



**HAL**  
open science

## Study of some spectral shape optimization problems.

Ataa Al Kheir

► **To cite this version:**

Ataa Al Kheir. Study of some spectral shape optimization problems.. General Mathematics [math.GM]. Université de Pau et des Pays de l'Adour, 2023. English. NNT : 2023PAUU3017 . tel-04439741

**HAL Id: tel-04439741**

**<https://theses.hal.science/tel-04439741>**

Submitted on 5 Feb 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# THÈSE

UNIVERSITE DE PAU ET DES PAYS DE L'ADOUR

École doctorale Sciences Exactes et leurs Applications (ED 211)

Présentée et soutenue le 4 juillet 2023

par **Ataa AL KHEIR**

pour obtenir le grade de docteur  
de l'Université de Pau et des Pays de l'Adour

**Spécialité : Mathématiques**

## L'étude de quelques problèmes d'optimisation de forme spectrale

### RAPPORTEURS

- M. Yannick PRIVAT
- M. Jimmy LAMBOLEY

Professeur des universités / Université de Strasbourg  
Professeur des universités / Sorbonne Université

### EXAMINATEURS

- M. Fabien CAUBET
- M. Charles-Edouard BREHIER  
(Le président du jury)
- Mme Aude RONDEPIERRE

Maître de conférences / Université de Pau et des Pays de l'Adour  
Professeur des universités / Université de Pau et des Pays de l'Adour

Professeur des universités / INSA Toulouse

### DIRECTEURS

- M. Marc DAMBRINE

Professeur des universités / Université de Pau et des Pays de l'Adour





**Resumé:** Nous étudions l'attitude de la solution du problème aux limites de Wentzell-Laplace par rapport aux déformations de forme. D'une part, nous prouvons la continuité de la solution sous une déformation Lipschitz d'un problème de valeurs limites uniformément Lipschitz domaines dans l'espace euclidien  $\mathbb{R}^d$ . D'autre part, nous démontrons la continuité de la solution sous la convergence  $W^{2,\infty}$  des domaines de  $(2, \infty)$ -Sobolev dans  $\mathbb{R}^d$ . Après cela, nous discutons de la convergence de la fonction distance signée dans  $W^{1,p}(B)$  avec  $1 \leq p \leq \infty$  par rapport à la convergence de Hausdorff des domaines moins réguliers, où  $B$  est une grande boule contenant la séquence des domaines  $(\Omega_n)_{n \in \mathbb{N}}$  et l'ensemble limite  $\Omega$ , et on peut en déduire les résultats suivants: le cas uniformément Lipschitz suffit pour prouver la convergence de la différentielle du premier ordre dans  $L^p$  avec  $1 \leq p < \infty$ ; alors que dans le cas  $p = \infty$ , nous devons supposer une régularité plus importante, qui est celle de la "positive reach". De plus, pour prouver la convergence de la différentielle du premier ordre de la fonction de projection sur  $\partial\Omega_n$  dans  $L^p$  avec  $1 \leq p \leq \infty$ , nous devons injecter la régularité de  $(2, \infty)$ -Sobolev avec la nécessité de convergence  $W^{2,\infty}$  de ces domaines.

Ensuite on considère le problème des valeurs propres associé. Afin d'illustrer le comportement du spectre, nous considérons un exemple explicite sur la boule dans  $\mathbb{R}^2$ , où nous trouvons explicitement les valeurs propres et leurs fonctions propres correspondantes. Après avoir montré l'analyticité des valeurs propres  $\lambda$  et de leurs fonctions propres correspondantes  $u$  dans un voisinage ouvert de  $t = 0$ , nous dérivons, au sens de Hadamard, les dérivées de forme du premier et du second ordre des valeurs propres à l'instant  $t = 0$ . Ensuite, nous montrons que la boule n'est pas une forme critique de ce problème de valeurs propres, avec et sans contrainte de volume. Enfin, en dimension deux, nous prouvons que, sous certaines contraintes et sous un champ de vecteurs de déformation  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$ , il n'y a pas de formes critiques pour ce problème de valeurs propres.

Nous avons essayé de montrer que si nous avons une valeur propre multiple  $\lambda$  dans  $\Omega$ , elle peut se diviser sous certaines déformations de la forme initiale. Enfin et surtout, nous discutons de la simplicité générique du spectre. Ces deux problèmes restent ouverts.

**Abstract:** We investigate the attitude of the solution of the Wentzell-Laplace boundary value problem with respect to shape deformations. On one hand, we prove the continuity of the solution under Lipschitz deformation of uniformly Lipschitz domains in the Euclidean space  $\mathbb{R}^d$ . On the other hand, we demonstrate the continuity of the solution under the  $W^{2,\infty}$  convergence of  $(2, \infty)$ -Sobolev domains in  $\mathbb{R}^d$ . After that, we discuss the convergence of the signed distance function in  $W^{1,p}(B)$  with  $1 \leq p \leq \infty$  with respect to the Hausdorff convergence of less regular domains, where  $B$  is a large ball containing the sequence of domains  $(\Omega_n)_{n \in \mathbb{N}}$  and the limit set  $\Omega$ , and we derive the following results: the uniformly Lipschitz case is enough for proving the convergence of the first order differential in  $L^p$  with  $1 \leq p < \infty$ ; while in case  $p = \infty$ , we need to assume more regularity, the "positive reach" one. Moreover, to prove the convergence of the first order differential of the projection function on  $\partial\Omega_n$  in  $L^p$  with  $1 \leq p \leq \infty$ , we have to inject the  $(2, \infty)$ -Sobolev regularity with the necessity of  $W^{2,\infty}$  convergence of these domains.

In a second step we consider the eigenvalues problem. In order to illustrate the behavior of the spectrum, we consider an explicit example about the ball in  $\mathbb{R}^2$ , where we find explicitly the eigenvalues and their corresponding eigenfunctions. After showing the analyticity of the eigenvalues  $\lambda$  and their corresponding eigenfunctions  $u$  in an open neighborhood of  $t = 0$ , we derive, in the sense of Hadamard, the first and second-order shape derivatives of the eigenvalues at time  $t = 0$ . After that, we show that the ball is not a critical shape of this eigenvalue problem, with and without a volume constraint. Finally, in dimension two, we prove that, with some constraints and under a deformation vector field  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$ , there are no critical shapes for this eigenvalue problem.

Finally, we try to show that if we have a multiple eigenvalue  $\lambda$  in  $\Omega$ , it can split under certain deformation of the initial shape. Last but not least, we discuss the generic simplicity of the spectrum. We provide reflections and elements of study for these two problems that remain open.

## Remerciements et Difficultés:

**Remerciement:** Dieu merci! Je souhaite partager ma thèse en mathématiques, du titre "L'Étude de quelques problèmes d'optimisation de forme spectrale", à l'Université de Pau et des pays de l'Adour, dans le "Laboratoire de mathématiques et leur application" (LMAP), sous la direction du professeur Marc DAMBRINE, que je remercie beaucoup pour sa patience et sa direction de mon programme de doctorat.

Aussi, je tiens à remercier toute ma famille qui a été le premier supporter pour moi dans ce long voyage. De plus, je n'oublierai jamais d'apprécier tous ceux qui m'ont aidé et encouragé afin de réaliser mon rêve.

**Difficultés générales:** Pour résumer ces trois ans et demi, j'ai commencé ma thèse en novembre 2019, qui se passait très bien jusqu'à la pandémie de Covid-19 qui a bloqué mon travail et qui a eu un énorme impact sur ma mobilité. De plus, il y avait quelques problèmes d'hospitalisation (surtout la première année), qui étaient difficiles à gérer en raison du changement de mode de vie dans le nouveau pays la France (je ne m'étais pas habitué à vivre seul).

**Difficultés mathématiques:** En fait, j'ai rencontré de nombreux obstacles au cours de mon programme de doctorat. Dès le début de la thèse, mon directeur Prof. Marc DAMBRINE m'a proposé d'étudier les éléments suivants :

1. La continuité des valeurs propres du problème aux valeurs propres de Wentzell-Laplace (P.V.P), par rapport à une variété de sens de convergence des domaines.
2. La régularité minimale sur les formes qu'il convient d'imposer pour valider ces résultats de continuité.
3. La simplicité et la simplicité générique du spectre.

En regardant la première partie, nous n'y sommes pas parvenus, nous avons donc étudié la continuité de la solution du problème aux limites de Wentzell-Laplace (P.L) avec la fonction fixe  $f \in L^2$  à droite côté de l'équation différentielle de volume. Il ne nous était pas clair comment transporter notre travail du P.L au P.V.P, donc nous ne savions pas comment bénéficier de la continuité de la solution du P.L (voir chapitre deux).

La deuxième question a été étudiée clairement dans le chapitre deux.

Concernant le dernier point, ils restent encore des problèmes ouverts (voir chapitre six).

Cependant, nous avons étudié les formes non critiques, et nous en avons tiré plusieurs résultats (le cas de la boule en plus des formes générales en dimension deux avec quelques contraintes), qui sont démontrés au chapitre cinq.

Remarquons que les chapitres cinq et six ne pourraient être étudiés sans l'analyse de la dérivée de forme étudiée aux chapitres trois et quatre.

Pour finir, les théorèmes et problèmes des chapitres cinq et six ont été traités à plusieurs reprises, dans lesquels la difficulté résidait dans le degré inconnu du calcul de dérivation que

nous devions atteindre (c'est-à-dire qu'on ne savait pas si les dérivées du premier ordre étaient suffisantes ou si nous avions pour passer aux dérivées du second et du troisième ordre). En fait, lors de notre procédure, nous pensions que c'était fait, mais malheureusement nous nous rendions encore compte de certaines erreurs de calcul, ce qui nous a amenés à refaire nos calculs et peut-être à réfléchir à différentes stratégies (ce qui était le plus gros obstacle), jusqu'à ce que nous atteignions la contradiction recherchée dans le résultat du chapitre cinq (sous certaines contraintes et atteindre le calcul des dérivées du troisième ordre), cependant, le chapitre six contient toujours ces problèmes ouverts.

# Contents

<b>1</b>	<b>Introduction and statement of the main results</b>	<b>7</b>
<b>2</b>	<b>Continuity of the solution under shape deformations</b>	<b>19</b>
2.1	Introduction and statement of the results . . . . .	19
2.2	Definitions and preliminaries . . . . .	23
2.3	Existence and uniqueness of the solution of (2.1) . . . . .	24
2.4	Extension operator and uniform bound estimations . . . . .	29
2.5	Interpretation of the limit . . . . .	30
2.5.1	The differential equation in the volume . . . . .	30
2.5.2	Localization in $\Omega$ (the limit set) . . . . .	31
2.5.3	Study of the surface term . . . . .	32
2.6	Study of the convergence of the signed distance and projection functions in $W^{1,p}$	33
2.6.1	$L^p$ convergence of the signed distance functions for uniformly Lipschitz domains, for $1 \leq p \leq \infty$ . . . . .	34
2.6.2	$W^{1,\infty}$ convergence of the signed distance functions for domains of positive reach . . . . .	34
2.6.3	$W^{1,p}$ convergence of the signed distance functions for uniformly Lipschitz domains, $1 \leq p < \infty$ . . . . .	36
2.6.4	$L^p$ convergence of the differential of the projection function for $(2, \infty)$ -Sobolev domains, for $1 \leq p \leq \infty$ . . . . .	38
2.7	The remaining convergence on the surface . . . . .	44
2.8	The proofs . . . . .	49
2.8.1	Proof of Theorem 1 . . . . .	49
2.8.2	Proof of Theorem 2 . . . . .	50
<b>3</b>	<b>Transition to the eigenvalue problem</b>	<b>51</b>
<b>4</b>	<b>Shape derivatives of the eigenvalues</b>	<b>55</b>
4.1	Introduction and statement of the results . . . . .	55
4.2	Prerequisites . . . . .	58
4.2.1	Notations and preliminaries for shape deformation . . . . .	58



4.2.2	Regularity of the eigenfunctions and eigenvalues with respect to the parameter . . . . .	62
4.2.3	Several useful lemmas . . . . .	66
4.3	The proofs . . . . .	69
4.3.1	Proof of Theorem 3 . . . . .	70
4.3.2	Proof of Theorem 4 . . . . .	71
4.3.3	Proof of Theorem 5 . . . . .	72
4.3.4	Proof of Theorem 6 . . . . .	73
4.3.5	Proof of Theorem 7 . . . . .	78
<b>5</b>	<b>Non-critical shapes</b>	<b>83</b>
5.1	Statement of the results . . . . .	83
5.2	Useful Lemma . . . . .	84
5.3	The Proofs . . . . .	86
5.3.1	Proof of Theorem 8 . . . . .	86
5.3.2	Proof of Corollary 1.1 . . . . .	89
5.3.3	Proof of Theorem 9 . . . . .	91
<b>6</b>	<b>Simplicity and generic simplicity</b>	<b>105</b>
6.1	Statement of the open problems . . . . .	106
6.2	Prerequisites . . . . .	106
6.3	The strategies . . . . .	107
6.3.1	Strategy for Open problem 1 . . . . .	107
6.3.2	Strategy for Open problem 2 . . . . .	122
<b>7</b>	<b>Appendices</b>	<b>125</b>
7.1	Appendix A. Differentiating in moving domains. . . . .	125
7.2	Appendix B. Differentiating on moving boundaries . . . . .	125
7.3	Appendix C. Some classical results on tangential differential calculus . . . . .	126
7.3.1	C.1. Notations and definitions . . . . .	126
7.3.2	C.2. Preliminary results . . . . .	127
7.3.3	C.3. Some commutation equations . . . . .	128

# Chapter 1

## Introduction and statement of the main results

Le problème aux limites de Wentzell-Laplace est un problème qui a été étudié au cours de la dernière décennie. Le problème aux valeurs propres de Wentzell-Laplace dans un domaine ouvert borné connexe  $\Omega \subset \mathbb{R}^d$  s'énonce ainsi: On cherche la valeur propre de l'opérateur de Laplace  $\lambda \in \mathbb{R}$  et la fonction propre  $u$  tels que

$$\begin{cases} -\Delta u = \lambda u & \text{dans } \Omega, \\ -\Delta_\tau u_n + \partial_{\mathbf{n}} u = 0 & \text{sur } \partial\Omega, \end{cases} \quad (1.1)$$

où  $\mathbf{n}$  est le vecteur normal unitaire extérieur sur  $\partial\Omega$  et  $\Delta_\tau := \operatorname{div}_\tau(\nabla_\tau)$  est l'opérateur de laplace-Beltrami défini sur  $\partial\Omega$ . On définit l'espace variationnel

$$\mathcal{H}_s(\Omega) := \{u \mid u \in H^1(\Omega), u|_{\partial\Omega} \in H^1(\partial\Omega)\},$$

équipé de la norme  $\|\cdot\|_{\mathcal{H}_s(\Omega)}$  défini par:

$$\|u\|_{\mathcal{H}_s(\Omega)} := \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)} + \|\nabla_\tau u\|_{L^2(\partial\Omega)}.$$

Le spectre du Laplacien avec les conditions de Wentzell est discret, c'est-à-dire que le spectre consiste en une suite croissante

$$\lambda_0(\Omega) = 0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \dots \rightarrow \infty \quad (1.2)$$

avec les fonctions propres orthonormées réelles correspondantes  $u_0, u_1, u_2, \dots$  dans  $L^2(\Omega)$ . La première valeur propre est zéro avec des constantes comme fonctions propres correspondantes. Comme d'habitude, nous adoptons la convention selon laquelle chaque valeur propre est répétée selon sa multiplicité. Ainsi, la première valeur propre d'intérêt est  $\lambda_1$ . Donnons une caractérisation variationnelle des valeurs propres: on définit sur  $\mathcal{H}_s(\Omega)$  les deux formes bilinéaires

$$A(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \nabla_{\tau} u \cdot \nabla_{\tau} v \, d\sigma, \quad B(u, v) = \int_{\Omega} uv, \quad (1.3)$$

où  $\nabla_{\tau}$  est le gradient tangentiel. Maintenant, les deux formes bilinéaires sont positives et la caractérisation variationnelle pour la  $k$ -ième valeur propre est

$$\lambda_k(\Omega) = \min \left\{ \frac{A(v, v)}{B(v, v)}, v \in \mathcal{H}_s(\Omega), \int_{\Omega} v u_i = 0, i = 0, \dots, k-1 \right\} \quad (1.4)$$

En particulier, lorsque  $k = 1$ , le minimum est pris sur les fonctions orthogonales aux fonctions propres associés à  $\lambda_0 = 0$ , c'est-à-dire des fonctions constantes.

En fait, ce problème a été fréquemment étudié dans la littérature, avec une version volumique et surfacique, en particulier au cours de la dernière décennie. Pour aller plus loin dans ce domaine, il a été étudié dans [20] la version volumique du problème aux valeurs propres de Wentzell-Laplace, où une inégalité de Faber-Krahn est donnée au Laplacien avec des conditions aux limites de Wentzell généralisées. Dans [11], la persistance du bien-posé du problème aux limites de Wentzell sous déformations de forme est étudiée. De plus, un problème aux valeurs propres extrémales pour l'opérateur de Wentzell-Laplace avec version de surface est étudié dans [16]. De plus, une inégalité de Weinstock en dimension supérieure est dérivée avec des conditions aux limites de Steklov dans [18], en particulier, les auteurs ont obtenu des inégalités isopérimétriques pour la première valeur propre de Wentzell de la version surfacique. Passant à la variété compacte  $n$ -dimensionnelle avec frontière, les valeurs propres de l'opérateur de Wentzell-Laplace de la version surfacique et des problèmes de Steklov du quatrième ordre sont étudiées dans [21], tandis que dans [22], quelques estimations des valeurs propres de l'opérateur de Wentzell-Laplace de la version surfacique sont dérivés. Plus tard, les auteurs de [19] se sont intéressés aux dérivées de forme des fonctionnelles aux valeurs propres pour certains problèmes scalaires, notamment les conditions aux limites de Dirichlet, Neumann, Robin et Wentzell (avec une version volumique et surfacique).

Dans cette thèse, nous nous intéressons à la version volumique car elle a été moins étudiée que celle en surface.

Dans le deuxième chapitre, nous donnons le problème aux limites de Wentzell-Laplace

$$\begin{cases} -\Delta u = f & \text{dans } \Omega, \\ -\Delta_{\tau} u + \partial_{\mathbf{n}} u = 0 & \text{sur } \partial\Omega, \end{cases} \quad (1.5)$$

avec  $f \in L^2(\Omega)$ , et soit  $B$  une boule contenant  $\Omega$ . Fixons  $\mathcal{H}(\Omega)$  comme

$$\mathcal{H}(\Omega) := \left\{ u \in \mathcal{H}_s(\Omega) \mid \int_{\partial\Omega} u = 0 \right\}$$

qui est fermé dans  $\mathcal{H}_s(\Omega)$  et qui hérite de la norme de  $\mathcal{H}_s(\Omega)$ . En fait, cet ensemble est nécessaire pour prouver l'existence et l'unicité de la solution (voir section 2.3) dans le chapitre deux.

En fait, le facteur de Wentzell  $\Delta_\tau u = \partial_n u$  est bien défini au sens faible (voir Définition 2.1) partout sur  $\partial\Omega$  si  $\Omega$  est uniformément Lipschitz.

Tout d'abord, nous étudions la continuité de la solution de (1.5) sous les déformations Lipschitz des domaines uniformément Lipschitz. Pour aller plus loin, on peut se référer à l'article de M. Dambrine et D. Kateb sur la "Persistence of wellposedness of Ventcel's border value problem under shape deformation" (voir [11]), où ils ont étudié la persistance de la solvabilité du problème aux limites sous déformation de domaine régulier.

De plus, nous traitons de la continuité de la solution de (1.5) par rapport à la convergence de  $W^{2,\infty}$  des domaines  $(2, \infty)$ -Sobolev (voir la définition des espaces de Sobolev dans Définition 2.4). Rappelons que le sens de convergence  $W^{2,\infty}$  des domaines signifie que nous avons la convergence des fonctions de distance signées en  $W^{2,\infty}$ .

Donnons maintenant le premier résultat de la thèse:

**Theorem 1 (continuité en ce qui concerne la déformation Lipschitz).** *Soit  $(\Omega_n)_n$  une suite de domaines ouverts uniformément Lipschitz et soit  $\Omega$  un domaine ouvert Lipschitz, qui sont inclus dans  $\mathbb{R}^d$ . Considérons le problème de Wentzell-Laplace (1.5). Soit  $u_n \in \mathcal{H}(\Omega_n)$  et  $u \in \mathcal{H}(\Omega)$  les solutions de (1.5) sur  $\Omega_n$  et  $\Omega$  respectivement. Supposons que la distance de Micheletti*

$$d(\Omega_n, \Omega) := \inf_{\Phi \in W^{1,\infty}(\mathbb{R}^d), \Omega_n = \Phi(\Omega)} \|\Phi\|_{W^{1,\infty}} \xrightarrow{n} 0,$$

alors  $u_n \xrightarrow{n} u$  dans  $\mathcal{H}(\Omega)$ .

Tout d'abord, nous devons montrer l'existence et l'unicité de la solution du problème (1.5). Pour prouver ce théorème, nous devons introduire une suite de champs de vecteurs Lipschitz  $(h_n)_{n \in \mathbb{N}}$ . On définit la suite d'applications  $(\Phi_n)_n$ , par  $\Phi_n : \mathbb{R}^d \rightarrow \mathbb{R}^d$  avec  $\Phi_n := Id + h_n$  de sorte que nous avons  $\Omega_n = \Phi_n(\Omega) = (Id + h_n)(\Omega)$ . En fait, nous divisons la preuve en trois étapes. Dans la première étape, on formule la formulation variationnelle de (1.5) en  $\Omega_n$ , puis on se ramène sur le domaine fixe  $\Omega$ , en utilisant l'inverse de la transformation  $\Phi_n$ . Nous définissons les fonctions de transport arrière  $v_n := u_n \circ \Phi_n$  et  $f := f_n \circ \Phi_n$ . La deuxième étape consiste à montrer que  $(v_n)_n$  est bornée dans  $\mathcal{H}(\Omega)$ , et ensuite de démontrer la convergence de  $v_n$  vers  $v$  (la solution sur  $\Omega$ ) dans  $\mathcal{H}(\Omega)$  dans la dernière étape.

Passons maintenant au deuxième résultat principal, qui est **indépendant** du théorème précédent:

**Theorem 2 (continuité en ce qui concerne la convergence de  $W^{2,\infty}$ ).** *Soit  $(\Omega_n)_n$  une suite des domaines  $(2, \infty)$ -Sobolev ouverts et soit  $\Omega$  un domaine  $(2, \infty)$ -Sobolev ouvert, qui sont inclus dans une boule  $B$  dans  $\mathbb{R}^d$ . Considérons le problème de Wentzell-Laplace (1.5), et soit  $u_n$*

$\in H^{\frac{3}{2}}(\Omega_n)$  la solution de (1.5) sur  $\Omega_n$ . Supposons que  $\Omega_n \xrightarrow{W^{2,\infty}} \Omega$ . Alors, il existe  $u \in H^{\frac{3}{2}}(\Omega)$  tel que  $u_n \xrightarrow{H^1(B)} u$  et  $u$  est la solution de (1.5) sur  $\Omega$ .

On remarque que le dernier théorème n'est pas valable pour la convergence de Hausdorff des ensembles, et nous donnons un contre-exemple en dimension deux (voir Remark 2.5).

La preuve du second théorème est beaucoup plus difficile que la précédente. En fait, le domaine  $\Omega$  devrait être au moins  $(2, \infty)$ -Sobolev, où un  $(2, p)$ -Sobolev avec  $1 \leq p \leq \infty$  est défini tel que la différentielle du second ordre de la fonction distance signée à la frontière  $D^2b_\Omega$  appartient à  $L^p_{loc}(U_h)$ , avec  $U_h$  est un voisinage tubulaire autour de  $\partial\Omega$  de rayon  $h > 0$ . Il est clair que nos ensembles seront en particulier des domaines Lipschitz ou dans la classe  $W^{1,\infty}$ . Cette remarque permet trouver des constantes de Lipschitz, dans certaines preuves de nos résultats.

Premièrement, nous devons montrer l'existence et l'unicité de la solution du problème (1.5). Ensuite, nous dérivons des estimations de bornes uniformes pour les solutions, puis nous montrons l'existence d'un opérateur d'extension de  $H^{\frac{3}{2}}(\Omega)$  dans  $H^{\frac{3}{2}}(B)$ . En effet, nous souhaitons étendre  $\mathcal{H}(\Omega)$  en  $\mathcal{H}(B)$ , mais ce n'est pas clair du tout. La difficulté avec l'ensemble  $\mathcal{H}$  est que la restriction à  $\Omega$  d'une fonction dans  $\mathcal{H}(B)$  n'a aucune raison d'appartenir à  $\mathcal{H}(\Omega)$ . Donc, on remarque qu'il est intéressant d'étudier la convergence dans l'espace  $H^{\frac{3}{2}}(\Omega)$ , grâce à la propriété de Necas (voir [4]) qui dit: *Si  $u \in H^1(\Omega)$  et  $u \in H^1(\partial\Omega)$  avec  $\Delta u = -f \in L^2(\Omega) \subset (H^{\frac{1}{2}}(\Omega))'$ , alors  $u \in H^{\frac{3}{2}}(\Omega)$  (avec une constante uniforme dans la norme de Lipschitz de  $\Omega$ ).* De plus, si  $(\Omega_n)_n$  est une suite des domaines bornés ouverts uniformément Lipschitz qui convergent vers  $\Omega$  dans une certaine topologie  $\mathcal{T}$ , et  $u_n$  étant la solution de problème (1.5) on  $\Omega_n$ , on prouve l'existence d'une sous-suite de  $(\tilde{u}_n)_{n \in \mathbb{N}}$ , avec  $\tilde{u}_n$  est l'extension de  $u_n$  dans  $B$ , qui converge faiblement dans  $H^{\frac{3}{2}}(B)$  vers une limite  $\tilde{u}$ . Par conséquent, on obtient la forte convergence en  $H^1(B)$  grâce à l'injection Compact de  $H^{\frac{3}{2}}(B)$  en  $H^1(B)$ . Définissez  $u$  comme étant la restriction de  $\tilde{u}$  à  $\Omega$ . On a  $u \in H^{\frac{3}{2}}(\Omega)$ . L'étape suivante montre que nous avons  $-\Delta u = f$  dans  $\Omega$ . La question est: est-ce que  $u$  satisfait la condition aux limites? Pour répondre, nous donnerons plusieurs lemmes. Dans Lemma 2.7, on montre que pour tout  $v \in W^{1,\infty}(B)$ , on a:

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v \longrightarrow - \int_{\partial\Omega} \partial_n u v.$$

Maintenant, pour continuer la preuve, nous devons prouver le Proposition 2.10 qui dit: pour tout  $v \in W^{2,\infty}(B)$ , nous avons:

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v \longrightarrow \int_{\partial\Omega} \nabla_\tau u \cdot \nabla_\tau v.$$

Pour franchir cet obstacle, nous avons besoin de l'hypothèse de la convergence de  $b_n$  à  $b$  dans  $W^{2,p}$ ,  $1 \leq p \leq \infty$ . En fait, un problème apparaîtra dans les différentielles du premier (cas  $p = \infty$ ) et du second ordre pour les formes uniformément Lipschitz. Par conséquent, nous

avons besoin de plus de régularité pour prouver que  $Db_n$  va à  $Db$  dans  $L^\infty(B)$ , comme les ensembles de "positive reach" (voir [7, Definition 6.1, p.303]) où il suffit d'avoir la convergence de Hausdorff de ces domaines, et encore plus régulier pour le calcul du second ordre comme les domaines  $(2, \infty)$ -Sobolev (ou simplement  $W^{2,\infty}$ ) avec la nécessité de la convergence  $W^{2,\infty}$  de ces domaines (pour plus de détails, on peut se référer à [7, Definition 9.1, p. 374]). En effet, pour éviter de travailler avec des mesures de Radon comme dans [7], [9], et [10], on aimerait faire une troncature autour des frontières des domaines pour se débarrasser des points singuliers en  $B$ ; en d'autres termes, nous essayons d'éloigner suffisamment les squelettes des frontières  $\partial\Omega_n$  et  $\partial\Omega$ , donc nous faisons les études dans un voisinage tubulaire uniforme  $U_h$ , pour un certain  $h > 0$  (voir sous-section 2.6.4). De plus, si  $Dp_n(x)$  et  $Dp(x)$  représentent les différentielles des fonctions de projection  $p_n$  et  $p$  au point  $x \in U_h$  sur  $\partial\Omega_n$  and  $\partial\Omega$ , respectivement, nous allons faire face à la convergence de  $Dp_n(x)$  vers  $Dp(x)$  dans lequel nous devons injecter la régularité de  $(2, \infty)$ -Sobolev avec la nécessité de convergence  $W^{2,\infty}$  de ces domaines, qui sera présenté dans la proposition 2.9. Cette proposition a pu être prouvée grâce à l'article de Gunther Leobacher et Alexander Steinicke (voir [8, Theorem C]), où ils ont étudié une formule explicite pour  $Dp(x)$ ,  $x \in U_h$ .

Dans la preuve du Theorem 2, nous utilisons l'unicité de la limite pour prouver que  $u$  est l'unique solution de (1.5). De plus, comme  $u$  est le seul point d'accumulation de la suite  $(u_n)_{n \in \mathbb{N}}$ , toute la suite converge vers  $u$  dans  $H^1(B)$ , ce qui termine la preuve.

Remarquons que les résultats du deuxième chapitre sont indépendants des chapitres suivants. En fait, dans cette thèse, nous nous intéressons à l'étude de l'attitude des valeurs propres du problème aux valeurs propres (1.1). Pour cela, nous souhaitons étudier le comportement des valeurs propres vis-à-vis des déformations de forme étudiées dans les Theorems 1 et 2. Cependant, il n'est pas clair comment transférer notre travail de la continuité de la solution du problème (1.5) à la continuité des valeurs propres du problème (1.1), en passant par la continuité des fonctions propres de (1.1).

Passant aux quatre derniers chapitres, on considère le problème des valeurs propres associé (1.1), et nous supposons que  $\Omega$  est assez lisse, c'est à dire  $\Omega$  est de classe  $C^2$ .

Afin d'illustrer le comportement du spectre, nous considérons un exemple explicite sur la boule dans  $\mathbb{R}^2$ , où nous trouvons explicitement les valeurs propres et leurs fonctions propres correspondantes.

**Proposition 1.1** *Soit  $\Omega = B_R$  un disque de rayon  $R$  en dimension deux; alors ses valeurs propres et ses fonctions propres pour l'opérateur de Wentzell-Laplacien sont:*

$$\lambda_{0,k} = \frac{j_{0,k}^2}{R^2}, \quad u_{0,k}(r, \theta) = J_0\left(\frac{j_{0,k}}{R}r\right), \quad k \geq 1, \quad (1.6)$$

$$\lambda_{n,k} = \frac{j_{n,k}^2}{R^2} \text{ valeur propre double}, \quad u_{n,k}(r, \theta) = \begin{cases} J_n\left(\frac{j_{n,k}}{R}r\right) \cos n\theta, \\ J_n\left(\frac{j_{n,k}}{R}r\right) \sin n\theta, \end{cases} \quad n, k \geq 1, \quad (1.7)$$

où  $j'_{n,k}$  est le  $k$ -ième zéro de la fonction de Bessel  $J'_n$  (la dérivée de  $J_n$ ).

Afin d'étudier la régularité des couples propres du problème aux valeurs propres (1.1), nous introduisons quelques notations fondamentales (voir sous-section 4.2.1 pour plus de détails):

On considère l'application  $t \mapsto \varphi_t = Id + tV$  où  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$  et  $t$  est assez petit. On note par

$$\Omega_t = \varphi_t(\Omega) = \{x + tV(x), \quad x \in \Omega\}.$$

Ensuite, nous étudions le transport des opérateurs de Laplace, Laplace-Beltrami et Dirichlet-à-Neumann des domaines mobiles vers un domaine fixe (au temps zéro). En fait, ces transports sont utilisés pour prouver la régularité des couples propres autour de zéro du problème de Wentzell-Laplace (1.1) dans le Theorem 12 (voir sous-section 4.2.2), où l'on suit la stratégie de M. Dambrine, D. Kateb et J. Lamboley dans [16, Theorem 3.4]. En fait, on a l'énoncé suivante:

Soit  $\Omega$  un domaine ouvert et lisse de  $\mathbb{R}^d$ . Supposons que  $\lambda$  est une valeur propre de multiplicité  $m$  de l'opérateur de Wentzell-Laplace. On suppose que  $\varphi_t = Id + tV$  pour un certain  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$  et notons  $\Omega_t = \varphi_t(\Omega)$ . Alors il existe  $m$  fonctions continues à valeurs réelles  $t \mapsto \lambda_i(t)$ ,  $i = 1, 2, \dots, m$  et  $m$  fonctions  $t \mapsto u_i^t \in H^{5/2}(\Omega)$  telles que les propriétés suivantes soient vérifiées

1.  $\lambda_i(0) = \lambda$ ,  $i = 1, \dots, m$ ,
2. les fonctions  $t \mapsto \lambda_i(t)$  et  $t \mapsto u_i^t$ ,  $i = 1, 2, \dots, m$  sont analytiques au voisinage de  $t = 0$ .
3. Les fonctions  $u_{i,t}$  définies par  $u_{i,t} \circ \varphi_t = u_i^t$  sont des fonctions propres normalisées associées à  $\lambda_i(t)$  sur le domaine mobile  $\Omega_t$ . Si l'on considère un sous-ensemble compact  $K$  tel que  $K \subset \Omega_t$  pour tout  $t$  suffisamment petit, alors  $t \mapsto (u_{i,t})|_K$  est aussi une fonction analytique de  $t$  dans un voisinage de  $t = 0$ .
4. Soit  $I \subset \mathbb{R}$  un intervalle tel que  $\bar{I}$  ne contienne que la valeur propre  $\lambda$  du problème de Wentzell de multiplicité  $m$ . Alors il existe un voisinage de  $t = 0$  tel que  $\lambda_i(t)$   $i = 1, \dots, m$  sont les seules valeurs propres de  $\Omega_t$  qui appartiennent à  $I$ .

Commençons maintenant à présenter nos principaux théorèmes du chapitre quatre. Tout d'abord, nous dérivons dans le Theorem 3 les variations locales des valeurs propres et des fonctions propres de notre problème (1.1):

**Theorem 3 (Équation caractéristique en  $u'$ ).** *Avec les notations du Theorem 12, si  $t \mapsto (\lambda(t), u_t)$  est l'un des chemins de paires propres lisses  $(\lambda_i(t), u_{i,t})$  de  $\Omega_t$  pour le problème de Wentzell (1.1), alors la dérivée de forme  $u' = (\partial_t u_t)|_{t=0}$  de la fonction propre  $u_t$  satisfait*

$$-\Delta u' = \lambda' u + \lambda u' \quad \text{dans } \Omega, \quad (1.8)$$

$$\begin{aligned} -\Delta_\tau u' + \partial_n u' &= \operatorname{div}_\tau(V_n \nabla_\tau u) + \Delta_\tau(V_n \partial_n u) \\ &\quad + \operatorname{div}_\tau(V_n (HId - 2D^2 b) \nabla_\tau u) + \lambda V_n u \quad \text{sur } \partial\Omega, \end{aligned} \quad (1.9)$$

où  $D^2 b$  est l'opérateur de Weingarten défini sur  $\partial\Omega$  et  $H$  est la somme des courbures principales de la frontière.

Soit  $\lambda$  une valeur propre simple de l'équation volumique de Wentzell-Laplace et soit  $u$  la valeur correspondante fonction propre normalisée. Nous donnons dans cette sous-section la formule explicite de la dérivée de forme de la valeur propre volumique de l'opérateur de Wentzell-Laplace associé à .

Sur  $\Omega_t = (Id + tV)(\Omega)$  avec  $t$  petit, il existe une valeur propre unique  $\lambda(t)$  près de  $\lambda$  qui est une valeur analytique fonction en respectant le paramètre  $t$ . La fonction propre associée  $u_t(x) = u(t, x)$  est solution de le problème . La dérivée de forme notée  $u'$  est la dérivée partielle  $\partial_t u(t, x)$  évaluée à  $t = 0$  et résout (1.1). Déduisons l'expression analytique de  $\lambda'(0)$ :

**Theorem 4 (Dérivée du premier ordre de valeur propre simple).** *Si  $(\lambda(t), u_t)$  est un couple propre (avec  $u_t$  normalisé) pour le problème de volume Wentzell avec l'hypothèse supplémentaire que  $\lambda(t)$  est simple alors l'application  $t \mapsto \lambda(t)$  est analytique et sa dérivée en  $t = 0$ , pour toute déformation  $V \in W^{3,\infty}(\mathbb{R}^d)$ , est*

$$\lambda'(0) = \int_{\partial\Omega} V_n (|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda |u|^2 + \nabla_\tau u^T (HId - 2D^2 b) \nabla_\tau u) \quad (1.10)$$

où  $D^2 b$  est l'opérateur de Weingarten défini sur  $\partial\Omega$  et  $H$  est la somme des courbures principales de la frontière.

Si nous supposons que  $\lambda(t)$  est une valeur propre de multiplicité  $m$ . Pour une déformation lisse  $t \mapsto \Omega_t$ , il y aura  $m$  valeurs propres proches de  $\lambda(t)$  (en comptant leurs multiplicités) pour les petites valeurs de  $t$ . Nous savons qu'un tel la valeur propre multiple n'est plus différentiable au sens classique. On est alors amené à calculer la dérivée directionnelle de  $t \mapsto \lambda_i(t)$  à  $t = 0$  où  $\lambda_i(t)$ ,  $j = 1, \dots, m$  sont donnés par le Theorem 12 (voir sous-section 4.2.2).



**Theorem 5 (Dérivée du premier ordre de valeur propre multiple).** Soit  $\lambda(t)$  une valeur propre multiple d'ordre  $m \geq 2$ . Alors chaque  $t \mapsto \lambda_i(t)$  pour  $i = 1, 2, \dots, m$  donné par le Théorème 12 a une dérivée proche de 0, et les valeurs de  $\lambda'_i(0)$ ,  $i = 1, 2, \dots, m$  sont les valeurs propres de la matrice  $M(V_n) = (M_{j,k})_{1 \leq j, k \leq m}$  définie par

$$M_{j,k} = \int_{\partial\Omega} V_n (\nabla_\tau u_j \cdot \nabla_\tau u_k - \partial_n u_j \partial_n u_k - \lambda u_j u_k + (HId - 2D^2b) \nabla_\tau u_j \cdot \nabla_\tau u_k) \quad (1.11)$$

où  $D^2b$  est l'opérateur de Weingarten défini sur  $\partial\Omega$  et  $H$  est la somme des courbures principales de la frontière.

Maintenant, nous étudions quelques calculs de dérivées du second ordre de forme. Commençons par dériver l'équation aux limites de  $u'' = \left( \frac{d^2 u}{dt^2} \right)_{|t=0}$  dans la direction de  $V \in W^{3,\infty}(\mathbb{R}^d)$ .

**Theorem 6 (Équation caractéristique en  $u''$ ).** Avec les notations du Theorem 12, si  $t \mapsto (\lambda(t), u_t)$  est l'un des chemins de paires propres lisses  $(\lambda_i(t), u_{i,t})$  de  $\Omega_t$  pour le problème de Wentzell (1.1), alors la dérivée seconde de forme  $u'' = (\partial_t^2 u_t)_{|t=0}$  de la fonction propre  $u_t$  satisfait

$$-\Delta u'' = \lambda'' u + 2\lambda' u' + \lambda u'' \quad \text{dans } \Omega,$$

$$\begin{aligned} -\Delta_\tau u'' + \partial_n u'' = & \operatorname{div}_\tau(A) + \Delta_\tau(B) + \nabla_\tau u' \cdot \nabla_\tau V_n + V_n \lambda' u + V_n \lambda u' + V_n \Delta_\tau u' \\ & + V_n' \lambda u + V_n \lambda' u + V_n \lambda u' + V_n \lambda \partial_n V_n u + V_n^2 \lambda u \partial_n u + V_n^2 H \lambda u \quad \text{sur } \partial\Omega. \end{aligned}$$

Où

$$\begin{aligned} A := & V_n [\nabla_\tau (\partial_n u') - 2D^2b \nabla_\tau u' + H \nabla_\tau u' + \nabla_\tau u' + \partial_n u \nabla_\tau V_n] \\ & + V_n [\partial_n V_n \nabla_\tau u + \partial_n u \mathcal{A} \nabla_\tau V_n + \mathcal{A} \nabla_\tau u' + \partial_n u \mathcal{A} \nabla_\tau V_n + \mathcal{A}' \nabla_\tau u] \\ & + V_n^2 [\nabla_\tau (\partial_n u) - 2D^2b \nabla_\tau u + H \nabla_\tau u + 2\mathcal{A} \nabla_\tau (\partial_n u) - 2D^2b \mathcal{A} \nabla_\tau u + \tilde{\mathcal{A}} \nabla_\tau u + H \mathcal{A} \nabla_\tau u] \\ & + \nabla_\tau V_n \partial_n u' + V_n' \nabla_\tau u + V_n' \mathcal{A} \nabla_\tau u + \partial_n V_n \mathcal{A} \nabla_\tau u, \end{aligned}$$

et

$$B := V_n' \partial_n u + V_n (\partial_n u' - \nabla_\tau V_n \cdot \nabla_\tau u + \partial_n V_n \partial_n u) - V_n^2 (\lambda u + \partial_n u + H \lambda u),$$

où  $D^2b$  est l'opérateur de Weingarten défini sur  $\partial\Omega$ ,  $H$  est la somme des courbures principales du bord,  $\mathcal{A} := (HId - 2D^2b)$  et on note  $\tilde{\mathcal{A}} := (\tilde{a}_{i,j})_{1 \leq i, j \leq d}$  la matrice dont les éléments sont les dérivées normales de  $a_{i,j}$ , où  $a_{i,j}$ ,  $i, j = 1, \dots, d$ , sont les composantes de la matrice  $\mathcal{A}$ .

Nous donnons dans le Theorem 7 une formule explicite pour la dérivée de forme au second ordre d'une valeur propre simple  $\lambda$  à  $t = 0$ , pour tout  $V \in W^{3,\infty}(\mathbb{R}^d)$ , c'est-à-dire :

**Theorem 7 (Dérivée du deuxième ordre de valeur propre simple).** *Soit  $(u_t, \lambda(t))$  un chemin lisse de paires propres de (1.1) sur  $\Omega_t$ . Si  $\lambda(t)$  est une valeur propre simple, alors la dérivée seconde de  $\lambda(t)$  à  $t = 0$  dans la direction  $V \in W^{3,\infty}(\mathbb{R}^d)$  est :*

$$\begin{aligned} \lambda''(0) &= \int_{\partial\Omega} (V'_n + V_n \partial_n V_n + V_n^2 H) [|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda|u|^2 + \mathcal{A} \nabla_\tau u \cdot \nabla_\tau u] \\ &+ \int_{\partial\Omega} V_n [2(Id + \mathcal{A}) \nabla_\tau u \cdot \nabla_\tau u' + 2(2Id + \mathcal{A}) \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u] \\ &+ \int_{\partial\Omega} V_n [-2\partial_n u \partial_n u' - \lambda'|u|^2 - 2\lambda u u' + \mathcal{A}' \nabla_\tau u \cdot \nabla_\tau u] \\ &+ \int_{\partial\Omega} V_n^2 \left[ 2(Id + \mathcal{A}) \nabla_\tau (\partial_n u) \cdot \nabla_\tau u - (2D^2 b + 2D^2 b \mathcal{A} - \tilde{\mathcal{A}}) \nabla_\tau u \cdot \nabla_\tau u + 2(1 + H) |\partial_n u|^2 \right] \end{aligned}$$

où  $D^2 b$  est l'opérateur de Weingarten défini sur  $\partial\Omega$ ,  $H$  est la somme des courbures principales du bord,  $\mathcal{A} := (HId - 2D^2 b)$  et on note  $\tilde{\mathcal{A}} := (\tilde{a}_{i,j})_{1 \leq i,j \leq d}$  la matrice dont les éléments sont les dérivées normales de  $a_{i,j}$ , où  $a_{i,j}$ ,  $i, j = 1, \dots, d$ , sont les composantes de la matrice  $\mathcal{A}$ .

Le théorème suivant sert à énoncer deux identités de Pohožaev.

**Theorem 8 (Identités de Pohožaev).** *Soit  $\Omega$  un domaine de classe  $C^2$  et soit  $u$  une fonction propre du problème (1.1). Nous avons ces deux identités de Pohožaev :*

$$\int_{\partial\Omega} (D^2 u + \partial_n u Id) X \cdot \nabla_\tau u + \int_{\partial\Omega} |\nabla_\tau u|^2 + \int_{\partial\Omega} |\partial_n u|^2 (X \cdot n) = 0 \quad (1.12)$$

où  $X$  est un champ vectoriel de translation, c'est-à-dire  $X(x) = x + \alpha$  pour tout  $x, \alpha \in \mathbb{R}^d$ , et  $D^2 u$  est la matrice Hessienne de  $u$ .

Aussi, nous avons

$$\int_{\partial\Omega} D^2 b \nabla_\tau u \cdot \nabla_\tau u + \int_{\partial\Omega} D^2 u \cdot n \cdot \nabla_\tau u + \int_{\partial\Omega} |\partial_n u|^2 = 0 \quad (1.13)$$

où  $D^2 b$  est la matrice hessienne de la fonction de distance signée  $b$  à  $\partial\Omega$ .

Donnons maintenant la définition d'une forme critique:

**Definition 1.1** *Un domaine  $\Omega$  est dit critique s'il existe une valeur propre  $\lambda$  telle que la dérivée  $\lambda'(\Omega, V)$ , dans toute direction  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$ , soit nulle.*

Le corollaire suivant montre la non-criticité des boules en toute dimension:

**Corollary 1.1** *Les boules dans  $\mathbb{R}^d$ , avec et sans contrainte de volume, ne sont pas critiques.*

Donnons maintenant ce théorème intéressant qui aborde le sujet des formes non critiques dans  $\mathbb{R}^2$ .

**Theorem 9 (Formes non critiques).** *Soit  $\Omega_0$  un domaine ouvert lisse borné critique de classe  $C^2$  dans  $\mathbb{R}^2$ . Supposons qu'il existe quelques points sur la frontière  $\partial\Omega_0$  où la courbure  $H$  est 2 et aussi quelques autres points où  $H \notin \{0, 2, \frac{2}{3}\}$ . Alors, sans contrainte de volume, il existe un champ vectoriel de déformation  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$  tel que pour tout petit  $t > 0$ ,  $\Omega_t = \Omega_0 + tV$  n'est pas critique.*

Remarquons que, sous l'hypothèse mentionnée sur la frontière  $\partial\Omega_0$  dans le théorème précédent, l'ensemble de ces formes  $\Omega_0$  peut être considéré comme grand dans l'ensemble des domaines admissibles dans  $\mathbb{R}^2$ , mais la question est: avons-nous de la densité?

En fait, pour prouver ce théorème, on raisonne par contradiction. On fixe donc un domaine critique initial  $\Omega_0$  de classe  $C^2$  à l'instant  $t = 0$ , et on suppose qu'il existe une petite période de temps  $\tau$  pendant laquelle ce domaine restera critique pour  $0 \leq t < \tau$ , pour tous les champs de vecteurs de déformation  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$ .

En fait, la finesse de cette preuve réside dans l'insuffisance des calculs de dérivées du second ordre. Il faut donc s'élever jusqu'aux calculs de dérivées d'ordre 3, pour finalement trouver une contradiction.

Donnons maintenant le premier problème ouvert suivant qui étudie la simplicité d'une valeur propre:

**Open problem 1 (Simplicité d'une valeur propre).** *Soit  $\Omega_0$  un domaine ouvert lisse borné de classe  $C^2$  dans  $\mathbb{R}^2$ . Suppose qu'il existe quelques points sur la frontière  $\partial\Omega_0$  où la courbure  $H$  est 2 et aussi quelques autres points où  $H \neq 2$ . Soit  $\lambda_i = \lambda_j = \lambda$  une valeur propre multiple dans  $\Omega_0$  à  $t = 0$ . Alors il existe un champ vectoriel de déformation  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$  tel que pour tout  $t > 0$ , les valeurs propres  $\lambda_i$  et  $\lambda_j$  sont distinctes en  $\Omega_t = \Omega_0 + tV$ .*

Toujours dans ce théorème, la dérivée du second ordre ne pouvait conduire à aucune contradiction. Par conséquent, pour approcher beaucoup plus et pour faciliter les choses, nous devons passer aux calculs de dérivées du troisième ordre. Mais nous n'avons toujours pas pu trouver de contradiction.

Pour terminer cette thèse, nous discutons de résultat de simplicité générique sur le spectre du (1.1) par rapport aux domaines  $\mathbb{R}^2$ . Nous commençons par clarifier la notion de généricité envisagée ci-dessous. Rappelons que l'ensemble des domaines bornés de  $\mathbb{R}^2$  avec  $C^2$  frontière notée  $\mathbb{D}_2^2$  peut être dotée de la topologie suivante: la base des voisinages ouverts est (essentiellement) donnée par les ensembles  $V(\Omega, \varepsilon)$  définis, pour tout domaine  $\Omega \in \mathbb{D}_2^2$  et  $\varepsilon > 0$  suffisamment petit, comme images de  $\Omega$  par  $\varphi = Id + U$ , avec  $U \in W^{3,\infty}(\Omega, \mathbb{R}^2)$  et  $\|U\|_{W^{3,\infty}} < \varepsilon$ . Alors

$\varepsilon$  est choisi tel que  $Id + U : \Omega \mapsto (Id + U)(\Omega)$  est un difféomorphisme. Comme le montre A.M. Micheletti dans [17], un voisinage  $V(\Omega, \varepsilon)$  de  $\Omega \in \mathbb{D}_2^2$  tel que défini précédemment est métrisable en utilisant une distance de type Courant, notée  $d_2$ , et chaque  $(V(\Omega, \varepsilon), d_2)$ , est complet et séparable. Pour tout domaine  $\Omega \in \mathbb{D}_2^2$ , nous utilisons  $\mathbb{D}_2^2(\Omega)$  pour désigner la variété de Banach obtenue comme l'ensemble des images  $(Id + U)(\Omega)$  par  $U \in W^{3,\infty}(\Omega, \mathbb{R}^2)$ , qui sont difféomorphes de  $\Omega$ . Une propriété  $(P)$  sera donc dite "générique par rapport à  $\Omega \in \mathbb{D}_2^2$ " si, pour tout  $\Omega \in \mathbb{D}_2^2$ , l'ensemble des domaines de  $\mathbb{D}_2^2(\Omega)$  où  $(P)$  est vrai contient une intersection dénombrable de sous-ensembles ouverts et denses de  $\mathbb{D}_2^2(\Omega)$ .

Donnons maintenant le deuxième problème ouvert suivant qui étudie la simplicité générique des valeurs propres:

**Open problem 2 (*Simplicité générique des valeurs propres*).** *Génériquement par rapport aux domaines  $\Omega \in \mathbb{D}_2^2$ , toutes les valeurs propres de (1.1) sont simples.*

En fait, nous suivons la stratégie classique de Y. Chitour, D. Kateb, R. Long dans [14] pour l'opérateur de Laplace avec conditions aux limites de Dirichlet, dans laquelle ils ont prouvé certaines propriétés génériques du spectre de l'opérateur de Stokes. Cette stratégie a notamment été appliquée avec succès dans [15] pour la simplicité générique de l'opérateur de Stokes dans un espace à deux dimensions. Dans ce qui suit, nous allons présenter les principaux concepts qu'ils ont utilisés. Pour cela, fixez un domaine  $\Omega_0 \in \mathbb{D}_2^2$ . On définit l'ensemble  $A_0 := \mathbb{D}_2^2(\Omega_0)$ .

Fixons un domaine  $\Omega_0$  et introduisons les ensembles suivants: pour  $n \geq 1$ , l'ensemble  $A_n$  contient toutes les images  $\Omega$  par la carte  $Id + U$  du domaine fixe  $\Omega_0$  pour que nous ayons  $\Omega = \Omega_0 + U$ ,  $U \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$ ,  $\Omega_0 + U \in A_0$  et les premières  $n$  valeurs propres de (1.1) sur  $\Omega_0 + U$  sont simples. De plus, nous définissons  $A$  comme l'intersection dénombrable de tous les  $A_n$ ,  $n \in \mathbb{N}$ . Notons que l'ensemble  $A$  sera l'ensemble des images  $\Omega \in A_0$ , tel que toutes les valeurs propres de (1.1) sur  $\Omega$  sont simples.

Ensuite, on applique le lemme de Baire sur les ensembles  $A_n \subset A_0$ . Nous affirmons qu'il suffit de montrer que les ensembles  $A_n$  sont ouverts dans  $A_0$  et que  $A_{n+1}$  est dense dans  $A_n$  pour tout  $n \geq 0$ . Après cela, si cette dernière est vraie, alors nous pouvons prouver ce qui suit : si  $A_{n+1}$  est dense dans  $A_n$  et  $A_n$  est dense dans  $A_0$ , alors  $A_{n+1}$  est également dense en  $A_0$ . Enfin, en appliquant le lemme de Baire aux ensembles  $A_n$ , on en déduit que  $A = \bigcap_n A_n$  est dense en  $A_0$ .

Mais la question repose sur la densité : peut-on prouver que l'ensemble  $A_{n+1}$  est dense dans  $A_n$  pour tout  $n \geq 0$  ?

Malheureusement, l'obstacle auquel nous serons confrontés dans ce problème est lié d'une manière ou d'une autre à la stratégie de Open problem 1. En fait, il n'est pas évident de garantir l'existence d'un point sur  $\partial\Omega$  où la courbure  $H$  soit 2. En effet, si on se restreint à ce cas, les ensembles  $A_n$  et  $A$  ne seront plus ouverts dans  $A_0$ , ce qui contredit la stratégie recherchée.

### Décomposition de thèse:

Le chapitre deux est décomposé de la manière suivante : La section 2.2 sert à introduire quelques notations, définitions et préliminaires. Dans la section 2.3, nous montrons l'existence et l'unicité de la solution du problème (1.5). La section 2.4 est pour dériver des estimations de bornes uniformes pour les solutions et montrer la existence d'un opérateur d'extension de  $H^{\frac{3}{2}}(\Omega)$  dans  $H^{\frac{3}{2}}(B)$ . En plus, la section 2.5 sert à prouver plusieurs lemmes utiles. Dans la section 2.6, nous discutons de la convergence de la fonction distance signée dans  $W^{m,p}$ , pour  $m = 0, 1$  et  $1 \leq p \leq \infty$ , par rapport à la régularité de nos formes données. Aussi dans cette section, on prouve la convergence de  $Dp_n(x)$  vers  $Dp(x)$  dans  $L^p(U_h)$ , avec les données nécessaires présentées ci-dessus. La section 2.7 est utilisée pour donner une proposition essentielle pour le preuve de Theorem 2. Enfin, dans la dernière section de ce chapitre, nous démontrons les Theorems 1 et 2.

Dans le chapitre trois, on considère le problème de valeurs propres associé, puis on démontre la Proposition 1.1.

Le chapitre quatre se décompose de la manière suivante: La section 4.2 est divisée en trois sous-sections. tout d'abord, la sous-section 4.2.1 sert à introduire quelques notations et principalement des résultats sur le problème des conditions aux limites de Wentzell-Laplace. Où dans sous-section 4.2.2, nous discutons de la régularité du couple propre du problème de Wentzell-Laplace. Dans la sous-section 4.2.3, nous donnons plusieurs lemmes techniques qui sont utiles dans la preuve de nos principaux résultats. Dans la dernière section du chapitre quatre, nous prouvons quelques résultats principaux de cette thèse. Tout d'abord, dans sous-section 4.3.1, nous étudions la variation locale des fonctions propres du problème, tandis que dans sous-section 4.3.2 et 4.3.3, nous énonçons les résultats des dérivées de forme des valeurs propres simples et multiples de (1.1). De plus, dans les sous-sections 4.3.4 et 4.3.5, nous étudions le calcul de la dérivée seconde de forme des valeurs propres du (1.1).

En chapitre cinq, Dans la section 5.2, on donne un lemme utile pour la preuve du Theorem 9. La dernière section de ce chapitre est décomposée en trois parties: dans la sous-section 5.3.1 nous dérivons deux identités de Pohožaev qui seront utilisées dans la sous-section 5.3.2, dans laquelle on montre que, avec et sans contrainte de volume, les boules de  $\mathbb{R}^d$  ne sont pas critiques. Enfin, dans la dernier sous-section, nous donnons une démonstration de la dernier résultat principal (Theorem 9).

Enfin, en chapitre six, en tant que problèmes ouverts (en dimension deux), il y a deux sections: Premièrement, nous donnons une stratégie en essayant de montrer que si nous avons une valeur propre multiple  $\lambda$  en  $\Omega$ , elle peut être séparé sous une certaine déformation de la forme initiale. Finalement, nous discutons de la simplicité générique du spectre.

## Chapter 2

# Continuity of the solution under shape deformations

In this chapter, We investigate the attitude of the solution of the Wentzell-Laplace boundary value problem with respect to shape deformations. On one hand, we prove the continuity of the solution under Lipschitz deformation of uniformly Lipschitz domains in the Euclidean space  $\mathbb{R}^d$ . On the other hand, we demonstrate the continuity of the solution under the  $W^{2,\infty}$  convergence of  $(2, \infty)$ -Sobolev domains in  $\mathbb{R}^d$ . After that, we discuss the convergence of the signed distance function in  $W^{1,p}(B)$  with  $1 \leq p \leq \infty$  with respect to the Hausdorff convergence of less regular domains, where  $B$  is a large ball containing the sequence of domains  $(\Omega_n)_{n \in \mathbb{N}}$  and the limit set  $\Omega$ , and we derive the following results: the uniformly Lipschitz case is enough for proving the convergence of the first order differential in  $L^p$  with  $1 \leq p < \infty$ ; while in case  $p = \infty$ , we need to assume more regularity, the "positive reach" one. Moreover, to prove the convergence of the first order differential of the projection function on  $\partial\Omega_n$  in  $L^p$  with  $1 \leq p \leq \infty$ , we have to inject the  $(2, \infty)$ -Sobolev regularity with the necessity of  $W^{2,\infty}$  convergence of these domains.

### 2.1 Introduction and statement of the results

Let  $\Omega$  be a Lipschitz open bounded domain in  $\mathbb{R}^d$ . Let us give the following boundary problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ -\Delta_\tau u + \partial_{\mathbf{n}} u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

with  $f \in L^2(\Omega)$ , and let  $B$  be a ball containing the set  $\Omega$ . The set  $\mathcal{H}_s$  is defined as in the introduction by

$$\mathcal{H}_s(\Omega) := \{u \mid u \in H^1(\Omega), u|_{\partial\Omega} \in H^1(\partial\Omega)\}$$

equipped with the graph norm  $\|\cdot\|_{\mathcal{H}_s(\Omega)}$  defined as:

$$\|u\|_{\mathcal{H}_s(\Omega)} := \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)} + \|\nabla_\tau u\|_{L^2(\partial\Omega)}.$$

Let's set  $\mathcal{H}(\Omega)$  as

$$\mathcal{H}(\Omega) := \{u \in \mathcal{H}_s(\Omega) \mid \int_{\partial\Omega} u = 0\}$$

which is closed in  $\mathcal{H}_s(\Omega)$  and inherits the norm from  $\mathcal{H}_s(\Omega)$ . In fact, this set is necessary in order to prove the existence and uniqueness of the solution (see section 2.3).

Let us give the first result concerning this chapter:

**Theorem 1 (continuity with respect to Lipschitz deformation).** *Let  $(\Omega_n)_n$  be a sequence of uniformly Lipschitz open domains and let  $\Omega$  be a Lipschitz open domain that are included in  $\mathbb{R}^d$ . Consider the Wentzell-Laplace problem (2.1). Let  $u_n \in \mathcal{H}(\Omega_n)$  and  $u \in \mathcal{H}(\Omega)$  be the solutions of (2.1) on  $\Omega_n$  and  $\Omega$ , respectively. Suppose that the Micheletti distance*

$$d(\Omega_n, \Omega) = \inf_{\Phi \in W^{1,\infty}(\mathbb{R}^d), \Omega_n = \Phi(\Omega)} \|\Phi\|_{W^{1,\infty}} \xrightarrow{n} 0,$$

then  $u_n \xrightarrow{n} u$  in  $\mathcal{H}(\Omega)$ .

**Remark 2.1** *To compare with, one can refer to M. Dambrine and D. Kateb article about the "Persistency of wellposedness of Ventcel's boundary value problem under shape deformations" (see [11]), where they studied the persistency of the solvability of the boundary value problem under **smooth** domain deformation.*

While the second theorem, that is **independent** of the previous one, is:

**Theorem 2 (continuity with respect to  $W^{2,\infty}$  convergence).** *Let  $(\Omega_n)_n$  be a sequence of  $(2, \infty)$ -Sobolev open domains and let  $\Omega$  be a  $(2, \infty)$ -Sobolev open domain, that are included in a ball  $B$  in  $\mathbb{R}^d$ . Consider the Wentzell-Laplace problem (2.1), and let  $u_n \in H^{\frac{3}{2}}(\Omega_n)$  be the solution of (2.1) on  $\Omega_n$ . Suppose that  $\Omega_n \xrightarrow{W^{2,\infty}} \Omega$ . Then there exists a limit  $u \in H^{\frac{3}{2}}(\Omega)$  such that  $u_n \xrightarrow{H^1(B)} u$  and  $u$  is the solution of (2.1) on  $\Omega$ .*

**Remark 2.2** *We remark that the last theorem is not valid for the Hausdorff convergence of the sets, and we give a counter example in dimension two (see Remark 2.5).*

First of all, we need to show the existence and uniqueness of the solution to the problem (2.1).

Now, In order to prove Theorem 2, we derive firstly some uniform bound estimates for the solution. Next, we wish to extend  $\mathcal{H}(\Omega)$  into  $\mathcal{H}(B)$ , but it is not clear at all. To overcome this obstacle, one can refer to the  $H^{\frac{3}{2}}(\Omega)$  space, thanks to the Necas property (see [4]) which

says: *If  $u \in H^1(\Omega)$  and  $u \in H^1(\partial\Omega)$  with  $\Delta u = -f \in L^2(\Omega) \subset (H^{\frac{1}{2}}(\Omega))'$ , then  $u \in H^{\frac{3}{2}}(\Omega)$*  (with a uniform constant in the Lipschitz norm of  $\Omega$ ), and it is easier to show the existence of an extension operator from  $H^{\frac{3}{2}}(\Omega)$  into  $H^{\frac{3}{2}}(B)$ . Moreover, if  $(\Omega_n)_n$  is a sequence of uniformly Lipschitz open bounded domains which converge to  $\Omega$  under certain topology  $\mathcal{T}$ ,  $u_n$  being the solution of problem (2.1) on  $\Omega_n$ , we prove the existence of a weakly convergent subsequence of  $\tilde{u}_n$  (the extension of  $u_n$  in  $B$ ) in  $H^{\frac{3}{2}}(B)$  to a limit  $\tilde{u}$ . Therefore, we obtain the strong convergence in  $H^1(B)$  thanks to the Compact injection of  $H^{\frac{3}{2}}(B)$  in  $H^1(B)$ . Actually, as  $\tilde{u}$  is the only accumulation point, the whole sequence converges to this limit. The next step shows that we have  $-\Delta u = f$  in  $\Omega$  where  $u \in H^{\frac{3}{2}}(\Omega)$  is the restriction of  $\tilde{u}$  to  $\Omega$ . The question now is: does  $u$  satisfy the boundary condition? To answer, we will give several lemmas (section 2.5) and propositions (subsection 2.6.4 and section 2.7), with the additional regularity  $(2, \infty)$ -Sobolev that is defined in Definition 2.4, where the convergence of  $\Omega_n$  toward  $\Omega$  should be in the  $W^{2, \infty}$  sense.

**The plan of this chapter is as follows (see Figure 2.1):** in order to study the existence and uniqueness of the solution of problem (2.1) in section 2.3, we define the set of  $\varepsilon$ -cone in Definition 2.3, then we conclude that a Lipschitz domain has this property thanks to Theorem 10. Furthermore, we can deduce in Corollary 2.1 that if  $\Omega$  is  $(2, \infty)$ -Sobolev domain then it also has the  $\varepsilon$ -cone property. Actually, we use the Lax-Milgram Theorem (see Proposition 2.1).

Concerning Theorem 1, the proof is in subsection 2.8.1.

For a proof of Theorem 2, we follow the following plan: After using Necas property stated above, we desire to derive several uniform estimates, with respect to domains in the set  $\mathcal{O}$  which is introduced in Definition 2.5, on the function  $u \in H^{\frac{3}{2}}(\Omega)$  (see Propositions 2.2 and 2.3). Then in Proposition 2.4, we show the existence of an extension operator from  $H^{\frac{3}{2}}(\Omega)$  to  $H^{\frac{3}{2}}(B)$ . After that, we give some uniform estimations (with respect to  $\varepsilon$ ) on the extended function  $\tilde{u}$  in  $H^{\frac{3}{2}}(B)$  (Corollary 2.2), and if we consider a sequence of uniformly Lipschitz domains  $(\Omega_n)_n$  that converges to  $\Omega$  under certain topology  $\mathcal{T}$ , we show the strong convergence of the sequence of solutions  $(\tilde{u}_n)_n$  to a limit  $\tilde{u}$  in  $H^1(B)$  (Corollary 2.3). In Lemma 2.3, we show that the restriction  $u \in H^{\frac{3}{2}}(\Omega)$  of the limit  $\tilde{u}$  to  $\Omega$  satisfies  $-\Delta u = f$ . But we still need to show that  $u$  solves the problem (2.1), i.e. does  $u$  satisfies  $-\Delta_\tau u = \partial_{\mathbf{n}} u$  on  $\partial\Omega$ ? To answer this question, we give several lemmas (from Lemma 2.4 to 2.7), in addition to Proposition 2.10 (see section 2.7). In fact, in this proposition, we need to deal with the convergence of the signed distance functions in  $W^{2,p}$ ,  $p \geq 1$ , in a uniform tubular neighborhood  $U_h$  for some  $h > 0$ , which always exists due to Proposition 2.8 (see subsection 2.6.4), and then Theorem 2 is proved.

In section 2.6, we are going to study the behavior of the convergence in  $W^{1,p}(B)$ , with  $1 \leq p \leq \infty$ , of the sequence of the oriented distance functions  $(b_n)_{n \in \mathbf{N}}$  or  $(b_{\Omega_n})_{n \in \mathbf{N}}$  with respect to the Hausdorff convergence of less regular domains (see Propositions 2.5, 2.6 and 2.7). Also, we study the convergence of the differential of the projection functions  $p_n$  on  $\partial\Omega_n$  in  $L^p$  for  $1 \leq p \leq \infty$  with respect to the  $W^{2, \infty}$  convergence of  $(2, \infty)$ -Sobolev domains (see Proposition 2.9).



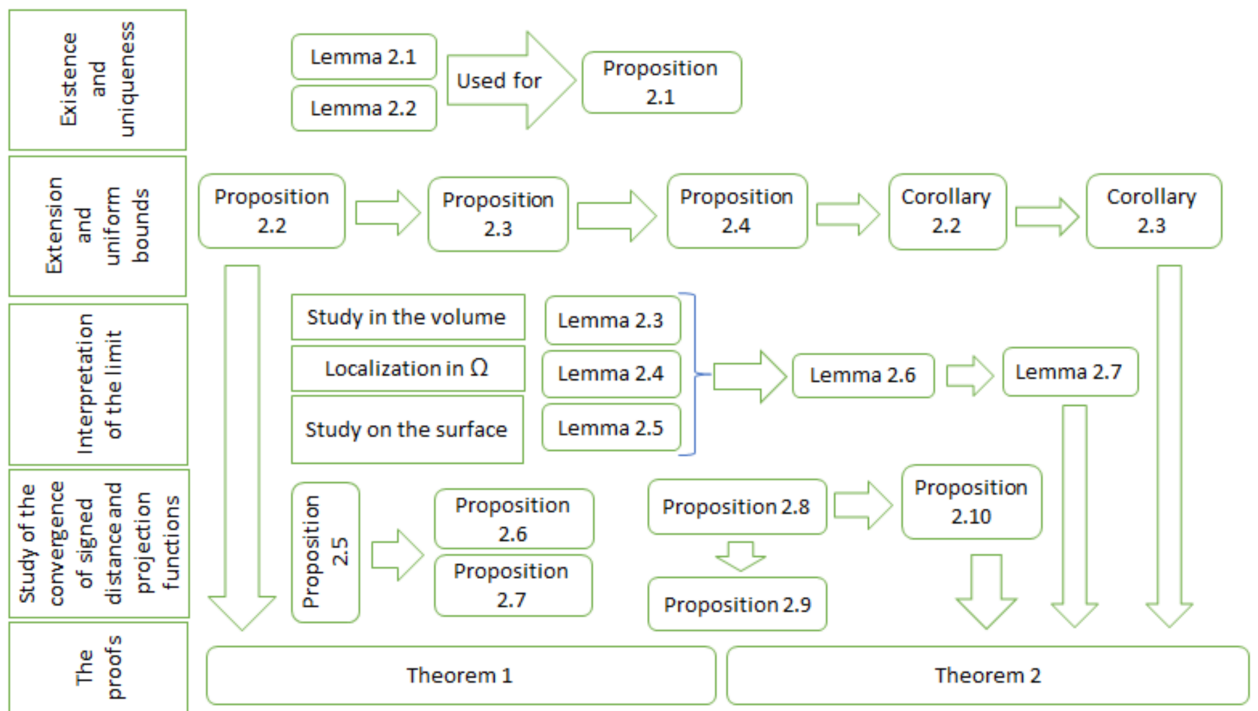


Figure 2.1: The map of Chapter two

## 2.2 Definitions and preliminaries

The first definition is for claiming that the normal derivative  $\partial_{\mathbf{n}}u$  is well defined in  $H^{-\frac{1}{2}}(\partial\Omega)$ :

**Definition 2.1** *If  $\Delta u \in L^2(\Omega)$ , then the normal derivative  $\partial_{\mathbf{n}}u$  is well defined in  $H^{-\frac{1}{2}}(\partial\Omega)$ , where  $\mathbf{n}$  denotes the outward unit normal vector field on the boundary, in the following weak sense:*

*Let us define the linear continuous form  $\mathbf{L}$  of  $H^{\frac{1}{2}}(\partial\Omega)$ :*

$$\forall v \in H^{\frac{1}{2}}(\partial\Omega), \quad \mathbf{L}(v) := \int_{\Omega} \Delta u \tilde{v} + \nabla u \cdot \nabla \tilde{v} = \int_{\partial\Omega} \partial_{\mathbf{n}}u v \quad (2.2)$$

where  $\tilde{v} := \gamma_0^{-1}(v) \in H^1(\Omega)$  is the inverse image of  $v \in H^{\frac{1}{2}}(\partial\Omega)$  by the linear bounded trace mapping  $\gamma_0$  from  $H^1(\Omega)$  onto  $H^{\frac{1}{2}}(\partial\Omega)$ . By Riesz representation theorem (see [5, Theorem 4.12, p.40]), there exists a unique  $y$  in  $(H^{\frac{1}{2}}(\partial\Omega))' = H^{-\frac{1}{2}}(\partial\Omega)$ , such that  $\mathbf{L}(v) = \langle y, v \rangle_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}$  for all  $v \in H^{\frac{1}{2}}(\partial\Omega)$ , then we can conclude from (2.2) that  $\partial_{\mathbf{n}}u = y$  in a weak sense on the boundary  $\partial\Omega$ .

◇

From the definition of  $\mathcal{H}(\Omega)$ ,  $\nabla_{\tau}u$  is defined on  $\partial\Omega$  in a weak sense. For the study of this chapter, we need to define an extension of  $\nabla_{\tau}u$  in  $B$ :

Let  $b_{\Omega}$  (or simply  $b$  if no confusion) be the signed distance function to the set  $\Omega$ , that is defined by

$$b_{\Omega}(x) = b(x, \partial\Omega) := \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega, \\ d(x, \partial\Omega) & \text{if } x \in \mathbb{R}^d \setminus \bar{\Omega}, \end{cases}$$

where  $d$  is the usual metric in  $\mathbb{R}^d$ , and for any  $x \in \mathbb{R}^d$ ,

$$d(x, \partial\Omega) = \inf_{y \in \partial\Omega} d(x, y) = \inf_{y \in \partial\Omega} |x - y|$$

One can refer to subsection 7.3.1 in Chapter seven for more details.

Now, as the skeletons of  $\Omega$  are of zero  $d$ -dimension, we deduce that  $b$  is differentiable almost everywhere in  $B$ , thus  $\nabla b \in L^2(B)$ . In addition, we have outside the skeletons: its gradient satisfies  $|\nabla b| = 1$  and  $\nabla b = \mathbf{n}$  on  $\partial\Omega$ , where  $\mathbf{n}$  is the outward normal vector field.

**Definition 2.2** *We define the extension vector  $\nabla_{\tau}u \in L^2(B)$  outside  $\partial\Omega$  as*

$$\nabla_{\tau}u := \nabla u - (\nabla u \cdot \nabla b) \nabla b.$$

◇

**Definition 2.3** For a fixed  $\varepsilon > 0$ , let us introduce the class of open sets:

$$O_\varepsilon = \{\Omega \text{ open set, } \Omega \text{ has the } \varepsilon\text{-cone property}\}.$$

◇

One can refer to [2, p.54] to see more details about this set. Moreover, we have the following theorem proved in [2, p.56].

**Theorem 10** An open set  $\Omega$  with a bounded boundary has the  $\varepsilon$ -cone property if and only if it has a Lipschitz boundary.

Let us now state a result in A. Henrot and M. Pierre book (see [2, Theorem 2.4.10, p.59]).

**Theorem 11** Let  $\Omega_n$  be a sequence of open sets in the class  $O_\varepsilon$ . Then there exists an open set  $\Omega \in O_\varepsilon$  and a subsequence  $\Omega_{n_k}$  that converges to  $\Omega$  in the sense of Hausdorff, in the sense of characteristic functions, and in the sense of compacts. Moreover,  $\Omega_{n_k}$  and  $\partial\Omega_{n_k}$  converge in the sense of Hausdorff respectively to  $\bar{\Omega}$  and  $\partial\Omega$ .

Let us introduce the following domains:

**Definition 2.4 (Sobolev domains).**

Given  $m > 1$  and  $p \geq 1$ , a subset  $\Omega$  of  $\mathbb{R}^d$  is said to be an  $(m, p)$ -Sobolev domain or simply a  $W^{m,p}$ -domain if  $\partial\Omega \neq \emptyset$  and there exists  $h > 0$  such that

$$b_\Omega \in W_{loc}^{m,p}(U_h(\partial\Omega)),$$

where  $U_h$  is a tubular neighborhood of  $\partial\Omega$  of radius  $h$ .

◇

We can deduce the following corollary:

**Corollary 2.1** If  $\Omega$  is a  $(2, \infty)$ -Sobolev open bounded domain, then it has the  $\varepsilon$ -cone property, with  $\varepsilon > 0$  is uniform on  $\partial\Omega$ .

## 2.3 Existence and uniqueness of the solution of (2.1)

In order to prove the existence and uniqueness of the solution of problem (2.1), we give the variational formulation of this problem:

For all  $v \in \mathcal{H}(\Omega)$ , we have

$$\begin{cases} \int_\Omega -\Delta u v = \int_\Omega f v, \\ \int_{\partial\Omega} -\Delta_\tau u v + \int_{\partial\Omega} \partial_{\mathbf{n}} u v = 0. \end{cases} \quad (2.3)$$

Hence, we get

$$\int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial\Omega} \nabla_{\tau} u \cdot \nabla_{\tau} v = \int_{\Omega} f v. \quad (2.4)$$

Actually, we aim to use the Lax-Milgram Theorem, by showing the continuity of  $A$  and  $B$  on  $\mathcal{H}(\Omega)$  (even  $\mathcal{H}_s(\Omega)$  is true), where

$$A(u, v) := \int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial\Omega} \nabla_{\tau} u \cdot \nabla_{\tau} v, \quad B(v) := \int_{\Omega} f v, \quad (2.5)$$

and that  $A$  is coercive on  $\mathcal{H}(\Omega)$ . Let us remark that the coercivity cannot be valid on  $\mathcal{H}_s(\Omega)$ , since  $A(1, 1) = 0$  and  $\|1\|_{\mathcal{H}_s(\Omega)} = \text{Volume}(\Omega) + \text{Perimeter}(\Omega)$ .

**Lemma 2.1** *A and B are continuous on  $\mathcal{H}(\Omega)$ .*

*Proof:* We have:

$$A(u, u) = \|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla_{\tau} u\|_{L^2(\Omega)}^2 \leq \|u\|_{\mathcal{H}(\Omega)}^2$$

and

$$|B(u)| \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{\mathcal{H}(\Omega)}$$

thanks to the Cauchy-Schwartz inequality. Therefore,  $A$  and  $B$  are continuous on  $\mathcal{H}(\Omega)$ .  $\square$

**Lemma 2.2** *A is coercive on  $\mathcal{H}(\Omega)$ .*

*Proof:* Let us divide our proof into five parts:

1. Set  $\lambda_{LB}$  to be the lowest non-zero eigenvalue of Laplacian-Beltrami operator, then  $\forall u \in H^1(\partial\Omega)$  with  $\int_{\partial\Omega} u = 0$ , we have

$$\|u\|_{L^2(\partial\Omega)} \leq \lambda_{LB} \|\nabla_{\tau} u\|_{L^2(\partial\Omega)} \quad (2.6)$$

since we have by Courant-Hilbert lemma

$$\lambda_{LB} = \inf_{u \in H^1(\partial\Omega), u \perp 1} \frac{\|\nabla_{\tau} u\|_{L^2(\partial\Omega)}}{\|u\|_{L^2(\partial\Omega)}}.$$

2. Since we have

$$\|u\|_{L^2(\partial\Omega)} \leq \lambda_{LB} \|\nabla_{\tau} u\|_{L^2(\partial\Omega)},$$

we get by interpolation between  $L^2(\partial\Omega)$  and  $H^{\frac{1}{2}}(\partial\Omega)$  that

$$\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C(\partial\Omega) \|\nabla_{\tau} u\|_{L^2(\partial\Omega)} \leq C(\partial\Omega) \|u\|_{H^1(\partial\Omega)}. \quad (2.7)$$

3. Let us split  $u = v + w$  where  $v$  solves

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = u & \text{on } \partial\Omega \end{cases} \quad (2.8)$$

and  $w$  solves

$$\begin{cases} \Delta w = \Delta u & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.9)$$

Now, for  $v \in H^{\frac{3}{2}}(\Omega)$  we have

$$\|v\|_{L^2(\Omega)} \leq \|v\|_{H^1(\Omega)} = \text{constant} \|v\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C(\varepsilon) \|v\|_{H^{\frac{1}{2}}(\partial\Omega)} = C(\varepsilon) \|u\|_{H^{\frac{1}{2}}(\partial\Omega)}, \quad (2.10)$$

where  $C(\varepsilon)$  is a constant that depends on the equivalence of norms between  $H^{\frac{1}{2}}(\partial\Omega)$  (the trace of  $H^1(\Omega)$ ) and the intrinsic Sobolev norm  $H^{\frac{1}{2}}$  defined as a double integral. In fact it depends also on the uniform Lipschitz constant or  $\varepsilon$ -cone property (for more details, see Maz'ya's book in Sobolev spaces [12]).

Thus, by (2.7) we get

$$\|v\|_{L^2(\Omega)} \leq C(\varepsilon) \|u\|_{H^1(\partial\Omega)}. \quad (2.11)$$

Concerning  $w$ , we have using integration by parts in  $\Omega$ :

$$-\int_{\Omega} w \Delta w = \int_{\Omega} |\nabla w|^2. \quad (2.12)$$

On the other hand, we have:

$$-\int_{\Omega} w \Delta w = -\int_{\Omega} w \Delta u = \int_{\Omega} \nabla u \cdot \nabla w \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)}. \quad (2.13)$$

Hence, combining between (2.12) and (2.13) we obtain

$$\|\nabla w\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)}. \quad (2.14)$$

But by Poincaré inequality we have:

$$\exists C_p = C_{\text{Poincaré}}, \text{ such that } \|w\|_{L^2(\Omega)} \leq C_p \|\nabla w\|_{L^2(\Omega)}. \quad (2.15)$$

Therefore, we get

$$\|w\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)}. \quad (2.16)$$

4. As  $u = v + w$ , we deduce that

$$\|u\|_{L^2(\Omega)}^2 \leq 2(\|v\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2) \leq 2(\|\nabla u\|_{L^2(\Omega)}^2 + C(\varepsilon)\|u\|_{H^1(\partial\Omega)}^2). \quad (2.17)$$

5. In this step, we will gather all the information above. So we have

$$\begin{aligned} \|u\|_{\mathcal{H}(\Omega)}^2 &= \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\partial\Omega)}^2 + \|\nabla_\tau u\|_{L^2(\partial\Omega)}^2 \\ &\leq 2 \left[ \|\nabla u\|_{L^2(\Omega)}^2 + C(\varepsilon) \left( \|u\|_{L^2(\partial\Omega)}^2 + \|\nabla_\tau u\|_{L^2(\partial\Omega)}^2 \right) \right] + \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\partial\Omega)}^2 + \|\nabla_\tau u\|_{L^2(\partial\Omega)}^2 \\ &\leq 3\|\nabla u\|_{L^2(\Omega)}^2 + 2C(\varepsilon) (\lambda_{LB}^2(\partial\Omega) + 1) \|\nabla_\tau u\|_{L^2(\partial\Omega)}^2 + (\lambda_{LB}^2(\partial\Omega) + 1) \|\nabla_\tau u\|_{L^2(\partial\Omega)}^2 \\ &= 3\|\nabla u\|_{L^2(\Omega)}^2 + (2C(\varepsilon) + 1) (\lambda_{LB}^2(\partial\Omega) + 1) \|\nabla_\tau u\|_{L^2(\partial\Omega)}^2 \\ &\leq \max [3, (2C(\varepsilon) + 1) (\lambda_{LB}^2(\partial\Omega) + 1)] \left( \|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla_\tau u\|_{L^2(\partial\Omega)}^2 \right) \end{aligned}$$

Therefore, there exists a constant  $\alpha := \max [3, (2C(\varepsilon) + 1) (\lambda_{LB}^2(\partial\Omega) + 1)]$  such that

$$\|u\|_{\mathcal{H}(\Omega)}^2 \leq \alpha \left( \|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla_\tau u\|_{L^2(\partial\Omega)}^2 \right). \quad (2.18)$$

If we set  $\beta := \frac{1}{\alpha}$

$$A(u, u) \geq \beta \|u\|_{\mathcal{H}(\Omega)}^2. \quad (2.19)$$

Hence  $A$  is coercive on  $\mathcal{H}(\Omega)$ .

□

**Remark 2.3** *The constant  $\alpha$  of coercivity of  $A$  depends only on:*

1.  $C(\varepsilon)$  the constant in the equivalence of norms between  $H^{\frac{1}{2}}(\partial\Omega)$  (seen as the trace of  $H^1(\Omega)$ ) and the intrinsic Sobolev norm  $H^{\frac{1}{2}}$  defined as a double integral. It is uniform on the class of  $\varepsilon$ -cone domains (see Maz'ya's book in Sobolev spaces [12]).

2. The first non-trivial eigenvalue of the Laplace-Beltrami operator on  $\partial\Omega$ .

In dimension two, it depends only on the perimeter and decreases with respect to it; if there exists  $\delta > 0$  such that  $\text{Perimeter}(\partial\Omega) \geq \delta > 0$ , then  $\lambda_{LB}(\partial\Omega)$  is bounded by a constant depending only on  $\delta$  and not  $\Omega$ .

In dimension three, bounds on curvature of  $\partial\Omega$  could be needed (to investigate).

Now, in order to end the proof of the existence and uniqueness of the solution, let us give this crucial proposition:

**Proposition 2.1 (Existence and uniqueness of the solution of (2.1)).** *Let  $\Omega$  be a Lipschitz domain. Then for any  $f \in \mathcal{H}'(\Omega)$  (the dual of  $\mathcal{H}(\Omega)$ ), problem (2.1) has a unique solution  $u$  in  $\mathcal{H}(\Omega)$ .*

**Proof:** In order to prove this proposition, we just have to apply Lax-Milgram Theorem. □

From the work above, we can show this proposition that will be useful in the next subsection:

**Proposition 2.2 (A priori bounds for  $u_\Omega$ ).** *For all  $f \in \mathcal{H}'(\Omega)$ , the solution  $u_\Omega$  of (2.1) on  $\Omega$  satisfies*

$$\|u_\Omega\|_{\mathcal{H}(\Omega)} \leq \alpha(\Omega)\|f\|_{\mathcal{H}'(\Omega)}. \quad (2.20)$$

Moreover, if  $f \in L^2(\Omega)$ , then we have

$$\|u_\Omega\|_{\mathcal{H}(\Omega)} \leq \alpha(\Omega)\|f\|_{L^2(\Omega)}. \quad (2.21)$$

**Proof:** Let  $f \in \mathcal{H}'(\Omega)$ . We use  $u_\Omega \in \mathcal{H}(\Omega)$  as a test function so that we get from (2.18)

$$\|u_\Omega\|_{\mathcal{H}(\Omega)}^2 \leq \alpha(\Omega)A(u_\Omega, u_\Omega) = \alpha(\Omega) b(u_\Omega) \leq \alpha(\Omega)\|f\|_{\mathcal{H}'(\Omega)}\|u_\Omega\|_{\mathcal{H}(\Omega)},$$

hence we get

$$\|u_\Omega\|_{\mathcal{H}(\Omega)} \leq \alpha(\Omega)\|f\|_{\mathcal{H}'(\Omega)}.$$

If  $f \in L^2(\Omega)$ , then we have

$$\|u_\Omega\|_{\mathcal{H}(\Omega)} \leq \alpha(\Omega)\|f\|_{L^2(\Omega)}.$$

□

## 2.4 Extension operator and uniform bound estimations

The difficulty with the set  $\mathcal{H}$  is that the restriction to  $\Omega$  of a function in  $\mathcal{H}(B)$  has no reason to belong to  $\mathcal{H}(\Omega)$ . Therefore, we consider the space  $H^{\frac{3}{2}}$  to overcome this difficulty. That's the reason we deal with the right hand side  $f$  to be in  $L^2(\Omega)$  and not just in  $\mathcal{H}'(\Omega)$ .

In this section, our goal is to show the existence of an extension operator from  $H^{\frac{3}{2}}(\Omega)$  into  $H^{\frac{3}{2}}(B)$ . Furthermore, we would like to study some uniform bound estimation in  $H^{\frac{3}{2}}(\Omega)$  and  $H^{\frac{3}{2}}(B)$  in order to show the existence of a weakly convergent sequence of solutions  $(u_n)_n$ , with respect to the convergence of a sequence of domains  $(\Omega_n)_n$  to the limit set  $\Omega$  in some sense, to a limit  $u$  in the extended set  $H^{\frac{3}{2}}(B)$ , and in particular we get the strong convergence of  $u_n$  to  $u$  in  $H^1(B)$ , thanks to the compact injection of  $H^{\frac{3}{2}}(B)$  into  $H^1(B)$ .

In order to set ourselves in the uniform case with respect to  $\Omega$ , we give this definition:

**Definition 2.5** *We set  $\mathcal{O}$  to be a class of domains  $\Omega \in O_\varepsilon$  such that there exists  $M > 0$  and  $\alpha(\Omega) \leq M$  (the constant  $M$  is uniform with respect to  $\Omega \in O_\varepsilon$ ). In other words, the domains in  $\mathcal{O}$  are considered to be uniformly Lipschitz with a uniform constant  $M$ .*

◇

In order to have an extension operator, we pass to  $H^{\frac{3}{2}}(\Omega)$  thanks to Necas property (see [4]):

If  $u \in H^1(\Omega)$  and  $u \in H^1(\partial\Omega)$  with  $\Delta u = -f \in L^2(\Omega) \subset (H^{\frac{1}{2}}(\Omega))'$ , then  $u \in H^{\frac{3}{2}}(\Omega)$  (with a uniform constant in the Lipschitz norm of  $\Omega$ ).

**Proposition 2.3** *On the set  $\mathcal{O}$ , there exist  $M > 0$  such that  $\|u\|_{H^{\frac{3}{2}}(\Omega)} \leq M$ .*

**Proof:** We use the uniform bound in  $\mathcal{H}(\Omega)$  and Necas property.

□

**Proposition 2.4** *Let  $\Omega \in \mathcal{O}$ . Then there exists an extension operator from  $H^{\frac{3}{2}}(\Omega)$  into  $H^{\frac{3}{2}}(B)$ .*

The following corollary investigates a uniform bound on the extended function  $\tilde{u}$  in  $H^{\frac{3}{2}}(B)$ :

**Corollary 2.2** *Let  $\Omega \in \mathcal{O}$ . Then there exists  $M > 0$  and  $\tilde{u}$  an extension in  $H^{\frac{3}{2}}(B)$  of  $u$  the solution of (2.1) in  $\Omega$ , such that*

$$\|\tilde{u}\|_{H^{\frac{3}{2}}(B)} \leq M. \quad (2.22)$$

As a consequence of Corollary 2.2, we have

**Corollary 2.3** *Assume, for a given topology  $\mathcal{T}$  on  $\mathcal{O}$ , that  $\Omega_n \xrightarrow{\mathcal{T}} \Omega$ , where  $\Omega_n$  and  $\Omega$  are included in  $B$  and belong to the set  $\mathcal{O}$ . Let  $u_n$  be the solution of (2.1) on  $\Omega_n$  and  $\tilde{u}_n$  its extension to  $B$ . Then  $\|\tilde{u}_n\|_{H^{\frac{3}{2}}(B)}$  is uniformly bounded. By consequence, there exists  $\tilde{u} \in H^{\frac{3}{2}}(B)$  such that*



$$\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{in } H^{\frac{3}{2}}(B). \quad (2.23)$$

In particular, we have the strong convergence in  $H^1(B)$ , in other words, we have

$$\tilde{u}_n \longrightarrow \tilde{u} \quad \text{in } L^2(B), \quad (2.24)$$

$$\nabla \tilde{u}_n \longrightarrow \nabla \tilde{u} \quad \text{in } L^2(B). \quad (2.25)$$

Set  $u$  to be the restriction of  $\tilde{u}$  to  $\Omega$ . One has  $u \in H^{\frac{3}{2}}(\Omega)$ . The question now is: does  $u$  solves (2.1) on  $\Omega$ ? To answer, we have to pass to the limit in the variational formulations.

**Remark 2.4** In the following, we will denote by  $u_n$  (respectively  $u$ ) the restriction to  $\Omega_n$  (respectively  $\Omega$ ) of  $\tilde{u}_n \in H^{\frac{3}{2}}(B)$  (respectively  $\tilde{u} \in H^{\frac{3}{2}}(B)$ ), where  $\Omega_n$  and  $\Omega$  are in  $\mathcal{O}$ .

## 2.5 Interpretation of the limit

In this section, we deal firstly with the volume case, so that we show that the limit  $u$  is a solution of the volume equation  $-\Delta u = f$ . But in order to prove that  $u$  satisfies the boundary conditions of (2.1), we give several lemmas, in addition to using Proposition 2.10. We will divide this section into many subsections.

### 2.5.1 The differential equation in the volume

**Lemma 2.3** Given a sequence of uniformly Lipschitz domains  $(\Omega_n)_n$  that converges to a Lipschitz domain  $\Omega$ , let  $(u_n)_n$  be a sequence of solutions of (2.1) defined in  $H^{\frac{3}{2}}(\Omega_n)$ , and let  $u$  be its limit in  $H^{\frac{3}{2}}(\Omega)$ . Then, for  $f \in L^2(\Omega)$ , we have on  $\Omega$ :

$$-\Delta u = f.$$

**Proof:** Let  $\varphi \in \mathcal{D}(\Omega)$ , we have:

$$\langle -\Delta u_n, \varphi \rangle_{\mathcal{D}'(\Omega_n) \times \mathcal{D}(\Omega)} = \langle f, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} \Rightarrow \langle \nabla u_n, \nabla \varphi \rangle_{\mathcal{D}'(\Omega_n) \times \mathcal{D}(\Omega)} = \langle f, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)}$$

$$\Rightarrow \langle \nabla u_n, \nabla \varphi \rangle_{L^2(B) \times L^2(\Omega)} = \langle f, \varphi \rangle_{L^2(B) \times L^2(\Omega)}$$

$$\Rightarrow \langle \nabla u, \nabla \varphi \rangle_{L^2(B) \times L^2(\Omega)} = \langle f, \varphi \rangle_{L^2(B) \times L^2(\Omega)} \quad \text{by (2.25)}$$

$$\Rightarrow \langle \nabla u, \nabla \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = \langle f, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} \Rightarrow \langle -\Delta u, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = \langle f, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)}.$$

□

### 2.5.2 Localization in $\Omega$ (the limit set)

As a consequence of Theorem 11, we can derive this lemma:

**Lemma 2.4** *Let  $(\Omega_n)_n$  be a sequence of uniformly Lipschitz open bounded sets that converges in the Hausdorff sense to a Lipschitz open bounded domain  $\Omega$ . Let  $f \in L^2(B)$  and  $v \in L^2(B)$ . Then we have*

$$\int_B \mathcal{X}_{\Omega_n} f v \longrightarrow \int_B \mathcal{X}_{\Omega} f v$$

where  $\mathcal{X}_{\Omega_n}$  and  $\mathcal{X}_{\Omega}$  are the classical characteristic functions defined in  $L^1(B)$ .

**Proof:** From Theorem 10, we deduce that  $\Omega_n \in \mathcal{O}_\varepsilon$ . Thus, we can apply Theorem 11 that provides the existence of a subsequence that converges in three different senses, in particular we have the characteristic convergence of  $\Omega_n$  to  $\Omega$  (i.e.  $|\Omega_n| \rightarrow |\Omega|$  or  $\Omega_n \rightarrow \Omega$  in  $L^1(B)$ ). Let  $f \in L^2(B)$  and  $v \in L^2(B)$ . Then we have:

$$\mathcal{X}_{\Omega_n} \xrightarrow{L^1(B)} \mathcal{X}_{\Omega} \quad (2.26)$$

Since  $\mathcal{X}_{\Omega_n}$  and the limit  $\mathcal{X}_{\Omega}$  take values in  $\{0, 1\}$ , we get

$$\mathcal{X}_{\Omega_n} \xrightarrow{L^\infty(B)} \mathcal{X}_{\Omega}. \quad (2.27)$$

Hence we have

$$\mathcal{X}_{\Omega_n} f v \xrightarrow{L^1(B)} \mathcal{X}_{\Omega} f v,$$

that is to say

$$\int_B \mathcal{X}_{\Omega_n} f v \longrightarrow \int_B \mathcal{X}_{\Omega} f v.$$

□

**Lemma 2.5** *Let  $(\Omega_n)_n$  be a sequence of uniformly Lipschitz open bounded sets that converges in the Hausdorff sense to a Lipschitz open bounded set  $\Omega$ . Let  $(u_n)_{n \geq 1}$  be a sequence of solutions of (2.1) defined in  $H^{\frac{3}{2}}(\Omega_n)$ , and let  $u$  be its limit in  $H^{\frac{3}{2}}(\Omega)$ . Then, for all  $v$  in  $W^{1,\infty}(B)$ , we have:*

$$\int_B \mathcal{X}_{\Omega_n} \nabla u_n \cdot \nabla v \longrightarrow \int_B \mathcal{X}_{\Omega} \nabla u \cdot \nabla v$$

where  $\mathcal{X}_{\Omega_n}$  and  $\mathcal{X}_{\Omega}$  are the classical characteristic functions defined in  $L^1(B)$ .

**Proof:** As  $\nabla u_n \rightarrow \nabla u$  and  $\mathcal{X}_{\Omega_n} \rightarrow \mathcal{X}_\Omega$  in  $L^2(B)$ , we get  $\nabla u_n \mathcal{X}_{\Omega_n} \rightarrow \nabla u \mathcal{X}_\Omega$  in  $L^1(B)$ . Since  $\nabla v \in L^\infty(B)$ , we deduce that  $\mathcal{X}_{\Omega_n} \nabla u_n \cdot \nabla v \rightarrow \mathcal{X}_\Omega \nabla u \cdot \nabla v$  in  $L^1(B)$ . Hence we obtain

$$\int_B \mathcal{X}_{\Omega_n} \nabla u_n \cdot \nabla v \rightarrow \int_B \mathcal{X}_\Omega \nabla u \cdot \nabla v.$$

□

### 2.5.3 Study of the surface term

**Lemma 2.6** *Let  $(\Omega_n)_n$  be a sequence of uniformly Lipschitz open bounded sets that converges in the Hausdorff sense to a Lipschitz open bounded domain  $\Omega$ . Let  $(u_n)_{n \geq 1}$  be a sequence of solutions of (2.1) defined in  $H^{\frac{3}{2}}(\Omega_n)$ , and let  $u$  be its limit in  $H^{\frac{3}{2}}(\Omega)$ . Then, for all  $v$  in  $W^{1,\infty}(B)$ , we have:*

$$\int_{\partial\Omega_n} \partial_{\mathbf{n}} u_n v \rightarrow \int_{\partial\Omega} \partial_{\mathbf{n}} u v.$$

**Proof:** We have:

$$\begin{aligned} \int_{\partial\Omega_n} \partial_{\mathbf{n}} u_n v &= \int_{\Omega_n} \Delta u_n v + \int_{\Omega_n} \nabla u_n \cdot \nabla v = - \int_{\Omega_n} f v + \int_{\Omega_n} \nabla u_n \cdot \nabla v \\ &= - \int_B \mathcal{X}_{\Omega_n} f v + \int_B \mathcal{X}_{\Omega_n} \nabla u_n \cdot \nabla v \\ &\rightarrow - \int_B \mathcal{X}_\Omega f v + \int_B \mathcal{X}_\Omega \nabla u \cdot \nabla v \quad \text{by Lemmas 2.4 and 2.5} \\ &= - \int_\Omega f v + \int_\Omega \nabla u \cdot \nabla v = \int_\Omega \Delta u v + \int_\Omega \nabla u \cdot \nabla v \quad \text{by Lemma 2.3} \\ &= \int_{\partial\Omega} \partial_{\mathbf{n}} u v. \end{aligned}$$

□

To end this subsection, we prove the following lemma which will be crucial in the proof of Theorem 2:

**Lemma 2.7** *Let  $(\Omega_n)_n$  be a sequence of uniformly Lipschitz open bounded sets that converges in the Hausdorff sense to a Lipschitz open bounded set  $\Omega$ . Let  $(u_n)_{n \geq 1}$  be a sequence of solutions of (2.1) defined in  $H^{\frac{3}{2}}(\Omega_n)$ , and let  $u$  be its limit in  $H^{\frac{3}{2}}(\Omega)$ . Then, for any  $v \in W^{1,\infty}(B)$ , we have:*

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v \rightarrow - \int_{\partial\Omega} \partial_{\mathbf{n}} u v.$$

**Proof:** The proof can be easily done. From the boundary conditions in (2.3), we have:

$$\int_{\partial\Omega_n} -\Delta_\tau u_n v + \int_{\partial\Omega_n} \partial_{\mathbf{n}} u_n v = 0.$$

Using integration by parts, we obtain

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v = - \int_{\partial\Omega_n} \partial_{\mathbf{n}} u_n v.$$

Applying the limit to the right-hand side as  $n$  tends to infinity, we get using Lemma 2.6

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v \longrightarrow - \int_{\partial\Omega} \partial_{\mathbf{n}} u v.$$

□

In order to complete the path concerning the surface terms, we still have to show that for all  $v$  in  $W^{2,\infty}(B)$ , we have

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v \longrightarrow \int_{\partial\Omega} \nabla_\tau u \cdot \nabla_\tau v$$

which will be proved in Proposition 2.10 (see section 2.7). In fact, to prove this proposition, we need to study locally the attitude of the convergence of the signed distance function  $b_n := b_{\Omega_n}$  to  $b := b_\Omega$  in  $W^{2,p}$ , with  $p \geq 1$ , in a uniform tubular neighborhood  $U_h$  of the boundaries for some  $h > 0$ , and actually we need the  $W^{2,\infty}$  convergence of  $(2, \infty)$ -Sobolev domains (here the  $(2, \infty)$ -Sobolev regularity has to be injected into the domains  $\Omega_n$  and  $\Omega$ , which is much smoother than the Lipschitz one).

## 2.6 Study of the convergence of the signed distance and projection functions in $W^{1,p}$

In this section, we are going to study the behavior of the convergence in  $W^{1,p}(B)$ , with  $1 \leq p \leq \infty$ , of the sequence of the oriented distance functions  $(b_n)_{n \in \mathbf{N}}$  or  $(b_{\Omega_n})_{n \in \mathbf{N}}$  with respect to the Hausdorff convergence of less regular domains (uniformly Lipschitz and positive reach regularities). After that, assuming that  $b_n$  converges toward  $b$  in  $W^{2,\infty}$ , we show the existence of a uniform tubular neighborhood  $U_h$  of the boundaries for some  $h > 0$ , for  $(2, \infty)$ -Sobolev domains. Also, we study the convergence of the differential of the projection functions  $p_n$  on  $\partial\Omega_n$  in  $L^p(U_h)$ ,  $1 \leq p \leq \infty$ , with the assumption of  $W^{2,\infty}$  convergence of  $(2, \infty)$ -Sobolev domains.

### 2.6.1 $L^p$ convergence of the signed distance functions for uniformly Lipschitz domains, for $1 \leq p \leq \infty$

**Proposition 2.5** *Let  $(\Omega_n)_n$  be a sequence of uniformly Lipschitz open bounded sets, that converges to a uniformly Lipschitz open bounded set  $\Omega$ , in the Hausdorff distance sense, and let  $B$  be a large ball containing  $\Omega_n$  and  $\Omega$ . Then we have:*

$$b_{\Omega_n} \xrightarrow{L^\infty(B)} b_\Omega.$$

In particular, for  $p \geq 1$  we have

$$b_{\Omega_n} \xrightarrow{L^p(B)} b_\Omega.$$

**Proof:** If  $p = \infty$ , the the proof is done for the complementary of  $\Omega_n$  and  $\Omega$  in M. Dambrine and B. Puig article (see [6], p.5, Proposition 2.7, Step 2.)

Moreover, if  $p < \infty$ , we have:

$$\|b_n - b\|_{L^p(B)}^p = \int_B |b_n(x) - b(x)|^p \leq \int_B (\sup_{x \in B} |b_n(x) - b(x)|)^p = \|b_n - b\|_{L^\infty(B)}^p \text{vol}(B) \xrightarrow{n} 0.$$

□

### 2.6.2 $W^{1,\infty}$ convergence of the signed distance functions for domains of positive reach

**Proposition 2.6** *Let  $(\Omega_n)_n$  be a sequence of open bounded sets of uniform positive reach (i.e. each skeleton is distant from its corresponding boundary) that converges to an open bounded domain  $\Omega$  of positive reach, in the Hausdorff distance sense. Let  $h_n$  be a positive real number and let  $U_h$  be a uniform tubular neighborhood, so that  $0 < h < h_n, \forall n \in \mathbb{N}$ , such that  $U_h \subset \cap_n U_{h_n}$ , where*

$$U_{h_n} := \{x \in B; |b_{\partial\Omega_n}(x)| \leq h_n\}.$$

Then we have

$$b_{\Omega_n} \xrightarrow{W^{1,\infty}(U_h)} b_\Omega.$$

**Proof:** First of all, by [7, Theorem 3.3, (vii), p.347], we compute the gradient of  $b_\Omega$ :

$$\nabla b_\Omega(x) = \left\{ \begin{array}{ll} \nabla d_\Omega(x) = \frac{x - p(x)}{|x - p(x)|}, & x \in \mathbb{R}^d \setminus \bar{\Omega}, \\ \nabla d_{\mathbb{C}\Omega}(x) = -\frac{x - p(x)}{|x - p(x)|}, & x \in \Omega \end{array} \right\}, \quad |\nabla b_\Omega(x)| = 1, \quad (2.28)$$

where  $d_\Omega$  is the distance function to  $\Omega$ , knowing that  $b_\Omega(x) = d_\Omega(x) - d_{\mathbb{C}\Omega}(x)$ , and  $p$  is the orthogonal projection of  $x \in U_h$  on  $\partial\Omega$ . Note that  $p(x)$  is unique in  $U_h$ . We will denote by  $p_n$  the projection on  $\partial\Omega_n$  instead of  $p_{\Omega_n}$ .

Then we have for all  $x \in U_h \setminus \cup_{n \in \mathbf{N}}(\partial\Omega_n \cup \partial\Omega)$ :

$$|\nabla b_{\Omega_n}(x) - \nabla b_\Omega(x)| = \left| \frac{x - p_n(x)}{|x - p_n(x)|} - \frac{x - p(x)}{|x - p(x)|} \right|.$$

Since the function  $\varphi(y) = \frac{y}{|y|}$  is Lipschitz continuous on  $\mathbb{R}^n \setminus \{0\}$ , we obtain

$$|\nabla b_{\Omega_n}(x) - \nabla b_\Omega(x)| \leq |x - p_n(x) - x + p(x)| = |p_n(x) - p(x)|. \quad (2.29)$$

Also, from Proposition 2.5, we get for all  $x \in U_h$ :

$$d(x, p_n(x)) = b_{\Omega_n}(x) \longrightarrow b_\Omega(x) = d(x, p(x)). \quad (2.30)$$

Now, since  $U_h$  is compact,  $\exists y \in U_h$ , and there exists a subsequence  $(p_k(x))_{k \in \mathbf{N}}$  of the sequence  $(p_n(x))_{n \in \mathbf{N}}$ , still denoted by  $(p_n(x))_{n \in \mathbf{N}}$ , such that  $p_n(x) \longrightarrow y$  in  $U_h$ . Then it remains to show that  $y \in \bar{\Omega}$  and  $y = p(x)$ .

**Step 1.** Setting  $y_n := p_n(x)$ , we have

$$d(y_n, \bar{\Omega}) \leq d(y_n, \bar{\Omega}_n) + d_H(\bar{\Omega}_n, \bar{\Omega}) = 0 + d_H(\bar{\Omega}_n, \bar{\Omega}) = d(\bar{\Omega}_n, \bar{\Omega}).$$

Therefore, passing to the limit, taking into consideration the Hausdorff convergence and from the continuity of the distance function  $d(\cdot, \bar{\Omega})$ , we get:

$$d(y, \bar{\Omega}) = \lim_n d(y_n, \bar{\Omega}) \leq \lim_n d(\bar{\Omega}_n, \bar{\Omega}) = 0.$$

So  $y \in \bar{\Omega}$ .

**Step 2.** If  $p_n(x) \longrightarrow y$ , then  $d(x, p_n(x)) \longrightarrow d(x, y)$  by continuity of the distance function. Thus by (2.30), we get  $d(x, y) = d(x, p(x))$  by the uniqueness of the limit. Since  $x \in U_h$ ,  $y \mapsto d(x, y)$  has the unique minimizer  $p(x)$  on  $\bar{\Omega}$ . Hence  $y = p(x)$ .

Therefore  $|p_n(x) - p(x)| \longrightarrow 0$  for all  $x \in U_h$ , this means that

$$\|p_n - p\|_{L^\infty(U_h)} \longrightarrow 0, \quad (2.31)$$

and by (2.29), we get

$$\|\nabla b_{\Omega_n} - \nabla b_{\Omega}\|_{L^\infty(U_h)} \longrightarrow 0. \quad (2.32)$$

Finally, by Proposition 2.5, we obtain the result.  $\square$

### 2.6.3 $W^{1,p}$ convergence of the signed distance functions for uniformly Lipschitz domains, $1 \leq p < \infty$

**Proposition 2.7** *Let  $(\Omega_n)_n$  be a sequence of uniformly Lipschitz open bounded sets in  $\mathbb{R}^d$ , that converges to a Lipschitz open bounded set  $\Omega$ , in the Hausdorff distance sense. Let  $B$  be a large ball containing all these domains. Then, we have for  $1 \leq p < \infty$*

$$b_{\Omega_n} \xrightarrow{W^{1,p}(B)} b_{\Omega}.$$

**Proof:** The proof here is much sharper than the precedent one. First of all, let us remark that since we don't have the positive reach condition, the points on the Skeletons (here the skeletons exist at least locally in a tabular neighborhood of  $\partial\Omega$ ) do not have a unique projection on the boundary  $p(x)$ . In other word, the function  $|p_n(x) - p(x)|$  is not defined on all  $B$ ; hence to overcome this problem, we look for a new procedure in the following:

Let  $x \in B \setminus \cup_{n \in \mathbb{N}}(\partial\Omega_n \cup \partial\Omega)$ . Set

$$\Theta_n := \arg \min_{y \in \partial\Omega_n} d(x, y) = \{y \in \partial\Omega_n \mid d(x, y) = d(x, \partial\Omega_n)\},$$

and

$$\Theta := \arg \min_{y \in \partial\Omega} d(x, y) = \{y \in \partial\Omega \mid d(x, y) = d(x, \partial\Omega)\},$$

and define the functions  $g_n$ ,  $n \in \mathbb{N}$ , on  $B$ , such that:

$$\forall x \in B, \quad g_n(x) := \inf_{y_n \in \Theta_n, y \in \Theta} d(y_n, y).$$

In fact, we have for  $x \in B \setminus \cup_{n \in \mathbb{N}}\partial\Omega_n \cup \partial\Omega$ ,

$$|\nabla b_n(x) - \nabla b(x)| \leq |p_n(x) - p(x)| = g_n(x).$$

Therefore, if we prove that  $g_n$  goes to zero pointwisely, and since  $|\nabla b_n(x) - \nabla b(x)| \leq |\nabla b_n(x)| + |\nabla b(x)| = 1 + 1 = 2$  which is integrable on  $B$ , then  $\nabla b_{\Omega_n} \xrightarrow{L^1(B)} \nabla b_{\Omega}$ , thanks to the dominated convergence theorem, and hence  $b_{\Omega_n} \xrightarrow{W^{1,1}(B)} b_{\Omega}$ . Now, by M. C. Delfour and

J.-P. Zolésio book (see [7 p.350, Theorem 4.1 (i)]), we have the convergence in  $W^{1,p}(B)$ , for all integers  $1 \leq p < \infty$ .

**Pointwise convergence:** From the Hausdorff converges of  $\Omega_n$  to  $\Omega$ , we have:

$$\forall \varepsilon > 0, \exists N_\varepsilon > 0, \text{ such that } \forall n \geq N_\varepsilon, d(\Omega_n, \Omega) \leq \varepsilon. \quad (2.33)$$

Then we get (see Figure 2.2):

1.  $B(p_n(x), \varepsilon) \cap \Omega_n \neq \emptyset$ , i.e.

$$\exists y_n \in \partial\Omega_n, \text{ such that } d(p_n(x), x) \leq d(y_n, x) \leq d(p(x), x) + \varepsilon \leq d(y, x) + \varepsilon.$$

2.  $B(p(x), \varepsilon) \cap \Omega_n \neq \emptyset$ , i.e.

$$\exists y \in \partial\Omega_n, \text{ such that } d(p(x), x) \leq d(y, x) \leq d(p_n(x), x) + \varepsilon \leq d(y_n, x) + \varepsilon.$$

Therefore, we have on one side:

$$|d(y_n, x) - d(y, x)| \leq \varepsilon. \quad (2.34)$$

On the other side, we get

$$|d(p_n(x), x) - d(p(x), x)| \leq \varepsilon,$$

that is equivalent to say

$$|d(\partial\Omega_n, x) - d(\partial\Omega, x)| \leq \varepsilon. \quad (2.35)$$

so, in order to finish the proof of the pointwise convergence, we have to prove that

$$\exists y_n \in \Theta_n, \exists y \in \Theta, \text{ such that } d(y_n, y) \longrightarrow 0. \quad (2.36)$$

To prove that, let  $(y_n)_{n \in \mathbb{N}}$ , with  $y_n \in \Theta_n$ , be a sequence in  $B$ . Since  $B$  is compact, there exists a subsequence  $(y_k)_k$  (still denoted by  $(y_n)_n$  since no confusion) of  $(y_n)_n$  in  $B$ , and there exist an element  $y$  in  $B$ , such that  $y_n \longrightarrow y$  in  $B$ . Now, it is sufficient to prove that  $y \in \partial\Omega$ .

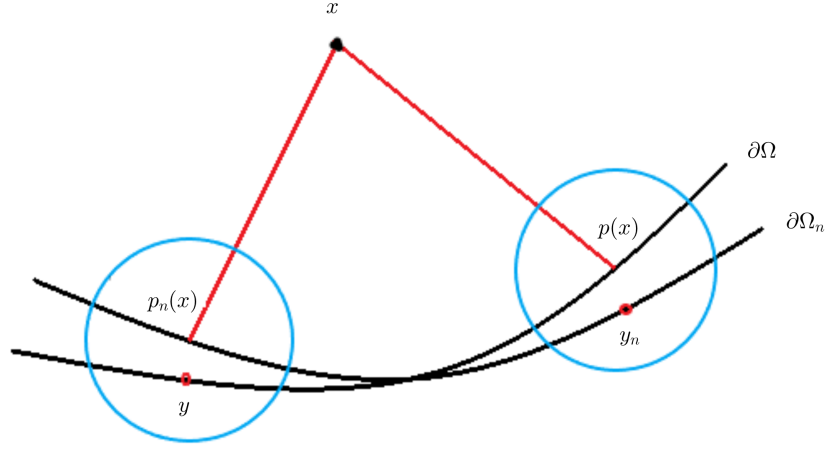
Indeed, we have:

$$\begin{aligned} d(x, y) &= \lim_n d(x, y_n) && \text{by estimation (2.34)} \\ &= \lim_n d(x, \partial\Omega_n) && \text{since } y_n \in \Theta_n \\ &= d(x, \partial\Omega) && \text{by estimation (2.35)} \end{aligned}$$

hence,  $y \in \partial\Omega$ , and the proof of (2.36) is finished.

□



Figure 2.2: The projection of point  $x$  on  $\partial\Omega_n$  and  $\partial\Omega$ .

#### 2.6.4 $L^p$ convergence of the differential of the projection function for $(2, \infty)$ -Sobolev domains, for $1 \leq p \leq \infty$

In the following, as mentioned in the introduction, we are going to deal with the second-order derivatives of the signed distance function. In fact, we need such an extra regularity on  $\Omega_n$  and  $\Omega$ , which is the  $(m, p)$ -Sobolev assumption, with  $m > 1$  and  $p \geq 1$ , in which via these regularity, the  $m$ -th order derivatives can be defined now locally. In particular, it is sufficient to choose  $m = 2$  to deal with the second-order derivatives. Let us remark that in order to avoid working with Radon measures as in [7], [9], and [10], we would like to make a truncation around the boundaries of the domains to get rid of the singular points of the signed distance functions in  $B$ . In other words, we separate the skeletons from the boundaries of  $\partial\Omega_n$  and  $\partial\Omega$ . Actually, we have to deal with the Hessian matrices  $D^2b_n$  and  $D^2b$  in a uniform tubular neighborhood  $U_h$ , which always exists (see Proposition 2.8). Moreover, If we set  $p_n$  and  $p$  to be the projection on  $\partial\Omega_n$  and  $\partial\Omega$ , respectively, we will study the convergence of  $Dp_n(x)$  to  $Dp(x)$  for all  $x \in U_h$ , which will be presented in Proposition 2.9. This proposition could be proved thanks to Gunther Leobacher and Alexander Steinicke paper (see [8, Theorem C]), where they investigated an explicit formula for  $Dp(x)$ ,  $x \in U_h$ .

To do so, let us introduce the notion of a tubular neighborhoods of  $\partial\Omega_n$  for all  $n \in \mathbb{N}$  (see Figure 2.3), then we show the existence of a uniform one, in the following proposition:

**Proposition 2.8** *Let  $(\Omega_n)_n$  be a sequence of  $(2, \infty)$ -Sobolev domains that converges toward a  $(2, \infty)$ -Sobolev domain  $\Omega$  in the  $W^{2, \infty}$  sense. For small  $h_n > 0$  and  $\forall n \in \mathbb{N}$ , we set*

$$U_{h_n} := \{x \in B; |b_{\partial\Omega_n}(x)| \leq h_n\}.$$

*Then there exists  $h > 0$ , such that  $h_n > h$ ,  $\forall n \in \mathbb{N}$ , and a uniform tubular neighborhood  $U_h$ , such that  $U_h \subset \cap_n U_{h_n}$ , so that we guarantee that the chosen domain  $U_h$  will not include any of*

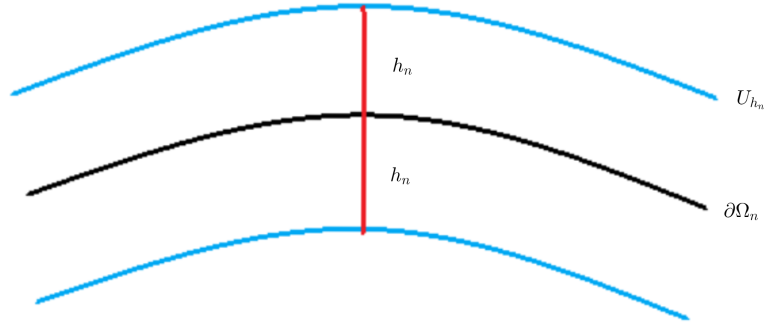


Figure 2.3: Tabular neighborhood of the boundary  $\partial\Omega_n$  locally.

the skeletons of  $\Omega_n$  and  $\Omega$ , for all  $n \in \mathbb{N}$ .

**Proof:** Suppose by contrary that there exists  $n_0 \in \mathbb{N}$  such that  $h_{n_0} < h$ , for all  $h$  small enough, then the skeleton of this set  $\Omega_{n_0}$  will approach from the boundary  $\partial\Omega_{n_0}$ , this means that the radius of curvature of this boundary at a point  $x_{n_0}$  is going to zero. This is equivalent to saying that: at least one of the principle curvature of this boundary at  $x_{n_0}$  is going to infinity. But we know that  $\Omega_{n_0}$  is converging to  $\Omega$  in the  $W^{2,\infty}$  sense, i.e. the eigenvalues of the Hessian matrix  $D^2b_{n_0}$  are converging toward those of  $D^2b$ . Hence, we get a contradiction. (see Figure 2.4).

□

**Remark 2.5** We remark that Proposition 2.8 is not true for the Hausdorff convergence of the domains (see a counter example in dimension two below).

**Counter example:** Let us give a counter example in dimension two for Hausdorff convergence of sets. Let  $\gamma : s \in \mathbb{R} \mapsto \mathbb{R}^2$  be a parametrization of the boundary  $\partial\Omega$ . We make a small perturbation  $\gamma_\varepsilon$ , for all  $\varepsilon > 0$  small enough, around a point  $x_0 \in \partial\Omega$  in the following way:

$$\gamma_\varepsilon(s) := \gamma(s) + \varepsilon \varphi \left( \frac{s - s_0}{\varepsilon} \right)$$

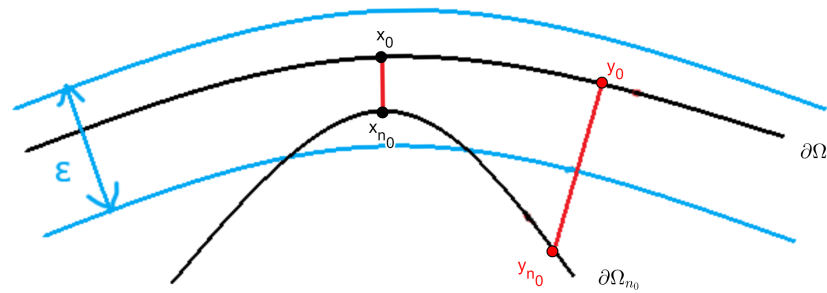


Figure 2.4: Curvatures of  $\partial\Omega_{n_0}$  and  $\partial\Omega$  at points  $x_{n_0}$  and  $x_0$  respectively.

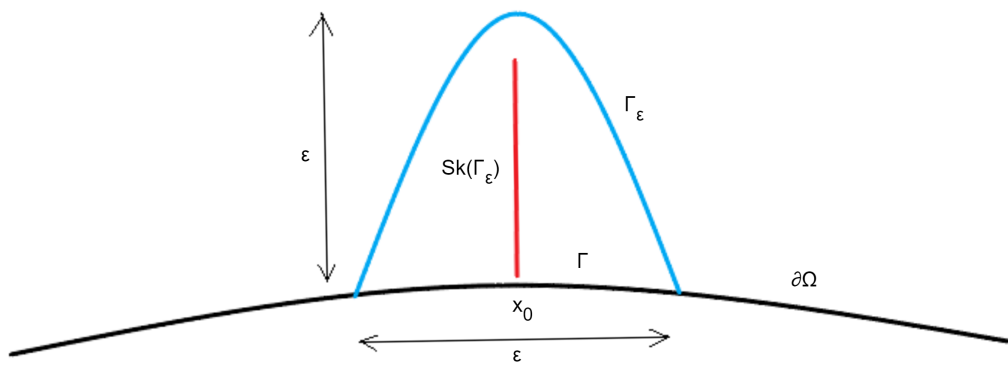


Figure 2.5: The perturbed domain

where  $\varphi$  is a mollifier and  $\gamma(s_0) = x_0$ . One can see that as  $\varepsilon$  goes to zero, the perturbed domain  $\Omega_\varepsilon$  converges to  $\Omega$  in the Hausdorff sense. But the distance  $h_\varepsilon$  between the skeleton  $Sk(\Gamma_\varepsilon)$  of the perturbed section  $\Gamma_\varepsilon$  of  $\partial\Omega_\varepsilon$  and the curve  $\Gamma_\varepsilon$  itself will tend to zero as  $\varepsilon \rightarrow 0$  (see Figure 2.5). This contradicts the fact that there exist  $h > 0$  such that  $h_\varepsilon > h$  for all  $\varepsilon > 0$ .

Now, let us make a uniform cut-off of size  $h$ .

**Definition 2.6** We define a  $C^\infty$  uniform cut-off function  $\psi$  as:

$$\psi := \ell \left( \frac{d}{h} \right)$$

where  $d$  is the distance function to the limit set  $\Omega$ , and  $\ell$  is a  $C^\infty$  function defined on  $\mathbb{R}^+$  as the following:

$$\ell(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{if } t \geq 1. \end{cases}$$

◇

Now, we prove the convergence of the sequence  $(Dp_n)_n$  to  $Dp$  in  $L^p(U_h)$  for  $p \geq 1$ , where  $Dp_n$  and  $Dp$  represent the differential of the projection functions  $p_n$  and  $p$ , respectively.

**Proposition 2.9** Let  $(\Omega_n)_n$  be a sequence of  $(2, \infty)$ -Sobolev open bounded domains in  $\mathbb{R}^d$ , that converges to a  $(2, \infty)$ -Sobolev open bounded domain  $\Omega$ , in the  $W^{2,\infty}$  sense. Then, for  $1 \leq p \leq \infty$ , we have

$$Dp_n \xrightarrow{L^p(U_h)} Dp,$$

where  $U_h$  is a uniform tubular neighborhood defined in subsection 2.6.4 and  $Dp_n \xrightarrow{L^p(U_h)} Dp$ ,  $1 \leq p \leq \infty$ , means that the  $ij$ -component  $D_n^{ij} \xrightarrow{L^p(U_h)} D^{ij}$  for all  $i, j = 1, \dots, d$ .

**Proof:** First of all, let us compute explicitly the matrix  $Dp(x)$  at the point  $x \in U_h$ . To do so, we refer to [8, Theorem C], where the differential of  $p$  at  $x \in U_h$  is computed as

$$Dp(x) = \left( Id_{T_{p(x)}(\partial\Omega)} - |x - p(x)| L_{p(x),v} \right)^{-1} P_{T_{p(x)}(\partial\Omega)}, \quad (2.37)$$

where  $v = |x - p(x)|^{-1}(x - p(x))$  and  $L_{p(x),v}$  is the shape operator in the direction  $v$  at  $p(x)$ , and  $P_{T_{p(x)}(\partial\Omega)}$  is the projection operator on  $T_{p(x)}(\partial\Omega)$ .

Let us explicit the shape and projection operator in local coordinates. We define the basis  $\mathcal{B}$  of  $\mathbb{R}^d$  around  $p(x)$  like this: the first  $d - 1$  vectors are the principle vectors of the tangent space  $T_{p(x)}(\partial\Omega)$  at  $p(x)$  and the last vector is the normal unit vector  $v$ . Then we have:

$$P_{T_{p(x)}(\partial\Omega)} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & 0 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix},$$

and the shape operator in the same basis can be written as

$$L_{p(x),v} = \begin{pmatrix} k_1(x) & 0 & \dots & \dots & 0 \\ 0 & k_2(x) & \dots & \dots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & k_{d-1}(x) & 0 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix},$$

where  $k_i(x)$ ,  $i = 1, \dots, d-1$ , are the principle curvatures of  $\partial\Omega$  at  $p(x)$ .

Hence, replacing  $|x - p(x)|$  by  $b(x)$ , we get

$$\begin{aligned} Dp(x) &= (Id_{T_{p(x)}(\partial\Omega)} - b(x)L_{p(x),v})^{-1}P_{T_{p(x)}(\partial\Omega)} \quad (2.38) \\ &= \begin{pmatrix} \frac{1}{1 - bk_1(x)} & 0 & \dots & \dots & 0 \\ 0 & \frac{1}{1 - bk_2(x)} & \dots & \dots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \frac{1}{1 - bk_{d-1}(x)} & 0 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}. \end{aligned}$$

Notice that the denominators  $1 - bk_i(x)$ ,  $i = 1, \dots, d$ , cannot be zero since we are working in a bounded volume  $U_h$ . Clearly, the matrix  $Dp(x)$  is diagonal with respect to the basis  $\mathcal{B}$ , so it is sufficient to study the diagonal terms. Therefore, if we denote by  $Dp^{ij}(x)$  the  $ij$ -th component of the matrix  $Dp(x)$ , we have for all  $i = j$ :

### 1) Convergence in $L^\infty(U_h)$ :

$$\|Dp_n^{ii} - Dp^{ii}\|_{L^\infty(U_h)} = \sup_{x \in U_h} |Dp_n^{ii}(x) - Dp^{ii}(x)| = \sup_{x \in U_h} \left| \frac{1}{1 - b_n k_{n,i}(x)} - \frac{1}{1 - bk_i(x)} \right|$$

$$\begin{aligned}
 &= \sup_{x \in U_h} \left| \frac{b_n k_{n,i}(x) - b k_i(x)}{(1 - b_n k_{n,i}(x))(1 - b k_i(x))} \right| = \sup_{x \in U_h} \left| \frac{b_n k_{n,i}(x) - b k_i(x) + b_n k_i(x) - b_n k_i(x)}{(1 - b_n k_{n,i}(x))(1 - b k_i(x))} \right| \\
 &\leq \sup_{x \in U_h} \left| \frac{(b_n - b)k_i(x)}{(1 - b_n k_{n,i}(x))(1 - b k_i(x))} \right| + \sup_{x \in U_h} \left| \frac{b_n(k_{n,i} - k_i)(x)}{(1 - b_n k_{n,i}(x))(1 - b k_i(x))} \right|
 \end{aligned}$$

In fact, since  $D^2 b_n$  tends to  $D^2 b$  in  $L^\infty(U_h)$ , we have the convergence of the eigenvalues of the Hessian matrix  $D^2 b_n$  toward those of  $D^2 b$  in  $L^\infty(U_h)$ , in other words, the principle curvatures  $k_{n,i}(x)$  for all  $i = 1, \dots, d$  converge uniformly toward  $k_i(x)$  (see Figure 2.4). Therefore, we have  $\|k_{n,i} - k_i\|_{L^\infty(U_h)} = \sup_{x \in U_h} |k_{n,i}(x) - k_i(x)| \xrightarrow{n} 0$ . In particular, the principle curvatures  $k_{n,i}$ ,  $i = 1, \dots, d$ , are uniformly bounded with respect to  $n \in \mathbb{N}$ . In addition, the oriented distance functions  $b_n$  go to  $b$  in  $L^\infty(U_h)$ , hence they are also uniformly bounded with respect to  $n \in \mathbb{N}$ . Therefore, we get:

$$\|Dp_n^{ii} - Dp^{ii}\|_{L^\infty(U_h)} \leq C_1 \|b_n - b\|_{L^\infty(U_h)} + C_2 \|k_{n,i} - k_i\|_{L^\infty(U_h)} \xrightarrow{n} 0$$

where  $C_1$  and  $C_2$  are two positive constants independent of  $n \in \mathbb{N}$ . Finally, we get the desired result:

$$Dp_n \xrightarrow{L^\infty(U_h)} Dp. \quad (2.39)$$

2) **Convergence in  $L^p(U_h)$ ,  $p < \infty$ :**

For all  $i = j$ , we have:

$$\begin{aligned}
 \|Dp_n^{ii} - Dp^{ii}\|_{L^p(U_h)}^p &= \int_{U_h} |Dp_n^{ii}(x) - Dp^{ii}(x)|^p \\
 &\leq \int_{U_h} (\sup_{x \in U_h} |Dp_n^{ii}(x) - Dp^{ii}(x)|)^p \\
 &= \|Dp_n^{ii} - Dp^{ii}\|_{L^\infty(U_h)}^p \text{vol}(U_h) \longrightarrow 0.
 \end{aligned}$$

Hence, we have it for all  $i, j = 1, \dots, d$ . Therefore, we get the result:

$$Dp_n \xrightarrow{L^p(U_h)} Dp. \quad (2.40)$$

□

## 2.7 The remaining convergence on the surface

In order to prove Theorem 2, we have to give the following crucial proposition:

**Proposition 2.10** *If  $u_n \in H^{\frac{3}{2}}(\Omega_n)$  and  $u \in H^{\frac{3}{2}}(\Omega)$  are defined as in Remark 2.4, then for all  $v \in W^{2,\infty}(B)$ , we have:*

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v \longrightarrow \int_{\partial\Omega} \nabla_\tau u \cdot \nabla_\tau v.$$

**Proof:** We begin by transferring our domains of the integrals to the large one  $B$  which is containing all of  $\Omega_n$  and  $\Omega$ , then we proceed by applying the truncation on the uniform domain  $U_h$ .

First of all, let us recall that the tangential vector  $\nabla_\tau u$  can be extended to the vector  $\nabla u - (\nabla u \cdot \nabla b) \nabla b$  in the uniform tubular neighborhood  $U_h$ . Now, without loss of generality, we choose the test function  $v \in W^{2,\infty}(B)$  so that the normal derivative vanishes on  $\partial\Omega$  (i.e. we get  $\nabla v = \nabla_\tau v$  in  $B$ ). Then, we have:

$$\int_{\partial\Omega} \nabla_\tau u \cdot \nabla_\tau v = \int_{\partial\Omega} [\nabla u - (\nabla u \cdot \nabla b) \nabla b] \cdot \nabla v = \int_{\partial\Omega} \nabla u \cdot \nabla v - \int_{\partial\Omega} (\nabla u \cdot \nabla b) \nabla b \cdot \nabla v$$

Now, as the normal vector  $\mathbf{n}$  is unitary and using the fact that  $\nabla b$  is the extension of  $\mathbf{n}$  in  $U_h$ , we obtain by applying the divergence theorem: for all  $v \in W^{2,\infty}(B)$ ,

$$\begin{aligned} \int_{\partial\Omega} \nabla u \cdot \nabla v &= \int_{\partial\Omega} \nabla u \cdot \nabla v (\mathbf{n} \cdot \mathbf{n}) = \int_{\partial\Omega} \psi (\nabla u \cdot \nabla v) (\mathbf{n} \cdot \mathbf{n}) \\ \int_{\partial\Omega} [\psi (\nabla u \cdot \nabla v) \mathbf{n}] \cdot \mathbf{n} &= \int_{\Omega} \operatorname{div} [\psi (\nabla u \cdot \nabla v) \mathbf{n}] = \int_{U_h} \operatorname{div} [\psi (\nabla u \cdot \nabla v) \nabla b] \end{aligned}$$

where the cut-off function  $\psi$  is introduced in Definition 2.6. Hence we get

$$\begin{aligned} \int_{\partial\Omega} \nabla u \cdot \nabla v &= \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b + \int_{U_h} \psi \nabla (\nabla u \cdot \nabla v) \cdot \nabla b + \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b \\ &= \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b + \int_{U_h} \psi (\nabla (\nabla u^T) \nabla v^T) \cdot \nabla b + \int_{U_h} \psi \nabla u^T \nabla (\nabla v^T) \nabla b + \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b \\ &= \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b + \int_{U_h} \psi \nabla v^T \nabla (\nabla u^T) \nabla b + \int_{U_h} \psi \nabla u^T \nabla (\nabla v^T) \nabla b + \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b \\ &= \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b + \int_{U_h} \psi \nabla v^T D^2 u \nabla b + \int_{U_h} \psi \nabla u^T D^2 v \nabla b + \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b. \end{aligned}$$

Also, we have:

$$\begin{aligned}
\int_{\partial\Omega} (\nabla u \cdot \nabla b) \nabla b \cdot \nabla v &= \int_{\partial\Omega} \psi (\nabla u \cdot \nabla b) \nabla v \cdot \mathbf{n} = \int_{\Omega} \operatorname{div} [\psi (\nabla u \cdot \nabla b) \nabla v] = \int_{U_h} \operatorname{div} [\psi (\nabla u \cdot \nabla b) \nabla v] \\
&= \int_{U_h} (\nabla u \cdot \nabla b) \nabla \psi \cdot \nabla v + \int_{U_h} \psi \nabla (\nabla u \cdot \nabla b) \cdot \nabla v + \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v \\
&= \int_{U_h} (\nabla u \cdot \nabla b) \nabla \psi \cdot \nabla v + \int_{U_h} \psi \nabla b^T \nabla (\nabla u^T) \nabla v + \int_{U_h} \psi \nabla u^T \nabla (\nabla b^T) \nabla v + \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v \\
&= \int_{U_h} (\nabla u \cdot \nabla b) \nabla \psi \cdot \nabla v + \int_{U_h} \psi \nabla b^T D^2 u \nabla v + \int_{U_h} \psi \nabla u^T D^2 b \nabla v + \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v.
\end{aligned}$$

Therefore, due to cancellation of some terms, we get:

$$\begin{aligned}
&\int_{\partial\Omega} \nabla_{\tau} u \cdot \nabla_{\tau} v = \\
&\int_{U_h} \psi \nabla u^T D^2 v \nabla b + \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b + \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b \\
&- \int_{U_h} \psi \nabla u^T D^2 b \nabla v - \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v - \int_{U_h} (\nabla u \cdot \nabla b) \nabla \psi \cdot \nabla v.
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
&\int_{\partial\Omega_n} \nabla_{\tau} u_n \nabla_{\tau} v - \int_{\partial\Omega} \nabla_{\tau} u \nabla_{\tau} v = \\
&\int_{U_h} \psi \nabla u_n^T D^2 v \nabla b_n - \int_{U_h} \psi \nabla u^T D^2 v \nabla b + \int_{U_h} \psi (\nabla u_n \cdot \nabla v) \Delta b_n - \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b \\
&- \int_{U_h} \psi \nabla u_n^T D^2 b_n \nabla v + \int_{U_h} \psi \nabla u^T D^2 b \nabla v - \int_{U_h} \psi (\nabla u_n \cdot \nabla b_n) \Delta v + \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v \\
&+ \int_{U_h} (\nabla u_n \cdot \nabla v) \nabla \psi \cdot \nabla b_n - \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b - \int_{U_h} (\nabla u_n \cdot \nabla b_n) \nabla \psi \cdot \nabla v + \int_{U_h} (\nabla u \cdot \nabla b) \nabla \psi \cdot \nabla v \\
&= (A_n - A) + (B_n - B) - (C_n - C) - (D_n - D) + (E_n - E) - (F_n - F).
\end{aligned}$$

Let us study the convergence of each of these six terms:



1) **Convergence of  $A_n - A$  to 0:**

We have:

$$\begin{aligned} & \int_{U_h} \psi \nabla u_n^T D^2 v \nabla b_n - \int_{U_h} \psi \nabla u^T D^2 v \nabla b \\ &= \int_{U_h} \psi \left[ \sum_{i=1}^d \sum_{j=1}^d \partial_i u_n \partial_{ij} v \partial_j b_n - \sum_{i=1}^d \sum_{j=1}^d \partial_i u \partial_{ij} v \partial_j b \right] \\ &= \sum_{i=1}^d \sum_{j=1}^d \int_{U_h} \psi [\partial_i u_n \partial_{ij} v \partial_j b_n - \partial_i u \partial_{ij} v \partial_j b]. \end{aligned}$$

As  $\partial_i u_n \xrightarrow{n} \partial_i u$  and  $\partial_j b_n \xrightarrow{n} \partial_j b$  in  $L^2(U_h)$ , we get  $\partial_i u_n \partial_j b_n \xrightarrow{n} \partial_i u \partial_j b$  in  $L^1(U_h)$ . Since  $\psi \partial_{ij} v \in L^\infty(U_h)$ , we obtain  $\psi \partial_i u_n \partial_{ij} v \partial_j b_n \xrightarrow{n} \psi \partial_i u \partial_{ij} v \partial_j b$  in  $L^1(U_h)$ , therefore we get

$$\int_{U_h} \psi \nabla u_n^T D^2 v \nabla b_n - \int_{U_h} \psi \nabla u^T D^2 v \nabla b = \sum_{i=1}^d \sum_{j=1}^d \int_{U_h} \psi [\partial_i u_n \partial_{ij} v \partial_j b_n - \partial_i u \partial_{ij} v \partial_j b] \xrightarrow{n} 0.$$

Hence, we get the aimed convergence:

$$A_n - A \xrightarrow{n} 0. \quad (2.41)$$

2) **Convergence of  $B_n - B$  to 0:**

In this point, we have:

$$\int_{U_h} \psi (\nabla u_n \cdot \nabla v) \Delta b_n - \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b = \int_{U_h} \psi [(\nabla u_n \cdot \nabla v) \Delta b_n - (\nabla u \cdot \nabla v) \Delta b].$$

As  $\nabla u_n \xrightarrow{n} \nabla u$  in  $L^2(U_h)$  (hence in  $L^1(U_h)$ ) and  $\psi \nabla v \in L^\infty(U_h)$ , we get  $\psi \nabla u_n \cdot \nabla v \xrightarrow{n} \psi \nabla u \cdot \nabla v$  in  $L^1(U_h)$ . Since  $\Delta b_n \xrightarrow{n} \Delta b$  in  $L^\infty(U_h)$ , we obtain  $\psi [(\Delta b_n \nabla u_n - \Delta b \nabla u) \cdot \nabla v] \xrightarrow{n} 0$  in  $L^1(U_h)$ , therefore we have

$$\int_{U_h} \psi (\nabla u_n \cdot \nabla v) \Delta b_n - \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b = \int_{U_h} \psi [(\Delta b_n \nabla u_n - \Delta b \nabla u) \cdot \nabla v] \xrightarrow{n} 0.$$

Thus, we get the desired convergence:

$$B_n - B \xrightarrow{n} 0. \quad (2.42)$$

3) **Convergence of  $C_n - C$  to 0:**

Applying the same procedure as before, we have:

$$\begin{aligned} & \int_{U_h} \psi \partial_i u_n \partial_{ij} b_n \partial_j v - \int_{U_h} \psi \partial_i u \partial_{ij} b \partial_j v \\ &= \int_{U_h} \psi \left[ \sum_{i=1}^d \sum_{j=1}^d \partial_i u_n \partial_{ij} b_n \partial_j v - \sum_{i=1}^d \sum_{j=1}^d \partial_i u \partial_{ij} b \partial_j v \right] \\ &= \sum_{i=1}^d \sum_{j=1}^d \int_{U_h} \psi [\partial_i u_n \partial_{ij} b_n \partial_j v - \partial_i u \partial_{ij} b \partial_j v]. \end{aligned}$$

As  $\partial_i u_n \xrightarrow{n} \partial_i u$  and  $\partial_{ij} b_n \xrightarrow{n} \partial_{ij} b$  in  $L^2(U_h)$ , we get  $\partial_i u_n \partial_{ij} b_n \xrightarrow{n} \partial_i u \partial_{ij} b$  in  $L^1(U_h)$ . Since  $\psi \partial_j v \in L^\infty(U_h)$ , we obtain  $\psi \partial_i u_n \partial_{ij} b_n \partial_j v \xrightarrow{n} \psi \partial_i u \partial_{ij} b \partial_j v$  in  $L^1(U_h)$ , thus we get

$$\int_{U_h} \psi \partial_i u_n \partial_{ij} b_n \partial_j v - \int_{U_h} \psi \partial_i u \partial_{ij} b \partial_j v = \sum_{i=1}^d \sum_{j=1}^d \int_{U_h} \psi [\partial_i u_n \partial_{ij} b_n - \partial_i u \partial_{ij} b] \partial_j v \xrightarrow{n} 0.$$

Therefore, we reach our target:

$$C_n - C \xrightarrow{n} 0. \quad (2.43)$$

4) **Convergence of  $D_n - D$  to 0:**

We have:

$$\int_{U_h} \psi (\nabla u_n \cdot \nabla b_n) \Delta v - \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v = \int_{U_h} \psi [\nabla u_n \cdot \nabla b_n - \nabla u \cdot \nabla b] \Delta v.$$

As  $\nabla u_n \xrightarrow{n} \nabla u$  and  $\nabla b_n \xrightarrow{n} \nabla b$  in  $L^2(U_h)$ , we get  $\nabla u_n \cdot \nabla b_n \xrightarrow{n} \nabla u \cdot \nabla b$  in  $L^1(U_h)$ . Since  $\psi \Delta v$  in  $L^\infty(U_h)$ , we obtain  $\psi [\nabla b_n \cdot \nabla u_n - \nabla b \cdot \nabla u] \Delta v \xrightarrow{n} 0$  in  $L^1(U_h)$ , therefore we have

$$\int_{U_h} \psi (\nabla u_n \cdot \nabla b_n) \Delta v - \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v = \int_{U_h} \psi [\nabla u_n \cdot \nabla b_n - \nabla u \cdot \nabla b] \Delta v \xrightarrow{n} 0.$$

Thus, we get:

$$D_n - D \xrightarrow{n} 0. \quad (2.44)$$

5) **Convergence of  $E_n - E$  to 0:**

In this point, we have:

$$\int_{U_h} (\nabla u_n \cdot \nabla v) \nabla \psi \cdot \nabla b_n - \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b = \int_{U_h} \nabla \psi \cdot [(\nabla u_n \cdot \nabla v) \nabla b_n - (\nabla u \cdot \nabla v) \nabla b].$$

As  $\nabla u_n \xrightarrow{n} \nabla u$  in  $L^2(U_h)$  (hence in  $L^1(U_h)$ ) and  $\nabla v \in L^\infty(U_h)$ , we get  $\nabla u_n \cdot \nabla v \xrightarrow{n} \nabla u \cdot \nabla v$  in  $L^1(U_h)$ . Since  $\nabla b_n \xrightarrow{n} \nabla b$  in  $L^\infty(U_h)$ , we get  $\nabla \psi \cdot \nabla b_n \xrightarrow{n} \nabla \psi \cdot \nabla b$  in  $L^\infty(U_h)$ . Hence, we obtain  $(\nabla u_n \cdot \nabla v) \nabla \psi \cdot \nabla b_n - (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b \xrightarrow{n} 0$  in  $L^1(U_h)$ . Therefore we get

$$\int_{U_h} (\nabla u_n \cdot \nabla v) \nabla \psi \cdot \nabla b_n - \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b \xrightarrow{n} 0.$$

Hence, we get the aimed convergence:

$$E_n - E \xrightarrow{n} 0. \quad (2.45)$$

6) **Convergence of  $F_n - F$  to 0:**

We have:

$$\int_{U_h} (\nabla u_n \cdot \nabla b_n) \nabla \psi \cdot \nabla v - \int_{U_h} (\nabla u \cdot \nabla b) \nabla \psi \cdot \nabla v = \int_{U_h} \nabla \psi \cdot \nabla v [\nabla u_n \cdot \nabla b_n - \nabla u \cdot \nabla b].$$

As  $\nabla u_n \xrightarrow{n} \nabla u$  in  $L^2(U_h)$  (hence in  $L^1(U_h)$ ) and  $\nabla b_n \xrightarrow{n} \nabla b$  in  $L^\infty(U_h)$ , we get  $\nabla u_n \cdot \nabla b_n \xrightarrow{n} \nabla u \cdot \nabla b$  in  $L^1(U_h)$ . Since  $\nabla \psi \cdot \nabla v \in L^\infty(U_h)$ , we obtain

$$\nabla \psi \cdot \nabla v [\nabla u_n \cdot \nabla b_n - \nabla u \cdot \nabla b] \xrightarrow{n} 0 \text{ in } L^1(U_h).$$

Therefore, we deduce that

$$\int_{U_h} \nabla \psi \cdot \nabla v [\nabla u_n \cdot \nabla b_n - \nabla u \cdot \nabla b] \xrightarrow{n} 0.$$

Hence, we get:

$$F_n - F \xrightarrow{n} 0. \quad (2.46)$$

□

## 2.8 The proofs

### 2.8.1 Proof of Theorem 1

Consider the sequence of vector fields  $(h_n)_{n \in \mathbb{N}}$ , with  $h_n \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\forall n \in \mathbb{N}$ . We define the sequence of diffeomorphisms  $(\Phi_n)_n$ , by  $\Phi_n : \mathbb{R}^d \mapsto \mathbb{R}^d$  with  $\Phi_n := Id + h_n$ , so that we have  $\Omega_n = \Phi_n(\Omega) = (Id + h_n)(\Omega)$ . The proof is divided into 3 steps:

#### Step 1. Transforming the solutions onto the fixed domain $\Omega$ :

First of all, let us give the variational formula of (2.1) on the moving domains  $\Omega_n$ :

$$\forall \varphi \in \mathcal{H}(\Omega_n), \quad \int_{\Omega_n} \nabla u_n \cdot \nabla \varphi + \int_{\partial \Omega_n} \nabla_\tau u_n \cdot \nabla_\tau \varphi = \int_{\Omega_n} f \varphi. \quad (2.47)$$

Then, we transform the solutions of (2.1) into a fixed domain  $\Omega$ , using the inverse of the transformation function  $\Phi_n$ . Let us set the following notations:

$$v_n := u_n \circ \Phi_n, \quad A_n := (D\Phi_n^{-1})^T D\Phi_n^{-1} J_n, \quad B_n := (D\Phi_n^{-1})^T D\Phi_n^{-1} J_n^\sigma,$$

where  $J_n$  and  $J_n^\sigma$  are the volume and the surface Jacobian, respectively. Then, transforming the integrals in equation (2.47) from  $\Omega_n$  into the fixed domain  $\Omega$ , we get:

$$\forall \varphi \in \mathcal{H}(\Omega), \quad \int_{\Omega} A_n \nabla v_n \cdot \nabla \varphi + \int_{\partial \Omega} B_n \nabla_\tau v_n \cdot \nabla_\tau \varphi = \int_{\Omega} J_n f \varphi. \quad (2.48)$$

#### Step 2. Uniform bound of $v_n$ in $\mathcal{H}(\Omega)$ :

By the uniform bound  $C$  shown in Proposition 2.2, we can deduce a uniform bound with respect to the parameter  $n \in \mathbb{N}$ , with the necessity of  $\Omega$  to be in  $\mathcal{O}$  defined in section 2.4, so that we have

$$\|v_n\|_{\mathcal{H}(\Omega)} \leq C.$$

#### Step 3. Showing the convergence of $v_n$ to $v$ (the solution on $\Omega$ ) in $\mathcal{H}(\Omega)$ :

Assume that it is not the case. Then there exists a subsequence  $(v_k)_{k \in \mathbb{N}}$  of the sequence  $(v_n)_{n \in \mathbb{N}}$ , and there exists  $\beta > 0$ , such that  $\|v_k - v\|_{\mathcal{H}(\Omega)} \geq \beta > 0$ . But we know that  $v_k$  is bounded in  $\mathcal{H}(\Omega)$ , hence there exists  $v_\infty \in \mathcal{H}(\Omega)$  and a subsequence  $(v_p)_{p \in \mathbb{N}}$  of the sequence  $(v_k)_{k \in \mathbb{N}}$ , such that  $v_p \rightharpoonup v_\infty$  weakly in  $\mathcal{H}(\Omega)$  as  $p \rightarrow \infty$ .

Now, writing (2.48) as

$$\forall \varphi \in \mathcal{H}(\Omega), \quad \int_{\Omega} A_p \nabla v_p \cdot \nabla \varphi + \int_{\partial \Omega} B_p \nabla_\tau v_p \cdot \nabla_\tau \varphi = \int_{\Omega} J_p f \varphi,$$

and passing to the limit when  $p \rightarrow \infty$ , we get:

$$\forall \varphi \in \mathcal{H}(\Omega), \quad \int_{\Omega} \nabla v_{\infty} \cdot \nabla \varphi + \int_{\partial\Omega} \nabla_{\tau} v_{\infty} \cdot \nabla_{\tau} \varphi = \int_{\Omega} f \varphi,$$

since  $A_p$  and  $B_p$  converge to  $Id$  and  $J_p$  goes to 1 when  $p \rightarrow \infty$ . Therefore,  $v_{\infty}$  is a solution of (2.1) in  $\mathcal{H}(\Omega)$ . By uniqueness of the solution and since  $v$  is the only accumulation point, it says that  $v_{\infty} = v$ . Thus  $\forall \beta > 0$ , we have  $\|v_p - v\|_{\mathcal{H}(\Omega)} \leq \beta$ , which leads to a contradiction.

### 2.8.2 Proof of Theorem 2

To start the proof, we have to gather Lemma 2.7 with Proposition 2.10, so that we get

$$- \int_{\partial\Omega} \partial_{\mathbf{n}} u \, v = \int_{\partial\Omega} \nabla_{\tau} u \cdot \nabla_{\tau} v,$$

by the uniqueness of the limit. Moreover, making integration by parts on the right-hand side, we obtain:

$$\forall v \in W^{2,\infty}(B), \quad \partial_{\mathbf{n}} u = \Delta_{\tau} u \quad \text{on } \partial\Omega.$$

Hence, we deduce that  $u$  is the unique solution of problem (2.1) in  $\Omega$ . Furthermore, as  $u$  is the only accumulation point of the sequence  $(u_n)_n$ , we deduce from Corollary 2.3 that the whole sequence converges strongly to  $u$  in  $H^1(B)$ .

## Chapter 3

# Transition to the eigenvalue problem

This chapter is an introduction for the remaining chapters where we study the eigenvalues problem for the Laplace operator with Wentzell boundary conditions defined on a connected open bounded subset  $\Omega$  of  $\mathbb{R}^d$  with  $C^2$  boundary,

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ -\Delta_\tau u + \partial_{\mathbf{n}} u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

where  $\lambda$  is an eigenvalue of the Laplace operator,  $\mathbf{n}$  is the outward unit normal vector on  $\partial\Omega$ ,  $\Delta_\tau := \operatorname{div}_\tau(\nabla_\tau)$  is the Laplace-Beltrami operator defined on  $\partial\Omega$ , and the corresponding eigenfunction  $u$  belongs to the variational space

$$\mathcal{H}_s(\Omega) := \{u \mid u \in H^1(\Omega), u|_{\partial\Omega} \in H^1(\partial\Omega)\}.$$

The spectrum of the Laplacian with volume Wentzell conditions is discrete, i.e the spectrum consists in an increasing sequence

$$\lambda_0(\Omega) = 0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \dots \rightarrow \infty \quad (3.2)$$

with corresponding real orthonormal eigenfunctions  $u_0, u_1, u_2, \dots$  in  $L^2(\Omega)$ . The first eigenvalue is zero with constants as corresponding eigenfunctions. As usual, we adopt the convention that each eigenvalue is repeated according to its multiplicity. Hence, the first eigenvalue of interest is  $\lambda_1$ . A variational characterization of the eigenvalues is available.

We define on  $\mathcal{H}_s(\Omega)$  the two bilinear forms

$$A(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \nabla_\tau u \cdot \nabla_\tau v \, d\sigma, \quad B(u, v) = \int_{\Omega} uv, \quad (3.3)$$

where  $\nabla_\tau$  is the tangential gradient. Now, the two bilinear forms are positive and the variational characterization for the  $k$ -th eigenvalue is

$$\lambda_k(\Omega) = \min \left\{ \frac{A(v, v)}{B(v, v)}, v \in \mathcal{H}_s(\Omega), \int_{\Omega} v u_i = 0, i = 0, \dots, k-1 \right\} \quad (3.4)$$

In particular, when  $k = 1$ , the minimum is taken over the functions orthogonal to the eigenfunctions associated to  $\lambda_0 = 0$ , i.e constant functions.

In order to illustrate the behaviour of the spectrum, we consider an explicit example: we study the eigenpairs of Wentzell-Laplacian operator in the case of a disk  $B_R$  of radius  $R$  centered at  $O$  in dimension two. We can state

**Proposition 1.1** *Let  $\Omega = B_R$  be a disk of radius  $R$  in dimension two; then its eigenvalues and eigenfunctions for the Wentzell-Laplacian operator are:*

$$\lambda_{0,k} = \frac{j_{0,k}^2}{R^2}, \quad u_{0,k}(r, \theta) = J_0\left(\frac{j_{0,k}}{R}r\right), \quad k \geq 1, \quad (3.5)$$

$$\lambda_{n,k} = \frac{j_{n,k}^2}{R^2} \text{ double eigenvalue}, \quad u_{n,k}(r, \theta) = \begin{cases} J_n\left(\frac{j_{n,k}}{R}r\right) \cos n\theta, \\ J_n\left(\frac{j_{n,k}}{R}r\right) \sin n\theta, \end{cases} \quad n, k \geq 1, \quad (3.6)$$

where  $j'_{n,k}$  is the  $k$ -th zero of the Bessel function  $J'_n$  (the derivative of  $J_n$ ).

**Proof:** Let us consider the disk  $B_R$  of radius  $R$  centered at  $O$ . In polar coordinates  $(r, \theta)$ , looking for an eigenfunction  $u$  of  $D$  of the kind  $u(r, \theta) = v(r)w(\theta)$  leads us to solve the ordinary differential equations:

$$w''(\theta) + kw(\theta) = 0, \quad w \text{ } 2\pi\text{-periodic}, \quad v''(r) + \frac{1}{r}v'(r) + \left(\lambda - \frac{k}{r^2}\right)v(r) = 0,$$

$$v'(0) = 0, \quad -w''(\theta)v(R) + v'(R)w(\theta) = 0.$$

The periodicity condition for the first one implies that  $k = n^2$  where  $n$  is an integer. Thus, the general solution of the first differential equation is

$$w(\theta) = a_n \cos n\theta + b_n \sin n\theta. \quad (3.7)$$

Then, replacing  $k$  by  $n^2$  in the second equation allows us to recognize the classical Bessel differential equation, for which its general solutions (bounded at 0 as our domain is a disk  $D = \{r < R\}$  and  $v'(0) = 0$ ) are the modified Bessel functions

$$v(r) = J_n(\sqrt{\lambda}r). \quad (3.8)$$

By the boundary conditions, we have  $v'(R) = -n^2v(R)$ . Therefore, we have

$$\sqrt{\lambda}J'_n(\sqrt{\lambda}R) + n^2J_n(\sqrt{\lambda}R) = 0. \quad (3.9)$$

Now, we use the following two relations about the Bessel functions:

$$J_n(\sqrt{\lambda}R) = \frac{\sqrt{\lambda}R}{2n} \left[ J_{n-1}(\sqrt{\lambda}R) + J_{n+1}(\sqrt{\lambda}R) \right], \quad \text{for } n \geq 1, \quad (3.10)$$

$$J'_n(\sqrt{\lambda}R) = \frac{1}{2} \left[ J_{n-1}(\sqrt{\lambda}R) - J_{n+1}(\sqrt{\lambda}R) \right], \quad (3.11)$$

We propagate by induction. For  $n = 0$ , By (3.9) we get  $J'_0(\sqrt{\lambda}R) = 0$ . Therefore, the eigenvalues  $\lambda_{0,k}$  belongs to the set  $\{\lambda \mid J'_0(\sqrt{\lambda}R) = 0\}$ , that is  $\lambda_{0,k} = \frac{j'^2_{0,k}}{R^2}$ ,  $k = 1, 2, \dots$ , where  $j'_{0,k}$  is the  $k$ -th zero of the Bessel function  $J'_0$ .

Let us assume that  $\lambda_{n,k} = \frac{j'^2_{n,k}}{R^2}$  is true for  $n$ , and we have to show that we have  $\lambda_{n+1,k} = \frac{j'^2_{n+1,k}}{R^2}$ . To do so, it is sufficient to prove that  $J'_{n+1}(\sqrt{\lambda}R) = 0$  for  $n + 1$ .

Now, since  $J'_n(\sqrt{\lambda}R) = 0$  for  $n$ , we obtain from (3.9) that  $J_n(\sqrt{\lambda}R) = 0$ . Hence, using the two equations (3.10) and (3.11), we have

$$\begin{pmatrix} \frac{\sqrt{\lambda}R}{2n} & \frac{\sqrt{\lambda}R}{2n} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} J_{n-1}(\sqrt{\lambda}R) \\ J_{n+1}(\sqrt{\lambda}R) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.12)$$

that is  $J_{n-1}(\sqrt{\lambda}R) = 0 = J_{n+1}(\sqrt{\lambda}R)$  for  $n$ . But we know that (3.9) is valid for  $n + 1$ , thus we have

$$\sqrt{\lambda}J'_{n+1}(\sqrt{\lambda}R) + (n+1)^2 J_{n+1}(\sqrt{\lambda}R) = 0, \quad (3.13)$$

and then we get  $J'_{n+1}(\sqrt{\lambda}R) = 0$  for  $n + 1$ .

Finally, we can deduce that the eigenvalues are  $\lambda_{n,k} = \frac{j'^2_{n,k}}{R^2}$ ,  $k = 1, 2, \dots$ , where  $j'_{n,k}$  is the  $k$ -th zero of the Bessel function  $J'_n$ . □





# Chapter 4

## Shape derivatives of the eigenvalues

In this chapter, we start by studying the analyticity of the eigenvalues  $\lambda$  and their corresponding eigenfunctions  $u$  in an open neighborhood of  $t = 0$ . Next, we derive, in the sense of Hadamard, the first and second-order shape derivatives of the eigenvalues at time  $t = 0$ .

### 4.1 Introduction and statement of the results

First of all, In order to study the regularity of the eigenpairs of the boundary problem (3.1), we introduce some fundamental notations (see subsection 4.2.1 for more details):

We consider the map  $t \mapsto \varphi_t = Id + tV$  where  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$  and  $t$  is small enough. We denote by

$$\Omega_t = \varphi_t(\Omega) = x + tV(x), \quad x \in \Omega.$$

After that, we study the transport of the Laplace, Laplace-Beltrami and the Dirichlet-to-Neumann operators from the moving domains onto a fixed one (at time zero). In fact, these transports are used in proving the regularity of the eigenpairs around zero of the Wentzell-Laplace problem (1.1) in Theorem 12 (see subsection 4.2.2), that says:

Let  $\Omega$  be an open smooth bounded domain of  $\mathbb{R}^d$ . Assume that  $\lambda$  is an eigenvalue of multiplicity  $m$  of the Wentzell-Laplace operator. We suppose that  $\varphi_t = Id + tV$  for some  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$  and denote  $\Omega_t = \varphi_t(\Omega)$ . Then there exists  $m$  real-valued continuous functions  $t \mapsto \lambda_i(t)$ ,  $i = 1, 2, \dots, m$  and  $m$  functions  $t \mapsto u_i^t \in H^{5/2}(\Omega)$  such that the following properties hold

1.  $\lambda_i(0) = \lambda$ ,  $i = 1, \dots, m$ ,
2. the functions  $t \mapsto \lambda_i(t)$  and  $t \mapsto u_i^t$ ,  $i = 1, 2, \dots, m$  are analytic in a neighborhood of  $t = 0$ .
3. The functions  $u_{i,t}$  defined by  $u_{i,t} \circ \varphi_t = u_i^t$  are normalized eigenfunctions associated to  $\lambda_i(t)$  on the moving domain  $\Omega_t$ . If one considers a compact subset  $K$  such that  $K \subset \Omega_t$  for all  $t$  small enough, then  $t \mapsto (u_{i,t})|_K$  is also an analytic function of  $t$  in a neighborhood of  $t = 0$ .

4. Let  $I \subset \mathbb{R}$  be an interval such that  $I$  contains only the eigenvalue  $\lambda$  of the Wentzell problem of multiplicity  $m$ . Then there exists a neighborhood of  $t = 0$  such that  $\lambda_i(t)$   $i = 1, \dots, m$  are the only eigenvalues of  $\Omega_t$  which belongs to  $I$ .

Actually, to prove Theorem 12, we follow the strategy of M. Dambrine, D. Kateb, and J. Lamboley in [16, Theorem 3.4].

Let us now begin to present our main theorems in this chapter. First, we derive in Theorem 3 the local variations of the eigenvalues and the eigenfunctions of our problem:

**Theorem 3 (Characteristic equation in  $u'$ ).** *With the notations of Theorem 12, if  $t \mapsto (\lambda(t), u_t)$  is one of the smooth eigenpair path  $(\lambda_i(t), u_{i,t})$  of  $\Omega_t$  for the Wentzell problem (1.1), then the shape derivative  $u' = (\partial_t u_t)|_{t=0}$  of the eigenfunction  $u_t$  satisfies*

$$-\Delta u' = \lambda' u + \lambda u' \quad \text{in } \Omega, \quad (4.1)$$

$$\begin{aligned} -\Delta_\tau u' + \partial_n u' &= \operatorname{div}_\tau(V_n \nabla_\tau u) + \Delta_\tau(V_n \partial_n u) \\ &\quad + \operatorname{div}_\tau(V_n (HId - 2D^2b) \nabla_\tau u) + \lambda V_n u \quad \text{on } \partial\Omega, \end{aligned} \quad (4.2)$$

where  $D^2b$  is the Weingarten operator defined on  $\partial\Omega$  and  $H$  is the sum of the principle curvatures of the boundary.

Let  $\lambda$  be a simple eigenvalue of the volume Wentzell-Laplace equation and let  $u$  be the corresponding normalized eigenfunction. We give in this subsection the explicit formula for the shape derivative of the volume eigenvalue of the Wentzell-Laplace operator.

On  $\Omega_t = (Id + tV)(\Omega)$  with  $t$  small, there is a unique eigenvalue  $\lambda(t)$  near  $\lambda$  which is an analytic function with respect to the parameter  $t$ . The associated eigenfunction  $u_t(x) = u(t, x)$  is the solution of the problem . The shape derivative denoted  $u'$  is the partial derivative  $\partial_t u(t, x)$  evaluated at  $t = 0$  and solves (1.1). Let us deduce the analytic expression of  $\lambda'(0)$ :

**Theorem 4 (First-order derivative of a simple eigenvalue).** *If  $(\lambda, u)$  is an eigenpair (with  $u$  normalized) for the volume Wentzell problem with the additional assumption that  $\lambda$  is simple then the application  $t \mapsto \lambda(t)$  is analytic and its derivative at  $t = 0$ , for all deformation  $V \in W^{3,\infty}(\mathbb{R}^d)$ , is*

$$\lambda'(0) = \int_{\partial\Omega} V_n (|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda |u|^2 + \nabla_\tau u^T (HId - 2D^2b) \nabla_\tau u) \quad (4.3)$$

where  $D^2b$  is the Weingarten operator defined on  $\partial\Omega$  and  $H$  is the sum of the principle curvatures of the boundary.

If we suppose that  $\lambda$  is an eigenvalue of multiplicity  $m$ . For smooth deformation  $t \mapsto \Omega_t$ , there will be  $m$  eigenvalues close to  $\lambda$  (counting their multiplicities) for small values of  $t$ . We know that such a multiple eigenvalue is no longer differentiable in the classical sense. We are

then led to compute the directional derivative of  $t \mapsto \lambda_i(t)$  at  $t = 0$  where  $\lambda_i(t)$ ,  $j = 1, \dots, m$  are given by Theorem 12 (see subsection 4.2.2).

**Theorem 5 (First-order derivative of a multiple eigenvalue).** *Let  $\lambda$  be a multiple eigenvalue of order  $m \geq 2$ . Then each  $t \mapsto \lambda_i(t)$  for  $i = 1, 2, \dots, m$  given by Theorem 12 has a derivative near 0, and the values of  $\lambda'_i(0)$ ,  $i = 1, 2, \dots, m$  are the eigenvalues of the matrix  $M(V_n) = (M_{j,k})_{1 \leq j,k \leq m}$  defined by*

$$M_{j,k} = \int_{\partial\Omega} V_n (\nabla_\tau u_j \cdot \nabla_\tau u_k - \partial_n u_j \partial_n u_k - \lambda u_j u_k + (HId - 2D^2b) \nabla_\tau u_j \cdot \nabla_\tau u_k) \quad (4.4)$$

where  $D^2b$  is the Weingarten operator defined on  $\partial\Omega$  and  $H$  is the sum of the principle curvatures of the boundary.

Now, we study some second-order shape derivative computations. Let us begin with deriving the characteristic equation in  $u'' = \left( \frac{d^2 u}{dt^2} \right)_{|t=0}$  in the direction of  $V \in W^{3,\infty}(\mathbb{R}^d)$ .

**Theorem 6 (Characteristic equation in  $u''$ ).** *With the notations of Theorem 12, if  $t \mapsto (\lambda(t), u_t)$  is one of the smooth eigenpair path  $(\lambda_i(t), u_{i,t})$  of  $\Omega_t$  for the Wentzell problem (1.1), then the second shape derivative  $u'' = (\partial_t^2 u_t)_{|t=0}$  of the eigenfunction  $u_t$  satisfies*

$$-\Delta u'' = \lambda'' u + 2\lambda' u' + \lambda u'' \quad \text{in } \Omega,$$

$$\begin{aligned} -\Delta_\tau u'' + \partial_n u'' &= \text{div}_\tau(A) + \Delta_\tau(B) + \nabla_\tau u' \cdot \nabla_\tau V_n + V_n \lambda' u + V_n \lambda u' + V_n \Delta_\tau u' \\ &\quad + V_n \lambda u + V_n \lambda' u + V_n \lambda u' + V_n \lambda \partial_n V_n u + V_n^2 \lambda u \partial_n u + V_n^2 H \lambda u \quad \text{on } \partial\Omega. \end{aligned}$$

Where

$$\begin{aligned} A &:= V_n [\nabla_\tau (\partial_n u') - 2D^2b \nabla_\tau u' + H \nabla_\tau u' + \nabla_\tau u' + \partial_n u \nabla_\tau V_n] \\ &\quad + V_n [\partial_n V_n \nabla_\tau u + \partial_n u \mathcal{A} \nabla_\tau V_n + \mathcal{A} \nabla_\tau u' + \partial_n u \mathcal{A} \nabla_\tau V_n + \mathcal{A}' \nabla_\tau u] \\ &\quad + V_n^2 [\nabla_\tau (\partial_n u) - 2D^2b \nabla_\tau u + H \nabla_\tau u + 2\mathcal{A} \nabla_\tau (\partial_n u) - 2D^2b \mathcal{A} \nabla_\tau u + \tilde{\mathcal{A}} \nabla_\tau u + H \mathcal{A} \nabla_\tau u] \\ &\quad + \nabla_\tau V_n \partial_n u' + V_n' \nabla_\tau u + V_n \mathcal{A} \nabla_\tau u + \partial_n V_n \mathcal{A} \nabla_\tau u, \end{aligned}$$

and

$$B := V_n' \partial_n u + V_n (\partial_n u' - \nabla_\tau V_n \cdot \nabla_\tau u + \partial_n V_n \partial_n u) - V_n^2 (\lambda u + \partial_n u + H \lambda u),$$

where  $D^2b$  is the Weingarten operator defined on  $\partial\Omega$ ,  $H$  is the sum of the principle curvatures of the boundary,  $\mathcal{A} := (HId - 2D^2b)$  and we denote by  $\tilde{\mathcal{A}} := (\tilde{a}_{i,j})_{1 \leq i,j \leq d}$  the matrix whose

elements are the normal derivatives of  $a_{i,j}$ , where  $a_{i,j}$ ,  $i, j = 1, \dots, d$ , are the components of the matrix  $\mathcal{A}$ .

We give in Theorem 7 an explicit formula for the second-order shape derivative of a simple eigenvalue  $\lambda$  at  $t = 0$ , for all  $V \in W^{3,\infty}(\mathbb{R}^d)$ , that is:

**Theorem 7 (Second-order derivative of a simple eigenvalue).** *Let  $(u_t, \lambda(t))$  be a smooth eigenpair path of (1.1) on  $\Omega_t$ . If  $\lambda(t)$  is a simple eigenvalue, then the second-order derivative of  $\lambda(t)$  at  $t = 0$  in direction  $V \in W^{3,\infty}(\mathbb{R}^d)$  is:*

$$\begin{aligned} \lambda''(0) = & \int_{\partial\Omega} (V'_n + V_n \partial_n V_n + V_n^2 H) [|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda |u|^2 + \mathcal{A} \nabla_\tau u \cdot \nabla_\tau u] \\ & + \int_{\partial\Omega} V_n [2(Id + \mathcal{A}) \nabla_\tau u \cdot \nabla_\tau u' + 2(2Id + \mathcal{A}) \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u] \\ & + \int_{\partial\Omega} V_n [-2\partial_n u \partial_n u' - \lambda' |u|^2 - 2\lambda u u' + \mathcal{A}' \nabla_\tau u \cdot \nabla_\tau u] \\ & + \int_{\partial\Omega} V_n^2 [2(Id + \mathcal{A}) \nabla_\tau (\partial_n u) \cdot \nabla_\tau u - (2D^2 b + 2D^2 b \mathcal{A} - \tilde{\mathcal{A}}) \nabla_\tau u \cdot \nabla_\tau u + 2(1 + H) |\partial_n u|^2] \end{aligned}$$

where  $D^2 b$  is the Weingarten operator defined on  $\partial\Omega$ ,  $H$  is the sum of the principle curvatures of the boundary,  $\mathcal{A} := (HId - 2D^2 b)$  and we denote by  $\tilde{\mathcal{A}} := (\tilde{a}_{i,j})_{1 \leq i, j \leq d}$  the matrix whose elements are the normal derivatives of  $a_{i,j}$ , where  $a_{i,j}$ ,  $i, j = 1, \dots, d$ , are the components of the matrix  $\mathcal{A}$ .

The organization of this chapter is summarized in Figure 4.1.

## 4.2 Prerequisites

### 4.2.1 Notations and preliminaries for shape deformation

We adopt the formalism of Hadamard's shape calculus and consider the map  $t \mapsto \varphi_t = Id + tV$  where  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$  and  $t$  is small enough. We denote by

$$\Omega_t = \varphi_t(\Omega) = \{x + tV(x), \quad x \in \Omega\}.$$

**Remark 4.1** *More generally the results and computations from this section are valid if  $t \mapsto \varphi_t$  satisfies:*

1.  $\varphi_0 = Id$ ,
2. for every  $t$  near 0,  $\varphi_t$  is a  $W^{3,\infty}$ -diffeomorphism from  $\Omega$  onto its image  $\Omega_t = \varphi_t(\Omega)$ .

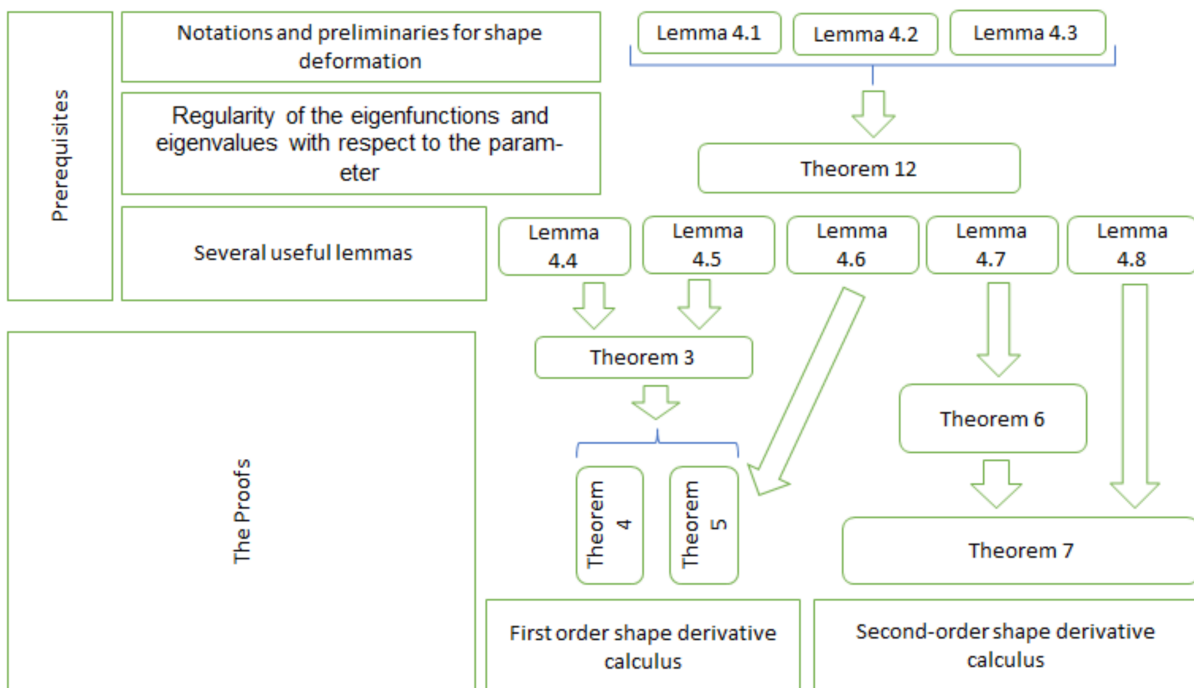


Figure 4.1: The map of Chapter four

3. The application  $t \mapsto \varphi_t$  is real-analytic near  $t = 0$ .

We need to introduce the surface jacobian  $w_t$  defined as

$$w_t(x) = \det(D\varphi_t(x)) \|(D\varphi_t(x)^T)^{-1} n(x)\|,$$

and the functions

$$A_t(x) = (D\varphi_t(x))^{-1}(D\varphi_t(x))^{-1}, \quad J_t(x) = \det(D\varphi_t(x))$$

$$\tilde{\mathcal{A}}_t(x) = J_t(x)A_t(x), \quad C_t(x) = w_t(x)A_t(x).$$

We have to study the transport of the considered eigenvalue problem on the deformed domain  $\Omega_t$ . To that end, we first rewrite the deformed equation on the fixed domain  $\Omega$  and its boundary  $\partial\Omega$ : we have to describe how are transported the Laplace-Beltrami and the Dirichlet-to-Neumann operators.

**Transport of the Dirichlet-to-Neumann map.** Let us consider the Dirichlet-to-Neumann operator defined on its natural space  $D_t : H^{1/2}(\partial\Omega_t) \mapsto H^{-1/2}(\partial\Omega_t)$ . It maps a function  $\phi_t$  in  $H^{1/2}(\partial\Omega_t)$  onto the normal derivative of its harmonic expansion in  $\Omega_t$ , that is to say  $D_t(\phi_t) = \partial_{n_t} u_t$ , where  $u_t$  solves the boundary value problem:

$$\begin{cases} -\Delta u_t = 0 & \text{in } \Omega_t, \\ u_t = \phi_t & \text{on } \partial\Omega_t. \end{cases} \quad (4.5)$$

To compute the quantity  $\mathcal{D}_t$  such that  $\mathcal{D}_t(\phi_t \circ \varphi_t) = [D_t(\phi_t)] \circ \varphi_t$ , we transport the boundary value problem (4.5) back on the domain  $\Omega$ .

To be more precise, we have the following result proved in [11].

**Lemma 4.1** *Given  $\psi \in H^{1/2}(\partial\Omega)$ , we denote  $v^t$  the solution of the boundary value problem*

$$\begin{cases} -\operatorname{div}(\tilde{\mathcal{A}}_t \nabla v^t) = 0 & \text{in } \Omega, \\ v^t = \psi & \text{on } \partial\Omega. \end{cases} \quad (4.6)$$

and then define  $\mathcal{D}_t\psi \in H^{-1/2}(\partial\Omega)$  as

$$\mathcal{D}_t\psi : f \in H^{1/2}(\partial\Omega) \mapsto \int_{\Omega} \tilde{\mathcal{A}}_t(x) \nabla v^t(x) \cdot \nabla E(f)(x) dx,$$

where  $E$  is a continuous extension operator from  $H^{1/2}(\partial\Omega)$  to  $H^1(\Omega)$ . Then the relation

$$(D_t\psi) \circ \varphi_t = \mathcal{D}_t[\psi \circ \varphi_t]$$

holds for all functions  $\psi \in H^{1/2}(\Omega_t)$ .

Setting  $u^t = u_t \circ \varphi_t$ , we check from the variational formulation, that the function  $u^t$  is the unique solution of the transported boundary value problem:

$$\begin{cases} -\operatorname{div}(\tilde{\mathcal{A}}_t \nabla u^t) &= 0 \text{ in } \Omega, \\ u^t &= \phi_t \circ \varphi_t \text{ on } \partial\Omega. \end{cases} \quad (4.7)$$

Hence, setting  $y = \varphi_t(x)$ ,  $x \in \Omega$  we get formally

$$D_t(\phi_t)(y) = \nabla u_t(y) \cdot n_t(y) = (D\varphi_t(x))^{-1} \nabla v_t(x) \cdot \frac{(D\varphi_t(x))^{-1} n(x)}{\|(D\varphi_t(x))^{-1} n(x)\|} = \frac{A_t(x) n(x) \cdot \nabla u^t(x)}{\|(D\varphi_t(x))^{-1} n(x)\|}.$$

Here again, we can give a sense to the co-normal derivative  $A_t n \cdot \nabla u^t$  thanks to the boundary value problem (4.7): this quantity is defined in a weak sense as the previous Dirichlet-to-Neumann operator  $D_t$ .

**Transport of the Laplace-Beltrami operator.** We recall now the expression of the transported Laplace-Beltrami operator, relying on the relation

$$\forall \phi \in H^2(\partial\Omega_t), \quad (\Delta_\tau \phi) \circ \varphi_t = \frac{1}{w_t(x)} \operatorname{div}_\tau [C_t(x) \nabla_\tau (\phi \circ \varphi_t)(x)] \text{ on } \partial\Omega.$$

Let us denote by  $\mathcal{L}_t$  the operator defined as

$$\mathcal{L}_t[\phi \circ \varphi_t](x) = \frac{1}{w_t(x)} \operatorname{div}_\tau \left\{ C_t(x) \nabla (\phi \circ \varphi_t)(x) - \frac{C_t(x) \nabla (\phi \circ \varphi_t)(x) \cdot n(x)}{A_t(x) n(x) \cdot n(x)} A_t(x) n(x) \right\}$$

for  $\phi \in H^{5/2}(\Omega_t)$ . In [11], they have showed the following lemma:

**Lemma 4.2** *The identity*

$$[\Delta_\tau \phi] \circ \varphi_t = \mathcal{L}_t[\phi \circ \varphi_t]$$

holds for all functions  $\phi$  belonging to  $H^{5/2}(\Omega_t)$ .

**Transport of the Laplace operator.** In the following, we transport back the Laplace operator  $\Delta$  defined on  $H^{5/2}(\Omega_t)$  into a stable one  $L_t$  that takes variables in  $H^{5/2}(\Omega)$ , where

$$\forall x \in \Omega, \quad L_t(x) := \frac{1}{J_t(x)} \operatorname{div} [\tilde{A}_t(x) \nabla \phi^t(x)]$$

and for  $\phi_t \in H^{5/2}(\Omega_t)$ ,  $\phi^t = \phi_t \circ \varphi_t$  is the back transport of  $\phi_t$  onto  $H^{5/2}(\Omega)$ .

Actually, we can prove the following lemma:

**Lemma 4.3** *The identity*

$$[\Delta \phi_t] \circ \varphi_t = L_t[\phi_t \circ \varphi_t]$$

holds for all functions  $\phi_t$  belonging to  $H^{5/2}(\Omega_t)$ .



**Proof:** Let  $\phi_t \in H^{5/2}(\Omega_t)$ . We fix a test function  $\psi_t \in H^{5/2}(\Omega_t)$ .

By the variational definition of the Laplace operator, if  $v_t$  is the trace of  $\phi_t$  on  $\partial\Omega_t$ , we get

$$\begin{aligned} \int_{\Omega_t} \Delta\phi_t(y)\psi_t(y) &= - \int_{\Omega_t} \nabla\phi_t(y) \cdot \nabla\psi_t(y) + \int_{\partial\Omega_t} \partial_n v_t(y)\psi_t(y) \\ &= - \int_{\Omega_t} \nabla\phi_t(y) \cdot \nabla\psi_t(y) + \int_{\partial\Omega_t} D_t\phi_t(y)\psi_t(y). \end{aligned}$$

By the change of variables  $y = \varphi_t(x)$ , we get integrals defined in the fixed domain  $\Omega$  and its boundary  $\partial\Omega$ . Therefore, thanks to the back transport of the Dirichlet-to-Neumann operator  $\mathcal{D}_t$ , we get

$$\begin{aligned} \int_{\Omega} ((\Delta\phi_t) \circ \varphi_t)(x)\psi^t(x)J_t(x) &= - \int_{\Omega} \nabla\phi^t(x) \cdot \nabla\psi^t(x)\tilde{A}_t(x) + \int_{\partial\Omega} \mathcal{D}_t(\phi_t \circ \varphi_t)(x)\psi^t(x)w_t(x) \\ &= - \int_{\Omega} \nabla\phi^t(x) \cdot \nabla\psi^t(x)\tilde{A}_t(x) + \int_{\partial\Omega} \frac{A_t(x) \mathbf{n}(x) \cdot \nabla\phi^t(x)}{\|(D\varphi_t(x))^{-1} \mathbf{n}(x)\|} \psi^t(x)w_t(x) \\ &= - \int_{\Omega} \nabla\phi^t(x) \cdot \nabla\psi^t(x)\tilde{A}_t(x) + \int_{\partial\Omega} \tilde{A}_t(x) \nabla\phi^t(x) \cdot \mathbf{n}(x)\psi^t(x). \end{aligned}$$

Thus, by the divergence theorem, we obtain

$$\begin{aligned} \int_{\Omega} ((\Delta\phi_t) \circ \varphi_t)(x)\psi^t(x)J_t(x) &= - \int_{\Omega} \nabla\phi^t(x) \cdot \nabla\psi^t(x)\tilde{A}_t(x) + \int_{\Omega} \operatorname{div} \left[ \tilde{A}_t(x) \nabla\phi^t(x)\psi^t(x) \right] \\ &= - \int_{\Omega} \nabla\phi^t(x) \cdot \nabla\psi^t(x)\tilde{A}_t(x) + \int_{\Omega} \operatorname{div} \left( \tilde{A}_t(x) \nabla\phi^t(x) \right) \psi^t(x) \\ &\quad + \int_{\Omega} \tilde{A}_t(x) \nabla\phi^t(x) \cdot \nabla\psi^t(x) \\ &= \int_{\Omega} \operatorname{div} \left( \tilde{A}_t(x) \nabla\phi^t(x) \right) \psi^t(x) = \int_{\Omega} L_t[\phi_t \circ \varphi_t](x)J_t(x)\psi^t(x). \end{aligned}$$

Hence, we get the result:

$$\forall \psi^t \in H^{5/2}(\Omega), \quad [\Delta\phi_t] \circ \varphi_t = L_t[\phi_t \circ \varphi_t].$$

□

#### 4.2.2 Regularity of the eigenfunctions and eigenvalues with respect to the parameter

We present now a standard theorem of the regularity of eigenvalues and associated eigenfunctions with respect to the parameter  $t$ . The section is a slight variation of a theorem due to

Ortega and Zuazua on the existence and regularity of eigenvalues and associated eigenfunctions in the case of the Stokes system [15]. The difficulty comes from the possible multiple eigenvalues. The main result is, for a fixed deformation field  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$ , the existence of smooth branches of eigenvalue. In other words, the eigenvalues are not regular when sorted in increasing order, but can be locally relabeled around the multiple point in order to remain smooth. The restriction is that this labeling depends on the deformation field  $V$  hence one cannot hope to prove Frechet-differentiability:

**Theorem 12** *Let  $\Omega$  be an open smooth bounded domain of  $\mathbb{R}^d$ . Assume that  $\lambda$  is an eigenvalue of multiplicity  $m$  of the Wentzell-Laplace operator. We suppose that  $\varphi_t = I_d + tV$  for some  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$  and denote  $\Omega_t = \varphi_t(\Omega)$ . Then there exists  $m$  real-valued continuous functions  $t \mapsto \lambda_i(t)$ ,  $i = 1, 2, \dots, m$  and  $m$  functions  $t \mapsto u_i^t \in H^{5/2}(\Omega)$  such that the following properties hold*

1.  $\lambda_i(0) = \lambda$ ,  $i = 1, \dots, m$ ,
2. the functions  $t \mapsto \lambda_i(t)$  and  $t \mapsto u_i^t$ ,  $i = 1, 2, \dots, m$  are analytic in a neighborhood of  $t = 0$ .
3. The functions  $u_{i,t}$  defined by  $u_{i,t} \circ \varphi_t = u_i^t$  are normalized eigenfunctions associated to  $\lambda_i(t)$  on the moving domain  $\Omega_t$ . If one considers a compact subset  $K$  such that  $K \subset \Omega_t$  for all  $t$  small enough, then  $t \mapsto (u_{i,t})|_K$  is also an analytic function of  $t$  in a neighborhood of  $t = 0$ .
4. Let  $I \subset \mathbb{R}$  be an interval such that  $\bar{I}$  contains only the eigenvalue  $\lambda$  of the Wentzell problem of multiplicity  $m$ . Then there exists a neighborhood of  $t = 0$  such that  $\lambda_i(t)$   $i = 1, \dots, m$  are the only eigenvalues of  $\Omega_t$  which belongs to  $I$ .

**Proof:** We follow the method used by M. Dambrine, D. Kateb and J. Lamboley (see [16, Theorem 3.4]), adapting ourselves to our volume Wentzell-Laplace problem, where we focus on the Laplace operator studied in the volume of  $\Omega_t$ . To start with, Let  $\lambda$  be an eigenvalue of multiplicity  $m$  and let  $u_1, \dots, u_m$  be the orthonormal eigenfunctions associated to  $\lambda$ . Let  $(\lambda(t), u_t)$  be an eigenpair satisfying

$$\begin{aligned} -\Delta u_t &= \lambda(t)u_t && \text{in } \Omega_t, \\ -\Delta_\tau u_t + \partial_n u_t &= 0 && \text{on } \partial\Omega_t. \end{aligned} \tag{4.8}$$

Setting  $u^t = u_t \circ \varphi_t$ , Lemma 4.3 (transport of the Laplace operator) shows that the system (4.8) above is equivalent to the following equation set in the volume

$$-L_t u^t = \lambda(t)J_t(x)u^t \quad \text{in } \Omega, \tag{4.9}$$

Consider the operator  $S(t)$  defined on  $H^{5/2}(\Omega)$  by

$$v \mapsto S(t)v = -L_t v \tag{4.10}$$

From the regularity assumption on  $\varphi_t$ , the operator  $A_t$  is analytic in a neighborhood of  $t = 0$ . Since  $J_t > 0$  for  $t$  small enough, we deduce that all the expressions involved in  $L_t$  are analytic in a neighborhood of  $t = 0$ . This enables us to conclude that  $S(t)$  is also analytic in a neighborhood of zero.

To show that the eigenvalues and the corresponding eigenfunctions are analytic in a neighborhood of zero, we apply the Lyapunov-Schmidt reduction in order to treat a problem on a finite dimensional space, namely the kernel of  $S(0) - \lambda I$ . To that end, we rewrite the problem (4.8) in the fixed domain  $\Omega$  as

$$S(t)(u^t) - \lambda(t)J_t(x)u^t = 0.$$

From the decomposition

$$(S(0) - \lambda)(u^t) = [(S(0) - S(t)) + [(\lambda(t) - \lambda)J_t + \lambda(J_t - 1)]]u^t,$$

$u^t$  is solution of the equation

$$(S(0) - \lambda)(u^t) = W(t, \lambda(t) - \lambda)u^t, \quad (4.11)$$

where we have set  $R(t) = S(0) - S(t) + \lambda(J_t - 1)$  and  $W(t, \alpha) = R(t) + \alpha J_t I$ . From the Lyapunov-Schmidt Theorem (see [15, Lemma 3-2, p. 999]), we obtain that  $S(0) - \lambda$  has a right inverse operator denoted by  $K$ . Hence the equation above implies that

$$u^t = KW(t, \lambda(t) - \lambda)u^t + \psi_t$$

where  $\psi_t \in \ker(S(0) - \lambda)$ , i.e  $\psi_t = \sum_{k=1}^m c_k(t)\phi_k$  where  $(\phi_k)$  is a basis of  $\ker(S(0) - \lambda)$ . Notice that  $I - KW(t, \lambda(t) - \lambda)$  is invertible on  $\ker(S(0) - \lambda I)$ , the inverse of his operator restricted to this kernel will be denoted by  $(I - KW(t, \lambda(t) - \lambda))^{-1}$  so that

$$u^t = (I - KW(t, \lambda(t) - \lambda))^{-1}\psi_t.$$

From (4.11),  $W(t, \lambda(t) - \lambda)u^t$  belongs to  $\text{Im}(S(0) - \lambda) = \ker^\perp(S(0) - \lambda)$  since  $S(0)$  is a Fredholm self-adjoint operator, and then

$$\sum_{k=1}^m c_k(t) \langle W(t, \lambda(t) - \lambda)(I - KW(t, \lambda(t) - \lambda))^{-1}\phi_k, \phi_i \rangle = 0, \quad i = 1, 2, \dots, m, \quad (4.12)$$

where  $\langle \cdot, \cdot \rangle$  denote the scalar product of  $L^2(\Omega)$ . This shows that a vector of coefficients  $C = (c_j)_{j=1, \dots, m} \neq 0$  is a solution if and only if the determinant of the  $m \times m$  matrix  $M(t, \lambda(t) - \lambda)$  with entries

$$M(t, \alpha)_{i,j} = \langle W(t, \alpha)(I - KW(t, \alpha))^{-1}\phi_j, \phi_i \rangle$$

satisfies

$$\det(M(t, \lambda(t) - \lambda)) = 0.$$

Hence  $\lambda(t)$  is an eigenvalue of our problem if and only if  $\det(M(t, \lambda(t) - \lambda)) = 0$ . Note that  $t \mapsto M(t, \lambda(t))$  is analytic around  $t = 0$ .

For small values of  $t$  the operator  $(I - KW(t, \alpha))^{-1}$  is well defined since  $I - KW(0, 0) = I$  and  $t \mapsto (I - KW(t, \alpha))^{-1}$  is analytic around  $t = 0$ . On the other hand, if  $\det M(t, \alpha) = 0$  then (4.12) has a nontrivial solution  $c_1(t), \dots, c_m(t)$  and this means that  $\lambda(t) = \lambda + \alpha$  is an eigenvalue of (4.8).

We focus now on  $\det(M(t, \alpha))$  for  $\alpha \in \mathbb{R}$ . From the fact that  $W(0, \alpha) = \alpha I$ , it comes that for sufficiently small values of  $\alpha$ , the operator  $I - KW(0, \alpha)$  is invertible on  $\ker(S(0) - \lambda I)$  and from the Von Neumann expansion we write

$$\langle W(0, \alpha)(I - KW(0, \alpha))^{-1}\phi_i, \phi_j \rangle = \alpha[\delta_{ij} + \sum_{k=1}^{\infty} \alpha^k \langle K^k \phi_i, \phi_j \rangle];$$

hence

$$\det(M(0, \alpha)) = \alpha^m + \sum_{i=1}^{\infty} \beta_i \alpha^{m+1} = \alpha^m (1 + \sum_{i=1}^{\infty} \beta_i \alpha^m).$$

Since  $\det(M(0, \alpha)) \neq 0$  is the restriction on  $t = 0$  of  $\det(M(t, \alpha))$ , we deduce from the Weierstrass preparation theorem that there is neighborhood of  $(0, 0)$  such that  $\det(M(t, \alpha))$  is uniquely representable as

$$\det(M(t, \alpha)) = P_m(t, \alpha)h(t, \alpha)$$

where

$$P_m(t, \alpha) = \alpha^m + \sum_{i=1}^{\infty} \beta_i \alpha^{m+1} = \alpha^m + \sum_{k=1}^m a_k(t) \alpha^{m-k}$$

and where

$$h(t, \alpha) \neq 0.$$

Furthermore, the coefficients  $a_k(t)$ ,  $k = 1, \dots, m$  are real and analytic in a neighborhood of  $t = 0$ . Then  $\det(M(t, \alpha)) = 0$  if and only if  $P_m(t, \alpha) = 0$ . If  $a_k(t)$ ,  $k = 1, \dots, m$  are the real roots of the polynomial, we take  $\lambda_1(t) = \lambda + \alpha_1(t)$  if  $\alpha_1(t)$  is not identically equal to zero.

We now have to find the  $(m - 1)$  other branches  $\lambda_i(t)$  and the corresponding eigenfunction  $u_{i,t}$  for  $i = 2, \dots, m$ . We use the idea of the deflation method by considering the operator

$$S_2(t) = S(t) - \lambda_1 P_1(t)$$

where  $P_1$  is the orthogonal projection on the subspace spanned by  $u_{1,t}$ . At  $t = 0$ , we obtain

$$S_2(0)u_j = S(0)u_j - \lambda \delta_{1j} u_j$$

in other terms  $S_2(0)u_j = \lambda u_j$ ,  $j = 2, \dots, m$  while  $S_2(0)u_1 = 0$ . This shows that  $\lambda$  is an eigenvalue of multiplicity  $m - 1$  of  $S_2(0)$  with eigenvalues  $u_2, \dots, u_m$ . One can show that these functions are the only linearly independent eigenfunctions associated to  $\lambda$ . Now we can apply the same recipe used before to the operator  $S_2$  instead of  $S$ . We then get a branch  $\lambda_2(t)$  such that  $t \mapsto \lambda_2(t)$  is analytic in a neighborhood of  $t = 0$ .

Iterating the process, we get at the end the  $m$ -branches  $\lambda_i(t)$ ,  $i = 1, \dots, m$  such that each branch is analytic in a neighborhood of  $t = 0$  and  $m$  corresponding eigenfunctions forming an orthonormal set of functions in  $H^{5/2}(\Omega_t)$ .

The proof of the last item follows the same lines as the proof of Ortega and Zuazua for the Stokes system, see [15]. □

### 4.2.3 Several useful lemmas

In this subsection, we will give several lemmas that are needed in our work. Beginning from the first order shape derivative calculations for a simple eigenvalues, passing through the multiple eigenvalue case with its corresponding orthonormal eigenfunctions in  $L^2(\Omega)$ , and reaching the second-order shape derivative computations for simple eigenvalues, where some additional derivation equations are derived.

**Lemma 4.4** *Let  $t \mapsto (\lambda(t), u_t)$  is a smooth eigenpair path on  $\Omega_t$  for the volume Wentzell problem (1.1), and  $u' = (\partial_t u_t)|_{t=0}$  is the derivative of the eigenfunction  $u$ , then for all  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$ , we have at  $t = 0$ :*

$$2 \int_{\Omega} u u' + \int_{\partial\Omega} |u|^2 V_n = 0, \quad (4.13)$$

$$\int_{\partial\Omega} \partial_n u u V_n = - \int_{\partial\Omega} \nabla_{\tau} V_n \nabla_{\tau} u u - \int_{\partial\Omega} |\nabla_{\tau} u|^2 V_n, \quad (4.14)$$

and

$$\int_{\partial\Omega} \nabla_{\tau} (V_n \partial_n u) \cdot \nabla_{\tau} u = \int_{\partial\Omega} \partial_n u \nabla_{\tau} V_n \cdot \nabla_{\tau} u + \int_{\partial\Omega} V_n \nabla_{\tau} (\partial_n u) \cdot \nabla_{\tau} u, \quad (4.15)$$

where  $V_n$  denotes the usual scalar product  $\langle V, n \rangle$ .

**Proof:** The first line comes from the fact that  $u_t$  is normalize in  $L^2(\Omega_t)$ , for all  $t > 0$ , i.e. we have

$$\int_{\Omega_t} |u_t|^2 = 1.$$

Making the shape derivative we obtain

$$\frac{d}{dt} \left( \int_{\Omega_t} |u_t|^2 \right)_{|t=0} = 0,$$

that is

$$2 \int_{\Omega} u u' + \int_{\partial\Omega} |u|^2 V_n = 0.$$

For the second line, it is sufficient to use the boundary condition  $\Delta_{\tau} u = \partial_n u$  in (1.1) and make integration by parts:

$$\int_{\partial\Omega} \partial_n u u V_n = \int_{\partial\Omega} \Delta_\tau u u V_n = - \int_{\partial\Omega} \nabla_\tau u \nabla_\tau (u V_n) = - \int_{\partial\Omega} \nabla_\tau V_n \nabla_\tau u u - \int_{\partial\Omega} |\nabla_\tau u|^2 V_n.$$

The last line is trivial. □

**Lemma 4.5** *Let  $t \mapsto (\lambda(t), u_t)$  is a smooth eigenpair path of  $\Omega_t$  for the volume Wentzell problem (1.1), and  $u' = (\partial_t u_t)|_{t=0}$  is the derivative of the eigenfunction  $u$ , then we have at  $t = 0$ :*

$$\lambda'(0) = \int_{\partial\Omega} \partial_n u u' - \partial_n u' u. \quad (4.16)$$

**Proof:** We have in  $\Omega$ :

$$-\Delta u' = \lambda u' + \lambda'(0)u.$$

Then, multiplying by  $u$ , and integrating over  $\Omega$  we get:

$$- \int_{\Omega} \Delta u' u = \lambda \int_{\Omega} u' u + \lambda' \int_{\Omega} |u|^2.$$

Using integration by parts twice and the normalization of  $u$ , we obtain

$$- \int_{\Omega} u' \Delta u + \int_{\partial\Omega} \partial_n u u' - \int_{\partial\Omega} \partial_n u' u = \lambda \int_{\Omega} u' u + \lambda'(0).$$

Since  $-\Delta u = \lambda u$  in  $\Omega$ , we obtain  $\lambda \int_{\Omega} u' u$  on both sides, which allow us to cancel this term. Therefore, we get the result

$$\lambda'(0) = \int_{\partial\Omega} \partial_n u u' - \partial_n u' u. \quad \square$$

**Lemma 4.6** *Let  $(u_i)_i$  and  $(u_j)_j$ ,  $i, j = 1, \dots, m$ , be two families of smooth eigenfunctions of (1.1) such that  $u_i$  and  $u_j$  are orthogonal in  $L^2(\Omega_t)$  for all  $i \neq j$  and  $\forall t$ , and denote by  $u'_i$  and  $u'_j$  the derivative of the eigenfunction  $u_i$  and  $u_j$  at  $t = 0$ , respectively. Then for all  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$ , we have at  $t = 0$ :*

$$\int_{\Omega} u'_i u_j + \int_{\Omega} u_i u'_j + \int_{\partial\Omega} u_i u_j V_n = 0.$$

**Proof:** We use the  $L^2(\Omega_t)$  orthogonality of  $u_i$  and  $u_j$  with each other:

$$\forall t > 0, \quad \int_{\Omega_t} u_i u_j = 0.$$

Applying the shape derivative, we get at  $t = 0$

$$\int_{\Omega} (u_i u_j)' + \int_{\partial\Omega} u_i u_j V_n = 0$$

that is

$$\int_{\Omega} u_i' u_j + \int_{\Omega} u_i u_j' + \int_{\partial\Omega} u_i u_j V_n = 0$$

which gives the result. □

In the following lemma, we give some derivation formulas:

**Lemma 4.7** *Let  $v$  be a  $C^2$  function defined on a  $C^2$  domain  $\Omega$ . Then we have the following derivation formulas on its boundary  $\partial\Omega$  for some deformation vector fields  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$ :*

$$(\partial_n v)' = \partial_n v' - \nabla_{\tau} v \nabla_{\tau} V_n, \quad (4.17)$$

and

$$(\nabla_{\tau} v)' = \nabla_{\tau} v' + (\nabla_{\tau} v \cdot \nabla_{\tau} V_n) n + \nabla_{\tau} V_n \partial_n v. \quad (4.18)$$

**Proof:** To start with the first equation, we write  $\partial_n v$  as  $\nabla v \cdot n$ , then we have

$$(\partial_n v)' = (\nabla v \cdot n)' = \nabla v' \cdot n + \nabla v \cdot n',$$

and using (7.2) we get

$$(\partial_n v)' = \nabla v' \cdot n - \nabla_{\tau} V_n \cdot \nabla v.$$

Again, by replacing  $\nabla v'$  by the tangential and normal components, we obtain

$$(\partial_n v)' = \nabla_{\tau} v' \cdot n + \partial_n v' n \cdot n - \nabla_{\tau} V_n \cdot \nabla_{\tau} v - (\partial_n v) n \cdot \nabla_{\tau} V_n = \partial_n v' - \nabla_{\tau} v \cdot \nabla_{\tau} V_n.$$

Moving on to the second equation, we remark that  $\nabla_{\tau} v = \nabla v - \partial_n v n$  on  $\partial\Omega$ . Therefore, we have

$$(\nabla_{\tau} v)' = (\nabla v - \partial_n v n)' = \nabla v' - (\partial_n v)' n - \partial_n v n',$$

and by (4.17), we get

$$(\nabla_\tau v)' = \nabla v' - (\partial_n v')n + (\nabla_\tau v \cdot \nabla_\tau V_n)n - \partial_n v n'.$$

Now, using (7.2), we obtain

$$(\nabla_\tau v)' = \nabla v' - (\partial_n v')n + (\nabla_\tau v \cdot \nabla_\tau V_n)n + \partial_n v \nabla_\tau V_n.$$

Finally, writing  $\nabla v'$  as  $(\partial_n v')n + \nabla_\tau v'$ , we get

$$(\nabla_\tau v)' = (\partial_n v')n + \nabla_\tau v' - (\partial_n v')n + (\nabla_\tau v \cdot \nabla_\tau V_n)n + \partial_n v \nabla_\tau V_n = \nabla_\tau v' + (\nabla_\tau v \cdot \nabla_\tau V_n)n + \partial_n v \nabla_\tau V_n.$$

□

The next Lemma is used to derive the second order shape derivative of the eigenvalues of (1.1):

**Lemma 4.8** *Let  $t \mapsto (\lambda(t), u_t)$  is a smooth eigenpair path on  $\Omega_t$  for the Wentzell-Laplace problem (1.1). Let  $u'$  and  $u''$  be the first and second-order derivatives of  $u$  at  $t = 0$ , respectively. Then we have at  $t = 0$ , for all deformation vector fields  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$ :*

$$\lambda''(0) = \int_{\partial\Omega} \partial_n u u'' - \int_{\partial\Omega} \partial_n u'' u + \int_{\partial\Omega} V_n \lambda' |u|^2. \quad (4.19)$$

**Proof:** We follow the same way as in lemma 4.5. In fact, we have in  $\Omega$ :

$$-\Delta u'' = \lambda'' u + 2\lambda' u' + \lambda u''$$

then, multiplying by  $u$ , and integrating over  $\Omega$  we get:

$$-\int_{\Omega} \Delta u'' u = \lambda \int_{\Omega} u'' u + \lambda'' \int_{\Omega} |u|^2 + 2\lambda' \int_{\Omega} u' u$$

using integration by parts twice and the normalization of the solution  $u$ ,

$$\lambda'' = -\int_{\Omega} u'' \Delta u + \int_{\partial\Omega} \partial_n u u'' - \int_{\partial\Omega} \partial_n u'' u - 2\lambda' \int_{\Omega} u' u - \int_{\Omega} \lambda u'' u$$

by (4.13) and since  $-\Delta u = \lambda u$  in  $\Omega$ , we get

$$\lambda''(0) = \int_{\partial\Omega} \partial_n u u'' - \int_{\partial\Omega} \partial_n u'' u + \int_{\partial\Omega} V_n \lambda' |u|^2.$$

□

### 4.3 The proofs

In this section, we prove all the main theorems, propositions and corollaries of this chapter.



### 4.3.1 Proof of Theorem 3

The expression of  $u'$  in the volume domain in the first line is trivial. To derive the boundary condition satisfied by  $u'$ , we use a test function  $\varphi_t$  defined in  $\Omega_t$  with  $\partial_n \varphi_t = 0$  and  $\varphi_t' = 0$  on  $\partial\Omega_t$ , then we get the following weak formulation valid for all  $t$  small enough:

$$\int_{\Omega_t} \nabla u(t, x) \cdot \nabla \varphi_t + \int_{\partial\Omega_t} \nabla_\tau u(t, x) \cdot \nabla_\tau \varphi_t = \lambda(t) \int_{\Omega_t} u(t, x) \varphi_t.$$

We take the derivative with respect to  $t$  and get at  $t = 0$ :

$$\frac{d}{dt} \left( \int_{\Omega_t} \nabla u(t, x) \cdot \nabla \varphi_t \right) \Big|_{t=0} + \frac{d}{dt} \left( \int_{\partial\Omega_t} \nabla_\tau u(t, x) \cdot \nabla_\tau \varphi_t \right) \Big|_{t=0} = \frac{d}{dt} \left( \lambda(t) \int_{\Omega_t} u(t, x) \varphi_t \right) \Big|_{t=0}.$$

Now, using Appendix A and making some integrations by parts, the first term of the left hand side can be computed:

$$\begin{aligned} \frac{d}{dt} \left( \int_{\Omega_t} \nabla u(t, x) \cdot \nabla \varphi_t \right) \Big|_{t=0} &= \int_{\Omega} (\nabla u \cdot \nabla \varphi)' + \int_{\partial\Omega} V_n \nabla u \cdot \nabla \varphi \\ &= \int_{\Omega} \nabla u' \cdot \nabla \varphi + \int_{\partial\Omega} V_n \nabla u \cdot \nabla \varphi = - \int_{\Omega} \Delta u' \varphi + \int_{\partial\Omega} \partial_n u' \varphi + \int_{\partial\Omega} V_n \nabla u \cdot \nabla \varphi \end{aligned}$$

but  $\nabla u = \nabla_\tau u + \partial_n u n$  on  $\partial\Omega$ , and  $\nabla \varphi = \nabla_\tau \varphi$  as  $\partial_n \varphi = 0$  on  $\partial\Omega$ . Then

$$\begin{aligned} \frac{d}{dt} \left( \int_{\Omega_t} \nabla u(t, x) \cdot \nabla \varphi_t \right) \Big|_{t=0} &= - \int_{\Omega} \Delta u' \varphi + \int_{\partial\Omega} \partial_n u' \varphi + \int_{\partial\Omega} V_n \nabla_\tau u \cdot \nabla_\tau \varphi \\ &= - \int_{\Omega} \Delta u' \varphi + \int_{\partial\Omega} \partial_n u' \varphi - \int_{\partial\Omega} \operatorname{div}_\tau (V_n \nabla_\tau u) \varphi. \end{aligned}$$

Where the second term is

$$\begin{aligned} \frac{d}{dt} \left( \int_{\partial\Omega_t} \nabla_\tau u(t, x) \cdot \nabla_\tau \varphi_t \right) \Big|_{t=0} &= \int_{\partial\Omega} (\nabla_\tau u \cdot \nabla_\tau \varphi)' + \int_{\partial\Omega} V_n [\partial_n (\nabla_\tau u \cdot \nabla_\tau \varphi) + H \nabla_\tau u \cdot \nabla_\tau \varphi] \\ &= \int_{\partial\Omega} (\nabla_\tau u)' \cdot \nabla_\tau \varphi + \int_{\partial\Omega} V_n [\partial_n (\nabla_\tau u \cdot \nabla_\tau \varphi) + H \nabla_\tau u \cdot \nabla_\tau \varphi]. \end{aligned}$$

By Equation (4.18) (see Lemma 4.7), that says

$$(\nabla_\tau u)' = \nabla_\tau u' + (\nabla_\tau u \cdot \nabla_\tau V_n) n + \nabla_\tau V_n \partial_n u,$$

we get

$$\frac{d}{dt} \left( \int_{\partial\Omega_t} \nabla_\tau u(t, x) \cdot \nabla_\tau \varphi_t \right) \Big|_{t=0} = \int_{\partial\Omega} \nabla_\tau u' \cdot \nabla_\tau \varphi + \int_{\partial\Omega} (\nabla_\tau V_n \cdot \nabla_\tau u) n \cdot \nabla_\tau \varphi$$

$$+ \int_{\partial\Omega} \partial_n u \nabla_\tau V_n \cdot \nabla_\tau \varphi + \int_{\partial\Omega} V_n [\partial_n (\nabla_\tau u \cdot \nabla_\tau \varphi) + H \nabla_\tau u \cdot \nabla_\tau \varphi]$$

Notice that we have

$$\int_{\partial\Omega} (\nabla_\tau V_n \cdot \nabla_\tau u) n \cdot \nabla_\tau \varphi = 0 \quad (4.20)$$

since  $n$  is orthogonal to  $\nabla_\tau \varphi$  in  $\mathbb{R}^d$ .

Moreover, using Proposition 7.5 in Appendix D, that says

$$\nabla_\tau \varphi \cdot \nabla_\tau (\partial_n u) + \nabla_\tau u \cdot \nabla_\tau (\partial_n \varphi) = 2 \nabla_\tau u^T D^2 b \nabla_\tau \varphi + \partial_n (\nabla_\tau u \cdot \nabla_\tau \varphi), \quad (4.21)$$

we get

$$\begin{aligned} \frac{d}{dt} \left( \int_{\partial\Omega_t} \nabla_\tau u(t, x) \cdot \nabla_\tau \varphi_t \right) \Big|_{t=0} &= \int_{\partial\Omega} \nabla_\tau u' \cdot \nabla_\tau \varphi + \int_{\partial\Omega} \partial_n u \nabla_\tau V_n \cdot \nabla_\tau \varphi \\ &+ \int_{\partial\Omega} V_n H \nabla_\tau u \cdot \nabla_\tau \varphi - 2 \int_{\partial\Omega} V_n \nabla_\tau u D^2 b \nabla_\tau \varphi + \int_{\partial\Omega} V_n \nabla_\tau (\partial_n u) \cdot \nabla_\tau \varphi + \int_{\partial\Omega} V_n \nabla_\tau (\partial_n \varphi) \cdot \nabla_\tau u \end{aligned}$$

by (4.15) and since  $\partial_n \varphi = 0$  on  $\partial\Omega$ , we get

$$\begin{aligned} \frac{d}{dt} \left( \int_{\partial\Omega_t} \nabla_\tau u(t, x) \cdot \nabla_\tau \varphi_t \right) \Big|_{t=0} &= \int_{\partial\Omega} \nabla_\tau u' \cdot \nabla_\tau \varphi + \int_{\partial\Omega} \nabla_\tau (V_n \partial_n u) \cdot \nabla_\tau \varphi + \int_{\partial\Omega} V_n \nabla_\tau u (H I_d - D^2 b) \nabla_\tau \varphi \\ &= \int_{\partial\Omega} (-\Delta_\tau u' - \Delta_\tau (V_n \partial_n u) - \operatorname{div}_\tau (V_n \nabla_\tau u (H I_d - 2D^2 b))) \varphi. \end{aligned}$$

On the other hand, the right hand side can be easily calculated by

$$\begin{aligned} \frac{d}{dt} \left( \lambda(t) \int_{\Omega_t} u(t, x) \varphi_t \right) \Big|_{t=0} &= \int_{\Omega_t} \frac{d}{dt} (\lambda(t) u(t, x) \varphi_t) \Big|_{t=0} + \int_{\partial\Omega} V_n \lambda(0) u(0, x) \varphi \\ &= \lambda'(0) \int_{\Omega} u \varphi + \lambda \int_{\Omega} u' \varphi + \lambda \int_{\partial\Omega} V_n u \varphi. \end{aligned}$$

To end the proof, it suffices to gather the relations.

### 4.3.2 Proof of Theorem 4

We start with the result of Theorem 3. Let us multiply the two sides of the boundary condition satisfied by  $u'$  (1.9) by the eigenfunction  $u$  and integrate over the boundary  $\partial\Omega$ , so that we get

$$\begin{aligned} \int_{\partial\Omega} \partial_n u' u - \int_{\partial\Omega} \Delta_\tau u' u' &= \int_{\partial\Omega} V_n \Delta_\tau u u + \int_{\partial\Omega} \Delta_\tau (V_n \partial_n u) u \\ &+ \int_{\partial\Omega} \operatorname{div}_\tau (V_n (HI_d - 2D^2b) \nabla_\tau u) u + \int_{\partial\Omega} \lambda V_n u u. \end{aligned} \quad (4.22)$$

Integrating by parts the second term on the left hand side twice, and knowing that  $\Delta_\tau u = \partial_n u$  on  $\partial\Omega$ , we obtain:

$$\begin{aligned} \int_{\partial\Omega} \partial_n u' u - \int_{\partial\Omega} \partial_n u u' &= \int_{\partial\Omega} V_n \Delta_\tau u u + \int_{\partial\Omega} \Delta_\tau (V_n \partial_n u) u \\ &+ \int_{\partial\Omega} \operatorname{div}_\tau (V_n (HI_d - 2D^2b) \nabla_\tau u) u + \int_{\partial\Omega} \lambda V_n u u. \end{aligned} \quad (4.23)$$

Using Lemma 4.5 and making some integration by parts on the right hand side, we obtain the result

$$\lambda'(0) = \int_{\partial\Omega} V_n (|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda |u|^2 + (HI_d - 2D^2b) \nabla_\tau u \cdot \nabla_\tau u).$$

### 4.3.3 Proof of Theorem 5

Let  $t \mapsto (u(t, x), \lambda(t) = \lambda(\Omega_t))$  be a smooth path of eigenpair of the Laplace-Wentzell problem, so that it satisfies

$$-\Delta u(t, x) = \lambda(t) u(t, x) \quad \text{in } \Omega_t, \quad (4.24)$$

$$-\Delta_\tau u(t, x) + \partial_n u(t, x) = 0 \quad \text{on } \partial\Omega_t. \quad (4.25)$$

We have proved that  $u' = \partial_t u(0, x)$  satisfies the volume equation (1.8) in  $\Omega$  and the boundary condition (1.9) on  $\partial\Omega$ . We use the decomposition of  $u = u(0, x)$  in the eigenspace as

$$u = \sum_{j=1}^m c_j u_j \quad (4.26)$$

for some  $c = (c_1, c_2, \dots, c_m)^T \neq 0$ . Multiplying the two sides equation of (4.26) by  $u_k$ , we get after some integration by parts the eigenvalue equation

$$\lambda'(0)c = Mc$$

where  $M = (M_{jk})_{1 \leq i, j \leq m}$  is defined by (1.11). From this, we deduce that the set of derivatives  $(\lambda'_i(0))_{i=1, \dots, m}$  is exactly the set of eigenvalues of the matrix  $M$ , which achieves the proof of the theorem.

#### 4.3.4 Proof of Theorem 6

The expression of  $u''$  in the volume domain in the first line is trivial. To derive the boundary condition satisfied by  $u''$ , we multiply (1.9) by a test function  $\varphi_t$  defined on  $\partial\Omega_t$  with  $\partial_n\varphi_t = 0$  and  $\varphi'_t = 0$ . Let us set  $\mathcal{A} := (HId - 2D^2b)$ . Then we get for all  $t$  small enough:

$$\begin{aligned} \int_{\partial\Omega_t} (-\Delta_\tau u' \varphi + \partial_n u') \varphi &= \int_{\partial\Omega_t} \operatorname{div}_\tau (V_n \nabla_\tau u) \varphi + \int_{\partial\Omega_t} \Delta_\tau (V_n \partial_n u) \varphi \\ &+ \int_{\partial\Omega_t} V_n \lambda u \varphi + \int_{\partial\Omega_t} \operatorname{div}_\tau (V_n \mathcal{A} \nabla_\tau u) \varphi. \end{aligned} \quad (4.27)$$

Let us do the shape derivative at time zero of each term of this equation. To simplify the notations, we set:

$$\begin{aligned} E(u, u', u'', \varphi) &:= \frac{d}{dt} \left( \int_{\partial\Omega_t} (-\Delta_\tau u' \varphi + \partial_n u') \varphi \right)_{|t=0} \\ A(u, u', \varphi) &:= \frac{d}{dt} \left( \int_{\partial\Omega_t} \operatorname{div}_\tau (V_n \nabla_\tau u) \varphi \right)_{|t=0} \\ B(u, u', \varphi) &:= \frac{d}{dt} \left( \int_{\partial\Omega_t} \Delta_\tau (V_n \partial_n u) \varphi \right)_{|t=0} \\ C(u, u', \varphi) &:= \frac{d}{dt} \left( \int_{\partial\Omega_t} V_n \lambda u \varphi \right)_{|t=0} \\ D(u, u', \varphi) &:= \frac{d}{dt} \left( \int_{\partial\Omega_t} \operatorname{div}_\tau (V_n \mathcal{A} \nabla_\tau u) \varphi \right)_{|t=0}. \end{aligned}$$

Thus we have:

$$E(u, u', u'', \varphi) = A(u, u', \varphi) + B(u, u', \varphi) + C(u, u', \varphi) + D(u, u', \varphi). \quad (4.28)$$

We are going to study each of the five terms of this equation:

**Study of  $E(u, u', u'', \varphi)$ :**

$$\begin{aligned} E(u, u', u'', \varphi) &:= \frac{d}{dt} \left( \int_{\partial\Omega_t} (-\Delta_\tau u' \varphi + \partial_n u') \varphi \right)_{|t=0} = \frac{d}{dt} \left( \int_{\partial\Omega_t} -\Delta_\tau u' \varphi \right)_{|t=0} + \frac{d}{dt} \left( \int_{\partial\Omega_t} \partial_n u' \varphi \right)_{|t=0} \\ &= \frac{d}{dt} \left( \int_{\partial\Omega_t} \nabla_\tau u' \cdot \nabla_\tau \varphi \right)_{|t=0} + \frac{d}{dt} \left( \int_{\partial\Omega_t} \partial_n u' \varphi \right)_{|t=0}. \end{aligned}$$

Applying Proposition 7.2 (see Appendix B), we get

$$\begin{aligned}
E(u, u', u'', \varphi) &= \int_{\partial\Omega} \frac{d}{dt} (\nabla_\tau u' \cdot \nabla_\tau \varphi) + \int_{\partial\Omega} V_n [\partial_n (\nabla_\tau u' \cdot \nabla_\tau \varphi) + H \nabla_\tau u' \cdot \nabla_\tau \varphi] \\
&\quad + \int_{\partial\Omega} \frac{d}{dt} (\partial_n u' \varphi) + \int_{\partial\Omega} V_n [\partial_n (\partial_n u' \varphi) + H \partial_n u' \varphi].
\end{aligned}$$

Since  $\partial_n \varphi = 0$  and  $\varphi' = 0$  on  $\partial\Omega$ , in addition to applying Proposition 7.5 (see Appendix C) and equation (4.17), we get

$$\begin{aligned}
E(u, u', u'', \varphi) &= \int_{\partial\Omega} (\nabla_\tau u')' \cdot \nabla_\tau \varphi + \int_{\partial\Omega} V_n [(\nabla_\tau (\partial_n u')) \cdot \nabla_\tau \varphi - 2D^2 b \nabla_\tau u' \cdot \nabla_\tau \varphi] + H \nabla_\tau u' \cdot \nabla_\tau \varphi \\
&\quad + \int_{\partial\Omega} (\partial_n u'' - \nabla_\tau u' \cdot \nabla_\tau V_n) \varphi + \int_{\partial\Omega} V_n \left[ \frac{\partial^2 u'}{\partial n^2} \varphi + H \partial_n u' \varphi \right].
\end{aligned}$$

Now, using equation (4.18) and Proposition 7.6 (see Appendix C), we obtain

$$\begin{aligned}
E(u, u', u'', \varphi) &= \int_{\partial\Omega} [\nabla_\tau u'' + \nabla_\tau V_n \partial_n u'] \cdot \nabla_\tau \varphi + \int_{\partial\Omega} V_n \nabla_\tau (\partial_n u') \cdot \nabla_\tau \varphi \\
&\quad - 2 \int_{\partial\Omega} V_n^2 D^2 b \nabla_\tau u' \cdot \nabla_\tau \varphi + \int_{\partial\Omega} V_n H \nabla_\tau u' \cdot \nabla_\tau \varphi + \int_{\partial\Omega} \partial_n u'' \varphi \\
&\quad - \int_{\partial\Omega} \nabla_\tau u' \cdot \nabla_\tau V_n \varphi + \int_{\partial\Omega} V_n [\Delta u' - \Delta_\tau u' - H \partial_n u'] \varphi + \int_{\partial\Omega} V_n H \partial_n u' \varphi.
\end{aligned}$$

Using the volume expression of  $u'$ , i.e  $-\Delta u' = \lambda' u + \lambda u'$ , we get

$$\begin{aligned}
E(u, u', u'', \varphi) &= \int_{\partial\Omega} \nabla_\tau u'' \cdot \nabla_\tau \varphi + \int_{\partial\Omega} \partial_n u' \nabla_\tau V_n \cdot \nabla_\tau \varphi + \int_{\partial\Omega} V_n \nabla_\tau (\partial_n u') \cdot \nabla_\tau \varphi \\
&\quad - 2 \int_{\partial\Omega} V_n^2 D^2 b \nabla_\tau u' \cdot \nabla_\tau \varphi + \int_{\partial\Omega} V_n H \nabla_\tau u' \cdot \nabla_\tau \varphi + \int_{\partial\Omega} \partial_n u'' \varphi \\
&\quad - \int_{\partial\Omega} \nabla_\tau u' \cdot \nabla_\tau V_n \varphi - \int_{\partial\Omega} V_n \lambda' u \varphi - \int_{\partial\Omega} V_n \lambda u' \varphi \\
&\quad - \int_{\partial\Omega} V_n \Delta_\tau u' \varphi - \int_{\partial\Omega} V_n H \partial_n u' \varphi + \int_{\partial\Omega} V_n H \partial_n u' \varphi.
\end{aligned}$$

Finally, in order to isolate the function  $\varphi$ , we make integration by parts to obtain

$$\begin{aligned}
E(u, u', u'', \varphi) &= - \int_{\partial\Omega} \operatorname{div}_\tau (\nabla_\tau u'') \varphi - \int_{\partial\Omega} \operatorname{div}_\tau (\nabla_\tau V_n \partial_n u') \varphi - \int_{\partial\Omega} \operatorname{div}_\tau (V_n \nabla_\tau (\partial_n u')) \varphi \\
&\quad + 2 \int_{\partial\Omega} \operatorname{div}_\tau (V_n D^2 b \nabla_\tau u') \varphi - \int_{\partial\Omega} \operatorname{div}_\tau (V_n H \nabla_\tau u') \varphi + \int_{\partial\Omega} \partial_n u'' \varphi \quad (4.29) \\
&\quad - \int_{\partial\Omega} \nabla_\tau u' \cdot \nabla_\tau V_n \varphi - \int_{\partial\Omega} V_n \lambda' u \varphi - \int_{\partial\Omega} V_n \lambda u' \varphi - \int_{\partial\Omega} V_n \Delta_\tau u' \varphi.
\end{aligned}$$

Now, let us move to the right hand side of equation (4.28):

**Study of  $A(u, u', \varphi)$ :**

$$A(u, u', \varphi) := \frac{d}{dt} \left( \int_{\partial\Omega_t} \operatorname{div}_\tau(V_n \nabla_\tau u) \varphi \right) \Big|_{t=0} = -\frac{d}{dt} \left( \int_{\partial\Omega_t} V_n \nabla_\tau u \cdot \nabla_\tau \varphi \right) \Big|_{t=0}.$$

Applying Proposition 7.2 (see Appendix B), we get

$$A(u, u', \varphi) = - \int_{\partial\Omega} \frac{d}{dt} (V_n \nabla_\tau u \cdot \nabla_\tau \varphi) - \int_{\partial\Omega} V_n [\partial_n (V_n \nabla_\tau u \cdot \nabla_\tau \varphi) - V_n \nabla_\tau u \cdot \nabla_\tau \varphi].$$

By equation (4.18) and using Proposition 7.5, and as  $\partial_n \varphi = 0$  and  $\varphi' = 0$ , we obtain

$$\begin{aligned} A(u, u', \varphi) &= - \int_{\partial\Omega} V_n' \nabla_\tau u \cdot \nabla_\tau \varphi - \int_{\partial\Omega} V_n \nabla_\tau u' \cdot \nabla_\tau \varphi - \int_{\partial\Omega} V_n \partial_n u \nabla_\tau V_n \cdot \nabla_\tau \varphi - \int_{\partial\Omega} V_n \partial_n V_n \nabla_\tau u \cdot \nabla_\tau \varphi \\ &\quad - \int_{\partial\Omega} V_n^2 \nabla_\tau (\partial_n u) \cdot \nabla_\tau \varphi + 2 \int_{\partial\Omega} V_n^2 D^2 b \nabla_\tau u \cdot \nabla_\tau \varphi - \int_{\partial\Omega} V_n^2 H \nabla_\tau u \cdot \nabla_\tau \varphi. \end{aligned}$$

Making integration by parts we obtain

$$\begin{aligned} A(u, u', \varphi) &= \int_{\partial\Omega} \operatorname{div}_\tau(V_n' \nabla_\tau u) \varphi + \int_{\partial\Omega} \operatorname{div}_\tau(V_n \nabla_\tau u') \varphi + \int_{\partial\Omega} \operatorname{div}_\tau(V_n \partial_n u \nabla_\tau V_n) \varphi \quad (4.30) \\ &\quad + \int_{\partial\Omega} \operatorname{div}_\tau(V_n \partial_n V_n \nabla_\tau u) \varphi + \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 \nabla_\tau (\partial_n u)) \varphi - 2 \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 D^2 b \nabla_\tau u) \varphi \\ &\quad + \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 H \nabla_\tau u) \varphi. \end{aligned}$$

**Study of  $B(u, u', \varphi)$ :**

$$B(u, u', \varphi) := \frac{d}{dt} \left( \int_{\partial\Omega_t} \Delta_\tau(V_n \partial_n u) \varphi \right) \Big|_{t=0} = -\frac{d}{dt} \left( \int_{\partial\Omega_t} \nabla_\tau(V_n \partial_n u) \cdot \nabla_\tau \varphi \right) \Big|_{t=0}.$$

Applying Proposition 7.2 (see Appendix B), we get

$$B(u, u', \varphi) = - \int_{\partial\Omega} \frac{d}{dt} (\nabla_\tau(V_n \partial_n u) \cdot \nabla_\tau \varphi) - \int_{\partial\Omega} V_n [\partial_n (\nabla_\tau(V_n \partial_n u) \cdot \nabla_\tau \varphi) + H \nabla_\tau(V_n \partial_n u) \cdot \nabla_\tau \varphi].$$

By equation (4.18) and Proposition 7.5, as  $\partial_n \varphi = 0$  and  $\varphi' = 0$ , we obtain

$$B(u, u', \varphi) = - \int_{\partial\Omega} \nabla_\tau((V_n \partial_n u)') \cdot \nabla_\tau \varphi - \int_{\partial\Omega} \partial_n(V_n \partial_n u) \nabla_\tau V_n \cdot \nabla_\tau \varphi - \int_{\partial\Omega} V_n \nabla_\tau(\partial_n(V_n \partial_n u)) \cdot \nabla_\tau \varphi$$

$$+2 \int_{\partial\Omega} V_n D^2 b \nabla_\tau (V_n \partial_n u) \cdot \nabla_\tau \varphi - \int_{\partial\Omega} V_n H \nabla_\tau (V_n \partial_n u) \cdot \nabla_\tau \varphi.$$

By equation (4.17) and integrating by parts the term  $\int_{\partial\Omega} V_n \nabla_\tau (\partial_n (V_n \partial_n u)) \cdot \nabla_\tau \varphi$ , we obtain

$$\begin{aligned} B(u, u', \varphi) &= - \int_{\partial\Omega} \nabla_\tau (V_n' \partial_n u) \cdot \nabla_\tau \varphi - \int_{\partial\Omega} \nabla_\tau (V_n \partial_n u') \cdot \nabla_\tau \varphi + \int_{\partial\Omega} \nabla_\tau (V_n \nabla_\tau V_n \nabla_\tau u) \cdot \nabla_\tau \varphi \\ &\quad - \int_{\partial\Omega} \partial_n (V_n \partial_n u) \nabla_\tau V_n \cdot \nabla_\tau \varphi + \int_{\partial\Omega} \partial_n (V_n \partial_n u) \nabla_\tau V_n \cdot \nabla_\tau \varphi + \int_{\partial\Omega} V_n (\partial_n (V_n \partial_n u)) \Delta_\tau \varphi \\ &\quad - \int_{\partial\Omega} V_n \mathcal{A} \nabla_\tau (V_n \partial_n u) \cdot \nabla_\tau \varphi. \end{aligned}$$

By expanding the terms and some cancellations, in addition to using Proposition 7.5, we obtain

$$\begin{aligned} B(u, u', \varphi) &= - \int_{\partial\Omega} \nabla_\tau (V_n' \partial_n u) \cdot \nabla_\tau \varphi - \int_{\partial\Omega} \nabla_\tau (V_n \partial_n u') \cdot \nabla_\tau \varphi + \int_{\partial\Omega} \nabla_\tau (V_n \nabla_\tau V_n \nabla_\tau u) \cdot \nabla_\tau \varphi \\ &\quad + \int_{\partial\Omega} V_n \partial_n V_n \partial_n u \Delta_\tau \varphi + \int_{\partial\Omega} V_n^2 [\Delta u - \Delta_\tau u - H \partial_n u] \Delta_\tau \varphi - \int_{\partial\Omega} V_n \partial_n u \mathcal{A} \nabla_\tau V_n \cdot \nabla_\tau \varphi \\ &\quad - \int_{\partial\Omega} V_n^2 \mathcal{A} \nabla_\tau (\partial_n u) \cdot \nabla_\tau \varphi. \end{aligned}$$

Now, by splitting the fifth term and since  $-\Delta u = \lambda u$ , we get

$$\begin{aligned} B(u, u', \varphi) &= - \int_{\partial\Omega} \nabla_\tau (V_n' \partial_n u) \cdot \nabla_\tau \varphi - \int_{\partial\Omega} \nabla_\tau (V_n \partial_n u') \cdot \nabla_\tau \varphi + \int_{\partial\Omega} \nabla_\tau (V_n \nabla_\tau V_n \nabla_\tau u) \cdot \nabla_\tau \varphi \\ &\quad + \int_{\partial\Omega} V_n \partial_n V_n \partial_n u \Delta_\tau \varphi - \int_{\partial\Omega} V_n^2 \lambda u \Delta_\tau \varphi - \int_{\partial\Omega} V_n^2 \Delta_\tau u \Delta_\tau \varphi - \int_{\partial\Omega} V_n^2 H \partial_n u \Delta_\tau \varphi \\ &\quad - \int_{\partial\Omega} V_n \partial_n u \mathcal{A} \nabla_\tau V_n \cdot \nabla_\tau \varphi - \int_{\partial\Omega} V_n^2 \mathcal{A} \nabla_\tau (\partial_n u) \cdot \nabla_\tau \varphi. \end{aligned}$$

Making integration by parts, we get

$$\begin{aligned} B(u, u', \varphi) &= \int_{\partial\Omega} \Delta_\tau (V_n' \partial_n u) \varphi + \int_{\partial\Omega} \Delta_\tau (V_n \partial_n u') \varphi - \int_{\partial\Omega} \Delta_\tau (V_n \nabla_\tau V_n \nabla_\tau u) \varphi \\ &\quad + \int_{\partial\Omega} \Delta_\tau (V_n \partial_n V_n \partial_n u) \varphi - \int_{\partial\Omega} \Delta_\tau (V_n^2 \lambda u) \varphi - \int_{\partial\Omega} \Delta_\tau (V_n^2 \Delta_\tau u) \varphi \\ &\quad - \int_{\partial\Omega} \Delta_\tau (V_n^2 H \partial_n u) \varphi + \int_{\partial\Omega} \operatorname{div}_\tau (V_n \partial_n u \mathcal{A} \nabla_\tau V_n) \varphi + \int_{\partial\Omega} \operatorname{div}_\tau (V_n^2 \mathcal{A} \nabla_\tau (\partial_n u)) \varphi. \end{aligned} \tag{4.31}$$

**Study of  $C(u, u', \varphi)$ :**

$$C(u, u', \varphi) := \frac{d}{dt} \left( \int_{\partial\Omega_t} V_n \lambda u \varphi \right) \Big|_{t=0}$$

Applying Proposition 7.2 (see Appendix B), we get

$$C(u, u', \varphi) = \int_{\partial\Omega} \frac{d}{dt} (V_n \lambda u \varphi) + \int_{\partial\Omega} V_n [\partial_n (V_n \lambda u \varphi) + H V_n \lambda u \varphi].$$

As  $\partial_n \varphi = 0$  and  $\varphi' = 0$ , we get

$$\begin{aligned} C(u, u', \varphi) &= \int_{\partial\Omega} V_n' \lambda u \varphi + \int_{\partial\Omega} V_n \lambda' u \varphi + \int_{\partial\Omega} V_n \lambda u' \varphi \\ &\quad + \int_{\partial\Omega} V_n \partial_n V_n \lambda u \varphi + \int_{\partial\Omega} V_n^2 \lambda \partial_n u \varphi + \int_{\partial\Omega} V_n^2 H \lambda u \varphi. \end{aligned} \quad (4.32)$$

**Study of  $D(u, u', \varphi)$ :** Denote by  $\tilde{\mathcal{A}} := (\tilde{a}_{i,j})_{1 \leq i, j \leq m}$  the matrix whose elements are the normal derivatives of  $a_{i,j}$ , where  $a_{i,j}$ ,  $i, j = 1, \dots, m$ , are the components of the matrix  $\mathcal{A}$ .

Thus, we have:

$$D(u, u', \varphi) := \frac{d}{dt} \left( \int_{\partial\Omega_t} \operatorname{div}_\tau (V_n \mathcal{A} \nabla_\tau u) \varphi \right) \Big|_{t=0} = - \frac{d}{dt} \left( \int_{\partial\Omega_t} V_n \mathcal{A} \nabla_\tau u \cdot \nabla_\tau \varphi \right) \Big|_{t=0}.$$

Applying Proposition 7.2 (see Appendix B), we get

$$D(u, u', \varphi) = - \int_{\partial\Omega} \frac{d}{dt} (V_n \mathcal{A} \nabla_\tau u \cdot \nabla_\tau \varphi) - \int_{\partial\Omega} V_n [\partial_n (V_n \mathcal{A} \nabla_\tau u \cdot \nabla_\tau \varphi) + H V_n \mathcal{A} \nabla_\tau u \cdot \nabla_\tau \varphi].$$

By equation (4.18) and Proposition 7.5, as  $\partial_n \varphi = 0$  and  $\varphi' = 0$ , we obtain

$$\begin{aligned} D(u, u', \varphi) &= - \int_{\partial\Omega} V_n' \mathcal{A} \nabla_\tau u \cdot \nabla_\tau \varphi - \int_{\partial\Omega} V_n \mathcal{A} \nabla_\tau u' \cdot \nabla_\tau \varphi - \int_{\partial\Omega} V_n \partial_n u \mathcal{A} \nabla_\tau V_n \cdot \nabla_\tau \varphi \\ &\quad - \int_{\partial\Omega} V_n \mathcal{A}' \nabla_\tau u \cdot \nabla_\tau \varphi - \int_{\partial\Omega_t} V_n \partial_n V_n \mathcal{A} \nabla_\tau u \cdot \nabla_\tau \varphi - \int_{\partial\Omega} V_n^2 \mathcal{A} \nabla_\tau (\partial_n u) \cdot \nabla_\tau \varphi \\ &\quad + 2 \int_{\partial\Omega} V_n^2 D^2 b \mathcal{A} \nabla_\tau u \cdot \nabla_\tau \varphi - \int_{\partial\Omega} V_n^2 \tilde{\mathcal{A}} \nabla_\tau u \cdot \nabla_\tau \varphi - \int_{\partial\Omega} V_n^2 H \mathcal{A} \nabla_\tau u \cdot \nabla_\tau \varphi. \end{aligned}$$

Making integrations by parts, we get



$$\begin{aligned}
D(u, u', \varphi) &= \int_{\partial\Omega} \operatorname{div}_\tau(V'_n \mathcal{A} \nabla_\tau u) \varphi + \int_{\partial\Omega} \operatorname{div}_\tau(V_n \mathcal{A} \nabla_\tau u') \varphi + \int_{\partial\Omega} \operatorname{div}_\tau(V_n \partial_n u \mathcal{A} \nabla_\tau V_n) \varphi \quad (4.33) \\
&+ \int_{\partial\Omega} \operatorname{div}_\tau(V_n \mathcal{A}' \nabla_\tau u) \varphi + \int_{\partial\Omega} \operatorname{div}_\tau(V_n \partial_n V_n \mathcal{A} \nabla_\tau u) \varphi + \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 \mathcal{A} \nabla_\tau (\partial_n u)) \varphi \\
&- 2 \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 D^2 b \mathcal{A} \nabla_\tau u) \varphi + \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 \tilde{\mathcal{A}} \nabla_\tau u) \varphi + \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 H \mathcal{A} \nabla_\tau u) \varphi.
\end{aligned}$$

#### 4.3.5 Proof of Theorem 7

First of all, we have seen in Lemma 4.8 that the expression of  $\lambda''$  contains the second order derivative  $u''$ . So in order to get rid of this term, we try to find an explicit formula that depends only on  $u$  and  $u'$ . To do so, we multiply the boundary condition equation obtained in Theorem 6 by a special test function which is chosen to be an eigenfunction  $u$  corresponding to eigenvalue  $\lambda$ , and then we make integration by parts to obtain the following equation:

$$\mathcal{E}(u, u', u'') = \mathcal{F}(u, u') - \mathcal{B}(u, u') - \mathcal{C}(u, u') + \mathcal{D}(u, u'), \quad (4.34)$$

where we pose:

$$\begin{aligned}
\mathcal{E}(u, u', u'') &:= E(u, u', u'', u), \\
\mathcal{A}(u, u') &:= A(u, u', u), \\
\mathcal{B}(u, u') &:= B(u, u', u), \\
\mathcal{C}(u, u') &:= C(u, u', u),
\end{aligned}$$

and

$$\mathcal{D}(u, u') := D(u, u', u).$$

We start with the left hand side term of equation (4.34):

#### Calculation of $\mathcal{E}(u, u', u'')$ :

We replace the function  $\varphi$  by an eigenfunction  $u$  in equation (4.29), so that we get:

$$\begin{aligned}
\mathcal{E}(u, u', u'') &= - \int_{\partial\Omega} \operatorname{div}_\tau(\nabla_\tau u'') u - \int_{\partial\Omega} \operatorname{div}_\tau(\nabla_\tau V_n \partial_n u') u - \int_{\partial\Omega} \operatorname{div}_\tau(V_n \nabla_\tau (\partial_n u')) u \\
&+ 2 \int_{\partial\Omega} \operatorname{div}_\tau(V_n D^2 b \nabla_\tau u') u - \int_{\partial\Omega} \operatorname{div}_\tau(V_n H \nabla_\tau u') u + \int_{\partial\Omega} \partial_n u'' u \\
&- \int_{\partial\Omega} (\nabla_\tau u' \cdot \nabla_\tau V_n) u - \int_{\partial\Omega} V_n \lambda' u u - \int_{\partial\Omega} V_n \lambda u' u - \int_{\partial\Omega} V_n \Delta_\tau u' u.
\end{aligned}$$

Making integration by parts, we get

$$\begin{aligned}\mathcal{E}(u, u', u'') &= \int_{\partial\Omega} \nabla_\tau u'' \cdot \nabla_\tau u + \int_{\partial\Omega} \partial_n u' \nabla_\tau V_n \cdot \nabla_\tau u + \int_{\partial\Omega} V_n \nabla_\tau (\partial_n u') \cdot \nabla_\tau u \\ &\quad - 2 \int_{\partial\Omega} V_n D^2 b \nabla_\tau u' \cdot \nabla_\tau u + \int_{\partial\Omega} V_n H \nabla_\tau u' \cdot \nabla_\tau u + \int_{\partial\Omega} \partial_n u'' u \\ &\quad - \int_{\partial\Omega} (\nabla_\tau u' \cdot \nabla_\tau V_n) u - \int_{\partial\Omega} V_n \lambda' u u - \int_{\partial\Omega} V_n \lambda u' u - \int_{\partial\Omega} V_n \Delta_\tau u' u.\end{aligned}$$

Let us remark that, using integration by parts, we have:

$$\int_{\partial\Omega} \nabla_\tau u'' \cdot \nabla_\tau u = - \int_{\partial\Omega} \operatorname{div}_\tau (\nabla_\tau u) u'' = - \int_{\partial\Omega} \Delta_\tau u u'' = - \int_{\partial\Omega} \partial_n u u'', \quad (4.35)$$

$$\int_{\partial\Omega} \partial_n u' \nabla_\tau V_n \cdot \nabla_\tau u = - \int_{\partial\Omega} V_n \operatorname{div}_\tau (\partial_n u' \nabla_\tau u) = - \int_{\partial\Omega} V_n \partial_n u' \Delta_\tau u - \int_{\partial\Omega} V_n \nabla_\tau (\partial_n u') \cdot \nabla_\tau u, \quad (4.36)$$

and

$$- \int_{\partial\Omega} V_n \Delta_\tau u' u = \int_{\partial\Omega} \nabla_\tau (V_n u) \cdot \nabla_\tau u' = \int_{\partial\Omega} (\nabla_\tau V_n \cdot \nabla_\tau u') u + \int_{\partial\Omega} V_n (\nabla_\tau u \cdot \nabla_\tau u'). \quad (4.37)$$

Therefore, by equations (4.35), (4.36) and (4.37), we get

$$\begin{aligned}\mathcal{E}(u, u', u'') &= - \int_{\partial\Omega} \partial_n u u'' - \int_{\partial\Omega} V_n \partial_n u' \Delta_\tau u - \int_{\partial\Omega} V_n \nabla_\tau (\partial_n u') \cdot \nabla_\tau u + \int_{\partial\Omega} V_n \nabla_\tau (\partial_n u') \cdot \nabla_\tau u \\ &\quad - 2 \int_{\partial\Omega} V_n D^2 b \nabla_\tau u' \cdot \nabla_\tau u + \int_{\partial\Omega} \partial_n u'' u + \int_{\partial\Omega} V_n H \nabla_\tau u' \cdot \nabla_\tau u - \int_{\partial\Omega} (\nabla_\tau u' \cdot \nabla_\tau V_n) u \\ &\quad - \int_{\partial\Omega} V_n \lambda' |u|^2 - \int_{\partial\Omega} V_n \lambda u' u + \int_{\partial\Omega} (\nabla_\tau V_n \cdot \nabla_\tau u') u + \int_{\partial\Omega} V_n (\nabla_\tau u \cdot \nabla_\tau u').\end{aligned}$$

Using Lemma 4.8, we obtain

$$\begin{aligned}\mathcal{E}(u, u', u'') &= -\lambda''(0) + \int_{\partial\Omega} V_n \lambda' |u|^2 - \int_{\partial\Omega} V_n \partial_n u' \Delta_\tau u - \int_{\partial\Omega} V_n \nabla_\tau (\partial_n u') \cdot \nabla_\tau u \\ &\quad + \int_{\partial\Omega} V_n \nabla_\tau (\partial_n u') \cdot \nabla_\tau u - 2 \int_{\partial\Omega} V_n D^2 b \nabla_\tau u' \cdot \nabla_\tau u + \int_{\partial\Omega} V_n H \nabla_\tau u' \cdot \nabla_\tau u \\ &\quad - \int_{\partial\Omega} (\nabla_\tau u' \cdot \nabla_\tau V_n) u - \int_{\partial\Omega} V_n \lambda' |u|^2 - \int_{\partial\Omega} V_n \lambda u' u \\ &\quad + \int_{\partial\Omega} (\nabla_\tau V_n \cdot \nabla_\tau u') u + \int_{\partial\Omega} V_n (\nabla_\tau u \cdot \nabla_\tau u').\end{aligned}$$

Finally, we can deduce that

$$\mathcal{E}(u, u', u'') = -\lambda''(0) + \int_{\partial\Omega} V_n(\nabla_\tau u' \cdot \nabla_\tau u) - \int_{\partial\Omega} V_n \partial_n u' \Delta_\tau u - \int_{\partial\Omega} V_n \lambda u' u + \int_{\partial\Omega} V_n \mathcal{A} \nabla_\tau u' \cdot \nabla_\tau u. \quad (4.38)$$

Let us now move to the other side of equation (4.34):

**Calculation of  $\mathcal{A}(u, u')$ :**

We replace  $\varphi$  by  $u$  in equation (4.30), so that we get

$$\begin{aligned} \mathcal{A}(u, u') &= \int_{\partial\Omega} \operatorname{div}_\tau(V_n' \nabla_\tau u)u + \int_{\partial\Omega} \operatorname{div}_\tau(V_n \nabla_\tau u')u + \int_{\partial\Omega} \operatorname{div}_\tau(V_n \partial_n u \nabla_\tau V_n)u \\ &+ \int_{\partial\Omega} \operatorname{div}_\tau(V_n \partial_n V_n \nabla_\tau u)u + \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 \nabla_\tau(\partial_n u))u - 2 \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 D^2 b \nabla_\tau u)u \\ &+ \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 H \nabla_\tau u)u. \end{aligned}$$

Making integration by parts, we get

$$\begin{aligned} \mathcal{A}(u, u') &= - \int_{\partial\Omega} V_n' \nabla_\tau u \cdot \nabla_\tau u - \int_{\partial\Omega} V_n \nabla_\tau u' \cdot \nabla_\tau u - \int_{\partial\Omega} V_n \partial_n u \nabla_\tau V_n \cdot \nabla_\tau u \\ &- \int_{\partial\Omega} V_n \partial_n V_n \nabla_\tau u \cdot \nabla_\tau u - \int_{\partial\Omega} V_n^2 \nabla_\tau(\partial_n u) \cdot \nabla_\tau u + 2 \int_{\partial\Omega} V_n^2 D^2 b \nabla_\tau u \cdot \nabla_\tau u \\ &- \int_{\partial\Omega} V_n^2 H \nabla_\tau u \cdot \nabla_\tau u. \end{aligned} \quad (4.39)$$

**Calculation of  $\mathcal{B}(u, u')$ :**

We replace  $\varphi$  by  $u$  in equation (4.31), so that we get

$$\begin{aligned} \mathcal{B}(u, u') &= \int_{\partial\Omega} \Delta_\tau(V_n' \partial_n u)u + \int_{\partial\Omega} \Delta_\tau(V_n \partial_n u')u - \int_{\partial\Omega} \Delta_\tau(V_n \nabla_\tau V_n \cdot \nabla_\tau u)u \\ &+ \int_{\partial\Omega} \Delta_\tau(V_n \partial_n V_n \partial_n u)u - \int_{\partial\Omega} \Delta_\tau(V_n^2 \lambda u)u - \int_{\partial\Omega} \Delta_\tau(V_n^2 \Delta_\tau u)u \\ &- \int_{\partial\Omega} \Delta_\tau(V_n^2 H \partial_n u)u + \int_{\partial\Omega} \operatorname{div}_\tau(V_n \partial_n u \mathcal{A} \nabla_\tau V_n)u + \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 \mathcal{A} \nabla_\tau(\partial_n u))u. \end{aligned}$$

Making integration by parts twice, we get

$$\begin{aligned}
\mathcal{B}(u, u') &= \int_{\partial\Omega} V_n' \partial_n u \Delta_\tau u + \int_{\partial\Omega} V_n \partial_n u' \Delta_\tau u - \int_{\partial\Omega} V_n \nabla_\tau V_n \cdot \nabla_\tau u \Delta_\tau u + \int_{\partial\Omega} V_n \partial_n V_n \partial_n u \Delta_\tau u \\
&\quad - \int_{\partial\Omega} V_n^2 \lambda u \Delta_\tau u - \int_{\partial\Omega} V_n^2 \Delta_\tau u \Delta_\tau u - \int_{\partial\Omega} V_n^2 H \partial_n u \Delta_\tau u - \int_{\partial\Omega} V_n \partial_n u \mathcal{A} \nabla_\tau V_n \cdot \nabla_\tau u \\
&\quad - \int_{\partial\Omega} V_n^2 \mathcal{A} \nabla_\tau (\partial_n u) \cdot \nabla_\tau u.
\end{aligned}$$

Remark that we have the following identity:

$$\begin{aligned}
& - \int_{\partial\Omega} V_n \nabla_\tau V_n \cdot \nabla_\tau u \Delta_\tau u = \int_{\partial\Omega} V_n \operatorname{div}_\tau (V_n \Delta_\tau u \nabla_\tau u) \\
& = \int_{\partial\Omega} V_n \nabla_\tau V_n \cdot \nabla_\tau u \Delta_\tau u + \int_{\partial\Omega} V_n^2 \partial_n u \Delta_\tau u + \int_{\partial\Omega} V_n^2 \nabla_\tau u \nabla_\tau (\Delta_\tau u).
\end{aligned} \tag{4.40}$$

Now, by (4.40) and using the boundary condition  $\Delta_\tau u = \partial_n u$ , we obtain

$$\begin{aligned}
\mathcal{B}(u, u') &= \int_{\partial\Omega} V_n' |\partial_n u|^2 + \int_{\partial\Omega} V_n \partial_n u' \partial_n u + \int_{\partial\Omega} V_n \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u + \int_{\partial\Omega} V_n^2 \nabla_\tau u \cdot \nabla_\tau (\partial_n u) \\
&\quad + \int_{\partial\Omega} V_n \partial_n V_n |\partial_n u|^2 - \int_{\partial\Omega} V_n^2 \lambda \partial_n u u - \int_{\partial\Omega} V_n^2 H |\partial_n u|^2 - \int_{\partial\Omega} V_n \partial_n u \mathcal{A} \nabla_\tau V_n \cdot \nabla_\tau u \\
&\quad - \int_{\partial\Omega} V_n^2 \mathcal{A} \nabla_\tau (\partial_n u) \cdot \nabla_\tau u.
\end{aligned}$$

By adding and subtracting the terms  $\int_{\partial\Omega} V_n^2 \nabla_\tau u \cdot \nabla_\tau (\partial_n u)$  and  $\int_{\partial\Omega} V_n^2 H |\partial_n u|^2$ , we get

$$\begin{aligned}
\mathcal{B}(u, u') &= \int_{\partial\Omega} V_n' |\partial_n u|^2 + \int_{\partial\Omega} V_n \partial_n u' \partial_n u + \int_{\partial\Omega} V_n \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u + 2 \int_{\partial\Omega} V_n^2 \nabla_\tau u \cdot \nabla_\tau (\partial_n u) \\
&\quad - \int_{\partial\Omega} V_n^2 \nabla_\tau u \cdot \nabla_\tau (\partial_n u) + \int_{\partial\Omega} V_n \partial_n V_n |\partial_n u|^2 - \int_{\partial\Omega} V_n^2 \lambda \partial_n u u - 2 \int_{\partial\Omega} V_n^2 H |\partial_n u|^2 \\
&\quad + \int_{\partial\Omega} V_n^2 H |\partial_n u|^2 - \int_{\partial\Omega} V_n \partial_n u \mathcal{A} \nabla_\tau V_n \cdot \nabla_\tau u - \int_{\partial\Omega} V_n^2 \mathcal{A} \nabla_\tau (\partial_n u) \cdot \nabla_\tau u.
\end{aligned}$$

Finally, using (4.40), we replace the term  $2 \int_{\partial\Omega} V_n^2 \nabla_\tau u \cdot \nabla_\tau (\partial_n u)$ , so that we get

$$\begin{aligned}
\mathcal{B}(u, u') &= \int_{\partial\Omega} V_n' |\partial_n u|^2 + \int_{\partial\Omega} V_n \partial_n u' \partial_n u - 3 \int_{\partial\Omega} V_n \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u - 2 \int_{\partial\Omega} V_n^2 |\partial_n u|^2 \\
&\quad - \int_{\partial\Omega} V_n^2 \nabla_\tau u \cdot \nabla_\tau (\partial_n u) + \int_{\partial\Omega} V_n \partial_n V_n |\partial_n u|^2 - \int_{\partial\Omega} V_n^2 \lambda \partial_n u u - 2 \int_{\partial\Omega} V_n^2 H |\partial_n u|^2 \\
&\quad + \int_{\partial\Omega} V_n^2 H |\partial_n u|^2 - \int_{\partial\Omega} V_n \partial_n u \mathcal{A} \nabla_\tau V_n \cdot \nabla_\tau u - \int_{\partial\Omega} V_n^2 \mathcal{A} \nabla_\tau (\partial_n u) \cdot \nabla_\tau u.
\end{aligned} \tag{4.41}$$

**Calculation of  $\mathcal{C}(u, u')$ :**

We replace  $\varphi$  by  $u$  in equation (4.32), so that we get

$$\begin{aligned} \mathcal{C}(u, u') &= \int_{\partial\Omega} V_n' \lambda |u|^2 + \int_{\partial\Omega} V_n \lambda' |u|^2 + \int_{\partial\Omega} V_n \lambda u' u + \int_{\partial\Omega} V_n \partial_n V_n \lambda |u|^2 \\ &\quad + \int_{\partial\Omega} V_n^2 \lambda \partial_n u u + \int_{\partial\Omega} V_n^2 H \lambda |u|^2. \end{aligned} \quad (4.42)$$

**Calculation of  $\mathcal{D}(u, u')$ :**

We replace  $\varphi$  by  $u$  in equation (4.33), so that we get

$$\begin{aligned} \mathcal{D}(u, u') &= \int_{\partial\Omega} \operatorname{div}_\tau(V_n' \mathcal{A} \nabla_\tau u) u + \int_{\partial\Omega} \operatorname{div}_\tau(V_n \mathcal{A} \nabla_\tau u') u + \int_{\partial\Omega} \operatorname{div}_\tau(V_n \partial_n u \mathcal{A} \nabla_\tau V_n) u \\ &\quad + \int_{\partial\Omega} \operatorname{div}_\tau(V_n \mathcal{A}' \nabla_\tau u) u + \int_{\partial\Omega} \operatorname{div}_\tau(V_n \partial_n V_n \mathcal{A} \nabla_\tau u) u + \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 \mathcal{A} \nabla_\tau (\partial_n u)) u \\ &\quad - 2 \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 D^2 b \mathcal{A} \nabla_\tau u) u + \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 \tilde{\mathcal{A}} \nabla_\tau u) u + \int_{\partial\Omega} \operatorname{div}_\tau(V_n^2 H \mathcal{A} \nabla_\tau u) u. \end{aligned}$$

Using integration by parts, we get

$$\begin{aligned} \mathcal{D}(u, u') &= - \int_{\partial\Omega} V_n' \mathcal{A} \nabla_\tau u \cdot \nabla_\tau u - \int_{\partial\Omega} V_n \mathcal{A} \nabla_\tau u' \cdot \nabla_\tau u - \int_{\partial\Omega} V_n \partial_n u \mathcal{A} \nabla_\tau V_n \cdot \nabla_\tau u \\ &\quad - \int_{\partial\Omega} V_n \mathcal{A}' \nabla_\tau u \cdot \nabla_\tau u - \int_{\partial\Omega} V_n \partial_n V_n \mathcal{A} \nabla_\tau u \cdot \nabla_\tau u - \int_{\partial\Omega} V_n^2 \mathcal{A} \nabla_\tau (\partial_n u) \cdot \nabla_\tau u \\ &\quad + 2 \int_{\partial\Omega} V_n^2 D^2 b \mathcal{A} \nabla_\tau u \cdot \nabla_\tau u - \int_{\partial\Omega} V_n^2 \tilde{\mathcal{A}} \nabla_\tau u \cdot \nabla_\tau u - \int_{\partial\Omega} V_n^2 H \mathcal{A} \nabla_\tau u \cdot \nabla_\tau u. \end{aligned} \quad (4.43)$$

In order to finish the proof, it is sufficient to replace the five terms in equation (4.34).

# Chapter 5

## Non-critical shapes

We start the chapter giving two Pohožaev's identities, which will be used for proving that the ball is not a critical shape of the eigenvalues problem (1.1), with and without a volume constraint.

Finally, in dimension two, we prove that, with some constraints and under a deformation vector field  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$ , there are no critical shapes for this eigenvalue problem. Actually, we argue by contradiction. So we fix an initial  $C^2$  critical domain  $\Omega_0$  at time  $t = 0$  with assuming that there exist some points on the boundary of the initial domain  $\partial\Omega_0$  where the curvature  $H$  is 2 and also some other points where  $H \notin \{0, 2, \frac{2}{3}\}$ , and we assume that there exists a small period  $\tau$  in which this domain will remain critical for  $0 \leq t < \tau$ , for all deformation vector fields  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$ .

In fact, the sharpness of this proof lies in the insufficiency of the second-order derivative computations. **Therefore, we need to rise up to third-order derivative calculations**, in order to find finally a contradiction.

### 5.1 Statement of the results

To start with, the first theorem allows us to state two Pohožaev's identities.

**Theorem 8 (Pohožaev's identities).** *Let  $\Omega$  be a  $C^2$  domain, and let  $u$  be an eigenfunction of problem (1.1).*

1. *Let  $X$  be a translation vector field, i.e  $X(x) = x + \alpha$  for all  $x, \alpha \in \mathbb{R}^d$ , and let  $D^2u$  be the Hessian matrix of  $u$ . We have*

$$\int_{\partial\Omega} (D^2u + \partial_n u Id) X \cdot \nabla_\tau u + \int_{\partial\Omega} |\nabla_\tau u|^2 + \int_{\partial\Omega} |\partial_n u|^2 (X \cdot n) = 0. \quad (5.1)$$

2. *If  $D^2b$  is the Hessian matrix of the signed distance function  $b$  to  $\partial\Omega$ , then we have*

$$\int_{\partial\Omega} D^2b \nabla_\tau u \cdot \nabla_\tau u + \int_{\partial\Omega} D^2u \cdot n \cdot \nabla_\tau u + \int_{\partial\Omega} |\partial_n u|^2 = 0. \quad (5.2)$$

Now, let us give the definition of a critical shape:

**Definition 5.1** *A domain  $\Omega$  is said to be critical if there exist an eigenvalue  $\lambda$  such that its derivative  $\lambda'(\Omega, V)$ , in any direction  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$ , is zero.*

The following corollary shows the non-criticality of the balls in any dimension:

**Corollary 1.1** *The balls in  $\mathbb{R}^d$ , with and without volume constraint, are not critical.*

Let us now give the last theorem in this chapter, which is about non-critical forms in  $\mathbb{R}^2$ .

**Theorem 9 (Non-critical shapes).** *Let  $\Omega_0$  be a  $C^2$  critical open bounded domain in  $\mathbb{R}^2$ . Assume that there exist some points on the boundary  $\partial\Omega_0$  where the curvature  $H$  is 2 and also some other points where  $H \notin \{0, 2, \frac{2}{3}\}$ . Then, without volume constraint, there is a deformation vector field  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$  such that for all small  $t > 0$ ,  $\Omega_t = \Omega_0 + tV$  is not critical.*

**Remark 5.1** *We remark that, under the mentioned assumption on the boundary  $\partial\Omega_0$  in the previous theorem, the set of these shapes  $\Omega_0$  can be considered to be large in the set of admissible domains in  $\mathbb{R}^2$ , but the question is: do we have density?*

**Remark:** We use third-order derivative calculations in order to prove this theorem.

**Comment:** I didn't manage to complete the proof without this restrictive assumption.

**Open question:** Is this assumption really needed?

In Figure 5.1, we give the plan of Chapter five.

## 5.2 Useful Lemma

The following lemma is needed to prove Theorem 9, in addition to some lemmas in Chapter four (subsection 4.2.3).

**Lemma 5.1** *Let  $\Omega$  be a  $C^2$  domain, and let  $(\lambda, u)$  be a smooth eigenpair on  $\Omega$  for the Wentzell-Laplace problem (1.1). Then we have the following derivation formula on its boundary  $\partial\Omega$  for some deformation vector fields  $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$ :*

$$\begin{aligned} [\nabla_\tau(\partial_n u)]' &= \nabla_\tau(\partial_n u') + \partial_n(\partial_n u')n - \Delta_\tau V_n \nabla_\tau u - \nabla_\tau V_n \Delta_\tau u \\ &\quad - \partial_n(\nabla_\tau V_n \cdot \nabla_\tau u)n - [\Delta u' - H' \partial_n u - (1 + H) \partial_n u' + (1 + H) \nabla_\tau V_n \cdot \nabla_\tau u]n \\ &\quad - \lambda \nabla_\tau V_n u - (1 + H) \nabla_\tau V_n \partial_n u. \end{aligned} \tag{5.3}$$

**Proof:** Let us divide the gradient into tangential and normal components:

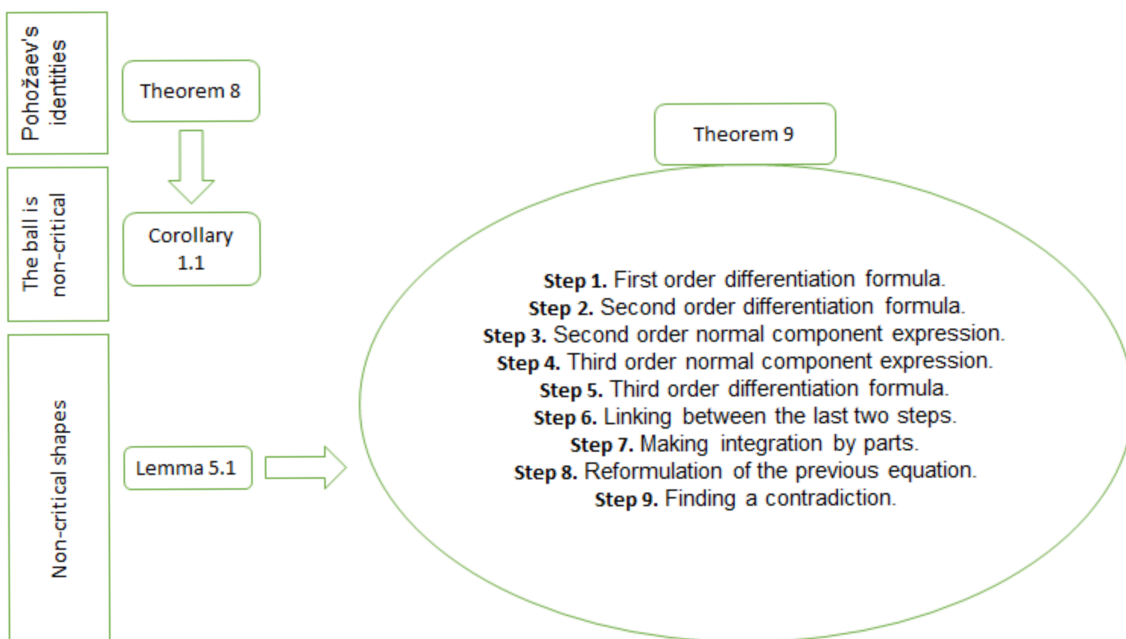


Figure 5.1: The map of Chapter five



$$\begin{aligned}
[\nabla_\tau(\partial_n u)]' &= [\nabla(\partial_n u) - \partial_n(\partial_n u)n]' = \nabla(\partial_n u)' - \left[ \frac{d^2 u}{dn^2} n \right]' \\
&= \nabla(\partial_n u') - \nabla(\nabla_\tau V_n \cdot \nabla_\tau u) - [(\Delta u - (1+H)\partial_n u)n]' \quad \text{by (4.17) and (7.4)} \\
&= \nabla_\tau(\partial_n u') + \partial_n(\partial_n u')n - \nabla_\tau(\nabla_\tau V_n \cdot \nabla_\tau u) - \partial_n(\nabla_\tau V_n \cdot \nabla_\tau u)n \\
&\quad - [\Delta u' - H'\partial_n u - (1+H)\partial_n u' + (1+H)\nabla_\tau V_n \cdot \nabla_\tau u]n \quad \text{by (4.17)} \\
&\quad - \lambda \nabla_\tau V_n u - (1+H)\nabla_\tau V_n \partial_n u \quad \text{by (7.2) and } -\Delta u = \lambda u.
\end{aligned}$$

□

## 5.3 The Proofs

### 5.3.1 Proof of Theorem 8

To write the first Pohožaev's identity, we multiply the boundary eigenvalue problem (1.1) in  $\Omega$  by  $X \cdot \nabla u$ , where  $u$  is an eigenfunction and  $X$  is defined on  $\mathbb{R}^d$  by  $X(x) := x + \alpha$  for all  $x, \alpha \in \mathbb{R}^d$ , so that we have

$$\int_{\Omega} \nabla u \cdot \nabla(X \cdot \nabla u) - \int_{\partial\Omega} \partial_n u (X \cdot \nabla u) = \lambda \int_{\Omega} u (X \cdot \nabla u). \quad (5.4)$$

Let us remark that  $\nabla(X \cdot \nabla u) = D(X)\nabla u + D^2 u X$ , where  $D(X)$  is the differential matrix of the vector  $X$  and  $D^2 u$  is the Hessian matrix of  $u$ . Since  $D(X) = Id$ , we get

$$\nabla(X \cdot \nabla u) = \nabla u + D^2 u X. \quad (5.5)$$

Now, using integration by parts on the boundary  $\partial\Omega$ , as  $\Delta_\tau u = \partial_n u$ , we get

$$\int_{\Omega} |\nabla u|^2 + \int_{\Omega} D^2 u X \cdot \nabla u + \int_{\partial\Omega} \nabla_\tau u \cdot \nabla_\tau (X \cdot \nabla u) = \lambda \int_{\Omega} u (X \cdot \nabla u). \quad (5.6)$$

Let us compute  $\nabla_\tau(X \cdot \nabla u)$ : We write  $\nabla_\tau(X \cdot \nabla u)$  as  $\nabla(X \cdot \nabla u) - \partial_n(X \cdot \nabla u)n$ . where  $n$  is the outward unit normal field on  $\partial\Omega$ . So we get

$$\begin{aligned}
\nabla_\tau(X \cdot \nabla u) &= \nabla(X \cdot \nabla u) - [\nabla(X \cdot \nabla u) \cdot n]n \\
&= \nabla u + D^2 u X - [\partial_n u + (D^2 u X) \cdot n]n \\
&= \nabla_\tau u + D^2 u X - (D^2 u X \cdot n)n
\end{aligned} \quad (5.7)$$

Now, let us remark that  $(D^2uX.n)n.\nabla_\tau u = 0$  since  $\nabla_\tau u$  is orthogonal to  $n$ . Therefore, replacing (5.7) in (5.6), we obtain

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} D^2uX.\nabla u - \lambda \int_{\Omega} u(X.\nabla u) &= - \int_{\partial\Omega} \nabla_\tau u.\nabla_\tau(X.\nabla u) \\ &= - \int_{\partial\Omega} |\nabla_\tau u|^2 - \int_{\partial\Omega} D^2uX.\nabla_\tau u. \end{aligned} \quad (5.8)$$

On the other hand, referring to (5.4), we have

$$\int_{\Omega} \nabla u.\nabla(X.\nabla u) - \lambda \int_{\Omega} u(X.\nabla u) = \int_{\partial\Omega} \partial_n u(X.\nabla u). \quad (5.9)$$

Hence, we get by expressing  $\nabla u$  as  $\nabla_\tau u + \partial_n un$ :

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} D^2uX.\nabla u - \lambda \int_{\Omega} u(X.\nabla u) &= \int_{\partial\Omega} \partial_n u(X.\nabla u) \\ &= \int_{\partial\Omega} \partial_n u(X.[\nabla_\tau u + \partial_n un]) \\ &= \int_{\partial\Omega} \partial_n u(X.\nabla_\tau u) + \int_{\partial\Omega} |\partial_n u|^2(X.n) \end{aligned} \quad (5.10)$$

Combining between (5.8) and (5.10), we obtain

$$\int_{\partial\Omega} (D^2u + \partial_n u Id)X.\nabla_\tau u + \int_{\partial\Omega} |\nabla_\tau u|^2 + \int_{\partial\Omega} |\partial_n u|^2(X.n) = 0. \quad (5.11)$$

Let us move to the second Pohožaev's identity. Since  $\Omega$  is of class  $C^2$ , there exists  $h > 0$ , such that the signed distance function  $b := b_\Omega$  belongs to  $C^2(U_h)$ , where  $U_h := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) < h\}$  is a tubular neighborhood. Let us make a truncation on  $U_h$  so that we set  $\tilde{b} = b\psi$  in  $U_h$ , where  $\psi$  is a  $C^\infty(\Omega)$  cut-off function defined by

$$\psi := \ell\left(\frac{d}{h}\right)$$

where  $d$  is the distance function to the boundary, and  $\ell$  is a  $C^\infty$  function defined on  $\mathbb{R}^+$  as the following:

$$\ell(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{if } t \geq 1. \end{cases}$$

Now, we have that  $\tilde{b} \in C^2(\mathbb{R}^d)$ . Furthermore, we have  $\tilde{b} = b$  in a neighborhood of  $\partial\Omega$ , in particular we get  $\tilde{b} = b$  on  $\partial\Omega$ .

To continue the proof, we multiply the boundary eigenvalue problem (1.1) in  $\Omega$  by  $\nabla\tilde{b} \cdot \nabla u$ , where  $u$  is an eigenfunction. Then, we integrate by parts, so that we get

$$\int_{\Omega} \nabla u \cdot \nabla(\nabla\tilde{b} \cdot \nabla u) - \int_{\partial\Omega} \partial_n u (\nabla\tilde{b} \cdot \nabla u) = \lambda \int_{\Omega} u (\nabla\tilde{b} \cdot \nabla u). \quad (5.12)$$

We remark that  $\nabla(\nabla\tilde{b} \cdot \nabla u) = D(\nabla\tilde{b})\nabla u + D^2u\nabla\tilde{b} = D^2\tilde{b}\nabla u + D^2u\nabla\tilde{b}$ , where  $D^2\tilde{b}$  and  $D^2u$  are the Hessian matrices of  $\tilde{b}$  and  $u$  respectively.

Now, using integration by parts on the boundary  $\partial\Omega$ , as  $\Delta_{\tau}u = \partial_n u$ , we get

$$\int_{\Omega} D^2\tilde{b}\nabla u \cdot \nabla u + \int_{\Omega} D^2u\nabla\tilde{b} \cdot \nabla u + \int_{\partial\Omega} \nabla_{\tau}u \cdot \nabla_{\tau}(\nabla\tilde{b} \cdot \nabla u) = \lambda \int_{\Omega} u (\nabla\tilde{b} \cdot \nabla u). \quad (5.13)$$

Let us compute  $\nabla_{\tau}(\nabla\tilde{b} \cdot \nabla u)$ : We write  $\nabla_{\tau}(\nabla\tilde{b} \cdot \nabla u)$  as  $\nabla(\nabla\tilde{b} \cdot \nabla u) - \partial_n(\nabla\tilde{b} \cdot \nabla u)n$ . where  $n$  is the outward unit normal field on  $\partial\Omega$ . So we get

$$\begin{aligned} \nabla_{\tau}(\nabla\tilde{b} \cdot \nabla u) &= \nabla(\nabla\tilde{b} \cdot \nabla u) - [\nabla(\nabla\tilde{b} \cdot \nabla u) \cdot n]n \\ &= D^2\tilde{b}\nabla u + D^2u\nabla\tilde{b} - [\nabla(\nabla\tilde{b} \cdot \nabla u) \cdot n]n \end{aligned} \quad (5.14)$$

Now, let us remark that  $[\nabla(\nabla\tilde{b} \cdot \nabla u) \cdot n]n \cdot \nabla_{\tau}u = 0$  since  $\nabla_{\tau}u$  is orthogonal to  $n$ . Therefore, replacing (5.14) in (5.13), we obtain

$$\begin{aligned} \int_{\Omega} D^2\tilde{b}\nabla u \cdot \nabla u + \int_{\Omega} D^2u\nabla\tilde{b} \cdot \nabla u - \lambda \int_{\Omega} u (\nabla\tilde{b} \cdot \nabla u) &= - \int_{\partial\Omega} \nabla_{\tau}u \cdot \nabla_{\tau}(\nabla\tilde{b} \cdot \nabla u) \\ &= - \int_{\partial\Omega} D^2\tilde{b}\nabla u \cdot \nabla_{\tau}u + D^2u\nabla\tilde{b} \cdot \nabla_{\tau}u \\ &= - \int_{\partial\Omega} D^2\tilde{b}\nabla_{\tau}u \cdot \nabla_{\tau}u - \int_{\partial\Omega} \partial_n u D^2\tilde{b} \cdot n \cdot \nabla_{\tau}u \\ &\quad - \int_{\partial\Omega} D^2u\nabla\tilde{b} \cdot \nabla_{\tau}u. \end{aligned} \quad (5.15)$$

On the other hand, referring to (5.12), we get by expressing  $\nabla u$  as  $\nabla_{\tau}u + \partial_n un$ :

$$\begin{aligned} \int_{\Omega} D^2\tilde{b}\nabla u \cdot \nabla u + \int_{\Omega} D^2u\nabla\tilde{b} \cdot \nabla u - \lambda \int_{\Omega} u (\nabla\tilde{b} \cdot \nabla u) &= \int_{\partial\Omega} \partial_n u (\nabla\tilde{b} \cdot \nabla u) \\ &= \int_{\partial\Omega} \partial_n u (\nabla\tilde{b} \cdot [\nabla_{\tau}u + \partial_n un]) \\ &= \int_{\partial\Omega} \partial_n u (\nabla\tilde{b} \cdot \nabla_{\tau}u) + \int_{\partial\Omega} |\partial_n u|^2 (\nabla\tilde{b} \cdot n). \end{aligned} \quad (5.16)$$

Combining between (5.15) and (5.16), we obtain

$$\begin{aligned} \int_{\partial\Omega} D^2\tilde{b}\nabla_\tau u \cdot \nabla_\tau u + \int_{\partial\Omega} \partial_n u D^2\tilde{b} n \cdot \nabla_\tau u + \int_{\partial\Omega} D^2 u \nabla\tilde{b} \cdot \nabla_\tau u + \int_{\partial\Omega} \partial_n u (\nabla\tilde{b} \cdot \nabla_\tau u) \\ + \int_{\partial\Omega} |\partial_n u|^2 (\nabla\tilde{b} \cdot n) = 0. \end{aligned} \quad (5.17)$$

As  $\tilde{b} = b$  on  $\partial\Omega$ , we deduce the following equation

$$\begin{aligned} \int_{\partial\Omega} D^2 b \nabla_\tau u \cdot \nabla_\tau u + \int_{\partial\Omega} \partial_n u D^2 b n \cdot \nabla_\tau u + \int_{\partial\Omega} D^2 u \nabla b \cdot \nabla_\tau u + \int_{\partial\Omega} \partial_n u (\nabla b \cdot \nabla_\tau u) \\ + \int_{\partial\Omega} |\partial_n u|^2 (\nabla b \cdot n) = 0. \end{aligned} \quad (5.18)$$

Let us remark that the vector  $D^2 b n = Dn \cdot n \equiv 0$  on  $\partial\Omega$  since  $n$  is unitary and  $Dn$  is symmetric (see [2, p.223]). Also since  $\nabla b = n$  on  $\partial\Omega$ , and as  $\nabla_\tau u$  is orthogonal to  $n$ , we obtain

$$\int_{\partial\Omega} D^2 b \nabla_\tau u \cdot \nabla_\tau u + \int_{\partial\Omega} D^2 u n \cdot \nabla_\tau u + \int_{\partial\Omega} |\partial_n u|^2 = 0. \quad (5.19)$$

### 5.3.2 Proof of Corollary 1.1

To start the proof, Let  $X \in W^{3,\infty}(\Omega, \mathbb{R}^2)$  be deformation field defined in Theorem 8. We are going to study both cases (simple and multiple cases).

1. **Simple case.** If  $\lambda$  is simple, then we will use the equation (1.10) in Theorem 4:

$$\lambda'(0) = \int_{\partial\Omega} V_n [|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda|u|^2 + (HId - 2D^2 b) \nabla_\tau u \cdot \nabla_\tau u].$$

2. **Multiple case.** If  $\lambda$  is of multiplicity  $h$ , then let us define the real numbers  $\varphi_{ij}$ ,  $i, j = 1, \dots, h$ , by

$$\varphi_{ij} := \int_{\partial\Omega} V_n \nabla_\tau u_i \nabla_\tau u_j - V_n \lambda u_i u_j - V_n \partial_n u_i \partial_n u_j + V_n (HId - 2D^2 b) \nabla_\tau u_i \cdot \nabla_\tau u_j.$$

Now, as the  $h \times h$  matrix  $M$  defined by  $(\varphi_{ij})_{i,j}$  is symmetric, then  $M$  is diagonalizable with a single eigenvalue  $\lambda'(0)$ . Therefore, we have

$$M = \lambda'(0)Id.$$

So we get:

$$\varphi_{ii} = \lambda'(0) \quad \text{for } 1 \leq i \leq h. \quad (5.20)$$

Hence, we obtain

$$\lambda'(0) = \int_{\partial\Omega} V_n [|\nabla_\tau u_i|^2 - |\partial_n u_i|^2 - \lambda|u_i|^2 + (HId - 2D^2b)\nabla_\tau u_i \cdot \nabla_\tau u_i], \quad i = 1, \dots, h. \quad (5.21)$$

In our case, the domain  $\Omega$  is a ball  $B$ , hence we have  $X.n = 1$ . Since  $X = (X.n)n$  on  $\partial B$ , then  $(D^2u + \partial_n u Id)X \cdot \nabla_\tau u = [(D^2u + \partial_n u Id)X.n]n \cdot \nabla_\tau u$  is zero due to orthogonality of the vector  $n$  to  $\nabla_\tau u$ . Thus, we get by (1.12)

$$\nabla_\tau u = 0 \text{ and } \partial_n u = 0 \text{ on } \partial B. \quad (5.22)$$

**First case.** If we suppose that  $B$  is a critical shape, without volume constraint, then we have:

$$\lambda'(0) = 0,$$

that is: for all  $V$  we have

$$\lambda|u|^2 = 0, \quad \text{on } \partial B.$$

Since  $\lambda > 0$ , we have for all  $V$ :

$$u = 0, \quad \text{on } \partial B.$$

By Holmgren's Theorem, we deduce that  $u$  is null in a neighborhood of  $\partial B$ . Thus, by analyticity, we get  $u = 0$  in the whole domain  $B$ , which is a contradiction since  $u$  is an eigenfunction of the problem (1.1).

**Second case.** On the other hand, if there is a volume constraint  $Vol(B) = \text{constant}$ , then there exists a Lagrangian constant  $\mu$  such that

$$\lambda'(0) = \mu Vol'(B). \quad (5.23)$$

So we have

$$\int_{\partial B} V_n [|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda|u|^2 + (HId - 2D^2b)\nabla_\tau u \cdot \nabla_\tau u] = \mu \int_{\partial B} V_n. \quad (5.24)$$

Therefore, by (5.22), we get for all  $V$

$$\lambda|u|^2 = \mu, \quad \text{on } \partial B.$$

Therefore, we deduce that  $u$  is constant on  $\partial B$ , which means that  $u$  is so in a neighborhood of  $\partial B$ , thanks to the Holmgren's Theorem. Thus, by analyticity, the eigenfunction  $u$  is constant in the whole domain  $B$ , which is impossible.

### 5.3.3 Proof of Theorem 9

Let us fix a  $C^2$  open bounded **critical** domain  $\Omega_0$  in  $\mathbb{R}^2$  such that there are some points on the boundary  $\partial\Omega_0$  where the curvature  $H$  is 2 and others where  $H \notin \{0, 2, \frac{2}{3}\}$ . Note that the branches  $t \mapsto \lambda(t)$  are those obtained in Theorem 12, which are analytic in the sense of Hadamard. Let  $u$  be the eigenfunction associated to  $\lambda$ .

In the following of this section and in order to avoid confusion, we will denote by  $\lambda(t)$  to be the shape functional  $\lambda(\Omega_t)$ , and by  $\lambda'(t)$  as the shape derivative  $\lambda'(\Omega_t, V)$  of  $\lambda(\Omega_t) = \lambda(\Omega_0 + tV)$ , at time  $t$  in the direction  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$ .

We want to prove that there exists  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$  and there exist a small real number  $\tau > 0$ , such that  $\lambda'(t) \neq 0$ , for  $0 < t < \tau$ .

We proceed by contradiction. We assume that, for all  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$ , we have

$$\lambda'(t) = 0, \quad 0 \leq t < \tau. \quad (5.25)$$

In the following, we will divide our proof into nine steps. Note that in the first step, we will divide our work into two cases: the first is for the simple eigenvalues and the second is for multiple ones. While in the other steps, the proof can be valid for both cases.

#### Step 1. (First order differentiation formula.)

We have two cases:

1. **Simple case.** If  $\lambda$  is simple, then from (5.25) we have for  $0 \leq t < \tau$ :

$$\int_{\partial\Omega_t} V_n (|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda|u|^2 + \nabla_\tau u^T (HId - 2D^2b) \nabla_\tau u) = 0. \quad (5.26)$$

Since we are in dimension two, that is  $\mathcal{A} := HId - 2D^2b = H - 2H = -H$ , we deduce that

$$\int_{\partial\Omega_t} V_n ((1 - H)|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda|u|^2) = 0. \quad (5.27)$$

Since (5.27) holds for all  $V$ , we have for all  $t$

$$(1 - H)|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda|u|^2 = 0 \quad \text{on } \partial\Omega_t. \quad (5.28)$$

Let us emphasize that (5.28) does not depend on  $V$ .

2. **Multiple case.** If  $\lambda$  is multiple of multiplicity  $m$ , then we have:

$$\lambda'(t) = \lambda'_i(t) = \lambda'_j(t) = 0, \quad i, j = 1, \dots, m, \quad 0 \leq t < \tau. \quad (5.29)$$

Let us define the real numbers  $\varphi_{ij}$  defined on  $\partial\Omega_t$ ,  $i, j = 1, \dots, m$  by

$$\varphi_{ij} := \int_{\partial\Omega_t} V_n \nabla_\tau u_i \nabla_\tau u_j - \lambda V_n u_i u_j - V_n \partial_n u_i \partial_n u_j + V_n \nabla_\tau u_i^T (H I_d - 2D^2 b) \nabla_\tau u_j.$$

Now, as the  $m \times m$  matrix  $M$  defined by  $(\varphi_{ij})_{i,j}$  is symmetric, then  $M$  is diagonalizable with a single eigenvalue  $\lambda'(t)$ . Therefore, we have

$$M = \lambda'(t) Id.$$

So we have:

$$\varphi_{ii} = \lambda'(t) \quad \text{for } 1 \leq i \leq m. \quad (5.30)$$

Hence, as we are in dimension two, we get

$$0 = \lambda'(t) = \int_{\partial\Omega_t} V_n ((1-H)|\nabla_\tau u_i|^2 - |\partial_n u_i|^2 - \lambda|u_i|^2), \quad i = 1, \dots, m. \quad (5.31)$$

Then for all  $V$ , we have on  $\partial\Omega_t$ ,  $0 \leq t < \tau$ :

$$(1-H)|\nabla_\tau u_i|^2 - |\partial_n u_i|^2 - \lambda|u_i|^2 = 0, \quad i = 1, \dots, m. \quad (5.32)$$

**Remark:** In the following steps, we will get rid of the index  $i = 1, \dots, m$ , since the proof is identical for all cases.

**Step 2. (Second order differentiation formula.)**

We differentiate the equation (5.28) with respect to  $t$ , so that we get on  $\partial\Omega_t$  for  $0 \leq t < \tau$ :

$$2(1-H)(\nabla_\tau u)'.\nabla_\tau u + \Delta_\tau V_n |\nabla_\tau u|^2 - 2\partial_n u (\partial_n u)' - \lambda'|u|^2 - 2\lambda u u' = 0. \quad (5.33)$$

Using (4.17) and (4.18), and since  $H' = -\Delta_\tau V_n$  (see Proposition 7.4 in Appendix C), we obtain on  $\partial\Omega_t$  for  $0 \leq t < \tau$

$$2(1-H)\nabla_\tau u'.\nabla_\tau u + 2(2-H)\partial_n u \nabla_\tau V_n.\nabla_\tau u + \Delta_\tau V_n |\nabla_\tau u|^2 - 2\partial_n u \partial_n u' - \lambda'|u|^2 - 2\lambda u u' = 0. \quad (5.34)$$

**Step 3. (Second order normal component expression.)**

In this step, we will follow the same strategy:

We set set  $J(t)$ , as:

$$J(t) := (1-H)|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda|u|^2. \quad (5.35)$$

Then from equation (5.32), we get  $J(t) = 0$ , on  $\partial\Omega_t$ , and thus we get  $\int_{\partial\Omega_t} J(t) = 0$ .

So we obtain for  $0 \leq t \leq \tau$ :

$$\frac{d}{dt} \left( \int_{\partial\Omega_t} J(t) \right) = 0.$$

By Proposition 7.2 (see Appendix B), we get

$$\int_{\partial\Omega_t} \frac{d}{dt} (J(t)) + \int_{\partial\Omega_t} V_n \partial_n (J(t)) + \int_{\partial\Omega_t} V_n H J(t) = 0.$$

Where the first and third terms will vanish thanks to (5.35) and the shape derivative of (5.35), therefore we deduce that the second term is zero. In other words, after computing the normal derivative of (5.35) in the direction  $V$ , we get:

$$\begin{aligned} \int_{\partial\Omega_t} V_n [2\nabla_\tau(\partial_n u)\nabla_\tau u - 2\nabla_\tau u^T D^2 b \nabla_\tau u + 2|\partial_n u|^2 + 2H|\partial_n u|^2 \\ - 2H\nabla_\tau(\partial_n u)\nabla_\tau u + 2H^2|\nabla_\tau u|^2 - \partial_n H|\nabla_\tau u|^2] = 0. \end{aligned} \quad (5.36)$$

So we deduce that for all  $V$ , we have on  $\partial\Omega_t$ :

$$\begin{aligned} 2\nabla_\tau(\partial_n u)\nabla_\tau u - 2\nabla_\tau u^T D^2 b \nabla_\tau u + 2|\partial_n u|^2 + 2H|\partial_n u|^2 \\ - 2H\nabla_\tau(\partial_n u)\nabla_\tau u + 2H^2|\nabla_\tau u|^2 - \partial_n H|\nabla_\tau u|^2 = 0. \end{aligned} \quad (5.37)$$

#### Step 4. (Third order normal component expression.)

Let us set

$$\mathcal{K}(t) := 2(1-H)\nabla_\tau u' \cdot \nabla_\tau u + 2(2-H)\partial_n u \nabla_\tau V_n \cdot \nabla_\tau u + \Delta_\tau V_n |\nabla_\tau u|^2 - 2\partial_n u \partial_n u' - \lambda'|u|^2 - 2\lambda_i u u'.$$

From equation (5.34), we get  $\mathcal{K}(t) = 0$  on  $\partial\Omega_t$ , for all  $0 \leq t < \tau$ , and thus we get  $\int_{\partial\Omega_t} \mathcal{K}(t) = 0$ .

Therefore, we have

$$\frac{d}{dt} \left( \int_{\partial\Omega_t} \mathcal{K}(t) \right) = 0, \quad 0 \leq t < \tau.$$

By Proposition 7.2 (see Appendix B), we get

$$\int_{\partial\Omega_t} \frac{d}{dt} (\mathcal{K}(t)) + \int_{\partial\Omega_t} V_n \partial_n (\mathcal{K}(t)) + \int_{\partial\Omega_t} V_n H \mathcal{K}(t) = 0,$$

Knowing that  $\mathcal{K}(t) = 0$  and  $\frac{d}{dt} (\mathcal{K}(t)) = 0$  on  $\partial\Omega_t$ , for all  $0 \leq t < \tau$ , we deduce that



$$\int_{\partial\Omega_t} V_n \partial_n(\mathcal{K}(t)) = 0. \quad (5.38)$$

Now, let us compute the normal derivative of  $\mathcal{K}(t)$  in the direction  $V$ . To do so, let us calculate the normal derivative of each term alone. Notice that in  $\mathcal{K}(t)$  we have six terms, thus we have:

**1. First term:**

$$\partial_n[2(1-H)\nabla_\tau u' \cdot \nabla_\tau u] = -2\partial_n H \nabla_\tau u' \cdot \nabla_\tau u + 2(1-H)[\partial_n(\nabla_\tau u' \cdot \nabla_\tau u)].$$

By Propositions 7.3 and 7.5 (see Appendix C), we get

$$\begin{aligned} \partial_n[2(1-H)\nabla_\tau u' \cdot \nabla_\tau u] &= 2H^2 \nabla_\tau u' \cdot \nabla_\tau u + 2(1-H)[\nabla_\tau(\partial_n u') \cdot \nabla_\tau u \\ &\quad + \nabla_\tau u' \cdot \nabla_\tau(\partial_n u) - 2H \nabla_\tau u' \cdot \nabla_\tau u]. \end{aligned} \quad (5.39)$$

**2. Second term:**

$$\begin{aligned} \partial_n[2(2-H)\partial_n u \nabla_\tau V_n \cdot \nabla_\tau u] &= 2(2-H) \frac{\partial^2 u}{\partial n^2} \nabla_\tau V_n \cdot \nabla_\tau u + 2\partial_n(2-H)\partial_n u \nabla_\tau V_n \cdot \nabla_\tau u \\ &\quad + 2(2-H)\partial_n u \partial_n[\nabla_\tau V_n \cdot \nabla_\tau u]. \end{aligned}$$

By Propositions 7.5 and 7.6 (see Appendix C), also knowing that  $\Delta u = -\lambda u$  and  $\Delta_\tau u = \partial_n u$ , we obtain

$$\begin{aligned} \partial_n[2(2-H)\partial_n u \nabla_\tau V_n \cdot \nabla_\tau u] &= 2(2-H)[- \lambda u - (1+H)\partial_n u] \nabla_\tau V_n \cdot \nabla_\tau u \\ &\quad + 2H^2 \partial_n u \nabla_\tau V_n \cdot \nabla_\tau u + 2(2-H)\partial_n u [\nabla_\tau(\partial_n V_n) \cdot \nabla_\tau u \\ &\quad + \nabla_\tau V_n \cdot \nabla_\tau(\partial_n u) - 2H \nabla_\tau V_n \cdot \nabla_\tau u]. \end{aligned} \quad (5.40)$$

**1 Third term:**

$$\partial_n[\Delta_\tau V_n |\nabla_\tau u|^2] = \partial_n(\Delta_\tau V_n) |\nabla_\tau u|^2 + \Delta_\tau V_n \partial_n(|\nabla_\tau u|^2).$$

By Proposition 7.5 (see Appendix C), we get

$$\partial_n[\Delta_\tau V_n |\nabla_\tau u|^2] = \partial_n(\Delta_\tau V_n) |\nabla_\tau u|^2 + \Delta_\tau V_n [2\nabla_\tau(\partial_n u) \cdot \nabla_\tau u - 2H |\nabla_\tau u|^2]. \quad (5.41)$$

**4. Fourth term:**

$$\partial_n[-2\partial_n u \partial_n u'] = -2 \frac{\partial^2 u}{\partial n^2} \partial_n u' - 2\partial_n u \frac{\partial^2 u'}{\partial n^2}.$$

By Proposition 7.6 (see Appendix C), also knowing that  $\Delta u = -\lambda u$ ,  $\Delta_\tau u = \partial_n u$  and  $\Delta u' = -\lambda' u - \lambda u'$ , we obtain

$$\partial_n[-2\partial_n u \partial_n u'] = -2[-\lambda u - (1+H)\partial_n u]\partial_n u' - 2\partial_n u[-\lambda' u - \lambda u' - \Delta_\tau u' - H\partial_n u']. \quad (5.42)$$

5. **Fifth term:**

$$\partial_n[-\lambda'|u|^2] = -2\lambda\partial_n u u. \quad (5.43)$$

6. **Sixth term:**

$$\partial_n[-2\lambda u u'] = -2\lambda\partial_n u u' - 2\lambda\partial_n u' u. \quad (5.44)$$

Now, after gathering these six terms, equation (5.38) will be:

$$\begin{aligned} 0 = & \int_{\partial\Omega_t} [-2HV_n\Delta_\tau V_n + V_n\partial_n(\Delta_\tau V_n)]|\nabla_\tau u|^2 + \int_{\partial\Omega_t} [8H^2 - 10H - 4]V_n\nabla_\tau V_n \cdot \nabla_\tau u \partial_n u \\ & + \int_{\partial\Omega_t} 2(1-H)V_n\nabla_\tau(\partial_n u') \cdot \nabla_\tau u + \int_{\partial\Omega_t} 2(1-H)V_n\nabla_\tau u' \cdot \nabla_\tau(\partial_n u) \\ & - \int_{\partial\Omega_t} 2(2-H)V_n\lambda\nabla_\tau V_n \cdot \nabla_\tau u u + \int_{\partial\Omega_t} 2(2-H)V_n\nabla_\tau(\partial_n V_n) \cdot \nabla_\tau u \partial_n u \\ & + \int_{\partial\Omega_t} 2(2-H)V_n\nabla_\tau V_n \cdot \nabla_\tau(\partial_n u)\partial_n u + \int_{\partial\Omega_t} 2V_n\Delta_\tau V_n\nabla_\tau(\partial_n u) \cdot \nabla_\tau u \\ & + \int_{\partial\Omega_t} 2V_n\Delta_\tau u' \partial_n u + \int_{\partial\Omega_t} 2(1+2H)V_n\partial_n u' \partial_n u + \int_{\partial\Omega_t} [6H^2 - 4H]V_n\nabla_\tau u \cdot \nabla_\tau u'. \end{aligned} \quad (5.45)$$

**Step 5. (Third order differentiation formula.)**

Now, let us differentiate (5.37) in the direction  $V$ , for  $0 \leq t < \tau$ , where we get three terms:

1. **First term:**

$$\begin{aligned} [2(1-H)\nabla_\tau(\partial_n u) \cdot \nabla_\tau u]' &= 2(1-H)'\nabla_\tau(\partial_n u) \cdot \nabla_\tau u \\ &+ 2(1-H)[\nabla_\tau(\partial_n u)]' \cdot \nabla_\tau u \\ &+ 2(1-H)\nabla_\tau(\partial_n u) \cdot [\nabla_\tau u]' \end{aligned} \quad (5.46)$$

Using (4.18), (5.3) and (7.3), as  $\mathbf{n} \cdot \nabla_\tau u = 0$ , we get

$$\begin{aligned}
[2(1-H)\nabla_\tau(\partial_n u) \cdot \nabla_\tau u]' &= 2\Delta_\tau V_n \nabla_\tau(\partial_n u) \cdot \nabla_\tau u \\
&+ 2(1-H)[\nabla_\tau(\partial_n u') - \Delta_\tau V_n \nabla_\tau u] \cdot \nabla_\tau u \\
&+ 2(1-H)[- \nabla_\tau V_n \Delta_\tau u - \lambda \nabla_\tau V_n u] \cdot \nabla_\tau u \\
&- 2(1-H)[(1+H)\nabla_\tau V_n \partial_n u] \cdot \nabla_\tau u \\
&+ 2(1-H)\nabla_\tau(\partial_n u) \cdot [\nabla_\tau u' + \nabla_\tau V_n \partial_n u]
\end{aligned} \tag{5.47}$$

## 2. Second term:

$$\begin{aligned}
[(2H(H-1) - \partial_n H)|\nabla_\tau u|^2]' &= [2H(H-1) - \partial_n H]'|\nabla_\tau u|^2 \\
&+ (2H(H-1) - \partial_n H)[|\nabla_\tau u|^2]'.
\end{aligned} \tag{5.48}$$

Using equations (4.17), (4.18), (7.1) and (7.3), we get

$$\begin{aligned}
[(2H(H-1) - \partial_n H)|\nabla_\tau u|^2]' &= [-2(H-1)\Delta_\tau V_n - 2H\Delta_\tau V_n + \partial_n(\Delta_\tau V_n) \\
&+ \nabla_\tau V_n \cdot \nabla_\tau H]|\nabla_\tau u|^2 \\
&+ 2H(3H-2)\nabla_\tau u \cdot [\nabla_\tau u' + \nabla_\tau V_n \partial_n u].
\end{aligned} \tag{5.49}$$

## 3. Third term:

$$[2(1+H)|\partial_n u|^2]' = [2(1+H)]'|\partial_n u|^2 + 2(1+H)[|\partial_n u|^2]'$$

Using (4.17) and (7.3), we get

$$[2(1+H)|\partial_n u|^2]' = -2\Delta_\tau V_n |\partial_n u|^2 + 4(1+H)\partial_n u [\partial_n u' - \nabla_\tau V_n \cdot \nabla_\tau u]. \tag{5.50}$$

Now, by gathering these three terms, and after integrating on  $\partial\Omega_t$ , for  $0 \leq t < \tau$ , we obtain in direction  $V$ :

$$\begin{aligned}
0 &= \int_{\partial\Omega_t} [-2HV_n \Delta_\tau V_n + V_n \partial_n(\Delta_\tau V_n) + V_n \nabla_\tau V_n \cdot \nabla_\tau H] |\nabla_\tau u|^2 \\
&+ \int_{\partial\Omega_t} [8H^2 - 6H - 8] V_n \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u + \int_{\partial\Omega_t} 2(1-H) V_n \nabla_\tau(\partial_n u') \cdot \nabla_\tau u \\
&+ \int_{\partial\Omega_t} 2(1-H) V_n \nabla_\tau u' \cdot \nabla_\tau(\partial_n u) - \int_{\partial\Omega_t} 2(1-H) V_n \lambda \nabla_\tau V_n \cdot \nabla_\tau u u \\
&+ \int_{\partial\Omega_t} 2(1-H) V_n \nabla_\tau V_n \cdot \nabla_\tau(\partial_n u) \partial_n u + \int_{\partial\Omega_t} 2V_n \Delta_\tau V_n \nabla_\tau(\partial_n u) \cdot \nabla_\tau u \\
&- \int_{\partial\Omega_t} 2V_n \Delta_\tau V_n |\partial_n u|^2 + \int_{\partial\Omega_t} 4(1+H) V_n \partial_n u' \partial_n u + \int_{\partial\Omega_t} [6H^2 - 4H] V_n \nabla_\tau u \cdot \nabla_\tau u'.
\end{aligned} \tag{5.51}$$

**Step 6. (Linking between the last two steps.)**

Now, we subtract (5.51) from (5.45), so that we get:

$$\begin{aligned}
0 &= \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau (\partial_n u) \partial_n u - \int_{\partial\Omega_t} 2V_n \lambda \nabla_\tau V_n \cdot \nabla_\tau u u - \int_{\partial\Omega_t} 2V_n \partial_n u' \partial_n u \quad (5.52) \\
&+ \int_{\partial\Omega_t} 2V_n \Delta_\tau u' \partial_n u - \int_{\partial\Omega_t} V_n (4H - 4) \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u - \int_{\partial\Omega_t} V_n \nabla_\tau H \cdot \nabla_\tau V_n |\nabla_\tau u|^2 \\
&+ \int_{\partial\Omega_t} 2V_n \Delta_\tau V_n |\partial_n u|^2 + \int_{\partial\Omega_t} (4 - 2H) \nabla_\tau (\partial_n V_n) \cdot \nabla_\tau u \partial_n u.
\end{aligned}$$

Multiplying the boundary condition (1.9) by  $\partial_n u V_n$  and integrating on  $\partial\Omega_t$ , we get:

$$\begin{aligned}
- \int_{\partial\Omega_t} 2V_n \partial_n u' \partial_n u &= - \int_{\partial\Omega_t} 2V_n \Delta_\tau u' \partial_n u - \int_{\partial\Omega_t} 2V_n \operatorname{div}_\tau (V_n \nabla_\tau u) \partial_n u \quad (5.53) \\
&- \int_{\partial\Omega_t} 2V_n \Delta_\tau (V_n \partial_n u) \partial_n u + \int_{\partial\Omega_t} 2V_n \operatorname{div}_\tau (V_n H \nabla_\tau u) \partial_n u \\
&- \int_{\partial\Omega_t} 2V_n \lambda V_n \partial_n u u.
\end{aligned}$$

Thus, by (5.53), we obtain

$$\begin{aligned}
0 &= \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau (\partial_n u) \partial_n u - \int_{\partial\Omega_t} 2V_n \lambda \nabla_\tau V_n \cdot \nabla_\tau u u - \int_{\partial\Omega_t} 2V_n \Delta_\tau u' \partial_n u \quad (5.54) \\
&- \int_{\partial\Omega_t} 2V_n \operatorname{div}_\tau (V_n \nabla_\tau u) \partial_n u - \int_{\partial\Omega_t} 2V_n \Delta_\tau (V_n \partial_n u) \partial_n u \\
&+ \int_{\partial\Omega_t} 2V_n \operatorname{div}_\tau (V_n H \nabla_\tau u) \partial_n u - \int_{\partial\Omega_t} 2V_n^2 \lambda \partial_n u u \\
&+ \int_{\partial\Omega_t} 2V_n \Delta_\tau u' \partial_n u - \int_{\partial\Omega_t} V_n (4H - 4) \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u \\
&+ \int_{\partial\Omega_t} 2V_n \Delta_\tau V_n |\partial_n u|^2 - \int_{\partial\Omega_t} V_n \nabla_\tau H \cdot \nabla_\tau V_n |\nabla_\tau u|^2 \\
&+ \int_{\partial\Omega_t} (4 - 2H) \nabla_\tau (\partial_n V_n) \cdot \nabla_\tau u \partial_n u.
\end{aligned}$$

**Step 7. (Making integration by parts).** In this step, we will replace some terms in (5.54) using integration by parts. Let us remark that we have:

$$\begin{aligned}
\int_{\partial\Omega_t} 2\lambda V_n^2 |\nabla_\tau u|^2 + \int_{\partial\Omega_t} 4\lambda V_n \nabla_\tau V_n \cdot \nabla_\tau uu &= \int_{\partial\Omega_t} 2\lambda V_n^2 |\nabla_\tau u|^2 + \int_{\partial\Omega_t} 2\lambda \nabla_\tau (V_n^2) \cdot \nabla_\tau uu \\
&= \int_{\partial\Omega_t} 2\lambda V_n^2 |\nabla_\tau u|^2 - \int_{\partial\Omega_t} 2\lambda V_n^2 \operatorname{div}_\tau (\nabla_\tau uu) \\
&= \int_{\partial\Omega_t} 2\lambda V_n^2 |\nabla_\tau u|^2 - \int_{\partial\Omega_t} 2\lambda V_n^2 \Delta_\tau uu \\
&\quad - \int_{\partial\Omega_t} 2\lambda V_n^2 |\nabla_\tau u|^2 \\
&= - \int_{\partial\Omega_t} 2\lambda V_n^2 \partial_n uu \quad \text{by } \Delta_\tau u = \partial_n u.
\end{aligned} \tag{5.55}$$

$$\begin{aligned}
\int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau (\partial_n u) \partial_n u &= \int_{\partial\Omega_t} 2\nabla_\tau (\partial_n u) \cdot \nabla_\tau V_n V_n \partial_n u = - \int_{\partial\Omega_t} 2\partial_n u \operatorname{div}_\tau (\nabla_\tau V_n V_n \partial_n u) \\
&= - \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau (\partial_n u) \partial_n u - \int_{\partial\Omega_t} 2V_n \Delta_\tau V_n |\partial_n u|^2 \\
&\quad - \int_{\partial\Omega_t} 2|\nabla_\tau V_n|^2 |\partial_n u|^2,
\end{aligned} \tag{5.56}$$

$$\begin{aligned}
- \int_{\partial\Omega_t} 2V_n \operatorname{div}_\tau (V_n \nabla_\tau u) \partial_n u &= \int_{\partial\Omega_t} 2V_n \nabla_\tau u \cdot \nabla_\tau (V_n \partial_n u) \\
&= \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u + \int_{\partial\Omega_t} 2V_n^2 \nabla_\tau u \cdot \nabla_\tau (\partial_n u),
\end{aligned} \tag{5.57}$$

$$\begin{aligned}
- \int_{\partial\Omega_t} 2V_n \Delta_\tau (V_n \partial_n u) \partial_n u &= \int_{\partial\Omega_t} 2\nabla_\tau (V_n \partial_n u) \cdot \nabla_\tau (V_n \partial_n u) \\
&= \int_{\partial\Omega_t} 2|\nabla_\tau V_n|^2 |\partial_n u|^2 + \int_{\partial\Omega_t} 4V_n \nabla_\tau V_n \cdot \nabla_\tau (\partial_n u) \partial_n u \\
&\quad + \int_{\partial\Omega_t} 2V_n^2 |\nabla_\tau (\partial_n u)|^2,
\end{aligned} \tag{5.58}$$

$$\begin{aligned}
\int_{\partial\Omega_t} 2V_n \operatorname{div}_\tau (V_n H \nabla_\tau u) \partial_n u &= - \int_{\partial\Omega_t} 2V_n H \nabla_\tau u \cdot \nabla_\tau (V_n \partial_n u) \\
&= - \int_{\partial\Omega_t} 2V_n H \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u - \int_{\partial\Omega_t} 2H V_n^2 \nabla_\tau u \cdot \nabla_\tau (\partial_n u),
\end{aligned} \tag{5.59}$$

and

$$\begin{aligned}
-\int_{\partial\Omega_t} V_n(4H-4)\nabla_\tau V_n \cdot \nabla_\tau u \partial_n u &= -\int_{\partial\Omega_t} \nabla_\tau V_n \cdot \nabla_\tau u V_n(4H-4)\partial_n u & (5.60) \\
&= \int_{\partial\Omega_t} V_n \operatorname{div}_\tau(V_n(4H-4)\partial_n u \nabla_\tau u) \\
&= \int_{\partial\Omega_t} V_n(4H-4)\nabla_\tau V_n \cdot \nabla_\tau u \partial_n u \\
&+ \int_{\partial\Omega_t} (4H-4)V_n^2 \nabla_\tau u \cdot \nabla_\tau(\partial_n u) \\
&+ \int_{\partial\Omega_t} (4H-4)V_n^2 |\partial_n u|^2 + \int_{\partial\Omega_t} 4V_n^2 \nabla_\tau H \cdot \nabla_\tau u \partial_n u.
\end{aligned}$$

Hence, using (5.55), (5.56), (5.57), (5.58), (5.59) and (5.60), we obtain

$$\begin{aligned}
0 &= -\int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau(\partial_n u) \partial_n u - \int_{\partial\Omega_t} 2V_n \Delta_\tau V_n |\partial_n u|^2 - \int_{\partial\Omega_t} 2|\nabla_\tau V_n|^2 |\partial_n u|^2 & (5.61) \\
&+ \int_{\partial\Omega_t} 2V_n \lambda \nabla_\tau V_n \cdot \nabla_\tau u u + \int_{\partial\Omega_t} 2\lambda V_n^2 \partial_n u u + \int_{\partial\Omega_t} 2\lambda V_n^2 |\nabla_\tau u|^2 - \int_{\partial\Omega_t} 2V_n \Delta_\tau u' \partial_n u \\
&+ \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u + \int_{\partial\Omega_t} 2V_n^2 \nabla_\tau u \cdot \nabla_\tau(\partial_n u) + \int_{\partial\Omega_t} 2|\nabla_\tau V_n|^2 |\partial_n u|^2 \\
&+ \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau(\partial_n u) \partial_n u + \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau(\partial_n u) \partial_n u + \int_{\partial\Omega_t} 2V_n^2 |\nabla_\tau(\partial_n u)|^2 \\
&- \int_{\partial\Omega_t} 2V_n H \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u - \int_{\partial\Omega_t} 2H V_n^2 \nabla_\tau u \cdot \nabla_\tau(\partial_n u) - \int_{\partial\Omega_t} 2\lambda V_n^2 \partial_n u u \\
&+ \int_{\partial\Omega_t} 2V_n \Delta_\tau u' \partial_n u + \int_{\partial\Omega_t} V_n(4H-4)\nabla_\tau V_n \cdot \nabla_\tau u \partial_n u + \int_{\partial\Omega_t} (4H-4)V_n^2 \nabla_\tau u \cdot \nabla_\tau(\partial_n u) \\
&+ \int_{\partial\Omega_t} (4H-4)V_n^2 |\partial_n u|^2 + \int_{\partial\Omega_t} 4V_n^2 \nabla_\tau H \cdot \nabla_\tau u \partial_n u + \int_{\partial\Omega_t} 2V_n \Delta_\tau V_n |\partial_n u|^2 \\
&- \int_{\partial\Omega_t} V_n \nabla_\tau H \cdot \nabla_\tau V_n |\nabla_\tau u|^2 + \int_{\partial\Omega_t} (4-2H)\nabla_\tau(\partial_n V_n) \cdot \nabla_\tau u \partial_n u.
\end{aligned}$$

After cancellation of some terms, we get

$$\begin{aligned}
0 &= \int_{\partial\Omega_t} 2\lambda V_n^2 |\nabla_\tau u|^2 + \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau u u + \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau (\partial_n u) \partial_n u \quad (5.62) \\
&+ \int_{\partial\Omega_t} 2V_n^2 |\nabla_\tau (\partial_n u)|^2 + \int_{\partial\Omega_t} V_n (2H - 2) \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u \\
&+ \int_{\partial\Omega_t} (2H - 2) V_n^2 \nabla_\tau u \cdot \nabla_\tau (\partial_n u) + \int_{\partial\Omega_t} (4H - 4) V_n^2 |\partial_n u|^2 + \int_{\partial\Omega_t} 4V_n^2 \nabla_\tau H \cdot \nabla_\tau u \partial_n u \\
&- \int_{\partial\Omega_t} V_n \nabla_\tau H \cdot \nabla_\tau V_n |\nabla_\tau u|^2 + \int_{\partial\Omega_t} (4 - 2H) \nabla_\tau (\partial_n V_n) \cdot \nabla_\tau u \partial_n u.
\end{aligned}$$

**Step 8. (Reformulation of (5.62).)** Let us reformulate (5.62) by isolating the terms containing  $V_n$ .

By integration by parts, we have

$$\int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau (\partial_n u) \partial_n u = \int_{\partial\Omega_t} \nabla_\tau (V_n^2) \cdot \nabla_\tau (\partial_n u) \partial_n u = - \int_{\partial\Omega_t} V_n^2 \operatorname{div}_\tau (\nabla_\tau (\partial_n u) \partial_n u). \quad (5.63)$$

$$\begin{aligned}
\int_{\partial\Omega_t} V_n (2H - 2) \nabla_\tau V_n \cdot \nabla_\tau u \partial_n u &= \int_{\partial\Omega_t} \nabla_\tau (V_n^2) \cdot \nabla_\tau u (H - 1) \partial_n u \quad (5.64) \\
&= - \int_{\partial\Omega_t} V_n^2 \operatorname{div}_\tau ((H - 1) \nabla_\tau u \partial_n u).
\end{aligned}$$

$$- \int_{\partial\Omega_t} V_n \nabla_\tau H \cdot \nabla_\tau V_n |\nabla_\tau u|^2 = - \int_{\partial\Omega_t} \frac{1}{2} \nabla_\tau (V_n^2) \cdot \nabla_\tau H |\nabla_\tau u|^2 = \int_{\partial\Omega_t} \frac{1}{2} V_n^2 \operatorname{div}_\tau (\nabla_\tau H |\nabla_\tau u|^2). \quad (5.65)$$

and

$$\int_{\partial\Omega_t} (4 - 2H) \nabla_\tau (\partial_n V_n) \cdot \nabla_\tau u \partial_n u = - \int_{\partial\Omega_t} \partial_n V_n \operatorname{div}_\tau ((4 - 2H) \nabla_\tau u \partial_n u) \quad (5.66)$$

Hence, replacing some terms using (5.55), (5.63), (5.64), (5.65) and (5.66), we get

$$\begin{aligned}
0 = & - \int_{\partial\Omega_t} \lambda V_n^2 \partial_n u u - \int_{\partial\Omega_t} \lambda V_n^2 |\nabla_\tau u|^2 - \int_{\partial\Omega_t} V_n^2 \operatorname{div}_\tau(\partial_n u \nabla_\tau(\partial_n u)) \\
& + \int_{\partial\Omega_t} 2V_n^2 |\nabla_\tau(\partial_n u)|^2 - \int_{\partial\Omega_t} V_n^2 \operatorname{div}_\tau((H-1)\partial_n u \nabla_\tau u) + \int_{\partial\Omega_t} (2H-2)V_n^2 \nabla_\tau u \cdot \nabla_\tau(\partial_n u) \\
& + \int_{\partial\Omega_t} (4H-4)V_n^2 |\partial_n u|^2 + \int_{\partial\Omega_t} 4V_n^2 \nabla_\tau H \cdot \nabla_\tau u \partial_n u + \int_{\partial\Omega_t} \frac{1}{2} V_n^2 \operatorname{div}_\tau(|\nabla_\tau u|^2 \nabla_\tau H) \\
& - \int_{\partial\Omega_t} \partial_n V_n \operatorname{div}_\tau[(4-2H)\partial_n u \nabla_\tau u].
\end{aligned} \tag{5.67}$$

**Step 9. (Finding a contradiction.)** In this last step, we are aiming to get a contradiction.

Let us remark that equation (5.67) is valid for any deformation field  $V$  at any time  $0 \leq t < \tau$ , hence, in particular, it is valid at  $t = 0$ . Let us set

$$\begin{aligned}
I(V) := & - \int_{\partial\Omega_0} \lambda V_n^2 \partial_n u u - \int_{\partial\Omega_0} \lambda V_n^2 |\nabla_\tau u|^2 - \int_{\partial\Omega_0} V_n^2 \operatorname{div}_\tau(\partial_n u \nabla_\tau(\partial_n u)) \\
& + \int_{\partial\Omega_0} 2V_n^2 |\nabla_\tau(\partial_n u)|^2 - \int_{\partial\Omega_0} V_n^2 \operatorname{div}_\tau((H-1)\partial_n u \nabla_\tau u) + \int_{\partial\Omega_0} (2H-2)V_n^2 \nabla_\tau u \cdot \nabla_\tau(\partial_n u) \\
& + \int_{\partial\Omega_0} (4H-4)V_n^2 |\partial_n u|^2 + \int_{\partial\Omega_0} 4V_n^2 \nabla_\tau H \cdot \nabla_\tau u \partial_n u + \int_{\partial\Omega_0} \frac{1}{2} V_n^2 \operatorname{div}_\tau(|\nabla_\tau u|^2 \nabla_\tau H)
\end{aligned} \tag{5.68}$$

and

$$J(V) := - \int_{\partial\Omega_0} \partial_n V_n \operatorname{div}_\tau[(4-2H)\partial_n u \nabla_\tau u]. \tag{5.69}$$

Now, if  $V^1$  and  $V^2$  are two given vector fields at time  $t = 0$  such that  $V_n^1 = V_n^2$  and  $\partial_n V_n^1 \neq \partial_n V_n^2$  on  $\partial\Omega_0$ , then we get

$$I(V^1) = I(V^2), \tag{5.70}$$

thus we obtain

$$I(V^1) = I(V^2). \tag{5.71}$$

But, we know from (5.67) that

$$(I+J)(V^1) = 0 = (I+J)(V^2), \tag{5.72}$$



then we deduce that

$$J(V^1) = J(V^2). \quad (5.73)$$

Therefore,  $J$  doesn't depend on  $\partial_n V_n$ , whatever  $V$  is chosen at time  $t = 0$ . But, as  $J$  is linear in  $\partial_n V_n$ , we deduce that  $J$  is linear and constant, and hence  $J$  is null, i.e we have:

$$\int_{\partial\Omega_0} \partial_n V_n \operatorname{div}_\tau[(4 - 2H)\partial_n u \nabla_\tau u] = 0. \quad (5.74)$$

Hence, for all  $V$  we have

$$\operatorname{div}_\tau[(4 - 2H)\partial_n u \nabla_\tau u] = 0 \quad \text{on } \partial\Omega_0. \quad (5.75)$$

Let us remark that since we are in dimension two, we set the following parameters:

$$s := \operatorname{arclength}(\partial\Omega_0), \text{ and } r := \operatorname{dist}(y, \partial\Omega_0), \quad \forall y \in \mathbb{R}^2,$$

so that we have  $\partial_n u = \partial_r u$ . Therefore, we have

$$\partial_s((4 - 2H)\partial_n u \partial_s u) = 0. \quad (5.76)$$

Then we get

$$(4 - 2H)\partial_n u \partial_s u = C_1 \quad (5.77)$$

where  $C_1$  is a constant real number on  $\partial\Omega_0$ . We remark that equation (5.77) is valid for  $H = 2$  (by the assumption of  $\partial\Omega_0$ ), therefore we deduce the constant  $C_1$  is zero on the whole boundary. Hence, there exist some subsets  $\Gamma$  of  $\partial\Omega_0$  with  $H \neq 2$  such that

$$\partial_n u \partial_s u = 0 \quad \text{on } \Gamma. \quad (5.78)$$

Now, if  $\partial_s u = 0$  on an interval  $\mathcal{U}$  of  $\Gamma$ , then restricting equation (5.28) on  $\mathcal{U}$ , that is to say

$$(1 - H)(\partial_s u)^2 - (\partial_n u)^2 - \lambda u^2 = 0 \quad \text{on } \mathcal{U}, \quad (5.79)$$

we deduce that  $\partial_n u$  and  $u$  are null on  $\mathcal{U}$ , as  $\lambda > 0$ . Therefore, by Holmgren's uniqueness theorem, we get that  $u$  is zero in a neighborhood of  $\mathcal{U}$ , and then in the whole domain  $\Omega_0$  by analyticity, which is impossible since  $u$  is an eigenfunction of (1.1).

On the other side, suppose that  $\partial_n u = 0$  on an interval  $\mathcal{V}$  of  $\Gamma$ . Applying equation (5.37) on  $\mathcal{V}$ , knowing that  $\partial_n H = -H^2$  (see Proposition 7.3), we obtain

$$2\partial_s(\partial_n u)\partial_s u - 2H(\partial_s u)^2 + 2(1+H)(\partial_n u)^2 - 2H\partial_s(\partial_n u)\partial_s u + 3H^2(\partial_s u)^2 = 0 \quad \text{on } \mathcal{V}. \quad (5.80)$$

Hence, we get  $H(3H - 2)(\partial_s u)^2 = 0$  on  $\mathcal{V}$ . Again, by the assumption, there exist some intervals  $\gamma$  in  $\mathcal{V}$  such that  $H \notin \{0, \frac{2}{3}\}$ ; so that we get  $\partial_s u = 0$  on  $\gamma$ , which is impossible by the above argument.

Theorem 9 is finally proved.



## Chapter 6

# Simplicity and generic simplicity

In the first section, we restate the open problems of this thesis. In the second section, we give some prerequisites that will be used in the last section, in which we give the strategies for the two open problems, in two subsections.

In the first subsection, we try to solve Open problem 1, that is showing that any shape in two-dimensional space holding a multiple eigenvalue in a certain time can be deformed in a way so that this eigenvalue will split in the deformed shape. Actually, we start by contradiction. We fix a  $C^2$  open bounded domain  $\Omega_0$  in  $\mathbb{R}^2$  such that there are some points on the boundary  $\partial\Omega_0$  where the curvature  $H$  is 2 and also some other points where  $H \neq 2$ , and we assume that  $\lambda_i = \lambda_j = \lambda$  is a multiple eigenvalue, of multiplicity  $h$ , in  $\Omega_0$  at  $t = 0$ . After that, we assume that there exists a small period  $\tau$  in which this eigenvalue will remain multiple in the new domain, for  $0 \leq t < \tau$ , and for all deformation vector fields  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$ .

Also in this theorem, the second-order derivative couldn't lead to any contradiction. Therefore, in order to approach much more and facilitate things, we need to progress to the third-order derivative calculations, but still couldn't find any contradiction.

In the second subsection, we move on to Open problem 2, so we study the generic simplicity of the eigenvalues of the Wentzell-Laplace boundary problem, in two-dimensional space. Actually, we almost use the same strategy applied in subsection 6.3.1, in which we tried to precede by supposing the contrast of the argument in order to find a contradiction.

In fact, we follow the classical strategy of Y. Chitour, D. Kateb, R. Long in [14] for the Laplace operator with Dirichlet boundary conditions, in which they have proved some generic properties of the spectrum of the Stokes operator. This strategy was in particular applied successfully in [15] for the generic simplicity of the Stokes operator in two space dimensions. In the following, we are going to introduce the main concepts that they have used. For that, fix a domain  $\Omega_0 \in \mathbb{D}_2^2$ , and we define the set  $A_0 := \mathbb{D}_2^2(\Omega_0)$  (for more details, refer to the introduction of the thesis).

We fix a domain  $\Omega_0$  and let us introduce the following sets: for  $n \geq 1$ , the set  $A_n$  contains all the images  $\Omega$  by the map  $Id + U$  of the fixed domain  $\Omega_0$  so that we have  $\Omega = \Omega_0 + U$ ,

$U \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$ ,  $\Omega_0 + U \in A_0$  and the first  $n$  eigenvalues of (1.1) on  $\Omega_0 + U$  are simple. Also, we set  $A$  as the countable intersection of all  $A_n$ ,  $n \in \mathbb{N}$ . We note that set  $A$  will be the set of the images  $\Omega \in A_0$ , such that all the eigenvalues of (1.1) on  $\Omega$  are simple. Then, we apply Baire's lemma on the sets  $A_n \subset A_0$ . We claim that it is sufficient to show that the sets  $A_n$  are open in  $A_0$  and that  $A_{n+1}$  is dense in  $A_n$  for all  $n \geq 0$ . After that, if the latter one is true, then we can prove the following: if  $A_{n+1}$  is dense in  $A_n$  and  $A_n$  is dense in  $A_0$ , then  $A_{n+1}$  is also dense in  $A_0$ . Finally, applying Baire's lemma to the sets  $A_n$ , we deduce that  $A = \bigcap_n A_n$  is dense in  $A_0$ .

But the question relies on the density: could we prove that the set  $A_{n+1}$  is dense in  $A_n$  for all  $n \geq 0$ ?

## 6.1 Statement of the open problems

Let us recall the two open problems here:

**Open problem 1 (*Simplicity of an eigenvalue*).** *Let  $\Omega_0$  be a  $C^2$  open bounded domain in  $\mathbb{R}^2$  such that there exist some points on the boundary  $\partial\Omega_0$  where the curvature  $H$  is 2 and also some other points where  $H \neq 2$ . Let  $\lambda_i = \lambda_j = \lambda$  be a multiple eigenvalue in  $\Omega_0$  at  $t = 0$ . Then there is a deformation vector field  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$  such that for all  $t > 0$ , the eigenvalues  $\lambda_i$  and  $\lambda_j$  are distinct in  $\Omega_t = \Omega_0 + tV$ .*

**Open problem 2 (*Generic simplicity of eigenvalues*).** *Generically with respect to domains  $\Omega \in \mathbb{D}_2^d$ , all eigenvalues of (1.1) are simple.*

## 6.2 Prerequisites

Now, we make precise the notion of genericity we use, in the following definition:

**Definition 6.1** *A property (P) is generic in  $\mathbb{D}_2^d$  if, for every  $\Omega \in \mathbb{D}_2^d$ , the set of domains of  $\mathbb{D}_2^d(\Omega)$  on which Property (P) holds true is residual i.e., contains a countable intersection of open and dense subsets of  $\mathbb{D}_2^d(\Omega)$ .*

◇

Actually, concerning the second open problem, we need to give the following two lemmas (that are essential for the genericity problem):

**Lemma 6.1 (*Baire's lemma*)** *Let  $X$  be a complete metric space and  $A_n$  be an open dense subset of  $X$  for all  $n \in \mathbb{N}$ . Then  $\bigcap_{n \in \mathbb{N}} A_n$  is dense in  $X$ .*

A direct consequence of the Baire's lemma is the following result, which is stated in [15].

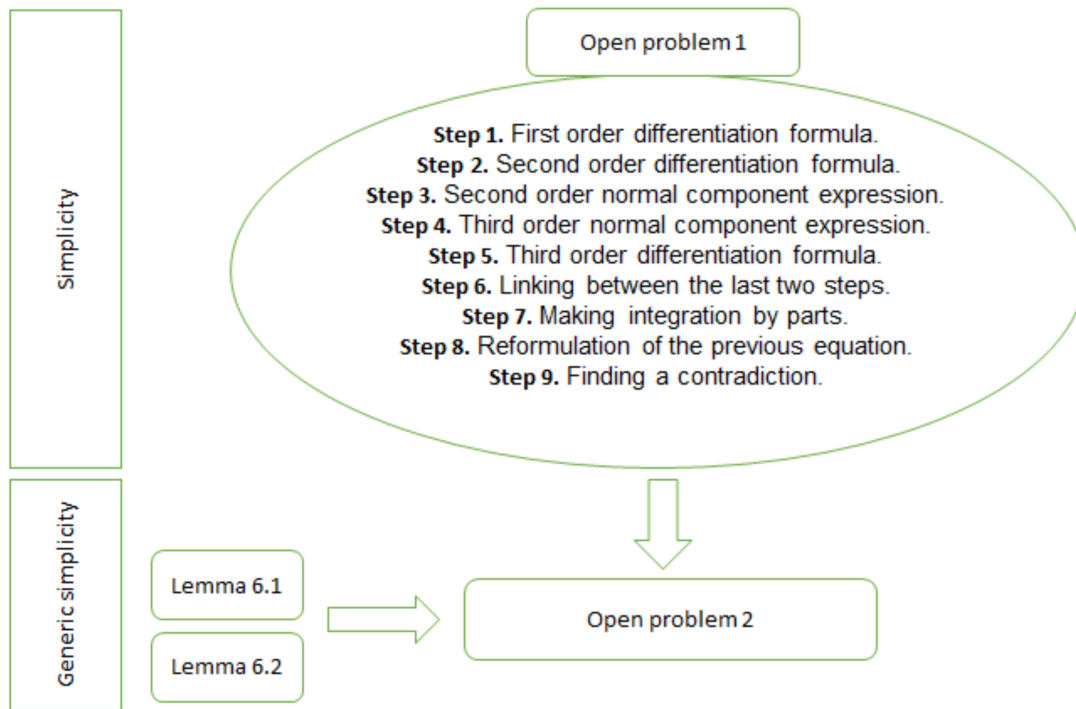


Figure 6.1: The plan of Chapter six

**Lemma 6.2** *Let  $X$  be a complete metric space and  $(A_n)_{n \geq 0}$  be a sequence of open subsets of  $X$  such that*

1.  $A_0 = X$ .
2.  $A_{n+1}$  is a dense subset of  $A_n$ , for all  $n \geq 0$ .

*Then  $\bigcap_{n=1}^{\infty} A_n$  is dense in  $X$ .*

In Figure 6.1, the plan of Chapter six is shown.

### 6.3 The strategies

#### 6.3.1 Strategy for Open problem 1

Let  $\Omega_0$  be a  $C^2$  open bounded domain in  $\mathbb{R}^2$  such that there are some points on the boundary  $\partial\Omega_0$  where the curvature  $H$  is 2 and also some other points where  $H \neq 2$ . Let  $\lambda_i = \lambda_j = \lambda$  be a

multiple eigenvalue, of multiplicity  $h$ , in  $\Omega_0$  at  $t = 0$ , and let  $u_1, u_2, \dots, u_h$  be the eigenfunctions associated to  $\lambda$ .

Note that the branches  $t \mapsto \lambda_i(t)$  are those obtained in Theorem 12, which are analytic in the sense of Hadamard. Therefore the eigenvalues  $\lambda_i$  are not necessarily ordered in an increasing way. In the following of this section and in order to avoid confusion, we will denote by  $\lambda(t)$  to be the shape functional  $\lambda(\Omega_t)$ , and by  $\lambda'(t)$  as the shape derivative  $\lambda'(\Omega_t, V)$  of  $\lambda(\Omega_t) = \lambda(\Omega_0 + tV)$ , at time  $t$  in the direction  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$ .

We want to prove that there exists  $V \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$  such that  $\lambda'_i(0) \neq \lambda'_j(0)$ , for each  $i \neq j$ ,  $i, j = 1, \dots, h$ . That will immediately imply that there exist a small enough real number  $\tau > 0$ , such that  $\Omega_t = \Omega_0 + tV$  for  $0 \leq t < \tau$ , and

$$\lambda_i(t) \neq \lambda_j(t), \quad \forall i, j = 1, \dots, h, \quad 0 \leq t < \tau.$$

To do so, we proceed by contradiction. If there exist  $i \neq j$ ,  $i, j = 1, \dots, h$  such that  $\lambda_i(t) = \lambda_j(t) = \lambda(t)$ , for all  $V \in W^{3,\infty}(\Omega, \mathbb{R}^2)$  in a neighborhood of  $t = 0$  (i.e. for  $0 \leq t < \tau$ ), then we have:

$$\lambda'_i(t) = \lambda'_j(t) = \lambda'(t), \quad \forall i, j = 1, \dots, h, \quad 0 \leq t < \tau. \quad (6.1)$$

We continue by dividing our strategy into nine steps. In the following, we almost follow the same strategy applied in the previous theorem, except for the last step.

**Step 1. (First order differentiation formula.)**

Let us define the real numbers  $\varphi_{ij}$  defined on  $\partial\Omega_t$ ,  $i, j = 1, \dots, h$ , by

$$\varphi_{ij} := \int_{\partial\Omega_t} V_n(1-H)\nabla_\tau u_i \nabla_\tau u_j - V_n \lambda u_i u_j - V_n \partial_n u_i \partial_n u_j.$$

Now, as the  $h \times h$  matrix  $M$  defined by  $(\varphi_{ij})_{i,j}$  is symmetric, then  $M$  is diagonalizable with a single eigenvalue  $\lambda'(t)$ . Therefore, we have

$$M = \lambda'(t)Id.$$

So we have:

$$\varphi_{ii} = \lambda'(t) \quad \text{for } 1 \leq i \leq h, \quad 0 \leq t < \tau. \quad (6.2)$$

Hence, we have for all  $V$ , with  $i, j = 1, \dots, h$  and  $0 \leq t < \tau$ :

$$(1-H)|\nabla_\tau u_i|^2 - |\partial_n u_i|^2 - \lambda|u_i|^2 = (1-H)|\nabla_\tau u_j|^2 - |\partial_n u_j|^2 - \lambda|u_j|^2. \quad (6.3)$$

Let us emphasize that (6.3) does not depend on  $V$ .

**Step 2. (Second order differentiation formula.)**

By making the shape derivatives of (6.3), after noticing that in dimension two we have  $H' = -\Delta_\tau V_n$ , we get on  $\partial\Omega_t$  for  $0 \leq t < \tau$ ,  $i, j = 1, \dots, h$ :

$$\begin{aligned} 0 = & 2(1 - H)[\nabla_\tau u'_i \cdot \nabla_\tau u_i - \nabla_\tau u'_j \cdot \nabla_\tau u_j] + (4 - 2H)\nabla_\tau V_n \cdot [\partial_n u_i \nabla_\tau u_i - \partial_n u_j \nabla_\tau u_j] \\ & - 2[\partial_n u_i \partial_n u'_i - \partial_n u_j \partial_n u'_j] - \lambda[|u_i|^2 - |u_j|^2] \\ & - 2\lambda[u_i u'_i - u_j u'_j] + \Delta_\tau V_n [|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2]. \end{aligned} \quad (6.4)$$

**Step 3. (Second order normal component expression.)**

Let us set  $\mathcal{J}_{ij}(t)$ ,  $i, j = 1, \dots, h$ , as :

$$\mathcal{J}_{ij}(t) := (1 - H)[|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2] - [|\partial_n u_i|^2 - |\partial_n u_j|^2] - \lambda[|u_i|^2 - |u_j|^2]. \quad (6.5)$$

From equation (6.3), we get  $\mathcal{J}_{ij}(t) = 0$ ,  $i, j = 1, \dots, h$ , on  $\partial\Omega_t$ , and thus we get  $\int_{\partial\Omega_t} \mathcal{J}_{ij} = 0$ ,  $i, j = 1, \dots, h$ .

So we obtain for  $0 \leq t < \tau$ :

$$\frac{d}{dt} \left( \int_{\partial\Omega_t} \mathcal{J}_{ij} \right) = 0, \quad i, j = 1, \dots, h.$$

By Proposition 7.2 (see Appendix B), we get

$$\int_{\partial\Omega_t} \frac{d}{dt} (\mathcal{J}_{ij}) + \int_{\partial\Omega_t} V_n \partial_n (\mathcal{J}_{ij}) + \int_{\partial\Omega_t} V_n H \mathcal{J}_{ij} = 0, \quad i, j = 1, \dots, h,$$

where the first and third terms will vanish thanks to (6.5) and the shape derivative of (6.5), therefore we deduce that the second term is zero. In other words, after computing the normal derivative of (6.5) in the direction  $V$  (for details about the computations, one can refer to the proof of Theorem 6), we get for all  $V$ :

$$\begin{aligned} & 2(1 - H)[\nabla_\tau (\partial_n u_i) \cdot \nabla_\tau u_i - \nabla_\tau (\partial_n u_j) \cdot \nabla_\tau u_j] + (2H(H - 1) - \partial_n H)[|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2] \\ & + 2(1 + H)[|\partial_n u_i|^2 - |\partial_n u_j|^2] = 0, \quad i, j = 1, \dots, h. \end{aligned} \quad (6.6)$$

**Step 4. (Third order normal component expression.)**

Let us set



$$\begin{aligned}
\mathcal{K}_{ij}(t) &:= 2(1 - H)[\nabla_\tau u'_i \cdot \nabla_\tau u_i - \nabla_\tau u'_j \cdot \nabla_\tau u_j] \\
&\quad + (4 - 2H)\nabla_\tau V_n \cdot [\partial_n u_i \nabla_\tau u_i - \partial_n u_j \nabla_\tau u_j] \\
&\quad - 2[\partial_n u_i \partial_n u'_i - \partial_n u_j \partial_n u'_j] - \lambda' [|u_i|^2 - |u_j|^2] - 2\lambda [u_i u'_i - u_j u'_j] \\
&\quad + \Delta_\tau V_n [|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2].
\end{aligned} \tag{6.7}$$

From equation (6.4), we get  $\mathcal{K}_{ij}(t) = 0$ ,  $i, j = 1, \dots, h$ , on  $\partial\Omega_t$ , for all  $0 \leq t < \tau$ , and thus we get  $\int_{\partial\Omega_t} \mathcal{K}_{ij}(t) = 0$ .

Therefore, we have

$$\frac{d}{dt} \left( \int_{\partial\Omega_t} \mathcal{K}_{ij}(t) \right) = 0, \quad 0 \leq t < \tau.$$

By Proposition 7.2 (see Appendix B), we get

$$\int_{\partial\Omega_t} \frac{d}{dt} (\mathcal{K}_{ij}(t)) + \int_{\partial\Omega_t} V_n \partial_n (\mathcal{K}_{ij}(t)) + \int_{\partial\Omega_t} V_n H \mathcal{K}_{ij}(t) = 0,$$

Knowing that  $\mathcal{K}_{ij}(t) = 0$  and  $\frac{d}{dt} (\mathcal{K}_{ij}(t)) = 0$ ,  $i, j = 1, \dots, h$ , on  $\partial\Omega_t$ , for all  $0 \leq t < \tau$ , we deduce that

$$\int_{\partial\Omega_t} V_n \partial_n (\mathcal{K}_{ij}(t)) = 0, \quad i, j = 1, \dots, h. \tag{6.8}$$

Now, we remark that one can refer to the proof of Theorem 9 (step 4) in order to compute the normal derivative of  $\mathcal{K}_{ij}(t)$  in the direction  $V$  and recognize the following equation:

$$\begin{aligned}
0 = & \int_{\partial\Omega_t} [-2HV_n\Delta_\tau V_n + V_n\partial_n(\Delta_\tau V_n)][|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2] \\
& + \int_{\partial\Omega_t} [8H^2 - 10H - 4]V_n\nabla_\tau V_n \cdot [\nabla_\tau u_i\partial_n u_i - \nabla_\tau u_j\partial_n u_j] \\
& + \int_{\partial\Omega_t} 2(1-H)V_n[\nabla_\tau(\partial_n u'_i) \cdot \nabla_\tau u_i - \nabla_\tau(\partial_n u'_j) \cdot \nabla_\tau u_j] \\
& + \int_{\partial\Omega_t} 2(1-H)V_n[\nabla_\tau u'_i \cdot \nabla_\tau(\partial_n u_i) - \nabla_\tau u'_j \cdot \nabla_\tau(\partial_n u_j)] \\
& - \int_{\partial\Omega_t} 2(2-H)V_n\lambda\nabla_\tau V_n \cdot [\nabla_\tau u_i u_i - \nabla_\tau u_j u_j] \\
& + \int_{\partial\Omega_t} 2(2-H)V_n\nabla_\tau(\partial_n V_n) \cdot [\nabla_\tau u_i\partial_n u_i - \nabla_\tau u_j\partial_n u_j] \\
& + \int_{\partial\Omega_t} 2(2-H)V_n\nabla_\tau V_n \cdot [\nabla_\tau(\partial_n u_i)\partial_n u_i - \nabla_\tau(\partial_n u_j)\partial_n u_j] \\
& + \int_{\partial\Omega_t} 2V_n[\Delta_\tau u'_i\partial_n u_i - \Delta_\tau u'_j\partial_n u_j] \\
& + \int_{\partial\Omega_t} 2V_n\Delta_\tau V_n[\nabla_\tau(\partial_n u_i) \cdot \nabla_\tau u_i - \nabla_\tau(\partial_n u_j) \cdot \nabla_\tau u_j] \\
& + \int_{\partial\Omega_t} 2(1+2H)V_n[\partial_n u'_i\partial_n u_i - \partial_n u'_j\partial_n u_j] \\
& + \int_{\partial\Omega_t} [6H^2 - 4H]V_n[\nabla_\tau u_i \cdot \nabla_\tau u'_i - \nabla_\tau u_j \cdot \nabla_\tau u'_j].
\end{aligned} \tag{6.9}$$

**Step 5. (Third order differentiation formula.)**

Now, let us differentiate (6.6) in the direction  $V$ , for  $0 \leq t < \tau$ , so that we get the following equation:

$$\begin{aligned}
& \frac{d}{dt} (2(1-H)[\nabla_\tau(\partial_n u_i) \cdot \nabla_\tau u_i - \nabla_\tau(\partial_n u_j) \cdot \nabla_\tau u_j]) \\
& + \frac{d}{dt} ((2H(H-1) - \partial_n H)[|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2]) \\
& + \frac{d}{dt} (2(1+H)[|\partial_n u_i|^2 - |\partial_n u_j|^2]) = 0, \quad i, j = 1, \dots, h.
\end{aligned} \tag{6.10}$$

Now, after computing the derivative of the three terms (see the proof of Theorem 9, step 5), we obtain by integrating over  $\partial\Omega_t$  in the direction  $V$ :

$$\begin{aligned}
0 = & \int_{\partial\Omega_t} [-2HV_n\Delta_\tau V_n + V_n\partial_n(\Delta_\tau V_n) + V_n\nabla_\tau V_n \cdot \nabla_\tau H][|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2] \quad (6.11) \\
& + \int_{\partial\Omega_t} [8H^2 - 6H - 8]V_n\nabla_\tau V_n \cdot [\nabla_\tau u_i\partial_n u_i - \nabla_\tau u_j\partial_n u_j] \\
& + \int_{\partial\Omega_t} 2(1-H)V_n[\nabla_\tau(\partial_n u'_i) \cdot \nabla_\tau u_i - \nabla_\tau(\partial_n u'_j) \cdot \nabla_\tau u_j] \\
& + \int_{\partial\Omega_t} 2(1-H)V_n[\nabla_\tau u'_i \cdot \nabla_\tau(\partial_n u_i) - \nabla_\tau u'_j \cdot \nabla_\tau(\partial_n u_j)] \\
& - \int_{\partial\Omega_t} 2(1-H)V_n\lambda\nabla_\tau V_n \cdot [\nabla_\tau u_i u_i - \nabla_\tau u_j u_j] \\
& + \int_{\partial\Omega_t} 2(1-H)V_n\nabla_\tau V_n \cdot [\nabla_\tau(\partial_n u_i)\partial_n u_i - \nabla_\tau(\partial_n u_j)\partial_n u_j] \\
& + \int_{\partial\Omega_t} 2V_n\Delta_\tau V_n[\nabla_\tau(\partial_n u_i) \cdot \nabla_\tau u_i - \nabla_\tau(\partial_n u_j) \cdot \nabla_\tau u_j] \\
& - \int_{\partial\Omega_t} 2V_n\Delta_\tau V_n[|\partial_n u_i|^2 - |\partial_n u_j|^2] + \int_{\partial\Omega_t} 4(1+H)V_n[\partial_n u'_i\partial_n u - \partial_n u'_j\partial_n u] \\
& + \int_{\partial\Omega_t} [6H^2 - 4H]V_n[\nabla_\tau u_i \cdot \nabla_\tau u'_i - \nabla_\tau u_j \cdot \nabla_\tau u'_j].
\end{aligned}$$

**Step 6. (Linking between the last two steps.)**

Now, we subtract (6.11) from (6.9), so that we get:

$$\begin{aligned}
0 = & \int_{\partial\Omega_t} 2V_n\nabla_\tau V_n \cdot [\nabla_\tau(\partial_n u_i)\partial_n u_i - \nabla_\tau(\partial_n u_j)\partial_n u_j] - \int_{\partial\Omega_t} 2V_n\lambda\nabla_\tau V_n \cdot [\nabla_\tau u_i u_i - \nabla_\tau u_j u_j] \\
& - \int_{\partial\Omega_t} 2V_n[\partial_n u'_i\partial_n u_i - \partial_n u'_j\partial_n u_j] + \int_{\partial\Omega_t} 2V_n[\Delta_\tau u'_i\partial_n u_i - \Delta_\tau u'_j\partial_n u_j] \quad (6.12) \\
& - \int_{\partial\Omega_t} V_n(4H-4)\nabla_\tau V_n \cdot [\nabla_\tau u_i\partial_n u_i - \nabla_\tau u_j\partial_n u_j] - \int_{\partial\Omega_t} V_n\nabla_\tau H \cdot \nabla_\tau V_n[|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2] \\
& + \int_{\partial\Omega_t} 2V_n\Delta_\tau V_n[|\partial_n u_i|^2 - |\partial_n u_j|^2] + \int_{\partial\Omega_t} (4-2H)\nabla_\tau(\partial_n V_n) \cdot [\nabla_\tau u_i\partial_n u_i - \nabla_\tau u_j\partial_n u_j].
\end{aligned}$$

Multiplying the boundary condition (1.9) by  $V_n\partial_n u_i$ ,  $i = 1, \dots, h$ , and integrating on  $\partial\Omega_t$ , we get:

$$\begin{aligned}
- \int_{\partial\Omega_t} 2V_n \partial_n u'_i \partial_n u_i &= - \int_{\partial\Omega_t} 2V_n \Delta_\tau u'_i \partial_n u_i - \int_{\partial\Omega_t} 2V_n \operatorname{div}_\tau (V_n \nabla_\tau u_i) \partial_n u_i \\
&\quad - \int_{\partial\Omega_t} 2V_n \Delta_\tau (V_n \partial_n u_i) \partial_n u_i + \int_{\partial\Omega_t} 2V_n \operatorname{div}_\tau (V_n H \nabla_\tau u_i) \partial_n u_i \\
&\quad - \int_{\partial\Omega_t} 2V_n^2 \lambda \partial_n u_i u_i, \quad i = 1, \dots, h.
\end{aligned} \tag{6.13}$$

Thus, by (6.13), we obtain

$$\begin{aligned}
0 &= \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot [\nabla_\tau (\partial_n u_i) \partial_n u_i - \nabla_\tau (\partial_n u_j) \partial_n u_j] - \int_{\partial\Omega_t} 2V_n \lambda \nabla_\tau V_n \cdot [\nabla_\tau u_i u_i - \nabla_\tau u_j u_j] \\
&\quad - \int_{\partial\Omega_t} 2V_n [\Delta_\tau u'_i \partial_n u_i - \Delta_\tau u'_j \partial_n u_j] - \int_{\partial\Omega_t} 2V_n [\operatorname{div}_\tau (V_n \nabla_\tau u_i) \partial_n u_i - \operatorname{div}_\tau (V_n \nabla_\tau u_j) \partial_n u_j] \\
&\quad - \int_{\partial\Omega_t} 2V_n [\Delta_\tau (V_n \partial_n u_i) \partial_n u_i - \Delta_\tau (V_n \partial_n u_j) \partial_n u_j] - \int_{\partial\Omega_t} 2V_n^2 \lambda [\partial_n u_i u_i - \partial_n u_j u_j] \\
&\quad + \int_{\partial\Omega_t} 2V_n [\operatorname{div}_\tau (V_n H \nabla_\tau u_i) \partial_n u_i - \operatorname{div}_\tau (V_n H \nabla_\tau u_j) \partial_n u_j] \\
&\quad + \int_{\partial\Omega_t} 2V_n [\Delta_\tau u'_i \partial_n u_i - \Delta_\tau u'_j \partial_n u_j] - \int_{\partial\Omega_t} V_n (4H - 4) \nabla_\tau V_n \cdot [\nabla_\tau u_i \partial_n u_i - \nabla_\tau u_j \partial_n u_j] \\
&\quad + \int_{\partial\Omega_t} 2V_n \Delta_\tau V_n [|\partial_n u_i|^2 - |\partial_n u_j|^2] - \int_{\partial\Omega_t} V_n \nabla_\tau H \cdot \nabla_\tau V_n [|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2] \\
&\quad + \int_{\partial\Omega_t} (4 - 2H) \nabla_\tau (\partial_n V_n) \cdot [\nabla_\tau u_i \partial_n u_i - \nabla_\tau u_j \partial_n u_j].
\end{aligned} \tag{6.14}$$

**Step 7. (Making integration by parts).** In this step, we will replace some terms in (6.14) using integration by parts. Let us remark that we have for all  $i = 1, \dots, h$ :

$$\begin{aligned}
\int_{\partial\Omega_t} 2\lambda V_n^2 |\nabla_\tau u_i|^2 + \int_{\partial\Omega_t} 4\lambda V_n \nabla_\tau V_n \cdot \nabla_\tau u_i u_i &= \int_{\partial\Omega_t} 2\lambda V_n^2 |\nabla_\tau u_i|^2 + \int_{\partial\Omega_t} 2\lambda \nabla_\tau (V_n^2) \cdot \nabla_\tau u_i u_i \\
&= \int_{\partial\Omega_t} 2\lambda V_n^2 |\nabla_\tau u_i|^2 - \int_{\partial\Omega_t} 2\lambda V_n^2 \operatorname{div}_\tau (\nabla_\tau u_i u_i) \\
&= \int_{\partial\Omega_t} 2\lambda V_n^2 |\nabla_\tau u_i|^2 - \int_{\partial\Omega_t} 2\lambda V_n^2 \Delta_\tau u_i u_i \\
&\quad - \int_{\partial\Omega_t} 2\lambda V_n^2 |\nabla_\tau u_i|^2 \\
&= - \int_{\partial\Omega_t} 2\lambda V_n^2 \partial_n u_i u_i \quad \text{by } \Delta_\tau u_i = \partial_n u_i,
\end{aligned} \tag{6.15}$$

$$\begin{aligned}
\int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau (\partial_n u_i) \partial_n u_i &= \int_{\partial\Omega_t} 2\nabla_\tau (\partial_n u_i) \cdot \nabla_\tau V_n V_n \partial_n u_i \\
&= - \int_{\partial\Omega_t} 2\partial_n u_i \operatorname{div}_\tau (\nabla_\tau V_n V_n \partial_n u_i) \\
&= - \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau (\partial_n u_i) \partial_n u_i - \int_{\partial\Omega_t} 2V_n \Delta_\tau V_n |\partial_n u_i|^2 \\
&\quad - \int_{\partial\Omega_t} 2|\nabla_\tau V_n|^2 |\partial_n u_i|^2,
\end{aligned} \tag{6.16}$$

$$\begin{aligned}
- \int_{\partial\Omega_t} 2V_n \operatorname{div}_\tau (V_n \nabla_\tau u_i) \partial_n u_i &= \int_{\partial\Omega_t} 2V_n \nabla_\tau u_i \cdot \nabla_\tau (V_n \partial_n u_i) \\
&= \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau u_i \partial_n u_i + \int_{\partial\Omega_t} 2V_n^2 \nabla_\tau u_i \cdot \nabla_\tau (\partial_n u_i),
\end{aligned} \tag{6.17}$$

$$\begin{aligned}
- \int_{\partial\Omega_t} 2V_n \Delta_\tau (V_n \partial_n u_i) \partial_n u_i &= \int_{\partial\Omega_t} 2\nabla_\tau (V_n \partial_n u_i) \cdot \nabla_\tau (V_n \partial_n u_i) \\
&= \int_{\partial\Omega_t} 2|\nabla_\tau V_n|^2 |\partial_n u_i|^2 + \int_{\partial\Omega_t} 4V_n \nabla_\tau V_n \cdot \nabla_\tau (\partial_n u_i) \partial_n u_i \\
&\quad + \int_{\partial\Omega_t} 2V_n^2 |\nabla_\tau (\partial_n u_i)|^2,
\end{aligned} \tag{6.18}$$

$$\begin{aligned}
\int_{\partial\Omega_t} 2V_n \operatorname{div}_\tau (V_n H \nabla_\tau u_i) \partial_n u_i &= - \int_{\partial\Omega_t} 2V_n H \nabla_\tau u_i \cdot \nabla_\tau (V_n \partial_n u_i) \\
&= - \int_{\partial\Omega_t} 2V_n H \nabla_\tau V_n \cdot \nabla_\tau u_i \partial_n u_i - \int_{\partial\Omega_t} 2H V_n^2 \nabla_\tau u_i \cdot \nabla_\tau (\partial_n u_i),
\end{aligned} \tag{6.19}$$

and

$$\begin{aligned}
-\int_{\partial\Omega_t} V_n(4H-4)\nabla_\tau V_n \cdot \nabla_\tau u_i \partial_n u_i &= -\int_{\partial\Omega_t} \nabla_\tau V_n \cdot \nabla_\tau u_i V_n(4H-4)\partial_n u_i & (6.20) \\
&= \int_{\partial\Omega_t} V_n \operatorname{div}_\tau(V_n(4H-4)\partial_n u_i \nabla_\tau u_i) \\
&= \int_{\partial\Omega_t} V_n(4H-4)\nabla_\tau V_n \cdot \nabla_\tau u_i \partial_n u_i \\
&\quad + \int_{\partial\Omega_t} (4H-4)V_n^2 \nabla_\tau u_i \cdot \nabla_\tau(\partial_n u_i) \\
&\quad + \int_{\partial\Omega_t} (4H-4)V_n^2 |\partial_n u_i|^2 + \int_{\partial\Omega_t} 4V_n^2 \nabla_\tau H \cdot \nabla_\tau u_i \partial_n u_i.
\end{aligned}$$

Hence, using (6.15), (6.16), (6.17), (6.18), (6.19) and (6.20), we obtain

$$\begin{aligned}
0 = & - \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot [\nabla_\tau(\partial_n u_i) \partial_n u_i - \nabla_\tau(\partial_n u_j) \partial_n u_j] - \int_{\partial\Omega_t} 2V_n \Delta_\tau V_n [|\partial_n u_i|^2 - |\partial_n u_j|^2] \\
& - \int_{\partial\Omega_t} 2|\nabla_\tau V_n|^2 [|\partial_n u_i|^2 - |\partial_n u_j|^2] + \int_{\partial\Omega_t} 2V_n \lambda \nabla_\tau V_n \cdot [\nabla_\tau u_i u_i - \nabla_\tau u_j u_j] \\
& + \int_{\partial\Omega_t} 2\lambda V_n^2 [\partial_n u_i u_i - \partial_n u_j u_j] + \int_{\partial\Omega_t} 2\lambda V_n^2 [|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2] \\
& - \int_{\partial\Omega_t} 2V_n [\Delta_\tau u'_i \partial_n u_i - \Delta_\tau u'_j \partial_n u_j] + \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot [\nabla_\tau u_i \partial_n u_i - \nabla_\tau u_j \partial_n u_j] \\
& + \int_{\partial\Omega_t} 2V_n^2 [\nabla_\tau u_i \cdot \nabla_\tau(\partial_n u_i) - \nabla_\tau u_j \cdot \nabla_\tau(\partial_n u_j)] + \int_{\partial\Omega_t} 2|\nabla_\tau V_n|^2 [|\partial_n u_i|^2 - |\partial_n u_j|^2] \\
& + \int_{\partial\Omega_t} 4V_n \nabla_\tau V_n \cdot [\nabla_\tau(\partial_n u_i) \partial_n u_i - \nabla_\tau(\partial_n u_j) \partial_n u_j] + \int_{\partial\Omega_t} 2V_n^2 [|\nabla_\tau(\partial_n u_i)|^2 - |\nabla_\tau(\partial_n u_j)|^2] \\
& - \int_{\partial\Omega_t} 2V_n H \nabla_\tau V_n \cdot [\nabla_\tau u_i \partial_n u_i - \nabla_\tau u_j \partial_n u_j] - \int_{\partial\Omega_t} 2\lambda V_n^2 [\partial_n u_i u_i - \partial_n u_j u_j] \\
& - \int_{\partial\Omega_t} 2H V_n^2 [\nabla_\tau u_i \cdot \nabla_\tau(\partial_n u_i) - \nabla_\tau u_j \cdot \nabla_\tau(\partial_n u_j)] + \int_{\partial\Omega_t} 2V_n [\Delta_\tau u'_i \partial_n u_i - \Delta_\tau u'_j \partial_n u_j] \\
& + \int_{\partial\Omega_t} V_n (4H - 4) \nabla_\tau V_n \cdot [\nabla_\tau u_i \partial_n u_i - \nabla_\tau u_j \partial_n u_j] + \int_{\partial\Omega_t} (4H - 4) V_n^2 [|\partial_n u_i|^2 - |\partial_n u_j|^2] \\
& + \int_{\partial\Omega_t} (4H - 4) V_n^2 [\nabla_\tau u_i \cdot \nabla_\tau(\partial_n u_i) - \nabla_\tau u_j \cdot \nabla_\tau(\partial_n u_j)] + \int_{\partial\Omega_t} 2V_n \Delta_\tau V_n [|\partial_n u_i|^2 - |\partial_n u_j|^2] \\
& + \int_{\partial\Omega_t} 4V_n^2 \nabla_\tau H \cdot [\nabla_\tau u_i \partial_n u_i - \nabla_\tau u_j \partial_n u_j] - \int_{\partial\Omega_t} V_n \nabla_\tau H \cdot \nabla_\tau V_n [|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2] \\
& + \int_{\partial\Omega_t} (4 - 2H) \nabla_\tau(\partial_n V_n) \cdot [\nabla_\tau u_i \partial_n u_i - \nabla_\tau u_j \partial_n u_j].
\end{aligned}$$

After cancellation of some terms, we get

$$\begin{aligned}
0 &= \int_{\partial\Omega_t} 2\lambda V_n^2 [|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2] + \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot [\nabla_\tau u_i u_i - \nabla_\tau u_j u_j] \\
&+ \int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot [\nabla_\tau(\partial_n u_i) \partial_n u_i - \nabla_\tau(\partial_n u_j) \partial_n u_j] \\
&+ \int_{\partial\Omega_t} 2V_n^2 [|\nabla_\tau(\partial_n u_i)|^2 - |\nabla_\tau(\partial_n u_j)|^2] + \int_{\partial\Omega_t} (4H - 4)V_n^2 [|\partial_n u_i|^2 - |\partial_n u_j|^2] \\
&+ \int_{\partial\Omega_t} V_n(2H - 2) \nabla_\tau V_n \cdot [\nabla_\tau u_i \partial_n u_i - \nabla_\tau u_j \partial_n u_j] \\
&+ \int_{\partial\Omega_t} 4V_n^2 \nabla_\tau H \cdot [\nabla_\tau u_i \partial_n u_i - \nabla_\tau u_j \partial_n u_j] - \int_{\partial\Omega_t} V_n \nabla_\tau H \cdot \nabla_\tau V_n [|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2] \\
&+ \int_{\partial\Omega_t} (2H - 2)V_n^2 [\nabla_\tau u_i \cdot \nabla_\tau(\partial_n u_i) - \nabla_\tau u_j \cdot \nabla_\tau(\partial_n u_j)] \\
&+ \int_{\partial\Omega_t} (4 - 2H) \nabla_\tau(\partial_n V_n) \cdot [\nabla_\tau u_i \partial_n u_i - \nabla_\tau u_j \partial_n u_j].
\end{aligned} \tag{6.21}$$

**Step 8. (Reformulation of (6.21).)** Let us reformulate (6.21) by isolating the terms containing  $V_n$ .

By integration by parts, we have for all  $i = 1, \dots, h$ :

$$\int_{\partial\Omega_t} 2V_n \nabla_\tau V_n \cdot \nabla_\tau(\partial_n u_i) \partial_n u_i = \int_{\partial\Omega_t} \nabla_\tau(V_n^2) \cdot \nabla_\tau(\partial_n u_i) \partial_n u_i = - \int_{\partial\Omega_t} V_n^2 \operatorname{div}_\tau(\nabla_\tau(\partial_n u_i) \partial_n u_i). \tag{6.22}$$

$$\begin{aligned}
\int_{\partial\Omega_t} V_n(2H - 2) \nabla_\tau V_n \cdot \nabla_\tau u_i \partial_n u_i &= \int_{\partial\Omega_t} \nabla_\tau(V_n^2) \cdot \nabla_\tau u_i (H - 1) \partial_n u_i \\
&= - \int_{\partial\Omega_t} V_n^2 \operatorname{div}_\tau((H - 1) \nabla_\tau u_i \partial_n u_i).
\end{aligned} \tag{6.23}$$

$$- \int_{\partial\Omega_t} V_n \nabla_\tau H \cdot \nabla_\tau V_n |\nabla_\tau u_i|^2 = - \int_{\partial\Omega_t} \frac{1}{2} \nabla_\tau(V_n^2) \cdot \nabla_\tau H |\nabla_\tau u_i|^2 = \int_{\partial\Omega_t} \frac{1}{2} V_n^2 \operatorname{div}_\tau(\nabla_\tau H |\nabla_\tau u_i|^2). \tag{6.24}$$

$$\int_{\partial\Omega_t} (4 - 2H) \nabla_\tau(\partial_n V_n) \cdot \nabla_\tau u_i \partial_n u_i = - \int_{\partial\Omega_t} \partial_n V_n \operatorname{div}_\tau((4 - 2H) \nabla_\tau u_i \partial_n u_i). \tag{6.25}$$

Hence, replacing some terms using (6.15), (6.22), (6.23), (6.24) and (6.25), we get



$$\begin{aligned}
0 = & - \int_{\partial\Omega_t} \lambda V_n^2 [\partial_n u_i u_i - \partial_n u_j u_j] - \int_{\partial\Omega_t} \lambda V_n^2 [|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2] \tag{6.26} \\
& - \int_{\partial\Omega_t} V_n^2 \operatorname{div}_\tau [\partial_n u_i \nabla_\tau (\partial_n u_i) - \partial_n u_j \nabla_\tau (\partial_n u_j)] + \int_{\partial\Omega_t} 2V_n^2 [|\nabla_\tau (\partial_n u_i)|^2 - |\nabla_\tau (\partial_n u_j)|^2] \\
& - \int_{\partial\Omega_t} V_n^2 \operatorname{div}_\tau [(H-1) \nabla_\tau u_i \partial_n u_i - (H-1) \nabla_\tau u_j \partial_n u_j] \\
& + \int_{\partial\Omega_t} (4H-4) V_n^2 [|\partial_n u_i|^2 - |\partial_n u_j|^2] + \int_{\partial\Omega_t} 4V_n^2 \nabla_\tau H \cdot [\nabla_\tau u_i \partial_n u_i - \nabla_\tau u_j \partial_n u_j] \\
& + \int_{\partial\Omega_t} (2H-2) V_n^2 [\nabla_\tau u_i \cdot \nabla_\tau (\partial_n u_i) - \nabla_\tau u_j \cdot \nabla_\tau (\partial_n u_j)] \\
& + \int_{\partial\Omega_t} \frac{1}{2} V_n^2 \operatorname{div}_\tau [\nabla_\tau H |\nabla_\tau u_i|^2 - \nabla_\tau H |\nabla_\tau u_j|^2] \\
& - \int_{\partial\Omega_t} \partial_n V_n \operatorname{div}_\tau [(4-2H) \partial_n u_i \nabla_\tau u_i - (4-2H) \partial_n u_j \nabla_\tau u_j].
\end{aligned}$$

**Step 9. (Finding a contradiction.)** In this last step, we are aiming to get a contradiction.

Let us remark that equation (6.26) is valid for any deformation fields  $V$  for any time  $0 \leq t < \tau$ , hence, in particular, it is valid at  $t = 0$ . Let us set

$$\begin{aligned}
I(V) := & - \int_{\partial\Omega_0} \lambda V_n^2 [\partial_n u_i u_i - \partial_n u_j u_j] - \int_{\partial\Omega_0} \lambda V_n^2 [|\nabla_\tau u_i|^2 - |\nabla_\tau u_j|^2] \tag{6.27} \\
& - \int_{\partial\Omega_0} V_n^2 \operatorname{div}_\tau [\partial_n u_i \nabla_\tau (\partial_n u_i) - \partial_n u_j \nabla_\tau (\partial_n u_j)] \\
& + \int_{\partial\Omega_0} 2V_n^2 [|\nabla_\tau (\partial_n u_i)|^2 - |\nabla_\tau (\partial_n u_j)|^2] \\
& - \int_{\partial\Omega_0} V_n^2 \operatorname{div}_\tau [(H-1) \nabla_\tau u_i \partial_n u_i - (H-1) \nabla_\tau u_j \partial_n u_j] \\
& + \int_{\partial\Omega_0} (4H-4) V_n^2 [|\partial_n u_i|^2 - |\partial_n u_j|^2] \tag{6.28} \\
& + \int_{\partial\Omega_0} 4V_n^2 \nabla_\tau H \cdot [\nabla_\tau u_i \partial_n u_i - \nabla_\tau u_j \partial_n u_j] \\
& + \int_{\partial\Omega_0} (2H-2) V_n^2 [\nabla_\tau u_i \cdot \nabla_\tau (\partial_n u_i) - \nabla_\tau u_j \cdot \nabla_\tau (\partial_n u_j)] \\
& + \int_{\partial\Omega_0} \frac{1}{2} V_n^2 \operatorname{div}_\tau [\nabla_\tau H |\nabla_\tau u_i|^2 - \nabla_\tau H |\nabla_\tau u_j|^2]
\end{aligned}$$

and

$$J(V) := - \int_{\partial\Omega_0} \partial_n V_n \operatorname{div}_\tau [(4 - 2H)\partial_n u_i \nabla_\tau u_i - (4 - 2H)\partial_n u_j \nabla_\tau u_j]. \quad (6.29)$$

Now, if  $V^1$  and  $V^2$  are two given vector fields at time  $t = 0$  such that  $V_n^1 = V_n^2$  and  $\partial_n V_n^1 \neq \partial_n V_n^2$  on  $\partial\Omega_0$ , then we get

$$I(V_n^1) = I(V_n^2), \quad (6.30)$$

thus we obtain

$$I(V^1) = I(V^2). \quad (6.31)$$

But, we know from (6.26) that

$$(I + J)(V^1) = 0 = (I + J)(V^2), \quad (6.32)$$

then we deduce that

$$J(V^1) = J(V^2). \quad (6.33)$$

Therefore,  $J$  doesn't depend on  $\partial_n V_n$ , whatever  $V$  is chosen at time  $t = 0$ . But, as  $J$  is linear in  $\partial_n V_n$ , we deduce that  $J$  is linear and constant, and hence  $J$  is null, i.e we have:

$$0 = \int_{\partial\Omega_0} \partial_n V_n \operatorname{div}_\tau [(4 - 2H)\partial_n u_i \nabla_\tau u_i - (4 - 2H)\partial_n u_j \nabla_\tau u_j]. \quad (6.34)$$

Hence, for all  $V$  we have on  $\partial\Omega_0$

$$\operatorname{div}_\tau [(4 - 2H)\partial_n u_i \nabla_\tau u_i - (4 - 2H)\partial_n u_j \nabla_\tau u_j] = 0. \quad (6.35)$$

Let us remark that since we are in dimension two, we set the following parameters:

$$s := \operatorname{arclength}(\partial\Omega_0), \text{ and } r := \operatorname{dist}(y, \partial\Omega_0), \quad \forall y \in \mathbb{R}^2.$$

Therefore, we have

$$\partial_s [(4 - 2H)\partial_r u_i \partial_s u_i - (4 - 2H)\partial_r u_j \partial_s u_j] = 0. \quad (6.36)$$

Then we get

$$(4 - 2H)[\partial_s u_i \partial_r u_i - \partial_s u_j \partial_r u_j] = C_1 \quad (6.37)$$

where  $C_1$  is a constant real number on  $\partial\Omega_0$ . We remark that (6.37) is valid for  $H = 2$  (by assumption), therefore we deduce the constant  $C_1$  is zero on the whole boundary. Hence, there exist some subsets  $\Gamma$  of  $\partial\Omega_0$  with  $H \neq 2$  such that

$$\partial_s u_i \partial_r u_i - \partial_s u_j \partial_r u_j = 0. \quad (6.38)$$

Now, let us remark that (6.38) is valid for any orthonormal eigenfunctions  $u_i$  and  $u_j$  in  $L^2(\partial\Omega_0)$ . Also we note that  $u_i - u_j$  and  $u_i + u_j$  belong to the eigenspace  $E_\lambda$  of  $\lambda$  and they are orthogonal in  $L^2(\partial\Omega_0)$  holding the same norm  $\|u_i - u_j\| = \|u_i + u_j\| = \sqrt{2}$ . Therefore, if we replace  $u_i$  by  $\frac{u_i - u_j}{\sqrt{2}}$  and  $u_j$  by  $\frac{u_i + u_j}{\sqrt{2}}$  in (6.38), we get

$$\partial_s(u_i - u_j) \partial_r(u_i - u_j) - \partial_s(u_i + u_j) \partial_r(u_i + u_j) = 0. \quad (6.39)$$

After expanding (6.39), we obtain

$$\partial_s u_i \partial_r u_j + \partial_s u_j \partial_r u_i = 0. \quad (6.40)$$

But we know from the boundary condition that  $\partial_r u = \partial_{ss}^2 u$ , this means that

$$\partial_s[\partial_s u_i \partial_s u_j] = 0. \quad (6.41)$$

Hence, we deduce that

$$\partial_s u_i \partial_s u_j = C_2. \quad (6.42)$$

where  $C_2$  is a constant real number on  $\Gamma$ . Our aim is to show that  $C_2 = 0$ .

Indeed, if  $C_2 = 0$ , then from (6.42), we deduce that either  $\partial_s u_i = 0$  on an interval  $\gamma_1 \subset \Gamma$  or  $\partial_s u_j = 0$  on an interval  $\gamma_2 \subset \Gamma$ , and hence we deduce that either  $\partial_r u_i = \partial_{ss}^2 u_i = 0$  on the interval  $\gamma_1 \subset \Gamma$  or  $\partial_r u_j = \partial_{ss}^2 u_j = 0$  on the interval  $\gamma_2 \subset \Gamma$ .

Now, as  $u_i - u_j$  and  $u_i + u_j$  belong to the eigenspace  $E_\lambda$  of  $\lambda$  and since these two eigenfunctions are orthogonal in  $L^2(\partial\Omega_0)$  with the same norm, we replace  $u_i$  and  $u_j$  by  $u_i - u_j$  and  $u_i + u_j$  in (6.3) respectively, to get

$$\begin{aligned} (1 - H)|\nabla_\tau(u_i - u_j)|^2 - |\partial_n(u_i - u_j)|^2 - \lambda|u_i - u_j|^2 \\ = (1 - H)|\nabla_\tau(u_i + u_j)|^2 - |\partial_n(u_i + u_j)|^2 - \lambda|u_i + u_j|^2 \quad \text{on } \partial\Omega_0. \end{aligned} \quad (6.43)$$

Expanding this equation, we get using (6.3):

$$(1 - H)\nabla_\tau u_i \cdot \nabla_\tau u_j - \partial_n u_i \partial_n u_j - \lambda u_i u_j = 0 \quad \text{on } \partial\Omega_0. \quad (6.44)$$

which can be written in the coordinates  $s$  and  $r$ :

$$(1 - H)\partial_s u_i \partial_s u_j - \partial_r u_i \partial_r u_j - \lambda u_i u_j = 0 \quad \text{on } \partial\Omega_0. \quad (6.45)$$

Therefore, by (6.45) we deduce that either  $u_i u_j = 0$  on  $\gamma_1$  or  $u_i u_j = 0$  on  $\gamma_2$ .

Now, we have four cases:

1. **First case.** If  $u_i = 0$  on  $\gamma_1^1 \subset \gamma_1$ , then  $\partial_r u_i = 0$  on  $\gamma_1^1$ . Therefore, by the Holmgren's uniqueness theorem, we get that  $u_i, i = 1, \dots, h$ , are zero in a neighborhood of  $\gamma_1^1$ , and then in the whole domain  $\Omega_0$  by analyticity, which is impossible since  $u_i, i = 1, \dots, h$ , are eigenfunctions of problem (1.1).
2. **Second case.** If  $u_j = 0$  on  $\gamma_1^2 \subset \gamma_1$ , then  $\partial_r u_j = 0$  on  $\gamma_1^2$ . Therefore, by the Holmgren's uniqueness theorem, we get that  $u_j, j = 1, \dots, h$ , are zero in a neighborhood of  $\gamma_1^2$ , and then in the whole domain  $\Omega_0$  by analyticity, which is impossible since  $u_j, j = 1, \dots, h$ , are eigenfunctions of problem (1.1).
3. **Third case.** If  $u_i = 0$  on  $\gamma_2^1 \subset \gamma_2$ , then  $\partial_r u_i = 0$  on  $\gamma_2^1$ . Therefore, by the Holmgren's uniqueness theorem, we get that  $u_i, i = 1, \dots, h$ , are zero in a neighborhood of  $\gamma_2^1$ , and then in the whole domain  $\Omega_0$  by analyticity, which is impossible since  $u_i, i = 1, \dots, h$ , are eigenfunctions of problem (1.1).
4. **Fourth case.** If  $u_j = 0$  on  $\gamma_2^2 \subset \gamma_2$ , then  $\partial_r u_j = 0$  on  $\gamma_2^2$ . Therefore, by the Holmgren's uniqueness theorem, we get that  $u_j, j = 1, \dots, h$ , are zero in a neighborhood of  $\gamma_2^2$ , and then in the whole domain  $\Omega_0$  by analyticity, which is impossible since  $u_j, j = 1, \dots, h$ , are eigenfunctions of problem (1.1).

Now, suppose that  $C_2 \neq 0$ , then there exist a real non-zero function  $\mu$  that depends on the position  $s$  along  $\Gamma$ , such that  $\partial_s u_i = \mu \partial_s u_j$ . Thus, by (6.42) we can have the two following identities

$$|\partial_s u_j|^2 = \frac{C_2}{\mu} \quad \text{and} \quad |\partial_s u_i|^2 = C_2 \mu. \quad (6.46)$$

Now, from (6.38) we can clearly deduce:

$$\partial_s u_i \partial_{ss}^2 u_i - \partial_s u_j \partial_{ss}^2 u_j = 0, \quad (6.47)$$

that is

$$\partial_s [|\partial_s u_i|^2 - |\partial_s u_j|^2] = 0. \quad (6.48)$$

Hence, we obtain

$$|\partial_s u_i|^2 - |\partial_s u_j|^2 = C_3. \quad (6.49)$$

where  $C_3$  is a constant real number on  $\Gamma$ . By (6.46) and (6.49), we get

$$C_2[\mu^2 - 1] = \mu C_3, \quad (6.50)$$

hence, we obtain

$$\frac{\mu^2 - 1}{\mu} = \frac{C_3}{C_2}, \quad (6.51)$$

this means that  $\frac{\mu^2 - 1}{\mu}$  is a constant function along  $\Gamma$ . But unfortunately, we still couldn't find any contradiction.

### 6.3.2 Strategy for Open problem 2

Let us discuss the strategy and the difficulties that we will face during our procedure:

We adapt ourselves to the classical strategy of Y. Chitour, D. Kateb, R. Long paper in [14] for the Laplace operator with Wentzell boundary conditions, in which they have proved the generic simplicity of the spectrum of the Stokes operator with Dirichlet boundary conditions in dimension three. This strategy was in particular applied successfully in J. H. Ortega and E. Zuazua paper (see [15]) for the generic simplicity of the spectrum of the Stokes operator with Dirichlet boundary conditions in two space dimensions. In this paper, we are going to introduce the main concepts that they have used. So, we fix a domain  $\Omega_0 \in \mathbb{D}_2^2$ . Let us define the set  $A_0 := \mathbb{D}_2^2(\Omega_0)$ , and for  $n \geq 1$ , we set

$$A_n := \{\Omega_0 + U, U \in W^{3,\infty}(\Omega_0, \mathbb{R}^2), \Omega_0 + U \in A_0 \\ \text{and the } n \text{ first eigenvalues of (1.1) on } \Omega_0 + U \text{ are simple}\}.$$

Set  $A := \bigcap_{n \in \mathbb{N}} A_n$ . Note that

$$A = \{\Omega_0 + U, U \in W^{3,\infty}(\Omega_0, \mathbb{R}^2), \Omega_0 + U \in A_0 \\ \text{and the eigenvalues of (1.1) on } \Omega_0 + U \text{ are simple}\}.$$

For seek of simplicity, we set  $\Omega := \Omega_0 + U$ , with  $U \in W^{3,\infty}(\Omega_0, \mathbb{R}^2)$ . Also, not to be confused, we will denote by  $\Omega_t$  as  $\Omega + tV$ , with  $V \in W^{3,\infty}(\Omega, \mathbb{R}^2)$  and  $t \geq 0$ . Note that the branches  $t \mapsto \lambda_i(t)$  are those obtained in Theorem 12, which are analytic in the sense of Hadamard. Therefore the eigenvalues  $\lambda_i$  are not necessarily ordered in an increasing way

in the definition of  $A_n$ . In the following of this section and in order to avoid confusion, we will denote by  $\lambda(t)$  to be the shape functional  $\lambda(\Omega_t)$ , and by  $\lambda'(t)$  as the shape derivative  $\lambda'(\Omega_t, V)$  of  $\lambda(\Omega_t) = \lambda(\Omega + tV)$ , at time  $t$  in the direction  $V$ .

The proof of the generic simplicity is based on the application of Baire's lemma (see Lemmas 6.1 and 6.2) on the sets  $A_n \subset A_0$ . We claim that it is sufficient to show that the sets  $A_n$  are open in  $A_0$  and that  $A_{n+1}$  is dense in  $A_n$  for all  $n \geq 0$ . Indeed, if the latter one is true, then we can prove by Lemma 6.2 the following: if  $A_{n+1}$  is dense in  $A_n$  and  $A_n$  is dense in  $A_0$ , then  $A_{n+1}$  is also dense in  $A_0$ . Finally, applying Baire's lemma (Lemma 6.1) to the sets  $A_n$ , we deduce that  $A = \bigcap_n A_n$  is dense in  $A_0$ .

Thus, it is sufficient to check these two facts:

**a)  $A_n$  is open in  $\mathbb{D}_2^2(\Omega_0)$ .**

It is clear that  $A_0$  is an open set. Now we analyze the case  $n \geq 1$ . Let  $\Omega \in \mathbb{D}_2^2(\Omega_0)$ . Then, by definition, the first  $n$  branches of the eigenvalues of (1.1),  $\lambda_i(\Omega)$  with  $i = 1, \dots, n$ , are simple. Therefore, we deduce that  $\lambda_i$ ,  $i = 1, \dots, n+1$ , are different (otherwise, if  $\lambda_n = \lambda_{n+1}$ , then  $\lambda_n$  is multiple). Let us define the function  $\delta : \mathbb{D}_2^2(\Omega_0) \mapsto \mathbb{R}^+$ , such that for all  $\Omega \in \mathbb{D}_2^2(\Omega_0)$

$$\delta(\Omega) := \min\{|\lambda_i - \lambda_j| : i, j = 1, \dots, n+1, i \neq j\} > 0.$$

Now, it is evident that the eigenvalues of (1.1) are continuous with respect to the deformation of  $\Omega$ . Also, we remark that the difference of continuous functions is still continuous, and the minimum of a finite number of functions is also continuous, which means that  $\delta$  is continuous on  $\mathbb{D}_2^2(\Omega_0)$ . Since  $\delta(\Omega) > 0$ , then the inverse image  $A_n$  of  $(0, +\infty)$  by  $\delta$  is open in  $\mathbb{D}_2^2(\Omega_0)$ .

**b)  $A_{n+1}$  is dense in  $A_n$ .**

We will prove that  $A_{n+1}$  is dense in  $A_n$ , for all  $n \geq 0$ ; in particular, for  $n = 0$  we will prove that  $A_1$  is dense in  $A_0 = \mathbb{D}_2^2(\Omega_0)$ . We proceed by contradiction. We assume that  $A_{n+1}$  is not dense in  $A_n$ ; then there exists  $\Omega \in A_n \setminus A_{n+1}$  and a neighborhood  $\mathcal{W}$  of  $\Omega$  such that  $\mathcal{W} \subset A_n \setminus A_{n+1}$ . Since  $\Omega \in A_n \setminus A_{n+1}$  we have that the first  $n$  branches of the eigenvalues of (1.1) are simple; that is,  $\lambda_i \neq \lambda_j$ ,  $\forall i, j = 1, \dots, n+1$ , if  $i \neq j$ .

Now, We can assume that  $\lambda_{n+1}$  is an eigenvalue with multiplicity  $h \geq 2$ ; that is,  $\lambda = \lambda_{n+1} = \dots = \lambda_{n+h}$  (in the case of  $n = 0$ , we have that the first eigenvalue has multiplicity  $h \geq 2$ ; that is,  $\lambda = \lambda_1 = \dots = \lambda_h$ ).

Let  $u_{n+1}, u_{n+2}, \dots, u_{n+h}$  be the eigenfunctions associated to  $\lambda$ . Let  $V \in W^{3,\infty}(\Omega, \mathbb{R}^2)$  be such that  $\Omega_t = \Omega + tV \in \mathcal{W}$ , for  $t > 0$ . We consider the eigenvalues  $\lambda_{n+i}(\Omega_t)$ , with  $i = 1, \dots, h$ ,

and the corresponding eigenfunctions  $u_{n+i}(\Omega_t)$  (or  $u_{n+i}(t)$  if there is no confusion) of (1.1), with  $i = 1, \dots, h$ .

We want to prove that there exists  $V \in W^{3,\infty}(\Omega, \mathbb{R}^2)$  such that  $\lambda'_{n+i}(\Omega, V) \neq \lambda'_{n+j}(\Omega, V)$ , for each  $i \neq j$ ,  $i, j = 1, \dots, h$ . That will immediately imply that there exist a small enough real number  $\tau > 0$ , such that

$$\lambda_{n+i}(\Omega + tV) \neq \lambda_{n+j}(\Omega + tV), \quad \forall i, j = 1, \dots, h, \quad 0 \leq t < \tau,$$

thus  $\Omega \in A_{n+1}$ .

To do so, we proceed again by contradiction. If there exist  $i \neq j$ ,  $i, j = 1, \dots, h$  such that  $\lambda_{n+i}(\Omega + tV) = \lambda_{n+j}(\Omega + tV) = \lambda_n(\Omega + tV)$ , for all  $V \in W^{3,\infty}(\Omega, \mathbb{R}^2)$  in a neighborhood of  $t = 0$ , i.e for  $0 \leq t < \tau$ , then we get

$$\lambda'_{n+i}(\Omega_t, V) = \lambda'_{n+j}(\Omega_t, V), \quad \forall i, j = 1, \dots, h, \quad 0 \leq t < \tau. \quad (6.52)$$

Now, we can repeat the same computations, as in the previous subsection, in order to obtain

$$(4 - 2H)[\partial_s u_{n+i} \partial_r u_{n+i} - \partial_s u_{n+j} \partial_r u_{n+j}] = C_1 \quad \text{on } \partial\Omega \quad (6.53)$$

where  $C_1$  is a constant real number on  $\partial\Omega$ , with  $\partial_s$  and  $\partial_r$  denote the tangential and the normal derivatives, respectively, on  $\partial\Omega$ .

Unfortunately, the obstacle that we will face in this problem is related somehow to the last step in the strategy of Open problem 1. In fact, it is not clear to guarantee the existence of a point on  $\partial\Omega$  where the curvature  $H$  is 2. Indeed, if we restrict ourselves to that case, the sets  $A_n$  and  $A$  will not be open in  $A_0$  anymore, which contradict the desired strategy.

## Chapter 7

# Appendices

### 7.1 Appendix A. Differentiating in moving domains.

Let us now state one of the main tools for differentiating on intervals of  $\mathbb{R}$  (see [2, Corollary 5.2.8, p.199]).

**Proposition 7.1** *Let us assume*

$$\varphi \in C^1([0, T]; W^{1, \infty}(\mathbb{R}^d)), \quad f \in C^1([0, T]; L^1(\mathbb{R}^d)) \cap C([0, T]; W^{1, 1}(\mathbb{R}^d))$$

*We set  $V(t, x) = \frac{\partial \varphi}{\partial t}(t, \varphi(t)^{-1}(x))$  for all  $x \in \Omega_t$ , where  $\Omega_t$  is a  $C^2$  domain. Then  $t \in [0, T) \rightarrow I(t)$  defined by*

$$I(t) := \int_{\Omega_t} f(t)$$

*is continuously differentiable on  $[0, T)$  and we have*

$$I'(t) = \int_{\Omega_t} \frac{\partial f}{\partial t}(t) + \operatorname{div}[fV](t) = \int_{\Omega_t} \frac{\partial f}{\partial t}(t) + \int_{\partial \Omega_t} (V \cdot n) f(t).$$

### 7.2 Appendix B. Differentiating on moving boundaries

We now state a classical proposition in shape optimization, which is proved in [2] (see [2, Proposition 5.4.4, p.215]).



**Proposition 7.2** *Assume  $\Omega$  is of class  $C^3$ , that  $t \in [0, T) \rightarrow \varphi(t) \in C^2$  is differentiable at 0 with  $\varphi(0) = I$ ,  $\varphi'(0) = V$ , and that  $t \rightarrow g(t) \in W^{1,1}(\mathbb{R}^d)$  is differentiable at 0 with  $g(0) \in W^{2,1}(\mathbb{R}^d)$ . Then,  $t \rightarrow G(t)$  defined by*

$$G(t) := \int_{\partial\Omega_t} g(t)$$

*is differentiable at 0 and we have the formula*

$$G'(t) = \int_{\partial\Omega} g'(0) + (V.n)[\partial_n g(0) + Hg(0)].$$

## 7.3 Appendix C. Some classical results on tangential differential calculus

### 7.3.1 C.1. Notations and definitions

We recall some essential notations and definitions that are needed for the computations of shape derivatives.

Given a smooth function  $f : \partial\Omega \mapsto \mathbb{R}$ , we define its tangential gradient  $\nabla_\tau$  as

$$\nabla_\tau f = \nabla \tilde{f} - (\nabla \tilde{f} \cdot \mathbf{n})\mathbf{n}$$

where  $\tilde{f}$  is any extension of  $f$  in a tubular neighborhood of  $\partial\Omega$ . An extension is easily obtained when  $\partial\Omega$  is smooth. The tangential gradient does not depend on the extension. It is also useful to define the tangential gradient as the normal projection of  $\nabla \tilde{f}$  to the tangent hyperplane of  $\partial\Omega$ ; in other words

$$\nabla_\tau f = \nabla \tilde{f} - n \otimes n \nabla \tilde{f}, \text{ on } \partial\Omega.$$

We also need the definition of the tangential divergence: For regular functions we define the surface Laplacian or Laplace-Beltrami operator as

$$\Delta_\tau f := \operatorname{div}_\tau(\nabla_\tau f).$$

We recall the definition of the oriented distance  $b_{\partial\Omega}$ :

$$b_{\partial\Omega}(x) := \left\{ \begin{array}{ll} d_\Omega(x) & \text{for } x \in \mathbb{R}^d \setminus \bar{\Omega} \\ -d_\Omega(x) & \text{for } x \in \Omega \end{array} \right\},$$

where the notation  $d_\Omega$  stands for the distance function for a subset  $\Omega \subset \mathbb{R}^d$ :

$$d_\Omega(x) = \inf_{y \in \Omega} |x - y|$$

We shall sometimes write  $b$  instead of  $b_{\partial\Omega}$ ; its gradient is an extension of the normal vector field  $\mathbf{n}$  in a neighborhood of  $\partial\Omega$ . Let  $D^2b$  be the Weingarten operator with entries  $(\nabla_\tau)_i n_j$  where  $n_j$  is the  $j$ -th component of  $\mathbf{n}$ . The normal vector is known to be in the kernel of  $D^2b$ , while the other eigenfunctions are tangential with the corresponding eigenvalues given by the principal curvatures of  $\partial\Omega$ . Let  $k_i, i = 1, \dots, d-1$  be the non zero eigenvalues of  $D^2b$ . We define the mean curvature  $H$  as

$$H = \sum_{i=1}^{d-1} k_i = \text{Tr}(D^2b) = \Delta b, \text{ on } \partial\Omega.$$

### 7.3.2 C.2. Preliminary results

We restate some propositions, in which one can refer to [16]. Let us start with the following proposition:

**Proposition 7.3** *Suppose that the boundary  $\partial\Omega$  is of class  $C^3$ . Then the normal derivative of the mean curvature  $H$  is*

$$\partial_n H = - \sum_{i=1}^{d-1} k_i^2. \quad (7.1)$$

**Shape derivative of the main curvature  $H$  and of the normal  $\mathbf{n}$**  in the direction of a velocity  $V$ :

**Proposition 7.4** *Let a surface  $\partial\Omega$  be of class  $C^2$ . The shape derivatives of the normal  $\mathbf{n}$  and of the mean curvature  $H$  in the direction of the velocity vector  $V$  are*

$$\mathbf{n}' = -\nabla_\tau V_n, \quad (7.2)$$

$$H' = -\Delta_\tau V_n, \quad (7.3)$$

where  $V_n = \langle V, \mathbf{n} \rangle$  denotes the normal component of the vector deformation  $V$ .

### 7.3.3 C.3. Some commutation equations

Let us state a useful proposition that is proved in [16]. Let  $f$  and  $g$  be two smooth functions defined on  $U$  a neighborhood of  $\partial\Omega$ ; the notation  $b$  stands for the oriented distance. Recall that its gradient is an extension of the normal field  $\mathbf{n}$  on  $\partial\Omega$ .

**Proposition 7.5** *Let  $f$  and  $g$  be two smooth functions defined on  $U$  a neighborhood of  $\partial\Omega$ . Then we have:*

$$\partial_n(\nabla_\tau f \cdot \nabla_\tau g) + 2\nabla_\tau f^T D^2 b \nabla_\tau g = \nabla_\tau(\partial_n f) \cdot \nabla_\tau g + \nabla_\tau(\partial_n g) \cdot \nabla_\tau f.$$

In the last proposition, we give an essential equation that is proved in [2, Proposition 5.4.12].

**Proposition 7.6** *Let  $\Omega$  be an open set of class  $C^2$  and  $u : \bar{\Omega} \rightarrow \mathbb{R}$  of class  $C^2$ . Then*

$$\Delta u = \Delta_\tau u + H \frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2} \quad \text{on } \partial\Omega. \quad (7.4)$$

# References

- [1] A. HENROT, *Extremum problems for eigenvalues of elliptic operators*, Frontiers in Mathematics, Birkhauser Basel, 2006.
- [2] A. HENROT, M. PIERRE, *Shape Variation and Optimization (A Geometrical Analysis)*, European Mathematical Society (Tracts in Mathematics 28), 2018.
- [3] L. C. EVANS, *Partial differential equations*, American Mathematical Society, volume 19, 1997.
- [4] C. AMROUCHE, M. MOUSSAOUI, *The Dirichlet and Neumann problems in Lipschitz and in  $C^{1,1}$  domains. Abstract*, arXiv:2204.02831v1, 2022
- [5] R. F. BASS, *Functional analysis*, 2014.
- [6] M. DAMBRINE, B. PUIG, *Oriented distance point of view on random sets*, ESAIM: Control, Optimisation and Calculus of Variations, 2020.
- [7] M. C. DELFOUR, J.-P. ZOLESIO, *Shapes and Geometries*, (Metrics, Analysis, Differential Calculus, and Optimization Second Edition), Society for Industrial and Applied Mathematics, 2011.
- [8] G. LEOBACHER, A. STEINICKE, *Existence, uniqueness and regularity of the projection onto differentiable manifolds*, Ann Glob Anal Geom 60, 559-587 (2021).
- [9] F. CAVALLETTI, A. MONDINO, *New formulas for the Laplacian of distance functions and applications*, Analysis and PDE, Vol. 13 (2020), No. 7, 2091-2147.
- [10] L. AMBROSIO, C. MANTEGAZZA, *Curvature and distance function from a manifold*, J. Geom. Anal., 8:723-748, 1998.
- [11] M. DAMBRINE, D. KATEB, *Persistency of wellposedness of Ventcel's boundary value problem under shape deformations*, J. Math. Anal. Appl., 394(1):129-138, 2012.
- [12] V. MAZ'YA, *Sobolev Spaces - with Applications to Elliptic Partial Differential Equations*, A Series of Comprehensive Studies in Mathematics, Volume 342, 2011.
- [13] H. SCHLICHTKRULL, *Curves and Surfaces, Lecture Notes for Geometry 1*, Department of Mathematics, University of Copenhagen, 2011.
- [14] Y. CHITOUR, D. KATEB, R. LONG, *Generic properties of the spectrum of the Stokes system with Dirichlet boundary condition in  $\mathbb{R}^3$* , Ann. I. H. Poincaré - AN 33 (2016) 119-167.
- [15] J. H. ORTEGA, E. ZUAZUA, *Generic Simplicity Of The Eigenvalues Of The Stokes System In Two Space Dimensions*, Advances in Differential Equations Volume 6, Number 8, August 2001, Pages 987-1023.

- [16] M. DAMBRINE, D. KATEB, J. LAMBOLEY, *An extremal eigenvalue problem for the Wentzell-Laplace operator*, 2014.
- [17] A.M. MICHELETTI, *Metrica per famiglie di domini limitati e proprietà generiche degli autovalori*, Ann. Sc. Norm. Super. Pisa 26 (1972) 683–694.
- [18] D. BUCUR, V. FERONE, C. NITSCH, C. TROMBETTI, *Weinstock inequality in higher dimensions*, arXiv:1710.04587v2 [math.AP], 2017.
- [19] F. CAUBET, M. DAMBRINE, R. MAHADEVAN, *Shape derivatives of eigenvalue functionals. Part one: scalar problems*, 2020.
- [20] J. KENNEDY, *A Faber-Krahn inequality for the Laplacian with Generalised Wentzell boundary conditions*, Journal of Evolution Equations 8 (2008) 557-582.
- [21] C. XIA, Q. WANG, *Eigenvalues of the Wentzell-Laplace operator and of the fourth order Steklov problems*, J. Differential Equations 264 (2018) 6486-6506.
- [22] F. DU, Q. WANG, C. XIA, *Estimates for eigenvalues of the Wentzell-Laplace operator*, Journal of Geometry and Physics 129 (2018) 25-33.