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## Stabilisation des équations des ondes

Meryem Kafnemer

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Thèse de doctorat



# Stabilization of wave equations

Thèse de doctorat de l'Institut Polytechnique de Paris  
préparée à l'École nationale supérieure de techniques avancées

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**MERYEM KAFNEMER**

Composition du Jury :

Enrique Zuazua Chair for Dynamics, Control and Numerics (Alexander von Humboldt-Professorship), University of Erlangen–Nuremberg (DDS)	Président
Judith Vancostenoble Assistant professor, Paul Sabatier University - Toulouse III (IMT)	Rapporteuse
Nguyen Hoai-Minh Professor, Sorbonne University (LJLL)	Rapporteur
Sonia Fliss Assistant professor, ENSTA Paris (UMA)	Examinatrice
Patrick Martinez Assistant professor, Paul Sabatier University - Toulouse III (IMT)	Examineur
Benmiloud Mebkhout Assistant professor, Abou bekr Belkaid university, Tlemcen	Examineur
Frédéric Jean Professor, ENSTA Paris (UMA)	Directeur de thèse
Yacine Chitour Professor, Paris-Saclay university (L2S, CentraleSupélec)	Co-directeur de thèse

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# Résumé en français

L'objectif principal de cette thèse est d'étudier la stabilité asymptotique des équations des ondes avec un terme d'amortissement localisé et des conditions aux limites de Dirichlet. Nous considérons le cadre hilbertien standard ainsi que des cadres plus généraux. Nous considérons également le problème non-linéaire ainsi que le problème linéaire, en essayant de répondre à chaque fois, aux questions sur le bien posé et la stabilité. Le problème général s'écrit comme suit

$$\begin{cases} z_{tt} - \Delta z = -a(x)g(z_t) & \text{for } (t, x) \in \mathbb{R}_+ \times \Omega, \\ z = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ z(0, \cdot) = z_0, \quad z_t(0, \cdot) = z_1, \end{cases}$$

où  $z$  est l'inconnue du problème,  $(z_0, z_1)$  sont les conditions initiales appartenant à l'espace fonctionnel  $L^p$  que l'on note  $X_p = W^{1,p}(\Omega) \times L^p(\Omega)$  avec  $p \in [1, \infty)$  qui sera défini plus tard avec plus de détails. Lorsque la dimension  $N \geq 2$  et  $p = 2$ , l'ensemble  $\Omega$  est un domaine borné  $C^2$  de  $\mathbb{R}^N$  et lorsque  $N = 1$  et  $p \in (1, \infty)$ ,  $\Omega$  est simplement un intervalle de  $\mathbb{R}$ .

La fonction  $g : \mathbb{R} \rightarrow \mathbb{R}$  est une fonction non décroissante de classe  $C^1$  vérifiant  $xg(x) > 0$  pour  $x \neq 0$ . Le terme  $-a(x)g(z_t)$  est appelé feedback, et il s'agit d'un amortissement dans ce cas. D'autres hypothèses, en particulier des hypothèses de croissance, sont généralement imposées sur  $g$  en fonction du contexte du problème.

La fonction  $a : \bar{\Omega} \rightarrow \mathbb{R}$  est une fonction continue non négative. Elle est appelée fonction de localisation car elle définit le sous-domaine de  $\Omega$  où l'amortissement agit. La fonction  $a$  est bornée inférieurement par une constante strictement positive  $a_0$  sur un sous-ensemble ouvert non vide  $\omega$  de  $\Omega$ . On peut voir le sous-ensemble  $\omega$  comme la région du domaine où le terme d'amortissement est actif. Dans le cas où

$\Omega \setminus \omega$  a une mesure de Lebesgue strictement positive, l'amortissement est dit *localisé* alors que lorsqu'il a une mesure nulle, c'est-à-dire que  $\Omega = \omega$ , l'amortissement est dit *global* car il est actif sur tout le domaine  $\Omega$ . La géométrie du domaine ainsi que la localisation de l'amortissement sont très importantes pour les problèmes de stabilisation des ondes. Des conditions géométriques sont imposées sur le domaine lorsque l'amortissement est localisé afin d'obtenir des résultats de stabilisation.

Dans le cas où  $p = 2$ , qui est le cadre fonctionnel hilbertien standard, l'énergie du système est définie par la quantité :

$$E_2(t) = \frac{1}{2} \int_{\Omega} (|z_t(t, x)|^2 + |\nabla z(t, x)|^2) dx.$$

Le problème a été largement étudié dans ce cadre. Le caractère bien posé a été établi à plusieurs reprises (voir par exemple [4], [24], [25]). Dafermos [11] et Haraux [14] ont prouvé la stabilisation forte dans le cas d'un  $g$  croissant, c'est à dire

$$\lim_{t \rightarrow +\infty} E_2(t) = 0, \tag{1}$$

la preuve est basée sur le principe d'invariance de Lasalle et nous pouvons nous référer à [4] pour ses détails.

Puisque nous savons grâce à cette preuve que l'énergie des solutions tend vers 0 lorsque  $t$  tend vers l'infini, on s'intéresse maintenant aux taux de décroissance, c'est-à-dire la vitesse à laquelle l'énergie tend vers 0.

Différents taux de stabilité ont été obtenus au cours des années, selon le type de problème (linéaire ou non-linéaire, localisé ou global) avec différentes hypothèses sur la non-linéarité  $g$ . Les références aux travaux antérieurs dans le cas  $p = 2$  et dans le contexte de cette thèse sont données dans l'introduction du chapitre 2.

Le meilleur taux de décroissance que l'on puisse espérer est la décroissance exponentielle. Afin d'obtenir ou du moins d'espérer une telle décroissance, des hypothèses géométriques doivent être imposées sur le domaine d'amortissement  $\omega$ . Ces hypothèses ont été caractérisées pour la première fois dans les travaux pionniers [32] sur les équations des ondes semi-linéaires et leur extension dans [22] et [21], où les conditions géométriques du multiplicateur (MGC) ont été caractérisées pour  $\omega$  comme des conditions suffisantes pour atteindre la stabilité exponentielle. Cependant, des conditions géométriques plus optimales (nécessaires et suffisantes) sont obtenues pour les équations

tions des ondes linéaires, elles sont introduites et détaillées dans [6] et [9]. Selon le type des conditions géométriques imposées, deux méthodes ont été développées pour étudier la décroissance exponentielle pour l'équation d'onde localement amortie : l'optique géométrique et la méthode des multiplicateurs. Nous nous intéressons dans cette thèse par la méthode des multiplicateurs.

Or, si le problème de la stabilisation des ondes lorsque  $p = 2$  est très courant et a été traité plusieurs fois sous différentes hypothèses, le même problème lorsque  $p \neq 2$  est beaucoup moins populaire et il y a très peu de travaux qui ont été faits dans ce cadre fonctionnel. Nous devons une partie importante de nos résultats dans ce cadre aux travaux pionniers de [16] et [10]. Toutes les références de ce cadre sont mentionnées dans l'introduction du chapitre 3.

La motivation derrière l'étude des problèmes de stabilisation des ondes dans un tel cadre général provient du domaine de l'automatique où l'on s'intéresse pas à l'énergie mais à l'amplitude des ondes, ce qui représente mathématiquement la norme  $L^\infty$  des solutions. Stabiliser les équations d'ondes dans un cadre  $L^\infty$  est donc très pratique puisque cela revient à stabiliser l'amplitude des solutions. Cela nous amène immédiatement à se poser la question en mathématiques sur ce qui se passe lorsque nous considérons un cadre général  $L^p$  avec  $1 \leq p \leq \infty$  et quel type de résultats de stabilisation pouvons-nous obtenir dans ce cas. Nous gardons également l'espoir d'obtenir une stabilisation dans le cas où  $p = \infty$  en passant par l'étude du problème pour tous les  $p \geq 1$ .

Il faut noter que l'étude d'un tel problème dans des cadres non-hilbertiens n'est pas toujours possible en dimensions supérieures, elle n'est possible qu'en dimension un. La raison en est que l'opérateur des ondes n'est pas toujours bien défini dans des espaces fonctionnels  $L^p$  pour des dimensions supérieures lorsque  $p \neq 2$ .

Avant d'aborder les différents problèmes de stabilisation, la première question à traiter dans un tel cadre est la question de la well-posedness. En effet, les techniques habituelles de preuve de bien-posé basées sur les espaces de Hilbert (opérateurs maximaux monotones par exemple) ne sont plus valables dans un cadre non-hilbertien. Nous devons faire preuve de créativité et utiliser de nouvelles techniques. Dans le cas linéaire, un argument basé sur la formule de D'Alembert et la théorie de point fixe est utilisé dans [10] et dans notre travail également pour prouver la well-posedness pour tous les  $p > 1$ . Dans le cas non linéaire, [16] prouve l'existence et l'unicité des solutions pour tous les  $p \geq 2$  en utilisant un argument basé sur l'existence et l'unicité des so-

lutions dans le cadre hilbertien et une fonctionnelle d'énergie équivalente mais avec l'hypothèse  $\frac{g(z_t)}{z_t} \in L_{loc}^\infty(\mathbb{R}_+, L^\infty(\Omega))$  qui n'est pas toujours satisfaite pour les solutions faibles. Nous proposons une preuve de l'existence des solutions faibles dans le cas où  $p \geq 2$  ainsi qu'une preuve de l'existence des solutions fortes dans le cas  $1 < p < \infty$ .

Nous pouvons maintenant poser des questions sur la stabilité et les taux de décroissance de l'énergie. La  $p$ -ième énergie  $E_p$  d'une solution a été introduite dans [14] comme une généralisation de l'énergie hilbertienne standard  $E_2$ . Il s'agit d'une énergie équivalente à l'énergie naturelle et elle est définie par :

$$E_p(t) = \frac{1}{p} \int_{\Omega} (|z_x(t, x) + z_t(t, x)|^p + |z_x(t, x) - z_t(t, x)|^p) dx.$$

Des estimées  $L^p$  ont également été prouvées dans [14], ce qui a conduit à prouver la décroissance polynomiale de l'énergie dans le cas non linéaire, avec  $g$  une fonction non décroissante de classe  $C^1$  se comportant comme  $ks|s|^r$ ,  $r, k > 0$ . Des résultats dans [14] ont été utilisés dans [5, 10] pour obtenir certains résultats de stabilité. Cette dernière référence s'appuie sur des techniques de Lyapunov pour des systèmes linéaires à variation temporelle pour prouver la stabilité exponentielle  $L^p$  pour  $p \geq 2$  du problème non-linéaire sous des hypothèses restrictives sur les données initiales (doivent appartenir aux espaces fonctionnels  $L^\infty$ ); d'autres résultats de stabilité ont été montrés dans la même référence en particulier la stabilité  $L^\infty$  mais toujours avec plus de restrictions sur les données initiales. Nous obtenons une décroissance exponentielle du problème linéaire pour tout  $1 < p < +\infty$ . Nous obtenons également une décroissance exponentielle dans le problème non linéaire dans le cas  $p > 1$  le long des solutions fortes.

Pour résumer, cette thèse traite trois problèmes de stabilisation des équations des ondes.

- **Premier problème:** Nous étudions la stabilité de l'équation des ondes avec un amortissement non-linéaire et localisé dans un cadre hilbertien standard en dimension deux. La preuve est basée sur les travaux de [25], auxquels nous rajoutons une localisation ainsi que des perturbations. Nous démontrons la stabilité exponentielle le long des solutions fortes en l'absence de perturbation ainsi qu'une sorte stabilité au sens Input-To-State par rapport aux perturbations considérées.
- **Deuxième problème:** Dans un deuxième travail, nous considérons un cadre fonctionnel plus général non forcément hilbertien, c-à-d un cadre  $L^p$  avec  $p \in (1, \infty)$ . Nous étudions la stabilité  $L^p$  de l'équation des ondes avec un amortissement linéaire et localisé. Cette étude n'est effectuée qu'en dimension un



car il n'est pas toujours possible de définir l'opérateur des ondes en dimensions supérieures lorsque  $p \neq 2$ . Nous démontrons la stabilité exponentielle du problème en généralisant les multiplicateurs du cadre hilbertien dans notre cadre plus général, avec des preuves différentes suivant que  $1 < p < 2$  ou  $p \geq 2$ . Nous démontrons également dans le même problème mais avec des cas particuliers d'un amortissement global et constant, une stabilité exponentielle dans le cas  $p = 1$  et  $p = \infty$ .

- **Troisième problème:** Dans un troisième travail, nous considérons la version non-linéaire du problème précédent: en nous basant sur une technique de linéarisation, nous nous ramenons à la preuve du problème linéaire pour démontrer la stabilité exponentielle du non-linéaire le long des solutions fortes et pour tout  $1 < p < \infty$ .

# Introduction

## 1.1 General Introduction

The main work of this thesis revolves around the asymptotic stability of the wave equation with a localized damping term and Dirichlet boundary conditions. We consider the standard Hilbertian framework as well as more general frameworks. We also consider the nonlinear problem as well as the linear problem, trying to answer each time, questions about the well-posedness and the stability. The general problem is written as follows

$$\begin{cases} z_{tt} - \Delta z = -a(x)g(z_t) & \text{for } (t, x) \in \mathbb{R}_+ \times \Omega, \\ z = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ z(0, \cdot) = z_0, \quad z_t(0, \cdot) = z_1, \end{cases} \quad (1.1)$$

where  $z$  is the unknown of the problem,  $(z_0, z_1)$  are the initial conditions and the pair belongs to an  $L^p$ -based functional space  $X_p = W^{1,p}(\Omega) \times L^p(\Omega)$  with  $p \in [1, \infty)$  that will be defined later with more details. When the dimension  $N \geq 2$  and  $p = 2$ , the set  $\Omega$  is a  $C^2$  bounded domain of  $\mathbb{R}^N$  and when  $N = 1$  and  $p \in (1, \infty)$ , then  $\Omega$  is simply an interval of  $\mathbb{R}$ .

The function  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^1$  non-decreasing function verifying  $xg(x) > 0$  for  $x \neq 0$ . The term  $-a(x)g(z_t)$  is called a feedback, and it is a damping in this case. More hypotheses, in particular growth hypotheses, are usually imposed on  $g$  depending on the context of the problem.

The function  $a : \overline{\Omega} \rightarrow \mathbb{R}$  is a continuous non-negative function. It is called a lo-

calization function because it defines where the damping is acting in the domain  $\Omega$  (when  $a$  is positive). More precisely,  $a$  is bounded from below by a positive constant  $a_0$  on some non-empty open subset  $\omega$  of  $\Omega$ . One can see the subset  $\omega$  as the region of the domain where the damping term is active. In the case where  $\Omega \setminus \omega$  has a positive Lebesgue measure, the damping is said to be *localized* whereas when it has a null measure, meaning that  $\Omega = \omega$ , the damping is said to be *global* because it is active on all the domain  $\Omega$ . The geometry of the domain as well as the localization of the damping are very important for wave stabilization problems. Geometrical conditions are imposed on the domain when the damping is localized in order to achieve stability results, more details about these conditions will follow in this introduction.

In the case where  $p = 2$ , which is the standard Hilbertian functional framework, the energy of the system is defined by the quantity:

$$E_2(t) = \frac{1}{2} \int_{\Omega} (|z_t(t, x)|^2 + |\nabla z(t, x)|^2) dx. \quad (1.2)$$

Problem (1.1) has been widely studied in this framework. The well-posedness has been established many times ( see for instance [4], [24], [25]). Dafermos [11] and Haraux [14] proved the strong stabilization in the case of an increasing  $g$ , i.e.

$$\lim_{t \rightarrow +\infty} E_2(t) = 0, \quad (1.3)$$

the proof is based on Lasalle invariance principle and we can refer to [4] for its details.

Since we know that the energy of solutions goes to 0 as  $t$  goes to infinity, the issue to be addressed regards the decay rate; how fast the energy goes to 0. Different stability rates have been obtained over the years, depending on whether the problem is non-linear with different hypotheses on the non-linearity  $g$  or linear ( $g = Id$ ), or whether the damping is localized or global ( $a = 1$ ). References to previous works in the case  $p = 2$  and in the context of this thesis are given in the introduction of Chapter 2.

The best decay rate one can hope for is the exponential decay when  $g$  is  $C^1$ . In order to obtain or at least hope for such decay, geometrical assumptions must be imposed on the damping domain  $\omega$ . These assumptions have been first put forward in the pioneering work [32] on semi-linear wave equations and its extension in [22] and [21], where the multiplier geometric conditions (MGC) have been characterized for  $\omega$  as sufficient conditions to achieve exponential stability. However, more optimal geomet-

rical conditions (necessary and sufficient) are obtained for linear equations, they are introduced and detailed in [6] and [9]. Depending on the type of the imposed geometrical conditions, two methods have been developed to study exponential decay for the locally damped wave equation:

*The method of geometric optics:* the one that gives necessary and sufficient geometrical conditions on the damping domain  $\omega$  for exponential stability, these conditions are not explicit but they allow us to get energy decay estimates under very general hypotheses. This method is based on microlocal analysis and it is adapted to linear problems. We are not interested in this method in the context of this thesis since we will be treating non-linear problems as well.

*The multiplier method:* the one that gives explicit sufficient geometric conditions on  $\omega$ . It is based on obtaining energy estimates and using Gronwall's inequalities. It consists on taking the main equation of the problem and multiplying it by different quantities called multipliers (usually the solution or different derivatives of the solution combined with localization functions when dealing with a localized damping). Each one of these multipliers will play a specific important role and they will lead when their results are combined together, to an energy estimate. This energy estimate proves potential exponential stability or even weaker rates through Gronwall's inequalities. There exists a generalization of this method, it is called the piecewise multiplier method which requires weaker conditions on the damping domain, called piecewise geometrical conditions (PMGC). They have the exact same concept as the multiplier geometrical conditions (MGC) but they use multiple observation points instead of one observation point in the classical (MGC). We will announce the geometrical conditions when needed in this thesis.

Using the multipliers method it has been proven that the energy decay rate of the locally damped linear problem is exponential (see [24]). It has also been shown that the globally damped non-linear problem is also exponentially stable in dimension two (see [25]) with growth hypothesis on  $g$ . We show in this thesis based on the precious remarks of [25] that the latter problem remains exponentially stable when the damping is localized.

Now as the stabilization problem of (1.1) when  $p = 2$  is very common and has been treated several times under different hypotheses, the same problem when  $p \neq 2$  is way less popular and there are very few studies and resources available in this research theme. We owe an important part of our results in this framework to the pioneering

work of [16] and [10]. All references of this framework relevant to the context of this thesis are mentioned in the introduction of Chapter 3.

The motivation behind studying wave stabilization problems in such general framework rises from the domain of automation where they are always interested in the amplitude of the waves, which represents mathematically the  $L^\infty$  norm of the solutions. Stabilizing wave equations in an  $L^\infty$  framework is then very practical since we are stabilizing the amplitude of the solutions. This immediately makes us consider as mathematicians the question of what happens when we consider a general framework  $L^p$  with  $1 \leq p \leq \infty$  and what kind of stabilization results can we obtain in this case. We also keep the hope of achieving stabilization in the case where  $p = \infty$  by studying the problem for all  $p \geq 1$ .

One should note that the study of such a problem in non-hilbertian frameworks is not possible in superior dimensions, it is only possible in one dimension. The reason is that the wave operator is not always well defined in  $L^p$  functional spaces.

Before getting to stabilization problems, the first issue to address into such framework is the question of the well-posedness. Since the usual well-posedness proof techniques based on Hilbert Spaces (maximal monotone operators for instance) are no longer valid in a non-hilbertian framework. We need to get creative and use new techniques. In the linear case, an argument based on D'Alembert formula and fixed point theory is used in [10] and in our work to prove the well-posedness for all  $p > 1$ . In the non-linear case, [16] proves the well-posedness for all  $p \geq 2$  using an argument based on the well-posedness in the Hilbertian framework and an equivalent energy functional but with the hypothesis  $\frac{g(z_t)}{z_t} \in L_{loc}^\infty(\mathbb{R}_+, L^\infty(\Omega))$  which is not always satisfied for weak solutions. We propose a well-posedness proof for weak solutions in the case  $p \geq 2$  and for strong solutions in the case  $1 < p < +\infty$ .

Once the well-posedness is established, we ask questions about the stability and the decay rate of the energy. The  $p$ -th energy  $E_p$  of a solution has been introduced in [14] as a generalization of the standard Hilbertian energy  $E_2$ . It is an equivalent energy to the natural energy and is defined by:

$$E_p(t) = \frac{1}{p} \int_{\Omega} (|z_x(t, x) + z_t(t, x)|^p + |z_x(t, x) - z_t(t, x)|^p) dx. \quad (1.4)$$

Very useful  $L^p$  estimates have also been proved in [14], which led to proving polynomial decay of the energy in the non-linear case, with  $g$  a non-decreasing  $C^1$  function behav-

ing like  $ks|s|^r$ ,  $r, k > 0$ . Some of the results in [14] have been used in [5, 10] to obtain some stability results. The latter reference relies on Lyapunov techniques for linear time varying systems to prove  $L^p$  exponential stability in the non-linear problem under restrictive hypotheses on initial data (imposed to belong to  $L^\infty$  functional spaces) and for  $p \geq 2$ ; other stability results have been shown in the same reference in particular  $L^\infty$  stability but always with more conditions on initial data. We obtain an exponential decay in the linear problem for all  $1 < p < +\infty$ . We also obtain an exponential decay in the non-linear problem in the case  $p \geq 2$  and in the case  $1 < p < 2$ .

## 1.2 Introduction to stabilization

The stability of Problem (1.1) is studied around the unique equilibrium point of the system which is  $z_e = 0$ . The equilibrium  $z_e$  is said to be *Stable* if whenever a solution starts out from initial data near the equilibrium point  $z_e$ , it always stays near  $z_e$ . More strongly, the equilibrium is said to be *asymptotically stable* (Lyapunov stable) if whenever a solution starts out from initial data near  $x_e$  it converges to  $x_e = 0$  as time progresses. If for all initial data in the state space, the solutions converge to the equilibrium, it is said to be *globally asymptotically stable*.

The stabilization on the other hand is just forcing the solutions of a problem to be stable by implementing some sort of control; an example would be stabilizing the usual wave equation by adding a global linear damping term to it.

**The stabilization problem:** The stabilization problem of (1.1) consists on using the feedback  $-a(x)g(z_t)$  which is a localized non-linear damping that depends on the solution  $z$  on any given time, to reduce the amplitude of the wave solutions over time to be as small as possible (as close to zero as possible). This translates to driving the energy of the solutions to 0 as time progresses. Our goal here is to obtain asymptotic stability of the equilibrium 0. The asymptotic stability is tied to the decay of the energy of the solutions toward 0. This decay to 0 is naturally wanted to be as fast as possible so that the stabilization is achieved in the smallest time possible. We usually aim to obtain an exponential decay but other slower decay rates can be obtained as well when the exponential one is difficult or impossible to obtain, for instance the energy can be found to go to 0 polynomially. Another important challenge in the stabilization problem is to impose as less hypotheses as possible on the feedback function  $g$  in theory which leaves us with more choices when choosing the feedback  $g$  in practice.

We assume next that well-posedness of solutions on  $\mathbb{R}_+$  has been established for every initial condition.

There are many types and notions of stability, we mainly focus in our work on exponential stability. However, before studying any type of stability it is always useful to prove first that the energy of a solution tends to 0 as time goes to infinity, which is known as strong stabilization.

**Definition 1.2.1. (*Strong stabilization*)**

Problem (1.1) is said to be strongly stable, if, for every initial data  $(z_0, z_1) \in X_p$ , the energy function  $t \mapsto E_p(t)$  of the corresponding solution tends to zero as  $t$  tends to infinity, Here  $p \in [1, +\infty)$  and  $E_p$  is defined by (1.4) when  $p \neq 2$  and by (1.2) when  $p = 2$ .

We give next a general definition of what we mean by an exponentially stable system.

**Definition 1.2.2. (*Exponential stability*)**

Problem (1.1) is said to be uniformly exponentially stable, if there exist two constants  $C, \gamma > 0$  such that the energy  $E_p$  of all solutions with initial data  $(z_0, z_1) \in X_p$  satisfies:

$$E_p(t) \leq CE_p(0)e^{-\gamma t}, \quad \forall t \geq 0,$$

where  $C$  and  $\gamma$  are independent of initial data  $(z_0, z_1)$ ,  $p \in [1, +\infty)$  and the energy  $E_p$  is defined by (1.4) when  $p \neq 2$  and by (1.2) when  $p = 2$ .

Another important stability notion that is often used for nonlinear problems is Input-To-State Stability (ISS), it has been first introduced in [30] and it is particularly useful when there are external inputs in the system (such as disturbances from Chapter 2). To define this notion we start by defining some key sets.

We define  $\mathcal{K}$  the set of continuous increasing functions  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  that vanish at zero,

$$\mathcal{K} = \{\gamma \in C(\mathbb{R}_+), \gamma(0) = 0 \text{ and } \gamma'(s) > 0 \quad \forall s > 0\}, \quad (1.5)$$

and  $\mathcal{L}$  the set of continuous decreasing functions that tends to 0 around infinity,

$$\mathcal{L} = \{\gamma \in C(\mathbb{R}_+), \gamma \text{ decreasing and } \lim_{t \rightarrow +\infty} \gamma(t) = 0\}. \quad (1.6)$$

We also define  $\mathcal{K}_\infty$  as the subset of  $\mathcal{K}$  of unbounded functions,

$$\mathcal{K}_\infty = \{\gamma \in \mathcal{K}, \gamma \text{ unbounded}\}. \quad (1.7)$$

Finally we define the set  $\mathcal{KL}$  as

$$\mathcal{KL} = \{\beta \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+), \beta(\cdot, t) \in \mathcal{K} \ \forall t \geq 0, \text{ and } \beta(r, \cdot) \in \mathcal{L} \ \forall r > 0\}. \quad (1.8)$$

Now consider the general system that we assume well-posed as the function  $f$  satisfies the necessary well-posedness hypotheses:

$$Z_t(t) = f(Z(t), U(t)), \quad Z(0) = Z_0, \quad (1.9)$$

where the solution  $Z$  and initial data  $Z_0 \in X$  belong to the normed vector space  $X$  which represents the state space, and  $U$  are some external inputs that belong to the normed vector space  $\mathcal{U}$ .

We can now give a simplified definition of the ISS in our context, this definition is fully inspired from Definition 1.6 in [28].

**Definition 1.2.3. (*Input-to-state stability*)**

System (1.9) is called input-to-state stable (ISS) if there exist functions  $\gamma \in \mathcal{K}_\infty$  and  $\beta \in \mathcal{KL}$  such that for all initial data  $Z_0 \in X$ , for all inputs  $U \in \mathcal{U}$  and all times  $t \geq 0$ , the solution  $Z \in X$  that starts in  $Z_0$  satisfies the following inequality

$$\|Z(t)\|_X \leq \beta(\|Z_0\|_X, t) + \gamma(\|U\|_{\mathcal{U}}). \quad (1.10)$$

The norm on  $X$  which is the norm in the functional space  $X_p$ ; in our case it will be taken to be  $(E_p)^{\frac{1}{p}}$  where  $E_p$  is defined by (1.4) when  $p \neq 2$  and by (1.2) when  $p = 2$ . We can also notice that  $Z(t)$  in our problem is  $(z(t, \cdot), z_t(t, \cdot))$ , and  $Z_0$  is  $(z_0, z_1)$ , where  $z$  is the solution of Problem (1.1).

**Remark 1.2.1.** All the stability notions defined above: strong stabilization, exponential stability, input-to-state stability imply global asymptotic stability.

## 1.3 Outline

The manuscript is organized as follows:



- In Chapter 2, we work in an  $L^2$  framework where we consider (1.1) in two dimensions and we take in consideration all the possible disturbances in the system. We prove in Section 2.3 the well-posedness of the problem using maximal monotone operators theory. We prove in Section 2.4 a weak ISS-type estimate using the multipliers method, the ISS assures exponential decay in the absence of disturbances.
- In Chapter 3, we consider the  $L^p$  framework. We establish the well-posedness for  $p > 1$  in Section 3.4. We prove in Section 3.5 an  $L^p$  exponential decay of the energy of the linear localized problem for  $p > 1$  using a generalized multiplier method. We also prove in Section 3.6 that the exponential decay remains valid for  $p = \infty$  and  $p = 1$  in some cases of a global constant damping.
- In Chapter 4, we consider the non-linear problem of the problem treated in Chapter 3. We prove the well-posedness in the space of strong solutions of the problem in Section 4.3 for  $1 \leq p \leq \infty$  using a density argument combined with the well-posedness already established in  $L^\infty$ . We also prove an exponential decay in Section 4.4 of the energy based on the proof of the linear case.

## 1.4 Main contributions

We summarize in this section the work that has been done in this thesis; we list and explain our main contributions.

The main goal of this thesis is to study wave equations stabilization problems. We study three main problems:

### First Problem:

We consider the general problem (1.1) in an  $L^2$  framework in two dimensions but we take in consideration two disturbances. The problem is given by

$$(\mathbf{P}_{\text{dis}}) \quad \begin{cases} z_{tt} - \Delta z = -a(x)g(z_t + d) - e, & \text{in } \mathbb{R}_+ \times \Omega, \\ z = 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ z(0, \cdot) = z_0, \quad z_t(0, \cdot) = z_1, \end{cases}$$

where  $e$  and  $d$  are the possible disturbances in the system which represent respectively the global disturbance that affects the whole dynamic and the damping disturbance. The two disturbances depends on the time variable and the space variable as well. We considered all the possible disturbances in the system, including disturbances in initial conditions and the localization function but with a simple change of variables we will always be studying the same problem given by  $(\mathbf{P}_{\text{dis}})$ .

We work under multiple hypotheses on the non-linearity  $g$  and on the two disturbances  $e$  and  $d$ . We also impose the famous geometrical conditions on the damping domain  $\omega$ . We list below the hypotheses under which we obtained the first contribution.

Hypotheses on the non-linearity  $g$ :

- The function  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^1$  non-decreasing function such that  $g(0) = 0$ ,  $g'(0) > 0$ ,  $g(x)x > 0$  for  $x \neq 0$ .
- We also impose a growth hypothesis on  $g$  by assuming that  $\exists C > 0$ ,  $\exists 1 < q < 5$ ,  $\forall |x| \geq 1$ ,  $|g(x)| \leq C|x|^q$ .
- A growth hypothesis is also required on the derivative of  $g$  since we will be manipulating strong solutions and therefore we will be differentiating Problem  $(\mathbf{P}_{\text{dis}})$ . We suppose that  $\exists C > 0$ ,  $\exists 0 < m < 4$ ,  $\forall |x| > 1$ ,  $|g'(x)| \leq C|x|^m$ .

Hypotheses on the localization function  $a$  and the damping domain  $\omega$ :

- The function  $a : \bar{\Omega} \rightarrow \mathbb{R}$  is a continuous function such that  $a \geq 0$  on  $\Omega$  and  $\exists a_0 > 0$ ,  $a \geq a_0$  on  $\omega$ .
- We impose what we called in the introduction the multiplier geometrical conditions (MGC) which require the existence of an observation point  $x_0 \in \mathbb{R}^2$  for which  $\omega$  contains the intersection of  $\Omega$  with an  $\epsilon$ -neighborhood of  $\Gamma(x_0) = \{x \in \partial\Omega, (x - x_0) \cdot \nu(x) \geq 0\}$ , where  $\nu$  is the unit outward normal vector for  $\partial\Omega$  and an  $\epsilon$ -neighborhood of  $\Gamma(x_0)$  is defined by  $\mathcal{N}_\epsilon(\Gamma(x_0)) = \{x \in \mathbb{R}^2 : \text{dist}(x, \Gamma(x_0)) \leq \epsilon\}$ .

Hypotheses on disturbances  $e$  and  $d$ :

- The disturbance function  $d : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  belongs to  $L^1(\mathbb{R}_+, L^2(\Omega))$  and satisfies that  $d(t, \cdot) \in H_0^1(\Omega) \cap L^{2q}(\Omega)$ ,  $\forall t \in \mathbb{R}_+$  and that  $t \mapsto \int_0^t \Delta d(s, \cdot) ds - d_t(t, \cdot) \in Lip(\mathbb{R}_+, H_0^1(\Omega))$ , where  $Lip$  denotes the space of Lipschitz continuous functions. We also impose that the following quantities  $\int_0^\infty \int_\Omega (|d|^2 + |d|^{2q}) dx dt$ ,  $\int_0^\infty \int_\Omega |d|^m (d_t)^2 dx dt$ ,  $\int_0^\infty \int_\Omega (d_t)^2 dx dt$ , and  $\int_0^\infty \left( \int_\Omega |d_t|^{2(\frac{p-1}{p-1})} dx \right)^{\frac{p-1}{p}} dt$ , are all finite, where  $p$  is a fixed real number so that, if  $0 < m \leq 2$ , then  $p > \frac{2}{m}$  and if  $2 < m < 4$ , then  $p \in (1, \frac{m}{m-2})$ .
- The disturbance function  $e : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  belongs to  $W^{1,1}(\mathbb{R}_+, L^2(\Omega))$  and satisfies that  $e \in Lip(\mathbb{R}_+, H_0^1(\Omega))$ ,  $e(0, \cdot) \in L^2(\Omega)$ , and  $\int_0^\infty \int_\Omega e^2 dx dt < \infty$ .

We prove the main theorem that states that:

For all initial data  $(z_0, z_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ , Problem  $(\mathbf{P}_{\text{dis}})$  has a unique strong solution  $z$ .

Furthermore, the following energy estimate holds:

$$E_2(t) \leq (C + C_z)E_2(0)e^{-\frac{t-1}{\sigma_z + \sigma}} + C_{d,e}(C_z + 1), \quad (2.26)$$

where the positive constant  $C_z$  depends only on the initial data and is a  $\mathcal{K}$ -function of the norms of initial data and the positive constant  $C_{d,e}$  depends only on the disturbances  $d$  and  $e$  and is  $\mathcal{K}$ -function of the different norms of  $e$  and  $d$ .

The theorem gives an ISS-type estimate but it fails to be a perfect one in the sense of Definition 1.2.3 since the estimated quantity  $E_2$  is the norm of a trajectory in the space  $H_0^1(\Omega) \times L^2(\Omega)$  while the constant  $C_z$  depends on the initial condition by its norm in the smaller space  $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$  there is then a sort of norm incompatibility which seems unavoidable when manipulating strong solutions. There is also another imperfection in the obtained ISS above that makes it weaker, which is the fact that the second term of the right hand side of the estimate is not a  $\mathcal{K}$ -function of the norms of the disturbances only but it involves initial data as well. However, we can get such a result if we have an extra assumption on  $g$ , typically  $g$  of growth at most linear at infinity (i.e.,  $q = 1$ ) with bounded derivative (i.e.,  $m = 0$ ). In particular, this covers the case of regular saturation functions (increasing bounded functions  $g$  with bounded derivatives).

Although the obtained estimate is a weak ISS, it still assures the exponential decay in the absence of the disturbances. Furthermore, in the case where the disturbances are both zero ( $d \equiv 0$  and  $e \equiv 0$ ), Theorem 2.3.1 holds without the growth hypothesis on

$g'$  and with no restrictions on  $q$  from the growth hypotheses on  $g$ . The latter can be then weakened to the following hypothesis

$$\exists C > 0, \exists q > 1, \forall |x| \geq 1, |g(x)| \leq C|x|^q.$$

It is clear that if  $g$  satisfies the last part of the condition above for  $0 \leq q \leq 1$ , it would still satisfy it for any  $q > 1$ .

The proof of the theorem treats the well-posedness first and then the energy estimate.

We prove the well-posedness in  $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$  by making a small change of variable  $y(t, x) = z(t, x) + \int_0^t d(s, x) ds$  for all  $(t, x) \in \mathbb{R}_+ \times \Omega$ , which leads to an equivalent simpler problem in  $y$  with no feedback disturbance. We transform the problem into an evolutionary problem and then we use maximal monotone operators theory to prove the existence and the uniqueness of strong solutions in  $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ .

As for the ISS-type estimate, we use the multipliers method inspired from [4] and [24] and we treat the nonlinear terms using the techniques introduced in [25].

We start by proving three technical lemmas. The first gives an expression of the energy derivative along strong solutions states that the energy of a strong solution satisfies

$$E_2(T) \leq E_2(S) + C_{d,e}, \quad \forall 0 \leq S \leq T,$$

where  $C_{d,e}$  is a constant that depends on the disturbances and is  $\mathcal{K}$ -function of the different norms of  $e$  and  $d$ . One can immediately notice that in the absence of disturbances we recover from the latter inequality that the energy is non-increasing. The second lemma states that for all time  $t \geq 0$ ,

$$\| -\Delta z(t, \cdot) + a(\cdot)g(z_t(t, \cdot) + d(t, \cdot)) + e(t, \cdot) \|_{L^2(\Omega)}^2 + \|z_t(t, \cdot)\|_{H_0^1(\Omega)}^2 \leq C_z + C_{d,e}.$$

This immediately give an upper bound of the  $H_0^1(\Omega)$  norm of  $z_t(t, \cdot)$  that is uniform with respect to time and that only depends on initial data and the disturbances (on initial data only in the absence of disturbances). We obtain such an important uniform bound by differentiating the main problem with respect to time and we consider the new problem of the unknown  $w = z_t$ . We prove an energy estimate for the solution  $w$  which leads to the inequality given by this second lemma. The third lemma gives an estimate of the  $L^q(\Omega)$  norm of  $z_t(t, \cdot)$  for all  $t \geq 0$  and for all  $q \geq 2$ . We use the

estimate of  $H_0^1(\Omega)$  norm of  $z_t(t, \cdot)$  from the previous lemma and we combine it with Gagliardo-Nirenberg theorem, we obtain an estimation of any  $L^q(\Omega)$  norm of  $z_t(t, \cdot)$ . This estimation is given by  $\|z_t(t, \cdot)\|_{L^q(\Omega)}^q \leq (C_z + C_{d,e}) E_2(t)$ . The lemmas can be seen as a generalization of the lemmas introduced in [25] that give similar results but without the disturbances.

We use these key lemmas along side with the multiplier method to prove that the energy  $E_2$  of a strong solution  $z$  of the disturbed problem satisfies:

$$\int_S^T E_2(t) dt \leq (C_z + C) E_2(S) + (1 + C_z) C_{d,e}, \quad \forall 0 \leq S \leq T.$$

We use several multipliers to achieve such an energy estimate, each multiplier will take us a step closer to obtain the right estimate. To define the multipliers we define three localization functions  $\psi$ ,  $\phi$  and  $\beta$ . These functions are defined using the subsets  $Q_i = \mathcal{N}_{\epsilon_i}[\Gamma(x_0)]$  for  $i = 0, 1, 2$ , where  $x_0 \in \mathbb{R}^2$  is an observation point,  $\epsilon$  is the same defined in the geometrical conditions and  $\epsilon_0, \epsilon_1$  and  $\epsilon_2$  are three positive real constants such that  $\epsilon_0 < \epsilon_1 < \epsilon_2 < \epsilon$ . Thanks to how  $Q_i$  are defined we are allowed to define the three localization smooth functions with compact supports.

$$\begin{cases} 0 \leq \psi \leq 1, \\ \psi = 0 \text{ on } Q_0, \\ \psi = 1 \text{ on } \bar{\Omega} \setminus Q_1, \end{cases} \quad \begin{cases} 0 \leq \phi \leq 1, \\ \phi = 1 \text{ on } Q_1, \\ \phi = 0 \text{ on } \mathbb{R}^2 \setminus Q_2, \end{cases} \quad \begin{cases} 0 \leq \beta \leq 1, \\ \beta = 1 \text{ on } Q_2 \cap \Omega, \\ \beta = 0 \text{ on } \Omega \setminus \omega. \end{cases}$$

We can now list the multipliers used in this problem:

- **First multiplier:**  $M(z) := k \nabla z + \frac{z}{2}$ , where  $k$  is a  $C^1$  vector field defined by  $k(x) := \psi(x)(x - x_0)$ .
- **Second multiplier:**  $\phi z$ .
- **Third multiplier:**  $v$ , where  $v$  is the solution of the following elliptic problem:

$$\begin{cases} \Delta v = \beta z & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying each time the problem by the multipliers defined above, integrating on  $\Omega \times \mathbb{R}$ , applying different manipulations, and then combining the results obtained using the three multipliers, we prove the required energy estimate. Finally, the ISS type-estimate is obtained as a direct result of an easily proved generalized Gronwall lemma.

**Second Problem:**

We consider next the  $L^p$  framework, which is a very recent framework and not much work is published about it. We consider the wave equation with a localized damping term and Dirichlet boundary conditions which is the linear case of the general problem (1.1) in one dimension. The problem is written as follows

$$(\mathbf{P}_{\text{lin}}) \quad \begin{cases} z_{tt} - z_{xx} + a(x)z_t = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \\ z(t, 0) = z(t, 1) = 0 & t \geq 0, \\ z(0, \cdot) = z_0, \quad z_t(0, \cdot) = z_1, \end{cases}$$

where this time initial data  $(z_0, z_1)$  belong to the phase space  $X_p$  defined for  $p \in [1, \infty)$  by  $X_p := W_0^{1,p}(0, 1) \times L^p(0, 1)$  and is equipped with the norm  $\|(u, v)\|_{X_p} := \left( \frac{1}{p} \int_0^1 (|u' + v|^p + |u' - v|^p) dx \right)^{\frac{1}{p}}$ , which is an equivalent norm to the natural norm that we usually take in such functional space. We also define the phase space for strong solutions  $Y_p$  defined by  $Y_p := (W^{2,p}(0, 1) \cap W_0^{1,p}(0, 1)) \times W_0^{1,p}(0, 1)$ , equipped with the norm  $\|(u, v)\|_{Y_p} := \left( \frac{1}{p} \int_0^1 (|u'' + v'|^p + |u'' - v'|^p) dx \right)^{\frac{1}{p}}$ .

We work under very general hypotheses, they're listed as follows:

*Hypotheses on the localization function  $a$  and the damping domain  $\omega$ :*

- The function  $a$  is continuous and non-negative on  $[0, 1]$  and satisfies that  $\exists a_0 > 0$ ,  $a \geq a_0$  on  $\omega = ]c, d[ \subset [0, 1]$ .
- the subset  $\omega$  is a non empty interval such that  $c = 0$  or  $d = 1$ , i.e.,  $\bar{\omega}$  contains a neighborhood of 0 or 1. There is no loss of generality in assuming  $d = 1$ , taking 0 as an observation point.

We actually do not need the second hypothesis on  $\omega$  (the geometrical condition) to prove the well-posedness or the stability. We only use it to simplify the work but the results still hold if the assumption that  $c = 0$  or  $d = 1$  is removed. The only thing that changes in this case is that we will be using a piecewise multiplier method instead of a simple one, i.e., we use both 0 and 1 as observation points (instead of simply 0 here) to obtain the required energy estimate.

We prove two main theorems under the above hypotheses. The first one concerns the well-posedness of Problem  $(\mathbf{P}_{\text{lin}})$  and it states that

Let  $p \in [1, \infty)$ . For any initial data  $(z_0, z_1) \in X_p$  (resp.  $Y_p$ ), there exists a unique weak (resp. strong) solution  $z$  such that

$$z \in L^\infty(\mathbb{R}_+, W_0^{1,p}(0, 1)) \cap W^{1,\infty}(\mathbb{R}_+, L^p(0, 1)),$$

$$\text{(resp. } z \in L^\infty(\mathbb{R}_+, W^{2,p}(0, 1) \cap W_0^{1,p}(0, 1)) \cap W^{1,\infty}(\mathbb{R}_+, W_0^{1,p}(0, 1)).)$$

Moreover, in both cases, the energy function  $t \mapsto E_p(t)$  associated with a solution is non-increasing.

The second theorem concerns the exponential stability and it states that

For all  $p \in (1, \infty)$ , the  $C^0$ -semigroup  $(S_p(t))_{t \geq 0}$  defining weak solutions of Problem  $(\mathbf{P}_{\text{lin}})$  is exponentially stable.

For the well-posedness, we prove for  $p > 1$  the existence and the uniqueness of solutions in  $X_p$  (weak solutions) as well as in  $Y_p$  (strong solutions). We use D'Alembert formula to transform the well-posedness problem into a fixed point problem. We prove the existence of solutions on some interval  $[0, T]$  and then we reproduce the reasoning on  $[T, 2T]$  and so on to establish the well-posedness for all  $t \geq 0$ .

As for the stability theorem, We start by rewriting the problem using the Riemann invariants  $\rho(t, x) = z_x(t, x) + z_t(t, x)$  and  $\xi(t, x) = z_x(t, x) - z_t(t, x)$ . We obtain an equivalent problem to  $(\mathbf{P}_{\text{lin}})$  along strong solutions. The new problem is given by

$$\begin{cases} \rho_t - \rho_x = -\frac{1}{2}a(x)(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0, 1), \\ \xi_t + \xi_x = \frac{1}{2}a(x)(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0, 1), \\ \rho(t, 0) - \xi(t, 0) = \rho(t, 1) - \xi(t, 1) = 0 & \forall t \in \mathbb{R}_+, \\ \rho_0 := \rho(0, \cdot) = z'_0 + z_1, \quad \xi_0 := \xi(0, \cdot) = z'_0 - z_1. \end{cases}$$

The energy  $E_p$  of a strong solution is then given for all  $t \geq 0$  in terms of  $\rho$  and  $\xi$  by

$$E_p(t) = \frac{1}{p} \int_0^1 (|\rho|^p + |\xi|^p) dx.$$

We prove using the new form of the problem that the energy of a strong solution is non-increasing and we give an explicit expression of its derivative, given by

$$E_p'(t) = -\frac{1}{2} \int_0^1 a(x)(\rho - \xi) (|\rho|^{p-1} - |\xi|^{p-1}) dx,$$

where  $|x|^{p-1} := \text{sgn}(x)|x|^{p-1}$ ,  $\forall x \in \mathbb{R}$ .

We prove strong stability in  $X_p$  using the energy functional  $E_p$  with a similar LaSalle type argument to the one that was used in the Hilbertian framework in [11].

To prove exponential stability, we generalize the multiplier method used in the  $L^2$  framework to the  $L^p$  framework where we use to obtain energy estimates, sets of multipliers instead of individual multipliers only. We use different multipliers and approaches when  $p \geq 2$  and  $1 < p < 2$ . In both cases we will be using to define the multipliers the functions  $\psi, \phi$  and  $\beta$  which are smooth with compact support and are defined as follows:

$$\begin{cases} 0 \leq \psi \leq 1, \\ \psi = 0 \text{ on } Q_0, \\ \psi = 1 \text{ on } (0, 1) \setminus Q_1, \end{cases} \quad \begin{cases} 0 \leq \phi \leq 1, \\ \phi = 1 \text{ on } Q_1, \\ \phi = 0 \text{ on } (0, 1) \setminus Q_2, \end{cases} \quad \begin{cases} 0 \leq \beta \leq 1, \\ \beta = 1 \text{ on } Q_2 \cap (0, 1), \\ \beta = 0 \text{ on } \mathbb{R} \setminus \omega. \end{cases}$$

In the case  $p \geq 2$ :

We define the two functions

$$\begin{aligned} f(y) &= (p-1) \int_0^y |s|^{p-2} ds = |y|^{p-1}, \quad \forall y \in \mathbb{R}, \\ F(y) &= \int_0^y f(s) ds = \frac{|y|^p}{p}, \quad \forall y \in \mathbb{R}, \end{aligned}$$

which allows to express the energy in terms of  $F$  by  $E_p(t) = \int_0^1 (F(\rho) + F(\xi)) dx$ .

We prove the following energy estimate for every  $(z_0, z_1) \in Y_p$

$$\forall 0 \leq S \leq T, \quad \int_S^T E_p(t) dt \leq C C_p E_p(S),$$

using the following three sets of multipliers:

- **First set of multipliers:**  $x \mapsto x\psi(x)f(\rho(t, x))$  and  $x \mapsto x\psi(x)f(\xi(t, x))$  for



every  $t \geq 0$ .

- **Second set of multipliers:**  $x \mapsto \phi(x)f'(\rho(t,x))z(t,x)$  and  $x \mapsto \phi(x)f'(\xi(t,x))z(t,x)$  for every  $t \geq 0$ , where  $f'(s) = (p-1)|s|^{p-2}$ .
- **Third multiplier:**  $x \mapsto v(t,x)$  for every  $t \geq 0$ , where  $v$  is the solution of the following elliptic problem defined for every  $t \geq 0$ :

$$\begin{cases} v_{xx} = \beta f(z) & x \in (0,1), \\ v(0) = v(1) = 0, \end{cases}$$

We multiply the problem expressed in Riemann invariants by the multipliers above. For the sets of multipliers, we multiply the problem by the two multipliers of each set and we sum them up to obtain the result that the set provides. We obtain in the end after combining the results of the three multipliers (two sets and one individual) the energy estimate that will immediately lead to the exponential stability using Gronwall lemma. Then, we use a density argument to extend the proof from strong to weak solutions since the constants obtained in the energy estimate do not depend on the  $Y_p$  norm of initial data.

In the case  $1 < p < 2$ :

We cannot use the same multipliers because the second set given by  $x \mapsto \phi(x)f'(\rho(t,x))z(t,x)$  and  $x \mapsto \phi(x)f'(\xi(t,x))z(t,x)$  for every  $t \geq 0$ , where  $f'(s) = (p-1)|s|^{p-2}$  is no longer well-defined since the power  $p-2$  is negative for  $1 < p < 2$ . This affects the whole process since the multipliers are logically connected and complete each other's results. Changing all the multipliers is then required. To do so, we kind of disturb the function  $f'$  that was used to define the second set of multipliers in the case  $p \geq 2$  with a positive number 1. We obtain the new key functions  $h$  and  $H$  for  $p \in (1,2)$ , which are defined on  $\mathbb{R}$  by:

$$\begin{aligned} h(y) &= (p-1) \int_0^y (|s|+1)^{p-2} ds = \operatorname{sgn}(y) [ (|y|+1)^{p-1} - 1 ], \\ H(y) &= \int_0^y h(s) ds = \frac{1}{p} [ (|y|+1)^p - 1 ] - |y|. \end{aligned}$$

One can see the functions  $h$  and  $H$  as sort of a perturbation of  $f$  and  $F$  to assure that the the multipliers are well defined around zero when  $p \in (1,2)$ .

We also modify the energy  $E_p$  by sort of a perturbed energy  $\mathcal{E}_p$  such that for every

$t \in \mathbb{R}_+$  and every solution of  $(\mathbf{P}_{\text{lin}})$ ,  $\mathcal{E}_p$  is defined by

$$\mathcal{E}_p(t) = \int_0^1 (H(\rho) + H(\xi)) \, dx.$$

The same way as for  $E_p$  we prove that  $t \mapsto \mathcal{E}_p(t)$  is non-increasing along strong solutions. We succeed to prove under the hypotheses of Problem  $(\mathbf{P}_{\text{lin}})$  for a fixed  $p \in (1, 2)$  that there exist positive constants  $C$  and  $C_p$  such that, for every  $(z_0, z_1) \in Y_p$   $E_p(0) \leq 1$ , we have the following energy estimate:

$$\forall 0 \leq S \leq T, \int_S^T \mathcal{E}_p(t) \, dt \leq C C_p \mathcal{E}_p(S).$$

The latter (perturbed) energy estimate is proved using the following sets of multipliers:

- **First set of multipliers**  $x \mapsto x\psi(x)h(\rho(t, x))$  and  $x \mapsto x\psi(x)h(\xi(t, x))$  for every  $t \geq 0$ ;
- **Second set of multipliers**  $x \mapsto \phi(x)h'(\rho(t, x))z(t, x)$  and  $x \mapsto \phi(x)h'(\xi(t, x))z(t, x)$  for every  $t \geq 0$ ;
- **Third multiplier**  $x \mapsto v(t, x)$  for every  $t \geq 0$ , where  $v$  is the solution of the following elliptic problem defined for every  $t \geq 0$ :

$$\begin{cases} v_{xx} = \beta h(z) & x \in (0, 1), \\ v(0) = v(1) = 0, \end{cases}$$

It is important to note that using the multipliers above, we face different difficulties to obtain certain estimates. These difficulties require a generalization of Young inequality since this last inequality has been widely used in the case  $p \geq 2$  but can no longer be used in the case  $1 < p < 2$  (at least for the second set of multipliers). To do so, we use Fenchel's inequality along with other estimates that we prove and that involve the functions  $h$ ,  $H$  and the Legendre transform of  $H$  denoted by  $\mathcal{H}$  to prove in the end that for all  $\eta > 0$ , for all  $a$ ,  $x \in \mathbb{R}$  we have the following key inequalities for  $1 < p < 2$  only:

$$\begin{aligned}
|ax| &\leq \frac{C_p}{\eta^2} H(a) + C_p \eta^p \mathcal{H}(x), \\
|ax| &\leq C_p \eta^p H(a) + \frac{C_p}{\eta^2} \mathcal{H}(x), \\
|ah(x)| &\leq \frac{C_p}{\eta^p} H(a) + C_p \eta^2 H(x).
\end{aligned}$$

This generalization along with other intermediate results, allows us to prove with the new multipliers, the required (perturbed) energy estimate. The proof clearly does not end here. Gronwall lemma only imply an exponential decay of the new 'perturbed' energy, which implies the existence of a constant  $\gamma_p > 0$  such that  $\mathcal{E}_p(t) \leq e^{1-\gamma_p t}$  for all  $t \geq 0$  and for solutions of  $(\mathbf{P}_{\text{lin}})$  satisfying  $E_p(0) \leq 1$ .

We deduce the exponential decay of the original energy by an argument based on picking initial conditions such that  $E_p(0) = 1$  and using the exponential decay inequality that has been proved for  $\mathcal{E}_p(t)$  to prove that  $E_p(t) \leq \frac{1}{2}$  for all time  $t$  greater or equal to a certain time  $t_p$  that depends on  $\gamma_p$ . This proves immediately when combined with the fact that  $t \mapsto (E_p(t))^{\frac{1}{p}}$  is a norm that  $\|S_p(t_p)\|_{X_p} < 1$ . This proves when combined with semigroups properties the exponential stability of the  $C_0$ -semigroup that defines the solutions of  $(\mathbf{P}_{\text{lin}})$ .

The ultimate goal of studying  $L^p$ -stability is actually to achieve the case  $p = \infty$  where the  $\infty$ -energy of a solution coincide with the altitude of the wave. We were not able to achieve that for the linear case with a non-constant localized damping. This is due to the fact that the constants obtained in the energy estimates in the case  $1 < p < 2$  and  $p \geq 2$  do not converge when letting  $p$  goes to one or infinity respectively. However, we are able to achieve that in a particular case, the case of a global constant damping  $a(x) = 2\alpha$  we prove the following proposition

For  $p = 1$  or  $p = \infty$ , the semi-group  $(S(t))_{t \geq 0}$  is exponentially stable for a global constant damping if  $\alpha \in (0, 2)$ .

The proof is based on considering  $p \in (1, \infty)$  and making the change of variable  $z(t, x) = e^{-\alpha t} v(t, x) \quad \forall x \in (0, 1), t \geq 0$  to obtain the equivalent problem

$$\begin{cases} v_{tt} - v_{xx} = \alpha^2 v & \text{in } \mathbb{R}_+ \times (0, 1), \\ v(t, 0) = v(t, 1) = 0 & t \geq 0, \\ v(0, \cdot) = z_0, \quad v_t(0, \cdot) = z_1 + \alpha z_0. \end{cases}$$

We easily prove a relationship between  $E_p$  and  $V_p$  energy, where  $V_p$  denotes the  $p$ th-energies associated with  $v$ . We obtain an explicit expression of  $V'_p$  the same way we obtained an explicit expression of  $E'_p$ . We use this expression with other previously established results to prove an energy estimate for  $V$  to which we use Gronwall's lemma to obtain an exponential decay for  $V$ . Due to the relationship between  $E_p$  and  $V_p$ , we obtain an exponential decay for  $E_p(t)$  with explicit constants. We conclude by letting  $p$  tend either to one or  $\infty$  and using an obvious density argument.

**Third problem:**

We study next the non-linear case of the second problem. The problem is given by

$$(\mathbf{P}_{nl}) \quad \begin{cases} z_{tt} - z_{xx} + a(x)g(z_t) = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \\ z(t, 0) = z(t, 1) = 0 & \text{for } t \geq 0, \\ z(0, \cdot) = z_0, \quad z_t(0, \cdot) = z_1 & \text{on } (0, 1), \end{cases}$$

We work under the following hypotheses

Hypotheses on the localization function  $a$  and the damping domain  $\omega$ :

(same as the second problem)

- The function  $a$  is continuous and non-negative on  $[0, 1]$  and satisfies that  $\exists a_0 > 0$ ,  $a \geq a_0$  on  $\omega = ]c, d[ \subset [0, 1]$ .
- The subset  $\omega$  is a non empty interval such that  $c = 0$  or  $d = 1$ , i.e.,  $\bar{\omega}$  contains a neighborhood of 0 or 1. There is no loss of generality in assuming  $d = 1$ , taking 0 as an observation point.

Just like in the second problem, we do not need the second hypothesis on  $\omega$ . We only use it to simplify computations. The results still hold if the geometric assumption is removed.

Hypotheses on the nonlinearity  $g$ :

$g : \mathbb{R} \mapsto \mathbb{R}$  a  $C^1$  non-decreasing function such that  $g(0) = 0$ ,  $g'(0) > 0$ , and  $g(x)x \geq 0 \quad \forall x \in \mathbb{R}$ .

We prove under these hypotheses the well-posedness theorem that states that

Suppose Hypotheses  $(\mathbf{H}_1)$  and  $(\mathbf{H}_2)$  are satisfied, then for all initial conditions  $(z_0, z_1) \in X_p$  with  $2 \leq p \leq \infty$ , we have the existence of a unique weak solution  $z$  such that

$$z \in L^\infty(\mathbb{R}_+, W_0^{1,p}(0, 1)) \text{ and } z_t \in L^\infty(\mathbb{R}_+, L^p(0, 1)).$$

Moreover, if  $(z_0, z_1) \in Y_p$  with  $1 \leq p \leq \infty$  then we have the existence of a unique strong solution  $z$  such that

$$z \in L^\infty(\mathbb{R}_+, W^{2,p}(0, 1) \cap W_0^{1,p}(0, 1)) \text{ and } z_t \in L^\infty(\mathbb{R}_+, W_0^{1,p}(0, 1)).$$

We also prove the following stability theorem

For  $1 < p < \infty$ . Given,  $(z_0, z_1) \in Y_p$ , there exists a constant  $C_p(z_0, z_1) > 0$  that depends on the norm of initial conditions in  $Y_p$  such that for all  $t \in \mathbb{R}_+$

$$E_p(t) \leq E_p(0)e^{1-C_p(z_0, z_1)t}.$$

For the well-posedness theorem, the result for strong and weak solutions for  $p = \infty$  has been proved in [10]. We prove the existence of weak solutions for  $2 \leq p < \infty$  by constructing a proof based on the known existence of solutions in  $X_\infty$ , combined with a density argument. Then we prove the existence of strong solutions for all  $1 \leq p < \infty$  by relying on the existence of solutions in  $X_\infty$  again and then proving an extra regularity of strong solutions in  $L^p$  by using techniques that were used in the  $L^2$  framework in [25].

As for the exponential decay of the energy along strong solutions, we combine some techniques used to study the first problem with the theory used to treat the second problem. The proof is based on using the work that has been already done in the linear case using the multipliers method to obtain a stabilization result for the non-linear problem. Indeed, we start by considering an intermediate problem:

$$\begin{cases} y_{tt} - y_{xx} + a(x)\theta(t, x)y_t = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \\ y(t, 0) = y(t, 1) = 0 & t \geq 0, \\ y(0, \cdot) = y_0, \quad y_t(0, \cdot) = y_1, \end{cases}$$

where  $a$  satisfies the same hypothesis as earlier and  $\theta : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$  is a non-negative continuous function such that

$$\exists \theta_1, \theta_2 > 0, \quad \theta_1 \leq \theta(t, x) \leq \theta_2 \quad \forall (t, x) \in \mathbb{R}_+ \times [0, 1].$$

The well-posedness of the intermediate problem can be treated the same way as the linear problem, which means that it is actually well-posed for all  $p > 1$ .

By following the exact same multiplier method used in the linear case, with very slight differences where we treat the occurrences of  $\theta$  using the hypothesis  $\theta_1 \leq \theta(t, x) \leq \theta_2$ ; we prove that for every  $(y_0, y_1) \in X_p$ , the energy along solutions in  $X_p$  of the intermediate problem decays exponentially to zero.

Then, to prove the exponential stability of the non-linear problem  $(\mathbf{P}_{\text{nl}})$ , we start by proving that the  $W^{1,p}(0, 1)$  norm of  $z_t(t, \cdot)$  is bounded by a constant that depends on the  $Y_p$  norm of initial data only which is a uniform bound with respect to time. This means that  $z_t(t, \cdot)$  is uniformly bounded in  $L^\infty(0, 1)$  and therefore the function  $g$  satisfies for some two positive constants  $\theta_1$  and  $\theta_2$  that  $\theta_1 \leq \frac{g(z_t(t, x))}{z_t} \leq \theta_2$ , for all  $t \geq 0, x \in (0, 1)$ . We denote  $\nu(z_t) = \frac{g(z_t(t, x))}{z_t}$  and we consider the problem

$$\begin{cases} y_{tt} - y_{xx} + a(x)\nu(z_t)y_t = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \\ y(t, 0) = y(t, 1) = 0 & t \geq 0, \\ y(0, \cdot) = y_0, \quad y_t(0, \cdot) = y_1, \end{cases}$$

which is nothing else but the intermediate problem with  $\theta(t, x) = \nu(z_t)$  since  $\nu(z_t)$  is seen as a function of  $t$  and  $x$  bounded uniformly from above and below by two positive constants. The exponential stability is known for this problem and it leads to the exponential stability of strong solutions for the non-linear  $(\mathbf{P}_{\text{nl}})$  by considering the particular solutions with initial data  $(y_0, y_1) = (z_0, z_1)$ .

Finally, we summarize the main contributions of the three problems in the following list:

- Studying the disturbed non-linear wave equation with localized damping in a Hilbertian framework in two dimensions
  - Proving the well-posedness (Theorem 2.3.1).
  - Proving a weak ISS-type estimate for strong solutions that includes an exponential decay in the absence of disturbances (Theorem 2.3.1).
- Studying the linear wave equation with localized damping in a non-Hilbertian  $L^p$  framework in one dimension
  - Establishing the well-posedness for  $p > 1$  (Theorem 3.4.1).

- Proving an exponential decay of the  $pth$ -energy for all  $p > 1$  (Theorem 3.5.1).
- Proving an exponential decay of the  $pth$ -energy for  $p = 1$  and  $p = \infty$  in some cases of a constant global damping (Proposition 3.6.1).
- Studying the non-linear wave equation with localized damping in a non-Hilbertian  $L^p$  framework in one dimension
  - Proving the well-posedness for weak solutions for all  $p \geq 2$  and for strong solutions for all  $1 \leq p \leq \infty$  (Theorem 4.3.1).
  - Proving an exponential decay of the  $pth$ -energy along strong solutions for all  $1 < p < \infty$  (Theorem 4.4.1).

# Weak input-to-state stability of the damped wave equation with localized and non-linear damping in two dimensions

*The work of this chapter has been published in the scientific paper [18].*

## 2.1 Introduction

Consider Problem (1.1) in two dimensions with added disturbances that we will define later on. The problem in this case is given by

$$\begin{cases} z_{tt} - \Delta z = -a(x)g(z_t + d) - e, & \text{in } \mathbb{R}_+ \times \Omega, \\ z = 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ z(0, \cdot) = z_0, \quad z_t(0, \cdot) = z_1, \end{cases} \quad (2.1)$$

where  $\Omega$  is a  $C^2$  bounded domain of  $\mathbb{R}^2$ ,  $d$  and  $e$  stand for a damping disturbance and a globally distributed disturbance for the wave dynamics respectively. The term  $-a(x)g(z_t + d)$  stands for the (perturbed) damping term, where  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^1$  non-decreasing function verifying  $\xi g(\xi) > 0$  for  $\xi \neq 0$  while  $a : \overline{\Omega} \rightarrow \mathbb{R}$  is a continuous non negative function which is bounded below by a positive constant  $a_0$  on some non-empty open subset  $\omega$  of  $\Omega$ . Here,  $\omega$  is the region of the domain where the damping term is active, more precisely, the region where the localization function  $a$  is bounded from below by  $a_0$ . As for the initial condition  $(z_0, z_1)$ , it belongs to the standard Hilbert space  $X_2 = H_0^1(\Omega) \times L^2(\Omega)$ .



In this chapter, our aim is to obtain input-to-state (ISS) type of results for Problem (2.1), i.e., estimates of the norm of the state  $u$  which, at once, show that trajectories tend to zero in the absence of disturbances and remain bounded by a function of the norms of the disturbances otherwise. One can refer to [30] for a thorough review of ISS results and techniques for finite dimension systems and to the recent survey [28] for infinite dimensional dynamical systems. In the case of the undisturbed dynamics, i.e., (2.1) with  $(d, e) \equiv (0, 0)$ , there is a vast literature regarding the stability of the corresponding system with respect to the origin, which is the unique equilibrium state of the problem. This in turn amounts to have appropriate assumptions on  $a$  and  $g$ , cf. [4] for extensive references. We will however point out the main ones that we need in order to provide the context of our work. To do so, we start by considering the energy of the system given by (1.2), which defines a natural norm on the space  $X_2 = H_0^1(\Omega) \times L^2(\Omega)$  given by

$$\|\cdot\|_{X_2} = (E_2)^{1/2} = \left( \frac{1}{2} \int_{\Omega} (|z_t(t, x)|^2 + |\nabla z(t, x)|^2) dx \right)^{\frac{1}{2}}. \quad (2.2)$$

Strong stabilization has been established in the early works [11] and [14], i.e., it is proved with an argument based on the Lasalle invariance principle that  $\lim_{t \rightarrow +\infty} E_2(t) = 0$  for every initial condition in  $H_0^1(\Omega) \times L^2(\Omega)$ . However, no decay rate of convergence for  $E_2$  is established since it requires in particular extra assumptions on  $g$  and  $\omega$ .

As a first working hypothesis, we will assume that  $g'(0) > 0$ , classifying the present work in those that aimed at establishing results of exponential convergence for strong solutions. We refer to [4] for the line of work where  $g$  is assumed to be super-linear in a neighborhood of the origin (typically of polynomial type). Note that, in most of these works (except for the linear case) the rate of exponential decay of  $E_2$  depends on the initial conditions. That latter fact in turn relies on growth conditions on  $g$  at infinity. Regarding the damping domain  $\omega$ , it satisfies the geometrical condition (MGC) that has been introduced in the general introduction and will be reminded later on when properly stating the problem.

In this chapter, our objective is to obtain results for non-linear damping terms and one should think of the nonlinearity  $g$  not only as a mean to provide more general asymptotic behavior at infinity than a linear one but also as modeling an uncertainty of the shape of the damping term. Dealing with nonlinearities justifies why microlocal techniques are not suited here and we will be using the multiplier method as presented e.g. in [21]. Many results have been established in the case where  $g'(0) = 0$ , for instance, decay rates for the energy are provided in [24] in the localized case but the non-linearity is to have a linear growth for large values of its arguments. Note that the

estimates as presented in [24] are not optimal in general, as for instance in the case of a power-like growth. For general optimal energy decay estimates and for general abstract PDEs, we refer the reader to [2] for a general formula for explicit energy decay estimates and to [3] for an equivalent simplified energy decay estimate with optimality results in the finite dimensional case. However, when it comes to working under the hypothesis  $g'(0) > 0$ , few general results are available. One can find a rather complete presentation of the available results in [25]. In particular, the proof of exponential stability along strong solutions has only been given for general nonlinearities  $g$ , in dimension two and in the special case of a non-localized damping with no disturbances requiring only one multiplier coupled with a judicious use of Gagliardo-Nirenberg's inequality. Our results generalize this finding in the absence of disturbances (even though it has been mentioned in [25] with no proof that this is the case). It has also to be noted that similar results are provided in [24] in the localized case but the nonlinearity is lower bounded by a linear function for large values of its arguments. That simplifies considerably some computations. Recall also that the purpose of [24] is instead to address issues when  $g'(0) = 0$  and to obtain accurate decay rates for  $E$ .

Hence a possible interest of the present chapter is the fact that it handles nonlinearities  $g$  so that  $g(v)/v$  tends to zero as  $|v|$  tends to infinity with a linear behavior in a neighborhood of the origin.

As for ISS purposes, this chapter can be seen as an extension to the infinite dimensional context of [23] where the nonlinearity is of the saturation type. Moreover, the present work extends to the dimension two the works [26] and [27], where this type of issues have been addressed by building appropriate Lyapunov functions and by providing results in dimension one. Here, we are not able to construct Lyapunov functions and we rely instead on energy estimates based on the multiplier method, showing how these estimates change when adding the two disturbances  $d$  and  $e$ . To develop that strategy, we must impose additional assumptions on  $g'$ , still handling saturation functions. As a final remark, we must recall that [25] contains other stability results in two directions. On one hand,  $g'$  can simply admit a (possibly) negative lower bound and on the other hand, the space dimension  $N$  can be larger than 2, at the price of more restrictive assumptions on  $g$ , in particular, by assuming quasi-linear lower bounds for its asymptotic behavior at infinity. One can readily extend the results of the present chapter in both directions by eventually adding growth conditions on  $g$ .

## 2.2 Preliminaries

**Lemma 2.2.1** ([17]). (*Young's inequality*)

Let  $p > 1$  and  $q = \frac{p}{p-1}$  its conjugate exponent. Then, for every  $A, B \in \mathbb{R}$  and  $\eta > 0$ , Young's inequality reads

$$|AB| \leq \eta^p \frac{|A|^p}{p} + \frac{|B|^q}{q\eta^q}. \quad (2.3)$$

**Theorem 2.2.1.** (*Gronwall integral lemma*)

Let  $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfy for some  $C_0, T > 0$ :

$$\int_t^{+\infty} E(s)ds \leq TE(t) + C_0, \quad \forall t \geq 0. \quad (2.4)$$

Then, the following estimate hold true

$$\int_t^{+\infty} E(s)ds \leq TE(0)e^{-\frac{t}{T}} + C_0, \quad \forall t \geq 0. \quad (2.5)$$

If in addition,  $t \mapsto E(t)$  is non-increasing, one has

$$E(t) \leq E(0)e^{1-\frac{t}{T}} + \frac{C_0}{T}, \quad \forall t \geq 0. \quad (2.6)$$

The proof is classical, cf. for instance [4].

**Theorem 2.2.2.** (*Generalized Gronwall lemma*)

Let  $F, h_1$  and  $h_2$  non negative functions defined on  $\mathbb{R}_+$  satisfying

$$\|h_1\|_1 := \int_0^\infty h_1(t)dt < \infty, \quad \|h_2\|_1 := \int_0^\infty h_2(t)dt < \infty,$$

and

$$F(T) \leq F(S) + C_3 + C_1 \int_S^T h_1(s)F^{\alpha_1}(s)ds + C_2 \int_S^T h_2(s)F^{\alpha_2}(s)ds, \quad \forall S \leq T, \quad (2.7)$$

where  $C_1, C_2, C_3$  are positive constants and  $0 \leq \alpha_1, \alpha_2 < 1$ . Then,  $F$  satisfies the following bound

$$\sup_{t \in [S, T]} F(t) \leq \max \left( 2(F(S) + C_3), (2\tilde{C})^{\frac{1}{1-\alpha}} \right), \quad \text{with } \tilde{C} := C_1\|h_1\|_1 + C_2\|h_2\|_1, \quad (2.8)$$

where  $\alpha := \max(\alpha_1, \alpha_2)$  if  $2\tilde{C} \geq 1$  or  $\alpha := \min(\alpha_1, \alpha_2)$  if  $2\tilde{C} < 1$ .

**Proof of Theorem 2.2.2:** Fix  $T \geq S \geq 0$ . For  $t \in [S, T]$  set  $Y(t)$  for the right-hand side of (2.7) applied at the pair of times  $S \leq t$ . It defines a non decreasing absolutely continuous function. Since  $F(t) \leq Y(t) \leq Y(T)$  for  $t \in [S, T]$ , one deduces that  $F_{S,T} := \sup_{t \in [S, T]} F(t)$  is finite for every  $t \in [S, T]$ . One gets from (2.7) that

$$F_{S,T} \leq F(S) + C_3 + \tilde{C} \max(F_{S,T}^{\alpha_1}, F_{S,T}^{\alpha_2}),$$

with the notations of (2.8). The latter follows at once by considering whether  $F(S) + C_3 > \tilde{C} \max(F_{S,T}^{\alpha_1}, F_{S,T}^{\alpha_2})$  or not. ■

**Theorem 2.2.3.** (*Gagliardo–Nirenberg interpolation inequality*)

Let  $\Omega \subset \mathbb{R}^N$  be a bounded Lipschitz domain,  $N \geq 1$ ,  $1 \leq r < p \leq \infty$ ,  $1 \leq q \leq p$  and  $m \geq 0$ . Then the inequality

$$\|v\|_p \leq C \|v\|_{m,q}^\theta \|v\|_r^{1-\theta} \quad \text{for } v \in W^{m,q}(\Omega) \cap L^r(\Omega) \quad (2.9)$$

holds for some constant  $C > 0$  and

$$\theta = \left( \frac{1}{r} - \frac{1}{p} \right) \left( \frac{m}{N} + \frac{1}{r} - \frac{1}{q} \right)^{-1}, \quad (2.10)$$

where  $0 < \theta \leq 1$  ( $0 < \theta < 1$  if  $p = \infty$  and  $m q = N$ ) and  $\|\cdot\|_p$  denotes the usual  $L^p(\Omega)$  norm and  $\|\cdot\|_{m,q}$  the norm in  $W^{m,q}(\Omega)$ .

The result is classical and has been taken from [25].

## 2.3 Statement of the problem and main result

In this section, we provide assumptions on the data needed to precisely define (2.1). We henceforth refer to (2.1) as the disturbed problem. Next, we state and comment the main results of this work and discuss possible extensions.

Throughout the chapter, the domain  $\Omega$  is a bounded open subset of  $\mathbb{R}^2$  of class  $C^2$ , the assumptions on  $g$  are the following.

**(H<sub>1</sub>)**: The function  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^1$  non-decreasing function such that

$$g(0) = 0, \quad g'(0) > 0, \quad g(x)x > 0 \quad \text{for } x \neq 0, \quad (2.11)$$

$$\exists C > 0, \exists 1 < q < 5, \forall |x| \geq 1, |g(x)| \leq C|x|^q, \quad (2.12)$$

$$\exists C > 0, \exists 0 < m < 4, \forall |x| > 1, |g'(x)| \leq C|x|^m. \quad (2.13)$$

**(H<sub>2</sub>)**: The localization function  $a : \bar{\Omega} \rightarrow \mathbb{R}$  is a continuous function such that

$$a \geq 0 \text{ on } \Omega \text{ and } \exists a_0 > 0, a \geq a_0 \text{ on } \omega. \quad (2.14)$$

In order to prove the stability of solutions, we impose a multiplier geometrical condition (MGC) on  $\omega$ . It is given by the following hypothesis.

**(H<sub>3</sub>)**: There exists an observation point  $x_0 \in \mathbb{R}^2$  for which  $\omega$  contains the intersection of  $\Omega$  with an  $\epsilon$ -neighborhood of

$$\Gamma(x_0) = \{x \in \partial\Omega, (x - x_0) \cdot \nu(x) \geq 0\}, \quad (2.15)$$

where  $\nu$  is the unit outward normal vector for  $\partial\Omega$  and an  $\epsilon$ -neighborhood of  $\Gamma(x_0)$  is defined by

$$\mathcal{N}_\epsilon(\Gamma(x_0)) = \{x \in \mathbb{R}^2 : \text{dist}(x, \Gamma(x_0)) \leq \epsilon\}. \quad (2.16)$$

Regarding the disturbances  $d$  and  $e$ , we make the following assumptions.

**(H<sub>4</sub>)**: the disturbance function  $d : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  belongs to  $L^1(\mathbb{R}_+, L^2(\Omega))$  and satisfies the following:

$$\begin{aligned} d(t, \cdot) &\in H_0^1(\Omega) \cap L^{2q}(\Omega), \\ \forall t \in \mathbb{R}_+, \quad t &\mapsto \int_0^t \Delta d(s, \cdot) ds - d_t(t, \cdot) \in Lip(\mathbb{R}_+, H_0^1(\Omega)), \end{aligned} \quad (2.17)$$

where  $Lip$  denotes the space of Lipschitz continuous functions. We also impose that the following quantities

$$\begin{aligned} C_1(d) &= \int_0^\infty \int_\Omega (|d|^2 + |d|^{2q}) dx dt, \quad C_2(d) = \int_0^\infty \int_\Omega |d|^m (d_t)^2 dx dt, \\ C_3(d) &= \int_0^\infty \int_\Omega (d_t)^2 dx dt, \quad C_4(d) = \int_0^\infty \left( \int_\Omega |d_t|^{2(\frac{p-1}{p-1})} dx \right)^{\left(\frac{p-1}{p}\right)} dt, \end{aligned} \quad (2.18)$$

are all finite, where  $p$  is a fixed real number so that, if  $0 < m \leq 2$ , then  $p > \frac{2}{m}$  and if  $2 < m < 4$ , then  $p \in (1, \frac{m}{m-2})$ .

**Remark 2.3.1.** The fact that  $d$  belongs to  $L^1(\mathbb{R}_+, L^2(\Omega))$  means that the following quantity is finite

$$C_5(d) = \int_0^\infty \|d\|_{L^2(\Omega)} dt, \quad (2.19)$$

which implies that the following quantity is also finite

$$C_6(d) = \int_0^\infty \int_\Omega |d| dx dt. \quad (2.20)$$

**(H<sub>5</sub>):** The disturbance function  $e : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  belongs to  $W^{1,1}(\mathbb{R}_+, L^2(\Omega))$  and satisfies the following

$$e \in Lip(\mathbb{R}_+, H_0^1(\Omega)), \quad e(0, \cdot) \in L^2(\Omega), \quad \bar{C}_1(e) = \int_0^\infty \int_\Omega e^2 dx dt < \infty. \quad (2.21)$$

**Remark 2.3.2.** The fact that  $e$  belongs to  $W^{1,1}(\mathbb{R}_+, L^2(\Omega))$  means that the following quantities are finite

$$\bar{C}_2(e) = \int_0^\infty \|e(t, \cdot)\|_{L^2(\Omega)} dt, \quad \bar{C}_3(e) = \int_0^\infty \|e_t(t, \cdot)\|_{L^2(\Omega)} dt. \quad (2.22)$$

**Remark 2.3.3.** In the rest of the chapter, we will use various symbols  $C$ ,  $C_z$  and  $C_{d,e}$  which are constants independent of the time  $t$ . However, it is important to stress that these symbols have specific dependence on other parameters of the problem. More precisely, the symbol  $C$  will be used to denote positive constants independent of initial conditions and disturbances, i.e., only depending on the domains  $\Omega, \omega$  and the functions  $a$  and  $g$ . The symbol  $C_z$  denotes a generic  $\mathcal{K}$ -function of the norms of the initial condition  $(z_0, z_1)$  and similarly the symbol  $C_{d,e}$  denotes a generic  $\mathcal{K}$ -function of the several quantities  $C_i(d)$  and  $\bar{C}_i(e)$ . Here  $\mathcal{K}$  denotes the set of continuous increasing functions  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\gamma(0) = 0$ , it has been defined in the introduction chapter of this thesis (see also [28]).

Moreover, in the course of intermediate computations, we will try to keep all the previous constants as explicit as possible in terms of the norms of the initial condition and the  $C_i(d)$  and  $\bar{C}_i(e)$  in order to keep track of the nature of generic constants. We will use the latter generic mainly in the statements of the results.

Before we state the main results, we define the notion of a strong solution of Problem (2.1). To do so, we start by giving an equivalent form of Problem (2.1) :

Define for every  $(t, x) \in \mathbb{R}_+ \times \Omega$ ,  $\bar{d}(t, x) = \int_0^t d(s, x) ds$ . We translate  $z$  in Prob-

lem (2.1) as  $y = z + \bar{d}$ , it is immediate to see that Problem (2.1) is equivalent to the following problem:

$$\begin{cases} y_{tt} - \Delta y + a(x)g(y_t) = \tilde{e}, & \text{in } \mathbb{R}_+ \times \Omega, \\ y = 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ y(0, \cdot) = y^0, \quad y_t(0, \cdot) = y^1, \end{cases} \quad (2.23)$$

where  $\tilde{e} = d_t - \Delta \bar{d} - e$ ,  $y^0 = z_0$  and  $y^1 = z_1 + d(0, \cdot)$ .

Define the unbounded operator

$$\begin{aligned} A : H = H_0^1(\Omega) \times L^2(\Omega) &\longrightarrow H, \\ (x_1, x_2) &\longmapsto (x_2, -\Delta x_1 + ag(x_2)), \end{aligned} \quad (2.24)$$

with domain

$$D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega).$$

For  $t \geq 0$ , set

$$Z(t) = \begin{pmatrix} z(t, \cdot) \\ z_t(t, \cdot) \end{pmatrix}, \quad Y(t) = \begin{pmatrix} y(t, \cdot) \\ y_t(t, \cdot) \end{pmatrix}, \quad D(t) = \begin{pmatrix} \bar{d}(t, \cdot) \\ d(t, \cdot) \end{pmatrix}, \quad G(t) = \begin{pmatrix} 0 \\ \tilde{e} \end{pmatrix}.$$

Notice that  $G \in Lip(\mathbb{R}_+, L^2(\Omega) \times H_0^1(\Omega))$ . Then Problem (2.23) can be written as

$$Y_t(t) = AY(t) + G(t), \quad Y(0) = Y_0 = \begin{pmatrix} y^0 \\ y^1 \end{pmatrix}. \quad (2.25)$$

A strong solution of (2.25) in the sens of [7] is a function  $Y \in C(\mathbb{R}_+, H)$ , absolutely continuous in every compact of  $\mathbb{R}_+$ , satisfying  $Y(t) \in D(A), \forall t \in \mathbb{R}_+$  and satisfying (2.25) almost everywhere in  $\mathbb{R}_+$ . On the other hand, the hypotheses satisfied by  $d$  imply that  $D(t) \in D(A)$  for every  $t \in \mathbb{R}_+$ . Since  $Z = Y - D$ , we can now give the following definition for a strong solution of Problem (2.1).

**Definition 2.3.1.** (Strong solution of Problem (2.1).)

A strong solution  $z$  of Problem (2.1) is a function  $z \in C^1(\mathbb{R}_+, L^2(\Omega)) \cap C(\mathbb{R}_+, H_0^1(\Omega))$  such that  $t \mapsto z_t(t, \cdot)$  is absolutely continuous in every compact of  $\mathbb{R}_+$ . For all  $t \in \mathbb{R}_+$ ,  $(z(t, \cdot), z_t(t, \cdot)) \in D(A)$  and  $z(t, \cdot)$  satisfies Problem (2.1) for almost all  $t \in \mathbb{R}_+$ .

We gather our findings in the following theorem regarding the disturbed system Problem (2.1).

**Theorem 2.3.1.** Suppose that Hypotheses  $(\mathbf{H}_1)$  to  $(\mathbf{H}_5)$  are satisfied. Then, given  $(z_0, z_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ , Problem (2.1) has a unique strong solution  $z$ . Furthermore, the following energy estimate holds:

$$E_2(t) \leq (C + C_z)E_2(0)e^{-\frac{t-1}{C_z+C}} + C_{d,e}(C_z + 1), \quad (2.26)$$

where the positive constant  $C_z$  depends only on the initial conditions and the positive constant  $C_{d,e}$  depends only on the disturbances  $d$  and  $e$ .

**Remark 2.3.4. (Comments and extensions)**

- Theorem 2.3.1 holds true if the Lipschitz assumptions in (2.17) and (2.21) are replaced by bounded variation ones.
- In the case where the disturbances are both zero ( $d \equiv 0$  and  $e \equiv 0$ ), Theorem 2.3.1 holds without the hypothesis on  $g'$  given by (2.13) (i.e. no restriction on  $q$  in (2.12)) and the hypothesis given by (2.12) can be then weakened to the following one

$$\exists C > 0, \exists q > 1, \forall |x| \geq 1, |g(x)| \leq C|x|^q.$$

It is clear that if  $g$  satisfies the last part of the condition above for  $0 \leq q \leq 1$ , it would still satisfy it for any  $q > 1$ .

- The geometrical condition MGC imposed in  $(\mathbf{H}_3)$  can be readily reduced to the weaker and more general MGC introduced in [22] and called piecewise MGC in [4].

**Remark 2.3.5.** Note that (2.26) is an ISS-type estimate but it fails to be a strict one (in the sense of Definition 1.2.3) for two facts. First of all, the estimated quantity  $E_2$  is the norm of a trajectory in the space  $H_0^1(\Omega) \times L^2(\Omega)$  while the constant  $C_z$  depends on the initial condition by its norm in the smaller space  $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ . This difference seems unavoidable since in the undisturbed case exponential decay can be proved only for strong solutions as soon as the nonlinearity  $g$  is not assumed to be bounded below at infinity by a linear function. As a matter of fact, it would be interesting to prove that strong stability is the best convergence result one could get for weak solutions, let say with damping functions  $g$  of saturation type functions and in dimension at least two.

The second difference lies in the second term in (2.26), namely it is not just a  $\mathcal{K}$ -function of the norms of the disturbances. We can get such a result if we have an extra assumption on  $g$ , typically  $g$  of growth at most linear at infinity (i.e.,  $q = 1$ )



with bounded derivative (i.e.,  $m = 0$ ). In particular, this covers the case of regular saturation functions (increasing bounded functions  $g$  with bounded derivatives).

We give now the proof of the well-posedness part of Theorem (2.3.1).

**Proof of the well-posedness:** The argument is standard since  $-A$ , where  $A$  is defined in (2.24), is a maximal monotone operator on  $H_0^1(\Omega) \times L^2(\Omega)$  (cf. for instance [15] for a proof). We can apply Theorem 3.4 combined with Propositions 3.2 and Propositions 3.3 in [7] to (2.25), which immediately proves the results of the well-posedness part. ■

**Remark 2.3.6.** In [25], the domain of the operator has been chosen as

$$\{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) : -\Delta u + g(v) \in L^2(\Omega)\}.$$

However, in dimension two, taking the domain of  $A$  in the case where  $d = e = 0$  as  $\{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) : -\Delta u + a(x)g(v) \in L^2(\Omega)\}$  or as  $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$  is equivalent. Indeed, using the hypothesis given by (2.12), we have that  $|g(v)| \leq C|v|^q$  for  $|v| < 1$ , which means when combining it with the fact that  $g(0) = 0$  that  $|g(v)| \leq C|v|^q + C|v|$  for all  $v$ . From Gagliardo-Nirenberg theorem (see in Appendix) we have for  $v \in H_0^1(\Omega)$  that

$$\|v\|_{L^{2q}(\Omega)}^{2q} \leq C\|v\|_{H_0^1(\Omega)}^{2q-2}\|v\|_{L^2(\Omega)}^2,$$

which means that

$$\begin{aligned} \|g(v)\|_{L^2(\Omega)}^2 &= \int_{\Omega} |g(v)|^2 dx \leq C \int_{\Omega} (|v|^q + |v|)^2 dx \leq C\|v\|_{L^{2q}(\Omega)}^{2q} + C\|v\|_{L^2(\Omega)}^2 \\ &\leq \|v\|_{H_0^1(\Omega)}^{2q-2}\|v\|_{L^2(\Omega)}^2 + C\|v\|_{L^2(\Omega)}^2 < +\infty \quad (\text{since } v \in H_0^1(\Omega)), \end{aligned}$$

i.e.,  $g(v) \in L^2(\Omega)$ . Then, by using Lemma 2.4.2 (with  $(d, e) \equiv (0, 0)$ ), we have that  $-\Delta u + ag(v) \in L^2(\Omega)$ , which means that  $\Delta u \in L^2(\Omega)$ . On the other hand,  $\|\Delta u\|_{L^2(\Omega)}$  is an equivalent norm to the norm of  $H^2(\Omega) \cap H_0^1(\Omega)$  and  $\Omega$  is of class  $C^2$  (the proof is a direct result of Theorem 4 of Section 6.3 in [12]). We can finally conclude that  $\{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) : -\Delta u + a(x)g(v) \in L^2(\Omega)\}$  is nothing else but  $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ .

## 2.4 Proof of the energy estimate

To prove the energy estimate given by (2.26), we are going to use the multiplier method combined with a Gronwall lemma and other technical lemmas given in this section. We will be referring to [24] and [25] in several computations since our problem is a generalization of their strategy to the case where the disturbances  $(d, e)$  are present.

We start with the following lemma stating that the energy  $E_2$  is bounded along trajectories of Problem (2.1).

**Lemma 2.4.1.** Under the hypotheses of Theorem 2.3.1, the energy of a strong solution of Problem (2.1), satisfies

$$E_2'(t) = - \int_{\Omega} az_t g(z_t + d) dx - \int_{\Omega} z_t e dx, \quad \forall t \geq 0. \quad (2.27)$$

Furthermore, there exist positive constants  $C$  and  $C_{d,e}$  such that

$$E_2(T) \leq CE_2(S) + C_{d,e}, \quad \forall 0 \leq S \leq T. \quad (2.28)$$

**Proof of Lemma 2.4.1:** Equation (2.27) follows after multiplying the first equation of (2.1) by  $z_t$  and performing standard computations. Notice that we do not have the dissipation of  $E_2$  since the sign of  $E_2'$  is not necessarily constant. To achieve (2.28), we first write

$$- \int_{\Omega} az_t g(z_t + d) dx = - \int_{|z_t| \leq |d|} az_t g(z_t + d) dx - \int_{|z_t| > |d|} az_t g(z_t + d) dx. \quad (2.29)$$

On one hand, from (2.11) and the fact that  $(z_t + d)$  and  $z_t$  have the same sign if  $|z_t| > |d|$ , we deduce that

$$- \int_{|z_t| > |d|} az_t g(z_t + d) dx \leq 0. \quad (2.30)$$

On the other hand, since  $g$  is non-decreasing, has linear growth in a neighborhood of

zero by (2.11), and satisfies (2.12), it follows that

$$\begin{aligned}
-\int_{|z_t| \leq |d|} a z_t g(z_t + d) dx &\leq C \int_{|z_t| \leq |d|} |d| |g(|2d|)| dx \leq C \int_{\Omega} |d| |g(|2d|)| dx \\
&\leq C \int_{|d| < 1} |d| |g(2d)| dx + C \int_{|d| \geq 1} |d| |g(2d)| dx \\
&\leq C \int_{|d| < 1} |d|^2 dx + C \int_{|d| \geq 1} |d|^{q+1} dx \\
&\leq C \int_{\Omega} (|d|^2 + |d|^{2q}) dx. \tag{2.31}
\end{aligned}$$

Combining (2.29), (2.30), (2.31) and (2.27), we obtain that

$$E'_2 \leq C \int_{\Omega} (|d|^2 + |d|^{2q}) dx - \int_{\Omega} z_t e dx dt. \tag{2.32}$$

Using Cauchy-Schwarz inequality,

$$\begin{aligned}
E'_2 &\leq C \int_{\Omega} (|d|^2 + |d|^{2q}) dx + \left( \int_{\Omega} |e|^2 dx dt \right)^{\frac{1}{2}} \left( \int_{\Omega} |z_t|^2 dx \right)^{\frac{1}{2}} \\
&\leq C \int_{\Omega} (|d|^2 + |d|^{2q}) dx + C \|e\|_{L^2(\Omega)} \sqrt{E_2},
\end{aligned}$$

then integrating between two arbitrary non negative times  $S \leq T$ , we get

$$E_2(T) \leq E_2(S) + CC_1(d) + C \int_S^T \|e\|_{L^2(\Omega)} \sqrt{E_2} dt,$$

which allows us to apply Theorem 2.2.2 and conclude that

$$E_2(T) \leq CE_2(S) + CC_1(d) + C\bar{C}_2(e)^2 = CE_2(S) + C_{d,e}.$$

Hence, the proof of Lemma 2.4.1 is completed. ■

**Remark 2.4.1.** In the absence of disturbances, in other words when  $d = e = 0$  we have that:

$$E'_2(t) = - \int_{\Omega} a z_t g(z_t) dx, \quad \forall t \geq 0, \tag{2.33}$$

and thus the energy  $E_2$  is non increasing by using (2.11). That latter fact simplifies the proof of exponential decrease in this case.

We provide now an extension of Lemma 2 in [25] to the context of Problem (2.1).

**Lemma 2.4.2.** Under the hypotheses of Theorem 2.3.1, for every solution of Problem (2.1) with initial conditions  $(z_0, z_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ , there exist explicit positive constants  $C_z$  and  $C_{d,e}$  such that

$$\forall t \geq 0, \quad \left\| -\Delta z(t, \cdot) + a(\cdot)g(z_t(t, \cdot) + d(t, \cdot)) + e(t, \cdot) \right\|_{L^2(\Omega)}^2 + \|z_t(t, \cdot)\|_{H_0^1(\Omega)}^2 \leq C_z + C_{d,e}. \quad (2.34)$$

**Proof of Lemma 2.4.2:** We set  $w := z_t$ , where  $u$  is the strong solution of Problem (2.1). We know that  $w(t) \in H_0^1(\Omega)$  for every  $t \geq 0$ . Moreover, it is standard to show that  $w(t)$  satisfies in the distributional sense the following problem:

$$\begin{cases} w_{tt} - \Delta w + ag'(w + d)(w_t + d_t) + e_t = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ w = 0, & \text{on } \partial\Omega \times \mathbb{R}_+, \\ w(0) = z_1, \quad w_t(0) = \Delta z_0 - g(z_1 + d(0)) - e(0). \end{cases} \quad (2.35)$$

Set  $E_{2,w}(t)$  to be the energy of  $w$  for all  $t \geq 0$ . It is given by

$$E_{2,w}(t) = \frac{1}{2} \int_{\Omega} (w_t^2(t, x) + |\nabla w(t, x)|^2) dx.$$

Using  $w_t$  as a test function in (2.35), then performing standard computations, we derive

$$E_{2,w}(t) - E_{2,w}(0) = - \int_0^t \int_{\Omega} (ag'(w + d)(d_t + w_t)w_t + e_t w_t) dx d\tau. \quad (2.36)$$

Let  $I := \int_0^t \int_{\Omega} a(\cdot)g'(w + d)(d_t + w_t)w_t dx d\tau$ . We split the domain  $\Omega$  in  $I$  according to whether  $|d_t| \leq |w_t|$  or not. Clearly the part corresponding to  $|d_t| \leq |w_t|$  is non negative since  $g' \geq 0$ ,  $a \geq 0$  and  $(d_t + w_t)$  and  $w_t$  have the same sign. From (2.13), one has the immediate estimate

$$g'(a + b) \leq C(1 + |a + b|^m) \leq C(1 + |a|^m + |b|^m), \quad \forall a, b \in \mathbb{R}.$$

Using the above, we can rewrite (2.36) as

$$\begin{aligned} & E_{2,w}(t) - E_{2,w}(0) \\ & \leq \int_0^t \int_{|d_t| > |w_t|} ag'(w + d)(d_t + w_t)w_t dx d\tau + \int_0^t \int_{\Omega} |e_t||w_t| dx d\tau \\ & \leq C \int_0^t \int_{\Omega} g'(w + d)d_t^2 dx d\tau + C \int_0^t \|e_t\|_{L^2(\Omega)} \sqrt{E_{2,w}} d\tau \\ & \leq C \int_0^t \int_{\Omega} (1 + |w|^m + |d|^m)d_t^2 dx d\tau + C \int_0^t \|e_t\|_{L^2(\Omega)} \sqrt{E_{2,w}} d\tau. \end{aligned} \quad (2.37)$$

Using Hölder's inequality,

$$\int_0^t \int_{\Omega} |w|^m d_t^2 dx d\tau \leq \int_0^t \left( \int_{\Omega} |w|^{pm} dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |d_t|^{2p'} dx \right)^{\frac{1}{p'}} d\tau, \quad (2.38)$$

with  $p$  defined in (2.18) and  $p' > 1$  is its conjugate exponent given by  $\frac{1}{p} + \frac{1}{p'} = 1$ . Thanks to the assumptions on  $p$ , one can use Gagliardo-Nirenberg's inequality for  $w$  to get

$$\left( \int_{\Omega} |w(t, x)|^{pm} dx \right)^{\frac{1}{p}} \leq C E_{2,w}(t)^{\frac{m\theta}{2}} E(t)^{\frac{(1-\theta)m}{2}}, \quad t \geq 0, \quad (2.39)$$

where  $\theta = 1 - \frac{2}{mp}$ . Combining (2.39), (2.38) and (2.37), it follows that

$$\begin{aligned} E_{2,w}(t) - E_{2,w}(0) &\leq C \int_0^t E_{2,w}^{\frac{m\theta}{2}} E^{\frac{(1-\theta)m}{2}} \int_{\Omega} (|d_t|^{2p'} dx)^{\frac{1}{p'}} d\tau \\ &\quad + \int_0^t \int_{\Omega} (1 + |d|^m) d_t^2 dx d\tau + C \int_0^t \|e_t\|_{L^2(\Omega)} \sqrt{E_{2,w}} d\tau. \end{aligned} \quad (2.40)$$

Note that  $\frac{m\theta}{2} < 1$ . Setting  $h_1(t) = \int_{\Omega} (|d_t|^{2p'} dx)^{\frac{1}{p'}}$ ,  $h_2(t) = \|e_t\|_{L^2(\Omega)}$  and using (2.28), (2.40) becomes

$$\begin{aligned} E_{2,w}(t) &\leq E_{2,w}(0) + C_2(d) + C_3(d) + (C_z + C_{d,e}) \int_0^t E_{2,w}^{\frac{m\theta}{2}} h_1(s) ds \\ &\quad + C \int_0^t h_2(s) \sqrt{E_{2,w}} ds. \end{aligned} \quad (2.41)$$

We know that

$$\int_0^{\infty} h_1(t) dt = C_4(d) < \infty, \quad \int_0^{\infty} h_2(t) dt = \bar{C}_3(e) < \infty. \quad (2.42)$$

We can now apply Theorem 2.2.2 on (2.41) with

$$\begin{aligned} S &= 0, \quad T = t, \quad \alpha_1 = \frac{m\theta}{2}, \quad \alpha_2 = \frac{1}{2}, \quad F(\cdot) = E_{2,w}(\cdot), \quad C_3 = C_2(d) + C_3(d), \\ C_1 &= C_z + C_{d,e}, \quad C_2 = C. \end{aligned}$$

We obtain the following bound for  $E_{2,w}(\cdot)$ :

$$E_{2,w}(t) \leq \max \left( 2(E_{2,w}(0) + C_2(d) + C_3(d)), (2\tilde{C})^{\frac{1}{1-\alpha}} \right), \quad (2.43)$$

where  $\tilde{C} := C_1 \|h_1\|_1 + C_2 \|h_2\|_1$  and  $\alpha := \max(\alpha_1, \alpha_2)$  if  $2\tilde{C} \geq 1$  or  $\alpha := \min(\alpha_1, \alpha_2)$  if  $2\tilde{C} < 1$ .

It is clear that  $\tilde{C} = (C_z + C_{d,e})C_4(d) + C\bar{C}_3(e) \leq C_z + C_{d,e}$ . One then rewrites (2.43) as

$$E_{2,w}(t) \leq 2(E_{2,w}(0) + C_2(d) + C_3(d)) + (C_z + C_{d,e})^{\frac{1}{1-\alpha}}. \quad (2.44)$$

Note that for  $t \geq 0$  one obviously has that

$$\begin{aligned} E_{2,w}(t) &= \frac{1}{2} \int_{\Omega} (w_t^2(t, x) + |\nabla w(t, x)|^2) dx \\ &= \frac{1}{2} \left( \|z_{tt}(t, \cdot)\|_{L^2(\Omega)}^2 + \|z_t(t, \cdot)\|_{H_0^1(\Omega)}^2 \right) \\ &= \| -\Delta z(t, \cdot) + a(\cdot)g(z_t(t, \cdot) + d(t, \cdot)) + e(t, \cdot) \|_{L^2(\Omega)}^2 + \|z_t(t, \cdot)\|_{H_0^1(\Omega)}^2. \end{aligned}$$

The conclusion of the lemma follows since, by taking into account (2.12), it is clear that  $E_{2,w}(0) \leq C_z + C_{d,e}$ . ■

We next provide the following important estimate based on Gagliardo-Nirenberg theorem:

**Lemma 2.4.3.** For all  $q > 2$ , a strong solution  $u$  of Problem (2.1) satisfies

$$\|z_t(t, \cdot)\|_{L^q(\Omega)}^q \leq (C_z + C_{d,e})E_2(t), \quad t \geq 0. \quad (2.45)$$

**Proof of Lemma 2.4.3:** We derive immediately from (2.34) that  $\|z_t\|_{H_0^1(\Omega)} \leq C_z + C_{d,e}$ . Then, using Gagliardo-Nirenberg's theorem, it follows that, for every  $t \geq 0$ ,

$$\|z_t(t, \cdot)\|_{L^q(\Omega)}^q \leq C \|z_t(t, \cdot)\|_{H_0^1(\Omega)}^{q-2} \|z_t(t, \cdot)\|_{L^2(\Omega)}^2 \leq (C_z + C_{d,e})E_2(t). \quad (2.46)$$
■

We have all the tools now to start the proof of the second part of Theorem 2.3.1. The stability result will be achieved as a direct consequence of the following proposition:

**Proposition 2.4.1.** Suppose that the hypotheses of Theorem (2.3.1) are satisfied, then the energy  $E_2$  of the strong solution  $z$  of Problem (2.1) with  $(z_0, z_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ , satisfies the following estimate:

$$\int_S^T E_2(t) dt \leq (C_z + C)E_2(S) + (1 + C_z)C_{d,e}, \quad (2.47)$$

where the positive constant  $C_z$  depends only on the initial condition, the positive constant  $C_{d,e}$  depends only on the disturbances  $d$  and  $e$  respectively and  $C$  is a positive real constant.

### Proof of Proposition 2.4.1

We now embark on an argument for Proposition 2.4.1. It is based on the use of several multipliers that we will apply to the partial differential equation of (2.1). For that purpose, we need to define several functions associated with  $\Omega$ .

Let  $(z_0, z_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ ,  $S \leq T$  two non negative times and  $x_0 \in \mathbb{R}^2$  an observation point. Define  $\epsilon_0$ ,  $\epsilon_1$  and  $\epsilon_2$  three positive real constants such that  $\epsilon_0 < \epsilon_1 < \epsilon_2 < \epsilon$  where  $\epsilon$  is the same as defined in (2.16). Using  $\epsilon_i$ , we define  $Q_i$  for  $i = 0, 1, 2$  as  $Q_i = \mathcal{N}_{\epsilon_i}[\Gamma(x_0)]$ .

Since  $(\overline{\Omega \setminus Q_1}) \cap \overline{Q_0} = \emptyset$ , we are allowed to define a function  $\psi \in C_0^\infty(\mathbb{R}^2)$  such that

$$\begin{cases} 0 \leq \psi \leq 1, \\ \psi = 1 \text{ on } \bar{\Omega} \setminus Q_1, \\ \psi = 0 \text{ on } Q_0. \end{cases} \quad (2.48)$$

We also define the  $C^1$  vector field  $k$  on  $\Omega$  by

$$k(x) := \psi(x)(x - x_0). \quad (2.49)$$

When the context is clear, we will omit the arguments of  $k$ .

We use the multiplier  $M(z) := k \nabla z + \frac{z}{2}$  to deduce the following first estimate:

**Lemma 2.4.4.** Under the hypotheses of Proposition 2.4.1, we have the following

inequality:

$$\begin{aligned}
\int_S^T E_2 dt \leq & \underbrace{\left[ \int_{\Omega} z_t M(z) dx \right]_S^T}_{\mathbf{T}_1} + C \underbrace{\int_S^T \int_{\Omega \cap Q_1} |\nabla z|^2 dx dt}_{\mathbf{T}_2} \\
& + \underbrace{\left| \int_S^T \int_{\Omega} ag(z_t + d)M(z) dx dt \right|}_{\mathbf{T}_3} \\
& + \underbrace{\left| \int_S^T \int_{\Omega} eM(z) dx dt \right|}_{\mathbf{T}_4} + C \underbrace{\int_S^T \int_{\omega} z_t^2 dx dt}_{\mathbf{T}_5}, \tag{2.50}
\end{aligned}$$

where  $k$  is defined in (2.49) and  $M(z)$  is the multiplier given by  $k \cdot \nabla z + \frac{z}{2}$ .

**Proof of Lemma 2.4.4.** The proof is based on multiplying Problem (2.1) by the multiplier  $M(z)$  and integrating on  $[S, T] \times \Omega$ . Then, we follow the steps that led to the proof of equation (3.15) in [24] except that we take  $\sigma = 0$  and  $\phi(t) = t$  in the beginning and we replace  $\rho(x, z_t)$  by  $a(x)g(z_t + d) + e$ . ■

**Remark 2.4.2.** From now on, whenever we refer to a proof in [24], we refer to the steps of the proof with the change of  $\sigma = 0$  and  $\phi(t) = t$  as well as replacing  $\rho(x, z_t)$  by  $a(x)g(z_t + d) + e$ .

The goal now is to estimate the terms  $T_1$  to  $T_5$ .

**Lemma 2.4.5.** Under the hypotheses of Proposition 2.4.1, there exists a positive constant  $C$  such that

$$T_1 \leq CE(S) + C_{d,e}. \tag{2.51}$$

**Proof of Lemma 2.4.5:** Exactly as the proof of equation (5.14) in [24] except that we use (2.28) in the very last step since we do not have the non-increasing of the energy here. We obtain (2.51). ■

The estimation of  $T_2$  requires more work and it is given in the following lemma:



**Lemma 2.4.6.** Under the hypotheses of Proposition 2.4.1,  $T_2$  is estimated by

$$\begin{aligned} T_2 \leq & C\eta_0 \int_S^T E_2 dt + \frac{C}{\eta_0} \int_S^T \int_\omega z_t^2 dx dt + \frac{1}{\eta_0} (C + C_z + C_{d,e}) E_2(S) \\ & + \frac{1}{\eta_0^5} (C_{d,e}C_z + C_{d,e}), \end{aligned} \quad (2.52)$$

where  $0 < \eta_0 < 1$  is an arbitrary real positive number to be chosen later.

**Proof of Lemma 2.4.6:** The argument requires a new multiplier, namely  $\phi z$ , where the function  $\phi \in C_0^\infty(\mathbb{R}^2)$  is defined by

$$\begin{cases} 0 \leq \phi \leq 1, \\ \phi = 1 \text{ on } Q_1, \\ \phi = 0 \text{ on } \mathbb{R}^2 \setminus Q_2. \end{cases} \quad (2.53)$$

Such a function  $\phi$  exists since  $\overline{\mathbb{R}^2 \setminus Q_2} \cap \overline{Q_1} = \emptyset$ . Using the multiplier  $\phi z$  and following the steps in the proof of Lemma 9 in [24], yields the following identity:

$$\begin{aligned} \int_S^T \int_\Omega \phi |\nabla z|^2 dx dt = & \int_S^T \int_\Omega \phi |z_t|^2 dx dt + \frac{1}{2} \int_S^T \int_\Omega \Delta \phi z^2 dx dt - \left[ \int_\Omega \phi z z_t dx \right]_S^T \\ & - \int_S^T \int_\Omega \phi z [a(x)g(z_t + d) + e] dx dt. \end{aligned} \quad (2.54)$$

Combining the fact that  $\Delta \phi$  is bounded and the definition of  $\phi$ , we derive from (2.54) that

$$\begin{aligned} T_2 \leq & \int_S^T \int_{\Omega \cap Q_2} |z_t|^2 dx dt + \underbrace{\left[ \int_{\Omega \cap Q_2} z z_t dx \right]_S^T}_{S_1} + C \underbrace{\int_S^T \int_{\Omega \cap Q_2} z^2 dx dt}_{S_2} \\ & + \underbrace{\int_S^T \int_\Omega |z a g(z_t + d)| dx dt}_{S_3} + \underbrace{\int_S^T \int_\Omega |z e| dx dt}_{S_4}. \end{aligned} \quad (2.55)$$

First, note that the first term of (2.55) is upper bounded by  $\int_S^T \int_\omega |z_t|^2 dx dt$  since  $\Omega \cap Q_2 \subset \omega$ . Left to estimate the other terms in the right-hand side of (2.55). We start by treating  $S_1$ . We easily get the following estimate by using Young and Poincaré inequalities:

$$\int_{\Omega \cap Q_2} |z z_t| dx \leq \frac{1}{2} \int_{\Omega \cap Q_2} |z|^2 dx + \frac{1}{2} \int_{\Omega \cap Q_2} |z_t|^2 dx \leq C E_2. \quad (2.56)$$

Using (2.28) with (2.56) we obtain the estimation of  $S_1$  given by

$$S_1 \leq CE_2(S) + C_{d,e}. \quad (2.57)$$

To estimate  $S_2$ , we introduce the last multiplier in what follows:

Since  $(\overline{\Omega \setminus \omega}) \cap (\overline{Q_2 \cap \Omega}) = \emptyset$ , there exists a function  $\beta \in C_0^\infty(\mathbb{R}^2)$  such that

$$\begin{cases} 0 \leq \beta \leq 1, \\ \beta = 1 \quad \text{on } Q_2 \cap \Omega, \\ \beta = 0 \quad \text{on } \Omega \setminus \omega. \end{cases} \quad (2.58)$$

For every  $t \geq 0$ , let  $v$  be the solution of the following elliptic problem:

$$\begin{cases} \Delta v = \beta z & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.59)$$

One can prove the following lemma:

**Lemma 2.4.7.** Under the hypotheses of Proposition 2.4.1 with  $v$  as defined in (2.59), it holds that

$$\begin{aligned} \|v\|_{L^2(\Omega)} &\leq C\|z\|_{L^2(\Omega)}, \quad \|v_t\|_{L^2(\Omega)}^2 \leq C \int_{\omega} |z_t|^2 dx, \quad \|\nabla v\|_{L^2(\Omega)} \leq C\|\nabla z\|_{L^2(\Omega)}, \quad (2.60) \\ \forall S \leq T \in \mathbb{R}_+, \quad &\int_S^T \int_{\Omega} \beta z^2 dx dt = \left[ \int_{\Omega} v z_t dx \right]_S^T \\ &+ \int_S^T \int_{\Omega} (-v_t z_t + v [ag(z_t + d) + e]) dx dt. \quad (2.61) \end{aligned}$$

**Proof of Lemma 2.4.7:** Equation 2.60 gathers standard elliptic estimates from the definition of  $v$  as a solution of (2.59) while (2.61) is obtained by using  $v$  as a multiplier for Problem (2.1). Steps of the proof are similar to the ones that led to equations (5.22), (5.25) and (5.26) in [24].

■

Since the non negative  $\beta$  is equal to 1 on  $Q_2$  and 0 on  $\mathbb{R}^2 \setminus \omega$ , it follows from (2.61) that

$$S_2 \leq \underbrace{\left[ \int_{\Omega} v z_t dx \right]_S^T}_{U_1} - \underbrace{\int_S^T \int_{\Omega} v_t z_t dx dt}_{U_2} + \underbrace{\int_S^T \int_{\Omega} v (ag(z_t + d) + e) dx dt}_{U_3}. \quad (2.62)$$

We estimate  $U_1$ ,  $U_2$  and  $U_3$ . We start by handling  $U_1$ . One has from Cauchy-Schwarz inequality, then (2.60) and Poincaré inequality that

$$\left| \int_{\Omega} v z_t dx \right| \leq \|v\|_{L^2(\Omega)} \|z_t\|_{L^2(\Omega)} \leq C \|\nabla z\|_{L^2(\Omega)} \|z_t\|_{L^2(\Omega)} \leq C E_2(t). \quad (2.63)$$

Using (2.63) and the fact that  $E$  is non-increasing, it is then immediate to derive that

$$|U_1| = \left| \left( \int_{\Omega} v z_t dx \right) (T) - \left( \int_{\Omega} v z_t dx \right) (S) \right| \leq C(E_2(T) + E_2(S)). \quad (2.64)$$

Finally, using (2.28) in (2.64), we obtain that

$$U_1 \leq C E_2(S) + C_{d,e}. \quad (2.65)$$

As for  $U_2$ , the use of Young inequality with an arbitrary real number  $0 < \eta_0 < 1$  yields

$$|U_2| \leq \int_S^T \int_{\Omega} \frac{1}{2\eta_0} |v_t|^2 dx dt + \int_S^T \int_{\Omega} \frac{\eta_0}{2} |z_t|^2 dx dt.$$

Then, we use (2.60) and the fact that  $0 \leq \beta \leq 1$  to conclude the following estimate:

$$U_2 \leq \frac{C}{\eta_0} \int_S^T \int_{\omega} u_t^2 dx dt + C\eta_0 \int_S^T E_2 dx dt, \quad (2.66)$$

where  $\eta_0$  is a positive real number to be chosen later.

Left to estimate  $U_3$ . We can rewrite it as the following:

$$\begin{aligned} U_3 = & \underbrace{\int_S^T \int_{|u_t+d|\leq 1} a(x) v g(z_t + d) dx dt}_{V_1} + \underbrace{\int_S^T \int_{|z_t+d|>1} a(x) v g(z_t + d) dx dt}_{V_2} \\ & + \underbrace{\int_S^T \int_{\Omega} a(x) z e dx dt}_{V_3}. \end{aligned} \quad (2.67)$$

We estimate the three terms  $V_1$ ,  $V_2$  and  $V_3$ . We start by estimating  $V_1$ . We have using Young inequality that

$$V_1 \leq C\eta_0 \int_S^T E_2 dt + \frac{1}{\eta_0} \int_S^T \int_{|z_t+d|\leq 1} |a g(z_t + d)|^2 dx dt. \quad (2.68)$$

The fact that  $g(0) = 0$  implies the existence of a constant  $C > 0$  such that  $|g(x)| \leq C|x|$

for all  $|x| \leq 1$ . Combining it with the fact that  $g(x)x \geq 0$ ,  $\forall x \in \mathbb{R}$ , it follows that

$$\begin{aligned} \int_S^T \int_{|z_t+d| \leq 1} |ag(z_t+d)|^2 dx dt &\leq \int_S^T \int_{|z_t+d| \leq 1} a(\cdot)(z_t+d)g(z_t+d) dx dt \\ &\leq \int_S^T \int_{\Omega} a(\cdot)(z_t+d)g(z_t+d) dx dt. \end{aligned} \quad (2.69)$$

Using (2.27) and Young inequality with  $0 < \eta_1 < 1$ ,

$$\begin{aligned} &\int_S^T \int_{\Omega} a(\cdot)(z_t+d)g(z_t+d) dx dt \\ &= \int_S^T \int_{\Omega} az_t g(z_t+d) dx dt + \int_S^T \int_{\Omega} adg(z_t+d) dx dt \\ &\leq \int_S^T \int_{\Omega} az_t g(z_t+d) dx dt + \int_S^T \int_{\Omega} z_t e dx dt - \int_S^T \int_{\Omega} z_t e dx dt \\ &\quad + C \int_S^T \int_{\Omega} |d||g(z_t+d)| dx dt \\ &\leq \int_S^T (-E_2') dt + \int_S^T \int_{\Omega} |z_t||e| dx dt + C \int_S^T \int_{\Omega} |d||g(z_t+d)| dx dt \\ &\leq E_2(S) + C\eta_1 \int_S^T E_2 dt + \frac{C}{\eta_1} \int_S^T \int_{\Omega} |e|^2 dx dt \\ &\quad + C \int_S^T \int_{\Omega} |d||g(z_t+d)| dx dt \\ &\leq E_2(S) + C\eta_1 \int_S^T E_2 dt + \frac{C}{\eta_1} \bar{C}_1(e) + C \int_S^T \int_{\Omega} |d||g(z_t+d)| dx dt. \end{aligned} \quad (2.70)$$

Left to estimate  $\int_S^T \int_{\Omega} |d||g(z_t+d)| dx dt$ , we proceed as the following:

$$\begin{aligned} &\int_S^T \int_{\Omega} |d||g(z_t+d)| dx dt \\ &= \int_S^T \int_{|z_t+d| \leq 1} |d||g(z_t+d)| dx dt + \int_S^T \int_{|z_t+d| > 1} |d||g(z_t+d)| dx dt \\ &\leq C \int_S^T \int_{|z_t+d| \leq 1} |d| dx dt + \frac{C}{\eta_1'} \int_S^T \int_{|z_t+d| > 1} |d|^2 dx dt \\ &\quad + \eta_1' \int_S^T \int_{|z_t+d| > 1} |g(z_t+d)|^2 dx dt \\ &\leq CC_6(d) + \frac{C}{\eta_1'} C_1(d) + C\eta_1' \int_S^T \int_{|z_t+d| > 1} |z_t+d|^{2q} dx dt \\ &\leq CC_6(d) + \frac{C}{\eta_1'} C_1(d) + C\eta_1' \int_S^T \int_{\Omega} |z_t|^{2q} + C\eta_1' \int_S^T \int_{\Omega} |d|^{2q} dx dt, \end{aligned} \quad (2.71)$$

where  $0 < \eta'_1 < 1$ . Then, using (2.45),

$$\begin{aligned} & \int_S^T \int_{\Omega} |d| |g(z_t + d)| \, dx \, dt \\ & \leq CC_6(d) + \frac{C}{\eta'_1} C_1(d) + \eta'_1 (C_z + C_{d,e}) \int_S^T E_2(t) \, dt + C\eta'_1 C_1(d) \\ & \leq \frac{1}{\eta'_1} C_{d,e} + \eta'_1 (C_z + C_{d,e}) \int_S^T E_2(t) \, dt. \end{aligned} \quad (2.72)$$

Combining (2.70) and (2.72),

$$\begin{aligned} & \int_S^T \int_{\Omega} a(\cdot)(z_t + d)g(z_t + d) \, dx \, dt \\ & \leq E_2(S) + (\eta_1 + \eta'_1 (C_z + C_{d,e})) \int_S^T E_2 \, dt + \frac{1}{\eta_1 \eta'_1} C_{d,e}. \end{aligned} \quad (2.73)$$

Combining now (2.73), (2.70) and (2.68), we obtain that

$$V_1 \leq C \left( \eta_0 + \frac{\eta'_1}{\eta_0} (C_z + C_{d,e}) + \frac{\eta_1}{\eta_0} \right) \int_S^T E_2 \, dt + \frac{C}{\eta_0} E_2(S) + \frac{1}{\eta_1 \eta_0 \eta'_1} C_{d,e}.$$

We take  $\eta_1 = \eta_0^2$  and  $\eta'_1 = \frac{\eta_0^2}{C_z + C_{d,e}}$  if  $C_z + C_{d,e} > 0$ . In that case,  $V_1$  would be estimated by

$$V_1 \leq C\eta_0 \int_S^T E_2 \, dt + \frac{C}{\eta_0} E_2(S) + \frac{1}{\eta_0^5} C_{d,e} (C_z + C_{d,e}). \quad (2.74)$$

If  $C_z = C_{d,e} = 0$ , the above equation holds true trivially.

**Remark 2.4.3.** With such a choice of  $\eta_1$  and  $\eta'_1$ , we have the following useful estimate obtained from (2.73):

$$\int_S^T \int_{\Omega} a(\cdot)(z_t + d)g(z_t + d) \, dx \, dt \leq E_2(S) + C\eta_0^2 \int_S^T E_2 \, dt + \frac{1}{\eta_0^4} (C_{d,e} C_z + C_{d,e}). \quad (2.75)$$

To estimate  $V_2$ , first notice that from Rellich-Kondrachov's theorem in dimension two (cf. [8]) that  $H^1(\Omega) \subset L^{q+1}(\Omega)$ , which means that  $\exists C > 0$  such that  $\|v\|_{L^{q+1}(\Omega)} \leq C\|v\|_{H^1(\Omega)}$ , adding to that the fact that  $v \in H_0^1(\Omega)$  and (2.60), it holds that

$$\|v\|_{L^{q+1}(\Omega)} \leq C\sqrt{E_2}. \quad (2.76)$$

Then, using Hölder inequality yields

$$V_2 \leq \int_S^T \left( \int_{|z_t+d|>1} (a|g(z_t+d)|)^{\frac{q+1}{q}} dx \right)^{\frac{q}{q+1}} \left( \int_{|z_t+d|>1} |v|^{q+1} dx \right)^{\frac{1}{q+1}} dt. \quad (2.77)$$

Combining (2.77) with the hypothesis given by (2.12), we get that

$$V_2 \leq C \int_S^T \left( \int_{|z_t+d|>1} a|z_t+d||g(z_t+d)| dx \right)^{\frac{q}{q+1}} \left( \int_{|z_t+d|>1} |v|^{q+1} dx \right)^{\frac{1}{q+1}} dt.$$

Using Young inequality for an arbitrary  $0 < \eta_2 < 1$ ,

$$\begin{aligned} V_2 &\leq C \int_S^T \left( \frac{1}{\eta_2^{\frac{q+1}{q}}} \int_{|z_t+d|>1} a(x)(z_t+d)g(z_t+d) dx + \eta_2^{q+1} \int_{\Omega} |v|^{q+1} dx \right) dt \\ &\leq C \int_S^T \left( \frac{1}{\eta_2^{\frac{q+1}{q}}} \int_{\Omega} a(x)(z_t+d)g(z_t+d) dx + \eta_2^{q+1} \int_{\Omega} |v|^{q+1} dx \right) dt \\ &\leq C \int_S^T \left( \frac{1}{\eta_2^{\frac{q+1}{q}}} \int_{\Omega} a(x)z_t g(z_t+d) dx + \frac{C}{\eta_2^{\frac{q+1}{q}}} \int_{\Omega} |d||g(z_t+d)| dx \right) dt \\ &\quad + C\eta_2^{q+1} \int_S^T \int_{\Omega} |v|^{q+1} dx dt. \end{aligned} \quad (2.78)$$

The previous inequality combined with (2.27) and (2.76) implies that

$$V_2 \leq C \int_S^T \left( \frac{1}{\eta_2^{\frac{q+1}{q}}} (-E_2') - \frac{1}{\eta_2^{\frac{q+1}{q}}} \int_{\Omega} z_t e dx + \frac{C}{\eta_2^{\frac{q+1}{q}}} \int_{\Omega} |d||g(z_t+d)| dx + \eta_2^{q+1} E_2^{\frac{q+1}{2}} \right) dt.$$

Then, using (2.28),  $E_2$  satisfies

$$\begin{aligned} \int_S^T E_2^{\frac{q+1}{2}} dt &= \int_S^T E_2^{\frac{q-1}{2}} E_2 dt \\ &\leq (CE_2(0) + C_{d,e})^{\frac{q-1}{2}} \int_S^T E_2 dt \\ &\leq (C_z + C_{d,e}) \int_S^T E_2 dt, \end{aligned} \quad (2.79)$$

which gives that

$$\begin{aligned} V_2 &\leq \frac{C}{\eta_2^{\frac{q+1}{q}}} E_2(S) + \eta_2^{q+1} (C_z + C_{d,e}) \int_S^T E_2 dt \\ &\quad + \frac{C}{\eta_2^{\frac{q+1}{q}}} \int_S^T \left( - \int_{\Omega} z_t e dx + \int_{\Omega} |d| |g(z_t + d)| dx \right) dt. \end{aligned}$$

We fix  $\eta_2 = \left( \frac{\eta_0}{C_z + C_{d,e}} \right)^{\frac{1}{q+1}}$ . It follows that

$$\begin{aligned} \eta_2^{q+1} (C_z + C_{d,e}) &= \eta_0, \\ \frac{C}{\eta_2^{\frac{q+1}{q}}} &= C \frac{(C_z + C_{d,e})^{\frac{1}{q}}}{\eta_0^{\frac{1}{q}}} \leq \frac{C}{\eta_0} (C_z^{\frac{1}{q}} + C_{d,e}^{\frac{1}{q}}) = \frac{1}{\eta_0^{\frac{1}{q}}} (C_z + C_{d,e}), \end{aligned}$$

which leads to

$$\begin{aligned} V_2 &\leq \frac{1}{\eta_0^{\frac{1}{q}}} (C_z + C_{d,e}) E_2(S) + \eta_0 \int_S^T E_2 dt \\ &\quad + \frac{1}{\eta_0^{\frac{1}{q}}} (C_z + C_{d,e}) \int_S^T \left( - \int_{\Omega} z_t e dx + \int_{\Omega} |d| |g(z_t + d)| dx \right) dt. \end{aligned} \quad (2.80)$$

To finish the estimation of  $V_2$ , we still have to handle the last two integral terms in (2.80).

On one hand, we have already estimated the term  $\int_S^T \int_{\Omega} |d| |g(z_t + d)| dx dt$  in (2.72).

We have immediately for some  $0 < \eta_3 < 1$  that

$$\begin{aligned} &(C_z + C_{d,e}) \int_S^T \int_{\Omega} |d| |g(z_t + d)| dx dt \\ &\leq \eta_3 (C_z + C_{d,e}) \int_S^T E_2 dt + \frac{1}{\eta_3} (C_{d,e} C_z + C_{d,e}). \end{aligned} \quad (2.81)$$

Choosing  $\eta_3$  to be equal to  $\frac{\eta_0^{\frac{q+1}{q}}}{C_z + C_{d,e}}$  implies that

$$\begin{aligned} \eta_3 (C_z + C_{d,e}) &= \eta_0^{\frac{q+1}{q}}, \\ \frac{1}{\eta_3} (C_{d,e} C_z + C_{d,e}) &\leq \frac{1}{\eta_0^{\frac{q+1}{q}}} (C_{d,e} C_z + C_{d,e}), \end{aligned}$$

which gives that

$$(C_z + C_{d,e}) \int_S^T \int_{\Omega} |d| |g(z_t + d)| dx dt \leq \eta_0^{\frac{q+1}{q}} \int_S^T E_2 dt + \frac{1}{\eta_0^{\frac{q+1}{q}}} (C_{d,e} C_z + C_{d,e}). \quad (2.82)$$

On the other hand, we have for  $0 < \eta_4 < 1$  that

$$(C_z + C_{d,e}) \int_S^T \int_{\Omega} z_t e dx dt \leq \eta_4 (C_z + C_{d,e}) \int_S^T E_2 dt + \frac{1}{\eta_4} (C_{d,e} C_z + C_{d,e}).$$

Using the same concept as before, we fix  $\eta_4 = \frac{\eta_0^{\frac{q+1}{q}}}{C_z + C_{d,e}}$ , we obtain that

$$(C_z + C_{d,e}) \int_S^T \int_{\Omega} z_t e dx dt \leq \eta_0^{\frac{q+1}{q}} \int_S^T E_2 dt + \frac{1}{\eta_0^{\frac{q+1}{q}}} (C_{d,e} C_z + C_{d,e}), \quad (2.83)$$

Combining (2.80), (2.82) and (2.83), we conclude that the estimation of  $V_2$  is given by

$$V_2 \leq \frac{1}{\eta_0^{\frac{q}{q+1}}} (C_z + C_{d,e}) E_2(S) + \eta_0 \int_S^T E_2 dt + \frac{1}{\eta_0^{\frac{q+2}{q}}} (C_{d,e} C_z + C_{d,e}). \quad (2.84)$$

As for  $V_3$ , we simply have when using (2.60) and Young inequality with  $\eta_0$  that

$$V_3 \leq C \eta_0 \int_S^T E_2 dt + \frac{C}{\eta_0} \bar{C}_1(e),$$

which means that

$$V_3 \leq C \eta_0 \int_S^T E_2 dt + \frac{1}{\eta_0} C_{d,e}. \quad (2.85)$$

To achieve an estimation of  $S_2$ , we just combine (2.65), (2.66) (2.74), (2.84) and (2.85) to get

$$\begin{aligned} S_2 &\leq C \eta_0 \int_S^T E_2 dt + \frac{C}{\eta_0} \int_S^T \int_{\omega} z_t^2 dx dt + \left( C + \frac{C}{\eta_0} + \frac{1}{\eta_0^{\frac{q}{q+1}}} (C_z + C_{d,e}) \right) E_2(S) \\ &\quad + \left( \frac{1}{\eta_0^{\frac{5}{q}}} + \frac{1}{\eta_0^{\frac{q+2}{q}}} \right) (C_{d,e} C_z + C_{d,e}) + \left( \frac{1}{\eta_0} + 1 \right) C_{d,e}. \end{aligned} \quad (2.86)$$

We can simplify the previous estimate by using the fact that  $0 < \eta_0 < 1$ . As a result,



(2.86) becomes

$$\begin{aligned} S_2 \leq & C\eta_0 \int_S^T E_2 dt + \frac{C}{\eta_0} \int_S^T \int_\omega z_t^2 dx dt + \frac{1}{\eta_0} (C + C_z + C_{d,e}) E_2(S) \\ & + \frac{1}{\eta_0^5} (C_{d,e}C_z + C_{d,e}). \end{aligned} \quad (2.87)$$

Regarding  $S_3$ , we follow the same steps we followed to get  $V_1 + V_2$ . It is possible because  $z$  satisfies the same result (2.76) as  $v$  from before. Hence, we obtain that

$$S_3 \leq C\eta_0 \int_S^T E_2 dt + \frac{1}{\eta_0} (C + C_z + C_{d,e}) E_2(S) + \frac{1}{\eta_0^5} (C_{d,e}C_z + C_{d,e}). \quad (2.88)$$

Finally, to estimate  $S_4$ , we simply have when using young inequality that

$$S_4 \leq \eta_0 \int_S^T E_2 dt + \frac{1}{\eta_0} C_{d,e}. \quad (2.89)$$

We complete the estimate of  $T_2$  in (2.55) by combining the estimations of  $S_1, S_2, S_3$  and  $S_4$ . Hence the proof of Lemma 2.4.6 is completed. ■

An estimate of  $T_3$  is provided in the next lemma:

**Lemma 2.4.8.** Under the hypotheses of Proposition 2.4.1, we have the following estimate:

$$\begin{aligned} T_3 \leq & C\eta_0 \int_S^T E_2 dt + \frac{1}{\eta_0} [C + (1 + C_{\eta_0})(C_z + C_{d,e})] E_2(S) \\ & + \frac{1}{\eta_0^5} (C_{\eta_0}^3 + 1) (C_{d,e}C_z + C_{d,e}), \end{aligned} \quad (2.90)$$

where  $0 < \eta_0 < 1$  is a positive arbitrary real number to be chosen later and  $C_{\eta_0}$  is an implicit positive constant that depends on  $\eta_0$  only.

**Proof of Lemma 2.4.8:** First, note that

$$T_3 \leq \frac{1}{2} S_3 + \underbrace{\int_S^T \int_\Omega |ag(z_t + d)\nabla z \cdot k| dx dt}_X. \quad (2.91)$$

We have already estimated  $S_3$  in (2.88). It remains to deal with  $X$ . Using Young

inequality implies that

$$\begin{aligned} X &\leq \frac{C}{\eta_0} \int_S^T \int_{\Omega} (a|g(z_t + d)|)^2 dx dt + C\eta_0 \int_S^T \int_{\Omega} |\nabla z|^2 dx dt \\ &\leq \frac{C}{\eta_0} \int_S^T \int_{\Omega} a|g(z_t + d)|^2 dx dt + C\eta_0 \int_S^T E_2 dt. \end{aligned} \quad (2.92)$$

Now, set  $R_1 > 1$  to be chosen later. We can rewrite the term  $\int_S^T \int_{\Omega} a|g(u_t + d)|^2 dx dt$  as

$$\begin{aligned} \int_S^T \int_{\Omega} a|g(z_t + d)|^2 dx dt &= \underbrace{\int_S^T \int_{|z_t+d| \leq R_1} a|g(z_t + d)|^2 dx dt}_{Y_1} \\ &\quad + \underbrace{\int_S^T \int_{|z_t+d| > R_1} a|g(z_t + d)|^2 dx dt}_{Y_2}. \end{aligned} \quad (2.93)$$

Since  $g(0) = 0$ , it holds that  $|g(x)| \leq C_{R_1}|x|$  for some constant  $C_{R_1}$  and for  $|x| < R_1$ . Combine it with (2.75), it follows that  $Y_1$  satisfies for some  $0 < \eta_5 < 1$

$$\begin{aligned} Y_1 &\leq C_{R_1} \int_S^T \int_{|z_t+d| \leq R_1} |ag(z_t + d)||z_t + d| dx dt \\ &\leq C_{R_1} \int_S^T \int_{\Omega} |ag(z_t + d)||z_t + d| dx dt \\ &\leq C_{R_1} E_2(S) + CC_{R_1} \eta_5^2 \int_S^T E_2 dt + \frac{C_{R_1}}{\eta_5^4} (C_{d,e} C_z + C_{d,e}). \end{aligned} \quad (2.94)$$

Taking  $\eta_5 = \frac{\eta_0}{\sqrt{CC_{R_1}}}$  leads to

$$Y_1 \leq C_{R_1} E_2(S) + \eta_0^2 \int_S^T E_2 dt + \frac{C_{R_1}^3}{\eta_0^4} (C_{d,e} C_z + C_{d,e}) \quad (2.95)$$

As for  $Y_2$ , we use (2.12) to obtain that

$$\begin{aligned}
Y_2 &\leq C \int_S^T \int_{|z_t+d|>R_1} |z_t + d|^{2q} dx dt \\
&\leq C \int_S^T \int_{|z_t+d|>R_1} |z_t|^{2q} dx dt + C \int_S^T \int_{|z_t+d|>R_1} |d|^{2q} dx \\
&\leq C \int_S^T \int_{|z_t+d|>R_1} \frac{|z_t + d|}{R_1} |z_t|^{2q} dx dt + C \int_S^T \int_{\Omega} |d|^{2q} dx dt \\
&\leq C \int_S^T \int_{\Omega} \frac{|z_t|}{R_1} |z_t|^{2q} dx dt + C \int_S^T \int_{\Omega} \frac{|d|}{R_1} |z_t|^{2q} dx dt + CC_1(d) \\
&\leq \frac{C}{R_1} \int_S^T \int_{\Omega} |z_t|^{2q+1} dx dt + \frac{C}{R_1^2} \int_S^T \int_{\Omega} |z_t|^{4q} dx dt + C_{d,e}.
\end{aligned}$$

Then, we use Lemma 2.4.3 as well as the fact that  $R_1 > 1$  to conclude that  $Y_2$  satisfies

$$Y_2 \leq \frac{1}{R_1} (C_z + C_{d,e}) \int_S^T E_2 dt + C_{d,e}.$$

We take  $R_1 = \frac{(C_z + C_{d,e})}{\eta_0^2}$ , we get the simplified estimate

$$Y_2 \leq \eta_0^2 \int_S^T E_2 dt + C_{d,e}. \quad (2.96)$$

**Remark 2.4.4.** For such a choice of  $R_1$ , and based on how  $C_{R_1}$  is defined, we can assume that  $C_{R_1}$  in (2.95) is a constant of the type  $C_{\eta_0}(C_z + C_{d,e})$ , where  $C_{\eta_0}$  is a positive constant that depends on  $\eta_0$  only.

Combining (2.92), (2.93), (2.95) and (2.96) implies that

$$X \leq C\eta_0 \int_S^T E_2 dt + \frac{C_{\eta_0}}{\eta_0} (C_{d,e} + C_z) E_2(S) + \frac{C_{\eta_0}^3}{\eta_0^5} (C_{d,e} C_z + C_{d,e}) + \frac{C_{d,e}}{\eta_0}. \quad (2.97)$$

Finally, we combine (2.91) and (2.97) with the estimation of  $S_3$ , we obtain (2.90). ■

We next seek to prove the upper bound of  $T_4$  that is given by the following lemma

**Lemma 2.4.9.** Under the hypotheses of Proposition 2.4.1, the following estimate holds:

$$T_4 \leq C\eta_0 \int_S^T E_2 dt + \frac{C}{\eta_0} C_{d,e}, \quad (2.98)$$

where  $0 < \eta_0 < 1$  is a positive constant to be chosen later.

**Proof of Lemma 2.4.9:** We have that

$$T_4 \leq \frac{1}{2} \int_S^T \int_{\Omega} |ez| dx dt + \int_S^T \int_{\Omega} |e\nabla z.k| dx dt. \quad (2.99)$$

On one hand, using Young inequality gives that

$$\int_S^T \int_{\Omega} |ez| dx dt \leq \eta_0 \int_S^T E_2 dt + \frac{C}{\eta_0} C_{d,e}. \quad (2.100)$$

On the other hand, it gives that

$$\int_S^T \int_{\Omega} |e\nabla z.k| dx dt \leq \eta_0 \int_S^T E_2 dt + \frac{C}{\eta_0} C_{d,e} \quad (2.101)$$

Combining (2.99), (2.100) and (2.101), we prove (2.98). ■

It remains to handle the last term  $T_5$ .

**Lemma 2.4.10.** Under the hypotheses of Proposition 2.4.1, we have the following estimation:

$$T_5 \leq \eta_0 \int_S^T E_2 dt + \bar{C}_{\eta_0} (C_z + C_{d,e}) E_2(S) + \frac{\bar{C}_{\eta_0}^3}{\eta_0^2} (C_{d,e} C_z + C_{d,e}) + C_{d,e}, \quad (2.102)$$

where  $0 < \eta_0 < 1$  is a positive constant to be chosen later and  $\bar{C}_{\eta_0}$  is an implicit positive constant that depends on  $\eta_0$  only.

**Proof of Lemma 2.4.10:** For every  $R_2 > 1$ , we have that

$$\begin{aligned} T_5 &\leq \frac{1}{a_0} \int_S^T \int_{\omega} a(x) z_t^2 dx dt \leq C \int_S^T \int_{\Omega} a(x) (z_t + d)^2 dx dt + C \int_S^T \int_{\Omega} a(x) d^2 dx dt \\ &\leq C \underbrace{\int_S^T \int_{|z_t+d| \leq R_2} a(x) (z_t + d)^2 dx dt}_{Z_1} + C \underbrace{\int_S^T \int_{|z_t+d| > R_2} a(x) (z_t + d)^2 dx dt}_{Z_2} + CC_1(d). \end{aligned} \quad (2.103)$$

On one hand, since  $g'(0) > 0$ , there exists  $\alpha_{R_2} > 0$  such that  $|g(x)| \geq \alpha_{R_2}|x|$  for

$|x| \leq R_2$ . Combining that with (2.75) yields for some  $0 < \eta_6 < 1$

$$\begin{aligned}
Z_1 &\leq \int_S^T \int_{|z_t+d| \leq R_2} a(x)(z_t+d)g(z_t+d) \frac{(z_t+d)}{g(z_t+d)} dx dt \\
&\leq \frac{1}{\alpha_{R_2}} \int_S^T \int_{|z_t+d| \leq R_2} a(x)(z_t+d)g(z_t+d) dx dt \\
&\leq \frac{1}{\alpha_{R_2}} \int_S^T \int_{\Omega} a(x)(z_t+d)g(z_t+d) dx dt \\
&\leq \frac{1}{\alpha_{R_2}} E_2(S) + C \frac{1}{\alpha_{R_2}} \eta_6^2 \int_S^T E_2 dt + \frac{1}{\alpha_{R_2}} \frac{1}{\eta_6^4} (C_{d,e} C_z + C_{d,e}).
\end{aligned}$$

We choose  $\eta_6 = \sqrt{\frac{\alpha_{R_2}}{C} \eta_0}$ , we obtain that

$$Z_1 \leq \frac{1}{\alpha_{R_2}} E_2(S) + \eta_0 \int_S^T E_2 dt + \frac{1}{\alpha_{R_2}^3 \eta_0^2} (C_{d,e} C_z + C_{d,e}).$$

As for  $Z_2$ , we have that

$$\begin{aligned}
Z_2 &\leq C \int_S^T \int_{|z_t+d| > R_2} |z_t|^2 dx dt + C \int_S^T \int_{|z_t+d| > R_2} |d|^2 dx dt \\
&\leq C \int_S^T \int_{|z_t+d| > R_2} \frac{|z_t+d|}{R_2} |z_t|^2 dx dt + CC_1(d) \\
&\leq C \int_S^T \int_{|z_t+d| > R_2} \frac{|z_t|^3}{R_2} dx dt + C \int_S^T \int_{|z_t+d| > R_2} \frac{|z_t|^2 |d|}{R_2} dx dt + CC_1(d) \\
&\leq \frac{C}{R_2} \int_S^T \int_{|z_t+d| > R_2} |z_t|^3 dx dt + \frac{C}{R_2^2} \int_S^T \int_{|z_t+d| > R_2} |z_t|^4 dx dt + CC_1(d). \tag{2.104}
\end{aligned}$$

We use Lemma 2.4.3 and the fact that  $R_2 > 1$ , we derive the following:

$$\begin{aligned}
&\frac{C}{R_2} \int_S^T \int_{|z_t+d| > R_2} |z_t|^3 dx dt + \frac{C}{R_2^2} \int_S^T \int_{|z_t+d| > R_2} |z_t|^4 dx dt \\
&\leq \left( \frac{C_z + C_{d,e}}{R_2} \right) \int_S^T E_2 dt. \tag{2.105}
\end{aligned}$$

We choose  $R_2 = \frac{(C_z + C_{d,e})}{\eta_0}$  and we combine (2.104) and (2.105) we have that

$$Z_2 \leq \eta_0 \int_S^T E_2 dt + C_{d,e}. \tag{2.106}$$

**Remark 2.4.5.** For such a choice of  $R_2$ , and based on how  $\alpha_{R_2}$  is defined, we can assume that  $\frac{1}{\alpha_{R_2}}$  is also a constant of the type  $\bar{C}_{\eta_0} (C_z + C_{d,e})$ , where  $\bar{C}_{\eta_0}$  is a constant

that depends on  $\eta_0$  only. As a result,  $Z_1$  is estimated by

$$Z_1 \leq \eta_0 \int_S^T E_2 dt + \bar{C}_{\eta_0}(C_z + C_{d,e})E_2(S) + \frac{\bar{C}_{\eta_0}^3}{\eta_0^2}(C_{d,e}C_z + C_{d,e}). \quad (2.107)$$

Combining (2.103), (2.106) and (2.107) and using (2.18) and (2.21), it follows that

$$T_5 \leq \eta_0 \int_S^T E_2 dt + \bar{C}_{\eta_0}(C_z + C_{d,e})E_2(S) + \frac{\bar{C}_{\eta_0}^3}{\eta_0^2}(C_{d,e}C_z + C_{d,e}) + C_{d,e},$$

which proves Lemma 2.4.10. ■

The estimation of  $T_5$  gives a direct estimation of the term  $\frac{C}{\eta_0} \int_S^T \int_{\omega} z_t^2 dx dt$  left in the estimation of  $T_2$ . We can easily manage to have that

$$\begin{aligned} \frac{1}{\eta_0} \int_S^T \int_{\omega} z_t^2 dx dt &\leq \eta_0 \int_S^T E_2 dt + \frac{1}{\eta_0} \bar{C}_{\eta_0^2}(C_z + C_{d,e})E_2(S) \\ &\quad + \frac{1}{\eta_0} \frac{\bar{C}_{\eta_0^2}^3}{\eta_0^4}(C_{d,e}C_z + C_{d,e}) + C_{d,e}. \end{aligned} \quad (2.108)$$

It is obtained by following the same steps that led to the estimation of  $T_5$  with replacing  $\eta_0$  by  $\eta_0^2$ .

We can finally finish the proof of Proposition 2.4.1: we combine the estimations of  $T_i$ ,  $i = 1, 2, 3, 4, 5$ , which are given by (2.51), (2.52), (2.90), (2.98) and (2.102) with (2.50), then we choose  $\eta_0$  such that  $C\eta_0 < 1$ , which means that the term  $C\eta_0 \int_S^T E_2(t) dt$  gets absorbed by  $\int_S^T E_2(t) dt$ . Then we use the fact that  $C_{d,e}E_2(S) \leq C_{d,e}(E_2(0) + C_{d,e}) = C_{d,e}C_z + C_{d,e}$  and the fact that the choice of  $\eta_0$  will be a constant  $C$ , we obtain (2.47). ■

**Proof of the energy estimate of Theorem 2.3.1:** Using the key result given by (2.47), we get at once from Theorem 2.2.1 that (2.5) holds true with  $T = C + C_z$  and  $C_0 = (1 + C_z)C_{d,e}$ . Using (2.28) for  $t \geq 1$  with  $T = t$  and  $S \in [t - 1, t]$  and integrating it over  $[t - 1, t]$ , one gets that

$$E_2(t) \leq C \int_{t-1}^t E_2(s) ds + C_{d,e} \leq C \int_{t-1}^{\infty} E_2(s) ds + C_{d,e}.$$

Combining the above with (2.4) yields (2.26) for  $t \geq 1$ . In turn, (2.28) with  $T \in [0, 1]$  and  $S = 0$  provides (2.26) for  $t \leq 1$ . The proof of Theorem 2.3.1 is then completed.

■

# $L^p$ -Asymptotic stability of 1D damped wave equations with localized and linear damping

*The work of this chapter has been published in the scientific paper [19].*

## 3.1 Introduction

This chapter is concerned with the asymptotic stability of the one dimensional wave equation with a localized damping term and Dirichlet boundary conditions. It is the linear case of the general problem (1.1) in one dimension. The problem is written as follows

$$\begin{cases} z_{tt} - z_{xx} + a(x)z_t = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \\ z(t, 0) = z(t, 1) = 0 & t \geq 0, \\ z(0, \cdot) = z_0, \quad z_t(0, \cdot) = z_1, \end{cases} \quad (3.1)$$

where  $z$  is the solution of the problem,  $(z_0, z_1)$  are the initial conditions and they all belong to an  $L^p$ -based functional space that will be defined later. The function  $a$  is a continuous non-negative function on  $[0, 1]$ , bounded from below by a positive constant on some non-empty open interval  $\omega$  of  $(0, 1)$ , which represents the region of the domain where the damping term is active.

Problem (3.1) has been widely studied in the case  $p = 2$  whether with a linear or a non-linear damping. Stability results are proved under a geometric condition imposed on the damping domain  $\omega$ : it is properly introduced in the early work [32] where the



semi-linear problem is considered even in higher dimension and the geometric condition is extended and characterized in [22]. Moreover, for linear problems there exist necessary and sufficient geometrical conditions for stabilization based on geometric optics methods (cf. the seminal work [6]). Strong stabilization, i.e., energy decay to zero for each trajectory, has been established in [11] and [14] with a LaSalle's invariance argument. For the linear localized damping case in higher dimensions, exponential stability has been established several times using different tools, in particular using the multiplier method which is the relevant method to our chapter context. We refer the reader to [21] for a complete presentation of the method as well as the tools associated to it. As for the stability results obtained by this method in this case, we refer for instance to [4] and [24] for detailed proofs and extended references. The non-linear problem on the other hand has been studied (for instance) in [25] with no localization and in [18] for a localized damping. We refer the reader to the excellent survey [4] for more references in the Hilbertian framework i.e. when  $p = 2$ .

As for more general functional frameworks, in particular  $L^p$ -based spaces with  $p \neq 2$ , few results exist and one reason is probably due to the fact that, in such non-Hilbertian framework, the semi-group associated with the d'Alembertian (i.e., the linear operator defining the wave equation) is not defined in general as soon as the space dimension is larger than or equal to two, see e.g., [29]). This is why most of the existing results focus on several stabilization issues only in one spatial dimension. Well-posedness results as well as important  $L^p$  estimates have been shown in [14], in particular the introduction of a  $p$ -th energy of a solution as a generalization of the standard  $E_2(t) = \int_0^1 \frac{z_x^2 + z_t^2}{2}$ . Some of these results have been used in [5, 10] recently. The latter reference relies on Lyapunov techniques for linear time varying systems to prove  $L^p$  exponential stability in the nonlinear problem under the hypothesis that initial data live in  $L^\infty$  functional spaces and with  $p \geq 2$  only; other stability results have been shown in the same reference in particular  $L^\infty$  stability but always with more conditions on initial data which creates a difference between the norms of trajectories and the norms of initial data used in their decay estimates.

In this chapter we extend the results existing in the case  $p = 2$  to the case  $p \in (1, \infty)$  by adapting the multiplier method to that issue. We first start by stating the problem and defining the appropriate  $L^p$  functional framework as well as the notion of solutions. We prove the well-posedness of the corresponding  $C^0$  semi-group of solutions using an argument inspired by [16] and [10]. As for stability issue, we prove that these semi-groups are indeed exponentially stable. Even though the argument depends on whether  $p \geq 2$  or  $p \in (1, 2)$ , it is another instance of the multiplier method, where the multipliers are expressed in terms of the Riemann invariant coordinates  $\rho = z_x + z_t$

and  $\xi = z_x - z_t$ . In particular, one of the multipliers in the case  $p = 2$  is equal to  $\phi(x)z$  with  $\phi$  a non negative function which is used to localize estimates inside  $\omega$ . If  $p \geq 2$ , this multiplier is replaced by the pair of functions  $\phi(x)z|\rho|^{p-2}$  and  $\phi(x)z|\xi|^{p-2}$ . Clearly, such multipliers cannot be used directly if  $p \in (1, 2)$  and must be modified, which yields to a more delicate treatment. In both cases, energy integral estimates are established following the standard strategy of the multiplier method and exponential stability is proved. For the two extremes cases  $p = 1$  and  $p = \infty$ , we are able to prove that the corresponding semi-groups are exponentially stable only for particular cases of global constant damping. However, we conjecture that such a fact should be true in case of any localized damping.

The chapter is divided into four sections, the first one being the introduction and the second one devoted to provide the main notations used throughout the chapter. Section 3 deals with the well-posedness issue and Section 4 contains the main result of the chapter, i.e. exponential stability of the  $C^0$  semi-group of solutions for  $p \in (1, \infty)$  as well as the partial result for  $p = 1$  and  $p = \infty$ . We gather in an appendix several technical results.

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## 3.2 Preliminaries

**Lemma 3.2.1.** Let  $p \in (1, \infty)$ . Then, for every  $v \in W_0^{1,p}(0, 1)$ , it holds the following Poincaré inequality

$$\int_0^1 |v(x)|^p dx \leq \frac{1}{p2^p} \int_0^1 |v'(x)|^p dx. \quad (3.2)$$

**Proof.** For  $x \in [0, \frac{1}{2}]$ , we have after using Hölder inequality that

$$|v(x)|^p = \left( \left| \int_0^x v'(s) ds \right| \right)^p \leq x^{\frac{p}{q}} \int_0^{\frac{1}{2}} |v'(x)|^p dx. \quad (3.3)$$

After integrating the previous between 0 and  $\frac{1}{2}$ , one gets

$$\int_0^{\frac{1}{2}} |v(x)|^p dx \leq \frac{1}{p2^p} \int_0^{\frac{1}{2}} |v'(x)|^p dx. \quad (3.4)$$

For  $x \in [\frac{1}{2}, 1]$ , we start from  $|v(x)| = |\int_1^x v'(s) ds|$  and get

$$\int_{\frac{1}{2}}^1 |v(x)|^p dx \leq \frac{1}{p2^p} \int_{\frac{1}{2}}^1 |v'(x)|^p dx, \quad (3.5)$$

and finally deduce (3.2) by adding (3.4) and (3.5). ■

The next lemma states a technical result used several times in the thesis. But before we state it, we need to define some notations and functions that will be used all along the the next chapters.

For  $r \geq 0$ , we introduce the following notation

$$\lfloor x \rfloor^r := \operatorname{sgn}(x)|x|^r, \quad \forall x \in \mathbb{R}, \quad (3.6)$$

where  $\operatorname{sgn}(x) = \frac{x}{|x|}$  for nonzero  $x \in \mathbb{R}$  and  $\operatorname{sgn}(0) = [-1, 1]$ . We have the following obvious formulas which will be repeatedly used later on:

$$\frac{d}{dx}(\lfloor x \rfloor^r) = r|x|^{r-1}, \quad \forall r \geq 1, x \in \mathbb{R}, \quad (3.7)$$

$$\frac{d}{dx}(|x|^r) = r\lfloor x \rfloor^{r-1}, \quad \forall r > 1, x \in \mathbb{R}. \quad (3.8)$$

We also introduce the function  $f$  by

$$f(s) = \lfloor s \rfloor^{p-1}, \quad \forall s \in \mathbb{R}. \quad (3.9)$$

and the function  $F(s) = \int_0^s f(\tau) d\tau$ , we have that

$$F(s) = \frac{|s|^p}{p}, \quad F' = f, \quad f'(s) = (p-1)|s|^{p-2}. \quad (3.10)$$

**Lemma 3.2.2.** For  $p > 1$ , there exists a positive constant  $C_p$  such that, for every real numbers  $a, b$  and  $\mu \in (0, 1)$  subject to  $|a - b| \geq \max(|a|, |b|)\mu$ , one has

$$F(a - b) \leq \frac{C_p}{\mu^{2-p}}(a - b)(f(a) - f(b)). \quad (3.11)$$

**Proof.** With no loss of generality, we can assume that  $\max(|a|, |b|) = |a| = R > |b|$  and have same sign. Indeed, if  $ab \leq 0$ , then  $|a - b|^p \leq 2^p R^p$  and  $(a - b)(f(a) - f(b)) \geq R^p$ , hence (3.11) is satisfied with  $C_p \geq 2^p$ . Set then  $\epsilon = 1 - \frac{b}{a}$  and  $\epsilon \in (0, 1)$ . Proving

(3.11) amounts to show that there exists  $C_p$  such that for every  $\epsilon, \mu \in (0, 1)$  with  $\epsilon \geq \mu$ , it holds

$$\epsilon^{p-1} \leq \frac{C_p}{\mu^{2-p}} |1 - (1 - \epsilon)^{p-1}|. \quad (3.12)$$

Clearly the inequality holds for  $\epsilon$  “far away” from zero for any  $C_p$  large enough (w.r.t. one) and hence it is enough to establish it for  $\epsilon$  close to zero. By linearizing  $(1 - \epsilon)^{p-1}$ , one must find  $C_p$  so that  $\epsilon^{p-2} \leq \frac{C_p}{\mu^{2-p}}$  which indeed holds true.

■

We remind the reader of the following classical result that is very useful to obtain important estimations.

**Lemma 3.2.3** ([17]). (*Fenchel’s inequality*)

Let  $a, b$  be two real numbers, and  $f$  any function then it holds that

$$|ab| \leq f(|a|) + f^*(|b|), \quad (3.13)$$

where  $f^*$  is the convex conjugate of  $f$  defined by the Legendre transform as follows

$$f^*(b) = \sup_{x \in \mathbb{R}} \{b \cdot x - f(x)\}, \quad b \in \mathbb{R}. \quad (3.14)$$

Moreover, if  $f$  is of class  $C^2$ , the derivative of  $f^*$  is given by

$$[f^*]'(y) = [f']^{-1}(y), \quad y \in \mathbb{R}. \quad (3.15)$$

Note that Young’s inequality is a particular case of Fenchel’s inequality, corresponding to the function  $f(a) = \frac{|a|^p}{p}$  for  $p > 1$ .

To be able to achieve some important estimations we consider, for  $p \in (1, 2)$ , the functions  $h$  and  $H$  defined on  $\mathbb{R}$ , by

$$h(y) = (p - 1) \int_0^y (|s| + 1)^{p-2} ds = \operatorname{sgn}(y) [(|y| + 1)^{p-1} - 1], \quad (3.16)$$

$$H(y) = \int_0^y h(s) ds = \frac{1}{p} [(|y| + 1)^p - 1] - |y|. \quad (3.17)$$

One can see the functions  $h$  and  $H$  as sort of a perturbation of  $f$  and  $F$ , this perturbation is assuring that the new functions are well defined around zero when  $p \in (1, 2)$ .

It will also allow us to generalize the multipliers of the case  $p \geq 0$  to the case  $p \in (1, 2)$ .

Consider the convex conjugate (cf. [17]) of  $H$  which we denote from now on by  $\mathcal{H}$ , it is defined as the Legendre transform of  $H$ , i.e.,

$$\mathcal{H}(s) := \sup_{y \in \mathbb{R}} \{sy - H(y)\}, \quad s \in \mathbb{R}. \quad (3.18)$$

Since  $H$  is of class  $C^2$  with invertible first derivative  $h$ , one has that

$$\mathcal{H}(h(x)) = \int_0^{h(x)} h^{-1}(s) ds = \int_0^x v h'(v) dv, \quad (3.19)$$

and

$$xh(x) = H(x) + \mathcal{H}(h(x)), \quad x \in \mathbb{R}. \quad (3.20)$$

The second equality in (3.19) is obtained using the change of variable  $v = h^{-1}(s)$  and (3.20) follows (for instance) by integration by part of the right-hand side of (3.20).

We are ready now and have all the tools to state some crucial lemmas for our subsequent work.

**Lemma 3.2.4.** For every  $p \in (1, 2)$ ,

- the function  $h$  is an odd increasing bijection from  $\mathbb{R}$  to  $\mathbb{R}$  with a continuous first derivative;
- the function  $H$  is even, of class  $C^2$  and strictly convex.

*Proof.* One has that, for every  $x \in \mathbb{R}$ ,

$$H''(x) = h'(x) = (p-1)(|x|+1)^{p-2} > 0. \quad (3.21)$$

It is clear that  $H''$  is continuous and positive, which proves the strict convexity. Also,  $h'$  being positive means that  $h$  is increasing on  $\mathbb{R}$  which gives the bijection. ■

**Lemma 3.2.5.** Let  $p \in (1, 2)$ . Then, the function  $h$ ,  $H$  and  $\mathcal{H}$  defined in (3.16), (3.17) and (3.18) satisfy the following relations:

(i) for every  $x \in \mathbb{R}$ , one has

$$xh(x) = H(x) + \mathcal{H}(h(x)). \quad (3.22)$$

(ii) for every  $x \in \mathbb{R}$ , it holds that

$$\frac{1}{2} x h(x) \leq H(x) \leq x h(x). \quad (3.23)$$

(iii) There exists a positive constant  $C_p$  only depending on  $p$  such that, for every  $x \in \mathbb{R}$ , one has

$$C_p x h(x) \leq \mathcal{H}(h(x)) \leq C_p x h(x). \quad (3.24)$$

$$C_p H(x) \leq \mathcal{H}(h(x)) \leq C_p H(x). \quad (3.25)$$

**Proof:** Thanks to the parity properties of  $h$  and  $H$ , it is enough to establish the several relations only for  $x \geq 0$ .

Item (i) is already proved in (3.20). As for Item (ii), the right inequality (3.23) is immediate since  $h$  is increasing. On the other hand, since  $p < 2$ , we have for all  $0 \leq s \leq x$  that

$$h'(x) = (p-1)(x+1)^{p-2} \leq (p-1)(s+1)^{p-2}. \quad (3.26)$$

Integrating between 0 and  $x$ , it follows that  $xh'(x) \leq h(x)$ , and then  $(xh)'(x) \leq 2h(x)$ , which yields the left inequality (3.23) after an integration between 0 and  $x$ .

As for the proof of Item (iii), it is clear that (3.25) follows from combining (3.23) and (3.24) and moreover, the right inequality in (3.24) is an immediate consequence of (3.22) since  $h(x) \geq 0$  for  $x \geq 0$ . The proof for the left inequality in (3.24) is divided in two cases and can be deduced at once from the following estimates.

(a) For every  $M > 0$  and real number  $x$  so that  $|x| \leq M$ , it holds

$$\begin{aligned} (p-1)(M+1)^{p-2} \frac{x^2}{2} &\leq \mathcal{H}(h(x)) \leq (p-1) \frac{x^2}{2}. \\ (p-1)(M+1)^{p-2} x^2 &\leq x h(x) \leq (p-1) x^2. \\ (p-1)(M+1)^{p-2} \frac{x^2}{2} &\leq H(x) \leq (p-1) \frac{x^2}{2}; \end{aligned} \quad (3.27)$$

(b) for every  $M > 0$  such that  $(1 + \frac{1}{M})^p < p$ , there exists a positive constant  $C_p$  only depending on  $p$  and  $M$  so that, for every real  $x$  verifying  $|x| > M$ , one has

$$\begin{aligned} & \left( \left(1 + \frac{1}{M}\right)^{p-1} - \left(\frac{1}{M}\right)^{p-1} \right) |x|^p \leq x h(x) \leq |x|^p, \\ & \frac{1}{2} \left( \left(1 + \frac{1}{M}\right)^{p-1} - \left(\frac{1}{M}\right)^{p-1} \right) |x|^p \leq H(x) \leq |x|^p, \\ & \left(1 - \frac{1}{p} \left(1 + \frac{1}{M}\right)^p\right) |x|^p \leq \mathcal{H}(h(x)) \leq \left(1 + \frac{1}{M}\right)^{p-1} |x|^p. \end{aligned} \quad (3.28)$$

**Remark 3.2.1.** Note that the condition  $(1 + \frac{1}{M})^p < p$  is only needed to get the third inequality of (3.28) only. Hence, the lower and upper bounds of  $xh(x)$  and  $H$  in (3.28) are valid for all  $M > 0$ .

In turn, the set of inequalities in Item (a) simply follows from the inequality

$$(M + 1)^{p-2} \leq (s + 1)^{p-2} \leq 1, \quad 0 \leq s \leq x \leq M,$$

and, after integrating between 0 and  $x$ , by the use of the equations (3.16), (3.17) and (3.19).

As for the set of inequalities in Item (b), one first uses the explicit expressions of  $xh(x)$  and  $H(x)$  given in (3.16), (3.17) to deduce that, for every  $x \geq 0$ ,

$$\begin{aligned} x h(x) &= x^p \left[ \left(1 + \frac{1}{x}\right)^{p-1} - \left(\frac{1}{x}\right)^{p-1} \right], \\ H(x) &= x^p \left( \frac{1}{p} \left[ \left(1 + \frac{1}{x}\right)^p - \left(\frac{1}{x}\right)^p \right] - \left(\frac{1}{x}\right)^{p-1} \right). \end{aligned}$$

Since  $p < 2$ , the function  $s \mapsto (1+s)^{p-1} - s^{p-1}$  is decreasing on  $[0, \frac{1}{M}]$  and then one gets the required bounds for  $xh(x)$  in (3.28). The upper and the lower bounds for  $H(x)$  in (3.28) are immediate and follow from combining the upper and the lower bounds of (3.23) and the bounds just established on  $xh(x)$ . Then the bounds for  $\mathcal{H}(h(x))$  in (3.28) are simply obtained by combining the previous estimates with the relation  $\mathcal{H}(h(x)) = xh(x) - H(x)$ .

■

The next lemma is a particular instance of Fenchel's inequality which is used repeatedly in the chapter.

**Lemma 3.2.6.** For  $p \in (1, 2)$ , there exist positive constants  $C_p$  such that, for every  $a, x \in \mathbb{R}$ , it holds that

$$|ax| \leq \frac{C_p}{\eta^2} H(a) + C_p \eta^p \mathcal{H}(x), \quad (3.29)$$

$$|ax| \leq C_p \eta^p H(a) + \frac{C_p}{\eta^2} \mathcal{H}(x), \quad (3.30)$$

and

$$|ah(x)| \leq \frac{C_p}{\eta^p} H(a) + C_p \eta^2 H(x). \quad (3.31)$$

**Proof:** Before proving the required inequalities, let us notice that one deduces from (3.27) and (3.28) that there exists constants  $C_p$  only depending on  $p \in (1, 2)$  such that

$$\begin{aligned} \text{if } |x| \leq C_p, \text{ then } C_p \frac{x^2}{2} &\leq H(x) \leq C_p \frac{x^2}{2}, \\ \text{if } |x| > C_p, \text{ then } C_p |x|^p &\leq H(x) \leq |x|^p. \end{aligned} \quad (3.32)$$

and that

$$\begin{aligned} \text{if } |x| \leq C_p, \text{ then } C_p x^2 &\leq \mathcal{H}(x) \leq C_p x^2, \\ \text{if } |x| > C_p, \text{ then } C_p |x|^p &\leq \mathcal{H}(x) \leq C_p |x|^p. \end{aligned} \quad (3.33)$$

Since  $h$  is convex, we apply Fenchel's inequality given in (3.13) with  $\frac{a}{\eta}$  and  $\eta x$  to obtain

$$|ax| \leq H\left(\frac{a}{\eta}\right) + \mathcal{H}(\eta x). \quad (3.34)$$

Since both  $\mathcal{H}$  and  $H$  are even functions, we assume with no loss of generality that both  $a$  and  $x$  are non negative.

Using the estimates for  $H$  and  $\mathcal{H}$  given in (3.32) and (3.33), we deduce that there exists a positive constant  $C_p$  only depending on  $p \in (1, 2)$  so that for every  $a \geq 0$  and  $\eta \in (0, 1)$ ,

$$\begin{aligned} H\left(\frac{a}{\eta}\right) &\leq C_p \max\left(\frac{1}{\eta^2}, \frac{1}{\eta^p}\right) H(a) \leq \frac{C_p}{\eta^2} H(a), \\ \mathcal{H}(\eta x) &\leq C_p \max(\eta^2, \eta^p) \mathcal{H}(x) \leq C_p \eta^p \mathcal{H}(x), \end{aligned} \quad (3.35)$$

and one immediately gets (3.29) from (3.34) and (3.35).



Using the following Fenchel's inequality instead of (3.34),

$$|ax| \leq H(\eta a) + \mathcal{H}\left(\frac{x}{\eta}\right), \quad (3.36)$$

with the same steps that led to (3.29), we obtain (3.30).

On the other hand, (3.31) follows from (3.29) when setting  $x = h(y)$  so that we obtain

$$|ah(y)| \leq \frac{C_p}{\eta^2} H(a) + C_p \eta^p \mathcal{H}(h(y)). \quad (3.37)$$

then we use (3.25) that states that  $\mathcal{H}(h(y)) \leq C_p H(y)$ .

■

As a corollary of the previous lemma, we have the following Poincaré-type of result.

**Corollary 3.2.1.** Let  $p \in (1, 2)$ . Then there exists a positive constant  $C_p$  such that, for every absolutely continuous function  $z : [0, 1] \rightarrow \mathbb{R}$  so that  $z(0) = 0$ , one has

$$\int_0^1 H(z(s)) ds \leq C_p \int_0^1 H(z'(s)) ds. \quad (3.38)$$

**Proof.** With no loss of generality, we can assume that the right-hand side of (3.38) is finite. One has for every  $x \in [0, 1]$

$$H(z(x)) = \int_0^x z'(s) h(z(s)) ds. \quad (3.39)$$

By applying (3.31), one gets that for every  $x \in [0, 1]$

$$H(z(x)) \leq \frac{C_p}{\eta^p} \int_0^1 h(z'(s)) ds + C_p \eta \int_0^1 h(z(s)) ds, \quad (3.40)$$

for every  $\eta > 0$  and positive constants  $C_p$  only depending on  $p$ . By integrating between 0 and 1 and then choosing appropriately  $\eta$  one concludes.

■

The following lemma is a useful extension of Lemma 3.2.2 with  $f, F$  replaced by  $h, H$ .

**Lemma 3.2.7.** For  $p > 1$ , there exists a positive constant  $C_p$  such that, for every real numbers  $a, b$  and  $\mu \in (0, 1)$  subject to  $|a - b| \geq \max(|a|, |b|)\mu$ , one has

$$H(a - b) \leq \frac{C_p}{\mu^{2-p}}(a - b)(h(a) - h(b)) \quad (3.41)$$

**Proof.** Thanks to (3.23), it is enough to prove the existence of  $C_p > 0$  so that

$$|h(a - b)| \leq \frac{C_p}{\mu^{2-p}}|h(a) - h(b)|, \quad (3.42)$$

for every  $a \geq b$ ,  $\mu \in (0, 1)$  such that  $|a - b| \geq \mu R$  where  $R = \max(|a|, |b|)$ . Assume first that  $ab \leq 0$ . Then the left-hand side of (3.42) is smaller than  $h(2R)$  while  $|h(a) - h(b)| \geq h(R)$ . Clearly  $h(2R) \leq 2h(R)$  since  $h$  is concave and hence (3.42) holds true in that case for any  $C_p \geq 2$ .

We next assume that  $a \geq b \geq 0$  and we consider  $c = a - b$  instead of  $b$ . The assumption on  $a, b$  reads  $c \geq \mu a$ . Equation (3.42) becomes

$$h(c) \leq \frac{C_p}{\mu^{2-p}}(h(a) - h(a - c)). \quad (3.43)$$

Note that the right-hand side of the above equation defines a decreasing function of  $a$ , once the other parameters are fixed. It is therefore enough to consider the case  $a = \frac{c}{\mu}$ . By replacing  $c$  by  $\frac{c}{\mu}$  in the explicit expression of  $h$ , we are led to prove the existence of  $C_p > 0$  so that

$$(\mu c + 1)^{p-1} - 1 \leq \frac{C_p}{\mu^{2-p}} [(c + 1)^{p-1} - ((1 - \mu)c + 1)^{p-1}], \quad (3.44)$$

for every  $c > 0$  and  $\mu \in (0, 1)$ . By applying the mean value theorem to both sides of the above equation and reordering the terms, (3.44) reads

$$\left( \frac{\mu + \mu\eta_2}{\eta_1 + 1} \right)^{2-p} \leq C_p, \quad (3.45)$$

for some  $\eta_1 \in (0, \mu c)$  and  $\eta_2 \in ((1 - \mu)c, c)$  both depending on  $c > 0$  and  $\mu$ . Assume first that  $\mu c \leq 1$ . Then clearly (3.44) holds true for any  $C_p \geq 2^{2-p}$  according to (3.45). If now  $\mu c > 1$ , then  $c > 1$  and since the left-hand side of (3.44) is smaller than  $(\mu c)^{p-1}$ , we are left to find  $C_p > 0$  such that

$$\left( \frac{\mu + \mu\eta_2}{\mu c} \right)^{2-p} \leq C_p. \quad (3.46)$$

The left-hand side of the above equation is again smaller than  $2^{2-p}$  and one concludes. ■

### 3.3 Statement and main notations of the problem

Consider Problem (3.1) where we assume that the following hypothesis are satisfied:

**(H<sub>6</sub>)**  $a : [0, 1] \rightarrow \mathbb{R}$  is a non-negative continuous function such that

$$\exists a_0 > 0, a \geq a_0 \text{ on } \omega = ]c, d[ \subset [0, 1], \quad (3.47)$$

where  $\omega$  is a non empty interval such that  $c = 0$  or  $d = 1$ , i.e.,  $\bar{\omega}$  contains a neighborhood of 0 or 1. There is no loss of generality in assuming  $d = 1$ , taking 0 as an observation point.

**Remark 3.3.1.** The results of this chapter still hold if the assumption that  $c = 0$  or  $d = 1$  is removed by using a piecewise multiplier method, i.e., we can use both 0 and 1 as observation points (instead of simply 0 here) to obtain the required energy estimate.

For  $p \in [1, \infty)$ , consider the functional spaces

$$X_p := W_0^{1,p}(0, 1) \times L^p(0, 1), \quad (3.48)$$

$$Y_p := (W^{2,p}(0, 1) \cap W_0^{1,p}(0, 1)) \times W_0^{1,p}(0, 1), \quad (3.49)$$

where  $X_p$  is equipped with the norm

$$\|(u, v)\|_{X_p} := \left( \frac{1}{p} \int_0^1 (|u' + v|^p + |u' - v|^p) dx \right)^{\frac{1}{p}}, \quad (3.50)$$

and the space  $Y_p$  is equipped with the norm

$$\|(u, v)\|_{Y_p} := \left( \frac{1}{p} \int_0^1 (|u'' + v'|^p + |u'' - v'|^p) dx \right)^{\frac{1}{p}}. \quad (3.51)$$

Initial conditions  $(z_0, z_1)$  for weak (resp. strong) solutions of (3.1) are taken in  $X_p$  (resp. in  $Y_p$ ), where the two concepts of solutions are precisely defined later in Definition 3.4.1.

For all  $(t, x) \in \mathbb{R}_+ \times (0, 1)$ , define the Riemann invariants

$$\rho(t, x) = z_x(t, x) + z_t(t, x), \quad (3.52)$$

$$\xi(t, x) = z_x(t, x) - z_t(t, x). \quad (3.53)$$

Along strong solutions of (3.1), we deduce that

$$\begin{cases} \rho_t - \rho_x = -\frac{1}{2}a(x)(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0, 1), \\ \xi_t + \xi_x = \frac{1}{2}a(x)(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0, 1), \\ \rho(t, 0) - \xi(t, 0) = \rho(t, 1) - \xi(t, 1) = 0 & \forall t \in \mathbb{R}_+, \\ \rho_0 := \rho(0, \cdot) = z'_0 + z_1, \quad \xi_0 := \xi(0, \cdot) = z'_0 - z_1, \end{cases} \quad (3.54)$$

with  $(\rho_0, \xi_0) \in W^{1,p}(0, 1) \times W^{1,p}(0, 1)$ .

We define for all  $t \geq 0$  the  $p$ th-energy of a (weak) solution of (3.1)  $E_p$  that has already been defined in the introduction Chapter but is reminded of here:

$$E_p(t) = \frac{1}{p} \int_0^1 (|z_x + z_t|^p + |z_x - z_t|^p) dx. \quad (3.55)$$

The energy  $E_p$  can be expressed in terms of  $\xi$  and  $\rho$  as

$$E_p(t) = \frac{1}{p} \int_0^1 (|\rho|^p + |\xi|^p) dx. \quad (3.56)$$

Before we state our results, we provide the following proposition (essentially inspired from [16]).

**Proposition 3.3.1.** Let  $p \in (1, \infty)$  and suppose that a strong solution  $z$  of (3.54) is defined on a non trivial interval  $I \subset \mathbb{R}_+$  containing 0, for some initial conditions  $(z_0, z_1) \in Y_p$ . For  $t \in I$ , define

$$\Phi(t) := \int_0^1 [\mathcal{F}(\rho) + \mathcal{F}(\xi)] dx, \quad (3.57)$$

where  $\mathcal{F}$  is a  $C^1$  convex function. Then  $\Phi$  is well defined for  $t \in I$  and satisfies

$$\frac{d}{dt} \Phi(t) = -\frac{1}{2} \int_0^1 a(x)(\rho - \xi)(\mathcal{F}'(\rho) - \mathcal{F}'(\xi)) dx \leq 0. \quad (3.58)$$

**Proof.** By the regularity assumptions,  $\rho(t, \cdot)$  and  $\xi(t, \cdot)$  are absolutely continuous functions. Formal differentiation, easy to justify a posteriori by the regularity of the

data, yields

$$\frac{d}{dt} \int_0^1 [\mathcal{F}(\rho) + \mathcal{F}(\xi)] dx = \int_0^1 [\rho_t \mathcal{F}'(\rho) + \xi_t \mathcal{F}'(\xi)] dx. \quad (3.59)$$

Using (3.54), one obtains that

$$\begin{aligned} & \frac{d}{dt} \int_0^1 (\mathcal{F}(\rho) + \mathcal{F}(\xi)) dx \\ &= \int_0^1 (\rho_x - \frac{1}{2}a(x)(\rho - \xi)) \mathcal{F}'(\rho) + (-\xi_x + \frac{1}{2}a(x)(\rho - \xi)) \mathcal{F}'(\xi) dx, \\ &= \int_0^1 [\mathcal{F}(\rho) - \mathcal{F}(\xi)]_x dx - \frac{1}{2} \int_0^1 a(x)(\rho - \xi)(\mathcal{F}'(\rho) - \mathcal{F}'(\xi)) dx, \\ &= -\frac{1}{2} \int_0^1 a(x)(\rho - \xi)(\mathcal{F}'(\rho) - \mathcal{F}'(\xi)) dx. \end{aligned} \quad (3.60)$$

Since  $\mathcal{F}$  is convex,  $\mathcal{F}'$  is non-decreasing, implying that  $(\rho - \xi)(\mathcal{F}'(\rho) - \mathcal{F}'(\xi)) \geq 0$  which gives the conclusion when combining it with (3.60). ■

**Corollary 3.3.1.** For  $(z_0, z_1) \in Y_p$ ,  $p \in [1, \infty)$ , suppose that the solution  $z$  of (3.1) exists on  $\mathbb{R}_+$ . Then the energy  $t \mapsto E_p(t)$  is non-increasing and, for  $t \geq 0$ ,

$$E'_p(t) = -\frac{1}{2} \int_0^1 a(x)(\rho - \xi) (|\rho|^{p-1} - |\xi|^{p-1}) dx. \quad (3.61)$$

**Proof.** For  $(z_0, z_1) \in Y_p$  and  $p > 1$ , we apply Proposition 3.3.1 with  $F(s) = \frac{|s|^p}{p}$ , which proves (3.61). The case  $p = 1$  is obtained by letting  $p$  tend to one. ■

### 3.4 Well-posedness

We start by recalling the classical representation formula for regular solutions of (3.1) given by the d'Alembert formula, cf. [31, Equation 8, page 36].

**Proposition 3.4.1.** Consider the following problem with an arbitrary source term  $g \in C^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$  and initial data  $z_0 \in C^2(\mathbb{R})$  and  $z_1 \in C^1(\mathbb{R})$ ,

$$\begin{cases} z_{tt}(t, x) - z_{xx}(t, x) + g(t, x) = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ z(0, \cdot) = z_0, \quad z_t(0, \cdot) = z_1. \end{cases} \quad (3.62)$$

Then the unique solution  $z$  of this problem is in  $C^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$  and is given for all  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$  by d'Alembert formula

$$z(t, x) = \frac{1}{2} [z_0(x+t) + z_0(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} z_1(s) ds + \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} g(s, \tau) d\tau ds. \quad (3.63)$$

In order to apply the above proposition to (3.1), we extend by a standard procedure (cf. [13, Exercise 4, section 4.3]) the following partial differential equation defined on  $\mathbb{R}_+ \times (0, 1)$

$$\begin{cases} z_{tt} - z_{xx} + g(t, x) = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \\ z(t, 0) = z(t, 1) = 0 & \forall t \in \mathbb{R}_+, \\ z(0, \cdot) = z_0, \quad z_t(0, \cdot) = z_1, \end{cases} \quad (3.64)$$

to an equivalent partial differential system defined on  $\mathbb{R}_+ \times \mathbb{R}$ . We first extend the data of the problem by considering  $\tilde{z}_0$ ,  $\tilde{z}_1$  and  $\tilde{g}$  the 2-periodic extensions to  $\mathbb{R}$  of the odd extensions of  $z_0$ ,  $z_1$  and  $g$  to  $[-1, 1]$ .

Using (3.63), we obtain then the expression of the solution  $z$  for the problem on  $\mathbb{R}_+ \times (0, 1)$ , which is the following

$$z(t, x) = \frac{1}{2} [\tilde{z}_0(x+t) + \tilde{z}_0(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} \tilde{z}_1(s) ds + \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} \tilde{g}(s, \tau) d\tau ds, \quad (3.65)$$

which clearly provides, for every  $t \geq 0$ , a 2-periodic odd function  $z(t, \cdot)$  on  $\mathbb{R}$ . We also have the expression of the derivatives

$$\begin{aligned} z_x(t, x) &= \frac{1}{2} [\tilde{z}'_0(x+t) + \tilde{z}'_0(x-t)] + \frac{1}{2} [\tilde{z}_1(x+t) - \tilde{z}_1(x-t)] \\ &\quad + \frac{1}{2} \int_0^t [\tilde{g}(s, x+(t-s)) - \tilde{g}(s, x-(t-s))] ds, \end{aligned} \quad (3.66)$$

and

$$\begin{aligned} z_t(t, x) &= \frac{1}{2} [\tilde{z}'_0(x+t) - \tilde{z}'_0(x-t)] + \frac{1}{2} [\tilde{z}_1(x+t) + \tilde{z}_1(x-t)] \\ &\quad + \frac{1}{2} \int_0^t [\tilde{g}(s, x+(t-s)) + \tilde{g}(s, x-(t-s))] ds. \end{aligned} \quad (3.67)$$

Before we proceed to the well-posedness of (3.1) in  $X_p$  (resp.  $Y_p$ ), we need to define the notion of its weak and strong solutions.

**Definition 3.4.1.** For  $(z_0, z_1)$  in  $X_p$  (resp.  $Y_p$ ), we say that (3.1) has a weak (resp. strong) solution  $z \in L^\infty(\mathbb{R}_+, W_0^{1,p}(0, 1)) \cap W^{1,\infty}(\mathbb{R}_+, L^p(0, 1))$  (resp.  $z \in L^\infty(\mathbb{R}_+, W^{2,p}(0, 1) \cap W_0^{1,p}(0, 1)) \cap W^{1,\infty}(\mathbb{R}_+, W_0^{1,p}(0, 1))$ ) given by the expression (3.65) in  $X_p$  (resp.  $Y_p$ ), if the source term  $g$  from (3.62) is given by  $g(t, x) = -a(x)z_t(t, x)$ .

**Theorem 3.4.1** (Well-posedness). Let  $p \in [1, \infty)$ . For any initial data  $(z_0, z_1) \in X_p$  (resp.  $Y_p$ ), there exists a unique weak (resp. strong) solution  $z$  such that

$$z \in L^\infty(\mathbb{R}_+, W_0^{1,p}(0, 1)) \cap W^{1,\infty}(\mathbb{R}_+, L^p(0, 1)), \quad (3.68)$$

$$\text{(resp. } z \in L^\infty(\mathbb{R}_+, W^{2,p}(0, 1) \cap W_0^{1,p}(0, 1)) \cap W^{1,\infty}(\mathbb{R}_+, W_0^{1,p}(0, 1)).) \quad (3.69)$$

Moreover, in both cases, the energy function  $t \mapsto E_p(t)$  associated with a solution is non-increasing.

**Proof.** The arguments for both items is adapted from that of [10, Theorem 1]. We prove the existence of an appropriate solution  $y$  of (3.67) by a standard fixed point argument. We proceed on some interval  $[0, T]$  for  $T > 0$  small enough independent on the initial condition. We can then reproduce the reasoning on  $[T, 2T]$  starting from the solution at  $t = T$  and so on to establish well-posedness for all  $t \geq 0$ .

Since  $\tilde{g}$  is 2-periodic function in space, it is natural to work in a space of functions that have the same features. Hence we denote by  $\mathcal{B}_T$  the space of functions that are defined on  $[0, T] \times \mathbb{R}$ , odd on  $[-1, 1]$  and 2-periodic in space and  $p$ -integrable. The space  $\mathcal{B}_T$  is equipped with the norm

$$\|y\|_{\mathcal{B}_T} = \sup_{t \in [0, T]} \|y(t, \cdot)\|_{L^p(0, 1)}, \quad (3.70)$$

which makes it a Banach Space. We define the mapping

$$\begin{aligned} F_T : \mathcal{B}_T &\longrightarrow \mathcal{B}_T \\ y &\longmapsto F_T(y), \end{aligned}$$

such that, for all  $(t, x) \in [0, T] \times \mathbb{R}$ , we have

$$\begin{aligned} F_T(y)(t, x) &= \frac{1}{2} [\tilde{z}_0'(x+t) - \tilde{z}_0'(x-t)] + \frac{1}{2} [\tilde{z}_1(x+t) + \tilde{z}_1(x-t)] \\ &+ \frac{1}{2} \int_0^t [\tilde{a}(x+(t-s))y(s, x+(t-s)) + \tilde{a}(x-(t-s))y(s, x-(t-s))] ds. \end{aligned} \quad (3.71)$$

Since  $a$  is bounded, it is clear that  $F_T$  is a contraction on  $\mathcal{B}_T$  for  $T > 0$  small enough, hence the existence of a fixed point to  $F_T$ , which is a (weak) solution of (3.1). It is also clear that  $T$  does not depend on the initial condition  $(z_0, z_1) \in X_p$ . As explained previously, this enables one to prove well-posedness in  $X_p$ .

As for the part regarding  $Y_p$ , the argument is similar to the previous one, after replacing  $\mathcal{B}_T$  by the space  $\mathcal{D}_T$  consisting of the functions defined on  $[0, T] \times \mathbb{R}$  which are odd on  $[-1, 1]$  and 2-periodic in space with  $p$ -integrable derivative with respect to  $x$ , equipped with the norm given by

$$\|y\|_{\mathcal{D}_T} = \sup_{t \in [0, T]} \|y(t, \cdot)\|_{W_0^{1,p}(0,1)}. \quad (3.72)$$

For  $(z_0, z_1) \in X_p$ ,  $p > 1$ , we get that  $t \mapsto E_p(t)$  is non increasing by the fact that  $Y_p$  is dense in  $X_p$ . For  $p = 1$ , we use the facts that  $X_p$  is dense in  $X_1$  for  $p > 1$  and the map  $p \mapsto E_p(t)$ , for a fixed trajectory and a fixed positive time  $t$ , is right-continuous.

■

**Remark 3.4.1.** Since (3.1) is linear and  $t \mapsto E_p(t)$  is non-increasing, the flow of its weak solutions defines a  $C^0$ -semigroup  $(S_p(t))_{t \geq 0}$  of contractions of  $X_p$ , for every  $p \in [1, \infty)$ .

## 3.5 Exponential stability

In this section, we aim to establish exponential stability for the  $C^0$ -semigroup  $(S_p(t))_{t \geq 0}$  defining the weak solutions of (3.1) for every  $p \in (1, \infty)$ . The argument relies on the multiplier method and is slightly different whether  $p \geq 2$  or not. Indeed, two multipliers involve the exponent  $p - 2$ , which becomes negative if  $p \in (1, 2)$ . In the latter case, one must modify all the multipliers to handle that situation.

Before starting describing such results, we have the following weaker general stability result for  $p \in [1, \infty)$ .

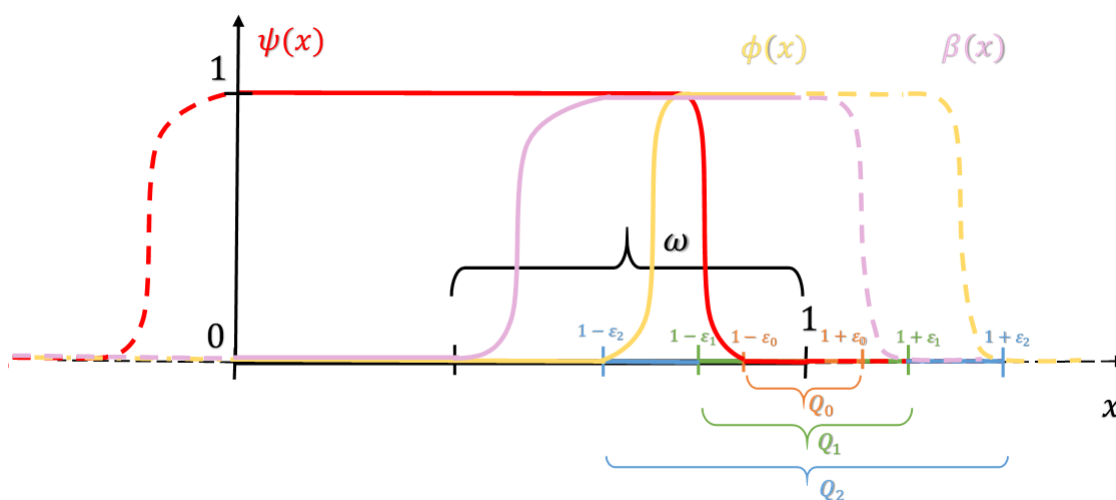
**Proposition 3.5.1** (Strong stability). Fix  $p \in [1, \infty)$  and suppose that Hypothesis  $(\mathbf{H}_1)$  is satisfied. Then, for every  $(z_0, z_1) \in X_p$ , the solution  $z(t, \cdot)$  of (3.1) starting at  $(z_0, z_1)$  tends to zero as  $t$  tends to infinity (in the sens that  $E_p(t)$  tends to 0 as  $t$  tends to infinity).

**Proof.** We follow the proof provided in the case  $p = 2$  in [11]: by a standard density argument, it is enough to establish the result for strong solutions of (3.1). The



latter is obtained by a LaSalle type argument using the energy function  $E_p$  and the fact that the set  $\{z(t, \cdot), t \geq 0\}$  is relatively compact in  $W_0^{1,p}(0, 1)$ , which is itself obtained by noticing that  $z_t$  is a weak solution of (3.1) with bounded energy  $E_p$ . ■

We introduce next some functions and notations which are common to the handling of both cases. Recalling that we have chosen  $x_0 = 0$  as an observation point, we consider for  $0 < \epsilon_0 < \epsilon_1 < \epsilon_2$ , the sets  $Q_i = (1 - \epsilon_i, 1 + \epsilon_i)$ ,  $i = 0, 1, 2$ , as well as the three smooth functions  $\psi, \phi$  and  $\beta$  already used in Chapter 2 in two dimensions. We redefine them in one dimension according to the next figure.



More precisely, the functions  $\psi, \phi$  and  $\beta$  are smooth with compact support and defined as follows:

$$\begin{cases} 0 \leq \psi \leq 1, \\ \psi = 0 \text{ on } Q_0, \\ \psi = 1 \text{ on } (0, 1) \setminus Q_1, \end{cases} \quad \begin{cases} 0 \leq \phi \leq 1, \\ \phi = 1 \text{ on } Q_1, \\ \phi = 0 \text{ on } (0, 1) \setminus Q_2, \end{cases} \quad \begin{cases} 0 \leq \beta \leq 1, \\ \beta = 1 \text{ on } Q_2 \cap (0, 1), \\ \beta = 0 \text{ on } \mathbb{R} \setminus \omega. \end{cases} \quad (3.73)$$

**Remark 3.5.1.** In the sequel, we will denote by  $C_p$  positive constants only depending on  $p$  and by  $C$  positive constants depending on  $a(\cdot)$  (typically through its upper bound  $A$  on  $[0, 1]$  and its lower bound  $a_0$  on  $\omega$ ), and on  $\psi, \phi$  and  $\beta$  (through bounds of their first derivatives over their supports).

Our main result is the following theorem:

**Theorem 3.5.1. (Exponential stability)** Fix  $p \in (1, \infty)$  and suppose that Hypothesis  $(\mathbf{H}_1)$  is satisfied. Then the  $C^0$ -semigroup  $(S_p(t))_{t \geq 0}$  is exponentially stable.

### 3.5.1 Case where $p \geq 2$

As usual, it is enough to prove Theorem 3.5.1 for strong solutions and then extend the result for weak solutions by a density argument. In turn, the theorem for strong solutions classically follows from the next proposition, cf. [4, Theorem 1.4.2] for instance.

**Proposition 3.5.2.** Fix  $p \in [2, \infty)$  and suppose that Hypothesis  $(\mathbf{H}_1)$  is satisfied. Then there exist positive constants  $C$  and  $C_p$  such that, for every  $(z_0, z_1) \in Y_p$ , it holds the following energy estimate:

$$\forall 0 \leq S \leq T, \quad \int_S^T E_p(t) dt \leq C C_p E_p(S), \quad (3.74)$$

where  $E_p(\cdot)$  denotes the energy of the solution of (3.1) starting at  $(z_0, z_1)$ .

The proof will be divided into four steps in subsections 3.5.1–3.5.1. We fix an arbitrary pair of times  $0 \leq S \leq T$  and a strong solution  $z(\cdot, \cdot)$  of (3.1) starting at  $(z_0, z_1) \in Y_p$ , and we consider three sets of multipliers:

(m1)  $x \mapsto x\psi(x)f(\rho(t, x))$  and  $x \mapsto x\psi(x)f(\xi(t, x))$  for every  $t \geq 0$ ;

(m2)  $x \mapsto \phi(x)f'(\rho(t, x))z(t, x)$  and  $x \mapsto \phi(x)f'(\xi(t, x))z(t, x)$  for every  $t \geq 0$ ;

(m3)  $x \mapsto v(t, x)$  for every  $t \geq 0$ , where  $v$  is the solution of the following elliptic problem defined for every  $t \geq 0$ :

$$\begin{cases} v_{xx} = \beta f(z) & x \in (0, 1), \\ v(0) = v(1) = 0, \end{cases} \quad (3.75)$$

where the function  $f$  is defined (3.9).

Note that we use the usual notation  $q = \frac{p}{p-1}$  for the conjugate exponent of  $p$ .

**Remark 3.5.2.** In the Hilbertian case  $p = 2$ , the classical multipliers as given in [4] are  $x\psi(x)z_x(t, x)$ ,  $x\phi(x)z(t, x)$  and  $v$  associated with  $p = 2$  (i.e.  $v_{xx} = \beta z$ ). Then, while clearly our third multiplier  $v$  is a straightforward extension of the Hilbertian case to any  $p \in [1, \infty)$ , the two sets of multipliers given in Items (m1) and (m2) seem to be new, even if those of Item (m2) are identical when  $p = 2$ .

**First set of multipliers**

The first step toward an energy estimate consists in obtaining an inequality that contains the expression of the energy  $E_p$  and, for this purpose, we use the first set of multipliers of Item (m1). We obtain the following lemma.

**Lemma 3.5.1.** Under the hypotheses of Proposition 3.5.2, we have the following estimate

$$\int_S^T E_p(t) dt \leq CC_p E_p(S) + C \underbrace{\int_S^T \int_{Q_1 \cap (0,1)} (F(\rho) + F(\xi)) dx dt}_{\mathbf{S}_4}, \quad (3.76)$$

where  $C_p$  denotes constants that depend on  $p$  only.

**Proof.** Multiplying the first equation of (3.54) by  $x \psi f(\rho)$  and integrating over  $[S, T] \times [0, 1]$ , we obtain that

$$\int_S^T \int_0^1 x \psi f(\rho) \left( \rho_t - \rho_x + \frac{1}{2} a(x)(\rho - \xi) \right) dx dt = 0. \quad (3.77)$$

Starting with  $\int_S^T \int_0^1 x \psi f(\rho) \rho_t dx dt$ , one has

$$\int_S^T \int_0^1 x \psi f(\rho) \rho_t dx dt = \int_0^1 x \psi \int_S^T (F(\rho))_t dt dx = \int_0^1 x \psi [F(\rho)]_S^T dx. \quad (3.78)$$

Regarding  $-\int_S^T \int_0^1 x \psi f(\rho) \rho_x dx dt$ , we use an integration by part with respect to  $x$  and we obtain

$$\begin{aligned} - \int_S^T \int_0^1 x \psi f(\rho) \rho_x dx dt &= - \int_S^T \int_0^1 x \psi (F(\rho))_x dx dt \\ &= \int_S^T \int_0^1 (x \psi)_x F(\rho) dx dt - \int_S^T [x \psi F(\rho)]_0^1 = \int_S^T \int_0^1 (x \psi)_x F(\rho) dx dt. \end{aligned} \quad (3.79)$$

By combining (3.78) and (3.79), we get

$$\int_S^T \int_0^1 (x \psi)_x F(\rho) dx dt + \int_0^1 x \psi [F(\rho)]_S^T dx = -\frac{1}{2} \int_S^T \int_0^1 x \psi(x) a(x) f(\rho) (\rho - \xi) dx dt. \quad (3.80)$$

We proceed similarly by multiplying the second equation of (3.54) by  $x \psi f(\xi)$  and,

following the same steps that yielded (3.80), we obtain that

$$\int_S^T \int_0^1 (x\psi)_x F(\xi) dx dt - \int_0^1 x\psi [F(\xi)]_S^T dx = -\frac{1}{2} \int_S^T \int_0^1 x\psi(x)a(x)f(\xi)(\rho - \xi) dx dt. \quad (3.81)$$

Summing up (3.80) and (3.81), we obtain

$$\begin{aligned} \int_S^T \int_0^1 (x\psi)_x (F(\rho) + F(\xi)) dx dt &= \int_0^1 x\psi [F(\xi)]_S^T dx - \int_0^1 x\psi [F(\rho)]_S^T dx \\ &\quad - \frac{1}{2} \int_S^T \int_0^1 a(x)x\psi(f(\xi) + f(\rho))(\rho - \xi) dx dt. \end{aligned} \quad (3.82)$$

Using the definition of  $\psi$ , we obtain

$$\begin{aligned} &\int_S^T \int_{(0,1)\setminus Q_1} (F(\rho) + F(\xi)) dx dt \\ &= - \int_S^T \int_{Q_1 \cap (0,1)} (x\psi)_x (F(\rho) + F(\xi)) dx dt + \int_0^1 x\psi [F(\xi)]_S^T dx \\ &\quad - \int_0^1 x\psi [F(\rho)]_S^T dx - \frac{1}{2} \int_S^T \int_0^1 a(x)x\psi(f(\xi) + f(\rho))(\rho - \xi) dx dt. \end{aligned} \quad (3.83)$$

We now complete the expression of the energy  $E_p$  in the left-hand side of the previous equality and, since  $\int_0^1 F(\xi) + F(\rho) dx = E_p$ , it follows that

$$\begin{aligned} \int_S^T E_p(t) dt &\leq \underbrace{\int_S^T \int_{Q_1 \cap (0,1)} |(1 - (x\psi)_x)| (F(\rho) + F(\xi)) dx dt}_{\mathbf{S}_1} \\ &\quad + \underbrace{\int_0^1 |x\psi| |[F(\rho) - F(\xi)]_S^T| dx}_{\mathbf{S}_2} + \underbrace{\frac{1}{2} \int_S^T \int_0^1 |a(x)x\psi| |(f(\rho) + f(\xi))| |\rho - \xi| dx dt}_{\mathbf{S}_3}. \end{aligned} \quad (3.84)$$

We start by estimating  $\mathbf{S}_1$ . Since  $|1 - (x\psi)_x| \leq C$ , we get

$$\begin{aligned} \int_S^T \int_{Q_1 \cap (0,1)} |1 - (x\psi)_x| (F(\rho) + F(\xi)) dx dt &\leq C \int_S^T \int_{Q_1 \cap (0,1)} (F(\rho) + F(\xi)) dx dt \\ &\leq C\mathbf{S}_4, \end{aligned} \quad (3.85)$$

where  $\mathbf{S}_4$  has been defined in (3.76).

As for  $\mathbf{S}_2$ , using the fact that  $|x\psi| < 1$  and the fact that  $t \mapsto E_p(t)$  is non increasing,

one gets the following upper bound for  $\mathbf{S}_2$ ,

$$\mathbf{S}_2 \leq E_p(S) + E_p(T) \leq 2E_p(S). \quad (3.86)$$

We finally estimate  $\mathbf{S}_3$ . Recall that  $q := \frac{p}{p-1}$  denotes the conjugate exponent of  $p$ . Using (2.3) in Lemma 2.2.1 with  $A = a(x)|\rho - \xi|$ ,  $B = |f(\xi)| + |f(\rho)|$  and  $\eta = \eta_1$  where  $\eta_1 > 0$  an arbitrary constant, it follows that

$$\begin{aligned} \mathbf{S}_3 &\leq C \int_S^T \int_0^1 a(x) |f(\rho) + f(\xi)| |\rho - \xi| dx dt \\ &\leq CC_p \eta_1^q \int_S^T \int_0^1 (F(\rho) + F(\xi)) dx dt + \frac{CC_p}{\eta_1^p} \int_S^T \int_0^1 a(x) |\rho - \xi|^p dx dt \\ &\leq CC_p \eta_1^q \int_S^T E_p(t) dt + \frac{CC_p}{\eta_1^p} \int_S^T \int_0^1 a(x) |\rho - \xi|^p dx dt. \end{aligned} \quad (3.87)$$

Set  $R = \max(|\rho|, |\xi|)$ . Then, for every  $0 < \mu_1 < 1$ , one has

$$\begin{aligned} \int_S^T \int_0^1 a(x) |\rho - \xi|^p dx dt &= \int_S^T \int_{|\rho - \xi| \geq R\mu_1} a(x) |\rho - \xi|^p dx dt \\ &\quad + \int_S^T \int_{|\rho - \xi| < R\mu_1} a(x) |\rho - \xi|^p dx dt. \end{aligned} \quad (3.88)$$

For the first integral term above, we have directly from Lemma 3.2.2 with  $a = \rho$ ,  $b = \xi$  that

$$\int_S^T \int_{|\rho - \xi| \geq R\mu_1} a(x) |\rho - \xi|^p dx dt \leq \frac{C_p}{\mu_1^{2-p}} \int_S^T (-E_p'(t)) dt \leq \frac{C_p}{\mu_1^{2-p}} E(S). \quad (3.89)$$

As for the second integral term in (3.88), we have that

$$\begin{aligned} \int_S^T \int_{|\rho - \xi| < R\mu_1} a(x) |\rho - \xi|^p dx dt &\leq \mu_1^p \int_S^T \int_{|\rho - \xi| < R\mu_1} a(x) R^p dx dt \\ &\leq C_p \mu_1^p \int_S^T \int_0^1 (F(\rho) + F(\xi)) dx dt \\ &\leq C_p \mu_1^p \int_S^T E_p(t) dt. \end{aligned} \quad (3.90)$$

Combining (3.88), (3.89) and (3.90), we obtain that

$$\int_S^T \int_0^1 a(x) |\rho - \xi|^p dx dt \leq C_p \mu_1^p \int_S^T E_p(t) dt + \frac{C_p}{\mu_1^{2-p}} E_p(S). \quad (3.91)$$

By combining (3.91) with (3.87), we obtain that

$$\mathbf{S}_3 \leq CC_p \left( \frac{\mu_1^p}{\eta_1^p} + \eta_1^q \right) \int_S^T E_p(t) dt + C \frac{C_p}{\eta_1^p \mu_1^{2-p}} E_p(S). \quad (3.92)$$

Gathering (3.84), (3.85), (3.86) and (3.92), it follows that

$$\begin{aligned} \int_S^T E(t) dt &\leq C \int_S^T \int_{Q_1 \cap (0,1)} (F(\rho) + F(\xi)) dx dt \\ &\quad + CC_p \left( \frac{\mu_1^p}{\eta_1^p} + C\eta_1^q \right) \int_S^T E_p(t) dt + \left( C \frac{C_p}{\eta_1^p \mu_1^{2-p}} + 2 \right) E_p(S). \end{aligned} \quad (3.93)$$

We can choose  $\eta_1 > 0$  and  $\mu_1 > 0$  such that

$$CC_p \left( \frac{\mu_1^p}{\eta_1^p} + \eta_1^q \right) < \frac{1}{2}, \quad (3.94)$$

which proves (3.76). ■

### Second pair of multipliers

The second set of multipliers given in Item (m2) is used to handle the term  $\mathbf{S}_4$  in (3.76) and it will lead us to the following lemma.

**Lemma 3.5.2.** Under the hypotheses of Proposition 3.5.2 with  $\phi$  as defined in (3.73), we have the following estimate

$$\mathbf{S}_4 \leq C \frac{C_p}{\eta_2^p} \underbrace{\int_S^T \int_{Q_2 \cap (0,1)} |z|^p dx dt}_{\mathbf{T}_5} + CC_p \eta_2^q \int_S^T E_p(t) dt + CC_p E_p(S), \quad (3.95)$$

where  $\eta_2$  is an arbitrary constant in  $(0, 1)$  and  $C$  and  $C_p$  are positive constants whose dependence is specified in Remark 3.5.1.

**Proof.** We multiply the first equation of (3.54) by  $\phi f'(\rho)z$ , where  $z$  is the solution of (3.1) and we integrate over  $[S, T] \times [0, 1]$  to obtain

$$\int_S^T \int_0^1 \phi f'(\rho)z(\rho_t - \rho_x + \frac{1}{2}a(x)(\rho - \xi)) dx dt = 0. \quad (3.96)$$

On one hand, we have that

$$\begin{aligned} \int_S^T \int_0^1 \phi f'(\rho) z \rho_t dx dt &= \int_S^T \int_0^1 \phi (f(\rho))_t z dx dt \\ &= - \int_0^1 \int_S^T \phi f(\rho) z_t dt dx + \int_0^1 \phi [f(\rho) z]_S^T dx. \end{aligned} \quad (3.97)$$

On the other hand, an integration by part with respect to  $x$  yields

$$- \int_S^T \int_0^1 \phi f'(\rho) z \rho_x dx dt = - \int_S^T \int_0^1 \phi z (f(\rho))_x dx dt = \int_S^T \int_0^1 (\phi z)_x f(\rho) dx dt, \quad (3.98)$$

and then

$$\int_S^T \int_0^1 (\phi z)_x f(\rho) dx dt = \int_S^T \int_0^1 \phi_x z f(\rho) dx dt + \int_S^T \int_0^1 \phi z_x f(\rho) dx dt. \quad (3.99)$$

Putting together (3.96), (3.97), (3.98) and (3.99)

$$\begin{aligned} & - \int_S^T \int_0^1 \phi f(\rho) z_t dx dt + \int_0^1 \phi [f(\rho) z]_S^T dx + \int_S^T \int_0^1 \phi_x z f(\rho) dx dt \\ & + \int_S^T \int_0^1 \phi z_x f(\rho) dx dt + \frac{1}{2} \int_S^T \int_0^1 \phi f'(\rho) z a(x) (\rho - \xi) dx dt = 0. \end{aligned} \quad (3.100)$$

Since  $\rho - 2z_t = z_x - z_t$ , we have that

$$\begin{aligned} & \int_S^T \int_0^1 \phi z_x f(\rho) dx dt - \int_S^T \int_0^1 \phi f(\rho) z_t dx dt \\ & = \int_S^T \int_0^1 \phi \rho f(\rho) dx dt - 2 \int_S^T \int_0^1 \phi z_t f(\rho) dx dt, \end{aligned} \quad (3.101)$$

it follows that

$$\begin{aligned} \int_S^T \int_0^1 \phi \rho f(\rho) dx dt &= 2 \int_S^T \int_0^1 \phi z_t f(\rho) dx dt - \int_0^1 \phi [f(\rho) z]_S^T dx \\ & - \int_S^T \int_0^1 \phi_x z f(\rho) dx dt - \frac{1}{2} \int_S^T \int_0^1 \phi f'(\rho) z a(x) (\rho - \xi) dx dt. \end{aligned} \quad (3.102)$$

We proceed similarly after multiplying the second equation of (3.54) by  $\phi f'(\xi) z$  and,

following the same steps that led to (3.102), we obtain that

$$\begin{aligned} \int_S^T \int_0^1 \phi \xi f(\xi) dx dt &= -2 \int_S^T \int_0^1 \phi z_t f(\xi) dx dt + \int_0^1 \phi [f(\xi) z dx]_S^T \\ &\quad - \int_S^T \int_0^1 \phi_x z f(\xi) dx dt - \frac{1}{2} \int_S^T \int_0^1 \phi f'(\xi) z a(x) (\rho - \xi) dx dt. \end{aligned} \quad (3.103)$$

We take the sum of (3.102) and (3.103) and get

$$\begin{aligned} \int_S^T \int_0^1 \phi (\rho f(\rho) + \xi f(\xi)) dx dt &= - \int_S^T \int_0^1 \phi_x z (f(\rho) + f(\xi)) dx dt \\ &\quad - \left[ \int_0^1 \phi (f(\rho) - f(\xi)) z dx \right]_S^T - \frac{1}{2} \int_S^T \int_0^1 \phi (f'(\rho) + f'(\xi)) z a(x) (\rho - \xi) dx dt \\ &\quad + 2 \int_S^T \int_0^1 \phi z_t (f(\rho) - f(\xi)) dx dt. \end{aligned} \quad (3.104)$$

Using the definition of  $\phi$  in (3.73) and the fact that  $2z_t = \rho - \xi$ , we derive that

$$\begin{aligned} \mathbf{S}_4 &\leq C \underbrace{\int_S^T \int_{Q_2 \cap (0,1)} |z (f(\rho) + f(\xi))| dx dt}_{\mathbf{T}_1} + C_p \underbrace{\left| \left[ \int_0^1 (f(\rho) - f(\xi)) z dx \right]_S^T \right|}_{\mathbf{T}_2} \\ &\quad + C_p \underbrace{\int_S^T \int_{Q_2 \cap (0,1)} |(f'(\rho) + f'(\xi)) z a(x) (\rho - \xi)| dx dt}_{\mathbf{T}_3} \\ &\quad + C_p \underbrace{\int_S^T \int_0^1 |\phi (\rho - \xi) (f(\rho) - f(\xi))| dx dt}_{\mathbf{T}_4}. \end{aligned} \quad (3.105)$$

We start by estimating  $\mathbf{T}_1$ . We have  $\mathbf{T}_1 \leq \mathbf{T}'_1$  where

$$\mathbf{T}'_1 := \int_S^T \int_{Q_2 \cap (0,1)} |z| (|f(\rho)| + |f(\xi)|) dx dt, \quad (3.106)$$

which gives when using (2.3) in Lemma 2.2.1 with  $A = |z|$  and  $B \in \{|f(\rho)|, |f(\xi)|\}$ ,

$$\mathbf{T}'_1 \leq \frac{C_p}{\eta_2^p} \int_S^T \int_{Q_2 \cap (0,1)} F(z) dx dt + C_p \eta_2^q \int_S^T E_p(t) dt, \quad (3.107)$$

where  $\eta_2 > 0$  is arbitrary.



To estimate  $\mathbf{T}_2$ , we have by Young's inequality recalled in Lemma 2.2.1 that

$$\begin{aligned} \left| \int_0^1 \phi (f(\rho) - f(\xi)) z dx \right| &\leq \int_0^1 |f(\rho)| |z| dx + \int_0^1 |f(\xi)| |z| dx \\ &\leq C_p \int_0^1 (F(\rho) + F(\xi)) dx + C_p \int_0^1 |z|^p dx. \end{aligned} \quad (3.108)$$

Using Poincaré's inequality,

$$\begin{aligned} \int_0^1 |z|^p dx &\leq C \int_0^1 |z_x|^p dx \leq C \int_0^1 |\rho + \xi|^p dx \\ &\leq CC_p \int_0^1 (|\rho|^p + |\xi|^p) dx \\ &\leq CC_p E_p(t). \end{aligned} \quad (3.109)$$

Combining (3.108) and (3.109) and the fact that  $t \mapsto E_p(t)$  is non increasing, it follows that

$$\mathbf{T}_2 \leq CC_p E_p(S). \quad (3.110)$$

As for  $\mathbf{T}_3$ , we first notice that for every  $(\rho, \xi) \in \mathbb{R}^2$ , one has

$$|(f'(\rho) + f'(\xi))(\rho - \xi)| \leq C_p (|f(\rho)| + |f(\xi)|). \quad (3.111)$$

It follows that  $\mathbf{T}_3 \leq C_p \mathbf{T}'_1$  which has been defined in (3.106) and which is upper bounded in (3.107).

Regarding  $\mathbf{T}_4$ , we use the fact that  $\phi(x) \leq Ca(x)$  for  $x \in [0, 1]$  to get

$$\mathbf{T}_4 \leq C \int_S^T \int_\omega a(x)(\rho - \xi)(f(\rho) - f(\xi)) dx dt \leq C \int_S^T (-E'_p(t)) dt \leq CE_p(S). \quad (3.112)$$

Combining (3.105), (3.107), (3.110) and (3.112), the estimate (3.95) is proved. ■

### Third multiplier

It remains to tackle the term  $\mathbf{T}_5$  appearing in (3.95). To handle it, we consider the multiplier introduced in Item (m3) and, in order to achieve future upper bounds, we will be needing estimates of the  $L^q$ -norms of  $v$  and  $v_t$ , where  $q = \frac{p}{p-1}$ , given in the

following lemma.

**Lemma 3.5.3.** For  $v$  as defined in (3.75), we have the following estimates:

$$\int_0^1 |v|^q dx \leq CC_p E_p(t). \quad (3.113)$$

$$\int_0^1 |v_t|^q dx \leq C_p \left( (p-2)\sigma E_p(t) + \frac{1}{\sigma^{p-2}} \int_0^1 \beta |z_t|^p dx \right), \quad (3.114)$$

where  $\sigma > 0$  is an arbitrary positive constant and  $C$  and  $C_p$  are positive constants whose dependence is specified in Remark 3.5.1.

**Proof.** From the definition of  $v$ , one gets

$$v(t, x) = -x \int_x^1 (1-s)\beta f(z) ds - (1-x) \int_0^x s\beta f(z) ds, \quad x \in [0, 1]. \quad (3.115)$$

One deduces that, by using Hölder's inequality,

$$|v(t, x)|^q \leq C \left( \int_0^1 |z|^{p-1} ds \right)^q \leq C \int_0^1 |z|^p ds, \quad x \in [0, 1]. \quad (3.116)$$

Poincaré's inequality yields  $\int_0^1 |z|^p ds \leq C \int_0^1 |z_x|^p ds$  and then (3.113) after integrating over  $x \in [0, 1]$  and the definition of  $E_p(t)$ .

Similarly, one has

$$v_t(t, x) = -x \int_x^1 (1-s)\beta z_t f'(z) ds - (1-x) \int_0^x s\beta z_t f'(z) ds, \quad x \in [0, 1]. \quad (3.117)$$

By using Hölder inequality and the fact that  $\beta$  is bounded by 1, one deduces that

$$|v_t(t, x)|^q \leq C_p \left( \int_0^1 \beta |z_t| |z|^{p-2} ds \right)^q \leq C_p \int_0^1 \beta |z_t|^q |z|^{q(p-2)} ds, \quad x \in [0, 1]. \quad (3.118)$$

If  $p = 2$ , we have  $q = 2$  and get (3.114) after integrating over  $x \in [0, 1]$ . For  $p > 2$ , we apply Young's inequality with the pair of conjugate exponents  $(p-1, \frac{p-1}{p-2})$  and conclude as for (3.113).

■

The next lemma shows the use of the third multiplier  $v$ .

**Lemma 3.5.4.** Under the hypotheses of Proposition 3.5.2, with  $v$  as defined in (3.75), we have the following estimate,

$$\underbrace{\int_S^T \int_{Q_2 \cap (0,1)} |z|^p dx dt}_{\mathbf{T}_5} \leq CC_p \left( \eta \int_S^T E_p(t) dt + \frac{1}{\eta^r} E_p(S) \right), \quad (3.119)$$

where  $r = \frac{2p^2-p-2}{p}$ ,  $\eta$  is any real number in  $(0, 1)$  and  $C$  and  $C_p$  are positive constants whose dependence is specified in Remark 3.5.1.

**Proof.** We multiply the first equation of (3.54) by  $v$

$$\int_S^T \int_0^1 v(\rho_t - \rho_x + \frac{1}{2}a(x)(\rho - \xi)) dx dt = 0. \quad (3.120)$$

First, an integration by part with respect to  $t$  gives

$$\int_0^1 \int_S^T v \rho_t dt dx = - \int_0^1 \int_S^T v_t \rho dt dx + \left[ \int_0^1 v \rho dx \right]_S^T. \quad (3.121)$$

Then, an integration by part with respect to  $x$  yields

$$- \int_0^1 v \rho_x dx = \int_0^1 v_x \rho dx = \int_0^1 v_x(z_x + z_t) dx = \int_0^1 v_x z_x dx + \int_0^1 v_x z_t dx. \quad (3.122)$$

We have that

$$\int_0^1 v_x z_x dx = - \int_0^1 v_{xx} z dx = - \int_0^1 \beta |z|^p dx, \quad (3.123)$$

which gives that

$$- \int_S^T \int_0^1 v \rho_x dx dt = - \int_S^T \int_0^1 \beta |z|^p dx dt + \int_S^T \int_0^1 v_x z_t dx dt. \quad (3.124)$$

Combining (3.120), (3.121) and (3.124), we obtain

$$\begin{aligned} \int_S^T \int_0^1 \beta |z|^p dx dt &= \int_S^T \int_0^1 v_x z_t dx dt - \int_0^1 \int_S^T v_t \rho dt dx + \left[ \int_0^1 v \rho dx \right]_S^T \\ &\quad + \frac{1}{2} \int_S^T \int_0^1 v a(x)(\rho - \xi) dx dt. \end{aligned} \quad (3.125)$$

We next multiply the second equation of (3.54) by  $v$  and, following the same steps

that yielded (3.125), we get

$$\begin{aligned} \int_S^T \int_0^1 \beta |z|^p dx dt &= - \int_S^T \int_0^1 v_x z_t dx dt + \int_0^1 \int_S^T v_t \xi dt dx - \left[ \int_0^1 v \xi dx \right]_S^T \\ &\quad + \frac{1}{2} \int_S^T \int_0^1 va(x)(\rho - \xi) dx dt. \end{aligned} \quad (3.126)$$

Now taking the sum of (3.125) and (3.126), we obtain

$$\begin{aligned} 2 \int_S^T \int_0^1 \beta |z|^p dx dt &= \int_0^1 \int_S^T v_t (\xi - \rho) dt dx - \left[ \int_0^1 v(\rho - \xi) dx \right]_S^T \\ &\quad + \int_S^T \int_0^1 va(x)(\rho - \xi) dx dt. \end{aligned} \quad (3.127)$$

Using the definition of  $\beta$ , we obtain

$$\begin{aligned} 2\mathbf{T}_5 &\leq \underbrace{\left| \left[ \int_0^1 v(\rho - \xi) dx \right]_S^T \right|}_{\mathbf{V}_1} + \underbrace{\int_S^T \int_0^1 |v_t| |(\xi - \rho)| dx dt}_{\mathbf{V}_2} \\ &\quad + \underbrace{\int_S^T \int_0^1 |va(x)(\rho - \xi)| dx dt}_{\mathbf{V}_3}. \end{aligned} \quad (3.128)$$

We start by estimating  $\mathbf{V}_1$ . For fixed  $t \in [S, T]$ , we have, by using (3.113)

$$\begin{aligned} \left| \int_0^1 v(\rho - \xi) dx \right| &\leq \int_0^1 (|v| |\rho| + |v| |\xi|) dx \leq 2 \frac{p-1}{p} \int_0^1 |v|^q dx \\ &\quad + \int_0^1 (F(\rho) + F(\xi)) dx \leq CC_p E_p(t), \end{aligned} \quad (3.129)$$

and hence, since  $E_p(T) \leq E_p(S)$ , we get

$$\mathbf{V}_1 \leq \left| - \left[ \int_0^1 v(\rho - \xi) dx \right]_S^T \right| \leq CC_p E_p(S). \quad (3.130)$$

Using Young's inequality, we have for every  $\eta > 0$

$$\begin{aligned} \mathbf{V}_2 &\leq \int_S^T \int_0^1 (|v_t| |\xi| + |v_t| |\rho|) dx dt \leq 2 \frac{p-1}{p\eta} \int_S^T \int_0^1 |v_t|^q dx dt \\ &\quad + \eta \int_S^T \int_0^1 (F(\rho) + F(\xi)) dx dt. \end{aligned} \quad (3.131)$$

From (3.114) and the fact that the definition of  $\beta$  implies  $\beta \leq C a$ , we get for every

$\sigma > 0$  that

$$\int_S^T \int_0^1 |v_t|^q \leq C_p \left( (p-2)\sigma \int_S^T E_p(t) dt + \frac{C}{\sigma^{p-2}} \int_S^T \int_0^1 a(x) |\rho - \xi|^p dx dt \right). \quad (3.132)$$

Using (3.91), we obtain for every  $\sigma, \mu_1 > 0$

$$\int_S^T \int_0^1 |v_t|^q \leq CC_p \left[ \left( (p-2)\sigma + \frac{\mu_1^p}{\sigma^{p-2}} \right) \int_S^T E_p(t) dt + \frac{1}{\mu_1 \sigma^{p-2}} E_p(S) \right]. \quad (3.133)$$

Combining (3.131) and (3.133), we obtain for every  $\eta, \sigma, \mu_1 > 0$

$$\mathbf{V}_2 \leq CC_p \left[ \left( (p-2)\frac{\sigma}{\eta} + \frac{\mu_1^p}{\sigma^{p-2}\eta} + \eta \right) \int_S^T E_p(t) dt + \frac{1}{\mu_1 \sigma^{p-2}\eta} E_p(S) \right]. \quad (3.134)$$

Choosing  $\sigma = \eta^2$  and  $\mu_1 = \eta^{2\frac{p-1}{p}}$ , one gets, for every  $\eta > 0$

$$\mathbf{V}_2 \leq CC_p \left( \eta \int_S^T E_p(t) dt + \frac{1}{\eta^r} E_p(S) \right). \quad (3.135)$$

Finally, we estimate  $\mathbf{V}_3$  in (3.128). Using Young's inequality, we have, for every  $\nu > 0$ ,

$$\mathbf{V}_3 \leq CC_p \left( \nu \int_S^T \int_0^1 |v|^q dx dt + \frac{1}{\nu} \int_S^T \int_0^1 a(x) |\rho - \xi|^p dx dt \right), \quad (3.136)$$

which yields by using (3.113) and (3.91), that for every  $\nu, \mu > 0$ , one has

$$\mathbf{V}_3 \leq CC_p \left( \left( \nu + \frac{\mu^p}{\nu} \right) \int_S^T E_p(t) dt + \frac{1}{\nu\mu} E_p(S) \right). \quad (3.137)$$

Choosing  $\mu^p = \nu$ , one gets that for every  $\eta > 0$

$$\mathbf{V}_3 \leq CC_p \left( \nu \int_S^T E_p(t) dt + \frac{1}{\nu^{1+\frac{2}{p}}} E_p(S) \right). \quad (3.138)$$

Combining (3.128), (3.130), (3.135) and (3.138) and taking  $\nu = \eta < 1$ , we obtain (3.119). ■

**End of the proof of Proposition 3.5.2**

Collecting (3.76), (3.95) and (3.119), we obtain for every positive  $\eta_2, \eta_3$  and  $\eta \in (0, 1)$  that

$$\int_S^T E_p(t) dt \leq CC_p \left[ \left( \eta_2^q + \frac{\eta}{\eta_2^p} \right) \int_S^T E_p(t) dt + \left( 1 + \frac{1}{\eta_2^p \eta^r} \right) E_p(S) \right]. \quad (3.139)$$

Taking  $\eta = \eta_2^{p+q}$  and fixing  $\eta_2$  so that  $2CC_p \eta_2^q = \frac{1}{2}$ , we immediately get (3.74). It is then standard to deduce that there exists  $\gamma_p > 0$  such that, for every  $(z_0, z_1) \in X_p$ , the energy  $E_p$  associated with of the solution  $z(t)$  of (3.1) starting at  $(z_0, z_1)$  satisfies the following,

$$E_p(t) \leq E_p(0) e^{1-\gamma_p t}, \quad t \geq 0. \quad (3.140)$$

That concludes the proof of Proposition 3.5.2.

**3.5.2 Case where  $1 < p < 2$** 

The main issue to prove Theorem 3.5.1 in the case  $p \in (1, 2)$  (when compared with the case  $p \in [2, \infty)$ ) is the trivial fact that  $p - 2 < 0$  and hence the weights  $f'(\rho)$  and  $f'(\xi)$  used in the multipliers of Items (m2) and (m3) may not be defined on sets of positive measure. As a consequence we cannot use these multipliers directly and we have to modify the functions  $f$  and  $F$ . This is why, we consider, for  $p \in (1, 2)$ , the functions  $h$  and  $H$  defined on (3.16) and (3.17).

It is clear that one has that  $|h(y)| \leq |f(y)|$  and  $|H(y)| \leq |F(y)|$  for every  $y \in \mathbb{R}$ . Finally, using the function  $h$ , we also modify the energy  $E_p$  by considering, for every  $t \in \mathbb{R}_+$  and every solution of (3.1), the function  $\mathcal{E}_p$  defined by

$$\mathcal{E}_p(t) = \int_0^1 (H(\rho) + H(\xi)) dx. \quad (3.141)$$

We start with an extension of Proposition 3.3.1.

**Lemma 3.5.5.** For every  $p \in (1, 2)$ , the energy  $t \mapsto \mathcal{E}_p(t)$  is non-increasing on  $\mathbb{R}_+$  along trajectories of (3.1).

**Proof.** This follows after using Proposition 3.3.1 with  $\mathcal{F} = \frac{H}{p}$  which admits a continuous first derivative by what precedes.

■

The proof of Theorem 3.5.1 in the case  $p \in (1, 2)$  relies on the following proposition which gives an estimate of the modified energy  $\mathcal{E}_p$  of a strong solution and which is similar to Proposition 3.5.2.

**Proposition 3.5.3.** Fix  $p \in (1, 2)$  and suppose that Hypothesis  $(\mathbf{H}_1)$  is satisfied. Then there exist positive constants  $C$  and  $C_p$  such that, for every  $(z_0, z_1) \in Y_p$  verifying

$$E_p(0) \leq 1, \quad (3.142)$$

we have the following energy estimate:

$$\forall 0 \leq S \leq T, \quad \int_S^T \mathcal{E}_p(t) dt \leq C C_p \mathcal{E}_p(S). \quad (3.143)$$

We next develop an argument for Proposition 3.5.3, which follows the lines of the proof of Proposition 3.5.2. The main idea consists in replacing  $f, F$  by  $h, H$  and to control all the constants  $C_p$  involved in these estimates in terms of  $p \in (1, 2)$ . We also provide a sketchy presentation where we only precise details specific to the present case.

We fix  $p \in (1, 2)$  and  $(z_0, z_1) \in Y_p$ . We recall that we have chosen  $x_0 = 0$  as an observation point and let  $0 < \epsilon_0 < \epsilon_1 < \epsilon_2$  with the corresponding sets  $Q_i = ]1 - \epsilon_i, 1 + \epsilon_i[$ ,  $i = 0, 1, 2$  as before.

As a consequence of (3.142) and Corollary 3.3.1 and standard estimates (such as the fact that  $\mathcal{E}_p \leq E_p$ ), one deduces that

$$|z(t, x)|^p + \mathcal{E}_p(t) \leq C_p, \quad \forall t \geq 0, x \in [0, 1], \quad (3.144)$$

where  $C_p$  is a positive constant that depends on  $p$  only.

In the case  $p \geq 2$ , we have used repeatedly Hölder's inequality for  $f, F$ , which is not adapted when dealing with  $h, H$  for the case  $p \in (1, 2)$ . Instead, one needs to consider the convex conjugate of  $H$  defined in (3.18) and use more generalized inequalities to handle this delicate case (see Section 3.2 for all the definitions and technical results).

### First pair of multipliers

For the first pair of multipliers, we change the function  $f$  in Item (m1) by the function  $h$  and hence use  $x\psi h(\rho)$ ,  $x\psi h(\xi)$ , where  $\psi$  is defined in (3.73).

**Lemma 3.5.6.** Under the hypotheses of Proposition 3.5.3, we have the following estimate

$$\int_S^T \mathcal{E}_p(t) dt \leq C C_p \mathcal{E}_p(S) + C \underbrace{\int_S^T \int_{Q_1 \cap (0,1)} (H(\rho) + H(\xi)) dx dt}_{\bar{\mathbf{S}}_4}. \quad (3.145)$$

*Proof.* Estimate 3.145 is obtained by following the exact same steps as those given to derive (3.76), with the difference that we use the function  $h$  instead of the function  $f$ . By multiplying the first equation of (3.54) by  $x\psi h(\rho)$  and the second one by  $x\psi h(\xi)$ , we perform the integrations by parts described to obtain (3.83) with the function  $f$  and, we are led to the similar equation

$$\begin{aligned} \int_S^T (H(\rho) + H(\xi)) dt &= \int_S^T \int_{Q_1 \cap (0,1)} (1 - (x\psi)_x) (H(\rho) + H(\xi)) dx dt \\ &+ \int_0^1 x\psi [H(\xi) - H(\rho)]_S^T dx - \frac{1}{2} \int_S^T \int_0^1 a(x)x\psi (h(\xi) + h(\rho))(\rho - \xi) dx dt, \end{aligned} \quad (3.146)$$

which yields that

$$\begin{aligned} \int_S^T \mathcal{E}_p(t) dt &\leq \underbrace{\int_S^T \int_{Q_1 \cap (0,1)} |(1 - (x\psi)_x)| (H(\rho) + H(\xi)) dx dt}_{\bar{\mathbf{S}}_1} \\ &+ \underbrace{\int_0^1 |x\psi| |[H(\xi) - H(\rho)]_S^T| dx}_{\bar{\mathbf{S}}_2} \\ &+ \underbrace{\frac{1}{2} \int_S^T \int_0^1 |a(x)x\psi| |h(\xi) + h(\rho)| |\rho - \xi| dx dt}_{\bar{\mathbf{S}}_3}. \end{aligned} \quad (3.147)$$

Using the fact that  $\psi_x$  is bounded, we get at once that

$$\bar{\mathbf{S}}_1 \leq C \int_S^T \int_{Q_1 \cap (0,1)} (H(\rho) + H(\xi)) dx dt \leq C \bar{\mathbf{S}}_4, \quad (3.148)$$

where  $\bar{\mathbf{S}}_4$  has been defined in (3.145). Using now the fact that  $|x\psi| \leq 1$  and the fact



that  $t \mapsto \mathcal{E}_p(t)$  is non increasing, it follows that

$$\bar{\mathbf{S}}_2 \leq \mathcal{E}_p(T) + \mathcal{E}_p(S) \leq 2\mathcal{E}_p(S). \quad (3.149)$$

As for  $\bar{\mathbf{S}}_3$ , we proceed as for the estimate of  $\mathbf{S}_3$  by first using (3.31) and Lemma 3.2.7 instead of Lemmas 2.2.1 and 3.2.2 respectively.

In particular, we have the following estimate, which extends (3.91) to the case  $p \in (1, 2)$  and which holds for every  $\mu_1 \in (0, 1)$ ,

$$\int_S^T \bar{\mathcal{E}}_p(t) dt \leq C C_p \mu_1^p \int_S^T \mathcal{E}_p(t) dt + C \frac{C_p}{\mu_1^{2-p}} \mathcal{E}_p(S), \quad (3.150)$$

where  $\bar{\mathcal{E}}_p(t)$  is defined by

$$\bar{\mathcal{E}}_p(t) = \int_0^1 a(x) H(z_t) dx. \quad (3.151)$$

## Second pair of multipliers

The goal of this subsection is to estimate  $\bar{\mathbf{S}}_4$ . To do so, we change the function  $f$  in Item (m2) by the function  $h$  and hence define the pair of multipliers:  $\phi h'(\rho)z, \phi h'(\xi)z$  where  $\phi$  is defined in (3.73).

**Lemma 3.5.7.** Under the hypotheses of Proposition 3.5.3 and for  $1 < p < 2$  with  $\phi$  as defined in (3.73), we have the following estimate:

$$\bar{\mathbf{S}}_4 \leq C \frac{C_p}{\eta_2^p} \underbrace{\int_S^T \int_{Q_2 \cap (0,1)} H(z) dx dt}_{\bar{\mathbf{T}}_5} + C C_p \eta_2^2 \int_S^T \mathcal{E}_p(t) dt + C C_p \mathcal{E}_p(S), \quad (3.152)$$

where  $\eta_2$  is an arbitrary constant in  $(0, 1)$  and  $C$  and  $C_p$  are positive constants whose dependence are specified in Remark 3.5.1. on  $p$ .

**Proof.** Estimate (3.152) is obtained by following the same steps as those given to derive (3.95), with  $h$  instead of  $f$ . By multiplying the first equation of (3.54) by  $\phi h'(\rho)z$  and the second one by  $\phi h'(\xi)z$ , where  $z$  is the solution of (3.1), we perform the integrations by parts described to obtain (3.104) with the function  $f$  and we are

led to the equation

$$\begin{aligned} \int_S \int_0^1 \phi (h(\rho)\rho + h(\xi)\xi) dx dt &= - \int_S \int_0^1 \phi_x z (h(\rho) + h(\xi)) dx dt \\ &+ \left[ \int_0^1 \phi (h(\xi) - h(\rho)) z dx \right]_S^T - \frac{1}{2} \int_S \int_0^1 \phi (h'(\rho) + h'(\xi)) z a(x) (\rho - \xi) dx dt \\ &+ 2 \int_S \int_0^1 \phi (h(\rho) - h(\xi)) (\rho - \xi) dx dt. \end{aligned} \quad (3.153)$$

According to (3.23), one has

$$h(\rho)\rho + h(\xi)\xi \geq h(\rho) + h(\xi), \quad \forall (\rho, \xi) \in \mathbb{R}^2. \quad (3.154)$$

Hence, also using the definition of  $\phi$ , it follows from (3.153) that

$$\begin{aligned} \bar{\mathbf{S}}_4 &\leq C \underbrace{\int_S \int_{Q_2 \cap (0,1)} |z (h(\rho) + h(\xi))| dx dt}_{\bar{\mathbf{T}}_1} + \underbrace{\left[ \int_{Q_2 \cap (0,1)} (h(\xi) - h(\rho)) z dx \right]_S^T}_{\bar{\mathbf{T}}_2} \\ &+ \frac{1}{2} \underbrace{\int_S \int_{Q_2 \cap (0,1)} (h'(\rho) + h'(\xi)) a(x) |z(\rho - \xi)| dx dt}_{\bar{\mathbf{T}}_3} \\ &+ 2 \underbrace{\int_S \int_{Q_2 \cap (0,1)} |(h(\rho) - h(\xi)) (\rho - \xi)| dx dt}_{\bar{\mathbf{T}}_4}, \end{aligned} \quad (3.155)$$

for some positive constant  $C$ . The above equation must be put in parallel with (3.105) where the term  $\bar{\mathbf{T}}_j$ ,  $1 \leq j \leq 4$  in (3.155) corresponds to the term  $\mathbf{T}_j$  in (3.105).

The term  $\bar{\mathbf{T}}_1$  is handled exactly as the term  $\mathbf{T}_1$  while using (3.31) instead of Lemma 2.2.1 in order to obtain

$$\bar{\mathbf{T}}_1 \leq \frac{C_p}{\eta_2^p} \int_S \int_{Q_2 \cap (0,1)} H(z) dx dt + C_p \eta_2^2 \int_S \mathcal{E}_p(t) dt, \quad (3.156)$$

where  $\eta_2 > 0$  is arbitrary.

We proceed similarly for the term  $\bar{\mathbf{T}}_2$  by using (3.31) instead of Young's inequality and Corollary 3.2.1 instead of the standard Poincaré inequality to obtain

$$\bar{\mathbf{T}}_2 \leq C_p \mathcal{E}_p(S). \quad (3.157)$$

The term  $\bar{\mathbf{T}}_4$  can also be treated identically as the term  $\mathbf{T}_4$  to obtain

$$\bar{\mathbf{T}}_4 \leq C\mathcal{E}_p(S). \quad (3.158)$$

We now turn to an estimate of  $\bar{\mathbf{T}}_3$  which differs slightly from that of  $\mathbf{T}_3$  because of the appearance of the function  $h'$ . Using (3.144) and the second equation in (3.27), one deduces that

$$(h'(\rho) + h'(\xi))|z| \leq C_p|h(z)|, \quad t \in [S, T], \quad x \in [0, 1], \quad (3.159)$$

where  $C_p$  is a positive constant only depending on  $p$ . One derives that

$$\bar{\mathbf{T}}_3 \leq C_p \int_S^T \int_{Q_2 \cap (0,1)} a(x)|h(z)|(|\rho| + |\xi|) dx dt. \quad (3.160)$$

Applying (3.31) to the above, we end up with an estimate of  $\bar{\mathbf{T}}_3$  by exactly the right-hand side of (3.156) and one concludes. ■

### Third multiplier

We finally turn to an estimation of the term  $\bar{\mathbf{T}}_5$  and, relying on the multiplier defined in Item (m3), we get after changing the function  $f$  by the function  $h$  the multiplier (still denoted)  $v$  solution of the following elliptic problem defined at every  $t \geq 0$  by

$$\begin{cases} v_{xx} = \beta h(z), & x \in [0, 1], \\ v(0) = v(1) = 0, \end{cases} \quad (3.161)$$

where  $\beta$  is defined in (3.73).

We will be needing the following estimates of  $v$  and  $v_t$  given in the next lemma.

**Lemma 3.5.8.** For  $v$  as defined in (3.161), we have the following estimates:

$$\int_0^1 \mathcal{H}(v) dx \leq CC_p \mathcal{E}_p(t), \quad (3.162)$$

$$\int_0^1 \mathcal{H}(v_t) dx \leq CC_p \bar{\mathcal{E}}_p(t), \quad (3.163)$$

where  $\bar{\mathcal{E}}_p(t)$  is defined in (3.151).

**Proof.** From the definition of  $v$ , one gets

$$v(t, x) = -x \int_x^1 (1-s)\beta h(z) ds - (1-x) \int_0^x s\beta h(z) ds, \quad x \in [0, 1]. \quad (3.164)$$

It immediately follows from the above that

$$\int_0^1 \mathcal{H}(v) dx \leq \mathcal{H} \left( C \int_0^1 \beta |h(z)| ds \right) \leq CC_p \mathcal{H} \left( \int_0^1 \beta |h(z)| ds \right), \quad (3.165)$$

where we have used (3.27) whether  $C_p \int_0^1 \beta |h(z)| ds \geq M$  or not. Since  $\mathcal{H}$  is (strictly) convex, one can apply Jensen's inequality to the right-hand side of the above equation to get that

$$\int_0^1 \mathcal{H}(v) dx \leq CC_p \int_0^1 \beta \mathcal{H}(h(z)) ds, \quad (3.166)$$

and one derives (3.162) by using (3.27) together with (3.144). Similarly, one has that

$$v_t(t, x) = -x \int_x^1 (1-s)\beta z_t h'(z) ds - (1-x) \int_0^x s\beta z_t h'(z) ds, \quad x \in [0, 1], \quad (3.167)$$

Upper bounding  $|g'(z)|$  by 1, one deduces that

$$\int_0^1 \mathcal{H}(v_t) dx \leq \mathcal{H} \left( C \int_0^1 \beta |z_t| \right) \leq CC_p \mathcal{H} \left( \int_0^1 \beta |z_t| \right) \leq CC_p \mathcal{H} \left( \int_0^1 a(x) |z_t| dx \right), \quad (3.168)$$

where we used the fact that  $\beta(x) \leq Ca(x)$  on  $[0, 1]$  and the convexity of  $\mathcal{H}$ .

Since  $\int_0^1 a(x) |z_t| dx = \int_{|z_t| \leq M} + \int_{|z_t| > M}$ , we have according to (3.27), (3.28) and Hölder's inequality that

$$\begin{aligned} \int_0^1 a(x) |z_t| dx &\leq CC_p \left( \int_0^1 a(x) h(|z_t|) dx + \int_0^1 a(x) h(|z_t|)^{\frac{1}{p}} dx \right) \\ &\leq CC_p \left( \int_0^1 a(x) h(|z_t|) dx + \bar{\mathcal{E}}_p^{\frac{1}{p}}(t) \right). \end{aligned} \quad (3.169)$$

By convexity of  $\mathcal{H}$ , we obtain after plugging the previous equation into (3.168) and using Jensen's inequality that

$$\int_0^1 \mathcal{H}(v_t) dx \leq CC_p \left( \int_0^1 a(x) \mathcal{H}(h(|z_t|)) dx + \mathcal{H}(\bar{\mathcal{E}}_p^{\frac{1}{p}}(t)) \right). \quad (3.170)$$

The first term in the right-hand side of the above inequality is clearly upper bounded

by  $CC_p \bar{\mathcal{E}}_p(t)$  thanks to (3.24). As for the second term, one has that

$$\mathcal{H}(\bar{\mathcal{E}}_p^{\frac{1}{p}}(t)) \leq CC_p h(y_*), \text{ where } y_* \text{ is defined by } h(y_*) = \bar{\mathcal{E}}_p^{\frac{1}{p}}(t). \quad (3.171)$$

By using (3.144), it follows that  $h(y_*) \leq CC_p$  and elementary computations using (3.27) and the fact that  $\bar{\mathcal{E}}_p \leq CC_p E_p$  yield that  $y_* \leq CC_p$ . Hence  $h(y_*) \geq CC_p y_*$ , i.e.,  $y_* \leq CC_p \bar{\mathcal{E}}_p^{\frac{1}{p}}(t)$ . Since  $g$  is convex and increasing on  $\mathbb{R}_+$ , one gets after using (3.27) that

$$\mathcal{H}(\bar{\mathcal{E}}_p^{\frac{1}{p}}(t)) \leq CC_p h\left(C_p \bar{\mathcal{E}}_p^{\frac{1}{p}}(t)\right) \leq CC_p h\left(\bar{\mathcal{E}}_p^{\frac{1}{p}}(t)\right) \leq CC_p \bar{\mathcal{E}}_p^{\frac{2}{p}}(t) \leq CC_p \bar{\mathcal{E}}_p(t), \quad (3.172)$$

where we have used repeatedly (3.144) and (3.27). This concludes the proof of (3.163). ■

We now use the multiplier  $v$  in (3.54) and we get the following result.

**Lemma 3.5.9.** Under the hypotheses of Proposition 3.5.3 with  $v$  as defined in (3.161), we have the following estimate

$$\bar{\mathbf{T}}_5 \leq CC_p \left( \eta^p \int_S^T \mathcal{E}_p(t) dt + \frac{1}{\eta^s} \mathcal{E}_p(S) \right), \quad (3.173)$$

where  $C_p$  is a positive number that depends on  $p$  only and  $\eta$  is any real number in  $(0, 1)$ .

**Proof.** Proceeding as in the proof of Lemma 3.5.4 to derive (3.128), we obtain

$$\begin{aligned} 2 \int_S^T \int_{Q_2 \cap (0,1)} zh(z) dx dt &\leq \underbrace{\left| \left[ \int_0^1 v(\rho - \xi) dx \right]_S^T \right|}_{\bar{\mathbf{V}}_1} + \underbrace{\int_S^T \int_0^1 |v_t| |(\xi - \rho)| dx dt}_{\bar{\mathbf{V}}_2} \\ &\quad + \underbrace{\int_S^T \int_0^1 |va(x)(\rho - \xi)| dx dt}_{\bar{\mathbf{V}}_3}. \end{aligned} \quad (3.174)$$

After using Fenchel's inequality (3.13) and (3.162), one gets the following estimate for  $\bar{\mathbf{V}}_1$

$$\bar{\mathbf{V}}_1 \leq CC_p \mathcal{E}_p(S). \quad (3.175)$$

As for  $\bar{\mathbf{V}}_2$ , we first apply (3.30) (corresponding to the adaptation to the case  $p \in (1, 2)$ )

of the use of Young's inequality in (3.131)) to get that

$$\bar{\mathbf{V}}_2 \leq C \frac{C_p}{\eta^q} \int_S \int_0^1 \mathcal{H}(v_t) + CC_p \eta^p \int_S \mathcal{E}_p(t) dt,$$

for every  $0 < \eta < 1$ . To handle the first integral term in the right-hand side of the above equation, we use (3.163) and (3.150) to get that

$$\bar{\mathbf{V}}_2 \leq C \frac{C_p}{\eta^q \mu^{2-p}} \mathcal{E}_p(S) + CC_p \left( \frac{\mu^p}{\eta^q} + \eta^p \right) \int_S \mathcal{E}_p(t) dt,$$

for every  $0 < \eta, \mu < 1$ . For  $\bar{\mathbf{V}}_3$ , we apply Fenchel's inequality, (3.162) and (3.150) to get that

$$\bar{\mathbf{V}}_3 \leq CC_p \left( \sigma^2 + \frac{\lambda^p}{\sigma^p} \right) \int_S \mathcal{E}_p(t) dt + C \frac{C_p}{\lambda^{2-p} \sigma^p} \mathcal{E}_p(S),$$

for every  $0 < \lambda, \sigma < 1$ . One chooses appropriately  $\lambda, \mu$  and  $\sigma$  in terms of  $\eta$  to easily conclude the proof of (3.173). ■

**End of the proofs of Proposition 3.5.3 and Theorem 3.5.1 in the case  $p \in (1, 2)$**  It is immediate to derive (3.143) by gathering (3.145), (3.152) and (3.173) with a constant  $C_p$  only depending on  $p$ . One deduces exponential decay of  $\mathcal{E}_p$  exactly of the type (3.140) with a constant  $\gamma_p > 0$  only depending on  $p$  for weak solutions verifying (3.142) for their initial conditions. Pick now any  $(z_0, z_1) \in X_p$  such that  $E_p(0) = 1$ . One deduces that for every  $t \geq 0$ ,

$$\mathcal{E}_p(t) \leq \mathcal{E}_p(0) e^{1-\gamma_p t} \leq e^{1-\gamma_p t}, \quad (3.176)$$

since  $\mathcal{E}_p \leq E_p$ . Set

$$\lambda_p := \left( \frac{p}{8} \right)^{\frac{1}{p}},$$

and let  $c_p$  be a positive constant such that

$$H(x) > c_p F(x), \text{ if } |x| > \lambda_p. \quad (3.177)$$

Note that such a constant  $c_p > 0$  exists according to the second equation in (3.28) and can be taken equal to  $\frac{p-1}{2}$ .

For every  $t \geq 0$  and  $x \in [0, 1]$ , let  $R(t, x) = \max(\rho(t, x), \xi(t, x))$ . It holds by

elementary computations that

$$\begin{aligned} \int_{R \leq \lambda_p} (F(\rho) + F(\xi)) dx &\leq \frac{1}{4}, \\ \int_{R > \lambda_p} (F(\rho) + F(\xi)) dx &< \frac{2}{c_p} \int_{R > \lambda_p} (H(\rho) + H(\xi)) dx. \end{aligned} \quad (3.178)$$

One deduces at once that

$$E_p(t) < \frac{1}{4} + \frac{2}{c_p} \mathcal{E}_p(t), \quad \forall t \geq 0. \quad (3.179)$$

Set

$$t_p := \frac{\left(1 + \ln\left(\frac{8}{c_p}\right)\right)}{\gamma_p}.$$

Then, using (3.176), it follows that  $E_p(t) \leq \frac{1}{2}$  if  $t \geq t_p$ . Since  $t \mapsto (E_p(t))^{\frac{1}{p}}$  is a norm on  $X_p$ , it implies that

$$\|S_p(t_p)\|_{X_p} = \sup_{E_p(0) \leq 1} E_p(t)^{\frac{1}{p}} \leq \left(\frac{1}{2}\right)^{\frac{1}{p}} < 1,$$

i.e., that the  $C^0$ -semi-group  $(S_p(t))_{t \geq 0}$  is exponentially stable for  $p \in (1, 2)$ .

**Remark 3.5.3.** From the argument, it is not difficult to see that  $\gamma_p$  is bounded above and  $c_p$  must tend to zero as  $p$  tends to one. That yields that our estimate for  $t_p$  tends to infinity as  $p$  tends to one. Hence it is not obvious how to use our line of proof to get exponential stability for  $p = 1$ .

### 3.6 Case of a global constant damping

Suppose now that we are dealing with a global constant damping, in other words  $\omega = (0, 1)$  and

$$a(x) \equiv 2\alpha, \quad \forall x \in (0, 1), \quad (3.180)$$

where  $\alpha$  is a positive constant. We then prove the following proposition.

**Proposition 3.6.1.** For  $p = 1$  or  $p = \infty$ , the semi-group  $(S(t))_{t \geq 0}$  is exponentially stable for a global constant damping if  $\alpha \in (0, 2)$ .

**Proof.** For every  $p \in (1, \infty)$ , we perform a change of unknown function, namely

$$z(t, x) = e^{-\alpha t} v(t, x), \quad x \in (0, 1), \quad t \geq 0,$$

where  $z$  is any solution of (3.1) starting at  $(z_0, z_1) \in X_p$ . Clearly  $v$  is a solution of

$$\begin{cases} v_{tt} - v_{xx} = \alpha^2 v & \text{in } \mathbb{R}_+ \times (0, 1), \\ v(t, 0) = v(t, 1) = 0 & t \geq 0, \\ v(0, \cdot) = z_0, \quad v_t(0, \cdot) = z_1 + \alpha z_0. \end{cases} \quad (3.181)$$

We use  $E_p$  and  $V_p$  to denote the  $p$ th-energies associated with  $z$  and  $v$  respectively. Since  $z_x = e^{-\alpha t} v_x$  and  $z_t = e^{-\alpha t} (v_t - \alpha v)$ , we get, after using Lemma 3.2.1 and the following inequality (cf. [1, Lemma 2.2])

$$|a + b|^p \leq 2^{p-1} (|a|^p + |b|^p), \quad \forall (a, b) \in \mathbb{R}^2, \quad (3.182)$$

that, for every  $t \geq 0$ ,

$$E_p(t) \leq e^{-\alpha p t} \left( 2^{p-1} V_p(t) + 2^p \alpha^p \int_0^1 |v(x)|^p dx \right) \leq e^{-\alpha p t} \left( 2^{p-1} + \frac{\alpha^p}{p^2} \right) V_p(t). \quad (3.183)$$

On the other hand, for strong solutions of (3.181), one has after applying Corollary 3.3.1 to  $v$  that, for every  $t \geq 0$ ,

$$V_p'(t) = \alpha^2 \int_0^1 v (|\rho|^{p-1} - |\xi|^{p-1}) dx,$$

which yields, by using Young's inequality, that

$$V_p'(t) \leq \alpha^2 \left( 2 \frac{\eta^p}{p} \int_0^1 |v|^p dx + \frac{p}{q\eta^q} V_p(t) \right), \quad (3.184)$$

for every  $\eta > 0$ . Using again Lemma 3.2.1 and the fact that  $v_x = \frac{1}{2}(\rho + \xi)$  together with (3.182), we obtain that, for every  $t \geq 0$ ,

$$V_p'(t) \leq \alpha^2 \left( \eta^p K_p + \frac{p}{q\eta^q} \right) V_p(t),$$

where we have set  $K_p := \frac{1}{p2^p}$ . The minimum with respect to  $\eta$  of  $\eta^p K_p + \frac{p}{q\eta^q}$  is equal to  $pK_p^{\frac{1}{p}}$ , and one gets by using Gronwall's lemma that

$$V_p(t) \leq V_p(0) e^{\alpha^2 p K_p^{\frac{1}{p}} t}. \quad (3.185)$$



Combining (3.183) and (3.185), one gets that, for every  $t \geq 0$ ,

$$E_p(t)^{\frac{1}{p}} \leq (2 + \alpha)^2 e^{M_\alpha t} E_p(0)^{\frac{1}{p}},$$

where

$$M_\alpha := -\alpha + \alpha^2 K_p^{\frac{1}{p}} = -\alpha \left( 1 - \frac{\alpha}{2p^{\frac{1}{p}}} \right).$$

One concludes easily by letting  $p$  tend either to one or  $\infty$  and using an obvious density argument.

■

# $L^p$ -asymptotic stability of 1D damped wave equations with nonlinear damping

This work has been published in the scientific preprint [20].

## 4.1 Introduction

In this chapter we consider the damped one dimension wave equation with nonlinear and localized damping and Dirichlet boundary conditions which yields the following nonlinear problem:

$$\begin{cases} z_{tt} - z_{xx} + a(x)g(z_t) = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \\ z(t, 0) = z(t, 1) = 0 & \text{for } t \geq 0, \\ z(0, \cdot) = z_0, z_t(0, \cdot) = z_1 & \text{on } (0, 1), \end{cases} \quad (4.1)$$

where initial data  $(z_0, z_1)$  belong to an  $L^p$  functional space to be defined later, where  $p \in [1, \infty)$ . The function  $a$  is a continuous non-negative function on  $[0, 1]$ , bounded from below by a positive constant on some non-empty open interval  $\omega$  of  $(0, 1)$ , which represents the region of the domain where the damping term is active. The nonlinearity  $g : \mathbb{R} \mapsto \mathbb{R}$  is a  $C^1$  non-decreasing function such that  $g(0) = 0$ ,  $g'(0) > 0$ , and  $g(x)x \geq 0 \quad \forall x \in \mathbb{R}$ .

The nonlinear problem (4.1) has already been studied several times in a hilbertian framework, i.e., with  $p = 2$ . The well-posedness is a classical result and is a consequence of the theory of maximal monotone operators (see for instance [25] for the

non-localized case and [18] for the localized case). Exponential stability was treated using the multiplier method that was generally presented in [21] and then used in different contexts of the linear problem, we refer to [4] and [24] for more details and extensive references in the Hilbertian framework. The same method was adopted and used in the nonlinear problem by [25] to prove the exponential stability in two dimensions with no localization and with a growth hypothesis on the nonlinearity  $g$ . Then the exponential stability for the localized case was established in [18] with the same hypotheses.

However, when it comes to the non-hilbertian framework, fewer results exist since it is still an unusual framework for the study of PDEs and has only been considered recently. As a result, we have less tools and techniques available to work with in such framework. The usual well-posedness proof techniques based on Hilbert Spaces (maximal monotone operators for instance) are no longer usable in this framework. Similarly, the multiplier method as known in the hilbertian framework to prove exponential stability does not work and requires at least to be generalized of the multipliers in order to achieve stability results.

The main results that exist and are relevant to our context are primarily gathered in [10], [16] and [19]. The  $p$ -th energy  $E_p$  of a solution that has been used in the three references has been introduced first in [16] as a generalization of the standard Hilbertian energy  $E_2$ . It is an equivalent energy to the natural energy in the  $L^p$  framework and is defined by:

$$E_p(t) = \frac{1}{p} \int_{\Omega} (|z_x(t, x) + z_t(t, x)|^p + |z_x(t, x) - z_t(t, x)|^p) dx. \quad (4.2)$$

The reference [16] also provides some useful energy estimates that were used for instance in [5], [10] and [19] to obtain stability results.

In the nonlinear case, [16] proves the well-posedness for all  $p \geq 2$  using an argument based on the well-posedness in the Hilbertian framework and an equivalent energy functional but with global growth conditions on  $g$ . Useful  $L^p$  estimates have also been provided in [16], which led to proving polynomial decay of the energy in the nonlinear case, with  $g$  a non-decreasing  $C^1$  function behaving like  $ks|s|^r$ ,  $r, k > 0$ . We extend in this chapter the results of [16] by proposing a well-posedness proof for weak solutions for all  $2 \leq p < \infty$  and a proof for strong solutions for all  $1 \leq p < \infty$ . The proof is based on a density argument combined with the well-posedness already established in  $L^\infty$  by [10]. We also extend the stability result by proving an exponential stability

with no additional growth hypotheses on the nonlinearity  $g$ .

Always in the nonlinear case but with a linearly bounded damping, an argument based on D'Alembert formula and fixed point theory is used in [10] to prove the well-posedness for all  $p \geq 2$ . The reference proves the existence of solutions in  $L^\infty$  framework for any nonlinear  $g$  satisfying the hypotheses mentioned in the beginning of the introduction. However, for the well-posedness in an  $L^p$  framework, the damping is supposed to be (uniformly) linearly bounded to be able to use the fixed point argument with D'Alembert formula. The latter reference relies on Lyapunov techniques for linear time varying systems along with estimates inspired from [16] to prove  $L^p$  semi-global exponential stability in the nonlinear problem under restrictive hypotheses on initial data (imposed to belong to  $L^\infty$  functional spaces) and for  $p \geq 2$ ; other stability results have been shown in the same reference in particular  $L^\infty$  stability but always with more conditions on initial data. Still in [10], They another stability result is obtained when the nonlinearity is (uniformly) linearly bounded using an interpolation for the initial data: semi-global exponential stability for  $q$  satisfying  $2 \leq q < p$  with initial data belonging to both  $L^p$  spaces and  $L^2$  is established. We extend the well-posedness results in  $L^\infty$  of [10] to  $L^p$  frameworks for  $p \geq 2$  and also their well-posedness results in  $L^p$  for  $p \geq 2$  by removing the assumption of a linearly bounded nonlinearity. The latter well-posedness result is also stated in our work for  $1 \leq p < 2$  for strong solutions. Additionally, we extend the stability results of [10] by providing a semi-global exponential stability result for strong solutions with no additional restrictions on initial data or the nonlinearity  $g$ .

The results of this chapter can also be seen as an extension to the nonlinear case for  $1 < p < \infty$  of [18] where the well-posedness was established for the linear problem for all  $p > 1$  using the argument based on D'Alembert formula from [10] and an exponential decay in the linear problem for all  $1 < p < +\infty$  was proved using a generalized multiplier method. The same reference also establishes an exponential stability when  $p = \infty$  and  $p = 1$  in some cases of a constant global damping.

We use the work that has already been done in the linear case in [19] alongside with some techniques from [25] and [10] to provide a proof for the exponential stability of strong solutions in the nonlinear case. The proof is based on a linearizing principle to reduce the study of the nonlinear problem to that of the linear problem.

The chapter is organized as follows: in Section 4.2 we properly state the problem with the functional framework and the considered hypotheses, we also rewrite the problem

using Riemann invariants and we prove that the  $p$ -th energy  $E_p$  is non-increasing. In Section 4.3, we give the proof of the well-posedness of the problem for  $1 \leq p < \infty$  by proving the existence and the uniqueness of weak solutions for  $2 \leq p < \infty$  and the existence and the uniqueness of strong solutions for  $1 \leq p < \infty$ . In Section 4.4 we prove the exponential decay of the energy by treating first an auxiliary linear problem and then by concluding the result for the nonlinear case.

## 4.2 Statement of the problem

Consider Problem (4.1) where we assume the following hypotheses satisfied:

**(H<sub>1</sub>)**  $a : [0, 1] \rightarrow \mathbb{R}$  is a non-negative continuous function such that

$$\exists a_0 > 0, a \geq a_0 \text{ on } \omega = (b, c) \subset [0, 1], \quad (4.3)$$

where  $\omega$  is a non empty interval such that  $b = 0$  or  $c = 1$ , i.e.,  $\bar{\omega}$  contains a neighborhood of 0 or 1. There is no loss of generality in assuming  $c = 1$ , taking 0 as an observation point.

**Remark 4.2.1.** Just like mentioned in Chapter 3, the hypothesis that  $\bar{\omega}$  contains a neighborhood of 0 or 1 can be removed. We impose it just for the sake of simplicity of computations.

**(H<sub>2</sub>)**  $g : \mathbb{R} \rightarrow \mathbb{R}$  a  $C^1$  non-decreasing function such that  $g(0) = 0$ ,  $g'(0) > 0$ , and

$$g(x)x \geq 0 \quad \forall x \in \mathbb{R}, \quad (4.4)$$

We redefine now the functional framework. For  $p \in [1, \infty]$ , consider the function spaces

$$X_p := W_0^{1,p}(0, 1) \times L^p(0, 1), \quad (4.5)$$

$$Y_p := (W^{2,p}(0, 1) \cap W_0^{1,p}(0, 1)) \times W_0^{1,p}(0, 1), \quad (4.6)$$

where  $X_p$  is equipped with the norm

$$\|(u, v)\|_{X_p} := \left( \frac{1}{p} \int_0^1 (|u' + v|^p + |u' - v|^p) dx \right)^{\frac{1}{p}}, \quad \text{if } 1 \leq p < \infty. \quad (4.7)$$

$$\|(u, v)\|_{X_\infty} := \|u' + v\|_\infty + \|u' - v\|_\infty, \quad \text{if } p = \infty, \quad (4.8)$$

and the space  $Y_p$  is equipped with the norm

$$\|(u, v)\|_{Y_p} := \left( \frac{1}{p} \int_0^1 (|u'' + v'|^p + |u'' - v'|^p) dx \right)^{\frac{1}{p}}, \quad \text{if } 1 \leq p < \infty. \quad (4.9)$$

$$\|(u, v)\|_{Y_\infty} := \|u'' + v'\|_\infty + \|u'' - v'\|_\infty, \quad \text{if } p = \infty. \quad (4.10)$$

$$E_\infty(t) = \|z_x(t, \cdot) + z_t(t, \cdot)\|_\infty + \|z_x(t, \cdot) - z_t(t, \cdot)\|_\infty \quad (4.11)$$

Initial conditions  $(z_0, z_1)$  for weak (resp. strong) solutions of (4.1) are taken in  $X_p$  (resp. in  $Y_p$ ).

**Definition 4.2.1.** The solutions of (4.1) are defined as follows.

(i) For all  $(z_0, z_1) \in X_p$ , the function

$$z \in L^\infty(\mathbb{R}_+, W_0^{1,p}(0, 1)) \cap W^{1,\infty}(\mathbb{R}_+, L^p(0, 1))$$

is said to be a weak solution of Problem (4.1) if it satisfies the problem in the dual sense (meaning that the equalities are taken in the weak topology of  $X_p$ ).

(ii) For all  $(z_0, z_1) \in Y_p$ , the function

$$z \in L^\infty(\mathbb{R}_+, W^{2,p}(0, 1) \cap W_0^{1,p}(0, 1)) \cap W^{1,\infty}(\mathbb{R}_+, W_0^{1,p}(0, 1))$$

is said to be a strong solution of Problem (4.1) if it satisfies the problem in the classical sense.

We define the Riemann invariants for all  $(t, x) \in \mathbb{R}_+ \times (0, 1)$  by

$$\rho(t, x) = z_x(t, x) + z_t(t, x), \quad (4.12)$$

$$\xi(t, x) = z_x(t, x) - z_t(t, x). \quad (4.13)$$

Along strong solutions of (4.1), we deduce that

$$\begin{cases} \rho_t - \rho_x = -a(x)g\left(\frac{\rho-\xi}{2}\right) & \text{in } \mathbb{R}_+ \times (0, 1), \\ \xi_t + \xi_x = a(x)g\left(\frac{\rho-\xi}{2}\right) & \text{in } \mathbb{R}_+ \times (0, 1), \\ \rho(t, 0) - \xi(t, 0) = \rho(t, 1) - \xi(t, 1) = 0 & \forall t \in \mathbb{R}_+, \\ \rho_0 := \rho(0, \cdot) = y'_0 + y_1, \quad \xi_0 := \xi(0, \cdot) = y'_0 - y_1, \end{cases} \quad (4.14)$$

with  $(\rho_0, \xi_0) \in W^{1,p}(0, 1) \times W^{1,p}(0, 1)$ .

The  $p$ th-energy  $E_p$  of a solution  $z$  defined can be written as

$$E_p(t) = \frac{1}{p} \int_0^1 (|\rho|^p + |\xi|^p) dx. \quad (4.15)$$

We prove the same proposition as Proposition 3.3.1 from Chapter 3.

**Proposition 4.2.1.** Let  $p \in [1, \infty)$  and suppose that a strong solution  $y$  of (4.1) exists and is defined on a non trivial interval  $I \subset \mathbb{R}_+$  containing 0, for some initial conditions  $(z_0, z_1) \in Y_p$ . For  $t \in I$ , define

$$\Phi(t) := \int_0^1 [\mathcal{F}(\rho) + \mathcal{F}(\xi)] dx, \quad (4.16)$$

where  $\rho, \xi$  are defined in (4.12) and  $\mathcal{F}$  is a  $C^1$  convex function. Then  $\Phi$  is well defined for  $t \in I$  and satisfies

$$\frac{d}{dt} \Phi(t) = - \int_0^1 a(x)g\left(\frac{\rho-\xi}{2}\right) (\mathcal{F}'(\rho) - \mathcal{F}'(\xi)) dx \leq 0. \quad (4.17)$$

**Proof.** The proof is similar to the proof of Proposition 3.3.1 from Chapter 3. We obtain by following the same steps that

$$\frac{d}{dt} \int_0^1 (\mathcal{F}(\rho) + \mathcal{F}(\xi)) dx = - \int_0^1 a(x)g\left(\frac{\rho-\xi}{2}\right) (\mathcal{F}'(\rho) - \mathcal{F}'(\xi)) dx. \quad (4.18)$$

Thanks to the convexity of  $\mathcal{F}$  and the hypothesis on  $g$  that states that  $g(x).x > 0$  for all  $x \neq 0$ , we conclude that

$$- \int_0^1 a(x)g\left(\frac{\rho-\xi}{2}\right) (\mathcal{F}'(\rho) - \mathcal{F}'(\xi)) dx \leq 0, \quad (4.19)$$

which concludes the proposition. ■

**Remark 4.2.2.** The previous proposition has been first introduced in [14] and reused in [10] to prove that the energy functional is non-increasing. Then it was improved in [19] by omitting the hypothesis that function  $\mathcal{F}$  should be even on top of being convex.

Before we state the next result, we remind the reader of the following notation

$$\forall r \geq 0, \quad \lfloor x \rfloor^r := \operatorname{sgn}(x)|x|^r, \quad \forall x \in \mathbb{R}, \quad (4.20)$$

where  $\operatorname{sgn}(x) = \frac{x}{|x|}$  for nonzero  $x \in \mathbb{R}$  and  $\operatorname{sgn}(0) = [-1, 1]$ . We have the following obvious formulas which will be repeatedly used later on:

$$\frac{d}{dx}(\lfloor x \rfloor^r) = r|x|^{r-1}, \quad \forall r \geq 1, \quad x \in \mathbb{R}, \quad (4.21)$$

$$\frac{d}{dx}(|x|^r) = r\lfloor x \rfloor^{r-1}, \quad \forall r > 1, \quad x \in \mathbb{R}. \quad (4.22)$$

**Corollary 4.2.1.** For  $(z_0, z_1) \in Y_p$ , one has that along strong solutions and for  $t \geq 0$ ,

$$E'_p(t) = - \int_0^1 a(x)g\left(\frac{\rho - \xi}{2}\right) (\lfloor \rho \rfloor^{p-1} - \lfloor \xi \rfloor^{p-1}) dx, \quad (4.23)$$

where  $\rho$  and  $\xi$  are the Riemann invariants defined in (4.12). Moreover, for  $(z_0, z_1) \in X_p$ , suppose that the solution  $z$  of (4.1) exists on  $\mathbb{R}_+$ . Then the energy  $t \mapsto E_p(t)$  is non-increasing.

The first part of the corollary is an immediate application of Proposition 3.3.1 while the second part is obtained by a standard density argument.

## 4.3 Well-posedness

Before addressing stabilization issues, we start by studying the well-posedness of the problem. One should note that our proof of the well-posedness in the linear problem is no longer applicable to the nonlinear problem. The reason is that we would find ourselves with a fixed-point problem for a map that does not necessarily map to itself. Hence, the requirement of a new proof for the nonlinear case.

**Theorem 4.3.1.** Suppose Hypotheses  $(\mathbf{H}_1)$  and  $(\mathbf{H}_2)$  are satisfied, then for all initial conditions  $(z_0, z_1) \in X_p$  with  $2 \leq p \leq \infty$ , we have the existence of a unique weak



solution  $z$  such that

$$z \in L^\infty(\mathbb{R}_+, W_0^{1,p}(0, 1)) \text{ and } z_t \in L^\infty(\mathbb{R}_+, L^p(0, 1)). \quad (4.24)$$

Moreover, if  $(z_0, z_1) \in Y_p$  with  $1 \leq p \leq \infty$  then we have the existence of a unique strong solution  $z$  such that

$$z \in L^\infty(\mathbb{R}_+, W^{2,p}(0, 1) \cap W_0^{1,p}(0, 1)) \text{ and } z_t \in L^\infty(\mathbb{R}_+, W_0^{1,p}(0, 1)). \quad (4.25)$$

**Proof:**

**Weak solutions:** For  $p = \infty$  the well-posedness has already been proved in [10]. Fix  $2 \leq p < +\infty$  and let  $(z_0, z_1) \in X_p$ .

Since  $X_\infty$  is dense in  $X_p$  for all  $2 \leq p < \infty$  and  $Z_0 = (z_0, z_1) \in X_p$ , there exists a sequence  $\{Z_0^n\}_n \subset X_\infty$  such that  $Z_0^n \rightarrow Z_0$  in  $X_p$ .

Since  $Z_0^n \in X_\infty$ , we have thanks to [10, Theorem 1] the existence of a unique solution  $Z^n = (z^n, z_t^n)$  such that  $(z^n, z_t^n) \in L^\infty(\mathbb{R}_+; W_0^{1,\infty}(0, 1)) \times W^{1,\infty}(\mathbb{R}_+; L^\infty(0, 1))$ . Moreover we have that for all  $t \geq 0$

$$\|(z^n, z_t^n)\|_{X_\infty} \leq 2 \max(\|z_0^n\|_{L^\infty(0,1)}, \|z_1^n\|_{L^\infty(0,1)}). \quad (4.26)$$

We prove now that, for every  $t_0 \geq 0$ , the sequence  $\{Z^n(t_0, \cdot)\}_n$  is a Cauchy sequence in  $X_p$ . Define for all  $(t, x) \in \mathbb{R}_+ \times (0, 1)$  the quantity  $e^{n,m}$  as

$$e^{n,m} = z^n - z^m, \quad (4.27)$$

which is a solution of the following problem

$$\begin{cases} e_{tt}^{n,m} - e_{xx}^{n,m} + a(x)(g(z^n) - g(z^m)) = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \\ e^{n,m}(t, 0) = e^{n,m}(t, 1) = 0 & t \geq 0, \\ e^{n,m}(0, \cdot) = z_0^n - z_0^m, \quad e_t^{n,m}(0, \cdot) = z_1^n - z_1^m. \end{cases} \quad (4.28)$$

The energy of  $e^{n,m}$  at time  $t$  is denoted by  $E_p(e^{n,m})(t)$  and is non-increasing. Indeed, if we use the Riemann invariants of Problem (4.28) denoted by  $\rho(e^{n,m})$ ,  $\xi(e^{n,m})$  given by  $\rho(e^{n,m}) = \frac{e_x^{n,m} + e_t^{n,m}}{2}$  and  $\xi(e^{n,m}) = \frac{e_x^{n,m} - e_t^{n,m}}{2}$  and rewrite the problem like Problem (4.14), then we follow the same steps in Proposition 4.2.1 and Corollary (4.2.1), we

obtain that along strong solutions of (4.28)

$$E_p(e^{n,m})'(t) = -2 \int_0^1 (g(z_t^n) - g(z_t^m)) (\mathcal{F}'(\rho(e^{n,m})) - \mathcal{F}'(\xi(e^{n,m}))) dx, \quad (4.29)$$

where  $\mathcal{F}$  is taken to be the convex function  $|\cdot|^p$ . By simple manipulations, it follows that

$$\begin{aligned} E_p(e^{n,m})'(t) &= -2 \int_0^1 \frac{(g(z_t^n) - g(z_t^m))}{z_t^n - z_t^m} (\mathcal{F}'(\rho(e^{n,m})) - \mathcal{F}'(\xi(e^{n,m}))) (\rho(e^{n,m}) - \xi(e^{n,m})) dx \leq 0, \end{aligned} \quad (4.30)$$

which confirms using a density argument that  $E_p(e^{n,m})$  is non-increasing along weak solutions. It follows that

$$E_p(z^n - z^m)(t) = E_p(e^{n,m})(t) \leq E_p(e^{n,m})(0) = E_p(z_0^n - z_0^m), \quad (4.31)$$

which gives that for every  $t_0 \geq 0$ ,  $\{Z^n(t_0, \cdot)\}_n$  is a Cauchy sequence in  $X_p$  since  $\{Z_0^n\}_n$  is a Cauchy sequence in  $X_p$ . It follows then that  $\{Z^n(t_0, \cdot)\}_n$  converges to a limit in  $X_p$  that we denote by  $Z(t_0, \cdot) = (z(t_0, \cdot), z_t(t_0, \cdot))$ , in particular for  $t_0 = 0$  we have that  $Z(0, \cdot) = Z_0(\cdot)$ . Note also that the convergence is uniform with respect to  $t_0 \geq 0$ . We define the function  $(t, x) \mapsto Z(t, x)$ , where  $Z(t_0, \cdot)$  is the limit of  $Z^n(t_0, \cdot)$  in  $X_p$  for every  $t_0 \geq 0$ .

We need to prove now that the limit  $Z$  is a weak solution of (4.1). Fix  $T > 0$  and denote  $\psi$  a test function that belongs to  $C^1([0, T] \times [0, 1])$  also verifying  $\psi(T, \cdot) = \psi(0, \cdot) \equiv 0$  and  $\psi(\cdot, 0) = \psi(\cdot, 1) \equiv 0$ . Define

$$A_T^n(\psi) = \int_0^T \int_0^1 (z_{tt}^n - z_{xx}^n) \psi dx dt. \quad (4.32)$$

We have that

$$A_T^n(\psi) - A_T^m(\psi) = \int_0^T \int_0^1 (z_{tt}^n - z_{xx}^n) \psi dx dt - \int_0^T \int_0^1 (z_{tt}^m - z_{xx}^m) \psi dx dt, \quad (4.33)$$

by integrating by part, it follows that

$$A_T^n(\psi) - A_T^m(\psi) = - \int_0^T \int_0^1 (z_t^n - z_t^m) \psi_t dx dt + \int_0^T \int_0^1 (z_x^m - z_x^n) \psi_x dx dt.$$

Using Holder's inequality,

$$\begin{aligned} |A_T^n(\psi) - A_T^m(\psi)| &\leq \left( \int_0^T \int_0^1 |z_t^n - z_t^m|^p dx dt \right)^{\frac{1}{p}} \left( \int_0^T \int_0^1 |\psi_t|^q dx dt \right)^{\frac{1}{q}} \\ &\quad + \left( \int_0^T \int_0^1 |z_x^n - z_x^m|^p dx dt \right)^{\frac{1}{p}} \left( \int_0^T \int_0^1 |\psi_x|^q dx dt \right)^{\frac{1}{q}}, \end{aligned} \quad (4.34)$$

which means that

$$|A_T^n(\psi) - A_T^m(\psi)| \leq T^{\frac{1}{p}} E_p(e^{n,m})^{\frac{1}{p}}(0) \left( \|\psi_t\|_{L^q((0,T) \times (0,1))} + \|\psi_x\|_{L^q((0,T) \times (0,1))} \right). \quad (4.35)$$

By a density argument we obtain that for all  $\psi$  in the space

$$\begin{aligned} \mathcal{X}_q^T = \{ \psi : [0, T] \times [0, 1] \mapsto \mathbb{R}_+ : (\psi, \psi_t) \in W^{1,q}((0, T) \times (0, 1)) \times L^q((0, T) \times (0, 1)), \\ \psi(T, \cdot) = \psi(0, \cdot) \equiv 0 \text{ and } \psi(\cdot, 0) = \psi(\cdot, 1) \equiv 0 \}, \end{aligned} \quad (4.36)$$

where  $q$  is the conjugate exponent of  $p$  and is equal to  $\frac{p}{p-1}$  for  $p \geq 2$  we have that

$$|A_T^n(\psi) - A_T^m(\psi)| \leq T^{\frac{1}{p}} E_p(e^{n,m})^{\frac{1}{p}}(0) \left( \|\psi_t\|_{L^q((0,T) \times (0,1))} + \|\psi_x\|_{L^q((0,T) \times (0,1))} \right), \quad (4.37)$$

and then

$$|A_T^n(\psi) - A_T^m(\psi)| \leq T^{\frac{1}{p}} E_p(e^{n,m})^{\frac{1}{p}}(0) \|\psi\|_{\mathcal{X}_q^T}, \quad (4.38)$$

which gives that  $\{A_T^n\}_n$  is a Cauchy sequence in  $(\mathcal{X}_q^T)'$ , the dual of  $\mathcal{X}_q^T$ , since  $\{Z_0^n\}_n$  is in  $X_p$ . We conclude then that  $\{A_T^n\}_n$  converges in  $(\mathcal{X}_q^T)'$ , i.e.,  $\{z_{tt}^n - z_{xx}^n\}_n$  converges to  $z_{tt} - z_{xx}$  as a linear functional on  $\mathcal{X}_q^T$ .

We also know that for all  $n \in \mathbb{N}$ ,

$$z_{tt}^n - z_{xx}^n = -g(z_t^n), \text{ on } \mathbb{R}_+ \times (0, 1), \quad (4.39)$$

which means that, the sequence of linear functionals  $\{-g(z_t^n)\}_n$  defined on  $\mathcal{X}_q^T$  also converges  $z_{tt}(t, \cdot) - z_{xx}(t, \cdot)$  in  $(\mathcal{X}_q^T)'$ .

We use now the existence of weak solutions in  $L^2$  framework (see [25], [18]). For  $(z_0, z_1) \in X_p$  with  $p \geq 2$ , we have the existence of a unique weak solution  $z \in W^{1,\infty}(\mathbb{R}_+, L^2(0, 1)) \cap L^\infty(\mathbb{R}_+, H_0^1(0, 1))$ . The solution  $z$  satisfies that for almost ev-

ery  $t \in \mathbb{R}_+$

$$z_{tt}(t, \cdot) - z_{xx}(t, \cdot) = -g(z_t(t, \cdot)) \quad \text{in } H^{-1}(0, 1). \quad (4.40)$$

In particular, for every  $T > 0$ ,  $z_{tt}(t, \cdot) - z_{xx}(t, \cdot) = -g(z_t)$  belongs to  $(\mathcal{X}_2^T)'$ . Since  $q \leq 2$ , one has that  $(\mathcal{X}_q^T)' \subset (\mathcal{X}_2^T)'$ , yielding in particular that  $z$  is a weak solution of System (4.1) in  $X_p$ .

**Strong solutions:** Take  $Z_0 = (z_0, z_1) \in Y_p$  for all  $1 \leq p < \infty$ , using Sobolev embeddings classical results, we have that  $Z_0 = (z_0, z_1) \in X_\infty$ .

We use [10, Theorem 1] to have the existence of a unique solution  $Z^n = (z, z_t)$  such that

$$(z, z_t) \in L^\infty(\mathbb{R}_+; W_0^{1,\infty}(0, 1)) \times W^{1,\infty}(\mathbb{R}_+; L^\infty(0, 1)).$$

Moreover we have the following inequality also proved in [10, Theorem 1] for all  $t \geq 0$

$$\|(z, z_t)\|_{X_\infty} \leq 2 \max(\|z'_0\|_{L^\infty(0,1)}, \|z_1\|_{L^\infty(0,1)}). \quad (4.41)$$

We are going to use Proposition 3.3.1 for  $w = z_t$ , where  $z$  is a solution of Problem (4.1). By differentiating (4.1) with respect to  $t$ , we obtain that  $w$  satisfies the following problem

$$\begin{cases} w_{tt} - w_{xx} = -4w_t g'(w) & \text{in } \mathbb{R}_+ \times (0, 1), \\ w(t, 0) = w(t, 1) = 0 & \forall t \in \mathbb{R}_+, \\ w(0, \cdot) = z_1, \quad w_t(0, \cdot) = z''_0 - g(z_1). \end{cases} \quad (4.42)$$

We define for all  $(t, x) \in \mathbb{R}_+ \times (0, 1)$  the Riemann invariants for (4.42),

$$u = w_x + w_t, \quad (4.43)$$

$$v = w_x - w_t. \quad (4.44)$$

Along strong solutions of (4.42), we have

$$\begin{cases} u_t - u_x = -2(u - v)g'(w) & \text{in } \mathbb{R}_+ \times (0, 1), \\ v_t + v_x = 2(u - v)g'(w) & \text{in } \mathbb{R}_+ \times (0, 1), \\ u(t, 0) - v(t, 0) = u(t, 1) - v(t, 1) = 0 & \forall t \in \mathbb{R}_+. \end{cases} \quad (4.45)$$

We define the  $p$ -th energy associated with  $w$  as

$$E_p(w)(t) = \frac{1}{p} \int_0^1 (|u|^p + |v|^p) dx. \quad (4.46)$$

By Corollary (4.2.1),  $E_p(w)$  is non-increasing along solutions  $w = z_t$ , which implies that

$$E_p(w)(t) \leq E_p(w)(0), \quad \text{for a.e. } t \geq 0. \quad (4.47)$$

Then, using the fact that  $E_p(w)$  is an equivalent energy to  $\frac{1}{p} \int_0^1 (|w_x|^p + |w_t|^p) dx$ , it follows that

$$\int_0^1 |w_x|^p dx \leq CpE_p(w)(0), \quad (4.48)$$

which means that

$$\|z_t\|_{W^{1,p}(0,1)} \leq (CpE_p(w)(0))^{\frac{1}{p}}. \quad (4.49)$$

This implies that  $(z, z_t) \in Y_p$  for all  $t \geq 0$  which yields the required regularity for a strong solution. ■

**Remark 4.3.1.** For strong solutions in the case  $p \geq 2$ , we can easily use the results that have been proved in [16] for  $p \geq 2$  to prove the well-posedness. Indeed, let  $(z_0, z_1) \in Y_p$ , with  $p \geq 2$  this implies that  $(z_0, z_1) \in (H^2(0, 1) \cap H_0^1(0, 1)) \times H_0^1(0, 1)$ . We have then the existence of a unique strong solution

$$z \in C(\mathbb{R}_+, H_0^1(0, 1)) \cap C^1(\mathbb{R}_+, L^2(0, 1)),$$

such that

$$(z(t, \cdot), z_t(t, \cdot)) \in (H^2(0, 1) \cap H_0^1(0, 1)) \times H_0^1(0, 1), \quad \forall t \in \mathbb{R}_+,$$

which means that  $z_t(t, \cdot) \in L^\infty(0, 1)$  we can then use [16, Corollary 2.3, item (ii)] that implies in our context that if  $(z_0, z_1) \in Y_p$  then the solution  $z \in L^\infty(\mathbb{R}_+, W^{2,p}(0, 1) \cap W_0^{1,p}(0, 1))$  and  $z_t \in L^\infty(\mathbb{R}_+, W_0^{1,p}(0, 1))$  which guarantees the well-posedness in  $Y_p$ .

**Remark 4.3.2.** The reason why the argument based on D'Alembert formula and fixed point theory that was used in [10] cannot be used in the nonlinear case without imposing the extra assumption that  $g$  is linearly bounded, is that the fixed point

argument cannot be used when we cannot prove that the nonlinearity maps the convex compact on which we define the fixed point formula to itself.

## 4.4 Exponential stability

In this section we are interested in the asymptotic stability of Problem (4.1). Our goal is to prove that the energy along strong solutions of Problem (4.1) is exponentially decreasing, with an exponential rate of decrease depending on the  $Y_p$ -norm of the initial data. (This property is usually referred as semi-global exponential stability in the control literature.) To do so, we plan to use the work that has already been done in the linear case to treat the nonlinear case. The main stability result that we achieve is given by the following theorem.

**Theorem 4.4.1.** Assume  $(\mathbf{H}_1)$  and  $(\mathbf{H}_2)$  satisfied and  $1 < p < \infty$ . Given  $(z_0, z_1) \in Y_p$ , there exists a constant  $C_p(z_0, z_1) > 0$  that depends on the norm of initial conditions in  $Y_p$  such that for all  $t \in \mathbb{R}_+$

$$E_p(t) \leq E_p(0)e^{1-C_p(z_0, z_1)t}. \quad (4.50)$$

**Remark 4.4.1.** The stability rate in Theorem 4.4.1 depends on initial conditions. That is due to the fact that we are only manipulating strong solutions in the proof and we need an estimate of the  $W^{1,p}(0, 1)$  norm of  $z_t(t, \cdot)$ .

### 4.4.1 Asymptotic stability of an auxiliary linear problem

To prove the exponential stability of Problem (4.1), we are going to start by considering the following auxiliary problem inspired from [10]

$$\begin{cases} y_{tt} - y_{xx} + a(x)\theta(t, x)y_t = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \\ y(t, 0) = y(t, 1) = 0 & t \geq 0, \\ y(0, \cdot) = y_0, \quad y_t(0, \cdot) = y_1, \end{cases} \quad (4.51)$$

where  $a$  satisfies Hypothesis  $(\mathbf{H}_1)$  and  $\theta$  satisfies:

$(\mathbf{H}_3)$   $\theta : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$  is a non-negative continuous function such that

$$\exists \theta_1, \theta_2 > 0, \quad \theta_1 \leq \theta(t, x) \leq \theta_2 \quad \forall (t, x) \in \mathbb{R}_+ \times [0, 1]. \quad (4.52)$$

**Remark 4.4.2.** The well-posedness of Problem (4.51) can be treated the same way as the linear problem was treated in [19].

We define the Riemann invariants for all  $(t, x) \in \mathbb{R}_+ \times (0, 1)$  by

$$\bar{\rho}(t, x) = y_x(t, x) + y_t(t, x), \quad (4.53)$$

$$\bar{\xi}(t, x) = y_x(t, x) - y_t(t, x). \quad (4.54)$$

Along strong solutions of (4.51), we deduce that

$$\begin{cases} \bar{\rho}_t - \bar{\rho}_x = -\frac{1}{2}a(x)\theta(t, x)(\bar{\rho} - \bar{\xi}) & \text{in } \mathbb{R}_+ \times (0, 1), \\ \bar{\xi}_t + \bar{\xi}_x = \frac{1}{2}a(x)\theta(t, x)(\bar{\rho} - \bar{\xi}) & \text{in } \mathbb{R}_+ \times (0, 1), \\ \bar{\rho}(t, 0) - \bar{\xi}(t, 0) = \bar{\rho}(t, 1) - \bar{\xi}(t, 1) = 0 & \forall t \in \mathbb{R}_+, \\ \bar{\rho}_0 := \bar{\rho}(0, \cdot) = y'_0 + y_1, \quad \bar{\xi}_0 := \bar{\xi}(0, \cdot) = y'_0 - y_1, \end{cases} \quad (4.55)$$

with  $(\bar{\rho}_0, \bar{\xi}_0) \in W^{1,p}(0, 1) \times W^{1,p}(0, 1)$ .

We consider the  $p$ th-energy  $E_p(y)$  of a solution  $y$ , defined on  $\mathbb{R}_+$  by

$$E_p(y)(t) = \frac{1}{p} \int_0^1 (|\bar{\rho}|^p + |\bar{\xi}|^p) dx. \quad (4.56)$$

We deduce from Proposition 3.3.1 the following proposition.

**Proposition 4.4.1.** Let  $p \in [1, \infty)$  and suppose that a strong solution  $y$  of (4.51) exists and is defined on a non trivial interval  $I \subset \mathbb{R}_+$  containing 0, for some initial conditions  $(y_0, y_1) \in Y_p$ . For  $t \in I$ , define

$$\Phi(t) := \int_0^1 [\mathcal{F}(\bar{\rho}) + \mathcal{F}(\bar{\xi})] dx, \quad (4.57)$$

where  $\bar{\rho}$  and  $\bar{\xi}$  are defined in (3.52) and  $\mathcal{F}$  is a  $C^1$  convex function. Then  $\Phi$  is well defined for  $t \in I$  and satisfies

$$\frac{d}{dt} \Phi(t) = -\frac{1}{2} \int_0^1 a(x)\theta(t, x)(\bar{\rho} - \bar{\xi})(\mathcal{F}'(\bar{\rho}) - \mathcal{F}'(\bar{\xi})) dx \leq 0. \quad (4.58)$$

**Proof.** The proof is the same as the proof of Proposition 3.3.1 where we replace  $g(z_t)$  by  $\theta z_t$ . ■

Similarly to Corollary (4.2.1), we have the following.

**Corollary 4.4.1.** If  $(y_0, y_1) \in Y_p$ , then we have along strong solutions that for  $t \geq 0$ ,

$$E_p(y)'(t) = -\frac{1}{2} \int_0^1 a(x)\theta(t, x)(\bar{\rho} - \bar{\xi}) (|\bar{\rho}|^{p-1} - |\bar{\xi}|^{p-1}) dx, \quad (4.59)$$

where  $\bar{\rho}$  and  $\bar{\xi}$  are defined in (4.53). Moreover, for  $(y_0, y_1) \in X_p$ , suppose that a weak solution  $y$  of (4.51) exists on  $\mathbb{R}_+$ . Then the energy  $t \mapsto E_p(y)(t)$  is non-increasing.

The main result of this section is given below.

**Theorem 4.4.2.** Fix  $p \in ]1, \infty)$  and suppose that Hypotheses **(H<sub>1</sub>)** and **(H<sub>3</sub>)** are satisfied. Then for every  $(y_0, y_1) \in X_p$ , the solution of (4.51) is exponentially stable.

To prove Theorem 4.4.2, we are going to follow the same steps of the proof of Theorem 3.5.1, all the computations remains the same with the only difference that  $a(x)$  is now replaced by  $a(x)\theta(t, x)$ . We have used three multipliers to treat the linear case in Chapter 3. The multipliers were slightly different in the case where  $1 < p < 2$  than the case where  $p \geq 2$  but in both cases we have the same potential occurrences of  $a(x)\theta(t, x)$ . As a result, we will only give the sketch of the proof in the case where  $p \geq 2$  and the case where  $1 < p < 2$  will be treated similarly. The sketch of proof that we provide lists the parts of the proof where  $\theta(t, x)$  occurs and how they are easily handled.

It is important to note that just like in Theorem 3.5.1, it is enough to prove Theorem 4.4.2 for strong solutions and then extend the result for weak solutions by a density argument.

#### 4.4.2 Case $p \geq 2$

The theorem for strong solutions and for  $p \geq 2$  follows directly from the next proposition by using Gronwall Lemma.

**Proposition 4.4.2.** Fix  $2 \leq p < +\infty$  and suppose that the hypotheses of Theorem 4.4.2 are satisfied. Then there exist positive constants  $C$  and  $C_p$  such that, for every  $(z_0, z_1) \in Y_p$ , it holds the following energy estimate:

$$\forall 0 \leq S \leq T, \quad \int_S^T E_p(t) dt \leq CC_p E_p(S). \quad (4.60)$$

To prove this key proposition, we divide the proof into steps, the result of each step is given by a key lemma and is obtained by using a specific multiplier. Before we



announce the lemmas, we remind the reader of the functions.

Let the function  $f$  defined by

$$f(s) = |s|^{p-1}, \quad \forall s \in \mathbb{R}. \quad (4.61)$$

and the function  $F(s) = \int_0^s f(\tau) d\tau$ , we have that

$$F(s) = \frac{|s|^p}{p}, \quad F' = f, \quad f'(s) = (p-1)|s|^{p-2}. \quad (4.62)$$

The multipliers that were used in Chapter 3 in the case  $p \geq 2$  are the following:

(m1)  $x \mapsto x\psi(x)f(\bar{\rho}(t, x))$  and  $x \mapsto x\psi(x)f(\bar{\xi}(t, x))$  for every  $t \geq 0$ ;

(m2)  $x \mapsto \phi(x)f'(\bar{\rho}(t, x))y(t, x)$  and  $x \mapsto \phi(x)f'(\bar{\xi}(t, x))y(t, x)$  for every  $t \geq 0$ ;

(m3)  $x \mapsto v(t, x)$  for every  $t \geq 0$ , where  $v$  is the solution of the following elliptic problem defined for every  $t \geq 0$ :

$$\begin{cases} v_{xx} = \beta f(y) & x \in (0, 1), \\ v(0) = v(1) = 0, \end{cases} \quad (4.63)$$

where  $\psi$ ,  $\phi$ ,  $\beta$  are the localization functions defined in (3.73) by

$$\begin{cases} 0 \leq \psi \leq 1, \\ \psi = 0 \text{ on } Q_0, \\ \psi = 1 \text{ on } (0, 1) \setminus Q_1, \end{cases} \quad \begin{cases} 0 \leq \phi \leq 1, \\ \phi = 1 \text{ on } Q_1, \\ \phi = 0 \text{ on } (0, 1) \setminus Q_2, \end{cases} \quad \begin{cases} 0 \leq \beta \leq 1, \\ \beta = 1 \text{ on } Q_2 \cap (0, 1), \\ \beta = 0 \text{ on } \mathbb{R} \setminus \omega. \end{cases}$$

**Lemma 4.4.1. (First set of multipliers)**

Under the hypothesis of Proposition 4.4.2, we have for all  $0 \leq S \leq T$  the following estimate:

$$\int_S^T E_p(y)(t) dt \leq CC_p E_p(y)(S) + C \underbrace{\int_S^T \int_{Q_1 \cap (0,1)} (F(\bar{\rho}) + F(\bar{\xi})) dx dt}_{\mathbf{S}_4}. \quad (4.64)$$

**Proof:** Multiplying the first equation of (4.55) by  $x\psi f(\bar{\rho})$  and then by  $x\psi f(\bar{\xi})$  and integrating over  $[S, T] \times [0, 1]$ , we obtain after doing the same manipulations that

led to (3.84), but instead we obtain

$$\begin{aligned}
\int_S^T E_p(y)(t) dt &\leq \underbrace{\int_S^T \int_{Q_1 \cap (0,1)} |(1 - (x\psi)_x)| (F(\bar{\rho}) + F(\bar{\xi})) dx dt}_{\mathbf{S}_1} \\
&\quad + \underbrace{\int_0^1 |x\psi| \left| [F(\bar{\rho}) - F(\bar{\xi})]_S^T \right| dx}_{\mathbf{S}_2} \\
&\quad + \underbrace{\frac{1}{2} \int_S^T \int_0^1 |a(x)\theta(t,x)x\psi| |(f(\bar{\rho}) + f(\bar{\xi}))| |\bar{\rho} - \bar{\xi}| dx dt}_{\mathbf{S}_3}. \quad (4.65)
\end{aligned}$$

The quantities  $\mathbf{S}_1$  and  $\mathbf{S}_2$  are denoted the same way as  $\mathbf{S}_1$  and  $\mathbf{S}_2$  from Lemma 3.5.1 and treated the same way to obtain (3.85) and (3.86). As for the estimation of  $\mathbf{S}_3$ , also denoted as  $\mathbf{S}_3$  from the proof of Lemma 3.5.1, nothing radically changes, we just follow the same computations as the one that led to (3.92) with taking in consideration  $\theta(t,x)$ , which gives the same estimation despite the presence of  $\theta(t,x)$ . This allows us to combine the estimations of  $\mathbf{S}_1$ ,  $\mathbf{S}_2$  and  $\mathbf{S}_3$  to obtain the same main result (3.76) of the first set of multipliers, which is given in our case by (4.64). ■

To estimate  $\mathbf{S}_4$ , we use the second set of multipliers.

**Lemma 4.4.2. (Second set of multipliers)**

Under the hypotheses of Proposition 4.4.2, we have for all  $0 \leq S \leq T$  the following estimate:

$$\mathbf{S}_4 \leq C \frac{C_p}{\eta^p} \underbrace{\int_S^T \int_{Q_2 \cap (0,1)} |y|^p dx dt}_{\mathbf{T}_5} + CC_p \eta^q \int_S^T E_p(y)(t) dt + CC_p E_p(y)(S), \quad (4.66)$$

where  $\eta > 0$  is arbitrary and  $q = \frac{p}{p-1}$ .

**Proof:** Multiplying the first equation of (4.55) by  $\phi f'(\bar{\rho})y$  and then by  $\phi f'(\bar{\xi})y$  and integrating over  $[S, T] \times [0, 1]$ , we obtain after doing the same manipulations that

led to (3.105), but instead we obtain

$$\begin{aligned}
\mathbf{S}_4 \leq & C \underbrace{\int_S^T \int_{Q_2 \cap (0,1)} |y| (|f(\bar{\rho})| + |f(\bar{\xi})|) dx dt}_{\mathbf{T}_1} + C_p \underbrace{\left[ \int_0^1 (f(\bar{\rho}) - f(\bar{\xi})) y dx \right]_S^T}_{\mathbf{T}_2} \\
& + C_p \underbrace{\int_S^T \int_{Q_2 \cap (0,1)} |(f'(\bar{\rho}) + f'(\bar{\xi})) y a(x) \theta(t, x) (\bar{\rho} - \bar{\xi})| dx dt}_{\mathbf{T}_3} \\
& + C_p \underbrace{\int_S^T \int_0^1 |\phi(\bar{\rho} - \bar{\xi}) (f(\bar{\rho}) - f(\bar{\xi}))| dx dt}_{\mathbf{T}_4}. \tag{4.67}
\end{aligned}$$

$\mathbf{T}_1$  and  $\mathbf{T}_2$  are denoted the same way as  $\mathbf{T}_1$  and  $\mathbf{T}_2$  in the proof of Lemma 3.5.2 and treated the same way to obtain (3.107) and (3.110). As for estimating  $\mathbf{T}_3$ , also denoted by  $\mathbf{T}_3$  in the proof of Lemma 3.5.2, we just use the fact that  $\theta_1 \leq \theta(t, x) \leq \theta_2$  to obtain

$$|a(x) \theta(t, x) (f'(\bar{\rho}) + f'(\bar{\xi})) (\bar{\rho} - \bar{\xi})| \leq C_p (|f(\bar{\rho})| + |f(\bar{\xi})|), \tag{4.68}$$

which implies that  $\mathbf{T}_3 \leq \mathbf{T}_1$  and hence it has the same estimation as  $\mathbf{T}_1$ .

Left to estimate  $\mathbf{T}_4$  which is also denoted by  $\mathbf{T}_4$  in the proof of Lemma 3.5.2. We follow the same steps that led to (3.112) but we use the fact that  $|\phi(x)| \leq Ca(x)\theta(t, x)$  for all  $t \in \mathbb{R}_+$  and  $x \in [0, 1]$ . We obtain the same estimation as (3.112).

We combine the estimations of  $\mathbf{T}_1$ ,  $\mathbf{T}_2$ ,  $\mathbf{T}_3$  and  $\mathbf{T}_4$ , we obtain the main result of the second set of multipliers, which is the same estimation (3.95) given in our case by (4.66). ■

Left to estimate now  $\mathbf{T}_5$ . To do so, we use one last multiplier.

**Lemma 4.4.3. (Third multiplier)**

Under the hypotheses of Proposition 4.4.2, we have for all  $0 \leq S \leq T$  the following estimate:

$$\underbrace{\int_S^T \int_{Q_2 \cap (0,1)} |y|^p dx dt}_{\mathbf{T}_5} \leq CC_p \left( \eta \int_S^T E_p(y)(t) dt + \frac{1}{\eta^r} E_p(y)(S) \right). \tag{4.69}$$

**Proof:** First, one should note that Lemma 3.5.3 remains valid here and it gives

an estimation of the  $L^q$  norms of  $v$  and  $v_t$ , where  $v$  is defined in (4.63).

Multiplying the first equation of (4.55) by  $v$  and integrating over  $[S, T] \times [0, 1]$ , we obtain after doing the same manipulations that led to (3.128), but instead we obtain

$$\begin{aligned}
2\mathbf{T}_5 \leq & \underbrace{\left| \left[ \int_0^1 v(\bar{\rho} - \bar{\xi}) dx \right]_S^T \right|}_{\mathbf{V}_1} + \underbrace{\int_S^T \int_0^1 |v_t| |(\bar{\xi} - \bar{\rho})| dx dt}_{\mathbf{V}_2} \\
& + \underbrace{\int_S^T \int_0^1 |va(x)\theta(t, x)(\bar{\rho} - \bar{\xi})| dx dt}_{\mathbf{V}_3}. \tag{4.70}
\end{aligned}$$

The quantities  $\mathbf{V}_1$ ,  $\mathbf{V}_2$  and  $\mathbf{V}_3$  are denoted the same way as  $\mathbf{V}_1$ ,  $\mathbf{V}_2$  and  $\mathbf{V}_3$  from the proof of Lemma 3.5.4. The quantity  $\mathbf{V}_1$  is treated the same way as (3.130). As for  $\mathbf{V}_2$ , we follow the same steps that led to (3.135) and we use the fact that  $\beta \leq C a(x)\theta(t, x)$  in (3.132). Finally for  $\mathbf{V}_3$ , nothing changes and we obtain (3.138) despite the presence of  $\theta(t, x)$  in this term. We combine the three estimations to obtain the main result of the third multiplier, which is the same estimation of  $\mathbf{T}_5$  given by (3.119), given in our case by (4.69). ■

***Proof of Proposition 4.4.2:***

Finally, as in [19, Section 4.1.4], by combining the results of the three multipliers (4.64), (4.66) and (4.69) and choosing  $\eta$  properly, we obtain the energy estimate given by (4.60). ■

### 4.4.3 Case $1 < p < 2$

We remind the reader that due to the presence of the power  $p - 2$  in the second set of multipliers in the case  $p \geq 2$ , it is not possible to use them directly in the case  $1 < p < 2$ . Therefore, we have to modify the functions  $f$  and  $F$  just like it was done in Chapter 3. We consider, for  $p \in (1, 2)$ , the functions already considered in Chapter

3,  $h$  and  $H$  defined on  $\mathbb{R}$ , by

$$h(y) = (p-1) \int_0^y (|s|+1)^{p-2} ds = \operatorname{sgn}(y) [(|y|+1)^{p-1} - 1], \quad (4.71)$$

$$H(y) = \int_0^y h(s) ds = \frac{1}{p} [(|y|+1)^p - 1] - |y|. \quad (4.72)$$

We also consider the modified energy  $\mathcal{E}_p$  defined by

$$\mathcal{E}_p(t) = \int_0^1 (H(\rho) + H(\xi)) dx. \quad (4.73)$$

The proof of Theorem 4.4.2 is a result of the following proposition by using the same argument from Section 3.5.2.

**Proposition 4.4.3.** Fix  $p \in (1, 2)$  and suppose that the hypotheses of Theorem 4.4.2 are satisfied. Then there exist positive constants  $C$  and  $C_p$  such that, for every  $(z_0, z_1) \in Y_p$  verifying

$$E_p(0) \leq 1, \quad (4.74)$$

we have the following energy estimate:

$$\forall 0 \leq S \leq T, \quad \int_S^T \mathcal{E}_p dt \leq CC_p \mathcal{E}_p, \quad (4.75)$$

where  $\mathcal{E}_p$  is defined in (3.141).

**Proof:** Just like the case  $p \geq 2$ , we follow the exact same steps in this case as the case  $1 < p < 2$  from Chapter 3, by using the multipliers

( $\bar{m}1$ )  $x \mapsto x\psi(x)h(\bar{\rho}(t, x))$  and  $x \mapsto x\psi(x)h(\bar{\xi}(t, x))$  for every  $t \geq 0$ ;

( $\bar{m}2$ )  $x \mapsto \phi(x)h'(\bar{\rho}(t, x))y(t, x)$  and  $x \mapsto \phi(x)h'(\bar{\xi}(t, x))y(t, x)$  for every  $t \geq 0$ ;

( $\bar{m}3$ )  $x \mapsto v(t, x)$  for every  $t \geq 0$ , where  $v$  is the solution of the following elliptic problem defined for every  $t \geq 0$ :

$$\begin{cases} v_{xx} = \beta h(y) & x \in (0, 1), \\ v(0) = v(1) = 0, \end{cases}$$

where  $\psi$ ,  $\phi$ ,  $\beta$  are the localization functions defined in (3.73) and  $g$  is the function defined in (4.71).

We take care of the presence of  $\theta$  just like we did in the previous proof since we have technically the same occurrences of  $\theta$  in the cases  $p \geq 2$  and  $1 < p < 2$ . We obtain the exponential decay similarly.

■

Now that we know that Problem (4.51) is exponentially stable for all  $1 < p < +\infty$ , we can now consider Problem (4.1).

#### 4.4.4 Asymptotic stability of the nonlinear problem

We conclude in this section by using what precedes, the stability of the nonlinear problem. However, to be able to finally conclude we need to state and prove some key lemmas first.

**Lemma 4.4.4.** Define the function the continuous function  $\nu$  defined for all  $x \in \mathbb{R}$  as

$$\begin{cases} \nu(x) = \frac{g(x)}{x} & \text{for } x \in \mathbb{R}^*, \\ \nu(0) = g'(0), \end{cases}$$

where  $g$  is defined in Hypothesis **(H<sub>2</sub>)**, then  $\nu$  has the following properties:

- $\nu(x) > 0$  for all  $x \neq 0$ .
- $\forall M > 0, \exists \nu_1(M), \nu_2(M) > 0$  such that  $\forall |x| \leq M, \nu_1(M) \leq \nu(x) \leq \nu_2(M)$ .

**Proof:** The proof is standard and is a direct result of Hypothesis **(H<sub>2</sub>)**. The first item is a direct result of (4.4). The second item is a result of  $g$  being  $C^1$ ,  $g(0) = 0$  and  $g'(0) \neq 0$ .

■

We next prove the following regularity lemma which is a generalization of [25, Lemma 2].

**Lemma 4.4.5.** Assume **(H<sub>1</sub>)** and **(H<sub>2</sub>)** satisfied and  $(z_0, z_1) \in Y_p$  with  $1 < p < +\infty$ , then the following estimate holds true for all  $t \geq 0$  and for a constant  $C_p(z_0, z_1) > 0$  that depends on the norm of initial conditions in  $Y_p$

$$\|z_t(t, \cdot)\|_{W^{1,p}(0,1)} \leq C_p(z_0, z_1). \quad (4.76)$$

**Proof:** The norm estimate (4.76) is a direct result of (4.49). The lemma is then concluded with  $C_p(z_0, z_1) = (C_p E_p(w)(0))^{\frac{1}{p}}$ , where  $w = z_t$ . ■

**Proof of Theorem 4.4.1:**

Consider an arbitrary strong solution  $z$  of Problem (4.1). Using the result of Lemma 4.4.5 which is  $z_t \in W^{1,p}(0, 1)$  with  $\|z_t(t, \cdot)\|_{W^{1,p}(0,1)}^p \leq C_p(z_0, z_1)$  for all  $t \in \mathbb{R}_+$  we deduce using the continuous embedding  $W^{1,p}(0, 1) \subset L^\infty(0, 1)$ , with a constant that does not depend on  $t$  (see [8, Theorem VIII.7]) that

$$\|z_t(t, \cdot)\|_{L^\infty(0,1)} \leq C \|z_t(t, \cdot)\|_{W^{1,p}(0,1)} \leq C_p(z_0, z_1) \quad \forall t \in \mathbb{R}_+. \quad (4.77)$$

Since  $g$  satisfies **(H<sub>2</sub>)** and  $z_t$  satisfies (4.77), we deduce the existence of two constants

$$C_p^1(z_0, z_1), C_p^2(z_0, z_1) > 0,$$

that depends on  $p$  and on the  $Y_p$ -norm of the initial data only such that for all  $t \geq 0$ , and  $x \in (0, 1)$ , it holds

$$C_p^1(z_0, z_1) \leq \nu(z_t) \leq C_p^2(z_0, z_1). \quad (4.78)$$

We consider the time-varying linear problem given by

$$\begin{cases} y_{tt} - y_{xx} + a(x)\nu(z_t)y_t = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \\ y(t, 0) = y(t, 1) = 0 & t \geq 0, \\ y(0, \cdot) = y_0, \quad y_t(0, \cdot) = y_1, \end{cases} \quad (4.79)$$

which is nothing else but the auxiliary problem (4.51) with  $\theta(t, x) = \nu(z_t)$  since  $\nu(z_t)$  is seen as a function of  $t$  and  $x$ . Moreover,  $\nu(z_t)$  satisfies Hypothesis **(H<sub>3</sub>)** with  $\theta_1 = C_p^1(z_0, z_1)$  and  $\theta_2 = C_p^2(z_0, z_1)$ .

By Theorem 4.4.2, the energy  $E_p(y)$  decays exponentially to zero along the solutions of (4.79). The key and trivial remark is that the strong solution  $z$  of Problem (4.1) considered at the beginning of the argument is the solution of (4.79) with initial data  $(z_0, z_1)$ . This concludes the proof of Theorem 4.4.1. ■

## Conclusion and perspectives

As a conclusion, we can say that we were able to treat and answer some of the questions in the domain of the stabilization of wave equations. However, there are many more questions that are still open and it would be interesting to answer them using maybe our work and the work of our fellow researchers before as a basis or even as a simple inspiration.

In Chapter 2, we studied the disturbed non-linear wave equation with localized damping in a Hilbertian framework in two dimensions. We achieved a weak ISS-type estimate for strong solutions. We believe it is possible to obtain a strong ISS (in the sense of Definition 1.2.3) for nonlinearities  $g$  of regular saturation type. It would be even more interesting to achieve a strong ISS for more general nonlinearities and not just saturation type.

It would also be interesting to prove an ISS-type estimate in higher dimensions; or at least an exponential decay in the absence of disturbances. Such a problem has been studied in [25] and an exponential decay in higher dimensions was proved but with a lower logarithmic-type of bounds on the non-linearity  $g$ , which remains a bit restrictive.

We worked with strong solutions only and so did [24] to obtain an exponential decay. One can also explore the different decay rates that weak solutions might offer or maybe prove that the strong stabilization is the best rate possible for such solutions.

In Chapter 3, we studied the linear wave equation with localized damping in a non-Hilbertian  $L^p$  framework in one dimension for all  $1 < p < \infty$ . It would be interesting to prove an  $L^p$  exponential decay in the case  $p = 1$  and  $p = \infty$ . We gave a proof of the



exponential decay in these two interesting cases but only in a very particular case of a constant global damping with a constant in  $(0, 4)$  (see Section 3.6). We believe that we can obtain an exponential stability for any constant global damping and not just for an interval of constants  $(0, 4)$ . It would be interesting to give a proof for this conjecture.

We also believe that obtaining an exponential decay for a global non-constant damping is achievable but since the multipliers method is clearly limited for  $p = 1$  and  $p = \infty$  when the damping is non-constant we recommend considering other methods, in particular the method of microlocal analysis since it gives the sharpest results for linear problems.

We also studied in Chapter 4 the non-linear case but in the case  $p \in (1, \infty)$  and gave a proof for the exponential decay along strong solutions only. We proved the existence of strong solutions in this non-linear problem for all  $1 \leq p \leq \infty$ ; but when it comes to weak solutions we only managed to prove the well-posedness for  $p \geq 2$  which opens the question for the existence of weak solutions for  $1 \leq p < 2$ .

Just like in the linear problem, it would be very interesting to achieve an exponential decay in the cases  $p = 1$  and  $p = \infty$  in the non-linear problem. It would also be interesting to consider weak solutions (at least for  $p \geq 2$ ) and explore the best decay rate that one can obtain.

Another interesting question that can be studied is trying to obtain ISS-type estimates in the  $L^p$  framework for the non-linear case by considering all the possible disturbances, just like we did with the same non-linear problem in the  $L^2$  framework in Chapter 2.

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**Titre :** Stabilization des équations des ondes

**Mots clés :** Stabilisation ; équation des ondes ; amortissement localisé ; cadre fonctionnel  $L^p$  ; stabilité Input-to-state ; stabilité exponentielle.

**Résumé :** Cette thèse traite trois problèmes liés à la stabilisation des équations des ondes. Nous considérons différents cadres fonctionnels et nous utilisons des techniques de démonstration basées sur la méthode des multiplicateurs. Tout d'abord, nous étudions la stabilité de l'équation des ondes avec un amortissement non-linéaire et localisé dans un cadre hilbertien standard en dimension deux. La preuve est basée sur les travaux de Martinez et Vancostenoble (2000), auxquels nous rajoutons une localisation ainsi que des perturbations. Nous démontrons la stabilité exponentielle le long des solutions fortes en l'absence de perturbation ainsi qu'une sorte de stabilité au sens Input-To-State par rapport aux perturbations considérées. Dans un deuxième travail, nous considérons un cadre fonctionnel plus général non forcément hilbertien, c-à-d un cadre  $L^p$  avec  $p \in (1, \infty)$ . Nous étudions la stabilité  $L^p$  de l'équation

des ondes avec un amortissement linéaire et localisé. Cette étude n'est effectuée qu'en dimension un car il n'est pas toujours possible de définir l'opérateur des ondes en dimensions supérieures lorsque  $p \neq 2$ . Nous démontrons la stabilité exponentielle du problème en généralisant les multiplicateurs du cadre hilbertien dans notre cadre plus général, avec des preuves différentes suivant que  $1 < p < 2$  ou  $p \geq 2$ . Nous démontrons également dans le même problème mais avec des cas particuliers d'un amortissement global et constant, une stabilité exponentielle dans le cas  $p = 1$  et  $p = \infty$ . Dans un troisième travail, nous considérons la version non-linéaire du problème précédent : en nous basant sur une technique de linéarisation, nous nous ramenons à la preuve du problème linéaire pour démontrer la stabilité exponentielle du non-linéaire le long des solutions fortes et pour tout  $1 < p < \infty$ .

**Title :** Stabilization of wave equations

**Keywords :** Stabilization ; wave equation ; localized damping,  $L^p$  framework ; Input-to-state stability ; exponential stability.

**Abstract :** This thesis focuses on three problems in the context of the stabilization of wave equations. We consider different frameworks and we use techniques based on the multipliers method. First, we study the stability of the wave equation with non-linear localized damping in a standard Hilbertian framework in two dimensions. The proof is based on the work Martinez and Vancostenoble (2000) to which we add a localization as well as disturbances. We prove the exponential stability of strong solutions in the absence of disturbances and also a weak Input-To-State stability property with respect to the considered disturbances. We next consider a more general functional framework, namely an  $L^p$  framework with  $p \in (1, \infty)$ . We study the  $L^p$  stability of the wave equation with a li-

near and localized damping in one dimension since it is not always possible to define the wave operator in higher dimensions when  $p \neq 2$ . We prove exponential stability of the problem by generalizing the multipliers of the Hilbertian framework in this new general framework, with a different proof for  $1 < p < 2$  and  $p \geq 2$ . We also prove in the same problem but with particular cases of a global constant damping, an exponential stability in the case  $p = 1$  and  $p = \infty$ . We consider next the nonlinear case of the previous problem : relying on a linearizing technique, we reduce that study to that of the linear problem case in order to prove exponential stability of the non-linear problem along strong solutions and for all  $1 < p < \infty$ .