Proof. Let $x \in \mathbb{R}^2$. Let $t > 0$.
One has that :

$$\begin{aligned} N(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2) &= \left(\sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2 \\ -t+x\leqslant n_1\leqslant t+x \\ -t+x\leqslant n_2\leqslant t+x}} 1 \right) \\ &= (\lfloor t+x \rfloor - \lceil -t+x \rceil + 1)^2. \end{aligned} \tag{5.166}$$

Furthermore, one has that :

$$\mathrm{Area}(t\mathrm{Rect}(1,1)) = 4t^2. \tag{5.167}$$

So, Equation (5.166) and Equation (5.167) and the definition of $\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)$ give us Equation (5.165). □

With Equation (5.165), one has that :

$$\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t} = (\lfloor t+x \rfloor - \lceil -t+x \rceil + 1 - 2t)\frac{(\lfloor t+x \rfloor - \lceil -t+x \rceil + 1 + 2t)}{t}. \tag{5.168}$$

Thanks to this last remark, the asymptotical study of $\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}$ with t distributed on $[0,T]$ according to the probability measure $\frac{1}{T}\rho(\frac{t}{T})dt$ is going to be reduced to the study of a simpler quantity. It is the object of the next subsection.

5.6.2 Reduction of the study of $\frac{\mathcal{R}(t\mathbf{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}$

The main object of this subsection is to prove the following proposition :

Proposition 51. *For every $g \in C_c(\mathbb{R})$,*

$$\int_{t=0}^{T}(g\left(\frac{\mathcal{R}(tRect(1,1)+(x,x),\mathbb{Z}^2)}{t}\right)-g\left(\Delta(t,x)\right))\frac{1}{T}\rho(\frac{t}{T})dt \underset{T\to\infty}{\to} 0 \tag{5.169}$$

where $\Delta(t,x)$ is defined by

$$\Delta(t,x)=4(\lfloor t+x\rfloor-\lceil -t+x\rceil+1-2t). \tag{5.170}$$

The proof is quite straightforward and lie on the definitions of $\lfloor\cdot\rfloor$ and of $\lceil\cdot\rceil$.

Proof. For every $t>0$, for every $x\in\mathbb{R}$,

$$t+x-1<\lfloor t+x\rfloor\leqslant t+x \tag{5.171}$$

and

$$-t+x\leqslant\lceil -t+x\rceil<-t+x+1. \tag{5.172}$$

From Equation (5.171) and Equation (5.172), one gets that :

$$\frac{4t-1}{t}<\frac{(\lfloor t+x\rfloor-\lceil -t+x\rceil+1+2t)}{t}\leqslant\frac{4t+1}{t} \tag{5.173}$$

So, from this last equation, one has that, when $t\to\infty$,

$$\frac{(\lfloor t+x\rfloor-\lceil -t+x\rceil+1+2t)}{t}\to 4. \tag{5.174}$$

So, one has that :

$$\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}-\Delta(t,x)\to 0 \tag{5.175}$$

when $t\to\infty$ because $(\lfloor t+x\rfloor-\lceil -t+x\rceil+1-2t)$ is bounded.
Now, the end of this proof is quite straightforward. Indeed, one has, for every $0<\kappa<\frac{1}{2}$:

$$\begin{aligned}|\int_{t=0}^{T}(g\left(\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}\right)-g\left(\Delta(t,x)\right))\frac{1}{T}\rho(\frac{t}{T})dt|\leqslant 2\|g\|_\infty\int_0^\kappa\rho(t)dt\\ +\int_{\kappa T}^{T}|g\left(\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}\right)-g\left(\Delta(t,x)\right)|\frac{1}{T}\rho(\frac{t}{T})dt\end{aligned} \tag{5.176}$$

and, because $g\in C_c(\mathbb{R})$, it is a uniformly continuous function and one has, from Equation (5.176), that

$$\limsup_{T\to\infty}\int_{\kappa T}^{T}|g\left(\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}\right)-g\left(\Delta(t,x)\right)|\frac{1}{T}\rho(\frac{t}{T})dt=0 \tag{5.177}$$

because, also, of Equation (5.175).
So, Equation (5.176) and Equation (5.177) give us that for every $0 < \kappa \leqslant \frac{1}{2}$:

$$\limsup_{T\to\infty} \left| \int_{t=0}^{T} (g\left(\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}\right) - g\left(\Delta(t,x)\right))\frac{1}{T}\rho(\frac{t}{T})dt\right| \leqslant 2\|g\|_\infty \int_0^\kappa \rho(t)dt. \tag{5.178}$$

By making κ go to 0, one gets the wanted result. □

Proposition 51 enables us to reduce the asymptotic study of $\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}$ to the asymptotic study of $\Delta(t,x)$. In the next subsection, we are going to show that $\Delta(t,x)$ converges in distribution and exhibit the limit distribution and its moments.

5.6.3 Convergence in distribution of $\Delta(t,x)$

The goal of this subsection is to prove the following proposition :

Proposition 52. *For all $x \in \mathbb{R}$, when $t \in [0,T]$ is distributed according to the probability density $\frac{1}{T}\rho(T\cdot)$ on $[0,T]$ then, when $T \to \infty$, $\Delta(t,x)$ converges in distribution. Furthermore, the limit distribution β has a compact support included in $[-2,4]$ and for every $k \in \mathbb{N}$, one has that*

$$\int_{x\in\mathbb{R}} x^k d\beta(x) = a_k$$

where

$$a_k = \frac{4^k(1+(-1)^k)(y^{k+1}+(1-y)^{k+1})}{2(k+1)}$$

with $y = |t_{2,0} - t_{1,0}|$ where $t_{2,0}$ is the first $t \geqslant 0$ such that $-t+x \in \mathbb{Z}$ and $t_{1,0}$ is the first $t \geqslant 0$ such that $t+x \in \mathbb{Z}$.

We are going to show this proposition in three times. The first time, and the next subsubsection is dedicated to it, consists in calculating the limit, when $T \to \infty$, of all entire moments of $\Delta(t,x)$ when $\rho = \mathbf{1}_{[0,1]}$. The second time consists in showing that these limits define a unique probability distribution over $\mathbb{R}$. The last subsection is dedicated to the conclusion of the proof. So, basically, we are going to use the method of moments to show Proposition 52.

Calculation of limits of moments of $\Delta(t,x)$

Before stating the main proposition of this section, we need to make some observations and put in place some notations.
Let us call $t_{1,0} < \cdots < t_{1,l}$ the different times $t \in [0,T]$ such that $t+x \in \mathbb{Z}$.
In the same way, let us call $t_{2,0} < \cdots < t_{2,h}$ the different times $t \in [0,T]$ such that $-t+x \in \mathbb{Z}$.
Let us observe that for every $i \in \{0,\cdots,l-1\}$, $t_{1,i+1} - t_{1,i} = 1$ and that for every $j \in \{0,\cdots,h-1\}$, $t_{2,j+1} - t_{2,j} = 1$.
As a consequence, one has necessarily that $t_{2,0} \in [t_{1,0}, t_{1,1}[$ or $t_{1,0} \in [t_{2,0}, t_{2,1}[$ and $h = l$

or $h = l - 1$ or $h = l + 1$.
Let us set :

$$y = |t_{1,0} - t_{2,0}|. \tag{5.179}$$

Then, the main proposition of this section is the following proposition :

Proposition 53. *For every $k \in \mathbb{N}$, when $\rho = \mathbf{1}_{[0,1]}$, one has that :*

$$\lim_{T\to\infty} \Big(\mathbb{E}((\Delta(t,x))^k)\Big) = a_k \tag{5.180}$$

where

$$a_k = \frac{4^k(1+(-1)^k)(y^{k+1}+(1-y)^{k+1})}{2(k+1)}. \tag{5.181}$$

The proof consists basically in cutting the interval $[0,T]$ into subintervals where all the quantities that intervene in the calculus can be expressed simply.

Proof. Let $k \geqslant 0$ and let us suppose that $\rho = \mathbf{1}_{[0,1]}$.
By symmetry, we can, and we will, also suppose that $t_{2,0} \in [t_{1,0}, t_{1,1}[$.
One has that :

$$-4 \leqslant \Delta(t,x) \leqslant 4 \tag{5.182}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$.
Consequently, we can suppose that

$$\mathbb{E}((\Delta(t,x))^k) = \sum_{i=0}^{h-1} \int_{t_{1,i}}^{t_{2,i}} \Delta(t,x)^k \frac{1}{T} dt + \int_{t_{2,i}}^{t_{1,i+1}} \Delta(t,x)^k \frac{1}{T} dt \tag{5.183}$$

even if it means neglecting the rest of the integral that is calculated on a union of two intervals of respective lengths at most 2 and so the corresponding term, because of Equation (5.182), is a $O(\frac{1}{T})$.
Let $i \in \{0, \cdots, h-1\}$.
Then one has :

$$\int_{t_{1,i}}^{t_{2,i}} \Delta(t,x)^k \frac{1}{T} dt = \int_{t_{1,i}}^{t_{2,i}} 4^k (t_{1,i} + t_{2,i} - 2t)^k \frac{1}{T} dt \tag{5.184}$$

according to the Equation (5.170).
So, one gets that :

$$\int_{t_{1,i}}^{t_{2,i}} \Delta(t,x)^k \frac{1}{T} dt = \frac{4^k}{2T(k+1)} (y^{k+1} - (-y)^{k+1}) \tag{5.185}$$

because for all $i \in \{0, \cdots, h-1\}$, $y = t_{2,0} - t_{1,0} = t_{2,i} - t_{1,i}$.
So, one gets that :

$$\int_{t_{1,i}}^{t_{2,i}} \Delta(t,x)^k \frac{1}{T} dt = \frac{4^k}{2T(k+1)} y^{k+1}(1+(-1)^k). \tag{5.186}$$

In a similar way, one gets that :

$$\int_{t_{2,i}}^{t_{1,i+1}} \Delta(t,x)^k \frac{1}{T} dt = \frac{4^k}{2T(k+1)} (1-y)^{k+1}(1+(-1)^k) \tag{5.187}$$

because $1 - y = t_{1,i+1} - t_{2,i}$.
So, with Equation (5.183), Equation (5.186) and Equation (5.187), one gets that :

$$\mathbb{E}((\Delta(t,x))^k) = \sum_{i=0}^{h-1} \frac{4^k}{2T(k+1)}(y^{k+1} + (1-y)^{k+1})(1 + (-1)^k). \tag{5.188}$$

By using the fact that $\lim_{T\to\infty} \frac{h}{T} = 1$, one gets from Equation (5.188) that :

$$\mathbb{E}((\Delta(t,x))^k) = \frac{4^k h}{2T(k+1)}(y^{k+1}+(1-y)^{k+1})(1+(-1)^k) \underset{T\to\infty}{\to} \frac{4^k}{2(k+1)}(y^{k+1}+(1-y)^{k+1})(1+(-1) \tag{5.189}$$

□

In the next subsubsection, we are going to see that there exists a unique probability distribution over $\mathbb{R}$ such that the entire moments are given by the a_k.

Existence and unicity of the distribution whose moments are given by the a_k

The main object of this subsubsection is to prove the following proposition :

Proposition 54. *There exists a unique probability distribution β over $\mathbb{R}$ such that for all $k \in \mathbb{N}$,*

$$\int_{x\in\mathbb{R}} x^k d\beta(x) = a_k.$$

To prove this proposition, we need to recall the following theorem.

Theorem 21. *Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence of real numbers such that the power series $\sum_{k\geqslant 0} \frac{\alpha_k}{k!} z^k$ has a positive radius of convergence. Then, there exists at most one probability measure β over $\mathbb{R}$ such that for all $k \in \mathbb{N}$,*

$$\alpha_k = \int_{x\in\mathbb{R}} x^k d\beta(x).$$

We can now prove the proposition 54.

Proof of Proposition 54. One has :

$$\frac{\frac{a_{k+1}}{(k+1)!}}{\frac{a_k}{k!}} \underset{k\to\infty}{\to} 0 \tag{5.190}$$

according to Equation (5.181).
So, the ratio test gives us that $\sum_{k\geqslant 0} \frac{a_k}{k!} z^k$ has a radius of convergence that is infinite.
As a consequence, Theorem 21 applies here and gives us that there is at most one probability measure β over $\mathbb{R}$ whose entire moments are given by the $(a_k)_{k\in\mathbb{N}}$.
Furthermore,

$$-4 \leqslant \Delta(t,x) \leqslant 4$$

for all $t \geqslant 0$ and for all $x \in \mathbb{R}$.
So, we have that for every $x \in \mathbb{R}$, $(\mathbb{P}_{\Delta(\cdot,x)})_{t\geqslant 0}$ is tight.
As a consequence, Prokhorov's theorem gives us the existence of the probability measure β whose entire moments are given by the $(a_k)_{k\in\mathbb{N}}$. □

Conclusion of the proof of Proposition 52

To conclude the proof of Proposition 52, we need to recall one theorem and to prove one lemma.

Theorem 22. *Let X be a real random variable characterized by its entire moments and $(X_n)_{n\in\mathbb{N}}$ be a sequence of real random variables such that for all $k \in \mathbb{N}$,*

$$E(X_n^k) \underset{n\to\infty}{\to} E(X^k).$$

Then the sequence (X_n) converges in distribution towards X.

The following lemma is in fact taken from [10] (but it was not formulated as it, see proof of theorem 4.3). We are going to give the proof of this lemma here for completeness.

Lemma 88. *Let F be a real measurable function from $\mathbb{R}_+$.*
Assume that there exists a probability measure μ over $\mathbb{R}$ such that for every $g \in C_b(\mathbb{R})$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g(F(t))dt = \int_{x\in\mathbb{R}} g(x)d\mu(x).$$

Then, for every probability density ρ over $[0,1]$, for every $g \in C_b(\mathbb{R})$, one has

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g(F(t))\rho(\frac{t}{T})dt = \int_{x\in\mathbb{R}} g(x)d\mu(x). \tag{5.191}$$

Proof. Assume first that ρ is a step-wise function consisting of a finite number of steps. By linearity of the Equation (5.191), it is enough, in this case, to prove (5.191) for the function

$$\rho(x) = \frac{1}{b-a}\mathbf{1}_{[a,b]}(x)$$

where $0 \leqslant a < b \leqslant 1$, *id est* a one-step function.
Let $g \in C_b(\mathbb{R})$.
Because of the assumption of Lemma 88, one has :

$$\begin{aligned}\frac{1}{T}\int_0^T g(F(t))\rho(\frac{t}{T})dt &= \frac{1}{T(b-a)}\int_{aT}^{bT} g(F(t))dt \\ &= \frac{1}{b-a}(b\frac{1}{bT}\int_0^{bT} g(F(t))dt - a\frac{1}{aT}\int_0^{aT} g(F(t))dt) \\ &\underset{T\to\infty}{\to} \frac{1}{b-a}(b\int_{x\in\mathbb{R}} g(x)d\mu(x) - a\int_{x\in\mathbb{R}} g(x)d\mu(x)) = \int_{x\in\mathbb{R}} g(x)d\mu(x)\end{aligned} \tag{5.192}$$

So we have Equation (5.191) in this case.
The general case follows now by using the fact that for every probability density ρ over $[0,1]$, for every $\epsilon > 0$, there exists ρ_ϵ, a step-wise function consisting of a finite number of steps, such that

$$\int_{x\in[0,1]} |\rho(x) - \rho_\epsilon(x)|dx \leqslant \epsilon.$$

□

We can now prove Proposition 52.

Proof of Proposition 52. Thanks to Lemma 88, we only need to prove Proposition 52 in the case where $\rho = \mathbf{1}_{[0,1]}$.
Furthermore, Proposition 53, Proposition 54 and Theorem 22 gives us that that $\Delta(\cdot, x)$ converges in distribution when $T \to \infty$ and the moments of the limit distribution is given by the a_k.
Finally, we recall that $-4 \leqslant \Delta(t, x) \leqslant 4$ for all $t \geqslant 0$, for all $x \in \mathbb{R}$ and so β has its support compact and included in $[-4, 4]$. □

5.6.4 Conclusion

We can now prove Theorem 16.

Proof of Theorem 16. Thanks to Lemma 88, we only need to prove Theorem 16 in the case where $\rho = \mathbf{1}_{[0,1]}$ and $a = 1$.
Proposition 51 gives us that, if $\Delta(\cdot, x)$ converges in distribution, then it is also the case of $\frac{\mathcal{R}(t\mathrm{Rect}(1,1)+(x,x),\mathbb{Z}^2)}{t}$ and the limit distribution is the same.
Finally, Proposition 52 gives the wanted result. □

Chapter 6

The particular case of $\mathbb{Z}^d$ and boxes

(Résumé français)Nous étudions l'erreur du nombre de points du réseau $\mathbb{Z}^d$ qui tombent dans un hypercube centré autour de 0 dilaté et translaté et dont les axes sont parallèles aux axes de coordonnées. Nous montrons que si t, le facteur de dilatation, est distribué selon la mesure de probabilité $\frac{1}{T}\rho(\frac{t}{T})dt$ avec ρ une densité de probabilité sur $[0,1]$, l'erreur, normalisée par t^{d-1}, converge en loi lorsque $T \to \infty$ dans le cas où la translation est de la forme $X = (x, \cdots, x)$ et dans le cas où les coordonnées de X sont indépendantes entre elles, indépendantes de t et distribuées selon la loi uniforme sur $[-\frac{1}{2}, \frac{1}{2}]$. Dans les deux cas, on calcule par ailleurs la fonction caractéristique de la loi limite.

(English abstract)We study the error of the number of points of the lattice $\mathbb{Z}^d$ that fall into a dilated and translated hypercube centred around 0 and whose axis are parallel to the axis of coordinates. We show that if t, the factor of dilatation, is distributed according to the probability measure $\frac{1}{T}\rho(\frac{t}{T})dt$ with ρ being a probability density over $[0,1]$ the error, when normalized by t^{d-1}, converges in distribution when $T \to \infty$ in the case where the translation is of the form $X = (x, \cdots, x)$ and in the case where the coordinates of X are independent between them, independent from t and distributed according to the uniform distribution over $[-\frac{1}{2}, \frac{1}{2}]$. In both cases, we compute the characteristic function of the limit distribution. study the existence of the moments of its distribution.

Contents

6.1 Introduction **190**

6.2 A bit of heuristic and plan of the paper **192**

6.3 Simplification of the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ **193**

6.3.1 An expression of $\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)$ 193

6.3.2 Reduction of the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ 194

6.4 Proof of Theorem 23 . . . . . **195**

6.4.1 Reduction of the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ 195

6.4.2 Computation of characteristic function 196
6.4.3 Conclusion 197
6.5 Proof of Theorem 24 198
6.5.1 Reduction of the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ 198
6.5.2 Computation of the characteristic function $\varphi_{\tilde{\Delta}(t,x_1)}$ 198
6.5.3 Conclusion 199

6.1 Introduction

Let P be a measurable subset of $\mathbb{R}^d$ of non-zero finite Lebesgue measure. We want to evaluate the following cardinal number when $t \to \infty$:

$$N(tP + X, L) = |(tP + X) \cap L|$$

where $X \in \mathbb{R}^d$, L is a lattice of $\mathbb{R}^d$ and $tP + X$ denotes the set P dilated by a factor t relatively to 0 and then translated by the vector X.
Under mild regularity conditions on the set P, one can show that :

$$N(tP + X, L) = t^d \frac{\mathrm{Vol}(P)}{\mathrm{Covol}(L)} + o(t^d)$$

where $o(f(t))$ denotes a quantity such that, when divided by $f(t)$, it goes to 0 when $t \to \infty$ and where $\mathrm{Covol}(L)$ is the volume of a fundamental set of the lattice L.
We are interested in the error term

$$\mathcal{R}(tP + X, L) = N(tP + X, L) - t^d \frac{\mathrm{Vol}(P)}{\mathrm{Covol}(L)}.$$

In the case where $d = 2$ and where P is the unit disk $\mathbb{D}^2$, Hardy's conjecture in [36] stipulates that we should have for all $\epsilon > 0$,

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^{\frac{1}{2}+\epsilon})$$

where $Y = O(X)$ means that there exists $D > 0$ such that $|Y| \leqslant D|X|$.
One of the result in this direction has been established by Iwaniec and Mozzochi in [43]. They have proven that for all $\epsilon > 0$,

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^{\frac{7}{11}+\epsilon}).$$

This result has been recently improved by Huxley in [42]. Indeed, he has proven that :

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^K \log(t)^\Lambda)$$

where $K = \frac{131}{208}$ and $\Lambda = \frac{18627}{8320}$.
In dimension 3, Heath-Brown has proven in [40] that :

$$\mathcal{R}(t\mathbb{D}^3, \mathbb{Z}^3) = O(t^{\frac{21}{16}+\epsilon}).$$

These last two results are all based on estimating what are called *exponential sums*. Furthermore, they only tackle "deterministic" cases.
Another result about this problem has been established by Bleher, Cheng, Dyson and Lebowitz in [11]. Let ρ be a probability density on $[0,1]$. They took interest in what is happening when the factor of dilatation t is distributed according to the probability measure $\frac{1}{T}\rho(\frac{t}{T})dt$ and when $T \to \infty$. Their result states as following :

Theorem ([11])**.** *There exists a probability density p on $\mathbb{R}$ such that for every piecewise continuous and bounded function $g : \mathbb{R} \longrightarrow \mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g\Big(\frac{\mathcal{R}(t\mathbb{D}^2,\mathbb{Z}^2)}{\sqrt{t}}\Big)\rho(\frac{t}{T})dt = \int_{\mathbb{R}} g(x)p(x)dx.$$

Furthermore p can be extended as an analytic function over $\mathbb{C}$ and satisfies that for every $\epsilon > 0$,

$$p(x) = O(e^{-|x|^{4-\epsilon}})$$

when $x \in \mathbb{R}$ and when $|x| \to \infty$.

We want to follow this approach on another problem. Namely, let us give $a > 0$ and let us define the following set

$$\mathcal{C}(a) = \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d \ |\forall i \in [1,d],\ |x_i| \leqslant a\}. \tag{6.1}$$

In that case, with ρ being a probability density over $[0,1]$, we want to study the possible convergence in distribution of the quantity $\frac{\mathcal{R}(t\mathcal{C}(a)+X,\mathbb{Z}^d}{t^{d-1}}$. We already proved such a result when the dimension d was equal to 2. Here, we are going to prove the two following theorems (that constitute a generalization of the previous result) :

Theorem 23. *Let x be a real number. When $t \in [0,T]$ is distributed according to the probability density $\frac{1}{T}\rho(\frac{\cdot}{T})$ on $[0,T]$ then, when $T \to \infty$, $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ converges in distribution with $X = (x, \cdots, x)$. Furthermore, the limit distribution has the following characteristic function*

$$\varphi(u) = \frac{\sin(d2^{d-1}uy) + \sin(d2^{d-1}u(1-y))}{d2^{d-1}u}$$

with $y = |t_{2,0} - t_{1,0}|$ where $t_{2,0}$ is the first $t \geqslant 0$ such that $-t + x \in \mathbb{Z}$ and $t_{1,0}$ is the first $t \geqslant 0$ such that $t + x \in \mathbb{Z}$. In fact, $y = |1 - 2\{x\}|$ where $\{x\}$ is the fractional part of x.

Remark. Instead of considering $\mathcal{C}(1)$ we can consider $\mathcal{C}(a)$ with $a > 0$: the result will still hold.
Remark. Theorem 23 can be extended to the general case where X is any vector of $\mathbb{R}^d$. It is just that the case $X = (x, \cdots, x)$ is more presentable and contains the idea of the proof in the general case. For more information, see the remark that follows the conclusion of the proof of Proposition 58.

Theorem 24. *Let us assume that $x_1, \cdots, x_d$ are independent random variables distributed according to the uniform distribution over $[-\frac{1}{2}, \frac{1}{2}]$. Let us assume also that $t \in [0,T]$ is distributed according to the probability density $\frac{1}{T}\rho(\frac{\cdot}{T})$ on $[0,T]$. Let us suppose that t and $x_1, \cdots, x_d$ are independent betweem them then, when $T \to \infty$,*

$\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ *converges in distribution where* $X=(x_1,\cdots,x_d)$.
Furthermore, the limit distribution has the following characteristic function :

$$\varphi(u)=(2\frac{1-\cos(2^{d-1}u)}{(2^{d-1}u)^2})^d.$$

Remark. Theorem 24 can be extended to other types of distributions. For example, in the case where the x_i are distributed according to a Poisson distribution, the convergence in distribution still holds and the characteristic function of the limit distribution is :

$$\varphi(u)=(\frac{\sin(u)}{u})^d.$$

Also, if the x_i are distributed according to a uniform distribution over $[-1,1]$, the convergence in distribution still holds and the characteristic function of the limit distribution is the same as in Theorem 24. By changing the distributions of the x_i, one only changes the result of Proposition 60 and the rest of the proof of Theorem 24 still holds.
We see that the normalization in the two cases addressed by these two theorems of the error $\mathcal{R}$ is of order t^{d-1}. Furthermore, the two cases studied here are two extreme cases : the case of Theorem 23 is a case where all the x_i are linked (in fact, they are all equal) whereas the case of Theorem 24 is a case where all the x_i are independent between them.
Before beginning, let us observe that it is enough to prove Theorem 23 and Theorem 24 in the case where $\rho=\mathbf{1}_{[0,1]}$ (see, for example, the proof of Theorem 4.2 in [10]).
So, in the rest of the article, we are going to suppose that $\rho=\mathbf{1}_{[0,1]}$.
In the next section we are going to give a bit of heuristic about Theorem 23 and Theorem 24.

6.2 A bit of heuristic and plan of the paper

First, let us say that the normalization of $\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)$ by t^{d-1} is quite natural. Indeed, to within a multiplicative factor, it corresponds to the surface measure of $\partial(t\mathcal{C}(1)+X)$.
This normalization appears when looking at the following expression of $\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)$:

$$\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}=\frac{\sum_{i=1}^d(2t)^{i-1}(\lfloor t+x_i\rfloor-\lceil -t+x_i\rceil+1-2t)\prod_{j=i+1}^d(\lfloor t+x_j\rfloor-\lceil -t+x_j\rceil+}{t^{d-1}}$$

with $X=(x_1,\cdots,x_d)$ (see Proposition 56 and Equation 6.7).

Plan of the paper. After having proved this expression of $\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)$, we show in section 3 that the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ can be reduced to the study of a simpler quantity which is $\Delta(t,X)$ (see Proposition 55 for the definition of $\Delta(t,X)$).
Then, in section 4, we give the proof of Theorem 23. In fact, the case of Theorem 23 corresponds to a case where the expression $\Delta(t,X)$ (and the expression of $\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)$) is simpler. This simple expression is used to compute the characteristic function of

$\Delta(t,X)$ (see Proposition 58). We conclude by using Levy's continuity theorem (see Theorem 25).
In section 5, we also compute the characteristic function of $\Delta(t,X)$ in the case of Theorem 24 (see Proposition 60). In fact, the key of Theorem 24 is the independence of the variables $t, x_1, \cdots, x_d$ which enables us to make this computation. Other computations, with other distributions for the variables $x_1, \cdots, x_d$, but always with the independence theorem, could be made. We, again, conclude by using Levy's continuity theorem.

The next section is dedicated to reduce $\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)$ when we study its asymptotical behaviour.

6.3 Simplification of the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$

The main object of this section is to prove the following proposition :

Proposition 55. *One has that :*

$$\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}} - \Delta(t,X) \underset{t\to\infty}{\to} 0 \tag{6.2}$$

where $\Delta(t,X)$ is defined by

$$\Delta(t,X) = 2^{d-1}\sum_{i=1}^{d}(\lfloor t+x_i\rfloor - \lceil -t+x_i\rceil + 1 - 2t) \tag{6.3}$$

with $X=(x_1,\cdots,x_d)$ and the convergence in Equation (6.2) is uniform in $X\in\mathbb{R}^d$.

It is a proposition that enables to do some reduction about the asymptotical study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$. The main idea is that, in this case, everything can be computed quite easily, it is only a matter of definitions.

6.3.1 An expression of $\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)$

The main object of this subsection is to prove the following proposition :

Proposition 56. *We have for every $X\in\mathbb{R}^d$, for every $t>0$, that*

$$\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d) = \prod_{i=1}^{d}(\lfloor t+x_i\rfloor - \lceil -t+x_i\rceil + 1) - (2t)^d \tag{6.4}$$

where $X=(x_1,\cdots,x_d)$.

The proof is quite straightforward.

Proof. Let $X = (x_1, \cdots, x_d) \in \mathbb{R}^d$. Let $t > 0$.
One has that :

$$N(t\mathcal{C}(1) + X, \mathbb{Z}^d) = \left(\sum_{\substack{(n_1,\cdots,n_d)\in\mathbb{Z}^d \\ \forall i\in[1,d],\ -t+x_i\leqslant n_i\leqslant t+x_i}} 1 \right)$$
$$= \prod_{i=1}^{d} (\lfloor t + x_i \rfloor - \lceil -t + x_i \rceil + 1). \tag{6.5}$$

according to Equation (6.1).
Furthermore, one has that :

$$\mathrm{Vol}(t\mathcal{C}(1) + X) = (2t)^d \tag{6.6}$$

So, Equation (6.5) and Equation (6.6) and the definition of $\mathcal{R}(t\mathcal{C}(1) + X, \mathbb{Z}^d)$ give us Equation (6.4). □

With Equation (6.4), one has that :

$$\frac{\mathcal{R}(t\mathcal{C}(1) + X, \mathbb{Z}^d)}{t^{d-1}} = \frac{\sum_{i=1}^{d} (2t)^{i-1} (\lfloor t + x_i \rfloor - \lceil -t + x_i \rceil + 1 - 2t) \prod_{j=i+1}^{d} (\lfloor t + x_j \rfloor - \lceil -t + x_j \rceil +}{t^{d-1}} \tag{6.7}$$

Thanks to this last remark, the asymptotical study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ with t distributed on $[0, T]$ according to the probability measure $\frac{1}{T}\rho(\frac{t}{T})dt$ is going to be reduced to the study of a simpler quantity. It is the object of the next subsection.

6.3.2 Reduction of the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$

The main object of this subsection is to prove Proposition 55. The proof is quite straightforward and lie on the definitions of $\lfloor \cdot \rfloor$ and of $\lceil \cdot \rceil$ and on Proposition 56.

Proof of Proposition 55. For every $t > 0$, for every $x \in \mathbb{R}$,

$$t + x - 1 < \lfloor t + x \rfloor \leqslant t + x \tag{6.8}$$

and

$$-t + x \leqslant \lceil -t + x \rceil < -t + x + 1. \tag{6.9}$$

From Equation (6.8) and Equation (6.9), one gets that :

$$2t - 1 < (\lfloor t + x \rfloor - \lceil -t + x \rceil + 1) \leqslant 2t + 1 \tag{6.10}$$

So, from this last equation and from Equation (6.7), one has that, when $t \to \infty$,

$$\left|\frac{\mathcal{R}(t\mathcal{C}(1) + X, \mathbb{Z}^d)}{t^{d-1}} - \Delta(t, X)\right| \leqslant \sum_{i=1}^{d-1} 2^{i-1} \frac{1}{t^{d-i}} = O(\frac{1}{t}). \tag{6.11}$$

So, one gets the wanted result. □

Thanks to Proposition 55, we see that the asymptotical study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ can be reduced to the study of $\Delta(t, X)$. We are going to use this fact in the next two sections.

6.4 Proof of Theorem 23

The main object of this section is to prove Theorem 23. We are going to use the reduction that was mentioned before (see Proposition 55). In the case of Theorem 23, the expression of $\Delta(t,X)$ is simple and the proof of Theorem 23 is only a matter of computation of a characteristic function.

6.4.1 Reduction of the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$

The main object of this subsection is to prove the following proposition :

Proposition 57. *For every $x \in \mathbb{R}$, for every $g \in C_c(\mathbb{R})$,*

$$\int_{t=0}^{T} \Big(g\left(\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}\right) - g\left(\Delta(t,X)\right)\Big)\frac{1}{T}dt \underset{T\to\infty}{\to} 0 \tag{6.12}$$

where $X=(x,\cdots,x)$ and where $\Delta(t,X)$ was defined in Proposition 55.

It should be noted in this case that

$$\Delta(t,X) = d2^{d-1}(\lfloor t+x\rfloor - \lceil -t+x\rceil + 1 - 2t). \tag{6.13}$$

The proof of Proposition 57 is quite straightforward and based on Proposition 55.

Proof. One has for every $0<\kappa<\frac{1}{2}$:

$$\begin{aligned} &|\int_{t=0}^{T}(g\left(\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}\right) - g\left(\Delta(t,X)\right))\frac{1}{T}dt| \leqslant 2\|g\|_\infty \int_0^\kappa dt \\ &+ \int_{\kappa T}^{T} |g\left(\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}\right) - g\left(\Delta(t,X)\right)|\frac{1}{T}dt \end{aligned} \tag{6.14}$$

and, because $g\in C_c(\mathbb{R})$, it is a uniformly continuous function and so one has, because of Proposition 55, that

$$\limsup_{T\to\infty} \int_{\kappa T}^{T} |g\left(\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}\right) - g\left(\Delta(t,X)\right)|\frac{1}{T}dt = 0 \tag{6.15}$$

So, Equation (6.14) and Equation (6.15) give us that for every $0<\kappa\leqslant\frac{1}{2}$:

$$\limsup_{T\to\infty} |\int_{t=0}^{T}(g\left(\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}\right) - g\left(\Delta(t,X)\right))\frac{1}{T}dt| \leqslant 2\|g\|_\infty \int_0^\kappa dt. \tag{6.16}$$

By making κ go to 0, one gets the wanted result. □

In the next subsection, we are going to compute the characteristic function of $\Delta(t,X)$ to within a multiplicative factor.

6.4.2 Computation of characteristic function

Before stating the main proposition of this section, we need to make some observations and put in place some notations.
Let us call $t_{1,0} < \cdots < t_{1,l}$ the different times $t \in [0,T]$ such that $t + x \in \mathbb{Z}$.
In the same way, let us call $t_{2,0} < \cdots < t_{2,h}$ the different times $t \in [0,T]$ such that $-t + x \in \mathbb{Z}$.
Let us observe that for every $i \in \{0, \cdots, l-1\}$, $t_{1,i+1} - t_{1,i} = 1$ and that for every $j \in \{0, \cdots, h-1\}$, $t_{2,j+1} - t_{2,j} = 1$.
As a consequence, one has necessarily that $t_{2,0} \in [t_{1,0}, t_{1,1}[$ or $t_{1,0} \in [t_{2,0}, t_{2,1}[$ and $h = l$ or $h = l-1$ or $h = l+1$.
Let us set :

$$y = |t_{1,0} - t_{2,0}| \tag{6.17}$$

and

$$\tilde{\Delta}(t,x) = (\lfloor t + x \rfloor - \lceil -t + x \rceil + 1 - 2t). \tag{6.18}$$

By the way, let us remark that for all $x \in \mathbb{R}$:

$$y = |1 - 2\{x\}| \tag{6.19}$$

where $\{x\}$ stands for the fractional part of the real x.
Then, with these notations, one has that :

Proposition 58. *For every $x \in \mathbb{R}$, one has that the characteristic function $\varphi_{\tilde{\Delta}(\cdot,x)}$ of $\tilde{\Delta}(t,x)$, with t being distributed according to $\frac{1}{T}\mathbf{1}_{[0,T]}dt$ satisfies that for every $u \in \mathbb{R}$,*

$$\varphi_{\tilde{\Delta}(\cdot,x)}(u) = \frac{h}{uT}(\sin(uy) + \sin(u(1-y))) + O(\frac{1}{T})$$

where the O is uniform in $x \in \mathbb{R}$.
As a consequence, when $T \to \infty$, for every $u \in \mathbb{R}$, one has that :

$$\varphi_{\tilde{\Delta}(\cdot,x)}(u) \underset{T\to\infty}{\to} \frac{\sin(uy) + \sin(u(1-y))}{u}.$$

The proof consists basically in cutting the interval $[0,T]$ into subintervals where all the quantities that intervene in the computation can be expressed simply.

Proof. By symmetry, we can, and we will, suppose that $t_{2,0} \in [t_{1,0}, t_{1,1}[$. Let $u \in \mathbb{R}$.
One has then that :

$$\mathbb{E}(e^{iu\tilde{\Delta}(t,x)}) = \sum_{i=0}^{h-1} \int_{t_{1,i}}^{t_{2,i}} e^{iu\tilde{\Delta}(t,x)} \frac{1}{T} dt + \int_{t_{2,i}}^{t_{1,i+1}} e^{iu\tilde{\Delta}(t,x)} \frac{1}{T} dt + O(\frac{1}{T}) \tag{6.20}$$

where the O corresponds to the rest of the integral that is calculated on a union of two intervals of respective lengths at most 2.
Let $i \in \{0, \cdots, h-1\}$.
Then one has :

$$\int_{t_{1,i}}^{t_{2,i}} e^{iu\tilde{\Delta}(t,x)} \frac{1}{T} dt = \int_{t_{1,i}}^{t_{2,i}} e^{iu(t_{1,i}+t_{2,i}-2t)} \frac{1}{T} dt \tag{6.21}$$

according to the Equation (6.18).
So, one gets that :

$$\int_{t_{1,i}}^{t_{2,i}} e^{iu\tilde{\Delta}(t,x)}\frac{1}{T}dt = \frac{\sin(uy)}{uT} \tag{6.22}$$

where one conveys that $\frac{sin(0)}{0} = 1$ and one has this last equation because for all $j \in \{0, \cdots, h-1\}$, $y = t_{2,0} - t_{1,0} = t_{2,j} - t_{1,j}$.
In a similar way, one gets that :

$$\int_{t_{2,i}}^{t_{1,i+1}} e^{iu\tilde{\Delta}(t,x)}\frac{1}{T}dt = \frac{\sin(u(1-y))}{uT} \tag{6.23}$$

because $1 - y = t_{1,i+1} - t_{2,i}$.
So, with Equation (6.20), Equation (6.22) and Equation (6.23), one gets that :

$$\mathbb{E}(e^{iu\tilde{\Delta}(t,x)}) = \sum_{i=0}^{h-1}\frac{\sin(uy)}{uT} + \frac{\sin(u(1-y))}{uT} + O(\frac{1}{T}) = \frac{h}{uT}(\sin(uy)+\sin(u(1-y))) + O(\frac{1}{T}) \tag{6.24}$$

By using the fact that $\lim_{T\to\infty}\frac{h}{T} = 1$, one gets from equation (6.24) that :

$$\mathbb{E}(e^{iu\tilde{\Delta}(t,x)}) \underset{T\to\infty}{\to} \frac{\sin(uy)+\sin(u(1-y))}{u}. \tag{6.25}$$

Remark. If one wanted to address the general case of X being any vector of $\mathbb{R}^d$, one would only need to adapt, in fact, Proposition 58. The basic idea, in the general case, is the same : the intervals $[0, T]$ must be partition in small intervals where we know how to calculate all the quantities $\lceil -t + x_i \rceil$ and $\lfloor t + x_j \rfloor$. To do so, we have to consider the instants between 0 and T when $-t + x_i$ is an integer and the instants between 0 and T when $t + x_j$ is an integer and consider the order between these instants. The rest follows by calculations. □

6.4.3 Conclusion

To conclude the proof of Theorem 24, we need to recall the Lévy's continuity theorem.

Theorem 25. *Let us give us $(X_n)_{n\geqslant 1}$ a sequence of real random variables and let us call $(\phi_n)_{n\geqslant 1}$ the associated sequence of their characteristic functions.*
Let us suppose that the sequence $(\varphi_n)_{n\geqslant 1}$ converges point wisely to some function φ.
Then, it is equivalent to say that there exists X a real random variable such that (X_n) converges in distribution towards X and to say that the function φ is continuous at the point $t = 0$.
Furthermore, if the last condition is realized, φ is the characteristic function of such a X.

We can now conclude the proof of Theorem 24.

Proof of Theorem 24. Because of Proposition 57, it is enough to study the asymptotic convergence in distribution, when $T \to \infty$, of the quantity $\Delta(t, X)$ with $X = (x, \cdots, x)$

and t being distributed according to the density $\frac{1}{T}\mathbf{1}_{[0,T]}(t)dt$.
The fact that

$$\Delta(t,X) = d2^{d-1}(\lfloor t+x \rfloor - \lceil -t+x \rceil + 1 - 2t) \tag{6.26}$$

and Proposition 58 and Theorem 25 give us that $\Delta(t,X)$ converges in distribution, when $T \to \infty$, and the characteristic function of the limit distribution is given by

$$\varphi(u) = \frac{\sin(d2^{d-1}uy) + \sin(d2^{d-1}u(1-y))}{d2^{d-1}u} \tag{6.27}$$

with $y = 1 - 2\{x\}$ according to Equation (6.19). □

6.5 Proof of Theorem 24

The main object of this section is to prove Theorem 24. We are going to use the reduction that was mentioned before (see Proposition 55). In the case of Theorem 24, the proof is only a matter of computation of the characteristic function of $\Delta(t,X)$ and here it can be easily dealt with thanks to the independence between the x_i and thanks to the independence between the x_i and t.

6.5.1 Reduction of the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$

The main object of this subsection is the following proposition :

Proposition 59. *One has that :*

$$\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}} - \Delta(t,X) \underset{T\to\infty}{\overset{\mathbb{P}}{\to}} 0 \tag{6.28}$$

when $T \to \infty$ and when t is distributed according to $\frac{1}{T}\mathbf{1}_{[0,T]}(t)dt$ and $X = (x_1, \cdots, x_d)$ is distributed according to $U([-\frac{1}{2},\frac{1}{2}])^{\otimes^d}$. $\underset{T\to\infty}{\overset{\mathbb{P}}{\to}}$ signifies that the convergence occurs in probability.

Proof. It is a direct consequence of Proposition 55. □

Because of the independence of the x_i between them and with t, it is convenient for us to calculate the characteristic function of $\tilde{\Delta}(t,x_1)$, $\varphi_{\tilde{\Delta}(t,x_1)}$ (because also the x_i are identically distributed). It is the object of the next subsection.

6.5.2 Computation of the characteristic function $\varphi_{\tilde{\Delta}(t,x_1)}$

The main object of this subsection is to prove the following proposition :

Proposition 60. *For x a real random variable distributed according to the probability measure $\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(x)dx$ and being independent from t, with t being distributed according to the probability measure $\frac{1}{T}\mathbf{1}_{[0,T]}(t)dt$, one has that the characteristic function of $\tilde{\Delta}(t,x)$, $\varphi_{\tilde{\Delta}(t,x)}$ satisfies that*

$$\varphi_{\tilde{\Delta}(t,x)}(u) = 2\frac{h}{T}\frac{1-\cos(u)}{u^2} + O(\frac{1}{T}) \tag{6.29}$$

As a consequence, one has that $\varphi_{\tilde{\Delta}(t,x)}(u) \underset{T\to\infty}{\to} 2\frac{1-\cos(u)}{u^2}$.

The proof is basically a computation that uses Proposition 58 :

Proof. According to Proposition 58 and because x and t are independent from one another, one has that

$$\varphi_{\tilde{\Delta}(t,x)}(u) = \mathbb{E}\left(\frac{h}{uT}(\sin(u|1-2\{x\}|) + \sin(u(1-|1-2\{x\}|)) + O(\frac{1}{T})\right) \tag{6.30}$$

because of Equation (6.19) and the O is uniform in x (h can be, and is, chosen so that it does not depend on x).
Two quick computations give us that :

$$\mathbb{E}\left(\sin(u|1-2\{x\}|)\right) = \frac{1-\cos(u)}{u} \tag{6.31}$$

and

$$\mathbb{E}\left(\sin(u(1-|1-2\{x\}|))\right) = \frac{1-\cos(u)}{u}. \tag{6.32}$$

So, with these last two equations and Equation (6.30), one has that :

$$\varphi_{\tilde{\Delta}(t,x)}(u) = 2\frac{h}{T}\frac{1-\cos(u)}{u^2} + O(\frac{1}{T}). \tag{6.33}$$

By using the fact that $\frac{h}{T} \to 1$ when $T \to \infty$, one gets finally that

$$\varphi_{\tilde{\Delta}(t,x)}(u) \underset{T\to\infty}{\to} 2\frac{1-\cos(u)}{u^2}. \tag{6.34}$$

□

6.5.3 Conclusion

We have now all the necessary tools to prove Theorem 24.

Proof of Theorem 24. Proposition 59 gives us that it is enough to prove $\Delta(t,X)$ converges in distribution, when $T \to \infty$. So, we are going to calculate the characteristic function of $\Delta(t,X)$.
One has, because t and the x_i are independent random variables :

$$\varphi_{\Delta(t,X)}(u) = \prod_{i=1}^{d} \varphi_{\tilde{\Delta}(t,x_i)}(2^{d-1}u) \tag{6.35}$$

and because $\Delta(t,X) = 2^{d-1}\sum_{i=1}^{d}\tilde{\Delta}(t,x_i)$.
Furthermore, the x_i are identically distributed according to the probability measure $\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$. So Proposition 60 and Equation (6.35) give us that :

$$\varphi_{\Delta(t,X)}(u) = \prod_{i=1}^{d} 2\frac{h}{T}\frac{1-\cos(2^{d-1}u)}{(2^{d-1}u)^2} + O(\frac{1}{T}). \tag{6.36}$$

So, by making T goes to ∞, one has that :

$$\varphi_{\Delta(t,X)}(u) \underset{T\to\infty}{\to} (2\frac{1-\cos(2^{d-1}u)}{(2^{d-1}u)^2})^d. \tag{6.37}$$

Theorem 25 gives us then the wanted result. $\square$

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Université Paris Cité

TITRE, RÉSUMÉ, MOTS-CLÉS ET PUBLICATIONS

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Convergence en loi de l'erreur dans le problème des points d'un réseau

Résumé - français

Dans cette thèse, nous nous intéressons au problème général suivant. Soit L un réseau unimodulaire de R^{d}. Soit S un ensemble mesurable de volume fini. Quel est le nombre N(S,L) de points qui appartiennent à la fois à L et à S ? Lorsque S est suffisamment régulier, on peut montrer que ce nombre N(S,L) est approchée par vol(S) à une erreur R(S,L) près. Le but de cette thèse est d'avoir une idée plus précise du comportement de l'erreur R(S,L) dans différentes situations.

Après avoir introduit le problème dans le chapitre 1, dans le chapitre 2 nous montrons que lorsque l'on prend pour S un parallélogramme P et lorsque l'on prend L unimodulaire et aléatoire, ainsi que X un vecteur de R^{2} aléatoire, on montre que R(t P +X,L)/\log(t) converge en loi, quand t tend vers l'infini, vers une loi de Cauchy centré.

Dans le chapitre 3, lorsque l'on prend pour S une ellipse E centrée en 0 et lorsque l'on prend L unimodulaire et aléatoire, on montre que R(tE,L)/sqrt(t) converge en loi, quand t tend vers l'infini, vers une loi non triviale que l'on explicite et dont on étudie la finitude des moments.

Dans le chapitre 4, on généralise le résultat précédent en prenant cette fois pour S un corps strictement convexe analytique C et en considérant, non plus t C, mais t C + X avec X un vecteur de R^{2} fixé.

Dans les chapitres précédents, on a suivi une approche qui s'inspire de celle de Kesten en prenant L aléatoire. Dans les deux derniers chapitres, on prend t aléatoire et c'est une approche qui s'inspire de celle de Bleher.

Plus précisément, dans le chapitre 5, on prend pour S un rectangle P de centre 0 dont les côtés sont parallèles aux axes de coordonnées. Puis on montre que, après moyennisation en X de l'erreur R(tP+X,L) au carré, correctement normalisée, que celle-ci converge en loi lorsque t est aléatoire et devient grand dans différents cas : lorsque le réseau L est admissible au sens de Skriganov ou typique. D'ailleurs, dans le cas typique, la normalisation, proche, d'une certaine manière, de log(t), est plus forte que dans le cas admissible, ce qui est attendu.

Enfin, dans le chapitre 6, on généralise un résultat qui avait aussi été obtenu dans le chapitre 5. Plus exactement, on montre que lorsque l'on prend un hypercube C de centre 0 dont les côtés sont parallèles aux axes de coordonnées et de longueur 2, R(t C + X, Z^{d})/t^{d-1} converge en loi vers une loi non triviale lorsque t est aléatoire et devient grand dans deux cas : lorsque X, vecteur de R^{d}, s'écrit X=(x,\cdots,x) et lorsque X s'écrit X=(x_{1},..., x_{d}) avec x_{1},...,x_{d} étant indépendants entre eux et de t et étant distribuées selon une loi uniforme sur [-1/2, 1/2].

Mots-clés - français

Géométrie des nombres, Dynamique, Espace homogène, Analyse harmonique, Probabilités

Description en anglais

Titre de la thèse - anglais

Convergence in distribution of the error term in the lattice point problem

Résumé - anglais

In this thesis, we are interested in the following general problem. Let L be a unimodular lattice of R^{d}. Let S be a measurable set of finite volume. What is the number N(S,L) of points that belong to both L and S? When S is sufficiently regular, it can be shown that this number N(S,L) is approximated by vol(S) to within an error R(S,L). The aim of this thesis is to get a better idea of the behaviour of the error R(S,L) in different situations.

After introducing the problem in Chapter 1, in Chapter 2 we show that when we take for S a parallelogram P and when we take L unimodular and random, as well as X a vector of R^{2} random, we show that R(t P +X,L)/log(t) converges in distribution, when t tends to infinity, to a centered Cauchy distribution.

In Chapter 3, when we take for S an ellipse E centred in 0 and when we take L unimodular and random, we show that R(tE,L)/sqrt(t) converges in distribution, when t tends to infinity, to a non-trivial distribution which we explicit and we study the finiteness of the moments of this last distribution.

In Chapter 4, we generalize the previous result by taking this time for S a strictly convex analytic body C and by considering, not t C, but t C + X with X a fixed vector of R^{2}.

In the previous chapters, we followed an approach inspired by that of Kesten by taking L random. In the last two chapters, we take t random and it is an approach inspired by Bleher's.

More precisely, in chapter 5, we take for S a rectangle P of centre 0 whose sides are parallel to the coordinate axes. Then we show that, after averaging in X the error R(tP+X,L) squared, correctly normalised, that it converges in distribution towards a positive constant when t is random and becomes large in different cases: when the lattice L is admissible in the sense of Skriganov or typical. Moreover, in the typical case, the normalisation, close in some way to log(t), is stronger than in the admissible case, which is expected.

Finally, in Chapter 6, we generalise a result that was also obtained in Chapter 5. More precisely, we show that when we take a hypercube C of centre 0 whose sides are parallel to the coordinate axes and of length 2, R(t C + X, Z^{d})/t^{d-1} converges to a non-trivial distribution when t is random and becomes large in two different cases: when X, a vector of R^{d}, is written as X=(x,\cdots,x) and when X is written as X=(x_{1},. ..., x_{d}) with x_{1},...,x_{d} being independent of each other and of t and being distributed according to a uniform distribution on [-1/2, 1/2].

Mots-clés - anglais

Geometry of Numbers, Dynamics, Homogeneous Space, Harmonic analysis, Probability