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Sorbonne Université

École doctorale de sciences mathématiques de Paris, centre 386

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– Théorie microlocale des faisceaux pour la transformation de Radon –

Par Mehdi Benchoufi

Thèse de doctorat de Mathématiques

Dirigée par Julien Grivaux, Pierre Schapira

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A Gennadi,

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Résumé

Le sujet de cette thèse est une approche microlocale de la transformation de Radon. Il s'agit d'appliquer à la dualité projective complexe et réelle les techniques initiées dans l'article fondateur de Sato-Kashiwara-Kawai de 1972 et de retrouver, reformuler, améliorer des travaux d'analyse plus classiques sur ce sujet, en particulier ceux de G. Henkin ou S. Gindikin. La dualité projective vue sous l'angle microlocal et faisceautique est apparue pour la première fois dans un travail important de J-L. Brylinski sur les faisceaux pervers, travail repris ensuite par D'Agnolo et Schapira dans le cadre des D-modules. Notre travail est de reprendre systématiquement cette étude avec les nouveaux outils de la théorie microlocale des faisceaux (théorie qui n'existait pas à l'époque de SKK72).

Ce travail se compose essentiellement de deux parties.

Dans la première, nous commençons par rappeler dans un cadre général la construction des transformations canoniques quantifiées, sous l'hypothèse de l'existence d'une section simple non-dégénérée (introduite sous un autre nom par J. Leray). Cette construction n'avait jamais été faite dans un cadre global hors du cas projectif. Nous montrons alors que ces transformations commutent à l'action des opérateurs microdifferentiels. Il s'agit là d' un résultat fondamental sans qu'aucune preuve consitante n'existe dans la littérature, ce résultat étant plus ou moins sous-entendu dans SKK72.

La deuxième partie de la thèse traite des applications à la transformation de Radon "classique". L'idée de base est que cette transformation échange support des hyperfonctions (modulo analyticité) et front d'onde analytique. Nous obtenons ainsi des théorèmes de prolongement ou d'unicité sur les ouverts linéellement concave. Nous obtenons aussi un théoréme des résidus pour les valeurs au bord de classes de cohomologie définies sur les cones de signatures $(1, n-1)$, clarifiant substantiellement des travaux de Cordaro-Gindikin-Trèves.

Mots Clés

Géométrie intégrale, Transformée de Radon, Analyse Algébrique, D-modules, E-modules, Transformations de Contact, Dualité Projective, Géométrie Symplectique, Lagrangien, Catégorie Dérivée.

Abstract

The subject of this thesis is a microlocal approach to the transformation of Radon. It is a question of applying to real and complex projective duality the techniques initiated in the founding article of Sato-Kashiwara-Kawai of 1972 and to find, reformulate, improve more classic analytical work on this subject, in particular those of G. Henkin or S. Gindikin. Projective duality seen from the microlocal and sheaf point of view appeared for the first time in an important work by J-L. Brylinski on perverse sheaves, work then taken up by D'Agnolo and Schapira in the framework of D-modules. Our work is to systematically resume this study with the new tools of the microlocal sheaf theory (theory which did not exist at the time of SKK72).

This work essentially consists of two parts.

In the first, we begin by recalling in a general framework the construction of quantized canonical transformations, under the hypothesis of the existence of a simple non-degenerate section (introduced under another name by J. Leray). This construction had never been done in a global framework outside the projective case. We then show that these transformations exchange the action of the microdifferential operators. This is a fundamental result without any consistent proof existing in the literature, this result being more or less implied in SKK72.

The second part of the thesis deals with the applications to the ‘ ‘classical ” Radon transform. The basic idea is that this transform exchanges the support of hyperfunctions (modulo analyticity) and the analytic wavefront set. We thus obtain theorems of continuation or uniqueness on linearly concave domain. We also get a residue theorem for the boundary values of finite cohomology classes defined on cones with $(1, n-1)$ signature, substantially clarifying the work of Cordaro-Gindikin-Trèves.

Keywords

Integral geometry, Radon transform, Algebraic Analysis, D-modules, E-modules, Contact transformations, Projective duality, Symplectic geometry, Lagrangian, Derived Category.

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Chapter 1

Introduction

1.1 Short introduction

In this thesis, we will study the *microlocal* Radon transform, understood as a quantization of projective duality, both in the real and the complex case.

In the real case, denote by P the real projective space (say of dimension n), by P^* its dual and by S the incidence relation:

$$S := \{(x, \xi) \in P \times P^*; \langle x, \xi \rangle = 0\}. \tag{1.1.1}$$

In this setting, there is a well-known correspondance between distributions on P and P^* due to Gelfand, Gindikin, Graev [GGG82] and to Helgason [Hel80]. However, it is known since the 70s under the influence of the Sato's school, that to well-understand what happens on real (analytic) manifolds, it may be worth to look at their complexification.

Hence, denote by $\mathbb{P}$ the complex projective space of dimension n, by $\mathbb{P}^*$ the dual projective space and by $\mathbb{S} \subset \mathbb{P} \times \mathbb{P}^*$ the incidence relation. Denoting by T^*X the cotangent bundle to a manifold X and $\dot{T}^*X$ the bundle T^*X with the zero section removed, we get the correspondence

$$\begin{tikzcd} & \dot{T}^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*) \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ \dot{T}^*\mathbb{P} \arrow[rr, "\sim"] & & \dot{T}^*\mathbb{P}^* \end{tikzcd} \tag{1.1.2}$$

This contact transformation induces an equivalence of categories between perverse sheaves modulo constant ones on the complex projective space and perverse sheaves modulo constant ones on its dual, as shown by Brylinski [Bry86], or between coherent $\mathcal{D}$-modules modulo flat connections, as shown by D'Agnolo-Schapira [DS94].

In this thesis, we will focus on the real projective duality for microfunctions. This is part of the continuation of the work of [Bry86] and [DS96]. We shall consider the

contact transform induced by (1.1.2)

$$\begin{tikzcd} & \dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)\cap(\dot{T}^*_P\mathbb{P}\times\dot{T}^*_{P^*}\mathbb{P}^*) \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ \dot{T}^*_P\mathbb{P} \arrow[rr, "\sim"] & & \dot{T}^*_{P^*}\mathbb{P}^* \end{tikzcd} \tag{1.1.3}$$

The above contact transformation leads to the well-known fact that the Radon transform establishes an isomorphism of sheaves of microfunctions on P and P^* (see [KKK86]).

In fact, we will work in the more general framework of integral transforms and then specialize our results to the case of projective duality. We will consider the following situation. Consider two complex manifolds X and Y of the same dimension, a closed submanifold Z of $X\times Y$, open subset $U\subset\dot{T}^*X$ and $V\subset\dot{T}^*Y$ and assume that the conormal bundle $\dot{T}^*_Z(X\times Y)$ induces a contact transformation

$$\begin{tikzcd} & \dot{T}^*_Z(X\times Y)\cap(U\times V^a) \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ \dot{T}^*X\supset U \arrow[rr, "\sim"] & & V\subset\dot{T}^*Y. \end{tikzcd} \tag{1.1.4}$$

Moreover, assume that X and Y are complexification of real analytic manifolds M and N respectively. Then, it is known that, under suitable hypotheses, one can quantize this contact transform and get an isomorphism between microfunctions of $U\cap T^*_MX$ and microfunctions on $V\cap T^*_NY$ [KKK86]. One of our main results will be the commutation of this isomorphism to the action of microdifferential operators. Although considered as well-known, the proof of this commutation does not appear clearly in the literature (see [SKK, p. 467]), and is far from being obvious. In fact, we will consider a more general setting, replacing sheaves of microfunctions with sheaves of the type $\mu hom(F,\mathscr{O}_X)$. Next, we will specialize our results to the setting of projective duality.

Finally, we will provide some applications of our results to the study of the real Radon transform of hyperfunctions. Precisely, we will get a Holmgren-type vanishing theorem describing the vanishing set of a hyperfunction whose wave-front set is subject to special geometrical conditions. From there, we will recover vanishing theorems of Kolm-Nagel [KN68] and Boman [Bom92]. Moreover, we will give an answer to a problem formulated by Henkin [Hen04], namely the description of the kernel of the real Radon transform in linearly concave domains. To treat this geometrical situation, a microlocal treatment will be peculiarly well-suited.

1.2 Classical results for Radon transform

Notations

We will use the langage of sheaves and $\mathcal{D}$-modules and we refer the reader to [KS90] and [Kas03] for a detailed developement of these topics.

We will denote by $\mathbf{k}$ a commutative unital ring of finite global dimension. Let X be a good topological space, i.e. Hausdorff, locally compact, countable at infinity, of finite cohomological dimension. We denote by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ the bounded derived category of the category of sheaves of $\mathbf{k}$ vector spaces on X and we shall freely make use of the six Gronthendieck operations. For a locally closed subset Z of X, we denote by $\mathbf{k}_Z$ the constant sheaf on Z with stalk $\mathbf{k}$, extended by 0 on $X \setminus Z$. For $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, we denote by

$$D'_X F := \mathrm{R}\mathscr{H}om\,(F, \mathbf{k}_X)$$

the dual of F. Assuming X is a manifold, we denote by $SS(F)$ the micro-support of F.

For manifolds M_i, M_j we denote by M_{ij} their cartesian product. We denote by q_i the projection $M_{ij} \to M_i$, p_i the projection $T^*M_{ij} \to T^*M_i$. We add a subscript ^{a} to p_j to denote by p_j^a the composition of p_j and the antipodal map on T^*M_j. Also, for diagonal embedding, we denote for instance by δ_2 the natural diagonal embedding: $M_{123} \hookrightarrow M_{1223}$.

On a complex manifold $(X, \mathcal{O}_X)$, we denote by d_X its complex dimension, by Ω_X the sheaf of holomorphic d_X-forms, by $\mathcal{D}_X$ the sheaf of rings of finite order holomorphic differential operators, by $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ the bounded derived category of the category of left $\mathcal{D}_X$-modules.

For a holomorphic line bundle $\mathcal{F}$ on X, one sets

$$\mathcal{F}^* := \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

the dual line bundle of $\mathcal{F}$, and

$$\mathcal{D}\mathcal{F} := \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}$$

the natural left $\mathcal{D}$-module attached to $\mathcal{F}$.

For a morphism $f : Y \to X$, we denote by $\underline{f}^{-1}$ and $\underline{f}_*$, the functors of inverse and direct images for $\mathcal{D}$-modules.

If Z is a closed complex submanifold of X of codimension d, we set

$$\mathscr{B}_{Z|X} := H^d_{[Z]}(\mathcal{O}_X)$$

(in the sense of algebraic cohomology). For S a complex submanifold of $X \times Y$, we denote by $\mathscr{B}^{(d_X,0)}_{S|X\times Y}$ the following $(\mathcal{D}_Y, \mathcal{D}_X)$-bimodule

$$\mathscr{B}^{(d_X,0)}_{S|X\times Y} := q_1^{-1}\Omega_X \otimes_{q_1^{-1}\mathcal{O}_X} \mathscr{B}_{S|X\times Y}$$

Integral transform for sheaves and $\mathcal{D}$-modules Let X, Y be complex manifolds and S a closed submanifold $X \times Y$. Consider the diagrams

(1.2.1)
$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y, \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{S} \arrow[dl, "g"'] \arrow[dr, "f"] & \\ Y & & X, \end{tikzcd} \qquad \begin{tikzcd} & X \times Y \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ X & & Y \end{tikzcd}$$

$\widetilde{S}$ being the image of S by the map $r : X \times Y \to Y \times X, (x, y) \mapsto (y, x)$. Let us fix some notations that we will detail later.

Let $G \in \mathrm{D^b}(\mathbb{C}_Y)$ and $K \in \mathrm{D^b}(\mathbb{C}_{X\times Y})$. The integral transform of G with respect to the kernel K is defined to be

$$\Phi_K(G) := \mathrm{R}q_{1!}(K \otimes q_2^{-1}G) \tag{1.2.2}$$

We will denote $\Phi_{\widetilde{S}}(G)$ the integral transform of G with respect to the kernel $\mathbb{C}_S[d_S - d_X]$.

For $\mathcal{M} \in \mathrm{D^b}(\mathcal{D}_X)$ with coherent cohomology, one defines the integral transform of $\mathcal{M}$,

$$\underline{\Phi}_S(\mathcal{M}) := \underline{g}_*\underline{f}^{-1}\mathcal{M}$$

Assuming $\mathcal{M}$ coherent and that f is non-characteristic for $\mathcal{M}$ and g is proper on $f^{-1}(\operatorname{supp}\mathcal{M})$, then we have the adjunction formula stated and proven in [DS94]

$$\mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}, \Phi_{\widetilde{S}}(G) \otimes \mathcal{O}_X)[d_X] \simeq \mathrm{RHom}_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{M}, G \otimes \mathcal{O}_Y)[d_Y] \tag{1.2.3}$$

We can use (1.2.3) to derive interesting computation of the integral transform of special $\mathcal{D}$-modules. Assume that X, Y are of equal dimension, and that f, g are smooth and proper. Let $\mathcal{F}$ and $\mathcal{G}$ be holomorphic line bundles. Then it is proven in [DS94] that, under specific hypotheses involving existence of global contact transformations, namely a global section of $q_1^{-1}\mathcal{F} \otimes_{q_1^{-1}\mathcal{O}_X} \mathscr{B}^{(d_X,0)}_{S|X\times Y} \otimes_{q_2^{-1}\mathcal{O}_Y} q_2^{-1}\mathcal{G}$, non-degenerate on some open subset of $T^*_S(X \times Y)$, and some additional topological conditions, there is an isomorphism

$$\underline{\Phi}_S\mathcal{D}\mathcal{F} \simeq \mathcal{D}\mathcal{G}$$

Applying (1.2.3) to the $\mathcal{D}$-module $\mathcal{M} = \mathcal{D}_X\mathcal{F}$, f being non-characteristic for $\mathcal{D}_X$ since f is smooth, we get the formula

$$\mathrm{R}\Gamma(X; \Phi_{\widetilde{S}}(G) \otimes \mathcal{F}^*) \simeq \mathrm{R}\Gamma(Y; G \otimes \mathcal{G}^*)$$

Radon transform for sheaves and $\mathcal{D}$-modules In the context of projective duality, we have the following situation

$$\begin{tikzcd} & T^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ T^*\mathbb{P} & & T^*\mathbb{P}^* \end{tikzcd}$$

Denoting by Λ the Lagrangian manifold $T^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*) \cap (\dot{T}^*\mathbb{P} \times \dot{T}^*\mathbb{P}^*)$, one can prove that $p_1|_\Lambda$, resp. $p_2|_\Lambda$, is an isomorphism on $\dot{T}^*\mathbb{P}$, resp. on $\dot{T}^*\mathbb{P}^*$. Then a theorem of [Bry86] asserts the equivalence of categories between perverse sheaves on $\mathbb{P}$ modulo constant sheaves and perverse sheaves on $\mathbb{P}^*$ modulo constant sheaves.

It is proven in [DS94] that there is an equivalence of categories between coherent $\mathcal{D}$-modules on $\mathbb{P}$ modulo flat connections and coherent $\mathcal{D}$-modules on $\mathbb{P}^*$ modulo flat

connections. At the heart of the proof, fixing an integer k such that $-n-1<k<0$, and denoting by $\mathcal{O}_{\mathbb{P}}(k)$ the $-k$-th power of the tautological line bundle, the following isomorphism of $\mathcal{D}$-modules is proven

$$\Phi_{\mathbb{S}}(\mathcal{O}_{\mathbb{P}}(k)) \simeq \mathcal{O}_{\mathbb{P}^*}(k^*) \tag{1.2.4}$$

with $k^* = -n-1-k$. From these results,we get the classical result, for $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}})$,

$$\mathrm{R}\Gamma(\mathbb{P}; F \otimes \mathcal{O}_{\mathbb{P}}(k)) \simeq \mathrm{R}\Gamma(\mathbb{P}^*; \Phi_{\mathbb{S}}(F) \otimes \mathcal{O}_{\mathbb{P}^*}(k^*))$$

Microlocalization of (1.2.4) Now, is there a microlocal counter-part of (1.2.4) ? At the germ level, the answer given in [KS90, Theo. 11.4.9] is positive.

Let us consider again the diagram (1.2.1) and related notations. It is proven in [SKK] that given a homogeneous symplectic isomorphism between open sets $U \subset T^*X$ and $V \subset T^*Y$, there is locally a (non unique) ring isomorphism between the sheaf of microdifferential operators on X and the sheaf of microdifferential operators on Y. This isomorphism is called a quantized contact transformation.

Also, let M, N be two real analytic manifolds, X, Y respective complexifications, and S a hypersurface of $X \times Y$. Consider the diagram:

$$\begin{tikzcd} & (U \times V^a) \cap T^*_S(X \times Y) \arrow[dl, "p_1"'] \arrow[dr, "p_{2^a}"] & \\ U \cap T^*_M X & & V \cap T^*_N Y \end{tikzcd} \tag{1.2.5}$$

Assume that p_1, p_{2^a} are isomorphisms. Then it is proven in [KKK86, Theo. 4.2.3] that there is a local isomorphism between the sheaf of microfunctions on $T^*_M X$ and the sheaf of microfunctions on $T^*_N Y$. We will prove in this work a global version of this isomorphism in the context of real projective duality. We will have to twist the sheaf of microfunctions by some power of the tautological line bundle.

Following [KS90], let X, Y be two complex manifolds of same dimension n, Λ be a closed complex Lagrangian submanifold of $T^*(X \times Y)$, such that the natural morphisms $T^*X \leftarrow \Lambda \rightarrow T^*Y$ are isomorphisms. Let $K \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X\times Y})$ be cohomologically constructible, simple with shift 0 along Λ and let $SS(K)$ be its micro-support. We suppose that $SS(K) \subset \Lambda$.

Let $p = (p_X, p_Y^a) \in \Lambda$ and consider a non-degenerate section $s \in H^0(\mu hom(K, \Omega_{X\times Y/Y})_p)$, where $\Omega_{X\times Y/Y} := \mathcal{O}_{X\times Y} \otimes_{q_1^{-1}\mathcal{O}_X} q_1^{-1}\Omega_X$. This defines a natural morphism

$$\Phi_{K[n]}(\mathcal{O}_X) \rightarrow \mathcal{O}_Y \tag{1.2.6}$$

in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y)$. This morphism is an isomorphism in the category $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y; p_Y)$, the localization of $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y)$ by the subcategory of sheaves whose singular support do not intersect p_Y. Moreover, given the isomorphism between microdifferential operators in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_X; p_X)$ and microdifferential operators in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y; p_Y)$ [SKK], the morphism (1.2.6) is compatible with their respective action on $\mathcal{O}_X$ in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_X; p_X)$ and $\mathcal{O}_Y$ in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y; p_Y)$ [KS90, Theo. 11.4.9, Coro. 11.4.8].

This is, at a germ level, a microlocal version of (1.2.4) with a precision related to the action of microdifferential operators. The main result of our work will be to give a global version of this result for an open set $U \subset \dot{T}^*Y$. From there, we will deduce a correspondence result between solutions of systems of microdifferential equations on the projective space and solutions of systems of microdifferential equations on its dual.

1.3 Announcement of the results

In this work, we will make the link between sheaf and $\mathcal{D}$-module theory and results related to the Radon transform, formulated or conjectured in the setting of several variable complex analysis. Contact transformations between cotangent bundles of the projective space and its dual play a key role to establish correspondance between relative behaviour of objects in P and objects in P^*: hyperfuctions, $\mathcal{D}$-modules, microfunctions, $\mathcal{E}$-modules. We will investigate specifically relations between microfunctions in $T^*_P\mathbb{P}$ and in $T^*_{P^*}\mathbb{P}^*$ not only as sheaves but as $\mathcal{E}$-modules. From there, we will establish an isomorphism between microfunction solutions of a system of micro-differential equations on the projective space and the counterpart solutions on its dual. As an application of sheaf theory, we will get a Holmgren-type vanishing theorem describing the vanishing set of a hyperfunction whose wave-front set is subject to special geometrical conditions. Moreover, we will describe the kernel of the real Radon transform in linearly concave domains as conjectured by Henkin [Hen04].

Results on the functor μhom

To establish our main results, we will need the following complement on the functor μhom.

For $(M_i)_{i=1,2,3}$, three manifolds, we consider the operation of composition of kernels:

$$\begin{aligned} \underset{2}{\circ} :\ & \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}}) \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{13}}) \\ & (K_1, K_2) \mapsto K_1 \underset{2}{\circ} K_2 := \mathrm{R}q_{13!}(q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2) \\ & \qquad\qquad \simeq \mathrm{R}q_{13!}\delta_2^{-1}(K_1 \boxtimes K_2). \end{aligned} \tag{1.3.1}$$

We define the composition of kernels on cotangent bundles (see [KS90, Prop. 4.4.11])

$$\begin{aligned} \overset{a}{\underset{2}{\circ}} :\ & \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*M_{12}}) \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*M_{23}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*M_{13}}) \\ & (K_1, K_2) \mapsto K_1 \overset{a}{\underset{2}{\circ}} K_2 := \mathrm{R}p_{13!}(p_{12^a}^{-1}K_1 \otimes p_{23}^{-1}K_2) \end{aligned} \tag{1.3.2}$$

Let F_i, G_i, H_i respectively in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}}), \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}}), \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{34}})$, $i = 1, 2$. Let U_i be an open subset of T^*M_{ij} $(i = 1, 2,\ j = i + 1)$ and set

$$U_3 = U_i \overset{a}{\underset{2}{\circ}} U_j = p_{13}(p_{12^a}^{-1}(U_1) \cap p_{23}^{-1}(U_2))$$

In [KS90], a canonical morphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*M_{13}})$ is constructed

$$\mu hom(F_1, F_2)|_{U_1} \overset{a}{\underset{2}{\circ}} \mu hom(G_1, G_2)|_{U_2} \to \mu hom(F_1 \underset{2}{\circ} G_1, F_2 \underset{2}{\circ} G_2)|_{U_3}. \tag{1.3.3}$$

We will see that the composition $\overset{a}{\circ}$ is associative and we will see also that the morphism (1.3.3) is compatible with associativity with respect to $\overset{a}{\circ}$.

Complex contact transformations

Consider now two complex manifolds X and Y of the same dimension n, open $\mathbb{C}^\times$-conic subsets U and V of $\dot{T}X$ and $\dot{T}Y$, respectively, Λ a smooth closed submanifold of $U \times V^a$ and assume that the projections $p_1|_\Lambda$ and $p_2^a|_\Lambda$ induce isomorphisms, hence a homogeneous symplectic isomorphism $\chi\colon U \xrightarrow{\sim} V$:

$$\begin{tikzcd}
 & \Lambda \subset U \times V^a \arrow[dl, "\sim", "p_1"'] \arrow[dr, "\sim", "p_2^a"] & \\
\dot{T}^*X \supset U \arrow[rr, "\sim", "\chi"'] & & V \subset \dot{T}^*Y
\end{tikzcd}$$

Let us consider a perverse sheaf L on $X \times Y$ satisfying

$$(p_1^{-1}(U) \cup p_2^{a\,-1}(V)) \cap \mathrm{SS}(L) \subset \Lambda$$

and a section s of $\mu hom(L, \Omega_{X\times Y/X})$ on Λ, where

$$\Omega_{X\times Y/X} := \mathcal{O}_{X\times Y} \otimes_{q_2^{-1}\mathcal{O}_Y} q_2^{-1}\Omega_Y$$

Recall that one denotes by $\mathscr{E}_X^{\mathbb{R}}$ sheaf of rings:

$$\mathscr{E}_X^{\mathbb{R}} := \mu hom(\mathbb{C}_{\Delta_X}, \Omega_{X\times X/X})[d_X] \tag{1.3.4}$$

and $\mathscr{E}_X$ the subsheaf of $\mathscr{E}_X^{\mathbb{R}}$ of finite order microdifferential operators. We will prove our main theorem:

Theorem 1.3.1. *Let $G \in \mathrm{D^b}(\mathbb{C}_Y)$ and assume to be given a section s of $\mu hom(L, \Omega_{X\times Y/X})$, non-degenerate on Λ.*

(i) *For $W \subset U$, $P \in \mathscr{E}_X(W)$, there is a unique $Q \in \mathscr{E}_Y(\chi(W))$ satisfying $P \cdot s = s \cdot Q$ (P, Q considered as sections of $\mathscr{E}_{X\times Y}$). The morphism induced by s*

$$\begin{aligned} \chi^{-1}\mathscr{E}_Y|_V &\to \mathscr{E}_X|_U \\ P &\mapsto Q \end{aligned}$$

is a ring isomorphism.

(ii) *We have the following isomorphism in $\mathrm{D^b}(\mathbb{C}_U)$*

$$\chi^{-1}\mu hom(G, \mathscr{O}_Y)|_V \xrightarrow{\sim} \mu hom(\Phi_{L[n]}(G), \mathscr{O}_X)|_U \tag{1.3.5}$$

(iii) *The isomorphism (1.3.5) is compatible with the action of $\mathscr{E}_Y$ and $\mathscr{E}_X$ on the left and right side of (1.3.5) respectively.*

We will see that the action of microdifferential operators in Theorem 1.3.1 (iii) is derived from the morphism (1.3.3) given the definition (1.3.4). We refer to (1.2.2) for the definition of $\Phi_{L[n]}$. The first part of this theorem is well-known, see [SKK], but the fact that isomorphism (1.3.5) is compatible with the action of microdifferential operators was announced for microfunctions in various papers but no detailed proof exists to our knowledge.

Projective duality for microfunctions

For M a real analytic manifold and X its complexification, we might be led to identify $T^*_M X$ with $i \cdot T^*M$. We denote by

$$\mathscr{A}_M := \mathscr{O}_X|_M$$

$$\mathscr{B}_M := \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X)$$

$$\mathscr{C}_M := \mu hom(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X)$$

the sheaves of real analytic functions, hyperfunctions, microfunctions, respectively.

We denote by sp the isomorphism

$$sp\colon \mathscr{B}_M \xrightarrow{\sim} \pi_*\mathscr{C}_M$$

Let U an open subset of M. For $u \in \mathscr{B}_M(U)$, we denote by $WF(u)$ its analytic wave front set:

$$WF(u) := supp(sp(u)) \subset T_U X$$

All through this thesis, we will quantize the contact transform associated with the Lagrangian submanifold $\dot{T}^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*)$, where $\mathbb{S}$ is the hypersurface of $\mathbb{P} \times \mathbb{P}^*$ defined by the incidence relation $\langle \xi, x\rangle = 0, (x, \xi) \in \mathbb{P} \times \mathbb{P}^*$.

We denote by $\dot{T}^*_P\mathbb{P}$, resp. $\dot{T}^*_{P^*}\mathbb{P}^*$, the conormal space to P in $\dot{T}^*\mathbb{P}$, resp. to P^* in $\dot{T}^*\mathbb{P}^*$, and we will construct and denote by χ the homogeneous symplectic isomorphism between $\dot{T}^*\mathbb{P}$ and $\dot{T}^*\mathbb{P}^*$.

For $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we denote by $\mathbb{C}_P(\varepsilon)$ the following sheaves: for $\varepsilon = 0$, we set

$$\mathbb{C}_P(0) := \mathbb{C}_P$$

for $\varepsilon = 1$, $\mathbb{C}_P(1)$ is the sheaf defined by the following exact sequence:

$$0 \to \mathbb{C}_P(1) \to q_!\mathbb{C}_{\widetilde{P}} \xrightarrow{tr} \mathbb{C}_P \to 0 \tag{1.3.6}$$

where q is the $2:1$ map from the universal cover $\widetilde{P}$ of P, to P and tr the integration morphism $tr : q_!\mathbb{C}_{\widetilde{P}} \simeq q_!q^!\mathbb{C}_P \to \mathbb{C}_P$.

Let an integer $p \in \mathbb{Z}$, $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we define the sheaves of real analytic functions, hyperfunctions on P resp. P^* twisted by some power of the tautological line bundle,

$$\mathscr{A}_P(\varepsilon, p) := \mathscr{A}_P \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{O}_{\mathbb{P}}(p) \otimes_{\mathbb{C}} \mathbb{C}_P(\varepsilon)$$

$$\mathscr{B}_P(\varepsilon,p) := \mathscr{B}_P \otimes_{\mathscr{A}_P} \mathscr{A}_P(\varepsilon,p) \simeq \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_{\mathbb{P}}\mathbb{C}_P, \mathscr{O}_{\mathbb{P}}(p)) \otimes \mathbb{C}_P(\varepsilon)$$

We define the sheaves of microfunctions on P resp. P^* twisted by some power of the tautological bundle,

$$\mathscr{C}_P(\varepsilon,p) := H^0(\mu\mathscr{hom}(D'_{\mathbb{P}}\mathbb{C}_P, \mathcal{O}_{\mathbb{P}}(p))) \otimes \mathbb{C}_P(\varepsilon)$$

and similarly with P^* instead of P. We notice that for n odd, $D'_{\mathbb{P}}\mathbb{C}_P \simeq \mathbb{C}_P(0) = \mathbb{C}_P$, and for n even $D'_{\mathbb{P}}\mathbb{C}_P \simeq \mathbb{C}_P(1)$.

For X, Y either the manifold $\mathbb{P}$ or $\mathbb{P}^*$, for any two integers p, q, we note $\mathcal{O}_{X\times Y}(p,q)$ the line bundle on $X \times Y$ with homogenity p in the X variable and q in the Y variable. We set

$$\begin{aligned}\Omega_{X\times Y/X}(p,q) &:= \Omega_{X\times Y/X} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{O}_{X\times Y}(p,q)\\ \mathscr{E}^{\mathbb{R}}_X(p,q) &:= \mu hom(\mathbb{C}_{\Delta_X}, \Omega_{X\times X/X}(p,q))[d_X]\end{aligned}$$

and we define accordingly $\mathscr{E}_X(p,q)$. Let us notice that $\mathscr{E}^{\mathbb{R}}_X(-p,p)$ is a sheaf of rings.

Let n be the dimension of P, (of course $n = d_{\mathbb{P}}$). For an integer k and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we note

$$k^* := -n-1-k$$

$$\varepsilon^* := -n-1-\varepsilon \ mod(2)$$

We have:

Theorem 1.3.2. (i) *Let k be an integer such that $-n-1 < k < 0$ and let s be a global non-degenerate section on $\dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)$ of $H^1(\mu hom(\mathbb{C}_{\mathbb{S}}, \Omega_{\mathbb{P}\times\mathbb{P}^*/\mathbb{P}^*}(-k,k^*)))$. For $P \in \mathscr{E}_{\mathbb{P}}(-k,k)$, there is a unique $Q \in \mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)$ satisfying $P\cdot s = s\cdot Q$. The morphism induced by s*

$$\chi_*\mathscr{E}_{\mathbb{P}}(-k,k) \to \mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)$$

$$P \mapsto Q$$

is a ring isomorphism.

(ii) *There exists such a non-degenerate section s.*

In fact, we will see that the non-degenerate section of Theorem 1.3.2 is provided by the Leray section.

Now, from classical adjunction formulas for $\mathscr{E}$-modules, we get a correspondance between solutions of systems of microdifferential equations on the projective space and solutions of systems of microdifferential equations on its dual. We will prove our main theorem

Theorem 1.3.3. *Let k be an integer such that $-n-1<k<0$. Let $\mathcal{N}$ be a coherent $\mathscr{E}_{\mathbb{P}}(-k,k)$-module and $F\in \mathrm{D}^{\mathrm{b}}(\mathbb{P})$. Then, we have an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\dot{T}^*\mathbb{P}^*})$*

$$\chi_*\mathrm{R}\mathscr{H}om_{\mathscr{E}_{\mathbb{P}}(-k,k)}(\mathcal{N},\mu hom(F,\mathcal{O}_{\mathbb{P}}(k)))\simeq$$
$$\mathrm{R}\mathscr{H}om_{\mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)}(\underline{\Phi}^{\mu}_{\mathbb{S}}(\mathcal{N}),\mu hom((\Phi_{\mathbb{C}_{\mathbb{S}}[-1]}F,\mathcal{O}_{\mathbb{P}^*}(k^*)))$$

where $\underline{\Phi}^{\mu}_{\mathbb{S}}$ it is the counterpart of $\underline{\Phi}_{\mathbb{S}}$ for $\mathscr{E}$-modules, and will be defined in Section 2.6.1.

Corollary 1.3.4. *Let k be an integer such that $-n-1<k<0$ and $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$. The section s of theorem 1.3.2 defines an isomorphism:*

$$\chi_*\mathscr{C}_P(\varepsilon,k)|_{\dot{T}^*_P\mathbb{P}}\simeq\mathscr{C}_{P^*}(\varepsilon^*,k^*)|_{\dot{T}^*_{P^*}\mathbb{P}^*}$$

Moreover, this morphism is compatible with the respective action of $\chi_*\mathscr{E}_{\mathbb{P}}(-k,k)$ and $\mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)$.

Applications

Application I : the wave-front set of hyperfunctions We will obtain a description of the kernel of the Radon transform in linearly concave domain, generalizing some results announed in [Hen04]. Consider the diagram:

(1.3.7)
$$\begin{tikzcd} & P\times P^* \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ P & & P^* \end{tikzcd}$$

For a set $A\subset P$, we set

$$\widehat{A}:=q_2(q_1^{-1}(A)\cap S)$$

and similarly for $B\subset P^*$. In some sense, $\widehat{A}$ represents the Radon transform of A.

Definition 1.3.5. A set $A\subset P$ is said to be linearly concave if there exists a set $B\subset P^*$ such that $A=\widehat{B}$. Moreover, if B is connected, A is said to be strongly linearly concave.

Let Ω be a non empty linearly concave domain, strictly contained in P. Let us denote by $\check{\Omega}$ the set:

$$P^*\setminus(\widehat{P\setminus\Omega})$$

i.e. the set of hyperplanes contained in Ω. As it will be seen in Section 4, S may be identified to P^*P the projectivization of the cotangent bundle of P. Let us denote by ρ_P the projection $\dot{T}^*P\to P^*P$ and similarly ρ_{P^*}. Let us consider the following situation:

$$\begin{tikzcd}
 & \dot{T}^*_S(P\times P^*) \arrow[dl, "p_1"'] \arrow[dr, "p_2^a"] & \\
\dot{T}^*P \arrow[rr, "\sim", "\chi"'] \arrow[d, "\rho_P"'] & & \dot{T}^*P^* \arrow[d, "\rho_{P^*}"] \\
 & P^*_S(P\times P^*) \arrow[dl, "\sim", "\widetilde{p}_1"'] \arrow[dr, "\sim", "\widetilde{p}_2"'] & \\
P^*P \arrow[d] & & P^*P^* \arrow[d] \\
P & & P^*
\end{tikzcd}$$

We will identify S to $P^*_S(P\times P^*)$. Let Ω be a non empty linearly concave domain strictly contained in P. The set $(\Omega\times\breve{\Omega})\cap S$ is decribed by:

$$(\Omega\times\breve{\Omega})\cap S=\{(x,\xi)\in P\times P^*; x\in\Omega, \widehat{\xi}\subset\Omega\}$$

We set

$$\mathbb{O}:=\rho_P^{-1}(\widetilde{p}_1((\Omega\times\breve{\Omega})\cap S))\subset\dot{T}^*P$$

We denote by $\mathbb{O}^*$ the set $\chi(\mathbb{O})\subset\dot{T}^*P^*$. We will see that $\mathbb{O}^*=\dot{T}^*_{\breve{\Omega}}\mathbb{P}^*$.

Definition 1.3.6. Let U an open subset of P. For a section f of $\mathscr{B}_P(\varepsilon,k)(U)$, we define the Radon transform $Rad(f)$ as the image of f by the sequence of morphisms:

$$Rad:\mathscr{B}_P(\varepsilon,k)(U)\longrightarrow\mathscr{C}_P(\varepsilon,k)(\dot{T}^*U)\xrightarrow{\simeq}\mathscr{C}_{P^*}(\varepsilon^*,k^*)(\chi(\dot{T}^*U))$$

For $f\in\mathscr{B}_P(\varepsilon,k)(\Omega)$, we will see that $Rad(f)|_{\mathbb{O}^*}$ may be viewed as an element of $\mathscr{B}_{P^*}(\varepsilon^*,k^*)/\mathcal{A}_{P^*}(\varepsilon^*,k^*)(\breve{\Omega})$. We will denote this element by $Rad(f)|_{\breve{\Omega}}$.

As an application of Theorem 1.3.1, we have

Proposition 1.3.7. *Let $f\in\mathscr{B}_P(\varepsilon,k)(\Omega)$. Then, we have*

$$WF(f)\cap\mathbb{O}=\chi^{-1}(WF(Rad(f)|_{\breve{\Omega}}))$$

In the case where $n=2$ and Ω is a strongly linealry concave domain, we denote by $\Omega_{\mathbb{C}}$ the union of complex lines, complexification of real lines contained in Ω. Let us consider an affine chart $\mathbb{R}^2$. Then, there are 2 connected components of $(\Omega_{\mathbb{C}}\setminus\Omega)\cap\mathbb{R}^2$, which we denote by $\Omega_\pm$. In this condition, we get the following corollary:

Corollary 1.3.8. *Assume that $WF(Rad(f)|_{\breve{\Omega}})=\varnothing$. Then, f is the boundary value of (f_+,f_-) respectively holomorphic in $\Omega_\pm$.*

Let V be a real finite dimensional vector space, V^* its dual, $\mathbb{V}$ its complexification, and set $\dot{V}^* = V^* \setminus \{0\}$. Let us denote by S the incidence hypersurface in $V \times V^*$. For an open set $U \subset V$, we set

$$\mathscr{B}(U) := \Gamma(U; \mathscr{B}_V)$$

We will deduce an analytic-continuation type theorem below from an Hölmgren-type theorem of Kashiwara, (whose proof is published in [Hör83, Th. 8.5.6']), stating that the normal cone of a support of a distribution is contained in its wave-front set. Let Ω be a convex open subset of V and $\omega \subset \Omega$ an open convex subset. Let us consider the set

$$Z = \overline{\{(x, i\xi) \in V \times i.\dot{V}^*; (x, \xi) \in S, H_{x,\xi} \cap \Omega \neq \varnothing, H_{x,\xi} \cap \omega = \varnothing\}}$$

where, for $(x, \xi) \in V \times \dot{V}^*$, we denote by $H_{x,\xi}$ the hyperplan passing through x and of conormal ξ.

Then one have the following theorem

Theorem 1.3.9. *Let $u \in \mathscr{B}(\Omega)$. Assume that $u = 0$ on ω and that $WF(u) \cap Z = \varnothing$. Then $u = 0$.*

Application II : residues of homolomorphic forms in tubes Let us denote by A the real affine space and by $\mathbb{A}$ its complexification. For q an integer such that $0 \leq q < n$, we denote by γ_q the cone of signature $(q, n-q)$ in A:

$$\gamma_q := \{(x_1, \dots x_n) \in A; \sum_{1 \leq j \leq q} x_j^2 < \sum_{q+1 \leq j \leq n} x_j^2\}$$

About a result announced by Henkin in [Hen04].

Let $\gamma \subset A$ be an open cone of signature $(1, n-1)$. Then, the complementary set of γ is the union of two closed convex cones, denoted by λ_1 and λ_2. Recall that we denote by $\mathscr{C}_A$ the sheaf of microfunctions on $T^*_A\mathbb{A}$ and by $\lambda_i^\circ, i = 1, 2$ the polar cones of λ_i considered here as closed subsets of $T^*_A\mathbb{A}$. We notice that

$$\lambda_{1,2}^\circ := \{(x_1, \dots x_n) \in A; x_1^2 \geq \sum_{2 \leq j \leq n} x_j^2, \pm x_1 \geq 0\}$$

Let $Int(\lambda^{i\circ})$, $i = 1, 2$ be the interior of the polar sets $\lambda^{i\circ}$.

Let K be a compact convex subset of A, we set as usual

$$\mathscr{B}(K) := \varinjlim_{U \supset K} \mathscr{B}(U)$$

The following Theorem brings a proof to a variant of a conjecture by Henkin in [Hen04], which was formulated, without the assumption that K is a convex compact, in the following way: the wave-front set of a hyperfunction has no intersection with $\lambda^{i\circ}, i = 1, 2$ if and only if it is the residue of a $\bar{\partial}$-closed $(0, n-2)$-form defined on $K + i \cdot \gamma$.

Theorem 1.3.10. *There exists a exact sequence*

$$0 \longrightarrow \varinjlim_{\gamma'} H^{n-2}(K+i\gamma';\mathcal{O}_{\mathbb{A}}) \xrightarrow{Res} \mathscr{B}(K) \longrightarrow$$
$$\longrightarrow \mathscr{C}_A(K+i\cdot Int(\lambda^{1\circ})) \bigoplus \mathscr{C}_A(K+i\cdot Int(\lambda^{2\circ})) \longrightarrow 0$$

where γ' ranges over the family of open convex cones of signature $(1,n-1)$, with $\gamma\subset\gamma'$.

Residue in cones of signature $(q,n-q)$

Proposition 1.3.11. *We have the natural morphism:*

$$H^{n-q-1}\mathrm{R}\Gamma_{A+i.\gamma_q}(\mathscr{O}_{\mathbb{A}}) \xrightarrow{Res} \mathscr{B}_A$$

This residue was constructed in [CGT95] through several variables complex analysis technics. Our approach provides a natural construction of this residue.

Residue diagram for Radon Transform

In the following, we see how cones of signature $(1,n-1)$ in $\mathbb{A}$ rise naturally from the Radon transform of strip-like sets in $\mathbb{P}^*$ and we explicit the diagram 1.3.8 showing how residues and Radon transforms commutes.

Let δ be a strictly positive real number. Let us consider the open subset U of $P^*\hookrightarrow\mathbb{P}^*$,

$$U := \{\xi\in A^*, (\sum_{2\le j\le n} \mid \xi_j \mid^2)^{\frac{1}{2}} < \delta\}$$

Let us set

$$Z := \widehat{U}\cap P$$
$$K := P\setminus Z$$

We will see that $\widehat{U}\setminus P$ is a cone of signature $(1,n-1)$. Let $\mathcal{L}_Z(\varepsilon^*)$ be the object of $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}^*})$, defined, up to an isomorphism, by the distinguished triangle:

$$\mathcal{L}_Z(\varepsilon^*) \longrightarrow \mathbb{C}_{P^*}(\varepsilon^*) \longrightarrow \mathbb{C}_{\widehat{K}}[n-1] \xrightarrow{+1}$$

Recalling that we denoted by χ the homogeneous symplectic isomorphism between $\dot{T}^*\mathbb{P}$ and $\dot{T}^*\mathbb{P}^*$, we have

Theorem 1.3.12. *Given a section s of $\mu hom(\mathbb{C}_{\mathbb{S}}[-1],\Omega_{\mathbb{P}\times\mathbb{P}^*/\mathbb{P}^*}(-k,k^*))$, the following diagram commutes:*

$$\begin{tikzcd}
\chi_* H^{n-2}(\mu hom(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon),\mathscr{O}_{\mathbb{P}}(k)))|_{\dot{T}^*\mathbb{P}} \arrow[r, "Res"] \arrow[d, "Rad"'] & \chi_* H^{n}(\mu hom(\mathbb{C}_{Z}(\varepsilon),\mathscr{O}_{\mathbb{P}}(k)))|_{\dot{T}^*\mathbb{P}} \arrow[d, "Rad"'] \\
H^{n-2}(\mu hom(\Phi_{\mathbb{S}}(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon)),\mathscr{O}_{\mathbb{P}^*}(k^*)))|_{\dot{T}^*\mathbb{P}^*} \arrow[r, "Res"'] & H^{n}(\mu hom(\mathcal{L}_Z(\varepsilon^*),\mathscr{O}_{\mathbb{P}^*}(k^*)))|_{\dot{T}^*\mathbb{P}^*}
\end{tikzcd} \tag{1.3.8}$$

Assuming that s is non-degenerate on $\dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)$, the vertical arrows Rad are isomorphisms.

Let us consider a complex neighborhood $\omega \subset \mathbb{A}^*$ of U such that, denoting by π the natural projection $\mathbb{A}^* \xrightarrow{\pi} A^*$, $\pi(\omega) = U$.

Proposition 1.3.13. *We have*

$$SS(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon)) \cap \chi^{-1}\dot{T}^*\omega = \varnothing \tag{1.3.9}$$

We deduce the following

Corollary 1.3.14. *We have*

$$\mu hom(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon), \mathscr{O}_{\mathbb{P}}(k))\big|_{\chi^{-1}\dot{T}^*\omega} \simeq 0$$

Chapter 2

Reminders on Algebraic Analysis and complements

In this chapter, we recall classical results of Algebraic Analysis, with the exception of section 2.7.

2.1 Sheaves

2.1.1 Notations for manifolds

(i) Let M_i $(i = 1, 2, 3)$ be manifolds. For short, we write $M_{ij} := M_i \times M_j$ $(1 \leq i, j \leq 3)$, $M_{123} = M_1 \times M_2 \times M_3$, $M_{1223} = M_1 \times M_2 \times M_2 \times M_3$, etc.

(ii) $\delta_{M_i} \colon M_i \to M_i \times M_i$ denote the diagonal embedding, and Δ_{M_i} the diagonal set of $M_i \times M_i$.

(iii) We will often write for short $\mathbf{k}_i$ instead of $\mathbf{k}_{M_i}$ and $\mathbf{k}_{\Delta_i}$ instead of $\mathbf{k}_{\Delta_{M_i}}$ and similarly with ω_{M_i}, etc., and with the index i replaced with several indices ij, etc.

(iv) We denote by π_i, π_{ij}, etc. the projection $T^*M_i \to M_i$, $T^*M_{ij} \to M_{ij}$, etc.

(v) For a fiber bundle $E \to M$, we denote by $\dot{E} \to M$ the fiber bundle with the zero-section removed.

(vi) We denote by q_i the projection $M_{ij} \to M_i$ or the projection $M_{123} \to M_i$ and by q_{ij} the projection $M_{123} \to M_{ij}$. Similarly, we denote by p_i the projection $T^*M_{ij} \to T^*M_i$ or the projection $T^*M_{123} \to T^*M_i$ and by p_{ij} the projection $T^*M_{123} \to T^*M_{ij}$.

(vii) We also need to introduce the maps p_{j^a} or p_{ij^a}, the composition of p_j or p_{ij} and the antipodal map a on T^*M_j. For example,

$$p_{12^a}((x_1, x_2, x_3; \xi_1, \xi_2, \xi_3)) = (x_1, x_2; \xi_1, -\xi_2).$$

(viii) We let $\delta_2 \colon M_{123} \to M_{1223}$ be the natural diagonal embedding.

2.1.2 Sheaves

We follow the notations of [KS90].

Let X be a good topological space, i.e. separated, locally compact, countable at infinity, of finite global cohomological dimension and let $\mathbf{k}$ be a commutative unital ring of finite global dimension.

For a locally closed subset Z of X, we denote by $\mathbf{k}_Z$, the sheaf, constant on Z with stalk $\mathbf{k}$, and 0 elsewhere.

We denote by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ the bounded derived category of the category of sheaves of $\mathbf{k}$-modules on X. If $\mathscr{R}$ is a sheaf of rings, we denote by $\mathrm{D}^{\mathrm{b}}(\mathscr{R})$ the bounded derived category of the category of left $\mathscr{R}$-modules.

Let Y be a good topological space and f a morphism $Y \to X$. We denote by $\mathrm{R}f_*, f^{-1}, \mathrm{R}f_!, f^!, \mathrm{R}\mathscr{H}om\,, \overset{\mathrm{L}}{\otimes}$ the six Grothendieck operations. We denote by $\boxtimes$ the exterior tensor product.

We denote by ω_X the dualizing complex on X, by $\omega_X^{\otimes -1}$ the sheaf-inverse of ω_X and by $\omega_{Y/X}$ the relative dualizing complex

$$\omega_{Y/X} := f^!(\mathbf{k}_X) \simeq \omega_Y \otimes f^{-1}(\omega_X^{\otimes -1})$$

In the following, we assume that X is a real manifold. Recall that $\omega_X \simeq \mathrm{or}_X\,[\dim X]$ where or_X is the orientation sheaf and $\dim X$ is the dimension of X. We denote by $\mathrm{D}_X(\bullet)$ resp. $\mathrm{D}'_X(\bullet)$ the duality functor

$$\begin{aligned}\mathrm{D}_X(\bullet) &= \mathrm{R}\mathscr{H}om\,(\bullet, \omega_X)\\ \mathrm{D}'_X(\bullet) &= \mathrm{R}\mathscr{H}om\,(\bullet, \mathbf{k}_X)\end{aligned}$$

For $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, we denote by $SS(F)$ its singular support, also called micro-support. For a a subset $Z \subset T^*X$, we denote by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X; Z)$ the localization of the category $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ by the full subcategory of objects whose micro-support is contained in $T^*X \setminus Z$.

For a closed submanifold M of X, we denote by

$$\nu_M \colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T_M X})$$

the functor of specialization along M,

$$\mu_M \colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T^*_M X})$$

the functor of microlocalization along M, and by

$$\mu hom \colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X) \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T^*X})$$

the functor of microcalization of $\mathrm{R}\mathscr{H}om$. The subscript $\mathbb{R}^+$ stands for conic objects of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*X})$, i.e. objects locally constant with respect to the natural action of $\mathbb{R}^+$ on the fibers of $T^*X \to X$. We recall that μ_M is the Fourier-Sato transform of ν_M and that

$$\mu_M(\bullet) = \mu hom(\mathbf{k}_M, \bullet)$$

Let M_i $(i = 1, 2, 3)$ be manifolds. We shall consider the operations of composition of kernels:

$$\begin{aligned} \underset{2}{\circ} :\ & \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}}) \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{13}}) \\ & (K_1, K_2) \mapsto K_1 \underset{2}{\circ} K_2 := \mathrm{R}q_{13!}(q_{12}^{-1} K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1} K_2) \\ & \qquad\qquad\qquad\qquad \simeq \mathrm{R}q_{13!}\delta_2^{-1}(K_1 \overset{\mathrm{L}}{\boxtimes} K_2) \end{aligned} \tag{2.1.1}$$

$$\begin{aligned} \underset{23}{\circ} :\ & \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}}) \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}}) \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{34}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{14}}) \\ & (K_1, K_2, K_3) \mapsto K_1 \underset{2}{\circ} K_2 \underset{3}{\circ} K_3 := \mathrm{R}q_{14!}({q_{12}}^{-1} K_1 \overset{\mathrm{L}}{\otimes} {q_{23}}^{-1} K_2 \overset{\mathrm{L}}{\otimes} {q_{34}}^{-1} K_3) \end{aligned} \tag{2.1.2}$$

Let us mention a variant of $\circ$:

$$\begin{aligned} \underset{2}{*} :\ & \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}}) \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{13}}) \\ & (K_1, K_2) \mapsto K_1 \underset{2}{*} K_2 := \mathrm{R}q_{13*}\big(q_2^{-1}\omega_2 \otimes \delta_2^!(K_1 \boxtimes K_2)\big) \end{aligned}$$

There is a natural morphism $K_1 \underset{2}{\circ} K_2 \to K_1 \underset{2}{*} K_2$.

We refer the reader to [KS90] for a detailed presentation of sheaves on manifolds.

2.2 $\mathcal{O}$-modules and $\mathcal{D}$-modules

We refer to [Kas03] for the notations and the main results of this section.

Let $(X, \mathcal{O}_X)$ be a complex manifold. We denote by d_X its complex dimension and by $\mathcal{D}_X$ the sheaf of rings of finite order holomorphic differential operators on X.

For an invertible $\mathcal{O}_X$-module $\mathcal{F}$, we note

$$\mathcal{F}^{\otimes -1} := \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

the inverse of $\mathcal{F}$. Denote by $\mathrm{Mod}(\mathcal{D}_X)$ the abelian category of left $\mathcal{D}_X$-modules and $\mathrm{Mod}(\mathcal{D}_X^{op})$ of right $\mathcal{D}_X$-modules. We denote by Ω_X the right $\mathcal{D}_X$-module of holomorphic d_X forms. There is an equivalence of category between left and right $\mathcal{D}_X$-modules provided by

$$\begin{aligned} \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{op}), \mathcal{M} \mapsto \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \\ \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{op}) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X),\ \mathcal{N} \mapsto \mathcal{N} \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1} \end{aligned}$$

Let $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ be the bounded derived category of the category of left $\mathcal{D}_X$-modules, $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ its full triangulated subcategory whose objects have coherent cohomology. Let us recall the notion of good $\mathcal{D}$-modules due to Kashiwara [Kas03].

Definition 2.2.1. An $\mathcal{O}_X$-module $\mathcal{F}$ is good if for any relatively compact open subset $U \subset\subset X$, there exists a small and filtrant category I and an inductive system $\{\mathcal{F}_i\}_{i \in I}$ of coherent $\mathcal{O}_U$-modules such that $\varinjlim_i \mathcal{F}_i \xrightarrow{\sim} \mathcal{F}_{|U}$. A coherent $\mathcal{D}_X$-module is good if it is good as an $\mathcal{O}_X$-module.

Let $\mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ be the triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$, whose objects have all cohomologies consisting in good $\mathcal{D}_X$-modules (see [Kas03] for a classical reference).

For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, the complex of holomorphic solutions of $\mathscr{M}$ is defined by

$$\mathrm{Sol}(\mathscr{M}) := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$$

Operations on $\mathcal{D}$-modules

We refer in the following to [Kas03]. Let $f : Y \to X$ be a morphism of complex manifolds. We denote by $\mathcal{D}_{Y\to X}$ the transfer bimodule, i.e. the $(\mathcal{D}_Y, f^{-1}\mathcal{D}_X)$-bimodule

$$\mathcal{D}_{Y\to X} := \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X$$

We shall be aware that the action of $\mathcal{D}_Y$ on $\mathcal{D}_{Y\to X}$ is not given by the action on $\mathcal{O}_Y$. We denote by $\mathcal{D}_{X\leftarrow Y}$ the $(f^{-1}\mathcal{D}_X, \mathcal{D}_Y)$-bimodule

$$\mathcal{D}_{X\leftarrow Y} := (\mathcal{D}_{Y\to X} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\Omega_X^{\otimes -1}$$

We define the pull-back and the direct image of $\mathcal{D}$-modules. Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$, $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y)$

$$\underline{f}^{-1}\mathcal{M} := \mathcal{D}_{Y\to X} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M}, \underline{f}_*\mathcal{N} := \mathrm{R}f_*(\mathcal{D}_{X\leftarrow Y} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{N})$$

The following results are proven in [KS96, Prop 2.4]. Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y)$, $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$, $G \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y)$. We assume that f is non-characteristic for $\mathcal{M}$. We have

$$\mathrm{R}f_*\mathrm{R}\mathscr{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathcal{M}, \mathcal{N}) \simeq \mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \underline{f}_*\mathcal{N}),$$

$$f^{-1}\mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq \mathrm{R}\mathscr{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathcal{M}, \mathcal{O}_Y),$$

$$\mathrm{R}f_!\mathrm{R}\mathscr{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathcal{M}, G \otimes \mathcal{O}_Y) \simeq \mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathrm{R}f_!G \otimes \mathcal{O}_X)$$

Assume $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$, $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Y)$. We no longer assume that f is non-characteristic for $\mathcal{M}$. We assume that f is proper on $supp(\mathcal{N})$. Then, we have

$$\mathrm{R}f_*\mathrm{R}\mathscr{H}om_{\mathcal{D}_Y}(\mathcal{N}, \underline{f}^{-1}\mathcal{M}[d_{Y/X}]) \simeq \mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\underline{f}_*\mathcal{N}, \mathcal{M}),$$

Specializing this isomorphism to the case $\mathcal{M} = F \otimes \mathcal{O}_X$, we get

$$\mathrm{R}f_!\mathrm{R}\mathscr{H}om_{\mathcal{D}_Y}(\mathcal{N}, f^{-1}F \otimes \mathcal{O}_Y)[d_{Y/X}] \simeq \mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\underline{f}_*\mathcal{N}, F \otimes \mathcal{O}_X)$$

2.3 $\mathscr{E}$-modules

We refer in the following to [SKK] (see also [Sch85] for an exposition). For a complex manifold X, one denotes by $\mathscr{E}_X$ the sheaf of filtered ring of finite order holomorphic microdifferential operators on T^*X. We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{E}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{E}_X)$ whose objects have coherent cohomology.

For $m \in \mathbb{Z}$, we denote by $\mathscr{E}_X(m)$ the abelian subgroup of $\mathscr{E}_X$ of microdifferential operators of order less or equal to m. For a section P of $\mathscr{E}_X$, we denote by $\sigma(P)$ the principal symbol of P.

Let π_X denote the natural projection $T^*X \to X$. Let us recall that $\mathscr{E}_X$ is flat over $\pi^{-1}(\mathcal{D}_X)$. To a $\mathcal{D}_X$-module $\mathscr{M}$, we associate an $\mathscr{E}_X$-module defined by

$$\mathscr{E}\mathscr{M} := \mathscr{E}_X \otimes_{\pi_X^{-1}\mathcal{D}_X} \pi_X^{-1}\mathscr{M}$$

Let us notice that

$$\mathscr{M} \simeq \mathscr{E}\mathscr{M}|_{T^*_X X}$$

To a morphism of manifolds $f\colon Y \to X$, we associate the diagram of natural morphisms:

$$\begin{tikzcd} TY \arrow[r, "Tf"] \arrow[dr, "\tau_Y"'] & Y \times_X TX \arrow[r, "f_\tau"] \arrow[d, "\tau"] & TX \arrow[d, "\tau_X"] \\ & Y \arrow[r, "f"] & X \end{tikzcd} \tag{2.3.1}$$

and the dual diagram

$$\begin{tikzcd} T^*Y \arrow[dr, "\pi_Y"'] & Y \times_X T^*X \arrow[l, "f_d"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"] & T^*X \arrow[d, "\pi_X"] \\ & Y \arrow[r, "f"] & X \end{tikzcd} \tag{2.3.2}$$

where f_d is the transposed of the tangent map $Tf\colon TY \to Y \times_X TX$.

We denote by $\mathscr{E}_{Y\to X}, \mathscr{E}_{X\leftarrow Y}$ the transfer bimodules and for $\mathcal{M}, \mathcal{N}$ objects of respectively $\mathrm{D}^{\mathrm{b}}(\mathscr{E}_X)$ and $\mathrm{D}^{\mathrm{b}}(\mathscr{E}_Y)$, one defines the functors $\underline{f}^{-1}_{\mathscr{E}}$ and $\underline{f}^{\mathscr{E}}_{*}$ by

$$\underline{f}^{-1}_{\mathscr{E}}(\mathcal{M}) := \mathrm{R}f_{d*}(\mathscr{E}_{Y\to X} \overset{\mathrm{L}}{\otimes}_{f_\pi^{-1}\mathscr{E}_X} f_\pi^{-1}\mathcal{M})$$

$$\underline{f}^{\mathscr{E}}_{*}(\mathcal{N}) := \mathrm{R}f_{\pi*}(\mathscr{E}_{X\leftarrow Y} \overset{\mathrm{L}}{\otimes}_{f_d^{-1}\mathscr{E}_Y} f_d^{-1}\mathcal{N})$$

Under specific assumptions, the extension of $\mathcal{D}$-modules to $\mathscr{E}$-modules commutes with the push forward and the pull-back operation of $\mathscr{E}$-modules ([SS94]).

Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$. We assume that f is non-characteristic for $\mathcal{M}$, then we have

$$\mathscr{E}(\underline{f}^{-1}\mathcal{M}) \simeq \underline{f}^{-1}_{\mathscr{E}}(\mathscr{E}\mathcal{M})$$

Let $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Y)$, and assume f is proper on $supp(\mathcal{N})$, then we have

$$\mathscr{E}(\underline{g}_*\mathcal{N}) \simeq \underline{g}^{\mathscr{E}}_*(\mathscr{E}\mathcal{N})$$

For $\mathscr{M}$ a left coherent $\mathscr{E}_X$-module generated by a section $u \in \mathscr{M}$, we denote by $\mathscr{I}_{\mathscr{M}}$ the annihilator left ideal of $\mathscr{E}_X$ given by:

$$\mathscr{I}_{\mathscr{M}} := \{P \in \mathscr{E}_X; Pu = 0\}$$

and by $\overline{\mathscr{I}}_{\mathscr{M}}$ the symbol ideal associated to $\mathscr{I}_{\mathscr{M}}$:

$$\overline{\mathscr{I}}_{\mathscr{M}} := \{\sigma(P); P \in \mathscr{I}_{\mathscr{M}}\}$$

Definition 2.3.1 ([Kas03])**.** Let $\mathscr{M}$ be a coherent $\mathscr{E}_X$-module generated by an element $u \in \mathscr{M}$. We say that $(\mathscr{M}, u)$ is a simple $\mathscr{E}_X$-module if $\overline{\mathscr{I}}_{\mathscr{M}}$ is reduced and $\overline{\mathscr{I}}_{\mathscr{M}} = \{\varphi \in \mathscr{O}_{T^*X}; \varphi|_{supp(\mathscr{M})} = 0\}$.

Consider two complex manifolds X and Y, open subsets U and V of $\dot{T}^*X$ and $\dot{T}^*Y$, respectively, and denote by p_1 and p_2 the projections $U \xleftarrow{p_1} U \times V^a \xrightarrow{p_2} V$. Let Λ be a smooth closed submanifold Lagrangian of $U \times V^a$. We will make use of the following result from [SKK, Th. 4.3.1], [Kas03, Prop. 8.5]:

Theorem 2.3.2 ([SKK],[Kas03])**.** *Let $(\mathscr{M}, u)$ be a simple $\mathscr{E}_{X\times Y}$-module defined on $U \times V^a$ such that* $\operatorname{supp}\mathscr{M} = \Lambda$. *Assume $\Lambda \to U$ is a diffeomorphism. Then, there is an isomorphism of $\mathscr{E}_X$-modules:*

$$\begin{array}{rcl} \mathscr{E}_X|_U & \xrightarrow{\sim} & (p_1|_{U\times V^a})_*\mathscr{M} \\ P & \mapsto & P \cdot u \end{array}$$

Assume that the projections $p_1|_\Lambda$ and $p_2^a|_\Lambda$ induce isomorphisms. We denote by χ the homogeneous symplectic isomorphism $\chi := p_2|_\Lambda \circ p_1|_\Lambda^{-1}$,

(2.3.3)
$$\begin{tikzcd} & \Lambda \subset U \times V^a \arrow[dl, "\sim", "p_1|_\Lambda"'] \arrow[dr, "\sim", "p_2^a|_\Lambda"] & \\ \dot{T}^*X \supset U \arrow[rr, "\sim", "\chi"'] & & V \subset \dot{T}^*Y \end{tikzcd}$$

Corollary 2.3.3. *Let $(\mathscr{M}, u)$ be a simple $\mathscr{E}_{X\times Y}$-module defined on $U \times V^a$. Assume* $\operatorname{supp}\mathscr{M} = \Lambda$. *Then, in the situation of (2.3.3), we have an anti-isomorphism of algebras*

$$\chi_*\mathscr{E}_X|_U \simeq \mathscr{E}_Y|_V$$

2.4 Hyperfunctions and microfunctions

Let M be a real analytic manifold and X a complexification of M. We denote by $\mathscr{A}_M$ the sheaf of complex valued real analytic functions on M:

$$\mathscr{A}_M := \mathscr{O}_X \otimes \mathbb{C}_M$$

by $\mathscr{B}_M$ the sheaf of hyperfunctions on M:

$$\mathscr{B}_M := \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X)$$

and by $\mathscr{C}_M$ the sheaf of microfunctions on $T^*_M X$:

$$\mathscr{C}_M := \mu hom(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X)$$

Let us notice that $\mathscr{B}_M$ and $\mathscr{C}_M$ are concentrated in degree 0. Let us denote by sp, the isomorphism

$$sp\colon \mathscr{B}_M \xrightarrow{\sim} \mathrm{R}\pi_{M*}\mathscr{C}_M \tag{2.4.1}$$

There is a natural action of the sheaf of microdifferential operators $\mathscr{E}_X$ on $\mathscr{C}_M$.

Given that, $\mathrm{D}'_X\mathbb{C}_M \simeq \omega_{M/X} \simeq \omega_M^{\otimes -1}$, and denoting by $\mathrm{or}_{X/M}$ the relative orientation sheaf, we get that

$$\begin{aligned}\mathscr{B}_M &\simeq \mathrm{R}\Gamma_M(\mathscr{O}_X)\otimes\omega_{M/X} \simeq H^n_M(\mathscr{O}_X)\otimes \mathrm{or}_{M/X},\\ \mathscr{C}_M &\simeq \mu_M(\mathscr{O}_X)\otimes\omega_{M/X} \simeq H^n(\mu_M(\mathscr{O}_X))\otimes \mathrm{or}_{M/X}\,.\end{aligned}$$

If Z is a closed complex submanifold of X of codimension d, we note

$$\mathscr{B}_{Z|X} := H^d_{[Z]}(\mathcal{O}_X)$$

the algebraic cohomology of $\mathcal{O}_X$ with support in Z.

2.5 Integral transforms for sheaves and $\mathcal{D}$-modules

2.5.1 Integral transforms for sheaves

Let X and Y be complex manifolds of respective dimension d_X, d_Y. Let S be a closed submanifold $X\times Y$ of dimension d_S. We set $d_{S/X} := d_S - d_X$. Consider the diagram of complex manifolds

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y, \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{S} \arrow[dl, "g"'] \arrow[dr, "f"] & \\ Y & & X \end{tikzcd} \tag{2.5.1}$$

where the second diagram is obtained by interchanging X and Y.

Let $F\in \mathrm{D^b}(\mathbb{C}_X)$, $G\in \mathrm{D^b}(\mathbb{C}_Y)$, we define

$$\Phi_S(F) := \mathrm{R}g_! f^{-1}F[d_{S/Y}], \qquad \Phi_{\widetilde{S}}(G) := \mathrm{R}f_! g^{-1}G[d_{S/X}]$$

$$\Psi_S(F) := \mathrm{R}g_* f^! F[d_{X/S}], \qquad \Psi_{\widetilde{S}}(G) := \mathrm{R}f_* g^! G[d_{Y/S}]$$

For $K \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X\times Y})$, and given the diagram $X \xleftarrow{q_1} X\times Y \xrightarrow{q_2} Y$, we define the Radon transform of F with kernel K

$$\Phi_K(F) := \mathrm{R}q_{2!}(K \otimes q_1^{-1}F)$$

When $K = \mathbb{C}_S[d_{S/Y}]$, we recover the functor Φ_S.

We have the adjunction formulae:

$$\mathrm{RHom}\,(\Phi_{\widetilde{S}}(G), F) \simeq \mathrm{RHom}\,(G, \Psi_S(F)),$$

$$\mathrm{R}\Gamma_c(X; \Phi_{\widetilde{S}}(G) \otimes F)[d_X] \simeq \mathrm{R}\Gamma_c(Y; G \otimes \Psi_S(F))[d_Y]$$

2.5.2 Integral transforms for $\mathcal{D}$-modules

Let X, Y be complex manifolds of equal dimension $n > 0$, and S a complex manifold. Consider again the situation (2.5.1).

We suppose

$$\left\{\begin{array}{l} f, g \text{ are smooth and proper,} \\ S \text{ is a complex submanifold of } X\times Y \text{ of codimension } c > 0 \end{array}\right. \tag{2.5.2}$$

Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$, $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y)$. Let us denote by $\widetilde{S}$ the image of S by the map $r : X\times Y \to Y\times X, (x,y) \mapsto (y,x)$. One sets

$$\underline{\Phi}_S(\mathcal{M}) := \underline{g}_*\underline{f}^{-1}\mathcal{M}, \qquad \underline{\Phi}_{\widetilde{S}}(\mathcal{N}) := \underline{f}_*\underline{g}^{-1}\mathcal{N}$$

Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, assume that f is non characteristic for $\mathcal{M}$ and let $G \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y)$. The functorial properties of inverse and direct image of $\mathcal{D}$-modules leads to the following adjonction formulae, proven in [DS94, Prop. 2.6.],

$$\begin{aligned} \Phi_S \mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) &\simeq \mathrm{R}\mathscr{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{M}, \mathcal{O}_Y) \\ \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}, \Phi_{\widetilde{S}}(G)\otimes\mathcal{O}_X)[d_X] &\simeq \mathrm{RHom}_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{M}, G\otimes\mathcal{O}_Y)[d_Y] \\ \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}\otimes\Phi_{\widetilde{S}}(G), \mathcal{O}_X)[d_X] &\simeq \mathrm{RHom}_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{M}\otimes G, \mathcal{O}_Y)[d_Y] \end{aligned} \tag{2.5.3}$$

Let us recall that we denote by Ω_X the sheaf of holomorphic n-forms and let

$$\mathscr{B}^{(n,0)}_{S|X\times Y} := q_1^{-1}\Omega_X \otimes_{q_1^{-1}\mathcal{O}_X} \mathscr{B}_{S|X\times Y}$$

This $(\mathcal{D}_Y, \mathcal{D}_X)$-bimodule allows the computation of $\underline{\Phi}_S$ because of the isomorphism, proven in [DS94, Prop 2.12]

$$\mathcal{D}_{Y\leftarrow S} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_S} \mathcal{D}_{S\to X} \xrightarrow{\sim} \mathscr{B}^{(n,0)}_{S|X\times Y}$$

leading to

$$\underline{\Phi}_S(\mathcal{M}) \simeq \mathrm{R}q_{2!}(\mathscr{B}^{(n,0)}_{S|X\times Y} \overset{\mathrm{L}}{\otimes}_{q_1^{-1}\mathcal{D}_X} q_1^{-1}\mathcal{M})$$

2.6 Microlocal integral transforms

2.6.1 Integral transforms for $\mathscr{E}$-modules

Let X, Y be complex manifolds and S is a closed submanifold of $X \times Y$. We consider again the diagram (2.5.1) under the hypothesis (2.5.2).

We define the functor

$$\mathrm{D}^{\mathrm{b}}(\mathcal{E}_X) \to \mathrm{D}^{\mathrm{b}}(\mathcal{E}_Y),\ \underline{\Phi}^{\mu}_S(\mathcal{M}) := \underline{g}^{\mathscr{E}}_* \underline{f}^{-1}_{\mathscr{E}} \mathcal{M}$$

We define the $\mathscr{E}_{X\times Y}$-module attached to $\mathscr{B}_{S|X\times Y}$,

$$\mathscr{C}_{S|X\times Y} := \mathscr{E}\mathscr{B}_{S|X\times Y}$$

and we consider the $(\mathscr{E}_Y, \mathscr{E}_X)$-bimodule

$$\mathscr{C}^{(n,0)}_{S|X\times Y} := \pi^{-1} q_1^{-1} \Omega_X \overset{\mathrm{L}}{\otimes}_{\pi^{-1} q_1^{-1} \mathcal{O}_X} \mathscr{C}_{S|X\times Y} \tag{2.6.1}$$

One can notice that

$$\mathscr{E}_Y \leftarrow {}_S\overset{\mathrm{L}}{\otimes}_{\mathscr{E}_S} \mathscr{E}_S \rightarrow X \xrightarrow{\sim} \mathscr{C}^{(n,0)}_{S|X\times Y}$$

and hence, we have

$$\underline{\Phi}^{\mu}_S(\mathcal{M}) \simeq \mathrm{R}p^a_{2!}(\mathscr{C}^{(n,0)}_{S|X\times Y} \overset{\mathrm{L}}{\otimes}_{p_1^{-1}\mathscr{E}_X} p_1^{-1}\mathcal{M}) \tag{2.6.2}$$

Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$. The functors $\underline{\Phi}^{\mu}_S$ and $\underline{\Phi}_S$ are linked through the following isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\dot{T}^*Y})$, (see [SS94])

$$\mathscr{E}(\underline{\Phi}_S(\mathcal{M})) \simeq \underline{\Phi}^{\mu}_S(\mathscr{E}\mathcal{M}) \tag{2.6.3}$$

2.6.2 Microlocal integral transform of the structure sheaf

Consider two open subsets U and V of T^*X and T^*Y, respectively and Λ a closed complex Lagrangian submanifold of $U \times V^a$

$$\begin{tikzcd} & U \times V^a \arrow[dl, "p_1"'] \arrow[dr, "p_2{}^a"] & \\ T^*X \supset U & & V \subset T^*Y \end{tikzcd} \tag{2.6.4}$$

As detailed in Section 11.4 of [KS90], let $K \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X\times Y})$, $SS(K)$ its micro-support and let us suppose that

(i) $p_1|\Lambda : \Lambda \to U$ and $p^a_2|\Lambda : \Lambda \to V$ are isomorphisms

(ii) K is cohomologically constructible

(iii) $(p_1^{-1}(U) \cup p_2^{-1}(V)) \cap SS(K) \subset \Lambda$

(iv) K is simple with shift 0 along Λ

Let $p = (p_X, p_Y^a) \in \Lambda$ and let us consider some section $s \in H^0(\mu hom(K, \Omega_{X\times Y/Y}))_p$, where $\Omega_{X\times Y/Y} := \mathcal{O}_{X\times Y} \otimes_{q_1^{-1}\mathcal{O}_X} q_1^{-1}\Omega_X$. The section s gives a morphism $K \to \Omega_{X\times Y/Y}$ in $\mathrm{D^b}(\mathbb{C}_{X\times Y}; p)$. Then, we get the sequence of morphisms

$$\begin{aligned}\Phi_{K[d_X]}(\mathcal{O}_X) &= \mathrm{R}q_{2!}(K[d_X] \otimes q_1^{-1}\mathcal{O}_X)\\ &\to \mathrm{R}q_{2!}(\Omega_{X\times Y/Y} \otimes q_1^{-1}\mathcal{O}_X[d_X])\\ &\to \mathrm{R}q_{2!}\Omega_{X\times Y/Y}[d_X]\\ &\to \mathcal{O}_Y\end{aligned} \tag{2.6.5}$$

the last morphism being the integration morphism. We have:

Theorem 2.6.1 ([KS90, Th.11.4.9])**.** *There exists $s \in H^0(\mu hom(K, \Omega_{X\times Y/Y}))_p$ such that the associated morphism $\Phi_{K[d_X]}(\mathcal{O}_X) \to \mathcal{O}_Y$ is an isomorphism in the category $\mathrm{D^b}(\mathbb{C}_Y; p_Y)$. Moreover, this morphism is compatible with the action of microdifferential operators on $\mathcal{O}_X$ in $\mathrm{D^b}(\mathbb{C}_X; p_X)$ and the action of microdifferential operators on $\mathcal{O}_Y$ in $\mathrm{D^b}(\mathbb{C}_Y; p_Y)$*

This is a microlocal version of (1.2.4) with a precision related to the action of microdifferential operators.

2.7 Complements on the functor μhom

2.7.1 Associativity for the composition of kernels

The next result is well-known although no proof is written down in the literature, to our knowledge.

Lemma 2.7.1. *Let M_1, M_2, M_3 be real manifolds, and K, L, M be objects respectively of $\mathrm{D^b}(\mathbf{k}_{M_{12}})$, $\mathrm{D^b}(\mathbf{k}_{M_{23}}), \mathrm{D^b}(\mathbf{k}_{M_{34}})$, then the composition of kernels $\circ$ defined in 2.1.1 is associative. We have the following isomorphism*

$$(K \underset{2}{\circ} L) \underset{3}{\circ} M \xrightarrow{\sim} K \underset{2}{\circ} (L \underset{3}{\circ} M) \tag{2.7.1}$$

such that for any $N \in \mathrm{D^b}(\mathbf{k}_{M_{45}})$, the diagram below commutes:

$$\begin{tikzcd}
((K \underset{2}{\circ} L) \underset{3}{\circ} M) \underset{4}{\circ} N \arrow[r] \arrow[d] & (K \underset{2}{\circ} L) \underset{3}{\circ} (M \underset{4}{\circ} N) \arrow[dd] \\
(K \underset{2}{\circ} (L \underset{3}{\circ} M)) \underset{4}{\circ} N \arrow[d] & \\
K \underset{2}{\circ} ((L \underset{3}{\circ} M) \underset{4}{\circ} N) \arrow[r] & K \underset{2}{\circ} (L \underset{3}{\circ} (M \underset{4}{\circ} N)).
\end{tikzcd} \tag{2.7.2}$$

Proof. Consider the following diagram

$$\begin{tikzcd}
& & & M_{1234} \arrow[dl, "q^3_{124}"'] \arrow[dr, "q^2_{134}"] \arrow[ddd, "q^{23}_{14}"] \arrow[ddll, bend right, "q^4_{12}"'] \arrow[ddrr, bend left, "q^1_{34}"] & & & \\
& & M_{124} \arrow[dr, "q^2_{14}"] \arrow[ddrr, "q^1_{24}"] & & M_{134} \arrow[dl, "q^3_{14}"'] \arrow[ddll, "q^4_{13}"'] & & \\
& M_{123} \arrow[dl, "q^3_{12}"'] \arrow[d, "q^2_{13}"] \arrow[drr, "q^1_{23}"] & & M_{14} & & M_{234} \arrow[d, "q^3_{24}"'] \arrow[dl, "q^2_{34}"] \arrow[dll, "q^4_{23}"'] & \\
M_{12} & M_{13} & & M_{23} & & M_{24} & M_{34}
\end{tikzcd}$$

where the thick squares are cartesian, and where for clarity we enforced the notation: the projection $M_{ijk} \to M_{ij}$ by q^k_{ij} (independently of order of appearence of the indices), and the projection $M_{ijkl} \to M_{ij}$ by q^{kl}_{ij}. We now have:

$$\begin{aligned}
\mathrm{R}q^3_{14!}({q^4_{13}}^{-1}(\mathrm{R}q^2_{13!}({q^3_{12}}^{-1}K \otimes {q^1_{23}}^{-1}L)) \otimes {q^1_{34}}^{-1}M) &\simeq \mathrm{R}q^3_{14!}(\mathrm{R}q^2_{134!}({q^{34}_{12}}^{-1}K \otimes {q^{14}_{23}}^{-1}L) \otimes {q^1_{34}}^{-1}M) \\
&\simeq \mathrm{R}q^{23}_{14!}({q^{34}_{12}}^{-1}K \otimes {q^{14}_{23}}^{-1}L \otimes {q^{12}_{34}}^{-1}M) \\
&:= K_1 \underset{2}{\circ} K_2 \underset{3}{\circ} K_3
\end{aligned}$$

The same way, we get the isomorphism

$$K_1 \underset{2}{\circ} K_2 \underset{3}{\circ} K_3 \simeq \mathrm{R}q^2_{14!}({q^4_{12}}^{-1}K \otimes ({q^1_{24}}^{-1}(\mathrm{R}q^3_{24!}({q^4_{23}}^{-1}L \otimes {q^2_{34}}^{-1}M))))$$

which proves the isomorphism (2.7.1). And, it follows immediately that given $N \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{45}})$, the diagram (2.7.2) commutes.

□

2.7.2 Associativity for the composition of μhom

We define the composition of kernels on cotangent bundles (see [KS90, section 3.6, (3.6.2)]).

$$\begin{aligned}
\overset{a}{\underset{2}{\circ}} :\ & \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*M_{12}}) \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*M_{23}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*M_{13}}) \\
& (K_1, K_2) \mapsto K_1 \overset{a}{\underset{2}{\circ}} K_2 := \mathrm{R}p_{13!}(p^{-1}_{12^a}K_1 \otimes p^{-1}_{23}K_2) \\
& \qquad\qquad \simeq \mathrm{R}p_{13^a!}(p^{-1}_{12^a}K_1 \otimes p^{-1}_{23^a}K_2).
\end{aligned} \tag{2.7.3}$$

There is a variant of the composition $\circ$, constructed in [KS14]:

$$\begin{aligned}
\underset{2}{*} :\ & \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}}) \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{13}}) \\
& (K_1, K_2) \mapsto K_1 \underset{2}{*} K_2 := \mathrm{R}q_{13*}\big(q_2^{-1}\omega_2 \otimes \delta_2^!(K_1 \overset{\mathrm{L}}{\boxtimes} K_2)\big).
\end{aligned} \tag{2.7.4}$$

There is a natural morphism for $K_1 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}})$ and $K_2 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}})$, $K_1 \underset{2}{\circ} K_2 \to K_1 \underset{2}{*} K_2$.

Let us state a theorem proven in [KS90, Prop. 4.4.11] refined in [KS14].

Theorem 2.7.2. *Let F_i, G_i, H_i respectively in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}}), \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}}), \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{34}})$, $i = 1, 2$. Let U_i be an open subset of T^*M_{ij} ($i = 1, 2$, $j = i + 1$) and set $U_3 = U_1 \overset{a}{\underset{2}{\circ}} U_2$. There exists a canonical morphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*M_{13}})$, functorial in F_1 (resp. F_2):*

$$\mu hom(F_1, F_2)|_{U_1} \overset{a}{\underset{2}{\circ}} \mu hom(G_1, G_2)|_{U_2} \to \mu hom(F_1 \underset{2}{*} G_1, F_2 \underset{2}{\circ} G_2)|_{U_3}. \tag{2.7.5}$$

and hence

$$\mu hom(F_1, F_2)|_{U_1} \overset{a}{\underset{2}{\circ}} \mu hom(G_1, G_2)|_{U_2} \to \mu hom(F_1 \underset{2}{\circ} G_1, F_2 \underset{2}{\circ} G_2)|_{U_3}. \tag{2.7.6}$$

We state the main theorem of this section.

Theorem 2.7.3. *Let F_i, G_i, H_i respectively in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}}), \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}}), \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{34}})$, $i = 1, 2$ then we have:*

(a)

$$\Big(\mu hom(F_1, F_2) \overset{a}{\underset{2}{\circ}} \mu hom(G_1, G_2)\Big) \overset{a}{\underset{3}{\circ}} \mu hom(H_1, H_2) \xrightarrow{\sim} \mu hom(F_1, F_2) \overset{a}{\underset{2}{\circ}} \Big(\mu hom(G_1, G_2) \overset{a}{\underset{3}{\circ}} \mu hom(H_1, H_2)\Big)$$

(b) *The above isomorphism is compatible with the composition $\circ$ in the sense that the following diagram commutes*

$$\begin{tikzcd}
(\mu hom(F_1, F_2) \overset{a}{\underset{2}{\circ}} \mu hom(G_1, G_2)) \overset{a}{\underset{3}{\circ}} \mu hom(H_1, H_2) \arrow[r, "\sim"] \arrow[d] & \mu hom(F_1, F_2) \overset{a}{\underset{2}{\circ}} (\mu hom(G_1, G_2) \overset{a}{\underset{3}{\circ}} \mu hom(H_1, H_2)) \arrow[d] \\
\mu hom(F_1 \underset{2}{\circ} G_1, F_2 \underset{2}{\circ} G_2) \overset{a}{\underset{3}{\circ}} \mu hom(H_1, H_2) \arrow[d] & \mu hom(F_1, F_2) \overset{a}{\underset{2}{\circ}} \mu hom(G_1 \underset{3}{\circ} H_1, G_2 \underset{3}{\circ} H_2) \arrow[d] \\
\mu hom((F_1 \underset{2}{\circ} G_1) \underset{3}{\circ} H_1, (F_2 \underset{2}{\circ} G_2) \underset{3}{\circ} H_2) \arrow[r, "\sim"] & \mu hom(F_1 \underset{2}{\circ} (G_1 \underset{3}{\circ} H_1), F_2 \underset{2}{\circ} (G_2 \underset{3}{\circ} H_2))
\end{tikzcd}$$

Proof. (a) This is a direct application of Lemma 2.7.1 with X, Y, Z taken to be respectively $T^*M_{12}, T^*M_{13}, T^*M_{34}$.

(b) We refer the reader to 6.2 in Appendix for a sketch of proof.

Chapter 3

Complex quantized contact transformations

3.1 Kernels on complex manifolds

Consider two complex manifolds X and Y of respective dimension d_X and d_Y. We shall follow the notations of Section 2.1.1.

For $K \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X\times Y})$, we recall that we defined the functor $\Phi_K : \mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$, $\Phi_K(G) = Rq_{1!}(K \otimes q_2^{-1}(G))$, for $G \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y)$. With regards to the notation of Section 2.1.1, let us notice that $\Phi_K(G)$ is $K \circ G$.

We set

$$\Omega_{X\times Y/X} := \mathcal{O}_{X\times Y} \otimes_{q_2^{-1}\mathcal{O}_Y} q_2^{-1}\Omega_Y$$

and

$$\mathscr{E}_X^{\mathbb{R}} := \mu hom(\mathbb{C}_{\Delta_X}, \Omega_{X\times X/X})[d_X]$$

We recall that $\mathscr{E}_X$ is a subring of $\mathscr{E}_X^{\mathbb{R}}$.

We recall the

Lemma 3.1.1. *There is a natural morphism*

$$\Omega_{X\times Y/X} \circ \mathscr{O}_Y\,[d_Y] \to \mathscr{O}_X.$$

Proof. We have

$$\begin{aligned}\Omega_{X\times Y/X} \circ \mathscr{O}_Y\,[d_Y] &= \mathrm{R}q_{1!}(\Omega_{X\times Y/X} \otimes q_1^{-1}\mathscr{O}_Y[d_Y])\\ &\to \mathrm{R}q_{1!}(\Omega_{X\times Y/X}[d_Y]) \xrightarrow{\int} \mathscr{O}_X,\end{aligned}$$

where the last arrow is the integration morphism on complex manifolds. □

The following Lemma will be useful for the proof of Lemma 3.1.3. Let us first denote by M_i $(i = 1, 2, 3, 4)$ four complex manifolds, $L_i \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{M_{i,i+1}})$, $1 \le i \le 3$. We set for short

$$d_i = \dim_{\mathbb{C}} M_i, d_{ij} = d_i + d_j, \Omega_{ij/i} = \Omega_{M_{ij}/M_i} = \Omega_{M_{ij}}^{(0,d_j)}.$$

Set for $1 \leq i \leq 3$,

$$\begin{aligned}
&K_i = \mu hom(L_i, \Omega_{i,j/i}[d_j]), \quad j = i+1 \\
&L_{ij} = L_i \circ L_j \quad j = i+1, \quad L_{123} = L_1 \circ L_2 \circ L_3, \\
&\widetilde{K}_{ij} = \mu hom(L_{ij}, \Omega_{i,j/i}\,[d_j] \circ \Omega_{j,k/j}\,[d_k]) \quad j = i+1, k = j+1 \\
&\widetilde{K}_{123} = \mu hom(L_{123}, \Omega_{12/1}[d_2] \circ \Omega_{23/2}[d_3] \circ \Omega_{34/3}[d_4]) \\
&K_{ij} = \mu hom(L_{ij}, \Omega_{i,k/i}[d_k]) \quad j = i+1, k = j+1, \\
&K_{123} = \mu hom(L_{123}, \Omega_{14/1}[d_4]).
\end{aligned}$$

We recall that we have the sequence of natural morphisms:

$$\begin{aligned}
\Omega_{i,j/i} \circ \Omega_{j,k/j} &= \mathrm{R}q_{i,k!}(q_{i,j}^{-1}\Omega_{i,j/i} \otimes q_{j,k}^{-1}\Omega_{j,k/j}) \\
&\to \mathrm{R}q_{i,k!}(\Omega_{i,j,k/i}) \\
&\to \Omega_{i,k/i}[-d_j]
\end{aligned} \tag{3.1.1}$$

Lemma 3.1.2. *The following diagram commutes:*

$$\begin{tikzcd}
 & K_1 \circ K_2 \circ K_3 \arrow[dl] \arrow[dr] \arrow[d, phantom, "A"] & \\
\widetilde{K}_{12} \circ K_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & \widetilde{K}_{123} \arrow[d] \arrow[dr, phantom, "C"] & K_1 \circ \widetilde{K}_{23} \arrow[l] \arrow[d] \\
K_{12} \circ K_3 \arrow[r] & K_{123} & K_1 \circ K_{23} \arrow[l]
\end{tikzcd}$$

Proof. Diagram labelled A commutes by the associativity of the functor μhom (see Theorem 2.7.3). Let us prove that Diagram B and C commute. Of course, it is enough to consider Diagram B. To make the notations easier, we assume that $M_1 = M_4 = \mathrm{pt}$. We are reduced to prove the commutativity of the diagram:

$$\begin{tikzcd}
\mu hom(L_2, \Omega_2 \circ \Omega_{2,3/2}[d_{23}]) \circ \mu hom(L_3, \mathscr{O}_3) \arrow[r] \arrow[d, "\int_2"] & \mu hom(L_{23}, \Omega_2 \circ \Omega_{2,3/2}[d_{23}] \circ \mathscr{O}_3) \arrow[d, "\int_2"] \\
\mu hom(L_2, \Omega_3[d_3]) \circ \mu hom(L_3, \mathscr{O}_3) \arrow[r] & \mu hom(L_{23}, \Omega_3[d_3] \circ \mathscr{O}_3)
\end{tikzcd}$$

For $F, F' \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{12})$, $G, G' \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{23})$, we saw in Theorem 2.7.3 (b) that the morphism $\mu hom(F, F') \circ \mu hom(G, G') \to \mu hom(F \circ G, F' \circ G')$ is functorial in F, F', G, G'. This fact applied to the morphism

$$\Omega_2 \circ \Omega_{2,3/2}[d_{23}] \to \Omega_3[d_3]$$

gives that the above diagram commutes and so diagram B commutes. $\square$

Let Z be a complex manifold and let $\Lambda \subset T^*(X \times Y)$ and $\Lambda' \subset T^*(Y \times Z)$ be two conic Lagrangian smooth locally closed complex submanifolds.

Let L, L', be perverse sheaves on $X \times Y$, $Y \times Z$, with microsupport $SS(L) \subset \Lambda$, $SS(L') \subset \Lambda'$ respectively. We set

$$L'' := L[d_Y] \circ L'$$

Assume that

$$p_2^a|_\Lambda : \Lambda \to T^*Y \text{ and } p_2|_{\Lambda'} : \Lambda' \to T^*Y \text{ are transversal} \tag{3.1.2}$$

and that

$$\text{the map } \Lambda \times_{T^*Y} \Lambda' \to \Lambda \circ \Lambda' \text{ is an isomorphism.} \tag{3.1.3}$$

Let us set

$$\mathscr{L} := \mu hom(L, \Omega_{X\times Y/X})$$

Note that $\mathscr{L} \in \mathrm{D^b}(T^*(X \times Y))$ is concentrated in degree 0. Indeed, it is proven in [KS90, Th. 10.3.12] that perverse sheaves are the ones which are pure with shift zero at any point of the non singular locus of their microsupport. On the other hand, Theorem 9.5.2 of [KS85] together with Definition 9.5.1 of [KS85] show that the latter verify the property that, when being applied $\mu hom(\bullet, \Omega_{X\times Y/X})$, they are concentrated in degree 0. Moreover, $\mathscr{L}$ is a $(\mathscr{E}_X, \mathscr{E}_Y)$-bimodule. Indeed, such actions come from morphism (2.7.6) and the integration morphism (3.1.1). We define similarly $\mathscr{L}'$ and $\mathscr{L}''$.

Now consider two open subsets U,V and W of $\dot{T}^*X$, $\dot{T}^*Y$, $\dot{T}^*Z$, respectively.

$K_{U\times V^a}$ the constant sheaf on $(U \times V^a) \cap \Lambda$ with stalk $H^0\mathrm{R}\Gamma(U \times V^a; \mathscr{L})$, extended by 0 elsewhere
$K'_{V\times W^a}$ the constant sheaf on $(V \times W^a) \cap \Lambda'$ with stalk $H^0\mathrm{R}\Gamma(V \times W^a; \mathscr{L}')$, extended by 0 elsewhere
$K''_{U\times W^a}$ the constant sheaf on $(U \times W^a) \cap \Lambda \circ \Lambda'$ with stalk $H^0\mathrm{R}\Gamma(U \times W^a; \mathscr{L}'')$, extended by 0 elsewhere

Let s, s' be sections of $\Gamma(U \times V^a; \mathscr{L})$ and $\Gamma(V \times W^a; \mathscr{L}')$ respectively. We define the product $s \cdot s'$ to be the section of $\Gamma(U \times W^a; \mathscr{L}'')$, image of 1 by the following sequence of morphisms

$$\begin{aligned}
\mathbb{C}_{\Lambda\circ\Lambda'} &\xleftarrow{\sim} \mathbb{C}_\Lambda \circ \mathbb{C}_{\Lambda'} \\
&:= \mathrm{R}p_{13!}(p_{12^a}^{-1}\mathbb{C}_\Lambda \otimes p_{23}^{-1}\mathbb{C}_{\Lambda'}) \\
&\to \mathrm{R}p_{13!}(p_{12^a}^{-1}K_{U\times V^a} \otimes p_{23}^{-1}K_{V\times W^a}) \\
&\to \mathrm{R}p_{13!}(p_{12^a}^{-1}\mu hom(L, \Omega_{X\times Y/X}) \otimes p_{23}^{-1}\mu hom(L', \Omega_{Y\times Z/Y})) \\
&:= \mu hom(L, \Omega_{X\times Y/X}) \circ \mu hom(L', \Omega_{Y\times Z/Y}) \to \mathscr{L}''
\end{aligned}$$

where the first isomorphism comes from the assumption 3.1.3.

Lemma 3.1.3. *Assume that conditions 3.1.2 and 3.1.3 are satisfied. Let s, s' be sections of $\Gamma(U \times V^a; \mathscr{L})$ and $\Gamma(V \times W^a; \mathscr{L}')$ respectively, and let $G \in \mathrm{D^b}(\mathbb{C}_Y)$, $H \in \mathrm{D^b}(\mathbb{C}_Z)$. Then,*

(i) *s defines a morphism*

$$\alpha_G(s) \colon \mathbb{C}_\Lambda \circ \mu hom(G, \mathscr{O}_Y)|_V \to \mu hom(L[d_Y] \circ G, \mathscr{O}_X)|_U$$

(ii) *Considering the morphism*

$$\alpha_H(s\cdot s')\colon \mathbb{C}_{\Lambda\circ\Lambda'}\circ \mu hom(H,\mathscr{O}_Z)|_W \to \mu hom(L[d_Y]\circ L'[d_Z]\circ H,\mathscr{O}_X)|_U$$

we have the isomorphism

$$\alpha_H(s\cdot s')\simeq \alpha_{L'[d_Z]\circ H}(s)\circ \Phi_{\mathbb{C}_\Lambda}(\alpha_H(s'))$$

Proof. (i) Given s and two objects $G_1, G_2\in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y)$, we have a morphism

$$\mathbb{C}_\Lambda\circ \mu hom(G_1,G_2)|_V \to \mu hom(L\circ G_1,\Omega_{X\times Y/X}\circ G_2)|_U$$

corrresponding to the composition of morphisms:

$$\begin{aligned}\mathrm{R}p_{1!}(\mathbb{C}_\Lambda\otimes p_{2^a}^{-1}\mu hom(G_1,G_2)|_V) &\to \mathrm{R}p_{1!}(K_{U\times V^a}\otimes p_{2^a}^{-1}\mu hom(G_1,G_2)|_V)\\ &\to \mathrm{R}p_{1!}(\mu hom(L,\Omega_{X\times Y/X})\otimes p_{2^a}^{-1}\mu hom(G_1,G_2)|_V)\\ &\to \mu hom(L\circ G_1,\Omega_{X\times Y/X}\circ G_2)|_U\end{aligned} \tag{3.1.4}$$

where the second morphism comes from the natural morphism $K_{U\times V^a}\to \mu hom(L,\Omega_{X\times Y/X})$. We conclude by choosing, $G_1=G$, $G_2=\mathcal{O}_Y$ and by using Lemma 3.1.1:

$$\mu hom(L\circ G_1,\Omega_{X\times Y/X}\circ \mathcal{O}_Y)\ \to\ \mu hom(L\circ G_1,\mathcal{O}_X[-d_Y])\ \xrightarrow{\sim}\ \mu hom(L[d_Y]\circ G_1,\mathcal{O}_X)$$

(ii) Let $H\in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_Z)$. We denote by $\mathscr{H}:=\mu hom(H,\mathscr{O}_Z)$. It suffices to prove that the following diagram commutes:

$$
\begin{tikzcd}
(\mathbb{C}_\Lambda \circ \mathbb{C}_{\Lambda'}) \circ \mathscr{H} \arrow[rr, "\simeq"] \arrow[d, "\simeq"'] \arrow[ddrr, dotted, "\Phi_{\mathbb{C}_\Lambda}(\alpha_H(s'))"] & & \mathbb{C}_\Lambda \circ (\mathbb{C}_{\Lambda'} \circ \mathscr{H}) \arrow[d] \\
\mathbb{C}_\Lambda \circ \mathbb{C}_{\Lambda'} \circ \mathscr{H} \arrow[d] \arrow[ddrr, dotted, "\alpha(s \cdot s')"] & & \mathbb{C}_\Lambda \circ K' \circ \mathscr{H} \arrow[d] \\
K \circ K' \circ \mathscr{H} \arrow[dd] & & \mathbb{C}_\Lambda \circ \mu hom(L', \Omega_{Y \times Z/Y}) \circ \mathscr{H} \arrow[d] \\
& A & \mathbb{C}_\Lambda \circ \mu hom(L' \circ H, \Omega_{Y \times Z/Y} \circ \mathscr{O}_Z) \arrow[d, "\int_Z"] \\
& & \mathbb{C}_\Lambda \circ \mu hom(L'[d_Z] \circ H, \mathscr{O}_Y) \arrow[d] \arrow[ddd, dotted, bend left, "\alpha_{L'[d_Z] \circ H}(s)"] \\
K \circ \mu hom(L', \Omega_{Y \times Z/Y}) \circ \mathscr{H} \arrow[rr, "\int_Z"] \arrow[d] & & K \circ \mu hom(L'[d_Z] \circ H, \mathscr{O}_Y) \arrow[d] \\
& B & \\
\mu hom(L, \Omega_{X \times Y/X}) \circ \mu hom(L', \Omega_{Y \times Z/Y}) \circ \mathscr{H} \arrow[r, "\int_Z"] \arrow[dd] & & \mu hom(L, \Omega_{X \times Y/X}) \circ \mu hom(L'[d_Z] \circ H, \mathscr{O}_Y) \arrow[d] \\
& C & \mu hom(L \circ L'[d_Z] \circ H, \Omega_{X \times Y/X} \circ \mathscr{O}_Y) \arrow[d, "\int_Y"] \\
\mu hom(L \circ L' \circ H, \Omega_{X \times Y/X} \circ \Omega_{Y \times Z/Y} \circ \mathscr{O}_Z) \arrow[rr, "\int_{Y,Z}"] & & \mu hom(L[d_Y] \circ L'[d_Z] \circ H, \mathscr{O}_X)
\end{tikzcd}
$$

where we omitted the subscript $U \times V^a$ and $V \times W^a$, H, $L'[d_Z] \circ H$ for $K_{U \times V^a}$, $K'_{V \times W^a}$, $\alpha_H, \alpha_{L'[d_Z] \circ H}$, respectively.

We know from Theorem 2.7.2 that the operation $\circ$ is functorial, so that diagram

A and B commute. For instance, diagram A decomposes this way:

$$\begin{tikzcd}
\mathbb{C}_\Lambda \circ \mathbb{C}_{\Lambda'} \circ \mathscr{H} \arrow[rr, "\simeq"] \arrow[d] & & \mathbb{C}_\Lambda \circ (\mathbb{C}_{\Lambda'} \circ \mathscr{H}) \arrow[d] \\
\mathbb{C}_\Lambda \circ K' \circ \mathscr{H} \arrow[r] \arrow[dd] & \mathbb{C}_\Lambda \circ \mu hom(L', \Omega_{Y\times Z/Y}) \circ \mathscr{H} \arrow[r, "\simeq"] \arrow[dd] & \mathbb{C}_\Lambda \circ (\mu hom(L', \Omega_{Y\times Z/Y}) \circ \mathscr{H}) \arrow[d, "\int_Y"] \\
& & \mathbb{C}_\Lambda \circ \mu hom(L'[d_Z] \circ H, \mathscr{O}_Y) \arrow[d] \\
K \circ K' \circ \mathscr{H} \arrow[r] & K \circ \mu hom(L', \Omega_{Y\times Z/Y}) \circ \mathscr{H} \arrow[r, "\int_Y"] & K \circ \mu hom(L'[d_Z] \circ H, \mathscr{O}_Y)
\end{tikzcd}$$

Besides, diagram C commutes by Lemma 3.1.2.

Finally, the bottom diagonal punctured line correponds to $\alpha(s \cdot s')$, since the following diagram commutes

$$\begin{tikzcd}
\mathbb{C}_{\Lambda\circ\Lambda'} \arrow[r, "\simeq"] \arrow[d] \arrow[dr, dotted, "\alpha(s\cdot s')"] & \mathbb{C}_\Lambda \circ \mathbb{C}_{\Lambda'} \arrow[d] \\
K''_{U\times W^a} \arrow[r] & \mu hom(L \circ L'[d_Y], \Omega_{X\times Z/X})
\end{tikzcd}$$

□

Remark 3.1.4. In the following of this thesis, unless necessary, we will omit the subsript for α.

Theorem 3.1.5. *Let $s \in \Gamma(U \times V^a; \mathscr{L})$, $G \in \mathrm{D^b}(\mathbb{C}_Y)$. Then,*
(i) s defines a morphism

$$\alpha(s)\colon \mathbb{C}_\Lambda \circ \mu hom(G, \mathscr{O}_Y)|_V \to \mu hom(L[d_Y] \circ G, \mathscr{O}_X)|_U. \tag{3.1.5}$$

(ii) Moreover, if $P \in \Gamma(U; \mathscr{E}_X)$ and $Q \in \Gamma(V; \mathscr{E}_Y)$ satisfy $P \cdot s = s \cdot Q$, then the diagram below commutes

$$\begin{tikzcd}
\mathbb{C}_\Lambda \circ \mu hom(G, \mathscr{O}_Y) \arrow[r, "\alpha(s)"] \arrow[d, "\Phi_{\mathbb{C}_\Lambda}(\alpha(Q))"'] & \mu hom(L[d_Y] \circ G, \mathscr{O}_X) \arrow[d, "\alpha(P)"] \\
\mathbb{C}_\Lambda \circ \mu hom(G, \mathscr{O}_Y) \arrow[r, "\alpha(s)"'] & \mu hom(L[d_Y] \circ G, \mathscr{O}_X).
\end{tikzcd} \tag{3.1.6}$$

Proof. (i) is already proven in Lemma 3.1.3.

(ii) With regards to the notation of Lemma 3.1.3, we consider the triplet of manifolds X, X, Y, $\Lambda = \mathbb{C}_{\Delta_X}$, $\mathscr{L} := \mu hom(\mathbb{C}_{\Delta_X}[-n], \Omega_{X\times X/X})$. Then, the assumption 3.1.2 is satisfied and noticing that $\Phi_{\mathbb{C}_{\Delta_X}} \simeq Id_X$, we conclude by Lemma 3.1.3 that

$$\alpha(P)\circ\alpha(s) \simeq \alpha(P\cdot s) \simeq \alpha(s\cdot Q) \simeq \alpha(s)\circ\Phi_{\mathbb{C}_\Lambda}(\alpha(Q))$$

□

3.2 Main theorem

In this section, we will apply Theorem 3.1.5 when we are given a homogeneous symplectic isomorphism. Consider two complex manifolds X and Y of the same dimension n, open subsets U and V of $\dot{T}^*X$ and $\dot{T}^*Y$, respectively, Λ a smooth closed Lagrangian submanifold of $U\times V^a$ and assume that the projections $p_1|_\Lambda$ and $p_2^a|_\Lambda$ induce isomorphisms, hence a homogeneous symplectic isomorphism $\chi\colon U \xrightarrow{\sim} V$:

(3.2.1)
$$\begin{tikzcd} & \Lambda\subset U\times V^a \arrow[dl, "\sim"', "p_1"] \arrow[dr, "\sim", "p_2^a"'] & \\ \dot{T}^*X\supset U \arrow[rr, "\sim", "\chi"'] & & V\subset\dot{T}^*Y \end{tikzcd}$$

We consider a perverse sheaf L on $X\times Y$ satisfying

(3.2.2)
$$(p_1^{-1}(U)\cup p_2^{a\,-1}(V))\cap \mathrm{SS}(L) = \Lambda.$$

and a section s in $\Gamma(U\times V^a; \mu hom(L, \Omega_{X\times Y/X}))$.

Let $G\in \mathrm{D^b}(\mathbb{C}_Y)$. From Theorem 3.1.5 (i), the left composition by s defines the morphism $\alpha(s)$ in $\mathrm{D^b}(\mathbb{C}_U)$:

(3.2.3)
$$\mathbb{C}_\Lambda\circ\mu hom(G,\mathscr{O}_Y)|_V \xrightarrow{\alpha(s)} \mu hom(L[n]\circ G,\mathscr{O}_X)|_U$$

The condition (3.2.2) implies that $\mathrm{supp}(\mu hom(L,\Omega_{X\times Y/X})|_{p_2^{a\,-1}(V)})\subset\Lambda$. Since, p_1 is an isomorphism from Λ to U and that $\chi\circ p_1|_\Lambda = p_2^a|_\Lambda$, we get a morphism in $\mathrm{D^b}(\mathbb{C}_U)$

(3.2.4)
$$\chi^{-1}\mu hom(G,\mathscr{O}_Y)|_V \xrightarrow{\alpha(s)} \mu hom(\Phi_{L[n]}(G),\mathscr{O}_X)|_U$$

Theorem 3.2.1. *Assume that the section s is non-degenerate on Λ. Then, for $G\in \mathrm{D^b}(\mathbb{C}_Y)$, we have the following isomorphism in $\mathrm{D^b}(\mathbb{C}_U)$*

(3.2.5)
$$\chi^{-1}\mu hom(G,\mathscr{O}_Y)|_V \xrightarrow{\sim} \mu hom(\Phi_{L[n]}(G),\mathscr{O}_X)|_U$$

Moreover, this isomorphism is compatible with the action of $\mathscr{E}_Y$ and $\mathscr{E}_X$ on the left and right side of (3.2.5) respectively.

Proof. Let us first prove the following lemma, whose proof is available at the level of germs in [KS90, Th. 11.4.9].

Let us prove that the morphism (3.2.4) is an isomorphism. Let L^* be the perverse sheaf $r^{-1}\mathrm{R}\mathscr{H}om\,(L,\omega_{X\times Y/Y})$ where r is the map $X\times Y\to Y\times X,(x,y)\mapsto(y,x)$. Let s' be a section of $\mu hom(L^*,\Omega_{Y\times X/Y})$, non-degenerate on $r(\Lambda)$, then we apply the same precedent construction to get a natural morphism

$$\chi_*\mu hom(\Phi_{L[n]}(G),\mathscr{O}_X)|_U\to\mu hom(\Phi_{L^*[n]}\circ\Phi_{L[n]}G,\mathscr{O}_Y)|_V\simeq\mu hom(\Phi_{L^*\circ L[n]}G,\mathscr{O}_Y)|_V$$

We know from [KS90, Th. 7.2.1] that $\mathbb{C}_{\Delta_X}\simeq L^*\circ L$, so that we get a morphism in $\mathrm{D^b}(\mathbb{C}_V)$

$$\chi_*\mu hom(\Phi_{L[n]}(G),\mathscr{O}_X)|_U\xrightarrow{\alpha(s')}\mu hom(G,\mathscr{O}_Y)|_V \tag{3.2.6}$$

We must prove that (3.2.4) and (3.2.6) are inverse to each other. By Lemma 3.1.3(ii), we get that the composition of these two morphisms is $\alpha(s'\cdot s)$, with $s'\cdot s\in\mathscr{E}_X$.

For any left $\mathscr{E}_X$-module $\mathscr{M}$, corresponds a right $\mathscr{E}_X$-module $\Omega_X\otimes_{\mathscr{O}_X}\mathscr{M}$. Fixing a non-degenerate form t_X of $\Omega_X|_U$ (resp. t_Y of $\Omega_Y|_V$), we apply now Theorem 2.3.3: s and s' are non-degenerate sections so that $(\mathscr{E}_{X\times Y}|_{U\times V^a},t_X\otimes s)$ and $(\mathscr{E}_{Y\times X}|_{V^a\times U},s'\otimes t_Y)$ are simple and so isomorphic to $(p_1^{-1}\mathscr{E}_X|_U,1)\simeq(p_2^{a-1}\mathscr{E}_Y|_V,1)$. Ω_X resp. Ω_Y being invertible $\mathscr{O}_X$-module resp. $\mathscr{O}_Y$-module, we get as well for the left-right $(\mathscr{E}_X|_U,\mathscr{E}_Y|_V)$ bi-module, resp. left-right $(\mathscr{E}_Y|_V,\mathscr{E}_X|_U)$ bi-module generated by s resp. s', that they are both isomorphic to $p_1^{-1}\mathscr{E}_X|_U\simeq p_2^{a-1}\mathscr{E}_Y|_V$.

Then, following the proof of [KS90, Th. 11.4.9], s and s', define ring isomorphisms associating to each $P\in\mathscr{E}_X(U)$, $P'\in\mathscr{E}_X(U)$, some $Q\in\mathscr{E}_Y(V)$, $Q'\in\mathscr{E}_Y(V)$, such that $P\cdot s=s\cdot Q$, $s'\cdot P'=Q'\cdot s'$, respectively. Hence, we get that $\alpha(s')\circ\alpha(s)$ is an automorphism $\mu hom(G,\mathscr{O}_Y)|_V$, defined by the left action of $s'\cdot s\in\mathscr{E}_X$. Hence, we can choose s' so that $\alpha(s')\circ\alpha(s)$ is the identity.

We are now in a position to prove Theorem 3.2.1: we constructed in the proof of the lemma, for each $P\in p_1^{-1}\mathscr{E}_X|_U$, some $Q\in p_{2^a}^{-1}\mathscr{E}_Y|_V$ such that $P\cdot s=s\cdot Q$ and we can apply Theorem 3.1.5 to conclude. □

Chapter 4

Radon transform for sheaves

We are going to apply the results of the last chapter to the case of projective duality.

4.1 Projective duality: geometry

4.1.1 Notations

We refer to the notations of the sections 1.3 and 2.1. We recall that we denote by

$$\begin{aligned} V,\ \mathbb{V},\ &\text{an } (n+1)\text{-dimensional real and complex vector space, respectively,} \\ P,\ \mathbb{P},\ &\text{the } n-\text{dimensional real and complex projective space, respectively,} \\ S,\ \mathbb{S},\ &\text{the real and complex incidence hypersurface in } P\times P^*,\ \mathbb{P}\times\mathbb{P}^*\text{, respectively.} \end{aligned}$$

When necessary, we will enforce the dimension by noting $\mathbb{P}_n$, resp. $\mathbb{P}_n^*$.

In this section, $\mathbf{k}$ denotes either the field $\mathbb{R}$ or $\mathbb{C}$.

Let X, Y be complex manifolds, we recall that we denote by q_1 and q_2 the respective projection of $X\times Y$ on each of its factor.

For $K\in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X\times Y})$, we recall that we defined the functor:

$$\begin{aligned} &\Phi_K\colon \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)\to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y) \\ &F\mapsto Rq_{2!}(K\otimes q_1^{-1}F) \end{aligned}$$

For an integer k and $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$, we note $k^*=-n-1-k$ and $\varepsilon^*=-n-1-\varepsilon \bmod(2)$. We refer to Section 1.3 for the definition of the sheaves of twisted microfunctions $\mathscr{C}_P(\varepsilon,k), \mathscr{C}_{P^*}(\varepsilon^*,k^*)$.

4.1.2 Geometry of projective duality

For a manifold X, we denote by P^*X the projectivization of the cotangent bundle of X. Let us prove the following:

Proposition 4.1.1. *There is an homogeneous complex symplectic isomorphism*

$$\dot{T}^*\mathbb{P} \simeq \dot{T}^*\mathbb{P}^* \tag{4.1.1}$$

and a contact isomorphism

$$P^*\mathbb{P} \simeq \mathbb{S} \simeq P^*\mathbb{P}^* \tag{4.1.2}$$

Proof. We have the natural morphism

$$\mathbb{V} \setminus \{0\} \xrightarrow{\rho} \mathbb{P}$$

According to 2.3.2, this morphism, after removing the zero section, induces the following diagram

$$\begin{tikzcd}
T^*(\mathbb{V} \setminus \{0\}) \arrow[dr] & \mathbb{V} \setminus \{0\} \times_{\mathbb{P}} T^*\mathbb{P} \arrow[l, "{}^t\rho'"'] \arrow[r] \arrow[d] & T^*\mathbb{P} \arrow[d] \\
& \mathbb{V} \setminus \{0\} \arrow[r] & \mathbb{P}
\end{tikzcd}$$

We notice that ${}^t\rho'$ is an immersion. Let us denote by $\mathbb{H}$, $\mathbb{H}^*$, the incidence hypersurfaces:

$$\mathbb{H} = \{(\xi, x) \in \mathbb{V}^* \times (\mathbb{V} \setminus \{0\}); \langle \xi, x \rangle = 0\}$$
$$\mathbb{H}^* = \{(x, \xi) \in \mathbb{V} \times (\mathbb{V}^* \setminus \{0\}); \langle x, \xi \rangle = 0\}$$

Noticing that for $x \in \mathbb{V} \setminus \{0\}$, ρ is constant along the fiber above $\rho(x)$, we see that ${}^t\rho'$ is an immersion into the incidence hypersurface $\mathbb{H}$. Besides, ${}^t\rho'$ is a morphism of fibered space and so, by a dimensional argument, we conclude that this immersion is also onto.

Removing the zero sections, we get the diagram

$$\begin{tikzcd}
T^*(\mathbb{V} \setminus \{0\}) \arrow[d, "\simeq"'] & \mathbb{H} \arrow[l, hook'] \arrow[d, "\simeq"'] & \mathbb{V} \setminus \{0\} \times_{\mathbb{P}} \dot{T}^*\mathbb{P} \arrow[l] \arrow[r] \arrow[d, "\simeq"'] & \dot{T}^*\mathbb{P} \\
T^*(\mathbb{V}^* \setminus \{0\}) & \mathbb{H}^* \arrow[l, hook'] & (\mathbb{V}^* \setminus \{0\}) \times_{\mathbb{P}^*} \dot{T}^*\mathbb{P}^* \arrow[l] \arrow[r] & \dot{T}^*\mathbb{P}^*
\end{tikzcd} \tag{4.1.3}$$

where the isomophism between $\mathbb{H}$ and $\mathbb{H}^*$ follows from the following symplectic isomorphism:

$$\dot{T}^*(\mathbb{V} \setminus \{0\}) \simeq \dot{T}^*(\mathbb{V}^* \setminus \{0\})$$
$$(x, \xi) \mapsto (\xi, -x)$$

Now, taking the quotient by the action of $\mathbb{C}^*$ on both sides of the isomorphism between $(\mathbb{V} \setminus \{0\}) \times_{\mathbb{P}} \dot{T}^*\mathbb{P}$ and $(\mathbb{V}^* \setminus \{0\}) \times_{\mathbb{P}^*} \dot{T}^*\mathbb{P}^*$, we get the isomorphism:

$$\dot{T}^*\mathbb{P} \simeq (\mathbb{V}^* \setminus \{0\}) \times_{\mathbb{P}^*} P^*\mathbb{P}^* \simeq \dot{T}^*\mathbb{P}^*$$

This gives (4.1.1).

Besides, passing to the quotient by the action of $\mathbb{C}^* \times \mathbb{C}^*$ on the two central columns of diagram (4.1.3), we get (4.1.2). $\square$

Proposition 4.1.2. *Consider the double fibrations*

$$\begin{tikzcd} & \dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*) \arrow[dl, "\sim"', "p_1"] \arrow[dr, "\sim", "p_2^a"'] & \\ \dot{T}^*\mathbb{P} \arrow[rr, "\sim", "\chi"'] & & \dot{T}^*\mathbb{P}^* \end{tikzcd} \tag{4.1.4}$$

Then, p_1 and p_2^a are isomorphisms and $\chi = p_2^a \circ p_1^{-1}$ is a homogeneous symplectic isomorphism.

Proof. In the proof Proposition 4.1.1, we have proved that $(\mathbb{V}\backslash\{0\})\times_{\mathbb{P}}\dot{T}^*\mathbb{P}$ is isomorphic to the incidence hypersurface of $(\mathbb{V}^*\setminus\{0\})\times(\mathbb{V}\setminus\{0\})$. Taking the quotient on the incidence hypersurface with respect to $\mathbb{V}$, we get the tautological line bundle over $\mathbb{S}$ with respect to the $\mathbb{V}$ variable. Hence, we have

$$\dot{T}^*\mathbb{P} \simeq (\mathbb{V}\setminus\{0\}\times\mathbb{P})\times_{\mathbb{S}}\mathbb{S}$$

and similarly, taking the quotient with respect to $\mathbb{V}^*$, we get tautological line bundle over $\mathbb{S}$ with respect to the $\mathbb{V}^*$ variable, we get

$$\dot{T}^*\mathbb{P}^* \simeq \mathbb{S}\times_{\mathbb{S}}(\mathbb{V}^*\setminus\{0\}\times\mathbb{P}^*)$$

But these tautological line bundles are both isomorphic to $\dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)$. □

Now, we are going to prove the following

Proposition 4.1.3. *The diagram 4.1.4 induces*

$$\begin{tikzcd} & \dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)\cap(\dot{T}^*_P\mathbb{P}\times\dot{T}^*_{P^*}\mathbb{P}^*) \arrow[dl, "\sim"', "p_1"] \arrow[dr, "\sim", "p_2^a"'] & \\ \dot{T}^*_P\mathbb{P} \arrow[rr, "\sim", "\chi"'] & & \dot{T}^*_{P^*}\mathbb{P}^* \end{tikzcd}$$

Proof. The embedding of P, P^* into their compexification $\mathbb{P}, \mathbb{P}^*$ induces an isomorphism

$$\dot{T}^*_P\mathbb{P} \simeq i\dot{T}^*P, \;\; \dot{T}^*_{P^*}\mathbb{P}^* \simeq i\dot{T}^*P^*$$

Besides, we have the isomoprhism

$$\dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)\cap(\dot{T}^*_P\mathbb{P}\times\dot{T}^*_{P^*}\mathbb{P}^*) \simeq i\dot{T}^*_S(P\times P^*)$$

Indeed, let $(z_0,\xi_0)\in P\times P^*$ and $\varphi\in(\dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*))_{z_0,\xi_0}$. In local coordinates $(z,\xi)\in\mathbb{P}\times\mathbb{P}^*$, φ vanishes on any vector (u,v) belonging to the kernel of the 1-form, $\omega=\xi dz+xd\xi$. Then $(\Re u,\Re v)\in TS$, and $(\operatorname{Im} u,\operatorname{Im} v)\in TS$. Assume now, $\varphi\in(\dot{T}^*_P\mathbb{P})_{z_0}\times(\dot{T}^*_{P^*}\mathbb{P}^*)_{\xi_0}$, then φ vanishes on vector of type $(\Re(u),\Re(v))$ and so is pure imaginary and vanishes on $(\operatorname{Im} u,\operatorname{Im} v)\in TS$. Hence, $\varphi\in i\dot{T}^*_S(P\times P^*)$. From Proposition 4.1.2, we get the isomorphisms:

$$\dot{T}^*_P\mathbb{P}\simeq i\dot{T}^*P\simeq i\dot{T}^*_S(P\times P^*)\simeq\dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)\cap(\dot{T}^*_P\mathbb{P}\times\dot{T}^*_{P^*}\mathbb{P}^*)\simeq\dot{T}^*_{P^*}\mathbb{P}^*$$

□

4.2 Projective duality for microdifferential operators

Let k, k' be integers and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. We follow the notations of the sections 1.3 and 4.2. We define similarly a twisted version of $\mathscr{B}^{(n,0)}_{\mathbb{S}|\mathbb{P}\times\mathbb{P}^*}$ and $\mathscr{C}^{(n,0)}_{\mathbb{S}|\mathbb{P}\times\mathbb{P}^*}$.

We set

$$\mathscr{B}^{(n,0)}_{\mathbb{S}}(k,k') := q_2^{-1}\mathcal{O}_{\mathbb{P}^*}(k') \otimes_{q_2^{-1}\mathcal{O}_{\mathbb{P}^*}} \mathscr{B}_{\mathbb{S}|\mathbb{P}\times\mathbb{P}^*} \otimes_{q_1^{-1}\mathcal{O}_{\mathbb{P}}} q_1^{-1}(\mathcal{O}_{\mathbb{P}}(k) \otimes_{\mathcal{O}_{\mathbb{P}}} \Omega_{\mathbb{P}})$$

and the $(\mathscr{E}_{\mathbb{P}}(-k,k), \mathscr{E}_{\mathbb{P}^*}(-k^*,k^*))$-module

$$\mathscr{C}^{(n,0)}_{\mathbb{S}|\mathbb{P}\times\mathbb{P}^*}(k,k') := \mathscr{E}\mathscr{B}^{(n,0)}_{\mathbb{S}|\mathbb{P}\times\mathbb{P}^*}(k,k')$$

We notice that $\mathscr{E}_{\mathbb{P}}(-k,k)$ is nothing but $\mathcal{O}_{\mathbb{P}}(-k)\mathcal{D} \otimes_{\pi_{\mathbb{P}}^{-1}\mathcal{D}_{\mathbb{P}}} \mathscr{E}_{\mathbb{P}} \otimes_{\pi_{\mathbb{P}}^{-1}\mathcal{D}_{\mathbb{P}}} \mathcal{D}\mathcal{O}_{\mathbb{P}}(k)$. According to the diagram 4.1.4, we denoted by χ the homogeneous symplectic isomorphism

$$\chi := p_2^a|_{\dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)} \circ p_1|^{-1}_{\dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)}$$

We have:

Theorem 4.2.1 ([DS96, p. 469])**.** *Assume $-n-1 < k < 0$. There exists a section s of $\mu hom(\mathbb{C}_{\mathbb{S}}[-1], \Omega_{\mathbb{P}\times\mathbb{P}^*/\mathbb{P}^*}(-k,k^*))$, non-degenerate on $\dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)$.*

Proof. From the exact sequence:

$$0 \longrightarrow \mathbb{C}_{(\mathbb{P}\times\mathbb{P}^*)\setminus\mathbb{S}} \longrightarrow \mathbb{C}_{\mathbb{P}\times\mathbb{P}^*} \longrightarrow \mathbb{C}_{\mathbb{S}} \longrightarrow 0 \tag{4.2.1}$$

we get the natural morphism

$$\begin{aligned}
\mathrm{R}\Gamma((\mathbb{P}\times\mathbb{P}^*)\setminus\mathbb{S}; \Omega_{\mathbb{P}\times\mathbb{P}^*/\mathbb{P}^*}(-k,k^*)) &\to \mathrm{R}\Gamma_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*; \Omega_{\mathbb{P}\times\mathbb{P}^*/\mathbb{P}^*}(-k,k^*))[1] \\
&\simeq \mathrm{R}\Gamma(\mathbb{P}\times\mathbb{P}^*; \mathrm{R}\mathscr{H}om\,(\mathbb{C}_{\mathbb{S}}; \Omega_{\mathbb{P}\times\mathbb{P}^*/\mathbb{P}^*}(-k,k^*)))[1] \\
&\simeq \mathrm{R}\Gamma(T^*(\mathbb{P}\times\mathbb{P}^*); \mu hom(\mathbb{C}_{\mathbb{S}}; \Omega_{\mathbb{P}\times\mathbb{P}^*/\mathbb{P}^*}(-k,k^*)))[1] \\
&\to \mathrm{R}\Gamma(\dot{T}^*(\mathbb{P}\times\mathbb{P}^*); \mu hom(\mathbb{C}_{\mathbb{S}}[-1]; \Omega_{\mathbb{P}\times\mathbb{P}^*/\mathbb{P}^*}(-k,k^*)))
\end{aligned}$$

Let $z = (z_0, ..., z_n)$ be a system of homogeneous coordinates on $\mathbb{P}$ and $\zeta = (\zeta_0, ..., \zeta_n)$ the dual coordinates on $\mathbb{P}^*$. As explained in [DS96], a non-degenerate section is provided by the Leray section, defined for $(z,\xi) \in (\mathbb{P}\times\mathbb{P}^*)\setminus\mathbb{S}$ by

$$s(z,\zeta) = \frac{\omega'(z)}{\langle z,\zeta\rangle^{n+1+k}} \tag{4.2.2}$$

where $\omega'(z)$ is the Leray form $\omega'(z) = \sum_{k=0}^{n}(-1)^k z_k dz_0 \wedge \ldots \wedge dz_{k-1} \wedge dz_{k+1} \wedge \ldots \wedge dz_n$, Leray [Ler59]. □

Let s be a section of $H^1(\mu hom(\mathbb{C}_{\mathbb{S}}[-1], \Omega_{\mathbb{P}\times\mathbb{P}^*/\mathbb{P}^*}(-k,k^*)))$, non-degenerate on $\dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)$.

Theorem 4.2.2. *Assume $-n-1 < k < 0$. Then, we have an isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\dot{T}^*\mathbb{P}})$

$$\underline{\Phi}^{\mu}_{\mathbb{S}}(\mathscr{E}_{\mathbb{P}}(-k,k)|_{\dot{T}^*\mathbb{P}}) \simeq \mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)|_{\dot{T}^*\mathbb{P}^*}$$
$$\chi_*\mathscr{E}_{\mathbb{P}}(-k,k)|_{\dot{T}^*\mathbb{P}} \simeq \mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)|_{\dot{T}^*\mathbb{P}^*}$$

Proof. Let $\mathcal{F}$, $\mathcal{G}$ be line bundles on $\mathbb{P}$, and $\mathbb{P}^*$ respectively. We know from [SKK] that a global non-degenerate section $s \in \Gamma(\dot{T}^*\mathbb{P} \times \dot{T}^*\mathbb{P}^*; \mathscr{C}^{(n,0)}_{\mathbb{S}|\mathbb{P}\times\mathbb{P}^*} \otimes_{p_1^{-1}\mathscr{E}_{\mathbb{P}}} \mathscr{E}\mathcal{F} \otimes_{p_2^{-1}\mathscr{E}_{\mathbb{P}^*}} \mathcal{G}^{\otimes -1}\mathscr{E})$ induces an isomorphism of $\mathscr{E}$-modules

$$\underline{\Phi}^{\mu}_{\mathbb{S}}(\mathscr{E}\mathcal{F}|_{\dot{T}^*\mathbb{P}}) \simeq \mathscr{E}\mathcal{G}|_{\dot{T}^*\mathbb{P}^*}$$

Now, let us set $\mathcal{F} = \mathcal{O}_{\mathbb{P}}(k)$, $\mathcal{G} = \mathcal{O}_{\mathbb{P}^*}(k^*)$. Then, 4.2.1 provides such a non-degenerate section in $\Gamma(\dot{T}^*\mathbb{P} \times \dot{T}^*\mathbb{P}^*; \mathscr{C}^{(n,0)}_{\mathbb{S}|\mathbb{P}\times\mathbb{P}^*} \otimes_{p_1^{-1}\mathscr{E}_{\mathbb{P}}} \mathscr{E}\mathcal{O}_{\mathbb{P}}(k) \otimes_{p_2^{-1}\mathscr{E}_{\mathbb{P}^*}} \mathcal{O}_{\mathbb{P}^*}(k^*)^{\otimes -1}\mathscr{E})$. So that, we have an isomorphism

$$\underline{\Phi}^{\mu}_{\mathbb{S}}(\mathscr{E}_{\mathbb{P}}(-k,k)|_{\dot{T}^*\mathbb{P}}) \simeq \mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)|_{\dot{T}^*\mathbb{P}^*}$$

On the other hand s is a non-degenerate section of $\mathscr{C}^{(n,0)}_{\mathbb{S}|\mathbb{P}\times\mathbb{P}^*}(-k,k^*)$, hence we can apply Theorem 2.3.3. Let us denote by

$$\mathscr{E}_{\mathbb{P}\times\mathbb{P}^*}(k,k^*) := \mathscr{E}_{\mathbb{P}}(-k,k) \boxtimes_{\pi^{-1}\mathcal{O}_{\mathbb{P}\times\mathbb{P}^*}} \mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)$$

Theorem 2.3.3 gives the following isomorphisms

$$\mathscr{E}_{\mathbb{P}}(-k,k)|_{\dot{T}^*\mathbb{P}} \simeq p_{1*}(\mathscr{E}_{\mathbb{P}\times\mathbb{P}^*}(k,k^*).s)|_{\dot{T}^*\mathbb{P}}$$
$$p_{2*}(\mathscr{E}_{\mathbb{P}\times\mathbb{P}^*}(k,k^*).s)|_{\dot{T}^*\mathbb{P}^*} \simeq \mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)|_{\dot{T}^*\mathbb{P}^*}$$

And so

$$\chi_*\mathscr{E}_{\mathbb{P}}(-k,k)|_{\dot{T}^*\mathbb{P}} \simeq \mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)|_{\dot{T}^*\mathbb{P}^*}$$

□

4.3 Projective duality for microfunctions

Let U and V be open sets of respectively $\dot{T}^*\mathbb{P}$ and $\dot{T}^*\mathbb{P}^*$ such that $\chi(U) = V$. Let us denote by Λ the conic lagragian set

$$\Lambda := (U \times V^a) \cap \dot{T}^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*)$$

In order to prove Proposition 4.3.3 below, we will use the following theorem proven in [KS90, Th. 7.2.1], stated inthere for smooth manifolds.

Theorem 4.3.1 ([KS90, Th. 7.2.1])**.** *Let $K \in \mathrm{D}^{\mathrm{b}}(\mathbb{P} \times \mathbb{P}^*)$ and assume that*

(i) K is cohomologically constructible

(ii) $(p_1^{-1}(U) \cup (p_2^a)^{-1}(V)) \cap SS(K) \subset \Lambda$

(iii) the natural morphism $\mathbb{C}_\Lambda \longrightarrow \mu\mathcal{hom}(K,K)|_\Lambda$ *is an isomorphism.*

Then for any $F_1, F_2 \in \mathrm{D}^{\mathrm{b}}(\mathbb{P};U)$*, the natural morphism*

$$\chi_*\mu\mathcal{hom}(F_1,F_2) \longrightarrow \mu\mathcal{hom}(\Phi_K(F_1),\Phi_K(F_2))$$

is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^*;V)$*.*

In the following, we will denote by K the object $\mathbb{C}_{\mathbb{S}}[n-1]$. In order to prove Proposition 4.3.3, we will need to compute $\Phi_K(\mathbb{C}_P(1))$. Though we provide an alternate proof, the following Lemma is also proved in [DS96].

Lemma 4.3.2. *We have*

$$\Phi_K(\mathbb{C}_P(1)) \simeq \begin{cases} \mathbb{C}_{P^*}(1) \text{ , for } n \text{ odd} \\ \mathbb{C}_{\mathbb{P}^*\setminus P^*}[1] \text{ , for } n \text{ even} \end{cases}$$

and

$$H^j(\Phi_K(\mathbb{C}_P(0))) \simeq \begin{cases} \mathbb{C}_{\mathbb{P}^*} \text{ , for } j=n-1 \\ \mathbb{C}_{\mathbb{P}^*\setminus P^*}, \text{ for } j=-1 \text{ and } n \text{ odd} \\ \mathbb{C}_{P^*}(1), \text{ for } j=0 \text{ and } n \text{ even} \\ 0 \text{ in any other case} \end{cases}$$

Proof. We will only treat the computation of $\Phi_K(\mathbb{C}_P(\varepsilon))$, $\varepsilon = 1$. We have:

$$\begin{aligned} K \otimes q_1^{-1}(\mathbb{C}_P(1)) &\simeq \mathbb{C}_{\mathbb{S}}[n-1] \otimes \mathbb{C}_{P\times\mathbb{P}^*}(1) \\ &\simeq \mathbb{C}_{(P\times\mathbb{P}^*)\cap\mathbb{S}}(1)[n-1] \end{aligned} \tag{4.3.1}$$

where $\mathbb{C}_{P\times\mathbb{P}^*}(1)$ denotes the exterior product $\mathbb{C}_P(1) \boxtimes \mathbb{C}_{\mathbb{P}^*}$, and $\mathbb{C}_{(P\times\mathbb{P}^*)\cap\mathbb{S}}(1) := \mathbb{C}_{P\times\mathbb{P}^*}(1) \otimes \mathbb{C}_{\mathbb{S}}$.

Let us compute the stalks of $\Phi_K(\mathbb{C}_P)$. For $\xi \in P^*$, we have

$$\Phi_K(\mathbb{C}_P)_\xi \simeq (Rq_{2!}(\mathbb{C}_{(P\times\mathbb{P}^*)\cap\mathbb{S}})[n-1])_\xi$$

Noticing that $(P\times\{\xi\})\cap\mathbb{S}$ is the $(n-1)$-dimensional real projective space, we have

$$\begin{aligned} R\Gamma_c(q_2^{-1}(\{\xi\});\mathbb{C}_{(P\times\mathbb{P}^*)\cap\mathbb{S}}) &\simeq R\Gamma_c(q_2^{-1}(\{\xi\});\mathbb{C}_{(P\times\{\xi\})\cap\mathbb{S}}) \\ &\simeq R\Gamma_c(\mathbb{P}_n;\mathbb{C}_{P_{n-1}}) \\ &\simeq R\Gamma_c(P_{n-1};\mathbb{C}_{P_{n-1}}) \end{aligned}$$

For $\xi \in \mathbb{P}^*\setminus P^*$, $(P\times\{\xi\})\cap\mathbb{S}$ is the intersection of two real projective spaces of dimension $(n-1)$, so that

$$\begin{aligned} R\Gamma_c(q_2^{-1}(\{\xi\});\mathbb{C}_{(P\times\mathbb{P}^*)\cap\mathbb{S}}) &\simeq R\Gamma_c(\mathbb{P}_n;\mathbb{C}_{P_{n-2}}) \\ &\simeq R\Gamma_c(P_{n-2};\mathbb{C}_{P_{n-2}}) \end{aligned}$$

We conclude by using real projective spaces cohomology classical computation: all constant sheaf cohomology groups vanish for degree ranging from 1 to $n-1$, and one have to distinguishes between n being even or odd.

For n even, we have

$$\begin{aligned} R\Gamma_c(P_{n-1};\mathbb{C}_{P_{n-1}}) &\simeq \mathbb{C}[0]\oplus\mathbb{C}[1-n] \\ R\Gamma_c(P_{n-2};\mathbb{C}_{P_{n-2}}) &\simeq \mathbb{C}[0] \end{aligned}$$

so that

$$\Phi_K(\mathbb{C}_P)_\xi \simeq \begin{cases} \mathbb{C}[0]\oplus\mathbb{C}[n-1], \text{ for } \xi\in P^* \\ \mathbb{C}[n-1], \text{ for } \xi\in\mathbb{P}^*\setminus P^* \end{cases}$$

For n odd, we have

$$\begin{aligned} R\Gamma_c(P_{n-1};\mathbb{C}_{P_{n-1}}) &\simeq \mathbb{C}[0] \\ R\Gamma_c(P_{n-2};\mathbb{C}_{P_{n-2}}) &\simeq \mathbb{C}[0]\oplus\mathbb{C}[2-n] \end{aligned}$$

so that

$$\Phi_K(\mathbb{C}_P)_\xi \simeq \begin{cases} \mathbb{C}[n-1], \text{ for } \xi\in P^* \\ \mathbb{C}[1]\oplus\mathbb{C}[n-1], \text{ for } \xi\in\mathbb{P}^*\setminus P^* \end{cases}$$

We recall that the sheaf of $\mathbb{C}_P(1)$ is defined by the following exact sequence

$$0\to\mathbb{C}_P(1)\to q_!\mathbb{C}_{\widetilde{P}}\xrightarrow{tr}\mathbb{C}_P\to 0$$

where $\widetilde{P}$ is the n-sphere, q is the $2:1$ fibered map $\widetilde{P}\to P$.

We note i the embedding $P\hookrightarrow\mathbb{P}$. We have the following diagram:

$$\begin{tikzcd}
 & \widetilde{P}\times\mathbb{P}^* \arrow[dl, "\widetilde{q}_1"'] \arrow[dr, "\widetilde{q}"] & & \\
\widetilde{P} \arrow[dr, "q"] & & P\times\mathbb{P}^* \arrow[dl, "q_1"'] \arrow[d, hook, "j"] \arrow[dr, "q_2"] & \\
 & P \arrow[d, hook, "i"] & \mathbb{P}\times\mathbb{P}^* \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \mathbb{P}^* \arrow[d, "id"] \\
 & \mathbb{P} & & \mathbb{P}^*
\end{tikzcd}$$

Let us compute $\Phi_K(i_*q_!\mathbb{C}_{\widetilde{P}})_\xi$, $\xi\in\mathbb{P}^*$. Still denoting $\mathbb{S}\cap(P\times\mathbb{P}^*)$ by $\mathbb{S}$ and denoting $\widetilde{q}^{-1}\mathbb{S}$ by $\widetilde{\mathbb{S}}$, we have

$$\begin{aligned} \Phi_K(i_*Rq_!\mathbb{C}_{\widetilde{P}}) &:= Rq_{2!}(\mathbb{C}_{\mathbb{S}}[n-1]\otimes q_1^{-1}i_*Rq_!\mathbb{C}_{\widetilde{P}}) \\ &\simeq Rq_{2!}(\mathbb{C}_{\mathbb{S}}[n-1]\otimes j_*q_1^{-1}Rq_!\mathbb{C}_{\widetilde{P}}) \\ &\simeq Rq_{2!}j_*(\mathbb{C}_{\mathbb{S}}[n-1]\otimes q_1^{-1}Rq_!\mathbb{C}_{\widetilde{P}}) \\ &\simeq Rq_{2!}j_*(\mathbb{C}_{\mathbb{S}}[n-1]\otimes R\widetilde{q}_!\widetilde{q}_1^{-1}\mathbb{C}_{\widetilde{P}}) \\ &\simeq Rq_{2!}j_*(\mathbb{C}_{\mathbb{S}}[n-1]\otimes R\widetilde{q}_!\mathbb{C}_{\widetilde{P}\times\mathbb{P}^*}) \\ &\simeq Rq_{2!}j_*R\widetilde{q}_!(\mathbb{C}_{\widetilde{\mathbb{S}}}[n-1]) \\ &\simeq R(q_2\circ j\circ\widetilde{q})_!(\mathbb{C}_{\widetilde{\mathbb{S}}}[n-1]) \end{aligned}$$

where we used that for the closed embedding i, j, the push-forward and proper direct image are isomorphic. Now, $\widetilde{\mathbb{S}} \cap (q_2 \circ j \circ \widetilde{q})^{-1}(\{\xi\})$ are the real $(n-1)$-sphere and the real $(n-2)$-sphere, when $\xi \in P^*$ and $\xi \in \mathbb{P}^* \setminus P^*$, respectively. From the well-known cohomology of spheres, we get that:

$$\Phi_K(Rq_!\mathbb{C}_{\widetilde{P}})_\xi \simeq \begin{cases} \mathbb{C}[0] \oplus \mathbb{C}[n-1], \text{ for } \xi \in P^* \\ \mathbb{C}[1] \oplus \mathbb{C}[n-1], \text{ for } \xi \in \mathbb{P}^* \setminus P^* \end{cases}$$

Consider the distinguished triangle,

$$\Phi_K(\mathbb{C}_P(1)) \longrightarrow \Phi_K(\mathrm{R}q_!\mathbb{C}_{\widetilde{P}}) \longrightarrow \Phi_K(\mathbb{C}_P) \xrightarrow{+1}$$

we deduce that, for n even,

$$\Phi_K(\mathbb{C}_P(1))_\xi \simeq \begin{cases} 0, \text{ for } \xi \in P^* \\ \mathbb{C}[1], \text{ for } \xi \in \mathbb{P}^* \setminus P^* \end{cases}$$

For n odd,

$$\Phi_K(\mathbb{C}_P(1))_\xi \simeq \begin{cases} \mathbb{C}[0], \text{ for } \xi \in P^* \\ 0, \text{ for } \xi \in \mathbb{P}^* \setminus P^* \end{cases}$$

So that, for n even, $\Phi_K(\mathbb{C}_P(1))$ is a locally constant sheaf, with stalks $\mathbb{C}$ on $\mathbb{P}^* \setminus P^*$, concentrated in degree 1, and for n odd, a locally constant sheaf with stalks $\mathbb{C}$ on P^*, concentrated in degree 0.

Now, to conclude, we refer to [DS96, Coro. 5.15], where it is shown in particular that $\mathrm{Hom}(\Phi_K(\mathbb{C}_P(1)), \mathbb{C}_P(1))$ is isomorphic to 0 for n even and $\mathbb{C}$ for n odd. And so, since there are only 2 locally constant sheaves on P^*, namely $\mathbb{C}_{P^*}$ and $\mathbb{C}_{P^*}(1)$, we conclude that for n odd, $\Phi_K(\mathbb{C}_P(1))$ is precisely $\mathbb{C}_P(1)$. Besides, for $n > 2$ and even, $\mathbb{P}^* \setminus P^*$ is contractible, so that $\Phi_K(\mathbb{C}_P(1))$ is precisely $\mathbb{C}_{\mathbb{P}^* \setminus P^*}[1]$.

□

Theorem 4.3.3. *Assume $-n-1 < k < 0$. Recall that any section $s \in \Gamma(\mathbb{P} \times \mathbb{P}^*; \mathscr{B}_{\mathbb{S}}^{(n,0)}(-k, k^*))$, defines a morphism in $\mathrm{D^b}(\mathbb{C}_{\dot{T}\mathbb{P}})$*

$$\chi_* \mathscr{C}_P(\varepsilon, k)|_{\dot{T}^*_P\mathbb{P}} \to \mathscr{C}_{P^*}(\varepsilon^*, k^*)|_{\dot{T}^*_{P^*}\mathbb{P}^*} \tag{4.3.2}$$

Assume s is non-degenerate on $\dot{T}^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*)$. Then (4.3.2) is an isomorphism. Moreover, there exists such a non-degenerate section.

Remark 4.3.4. (i) This is a refinement of a general theorem of [SKK] and is a microlocal version of Theorem 5.17 in [DS96].

(ii) The classical Radon transform deals with the case where $k = -n$, $k^* = -1$.

Proof. We will deal with the case $\varepsilon = 1$ and n even, the complementary cases being proven the same way. Let us apply Theorem 4.3.1 in the following particular case

- $U = \dot{T}^*\mathbb{P}$, $V = \dot{T}^*\mathbb{P}^*$, $\Lambda = \dot{T}^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*)$.

- K is $\mathbb{C}_{\mathbb{S}}[n-1]$.
- $F_1 = \mathbb{C}_P(1)$ and $F_2 = \mathcal{O}_{\mathbb{P}}(k)$

K verifies conditions (i),(ii),(iii) of Theorem 4.3.1

(i) is fulfilled as the constant sheaf on a closed submanifold of a manifold is cohomologically constructible.

(ii) is fulfilled since $SS(\mathbb{C}_{\mathbb{S}})$ is nothing but $T^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)$.

(iii) $\mathbb{C}_{T^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)} \longrightarrow \mu\mathcal{hom}(\mathbb{C}_{\mathbb{S}},\mathbb{C}_{\mathbb{S}})$ is an isomorphism on $T^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)$ (this follows from the fact that for a closed submanifold Z of a manifold X, $\mu_Z(\mathbb{C}_Z) \xrightarrow{\sim} \mathbb{C}_{T^*_Z X}$, see [KS90, Prop. 4.4.3]).

By a fundamental result in [DS96, Th 5.17], we know that for $-n-1<k<0$, a section $s \in \Gamma(\mathbb{P}\times\mathbb{P}^*;\mathscr{B}^{(n,0)}_{\mathbb{S}}(-k,k^*))$, non-degenerate on $\dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)$, induces an isomorphism

$$\Phi_K(\mathcal{O}_{\mathbb{P}}(k)) \simeq \mathcal{O}_{\mathbb{P}^*}(k^*)$$

(see Appendix 6.1.1 for a sketch of the proof). Formula (4.2.2) provides an example of such a non-degenerate section. Hence, applying Lemma 4.3.2, Theorem 4.3.1 gives:

$$\chi_*\mu\mathcal{hom}(\mathbb{C}_P(1),\mathcal{O}_{\mathbb{P}}(k))|_{\dot{T}^*\mathbb{P}} \simeq \mu\mathcal{hom}(\mathbb{C}_{\mathbb{P}^*\setminus P^*}[1],\mathcal{O}_{\mathbb{P}^*}(k^*))|_{\dot{T}^*\mathbb{P}^*}$$

We have the exact sequence:

$$0 \longrightarrow \mathbb{C}_{\mathbb{P}^*\setminus P^*} \longrightarrow \mathbb{C}_{\mathbb{P}^*} \longrightarrow \mathbb{C}_{P^*} \longrightarrow 0 \tag{4.3.3}$$

Now, for any $F \in \mathrm{D^b}(\mathbb{C}_{\mathbb{P}^*})$, we have

$$supp(\mu hom(\mathbb{C}_{\mathbb{P}^*},F)|_{\dot{T}^*\mathbb{P}^*}) \subset (SS(\mathbb{C}_{\mathbb{P}^*}) \cap \dot{T}^*\mathbb{P}^*) \cap SS(F) = \varnothing$$

and hence,

$$\mu hom(\mathbb{C}_{\mathbb{P}^*},F)|_{\dot{T}^*\mathbb{P}^*} \simeq 0$$

Applying the μhom functor to 4.3.3, we get

$$\mu hom(\mathbb{C}_{\mathbb{P}^*\setminus P^*},F)|_{\dot{T}^*\mathbb{P}^*}[-1] \simeq \mu hom(\mathbb{C}_{P^*},F)|_{\dot{T}^*\mathbb{P}^*}$$

Hence, we have proved in particular that

$$\chi_*\mu\mathcal{hom}(\mathbb{C}_P(1),\mathcal{O}_{\mathbb{P}}(k))|_{\dot{T}^*_P\mathbb{P}} \simeq \mu\mathcal{hom}(\mathbb{C}_{P^*},\mathcal{O}_{\mathbb{P}^*}(k^*))|_{\dot{T}^*_{P^*}\mathbb{P}^*}$$

□

4.4 Main results

We follow the notations of Section 1.3 and Section 2.1.

Let us consider the situation (4.1.4), where we denoted by χ the homogeneous symplectic isomorphism between $\dot{T}^*\mathbb{P}$ and $\dot{T}^*\mathbb{P}^*$ through $\dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)$. We set

$$L := \mathbb{C}_{\mathbb{S}}[-1]$$

Then L is a perverse sheaf satisfying

$$(p_1^{-1}(\dot{T}^*\mathbb{P}) \cup p_2^{a-1}(\dot{T}^*\mathbb{P}^*)) \cap \mathrm{SS}(L) = \dot{T}^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*) \tag{4.4.1}$$

Recall Theorem 4.2.1, and let s be a section of $\mu hom(\mathbb{C}_{\mathbb{S}}[-1], \Omega_{\mathbb{P}\times\mathbb{P}^*/\mathbb{P}^*}(-k, k^*))$, non-degenerate on $\dot{T}^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*)$. We are in situation to apply Theorem 3.2.1.

Theorem 4.4.1. *Let $G \in \mathrm{D^b}(\mathbb{C}_{\mathbb{P}^*})$, k an integer. Assume $-n-1 < k < 0$. Then, we have an isomorphism in $\mathrm{D^b}(\mathbb{C}_{\dot{T}^*\mathbb{P}})$:*

$$\chi^{-1}\mu hom(G, \mathscr{O}_{\mathbb{P}^*}(k^*)) \xrightarrow{\sim} \mu hom(\Phi_{\mathbb{C}_{\mathbb{S}}[n-1]}(G), \mathscr{O}_{\mathbb{P}}(k)) \tag{4.4.2}$$

This isomorphism is compatible with the action of $\mathscr{E}_{\mathbb{P}^*}(-k^*, k^*)$ and $\mathscr{E}_{\mathbb{P}}(-k, k)$ on the left and right side of (4.4.2) respectively.

Proof. The isomorphism is directly provided by Theorem 3.2.1 in the situation where, using the notation inthere, $U = \dot{T}^*\mathbb{P}$, $V = \dot{T}^*\mathbb{P}^*$ and $\Lambda = \dot{T}^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*)$ and where we twist by homogenous line bundles of $\mathbb{P}$, $\mathbb{P}^*$ as explained below.

Let us adapt (3.2.5) by taking into account the twist by homogeneous line bundles. We follow the exact same reasoning than sections of 3.1 and 3.2.

We have the natural morphism

$$\Omega_{\mathbb{P}^*\times\mathbb{P}/\mathbb{P}}(-k^*, k) \circ \mathscr{O}_{\mathbb{P}^*}(k^*)\,[n] \to \mathscr{O}_{\mathbb{P}}(k).$$

Indeed, we have

$$\begin{aligned}\Omega_{\mathbb{P}^*\times\mathbb{P}/\mathbb{P}}(-k^*, k) \circ \mathscr{O}_{\mathbb{P}^*}(k^*)\,[n] &= \mathrm{R}q_{1!}(\mathcal{O}_{\mathbb{P}^*\times\mathbb{P}}(-k^*, k) \otimes_{q_2^{-1}\mathcal{O}_{\mathbb{P}^*}} q_2^{-1}\Omega_{\mathbb{P}^*} \otimes q_2^{-1}\mathscr{O}_{\mathbb{P}^*}(k^*)[n]) \\ &\to \mathrm{R}q_{1!}(\mathcal{O}_{\mathbb{P}^*\times\mathbb{P}}(k, 0) \otimes_{q_2^{-1}\mathcal{O}_{\mathbb{P}^*}} q_2^{-1}\Omega_{\mathbb{P}^*})[n] \xrightarrow{\int} \mathscr{O}_{\mathbb{P}}(k)\end{aligned}$$

Given this morphism and considering $\mathscr{L} := \mu hom(\mathbb{C}_{\mathbb{S}}[-1], \Omega_{\mathbb{P}^*\times\mathbb{P}/\mathbb{P}}(-k^*, k))$, we mimic the proof of Theorem 3.1.5 so that for a section s of $\mathscr{L}$ on $\dot{T}^*\mathbb{P} \times \dot{T}^*\mathbb{P}^{*a}$ and for $P \in \Gamma(\dot{T}^*\mathbb{P}; \mathscr{E}_{\mathbb{P}}(-k, k))$ and $Q \in \Gamma(\dot{T}^*\mathbb{P}^*; \mathscr{E}_{\mathbb{P}^*}(-k^*, k^*))$ satisfying $P \cdot s = s \cdot Q$, the diagram below commutes:

(4.4.3)

$$\begin{tikzcd}
\mathbb{C}_{\mathbb{S}} \circ \mu hom(G, \mathscr{O}_{\mathbb{P}^*}(k^*))|_{\dot{T}^*_{P^*}\mathbb{P}^*} \arrow[r, "\alpha(s)"'] \arrow[d, "\Phi_{\mathbb{C}_{\mathbb{S}}}(\alpha(Q))"'] & \mu hom(\mathbb{C}_{\mathbb{S}}[n-1] \circ G, \mathscr{O}_{\mathbb{P}}(k))|_{\dot{T}^*_{P}\mathbb{P}} \arrow[d, "\alpha(P)"'] \\
\mathbb{C}_{\mathbb{S}} \circ \mu hom(G, \mathscr{O}_{\mathbb{P}^*}(k^*))|_{\dot{T}^*_{P^*}\mathbb{P}^*} \arrow[r, "\alpha(s)"'] & \mu hom(\mathbb{C}_{\mathbb{S}}[n-1] \circ G, \mathscr{O}_{\mathbb{P}}(k))|_{\dot{T}^*_{P}\mathbb{P}}.
\end{tikzcd}$$

From there, given a non-degenerate section of $\mathscr{L}$ on $\dot{T}^*_S(\mathbb{P}\times\mathbb{P}^*)$, Theorem 3.2.1 gives the compatible action of micro-differential operators on each side of the isomorphism (4.4.2)

$$\chi^{-1}\mu hom(G, \mathscr{O}_{\mathbb{P}^*}(k^*))|_{\dot{T}^*_{P^*}\mathbb{P}^*} \xrightarrow{\sim} \mu hom(\Phi_{\mathbb{C}_{\mathbb{S}}[n-1]}(G), \mathscr{O}_{\mathbb{P}}(k))|_{\dot{T}^*_{P}\mathbb{P}}$$

It remains to exhibit a non-degenerate section so that, for $P \in \Gamma(\dot{T}^*\mathbb{P}; \mathscr{E}_{\mathbb{P}}(-k,k))$, there is $Q \in \Gamma(\dot{T}^*\mathbb{P}^*; \mathscr{E}_{\mathbb{P}^*}(-k^*,k^*))$ such that $P \cdot s = s \cdot Q$. Precisely, s is given by Proposition 4.2.1. □

Specializing the above proposition, we get

Corollary 4.4.2. *Let $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. In the situation of Proposition 4.4.1, we have the isomorphism, compatible with the respective action of $p_1^{-1}\mathscr{E}_{\mathbb{P}}(-k,k)$ and $p_2^{a-1}\mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)$*

$$\chi_*\mathscr{C}_P(\varepsilon,k)|_{\dot{T}^*_P\mathbb{P}} \simeq \mathscr{C}_{P^*}(\varepsilon^*,k^*)|_{\dot{T}^*_{P^*}\mathbb{P}^*}$$

Proof. This is an immediate consequence of Proposition 4.4.1, where we consider the special case $G = \mathbb{C}_{P^*}(\varepsilon^*)$. Indeed, we have, from Lemma 4.3.2, the isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}^*}; \dot{T}^*\mathbb{P}^*)$

$$\mathbb{C}_{\mathbb{S}}[n-1] \circ \mathbb{C}_{P^*}(\varepsilon^*) \simeq \mathbb{C}_P(\varepsilon)$$

□

We can state now

Corollary 4.4.3. *Let k be an integer. Let $\mathcal{N}$ be a coherent $\mathscr{E}_{\mathbb{P}}(-k,k)$-module and $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{P})$. Assume $-n-1 < k < 0$. Then, we have an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\dot{T}^*\mathbb{P}})$*

$$\chi_*\mathrm{R}\mathscr{H}om_{\mathscr{E}_{\mathbb{P}}(-k,k)}(\mathcal{N}, \mu hom(F, \mathcal{O}_{\mathbb{P}}(k))) \simeq$$

$$\mathrm{R}\mathscr{H}om_{\mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)}(\underline{\Phi}^{\mu}_{\mathbb{S}}(\mathcal{N}), \mu hom((\Phi_{\mathbb{C}_{\mathbb{S}}[n-1]}F, \mathcal{O}_{\mathbb{P}^*}(k^*)))$$

Proof. It suffices to prove this statement for finite free $\mathscr{E}_{\mathbb{P}}(-k,k)$-modules, which in turn can be reduced to the case where $\mathcal{N} = \mathscr{E}_{\mathbb{P}}(-k,k)$. By Theorem 4.2.2, we have

$$\underline{\Phi}^{\mu}_{\mathbb{S}}(\mathscr{E}_{\mathbb{P}}(-k,k)|_{\dot{T}^*\mathbb{P}}) \simeq \mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)|_{\dot{T}^*\mathbb{P}^*}$$

Then, by applying Proposition 4.4.1, we have

$$\chi_*\mu hom(F, \mathcal{O}_{\mathbb{P}}(k))|_{\dot{T}^*\mathbb{P}^*} \simeq \mathrm{R}\mathscr{H}om_{\mathscr{E}_{\mathbb{P}^*}(-k^*,k^*)}(\mathscr{E}_{\mathbb{P}^*}(-k^*,k^*), \mu hom((\Phi_{\mathbb{C}_{\mathbb{S}}[n-1]}F, \mathcal{O}_{\mathbb{P}^*}(k^*)))|_{\dot{T}^*\mathbb{P}^*}$$

which proves the corollary. □

Chapter 5

Applications

5.1 Geometrical preliminaries

5.1.1 Radon transform for sets

Let q_1, q_2 denote the first and second projection of $P \times P^*$ on each of its factor. For A and B subsets of P and P^*, we denote by $\widehat{A}$ and $\widehat{B}$ the subsets of P^* and P respectively

$$\widehat{A} := q_2(q_1^{-1}(A) \cap S)$$
$$\widehat{B} := q_1(q_2^{-1}(A) \cap S)$$

Of course, for $\xi \in P^*$, $\widehat{\{\xi\}}$ is the hyperplane of P associated to ξ. We denote by $\check{A}$ the set

$$\check{A} := P^* \setminus (\widehat{P \setminus A})$$

and similarly for $\check{B}$.

We will say that A is affine if it is contained in a local chart of P. If A is affine, we denote by $Conv(A)$ the convex hull of A.

We have the following description of $\widehat{A}$ and $\check{A}$:

Lemma 5.1.1. *For $A \subset P$, we have*

(i) $\widehat{A} = \{\xi \in P^*; \widehat{\{\xi\}} \cap A \neq \varnothing\}$

(ii) $\check{A} = \{\xi \in P^*; \widehat{\{\xi\}} \subset A\}$

Proof. (i) By definition, $\xi \in \widehat{A}$ if and only if there exists $x \in A$, such that $(x, \xi) \in S$, and so if and only if there exists $x \in A$, such that $x \in \widehat{\{\xi\}}$.

(ii) Indeed, from (i), we have that $\xi \in \check{A}$ if and only if $\widehat{\{\xi\}} \cap (P \setminus A) = \varnothing$. □

We have the following properties.

Lemma 5.1.2. *For $A \subset P$, the following statements hold*

(i) *If $A \neq \varnothing$, then $\widehat{\widehat{A}} = P$.*

Assume now that A is affine and connected. Then:

(ii) $\widehat{A} = \widehat{Conv(A)}$.

(iii) $\overset{\vee}{\widehat{A}} = Conv(A)$.

(iv) $\overset{\widehat{\vee}}{\widehat{A}} = \widehat{A}$.

Proof. (i) Let x, y be two distinct element of P with $x \in A$. Let $H \subset P$ be any hyperplane containing both x and y. Setting $H = \widehat{\{\xi\}}$ for $\xi \in P^*$, we have $\{\xi\} \subset \widehat{\{x\}}$. Since, $y \in \widehat{\{\xi\}}$, we get that $y \in \widehat{\widehat{\{x\}}}$.

(ii) Assume A is a subset of E, a local chart of P. Let $\xi \in \widehat{Conv(A)}$ and denote by H be the hyperplan $\widehat{\{\xi\}}$. There exists $z \in Conv(A)$ such that $z \in H$. Let $x, y \in A$ be such that $z \in [x, y]$. Since A is connected, $[x, y] \cap H \neq \varnothing$ implies $H \cap A \neq \varnothing$, so that we have $\xi \in \widehat{A}$.

We have thus proved that, for any $x, y \in A$, and any $z \in [x, y]$, $\widehat{z} \subset \widehat{A}$, so that $\widehat{Conv(A)} \subset \widehat{A}$. Now since,$\widehat{\ }$is increasing in the sense that $A \subset B$ implies $\widehat{A} \subset \widehat{B}$, we have $\widehat{A} \subset \widehat{Conv(A)}$.

(iii) The inclusion $Conv(A) \subset \overset{\vee}{\widehat{A}}$ follows immedialtely from (ii) and lemma 5.1.1 (ii). Let us prove the converse inclusion. Let $x \notin Conv(A)$ and consider H a separation hyperplane of $\{x\}$ and A. There exists $\xi \in P^*$ such that $\widehat{\{\xi\}} = H$. Again by Lemma 5.1.1 (ii), it suffices to prove that $\{\xi\} \notin \widehat{A}$. Indeed, in that case, $\widehat{\{x\}} \cap (P^* \setminus \widehat{A}) \neq \varnothing$ since it would contain $\{\xi\}$ implying that $x \notin \overset{\vee}{\widehat{A}}$.

Suppose that $\xi \in \widehat{A}$. Then, then there exists $y \in A$ such that $\xi \in \widehat{\{y\}}$. However, the latter fact means that $y \in H$, implying that $H \cap A \neq \varnothing$, which leads to a contradiction.

(iv) Follows from (ii) and (iii). □

Remark 5.1.3. (i) Let H_∞ be a hyperplane in P. For any non empty affine set $A \subset P \setminus H_\infty$, we have $\overset{\vee}{A} = \varnothing$. Indeed, for any $\xi \in P^*$, $\widehat{\xi}$ has non empty intersection with H_∞.

(ii) For two sets $A, B \subset P$, we have the obvious relation $\widehat{A \cup B} = \widehat{A} \cup \widehat{B}$. However, the $\widehat{\ }$ operation does not commute with the set-intersection operation. For instance, consider x, y distinct elements of P, $\widehat{\{x\}} \cap \widehat{\{y\}}$ is non empty since it contains any $\xi \in P^*$ whose associated hyperplane contains x and y.

In the following, we will be essentially interested in linearly concave domains.

Definition 5.1.4. A set $A \subset P$ is linearly concave if there exists a set $B \subset P^*$ such that $A = \widehat{B}$. Moreover, if B is connected, A is said to be strongly linearly concave.

We have the simple lemma:

Lemma 5.1.5. *Let A be a subset of P. Then A is a linearly concave if and only if $\widehat{\widehat{A}} = A$.*

Proof. Let $B \subset P^*$ such that $A = \widehat{B}$. By lemma 5.1.2 (iv), we have the follwing equalities:

$$\widehat{\widehat{A}} = \widehat{\widehat{\widehat{B}}} = \widehat{B} = A$$

The converse statement is obvious. □

5.1.2 Sheaves associated with some locally closed subsets

Let X, Y be two real analytic manifolds of dimension n, S a closed submanifold of $X \times Y$ of codimension c. We note $X \xleftarrow{q_1} X \times Y \xrightarrow{q_2} Y$ the projections, j the closed embedding $S \hookrightarrow X \times Y$. We denote by f, g the restrictions of q_1, q_2 to S:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

We assume that f and g are smooth and proper. For $A \subset X$, $B \subset Y$, we still note $\widehat{A} := g(f^{-1}(A))$, $\widehat{B} := f(g^{-1}(B))$.

Definition 5.1.6. [DS96] Let Z be a locally closed subset of X. One says that Z is S-trivial if for any $y \in \widehat{Z}$, $g^{-1}(\{y\})$ has trivial cohomology in the sense that

$$\mathbf{k} \xrightarrow{\sim} \mathrm{R}\Gamma(g^{-1}(\{y\}); \mathbf{k}_{f^{-1}(Z)}) \tag{5.1.1}$$

Refering to the notations of section 2.5, recall that for $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$, we defined

$$\Phi_S(F) := \mathrm{R}g_! f^{-1} F[d_{S/Y}]$$

Lemma 5.1.7. *For $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, we have*

$$\Phi_S(F) \simeq \mathrm{R}q_{2!}(\mathbf{k}_S \otimes q_1^{-1} F)[n-c]$$

Proof. We have

$$\begin{aligned} \mathrm{R}g_! f^{-1} F &\simeq \mathrm{R}g_!(j^{-1} q_1^{-1} F) \\ &\simeq \mathrm{R}q_{2!}(\mathrm{R}j_! j^{-1}) q_1^{-1} F \\ &\simeq \mathrm{R}q_{2!}((q_1^{-1} F)_{j_* S}) \\ &\simeq \mathrm{R}q_{2!}(\mathbf{k}_{j_* S} \otimes q_1^{-1} F) \end{aligned}$$

□

Here are some conditions under which the Radon transform of locally constant sheaves on some locally closed set $A \subset X$, is the constant sheaf on $\widehat{A}$ with some shift in degrees.

Lemma 5.1.8 ([DS94, Lem 2.8]). (i) *Let U be an S-trivial open subset of X, then*

$$\Phi_S(\mathbf{k}_U) \simeq \mathbf{k}_{\widehat{U}}[c-n] \tag{5.1.2}$$

(ii) *Let K be a S-trivial compact subset of X, then*

$$\Phi_S(\mathbf{k}_K) \simeq \mathbf{k}_{\widehat{K}}[n-c] \tag{5.1.3}$$

□

5.2 An application of the Holmgren-Kashiwara theorem

Let M be a real analytic manifold, X a complexification of M. In the sequel, we identify the conormal bundle $T^*_M X$ and iT^*M. We denote by $\mathscr{B}_M$ the sheaf of Sato's hyperfunctions on M, by $\mathscr{B}(M)$ the vector space $\Gamma(M;\mathscr{B}_M)$, and for $u \in \mathscr{B}(M)$, we denote by $WF(u)$ its analytic wave front set, a closed conic subset of $T^*_M X$. Let us first recall an unpublished result due to M. Kashiwara, proven in Hormander [Hör83] Th. 8.5.6.

Theorem 5.2.1. *Let M be a real analytic manifold and let $f : M \to \mathbb{R}$ be a $\mathscr{C}^1$ function. Let $x_0 \in M$ with $f(x_0) = 0$ and $df(x_0) \neq 0$ and let $U = \{x \in M; f(x) < 0\}$. Let $u \in \mathscr{B}(M)$ satisfying $u|U = 0$ and $x_0 \in supp(u)$. Then $\pm i.df(x_0) \in WF(u)$.*

From this result, we deduce the following. Let V be a real finite dimensional vector space, V^* its dual, and set $\dot{V}^* = V^* \setminus \{0\}$. We denote by S the incidence variety of $V \times V^*$. Let $\omega \subset \Omega$ be convex open subsets of V. For $(x,\xi) \in V \times \dot{V}^*$, we denote by $H_{x,\xi}$ the hyperplane passing through x and of conormal ξ. Let us consider the set

$$Z = \overline{\{(x, i\xi) \in V \times i\dot{V}^*; (x,\xi) \in S, H_{x,\xi} \cap \Omega \neq \varnothing, H_{x,\xi} \cap \omega = \varnothing\}}$$

Then one have the following theorem

Theorem 5.2.2. *Let $u \in \mathscr{B}(\Omega)$ and assume that $u = 0$ on ω and that $WF(u) \cap Z = \varnothing$. Then $u = 0$.*

Proof. We will mimic the proof of [Hör83, Th. 8.6.8]. Let $y \in \Omega$ and $x \in \omega$ and denote by I the segment joining y and x. Then, as explicited in [Hör83, Th. 8.6.8], one can find an open convex set $X \Subset \omega$ such that for any $(z,\xi) \in WF(u) \cap Z$ such that $H_{z,\xi} \cap I \neq \varnothing$, then $H_{z,\xi} \cap X \neq \varnothing$. For $t \in [0,1]$, let us denote by Y_t the convex hull of X and the point $y_t = tx + (1-t)y$. For small enough t, $y_t \in X$, so that $Y_t = X$ and $u|_{Y_t} = 0$. Let T be $\sup_{t\in[0,1]}\{u|_{Y_t} = 0\}$. We want to prove that $T = 1$. We have that $u|_{Y_T} = 0$ with $y_T \notin \omega$. Let us consider a supporting hyperplane H of Y_T. If $y_T \in H$, then $WF(u) \cap Z = \varnothing$, since H does not intersect X. If $y_T \notin H$, then $H \cap \overline{Y_T} \Subset \overline{U}$. It follows from [Hör83, Th. 8.5.8] and Theorem 5.2.1 that $\partial Y_T \cap supp(u) = \varnothing$, and so that $T = 1$ and u vanishes in the neighborhood of y.

□

We are going to apply Theorem 5.2.2 to prove a refinement of the "edge-of-the-wedge" theorem, due to Kolm and Nagel [KN68], (we refer also to Vladimirov [Vla61], Martineau [Mar68]).

Let K be a convex compact set of V and let $a, b \in V$, we define

$$\Gamma_K^a := V \setminus \widehat{\widehat{\{a\}} \cap \widehat{K}}$$

which is the affine cone, union of hyperplanes passing through a with empty intersection with K.

Let A be a subset of V. We define

$$\Gamma_K^{a,b}, \text{ the connected component of } \Gamma_K^a \cap \Gamma_K^b \text{ containing the segment }]a, b[,$$

$$Conv_K A = \cup_{a,b \in A} \overline{\Gamma_K^{a,b}},$$

$$Z = \overline{\{(x, i\xi) \in V \times i\dot{V}^*, (x, \xi) \in S, H_{x,\xi} \cap K = \varnothing\}}$$

Corollary 5.2.3. *Let ω be an open convex set in $V \setminus K$. Let $u \in \mathscr{B}(V)$ such that $u|_\omega = 0$. We assume that $WF(u) \cap Z = \varnothing$. Then u vanishes on $Conv_K(\omega)$.*

Proof. This is a direct application of theoreom 5.2.2 where we set $\Omega = \overset{\circ}{Conv}_K(\omega)$ since we verify that for any $x \in Conv_K(\omega)$ and $(x, i\xi) \in Z$, $H_{x,\xi} \cap \omega = \varnothing$. □

5.3 Hyperfunctions whose Radon transfom vanishes in linearly concave domains

5.3.1 Notations

Let us fix some notations. Let k be an integer, $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. We will assume in this chapter that $0 < k < -n-1$. For a real analytic manifold M and its complexfication X, let π_M denote the projection $T_M^*X \to M$. Denote by P^*M the projectivisation of the cotangent bundle of M, and by ρ_M the projection $\dot{T}^*M \to P^*M$.

As usual, P and P^* will denote the real dimension projective space and its dual, and S the incidence submanifold of $P \times P^*$. We denote by $\widetilde{p}_1$ and $\widetilde{p}_2$ the isomorphisms $P^*P \simeq P_S^*(P \times P^*) \simeq P^*P^*$, and we will identify S to $P_S^*(P \times P^*)$ (cf. remark following

Proposition 4.1.1). Let us consider the following situation:

$$\begin{tikzcd}
 & \dot{T}^*_S(P \times P^*) \arrow[dl, "p_1"'] \arrow[dr, "p_2^a"] & \\
\dot{T}^*P \arrow[rr, "\sim", "\chi"'] \arrow[d, "\rho_P"'] & & \dot{T}^*P^* \arrow[d, "\rho_{P^*}"] \\
 & P^*_S(P \times P^*) \arrow[dl, "\sim", "\widetilde{p}_1"'] \arrow[dr, "\sim", "\widetilde{p}_2"'] & \\
P^*P \arrow[d] & & P^*P^* \arrow[d] \\
P & & P^*
\end{tikzcd}$$

Let Ω be a non empty linearly concave domain strictly contained in P. We denote by

$$\mathbb{O} := \rho_P^{-1}(\widetilde{p}_1((\Omega \times \breve{\Omega}) \cap S)) \subset \dot{T}^*P$$

$$\mathbb{O}^* := \rho_{P^*}^{-1}(\widetilde{p}_2((\Omega \times \breve{\Omega}) \cap S)) \subset \dot{T}^*P^*$$

We notice that

$$\mathbb{O} = \rho_P^{-1}(\widetilde{p}_1(\{(x,\xi) \in S; x \in \Omega, \{x\} \subset \widehat{\xi} \subset \Omega\}))$$

$$\mathbb{O}^* = \chi(\mathbb{O})$$

In the following, we will identify $i.T^*P \simeq T^*_P\mathbb{P}$ and $i.T^*P^* \simeq T^*_{P^*}\mathbb{P}^*$.

5.3.2 Application of projective duality for microfunctions

We shall follow the notations of Section 1.3. Let k be an integer with $-n-1 < k < 0$. In all this section, we will set

$$s \in \Gamma(\dot{T}^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*); \mu hom(\mathbb{C}_{\mathbb{S}}[-1], \Omega_{\mathbb{P}\times\mathbb{P}^*/\mathbb{P}^*}(-k,k^*)))$$

Assume s is non-degenerate on $\dot{T}^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*)$. Theorem 4.2.1 provides such a section.

Let U an open subset of P. Let f be a section of $\mathscr{B}_P(\varepsilon,k)(U)$. As defined in 1.3.6, $Rad(f)$ is the image of f by the sequence of morphisms:

$$\mathscr{B}_P(\varepsilon,k)(U) \longrightarrow \mathscr{C}_P(\varepsilon,k)(\dot{T}^*_U\mathbb{P}) \xrightarrow{\simeq} \mathscr{C}_{P^*}(\varepsilon^*,k^*)(\chi(\dot{T}^*_U\mathbb{P}))$$

Lemma 5.3.1. *Let $\Omega \subsetneq P$ be a non empty linearly concave domain. Then,*

(i) $\pi_P(\mathbb{O}) = \Omega$

(ii) $\mathbb{O}^* = \dot{T}^*_{\breve{\Omega}}\mathbb{P}^*$

Proof. (i) By lemma 5.1.5, we have $\Omega = \widehat{\breve{\Omega}}$, so that for any $x \in \Omega$, there exists an element $\xi \in \breve{\Omega}$ such that $x \in \widehat{\xi}$ and $\widehat{\xi} \subset \Omega$. And so, $x \in \pi_P(\mathbb{O})$

(ii) For $\xi \in \breve{\Omega}$, the fiber $\mathbb{O}^*_\xi$ is $\rho^{-1}_{P^*}(\widetilde{p}_2(\{(\xi, x) \in P^*P^*; \{x\} \subset \widehat{\xi} \subset \Omega\}))$ which is $\dot{T}^*P^*_\xi$. So, after identifying In the following $i.T^*P^* \simeq T^*_{P^*}\mathbb{P}^*$, we get $\mathbb{O}^* = \dot{T}^*_{\breve{\Omega}}\mathbb{P}^*$. □

Remark 5.3.2. Let $f \in \mathscr{B}_P(\varepsilon, k)(\Omega)$. It follows immediately from (ii) of the above lemma that $Rad(f)|_{\mathbb{O}^*}$ may be viewed as an element of $\mathscr{B}_{P^*}(\varepsilon^*, k^*)/\mathcal{A}_{P^*}(\varepsilon^*, k^*)(\breve{\Omega})$. We will denote this element by $Rad(f)|_{\breve{\Omega}}$.

We have:

Proposition 5.3.3. *Let $\Omega \subsetneq P$ be a non empty linearly concave domain and let $f \in \mathscr{B}_P(\varepsilon, k)(\Omega)$. Then, we have*

$$WF(f) \cap \mathbb{O} = \chi^{-1}(WF(Rad(f)|_{\breve{\Omega}}))$$

Proof. For any integer $-n-1 < k < 0$, Theorem 4.3.3 gives a sheaf isomorphism between $\chi_*\mathscr{C}_P(\varepsilon, k)|_{\dot{T}^*_P\mathbb{P}}$ and $\mathscr{C}_{P^*}(\varepsilon^*, k^*)|_{\dot{T}^*_{P^*}\mathbb{P}^*}$. Thus, we have an isomorphism of $\mathbb{C}$-modules $\mathscr{C}_P(\varepsilon, k)|_{\dot{T}^*_P\mathbb{P}}(\mathbb{O}) \simeq \mathscr{C}_{P^*}(\varepsilon^*, k^*)|_{\dot{T}^*_{P^*}\mathbb{P}^*}(\mathbb{O}^*)$. Recalling the isomorphism sp in 2.4.1, we have

$$WF(f) \cap \mathbb{O} = supp(sp(f)) \cap \mathbb{O} = \chi^{-1}(WF(Rad(f)) \cap \mathbb{O}^*) = \chi^{-1}(WF(Rad(f)|_{\breve{\Omega}}))$$

□

In the particular case of dimension $n = 2$, we denote by $\Omega_{\mathbb{C}}$, the union of complexification of the real lines contained in Ω. Assume Ω is strongly linearly concave domain and let us consider an affine chart $\mathbb{R}^2$. Then, there are 2 connected components of $(\Omega_{\mathbb{C}} \setminus \Omega) \cap \mathbb{R}^2$, which we denote by $\Omega_\pm$. In this condition, we deduce immediatly:

Corollary 5.3.4. *Assume that $n = 2$ and that $WF(Rad(f)|_{\breve{\Omega}}) = \varnothing$. Then, f is the boundary value of (f_+, f_-) respectively holomorphic in $\Omega_\pm$.*

Remark 5.3.5. We recover a variant of a result of [Ben11, Th 1.4]. Indeed, it was proven therein that, if the Radon transform of a function f in $\mathbb{R}^2$ with $\mathcal{O}(\frac{1}{|x|^2})$ growth at infinity, vanishes in a strongly linearly concave domain, then f is the boundary value of holomorphic functions, verifying in their respective domains some growth conditions at $\mathcal{O}(\frac{1}{|z|^2})$ rate at infinity.

5.4 Radon transform for quadratic cones

5.4.1 Residues of forms on cones of signature $(1, n-1)$

Notations

Let $n \geq 2$ and let A be an affine chart of P. For a cone γ in A, we denote by γ° the polar cone of γ:

$$\gamma^\circ := \{x \in A; \langle x, u\rangle \geq 0, \forall u \in \gamma\}$$

Let $x = (x_1, ..., x_n)$ be a system of linear coordinates on A, $u = (u_1, ..., u_n)$ the dual coordinates on A^*. For q an integer such that $0 \leq q < n$, we denote by γ_q the cone of signature $(q, n-q)$ in A:

$$\gamma_q := \{(x_1, \dots x_n) \in A; \sum_{1\leq j\leq q} x_j^2 < \sum_{q+1\leq j\leq n} x_j^2\} \tag{5.4.1}$$

$$\lambda_q := A \setminus \gamma_q$$

and

$$\gamma_q^* := \{(u_1, \dots u_n) \in A^*; \sum_{q+1\leq j\leq n} u_j^2 < \sum_{1\leq j\leq q} u_j^2\}$$

$$\lambda_q^* := A^* \setminus \gamma_q^*$$

We denote by λ_q the complementary of γ_q in A and similarly for λ_q^* in A^*.

Let us denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_{A^*})$, the subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{A^*})$ consisting of conic objects with respect to the natural action of $\mathbb{R}^+$. For $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_{A^*})$, we denote by $F^\wedge \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_A)$ its Fourier-Sato transform.

Topological boundary morphism for cones of signature $(q, n-q)$

Proposition 5.4.1. *There is a natural morphism in* $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_A)$

$$\mathbb{C}_{\{0\}} \to \mathbb{C}_{\gamma_q}[q+1] \tag{5.4.2}$$

Proof. We have a natural mophism

$$\mathbb{C}_{A^*} \to \mathbb{C}_{\lambda_q^*}$$

Let us apply on both sides the Fourier-Sato functor using [KS97, Lem. 6.2.1], we get

$$(\mathbb{C}_{A^*})^\wedge \simeq \mathbb{C}_{\{0\}}[-n]$$
$$(\mathbb{C}_{\lambda_q^*})^\wedge \simeq \mathbb{C}_{\lambda_q}[-n+q]$$

Hence, we get the natural morphism

$$\mathbb{C}_{\{0\}} \to \mathbb{C}_{\lambda_q}[q]$$

Given the distinguished triangle,

$$\mathbb{C}_{\gamma_q} \longrightarrow \mathbb{C}_A \longrightarrow \mathbb{C}_{\lambda_q} \xrightarrow{+1}$$

we have:

$$\mathbb{C}_{\{0\}} \to \mathbb{C}_{\lambda_q}[q] \to \mathbb{C}_{\gamma_q}[q+1]$$

□

Remark 5.4.2. The case of cones of signature $(1, n-1)$ can be handled in a more constructive way. Indeed, the complementary set λ_1 of γ_1 is the union of two closed convex sets, let us say λ_1^1 and λ_1^2.

We have the two distinguished triangles:

$$\mathbb{C}_{\lambda_1^1 \cup \lambda_1^2} \longrightarrow \mathbb{C}_{\lambda_1^1} \bigoplus \mathbb{C}_{\lambda_1^2} \longrightarrow \mathbb{C}_{\{0\}} \xrightarrow{+1} \tag{5.4.3}$$

$$\mathbb{C}_{\gamma_1} \longrightarrow \mathbb{C}_A \longrightarrow \mathbb{C}_{\lambda_1^1 \cup \lambda_1^2} \xrightarrow{+1} \tag{5.4.4}$$

From the triangles (5.4.3), (5.4.4), we get the morphisms:

$$\mathbb{C}_{\{0\}} \to \mathbb{C}_{\lambda_1^1 \cup \lambda_1^2}[1]$$
$$\mathbb{C}_{\lambda_1^1 \cup \lambda_1^2} \to \mathbb{C}_{\gamma_1}[1]$$

Hence, we get:

$$\mathbb{C}_{\{0\}} \to \mathbb{C}_{\gamma_1}[2] \tag{5.4.5}$$

Application to hyperfunctions with wave-front set in (n-1,1)-signature cones

We recall that we denote by $\mathscr{C}_A$ the sheaf of microfunctions on $T_A^*\mathbb{A}$. For K be a compact convex subset of A, we set

$$\mathscr{B}(K) := \varinjlim_{U \supset K} \mathscr{B}(U)$$

In the following, let us denote by $\gamma = \gamma_1$ the cone of signature $(1, n-1)$ defined by 5.4.1, and by $\lambda = \lambda_1$ its complementary set in A, union of two closed convex sets, λ^1 and λ^2. Let $Int(\lambda^{i\circ})$, $i = 1, 2$ be the interior of the polar sets $\lambda^{i\circ}$.

Lemma 5.4.3. *Let W be an open convex neighborhood of K in A. There exists a natural exact sequence*

(5.4.6)

$$0 \longrightarrow H^{n-2}(W + i\gamma; \mathcal{O}_{\mathbb{A}}) \longrightarrow \mathscr{B}(W) \longrightarrow H^n_{W+i\lambda^1}(\mathbb{A}; \mathcal{O}_{\mathbb{A}}) \bigoplus H^n_{W+i\lambda^2}(\mathbb{A}; \mathcal{O}_{\mathbb{A}}) \longrightarrow 0$$

Proof. From (5.4.3), we get

$$\mathbb{C}_{W+i(\lambda_1^1\cup\lambda_1^2)} \longrightarrow \mathbb{C}_{W+i\lambda_1^1} \bigoplus \mathbb{C}_{W+i\lambda_1^2} \longrightarrow \mathbb{C}_W \xrightarrow{+1}$$

Applying the functor $\mathrm{RHom}\,(\bullet, \mathcal{O}_{\mathbb{A}})$ to this distinguished triangle, we get the long exact sequence:

$$(5.4.7)\cdots \longrightarrow H^{n-1}_{W+i\lambda^1}(\mathbb{A};\mathcal{O}_{\mathbb{A}}) \bigoplus H^{n-1}_{W+i\lambda^2}(\mathbb{A};\mathcal{O}_{\mathbb{A}}) \longrightarrow H^{n-1}_{W+i(\lambda^1\cup\lambda^2)}(\mathbb{A};\mathcal{O}_{\mathbb{A}}) \longrightarrow$$

$$H^n_W(\mathbb{A};\mathcal{O}_{\mathbb{A}}) \longrightarrow H^n_{W+i\lambda^1}(\mathbb{A};\mathcal{O}_{\mathbb{A}}) \bigoplus H^n_{W+i\lambda^2}(\mathbb{A};\mathcal{O}_{\mathbb{A}}) \longrightarrow H^n_{W+i(\lambda^1\cup\lambda^2)}(\mathbb{A};\mathcal{O}_{\mathbb{A}}) \longrightarrow \cdots$$

From the distinguished triangle (5.4.4), and taking into account that $W + iA$ is convex, $\mathrm{R}\mathscr{H}om\,(\mathbb{C}_{W+iA}, \mathcal{O}_{\mathbb{A}}) \simeq \mathcal{O}_{W+iA}$ is concentrated in degree 0, and we get the isomorphism

$$H^{n-2}(W + i\gamma;\mathcal{O}_{\mathbb{A}}) \xrightarrow{\sim} H^{n-1}_{W+i(\lambda^1\cup\lambda^2)}(\mathbb{A};\mathcal{O}_{\mathbb{A}}) \tag{5.4.8}$$

and

$$H^{n-1}(W + i\gamma;\mathcal{O}_{\mathbb{A}}) \xrightarrow{\sim} H^{n}_{W+i(\lambda^1\cup\lambda^2)}(\mathbb{A};\mathcal{O}_{\mathbb{A}})$$

Since, γ is a cone of signature $(1, n-1)$, $H^{n-1}(W + i\gamma;\mathcal{O}_{\mathbb{A}}) \simeq 0$, and so

$$H^{n}_{W+i(\lambda^1\cup\lambda^2)}(\mathbb{A};\mathcal{O}_{\mathbb{A}}) \simeq 0 \tag{5.4.9}$$

Noticing that $\mathrm{R}\Gamma_{W+i\lambda^i}(\mathcal{O}_{\mathbb{A}})$, $i = 1, 2$, is concentrated in degree n, and applying 5.4.8, 5.4.9 to 5.4.7, we get the exact sequence (5.4.6) □

Theorem 5.4.4. *Let K be a compact convex subset of A. There exists a natural exact sequence*

$$0 \longrightarrow \varinjlim_{\gamma'} H^{n-2}(K + i\gamma';\mathcal{O}_{\mathbb{A}}) \xrightarrow{Res} \mathscr{B}(K) \longrightarrow$$
$$\mathscr{C}_A(K + i\cdot Int(\lambda^{1\circ})) \bigoplus \mathscr{C}_A(K + i\cdot Int(\lambda^{2\circ})) \longrightarrow 0$$

where γ' ranges over the open convex cones of signature $(1, n-1)$, with $\gamma \subset \gamma'$.

Proof. Let W be an open neighborhood of K in A.

Following [KS90, Th.4.3.2], we know that for an open convex cone $V \subset T^*_W\mathbb{A}$, we have the isomorphism

$$\mathscr{C}_A(V) \simeq \varinjlim_{Z,U} H^n_{Z\cap U}(U;\mathcal{O}_{\mathbb{A}})$$

where Z and U range over respectively, the closed sets whose normal cone is contained in the polar set of V, and the open subsets U of $\mathbb{A}$ such that $\pi(V) = U \cap W$, π being the natural projection of $T^*\mathbb{A}$ onto $\mathbb{A}$.

Let U be an open neighborhood of K in $\mathbb{A}$, γ' an open cone of signature $(1, n-1)$, with $\gamma' \supset \gamma$ and $\lambda^{i'}$, $i = 1, 2$, the two convex closed sets the union of which, is the complementary of γ'. Then, we get by the same steps as above the exact sequence:

$$0 \longrightarrow H^{n-2}((W + i.\gamma') \cap U; \mathcal{O}_{\mathbb{A}}) \longrightarrow \mathscr{B}(W) \longrightarrow$$
$$H^n_{(W+i.\lambda^{1'})\cap U}(\mathbb{A}; \mathcal{O}_{\mathbb{A}}) \bigoplus H^n_{(W+i.\lambda^{2'})\cap U}(\mathbb{A}; \mathcal{O}_{\mathbb{A}}) \longrightarrow 0$$

Passing to the inductive limit on $W \supset K$ in A, U complex neighborhood of K, and γ' ranges over the open convex cones of signature $(1, n-1)$, with $\gamma \subset \gamma'$, which consititutes a fundamental system of neighborhoods, we deduce from the above sequence the following exact sequence:

$$0 \longrightarrow \varinjlim_{\gamma', U} H^{n-2}((K + i\gamma') \cap U; \mathcal{O}_{\mathbb{A}}) \longrightarrow \mathscr{B}(K) \longrightarrow$$
$$\mathscr{C}_A(K + i.Int(\lambda^{1\circ})) \bigoplus \mathscr{C}_A(K + i.Int(\lambda^{2\circ})) \longrightarrow 0$$

Indeed,

$$\begin{aligned}
\mathscr{C}_A(K + i.Int(\lambda^{i\circ})) &\simeq \varinjlim_{W \supset K} \mathscr{C}_A(W + i.Int(\lambda^{i\circ})) \\
&\simeq \varinjlim_{W \supset K, \gamma', U} H^n_{(\overline{W}+i.\lambda'_i)\cap U}(\mathbb{A}; \mathcal{O}_{\mathbb{A}}) \\
&\simeq \varinjlim_{W \supset K, \gamma', U} H^n_{(W+i.\lambda'_i)\cap U}(\mathbb{A}; \mathcal{O}_{\mathbb{A}})
\end{aligned}$$

□

Remark 5.4.5. Theorem 5.4.4 brings a proof to a variant of a conjecture by Henkin in [Hen04], which was formulated, without the assumption that K is a convex compact, in the following way: the wave-front set of a hyperfunction has no intersection with $\lambda^{i\circ}, i = 1, 2$ if and only if it is the residue of a $(0, n-2)$-form defined on $K + i\gamma$ and $\bar{\partial}$-closed inthere.

5.4.2 Residue diagram for Radon Transform and application

Residue on cones of signature $(q, n-q)$

Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}})$ and suppose we are given a topological boundary morphism

$$\mathbb{C}_P(\varepsilon)[-n] \to F$$

Then, we get the analytic boundary morphism

$$\mu hom(F, \mathscr{O}_{\mathbb{P}}(k)) \to \mu hom(\mathbb{C}_P(\varepsilon)[-n], \mathscr{O}_{\mathbb{P}}(k)) := \mathscr{C}_P(\varepsilon, k) \tag{5.4.10}$$

Let k be an integer with $-n-1 < k < 0$. Let us set,

$$s \in \Gamma(\dot{T}^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*); \mu hom(\mathbb{C}_{\mathbb{S}}[-1], \Omega_{\mathbb{P}\times\mathbb{P}^*/\mathbb{P}}(-k, k^*)))$$

The following diagram commutes

$$\begin{tikzcd} \chi_*\mu hom(F,\mathscr{O}_{\mathbb{P}}(k))|_{\dot{T}^*\mathbb{P}} \arrow[r, "Res"] \arrow[d, "Rad"'] & \chi_*\mathscr{C}_P(\varepsilon,k)|_{\dot{T}^*\mathbb{P}} \arrow[d, "Rad"] \\ \mu hom(\Phi_{\mathbb{S}}(F),\mathscr{O}_{\mathbb{P}^*}(k^*))|_{\dot{T}^*\mathbb{P}^*} \arrow[r, "Res"'] & \mathscr{C}_{P^*}(\varepsilon^*,k^*)|_{\dot{T}^*\mathbb{P}^*} \end{tikzcd} \tag{5.4.11}$$

Assume s is non-degenerate on $\dot{T}^*_{\mathbb{S}}(\mathbb{P}\times\mathbb{P}^*)$, then the vertical arrows *Rad* above are isomorphisms.

Throughout the rest of this section, we will set s the section of 4.2.1, which provides such a non-degenerate section.

We recall we denoted by A an affine chart of P.

Proposition 5.4.6. *We have the natural morphism:*

$$H^{n-q-1}\mathrm{R}\Gamma_{A+i.\gamma_q}(\mathscr{O}_{\mathbb{A}}) \xrightarrow{Res} \mathscr{B}_A$$

Proof. We already constructed the natural morphism (5.4.2):

$$\mathbb{C}_{\{0\}} \to \mathbb{C}_{\gamma_q}[q+1] \tag{5.4.12}$$

We deduce the morphism

$$\mathbb{C}_A \to \mathbb{C}_{A+i.\gamma_q}[q+1]$$

□

Remark 5.4.7. The residue 5.4.12 was constructed in [CGT95] through several variables complex analysis technics. Our approach provides a natural construction of this residue.

The special case of cones of signature $(1, n-1)$

Throughout the end of this section, we will see how cones of signature $(1, n-1)$ in $\mathbb{A}$ rise naturally from the Radon transform of strip-like sets in $\mathbb{P}^*$. We will then provide the residue diagram 5.4.17.

Setting the geometric context Let $\delta > 0$ be a real number. Let $\mathbb{A}$ be the affine chart relative to $\{(z_0 : z_1 : \ldots : z_n) \in \mathbb{P}; z_1 = 1\}$, and $\mathbb{A}^*$ the affine chart related to $\{(\xi_0 : \xi_1 : \ldots : \xi_n) \in \mathbb{P}^*; \xi_0 = 1\}$. We set:

$$\mathbb{H} := \{(z_0 : z_1 : \ldots : z_n) \in \mathbb{P}; z_1 = 0\} = \mathbb{A} \cap P$$
$$A^* := \mathbb{A}^* \cap P^*$$
$$H := \{(x_0 : x_1 : \ldots : x_n) \in P; x_1 = 0\}$$

We will consider the following sets

$$U := \{\xi \in A^*; (\sum_{2\le j\le n} \xi_j^2)^{\frac{1}{2}} < \delta\} \hookrightarrow \mathbb{P}^* \tag{5.4.13}$$

$$Z := \widehat{U} \cap P \tag{5.4.14}$$

We have

Lemma 5.4.8. *The following assertions hold*

(i) *U is $\mathbb{S}$-trivial,*

(ii) *$\widehat{U} \setminus \mathbb{H}$ is a cone of signature $(1, n-1)$,*

(iii) *There exists an affine $\mathbb{S}$-trivial compact set $K \subset P$ such that $Z = P \setminus K$*

Proof. (i) is related to a more general and obvious fact: convex sets, either open or closed, are $\mathbb{S}$-trivial.

(ii) Let $z \in \widehat{U} \setminus \mathbb{H}$. There exists $\xi \in U$ such that $\langle z, \xi \rangle = 0$, so that $z_0 + \xi_1 + \sum_{2 \leq j \leq n} z_j \xi_j = 0$. Taking the imaginary part, and denoting $y = \mathrm{Im}(z)$, we have

$$| y_0 | = \sum_{2 \leq j \leq n} | y_j \xi_j | < \delta (\sum_{2 \leq j \leq n} y_j^2)^{\frac{1}{2}} \tag{5.4.15}$$

Let γ be the cone of signature $(1, n-1)$

$$\gamma := \{ y = (y_0, y_2 \ldots, y_n) \in A ; | y_0 | < \delta (\sum_{2 \leq j \leq n} y_j^2)^{\frac{1}{2}} \}$$

We have

$$\widehat{U} \setminus \mathbb{H} = A + i\gamma$$

Indeed, (5.4.15) proves that $\widehat{U} \setminus \mathbb{H} \subset A + i\gamma$. For the converse inclusion, let $z = x + iy \in A + i\gamma$. Let us choose $\xi \in A^*$ such that

$$\xi_1 = -z_0 - \sum_{2 \leq j \leq n} z_j \xi_j$$
$$\xi_j = \frac{y_0 . y_j}{\sum_{2 \leq j \leq n} y_j^2}, j \neq 2$$

Then $\xi \in U$ and $\langle z, \xi \rangle = 0$.

(iii) Let us notice that $Z \cap H$ is described by $\{ (x_0, x_2, \ldots, x_n) \in A ; \sum_{2 \leq j \leq n} x_j^2 > \frac{x_0^2}{\delta^2} \}$, the complement of which in H is a ball of radius $\frac{1}{\delta}$. Indeed, for $x \in Z \cap H$ and any $\xi \in U$ such that $(x, \xi) \in S$, we have:

$$| x_0 | = \sum_{2 \leq j \leq n} | x_j \xi_j | < \delta (\sum_{2 \leq j \leq n} x_j^2)^{\frac{1}{2}}$$

□

The Residue diagram Let $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. We recall the defintion of Z by 5.4.14, and K in 5.4.8 (iii).

Lemma 5.4.9. *There is a natural morphism in* $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}^*})$

$$\mathbb{C}_{P^*}(\varepsilon^*) \to \mathbb{C}_{\widehat{K}}[n-1]$$

Denoting by $\mathcal{L}_Z(\varepsilon^*)$ *the object in* $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}^*})$ *defined up to an isomorphism by the distinguished triangle:*

$$\mathcal{L}_Z(\varepsilon^*) \longrightarrow \mathbb{C}_{P^*}(\varepsilon^*) \longrightarrow \mathbb{C}_{\widehat{K}}[n-1] \xrightarrow{+1} \tag{5.4.16}$$

we have in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}^*}; \dot{T}^*\mathbb{P}^*)$

$$\Phi_{\mathbb{S}}(\mathbb{C}_Z(\varepsilon^*)) \simeq \mathcal{L}_Z(\varepsilon^*)$$

Proof. Let us compute $\Phi_{\mathbb{S}}(\mathbb{C}_Z(\varepsilon))$.

We have the exact sequence

$$0 \longrightarrow \mathbb{C}_Z(\varepsilon) \longrightarrow \mathbb{C}_P(\varepsilon) \longrightarrow \mathbb{C}_K(\varepsilon) \longrightarrow 0$$

Applying Lemma 4.3.2 and Lemma 5.1.8, we get the distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}^*}; \dot{T}^*\mathbb{P}^*)$:

$$\Phi_{\mathbb{S}}(\mathbb{C}_Z(\varepsilon^*)) \longrightarrow \mathbb{C}_{P^*}(\varepsilon^*) \longrightarrow \mathbb{C}_{\widehat{K}}[n-1] \xrightarrow{+1}$$

□

Theorem 5.4.10. *Given a section* s *of* $H^1(\mathbb{P} \times \mathbb{P}^*); \mu hom(\mathbb{C}_{\mathbb{S}}[-1], \Omega_{\mathbb{P}\times\mathbb{P}^*/\mathbb{P}^*}(-k, k^*))$ *The following diagram commutes:*

(5.4.17)

$$\begin{tikzcd}
\chi_* H^{n-2}(\mu hom(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon), \mathscr{O}_{\mathbb{P}}(k)))|_{\dot{T}^*\mathbb{P}} \arrow[r, "Res"] \arrow[d, "Rad"'] & \chi_* H^{n}(\mu hom(\mathbb{C}_{Z}(\varepsilon), \mathscr{O}_{\mathbb{P}}(k)))|_{\dot{T}^*\mathbb{P}} \arrow[d, "Rad"'] \\
H^{n-2}(\mu hom(\Phi_{\mathbb{S}}(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon)), \mathscr{O}_{\mathbb{P}^*}(k^*)))|_{\dot{T}^*\mathbb{P}^*} \arrow[r, "Res"'] & H^{n}(\mu hom(\mathcal{L}_Z(\varepsilon^*), \mathscr{O}_{\mathbb{P}^*}(k^*)))|_{\dot{T}^*\mathbb{P}^*}
\end{tikzcd}$$

Assuming that s *is non-degenerate on* $\dot{T}^*_{\mathbb{S}}(\mathbb{P} \times \mathbb{P}^*)$*, the vertical arrows Rad are isomorphisms.*

Proof. Since $\widehat{U} \setminus Z$ is a cone of signature $(1, n-1)$, (5.4.2) gives a morphism:

$$\mathbb{C}_A \to \mathbb{C}_{\widehat{U}\setminus Z}[2]$$

and so,

$$\mathbb{C}_Z \to \mathbb{C}_{\widehat{U}\setminus A}[2] \tag{5.4.18}$$

Let us now apply (5.4.11) 5.4.9. The following diagram commutes:

(5.4.19)

$$\begin{array}{ccc}
\chi_* H^{n-2}(\mu hom(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon), \mathscr{O}_{\mathbb{P}}(k)))|_{\dot{T}^*\mathbb{P}} & \xrightarrow{Res} & \chi_* H^{n}(\mu hom(\mathbb{C}_{Z}(\varepsilon), \mathscr{O}_{\mathbb{P}}(k)))|_{\dot{T}^*\mathbb{P}} \\
\simeq \downarrow & & \downarrow \simeq \\
H^{n-2}(\mu hom(\Phi_{\mathbb{S}}(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon)), \mathscr{O}_{\mathbb{P}^*}(k^*)))|_{\dot{T}^*\mathbb{P}^*} & \xrightarrow[Res]{} & H^{n}(\mu hom(\mathcal{L}_Z(\varepsilon^*), \mathscr{O}_{\mathbb{P}^*}(k^*)))|_{\dot{T}^*\mathbb{P}^*}
\end{array}$$

□

Application to Radon transform for microfunctions residues

Now, let us consider a complex neighborhood $\omega \subset \mathbb{A}^*$ of U such that, denoting by π the natural projection $\mathbb{A}^* \xrightarrow{\pi} A^*$, $\pi(\omega) = U$.

Proposition 5.4.11. *We have*

$$SS(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon)) \cap \chi^{-1}\dot{T}^*\omega = \varnothing \tag{5.4.20}$$

Proof. For any $\xi \in \omega$, the set $\widehat{\{\xi\}} \cap (\widehat{U} \setminus A)$ is homeomorphic to $\mathbb{P}_{n-1} \setminus A_{n-1}$. Hence, the stalks of $\Phi_{\mathbb{S}}(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon))$ are constant over ω. Since ω is simply connected, there is $L \in \mathrm{D}^{\mathrm{b}}(\mathfrak{mod}(\mathbb{C}))$ such that $\Phi_{\mathbb{S}}(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon))|_{\omega}$ is is the constant sheaf L_{ω}.

Hence, we have

$$SS(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon)) \cap \chi^{-1}\dot{T}^*\omega = \chi^{-1}(SS(\Phi_{\mathbb{S}}(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon))) \cap \dot{T}^*\omega) = \varnothing$$

□

As an immediate corollary,

Corollary 5.4.12. *We have*

$$\mu hom(\mathbb{C}_{\widehat{U}\setminus A}(\varepsilon), \mathscr{O}_{\mathbb{P}}(k))|_{\chi^{-1}\dot{T}^*\omega} \simeq 0$$

Chapter 6

Appendix

6.1 Radon transform of the structure sheaf

6.1.1 Notations

In the following, let X, Y be two complex manifolds of dimension n, S a closed submanifold a codimension c, we set

$$\Lambda := T^*_S(X \times Y) \cap \dot{T}^*X \times \dot{T}^*Y)$$

$$\tilde{S} = r(S) \subset Y \times X$$

where r is the natural morphism $X \times Y \to Y \times X$, $(x, y) \mapsto (y, x)$. We denote by $X \xleftarrow{q_1} X \times Y \xrightarrow{q_2} Y$ the projections. We denote by f, g the restrictions of q_1, q_2 to S:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y, \end{tikzcd} \qquad \begin{tikzcd} & \Lambda \arrow[dl, "p_1|_\Lambda"'] \arrow[dr, "p_2^a|_\Lambda"] & \\ \dot{T}^*X & & \dot{T}^*Y \end{tikzcd}$$

In the following, we also assume that,

$$\begin{cases} \text{(i) f and g are smooth and proper} \\ \text{(ii) } p_1|_\Lambda \text{ and } p_2^a|_\Lambda \text{ are isomorphisms} \end{cases}$$

For $A \subset X$, $B \subset Y$, we note $\widehat{A} := g(f^{-1}(A))$, $\widehat{B} := f(g^{-1}(B))$.
We denote by $\mathscr{B}_{S|X\times Y}$ the $\mathscr{D}_{X\times Y}$-module $H^d_{[S]}(\mathcal{O}(X \times Y))$ and we set

$$\mathscr{B}^{(n,0)}_{S|X\times Y} := q_1^{-1}\Omega_X \otimes_{q_1^{-1}\mathcal{O}_X} \mathscr{B}_{S|X\times Y}$$

For $\mathscr{F}, \mathscr{G}$ line bundles over X and Y respectively, one denotes by

$$\mathscr{B}^{(n,0)}_{S|X\times Y}(\mathscr{F}, \mathscr{G}) := q_2^{-1}\mathscr{G}\mathscr{D} \otimes_{q_2^{-1}\mathscr{D}_Y} \mathscr{B}_{S|X\times Y} \otimes_{q_1^{-1}\mathscr{D}_X} q_1^{-1}\mathscr{D}(\mathscr{F} \otimes_{\mathcal{O}_X} \Omega_X)$$

where $\mathscr{G}\mathscr{D}$ and $\mathscr{D}\mathscr{F}$ are the natural $\mathscr{D}_Y$-modules and $\mathscr{D}_X$-modules associated with $\mathscr{G}$ and $\mathscr{F}$ respectively.

For $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$, $G \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y)$, we set

$$\Phi_S(F) := \mathrm{R}g_! f^{-1} F[n-c]$$

$$\Phi_{\tilde{S}}(G) := \mathrm{R}f_! g^{-1} G[n-c]$$

Let $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ be the derived category of $\mathscr{D}_X$-modules. For $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$, and $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)$, we set

$$\underline{\Phi}_S(\mathscr{M}) := \underline{g}_* \underline{f}^{-1} \mathscr{M}$$

$$\underline{\Phi}_{\tilde{S}}(\mathscr{N}) := \underline{f}_* \underline{g}^{-1} \mathscr{N}$$

It can be seen easily that

$$\underline{\Phi}_S(\mathscr{M}) \simeq \mathrm{R}q_{2!}(\mathscr{B}^{(n,0)}_{S|X\times Y} \otimes_{q_1^{-1}\mathscr{D}_X} q_1^{-1}\mathscr{M})$$

Denoting by $\pi : T^*X \to X$ the natural projection, one associates to any $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$, the $\mathscr{E}_X$-module

$$\mathscr{E}\mathscr{M} := \mathscr{E}_X \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\mathscr{M}$$

One can define a microlocal version of $\underline{\Phi}_S$. For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{E}_X)$, we set

$$\underline{\Phi}^{\mu}_S(\mathscr{M}) := \mathrm{R}p^a_{2!}(\mathscr{C}^{(n,0)}_{S|X\times Y} \otimes_{p_1^{-1}\mathscr{E}_X} p_1^{-1}\mathscr{M})$$

where $\mathscr{C}^{(n,0)}_{S|X\times Y} := q_1^{-1}\Omega_X \otimes_{q_1^{-1}\mathcal{O}_X} \mathscr{C}_{S|X\times Y}$ and $\mathscr{C}_{S|X\times Y} := \mathscr{E}_X \mathscr{B}_{S|X\times Y}$.

6.1.2 Integral transform for $\mathcal{D}$-modules associated to line bundles

There is an isomorphism

$$\Gamma(X\times Y; \mathscr{B}^{(n,0)}_{S|X\times Y}(\mathscr{F}, \mathscr{G}^*)) \simeq \mathrm{Hom}_{\mathscr{D}_Y}(\mathscr{D}\mathscr{G}, \underline{\Phi}_S \mathscr{D}\mathscr{F})$$

So that to any $s \in \Gamma(X\times Y; \mathscr{B}^{(n,0)}_{S|X\times Y}(\mathscr{F}, \mathscr{G}^*))$, one can associate a $\mathscr{D}_Y$ morphism $\alpha(s) : \mathscr{D}\mathscr{G} \to \underline{\Phi}_S \mathscr{D}\mathscr{F}$. We then have

Theorem 6.1.1 ([DS96]). *For s a non-degenerate section of $\Gamma(X\times Y; \mathscr{B}^{(n,0)}_{S|X\times Y}(\mathscr{F}, \mathscr{G}^*))$ over Λ, the induced morphism $H^0(\alpha(s)) : \mathscr{D}\mathscr{G} \to H^0(\underline{\Phi}_S \mathscr{D}\mathscr{F})$ is a $\mathscr{D}_Y$-linear isomorphism.*

Proof. We reproduce the proof of [DS96]. It is shown in [SKK] that this morphism is an isomorphism on $\dot{T}^*Y$. It can be shown that $\alpha(s)$ induces an $\mathscr{E}_Y$-linear morphism $\mathscr{E}\mathscr{G} \to \underline{\Phi}^{\mu}_S(\mathscr{E}\mathscr{F})$. Let π_Y be the projection $T^*Y \to Y$, we have the following distinguished triangle for conic objects of $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{T^*Y})$

$$\mathrm{R}\pi_{Y!}(.) \longrightarrow \mathrm{R}\pi_{Y*}(.) \longrightarrow \mathrm{R}\dot{\pi}_{Y*}(.) \xrightarrow{+1}$$

So that we have

$$\begin{tikzcd}
\mathrm{R}\pi_{Y!}(\mathscr{E}\mathscr{G}) \arrow[r] \arrow[d, "\tilde{\alpha}(s)"] & \mathrm{R}\pi_{Y*}(\mathscr{E}\mathscr{G}) \arrow[r] \arrow[d, "\alpha(s)"] & \mathrm{R}\dot{\pi}_{Y*}(\mathscr{E}\mathscr{G}) \arrow[r, "+1"] \arrow[d, "\dot{\alpha}(s)"] & {} \\
\mathrm{R}\pi_{Y!}(\underline{\Phi}^{\mu}_S(\mathscr{E}\mathscr{F})) \arrow[r] & \mathrm{R}\pi_{Y*}(\underline{\Phi}^{\mu}_S(\mathscr{E}\mathscr{F})) \arrow[r] & \mathrm{R}\dot{\pi}_{Y*}(\underline{\Phi}^{\mu}_S(\mathscr{E}\mathscr{F})) \arrow[r, "+1"] & {}
\end{tikzcd} \tag{6.1.1}$$

It suffices to prove that $\alpha(s)$ is an isomorphism at the 0 cohomology degree. From the theory of [SKK], we know that $\dot{\alpha}(s)$ is an isomorphism. It is easily seen that

$$\mathrm{R}\pi_{Y!}(\underline{\Phi}^{\mu}_S(\mathscr{E}\mathscr{F})) \simeq \mathrm{R}g_!(\mathcal{O}^{(n,0)}_{X\times Y}|_S \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathscr{F})[-c]$$

and

$$\mathrm{R}\pi_{Y!}(\mathscr{E}\mathscr{G}) \simeq \mathcal{O}^{(n,0)}_{Y\times Y}|_Y \otimes_{\mathcal{O}_Y} \mathscr{G}[-n]$$

The 0-degree cohomology of these last two terms are null since $c > 0$. So the long exact sequence induced by 6.1.1 gives

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \pi_{Y*}(\mathscr{E}\mathscr{G}) \arrow[r] \arrow[d, "H^0(\alpha(s))"] & H^0(\mathrm{R}\dot{\pi}_{Y*}(\mathscr{E}\mathscr{G})) \arrow[r, "+1"] \arrow[d, "\simeq"] & H^{-n+1}(\mathcal{O}^{(n,0)}_{Y\times Y}|_Y \otimes_{\mathcal{O}_Y} \mathscr{G}) \arrow[d, "H^1(\tilde{\alpha}(s))"] \\
0 \arrow[r] & \pi_{Y*}(\Phi_S(\mathscr{E}\mathscr{F})) \arrow[r] & H^0(\mathrm{R}\dot{\pi}_{Y*}(\underline{\Phi}^{\mu}_S(\mathscr{E}\mathscr{F}))) \arrow[r, "+1"] & H^{-c+1}(\mathrm{R}g_!(\mathcal{O}^{(n,0)}_{X\times Y}|_S \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathscr{F}))
\end{tikzcd}$$

The result is immediate for $n, c > 1$ and we refer to [DS96] for the cases $n = 1$ or $n > 1, c = 1$. □

Let $n > 0$ be the dimension of the complex projective space. Let $k \in \mathbb{Z}$ such that $-n-1 < k < 0$. We denote by $k^* = -n-1-k$. Specializing theorem 6.1.1 for the projective space and its dual, we get

Corollary 6.1.2 ([DS96])**.** *There is an isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}^*})$*:*

$$\Phi_{\mathbb{S}}(\mathcal{O}_{\mathbb{P}}(k)) \simeq \mathcal{O}_{\mathbb{P}^*}(k^*)$$

6.2 Associativity for the composition of μhom

We will make use of the notations defined in section 2.1.1 and prove Theorem 2.7.3.

Theorem. Let F_i, G_i, H_i respectively in $\mathrm{D^b}(\mathbf{k}_{M_{12}}), \mathrm{D^b}(\mathbf{k}_{M_{23}}), \mathrm{D^b}(\mathbf{k}_{M_{34}})$, $i = 1, 2$ then the following isomorphism

$$\Big(\mu hom(F_1, F_2) \overset{a}{\underset{2}{\circ}} \mu hom(G_1, G_2)\Big) \overset{a}{\underset{3}{\circ}} \mu hom(H_1, H_2) \xrightarrow{\sim} \mu hom(F_1, F_2) \overset{a}{\underset{2}{\circ}} \Big(\mu hom(G_1, G_2) \overset{a}{\underset{3}{\circ}} \mu hom(H_1, H_2)\Big)$$

is compatible with the $\circ$ bifunctor in the following sense

$$\begin{tikzcd}
(\mu hom(F_1, F_2) \overset{a}{\underset{2}{\circ}} \mu hom(G_1, G_2)) \overset{a}{\underset{3}{\circ}} \mu hom(H_1, H_2) \arrow[r, "\sim"] \arrow[d] & \mu hom(F_1, F_2) \overset{a}{\underset{2}{\circ}} (\mu hom(G_1, G_2) \overset{a}{\underset{3}{\circ}} \mu hom(H_1, H_2)) \arrow[d] \\
\mu hom(F_1 \underset{2}{\circ} G_1, F_2 \underset{2}{\circ} G_2) \overset{a}{\underset{3}{\circ}} \mu hom(H_1, H_2) \arrow[d] & \mu hom(F_1, F_2) \overset{a}{\underset{2}{\circ}} \mu hom(G_1 \underset{3}{\circ} H_1, G_2 \underset{3}{\circ} H_2) \arrow[d] \\
\mu hom((F_1 \underset{2}{\circ} G_1) \underset{3}{\circ} H_1, (F_2 \underset{2}{\circ} G_2) \underset{3}{\circ} H_2) \arrow[r, "\sim"] & \mu hom(F_1 \underset{2}{\circ} (G_1 \underset{3}{\circ} H_1), F_2 \underset{2}{\circ} (G_2 \underset{3}{\circ} H_2))
\end{tikzcd}$$

Proof. To lighten the notation, we won't mention in the following the subscript of $\underset{i}{\circ}, \overset{a}{\underset{i}{\circ}}$, and simply denote it $\circ$. Let us first recall the construction of the natural morphism in $\mathrm{D^b}(\mathbf{k}_{T^*M_{13}})$

Lemma 6.2.1 ([KS90, Prop. 4.4.11])**.** *There is a natural morphism*

$$\mu hom(F_1, F_2) \overset{a}{\underset{2}{\circ}} \mu hom(G_1, G_2) \to \mu hom(F_1 \underset{2}{\circ} G_1, F_2 \underset{2}{\circ} G_2)$$

Proof. Let us consider the following situation:

(6.2.1)
$$\begin{tikzcd}
 & T^*M_{12} \times_{M_2} T^*M_{23} \arrow[d, "{}^t j'"] \arrow[ddl, "\tilde{p}_{12}"'] \arrow[ddr, "\tilde{p}_{23}"] & T^*M_{123} \arrow[l, hook', "\delta"'] \arrow[d, "p^2_{13}"'] \\
 & T^*M_{123} \arrow[d, "p_{13}"] & M_2 \times T^*M_{13} \arrow[l, "{}^t p'_{13}"] \arrow[dl, "p_{13\pi}"] \arrow[dr] \\
T^*M_{12} & T^*M_{13} & T^*M_{23}
\end{tikzcd}$$

Let us denote by j the diagonal inclusion $M_{123} \overset{j}{\hookrightarrow} M_{12} \times M_{23}$, and by p_{13} the projection $T^*M_{123} \xrightarrow{p_{13}} T^*M_{13}$. We have the natural morphisms :

$$T^*M_{123} \xleftarrow{{}^t j'} T^*M_{12} \times_{M_2} T^*M_{23} \xleftarrow{\delta} T^*M_{123}$$

$$T^*M_{123} \xleftarrow{{}^t p'_{13}} M_2 \times T^*M_{13} \xrightarrow{p_{13\pi}} T^*M_{13}$$

Let Y, X, S be 3 manifolds and let $G_i \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$, $F_i \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, $i = 1, 2$. Assume we are given morphims of manifolds $X \to S, Y \to S$ and that the fiber product $X \times_S Y$ is submanifold of $X \times Y$. Let us denote by j the embedding $X \times_S Y \overset{j}{\hookrightarrow} X \times Y$, and consider the induced natural morphisms

$$T^*(X \times_S Y) \xleftarrow{{}^t j'} T^*X \times_S T^*Y \xrightarrow{j_\pi} T^*X \times T^*Y$$

It is proven in [KS90, Prop 4.4.8.], that there is the natural morphism

$$\mathrm{R}^t j'_!(\mu hom(F_1, F_2) \boxtimes_S \mu hom(G_1, G_2)) \longrightarrow \mu hom(F_1 \boxtimes_S G_1, F_2 \boxtimes_S G_2)$$

In the case where $X = M_{12}, Y = M_{23}, S = M_2$, we get

$$\mathrm{R}^t j'_!(\mu hom(F_1, F_2) \boxtimes_{M_2} \mu hom(G_1, G_2)) \longrightarrow \mu hom(F_1 \boxtimes_{M_2} G_1, F_2 \boxtimes_{M_2} G_2) \tag{6.2.2}$$

Now, if we apply $\mathrm{R}^t p_{13\pi!} \circ^t p_{13}'^{-1}$ to both side of the latter morphism, we get the morphism

$$\mathrm{R}^t p_{13\pi!} \circ^t p_{13}'^{-1} \circ \mathrm{R}^t j'_!(\mu hom(F_1, F_2) \boxtimes_{M_2} \mu hom(G_1, G_2)) \longrightarrow$$
$$\mathrm{R}^t p_{13\pi!} \circ^t p_{13}^{-1}(\mu hom(F_1 \boxtimes_{M_2} G_1, F_2 \boxtimes_{M_2} G_2))$$

so that

$$\begin{aligned} &\mathrm{R}^t p_{13\pi!} \circ^t p_{13}'^{-1} \circ \mathrm{R}^t j'_!(\mu hom(F_1, F_2) \boxtimes_{M_2} \mu hom(G_1, G_2)) \\ &\quad \xrightarrow{\sim} \mathrm{R}^t p_{13\pi!} \circ \mathrm{R}p^2_{13!} \circ \delta^{-1}(\mu hom(F_1, F_2) \boxtimes_{M_2} \mu hom(G_1, G_2)) \\ &\quad \xrightarrow{\sim} \mathrm{R}p_{13!} \circ \delta^{-1}(\mu hom(F_1, F_2) \boxtimes_{M_2} \mu hom(G_1, G_2)) \\ &\quad \xrightarrow{\sim} \mathrm{R}p_{13!}(p_{12}^{-1} \mu hom(F_1, F_2) \otimes p_{23}^{-1} \mu hom(G_1, G_2)) \\ &\quad := \ \mu hom(F_1, F_2) \circ \mu hom(G_1, G_2) \end{aligned} \tag{6.2.3}$$

and on the other hand, from (6.2.2) and from [KS90, Prop. 4.4.7.(ii)]

$$\begin{aligned} &\mathrm{R}^t p_{13\pi!} \circ^t p_{13}'^{-1} \circ \mathrm{R}^t j'_!(\mu hom(F_1, F_2) \boxtimes_{M_2} \mu hom(G_1, G_2)) \\ &\quad \to \ \mathrm{R}^t p_{13\pi!} \circ^t p_{13}'^{-1}(\mu hom(F_1 \boxtimes_{M_2} G_1, F_2 \boxtimes_{M_2} G_2)) \\ &\quad \to \ \mu hom(\mathrm{R}p_{13!}(F_1 \boxtimes_{M_2} G_1), \mathrm{R}p_{13!}(F_2 \boxtimes_{M_2} G_2)) \\ &\quad \xrightarrow{\sim} \mu hom(F_1 \circ G_1, F_2 \circ G_2) \end{aligned} \tag{6.2.4}$$

so that we constructed a natural morphism

$$\mu hom(F_1, F_2) \circ \mu hom(G_1, G_2) \to \mu hom(F_1 \circ G_1, F_2 \circ G_2) \tag{6.2.5}$$

□

Considering the following diagram, with j the embedding $M_{1234} \hookrightarrow M_{12} \times M_{23} \times M_{34}$ and j_{23} the natural diagonal embedding $T^*M_{1234} \hookrightarrow T^*M_{12} \times T^*M_{23} \times T^*M_{34}$

(6.2.6)
$$\begin{tikzcd}
T^*M_{1234} \arrow[r, hook, "j_{23}"'] \arrow[d, "p_{14}^{23}"'] & T^*M_{12} \times_{M_2} T^*M_{23} \times_{M_3} T^*M_{34} \arrow[d, "{}^t j'"'] \arrow[dl, "{}^t p'_{14}"'] \arrow[dr, "\tilde{p}_{34}"] \arrow[dd, bend right, "p_{14}"] \\
M_2 \times M_3 \times T^*M_{14} \arrow[r, "\tilde{p}_{12}"] \arrow[dl, "p_{13}"'] \arrow[ddr, "p_{14\pi}"'] & T^*M_{1234} \arrow[dl, "\tilde{p}_{23}"] \arrow[dr, "p_{34}"] \arrow[dd, bend left, "p_{14}"] \\
T^*M_{12} & T^*M_{23} & T^*M_{34} \\
& T^*M_{14} &
\end{tikzcd}$$

Lemma 6.2.2. *We have the isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*M_{13}})$

$$\begin{aligned}
&(\mu hom(F_1, F_2) \circ \mu hom(G_1, G_2)) \circ \mu hom(H_1, H_2) \\
&\simeq \mathrm{R}p_{14\pi!} \circ^t p'^{-1}_{14} \circ \mathrm{R}^t j'_!(\tilde{p}_{12}^{-1} \mu hom(F_1, F_2) \otimes \tilde{p}_{23}^{-1} \mu hom(G_1, G_2) \otimes \tilde{p}_{34}^{-1} \mu hom(H_1, H_2))
\end{aligned}$$

Proof. From Lemma 2.7.1, we have

$$\begin{aligned}
&(\mu hom(F_1, F_2) \circ \mu hom(G_1, G_2)) \circ \mu hom(H_1, H_2) \\
&\simeq \mathrm{R}p_{14!}(p_{12}^{-1} \mu hom(F_1, F_2) \otimes p_{23}^{-1} \mu hom(G_1, G_2) \otimes p_{34}^{-1} \mu hom(H_1, H_2))
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{R}p_{14!}(p_{12}^{-1} \mu hom(F_1, F_2) \otimes p_{23}^{-1} \mu hom(G_1, G_2) \otimes p_{34}^{-1} \mu hom(H_1, H_2)) \\
&\simeq \mathrm{R}p_{14\pi!} \circ^t p'^{-1}_{14} \circ \mathrm{R}^t j'_!(\tilde{p}_{12}^{-1} \mu hom(F_1, F_2) \otimes \tilde{p}_{23}^{-1} \mu hom(G_1, G_2) \otimes \tilde{p}_{34}^{-1} \mu hom(H_1, H_2))
\end{aligned}$$

□

We decompose j in $k \circ h$, as follows:

$$T^*M_{1234} \xhookrightarrow{h} T^*M_{123} \times T^*M_{34} \xhookrightarrow{k} T^*M_{12} \times T^*M_{23} \times M_{34}$$

Let us denote by $p_{3\pi}$ and ${}^t p'_3$ the morphisms

$$T^*M_{134} \xleftarrow{{}^t p'_3} M_3 \times T^*M_{14} \xrightarrow{p_{3\pi}} T^*M_{14}$$

Lemma 6.2.3. *We have the isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*M_{14}})$

(6.2.7)
$$\begin{aligned}
&\mathrm{R}p_{14\pi!} \circ^t p'^{-1}_{14} \circ \mathrm{R}^t j'_! \circ j_\pi^{-1} \\
&\qquad \simeq \mathrm{R}p_{3\pi!} \circ^t p'^{-1}_3 \circ \mathrm{R}p_{134\pi!} \circ^t p'^{-1}_{134} \circ \mathrm{R}^t h'_! \circ h_\pi^{-1} \circ \mathrm{R}^t k'_! \circ k_\pi^{-1}
\end{aligned}$$

Proof. □

We denote by l the inclusion $T^*M_{134} \hookrightarrow T^*M_{1334}$ and by p_{1334} the projection $T^*M_{12334} \to T^*M_{1334}$.

Lemma 6.2.4. *We have the isomorphism in* $\mathrm{D^b}(\mathbf{k}_{T^*M_{14}})$

$$\begin{aligned}&\mathrm{R}p_{3\pi!} \circ^t p_3'^{-1} \circ \mathrm{R}p_{134\pi!} \circ^t p_2'^{-1} \circ \mathrm{R}^t h'_! \circ h_\pi^{-1} \circ \mathrm{R}^t k'_! \circ k_\pi^{-1}\\ &\simeq \mathrm{R}p_{3\pi!} \circ^t p_3'^{-1} \circ \mathrm{R}^t l'_! \circ l_\pi^{-1} \circ \mathrm{R}p_{1334\pi!} \circ^t p_{1334}'^{-1} \circ \mathrm{R}^t k'_! \circ k_\pi^{-1}\end{aligned}$$

After having "separating the variables", the associativity of the diagram of Theorem 6.2 comes essentially from the following lemma:

Lemma 6.2.5. *The following diagram of canonical morsphims commutes:*

$$\begin{tikzcd}
(\nu_M(F) \boxtimes \nu_N(G)) \boxtimes \nu_L(H) \arrow[r, "\sim"] \arrow[d] & \nu_M(F) \boxtimes (\nu_N(G)) \boxtimes \nu_L(H)) \arrow[d] \\
(\nu_{M\times N}(F \boxtimes G)) \boxtimes \nu_L(H) \arrow[d] & \nu_M(F) \boxtimes (\nu_{N\times L}(G \boxtimes H)) \arrow[d] \\
\nu_{M\times N\times L}((F \boxtimes G) \boxtimes H) \arrow[r, "\sim"] & \nu_{M\times N\times L}(F \boxtimes (G \boxtimes H))
\end{tikzcd} \tag{6.2.8}$$

Proof. The proof relies on the following description of the speacialization sheaf given in [KS90] th.4.2.3. Let V a conic open subet of $T_M X$, then for an integer j

$$H^j(V; \nu_M F) \simeq \varinjlim_U H^j(U; F)$$

where U range through the open subsets of X such that $C_M(X \setminus U) \cap V = \varnothing$.

The natural morphism $\nu_M(F) \boxtimes \nu_N(G) \to \nu_{M\times N}(F \boxtimes G)$ is exposed in [KS90, Prop. 4.2.6]. We are going to explicit this morphism at the germ level and it will then be sufficient to prove that the diagram 6.2.8 commutes at the germ level. Let p, p' respectively be in $T_M X, T_N Y$ and let U resp. U' be open subsets of X resp Y, such that $p \notin C_M(X \setminus U)$, resp. $p' \notin C_N(Y \setminus U')$. For a topological space Z, we note a_Z the natural morphism $Z \to \{\mathrm{pt}\}$.

We have $(\nu_M(F) \boxtimes \nu_N(G))_{p\times p'} \simeq (\nu_M(F))_p \otimes (\nu_N(G))'_p$, so that

$$\begin{aligned}H^j((\nu_M(F) \boxtimes \nu_N(G))_{p\times p'}) &\simeq H^j((\nu_M(F))_p \otimes (\nu_N(G))'_p)\\ &\simeq \textstyle\bigoplus_{k+l=j} H^k((\nu_M(F))_p) \otimes H^l((\nu_N(G))'_p)\\ &\simeq \varinjlim_{U,U'} \textstyle\bigoplus_{k+l=j} H^k(\mathrm{R}a_{U*}F) \otimes H^l(\mathrm{R}a_{U'*}G)\\ &\to \varinjlim_{U,U'} H^j(\mathrm{R}a_{U\times U'*}(F \boxtimes G))\\ &\to H^j(\nu_{M\times N}(F \boxtimes G)_{p\times p'})\end{aligned}$$

Now, let $p'' \in T_L Z$ and the open subsets U'' such that $p'' \notin C_L(Z \setminus U'')$, the diagram 6.2.8 corresponds to nothing but the commutation of inductive limits when it is taken first through the family of open sets U, U' then $U \times U', U''$, or taken first through the family of open sets U', U'' then $U, U' \times U''$ (we notice that for such families U, U', $(p, p') \notin C_{M\times N}(X \setminus (U \times U')))$. □

□

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