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Grégoire Sergeant-Perthuis

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Grégoire Sergeant-Perthuis. Intersection property, interaction decomposition, regionalized optimization and applications.. General Mathematics [math.GM]. Université Paris Cité, 2021. English. NNT : 2021UNIP7390 . tel-04452954

HAL Id: tel-04452954

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Université de Paris

École doctorale 386

Laboratoire IMJ-PRG

Intersection property, interaction decomposition, regionalized optimization and applications.

Par Grégoire Sergeant-Perthuis

Thèse de doctorat de Mathématiques

Dirigée par Daniel Bennequin

Présentée et soutenue publiquement le 28-05-21

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Titre: Propriété d'intersection, décomposition d'interaction, optimisation régionalisée et applications.

Mots clés: Problème des marginales, Mesures de Gibbs, Probabilités et théorie des catégories, Energie libre de Bethe, Algorithme de passage de messages, Décomposition d'interaction, Décomposition en chaos

Résumé

Replacer des constructions probabilistes et statistiques dans le cadre de la théorie des catégories est un moyen de les unifier et de les étendre tout en les rendant plus modulables pour des applications concrètes; c'est aussi une manière d'identifier ce qu'elles ont de topologique. Les pionnier dans cette direction sont Giry [1] et Lawvere [2], et plus récemment Fritz [3], Perrone [4] et Spivak [5]. Parallèlement l'approche topologique de la théorie de l'information trouve son origine dans les travaux de Čensov [6] et plus récemment Baez, Fritz, Leinster [7], Gromov [8], Marcolli, Manin [9]- [10], Baudot, Bennequin [11], Vigneaux [12] ont contribué grandement dans cette direction.

La première partie de cette thèse est consacrée à l'étude du problème des marginales et à la description des mesures de Gibbs des systèmes statistiques. Ces problèmes peuvent être reformulés et étendus dans le cadre de la théorie des catégories, respectivement dans la catégorie des espaces mesurables et dans celles des noyaux de probabilité comme initiés par Giry dans *A categorical approach to probability theory.* Dans le premier cas on définit alors le problème des marginales pour des préfaisceaux d'un poset, un ensemble muni d'un ordre partiel, dans la catégorie des espaces mesurables et dans le second on définit des mesures de Gibbs pour des foncteurs, F, d'un poset dans la catégorie des noyaux de probabilité pour lesquels il existe un préfaisceaux de même source dans la catégorie des espaces mesurable dont les morphismes sont des rétractions des morphismes de F. Nous n'avons pas connaissance de travaux antérieurs qui proposent une présentation catégorique des mesures de Gibbs; ce point de vue permet d'exhiber de grandes similarités entre le problème des marginales et la caractérisation des mesures de Gibbs, les deux ayant la même interprétation topologique.

Une contribution importante au problème des marginales est le résultat de Kellerer [13] pour le problème des marginales linéarisé qui repose sur une construction algébrique que Lauritzen nomme la décomposition en sous-espaces d'interaction dans son livre de référence *Graphical Models* [14] (voir [15] pour une présentation historique et le lien avec les modèles graphiques). Dans ce travail, nous étendons cette construction, que nous appellerons décomposition d'interaction, dans plusieurs directions ce qui permet d'obtenir de nouveaux résultats pour le prob-

lème des marginales et la caractérisation des mesures de Gibbs dans les contextes énoncés précédemment. Les décompositions d'interaction n'existent pas toujours, c'est pourquoi nous introduisons plusieurs conditions intrinsèques qui caractérisent leur existence: ce sont les propriétés d'intersection, nommées ainsi en référence à la propriété d'intersection pour les graphoïdes.

Dans une autre direction on se propose, dans le même esprit que Yedidia, Freeman, Weiss dans leur article de référence [16], de donner une procédure pour reconstruire des problèmes d'optimisation globaux à partir de problèmes locaux; nous appellerons cette procédure l'optimisation régionalisée en référence à leur généralisation de l'énergie libre de Bethe, les approximations de l'énergie libre basées sur des régions. Une application de l'étude de cette procédure est la formulation d'une Analyse en Composantes Principales adaptées à des données filtrées telles que les séries temporelles; une autre application est l'extension des approximations de l'énergie libre basées sur des régions de Yedidia, Freeman, Weiss aux collections de mesures de probabilités dont les règles de compatibilités sont données par un diagramme dans la catégorie des noyaux de probabilités. Nous pensons qu'une telle procédure trouvera de nombreuses applications concrètes notamment pour la modélisation des mouvements du ribosomes pendant la traduction de l'ARN messager [17] mais aussi pour l'expression des émotions [18]. Des propriétés de la décomposition d'interaction permettent de donner un algorithme très simple pour trouver les points critiques de la fonction de coût sous-jacente à l'optimisation régionalisée.

Title: Intersection property, interaction decomposition, regionalized optimization and applications

Keywords: Marginal problem, Gibbs measures, Categorical Probability, Bethe Free Energy, Belief Propagation, Interaction Decomposition, Chaos Decomposition

Summary

Replacing constructions and properties of probability theory and statistics in a categorical setting is a way to unify and extend them, making them more practical for concrete applications; it is also a way to identify their topological nature. The pionners in that direction are Giry [1] and Lawvere [2] and recent advances in that direction are found, for example, in the work of Fritz [3], Perrone [4] and Spivak [5]. Similarly a topological understanding of information theory can be traced back to Čensov [6] and recently Baez, Fritz, Leinster [7], Gromov [8], Marcolli, Manin [9]- [10], Baudot, Bennequin [11], Vigneaux [12] have contributed greatly in that direction.

In this dissertation we first focus on two problems in probability theory, the marginal problem and the description of Gibbs (probability) measures of statistical systems. These problems can be reformulated in a categorical setting, respectively in the category of measurable spaces and the category of probability kernels as initiated by Giry in *A categorical approach to probability theory.* This allow us to define the marginal problem for cofunctors from a poset, a set provided with a partial order, to the category of measurable spaces and to define Gibbs measures for functors, F, from a poset to the category of probability kernels for which one can find a cofunctor in the category of measurable spaces with morphisms that are retractions of the morphisms of F. We believe that it is the first time that a categorical presentation of Gibbs measures is attempted and doing so exhibits great similarities between the marginal problem and the characterization of Gibbs measures, in particular they both have the same topological interpretation.

An important contributions to the marginal problem is Kellerer's result [13] for the linearized marginal problem and it relies on an algebraic construction coined as the decomposition into interaction subspaces in Lauritzen's reference book *Graphical Models* [14] (see [15] for a historical presentation and its relation with graphical models). We extend this construction, that we will now call the interaction decomposition, in several directions which allows us to get new results for the marginal problem and for the characterization of Gibbs measures. Interaction decompositions do not always exist and several intrinsic conditions characterize their existence: we named them intersection properties in reference to the intersection property for graphoids.

In an other direction we follow the spirit of the reference article by Yedidia, Freeman, Weiss [16] and give a way to reconstruct a global optimization problem from local ones; we name this procedure the regionalized optimization as they called their generalization of the Bethe free energy, the region-based free energy approximations. As an application we formulate a version of the Principal Component Analysis adapted to filtered data such as time series and give an extension of the region-based free energy approximations proposed in [16] to collections of probability distribution that respect compatibility conditions given by a diagram in the category of probability kernels. We believe that such extensions will be of great use in applications, for example for modeling the movements of the ribosome during translation of messenger RNA [17] or the expression of emotions [18]. Results on the interaction decomposition allows us to give a simple algorithm for finding critical points of the cost function underlying regionalized optimization.

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Chapter 1

Introduction

Replacing constructions and properties of probability theory and statistics in a categorical setting is a way to unify and extend them, making them more practical for concrete applications; it is also a way to identify their topological nature. The pionners in that direction are Giry [1] and Lawvere [2] and recent advances in that direction are found, for example, in the work of Fritz [3], Perrone [4] and Spivak [5]. Similarly a topological understanding of information theory can be traced back to Čensov [6] and recently Baez, Fritz, Leinster [7], Gromov [8], Marcolli, Manin [9]- [10], Baudot, Bennequin [11], Vigneaux [12] have contributed greatly in that direction.

In this dissertation we first focus on two problems in probability theory, the marginal problem and the description of Gibbs (probability) measures of a statistical system. These problems can be reformulated in a categorical setting, respectively in the category of measurable spaces and the category of probability kernels as initiated by Giry in *A categorical approach to probability theory*. This allow us to define the marginal problem for cofunctors from a poset, a set provided with a partial order, to the category of measurable spaces and to define Gibbs measures for functors, F, in the category of probability kernels for which one can find a cofunctor in the category of measurable spaces with morphisms that are retractions of the morphisms of F. We believe that it is the first time that a categorical presentation of Gibbs measures is attempted and doing so exhibits great similarities between the marginal problem and the characterization of Gibbs measures, in particular they both have the same topological interpretation. We will discuss these problems and the categorical setting in greater details in the two next sections.

An important contributions to the marginal problem is Kellerer's result [13] for the linearized marginal problem and it relies on an algebraic construction coined as the decomposition into interaction subspaces in Laurtizen's reference book *Graphical Models* [14] (see [15] for a historical presentation). We extend this construction,

that we will now call the interaction decomposition, in several directions which allows us to get new results for the marginal problem and for the characterization of Gibbs measures.

In an other direction we follow the spirit of the reference article by Yedidia, Freeman, Weiss [16] and give a way to reconstruct a global optimization problem from local ones; we name this procedure the regionalized optimization as they called their generalization of the Bethe free energy, the region-based free energy approximations. As an application we formulate a version of the Principal Component Analysis adapted to filtered data such as time series and give an extension of the region-based free energy approximations proposed in [16] to collections of probability distribution that respect compatibility conditions given by a diagram in the category of probability kernels. We believe that such extensions will be of great use in applications, for example for modeling the movements of the ribosome during translation of messenger RNA [17] or the expression of emotions [18]. Results on the interaction decomposition allows us to give a simple algorithm for finding critical points of the cost function underlying regionalized optimization.

1.1 Functors in the category of measurable spaces and marginal problem

For any measurable space X let $\mathbb{P}(X)$ denote the set of probability measures over X and for any measurable application $f : X \to Y$ let the pushforward $f_* : \mathbb{P}(X) \to \mathbb{P}(Y)$ be denoted as $\mathbb{P}(f)$. We will note **Mes** the category of measurable spaces with measurable applications as morphism. $\mathbb{P} : \mathbf{Mes} \to \mathbf{Mes}$ is an endofunctor.
Let I be a finite set and $(E_i, i \in I)$ be a collection of finite sets; let for any $a \subseteq I$, $E_a = \prod_{i \in a} E_i$, let for $b \subseteq I$ such that $b \leq a$, $p^a_b : E_a \to E_b$ be the projection of E_a onto E_b. p defines a cofunctor from $\mathscr{P}(I)$, the powerset of I, to **Mes**. The Kolmogorov extension theorem states that,

$$\lim \mathbb{P}(p) = \mathbb{P}(\prod_{i \in I} E_i) \tag{1.1.1}$$

One can ask if there is an analogous to the Kolmogorov extension theorem when the source of the cofunctor is a subposet of $\mathscr{P}(I)$, $\mathscr{A} \subseteq \mathscr{P}(I)$; this is the marginal problem as we will explain now.

For any $a \in \mathscr{P}(I)$ the a-factor spaces are the cylindric functions $\mathbb{R}^{E_a}$ seen as a subset of $\mathbb{R}^E$, let us note $\mathrm{U}(a) = \mathbb{R}^{E_a}$ and U_1 the a-factor spaces in order to distinguish both spaces; in particular $\mathrm{U}(I) = \mathbb{R}^E$. By convention $\mathrm{U}(\emptyset) = \mathbb{R}$ and let us remark that $\mathrm{U}(a)$ is also $L^\infty(E_a)$. $\mathrm{U}(a)$ is a finite dimensional vector space and we will denote $\mathrm{U}(a)^*$ its dual. Let $a \in \mathscr{A}$, $\mathbb{P}(p_a) : \mathbb{P}(\prod E_i) \to \mathbb{P}(E_a)$ always factors

through $\lim \mathbb{P}(p|_{\mathscr{A}})$, let us denote this factorization as $\phi : \mathbb{P}(\prod E_i) \to \lim_{\mathscr{A}} \mathbb{P}(p|_{\mathscr{A}})$, in other words there is $\eta_a : \lim \mathbb{P}(p|_{\mathscr{A}}) \to \mathbb{P}(E_a)$ such that,

$$\mathbb{P}(p_a) = \eta_a \phi \tag{1.1.2}$$

The general class of problems coined as 'marginal problem' is to characterize ϕ and in particular to know its image; this is a non linear problem as there is the constraint that the densities of the laws are positive and sum to one. A simpler problem is the associated linear problem, or linear marginal problem, where one allows measures to be signed; in this case the cofunctor one considered is defined as, for any $a \in \mathscr{A} \subseteq \mathscr{P}(I)$,

$$\mathbb{M}(p)(a) = \{m \in \mathrm{U}(a) : \sum_{x_a \in E_a} m(x_a) = 1\} \tag{1.1.3}$$

and for any $b \in \mathscr{A}$ such that $b \subseteq a$, any $m \in \mathbb{M}(p)(a)$ and any $y_b \in E_b$,

$$\mathbb{M}(p_b^a)(m)(y_b) = \sum_{\substack{x_a \in E_a : \\ x_b = y_b}} m(x_a) \tag{1.1.4}$$

$\mathbb{M}(p_b^a)$ is simply a restriction of $L^\infty(p_b^a)^* : \mathrm{U}(a)^* \to \mathrm{U}(b)^*$ on spaces of signed measures that sum to 1 after identification of $\mathrm{U}(a)^*$ to $\mathrm{U}(a)$ by the canonical scalar product defined as follows, for any $m, m_1 \in \mathrm{U}(a)$,

$$\langle m, m_1 \rangle = \sum_{x_a \in E_a} m(x_a) m_1(x_a) \tag{1.1.5}$$

$\lim_{\mathscr{A}} \mathbb{M}(p|_{\mathscr{A}})$ is an affine subspace of $\bigoplus_{a \in \mathscr{A}} \mathrm{U}(a)$ and therefore $\psi : \mathbb{M}(\prod E_i) \to \lim_{\mathscr{A}} \mathbb{M}(p|_{\mathscr{A}})$ is an affine map. Characterizing this map means characterizing its image and kernel and in particular their dimension. Let us now recall some classical results on the marginal problem.

A poset has conditional meet when for any $b, c \in \mathscr{A}$ such that there is $a \in \mathscr{A}$ with $b \leq a$, $c \leq a$ then their meet, $b \wedge c$, exists. A celebrated result for the marginal problem is due to Kellerer [13] and states that if $\mathscr{A}$ has conditional meet then ψ is surjective. This result is based on a central construction in Graphical models which is the decomposition into interaction subspaces of U. In fact there are several such decompositions [14, 19–21], let us give the one that can be found in Lauritzen's reference book *Graphical Models*.

Let $S_a = \mathrm{U}_1(a) \cap \bigcap_{b \subsetneq a} \mathrm{U}_1(b)^\perp$ then for any $a \in \mathscr{A}$,

$$\mathrm{U}_1(a) = \bigoplus_{b \leq a}^{\perp} S_b \tag{1.1.6}$$

where for any $b \in \mathscr{A}$, such that $b \neq a$, $\langle S_a, S_b \rangle = 0$. Furthermore one can give an explicit expression of the projections on each S_a. For $b \leq a$ let us denote the inclusion $\mathrm{U}_1(b) \to \mathrm{U}_1(a)$ as j_{ab}, its adjoint $\pi_b^a = j_{ab}^\dagger$ is the following conditional mean, for any $y \in E_b$,

$$\pi_b^a(g)(y) = \frac{1}{|E_{a\setminus b}|} \sum_{x \in E_{a\setminus b}} g(x, y) \tag{1.1.7}$$

and let us denote π_b^I simply as π_b. Then, the projections on each S_a, denoted s_a is such that,

$$s_a = \sum_{b \leq a} (-1)^{|a\setminus b|} \pi_b \tag{1.1.8}$$

In *A Note on Nearest-Neighbour Gibbs and Markv Probabilities* [19], Speed traces back the first appearance of such a decomposition to Kellerer [13], Asmussen, Davidson [22] and Haberman [23]. In this PhD we extend the interaction decomposition to cofunctors from a poset to **Mes** which allows us to solve the linearized marginal problem for decomposable cofunctors.

More precisely, a cofunctor G from $\mathscr{A}$ to **Mes** can be sent to the functor $L^\infty G$ from $\mathscr{A}$ to **Vect** defined as for any $a \in \mathscr{A}$,

$$L^\infty G(a) = L^\infty(G(a)) \tag{1.1.9}$$

and for $b \in \mathscr{A}$ such that $b \leq a$ and $f \in L^\infty(G(b))$,

$$L^\infty G_a^b(f) = f \circ G_b^a \tag{1.1.10}$$

We we will say that a functor G from $\mathscr{A}$ to **Vect** is decomposable when there is a collection of vector spaces $(S_a, a \in \mathscr{A})$ such that for any $a \in \mathscr{A}$,

$$G(a) \cong \bigoplus_{b \leq a} S_b \tag{1.1.11}$$

Not all functors from a poset to **Vect** are decomposable so we give an intrinsic condition, that we shall call the intersection property, for functors to be decomposable (see Chapter 4), let us now state this condition.

Any decomposable functor G from a poset $\mathscr{A}$ to **Vect** is such that for any $a, b \in \mathscr{A}$ such that $b \leq a$, G_a^b is injective therefore we will restrict our attention to such functors that we will call injective. One shows that for any injective functor, G, there is a vector spaces V such that G is isomorphic to an increasing function, U, from a poset $\mathscr{A}$ to the poset of vector subspaces, denoted as $\mathbf{Gr}\, V$. We will therefore state the intrinsic condition for increasing functions from $\mathscr{A}$ to $\mathbf{Gr}\, V$.

A subposet $\mathscr{B}$ of $\mathscr{A}$ is a lower set if for any $a \in \mathscr{B}$ and $b \in \mathscr{A}$ such that $b \leq a$ then $b \in \mathscr{B}$; let us denote $\mathscr{U}(\mathscr{A})$ the set of lower sets of $\mathscr{A}$.

Definition (Intersection property)**.** Let $\mathscr{A}$ be any poset and V a vector space, an increasing function U from $\mathscr{A}$ to $\mathbf{Gr}\, V$ is said to verify the intersection property (I) if and only if,

$$\forall \mathscr{B}, \mathscr{C} \in \mathscr{U}(\mathscr{A}), \quad \sum_{b \in \mathscr{B}} \mathrm{U}(b) \cap \sum_{c \in \mathscr{C}} \mathrm{U}(c) \subseteq \sum_{a \in \mathscr{B} \cap \mathscr{C}} \mathrm{U}(a) \tag{I}$$

In Chapter 4 we prove that when the poset $\mathscr{A}$ is well founded, i.e. that any strictly decreasing chain terminates, satisfying the intersection property is equivalent to being decomposable (Theorem 3.5.1).

Our original motivation for introducing the intersection property was to extend the Bayesian intersection property to factorization spaces that are defined as, when I is finite, the collection of subspaces indexed by $\mathscr{A} \subseteq \mathscr{P}(I)$,

$$\mathscr{F}_{\mathscr{A}} = \exp \sum_{a \in \mathscr{A}} \mathrm{U}_1(a) \tag{1.1.12}$$

This approach is detailed in Chapter 2. A particular case of the intersection property appears in Lauritzen's *Graphical Models* [14] in Appendix B Proposition B.5 as a consequence of the decomposition into interaction subspaces. The proof we give of this result holds in a more general setting and is a direct one that does not rely on the decomposition into interaction subspaces; as a consequence we show that the celebrated Hammersley-Clifford theorem reduces to a statement on graphs and that it can be stated without any finiteness condition on the number of variables nor on the space in which these variables take their values.

A cofunctor G, from a poset $\mathscr{A}$ to **Mes** such that $L^\infty G$ is decomposable will be called decomposable. We show in Chapter 7 that for decomposable cofunctors the dimension of $\operatorname{colim} L^\infty G$ is know which allows us to solve the linearized marginal problem for G. In chapter 5 we extends the characterization for decomposable functors from a poset $\mathscr{A}$ to **Vect** to functors to the category of Hilbert spaces.

1.2 Specifications in the category of probability kernels and Gibbs measures

In Statistical Physics one is interested in characterizing the set of infinite volume Gibbs probability measures which is the convex hull of the different phases the physical system can be in. It is in fact similar to the marginal problem where instead of considering the pushforward of a probability law with respect to a measurable applications one considers it with respect to a probability kernels. Let us

briefly recall what probability kernels, specifications and Gibbs measures are in the particular case of the Ising model in order to introduce these concepts. In Chapter 8 we detail these definition following standard presentations such as [24] and propose an extension of these concepts to a much more general setting.

Definition (Probability Kernel Category)**.** Let $(E, \mathscr{E})$, $(E_1, \mathscr{E}_1)$ be two measurable spaces, a kernel from $(E, \mathscr{E})$ to $(E, \mathscr{E}_1)$, denoted as $E \to E_1$, is a function $\pi : \mathscr{E}_1 \times E \to [0, 1]$ such that:

1. $\pi(.|\omega)$ is a probability measure on $(E_1, \mathscr{E}_1)$ for all $\omega \in E$
2. $\pi(A|.)$ is measurable for any $A \in \mathscr{E}_1$.

One can compose probability kernels as follows: let $\pi : E \to E_1$, $\pi_1 : E_1 \to E_2$, for any $A \in \mathscr{E}_2$ and $\omega \in E$,

$$\pi_1 \circ \pi(A|\omega) = \int \pi_1(A|\omega_1)\pi(d\omega_1|\omega) \tag{1.2.1}$$

The collection of measurable spaces and of probability kernels defines a category that we will denote as **Kern**.

Any kernel $\pi : X \to Y$ induces a continuous linear map $L^\infty(\pi) : L^\infty(Y) \to L^\infty(X)$ as follows, for any $f \in L^\infty(Y)$ and $x \in X$,

$$L^\infty\pi(f)(x) = \int f(y)\pi(dy|x) \tag{1.2.2}$$

Similarly one can define $\mathbb{P}(\pi) : \mathbb{P}(X) \to \mathbb{P}(Y)$ as follows, for any $P \in \mathbb{P}(X)$, $\mathbb{P}(\pi)(P) = \pi \circ P$.

A celebrated example of statistical system is the Ising model, $I = \mathbb{Z}^2$ and $E = \prod_I \{-1, 1\}$; let G be the graph that has as vertices $\mathbb{Z}^2$ and as edges A neighbors in I,

$$\{i, j\} \in A \iff |i - j| \leq 1 \tag{1.2.3}$$

Let a be any finite subset of I, let $\overline{a}$ be its complementary in I; one associates the following probability kernel $\pi_a : E_{\overline{a}} \to E$ defined as for any $\omega \in \{-1, 1\}^{\mathbb{Z}}$ and $\omega'_{\overline{a}} \in \{-1, 1\}^{\overline{a}}$,

$$\pi_a(\omega|\omega'_{\overline{a}}) \cong \exp(-\beta \sum_{\{i,j\}\cap a \neq \emptyset} \omega_i\omega_j)1[\omega_{\overline{a}} = \omega'_{\overline{a}}] \tag{1.2.4}$$

Kernels encode border conditions or more precisely conditional expectations with respect to a border condition.

These kernels are compatible, for any finite subsets $a, b \subseteq I$ such that $b \subseteq a$, then,

$$\int \pi_a(.|\omega_{\overline{a}})\pi_b(d\omega|\omega'_{\overline{b}}) = \pi_b(.|\omega'_{\overline{b}}) \tag{1.2.5}$$

and are proper, i.e. that for any $\mathscr{E}$-measurable set, A, and any $\mathscr{E}_{\overline{a}}$-measurable set, B, then,

$$\pi_a(A \cap B|\omega'_{\overline{a}}) = \pi_a(A|\omega'_{\overline{a}})1[\omega'_{\overline{a}} \in B] \tag{1.2.6}$$

A collection of compatible and proper kernels is called a specification.

Infinite volume Gibbs measures are measures in $\mathbb{P}(E)$ that satisfy the following Dobrushin, Lanford et Ruelle condition, for any $A \in \mathscr{E}$,

$$\int \mu(d\omega)\pi(A|\omega_{\overline{a}}) = \mu(A) \tag{1.2.7}$$

In [24] Gibbs measures are defined for specification of $\Omega = \prod_{i \in I} E$ where I is countably infinite and E is a measurable space, the definition can be extended to product of different measurable sets.

A special class of specifications are the ones built from a product measure on E as we will detail now. Suppose that I is a finite set and $(E_i, i \in I)$ is a collection of finite sets; let $(P_i \in \mathbb{P}(E_i), i \in I)$ be a collection of probability measures, let $a, b \in \mathscr{A}$ such that $b \subseteq a$, let $\omega_a \in E_a$, let $\omega' \in E_b$ and let,

$$\pi^a_b(\omega_a|\omega'_b) = \prod_{i \in a \setminus b} P_i(\omega_i)1[\omega_b = \omega'_b] \tag{1.2.8}$$

For these specifications, $L^\infty\pi$ is isomorphic to the presheaf defined by the projection $\bigoplus_{c \leq a} S_c \to \bigoplus_{c \leq b} S_c$ for $b \leq a$ and for some collection $(S_a \subseteq L^\infty(E_a), a \in \mathscr{A})$; we call such specification decomposable. In Chapter 8, we propose an extension of the definition of specifications and Gibbs measures to presheaves F from a poset $\mathscr{A}$ to **Kern** which allows for probability kernels defined on spaces that are not product spaces. In Chapters 4,8, we give an intrinsic condition for presheaves from $\mathscr{A}$ to **Vect** to be decomposable and characterize the Gibbs measures of decomposable specifications.

1.3 Invariants for functors in Mes and specifications

As we explained above functors and presheaves from a poset to **Mes** or **Kern** can be seen as functors into **Vect** by composition with the L^∞ presheaf. The limit and colimit functors from $\mathbf{Vect}^{\mathscr{A}}$ to **Vect** have a derived functor, which allows us

to define invariants for functors to **Mes** and more generally for specifications. In Chapters 6, 7 we show that decomposable functors and decomposable specifications are acyclic, in other words that the cohomology groups of order greater or equal to 1 vanish. In Chapter 6, which is a collective work with Daniel Bennequin, Olivier Peltre and Juan Pablo Vigneaux, we explore an other direction and define what decomposable sheaves over a topological space are, we call them extra-fine sheaves, and show that they are acyclic for the Čech cohomology. This allows us to interprete Kellerer's result on the marginal problem as the long exact sequence induced by the inclusion between two presheaves. More generally, as explained in Chapter 8 we believe that the study of decomposable objects (functors, specification, extrafine sheaves) is a first step toward developing new algebraic tools for the characterization of Gibbs measures.

1.4 Regionalized Optimization

Chapter 9 is a continuation and an application of the results of the previous section. We propose a way to reconstruct a global optimization problem from local ones following the spirit of the reference article *Constructing Free Energy Approximations and Generalized Belief Propagation algorithms* [16] by Yedidia, Freeman, Weiss. They propose a regionalized version of entropy as we will detail now. Let $\mathscr{A}$ be a finite poset, the ζ function of the poset is defined as follows,

$$\begin{array}{rcccc} \zeta & : & \bigoplus_{a\in\mathscr{A}}\mathbb{Z} & \to & \bigoplus_{a\in\mathscr{A}}\mathbb{Z} \\ & & \lambda & \mapsto & (\sum_{b\leq a}\lambda_b, a\in\mathscr{A}) \end{array} \tag{1.4.1}$$

It is invertible and it's inverse is the Mobius function of $\mathscr{A}$, denotes as μ (see [25]). Let us recall that for any finite measurable space X, and any probability density $P \in \mathbb{P}(X)$, the entropy of P, when the Boltzmann constant is set to 1, is

$$S(P) = -\sum_{x\in X} P(x)\ln P(x) \tag{1.4.2}$$

The free energy with respect to a Hamiltonian $H \in L^\infty(X)$ and at temperature T is defined as,

$$F(P) = \mathbb{E}_P[H] - TS(P) \tag{1.4.3}$$

Let I be a finite set and let $E = \prod_{i\in I} E_i$ be a product of finite sets and $\mathscr{A}$ a subposet of $\mathscr{P}(I)$. Yedidia, Freeman, Weiss consider a cost function for the sections, $P = (P_a \in \mathbb{P}(E_a), a \in \mathscr{A})$, of the marginal presheaf $(\mathbb{P}(p)^a_b | a, b \subseteq I, b \subseteq a)$, built from the entropy of each probability P_a,

$$F_{Bethe}(P) = \sum_{a \in \mathscr{A}} \sum_{b \leq a} \mu(a,b) \left(\beta \mathbb{E}_{P_a}[H_a] - S(P_a)\right) \tag{1.4.4}$$

with $(H_a \in \mathrm{U}(a), a \in \mathscr{A})$ a collection of Hamiltonians. If one sets β as $\frac{1}{T}$ one see that F_{Bethe} is a weighted sum of free energies. As remarked by Olivier Peltre [26], [27] the critical points $p^* \in \lim \mathbb{P}(p)$ of this free energy are such that there is a collection $(m_{a \to b} \in \bigoplus_{\substack{a,b: \\ b \subseteq a}} \mathrm{U}(b))$ and a collection $(\lambda_a \in \mathbb{R}, a \in \mathscr{A})$ such that for any $a \in \mathscr{A}$,

$$-\ln(p_a^*) = H_a + \lambda_a + \sum_{b \leq a} \left(\sum_{c \leq b} j_{ac} m_{b \to c} - \sum_{c \geq b} j_{ab} m_{c \to b} \right) \tag{1.4.5}$$

We give a general framework for regionalizing cost functions. This framework allows for an extension of results of Yedidia, Freeman, Weiss to presheaves from a finite poset to **Kern** but also to other cost function; for example we derive a regionalized version of the PCA and give an explicit expression for gradient descent when the limit is taken is over decomposable functors.

1.4.1 Independent result

The last chapter of this thesis is independent from the others. In Chapter 10 we make a statistical study of the connectome of the ribosome in the three kingdoms (Bacteria, Archaea, Eukarya), introducing a conditional test for bootstraping. It is a statistical test for cases where a closed-form expression of the law is unkown and when ressampling it is the only option one has. One hope for the future is to enlarge the graph of proteins connections with biochemical data, giving hamiltonians, and energy functions, that could be studied by the methods of the thesis for a better understanding of the information flow in the network. This should make more precise the conjecture that the protein network supports a control of the movements of the ribosome during translation of ARNm.

Chapter 2

Bayesian/Graphoid intersection property for factorisation models.

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Abstract

We remark that Pearl's Graphoid intersection property, also called intersection property in Bayesian networks, is a particular case of a general intersection property, in the sense of intersection of coverings, for factorization spaces coined under several names: factorization models, factor graphs or by Lauritzen in his reference book *Graphical Models* as hierarchical model subspaces. We give a direct and

new proof of the Hammersley-Clifford theorem transposing and reducing it to a corresponding statement for graphs, justifying formally the geometric intuition of independency, and we extend it to non finite graphs. This intersection property is derived in the most general setting and is the starting point for a generalization of the decomposition into interaction subspaces to collections of vector spaces (Chapter 3).

2.1 Introduction

2.1.1 Intersection property

Intersection property and graphoids

To describe the structure of dependencies of a set of random variables, as well said by Judea Pearl in Chapter 3 of [28], one can introduce a ternary operator corresponding to the conditional independence:

"The notion of informational relevance is given [...] through the device of conditional independence, which successfully captures our intuition about how dependencies should change in response to news facts".

For any three random variables with discrete values, we will note $X \perp\!\!\!\perp Y|_P Z$[1] the fact that X is independent of Y conditionally to Z (see Section 2.5.2 Equation 2.5.2).

The intersection property in Bayesian networks[2], is the following proposition.

Proposition 2.5.1 (Intersection property). Let W, X, Y, Z be four random variables that take values in a finite set and for which the probability density $P_{W,X,Y,Z}$ is strictly positive, then,

$$X \perp\!\!\!\perp Y|(Z,W) \text{ and } X \perp\!\!\!\perp W|(Z,Y) \implies X \perp\!\!\!\perp (Y,W)|Z \tag{2.1.1}$$

Semi-graphoids and graphoids were introduced to give a formal set of axioms, on $\perp\!\!\!\perp$, for conditional independence (see [30], [28]); in this context Proposition 2.5.1 is called the intersection axiom.

Definition 2.1.1 (Semi-graphoid, graphoid [28] [14])**.** A semi-graphoid structure on a collection $I = \coprod_J \{X_j\}$ is a ternary relation on subsets of I, that we shall note as $X \perp\!\!\!\perp Y|Z$, such that, for any, X, Y, Z, W, disjoint subsets of I,

[1] P will be omitted from now on, as in literature.

[2] As found in [29] (Chapter 2 Proposition 2.12) or [28] (Chapter 3 Theorem 1).

1. if $X \perp\!\!\!\perp Y|Z$ then $Y \perp\!\!\!\perp X|Z$;
2. if $X \perp\!\!\!\perp Y|Z$ and $U \subseteq X$, then $U \perp\!\!\!\perp Y|Z$;
3. if $X \perp\!\!\!\perp Y|Z$ and $U \subseteq X$, then $X \perp\!\!\!\perp Y|Z \cup U$;
4. if$X \perp\!\!\!\perp Y|Z$ and $X \perp\!\!\!\perp W|Y \cup Z$ then $X \perp\!\!\!\perp W \cup Y|Z$.

It is a graphoid if furthermore it satisfies the intersection axiom.

Factorisation spaces

Let us suppose that X, Y, Z take values respectively in finite spaces Ω_X, Ω_Y, Ω_Z. The fact that X is independent of Y conditionally to Z can be restated as a factorization property on $P_{X,Y,Z}$; for simplicity let us assume that $P_{X,Y,Z}$ is strictly positive, then $X \perp\!\!\!\perp Y|Z$ if and only if for any $(x, y, z) \in \Omega_X \times \Omega_Y \times \Omega_Z$,

$$P_{X,Y,Z}(x, y, z) = \frac{P_{X,Z}(x, z)P_{Y,Z}(y, z)}{P_Z(z)} \tag{2.1.2}$$

where $P_{X,Z}$, $P_{Y,Z}$, P_Z are respectively the marginal probabilities of (X, Z), (Y, Z) and Z.

If one notes $\mathscr{F}_{Y,Z}$ the set of strictly positive functions on $\Omega_X \times \Omega_Y \times \Omega_Z$ that only depend on (Y, Z) and $\mathscr{F}_{X,Z}\mathscr{F}_{Y,Z}$ the set of functions that are the product of a function of $\mathscr{F}_{X,Z}$ and a function of $\mathscr{F}_{Y,Z}$, then the intersection property can be restated as, for any strictly positive probability law $P_{X,Y,Z,W}$,

$$P_{X,Y,Z,W} \in \mathscr{F}_{W,X,Z}\mathscr{F}_{W,Y,Z} \cap \mathscr{F}_{X,Y,Z}\mathscr{F}_{W,Y,Z} \implies P_{X,Y,Z,W} \in \mathscr{F}_{X,Z}\mathscr{F}_{W,Y,Z} \tag{2.1.3}$$

We are interested in generalizing this result to intersections of factorization spaces that we will define now.

Definition 2.1.2 (Factorization space)**.** Let I be a finite set, let $\mathscr{A} \subseteq \mathscr{P}(I)$, where $\mathscr{P}(I)$ is the set of subsets of I. Let $(E_i, i \in I)$ be a collection of sets, let $E_a = \prod_{i \in a} E_i$ for any $a \in \mathscr{P}(I)$; for $x \in E_I$, we will denote x_a its projection onto E_a. The factorization space over $\mathscr{A}$ is defined as follows,

$$\mathscr{F}_{\mathscr{A}} = \{f \in \mathbb{R}_{>0}^{E_I} : \quad \exists (f_a \in \mathbb{R}_{>0}^{E_a}, a \in \mathscr{A}), \forall x \in E_I \ f = \prod_{a \in \mathscr{A}} f_a(x_a)\} \tag{2.1.4}$$

Notation 2.1.1. From now on we shall note $\mathscr{P}\mathscr{P}(I)$ as $\mathscr{P}^2(I)$.

One can extend the previous definition to the case where $\mathscr{A}$ is non finite. To do so let us introduce a notation; for any $\mathscr{A} \subseteq \mathscr{P}(I)$, let,

$$\hat{\mathscr{A}} = \{a \in \mathscr{P}(I) : \ \exists b \in \mathscr{A}, \ a \leq b\} \tag{2.1.5}$$

Notation 2.1.2. $\hat{\mathscr{A}}$ is called the lower set of $\mathscr{A}$ and the set of lower sets will be denoted as $\mathscr{U}(\mathscr{P}(I))$, i.e. $\mathscr{U}(\mathscr{P}(I)) = \{\hat{\mathscr{A}} | \mathscr{A} \subseteq \mathscr{P}(I)\}$.

Definition 2.1.3 (Generalized factorization spaces)**.** Let I be any set and let $\mathscr{A} \subseteq \mathscr{P}(I)$, any $f \in \mathbb{R}_{>0}^{E_I}$ is in $\mathscr{F}_{\mathscr{A}}$ if and only if there is $n \in \mathbb{N}$, a collection $(a_k \in \hat{\mathscr{A}}, k \in [1, n])$ and a collection $(f_k \in \mathbb{R}^{E_{a_k}})$ such that for any $k \in [1, n]$, $|a_k| < \infty$ and for any $x \in E_I$,

$$f(x) = \prod_{k \in [1,n]} f_k(x_{a_k}) \tag{2.1.6}$$

In particular,

$$\mathscr{F}_{\mathscr{A}} = \mathscr{F}_{\hat{\mathscr{A}}} \tag{2.1.7}$$

Main theorem

The result we want to emphasize in this document is that an intersection property still holds for factorization spaces.

Theorem 2.4.1 (Intersection property for factorization spaces)*.* Let I be any set, let $(E_i, i \in I)$ be any collection of sets. For any family $(\mathscr{A}_j)_{j \in J}$ of elements of $\mathscr{P}^2(I)$,

$$\bigcap_{j \in J} \mathscr{F}_{\hat{\mathscr{A}}_j} = \mathscr{F}_{\bigcap_{j \in J} \hat{\mathscr{A}}_j}. \tag{2.1.8}$$

A particular case of the intersection property appears in Lauritzen's *Graphical Models* [14] in Appendix B Proposition B.5 as a consequence of the decomposition into interaction subspaces, that we will introduce in the next subsection. The proof we give of this result holds in a more general setting and is a direct one that does not rely on the decomposition into interaction subspaces. In fact in [31] we show the converse statement that Equation 2.1.8 is a structure property that characterizes collections of vector spaces that can be decomposed into direct sums of subspaces, similarly to the decomposition into interaction subspaces, in other words satisfying the intersection property implies that this collection has such decomposition.

A direct consequence of Theorem 2.4.1 is that there is a complete lattice morphism between $(\mathscr{U}(\mathscr{P}(I)), \subseteq)$ and factorization spaces. This remark enable us to prove the Hammersley-Clifford Theorem in a direct and novel manner, pushing properties of the graph of dependencies directly on its graphical model, that we will now sketch and allows us to give a generalization of the Hammersley-Clifford theorem.

2.1.2 Hammersley-Clifford Theorem

Graphical models: Markov fields and Gibbs fields

A graphical model is a way to express the interactions of random variables through the properties of a graph. For example, let I be a finite set and let $(X_i, i \in I)$ be a collection of random variables. Let us associate to each random variable a vertex of an undirected graph $G = (I, A)$, where I is its set of vertices and A its set of edges; one could say that two random variables are in interaction if their vertices are nearest-neighbors in G and expect that there is a collection $(f_a \in \mathbb{R}^{E_a}, a \in A)$ such that,

$$\ln P = \sum_{a \in A} f_a \tag{2.1.9}$$

This is an example of a Gibbs state with respect to a potential. The adjacent elements of $i \in G$ will be denoted as ∂i.

Definition 2.1.4 (Gibbs States)**.** Let I be a finite set and $(E_i, i \in I)$ be a collection of finite sets, let $\mathscr{A} \in \mathscr{P}^2(I)$ and let $\Phi = (\phi_a \in \mathbb{R}^{E_a}, a \in \mathscr{A})$ be a collection of interactions, which we shall call a potential; a Gibbs state with respect to a potential Φ is defined as follows, for any $x \in E$,

$$P(x) = \frac{e^{\sum_{a \in \mathscr{A}} \phi_a(x_a)}}{\sum_{y \in E_I} e^{\sum_{a \in \mathscr{A}} \phi_a(y_a)}} \tag{2.1.10}$$

Remark 2.1.1. Any probability law on E is a Gibbs state; furthermore if there is a potential $\Phi = (\phi_a \in \mathbb{R}^{E_a}, a \in \mathscr{A})$ such that a probability law P is a Gibbs state with respect to Φ, then P is in the factorization space over $\mathscr{A}$.

There is an other way to specify the interactions of the random variables from the properties of a graph. For example on can imagine that if two vertices v, u are connected only through a third vertex k, i.e. any path from v to u pass by k, this would mean that the corresponding random variables, X_v, X_u are dependent only through X_k, i.e. that,

$$X_v \perp\!\!\!\perp X_u | X_k \tag{2.1.11}$$

This is a particular case of spatial Markov property for the probability law of the random variables. There are several, a priori, different way to translate conditional connectedness properties of the graph into conditional independence properties, let us define two of such. Let for $a \subseteq I$, X_a denote $(X_i)_{i \in a} = X_{|a}$.

Definition 2.1.5 (Markov properties)**.** Let $G = (I, A)$ be a finite graph, a strictly positive probability P_X on a finite set $E = \prod_{i \in I} E_i$ obeys,

1. (P) the pairwise Markov property relative to G, if for any pair (i, j) of non-adjacent vertices

$$X_i \perp\!\!\!\perp X_j | X_{I \setminus \{i,j\}}.$$

2. (L) the local Markov property relative to G, if for any vertex $i \in V$,

$$X_i \perp\!\!\!\perp X_{I \setminus (i \cup \partial i)} | X_{\partial i}$$

And we call the respective sets $P(G)$, $L(G)$.

As we will see the Hammersley-Clifford theorem asserts that the two points of view for reading the interactions from a graph, the Gibbs state and Markov property point of views, are in fact equivalent for a strictly positve probability law. One of the ways to prove the Hammersley-Clifford theorem is to build a decomposition into interaction subspaces of the factorisation spaces [19], we shall therefore give a brief presentation of this decomposition even though we shall not be using it in the rest of this chapter.

The decomposition into interaction subspaces

Let I be a finite set and let $(E_i, i \in I)$ be a collection of finite sets. Let us consider the canonical scalar product on $\mathbb{R}^{E_I}$, i.e. for any $f, g \in \mathbb{R}^{E_I}$,

$$\langle f, g \rangle = \sum_{x \in E_I} f(x) g(x) \tag{2.1.12}$$

Let for any $\mathscr{A} \in \mathscr{P}^2(I)$,

$$U(\mathscr{A}) = \ln \mathscr{F}_{\mathscr{A}} \tag{2.1.13}$$

Theorem 2.1.1 (Decomposition into interaction subspaces)**.** *There is a collection of vector subspaces of* $\mathbb{R}^{E_I}$*,* $(S_a, a \subseteq \mathscr{P}(I))$*, such that, for any* $a \subseteq \mathscr{P}(I)$*,*

$$U(\{a\}) = \bigoplus_{b \subseteq a} S_b \tag{2.1.14}$$

and any two S_a, S_b*, with* $a \neq b$*, are orthogonal to one another.*

Several proofs of this result can be found in [19].

A new proof of the Hammersley-Clifford Theorem

The Hammersley-Clifford theorem states that any Markov condition for a strictly positive probability law can be restated as a condition on the locality of the interactions of its potential, in other words Markov conditions correspond to some factorization spaces.

Let $G = (I, A)$ be a graph; a clique of G is a subset of G such that every two distinct vertices are adjacent. We will note $\mathscr{C}$ the set of its cliques.

Theorem 2.1.2 (Hammersley-Clifford)**.** *Let* $G = (I, A)$ *be a finite graph. For all* P_X *strictly positive probability law on a finite set* $\prod_{i \in I} E_i$*,*

$$P_X \in P(G) \iff P_X \in L(G) \iff P_X \in \mathscr{F}_{\mathscr{C}}. \tag{2.1.15}$$

The intersection property for factorization spaces enables us to bring back the proof of the Hammersley-Clifford theorem to a general property on graphs, let us sketch the proof that will will present in more details in this document.

Let (i, j) be a pair of elements of I, let $[i, j] = \{I \setminus \{i\}), I \setminus \{j\})\}$ and let,

$$\mathscr{A}_P = \bigcap_{(i,j):\ i \notin \partial j} \widehat{[i, j]} \tag{2.1.16}$$

Similarly for all $i \in I$, let $[i] = \{I \setminus i, i \cup \partial i\}$ and let $\mathscr{A}_L = \bigcap_i \hat{[i]}$.

By remarking that,

$$\hat{\mathscr{A}_L} = \hat{\mathscr{A}_P} = \mathscr{C} \tag{2.1.17}$$

and applying the intersection property for factorisation spaces one ends the proof.

2.1.3 Structure of this document

In Section 2.2, we will give some general properties on partial coverings and their natural order making it a preorder with join and meet. Proposition 2.2.3 states that there is an increasing function between the preorder set of partial coverings and the poset of factorization spaces that preserve the join.

In Section 2.3 we prove the intersection property (Theorem 2.3.1) and as a consequence we show that the increasing function also preserves meets. In this section we do not assume the $(E_i, i \in I)$ to be finite, however we assume I to be finite.

In the next section, Section 2.4, we extend the intersection property to any sets I, Theorem 2.4.1.

Finally in Section 2.5 we give apply the previous theorems giving new proofs for classical results around factorization spaces that allow us to extend them, in particular we give a generalization of the Hammersley-Clifford theorem.

2.2 Order on partial coverings and factorisation spaces

In this section I is a finite set, let $E = \prod_{i \in I} E_i$ be a product of any sets. For $a, b \in \mathscr{P}(I)$ such that $b \subseteq a$ let $\pi^a_b : E_a \to E_b$ be the projection of E_a onto E_b where by convention $\pi^a_\emptyset : E_a \to *$ is the projection on the set with one element $*$; for $x \in E$, we shall note $\pi_a(x)$ as x_a. In particular $\mathscr{F}_\emptyset$ is the set of strictly positive constant functions and for any $a \in \mathscr{P}(I)$ we note $\mathscr{F}_{\{a\}}$ as $\mathscr{F}_a$.

$\mathbb{R}_{>0}$ can be seen as a vector space for the product law and the exponentiation and similarly for the product of these spaces. In this section we keep the, unusual, product convention to stay closer to the spirit of factorization.

2.2.1 Order on partial coverings

Any subset $\mathscr{A} \subseteq \mathscr{P}(I)$ can be seen as a partial covering of I of support $\cup_{a \in \mathscr{A}} a$. The order for partial covering that we will now introduce is a direct extension of the usual one for coverings.

Definition 2.2.1. Let us define an intersection $\sqcap$ and a relation R on $\mathscr{P}^2(I)$. For all $\mathscr{A}, \mathscr{B} \in \mathscr{P}^2(I)$,

$$\mathscr{A} \text{ R } \mathscr{B} \quad \Longleftrightarrow \quad \forall a \in \mathscr{A}, \quad \exists b \in \mathscr{B}, \quad a \subseteq b \tag{2.2.1}$$

$$\mathscr{A} \sqcap \mathscr{B} = \{a \cap b \quad | a \in \mathscr{A}, b \in \mathscr{B}\} \tag{2.2.2}$$

Proposition 2.2.1. *R is pre-order that we will note $\leq$ and for $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D} \in \mathscr{P}^2(I)$,*

$$\mathscr{A} \sqcap \mathscr{B} = \mathscr{B} \sqcap \mathscr{A}, \quad (\mathscr{A} \cup \mathscr{B}) \sqcap \mathscr{C} = (\mathscr{A} \sqcap \mathscr{C}) \cup (\mathscr{B} \sqcap \mathscr{C}), \quad \mathscr{A} \sqcap \mathscr{B} \leq \mathscr{A} \quad . \tag{2.2.3}$$

$$[\mathscr{A} \leq \mathscr{C} \wedge \mathscr{B} \leq \mathscr{D}] \implies \mathscr{A} \cup \mathscr{B} \leq \mathscr{C} \cup \mathscr{D}.^3 \tag{2.2.4}$$

$$[\mathscr{A} \leq \mathscr{C} \wedge \mathscr{B} \leq \mathscr{D}] \implies \mathscr{A} \sqcap \mathscr{B} \leq \mathscr{C} \sqcap \mathscr{D}. \tag{2.2.5}$$

Proof. Let $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathscr{P}^2(I)$. For all $a \in \mathscr{A}$, $a \subseteq a$. Therefore $\mathscr{A} \leq \mathscr{A}$. Assume, $\mathscr{A} \leq \mathscr{B}$ and $\mathscr{B} \leq C$, then,

$$\forall a \in \mathscr{A}, \quad \exists b \in \mathscr{B}, a \subseteq b \qquad \forall b \in \mathscr{B}, \quad \exists c \in \mathscr{C}, b \subseteq c \quad .$$

For $a \in \mathscr{A}$ there is $b \in \mathscr{B}$ and $c \in \mathscr{C}$ such that $a \subseteq b \subseteq c$ so $a \subseteq c$ and $[\mathscr{A} \leq \mathscr{B} \wedge \mathscr{B} \leq \mathscr{C}] \implies \mathscr{A} \leq \mathscr{C}$. Therefore $\leq$ is a pre-order.

$$\exists a \in \mathscr{A}, \exists b \in \mathscr{B}, \quad x = a \cap b \iff \exists a \in \mathscr{B}, \exists b \in \mathscr{A}, x = a \cap b$$

So $\mathscr{A} \sqcap \mathscr{B} = \mathscr{B} \sqcap \mathscr{A}$.

$$(\mathscr{A} \cup \mathscr{B}) \sqcap \mathscr{C} = \bigcup_{(a,c) \in (\mathscr{A} \cup \mathscr{B}) \times \mathscr{C}} \{a \cap c\} = \bigcup_{\substack{(a,c) \in \mathscr{A} \times \mathscr{C} \\ or \ (a,c) \in \mathscr{B} \times \mathscr{C}}} \{a \cap c\} = \bigcup_{(a,c) \in \mathscr{A} \times \mathscr{C}} \{a \cap c\} \quad \cup \bigcup_{(b,c) \in \mathscr{B} \times \mathscr{C}} \{b \cap c\} \quad .$$

So $(\mathscr{A} \cup \mathscr{B}) \sqcap \mathscr{C} = (\mathscr{A} \sqcap \mathscr{C}) \cup (\mathscr{B} \sqcap \mathscr{C})$.

Let $c \in \mathscr{A} \sqcap \mathscr{B}$ then there is $a \in A$, $b \in \mathscr{B}$ such that, $c \subseteq a \cap b \subseteq a$. So, $[\mathscr{A} \sqcap \mathscr{B} \leq \mathscr{A}] \wedge [\mathscr{A} \sqcap \mathscr{B} \leq \mathscr{B}]$.

Assume $\mathscr{A} \leq \mathscr{C}$ and $\mathscr{B} \leq \mathscr{D}$ then for all $a \in \mathscr{A}$ there is $c \in \mathscr{C}$ such that $a \subseteq c$, for all $b \in \mathscr{B}$ there is $d \in \mathscr{D}$ such that $b \subseteq d$. So for $x \in \mathscr{A} \cup \mathscr{B}$ there is $c \in \mathscr{C}$ such that $x \subseteq c$ or $d \in \mathscr{D}$ such that $x \subseteq d$. However c and $d \in \mathscr{C} \cup \mathscr{D}$ so $\mathscr{A} \cup \mathscr{B} \leq \mathscr{C} \cup \mathscr{D}$. The last is proven the same way noting that $a \subseteq c$, $b \subseteq d$ implies $a \cap b \subseteq c \cap d$.

□

[3] $\wedge$ is the logic operator "and".

Definition 2.2.2. Let us introduce the usual equivalence relation for a pre-order [4], for all $\mathscr{A}, \mathscr{B} \in \mathscr{P}^2(I)$,

$$\mathscr{A} \sim \mathscr{B} \quad \Longleftrightarrow \quad [\mathscr{A} \leq \mathscr{B}] \wedge [\mathscr{B} \leq \mathscr{A}]. \tag{2.2.6}$$

Let $q : \mathscr{P}^2(I) \to J$, with J any poset, be a pre-order morphism, in the sense that for any $a, b \in \mathscr{P}^2(I)$ such that $a \leq b$, $q(a) \leq q(b)$. q is said to preserve the equivalence relation when for all $\mathscr{A}, \mathscr{B} \in \mathscr{P}^2(I)$, $[\mathscr{A} \sim \mathscr{B} \implies q(\mathscr{A}) = q(\mathscr{B})]$. In what follows we supposre that q preserves the equivalence relation.

If, for any $f : \mathscr{P}^2(I) \to K$, with K a poset, that is a pre-order morphism and that preserves the equivalence relation, there is a unique $\overline{f}$ that is a poset morphism such that $f = \overline{f} \circ q$, then we will say that q verifies the universal property (P).

Let us note $\mathscr{P}^2(I)/\sim$ as $\overline{\mathscr{P}^2(I)}$.

Proposition 2.2.2. *If two pre-order morphism, $p_1 : \mathscr{P}^2(I) \to J$, $p_2 : \mathscr{P}^2(I) \to K$, that preserve the equivalence relation, verify the universal property (P), then there is a poset isomorphism between J and K.*

Let us define p as,

$$\begin{array}{rcccc} p & : & \mathscr{P}^2(I) & \to & \overline{\mathscr{P}^2(I)} \\ & & A & \mapsto & [A] \end{array}$$

There is a unique order $\overline{\leq}$ on $\overline{\mathscr{P}^2(I)}$ such that $p : (\mathscr{P}^2(I), \leq) \to (\overline{\mathscr{P}^2(I)}, \overline{\leq})$ is a pre-order morphism and verifies (P). It verifies for all $\mathscr{A}, \mathscr{B} \in \mathscr{P}^2(I)$,

$$[\mathscr{A}] \overline{\leq} [\mathscr{B}] \iff \mathscr{A} \leq \mathscr{B}. \tag{2.2.7}$$

Furthermore one can define a union on $\overline{\mathscr{P}^2(I)}$ and an intersection such for all $\mathscr{A}$,$\mathscr{B}$,

$$[\mathscr{A} \cup \mathscr{B}] = [\mathscr{A}] \cup [\mathscr{B}], \quad [\mathscr{A} \sqcap \mathscr{B}] = [\mathscr{A}] \sqcap [\mathscr{B}] \quad . \tag{2.2.8}$$

Equations 2.2.3,2.2.4,2.2.5 stay true on $\overline{\mathscr{P}^2(I)}$. Let us recall them, $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D} \in \overline{\mathscr{P}^2(I)}$,

$$\mathscr{A} \sqcap \mathscr{B} = \mathscr{B} \sqcap \mathscr{A}, \quad (\mathscr{A} \cup \mathscr{B}) \sqcap \mathscr{C} = (\mathscr{A} \sqcap \mathscr{C}) \cup (\mathscr{B} \sqcap \mathscr{C}), \quad \mathscr{A} \sqcap \mathscr{B} \leq \mathscr{A} \quad . \tag{2.2.9}$$

[4] See E.III.3 [32].

$$[\mathscr{A} \leq \mathscr{C} \wedge \mathscr{B} \leq \mathscr{D}] \implies \mathscr{A} \cup \mathscr{B} \leq \mathscr{C} \cup \mathscr{D}. \tag{2.2.10}$$

$$[\mathscr{A} \leq \mathscr{C} \wedge \mathscr{B} \leq \mathscr{D}] \implies \mathscr{A} \sqcap \mathscr{B} \leq \mathscr{C} \sqcap \mathscr{D}. \tag{2.2.11}$$

Proof. Let $p_1 : \mathscr{P}^2(I) \to J$, $p_2 : \mathscr{P}^2(I) \to K$, that preserve the equivalence relation, verify the universal property (P). Then there is $\overline{p_1}$, $\overline{p_2}$, two poset morphisms, such that $p_1 = \overline{p_1} \circ p_2$, $p_2 = \overline{p_2} \circ p_1$. So $p_1 = \overline{p_1} \circ \overline{p_2} \circ p_1$, in other words the following diagram commutes:

$$\begin{tikzcd} \mathscr{P}^2(I) \arrow[r, "p_1"] \arrow[dr, "p_1"'] & J \arrow[d, "\overline{p_1} \circ \overline{p_2}"] \\ & J \end{tikzcd} \tag{2.2.12}$$

But $p_1 = \mathrm{id} \circ p_1$, therefore by the unicity statement in (P), $\overline{p_1} \circ \overline{p_2} = \mathrm{id}$. One also has that $p_2 = \overline{p_2} \circ \overline{p_1} \circ p_2$, so $\overline{p_2} \circ \overline{p_1} = \mathrm{id}$. Therefore $\overline{p_1}$ is a poset isomorphism between J and K.

Le us define the following relation for $x, y \in \overline{\mathscr{P}^2(I)}$,

$$x \overline{\leq} y \quad \iff \quad \exists \mathscr{A}, \exists \mathscr{B}, \quad x = [\mathscr{A}] \quad \wedge \; y = [\mathscr{B}] \quad \wedge \quad \mathscr{A} \leq \mathscr{B} \quad . \tag{2.2.13}$$

$(\overline{\mathscr{P}^2(I)}, \overline{\leq})$ is a poset (see E.III.3 [32]).

Let $f : \mathscr{P}^2(I) \to K$, with K a poset, be a pre-order morphism that preserves the equivalence relation. By the universal property for the quotient map,there is a unique $\overline{f}$ such that $f = \overline{f} \circ p$. For $\mathscr{A}, \mathscr{B} \in \mathscr{P}^2(I)$, suppose $[\mathscr{A}] \overline{\leq} [\mathscr{B}]$, then $\mathscr{A} \leq \mathscr{B}$ and $f(\mathscr{A}) \leq f(\mathscr{B})$. $\overline{f}([\mathscr{A}]) = f(\mathscr{A})$ and $\overline{f}([\mathscr{B}]) = f(\mathscr{B})$, so $\overline{f}([\mathscr{A}]) \leq \overline{f}([\mathscr{B}])$. Therefore $\overline{f}$ is a poset morphism[5].

Suppose that there are two orders $\leq_1$ and $\leq_2$ on $\overline{\mathscr{P}^2(I)}$ such that $p : (\mathscr{P}^2(I), \leq) \to (\overline{\mathscr{P}^2(I)}, \leq_1)$ and $p : (\mathscr{P}^2(I), \leq) \to (\overline{\mathscr{P}^2(I)}, \leq_2)$ are pre-order morphism and verify (P). Then there is $\overline{p}$, a poset isomorphism, such that $p = \overline{p} \circ p$. But by the universal property for the quotient map, $\overline{p} = \mathrm{id}$. Therefore $\mathrm{id} : (\overline{\mathscr{P}^2(I)}, \leq_1) \to (\overline{\mathscr{P}^2(I)}, \leq_2)$ is a poset isomorphism. For all $x, y \in \overline{\mathscr{P}^2(I)}$,

$$x \leq_1 y \iff x \leq_2 y \quad .$$

So $\leq_1 = \leq_2$.

[5]The quotient map p is surjective.

Let $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D} \in \mathscr{P}^2(I)$, such that $\mathscr{A} \sim \mathscr{C}$, $\mathscr{B} \sim \mathscr{D}$, then by property Eq 2.2.4, $\mathscr{A} \cup \mathscr{B} \leq \mathscr{C} \cup \mathscr{D}$ and $\mathscr{C} \cup \mathscr{D} \leq \mathscr{A} \cup \mathscr{B}$, so $\mathscr{A} \cup \mathscr{B} \sim \mathscr{C} \cup \mathscr{D}$.

Similarly, by property Eq 2.2.5 $\mathscr{A} \sqcap \mathscr{B} \leq \mathscr{C} \sqcap \mathscr{D}$ and $\mathscr{C} \sqcap \mathscr{D} \leq \mathscr{A} \sqcap \mathscr{B}$, so $\mathscr{A} \sqcap \mathscr{B} \sim \mathscr{C} \sqcap \mathscr{D}$. Therefore the union and intersection given by Eq 2.2.8 are well defined.

For any $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D} \in \mathscr{P}^2(I)$,

$$[\mathscr{A}] \sqcap [\mathscr{B}] = [\mathscr{A} \sqcap \mathscr{B}] = [\mathscr{B}] \sqcap [\mathscr{A}] = [\mathscr{B} \sqcap \mathscr{A}].$$

$$([\mathscr{A}] \cup [\mathscr{B}]) \sqcap [\mathscr{C}] = [(\mathscr{A} \cup \mathscr{B}) \sqcap \mathscr{C}] = [(\mathscr{A} \sqcap \mathscr{C}) \cup (\mathscr{B} \sqcap \mathscr{C})] = ([\mathscr{A}] \sqcap [\mathscr{C}]) \cup ([\mathscr{B}] \sqcap [\mathscr{C}]).$$

$$[\mathscr{A} \sqcap \mathscr{B}] \leq [\mathscr{A}].$$

Therefore, $[\mathscr{A}] \sqcap [\mathscr{B}] \leq [\mathscr{A}]$. And one proceeds similarly for the two other properties.

□

We will now also note $\overline{\leq}$ as $\leq$.

Example 2.2.1. Consider $I = \{1, 2, 3, 4\}$. $\{\{1, 2\}, \{1, 3\}\} \leq \{I\}$ and this is true for any element of $\mathscr{P}^2(I)$.

$$\{\{1, 2\}, \{1, 3\}\} \cup \{\{2\}\} = \{\{1, 2\}, \{1, 3\}, \{2\}\} \sim \{\{1, 2\}, \{1, 3\}\}.$$

$$\{\{1, 2, 4\}, \{1, 3\}\} \sqcap \{\{2, 4\}, \{2, 3\}\} = \{\{2, 4\}, \{2\}, \emptyset, \{3\}\} \sim \{\{2, 4\}, \{3\}\}.$$

Remark 2.2.1. By construction, any section of p induces a poset isomorphism. For example the application that sends $\mathscr{A}$ to its lower set induces a section $s : [\mathscr{A}] \mapsto \hat{\mathscr{A}}$ of p; $\widehat{\mathscr{P}^2(I)}$ and $p_{|\,\mathscr{U}(\mathscr{P}(I))}$ is a poset isomorphism. On $\mathscr{U}(\mathscr{P}(I))$, $\leq$ is equal to the inclusion $\subseteq$ and $\sqcap = \cap$.

2.2.2 Increasing function from $\mathscr{P}^2(I)$ to the poset of factorisation spaces

Let us denote $\mathscr{F}(I)$ the poset of factorization spaces.

Proposition 2.2.3. *Let,*

$$\begin{array}{rcccl} \Phi & : & \mathscr{P}^2(I) & \to & \mathscr{F}(I) \\ & & \mathscr{A} & \mapsto & \mathscr{F}_{\mathscr{A}} \end{array}$$

$$\begin{array}{rcccl} \overline{\Phi} & : & \overline{\mathscr{P}^2(I)} & \to & \mathscr{F}(I) \\ & & [\mathscr{A}] & \mapsto & \mathscr{F}_{\mathscr{A}} \end{array}$$

$\overline{\Phi} : (\overline{\mathscr{P}^2(I)}, \leq) \to (\mathscr{F}(I), \subseteq)$ *is a poset morphism. For all* $\mathscr{A}, \mathscr{B} \in \mathscr{P}^2(I), \Phi(\mathscr{A} \cup \mathscr{B}) = \Phi(\mathscr{A}).\Phi(\mathscr{B})$, $\overline{\Phi}([\mathscr{A}] \cup [\mathscr{B}]) = \overline{\Phi}([\mathscr{A}]).\overline{\Phi}([\mathscr{B}])$.

If for all $i \in I$, $|E_i| \geq 2$ *then* $\overline{\Phi}$ *is injective and is a poset isomorphism.*

Let us remark that for all $a, b \subseteq I$ such that $a \subseteq b$, $\mathscr{F}_a \subseteq \mathscr{F}_b$ and that for all $a \in \mathscr{A}$, $\mathscr{F}_a \subseteq \mathscr{F}_{\mathscr{A}}$.

Indeed, $\pi_a = \pi_a^b \circ \pi_b$ so for all $f : E_a \to \mathbb{R}_{>0}$, $f \circ \pi_a = (f \circ \pi_a^b) \circ \pi_b$, so $f \circ \pi_a \in \mathscr{F}_b$. Let us note 1 the constant function equal to 1. For all $a \subseteq I$, $1 \in \mathscr{F}_\emptyset \subseteq \mathscr{F}_a$. For $a \in \mathscr{A}$, $f \in \mathscr{F}_a$, $f = f \prod_{b \in \mathscr{A} \setminus \{a\}} 1$, so $f \in \mathscr{F}_{\mathscr{A}}$.

Let us now prove Proposition 2.2.3.

Proof. Let $\mathscr{A}, \mathscr{B} \in \mathscr{P}^2(I)$ such that $\mathscr{A} \leq \mathscr{B}$ and $f \in \mathscr{F}_{\mathscr{A}}$ such that $f = \prod_{a \in \mathscr{A}} g_a$. For all $a \in \mathscr{A}$ there is $b(a) \in \mathscr{B}$ such that $a \subseteq b(a)$, so $g_a \in \mathscr{F}_{b(a)} \subseteq \mathscr{F}_{\mathscr{B}}$ and $\prod_{a \in \mathscr{A}} g_a \in \mathscr{F}_{\mathscr{B}}$ as $\mathscr{F}_{\mathscr{B}}$ is a vector space. So $\mathscr{F}_{\mathscr{A}} \leq \mathscr{F}_{\mathscr{B}}$.

Let $\mathscr{A} \leq \mathscr{B}$ and $\mathscr{B} \leq \mathscr{A}$ then $\mathscr{F}_{\mathscr{A}} \subseteq \mathscr{F}_{\mathscr{B}}$ and $\mathscr{F}_{\mathscr{B}} \subseteq \mathscr{F}_{\mathscr{A}}$, then $\mathscr{F}_{\mathscr{A}} = \mathscr{F}_{\mathscr{B}}$ and $\overline{\Phi}$ is well defined and is a poset morphism.

For all $\mathscr{A}, \mathscr{B} \in \mathscr{P}^2(I)$, $\Phi(\mathscr{A})$ and $\Phi(\mathscr{B})$ are subspaces of $\Phi(\mathscr{A} \cup \mathscr{B})$ so $\Phi(\mathscr{A}).\Phi(\mathscr{B}) \subseteq \Phi(\mathscr{A} \cup \mathscr{B})$. For all $a \in \mathscr{A} \cup \mathscr{B}$, $\mathscr{F}_a \subseteq \Phi(\mathscr{A}).\Phi(\mathscr{B})$; $\Phi(\mathscr{A}).\Phi(\mathscr{B})$ also being a vector space, $\Phi(\mathscr{A} \cup \mathscr{B}) \subseteq \Phi(\mathscr{A}).\Phi(\mathscr{B})$.

If for all $i \in I$, $|E_i| \geq 2$, Corollary 2 in [20] stipulates that $\mathscr{F}_{\mathscr{A}} = \mathscr{F}_{\mathscr{B}}$ if and only if $\hat{\mathscr{A}} = \hat{\mathscr{B}}$ but the proof of this results shows that if $\mathscr{F}_{\mathscr{A}} \subseteq \mathscr{F}_{\mathscr{B}}$ then $\hat{\mathscr{A}} \leq \hat{\mathscr{B}}$. So $\Phi_{|\mathscr{P}^{\hat{2}}(I)}$ is injective therefore so is $\overline{\Phi}$ by remark 2.2.1. Furthermore $\overline{\Phi}([\mathscr{A}]) \subseteq \overline{\Phi}([\mathscr{B}])$ implies $[\mathscr{A}] \leq [\mathscr{B}]$, so $\overline{\Phi}$ is a poset isomorphism.

□

Remark 2.2.2. Proposition 2.2.3 is a very general property for any increasing function Γ from any poset $(\mathscr{A}, \leq)$ to $\mathbf{Gr}\, V$ the set of vector subspaces of a vector space V. Indeed let $\mathscr{U}, \mathscr{V} \in \mathscr{P}(\mathscr{A})$, $\sum_{a \in \mathscr{U}} \Gamma(a) + \sum_{b \in \mathscr{V}} \Gamma(b) = \sum_{a \in \mathscr{U} \cup \mathscr{V}} \Gamma(a)$, and if $\mathscr{U} \leq \mathscr{V}$,

in the same sense than in Definition 2.2.2, then $\sum_{a\in\mathscr{U}} \Gamma(a) \subseteq \sum_{a\in\mathscr{V}} \Gamma(a)$. We enounced it as a proposition in order to clarify the presentation, as we use it as a know fact in later proofs.

2.3 Intersection property for factorisations on finite posets

In this section we still assume that I is finite. For $a, b, c \subseteq I$ such that $b \cup c = a$ and $b \cap c = \emptyset$, the map $\pi^a_{(c,b)} : E_a \to E_b \times E_c$ is a bijection. We will note for $u \in E_b, v \in E_c$, ${\pi^a_{(c,d)}}^{-1}(u, v)$ as uv, in particular for $x \in E$, $x_a = \pi^a_b(x_a)\pi^a_c(x_a) = x_b x_c$. Thus we can also write, for any $a, b \subseteq I$, $x_a = x_{a\cap b} x_{a\cap \bar{b}}$.

Lemma 2.3.1. *Let $a \subseteq I$, $\mathscr{B} \in \mathscr{P}^2(I)$,*

$$\mathscr{F}_a \cap \mathscr{F}_{\mathscr{B}} \subseteq \mathscr{F}_{\{a\}\sqcap\mathscr{B}}. \tag{2.3.1}$$

Proof. Let $f \in \mathscr{F}_a$ and $(g_b)_{b\in\mathscr{B}} \in \prod_{b\in\mathscr{B}} \mathscr{F}_b$ such that for all $x \in E$,

$$f(x) = \prod_{b\in\mathscr{B}} g_b(x) \quad . \tag{2.3.2}$$

There are $f_a, (\tilde{g}_b)_{b\in\mathscr{B}}$ such that for all $x \in E$, $b \in \mathscr{B}$, $f(x) = f_a(x_a)$, $g_b(x) = \tilde{g}_b(x)$.

For all $x \in E$,

$$f_a(x_a) = \prod_{b\in B} \tilde{g}_b(x_{b\cap a} x_{b\cap \overline{a}})$$

Let $c_{\overline{a}} \in E_{\overline{a}}$ then, $\pi_a(x_a c_{\overline{a}}) = x_a$ and $\pi_b(x_a c_{\overline{a}}) = (x_{b\cap a} c_{b\cap\overline{a}})$. So,

$$f_a(x_a) = \prod_{b\in B} \tilde{g}_b(x_{b\cap a} c_{b\cap\overline{a}})$$

Let us pose for all $b \in \mathscr{B}$, $g_{1,b}(x_{b\cap a}) = \tilde{g}_b(x_{b\cap a} c_{b\cap\overline{a}})$, then $f = \prod_{b\in B} g_{1,b} \circ \pi_{b\cap a}$.

□

Theorem 2.3.1 (Intersection property)**.** *Let I be a finite set and let $(E_i)_{i\in I}$ be family of non necessarily finite sets.*

For $\mathscr{A}, \mathscr{B} \in \mathscr{P}^2(I)$, $f \in \mathbb{R}^E_{>0}$, $(f_a)_{a\in\mathscr{A}} \in \prod_{a\in\mathscr{A}} \mathbb{R}^{E_a}_{>0}$ and $(g_b)_{b\in\mathscr{B}} \in \prod_{b\in\mathscr{B}} \mathbb{R}^{E_b}_{>0}$ such that, for all $x \in E$,

$$f(x) = \prod_{a \in \mathscr{A}} f_a(x_a) = \prod_{b \in \mathscr{B}} g_b(x_b) \quad .$$

There is $(h_{a,b})_{(a,b)\in\mathscr{A}\times\mathscr{B}} \in \prod_{(a,b)\in\mathscr{A}\times\mathscr{B}} \mathbb{R}_{>0}^{E_{a\cap b}}$ *such that for all* $x \in E$,

$$f(x) = \prod_{(a,b)\in\mathscr{A}\times\mathscr{B}} h_{a,b}(x_{a\cap b}).$$

Equivalently,

$$\mathscr{F}_{\mathscr{A}} \cap \mathscr{F}_{\mathscr{B}} \subseteq \mathscr{F}_{\mathscr{A}\sqcap\mathscr{B}} \quad . \tag{2.3.3}$$

Proof. For $\mathscr{A}, \mathscr{B} \in \mathscr{P}^2(I)$, $\mathscr{A} \sqcap \mathscr{B} \leq \mathscr{A}$, $\mathscr{A} \sqcap \mathscr{B} \leq \mathscr{B}$. Therefore by Proposition.(2.2.3) $\mathscr{F}_{\mathscr{A}\sqcap\mathscr{B}} \subseteq \mathscr{F}_{\mathscr{A}} \cap \mathscr{F}_{\mathscr{B}}$.

Let us prove the other inclusion by induction on $|\mathscr{A}|$.

$|\mathscr{A}| = 1$ is the previous Lemma.2.3.1.

Suppose that for all $\mathscr{A}, \mathscr{B} \in \mathscr{P}^2(I)$ such that $|\mathscr{A}| = n \in \mathbb{N}$, $\mathscr{F}_{\mathscr{A}} \cap \mathscr{F}_{\mathscr{B}} \subseteq \mathscr{F}_{\mathscr{A}\sqcap\mathscr{B}}$.

Let $\mathscr{A} \in \mathscr{P}^2(I)$, $|\mathscr{A}| = n+1$. Take $\alpha \in \mathscr{A}$, $|\mathscr{A} \setminus \{\alpha\}| = n$. Pose $\mathscr{C} = \mathscr{A} \setminus \{\alpha\}$. Let $f \in \mathscr{F}_{\mathscr{A}} \cap \mathscr{F}_{\mathscr{B}}$, then there is $h_1 \in \mathscr{F}_{\alpha}$, $f_1 \in \mathscr{F}_{\mathscr{C}}$, $g \in \mathscr{F}_{\mathscr{B}}$ such that $f = h_1.f_1 = g$.

So $h_1 = \frac{g}{f_1}$ and $h_1 \in \mathscr{F}_{\mathscr{C}}.\mathscr{F}_{\mathscr{B}}$. So by Proposition.(2.2.3), $h_1 \in \mathscr{F}_{\mathscr{C}\cup\mathscr{B}}$. Then by Lemma.2.3.1 $h_1 \in \mathscr{F}_{(\mathscr{C}\cup\mathscr{B})\sqcap\{\alpha\}}$. But $(\mathscr{C} \cup \mathscr{B}) \sqcap \{\alpha\} = (\mathscr{C} \sqcap \{\alpha\}) \cup (\mathscr{B} \sqcap \{\alpha\})$.

So $h_1 \in \mathscr{F}_{\mathscr{C}\sqcap\{\alpha\}}.\mathscr{F}_{\mathscr{B}\sqcap\{\alpha\}}$. Furthermore $f_1 \in \mathscr{F}_{\mathscr{C}}$ so $f = h_1.f_1 \in \mathscr{F}_{\mathscr{C}}.\mathscr{F}_{\mathscr{C}\sqcap\{\alpha\}}.\mathscr{F}_{\mathscr{B}\sqcap\{\alpha\}}$. But $\mathscr{F}_{\mathscr{C}\sqcap\{\alpha\}} \subseteq \mathscr{F}_{\mathscr{C}}$ so $\mathscr{F}_{\mathscr{C}}.\mathscr{F}_{\mathscr{C}\sqcap\{\alpha\}} \subseteq \mathscr{F}_{\mathscr{C}}$ (it is even equal).

So there is $f_2 \in \mathscr{F}_{\mathscr{C}}$, $h_2 \in \mathscr{F}_{\mathscr{B}\sqcap\{\alpha\}}$ such that $g = h_2.f_2$. Therefore $f_2 = \frac{g}{h_2}$. But $\mathscr{F}_{\mathscr{B}\sqcap\{\alpha\}} \subseteq \mathscr{F}_{\mathscr{B}}$ so $f_2 \in \mathscr{F}_{\mathscr{B}}$.

Therefore by the induction hypothesis, $f_2 \in \mathscr{F}_{\mathscr{C}\sqcap\mathscr{B}}$, and so $f \in \mathscr{F}_{\mathscr{B}\sqcap\{\alpha\}}\mathscr{F}_{\mathscr{C}\sqcap\mathscr{B}}$. One remarks that $(\{\alpha\} \sqcap \mathscr{B}) \cup (\mathscr{C} \sqcap \mathscr{B}) = \mathscr{A} \sqcap \mathscr{B}$ so $f \in \mathscr{F}_{\mathscr{A}\sqcap\mathscr{B}}$. Which ends the proof by induction.

□

Corollary 2.3.1. *For all* $\mathscr{A}, \mathscr{B} \in \mathscr{P}^2(I)$,

$$\mathscr{F}_{\mathscr{A}} \cap \mathscr{F}_{\mathscr{B}} = \mathscr{F}_{\mathscr{A}\sqcap\mathscr{B}} = \mathscr{F}_{\hat{\mathscr{A}}\cap\hat{\mathscr{B}}} \quad . \tag{2.3.4}$$

Which can be rewritten as, for all $\mathscr{A}, \mathscr{B} \in \overline{\mathscr{P}^2(I)}$,

$$\overline{\Phi}(\mathscr{A} \sqcap \mathscr{B}) = \overline{\Phi}(\mathscr{A}) \cap \overline{\Phi}(\mathscr{B}) \quad . \tag{2.3.5}$$

Proof. $\mathscr{A} \sqcap \mathscr{B} \leq \mathscr{A}$ and $\mathscr{A} \sqcap \mathscr{B} \leq \mathscr{B}$ therefore $\mathscr{F}_{\mathscr{A}\sqcap\mathscr{B}} \subseteq \mathscr{F}_{\mathscr{A}}$ and $\mathscr{F}_{\mathscr{A}\sqcap\mathscr{B}} \subseteq \mathscr{F}_{\mathscr{B}}$.

□

2.4 Extension for infinite posets

In this section I is any set; let us now use the summation convention instead of the product one. We would like to give a similar definition of $\mathscr{F}_{\mathscr{A}}$ but for infinite posets. If for any $\mathscr{A} \subseteq \mathscr{P}(I)$, we defined $U(\mathscr{A})$ as $\sum\limits_{\substack{a\in\mathscr{A}\\ |a|<+\infty}} U(a)$ then $U(I) = 0$. One needs to consider only lower sets in $\mathscr{U}(\mathscr{P}(I))$.

Let us call $U = U_{\mathscr{P}(I)}$ and $U(I)$ the poset constituted of the $U_{\mathscr{A}}$; let Ψ be such that,

$$\begin{array}{rcccl} \Psi & : & \mathscr{U}(\mathscr{P}(I)) & \to & U(I) \\ & & \mathscr{A} & \mapsto & U_{\mathscr{A}} \end{array} \tag{2.4.1}$$

In particular,

$$U(\mathscr{A}) = \ln \mathscr{F}_{\mathscr{A}} \tag{2.4.2}$$

Corollary 2.4.1. *For all* $\mathscr{A}, \mathscr{B} \in \mathscr{U}(\mathscr{P}(I))$,

$$U_{\mathscr{A}} \cap U_{\mathscr{B}} = U_{\mathscr{A}\cap\mathscr{B}}. \tag{2.4.3}$$

Proof. Let $f \in U_{\mathscr{A}} \cap U_{\mathscr{B}}$. There are by definition, $\mathscr{C}_1 \subseteq \mathscr{A}$, $\mathscr{C}_2 \subseteq \mathscr{B}$, that are of finite cardinal, such that $f \in U_{\mathscr{C}_1}$ and $f \in U_{\mathscr{C}_2}$. By Corollary 2.3.1, $f \in U_{\mathscr{C}_1\sqcap\mathscr{C}_2}$. As $\mathscr{C}_1 \sqcap \mathscr{C}_2 \subseteq \mathscr{A} \cap \mathscr{B}$, $f \in U_{\mathscr{A}\cap\mathscr{B}}$. □

We will now show that a stronger version of Corollary.(2.4.1) holds for the intersection on any family of elements of $\mathscr{U}(\mathscr{P}(I))$.

Theorem 2.4.1. *For any family* $(\mathscr{A}_j)_{j\in J}$ *of elements of* $\mathscr{U}(\mathscr{P}(I))$,

$$\bigcap_{j\in J} U_{\mathscr{A}_j} = U_{\bigcap\limits_{j\in J}\mathscr{A}_j}. \tag{2.4.4}$$

Before giving a proof of this result, let us first state the following lemma,

Lemma 2.4.1. *Let V_1, V_2 be two vector subspaces of U. If for any finite $a \in \mathscr{P}(I)$,*

$$V_1 \cap U_a \subseteq V_2 \cap U_a. \tag{2.4.5}$$

Then,

$$V_1 \subseteq V_2 \tag{2.4.6}$$

Proof. Let $v \in V_1$, there is a finite collection of finite subsets of I, $(a_k)_{1\leq k\leq n}$, such that, $v \in \sum\limits_{1\leq k\leq n} U_{a_k}$.

Therefore $v \in U_{(\bigcup\limits_{1\leq k\leq n} a_k)}$. But $\bigcup\limits_{1\leq k\leq n} a_k$ is of finite cardinal. So $v \in V_2 \cap U_{(\bigcup\limits_{1\leq k\leq n} a_k)} \subseteq V_2$.

Therefore $V_1 \subseteq V_2$. □

A direct consequence of Lemma.(2.4.1) is that if for any finite $a \in \mathscr{P}(I)$,

$$V_1 \cap U_a = V_2 \cap U_a. \tag{2.4.7}$$

Then $V_1 = V_2$.

Proof of the Theorem.(2.4.1). Let $(\mathscr{A}_j)_{j\in J}$ be a family of elements of $\mathscr{U}(\mathscr{P}(I))$. Let $a \subseteq I$ of finite cardinal.

$\bigcap\limits_{j\in J} U_{\mathscr{A}_j} \cap U_a = \bigcap\limits_{j\in J} \Big(U_{\mathscr{A}_j} \cap U_a\Big)$.

But, $U_{\mathscr{A}_j} \cap U_a = U_{\mathscr{A}_j \cap \widehat{\{a\}}}$. And $\{U_{\mathscr{A}_j \cap \widehat{\{a\}}} : \quad j \in J\}$ is finite, so $\bigcap\limits_{j\in J} \Big(U_{\mathscr{A}_j} \cap U_a\Big)$ can be rewritten as a finite intersection and by Corollary (2.4.1),

$$\bigcap_{j\in J} \Big(U_{\mathscr{A}_j} \cap U_a\Big) = U_{\bigcap\limits_{j\in J} (\mathscr{A}_j \cap \widehat{\{a\}})} \subset U_{\bigcap\limits_{j\in J} \mathscr{A}_j} \cap U_a \quad .$$

By Lemma.(2.4.1),

$$\bigcap_{j\in J} U_{\mathscr{A}_j} \subseteq U_{\bigcap\limits_{j\in J} \mathscr{A}_j} \quad .$$

The other inclusion is always true (Remark (2.2.1)) as for any $i \in J$, $\bigcap\limits_{j\in J} \mathscr{A}_j \subseteq \mathscr{A}_i$. □

Remark 2.4.1. This proposition can also be stated in terms of the $\mathscr{F}_{\mathscr{A}}$ by taking the exponential:

$$\bigcap_{j\in J} \mathscr{F}_{\mathscr{A}_j} = \mathscr{F}_{\bigcap_{j\in J} \mathscr{A}_j}. \tag{2.4.8}$$

2.5 Applications

2.5.1 Minimal factorisation

In [20] a proof of the existence of a minimum factorization[6] is given, based on the existence of a decomposition into interaction subspaces, when E is finite and I finite. Let us give a proof of this result using Theorem 2.4.1 so without assuming that I nor E are finite.

Corollary 2.5.1. *Let I be any set and $E = \prod_{i\in I} E_i$ be the product of any collection of sets; for all $f \in \mathscr{F}$ let us call $\mathscr{F}(f) = \{\mathscr{F}_{\mathscr{A}} | \quad f \in \mathscr{F}_{\mathscr{A}}\}$. $\mathscr{F}(f)$ admits a minimum and we say that f admits a minimum decomposition.*

Proof. Let $\mathscr{M}(f) = \{\mathscr{A} \in \mathscr{U}(\mathscr{P}(I))| \quad f \in \mathscr{F}_{\mathscr{A}}\}$. From Theorem 2.4.1, one has that,

$$\bigcap_{\mathscr{A}\in\mathscr{M}(f)} \mathscr{F}_{\mathscr{A}} = \mathscr{F}_{\bigcap_{\mathscr{A}\in\mathscr{M}(f)} \mathscr{A}}.$$

Any K $\in \mathscr{F}(f)$ contains $\bigcap_{\mathscr{A}\in\mathscr{M}(f)} \mathscr{F}_{\mathscr{A}}$, therefore $\mathscr{F}_{\bigcap_{\mathscr{A}\in\mathscr{M}(f)} \mathscr{A}}$ is the minimum of $\mathscr{F}(f)$.

□

2.5.2 Markov properties and Hammersley-Clifford

Let us consider four random variables W, X, Y, Z taking values respectively in E_0, E_1, E_2, E_3 **finite sets**, with strictly positive joint law. Let us recall the law of X conditionally to Y,

$$\forall (x, y) \in E_1 \times E_2, \quad \mathbb{P}_{X|Y}(x, y) = \frac{\mathbb{P}_{X,Y}(x, y)}{\mathbb{P}_Y(y)} \tag{2.5.1}$$

Conditional independence is usually defined as follows,

$$X \perp\!\!\!\perp Y|Z \quad \Longleftrightarrow \quad \forall (x, y, z) \in E_1\times E_2\times E_3, \quad \mathbb{P}_{(X,Y)|Z}(x, y, z) = \mathbb{P}_{X|Z}(x, z)\mathbb{P}_{Y|Z}(y, z) \tag{2.5.2}$$

[6] In a poset $\mathscr{A}$, $a \in \mathscr{A}$ is said to be a minimum if any $b \in \mathscr{A}$ is such that $a \leq b$.

Let us pose $I = \{0,1,2,3\}$ we identify $\prod_{i\in I} E_i$ with $E_0 \times E_1 \times E_2 \times E_3$ by the following $x \mapsto (x(0), x(1), x(2), x(3))$ and then $\mathscr{F}_{\mathscr{A}}$ to sets in $\mathbb{R}_{>0}^{E_0\times E_1\times E_2\times E_3}$. Let $a = \{1,3\}$, $b = \{2,3\}$ and $\mathscr{A} = \{a, b\}$,

$$X \perp\!\!\!\perp Y|Z \quad \Longleftrightarrow \quad \mathbb{P}_{X,Y,Z} \in \mathscr{F}_{\mathscr{A}}. \tag{2.5.3}$$

Proposition 2.5.1 (Bayesian or Graphoid intersection property)**.**

$$(X \perp\!\!\!\perp Y|(Z,W)) \wedge (X \perp\!\!\!\perp W)|(Z,Y) \implies X \perp\!\!\!\perp (Y,W)|Z. \tag{2.5.4}$$

Proof. Let $a = \{0,1,3\}$, $b = \{0,2,3\}$, $c = \{1,2,3\}$, $d = \{1,3\}$ and $\mathscr{A} = \{a,b\}$, $\mathscr{B} = \{b,c\}$, $\mathscr{C} = \{b,d\}$. $\mathscr{A} \sqcap \mathscr{B} \equiv \{a \cap c, b\} = \{d, b\}$ so $\mathscr{F}_{\mathscr{A}} \cap \mathscr{F}_{\mathscr{B}} \subseteq \mathscr{F}_{\mathscr{C}}$. □

Corollary 2.5.2. *(Hammersley-Clifford)*

Let $G = (I, A)$ be a finite graph. For all strictly positive probability law, P_X, on a finite E,

$$P_X \in P(G) \iff P_X \in L(G) \iff P_X \in U_{\mathscr{C}} \tag{2.5.5}$$

For any pair (i,j) of elements of I and for all probability law P,

$$X_i \perp\!\!\!\perp X_j|X_{I\setminus\{i,j\}} \iff P_X \in U_{[i,j]} \tag{2.5.6}$$

Let $\mathscr{A}_P = \bigsqcap_{(i,j):\, i\notin\partial j} [i,j]$. Similarly, for all $i \in I$,

$$X_i \perp\!\!\!\perp X_{I\setminus(i\cup\partial i)}|X_{\partial i} \iff \mathbb{P}_X \in U_{[i]} \tag{2.5.7}$$

Let $\mathscr{A}_L = \bigsqcap_i [i]$. The following lemma is the version of the Hammersley-Clifford on graphs that we will then translate to graphical models by applying Ψ.

Lemma 2.5.1.

$$\hat{\mathscr{A}_L} = \hat{\mathscr{A}_P} = \mathscr{C} \tag{2.5.8}$$

Proof. Firstly, $\hat{\mathscr{A}_L} = \bigcap_{(k,l):\, k\notin\partial l} \widehat{[k,l]}$. Let $a \in \hat{\mathscr{A}_L}$ and assume that a is not a clique. So there is $i, j \in a$ such that $i \notin \partial j$. But $a \in \widehat{[i,j]}$, so $a \subseteq i \cup (I \setminus \{i,j\})$ or $a \subseteq j \cup (I \setminus \{i,j\})$. It is not possible as any of these two sets separate i and j. So a must be a clique. In other words, $\{i,j\} \subseteq a$ but $\{i,j\} \not\subseteq i \cup (I \setminus \{i,j\})$ and $\{i,j\} \not\subseteq j \cup (I \setminus \{i,j\})$ ($\{i,j\} \notin \widehat{[i,j]}$). So if a is not a clique of G, $a \notin \hat{\mathscr{A}_L}$.

Suppose a is a clique of G. Let $i, j \in I$ such that $i \notin \partial j$. $i \cup (I \setminus \{i,j\})$ and $j \cup (I \setminus \{i,j\})$ separate i, j. So a clique most be in only one of the two sets. To be

more formal, for any subset a of I, there is $b \subseteq I \setminus \{i,j\}$, such that $a = b$ or $a = b \cup i$ or $a = b \cup j$ or $a = b \cup \{i,j\}$. As a is a clique $\{i,j\} \not\subseteq a$. So there is $b \subseteq I \setminus \{i,j\}$, such that $a = b$ or $a = b \cup i$ or $a = b \cup j$. Which is equivalent to saying that $a \in [\hat{i,j}]$.

So we proved that,

$$\hat{\mathscr{A}_P} = \mathscr{C}.$$

For the local case, $\hat{\mathscr{A}_L}$, one has to remark that a is a clique of G if and only if for all $i \in a$, $a \subseteq \{i, \partial i\}$ (for exemple see slide 6 [33]).

□

Proof of Corollary 2.5.2. Let us remark that $P_X \in P(G)$ if and only if $P_X \in \bigcap_{(i,j):\, i \notin \partial j} U_{[i,j]}$ and similarly $P_X \in L(G)$ if and only if $P_X \in \bigcap_{i \in I} U_{[i]}$.

As P_X is strictly positive, by Corollary 2.3.1,

$$P_X \in P(G) \iff P_X \in U_{\mathscr{A}_P} \iff P_X \in U_{\mathscr{C}}. \tag{2.5.9}$$

$$P_X \in L(G) \iff P_X \in G_{\mathscr{A}_L} \iff P_X \in U_{\mathscr{C}}. \tag{2.5.10}$$

□

Similarly, when $G = (I, D)$ is any graph and $(E_i)_{i \in I}$ is any collection of sets, Lemma 2.5.1 still holds and one has the following result which extends the Hammersley-Clifford theorem.

Corollary 2.5.3.

$$\bigcap_{(i,j): i \notin \partial j} \mathscr{F}_{\widehat{[i,j]}} = \bigcap_{i \in I} \mathscr{F}_{\widehat{[i]}} = \mathscr{F}_{\mathscr{C}}. \tag{2.5.11}$$

Chapter 3

Intersection property and interaction decomposition

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Abstract

The decomposition into interaction subspaces is a hierarchical decomposition of the spaces of cylindric functions of a finite product space, also called factor spaces. It is an important construction in graphical models and a standard way to prove the Hammersley-Clifford theorem that relates Markov fields to Gibbs fields. We define an intersection of sum property, or simply intersection property, and show that it characterizes collections of vector subspaces over a poset that can be hierarchically decomposed into direct sums, giving therefore a general setting for such construction to hold. The intersection property is the Bayesian intersection property, we introduced in the previous chapter, when specified to factor space which, under this new perspective on the interaction decomposition, appears to be a structure property. An application is the extension of the interaction decomposition for any product of any set.

3.1 Introduction

3.1.1 Motivation

For a finite set I and a finite product space $E = \prod_{i \in I} E_i$, the factor subspaces $\mathrm{U}(a) = \mathbb{R}^{E_a}$, seen as a subspace of $\mathbb{R}^E$, with $a \subseteq I$ and $E_a = \prod_{i \in a} E_i$, can be decomposed into direct sums of a collection of subspaces $(S_a \subseteq \mathbb{R}^E, a \subseteq I)$, called interaction subspaces ([19], [14]),

$$\forall a \subseteq I \;\; \mathrm{U}(a) = \bigoplus_{\substack{b \in \mathscr{P}(I) \\ b \subseteq a}} S_b \tag{3.1.1}$$

$\mathscr{P}(I)$ is a partially ordered set, or poset, and it can also be seen as a category with only one morphism between each of its elements $b \to a$, every times that $b \subseteq a$. In particular one can see U as a functor from a poset $\mathscr{A} = \mathscr{P}(I)$ to the category of vector spaces **Vect**. In this chapter we characterize functors for which such decomposition exists.

Firstly we must give a definition of what a decomposition for functors over a poset would be.

Definition (Decomposable functor)**.** A functor $G : \mathscr{A} \to \mathbf{Vect}$ is decomposable if and only if there is a collection of vector spaces $(S_a, a \in \mathscr{A})$ such that for any $a \in \mathscr{A}$

$$G(a) \cong \sum_{b \leq a} S_a \tag{3.1.2}$$

and for $b \in \mathscr{A}$ with $b \leq a$, G_a^b is isomorphic to the inclusion $\bigoplus_{c \leq b} S_c \to \bigoplus_{c \leq a} S_c$. We will call $(S_a, a \in \mathscr{A})$ a decomposition of G.

If a functor is decomposable then its morphisms $(G_b^a | a, b \in \mathscr{A}, b \leq a)$ are injective; we will call such functors as injective. We show (Proposition 3.2.1) that every injective functor is isomorphic to an increasing functions from a poset $\mathscr{A}$ to the poset of vector subspaces of a vector space, seen as a category; we therefore restrict our attention to the latter for this chapter.

Notation 3.1.1. For $n \in \mathbb{N}$, we will note $[n]$ the integer interval $[0, n]$ and a collection of elements over a set E, $(f_x)_{x \in E}$, will also be referred to as $f_x, x \in E$.
In this document $\mathscr{A}$ will denote a poset and V a vector space. Let $a, b \in \mathscr{A}$, the interval $[a, b]$ is the $c \in \mathscr{A}$ such that $a \leq c \leq b$.

Notation 3.1.2. $\hom(\mathscr{A}, \mathbf{Gr}\, M)$ will be the set of increasing functions between a poset $\mathscr{A}$ and the poset of sub-modules of a module M; when M is a vector space, $\mathbf{Gr}\, V$ is also called the Grassmannian of V.

Remark 3.1.1. As any poset is a category, in particular $\hom(\mathscr{A}, \mathbf{Gr}\, V)$ is a functor from $\mathscr{A}$ to $\mathbf{Gr}\, V$.

3.1.2 Main results of this chapter

Well-founded poset

The main result of this chapter holds for well-founded poset, let us now recall what a well-founded poset is.

Definition 3.1.1. A poset $\mathscr{A}$ is well-founded if any chain of $\mathscr{A}$ has a minimal element. This condition is often stated as the descending chain condition: every strictly decreasing sequences of elements of $\mathscr{A}$ terminates.

Remark 3.1.2. Any non-empty subposet of a well-founded poset has at least one minimal element.

Proposition 3.1.1. *Let $\mathscr{A}$ be a well-founded poset. To show that a property P holds for any $a \in \mathscr{A}$ is suffices to show that,*

$$\forall a \in \mathscr{A}, \quad [\forall b \in \mathscr{A}, b < a, \quad P(b)] \implies P(a) \tag{3.1.3}$$

Proof. Let $\mathscr{A}$ be well-founded. Let us assume that,

$$\forall a \in \mathscr{A}, \quad [\forall b \in \mathscr{A}, b < a, \quad P(b)] \implies P(a). \tag{3.1.4}$$

Let $\mathscr{B}$ be the set of elements of $\mathscr{A}$ that do not verify P, in other words,

$$\mathscr{B} = \{a \in \mathscr{A} | \quad \neg P(a)\} \tag{3.1.5}$$

Let us suppose that $\mathscr{B} \neq \emptyset$. Let b be a a minimal element of $\mathscr{B}$. For any $c < b$, $P(c)$ holds. But by hypothesis this implies that $P(b)$ holds. This is a contradiction. Therefore $\mathscr{B}$ is empty, which ends the proof. □

Remark 3.1.3. Proposition (3.1.1) is a generalization of the proof by induction and we will refer to it as the second form of proof by induction in this document.

Intersection property

The intrinsic condition that characterizes the decomposable functors is the following.

Definition (Intersection property)**.** Let $\mathscr{A}$ be any poset, an increasing function $\mathrm{U} \in hom(\mathscr{A}, \mathbf{Gr}\, V)$ is said to verify the intersection property (I) if and only if,

$$\forall \mathscr{B}, \mathscr{C} \in \mathscr{U}(\mathscr{A}), \quad \sum_{b \in \mathscr{B}} \mathrm{U}(b) \cap \sum_{c \in \mathscr{C}} \mathrm{U}(c) \subseteq \sum_{a \in \mathscr{B} \cap \mathscr{C}} \mathrm{U}(a) \tag{I}$$

The intersection property is exactly the one we exhibited in the previous chapter, in the context of generalized factor spaces. For well-founded poset there is a subset of conditions of (I) that is sufficient to prove decomposability and that is therefore equivalent to (I).

Definition. $\mathrm{U} \in \mathrm{hom}(\mathscr{A}, \mathbf{Gr}\, V)$ is said to verify the property (C) if and only if,

$$\forall a \in \mathscr{A}, \sum_{b : b \leq a} \mathrm{U}(b) \cap \sum_{b : a \not\leq b} \mathrm{U}(b) \subseteq \sum_{b : b \lneq a} \mathrm{U}(b) \tag{C}$$

Showing that an increasing collection of vector subspaces satisfies (C) is easier than showing (I).

Equivalence theorem

Theorem 3.5.1. Let $\mathscr{A}$ be a well-founded poset and V any vector space. $\mathrm{U} \in \mathrm{hom}(\mathscr{A}, \mathbf{Gr}\, V)$ is decomposable if and only if U verifies (C) or (I).

3.1.3 Structure of this chapter

We choose to focus the first part of this chapter to proving the equivalence theorem for the simpler case of increasing function over a finite poset (Sections 3.3,3.4). We will then use this result to show this theorem in the general setting (Section 3.5).

Being decomposable is not functorial, in other words let $\phi : \mathscr{B} \to \mathscr{A}$ be an increasing function and $\mathrm{U} \in \hom(\mathscr{B}, \mathbf{Gr}\, V)$ then $\phi_! \mathrm{U}$ is in general not decomposable. In Section 3.6 we give some cases where the pushforward of a decomposable functor is decomposable. Finally we apply the results of the previous chapter and the equivalence theorem of this chapter to give a generalized version of the interaction decomposition factor spaces.

3.2 Colimits of injective functors over a poset

Definition 3.2.1 (Injective functor)**.** A functor G from a poset $\mathscr{A}$ to the category of (R)-modules **Mod** will be called injective if for any $a, b \in \mathscr{A}$ such that $b \leq a$, G_a^b is a monomorphism, i.e. is injective.

Proposition 3.2.1. *Let G be an injective functor from any poset $\mathscr{A}$ to the category of (R)-modules **Mod**. Let $(G_a, a \in \mathscr{A}, \operatorname{colim} G)$ be the initial cocone over G; then G_a is injective for any $a \in \mathscr{A}$.*

Proof. Let $v \in G(a)$ be such that $[v_a] = 0$; therefore there are $m \in \mathbb{N}$, $m_1 \in \mathbb{N}$, a collection $(b_i \leq a, i \in [m])$ of elements of $\mathscr{A}$, a collection $(c_i \geq a, i \in [m_1]$, a collection $(u_i \in G(b_i), i \in [m])$, a collection $(w_i \in G(a), i \in [m_1])$ such that,

$$v \times a = \sum_{i \in [m]} (G_a^{b_i}(u_i) \times a - u_i \times b_i) + \sum_{i \in [m_1]} (G_{c_i}^a(w_i) \times c_i - w_i \times a) \tag{3.2.1}$$

Therefore there is a finite set B of elements strictly less than a and a finite set C of elements strictly greater than a, there are two collections $(u_b \in G(b), b \in B)$ and $(w_c \in G(a), c \in C)$ such that,

$$v \times a = \sum_{b \in B} (G_a^b(u_b) \times a - u_b \times b) + \sum_{c \in C} (G_c^a(w_c) \times c - w_c \times a \tag{3.2.2}$$

As $B \cap C = \emptyset$, the projections on $b \in B$ and $c \in C$ gives $u_b = 0$ and $w_c = 0$. Therefore $v = 0$.

□

Corollary 3.2.1. *Let $G : \mathscr{A} \to$ **Mod** be an injective functor there is a module M such that G is isomorphic to $\tilde{G} \in \hom(\mathscr{A}, M)$*

Proof. Let $M = \operatorname{colim}_a G(a)$ and for any $a \in \mathscr{A}$ let $\tilde{G}(a) = \operatorname{im} G_a$ for any $a \in \mathscr{A}$, by Proposition 3.2.1, $\tilde{G}$ is isomorphic to G. □

In what follows, without loss of generality, we consider functors in $\hom(\mathscr{A}, \mathbf{Gr}(V))$.

3.3 Decomposability

Definition 3.3.1 (Decomposabile injective functor)**.** $S_a, a \in \mathscr{A}$ is a decomposition of $\mathrm{U} \in \hom(\mathscr{A}, \mathbf{Gr}\, V)$ if and only if,

1. for all $a \in \mathscr{A}$, $S_a \in \mathbf{Gr}\, V$.
2. $\sum\limits_{a \in \mathscr{A}} S_a$ is a direct sum in V; in other words,

$$\begin{array}{rcccl} p & : & \bigoplus\limits_{a \in \mathscr{A}} S_a & \to & V \\ & & (v_a)_a & \mapsto & \sum\limits_{a \in \mathscr{A}} v_a \end{array} \tag{3.3.1}$$

   is injective.

3. for all $a \in \mathscr{A}$, $\mathrm{U}(a) = \sum\limits_{b \leq a} S_b$.

U is then said to be decomposable.

Remark 3.3.1. When p in Equation (3.3.1) is injective, one says that $\sum\limits_{a \in \mathscr{A}} S_a$ is in direct sum in V and it is noted $\sum_{a \in \mathscr{A}} S_a = \bigoplus_{a \in \mathscr{A}} S_a$; with this notation the previous definition can be restated as $\sum\limits_{a \in \mathscr{A}} S_a = \bigoplus\limits_{a \in \mathscr{A}} S_a$ and $\forall a \in \mathscr{A}, \mathrm{U}(a) = \bigoplus\limits_{b \leq a} S_b$.

Remark 3.3.2. Let $V = \mathbb{R}^2$ and a_1, a_2, a_3, three lines in V that are pairwise different. Let $\mathscr{A} = \{a_1, a_2, a_3\}$ and $\mathrm{U} : \mathscr{A} \overset{i}{\hookrightarrow} \mathbf{Gr}\, V$ be the inclusion map. U is not decomposable: if it were, $\dim \sum\limits_{a \in \mathscr{A}} v_a = 3$.

We would like to find a condition for which an increasing function $\mathrm{U} : \mathscr{A} \to \mathbf{Gr}\, V$ is decomposable. The main result of this section is to show that U is decomposable if an only if it verifies a certain intersection of sums property.

Definition 3.3.2 (Order-embedding)**.** Let $\mathscr{A}, \mathscr{B}$ be two posets and let $f : \mathscr{A} \to \mathscr{B}$ be an increasing function. f is an order-embedding if for all $a_1, a_1 \in \mathscr{B}$,

$$f(a_1) \leq f(a_2) \quad \Longrightarrow \quad a_1 \leq a_2$$

An order-embedding function is always injective.

Definition 3.3.3 (Intersection property)**.** Let $\mathscr{A}$ be any poset, an increasing function $\mathrm{U} \in hom(\mathscr{A}, \mathbf{Gr}\, V)$ is said to verify the intersection property (I) if and only if,

$$\forall \mathscr{B}, \mathscr{C} \in \mathscr{U}(\mathscr{A}), \quad \sum_{b \in \mathscr{B}} \mathrm{U}(b) \cap \sum_{c \in \mathscr{C}} \mathrm{U}(c) \subseteq \sum_{a \in \mathscr{B} \cap \mathscr{C}} \mathrm{U}(a) \tag{I}$$

Let $\mathscr{B} \subseteq \mathscr{A}$ be a subposet of $\mathscr{A}$, we will note $\mathrm{U}(\mathscr{B}) = \sum_{b \in \mathscr{B}} \mathrm{U}(b)$. Let us consider on $\mathscr{P}(\mathscr{A})$ the following order,

$$\forall \mathscr{B}_1, \mathscr{B}_2 \in \mathscr{P}(\mathscr{A}) \quad \mathscr{B}_1 \leq \mathscr{B}_2 \quad \Longleftrightarrow \quad \hat{\mathscr{B}}_1 \subseteq \hat{\mathscr{B}}_2. \tag{3.3.2}$$

Remark 3.3.3. We have extended U to a poset morphism between the set of subposets of $\mathscr{A}$, $(\mathscr{P}(\mathscr{A}), \leq)$, to $\mathbf{Gr}\, V$ (Section 2.4); in particular, if one calls i the following application,

$$\begin{array}{rcccl} i & : & \mathscr{A} & \to & \mathscr{U}(\mathscr{A}) \\ & & a & \mapsto & \widehat{\{a\}} \end{array}$$

the extension $(i_! \, \mathrm{U}(\mathscr{B}) = \mathrm{U}(\mathscr{B}), \mathscr{B} \in \mathscr{U}(\mathscr{A}))$ of U to $\mathscr{U}(\mathscr{A})$ is the left adjoint of i^*, the inverse image of U by i.

Notation 3.3.1. Several subposets of $\mathscr{A}$ will play an important role in the following development. Let us note,

$$\begin{aligned} \hat{a} &= \{b \in \mathscr{A} | b \leq a\} \\ \overset{\neg\vee}{a} &= \{b \in \mathscr{A} | a \not\leq b\} \\ \hat{a}^* &= \{b \in \mathscr{A} | b \lneq a\} \end{aligned} \tag{3.3.3}$$

Definition 3.3.4. $\mathrm{U} \in \mathrm{hom}(\mathscr{A}, \mathbf{Gr}\, V)$ is said to verify the property (C) if and only if,

$$\forall a \in \mathscr{A}, \mathrm{U}(a) \cap \mathrm{U}(\overset{\neg\vee}{a}) \subseteq \mathrm{U}(\hat{a}^*) \tag{C}$$

Remark 3.3.4. Let V be a vector space and V_1 a vector subspace of V, we will note the quotient map with respect to V_1, $\pi : V \to V/V_1$ as $\mod V_1$.

Let V_1, V_2, V_3 be three vector subspaces of V, V_1 is independent of V_2 conditionally to V_3, denoted as $V_1 \perp\!\!\!\perp V_2 | V_3$ if and only if $(V_1 \mod V_3) \cap (V_2 \mod V_3) = 0$. Condition (C) can be rewritten in terms of conditional independence properties:

$$\mathrm{U}(a) \perp\!\!\!\perp \mathrm{U}(\overset{\neg\vee}{a}) \Big| \, \mathrm{U}(\hat{a}^*) \tag{3.3.4}$$

Remark 3.3.5. Condition (C) is a subset of condition of (I) and therefore (I) implies (C).

Let us now show that decomposability implies a stronger version of intersection property.

Definition 3.3.5 (Strong intersection property)**.** An increasing function $\mathrm{U} : \mathscr{A} \to \mathbf{Gr}\, V$ is said to verify the strong intersection property (sI) if and only if for any family $(\mathscr{A}_j)_{j\in J}$ of elements of $\mathscr{U}(\mathscr{A})$,

$$\bigcap_{j\in J} \mathrm{U}(\mathscr{A}_j) = \mathrm{U}(\bigcap_{j\in J} \mathscr{A}_j). \tag{sI}$$

Remark 3.3.6. The motivation for this definition can be found in Chapter 2 Theorem 2.4.1. It is a natural extension of the intersection property (I); (sI) implies (I). Consider $\mathscr{A} = \mathbb{N}$ and let V be a vector space, let for any $a \in \mathscr{A}$ $\mathrm{U}(a) = V$; U verifies (I) but not (sI) therefore condition (sI) is in general strictly stronger than condition (I).

Proposition 3.3.1. *Let* $\mathrm{U} \in \hom(\mathscr{A}, \boldsymbol{Gr}\, V)$*, with* $\mathscr{A}$ *any poset. If* U *is decomposable then* U *verifies* (sI)*.*

Proof. By hypothesis, for any $v \in \mathrm{U}(\mathscr{A})$ there is a unique collection $s_a(v), a \in \mathscr{A}$, with $s_a(v) \in s_a$ such that $v = \sum_{a\in\mathscr{A}} s_a(v)$.

Let $(\mathscr{A}_j)_{j\in J}$ be a collection of elements of $\mathscr{U}(\mathscr{A})$, and $v \in \bigcap_{j\in J} \mathrm{U}(\mathscr{A}_j)$.

For any $j \in J$, $a \notin \mathscr{A}_j$, $s_a(v) = 0$. Therefore, for any $a \notin \bigcap_{j\in J} \mathscr{A}_j$, $s_a(v) = 0$. Therefore $v = \sum_{a\in \bigcap_{j\in J} \mathscr{A}_j} s_a(v)$.

$\bigcap_{j\in J} \mathscr{A}_j \in \mathscr{U}(\mathscr{A})$ and $\mathrm{U}(\bigcap_{j\in J} \mathscr{A}_j) = \sum_{a\in \bigcap_{j\in J} \mathscr{A}_j} s_a$, so $v \in \mathrm{U}(\bigcap_{j\in J} \mathscr{A}_j)$. The other inclusion is true whether or not U verifies (sI).

□

The rest of the chapter we will dedicated to proving that, under very general assumptions on the poset, if U satisfies the intersection property then it is decomposable.

3.4 Intersection Property implies decomposability over finite posets

Lemma 3.4.1. *Let* $(\mathrm{U}(x), x \in E)$ *be a any collection of vector subspaces of* V *and* $z \in E$.

$\sum_{x\in E} \mathrm{U}(a) \simeq \bigoplus_{x\in E} \mathrm{U}(a)$ *if and only if, for any* $x \in E$ *such that* $x \neq z$,

$$\mathrm{U}(x) \cap \sum_{\substack{y\in E\\ y\neq x}} \mathrm{U}(y) = 0 \tag{3.4.1}$$

Proof. Let $x_i, i \in [n]$ be a finite collection of elements of X. Let, $v_i \in \mathrm{U}(a_i), i \in [n]$, $v_z \in \mathrm{U}(z)$, such that $\sum_{i\in[n]} v_i + v_z = 0$. Then for any $i \in [n]$, $v_i = -\sum_{j\neq i} v_j$, so $v_i \in \left(\sum_{j\neq i} \mathrm{U}(x_j)\right) \cap \mathrm{U}(x_i)$. Therefore by hypothesis for any $i \in [n]$, $v_i = 0$. And so $v_z = 0$.

□

Theorem 3.4.1. *Let* $\mathrm{U} \in \hom(\mathscr{A}, \boldsymbol{Gr}V)$, *with* $\mathscr{A}$ *finite and* V *any vector space.* U *is decomposable if and only if* U *verifies (C).*

Proof. The necessary condition is a direct consequence of Proposition (3.3.1).

<u>Sufficient condition:</u> Let us prove inductively on the height of the posets that property (C) implies U to be decomposable.

If $h(\mathscr{A}) = 0$, then $\mathscr{A} = \emptyset$. $\mathrm{U} = 0$ is decomposable[1].

Assume that for any poset of height lower than $n \in \mathbb{N}$, property (C) implies U decomposable. Let $\mathscr{A}$ be height $n+1$. For any $a \in \mathscr{A}$, let $W_a = \mathrm{U}(\hat{a}^*)$

Let $\mathscr{M}$ be the set of maximal elements of $\mathscr{A}$.

For $a \in \mathscr{M}$, let s_a be a supplementary of W_a, in other words, $\mathrm{U}(a) = s_a \oplus W_a$ and let[2] $\mathscr{B} = \mathscr{A} \setminus \mathscr{M}$.

Let us show that $((s_a)_{a\in\mathscr{M}}, \mathrm{U}(\mathscr{B}))$ verify the hypotheses of Lemma.(3.4.1).

[1]By convention $\sum_{i\in\emptyset} V_i = 0$

[2]Let us remark that $\mathscr{B}$ is full.

Let $a \in \mathscr{M}$, $v \in s_a \cap \left(\sum_{\substack{b \in \mathscr{M} \\ b \neq a}} s_b + \mathrm{U}(\mathscr{B}) \right)$.

As a is a maximal element in $\mathscr{A}$, $\overset{\neg\vee}{a} = \mathscr{A} \setminus a$. In particular[3], $\sum_{\substack{b \in \mathscr{M} \\ b \neq a}} s_b + \mathrm{U}(\mathscr{B}) \subseteq \mathrm{U}(\mathscr{A} \setminus a)$.

Therefore, $v \in \mathrm{U}(\hat{a}^*)$. However by construction, $s_a \cap \mathrm{U}(\hat{a}^*) = 0$. So $v = 0$.

Therefore by Lemma.(3.4.1), $\sum_{a \in \mathscr{M}} s_a + \mathrm{U}(\mathscr{B}) \simeq \bigoplus_{a \in \mathscr{M}} s_a \oplus \mathrm{U}(\mathscr{B})$.

Furthermore $\mathrm{U}(\mathscr{A}) = \sum_{a \in \mathscr{A}} \mathrm{U}(a) = \sum_{a \in \mathscr{M}} \mathrm{U}(a) + \mathrm{U}(\mathscr{B})$. Let us recall that, for $a \in \mathscr{M}$, $\mathrm{U}(a) \subseteq s_a + \mathrm{U}(\mathscr{B})$. So $\mathrm{U}(\mathscr{A}) \subseteq \sum_{a \in \mathscr{M}} s_a + \mathrm{U}(\mathscr{B})$. The other inclusion is also true so, $\mathrm{U}(\mathscr{A}) = \sum_{a \in \mathscr{M}} s_a + \mathrm{U}(\mathscr{B})$.

Therefore, $\mathrm{U}(\mathscr{A}) \simeq \bigoplus_{a \in \mathscr{M}} s_a \oplus \mathrm{U}(\mathscr{B})$. Furthermore, $\mathscr{B}$ is of height n so $\mathrm{U}(\mathscr{B})$ is decomposable. Therefore there is $s_b, b \in \mathscr{B}$ such that $\sum_{b \in \mathscr{B}} s_b \simeq \bigoplus_{b \in \mathscr{B}} s_b$ and for any $b \in \mathscr{B}$, $\mathrm{U}(b) = \sum_{\substack{c \in \mathscr{B} \\ c \leq b}} s_b$.

Hence $\sum_{a \in \mathscr{A}} s_a \simeq \bigoplus_{a \in \mathscr{A}} s_a$.

Furthermore as $\mathscr{B} \in \mathscr{U}(\mathscr{A})$, $\mathrm{U}(b) = \sum_{\substack{c \in \mathscr{B} \\ c \leq b}} s_b = \sum_{\substack{c \in \mathscr{A} \\ c \leq b}} s_b$, which shows that U is decomposable and ends the induction.

□

Corollary 3.4.1. *Let* $\mathrm{U} \in \hom(\mathscr{A}, \boldsymbol{Gr}\, V)$*, with* $\mathscr{A}$ *finite and* V *any vector space.* U *is decomposable if and only if* U *verifies (I).*

Proof. If U verifies (I) then it verifies (C), as (C) requires a subset of the conditions required for (I) to hold. □

Corollary 3.4.2. *If* $\mathscr{A}$ *is finite, condition (I) and (C) are equivalent.*

Proof. We just saw that if $\mathrm{U} \in \hom(\mathscr{A}, \mathbf{Gr}\, V)$ verifies (I) it verifies (C). If U satisfies (C) then U is decomposable by Theorem.(3.4.1). Therefore, by Proposition.(3.3.1), U satisfies (C). □

[3]The following statement is even an equality.

3.5 Decomposition when $\mathscr{A}$ is not finite

3.5.1 Counter example and predecompositions

Remark 3.5.1. Let $\mathscr{A} = \mathbb{N}$ and V a vector space. For any $a \in \mathscr{A}$ let $\mathrm{U}(a) = V$. U verifies (I). This example is a counter example to Theorem (3.4.1) for non finite posets, in other words for infinite posets (I) does not necessarily imply U decomposable.

Let us first state what always holds when U has the intersection property.

Definition 3.5.1. For $a \in \mathscr{A}$, $\pi_a : \mathrm{U}(a) \twoheadrightarrow \mathrm{U}(a)/\,\mathrm{U}(\hat{a}^*)$ is surjective. Let s_a be a section of π_a and let us also note S_a the image of this section: $S_a = s_a(\mathrm{U}(a)/\,\mathrm{U}(\hat{a}^*))$. We call any such collection $S_a, a \in \mathscr{A}$ a pre-decomposition of U.

Proposition 3.5.1. *Let* $\mathrm{U} \in \hom(\mathscr{A}, \boldsymbol{Gr}\,V)$ *decomposable, and* $s_a, a \in \mathscr{A}$ *a decomposition of* U. $s_a, a \in \mathscr{A}$ *is a pre-decomposition of* U.

Proof. Let $s_a, a \in \mathscr{A}$ be a decomposition of U. Then for any $v \in \mathrm{U}(a)$, there is a unique $u \in s_a$ and $w \in \mathrm{U}(\hat{a}^*)$ such that $v = u + w$. Let $s_a(v) = u$.

For any $v \in \mathrm{U}(\hat{a}^*)$, $s_a(v) = 0$. Therefore s_a factorizes through π_a and $s_a = s_a^{'} \circ \pi_a$. Furthermore for any $v \in \mathrm{U}(a)$,

$$s_a^{'}(v \mod \mathrm{U}(\hat{a}^*)) \mod \mathrm{U}(\hat{a}^*) = [s_a(v)] = [v] \tag{3.5.1}$$

Therefore $s_a^{'}$ is a section of π_a, which ends the proof. □

Remark 3.5.2. $\mathrm{U} \in \hom(\mathscr{A}, \mathbf{Gr}\,V)$ is said to verify the property (C1) if and only if for any $n \in \mathbb{N}$, $a \in \mathscr{A}$ and finite collection $a_i, i \in [n]$ such that for any $i \in [n], a_i \in \overset{\neg\vee}{a}$,

$$\mathrm{U}(a) \cap \mathrm{U}(\bigcup_{i \in [n]} \hat{a}_i) \subseteq \mathrm{U}(\hat{a}^*) \tag{C1}$$

By construction (C) and (C1) are equivalent. In practice one proves (C1) in order to show (C) which under certain assumptions on $\mathscr{A}$ (Corollary (3.5.3)) implies (I).

Proposition 3.5.2. *Let* $\mathrm{U} \in \hom(\mathscr{A}, \boldsymbol{Gr}\,V)$ *and* $(S_a, a \in \mathscr{A})$ *a pre-decomposition of* U. *Let us suppose that* U *verifies (C), then,*

$$\sum_{a \in \mathscr{A}} S_a \simeq \bigoplus_{a \in \mathscr{A}} S_a \tag{3.5.2}$$

Proof. Let us prove by induction on n that for any collection $a_i, i \in [n]$ of elements of $\mathscr{A}$, $\sum\limits_{i\in I} S_{a_i} \simeq \bigoplus\limits_{i\in I} S_{a_i}$. We will use the second form of proof by induction (see Remark (3.1.3)).

Let $n \in \mathbb{N}$, and suppose that for any $n' < n$ and any collection $a_i, i \in [n']$ of elements of $\mathscr{A}$, $\sum\limits_{i\in[n']} S_{a_i} \simeq \bigoplus\limits_{i\in I} S_{a_i}$.

Let $a_i, i \in [n]$ be a collection of elements of $\mathscr{A}$.

$\{a_i | i \in [n]\}$ is a finite poset. Let $\mathscr{M}$ be the set of its maximal elements and respectively $J = a^{-1}(\mathscr{M})$. Let $\overline{\mathscr{M}} = a([n] \setminus J)$.

Let $i \in J$. For any $j \neq i$, $a_j \in \overset{\neg\vee}{a}_i$. Furthermore, any $b \in \overline{M}$ is in any $\overset{\neg\vee}{a}_j$. Indeed, let $b \in \overline{M}$, then there is $j \in J$ such that $b \leq a_j$; if there is $k \in J$ such that $a_k \leq b$, then $a_k \leq a_j$ which is contradictory with a_j being a maximal element.

Therefore, $\sum\limits_{\substack{j\in J\\ j\neq i}} S_{a_j} + \mathrm{U}(\overline{M}) \subseteq \sum\limits_{b\in \overset{\neg\vee}{a}} \mathrm{U}(b)$ and $S_{a_i} \cap \left(\sum\limits_{\substack{j\in J\\ j\neq i}} S_{a_j} + \mathrm{U}(\overline{M}) \right) = 0$.

Therefore by Lemma (3.4.1), $\sum\limits_{i\in J} S_{a_i} + \mathrm{U}(\overline{M}) \simeq \bigoplus\limits_{i\in J} S_{a_i} \oplus \mathrm{U}(\overline{M})$.

$|[n] \setminus J| < n + 1$ then by induction $\sum\limits_{i\in[n]\setminus J} S_{a_i} \simeq \bigoplus\limits_{i\in[n]\setminus J} S_{a_i}$.

Finaly, $\sum\limits_{i\in[n]\setminus J} S_{a_i} \subseteq \mathrm{U}(\overline{M})$, therefore $\sum\limits_{i\in[n]} S_{a_i} \simeq \bigoplus\limits_{i\in[n]} S_{a_i}$. Which ends the proof.

□

Corollary 3.5.1. $\mathrm{U} \in \hom(\mathscr{A}, \boldsymbol{Gr}V)$ *is decomposable if and only if any pre-decomposition of* U *is a decomposition.*

Remark 3.5.3. If U has one pre-decomposition that is a decomposition, then any pre-decomposition is a decomposition.

Remark 3.5.4. Corollary (3.5.1) is in fact the easiest way to build decompositions and move decompositions around through increasing functions, as example, one can see the proof of Proposition (3.6.3).

We therefore need to find a condition on $\mathscr{A}$ for which any U that verifies (C) also verifies that for all $a \in \mathscr{A}$, $\mathrm{U}(a) = \sum_{b\in\hat{a}} S_b$. A simple way to do so would be to reiterate the proof of Theorem (3.4.1) but in a more general setting. We would like to have a stronger version of the proof by induction and one can do so if $\mathscr{A}$ is well-founded.

3.5.2 Well-founded posets

A case of well-founded posets that we will often meet are graded posets.

Definition 3.5.2. Let $\mathscr{A}$ be a poset. Assume that there is $r : \mathscr{A} \to \mathbb{N}$ a strictly increasing function, in other words,

$$\forall a, b \in \mathscr{A}, a < b \implies r(a) < r(b) \tag{3.5.3}$$

and that for any $a, b \in \mathscr{A}$, such that $a < b$ and such that there is no $c \in \mathscr{A}$ such that $a < c < b$, $r(b) = r(a) + 1$.

Then one says that $\mathscr{A}$ is graded and r is called the rank of $\mathscr{A}$.

Proposition 3.5.3. *Let $\mathscr{A}$ be a poset, $\mathscr{B}$ a well-founded posed and $r : \mathscr{A} \to \mathscr{B}$ a stricly increasing function. Then $\mathscr{A}$ is well-founded.*

Therefore any graded poset is well-founded.

Proof. Let $a_i, i \in \mathbb{N}$ be a strictly decreasing sequence of elements of $\mathscr{A}$. $r(a_i)$ is a strictly decreasing sequence sequence of elements of $\mathscr{B}$. Therefore there is $i \in \mathbb{N}$ such that, $\forall j \neq i$, $r(a_j) > r(a_i)$.

Therefore for any $j \in \mathbb{N}$, $j \neq i$, $\neg(a_j < a_i)$. As $a_i, i \in N$ is a chain it is a total order. Therefore for any $j \in \mathbb{N}$, $j \neq i$, $a_j \geq a_i$, and as $a_j \neq a_i$, one has that $a_j > a_i$. Therefore any $j > i$ is strictly greater than i, which is contradictory.

There is no infinite strictly descending chain in $\mathscr{A}$.

□

3.5.3 Main Theorem for Well-founded posets

Theorem 3.5.1. *Let $\mathscr{A}$ be a well-founded poset and V any vector space.* $\mathrm{U} \in \hom(\mathscr{A}, \boldsymbol{Gr}\, V)$ *is decomposable if and only if* U *verifies (C).*

Proof. The necessary condition is a direct consequence of Proposition (3.3.1).

Let $\mathrm{U} \in \hom(\mathscr{A}, \mathbf{Gr}\, V)$, and $s_a, a \in \mathscr{A}$ be a pre-decomposition of U.

Let us show by induction on $a \in \mathscr{A}$, that for any $a \in \mathscr{A}$, $\mathrm{U}(a) = \sum_{b \leq a} s_a$. We will use the second form of proof by induction (see Remark.(3.1.3)).

Let us suppose that for any $b < a$, $\mathrm{U}(b) = \sum_{c \leq b} s_b$.

By construction, $\mathrm{U}(a) = s_a + \mathrm{U}(\hat{a}^*)$.

By hypothesis, for any $b < a$, $\mathrm{U}(b) = \sum_{c \leq b} s_c$. Therefore $\mathrm{U}(\hat{a}^*) = \sum_{b<a} \sum_{c \leq b} s_c \subseteq \sum_{b<a} s_b$. The other inclusion holds by definition of a pre-decomposition.

Therefore, $\mathrm{U}(a) = \sum_{b \leq a} s_a$.

As $\mathscr{A}$ is well-founded we can conclude (Proposition (3.1.1)) that for any $a \in \mathscr{A}$, $\mathrm{U}(a) = \sum_{b \leq a} s_a$.

Furthermore by Proposition (3.5.2),

$$\sum_{a \in \mathscr{A}} s_a \simeq \bigoplus_{a \in \mathscr{A}} s_a \tag{3.5.4}$$

Therefore, U is decomposable.

□

Corollary 3.5.2. *Let* $\mathrm{U} \in \hom(\mathscr{A}, \boldsymbol{Gr}\, V)$*, with* $\mathscr{A}$ *well-founded and* V *any vector space.*

U *is decomposable if and only if* U *verifies (I).*

Proof. If U verifies (I) then it verifies (C) as (C) requires a subset of the conditions required for (I) to hold. □

Corollary 3.5.3. *If* $\mathscr{A}$ *is well-founded, condition (I) and (C) are equivalent.*

Proof. We just saw that if $\mathrm{U} \in \hom(\mathscr{A}, \mathbf{Gr}\, V)$ verifies (I) it verifies (C). If U satisfies (C) then U is decomposable by Theorem (3.5.1). Therefore, by Proposition (3.3.1), U satisfies (I). □

3.6 A bit more around intersection and decompositions

3.6.1 A condition equivalent to (I)

If one considers any poset $\mathscr{A}$ it is not clear whether condition (C) implies (I) however there is still a property easier to verify than property (I) that is equivalent to (I), for any poset $\mathscr{A}$.

Definition 3.6.1. Let $\mathscr{A}$ be any poset, V any vector space. $\mathrm{U} \in \hom(\mathscr{A}, \mathbf{Gr}\, V)$ is said to verify the property (C2) if and only if for any $n \in \mathbb{N}$, $a \in \mathscr{A}$ and finite collection $(a_i, i \in [n])$,

$$\mathrm{U}(a) \cap \mathrm{U}(\bigcup_{i\in[n]} \hat{a}_i) \subseteq \mathrm{U}(\hat{a} \cap \bigcup_{i\in[n]} \hat{a}_i) \tag{C2}$$

Proposition 3.6.1. *For any poset $\mathscr{A}$ condition (I) and (C2) are equivalent.*

Proof. The proof of the sufficient condition is an adaptation of the one of Theorem 2.3.1 (Lemma 2.3.1 is in fact condition (C2)).

Let $\mathrm{U} \in \hom(\mathscr{A}, \mathbf{Gr}\, V)$.

Let $n_1 \in \mathbb{N}$, $a_i, i \in [n_1]$ be a collection of elements of $\mathscr{A}$. Let us prove by induction on n that for any $b_i, i \in [n]$ collection of elements of $\mathscr{A}$,

$$\mathrm{U}(\bigcup_{i\in[n_1]} \hat{a}_i) \cap \mathrm{U}(\bigcup_{i\in[n]} \hat{b}_i) \subseteq \mathrm{U}(\bigcup_{i\in[n_1]} \hat{a}_i \cap \bigcup_{i\in[n]} \hat{b}_i) \tag{3.6.1}$$

We will use the second form of proof by induction (see Remark (3.1.3)).

Let us assume that for any $m < n$, and any $c_i, i \in [m]$ collection of elements of $\mathscr{A}$,

$$\mathrm{U}(\bigcup_{i\in[n_1]} \hat{a}_i) \cap \mathrm{U}(\bigcup_{i\in[m]} \hat{c}_i) \subseteq \mathrm{U}(\bigcup_{i\in[n_1]} \hat{a}_i \cap \bigcup_{i\in[m]} \hat{c}_i) \tag{3.6.2}$$

Let $b_i, i \in [n]$ a collection of elements of $\mathscr{A}$.

Let $i \in [n]$, and $v \in \mathrm{U}(\bigcup_{i\in[n]} \hat{b}_i) \cap \mathrm{U}(\bigcup_{i\in[n_1]} \hat{a}_i)$, there is $v_i, i \in [n]$ such that for any $i \in [n], v_i \in \mathrm{U}(b_i)$ and $u_i, i \in [n_1]$ such for any $i \in [n_1], u_i \in \mathrm{U}(a_i)$, such that,

$$v = \sum_{j\in[n]} v_j = \sum_{j\in[n_1]} u_j$$

One has that,

$$v_i = \sum_{j\in[n_1]} u_j - \sum_{\substack{j\in[n]\\ j\neq i}} v_j$$

Pose $\mathscr{A}_1 = \bigcup_{j\in[n_1]} \hat{a}_j$, $\mathscr{B}_1 = \bigcup_{\substack{j\in[n]\\ j\neq i}} \hat{b}_j$.

Therefore, $v_i \in \mathrm{U}(b_i) \cap \mathrm{U}(\mathscr{A}_1 \cup B_1)$ and so $v_i \in \mathrm{U}(\hat{b}_i \cap (\mathscr{A}_1 \cup B_1))$.

Therefore, $v_i \in \mathrm{U}((\hat{b}_i \cap \mathscr{A}_1) \cup (\hat{b} \cap B_1)) = \mathrm{U}(\hat{b}_i \cap \mathscr{A}_1) + \mathrm{U}(\hat{b}_i \cap \mathscr{B}_1)$ and in particular there is $w_1 \in \mathrm{U}(\hat{b}_i \cap \mathscr{A}_1)$, $w_2 \in \mathrm{U}(\mathscr{B}_1)$ such that $v_i = w_1 + w_2$.

Furthermore, $\sum\limits_{\substack{j\in[n]\\ j\neq i}} v_j + w_2 = \sum\limits_{j\in[n_1]} u_j - w_1$.

But $\sum\limits_{\substack{j\in[n]\\ j\neq i}} v_j + w_2 \in \mathrm{U}(\mathscr{B}_1)$ and $\sum\limits_{j\in[n_1]} u_j - w_1 \in \mathrm{U}(\mathscr{A}_1)$.

One also has that $|\{b_j| \quad j \neq i\}| < n$ then by the induction hypothesis $\sum\limits_{\substack{j\in[n]\\ j\neq i}} v_j + w_2 \in \mathrm{U}(\mathscr{A}_1 \cap \mathscr{B}_1)$.

Therefore $v \in \mathrm{U}(\bigcup\limits_{i\in[n]} \hat{b}_i \cap \bigcup\limits_{i\in[n_1]} \hat{a}_i)$, which ends the proof.

□

3.6.2 Fonctoriality of decomposability

Notation 3.6.1. Let $\mathscr{A}$ be a poset and $\mathscr{B} \subset \mathscr{A}$. Let U be an increasing function from $\mathscr{A}$ to $\mathbf{Gr}\, V$. Let us note $\mathrm{U}|_{\mathscr{B}} \in \hom(\mathscr{B}, \mathbf{Gr}\, V)$, the restriction of U to $\mathscr{B}$.

Proposition 3.6.2. *Let $\mathscr{A}$ be a poset and $\mathscr{B} \in \mathscr{U}(\mathscr{A})$, let $\mathrm{U} \in \hom(\mathscr{A}, \boldsymbol{Gr}V)$ decomposable, then $\mathrm{U}|_{\mathscr{B}}$ is decomposable. If U verifies (I) so does $\mathrm{U}|_{\mathscr{B}}$.*

Proof. Suppose that U is decomposable, let $(S_a, a \in \mathscr{A})$ be a decomposition of U, then for any $a \in \mathscr{B}$,

$$\mathrm{U}(a) = \bigoplus_{\substack{b\in\mathscr{A}\\ b\leq a}} S_a = \bigoplus_{\substack{b\in\mathscr{B}\\ b\leq a}} S_a \tag{3.6.3}$$

Suppose that U satisfies (I). Any $\mathscr{C} \in \mathscr{U}(\mathscr{B})$ is also in $\mathscr{U}(\mathscr{A})$ therefore $\mathrm{U}|_{\mathscr{B}}$ is decomposable. □

Proposition 3.6.3. *Let $f : \mathscr{A} \to \mathscr{B}$ be an order-embedding and $\mathrm{U} \in \hom(\mathscr{A}, \boldsymbol{Gr}V)$.*

U is decomposable if and only if $f_*\,\mathrm{U}$ is decomposable.

Furthermore, any decomposition of $f_*\,\mathrm{U}$ induces, by restriction to $f(\mathscr{A})$, a decomposition of U; in other words, if $s_b^{\mathscr{B}}, b \in \mathscr{B}$ is a decomposition of $f_*\,\mathrm{U}$ then

$s_a^{\mathscr{A}} = s_{f(a)}^{\mathscr{B}}, a \in \mathscr{A}$ *is a decomposition of* U.

Any decomposition of U *can be extended in a decomposition of* $f_* \mathrm{U}$. *Let* $s_a^{\mathscr{A}}, a \in \mathscr{A}$ *be a decomposition of* U *then,*

$$s_b^{\mathscr{B}} = \begin{cases} s_a^{\mathscr{A}} & \text{if } \exists a \in \mathscr{A}, \quad b = f(a) \\ 0 & \text{otherwise} \end{cases} \tag{3.6.4}$$

is a decomposition of $f_* \mathrm{U}$.

Proof. Let $s_b^{\mathscr{B}}, b \in \mathscr{B}$ be a pre-decomposition of $f_* \mathrm{U}$. As for any $a \in \mathscr{A}$, $f_* \mathrm{U}(f(a)) = \mathrm{U}(a)$ and by definition, $\mathrm{U}(\hat{a}^*) = f_* \mathrm{U}(\overset{\circ}{f(a)})$, one has by construction that $s_a^{\mathscr{A}} = s_{f(a)}^{\mathscr{B}}, a \in \mathscr{A}$ is a pre-decomposition of U.

Assume U is decomposable, then (Corollary (3.5.1)) for any $a \in \mathscr{A}$, $f_* \mathrm{U}(f(a)) = \mathrm{U}(a) = \sum_{a_1 \in \hat{a}} s_{a_1}^{\mathscr{A}}$ and $\mathrm{U}(f(a)) = \sum_{b_1 \in f(\hat{a})} s_{b_1}^{\mathscr{B}}$. One has that $\sum_{b_1 \in f(\hat{a})} s_{b_1}^{\mathscr{B}} \subseteq \sum_{b_1 \in \widehat{f(a)}} s_{b_1}^{\mathscr{B}}$.

Furthermore, for any $b \in \mathscr{B}$, $f_* \mathrm{U}(b) = \sum_{a : i(a) \leq b} \mathrm{U}(a)$. Then one has that, $f_* \mathrm{U}(b) \subseteq \sum_{a : i(a) \leq b} s_a^{\mathscr{A}} \subseteq \sum_{b_1 \in \hat{b}} s_{b_1}^{\mathscr{B}}$.

As, by Proposition (3.5.2), $\sum_{b \in \mathscr{B}} s_b^{\mathscr{B}} = \bigoplus_{b \in \mathscr{B}} s_b^{\mathscr{B}}$, we can conclude that $f_* \mathrm{U}$ is decomposable.

If $f_* \mathrm{U}$ is decomposable, then $s_b^{\mathscr{B}}, b \in \mathscr{B}$ is a decomposition of $i_* \mathrm{U}$ (Corollary (3.5.1)). One remarks that,

$$s_b^{\mathscr{B}} = \begin{cases} s_a^{\mathscr{A}} & \text{if } \exists a \in \mathscr{A}, \quad b = f(a) \\ 0 & \text{otherwise} \end{cases} \tag{3.6.5}$$

Let $a \in \mathscr{A}$, $\mathrm{U}(a) = \sum_{b \in \widehat{f(a)}} s_b^{\mathscr{B}}$. Furthermore from Equation (3.6.5), one can conclude that $\sum_{b \in \widehat{f(a)}} s_b^{\mathscr{B}} = \sum_{a_1 \in \hat{a}} s_{a_1}^{\mathscr{A}}$.

Therefore $\mathrm{U}(a) = \sum_{a \in \hat{a}} s_a^{\mathscr{A}}$ and by Proposition (3.5.2) one concludes that U is decomposable. □

Corollary 3.6.1. $\mathrm{U} \in \hom(\mathscr{A}, \boldsymbol{Gr} V)$ *is decomposable if and only if* $i_* \mathrm{U} \in \hom(\mathscr{U}(\mathscr{A}), \boldsymbol{Gr} V)$ *is decomposable.*

Proof. i is an order-embedding, so one can conclude thanks to Proposition (3.6.3). □

Corollary 3.6.2. *Let* $\mathrm{U} \in \hom(\mathscr{A}, \boldsymbol{Gr}V)$ *be decomposable. Let* $\mathscr{B} \subseteq \mathscr{U}(\mathscr{A})$*, such that* $i(\mathscr{A}) \subseteq \mathscr{B}$*. Then* $i_* \mathrm{U}|_{\mathscr{B}}$ *is decomposable.*

Proof. i restricted on its codomain to $\mathscr{B}$ is an order-embedding. □

3.7 Interaction decomposition for collections of random variables

Let us first recall some results from Chapter 2. Let I denote a finite set. The poset $\mathscr{A}$ will be $\mathscr{P}(I)$. Let for all $i \in I$, E_i be **any** non empty set. $E = \prod_{i \in I} E_i$ is a set of functions on I. For $x \in E$, one has that $pr_i(x) = x(i)$, and for $a \subseteq I$ non empty, we will note $x_{|a}$ as x_a. We will call $E_a = \prod_{i \in a} E_i$ and,

$$\begin{array}{rcccl} \pi_a & : & E & \to & E_a \\ & & x & \mapsto & x_a \end{array}$$

Let $*$ be a given singleton. Then there is only one application of domain E to $*$ that we call $\pi_\emptyset$; we pose $x_\emptyset = \pi_\emptyset(x)$. Pose $\mathscr{F} = \mathbb{R}^E$.

Definition 3.7.1 (Factor subspaces)**.** For any $a \in \mathscr{P}(I)$, let $\mathrm{U}(a)$ be the vector subspace of $\mathscr{F}$ constituted of functions f that can be factorised by π_a, in other words there is $\tilde{f}$ such that $f = \tilde{f} \circ \pi_a$.

As explained in Section 2.3, U is an increasing function of $\hom(\mathscr{P}(I), \mathbf{Gr}(\mathscr{F}))$, furthermore Theorem 2.3.1 asserts that for any $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{U}(\mathscr{P}(I))$,

$$\mathrm{U}(\mathscr{B}_1) \cap \mathrm{U}(\mathscr{B}_2) \subseteq \mathrm{U}(\mathscr{B}_1 \cap \mathscr{B}_2) \tag{3.7.1}$$

Corollary 3.7.1. *[Decomposition of interactions]* U *is decomposable.*

Proof. As the intersection property (I) (Equation (3.7.1)) holds for U, by Theorem (3.4.1), U is decomposable.

□

Remark 3.7.1. We will call any choice of decomposition of U a decomposition of interactions. In Appendix B Proposition B.4 of [14] a decomposition is chosen with respect to the euclidian scalar product, and other ones can be found in [20].

Let now I be any set. Let $E = \prod_{i \in I} E_i$ for any collection of sets $E_i, i \in I$. Let us note $\mathscr{P}_f(I)$ the set of subsets of I which are finite and once again $\mathscr{F} = \mathbb{R}^E$.

Corollary 3.7.2. $\mathscr{P}_f(I)$ *is well-founded.*

Proof. The cardinal function on $\mathscr{P}_f(I)$ to $\mathbb{N}$, $|.| : a \to |a|$ is a rank for $\mathscr{P}_f(I)$ and $\mathscr{P}_f(I)$ is graded. Therefore, Corollary (3.5.3), $\mathscr{P}_f(I)$ is well-founded. □

For $a \in \mathscr{P}_f(I)$. $\mathrm{U}(a)$ will still be the set of functions from E to $\mathbb{R}$ that factorise through π_a (Section 2.4) and $\mathrm{U} \in \hom(\mathscr{P}_f(I), \mathbf{Gr}(\mathscr{F}))$.

Proposition 3.7.1. U *is decomposable.*

Proof. Corollary 2.4.1 states that U verifies the intersection property (I) therefore as $\mathscr{P}_f(I)$ is well-founded, Corollary (3.5.2) tells us that U is decomposable. □

Let us conclude this section by remarking that one can extend U to $\mathscr{P}(I)$ and that this extension is decomposable even though $\mathscr{P}(I)$ is not necessarily well-founded.

Definition 3.7.2. (Generalized Factor subspaces)
One can always inject $\mathscr{P}(I)$ in $\mathscr{U}(\mathscr{P}_f(I))$ by the increasing function[4],

$$\begin{array}{rcccc} j & : & \mathscr{P}(I) & \to & \mathscr{U}(\mathscr{P}_f(I)) \\ & & a & \mapsto & \{b \in \mathscr{P}_f(I) | b \leq a\} \end{array}$$

For $a \in \mathscr{P}(I)$, let $H(a) = i_* \mathrm{U}(j(a))$. This definition of H follows from Definition 4 [34] and Proposition 4 [34] ensures $H \in \hom(\mathscr{P}(I), \mathbf{Gr}(\mathscr{F}))$.

Proposition 3.7.2. *H is decomposable.*

Proof. Let us identify $\mathscr{P}(I)$ with $j(\mathscr{P}(I))$, then H can be identified to $i_* \mathrm{U}_{|j(\mathscr{P}(I))}$. As $i(\mathscr{P}_f(I)) \subseteq j(\mathscr{P}(I))$, by Corollary (3.6.2), H is decomposable.

□

3.8 Conclusion

We would like to conclude by opening on a possible extension of Theorem (3.5.1). And to do so let us first remark that for any poset $\mathscr{A}$ and vector space V, if one considers for any $a \in \mathscr{A}$, $\mathrm{U}(a) = 0$, U is decomposable. Therefore the condition we imposed on the poset are not the optimal ones as there are some $\mathrm{U} \in \hom(\mathscr{A}, \mathbf{Gr}(V))$ that do not verify them but still are decomposable.
As for now we do not know how (sI) relates to U being decomposable other than by 3.3.1.

[4]Let us recal that $\hat{a}$ is not defined as $a \notin \mathscr{P}_f(I)$. Furthermore j is even an order-embedding

Chapter 4

Interaction decomposition for presheaves

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Abstract

Consider a collection of vector subspaces of a given vector space and a collection of projectors on these vector spaces, can we decompose the vector space into a product of vector subspaces such that the projectors are isomorphic to projections? We provide an answer to this question by extending the relation between the intersection property and the interaction decomposition ([31]) to the projective case. This enables us to classify the decompositions of interactions for factor spaces. We then extend these results for presheaves from a poset to the category of modules by adding the data of a section functor when it exists.

4.1 Introduction

4.1.1 Motivation

In the previous chapter we started from the decomposition into interaction subspaces of factor spaces and looked for the general setting and a characterization of such construction. Let us consider once again this construction but this time with a different perspective from the previous chapter. Let I and $E = \prod_{i\in I} E_i$ be a finite sets; for $a \subseteq I$, let $\pi_a : \mathbb{R}^E \to \mathbb{R}^E$ be the orthogonal projection on the $a-$factor space with respect to the canonical scalar product on $\mathbb{R}^E$; in Lauritzen's proof of the decomposition into interaction subspaces, in Appendix B of his reference book Graphical Models ([14]) it is stated that if $(s_a, a \in \mathscr{A})$ is such that for any $a \in \mathscr{A}$,

$$\pi_a = \sum_{b\leq a} s_b \tag{4.1.1}$$

then for any $a, b \in \mathscr{A}$,

$$s_a s_b = 1[a = b]s_a \tag{4.1.2}$$

In particular, from the previous equation it is easy to show that for any $a, b \in \mathscr{A}$ such that $b \leq a$,

$$\pi_a \pi_b = \pi_{a \cap b} \tag{I}$$

We will say that $(\pi_a, a \in \mathscr{A})$ is decomposable and that $(s_a, a \in \mathscr{A})$ is its canonical decomposition. In this chapter we prove that condition (I) implies that a collection of projectors is decomposable. More generaly any collection of projectors $(\pi_a, a \in \mathscr{A})$ that is such that the collection $(s_a, a \in \mathscr{A})$ defined by Equation 4.1.1 satisfies Equation 4.1.2 will be called decomposable.

4.1.2 Main results

Decomposable presheaf

The collection of projectors we are interested in can be encoded by a presheaf, F, from a poset $\mathscr{A}$ to the category of vector spaces $(k)-\mathbf{Vect}$ with respect to functor $G : \mathscr{A} \to \mathbf{Vect}$, with k a commutative field, such that F is a retraction of G. In most generality we consider couples of functor/ presheaf, (G, F), from a poset $\mathscr{A}$ to the category of $R-$modules, where R is a commutative ring.

Definition (Category **Split**)**.** Let $\mathbf{C}$ be any category, let $\mathbf{Split}(\mathbf{C})$ be the subcategory of $\mathbf{C} \times \mathbf{C}^{op}$ that has as objects (M, M) with M an obejct of $\mathbf{C}$ $(R-)$ and for any two objects of $\mathbf{C}$,M, M_1, the morphism of **Split** are given by

$$\mathbf{Split}(\mathbf{C})\left((M, M), (M_1, M_1)\right) = \{(s, r) \in \mathbf{Mod}(V, V_1) \times \mathbf{Mod}(V_1, V) : rs = \mathrm{id}\} \tag{4.1.3}$$

A functor from a poset $\mathscr{A}$ to $\mathbf{Split}(\mathbf{C})$ can be encoded by a couple of functor/presheaf, (G, F), from a poset $\mathscr{A}$ to $\mathbf{C}$, such that for any $a, b \in \mathscr{A}$ such that $b \leq a$,

$$F_b^a G_a^b = \mathrm{id} \tag{4.1.4}$$

We will consider two categories, the category of modules **Mod** and its subcategory for which all morphisms are isomporhisms. For a collection of functors $(G_a : \mathscr{A} \to \mathbf{Mod}, a \in \mathscr{A})$ form $\mathscr{A}$ to **Mod** and a collection of presheaves $(F_a : \mathscr{A} \to \mathbf{Mod}, a \in \mathscr{A})$ we will denote $\prod_a G_a 1[a \leq .]$ and respectively $\prod_a F_a 1[a \leq .]$ the functors and presheaves defined as follows, for any $a \in \mathscr{A}$,

$$\prod_b G_b 1[b \leq .](a) = \prod_{b \leq a} G_b(a) \tag{4.1.5}$$

$$\prod_b F_b 1[b \leq .](a) = \prod_{b \leq a} F_b(a) \tag{4.1.6}$$

and the morphisms are respectively the associated inclusions and projections, in other words for $b, d \in \mathscr{A}$ such that $b \leq a$, for $v \in \bigoplus_{c \leq b} G_c(b)$ and $w \in \bigoplus_{c \leq a} G_c(a)$ then,

$$\prod_c G_c 1[c \leq .]_a^b(v)(d) = G_{ca}^{\ b}(v_c) 1[d \leq b] \tag{4.1.7}$$

$$\prod_c F_c 1[c \leq .]_b^a(w)(d) = F_{cb}^{\ a}(w_c) 1[d \leq b] \tag{4.1.8}$$

Definition 4.1.1 (Core of a category)**.** The core of a category $\mathbf{C}$, $\operatorname{core}\mathbf{C}$, is the subcategory of $\mathbf{C}$ that has the sames objects than $\mathbf{C}$ but whose morphisms are the isomorphisms of $\mathbf{C}$.

We will note **Split**(**Mod**) simply as **Split**.

Definition (Decomposable functors to **Split**)**.** Let H be a functor from $\mathscr{A}$ to **Split**. H is decomposable when there is a collection $((G_a, F^a), a \in \mathscr{A})$ of functors from $\mathscr{A}$ to **Split**(core **Mod**) such that,

$$H \cong (\prod_c G_c 1[c \leq .], \prod_c F_c 1[c \leq .]) \tag{4.1.9}$$

When H is decomposable we shall call $(\prod_{a \in \mathscr{A}} G_a 1[a \leq .], \prod_{a \in \mathscr{A}} F^a 1[a \leq .])$ its decomposition and note it as $(\prod_{a \in \mathscr{A}} S_a, \prod_{a \in \mathscr{A}} S^a)$.

4.1.3 Intersection property for functors from $\mathscr{A}$ to Split

In order to state the intersection property we must consider posets that are such that $\hat{a}$ is finite for any $a \in \mathscr{A}$; we will note the class of these posets as $\hat{\mathscr{P}}_f$.

Definition (Intersection property)**.** Let (G, F) be a functor from $\mathscr{A} \in \hat{\mathscr{P}}_f$ to **Split**. For any $\alpha, a \in \mathscr{A}$ such that $a \leq \alpha$, let $\pi_{\alpha a}^{\alpha\alpha} = G_\alpha^a F_a^\alpha$, $\pi_{\alpha a}^{\alpha\alpha}$ is a projector. For a given α, we shall denote this collection as π^α. Let for any $\alpha \in \mathscr{A}$, $(s_a^\alpha, a \leq \alpha)$ be characterized by,

$$\pi_a^\alpha = \sum_{b \leq a} s_a^\alpha \tag{4.1.10}$$

(G, F) is said to satisfy the intersection property for any $\alpha \in \mathscr{A}$ and any $a, b \leq \alpha$,

$$\pi_a^\alpha \pi_b^\alpha = \sum_{\substack{c : c \leq a \\ c \leq b}} s_c \tag{I}$$

When the poset $\mathscr{A}$ is meet semi-lattice for any two elements $a, b \in \mathscr{A}$ there is meet $a \wedge b$; we will note this meet as $a \cap b$. In this case, (I) becomes, for any $\alpha \in \mathscr{A}$ and any $a, b \leq \alpha$,

$$\pi_a^\alpha \pi_b^\alpha = \pi_{a\cap b}^\alpha \tag{4.1.11}$$

Main theorem

Theorem 4.4.1. Let (G, F) be a functor from $\mathscr{A} \in \hat{\mathscr{P}}_f$ to **Split**, (G, F) statisfies the intersection property if and only if (G, F) is decomposable.

A corollary of this equivalence theorem is that one can generalize the interaction decomposition for factor spaces by characterizing the probability distributions for which one can build a decomposition.

Corollary (Interaction Decomposition for factor spaces)**.** *Let I be a finite set, $(E_i, i \in I)$ a collection of finite sets, and $\mathbb{P}$ a probability measure on E, $(E_a[.|\mathscr{F}_a], a \in \mathscr{P}(I))$ is decomposable if and only if $\mathbb{P}$ is a product measure, i.e there is $(p_i \in \mathbb{P}(E_i), i \in I)$ such that $\mathbb{P} = \bigotimes_{i\in I} p_i$.*

4.1.4 Structure of this document

This chapter is composed of mainly two parts. We first motivate the framework for decomposability in the context of collections of projectors and prove the equivalence theorem between intersection property and decomposability in this context (Section 4.24.3). This allows us to characterize the probability distribution that induces, by conditioning, a decomposable collection of projectors.

In the second part of this chapter we extend these results to functors (G, F) from $\mathscr{A} \in \hat{\mathscr{P}}_f$ to **Split** (Section 4.4). In order to state the intersection property we show that, by nesting in each element $\alpha \in \mathscr{A}$ the poset $\hat{\alpha}$ and by bringing back $F|_{\hat{\alpha}}$ in $F(a)$, we can use the definitions and results for collection of projectors in $F(\alpha)$ to extend them to functors to **Split**. We then show that from the decomposition that can be build for each α we can extract a decomposition for (G, F) proving therefore the main theorem of this chapter.

4.2 Decomposability for collections of projectors

4.2.1 Projections on factor spaces

As the interaction decomposition in [14] is stated for collection of random variables, let us first give some reasons for why solving this question can be interesting in probability; let I be a finite set, it indexes random variables, let for all $i \in I$, E_i be a finite set with the discrete σ-algebra, in which the i-th random variable takes its values, $E = \prod_{i \in I} E_i$ is the configuration space; for $\omega \in E$, one has that $pr_i(\omega) = \omega(i)$, and for $a \subseteq I$ non empty, we will note $\omega_{|a}$ as x_a. We will call $E_a = \prod_{i \in a} E_i$ and,

$$\begin{array}{rcccl} p_a & : & E & \to & E_a \\ & & x & \mapsto & x_a \end{array}$$

Let $*$ be a given singleton. Then there is only one application of domain E to $*$ that we call $\pi_\emptyset$; we pose $\omega_\emptyset = p_\emptyset(\omega)$. The σ-algebra one considers implicitly on $\prod_{i \in I} E_i$ will be the Borel algebra with respect to the product topology, i.e. the smallest algebra that makes the projections, for any $a \subseteq I$, π_a measurable, here as the E_i are finite and I is finite it coincides with the dicrete σ-algebra on E. Let us denote $\mathscr{F}_a$ the smallest σ-algebra that makes p_a measureable, i.e generated by the cylindric events $\{\omega_a\} := \{\omega_1 \in E \ : \omega_{1a} = \omega_a\}$.

Notation 4.2.1. For any measurable space $(E, \mathscr{F})$ let us denote $\mathscr{M}(E, \mathscr{F})$ the set of measurable function and $\mathscr{M}_b(E, \mathscr{F})$ the set of bounded measurable functions, i.e. $\mathscr{L}^\infty(E, \mathscr{F})$. Let $\mathbb{P}(E)$ be the set of probability measures of E.

Notation 4.2.2. We shall note the image measure $p_{a*}\mathbb{P}$, i.e. the marginalisation of $\mathbb{P}$ over $E_{\overline{a}}$, as $\mathbb{P}_a$.

Definition 4.2.1. For any $a \in \mathscr{P}(I)$, let $V(a)$ be the vector subspace of V constituted of functions f that can be factorised by p_a, in other words there is $\tilde{f}$ such that $f = \tilde{f} \circ p_a$. $V(a)$ is called the a-factor space, $V(a) = \mathscr{M}_b(E, \mathscr{F}_a)$.

As E is finite one can define for a probability $\mathbb{P}$ its support,

$$\operatorname{Supp} \mathbb{P} = \{\omega \in E \quad : \mathbb{P}(\omega) \neq 0\}$$

Let $\mathbb{P}$ be the probability measure on E associated to the collection of random variables I, for any $a \subseteq I$, let $\pi_a : V \to V(a)$ be such that $\pi_a = E_a[\quad |\mathscr{F}_a]$ the conditional expectation with respect to the a factor space; we take as convention that for any cylinder events if $\mathbb{P}(\{\omega_a\}) = 0$, $E_a[\quad |\mathscr{F}_a](\omega) = 0$; therefore for any $a \in \mathscr{A}$ and $f \in \mathscr{M}_b(E)$,

$$E[f|\mathscr{F}_a] = 1[\in \operatorname{Supp} \mathbb{P}_a] \sum_{\omega' : \omega'_a = \omega_a} \frac{\mathbb{P}(\omega')}{\mathbb{P}_a(\omega_a)} f(\omega') \tag{4.2.1}$$

4.2.2 Collection of projectors and functoriality

Notation 4.2.3. We shall note the set of endomorphisms of V as $\mathscr{L}(V)$. For $\pi = (\pi_i \in \mathscr{L}(V), i \in I)$ the collection $(\operatorname{im} \pi_i, i \in I)$ will be called the image of π and denoted as $\operatorname{im} \pi$.

Proposition 4.2.1. *Let π be a collection of projectors of V over $\mathscr{A}$. $\operatorname{im} \pi \in hom(\mathscr{A}, \boldsymbol{Gr} V)$ if and only if for any $a, b \in \mathscr{A}$ such that $b \leq a$, $\pi_a \pi_b = \pi_b$:*

Proof. Assume that $\operatorname{im} \pi_b \subseteq \operatorname{im} \pi_a$ for $b \leq a$ then $\pi_a \pi_b = \pi_b$; suppose $\pi_a \pi_b = \pi_b$ then $\operatorname{im} \pi_b \subseteq \operatorname{im} \pi_a$. □

Remark 4.2.1. When $\operatorname{im} \pi \in hom(\mathscr{A}, \mathbf{Gr} V)$, let for $b \leq a$, $G(a) = \operatorname{im} \pi_a$ and $G_a^b = i_{\operatorname{im} \pi_a}^{\operatorname{im} \pi_b}$; G is a functor from $\mathscr{A}$ to $\mathbf{Gr} V$, where $\mathbf{Gr} V$ has as morphisms inclusions; we shall call $G(\pi)$ the canonical functor associated to π.

Notation 4.2.4. For V a vector space, we shall note $\mathbf{sub}(V)$ the subcategory of subvector spaces of V; morphisms are any linear transformations from one subspace to another.

Definition 4.2.2. Let $\mathscr{A}$ be a poset, $\pi = (\pi_a, a \in \mathscr{A})$ a collection of endomorphisms of V. If for any $a, b \in \mathscr{A}$ such that $b \leq a$ there is $f_b^a \in \mathscr{L}(V)$ such that,

$$\pi_b = f_b^a \circ \pi_a \tag{4.2.2}$$

we will say that π is presheafable in $\mathbf{sub}(V)$.

Definition 4.2.3. For any poset $\mathscr{A}$, we shall call the nerve of $\mathscr{A}$, denoted as $N(\mathscr{A})$, the set of strictly increasing sequence of $\mathscr{A}$; in particular we shall call $N(\mathscr{A})_n$ the set of strictly increasing sequence of n-elements.

Example 4.2.1. $N(\mathscr{A})_2 = \{(a, b) \in \mathscr{A}^2 : a < b\}$.

Notation 4.2.5. Let X, A, Y, B be four sets such that $A \subseteq X$ and $B \subseteq Y$. Let $f : X \to Y$ be a function such that $f(A) \subset B$, we shall note $f|_A^B$ is restriction on A and B.

Proposition 4.2.2. *Let π be presheafable in $\boldsymbol{sub}(V)$ and $(f_a^b, (a, b) \in N(\mathscr{A})_1)$ a collection that satisfies Equation (4.2.2); let $F(\pi)_a^a = id$ and $F(\pi)_b^a = f_b^a|_{\operatorname{im} \pi_b}^{\operatorname{im} \pi_a}$ when $b \leq a$, $F(\pi)$ is a presheaf from $\mathscr{A}$ to $\boldsymbol{sub}(V)$ that we shall call the canonical presheaf associated to π.*

Proof. For any $(a,b) \in N(\mathscr{A})_1$, $\operatorname{im} f_b^a(\operatorname{im}\pi_a) \subseteq \operatorname{im}\pi_b$ therefore F_a^b is well defined. Furthermore for $a \geq b \geq c$ and $v \in V$, $F_c^b F_b^a \pi_a(v) = f_c^b \pi_b(v) = \pi_c(v) = F_c^a \pi_a(v)$; as π_a is surjective on its image, $F_c^b F_b^a = F_c^a$.

Let $(f_{1_a}^{\ b}, (a,b) \in N(\mathscr{A})_1)$ that satisfies Equation (4.2.2), for $b \leq a$, $f_b^a \pi_a = \pi_b = f_{1_b}^{\ a} \pi_b$ therefore $F_b^a = F_{1_b}^{\ a}$ which justifies why we can call F canonical.

□

Proposition 4.2.3. *Let $\mathscr{A}$ be a poset, $\pi = (\pi_a, a \in \mathscr{A})$ a collection of projectors of V. π is presheafable in $\boldsymbol{sub}(V)$ if an only if for any for any $a, b \in N(\mathscr{A})_1$, $\pi_a \pi_b = \pi_a$.*

Proof. Let us assume that π is presheafable, and let $(f_a^b, (a,b) \in N(\mathscr{A})_1)$ be such that Equation (4.2.2) holds for any $(a,b) \in N(\mathscr{A})_1$. Let $(a,b) \in N(\mathscr{A})_1$ and $v \in \operatorname{im}\pi_b$ then, $\pi_a \pi_b = f_a^b \pi_b \pi_b = f_a^b \pi_b = \pi_a$; let π be such that for $(a,b) \in N(\mathscr{A})_1$, $\pi_a \pi_b = \pi_a$ then by definition π is presheafable ($f_a^b = \pi_a$). □

Remark 4.2.2. A presheafable collection of projectors over $\mathscr{A}$, doesn't generally satisfy $\operatorname{im}\pi \in hom(\mathscr{A}, \mathbf{Gr}\, V)$ and a collection of projectors such that $\operatorname{im}\pi \in hom(\mathscr{A}, \mathbf{Gr}\, V)$ is not necessarily presheafable: let π_1, π_2 two projectors, $\pi_1\pi_2 = \pi_2$ is equivalent to having $\operatorname{im}\pi_2 \subseteq \operatorname{im}\pi_1$, as one can rewrite $\pi_1\pi_2 = \pi_2$ as $(id - \pi_1)\pi_2 = 0$; $\pi_1\pi_2 = \pi_1$ is equivalent to $\ker\pi_2 \subseteq \ker\pi_1$ as one can rewrite $\pi_1\pi_2 = \pi_1$ as $\pi_1(id - \pi_2) = 0$ which says that $\operatorname{im}(id - \pi_2) \subseteq \ker\pi_1$ and $\operatorname{im}(id - \pi_2) = \ker\pi_2$.

Proposition 4.2.4. *Let I be a finite set, $E = \prod_{i \in I} E_i$, $\mathbb{P}$ a probability measure on E; $(\pi_a = E[\ \ |\mathscr{F}_a], a \in \mathscr{P}(I))$ is such that $\operatorname{im}\pi \in hom(\mathscr{A}, \boldsymbol{Gr}\, V)$ and is presheafable.*

Proof. For any $f \in \mathscr{M}_b(E)$ and $a, b \in \mathscr{A}$ such that $b \leq a$, $E[E[f|\mathscr{F}_a]|\mathscr{F}_b] = E[f|\mathscr{F}_b]$ therefore by Proposition 4.2.3 π is presheafable; for $f \in \mathscr{F}_b$, $E[f|\mathscr{F}_a] = f1[. \in \operatorname{Supp}\mathbb{P}_b]$, $E[E[f|\mathscr{F}_b]|\mathscr{F}_a] = E[f|\mathscr{F}_b]1[. \in \operatorname{Supp}\mathbb{P}_b] = E[f|\mathscr{F}_b]$, therefore by Proposition 4.2.1. □

4.2.3 Decomposable collections of projectors implies decomposable collection of vector spaces

Notation 4.2.6. Let $\mathscr{A}$ be a poset, let $\mathscr{A}^+ = \mathscr{A} \oplus 1$ be the poset sum of $\mathscr{A}$ and the one element poset 1, i.e. any $a \in \mathscr{A}^+$ is in $\mathscr{A}$ or is equal to 1 and 1 is a final element.

Definition 4.2.4. Let V be a vector space, $\mathscr{A}$ be a finite poset and $(\pi_a, a \in \mathscr{A})$ a collection of endomorphisms of V. Let $\mathrm{U}(\pi)$ be the application from $\mathscr{A}^+$ to $\mathbf{Gr}\, V$ such that for any $a \in \mathscr{A}$, $\mathrm{U}(a) = \operatorname{im}\pi(a)$, $\mathrm{U}(1) = V$. We say that $(\pi_a, a \in \mathscr{A})$ is

decomposable if and only if there is a collection of vector subspaces of V, $(S_a, a \in \mathscr{A}^+)$ such that

1. $\bigoplus_{a\in\mathscr{A}^+} i_V^{S_a} : \bigoplus_{a\in\mathscr{A}^+} S_a \to V$ is an isomorphism; let us recall that for any $w = (w_a, a \in \mathscr{A}^1) \in \bigoplus_{a\in\mathscr{A}^+} S_a$, $\bigoplus_{a\in\mathscr{A}^+} i_V^{S_a}(w) = \sum_{a\in\mathscr{A}^+} w_a$.
2. for any $a \in \mathscr{A}$, $\pi_a(v) = \sum_{b\leq a} w_b$.

We shall note $\bigoplus_{a\in\mathscr{A}^+} i_V^{S_a}$ as ϕ; we shall say that $(S_a, a \in \mathscr{A})$ is the decomposition of π.

For I finite, $(E_i, i \in I)$ a collection of finite sets and $\mathbb{P} \in \mathbb{P}(E)$, we shall say that $(E[.|\mathscr{F}_a], a \in \mathscr{P}(I))$ admits an interaction decomposition if it is decomposable.

Let us recall what a decomposable collection of vector spaces is (Definition 2.1 3).

Definition 4.2.5. $(S_a, a \in \mathscr{A})$ is a decomposition of $\mathrm{U} \in \hom(\mathscr{A}, \mathbf{Gr}(V))$ if and only if,

1. for all $a \in \mathscr{A}$, $S_a \in \mathbf{Gr}(\mathrm{U}(a))$.
2. for all $a \in \mathscr{A}$, $\bigoplus_{b\leq a} i_{U(a)}^{S_b} : \bigoplus_{b\in\hat{a}} S_b \to \mathrm{U}(a)$ are isomorphisms and $\bigoplus_{a\in\mathscr{A}} i_V^{S_a}$ is an isomorphism on its image.

We shall note $i_{U(a)}^{S_b}$ as ϕ_a; we shall say that U is decomposable.

Proposition 4.2.5. *Let $\pi = (\pi_a, a \in \mathscr{A})$ be decomposable and $(S_a, a \in \mathscr{A}^+)$ the decomposition of π, let for any $a \in \mathscr{A}^+$, $\phi_a : \bigoplus_{b\leq a} S_a \to \mathrm{U}(\pi)(a)$ be such that $\phi_a(\bigoplus_{b\leq a} w_b) = \sum_{b\leq a} w_b$. Then $(\phi_a, a \in \mathscr{A})$ is a natural transformation from $\bigoplus S^a$ to $G(\pi)$ and a natural transformation from $\bigoplus S_a$ to $F(\pi)$, it is also an isomorphism, $\mathrm{U}(\pi)$ is decomposable and $(\pi_a, a \in \mathscr{A})$ is presheafable.*

Proof. For any $v \in \bigoplus_{b\in\mathscr{A}^+} S_b$, and $a \in \mathscr{A}$, $\phi^{-1}\pi_a(\phi(v)) = i^{\bigoplus_{b\leq a} S_b}_{\bigoplus_{b\in\mathscr{A}^+} S_b} pr^{\bigoplus_{b\in\mathscr{A}^+} S_b}_{\bigoplus_{b\leq a} S_b}(v)$ therefore $\phi(i^{\bigoplus_{b\leq a} S_b}_{\bigoplus_{b\in\mathscr{A}^+} S_b} \bigoplus_{b\leq a} S_b) = \mathrm{U}(a)$ and $\phi_a = \phi i^{\bigoplus_{b\leq a} S_b}_{\bigoplus_{b\in\mathscr{A}^+} S_b}|^{\mathrm{U}(a)}$ is well defined and is an isomorphism; from this remark one can conclude that $(\phi_a, a \in \mathscr{A})$ is a natural transformation from $\bigoplus S^a$ to $G(\pi)$ and a natural transformation from $\bigoplus S_a$ to $F(\pi)$, it is also an isomorphism, $\mathrm{U}(\pi)$ is decomposable. $(\pi_a, a \in \mathscr{A})$ is presheafable because $\mathrm{pr}^{\bigoplus_{b\in\mathscr{A}^+} S_b}_{\bigoplus_{b\leq a} S_b}$ is presheafable.

□

Definition 4.2.6 (Meet semi-lattice)**.** Let $\mathscr{A}$ be a poset, $a, b \in \mathscr{A}$. We shall say that $\mathscr{A}$ has a meet for (a, b) when there is d such that,

$$\forall c \in \mathscr{A}, \quad c \leq a \quad \& \quad c \leq b \implies c \leq d$$

d is unique and we shall note it $a \cap b$. We will call meet semi-lattice any poset that has all meets for any couple.

Proof. Let d and d_1 be two intersections for a, b, then $d \leq a$, $d \leq b$ and $d \leq d_1$ and by exchanging d and d_1 one gets $d_1 \leq d$ and therefore $d = d_1$. □

Example 4.2.2. Let I be any set then $\mathscr{P}(I)$ is a meet semi-lattice.

4.2.4 Necessary condition

Proposition 4.2.6. *Let $(\pi_b, b \in \mathscr{A})$ be decomposable and $(S_b, b \in \mathscr{A})$ its decomposition; let $v \in V$, $w \in \bigoplus_{b \in \mathscr{A}^+} S_b$ such that $v = \sum_{b \in \mathscr{A}^+} w_b$; for any $a \in \mathscr{A}^+$ pose $s_a : V \to V$ as $s_a(v) = w_a$. For any $a, b \in \mathscr{A}$, $s_a s_b = \delta_a(b) s_a$ and,*

$$\pi_a \pi_b = \sum_{\substack{c \leq a \\ c \leq b}} s_c$$

If $\mathscr{A}$ is a meet semi-lattice then,

$$\forall a, b \in \mathscr{A} \quad \pi_a \pi_b = \pi_{a \cap b} \tag{I}$$

Proof. By definition $s_a s_b = \delta_a(b) s_b$,

$$\pi_a \pi_b = \sum_{d \leq a} s_d (\sum_{c \leq b} s_c) = \sum_{d \leq a} \sum_{c \leq b} \delta_d(c) s_c = \sum_{\substack{c \leq a \\ c \leq b}} s_c$$

When $\mathscr{A}$is a meet semi-lattice $\pi_{a \cap b} = \sum_{c \leq a \cap b} s_c = \sum_{\substack{c \leq a \\ c \leq b}} s_c$.

□

We have defined what decomposability for a collection of endomorphisms is and shown that decomposable collections satisfy a meet or intersection property (I) when the poset is a meet semi-lattice. In the next section we will show that collections that satisfy the intersection property are decomposable and will characterize the probability measures, $\mathbb{P}$, for which $(E_{\mathbb{P}}[.|\mathscr{F}_a], a \in \mathscr{P}(I))$ admits an iteraction decomposition, i.e is decomposable.

4.3 Necessary and sufficient condition for the interaction decomposition of projectors from a finite poset to Vect

4.3.1 Zeta function and Mobius inversion with coefficient in modules

Definition 4.3.1. A poset $\mathscr{A}$ is locally finite if for any $b \leq a$, $[b, a] = \{c \in \mathscr{A} : b \leq c \leq a\}$ is finite.

Definition 4.3.2 (Zeta function)**.** Let $\mathscr{A}$ be a locally finite poset, let $\zeta : \bigoplus_{a \in \mathscr{A}} \mathbb{Z} \to \bigoplus_{a \in \mathscr{A}} \mathbb{Z}$ be such that for any $\lambda \in \bigoplus_{a \in \mathscr{A}} \mathbb{Z}$ and $a \in \mathscr{A}$,

$$\zeta(\lambda)(a) = \sum_{b \leq a} \lambda_b$$

ζ is the zeta function of $\mathscr{A}$.

Proposition 4.3.1 (Mobius inversion)**.** *The zeta function of a locally fintie poset $\mathscr{A}$ is invertible, we shall note μ its inverse and there is $f : \mathscr{A} \times \mathscr{A} \to \mathbb{Z}$ such that for any $\lambda \in \bigoplus_{a \in \mathscr{A}} \mathbb{Z}$ and $a \in \mathscr{A}$,*

$$\mu(\lambda)(a) = \sum_{b \leq a} f(a, b) \lambda_b$$

We shall note f as $\mu_{\mathscr{A}}$.

Proof. By applying Poposition 2 [25].

□

Definition 4.3.3. Let $\mathscr{A}$ be a locally finite poset and M a (R-)module; let $M_{\mathscr{A}} = \bigoplus_{a \in \mathscr{A}} M$. The *zeta* function of $\mathscr{A}$ with values in M, $\zeta_{\mathscr{A}}(M) : M_{\mathscr{A}} \to M_{\mathscr{A}}$, is such that for any $m \in M_{\mathscr{A}}$,

$$\zeta_{\mathscr{A}}(M)(m)(a) = \sum_{b \leq a} m_a \tag{4.3.1}$$

We shall note $\zeta_{\mathscr{A}}(M)$ as $\zeta_{\mathscr{A}}$ making the reference to M implicit.

Proposition 4.3.2. *Let $\mathscr{A}$ be a locally finite poset, M a module, $\zeta_{\mathscr{A}}(M)$ is invertible, we shall call its inverse the Mobius function with values in M and note is as $\mu_{\mathscr{A}}(M) : M_{\mathscr{A}} \to M_{\mathscr{A}}$. Furthermore,for any $m \in M_{\mathscr{A}}$ and $a \in \mathscr{A}$,*

$$\mu_{\mathscr{A}}(m)(a) = \sum_{b \leq a} \mu_{\mathscr{A}}(a, b) m_a \tag{4.3.2}$$

Proof. For any $m \in M$ and $b \in \mathscr{A}$, $\zeta_{\mathscr{A}}\mu_{\mathscr{A}}(m)(b) = \sum_{c \leq b}\sum_{a \leq c} \mu_{\mathscr{A}}(c,a)m_a$ and $\sum_a \sum_{c:a \leq c \leq b} \mu_{\mathscr{A}}(c,a)m_a = \sum_a \delta_b(a)m_a = m_b$ and similarly $\mu_{\mathscr{A}}\zeta_{\mathscr{A}} = id$.

□

Definition 4.3.4. Let $\mathscr{A}$ be a finite poset, let $(\pi_a, a \in \mathscr{A})$ be a collection of endomorphism of V. Let $\Pi((\pi_a, a \in \mathscr{A})) : V \to V_{\mathscr{A}}$ be such that for any $v \in V$, $\Pi(v)(a) = \pi_a(v)$. For any $a \in \mathscr{A}$ and $v \in V$ let $s_a(v) = \mu_{\mathscr{A}} \circ \Pi(v)(a)$, as $\zeta_{\mathscr{A}}\mu_{\mathscr{A}}\Pi = \Pi$,

$$\pi_a(v) = \sum_{b \leq a} s_b(v) \tag{4.3.3}$$

Remark 4.3.1. For $\mathscr{A}$ locally finite, $(s_a(v), a \in \mathscr{A})$ is in general not in $V_{\mathscr{A}}$, therefore we decide to restrict our attention to $\mathscr{A}$ finite for the moment.

4.3.2 Zeta function and functoriality

Notation 4.3.1. For any poset, let $\mathscr{U}(\mathscr{A})$ denote the set of lower-sets of $\mathscr{A}$, i.e subsets of $\mathscr{A}$ such that for any $a \in \mathscr{A}$,$b \in \mathscr{B}$ such that $a \leq b$ one has that $a \in \mathscr{B}$. We shall also call $\mathscr{U}(\mathscr{A})$ the poset topology of $\mathscr{A}$. For $a \in \mathscr{A}$, let $\hat{a} = \{b \in \mathscr{A} : b \leq a\}$, $\hat{a} \in \mathscr{U}(\mathscr{A})$.

Proposition 4.3.3. *Let $\mathscr{A}$ a locally finite poset, if $\mathscr{B} \in \mathscr{U}(\mathscr{A})$, then one has the following commutation relations,*

$$\zeta_{\mathscr{B}} pr^{\mathscr{A}}_{\mathscr{B}} = pr^{\mathscr{A}}_{\mathscr{B}} \zeta_{\mathscr{A}} \tag{4.3.4}$$

$$\mu_{\mathscr{B}} pr^{\mathscr{A}}_{\mathscr{B}} = pr^{\mathscr{A}}_{\mathscr{B}} \mu_{\mathscr{A}} \tag{4.3.5}$$

Proof. Let $\mathscr{B} \in \mathscr{U}(\mathscr{A})$, $b \in \mathscr{B}$ and $v \in V_{\mathscr{A}}$,

$$\zeta_{\mathscr{B}}(pr^{\mathscr{A}}_{\mathscr{B}}(v))(b) = \sum_{\substack{c \in \mathscr{B} \\ c \leq b}} v_c = \sum_{\substack{c \in \mathscr{A} \\ c \leq b}} v_c = \zeta_{\mathscr{A}}(v)(b)$$

Therefore $\zeta_{\mathscr{B}}\mathrm{pr}^{\mathscr{A}}_{\mathscr{B}} = \mathrm{pr}^{\mathscr{A}}_{\mathscr{B}}\zeta_{\mathscr{A}}$, therefore $\mathrm{pr}^{\mathscr{A}}_{\mathscr{B}} = \mu_{\mathscr{B}}\mathrm{pr}^{\mathscr{A}}_{\mathscr{B}}\zeta_{\mathscr{A}}$ and $\mathrm{pr}^{\mathscr{A}}_{\mathscr{B}}\mu_{\mathscr{A}} = \mu_{\mathscr{B}}\mathrm{pr}^{\mathscr{A}}_{\mathscr{B}}$.

□

Definition 4.3.5. Let $\mathscr{A}$ be a any poset and $\mathscr{B} \in \mathscr{U}(\mathscr{A})$,

$$W_{\mathscr{A}}(\mathscr{B}) = \{v \in V_{\mathscr{A}} : \forall a, c \in \mathscr{A}, \quad \hat{a} \cap \mathscr{B} = \hat{c} \cap \mathscr{B} \implies v_a = v_c\} \tag{4.3.6}$$

Example 4.3.1. Let $\mathscr{A} = \{0, 1, 1'\}$ where $0 \leq 1$ and $0 \leq 1'$; $W_{\hat{0}} = \{(v, v, v) : \quad v \in V\}$.

Proposition 4.3.4. *Let $\mathscr{A}$ be a locally finite poset, for any $\mathscr{B} \in \mathscr{U}(\mathscr{A})$, $\zeta_{\mathscr{A}}(V_{\mathscr{B}}) \subseteq W_{\mathscr{A}}(\mathscr{B})$. If there is $b \in \mathscr{A}$ such that $\mathscr{B} = \hat{b}$ and $\mathscr{A}$ has all its intersections $\zeta_{\mathscr{A}}(V_{\mathscr{B}}) = W_{\mathscr{B}}$*

Proof. Let $v \in V_{\mathscr{B}}$, for any $a \in \mathscr{A}$, $\zeta_{\mathscr{A}}(v)(a) = \sum\limits_{c \in \hat{a} \cap \mathscr{B}} v_a$. Therefore if $\hat{a} \cap \mathscr{B} = \hat{b} \cap \mathscr{B}$, $\zeta_{\mathscr{A}}(v)(a) = \zeta_{\mathscr{A}}(v)(b)$.

Let $u \in W_{\mathscr{A}}(\hat{b})$ and $v \in V_{\mathscr{A}}$ such that $u = \zeta_{\mathscr{A}}(v)$, for any $a \in \mathscr{A}$, $\zeta_{\mathscr{A}}(v1[. \in \hat{b}])(a) = \sum\limits_{c \in \hat{a} \cap \hat{b}} v_c = u_{a \cap b} = u_a = \zeta_{\mathscr{A}}(v)(a)$; therefore $v = v1[. \in \hat{b}]$.

□

Remark 4.3.2. Let us remark that $\zeta_{\mathscr{A}}(V_{\mathscr{B}})$ is not equal to $W_{\mathscr{A}}(\mathscr{B})$ in general; let us consider $\mathscr{A} = \{a, b, c, d\}$, with, $a \leq c$, $a \leq d$, $b \leq c$, $b \leq d$. Let $\mathscr{B} = \{a, b\}$, asking for $v \in W_{\mathscr{A}}(\mathscr{B})$ is the same than asking for $v_c = v_d$, which can always be fullfilled: let $u = (s_a, s_b, s_c, s_d)$ elements in V such that $s_c = s_d \neq 0$, then $\zeta(u) \in W_{\mathscr{A}}(\mathscr{B})$ but $u \notin V_{\mathscr{B}}$.

4.3.3 Intersection property implies decomposability for collection of projectors over a finite posets

For finite semi-lattices

Definition 4.3.6. Let $\mathscr{A}$ be a non necessarily finite join semi-lattice, let V be a vector spaces, a collection $(\pi_a \in \mathscr{L}(V), a \in \mathscr{A})$ is said to verify the intersection property if,

$$\forall a, b \in \mathscr{A}, \quad \pi_a \pi_b = \pi_{a \cap b} \tag{I}$$

Example 4.3.2. Let $\mathscr{A} = \{0, 1, 1'\}$, with $0 \leq 1$, $0 \leq 1'$; let $(V, <, >)$ be a Hilbert space, such that there are three closed subspaces $S_0, S_1, S_{1'}$ such that $V = S_0 \oplus_\perp S_1 \oplus_\perp S_{1'}$; let $V_0 = S_0$, $V_1 = S_0 \oplus S_1$, $V_{1'} = S_0 \oplus S_{1'}$. Let $\pi_0, \pi_1, \pi_{1'}$ be the orthogonal projection on respectively $V_0, V_1, V_{1'}$, then $\pi_1 \pi_{1'} = \pi_0 = \pi_{1 \cap 1'}$.

Proposition 4.3.5. *Let $\mathscr{A}$ be a finite join semi-lattice, and let $(\pi_a \in \mathscr{L}(V), a \in \mathscr{A})$ that satisfies the intersection property, then for any $a, b \in \mathscr{A}$,*

$$s_b \pi_a = 1[b \leq a] s_b$$

Proof. Let us remark that for any $b, c \in \mathscr{A}$ such that $c \leq b$, $\Pi_{\mathscr{A}} \circ \pi_b(c) = \Pi_{\mathscr{A}}(c)$; therefore for any $b \leq a$,

$$s_b \pi_a = \mathrm{pr}_b^{\hat{a}} \mathrm{pr}_{\hat{a}}^{\mathscr{A}} \mu_{\mathscr{A}} \Pi_{\mathscr{A}} \pi_a = \mathrm{pr}_b^{\hat{a}} \mu_{\hat{a}} \mathrm{pr}_{\hat{a}}^{\mathscr{A}} \Pi_{\mathscr{A}} \pi_a$$

and as noted just before $\mathrm{pr}_{\hat{a}}^{\mathscr{A}} \Pi_{\mathscr{A}} \pi_a = \mathrm{pr}_{\hat{a}}^{\mathscr{A}} \Pi_{\mathscr{A}}$ so $s_b \pi_a = s_b$. Furthermore for any $b \in \mathscr{A}$, $\pi_b \pi_a = \pi_{a \cap b} = \pi_{a \cap b} \pi_a$, therefore $\mathrm{im}\, \Pi_{\mathscr{A}} \pi_a \subseteq W_{\hat{a}}$ and so $s_b \pi_a = 1[b \leq a] s_b \pi_a$ and therefore $s_a \pi_b = 1[a \leq b] s_a$.

□

Proposition 4.3.6. *Let $\mathscr{A}$ be a finite join semi-lattice, V a vector space and $(\pi_a \in \mathscr{L}(V), a \in \mathscr{A})$ a collection that verifies the intersection property (I). Then for any $a, b \in \mathscr{A}$,*

$$s_a s_b = \delta_a(b) s_a \tag{4.3.7}$$

Proof. For any $a, b \in \mathscr{A}$,

$$s_a s_b = s_a \sum_{c \leq b} \mu(b, c) \pi_c(v) = \sum_{c \leq b} \mu(b, c) s_a \pi_c(v)$$

$$s_a s_b = \sum_{c \leq b} \mu(b, c) 1[a \leq c] s_a = s_a \sum_{a \leq c \leq b} \mu(b, c)$$

and $\sum_{a \leq c \leq b} \mu(b, c) = \delta_a(b)$, by definition of μ.

□

Remark 4.3.3. Proposition 4.3.5 simply relies on two remarks: firstly that for any $\mathscr{A}$ finite and $(\pi_a \in \mathscr{L}(V), a \in \mathscr{A})$, if for any $b \in \mathscr{A}$, $\mu_{\mathscr{A}}(\mathrm{im}\, \Pi \circ \pi_b) \subseteq V_{\hat{b}}$ then for any $a, b \in \mathscr{A}$ such that $b \leq a$, $s_b \pi_a = 1[b \leq a] s_b \pi_a$; secondly that if $(\pi_a, a \in \mathscr{A})$ is a collection of projectors, for any $b \leq a$, $s_b \pi_a = s_b$, as shown in the proof of 4.3.5. For this reason it seems natural to define as extension of the intersection property for a collection of projectors of V over any finite poset to be that $\mu_{\mathscr{A}}(\mathrm{im}\, \Pi \circ \pi_b) \subseteq V_{\hat{b}}$; this would still allow us to get Theorem 4.3.6. Showing (I) when the poset is a join semi-lattice is however much easier than when it isn't.

For finite lattices

Definition 4.3.7. [Intersection property] Let $\mathscr{A}$ be a finite poset, we shall say that a collection of projectors of V, $(\pi_a, a \in \mathscr{A})$, satisfies the intersection property if,

$$\forall b \in \mathscr{A}, \quad \mu_{\mathscr{A}}(\mathrm{im}\, \Pi \circ \pi_b) \subseteq V_{\hat{b}} \tag{I}$$

Lemma 4.3.1. *Condition (I) is equivalent to asking that for any $v \in V$ and any $a, b \in \mathscr{A}$, $\pi_a \pi_b = \sum\limits_{c \in \hat{a} \cap \hat{b}} s_c$, and for $\mathscr{A}$ a finite join semi-lattice $\pi_{a \cap b} = \sum_{c \in \hat{a} \cap \hat{b}} s_c$.*

Proposition 4.3.7. *Let $\mathscr{A}$ be a finite poset, $(\pi_a, a \in \mathscr{A})$ a collection of projectors of V that verifies the intersection property; for any $a, b \in \mathscr{A}$,*

$$s_a s_b = \delta_a(b) s_a \tag{4.3.8}$$

Proof. As $\mu_{\mathscr{A}}(\operatorname{im} \Pi \circ \pi_b) \subseteq V_{\hat{b}}$, for any $a \in \mathscr{A}$ and $v \in V$, $(s_b(\pi_a(v)) \in V_{\mathscr{B}}$, therefore for any $b \in \mathscr{B}$, $s_b \pi_a = 1[b \leq a] s_b \pi_a$. For any $b, c \in \mathscr{A}$ such that $c \leq b$, $\Pi_{\mathscr{A}} \circ \pi_b(c) = \Pi_{\mathscr{A}}(c)$; therefore for any $b \leq a$,

$$s_b \pi_a = \mathrm{pr}_b^{\hat{a}} \mathrm{pr}_{\hat{a}}^{\mathscr{A}} \mu_{\mathscr{A}} \Pi_{\mathscr{A}} \pi_a = \mathrm{pr}_b^{\hat{a}} \mu_{\hat{a}} \mathrm{pr}_{\hat{a}}^{\mathscr{A}} \Pi_{\mathscr{A}} \pi_a = \mathrm{pr}_{\hat{a}}^{\mathscr{A}} \Pi_{\mathscr{A}}$$

Therefore $s_b \pi_a = s_b$; the end of the proof is exactly the same than the one of Theorem 4.3.6.

□

Remark 4.3.4. Let us call projectors of $(R-)$modules endomorphims π of an module M such that $\pi^2 = \mathrm{id}$; it is possible to replace the vector space V by an $(R-)$modules M in the definition of the intersection property (Definition 4.3.7) and the proof of Proposition 4.3.7 still holds for an $(R)-$module M instead of a $(k-)$vector space V. We choose to start by a presentation of the equivalence theorem in the cases of vectors spaces as one usually encounters projectors as endomorphisms of vector spaces; if π is a projector of a module M then it splits, i.e. it admits a section, as its image and kernel are in direct sum, this does not happen in general for modules.

Corollary 4.3.1. *Let $\mathscr{A}$ be a finite poset, V a vector space and $(\pi_a, a \in \mathscr{A})$ a collection of projectors of V that verifies the intersection property (I). Let for $a \in \mathscr{A}$, $S_a = \operatorname{im} s_a$, let $S(\mathscr{A}) = \bigoplus_{a \in \mathscr{A}} S_a$ and $S(\hat{a}) = \bigoplus_{b \leq a} S_b$ then $\zeta_{\mathscr{A}}|_{\operatorname{im} \pi}^{S(\mathscr{A})}$ and $\zeta_{\hat{a}}|_{\operatorname{im} \pi_a}^{S(\hat{a})}$ are isomorphisms.*

Proof. For any collection $(\pi_a \in \mathscr{L}(V), a \in \mathscr{A})$ one has that $\operatorname{im} \zeta_{\mathscr{A}} \subseteq \bigoplus\limits_{b \in \mathscr{A}} S_b$ and for $a \in \mathscr{A}$, $\operatorname{im} \zeta_{\hat{a}} \subseteq \bigoplus\limits_{b \leq a} S_b$; when $(\pi_a, a \in \mathscr{A})$ is a collection of projectors that verifies (I), from Proposition 4.3.7, for any $b \leq a$,$\pi_a s_b = \sum\limits_{c \leq a} s_c s_b = s_b$ and $S_b \subseteq \operatorname{im} \pi_a$, so $\operatorname{im} \zeta_{\mathscr{A}} = \bigoplus\limits_{b \in \mathscr{A}} S_b$ and $\operatorname{im} \zeta_{\hat{a}} = \bigoplus\limits_{b \leq a} S_b$. As $\zeta_{\mathscr{A}}$, $\zeta_{\hat{a}}$ are injective (Proposition 4.3.1), $\zeta_{\mathscr{A}}|_{\operatorname{im} \pi}^{\bigoplus_{b \in \mathscr{A}} S_b}$ and $\zeta_{\hat{a}}|_{\operatorname{im} \pi_a}^{\bigoplus_{b \leq a} S_b}$ are isomorphisms.

□

Theorem 4.3.1 (Intersection property and decomposability)**.** *Let $\mathscr{A}$ be a finite poset, V a vector space; a collection of projections of V, $(\pi_a, a \in \mathscr{A})$, satisfies the intersection property if and only if it is decomposable.*

Proof. If $(\pi_a, a \in \mathscr{A})$ is decomposable Proposition 4.2.6 and Remark 4.3.1 imply that it satisfies (I).

For any $v \in V$, let $s_1(v) = v - \sum_{a \in \mathscr{A}} s_a(v)$, for any $a \in \mathscr{A}$, $s_a s_1(v) = s_a(v) - s_a(v) = 0 = s_1 s_a(v)$. Let $S_1 = \operatorname{im} s_1$ then by Proposition 4.3.7, $(S_b, b \in \mathscr{A}^+)$ is a decomposition of U and for any $v \in V$, and $a \in \mathscr{A}^+$, $\pi_a(v) = \sum_{b \leq a} s_a(v)$, where $\pi_1 = id$.

□

4.3.4 Characterizing interaction decompositions for factor spaces

Corollary 4.3.2 (Interaction Decomposition for factor spaces)**.** *Let I be a finite set, $(E_i, i \in I)$ a collection of finite sets, and $\mathbb{P}$ a probability measure on E, $(E_a[.|\mathscr{F}_a], a \in \mathscr{P}(I))$ is decomposable if and only if $\mathbb{P}$ is a product measure, i.e there is $(p_i \in \mathbb{P}(E_i), i \in I)$ such that $\mathbb{P} = \bigotimes_{i \in I} p_i$.*

Proof. $\mathscr{P}(I)$ possesses all its intersections; if $(\pi_a = E_a[.|\mathscr{F}_a], a \in \mathscr{P}(I))$ is decomposable, then by Corollary 4.3.1, for any $a, b \in \mathscr{A}$, $\pi_a \pi_b = \pi_{a \cap b}$. Therefore for any $i \in I$ and $f \in \mathscr{M}_b(E, \mathscr{F})$, $\pi_{\{i\}} \pi_{\{i\}^c}(f) = \pi_\emptyset(f) = E[f]$, and so $\mathbb{P}$ is a product measure.

Let $\mathbb{P}$ be a product measure, then for any $a, b \in \mathscr{A}$, $\pi_a \pi_b = \pi_{a \cap b}$ and by Theorem 4.3.6 $(E_a[.|\mathscr{F}_a], a \in \mathscr{P}(I))$ is decomposable. □

4.4 Interaction Decomposition for presheaves in Mod

4.4.1 The idea behind the extension and the idea of the proof

We saw that collections of projectors can be encoded by a presheaf in Proposition 4.2.2; in this section we will extend the previous results to presheaves. Let us consider a presheaf $F : \mathscr{A} \to \mathbf{Vect}$ from a finite poset $\mathscr{A}$ to the category of vector spaces, in order to use the previous results (Proposition 4.3.7, Theorem 4.3.1) we

must have a way to bring back the spaces $F(b)$ into $F(a)$ for $a, b \in \mathscr{A}$ such that $b \leq a$ in such a way that F^a_b seen inside $F(a)$ is a projector. If one considers a morphism $\pi : V \to V_1$ between two vector spaces then for any section, s, of π, $s\pi$ is projector. It turns out that when considering the additional data of a functor $G : \mathscr{A} \to \mathbf{Vect}$ that is a section of F, in other words for any $b \leq a$, $F^a_b G^b_a = \mathrm{id}$, one can characterize intrinsically presheaves over a poset that are isomorphic to the presheaf of a projections of a direct sum and there is a canonical way to define this ismorphism for such presheaves. In this context there is no reason to restrict our attention to vector spaces, as every F^a_b is a split epimorphism by construction, i.e it admits a section, and we consider instead presheaves in $R-$ modules where R is a commutative ring.

4.4.2 The category of splittings

Definition 4.4.1 (Category **Split**)**.** Let $\mathbf{C}$ be any category, $\mathbf{Split}(\mathbf{C})$ is the subcategory of $\mathbf{C} \times \mathbf{C}^{op}$ that has as objects (M, M) with M an object of $\mathbf{C}$ and for any M, M_1 two objects of $\mathbf{C}$, $\mathbf{Split}\left((M, M), (M_1, M_1)\right)$ are couples of morphisms, (s, r), with $s : M \to M_1$, $r : M_1 \to M$ such that,

$$rs = \mathrm{id} \tag{4.4.1}$$

Proof. Let $(s, r) \in \mathbf{Split}\left((M, M), (M_1, M_1)\right)$, $(s_1, r_1) \in \mathbf{Split}\left((M_1, M_1), (M_2, M_2)\right)$, $rr_1s_1s = \mathrm{id}$.

□

Notation 4.4.1. When shall note $\mathbf{Split}(\mathbf{Mod})$ simply as $\mathbf{Split}$.

Remark 4.4.1. Let π_1, π_2 be the two projections from respectively $\mathbf{Split} \to \mathbf{Mod}$, $\mathbf{Split} \to \mathbf{Mod}^{op}$ defined as $\pi_1(V, V) = \pi_2(V, V) = V$ for V and object of $\mathbf{Split}$ and for a morphism of $\mathbf{Split}$, $\pi_1(s, r) = s$, $\pi_2(s, r) = r$. Any functor H from a poset $\mathscr{A}$ to $\mathbf{Mod} \times \mathbf{Mod}^{op}$ defines a couple of functor/presheaf $(\pi_1 H, \pi_2 H)$ and for any couple of functor/presheaf (G, F) there is a unique a functor from $\mathscr{A}$ to $\mathbf{Mod} \times \mathbf{Mod}^{op}$, H, such that $\pi_1 H = G$, $\pi_2 H = F$; similarly any functor H from a poset $\mathscr{A}$ to $\mathbf{Split}$ defines a couple of functor/presheaf $(\pi_1 H, \pi_2 H)$, any couple (G, F) of functor/presheaf from $\mathscr{A}$ to $\mathbf{Mod}$ defines a functor from $\mathscr{A}$ to $\mathbf{Split}$ if and only if for any $a, b \in \mathscr{A}$ such that $b \leq a$, $F^a_b G^b_a = \mathrm{id}$. From now on when we refer to a functor from $\mathscr{A} \to \mathbf{Split}$ we shall refer to its couple $(\pi_1 H, \pi_2 H)$.

Notation 4.4.2. Let $\mathbf{C}, \mathbf{C}_1$ be two categories we will denote the category of functors from $\mathbf{C}$ to $\mathbf{C}_1$ as $\mathbf{C}_1^{\mathbf{C}}$.

Notation 4.4.3. Let $\mathbf{C}$, $\mathbf{C}_1$, $\mathbf{C}_2$ be three categories, let $F, G : \mathbf{C} \to \mathbf{C}_1$ be two functor, $H : \mathbf{C}_1 \to \mathbf{C}_2$ a functor and $\phi : F \to G$ a natural transformation. We shall note $H \star \phi$ their whiskering (Appendix A Definition A.3.5 [35]).

Remark 4.4.2. Let $\mathbf{C}$ be any category, let (G, F), (G_1, F_1) be two functors from $\mathscr{A}$ to $\mathbf{Split}(\mathbf{C})$ and let $\rho = (\phi, \psi) \in \mathbf{Split}(\mathbf{C})^{\mathscr{A}}[(G, F), (G_1, F_1)]$ be a natural transformation of these functors. For any $a \in \mathscr{A}$, $\psi_a \phi_a = \mathrm{id}$ and $\phi \in \mathbf{C}^{\mathscr{A}}(G, G_1)$, $\psi \in \mathbf{C}^{\mathscr{A}}(F_1, F)$, in other words $\pi_1 \star \rho$ is a natural transformation from G to G_1 and $\pi_2 \star \rho$ is a natural transformation from F_1 to F. Now if ϕ or ψ was an isomorphism then (ϕ, ϕ^{-1}) would be a morphisms from (G, F) to (G_1, F_1) and respectively (ψ^{-1}, ψ); furthermore if ϕ is an isomorphism it also a natural tranformation from F to F_1 and respectively if ψ is an isomorphism it is a natural transformation from G to G_1; when one of the two options is true the reference to its inverse will be implicit, for example we will just say that ϕ is a natural transformation from (G, F) to (G_1, F_1).

Proposition 4.4.1. *Let $(G, F), (G_1, F_1)$ be two functors from $\mathscr{A}$ to **Split** and $\phi : (G, F) \to (G_1, F_1)$ be a natural transformation. Let $\operatorname{im} \phi = (\operatorname{im} \pi_1 \star \phi, \operatorname{im} \pi_2 \star \phi)$, $\operatorname{im} \phi$ is a functor from $\mathscr{A}$ to **Split**.*

Proof. Let $g = \operatorname{im} \pi_1 \star \phi$, $f = \operatorname{im} \pi_2 \star \phi$, for any $a, b \in \mathscr{A}$ such that $a \geq b$, and $v \in g(a)$, $f_b^a g_a^b(v) = F_{1_b}^{\ a} G_{1_a}^{\ b}(v) = v$.

□

Proposition 4.4.2. *Let I be any set and $((G_i, F_i), i \in I)$ a collection of functors from $\mathscr{A}$ to **Split**; $(\bigoplus_{i \in I} G_i, \bigoplus_{i \in I} F_i)$ and $(\prod_{i \in I} G_i, \prod_{i \in I} F_i)$ are functors from $\mathscr{A}$ to **Split**.*

Proof. Let $g = \prod_{i \in I} G_i$, $f = \prod_{i \in I} F_i$, let $a, b \in \mathscr{A}$ such that $b \leq a$, let $v \in \prod_{i \in I} G_i(a)$, for any $j \in I$, $f_b^a g_a^b(v)(j) = F_{j_b}^{\ a} G_{j_a}^{\ b}(v_j) = v_j$. □

Proposition 4.4.3. *Let (G, F) be a functor from a poset $\mathscr{A}$ to **Split**, for any $a \geq b \geq c$,*

$$F_b^a G_a^c = G_b^c$$

Proof. $F_b^a G_a^c = F_b^a G_a^b G_b^c = G_b^c$.

□

Proposition 4.4.4. *Let (G, F) be a functor from $\mathscr{A}$ to **Split** and (G_1, F_1) a functor from $\mathscr{A}$ to $\mathbf{Mod} \times \mathbf{Mod}^{op}$; if there is an epimorphism ϕ from (G, F) to (G_1, F_1) then (G_1, F_1) is a functor from $\mathscr{A}$ to **Split**.*

Proof. Let $a, b \in \mathscr{A}$ such that $b \leq a$, one has that $F_{1_b}^{\ a} G_{1_a}^{\ b} \phi_b = \phi_b F_b^a G_a^b = \phi_b$ and as ϕ_b is an epimorphism, $F_{1_b}^{\ a} G_{1_a}^{\ b} = \mathrm{id}$. □

4.4.3 Decomposable functors to Split

Definition 4.4.2. Let $\mathscr{A}$ be any poset and $G : \mathscr{A} \to \mathbf{Mod}$ be a functor. For any $a, b, c \in \mathscr{A}$, let $G1[a \leq .](b) = G(b)$ if $b \geq a$ and otherwise $G1[a \leq .](b) = 0$; for $a \leq c \leq b$, $G1[a \leq .]^c_b = G^c_b$ if $a \not\leq c$ and $c \leq b$, $G1[a \leq .]^c_b = 0$ (as 0 is initial in **Mod**). $G1[a \leq .]$ is a functor from $\mathscr{A}$ to **Mod**.

Similarly let F be a presheaf from $\mathscr{A}$ to **Mod**, let for any $a, b \in \mathscr{A}$ and $a \leq b$, $F1[a \leq .](b) = F(b)$ and otherwise $F1[a \leq .](b) = 0$, for $a \leq c \leq b$, $F1[a \leq .]^b_c = F^b_c$. $F1[a \leq .]$ is a presheaf.

Proof. Let $a, b, c, d \in \mathscr{A}$ such that $d \leq c \leq b$, if $a \leq d$ then $G1[a \leq .]^c_b G1[a \leq .]^d_c = G^c_b G^d_c = G^d_b = G1[a \leq .]^d_b$ and $F1[a \leq .]^c_d F1[a \leq .]^b_c = F^c_d F^b_c = F1[a \leq .]^b_d$; if $a \not\leq d$ then $G1[a \leq .]^c_b G1[a \leq .]^d_c = G1[a \leq .]^c_b 0 = G1[a \leq .]^d_b$, $F1[a \leq .]^c_d F1[a \leq .]^b_c = 0F1[a \leq .]^b_c = F1[a \leq .]^b_d$.

□

Remark 4.4.3. Let G be functor from a poset $\mathscr{A}$ to a core(**Mod**), let $(G^a : G(a) \to \operatorname{colim} G, a \in \mathscr{A})$ be the colimit cone over G; $(G^a, a \in \mathscr{A})$ is a monomorphism from G to the constant functor $\operatorname{colim} G$. Therefore G is isomorphic to the image of $(G^a, a \in \mathscr{A})$, that we denote as G_1. Let us call the connected components of $\mathscr{A}$ the equivalence classes for the equivalence relation generated by the order on $\mathscr{A}$; for any connected component of $\mathscr{A}$, C, there is a module M_C such that $G_1|_C$ is the constant functor M_C.

Definition (Decomposable functors to **Split**)**.** Let H be a functor from $\mathscr{A}$ to **Split**. H is decomposable when there is a collection $((G_a, F^a), a \in \mathscr{A})$ of functors from $\mathscr{A}$ to **Split**(core **Mod**) such that,

$$H \cong (\prod_a G_a 1[a \leq .], \prod_a F_a 1[a \leq .]) \tag{4.4.2}$$

When H is decomposable we shall call $(\prod_{a \in \mathscr{A}} G_a 1[a \leq .], \prod_{a \in \mathscr{A}} F^a 1[a \leq .])$ its decomposition and note it as $(\prod_{a \in \mathscr{A}} S_a, \prod_{a \in \mathscr{A}} S^a)$.

Remark 4.4.4. A presheaf from $\mathscr{A}$ to **Mod** is decomposable when there is a collection of presheaves, W^a, from $\mathscr{A}$ to **Split**(core **Mod**) such that F is isomorphic to $\prod_{a \in \mathscr{A}} W^a 1[a \leq .]$; in other words a presheaf is decomposable when there is a section G of F such that $(G, F) : \mathscr{A} \to \mathbf{Split}$ is decomposable. In the example of the interaction decomposition for factor spaces, G is the poset of factor subspaces indexed by the parts of a finite set and the presheaves that are considered are the projections on the subspaces for a given scalar product; decomposability of G is the subject of Chapter 2, decomposable couples (G, F) are seen as specific projections

on G. This is the reason why we think that the good definition of decomposable presheaves is the definition of decomposable functors from a poset to **Split**.

Corollary 4.4.1. *Let H be a decomposable functor from $\mathscr{A}$ to **Split** and $(\prod_{a\in\mathscr{A}} S_a, \prod_{a\in\mathscr{A}} S^a)$ its decomposition, $(\prod_{a\in\mathscr{A}} S_a, \prod_{a\in\mathscr{A}} S^a)$ is a functor from $\mathscr{A}$ to **Split** and for any $a \in \mathscr{A}$, (S_a, S^a) is too.*

Proof. By Proposition 4.4.4, $(\prod_{a\in\mathscr{A}} S_a, \prod_{a\in\mathscr{A}} S^a)$ is a functor from $\mathscr{A}$ to **Split**. Let $a, b, c \in \mathscr{A}$ such that $c \leq b$ and $v \in \prod_{d\in\mathscr{A}} S^d(c)$, $\prod_{d\in\mathscr{A}} S^{d^b}{}_c \prod_{d\in\mathscr{A}} S_{d^b}^c(v)(a) = v_a = S^{ab}{}_c S_{ab}^c(v_a)$. □

Remark 4.4.5. Let H be a decomposable functor from $\mathscr{A}$ to **Split** and $(\prod_{a\in\mathscr{A}} S_a, \prod_{a\in\mathscr{A}} S^a)$ its decomposition, for any $a, b, c \in \mathscr{A}$ such that $c \leq b$, $S_{ab}^{c-1} = S^{ab}{}_c$.

4.4.4 Bringing back $F|_{\hat{a}}$ in $F(a)$

Proposition 4.4.5. *Let $\mathscr{A}$ be a poset and (G, F) a functor from $\mathscr{A}$ to **Split**, for any $a, b, c \in \mathscr{A}$ such that $c \leq b \leq a$,*

$$F_c^b(kerG_a^b) \subseteq \ker G_a^c \qquad F_b^a(\operatorname{im} G_a^c) = \operatorname{im} G_b^c \tag{4.4.3}$$

Proof. For any $a, b, c \in \mathscr{A}$ such that $c \leq b \leq a$, $F_b^a G_a^b = \mathrm{id}$, therefore $\ker G_a^b = 0$; $F_b^a G_a^c = G_b^c$ (Proposition 4.4.3) therefore $F_b^a(\operatorname{im} G_a^c) = \operatorname{im} G_b^c$.

□

Definition 4.4.3. Let $\mathscr{A}$ be a poset and (G, F) a functor from $\mathscr{A}$ to **Split**, let $R(\alpha, a) = \operatorname{im} G_\alpha^a$, let $\alpha \geq b \geq c$, let us call $R_{\alpha c}^{\alpha b} : \operatorname{im} G_\alpha^b \to \operatorname{im} G_\alpha^c$ the unique morphism that satisfies $R_{\alpha c}^{\alpha b} G_\alpha^b|^{R(\alpha,b)} = G_\alpha^c|^{R(\alpha,c)} F_c^b$ and for $\alpha \geq \beta \geq a$, let $R_{\beta a}^{\alpha a} : R(\alpha, a) \to R(\beta, a)$ be such that $R_{\beta a}^{\alpha a} = F_\beta^\alpha|_{R(\alpha,a)}^{R(\beta,a)}$.

For $\alpha \geq \beta \geq a \geq b$, $R(\alpha, b) \subseteq R(\alpha, a)$, we shall note the inclusion as $L_{\alpha a}^{\alpha b}$; let $L_{\alpha a}^{\beta a} = G_\alpha^\beta|_{R(\beta,a)}^{R(\alpha,a)}$.

We shall call $L(G, F)$ the left coupling of (G, F) and $R(G, F)$ its right coupling.

Proof. By Proposition 4.4.5 R is well defined. For any $a \geq b \geq c$, $G_a^b G_b^c = G_a^c$, therefore $R(a, c) = \operatorname{im} G_a^c \subseteq \operatorname{im} G_a^b = R(a, b)$, and $G_a^b(\operatorname{im} G_b^c) = \operatorname{im} G_a^c$ and L is well defined. □

Remark 4.4.6. Let (G, F) be a functor from $\mathscr{A}$ to **Split**, for any $\alpha \in \mathscr{A}$, $G_{|\hat{\alpha}}$ induces a functor monomorphism $G_\alpha^{\hat{\alpha}} : G_{|\hat{\alpha}} \to G(a)$ and a presheaf monomorphism $G_\alpha^{\hat{\alpha}F} : F_{\hat{\alpha}} \to F(\alpha)$; $L(\alpha, .) = \operatorname{im} G_\alpha^{\hat{\alpha}}$, $R(\alpha, .) = \operatorname{im} G_\alpha^{\hat{\alpha}F}$. Furthermore, $G_\alpha^{\hat{\alpha}}|^{L(\alpha,.)}$ and $G_\alpha^{\hat{\alpha}F}|^{R(\alpha,.)}$ are isomorphisms.

Proposition 4.4.6. *Let (G, F) be a functor from $\mathscr{A}$ to **Split**, L its left coupling, R its right coupling; for any $\alpha, \beta, a, b \in \mathscr{A}$ such that $\alpha \geq \beta \geq a \geq b$,*

$$L^{\beta a}_{\alpha a} L^{\beta b}_{\beta a} = L^{\alpha b}_{\alpha a} L^{\beta b}_{\alpha b} \tag{4.4.4}$$

$$R^{\beta a}_{\beta b} R^{\alpha a}_{\beta a} = R^{\alpha b}_{\beta b} R^{\alpha a}_{\alpha b} \tag{4.4.5}$$

Proof. Let $\alpha \geq \beta \geq a \geq b$, for any $v \in R(\beta, b)$, $L^{\beta a}_{\alpha a} L^{\beta b}_{\beta a}(v) = G^{\beta}_{\alpha}(v) = L^{\alpha b}_{\alpha a} L^{\beta b}_{\alpha b}(v)$.

For $v \in F(a)$, $v_1 \in F(b)$, $R^{\alpha a}_{\beta a} G^a_\alpha(v) = F^\alpha_\beta G^a_\alpha(v) = G^a_\beta(v)$ and $R^{\alpha b}_{\beta b} G^b_\alpha(v_1) = G^b_\beta(v_1)$; $R^{\beta a}_{\beta b} R^{\alpha a}_{\beta a} G^a_\alpha(v) = R^{\beta a}_{\beta b} G^a_\beta(v)$, $R^{\alpha b}_{\beta b} R^{\alpha a}_{\alpha b} G^a_\alpha(v) = R^{\alpha b}_{\beta b} G^b_\alpha F^a_b(v) = G^b_\beta F^a_b(v)$, by construction $G^b_\beta F^a_b(v) = R^{\beta a}_{\beta b} G^a_\beta$.

$$\begin{tikzcd}
F(a) \arrow[r, "G^a_\alpha|^{R(\alpha,a)}"'] \arrow[d, "F^a_b"] & R(\alpha, a) \arrow[r, "R^{\alpha a}_{\beta a}"] \arrow[d, "R^{\alpha a}_{\alpha b}"] & R(\beta, a) \arrow[d, "R^{\beta a}_{\beta b}"] \\
F(b) \arrow[r, "G^b_\alpha|^{R(\alpha,b)}"'] & R(\alpha, b) \arrow[r, "R^{\alpha b}_{\beta b}"'] & R(\beta, b)
\end{tikzcd} \tag{4.4.6}$$

□

Definition 4.4.4. Let $\mathscr{A}_1$ be the subposet of $\mathscr{A} \times \mathscr{A}$ constituted of couples (α, a) such that $a \leq \alpha$.

Proposition 4.4.7. *Let $\mathscr{A}$ be a poset, $\boldsymbol{C}$ be any category; let $M_1 = \{((\alpha, a), (\alpha, b)) : (\alpha, a), (\alpha, b) \in \mathscr{A}_1$ and $a \geq b\}$, $M_2 = \{((\alpha, a), (\beta, a)) : (\alpha, a), (\beta, a) \in \mathscr{A}_1$ and $\alpha \geq \beta\}$, and let $(G^i_j; i, j \in M_1 \cup M_2 : i \leq j)$ be such that for any $(\alpha, a), (\alpha, b), (\alpha, c) \in \mathscr{A}_1$ such that $(\alpha, a) \geq (\alpha, b) \geq (\alpha, c)$,*

$$G^{\alpha b}_{\alpha a} G^{\alpha c}_{\alpha b} = G^{\alpha c}_{\alpha a}$$

for any $(\alpha, a), (\beta, a), (\gamma, a)$ such that $(\alpha, a) \geq (\beta, a) \geq (\gamma, a)$,

$$G^{\beta a}_{\alpha a} G^{\gamma a}_{\beta a} = G^{\gamma a}_{\alpha a}$$

and for any $(\alpha, a), (\alpha, b), (\beta, a), (\beta, b)$ such that $(\alpha, a) \geq (\alpha, b) \geq (\beta, b)$ and $(\alpha, a) \geq (\beta, a) \geq (\beta, b)$, i.e $\alpha \geq \beta \geq a \geq b$,

$$G^{\beta a}_{\alpha a} G^{\beta b}_{\beta a} = G^{\alpha b}_{\alpha a} G^{\beta b}_{\alpha b}$$

Then G extends into a unique functor $G_1 : \mathscr{A}_1 \to \boldsymbol{C}$, we shall also denote this extension as G.

Proof. Let us remark that for any $(\alpha, a), (\beta, b) \in \mathscr{A}_1$ such that $(\alpha, a) \geq (\beta, b)$, then $(\alpha, a) \geq (\alpha, b) \geq (\beta, b)$. Let $G_1 : \mathscr{A}_1 \to \mathbf{C}$ be a functor such that for any $(\alpha, a) \geq (\alpha, b)$, ${G_1}_{\alpha b}^{\alpha a} = G_{\alpha b}^{\alpha a}$ and for any $(\alpha, a) \geq (\beta, a)$, ${G_1}_{\alpha a}^{\beta a} = G_{\alpha a}^{\beta a}$, then for any $(\alpha, a) \geq (\beta, b)$, ${G_1}_{\alpha a}^{\beta b} = G_{\alpha a}^{\alpha b} G_{\alpha b}^{\beta b}$. Therefore there can be only one functor that extends G to $\mathscr{A}_1$.

Let for any $(\alpha, a) \geq (\beta, b)$, ${G_1}_{\alpha a}^{\beta b} = G_{\alpha a}^{\alpha b} G_{\alpha b}^{\beta b}$.

$$\begin{tikzcd}
G(\beta, b) \arrow[d, "G_{\alpha b}^{\beta b}"'] \arrow[dr, "{G_1}_{\alpha a}^{\beta b}"] & \\
G(\alpha, b) \arrow[r, "G_{\alpha a}^{\alpha b}"'] & G(\alpha, a)
\end{tikzcd} \tag{4.4.7}$$

For any $(\alpha, a) \geq (\beta, b) \geq (\gamma, c)$,

$${G_1}_{\alpha a}^{\beta b} {G_1}_{\beta b}^{\gamma c} = G_{\alpha a}^{\alpha b} G_{\alpha b}^{\beta b} G_{\beta b}^{\beta c} G_{\beta c}^{\gamma c} = G_{\alpha a}^{\alpha b} G_{\alpha b}^{\alpha c} G_{\alpha c}^{\beta c} G_{\beta c}^{\gamma c} = G_{\alpha a}^{\alpha c} G_{\alpha c}^{\gamma c}$$

$$\begin{tikzcd}
G(\gamma, c) \arrow[d, "G_{\beta c}^{\gamma c}"'] \arrow[dr, "{G_1}_{\beta b}^{\gamma c}"] & & \\
G(\beta, c) \arrow[r, "G_{\beta b}^{\beta c}"] \arrow[d, "G_{\alpha c}^{\beta c}"'] & G(\beta, b) \arrow[d, "G_{\alpha b}^{\beta b}"'] \arrow[dr, "{G_1}_{\alpha a}^{\beta b}"] & \\
G(\alpha, c) \arrow[r, "G_{\alpha b}^{\alpha c}"'] & G(\alpha, b) \arrow[r, "G_{\alpha a}^{\alpha b}"'] & G(\alpha, a)
\end{tikzcd} \tag{4.4.8}$$

□

Remark 4.4.7. Let $\mathscr{A}$ be a poset, if one applies Proposition 4.4.7 for $\mathscr{A}^{op}$ then one extends G to a presheaf.

Corollary 4.4.2. *Let (G, F) be a functor from $\mathscr{A}$ to **Split**, let (L, R) be its left and right coupling; (L, R) has a unique extension into a functor from $\mathscr{A}$ to **Split**.*

Proof. By construction for any $(\alpha, a), (\beta, a), (\gamma, a) \in \mathscr{A}_1$ such that $(\alpha, a) \geq (\beta, a) \geq (\gamma, a)$, $L_{\alpha a}^{\beta a} L_{\beta a}^{\gamma a} = L_{\alpha a}^{\gamma a}$, and for any $(\alpha, a), (\alpha, b), (\alpha, c) \in \mathscr{A}_1$ such that $(\alpha, a) \geq (\alpha, b) \geq (\alpha, c)$, $L_{\alpha a}^{\alpha b} L_{\alpha b}^{\alpha c} = L_{\alpha a}^{\alpha c}$. Therefore by Proposition 4.4.6 and Proposition

4.4.7, L extends into a unique functor from $\mathscr{A}_1$ to **Mod**.

By construction for any $(\alpha, a), (\beta, a), (\gamma, a) \in \mathscr{A}_1$ such that $(\alpha, a) \geq (\beta, a) \geq (\gamma, a)$, $R^{\beta a}_{\gamma a} R^{\alpha a}_{\beta a} = R^{\alpha a}_{\gamma a}$. Let $(\alpha, a), (\alpha, b), (\alpha, c) \in \mathscr{A}_1$ such that $(\alpha, a) \geq (\alpha b) \geq (\alpha c)$ and $v \in G(a)$, $R^{\alpha b}_{\alpha c} R^{\alpha a}_{\alpha b} G^a_\alpha(v) = R^{\alpha b}_{\alpha c} G^b_\alpha F^a_b(v) = G^c_\alpha F^a_c(v) = R^{\alpha a}_{\alpha c} G^a_\alpha(v)$; as G^a_α is surjective, $R^{\alpha b}_{\alpha c} R^{\alpha a}_{\alpha b} = R^{\alpha a}_{\alpha c}$. Therefore by Proposition 4.4.6 and Proposition 4.4.7, R extends into a unique presheaf from $\mathscr{A}_1$ to **Mod**.

For $\alpha \geq a \geq b$ and $v \in G(b)$, $R^{\alpha a}_{\alpha b} L^{\alpha b}_{\alpha a} G^b_\alpha(v) = R^{\alpha a}_{\alpha b} G^a_\alpha G^b_a(v) = G^b_\alpha F^a_b G^b_a(v) = G^b_\alpha(v)$, as G^b_α is surjective, $R^{\alpha a}_{\alpha b} L^{\alpha b}_{\alpha a} = \mathrm{id}$; for $\alpha \geq \beta \geq a$ and $v \in R(\beta, a)$, $R^{\alpha a}_{\beta a} L^{\beta a}_{\alpha a}(v) = F^\alpha_\beta G^\beta_\alpha(v) = v$; therefore for any $(\alpha, a) \geq (\beta, b)$,

$$R^{\alpha a}_{\beta b} L^{\beta b}_{\alpha a} = R^{\alpha b}_{\beta b} R^{\alpha a}_{\alpha b} L^{\alpha b}_{\alpha a} L^{\beta b}_{\alpha b} = R^{\alpha b}_{\beta b} L^{\beta b}_{\alpha b} = \mathrm{id} \tag{4.4.9}$$

□

Remark 4.4.8. For (G, F) a functor from $\mathscr{A}$ to **Split**, and $(\alpha, a) \in \mathscr{A}_1$, $L^{aa}_{\alpha\alpha} = G^a_\alpha$, $R^{\alpha\alpha}_{aa} = F^\alpha_a$. Indeed, let us recall that $R(a, a) = F(a)$, $R^{\alpha\alpha}_{\alpha a} = L^{aa}_{\alpha a} F^\alpha_a$ and $R^{\alpha\alpha}_{aa} = R^{\alpha a}_{aa} R^{\alpha\alpha}_{\alpha a} = R^{\alpha a}_{aa} L^{aa}_{\alpha a} F^\alpha_a$; $R^{\alpha a}_{aa} = F^\alpha_a|_{R(\alpha,a)} = F^\alpha_a L^{\alpha a}_{\alpha\alpha}$; so $R^{\alpha\alpha}_{aa} = F^\alpha_a L^{\alpha a}_{\alpha\alpha} L^{aa}_{\alpha a} F^\alpha_a = F^\alpha_a L^{aa}_{\alpha\alpha} F^\alpha_a = F^\alpha_a G^a_\alpha F^\alpha_a = F^\alpha_a$.

Corollary 4.4.3. *Let (G, F) be a functor from $\mathscr{A}$ to **Split**, for any $\alpha, \beta, a \in \mathscr{A}$ such that $\alpha \geq \beta \geq a$, $L^{\beta a}_{\alpha a}$ is an isomorphism, its inverse is $R^{\alpha a}_{\beta a}$.*

Proof. For any $\alpha \geq \beta \geq a$, $L^{\beta a}_{\alpha a} = G^\beta_\alpha|^{\mathrm{im}\, G^a_\alpha}_{\mathrm{im}\, G^a_\beta}$ and as G^β_α is injective and $G^\beta_\alpha G^a_\beta = G^a_\alpha$, $\mathrm{im}\, L^{\beta a}_{\alpha a} = \mathrm{im}\, G^a_\alpha = L(\alpha, a)$; therefore $L^{\beta a}_{\alpha a}$ is an isomorphism. Furthermore by Corollary 4.4.2 $R^{\alpha a}_{\alpha b}$ is the inverse of $L^{\alpha b}_{\alpha a}$.

□

Definition 4.4.5. Let $\mathscr{A}$ be any poset, let (G, F) be a functor from $\mathscr{A}$ to **Split**. For any $(\alpha, a) \in \mathscr{A}_1$, let $V(\alpha, a) = \prod_{b \leq a} G(\alpha)$ (which in the previous section we would note as $G(\alpha)_{\hat{a}}$). For any α, β, a, b such that $\alpha \geq \beta \geq a \geq b$ let ${V_r}^{\alpha a}_{\alpha b} = pr^{\prod_{c \in \hat{a}} G(\alpha)}_{\prod_{c \in \hat{b}} G(\alpha)}$, $V^{\alpha b}_{l\alpha a} = i^{\prod_{c \in \hat{b}} G(\alpha)}_{\prod_{c \in \hat{a}} G(\alpha)}$, ${V_r}^{\alpha a}_{\beta a} : V(\alpha, a) \to V(\beta, a)$ be such that for any $v \in V(\alpha, a)$ and $c \leq a$, ${V_r}^{\alpha a}_{\beta a}(v)(c) = F^\alpha_\beta(v_c)$, $V^{\beta a}_{l\alpha a} : V(\beta, a) \to V(\alpha, a)$ be such that for any $v \in V(\beta, a)$ and $c \leq a$, $V^{\alpha a}_{l\beta a}(v)(c) = G^\beta_\alpha(v_c)$.

Proposition 4.4.8. *Let $\mathscr{A}$, (G, F) be a functor from $\mathscr{A}$ to **Split**, $(\prod_{a \in \mathscr{A}} G1[a \leq .], \prod_{a \in \mathscr{A}} F1[a \leq .])$ is a functor from $\mathscr{A}$ to **Split**. (V_l, V_r) extends into a unique functor form $\mathscr{A}_1$ to **Split**.*

Proof. Let $b, c \in \mathscr{A}$ such that $c \leq b$, let us note $\prod_{a\in\mathscr{A}} F1[a \leq .]_c^b$ as F and $\prod_{a\in\mathscr{A}} G1[a \leq .]_b^c$ as G; for any $v \in \prod_{a\in\mathscr{A}} G1[a \leq .](c)$, and $a \in \mathscr{A}$, $FG(v)(a) = F_c^b G_b^c(v_a)1[a \leq b]1[a \leq c] = v_a 1[a \leq c] = \mathrm{id}_{\prod_{a\in\mathscr{A}} G1[a\leq.](c)}(v)(a)(\simeq \mathrm{id}_{\prod\limits_{\substack{a_1\in\mathscr{A}\\ a_1\leq c}} G(c)}(v)(a))$ (this also shows that $(G1[a \leq .], F1[a \leq])$ is a functor from $\mathscr{A}$ to **Split**).

For $\alpha \geq a \geq b \geq c$, $V_{l\alpha a}^{\alpha b} V_{l\alpha b}^{\alpha c} = V_{l\alpha a}^{\alpha c}$, $V_{r\alpha c}^{\alpha b} V_{r\alpha b}^{\alpha a} = V_{r\alpha c}^{\alpha a}$. For any $\alpha \geq \beta \geq \gamma \geq a$, any $v \in V(\alpha, c)$, any $c \leq a$, $V_{l\alpha a}^{\beta a} V_{l\beta a}^{\gamma a}(v)(c) = G_\alpha^\beta G_\alpha^\gamma(v_c) = G_\alpha^\gamma(v_c) = V_{l\alpha a}^{\gamma a}(v)(c)$ and for any $v \in V(\alpha, a)$, and $c \leq a$, $V_{r\gamma a}^{\beta a} V_{r\beta a}^{\alpha a}(v)(c) = F_\gamma^\beta F_\beta^\alpha(v_c) = F_\gamma^\alpha(v_c) = V_{r\gamma a}^{\alpha a}(v)(c)$.

Let $\alpha \geq \beta \geq a \geq b$, $v \in V(\beta, b)$; for any $c \leq a$, $V_{l\alpha a}^{\beta a} V_{l\beta a}^{\beta b}(v)(c) = G_\alpha^\beta(V_{l\beta a}^{\beta b}(v)(c)) = G_\alpha^\beta(v_c 1[c \leq b]) = G_\alpha^\beta(v_c)1[c \leq b]$ and $V_{l\alpha a}^{\alpha b} V_{l\alpha b}^{\beta b}(v)(c) = V_{l\alpha b}^{\beta b}(v)(c)1[c \leq b] = G_\alpha^\beta(v_c)1[c \leq b]$; similarly one has that $V_{r\beta b}^{\alpha b} V_{r\alpha b}^{\alpha a} = V_{r\beta b}^{\beta a} V_{r\beta a}^{\alpha a}$. Therefore by Proposition 4.4.7 V_l extends to a functor from $\mathscr{A}_1$ to **Mod** and V_r to a presheaf from $\mathscr{A}_1$ to **Mod**.

For any $\alpha \geq \beta \geq a$, $v \in V(\beta, a)$, $c \leq b$, $V_{r\beta a}^{\alpha a} V_{l\alpha a}^{\beta a}(v)(c) = F_\beta^\alpha G_\alpha^\beta(v_c) = v_c$; for any $\alpha \geq a \geq b$, $V_{r\alpha b}^{\alpha a} V_{l\alpha a}^{\alpha b} = \mathrm{id}$, therefore for any $(\alpha, a) \geq (\beta, b)$, $V_{r\beta b}^{\alpha a} V_{l\alpha a}^{\beta b} = V_{r\beta b}^{\alpha b} V_{r\alpha b}^{\alpha a} V_{l\alpha a}^{\alpha b} V_{l\alpha b}^{\beta b} = \mathrm{id}$.

□

Until now in this subsection there was no constraint on $\mathscr{A}$, in order to be able to define $\zeta_{\hat{\alpha}}(G(\alpha))$ on $V(\alpha, \alpha)$ we will have to assume that $\hat{\alpha}$ is finite for any $\alpha \in \mathscr{A}$.

Notation 4.4.4. The class of posets that are such that $\hat{a}$ is finite for any $a \in \mathscr{A}$ will be denoted as $\hat{\mathscr{P}}_f$.

Remark 4.4.9. Let $\mathscr{A} \in \hat{\mathscr{P}}_f$ and $(G_a, a \in \mathscr{A})$ a collection of functors from $\mathscr{A}$ to **Mod**, $\prod_{a\in\mathscr{A}} G_a 1[a \leq .] = \bigoplus\limits_{a\in\mathscr{A}} G_a 1[a \leq .]$, indeed for any $b \in \mathscr{A}$, $\prod\limits_{a:a\leq b} G_a(b) = \bigoplus\limits_{a:a\leq b} G_a(b)$.

Proposition 4.4.9. *Let (G, F) be a functor from $\mathscr{A} \in \hat{\mathscr{P}}_f$ to **Split**, for any $(\alpha, a) \in \mathscr{A}_1$, let $\zeta(\alpha, a) = \zeta_{\hat{a}}(G(\alpha)) : V(\alpha, a) \to V(\alpha, a)$ and $\mu(\alpha, a) = \mu_{\hat{a}}(G(\alpha)) : V(\alpha, a) \to V(\alpha, a)$; ζ, $\mu \in \boldsymbol{Mod}^{\mathscr{A}^{op}}(V_r, V_r)$.*

Lemma 4.4.1. *Let V,V_1 be two modules, $\mathscr{A}$ a finite poset, and $l : V \to V_1$ a linear application; let $L : V_{\mathscr{A}} \to V_{1\mathscr{A}}$ be such that $L(v)(a) = l(v_a)$. $\zeta_{\mathscr{A}}(V_1)L = L\zeta_{\mathscr{A}}(V)$ and $\mu_{\mathscr{A}}(V_1)L = L\mu_{\mathscr{A}}(V)$.*

Proof. For any $v \in V_{\mathscr{A}}$, $a \in \mathscr{A}$, $L(\zeta_{\mathscr{A}}(V)(v))(a) = l(\sum_{b \leq a} v_b) = \sum_{b \leq a} l(v_b) = \zeta_{\mathscr{A}}(V_1)(L(v))(a)$. Futhermore $\mu_{\mathscr{A}}(V_1)\zeta_{\mathscr{A}}(V_1)L\mu_{\mathscr{A}}(V) = \mu_{\mathscr{A}}(V_1)L\zeta_{\mathscr{A}}(V)\mu_{\mathscr{A}}(V)$ so $\mu_{\mathscr{A}}(V_1)L = L\mu_{\mathscr{A}}(V)$. □

Proof. Proof of Proposition 4.4.9.

For any $\alpha \geq a \geq b$, by Proposition 4.3.3 $\zeta(\alpha, b)V_{r\alpha b}^{\alpha a} = V_{r\alpha b}^{\alpha a}\zeta(\alpha, a)$. By Lemma 4.3.3, for any $\alpha \geq \beta \geq a$, $\zeta(\beta, a)V_{r\beta a}^{\alpha a} = V_{r\beta a}^{\alpha a}\zeta(\alpha, a)$. Therefore for any $(\alpha, a), (\beta, b) \in \mathscr{A}_1$ such that $(\alpha, a) \geq (\beta, b)$, $V_{r\beta b}^{\alpha a}\zeta(\alpha, a) = V_{r\beta b}^{\alpha b}V_{\alpha b}^{\alpha a}\zeta(\alpha, a) = V_{r\beta b}^{\alpha b}\zeta(\alpha, b)V_{r\alpha b}^{\alpha a} = \zeta(\beta, b)V_{r\beta b}^{\alpha b}V_{r\alpha b}^{\alpha a}$. Therefore $V_{r\beta b}^{\alpha a}\mu(\alpha, a) = \mu(\beta, b)V_{r\beta b}^{\alpha a}$. □

Remark 4.4.10. ζ, μ are in general not natural transformations from V_l to V_l because for $\alpha \geq a \geq b$, $V_{l\alpha a}^{\alpha b}\zeta(\alpha, b) \neq \zeta(\alpha, a)V_{l\alpha a}^{\alpha b}$.

Notation 4.4.5. For any $\alpha \in \mathscr{A}$, we shall note $\zeta(\alpha, \alpha)$ as ζ^{α}.

4.4.5 Intersection for functors from $\mathscr{A}$ to Split

Definition 4.4.6 (Intersection property)**.** Let (G, F) be a functor from $\mathscr{A} \in \hat{\mathscr{P}}_f$ to **Split**. For any $(\alpha, a) \in \mathscr{A}_1$, let $\pi_{\alpha a}^{\alpha\alpha} = L_{\alpha\alpha}^{\alpha a}R_{\alpha a}^{\alpha\alpha}$, $\pi_{\alpha a}^{\alpha\alpha}$ is a projector. For a given α, we shall denote this collection as π^{α}.

(G, F) is said to satisfy the intersection property for $\alpha \in \mathscr{A}$ if π^{α} satisfies the intersection property (I') and is said to satisfy the intersection property if is satisfies it for any $\alpha \in \mathscr{A}$.

Proof. As (L, R) is a functor from $\mathscr{A}_1$ to **Split** (Corollary 4.4.2) $\pi_{\alpha a}^{\alpha\alpha 2} = L_{\alpha\alpha}^{\alpha a}R_{\alpha a}^{\alpha\alpha}L_{\alpha\alpha}^{\alpha a}R_{\alpha a}^{\alpha\alpha} = L_{\alpha\alpha}^{\alpha a}R_{\alpha a}^{\alpha\alpha}$. □

Remark 4.4.11. Let (G, F) be a functor from $\mathscr{A}$ to **Split**; let $(\alpha, a) \in \mathscr{A}_1$, let us remark that $\pi_{\alpha a}^{\alpha\alpha} = L_{\alpha\alpha}^{\alpha a}L_{\alpha a}^{aa}F_a^{\alpha} = L_{\alpha\alpha}^{aa}F_a^{\alpha} = G_{\alpha}^{a}F_a^{\alpha}$. Therefore the intersection property is equivalent to for any $(\alpha, a), \in \mathscr{A}_1$, for any $v \in G(\alpha)$, $\operatorname{im}(G_{\alpha}^{b}F_b^{\alpha}G_{\alpha}^{a}F_a^{\alpha}, b \in \hat{\alpha}) \subseteq \operatorname{im}\zeta^{\alpha}V_{l\alpha\alpha}^{\alpha a}$. If $\mathscr{A}$ has all its intersections, the intersection property is equivalent to for any $(\alpha, a), (\alpha, b) \in \mathscr{A}_1$, $G_{\alpha}^{b}F_b^{\alpha}G_{\alpha}^{a}F_a^{\alpha} = G_{\alpha}^{b\cap a}F_{b\cap a}^{\alpha}$.

Remark 4.4.12. Let (G, F) be a functor from $\mathscr{A}$ to **Split**, for any $\alpha \in \mathscr{A}$, $(\pi_{\alpha,a}^{\alpha,\alpha}, a \in \hat{\alpha})$ is presheafable as for any $v \in R(\alpha, \alpha)$ and $(\alpha, a), (\alpha, b) \in \mathscr{A}$, such that $b \leq a$, $\pi_{\alpha b}^{\alpha\alpha}(v) = \pi_{\alpha b}^{\alpha a}\pi_{\alpha a}^{\alpha\alpha}(v)$.

Remark 4.4.13. Let (G, F) be a functor from $\mathscr{A}$ to **Split**, for any $(\alpha, a) \in \mathscr{A}_1$, $\operatorname{im}\pi_{\alpha a}^{\alpha\alpha} = R(\alpha, a)$.

Proposition 4.4.10. *Let (G, F) be a functor from $\mathscr{A}$ to **Split**, if (G, F) verifies the intersection property for α then it verifies it for any $\beta \leq \alpha$.*

4.4.6 Equivalence between intersection property and decomposability

Proposition 4.4.11. *Let (G,F) be a functor from $\mathscr{A}$ to **Split**, for any $(\alpha,a)\in\mathscr{A}_1$ let us denote $R|_{(\alpha,\hat{a})}$ as $R^{\alpha,\hat{a}}$; $(R^{\alpha a}_{\alpha b}, b\leq a)$ is a natural transformation from $R(\alpha,a)$ to $R^{\alpha,\hat{a}}$, let us denote $\phi(\alpha,a): R(\alpha,a)\to \lim\limits_b R^{\alpha,\hat{a}}(b)$ its limit. One has that $(\lim\limits_b R^{\alpha,\hat{a}}(b),(\alpha,a)\in\mathscr{A}_1)$ is a subobject of V_r that we shall note as M and $\phi: R\to M$ is a natural transformation, furthermore it is an isomorphism.*

Proof. By definition $\lim\limits_b R^{\alpha,\hat{a}}(b)\subseteq V(\alpha,a)$; let $\alpha,a,b,c,c_1\in\mathscr{A}$ such that $\alpha\geq a\geq b\geq c\geq c_1$ and $v\in\lim\limits_c R^{\alpha,\hat{a}}(c)$, one has that $R^{\alpha c}_{\alpha c_1}({V_r}^{\alpha a}_{\alpha b}(v)(c)) = R^{\alpha c}_{\alpha c_1}(v_c) = v_{c_1}$. Let $\alpha\geq\beta\geq a\geq b\geq b_1$ and $v\in\lim\limits_c R^{\alpha,\hat{a}}(c)$, one has that ${V_r}^{\alpha a}_{\beta a}(v)(b)\in R(\beta,b)$ and $R^{\beta b}_{\beta b_1}({V_r}^{\alpha a}_{\beta a}(v)(b)) = R^{\beta b}_{\beta b_1}F^{\alpha}_{\beta}(v_b) = R^{\beta b}_{\beta b_1}R^{\alpha b}_{\beta b}(v_b) = R^{\alpha b_1}_{\beta b_1}R^{\alpha b}_{\alpha b_1}(v_b) = R^{\alpha b_1}_{\beta b_1}(v_{b_1}) = F^{\alpha}_{\beta}(v_{b_1})$ and therefore $R^{\beta b}_{\beta b_1}({V_r}^{\alpha a}_{\beta a}(v)(b)) = {V_r}^{\alpha a}_{\beta a}(v)(b_1)$. For any $(\alpha,a),(\beta,b)\in\mathscr{A}_1$ such that $(\alpha,a)\geq(\beta,b)$, ${V_r}^{\alpha a}_{\beta b}(\lim\limits_c R^{\alpha,\hat{a}}(c)) = {V_r}^{\alpha b}_{\beta b}{V_r}^{\alpha a}_{\alpha b}(\lim\limits_c R^{\alpha,\hat{a}}(c))\subseteq {V_r}^{\alpha b}_{\beta b}(\lim\limits_c R^{\alpha,\hat{b}}(c))\subseteq \lim\limits_c R^{\beta,\hat{b}}(c))$.

Let $\alpha\geq a\geq b\geq c$ and $v\in R(\alpha,a)$, $M^{\alpha a}_{\alpha b}\phi(\alpha,a)(v)(c) = R^{\alpha a}_{\alpha c}(v) = R^{\alpha b}_{\alpha c}R^{\alpha a}_{\alpha b}(v) = \phi(\alpha,b)R^{\alpha a}_{\alpha b}(v)(c)$. For $\alpha\geq\beta\geq a\geq b$, $M^{\alpha a}_{\beta a}\phi(\alpha,a)(v)(b) = F^{\alpha}_{\beta}(R^{\alpha a}_{\alpha b}(v)) = R^{\beta a}_{\beta b}R^{\alpha a}_{\beta a}(v) = \phi(\alpha,a)R^{\alpha a}_{\beta a}(v)(b)$. Therefore for any $(\alpha,a),(\beta,b)\in\mathscr{A}_1$ such that $(\alpha,a)\geq(\beta,b)$, $M^{\alpha a}_{\beta b}\phi(\alpha,a) = \phi(\beta,b)R^{\alpha a}_{\beta b}$.

For any $(\alpha,a)\in\mathscr{A}_1$, (α,a) is initial in $(\alpha,\hat{a})$, therefore $\phi(\alpha,a)$ is an isomorphism. therefore so is ϕ.

□

Proposition 4.4.12. *Let (G,F) be a functor from $\mathscr{A}\in\hat{\mathscr{P}}_f$ to **Split**, if (G,F) satisfies the intersection property, $j=\mu i^M_{V_r}\phi\in\textbf{\textit{Split}}^{\mathscr{A}_1}((L,R),(V_l,V_r))$ is a monomorphism.*

Proof. Let $(\alpha,a)\in\mathscr{A}_1$ and $v\in R(\alpha,a)$, for any $b\leq a$, $j(\alpha,\alpha)L^{\alpha a}_{\alpha\alpha}(v)(b) = \mu(\alpha,\alpha)(R^{\alpha\alpha}_{\alpha c}(v), c\leq\alpha)(b) = \mu(\alpha,a)(R^{\alpha a}_{\alpha c}(v), c\leq a)(b)$ as μ is a endomorphism of V_r. Furthermore as $R(\alpha,a) = \operatorname{im}\pi^{\alpha\alpha}_{\alpha a}$, $\phi(\alpha,\alpha)L^{\alpha a}_{\alpha\alpha}(v) = \phi(\alpha,\alpha)(v)\in\operatorname{im}\zeta^{\alpha}{V_l}^{\alpha a}_{\alpha\alpha}$, therefore for any $b\not\leq a$, $\mu(\alpha,\alpha)(\phi(\alpha,\alpha)(v))(b) = 0$ and so $j(\alpha,\alpha)L^{\alpha a}_{\alpha\alpha} = {V_l}^{\alpha a}_{\alpha\alpha}j(\alpha,a)$. Therefore $j(\alpha,\alpha)L^{\alpha a}_{\alpha\alpha} = {V_l}^{\alpha a}_{\alpha\alpha}j(\alpha a)$.

Let $b\leq a$, ${V_l}^{\alpha a}_{\alpha\alpha}j(\alpha,a)L^{\alpha b}_{\alpha a} = j(\alpha,\alpha)L^{\alpha b}_{\alpha\alpha} = {V_l}^{\alpha b}_{\alpha\alpha}j(\alpha,b) = {V_l}^{\alpha a}_{\alpha\alpha}{V_l}^{\alpha b}_{\alpha a}j(\alpha,b)$. ${V_l}^{\alpha a}_{\alpha\alpha}$ is a monomorphism therefore, $j(\alpha,a)L^{\alpha b}_{\alpha a} = {V_l}^{\alpha b}_{\alpha a}j(\alpha b)$.

For $\alpha \geq \beta \geq a$, $j(\alpha,a)L^{\beta a}_{\alpha a} = V_{l\,\alpha a}^{\beta a} j(\beta,a)$ holds even if (G,F) does not satisfy the intersection property.

ϕ is a isomorphism from R to M, $i^M_{V_r}$ is a monomorphism and μ is an isomorphism from V_r to V_r therefore j is a monomorphism. $\square$

Proposition 4.4.13. *Let (G,F) be a functor from $\mathscr{A}$ to **Split**, let $b \in \mathscr{A}$. For any $(\alpha,a) \in \mathscr{A}_1$ if $a \geq b$ let $G_b(\alpha,a) = G(\alpha)$ and if not $G_b(\alpha,a) = 0$; let $\alpha \geq a \geq a_1$, if $a_1 \geq b$, let $G_{b\,\alpha a_1}^{\alpha a} = G^{b\,\alpha a_1}_{\alpha a} = \mathrm{id}$, if not $G_{b\,\alpha a_1}^{\alpha a} = G^{b\,\alpha a_1}_{\alpha a} = 0$, let $\alpha \geq \beta \geq a$, if $a \geq b$ let $G^{b\,\alpha a}_{\beta a} = F^{\alpha}_{\beta}$ and $G_{b\,\alpha a}^{\beta a} = G^{\beta}_{\alpha}$, if not, $G_{b\,\beta a}^{\alpha a} = 0 = G^{b\,\beta a}_{\alpha a}$. (G^b, G_b) extend into a unique functor from $\mathscr{A}_1$ to **Split**. For any $(\alpha,a) \in \mathscr{A}_1$, if $a \geq b$ let $pr_b(\alpha,a) : V(\alpha,a) \to G(\alpha)$ be such that for any $v \in V(\alpha,a)$, $pr_b(v) = v_b$, if $a \not\geq b$, $pr_b(v) = 0$. $pr_b \in \textbf{\textit{Split}}^{\mathscr{A}_1}((V_l,V_r),(G_b,G^b))$.*

Proof. Let $b \in \mathscr{A}$, $\alpha \in \mathscr{A}$, $G_b|_{(\alpha,\hat{\alpha})} = G(\alpha)1[(\alpha,b) \leq .]$ (if $b \not\leq \alpha$), $1[(\alpha,b) \leq .] = 0$ as the relation is taken in $\mathscr{A}_1$), and $G^b|_{(\alpha,\hat{\alpha})^{op}} = G(\alpha)1[a \leq .]$, therefore for $\alpha \geq a \geq a_1 \geq a_2$, $G_{b\,\alpha a}^{\alpha a_1} G_{b\,\alpha a_1}^{\alpha a_2} = G_{b\,\alpha a}^{\alpha a_2}$, $G^{b\,\alpha a_1}_{\alpha a_2} G^{b\,\alpha a}_{\alpha a_1} = G^{b\,\alpha a}_{\alpha a_2}$.

Let $\alpha \geq \beta \geq \gamma \geq a$, if $a \geq b$, $G_{b\,\alpha a}^{\beta a} G_{b\,\beta a}^{\gamma a} = G^{\beta}_{\alpha} G^{\gamma}_{\beta} = G_{b\,\alpha a}^{\gamma a}$, $G^{b\,\beta a}_{\gamma a} G^{b\,\alpha a}_{\beta a} = F^{\beta}_{\gamma} F^{\alpha}_{\beta} = G^{b\,\alpha a}_{\gamma a}$, and if $a \not\geq b$, $G_{b\,\alpha a}^{\beta a} G_{b\,\beta a}^{\gamma a} = 0 = G_{b\,\alpha a}^{\gamma a}$, $G^{b\,\beta a}_{\gamma a} G^{b\,\alpha a}_{\beta a} = 0 = G^{b\,\alpha a}_{\gamma a}$.

Furthermore for $\alpha \geq \beta \geq a \geq c$, if $c \geq b$, $G_{b\,\alpha a}^{\beta a} G_{b\,\beta a}^{\beta c} = G^{\beta}_{\alpha} = G_{b\,\alpha a}^{\alpha c} G_{b\,\alpha b}^{\beta c}$, $G^{b\,\beta a}_{\beta c} G^{b\,\alpha a}_{\beta a} = F^{\alpha}_{\beta} = G^{b\,\alpha c}_{\beta c} G^{b\,\alpha a}_{\alpha c}$. Therefore by Proposition 4.4.7, G_a, G^a extend respectively to a functor and a presheaf from $\mathscr{A}_1$ to **Mod**.

Let $\alpha \geq a \geq a_1$, if $b \leq a_1$, $G^{b\,\alpha a}_{\alpha a_1} G_{b\,\alpha a}^{\alpha a_1} = \mathrm{id}$ if $b \not\leq a_1$, $G^{b\,\alpha a}_{\alpha a_1} G_{b\,\alpha a}^{\alpha a_1} : 0 \to 0$ therefore $G^{b\,\alpha a}_{\alpha a_1} G_{b\,\alpha a}^{\alpha a_1} = \mathrm{id}$; let $\alpha \geq \beta \geq a$, $G^{b\,\alpha,a}_{\beta a,} G_{b\,\alpha,a}^{\beta,a} = \mathrm{id}$; therefore (G_b, G^b) is a functor from $\mathscr{A}_1$ to **Split**.

Let $\alpha \geq a \geq a_1$, if $b \leq a_1$, $pr_b(\alpha,a) V_{l\,\alpha a}^{\alpha a_1} = pr_b(\alpha,a_1) = G_{b\,\alpha a}^{\alpha a_1} pr_b(\alpha,a_1)$, $pr_b(\alpha,a_1) V_{r\,\alpha a_1}^{\alpha a} = pr_b(\alpha,a) = G^{b\,\alpha a}_{\alpha a_1} pr_b(\alpha,a)$, and if $b \not\leq a_1$, for any $v \in V_l(\alpha,a_1)$, $G_{b\,\alpha a}^{\alpha a_1} pr_b(\alpha,a_1)(v) = 0 = v_b 1[b \leq a_1] = pr_b(\alpha,a) V_{l\,\alpha a}^{\alpha a_1}$ and $pr_b(\alpha,a_1) V_{r\,\alpha a_1}^{\alpha a} = 0 = G_{b\,\alpha a_1}^{\alpha a} pr_b(\alpha,a)$. For $\alpha \geq \beta \geq a$, if $b \leq a$, $pr_b(\alpha,a) V_{l\,\alpha a}^{\beta a} = G^{\beta}_{\alpha} pr_b(\beta,a) = G_{b\,\alpha a}^{\beta a} pr_b(\beta,a)$ and $pr_b(\beta,a) V_{r\,\beta a}^{\alpha a} = F^{\alpha}_{\beta} pr_b(\alpha,a) = G^{b\,\alpha a}_{\beta a} pr_b(\alpha,a)$; if $b \not\leq a$, $pr_b(\beta,a) V_{r\,\beta a}^{\alpha a} = 0 = G^{b\,\alpha a}_{\beta a} pr_b(\alpha,a)$ and $pr_b(\alpha,a) V_{l\,\alpha a}^{\beta a} = 0 = G_{b\,\alpha a}^{\beta a} pr_b(\beta,a)$.

$\square$

Definition 4.4.7. Let (G,F) be a functor from $\mathscr{A} \in \hat{\mathscr{P}}_f$ to **Split**. For any $a \in \mathscr{A}$ let $(S_a, S^a) = \mathrm{im}\, pr_a \circ j : \mathscr{A}_1 \to \textbf{Split}$

Proposition 4.4.14. *Let (G,F) a functor from $\mathscr{A} \in \hat{\mathscr{P}}_f$ to **Split** that satisfies the intersection property, then $j|^{\operatorname{im} j}(L,R) \to (\bigoplus_{a\in\mathscr{A}_1} S_a, \bigoplus_{a\in\mathscr{A}_1} S^a)$ is an isomorphism. We shall note $j|^{\operatorname{im} j}$ as ψ.*

Proof. From Proposition 4.4.12 j is a monomorphism furthermore from Corollary 4.3.1 for any $(\alpha,a) \in \mathscr{A}_1$, $\operatorname{im} j(\alpha,a) = \bigoplus_{b\leq a} S_b(\alpha,a) = \bigoplus_{b\in\mathscr{A}} S_b(\alpha,a)$. □

Theorem 4.4.1. *Let (G,F) be a functor from $\mathscr{A} \in \hat{\mathscr{P}}_f$ to **Split**, (G,F) statisfies the intersection property if and only if (G,F) is decomposable.*

Proof. Let (G,F) be a functor from $\mathscr{A} \in \hat{\mathscr{P}}_f$ to **Split** that satisfies the intersection property; let us recall that (L,R) is a functor from $\mathscr{A}_1$ to **Split** (Corollary 4.4.2) and that for any $\alpha \geq \beta \geq a$, $L^{\beta a}_{\alpha a}$ is an isomorphism. Let us recall that for any $\alpha \geq \beta \geq a$, $\bigoplus_{b\in\mathscr{A}} S_b{}^{\beta a}_{\alpha a}\psi(\beta,a) = \psi(\alpha,a)L^{\beta a}_{\alpha a}$, where $\psi(\alpha,a)$, $\psi(\beta,a)$ (Proposition 4.4.14), $L^{\alpha a}_{\alpha a}$ are isomorphisms; therefore $\bigoplus_{b\in\mathscr{A}} S_b{}^{\beta a}_{\alpha a}$ is an isomorphism and as $(\bigoplus_{b\in\mathscr{A}} S_b, \bigoplus_{b\in\mathscr{A}} S^b)$ is a functor from $\mathscr{A}_1$ to **Split** its inverse is $\bigoplus_{b\in\mathscr{A}} S^b{}^{\alpha a}_{\beta a}$ and $S_a{}^{\beta a}_{\alpha a}$ is an isomorphism (its inverse is $S^a{}^{\alpha a}_{\beta a}$); for $\alpha \geq a \geq a_1 \geq b$, $S_a{}^{\alpha a_1}_{\alpha a}$ is also an isomorphism.

Let us remark that $\{(a,a) \in A_1 : a \in \mathscr{A}\}$ is isomorphic to $\mathscr{A}$; let for any $a \in \mathscr{A}$, $C^a = S^a|_{\mathscr{A}}$ and $C_a = S_a|_{\mathscr{A}}$; by definition (C^a, C_a) is a functor from $\mathscr{A}$ to **Split**; for any $a_2 \geq a_1 \geq a$, $C_a{}^{a_1}_{a_2} = S_a{}^{a_1 a_1}_{a_2,a_2} = S_a{}^{a_2 a_1}_{a_2 a_2} S_a{}^{a_1 a_1}_{a_2 a_1} = S_a{}^{a_1 a_1}_{a_2 a_1}$ is an isomorphism. Furthermore $(G,F) \cong (L,R)|_{\mathscr{A}}$ (Remark 4.4.8), therefore (G,F) is decomposable and its decomposition is $(\bigoplus_{a\in\mathscr{A}} C^a, \bigoplus_{a\in\mathscr{A}} C_a)$.

Let $\mathscr{A} \in \hat{\mathscr{P}}_f$, $(G,F) = (\bigoplus_{a\in\mathscr{A}} C^a, \bigoplus_{a\in\mathscr{A}} C_a)$, let $\alpha \in \mathscr{A}$, for any $a \leq \alpha$ and $c \leq \alpha$ and $v \in G(\alpha)$, $\pi^\alpha{}_a(v)(c) = G^a_\alpha F^\alpha_a(v)(c) = v_c 1[c \leq a]$. Let us denote π^α simply as π and $\mu_{\hat{\alpha}}$ as μ; for any $v \in \operatorname{im}\Pi(\pi)$ and $a \in \hat{\alpha}$, $\mu(v)(a) = \sum_{b\leq a} \mu(a,b) \bigoplus_{c\leq b} v_c = \bigoplus_{c\leq a} \sum_{\substack{b:\\ c\leq b\leq a}} \mu(a,b) v_c = v_a \times a$.

Furthermore for $a \leq \alpha$ and $b \leq \alpha$ and $c \leq \alpha$ and $v \in G(\alpha)$, $\pi_a\pi_b(v)(c) = v_c 1[c \leq a \& c \leq b]$, $\pi_a\pi_b(v) = \sum_{\substack{c\leq a\\ c\leq b}} v_c \times c = \sum_{\substack{c\leq a\\ c\leq b}} \mu(v)(c)$. Therefore (G,F) satisfies the intersection property.

□

Chapter 5

Interaction decomposition for Hilbert spaces

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5.1 Introduction

Similar constructions appear in statistical physics and for Graphical models [15], namely what we shall call an interaction decomposition and this is because both are interested in the same objects, namely Gibbs states. Indeed, in order to capture the interactions between a finite number of random variables, $(X_i \in E_i, i \in I)$, when modelizing a phenomenon, one can introduce the notion of potential.

Definition 5.1.1 (Potential and Gibbs state)**.** Let $(E_i, i \in I)$ be a collection of finite sets over a finite set I, let for $a \subseteq I$, $E_a = \prod_{i \in a} E_a$. A potential $\Phi = (\phi_a \in \mathbb{R}^{E_a}, a \in \mathscr{P}(I))$ is a collection of cylindric functions.

One can associtate to any potential a probability law as follows,

$$P = \frac{e^{\sum_{a \subseteq I} \phi_a}}{\sum_{x \in E_I} e^{\sum_{a \subseteq I} \phi_a(x)}} \tag{5.1.1}$$

We shall refer to such a potential as a Gibbs state with respect to the potential Φ.

Let us note for any $\mathscr{A} \subseteq \mathscr{P}(I)$,

$$H_{\mathscr{A}} = \{\sum_{a \in \mathscr{A}} \phi_a | \ \forall a \in \mathscr{A}, \ \phi_a \in \mathbb{R}^{E_a}\} \tag{5.1.2}$$

These are called hierarchical model subspaces by Lauritzen in this reference book on Graphical Models [14] and the associated probability laws live in what we have called in Chapter 2 a factorisation space, denoted as $G_{\mathscr{A}}$ for $\mathscr{A} \subseteq \mathscr{P}(I)$.

5.1.1 Interaction decomposition in Graphical models

In this chapter we focus once again on the interaction decomposition but this time focussing on the fact that it is a way to decompose the space of all random variables into **orthogonal** bits from which one can rebuild the hierarchical model subspaces. Let I be a finite set and let $E = \prod_{i \in I} E_i$ be a finite set; let us consider the canonical scalar product on $\mathbb{R}^{E_I}$, i.e. for any $f, g \in \mathbb{R}^{E_I}$,

$$\langle f, g \rangle = \sum_{x \in E_i} f(x) g(x) \tag{5.1.3}$$

Then the following result holds,

Theorem (Interaction decomposition for hierarchical model subspaces)**.** *The space of random variables, $\mathbb{R}^{E_I}$, can be decomposed into an othogonal direct sum of vector subspaces, $(S_a, a \subseteq I)$,*

$$\mathbb{R}^{E_I} = \bigoplus_{a \subseteq I} S_a \tag{5.1.4}$$

such that for any lower set $\mathscr{A} \subseteq \mathscr{P}(I)$,

$$H_{\mathscr{A}} = \bigoplus_{a \in \mathscr{A}} S_a \tag{5.1.5}$$

For $a \subseteq I$, let us note s_a the orthogonal projections onto S_a.

The interaction decomposition for hierarchical model subspaces is very useful when one wants to test if a (strictly positive) probability distribution is in a factorisation space, i.e. for $\mathscr{A} \subseteq \mathscr{P}(I)$,

$$P \in G_{\mathscr{A}} \iff \forall a \notin \mathscr{A}, \ s_a(\ln P) = 0 \tag{5.1.6}$$

5.1.2 Chaos decomposition in statistical physics

A simpler, and older case of the interaction decomposition is the chaos decomposition for a $\mathbb{N}$-filtered collection of subspaces of the space of all random variables, $(H_n, n \in \mathbb{N})$, with $H_n \subseteq H_m$ for $n \leq m$. The space of random variables is $L^2(\mathbb{R}^I, P)$ where P is a Gaussian mesure and $I = \mathbb{Z}^d$. Let $n \in \mathbb{N}$, it represents a degree, let for any integer $m \leq n$, any $x : [1, m] \to I$, and any $\phi \in \mathbb{R}^I$,

$$\Psi(x)(\phi) = \prod_{i \in [1,n]} \phi(x_i) \tag{5.1.7}$$

Then, H_n is the Hilbert subspace generated by the $\Psi(x)$ of degree at least n and indeed $H_n \subseteq H_{n+1}$.

The construction of a decomposition of the $(H_n, n \in \mathbb{N})$ relies on the Hermite-Ito polynomials for Gaussian fields (see Section 5.2). The aim of this construction is to define the space of observables, of potentials, when the number of random variables I is not finite. It is an essential construction in the field of statistical physics at the thermodynamic limit.

5.1.3 Generalizing the Interaction decomposition and intersection property

We want to unify both of the previous constructions into a unique algebraic framework. Both of these constructions are a decomposition of a collection of Hilbert subspaces of a given Hilbert space H, indexed on a partially ordered set. In greater generality these collections are functors from a poset to the category of Hilbert spaces with isometries as morphisms. Not all these functors admit a decomposition, the aim of this chapter is to prove that the ones that do are exactly those that that satisfy the intersection property.

5.1.4 Main results of this document

Prerequisites

In order to state the main results of this chapter we need firstly to recall some concepts.

Definition 5.1.2 (Categories)**.** We shall consider several categories; the category that has as objects Hilbert spaces and as morphism continuous linear applications will be noted **Hilb**, when the morphisms are isometries it is **IHilb**; when the objects are pre-Hilbert spaces and the morphisms are continuous linear applications

the corresponding category is **PHilb**; finaly the category that has as object pre-Hilbert spaces and as morphism isometries shall be denoted as **IPhilb** and the category that has as objects vector spaces and morphism linear applications shall still be noted as **Vect**. The reference to the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is made implicit.

Proposition 5.A.1 (Completion). Let H be a pre-Hilbert space, there is a Hilbert space that we shall note as $\mathscr{C}\, H$ and a continuous injective linear application $\eta_H : H \to \mathscr{C}\, H$ such that for any Hilbert spaces H_1 and any continuous linear application (of **PHilb**), $\phi : H \to H_1$, there is a unique continuous linear application $\mathscr{C}\, \phi : \mathscr{C}\, H \to H_1$ such that $\mathscr{C}\, \phi \eta_H = \phi$. $\mathscr{C}\, H$ is called the completion of H.

Definition 5.1.3 (Poset extension)**.** Let $\mathscr{A}_2$ be the subposet of $\mathscr{A} \times \mathscr{U}(\mathscr{A})$ constituted of couples $(\alpha, \mathscr{B})$ such that $\mathscr{B} \subseteq \hat{\alpha}$.

Corollary 5.A.3 (Sub as join semi-lattice). For the category **IHilb**, and for a given Hilbert space H, the "poset" $\mathrm{Sub}(H)$ of subobjects of H has a join, $\bigvee_{i \in I} H_i$, for any collection, $(H_i, i \in I)$, of objects of $\mathrm{Sub}(H)$.

Definition of decomposability

Definition (Decomposability for funtors in **IHilb**)**.** Let $\mathscr{A}$ be any poset, a functor $G : \mathscr{A} \to \mathbf{IHilb}$ is decomposable if there is a collection of functors $(S_a : \mathscr{A} \to \mathrm{core}\,\mathbf{IHilb}, a \in \mathscr{A})$ such that,

$$G \cong \mathscr{C} \bigoplus_{a \in \mathscr{A}} S_a 1[a \leq .] \tag{5.1.8}$$

i.e. there is a natural transformation, $(\phi_a, a \in \mathscr{A})$, where for any $a \in \mathscr{A}$, ϕ_a is an isometric isomorphisms from $G(a)$ to $\mathscr{C} \bigoplus_{b \leq a} S_b(a)$.

Definition of the intersection property

Definition (Intersection property)**.** Let $\mathscr{A}$ be any poset and G be a functor from $\mathscr{A}$ to **IHilb**. For any $(\alpha, \mathscr{B}) \in \mathscr{A}_2$ let us note $\pi^\alpha(\mathscr{B})$ the orthogonal projection of $G(\alpha)$ onto $G_\alpha^{\mathscr{B}} = \bigvee_{a \in \mathscr{B}} G_\alpha^a$, i.e.,

$$\pi^\alpha(\mathscr{B}) = \bigvee_{a \in \mathscr{B}} G_\alpha^a \left(\bigvee_{a \in \mathscr{B}} G_\alpha^a \right)^\dagger \tag{5.1.9}$$

We shall say that G satisfies the intersection property if for any

$$\forall (\alpha, a), (\alpha, b) \in \mathscr{A}_1, \quad \pi^\alpha(\hat{a} \cap \hat{b}) = \pi^\alpha(\hat{a}) \pi^\alpha(\hat{b}) \tag{I'}$$

When $\mathscr{A}$ is a meet-semilattice, the intersection property can be restated as,

$$\pi^\alpha(\widehat{a \wedge b}) = \pi^\alpha(\hat{a}) \pi^\alpha(\hat{b}) \tag{5.1.10}$$

Theorem

The main result of this chapter is the equivalence between the intersection property and the interaction decomposition.

Theorem 5.4.1 (Main theorem). Let $\mathscr{A}$ be a well founded poset and G be a functor from $\mathscr{A}$ to **IHilb**, G is decomposable if and only if it statifies the intersection property.

In order to prove this Theorem we must first prove it in the particular case of an increasing collection of Hilbert subspace of a given Hilbert space. In this context the definition of interaction decomposition is simpler.

Let H be a Hilbert space and $(K_i, i \in I)$ be an orthogonal collection of Hilbert subspaces of H over any set I; we shall note the closure of the sum of such collection of subspaces as,

$$\sum_{i \in I} K_i = \overset{\perp}{\bigoplus_{i \in I}} K_i \tag{5.1.11}$$

Definition (Decomposable collection of Hilbert subspaces)**.** Let $\mathscr{A}$ be any poset, let H be a Hilbert space and let $(H_a \subseteq H, a \in \mathscr{A})$ be a collection of Hilbert subspaces of H. $(H_a, a \in \mathscr{A})$ is said to be decomposable if there is a collection of Hilbert subspaces of H, $(S_a \subseteq H, a \in \mathscr{A}^+)$, such that for any $a \in \mathscr{A}$,

$$H = \overset{\perp}{\bigoplus_{a \in \mathscr{A}^+}} S_a \tag{5.1.12}$$

and for any $a \in \mathscr{A}$,

$$H_a = \overset{\perp}{\bigoplus_{b \leq a}} S_b \tag{5.1.13}$$

We shall call $(S_a, a \in \mathscr{A}^+)$ a decomposition of $(H_a, a \in \mathscr{A})$.

The simpler version of the Theorem 5.4.1 is as follows.

Theorem 5.3.1. Let $\mathscr{A}$ be a well-founded poset, let H be a Hilbert space, let $(H_a \in \mathbf{Gr}\, H, a \in \mathscr{A})$ be a collection of subspaces of H; $(H_a \in \mathbf{Gr}\, H, a \in \mathscr{A})$ is decomposable if and only if it satisfies the intersection property.

Which can be stated, when $\mathscr{A}$ is a meet-semilattice, as follows.

Corollary 5.3.1. Let $\mathscr{A}$ be a well-founded meet semilattice. Let H be a Hilbert space and $(H_a \in \mathbf{Gr}\, H, a \in \mathscr{A})$ be a collection of subspace of H such that for any $a, b \in \mathscr{A}$,

$$\pi_a \pi_b = \pi_{a \wedge b} \tag{5.1.14}$$

where π_a is the orthogonal projection of H onto H_a for any $a \in \mathscr{A}$.

Then, $(H_a, a \in \mathscr{A})$ is decomposable.

Furthermore any decomposable functor is in fact isometrically isomorphic to an increasing collection of Hilbert subspaces of a given Hilbert space.

Proposition 5.4.3. Let $\mathscr{A}$ be any poset and G be a functor from $\mathscr{A}$ to **IHilb**. If G is decomposable then there is H a Hilbert space and a functor $\mathscr{U} : \mathscr{A} \to \mathbf{Gr}\, H$ such that $G \cong \mathscr{U}$.

5.1.5 Structure of this chapter

We start by recalling what the chaos decomposition (Section 5.2). This motivates our study of decomposable collections of Hilbert subspaces (Section 5.3). In Section 5.3.2 we then remark that there is a unique decomposition for decomposable collections of subspaces, then we define the intersection property in this context (Section 5.3.3) and the equivalence theorem for collections of Hilbert subspaces (Section 5.3.4).

In the second part, Section 5.4, of this chapter we explain how decomposability for collections of Hilbert subspaces trnaslates in the context of functor to **IHilb** (Section 5.4.1). In Section 5.4.2 we discuss why the result of the first are not enough to prove the general equivalence theorem and the rest of Section 5.4 is dedicated to the proof of the equivalence the intersection property and decomposability for functors from a well-founded poset to **IHilb**.

Finaly in Section 5.5 we explain how the interaction decomposition of Chapter 4 and the one we propose here are related.

5.2 Interaction decomposition and Chaos decomposition

As in the previous chapter we will start by reconsidering the interaction for factor spaces taking again an other perspective on this construction.

Definition 5.2.1. Let H be a Hilbert space, we shall call the Grassmannian of H in **Hilb**, noted $\mathbf{Gr}_{\mathbf{Hilb}}\, H$ or simply $\mathbf{Gr}\, H$, the set of all Hilbert subspaces of H,

i.e. closed vector subspaces of H. Respectively $\mathbf{Gr}_{\mathbf{Vect}}\, H$ for the vector subspaces of H.

Definition 5.2.2. Let H be a Hilbert space, let $\mathscr{A}$ be any poset, let $(W_a \in \mathbf{Gr}\, H, a \in \mathscr{A})$ be a collection of subspaces of H; if for any $a, b \in \mathscr{A}$ such that $a \neq b$, W_a, W_b are othogonal, then we shall note the closure of $\sum_{a\in\mathscr{A}} W_a$ as $\overset{\perp}{\bigoplus}_{a\in\mathscr{A}} W_a$.

One can define the sum of a collection of pre-Hilbert space but it is not universal (see Appendix B Definition 5.A.4, Proposition 5.A.6). Let us recall that $\mathscr{C} : \mathbf{Philb} \to \mathbf{Hilb}$ is the completion functor that is left adjoint to the forgetful functor U (Appendix B Proposition 5.A.1, 5.A.4)

Furthermore for the category **IHilb** and for a given Hilbert space H, the "poset" $\mathrm{Sub}(H)$ of subobject of H has a join for any set I of objects of $\mathrm{Sub}(H)$ (Appendix B Corollary 5.A.3). Given a collection $(\phi_i, i \in I)$ of monomorphism that have as codomain H, $\bigvee_{i\in I} \phi_i$ can be represented as $\overline{\sum_{i\in I} \mathrm{im}\, \phi_i} \hookrightarrow H$, where $\mathrm{im}\, \phi_i$ is the image of ϕ_i in **Vect** but is also a Hilbert space as ϕ_i is an isometry for any $i \in I$. In particular for collections of Hilbert subspace $(H_i \subseteq H, i \in I)$, a representant of $\bigvee_{i\in I} H_i$ is $\overline{\sum_{i\in I} H_i}$, where H_i is implicitly identified to its inclusion in H.

Remark 5.2.1. Let $\mathscr{A}$ be any poset and let $(W_a \in \mathbf{Gr}\, H, a \in \mathscr{A})$ be a collection of subspaces of a Hilbert space H such that $\overline{\sum_{a\in\mathscr{A}} W_a} = \overset{\perp}{\bigoplus}_{a\in\mathscr{A}} W_a$ then

$$\mathscr{C} \bigoplus_{a\in\mathscr{A}} W_a \cong \overset{\perp}{\bigoplus}_{a\in\mathscr{A}} W_a = \bigvee_{a\in\mathscr{A}} W_a \tag{5.2.1}$$

Theorem 5.2.1 (Decomposition into interaction subspaces)**.** *Let I be a finite set and $(H_a, a \subseteq I)$ be the collection of factor spaces; for any $a \subseteq I$, let $S_a = H_a \cap \left(\sum_{b\subsetneq a} H_b\right)^{\perp}$, then*

$$H_a = \overset{\perp}{\bigoplus_{b\leq a}} S_b \tag{5.2.2}$$

In Speed's review on the decomposition into interaction subspaces [19] the proof of this result is done by induction and in Lauritzen's reference book on Graphical models (Appendix B [14]) the explicit expression of the projections on the interaction spaces is given, which is for any $a \subseteq I$,

$$s_a = \sum_{b\leq a} (-1)^{|a\setminus b|} \pi_b \tag{5.2.3}$$

We would like in this document to extend the interaction decomposition to Hilbert spaces. At first sight one could think that it is a particular case of an interaction decomposition compatible with a collection of projectors; one could argue that here Equation 5.2.3 is simply the canonical decomposition with respect to the collection of projectors $(\pi_a, a \in \mathscr{A})$ as explained in Chapter 4. However it is not the case as we shall discuss in Section 5; the two points of view differ and as a consequence the results we shall present in Section 3 requiere much weaker constraints on the poset $\mathscr{A}$ than the one needed for collections of projectors.

There is an other example of such decomposition for a $\mathbb{N}$-filtered collection of subspaces of the space of all random variables which is the chaos decomposition. We will follow the presentation given in Sinai's *Theory of Phase Transition: rigorous results* [36]. This construction makes it possible to characterize the space of potentials. Let us recall how one can define potentials for a non finite but countable I, and with $E_i = \mathbb{R}$ for $i \in I$.

Notation 5.2.1. For $m \in \mathbb{N}$, we note $[m]$ for $[1, ..., m]$.

Let $I = \mathbb{Z}^d$, $E = \mathbb{R}^I$; for any $\phi \in E$ and $x : [m] \to I$, let $\phi(x) = \prod_{i \in [m]} \phi(x_i)$. Let us note the set of maps of $I^{[m]}$ as $\mathscr{A}_m$.

Let us note $\Phi(x)$ the random variable that sends $\phi \in \mathbb{R}^I$ to $\phi(x)$.

Notation 5.2.2. By convention $[0] = \emptyset$ and $\prod_{i \in \emptyset} \phi(x_i) = 1$; therefore $\mathscr{A}_0$ has one element.

Let us assume that P is a Gaussian distribution on E, for any $x \in \mathscr{A}$, $\Phi(x) \in L^2(E, P)$; for a collection $(v_j \in L^2(E, P), j \in J)$, we shall note $\langle v_j, j \in J \rangle$ the Hilbert subspace of $L^2(E, P)$ generated by these elements, i.e. the smallest Hilbert subspace of $L^2(E, P)$ that contains them, its explicit expression is $\overline{\sum_{j \in J} \mathbb{K} v_j}$ when $L^2(E, P)$ is a $\mathbb{K}$-Hilbert space.

For any $m \in \mathbb{N}$, let $H(m) = \langle \Psi(x), x \in \bigcup_{k \leq m} \mathscr{A}_m \rangle$ and $H = L^2(E, P)$.

Lemma 5.2.1.

$$H = \overline{\sum_{m \in \mathbb{N}} H_m} \tag{5.2.4}$$

Proof. Theorem 2 [37]

□

Proposition 5.2.1. *For $m \in \mathbb{N}$, let*

$$S_m = H(m) \cap H(m-1)^{\perp} \tag{5.2.5}$$

then,

$$H(m) = \overset{\perp}{\bigoplus}_{k \leq m} S_m \tag{5.2.6}$$

Furthermore $L^2(E, P) = \overset{\perp}{\bigoplus}_{m \in \mathbb{N}} S_m$

Proof. Let us show the first statement by induction.

$H(\emptyset) = \mathbb{K} = S(\emptyset)$. Let us assume that for some $m \geq 0$, $H(m) = \overset{\perp}{\bigoplus}_{0 \leq k \leq m} S_k$; one has that $H(m+1) = S_{m+1} \bigoplus^{\perp} H(m)$ therefore,

$$H(m+1) = \overset{\perp}{\bigoplus_{0 \leq k \leq m+1}} S_k \tag{5.2.7}$$

which ends the proof by induction.

Therefore $\sum_{m \in \mathbb{N}} H_m = \bigoplus_{m \in \mathbb{N}} S_n$ and as $H = \overline{\sum_{m \in \mathbb{N}} H_m}$ (Lemma 5.2.1) therefore $H = \overset{\perp}{\bigoplus}_{m \in \mathbb{N}} S_n$.

□

Definition 5.2.3. [Definition 4.7 [36]]

Let P be a Gaussian distribution on E, let $m \in \mathbb{N}$, let $x \in \mathscr{A}_m$; $\Psi(x)$ has a unique orthogonal decomposition, in other words there is a unique couple (u_1, u_2) such that $\Psi(x) = u_1 + u_2$ with $u_1 \in H(m-1)^{\perp}$, and $u_2 \in H(m-1)$ with $\langle u_1, u_2 \rangle = 0$.

We shall note u_1 as $: \Psi(x) :$ and call it the Hermite-Ito polynomial of $\Psi(x)$.

Proposition 5.2.2. *For any $m \in \mathbb{N}$ and $x \in \mathscr{A}_m$, then $S_m(\Psi(x)) =: \Psi(x) :$, its Hermite-Ito polynomial.*

Proof. By construction,

$$\Psi(x) = S_m(\Psi(x)) + \sum_{k<m} S_k(\Psi(x)) \tag{5.2.8}$$

and therefore, $S_m(\Psi(x)) =: \Psi(x) :$

□

Instead of introducing directly the main equivalence theorem of this chapter for the most general setting of functors in **IHilb** we decided, in coherence with the previous chapters of this thesis, rather to start by restricting our attention to collections of Hilbert subspaces which will serve as a motivation and justification for the general setting.

5.3 Necessary and sufficient condition for the interaction decomposition to hold for a collection of Hilbert subspaces

5.3.1 Decomposability for collections of Hilbert subspaces

As stated in the last section, a common way to state an interaction decomposition for factor subspaces is to use the canonical scalar product on $\mathbb{R}^E = L^2(E)$, where E is a finite set. We shall state this result in a more general context by giving a necessary and sufficient condition for a collection of Hilbert subspaces over a well founded poset (Chapter 3 Definition 3.1.1) to be decomposable.

Let us start by defining what an interaction decomposition for a collection of Hilbert subspaces of a given Hilbert space is.

Definition 5.3.1 (Decomposable collection of Hilbert subspaces)**.** Let $\mathscr{A}$ be any poset, let H be a Hilbert space and let $(H_a, a \in \mathscr{A}) \in \hom(\mathscr{A}, \mathbf{Gr}\, H)$ be an increasing collection of subspaces of H. $(H_a, a \in \mathscr{A})$ is said to be decomposable if there is a collection of subspaces of H, $(S_a \in \mathbf{Gr}\, H, a \in \mathscr{A}^+)$, such that for any $a \in \mathscr{A}$,

$$H = \overset{\perp}{\bigoplus_{a \in \mathscr{A}^+}} S_a \tag{5.3.1}$$

and such that for any $a \in \mathscr{A}$,

$$H_a = \overset{\perp}{\bigoplus_{b \leq a}} S_b \tag{5.3.2}$$

We shall call $(S_a, a \in \mathscr{A}^+)$ a decomposition of $(H_a, a \in \mathscr{A})$; by convention $H_1 = H$.

5.3.2 Characterizing the decompositions

Proposition 5.3.1. *Let H be a Hilbert space and let $(H_a \in \boldsymbol{Gr}\, H, a \in \mathscr{A})$ be a decomposable collection of Hilbert subspaces of H; let $(S_a, a \in \mathscr{A}^+)$ be a decomposition of $(H_a, a \in \mathscr{A})$, then for any $a \in \mathscr{A}^+$*

$$S_a = H_a \cap \bigcap_{b \lneq a} H_b^{\perp} \tag{5.3.3}$$

Lemma 5.3.1. *Let I be any set and $(V_i, i \in I)$ a collection of vector subspaces, that are not necessarily closed, of a Hilbert space H, then*

$$\overline{\sum_{i \in I} V_i} = \overline{\sum_{i \in I} \overline{V_i}} \tag{5.3.4}$$

Proof. By definition $\overline{\sum_{i \in I} V_i} \subseteq \overline{\sum_{i \in I} \overline{V_i}}$; for any $i \in V_i$, $V_i \subseteq \overline{\sum_{i \in I} V_i}$ and therefore $\overline{\sum_{i \in I} \overline{V_i}} \subseteq \overline{\sum_{i \in I} V_i}$. □

Remark 5.3.1. The sum in Lemma 5.3.1 can be rewritten as,

$$\mathscr{C} \bigvee V_i = \bigvee_{i \in I} \mathscr{C}\, V_i \tag{5.3.5}$$

where V_i are seen as pre-Hilbert spaces, and which is a consequence of $\mathscr{C}$ being left adjoint to the U.

Proof of Proposition 5.3.1

Proof. let $(H_a \in \mathbf{Gr}\, H, a \in \mathscr{A})$ be a decomposable collection of Hilbert subspaces of H.

Let us notice that $\bigcap_{b \lneq a} H_b^{\perp} = \left(\sum_{b \lneq a} H_b \right)^{\perp}$. Furthermore,

$$\overline{\sum_{b \lneq a} H_b} = \overline{\sum \overline{\sum_{b \lneq a c \leq b} S_c}} \tag{5.3.6}$$

By Lemma 5.3.1 $\overline{\sum_{b \lneq a} H_b} = \overline{\sum \sum_{b \lneq a c \leq b} S_c}$ and one remarks that $\sum \sum_{b \lneq a c \leq b} S_c = \sum_{b \lneq a} S_b$. Futhermore, as $H_a = \overset{\perp}{\bigoplus_{b \leq a}} S_a$ one has that S_a is orthogonal to $\sum_{b \lneq a} S_b$; therefore, $S_a = H_a \cap \left(\overline{\sum_{b \lneq a} H_b} \right)^{\perp}$ and so $S_a = H_a \cap \bigcap_{b \lneq a} H_b^{\perp}$. □

Remark 5.3.2. A consequence of Proposition 5.3.1 is that if $(H_a, a \in \mathscr{A})$ is decomposable then it has a unique decomposition, i.e. if for any $a \in \mathscr{A}^+$, $H_a = \overset{\perp}{\underset{b \leq a}{\bigoplus}} S_a = \overset{\perp}{\underset{b \leq a}{\bigoplus}} S_{1a}$ then for any $a \in \mathscr{A}$, $S_a = S_{1a}$.

Notation 5.3.1. Let $(H_a \in \mathbf{Gr}\, H, a \in \mathscr{A})$ be a collection of Hilbert subspaces of a Hilbert space H, for $a \in \mathscr{A}^+$ we shall note $s_a^\perp$ the orthogonal projection onto $S_a = H_a \cap \bigcap_{b \lneq a} H_b^\perp$

5.3.3 Intersection property

Definition 5.3.2. Let H be a Hilbert space, let $\mathscr{A}$ be any poset and let $(H_a \in \mathbf{Gr}\, H, a \in \mathscr{A})$ be an increasing collection of subspace of H; for any subposet $\mathscr{B}$ of $\mathscr{A}$ we shall denote $H(\mathscr{B})$ the completion of $\sum_{b \in \mathscr{B}} H_b$, i.e. $\overline{\sum_{b \in \mathscr{B}} H_b}$ (Appendix B Proposition 5.A.5). We shall denote $\pi_{\mathscr{B}}$ the othogonal projection on $H(\mathscr{B})$ and if $\mathscr{B} = \hat{a}$ with $a \in \mathscr{A}$ we shall simply note it as π_a.

Proposition 5.3.2. *Let H be a Hilbert space, let I be any set and let $(V_i, i \in I)$ be vector subspaces of H, i.e pre-Hilbert subspace of H. If $H = \overset{\perp}{\bigoplus}_{i \in I} V_i$ then for any $J, J_1 \subseteq I$ disjoint subsets of I,*

$$H = \overset{\perp}{\bigoplus}_{j \in J} V_j \oplus^\perp \overset{\perp}{\bigoplus}_{j \in J_1} V_j \tag{5.3.7}$$

Proof. Let $V = \overset{\perp}{\bigoplus}_{j \in J} V_j$, $U = \overset{\perp}{\bigoplus}_{j \in J_1} V_j$, one has that $H = V + U$ by Lemma 5.3.1 and bacause a finite sum of closed spaces is a closed.

Let $v \in V$ and $u \in U$ $\langle v, u \rangle = 0$. Let $v \in \overline{V}$ and $u \in \overline{U}$, then there is $v_n \in V, n \in \mathbb{N}$ and $u_n \in U, n \in \mathbb{N}$ such that $\lim_{n \to \infty} v_n = v$ and $\lim_{n \to \infty} u_n = u$. There is $N \in \mathbb{N}$ such that for any $n \geq N$,

$$|\langle u, v \rangle| = |\langle u, v \rangle - \langle u_n, v_n \rangle| \leq 2\|v - v_n\| \|u - u_n\| \tag{5.3.8}$$

Therefore $\langle u, v \rangle = 0$. □

Definition 5.3.3 (Intersection property)**.** Let $\mathscr{A}$ be any poset, let H be a Hilbert space, let $(H_a \in \mathbf{Gr}\, H, a \in \mathscr{A})$ be an increasing collection of subspace of H; $(H_a \in \mathbf{Gr}\, H, a \in \mathscr{A})$ is said to verify the intersection property if,

$$\forall a, b \in \mathscr{A} \quad \pi(\hat{a} \cap \hat{b}) = \pi_a \pi_b \tag{I}$$

5.3.4 Theorem: equivalence between interaction decomposition and intersection property

Lemma 5.3.2. *Let $\mathscr{A}$ be any poset, let H be a Hilbert space, let $(H_a \in \boldsymbol{Gr}\, H, a \in \mathscr{A})$ be an increasing collection of subspace of H; if $(H_a, a \in \mathscr{A})$ is decomposable, then $(H_a, a \in \mathscr{A})$ satisfies the intersection property.*

Proof. Let $(H_a \in \mathbf{Gr}\, H, a \in \mathscr{A})$ be a decomposable collection of subspace of H; let $(S_a, a \in \mathscr{A})$ be its decomposition. Let $v \in \bigoplus_{c \leq a} S_c$ and $u \in S_b$,

$$\langle v, \pi_a(u) \rangle = \langle v, u \rangle = 1[b \leq a] \langle v, u \rangle \tag{5.3.9}$$

Furthermore as $\sum_{c \leq a} S_c$ is dense in H_a, one has that for any $v \in H_a$, $\langle v, \pi_a(u) \rangle = 1[b \leq a] \langle v, u \rangle$; therefore $\pi_a s_b^{\perp} = 1[b \leq a] s_b^{\perp}$.

Let $v \in \bigoplus_{c \notin \hat{a} \cap \hat{b}} S_c$, and $u \in H$ then by Equation (5.3.9) one has that,

$$\langle \pi_a \pi_b u, v \rangle = \langle u, \pi_b \pi_a v \rangle = 0 \tag{5.3.10}$$

Therefore by Proposition 5.3.2, $\pi_a \pi_b u \in \overset{\perp}{\bigoplus}_{c \in \hat{a} \cap \hat{b}} S_c = H(\hat{a} \cap \hat{b})$; furthermore for any $v \in H(\hat{a} \cap \hat{b})$ as $v \in H(a) \cap H(b)$,

$$\langle u, v \rangle = \langle \pi_b u, v \rangle = \langle \pi_b u, \pi_a v \rangle = \langle \pi_a \pi_b u, v \rangle \tag{5.3.11}$$

Therefore, $\pi_a \pi_b = \pi(\hat{a} \cap \hat{b})$. □

Lemma 5.3.3. *Let $\mathscr{A}$ be any poset, let H be a Hilbert space, let $(H_a \in \boldsymbol{Gr}\, H, a \in \mathscr{A})$ be an increasing collection of subspace of H that satisfies the intersection property, then for any $a, b \in \mathscr{A}^{+1}$,*

$$s_a^{\perp} s_b^{\perp} = \delta_a(b) s_a^{\perp} \tag{5.3.12}$$

Proof. If $a \neq b$ then $\hat{a} \cap \hat{b} \subsetneq \hat{b}$; for any $v, u \in H$,

$$\langle s_a^{\perp}(v), s_b^{\perp}(u) \rangle = \langle \pi_a s_a^{\perp}(v), \pi_b s_b^{\perp}(u) \rangle = \langle \pi_b \pi_a s_a^{\perp}(v), s_b^{\perp}(u) \rangle \tag{5.3.13}$$

Therefore,

$$\langle s_a^{\perp}(v), s_b^{\perp}(u) \rangle = \langle \pi(\hat{a} \cap \hat{b}) s_a^{\perp}(v), s_b^{\perp}(u) \rangle \tag{5.3.14}$$

As $\pi(\hat{a} \cap \hat{b}) s_a^{\perp}(v) \in \overline{\sum_{c \lneq b} H_c}$, by construction

$$\langle \pi(\hat{a} \cap \hat{b}) s_a^{\perp}(v), s_b^{\perp}(u) \rangle = 0 \tag{5.3.15}$$

and so,

$$\langle s_a^{\perp} s_b(v), u \rangle = 0 \tag{5.3.16}$$

As $s_a^{\perp}$ is a projector, ${s_a^{\perp}}^2 = s_a^{\perp}$. □

Theorem 5.3.1. *Let $\mathscr{A}$ be a well-founded poset, let H be a Hilbert space, let $(H_a \in \boldsymbol{Gr}\, H, a \in \mathscr{A})$ be an increasing collection of subspaces of H; $(H_a \in \boldsymbol{Gr}\, H, a \in \mathscr{A})$ is decomposable if and only if it satisfies the intersection property.*

Proof. The necessary condition is Lemma 5.3.2.

Let us show by transfinite induction on $\mathscr{A}$ (Chapter 3 Proposition 3.1.1) that for any $a \in \mathscr{A}$, $H_a = \overline{\sum_{b \leq a} S_b}$.

Let $a \in \mathscr{A}$, let for any $b < a$, $H_b = \overline{\sum_{c \leq b} S_b}$, then by construction, $H_a = S_a \overset{\perp}{\bigoplus} \overline{\sum_{b \lneq a} H_b}$. By Lemma 5.3.1, $\overline{\sum_{b \lneq a} H_b} = \overline{\sum_{c \lneq a} S_c}$. This ends the proof by transfinite induction.

Lemma 5.3.3 enable us to conclude that the sum is an othogonal one, which ends the proof.

□

Remark 5.3.3. Once one has exhibited the good framework, i.e. the condition on the subspaces (Intersection property) and the condition on the poset (well founded), the proof of Theorem 5.3.1 is essentially repeating the one given in the particular case of factor spaces for the canonical scalar product as presented in Lemma 2.1 [19]. However if one does not consider that the ambient space is a Hilbert space the methods for showing that the interaction decomposition exists if and only if the intersection property is satisfied are very different from the one present above as we will explain in Section 5.4.2.

5.3.5 Simplification for meet semi-lattices

Corollary 5.3.1. *Let $\mathscr{A}$ be a well-founded meet semi-lattice. Let H be a Hilbert space and $(H_a \in \boldsymbol{Gr}\, H, a \in \mathscr{A})$ be a collection of subspace of H such that for any $a, b \in \mathscr{A}$,*

$$\pi_a \pi_b = \pi_{a \wedge b} \tag{5.3.17}$$

Then $(H_a, a \in \mathscr{A})$ is decomposable.

Proof. When $\mathscr{A}$ is a semi-lattice $\widehat{a \cap b} = \hat{a} \cap \hat{b}$, therefore one concludes by Theorem 5.3.1. □

5.4 Extension to IHilb

5.4.1 Decomposability for functors in IHilb

Let H be a Hilbert space and $\mathrm{U} : \mathscr{A} \to \mathbf{Gr}\, H$ be a poset morphism; the poset $\mathbf{Gr}\, H$ can be identified to the category that has as objects the Hilbert subspaces of H and as morphisms the inclusion maps; therefore U is a functor from $\mathscr{A}$ into this category. Furthermore the inclusions are in fact isometries, so $\mathbf{Gr}\, H$ is a subcategory of **IHilb**. We shall therefore consider in what follows, functors from a poset $\mathscr{A}$ to **IHilb**.

Let us now try to find what is the good notion of decomposability for functors from a poset to **IHilb**; in order to do so let us reformulate decomposability for functors from $\mathscr{A}$ to $\mathbf{Gr}\, H$.

Let $\mathscr{A}$ be a poset, let $\mathbf{C}$ be any category; any object of A of $\mathbf{C}$ induces a constant functor $iA : \mathscr{A} \to \mathbf{C}$, that we shall note as A.

For any collection of pre-Hilbert spaces, $(H_i, i \in I)$, one can define what we called its direct sum $\bigoplus_{i\in I} H_i$ (Appendix B Definition 5.A.4) and for any collection of pre-Hilbert spaces $(K_i, i \in I)$ and for any collection of morphism of **Philb**, $(\phi_i : H_i \to H_{1i}, i \in I)$, we shall call ϕ the following morphism,

$$\begin{array}{rccc} \phi & : & \bigoplus_{i\in I} H_i & \to & \bigoplus_{i\in I} K_i \\ & & (v_i, i \in I) & \mapsto & (\phi_i(v_i), i \in I) \end{array} \tag{5.4.1}$$

ϕ is continuous and if $(\phi_i, i \in I)$ are isometries so is ϕ.

Proposition 5.4.1. *Let I be any set and let $(G_i, i \in I)$ be a collection of functors from a poset $\mathscr{A}$ to **IPhilb**; let for any $a \in \mathscr{A}$,*

$$(\bigoplus_{i\in I} G_i)(a) = \bigoplus_{i\in I} (G_i(a)) \tag{5.4.2}$$

and for any $a, b \in \mathscr{A}$ such that $b \leq a$,

$$(\bigoplus_{i\in I} G_i)_a^b = \bigoplus_{i\in I} G_{ia}^{\ b} \tag{5.4.3}$$

*Then $\bigoplus_{i\in I} G_i$ is a functor from $\mathscr{A}$ to **IPhilb**.*

Proof. Let U_1 : **Philb** $\to$ **Vect** be the forgetful functor, let $a, b, c \in \mathscr{A}$ such that $c \leq b \leq a$ then $U_1(\bigoplus_{i\in I} G_{ia}^{\ b}) = \bigoplus_{i\in I} G_{ia}^{\ b}$, where the direct sum is taken in **Vect**; therefore $U_1 \bigoplus_{i\in I} G_{ia}^{\ b} \circ U_1 \bigoplus_{i\in I} G_{ib}^{\ c} = U_1 \bigoplus_{i\in I} G_{ia}^{\ c}$ and $\bigoplus_{i\in I} G_{ia}^{\ b} \bigoplus_{i\in I} G_{ib}^{\ c} =$

$\bigoplus_{i\in I} G_{i_a}^{c}$.

For any $a, b \in \mathscr{A}$ such that $a \geq b$, $\bigoplus_{i\in I} G_{i_a}^{b}$ is an isometry as, for any $v \in \bigoplus_{i\in I} G_i(b)$,

$$\langle \bigoplus_{i\in I} G_{i_a}^{b}(v), \bigoplus_{i\in I} G_{i_a}^{b}(v)\rangle = \sum_{i\in I} \langle G_{i_a}^{b}(v_i), G_{i_a}^{b}(v_i)\rangle \tag{5.4.4}$$

□

Proposition 5.4.2. *Let $\mathscr{A}$ be any poset, let $H : \mathscr{A} \to \boldsymbol{Gr}H$ be a functor; $(H_a, a \in \mathscr{A})$ is decomposable in the sense of Definition 5.3.1 if and only if H is isometrically isomorphic to $\mathscr{C} \bigoplus_{a\in\mathscr{A}} iS_a 1[a \leq .]$, where $(S_a, a \in \mathscr{A}^+)$ is the decomposition of $(H_a, a \in \mathscr{A})$.*

Proof. Let $(H_a, a \in \mathscr{A})$ be decomposable, then for any $a \in \mathscr{A}$,

$$H_a = \mathscr{C} \bigoplus_{b\leq a} S_b = \mathscr{C} \bigoplus_{b\in\mathscr{A}} S_b(a) 1[b \leq a] \tag{5.4.5}$$

In other words, the application that sends any $v \in H_a$ to $\bigoplus_{b\leq a} s_b^{\perp}(v)$ is an isomorphic isometry. Furthermore for $b \in \mathscr{A}$ such that $b \leq a$, the inclusion of $H(b)$ to $H(a)$ corresponds, up to this isomophic isometry, to $\mathscr{C} \bigoplus_{c\in\mathscr{A}} iS_c 1[c \leq .]_a^b$.

The other way around, one only needs to remark that when H_a is isometrically isomorphic to $\bigoplus_{b\leq a} S_b$ then, for any $b, c \in \mathscr{A}$ such that $b \leq a$ and $c \leq a$, and for $u \in S_b$, $v \in S_c$, then

$$\langle u, v\rangle = \delta_b(c)\langle u, v\rangle \tag{5.4.6}$$

□

As for any $a \in \mathscr{A}$, S_a is a functor from $\mathscr{A}$ to **IHilb**, Propositions 5.4.1, 5.4.2 unables us to conclude that decomposability can be directly stated for functor in **IHilb** as that that it is equal to a decomposition up to isomorphism; lets make this more formal.

Definition 5.4.1 (Decomposability for functors in **IHilb**)**.** Let $\mathscr{A}$ be any poset, a functor $G : \mathscr{A} \to$ **IHilb** is decomposable if there is a collection of functors $(S_a : \mathscr{A} \to \operatorname{core}\mathbf{IHilb}, a \in \mathscr{A})$ such that,

$$G \cong \mathscr{C} \bigoplus_{a\in\mathscr{A}} S_a 1[a \leq .] \tag{5.4.7}$$

i.e. there is a natural transformation, $(\phi_a, a \in \mathscr{A})$, where for any $a \in \mathscr{A}$, ϕ_a is an isometric isomorphism from $G(a)$ to $\mathscr{C} \bigoplus_{b \leq a} S_b(a)$. $(S_a 1[a \leq .], a \in \mathscr{A})$ will be called a decomposition of G.

Proposition 5.4.3. *Let $\mathscr{A}$ be any poset and G be a functor from $\mathscr{A}$ to **IHilb**. If G is decomposable then there is H a Hilbert space and a functor $\mathscr{U} : \mathscr{A} \to \boldsymbol{Gr}\, H$ such that $G \cong \mathscr{U}$.*

Proof. Let G be decomposable and let $(S_a, a \in \mathscr{A})$ be a decomposition of $c \in \mathscr{A}$, then $S_c(c)1[c \leq .] \cong S_c$ therefore $\mathscr{C} \bigoplus_{c \in \mathscr{A}} S_c \cong \bigoplus_{c \in \mathscr{A}} S_c(c)1[c \leq .]$ and for any $a \in \mathscr{A}$, $\cong \bigoplus_{c \leq a} S_c(c)$ is a subspace of $\bigoplus_{c \in \mathscr{A}} S_c(a)$ which ends the proof.

□

5.4.2 No short-cut

If any functor from $\mathscr{A}$ to **IHilb** was isometrically isomorphic to a functor from $\mathscr{A} \to \mathbf{Gr}\, H$ for some Hilbert space H, the last section would give an answer to when one can say that a functor in **IHilb** is decomposable; however we will now explain why it is not the case.

Notation 5.4.1. Let us note V the forgetful functor from **IHilb** to **Vect**.

Proposition 5.4.4. *Let $\mathscr{A}$ be any poset and $G : \mathscr{A} \to$ **IHilb** be a functor, then $\eta : VG \to \operatorname{colim}_{a \in \mathscr{A}} VG$ is a monomorphism of functors of **Vect**.*

Proof. For any $a, b \in \mathscr{A}$ such that $b \leq a$, ${G_a^b}^\dagger G_a^b = \mathrm{id}$; therefore G_a^b is injective and so is VG_a^b, therefore by Proposition 3.2.1, $\eta_a : G(a) \to \operatorname*{colim}_{a \in \mathscr{A}} VG$ is injective.

□

For any pre-Hilbert space H and any closed pre-Hilbert subspace H_1 of H, one can define the pre-Hilbert space H/H_1 (see Appendix Proposition 5.A.7).

Notation 5.4.2. Let G be a functor from a category $\mathbf{C}$ to **Vect**; we shall note $\ker G$ the vector subspace of $\bigoplus_{a \in \mathscr{A}} G(a)$ generated by the collection $(G_a^b(v_b) \times a - v_b \times b | (b \leq a, v_b \in G(b)))$.

For a functor $G : \mathscr{A} \to$ **IHilb**, $\eta : G \to \operatorname{colim}_{a \in \mathscr{A}} VG$ induces a morphism $\rho : G \to \bigoplus G(a)/\overline{\ker G}$, where $\overline{\ker G} \subseteq \bigoplus G(a)$. By composition with the injection from $\bigoplus G(a)/\overline{\ker G}$ to $\mathscr{C}(\bigoplus G(a)/\overline{\ker G})$, it induces a morphism $\psi : G \to \mathscr{C}(\bigoplus G(a)/\overline{\ker G})$. Let us remark that by Proposition 5.A.9 and the fact that $\mathscr{C} \ker G = \mathscr{C}\, \overline{\ker G}$, one has that,

$$\mathscr{C}(\bigoplus G(a)/\overline{\ker G}) = \mathscr{C} \bigoplus G(a)/\mathscr{C} \ker G \tag{5.4.8}$$

In general ψ is not injective and even when it is, it is not an isometry.

Indeed in general $\ker G$ is not closed in $\bigoplus_{a\in\mathscr{A}} G(a)$ and we shall now give a counter example. Let $\mathscr{A} = \mathbb{N}$ and H a Hilbert space, for any $n \in \mathbb{N}$ let $G(n) = H$ and for any $m \geq n$ let $G^n_m = \mathrm{id}$. The orthogonal to $\ker G$ in $\mathscr{C} \bigoplus_{n\in\mathbb{N}} H$ is 0 as we shall show. Any collection $(v_n, n \in \mathbb{N})$ is in $\ker G^\perp$ if and only if, for any $u \in H$ and $n \in \mathbb{N}$,

$$\langle v_n, u\rangle = \langle v_{n+1}, u\rangle \tag{5.4.9}$$

Therefore $(v_n, n \in \mathbb{N})$ is a constant sequence; however $\lim_{n\to\infty} ||v_n|| = 0$ therefore $(v_n, n \in \mathbb{N}) = 0$. Then $\overline{\ker G} = \mathscr{C} \bigoplus_{a\in\mathscr{A}} G(a)$ and $\ker G$ is dense in $\bigoplus_{a\in\mathscr{A}} G(a)$; finaly let $v \in H$; pose $v_0 = v$ and $v_n = 0$ for $n \in \mathbb{N}$, v is not in $\ker G$. In this example, $\psi_a = 0$ for any $a \in \mathscr{A}$.

Let us now show that even when ψ is injective, it is in generale not a morphism of functor from $\mathscr{A}$ to **IHilb**.

Proposition 5.4.5. *Let $\mathscr{A}$ be any poset, let us consider a funtor $G : \mathscr{A} \to$ **IHilb**. $\mathscr{C}\, G/\mathscr{C} \ker G$ is isometrically isomorphic to $(\mathscr{C} \ker G)^\perp \subseteq \mathscr{C} \bigoplus_{a\in\mathscr{A}} G(a)$.*

Proof. Let $p : \mathscr{C} \bigoplus_{a\in\mathscr{A}} G(a) \to (\mathscr{C} \ker G)^\perp$ be the orhogonal projection on $(\mathscr{C} \ker G)^\perp$; by definition it is surjective furthermore for any $v \in \mathscr{C} \bigoplus_{a\in\mathscr{A}} G(a)$,

$$\|[v]\| = \inf_{u\in\ker G} \|v + u\| = \inf_{u\in\overline{\ker G}} \|v + u\| \tag{5.4.10}$$

and,

$$\inf_{u\in\overline{\ker G}} \|v + u\| = \|p(v)\| \tag{5.4.11}$$

which ends the proof.

□

Let G be a functor from $\mathscr{A}$ to **IHilb**, let $p : \mathscr{C} \bigoplus_{a\in\mathscr{A}} G(a) \to \mathscr{C} \ker G^\perp$ be the orhogonal projection on $(\mathscr{C} \ker G)^\perp \subseteq \mathscr{C} \bigoplus_{a\in\mathscr{A}} G(a)$. Let $a, b \in \mathscr{A}$ with $b < a$, let $v \in G(b)$,then

$$\langle v, v \times b - G^b_a(v) \times a\rangle = \|v\|^2 \tag{5.4.12}$$

Therefore v is not in $(\mathscr{C} \ker G)^\perp$ unless $||v|| = 0$ and if $||v|| \neq 0$, by Proposition 5.4.5,

$$\|\psi_b(v)\| = \|p(v)\| < \|v\| \tag{5.4.13}$$

Therefore if $\mathscr{A}$ has at least two elements, $a, b \in \mathscr{A}$, such that $b < a$, then the morphisms of ψ are not morphisms of **IHilb**, i.e. are not isometries; in fact the only case where ψ is an isometry is when the poset is a disjoint union of elements I, i.e. for $x, y \in I$,

$$x \leq y \text{ if and only if } x = y \tag{5.4.14}$$

and in this case the colimit is a direct sum of Hilbert spaces.

This is the reason why, contrarily to the decomposition in **Vect** as discussed in Chaptre 3, we can't directly apply Theorem 5.3.1 to extend this theorem to **IHilb**. We shall therefore follow the (longer) path followed in Chapter 4 to show the an extension still holds.

5.4.3 Extension to $\mathscr{A}_1$

Let us first give the idea of the construction we will present in what follows. We know the Theorem 5.3.1 holds for subpaces of a given Hilbert space, therefore for H a functor from a poset $\mathscr{A}$ to **IHilb**, we will want to nest the Hilbert spaces $(H_a, a \leq \alpha)$ inside a given Hilbert space H_α when this Theorem holds. What we will show is that nesting is functorial in **IHilb** and that the decomposition of each H_α are connected to each other in such way that we can rebuild a decomposition (in the sense of Section 2) for H.

Remark 5.4.1. Let us recall that if ϕ is an isometry then $\operatorname{im}\phi$, in the sense of its image in **Vect**, is a Hilbert space.

For any poset $\mathscr{A}$, any $\alpha \in \mathscr{A}$ and for G a functor from $\mathscr{A}$ to **IHilb**, let us recall that $G_\alpha^{\hat{\alpha}} : G|_{\hat{\alpha}} \to G(a)$ is the monomorphism induces by G. $(G, G^\dagger)$ is a functor from $\mathscr{A}$ to **Split**(**Vect**); let L be its left coupling.

Proposition 5.4.6. *Let G be a functor from $\mathscr{A}$ to **IHilb**, let $\alpha \in \mathscr{A}$, then $G_\alpha^{\hat{\alpha}}|^{L(\alpha,.)}$ is an isomorphism.*

Proof. Let us note $H = G_\alpha^{\hat{\alpha}}|^{L(\alpha,.)}$ for any $a \leq \alpha$, $H(a)$ is surjective and an isometry, therefore it is an isomorphism

□

Proposition 5.4.7. *Let G be a functor from $\mathscr{A}$ to **IHilb**, let L be its left coupling; L is a functor from $\mathscr{A}_1$ to **IHilb**.*

Proof. Let $\mathscr{A}$ be any poset and $G : \mathscr{A} \to$ **IHilb** be a functor from $\mathscr{A}$ to **IHilb** ; let $(\alpha, a), (\beta, a) \in \mathscr{A}_1$ be such that $\alpha \geq \beta$, the following diagram commutes

$$\begin{tikzcd} G(a) \arrow[r, "\mathrm{id}"] \arrow[d, "G^a_\beta"'] & G(a) \arrow[d, "G^a_\alpha"] \\ G(\beta) \arrow[r, "G^\beta_\alpha"] & G(\alpha) \end{tikzcd} \tag{5.4.15}$$

and by Proposition 5.4.6, $G^a_\alpha|^{L(\alpha,a)}$ and $G^a_\beta|^{L(\beta,a)}$ are isomorphims, therefore $L^{\beta a}_{\alpha a} = G^\beta_\alpha|^{L(\beta,a)}_{L(\alpha,a)}$ is an isomorphism of **IHilb**. □

5.4.4 Extension to $\mathscr{A}_2$

Definition 5.4.2 (Complete join-lattice)**.** A complete join-lattice, $\mathscr{A}$, is a poset that has a join for any subset of the poset, i.e. for any set I and collection $(a_i \in \mathscr{A}, i \in I)$ there is $\vee_{i\in I} a_i$ such that for all $c \in \mathscr{A}$ such that,

$$\forall i \in I \quad c \geq a_i \tag{5.4.16}$$

then one has that $\vee_{i\in I} a_i \leq c$.

Remark 5.4.2. Let G be a functor from $\mathscr{A}$ to **IHilb**, for any $a, b \in \mathscr{A}$ such that $a \geq b$, as G^b_a is injective, it is a suboject of $G(a)$ (Appendix B Definition 5.A.5, Proposition 5.A.2) and a subspace of $G(a)$ up to isomorphism.

Definition 5.4.3. Let $\mathscr{A}$ be any poset and let G be a functor from $\mathscr{A}$ to **IHilb**; we shall note for any $\mathscr{B} \in \mathscr{U}(\alpha)$,

$$G^\alpha(\mathscr{B}) = \bigvee_{a\in\mathscr{B}} G^a_\alpha \tag{5.4.17}$$

and for any $\mathscr{B}_1 \in \mathscr{U}(\alpha)$ such that $\mathscr{B}_1 \subseteq \mathscr{B}$, we shall note the inclusion $G^\alpha(\mathscr{B}_1) \leq G^\alpha(\mathscr{B})$ as $\hat{G}^{\alpha\mathscr{B}_1}_{\alpha\mathscr{B}}$. We shall also denote $G^\alpha(\mathscr{B})$ as $\hat{G}(\alpha, \mathscr{B})$.

By Annex B Corollary 5.A.3, a representant of $G^\alpha(\mathscr{B})$ is $\overline{\sum_{b\in\mathscr{B}} L(\alpha, b)}$ which reduces to $L(\alpha, b)$ when $\mathscr{B} = \hat{b}$. From now on we will consider that $G^\alpha(\mathscr{B})$ is $\overline{\sum_{b\in\mathscr{B}} L(\alpha, b)}$.

For $\alpha, \beta \in \mathscr{A}$ such that $\alpha \geq \beta$ and for $\mathscr{B} \in \mathscr{U}(\beta)$, let,

$$\hat{G}^{\beta\mathscr{B}}_{\alpha\mathscr{B}} = G^\beta_\alpha|^{\hat{G}(\beta,\mathscr{B})}_{\hat{G}(\alpha,\mathscr{B})} \tag{5.4.18}$$

Remark 5.4.3. Let us recall that colimits do not exist in general in **IHilb** (Appendix B Proposition 5.A.6); if $\mathscr{A}$ is a poset, $a \in \mathscr{A}$, $\mathscr{B} \in \mathscr{U}(\mathscr{A})$ and $G : \mathscr{A} \to$ **IHilb** a functor from $\mathscr{A}$ to **IHilb** then $\bigvee_{b\in\mathscr{B}} G^b_a$ is in general not a colimit of $G|_{\hat{a}\setminus a}$

Proposition 5.4.8. *For any poset $\mathscr{A}$ and functor $G : \mathscr{A} \to$ **IHilb**, $\hat{G}$ extends to a unique functor from $\mathscr{A}_2 \to$ **IHilb**.*

Lemma 5.4.1. *Let $\mathscr{A}$ be a poset, $\boldsymbol{C}$ be any category; let $M_1 = \{((\alpha, \mathscr{B}), (\alpha, \mathscr{B}_1)) : (\alpha, \mathscr{B}), (\alpha, \mathscr{B}_1) \in \mathscr{A}_2$ and $\mathscr{B} \geq \mathscr{B}_1\}$, $M_2 = \{((\alpha, \mathscr{B}), (\beta, \mathscr{B})) : (\alpha, \mathscr{B}), (\beta, \mathscr{B}) \in \mathscr{A}_2$ and $\alpha \geq \beta\}$, and let $(G^i_j; i, j \in M_1 \cup M_2 : i \leq j)$ be such that for any $(\alpha, \mathscr{B}), (\alpha, \mathscr{B}_1), (\alpha, \mathscr{B}_2) \in \mathscr{A}_1$ such that $(\alpha, \mathscr{B}) \geq (\alpha, \mathscr{B}_1) \geq (\alpha, \mathscr{B}_2)$,*

$$G^{\alpha\mathscr{B}_1}_{\alpha\mathscr{B}} G^{\alpha\mathscr{B}_2}_{\alpha\mathscr{B}_1} = G^{\alpha\mathscr{B}_2}_{\alpha\mathscr{B}_1} \tag{5.4.19}$$

for any $(\alpha, \mathscr{B}), (\beta, \mathscr{B}), (\gamma, \mathscr{B})$ such that $(\alpha, \mathscr{B}) \geq (\beta, \mathscr{B}) \geq (\gamma, \mathscr{B})$,

$$G^{\beta\mathscr{B}}_{\alpha\mathscr{B}} G^{\gamma\mathscr{B}}_{\beta\mathscr{B}} = G^{\gamma\mathscr{B}}_{\alpha\mathscr{B}} \tag{5.4.20}$$

and for any $(\alpha, \mathscr{B}), (\alpha, \mathscr{B}_1), (\beta, \mathscr{B}), (\beta, \mathscr{B}_1)$ such that $(\alpha, \mathscr{B}) \geq (\alpha, \mathscr{B}_1) \geq (\beta, \mathscr{B}_1)$ and $(\alpha, \mathscr{B}) \geq (\beta, \mathscr{B}) \geq (\beta, \mathscr{B}_1)$, i.e $\hat{\alpha} \geq \hat{\beta} \geq \mathscr{B} \geq \mathscr{B}_1$,

$$G^{\beta\mathscr{B}}_{\alpha\mathscr{B}} G^{\beta\mathscr{B}_1}_{\beta\mathscr{B}} = G^{\alpha\mathscr{B}_1}_{\alpha\mathscr{B}} G^{\beta\mathscr{B}_1}_{\alpha\mathscr{B}_1} \tag{5.4.21}$$

Then G extends into a unique functor $G_1 : \mathscr{A}_2 \to \boldsymbol{C}$, which shall also be denote as G.

Proof. The proof is essentially rewriting the proof of Lemma 4.4.7 replacing c by $\mathscr{B}_2$, b by $\mathscr{B}_1$ and a by $\mathscr{B}$. □

Proof of Proposition 5.4.8.

Proof. By Lemma 5.4.1 $\hat{G}$ is a functor and for any $(\beta, \mathscr{B}_1)$, $(\alpha, \mathscr{B}) \in \mathscr{A}_2$ such that $(\beta, \mathscr{B}_1) \leq (\alpha, \mathscr{B})$, $\hat{G}^{\beta\mathscr{B}_1}_{\beta\mathscr{B}}$ is an inclusion therefore an isometry and $\hat{G}^{\beta\mathscr{B}}_{\alpha\mathscr{B}}$ is a restriction of G^{β}_{α}, which is an isometry, on its domain and codomain (it is even an isomorphism); therefore $\hat{G}^{\beta\mathscr{B}_1}_{\alpha\mathscr{B}}$.

□

Remark 5.4.4. If one notes $j : \mathscr{A} \to \mathscr{A}_1$ the increasing function such that $j(a) = (a, a)$ and $k : \mathscr{A} \to \mathscr{A}_2$ the increasing function such that $k(a) = (a, \hat{a})$ then

$$Lk = \hat{G}j \cong G \tag{5.4.22}$$

5.4.5 Intersection property

Remark 5.4.5. Let $\mathscr{A}$ be any poset and G a functor from $\mathscr{A}$ to **IHilb**. Let R be defined as for any $(\alpha, a), (\beta, b) \in \mathscr{A}_1$ such that $(\alpha, a) \geq (\beta, b)$,

$$R^{\alpha a}_{\beta b} = L^{\beta b\,\dagger}_{\alpha a} \tag{5.4.23}$$

Then F is a presheaf.

Similarly let $\hat{F}$ be defined as for any $(\alpha, \mathscr{B}), (\beta, \mathscr{B}_1) \in \mathscr{A}_2$ such that $(\alpha, \mathscr{B}) \geq (\beta, \mathscr{B}_1)$,

$$\hat{F}^{\alpha\mathscr{B}}_{\beta\mathscr{B}_1} = \hat{G}^{\alpha\mathscr{B}}_{\beta\mathscr{B}_1}{}^{\dagger} \tag{5.4.24}$$

Then $\hat{F}$ is a presheaf.

Definition 5.4.4 (Intersection property)**.** Let $\mathscr{A}$ be any poset and G be a functor from $\mathscr{A}$ to **IHilb**. For any $(\alpha, \mathscr{B}) \in \mathscr{A}_2$ let us note $\pi^\alpha(\mathscr{B})$ the orthogonal projection of $G(\alpha)$ onto $\hat{G}(\alpha, \mathscr{B})$.

We shall say that G satisfies the intersection property if for any

$$\forall (\alpha, a), (\alpha, b) \in \mathscr{A}_1, \quad \pi^\alpha(\hat{a} \cap \hat{b}) = \pi^\alpha(\hat{a})\pi^\alpha(\hat{b}) \tag{I'}$$

Proposition 5.4.9. *Let G be a functor from a poset $\mathscr{A}$ to **IHilb**, G satisfies the intersection property if and only if*

$$\forall \alpha, a, b \in \mathscr{A} \ \ s.t. \ a \leq \alpha, b \leq \alpha, \quad G^a_\alpha G^{a\,\dagger}_\alpha G^b_\alpha G^{b\,\dagger}_\alpha = \hat{G}^{\alpha\hat{a}\cap\hat{b}}_{\alpha\hat{\alpha}} \hat{G}^{\alpha\hat{a}\cap\hat{b}\dagger}_{\alpha\hat{\alpha}} \tag{5.4.25}$$

Proof. For any $(\alpha, b) \in \mathscr{A}_1$, $\phi_b : G(b) \to G(\alpha, b)$ is an isomorphism of **IHilb** and $G^b_\alpha = \phi_\alpha G^{\alpha b}_{\alpha\alpha} \phi_b^{-1}$; as $G^{\alpha b}_{\alpha\alpha}$ is an inclusion, the orthogonal projection $\pi^\alpha(b) = G^{\alpha b}_{\alpha\alpha}{}^{\dagger}$. One then has that,

$$G^b_\alpha G^{b\,\dagger}_\alpha = \phi_\alpha G^b_\alpha \phi_b^{-1} \phi_b^\dagger G^{b\,\dagger}_\alpha \phi_\alpha^{-1\dagger} \tag{5.4.26}$$

and as ϕ_b is an isometry and $\phi_\alpha = id$,

$$G^b_\alpha G^{b\,\dagger}_\alpha = \ G^b_\alpha G^{b\,\dagger}_\alpha \tag{5.4.27}$$

which ends the proof.

□

Proposition 5.4.10. *Let $\mathscr{A}$ be any poset and $G : \mathscr{A} \to$ **IHilb** be a functor from $\mathscr{A}$ to **IHilb**; let us assume that for some $\alpha \in \mathscr{A}$, and for any $a, b \in \mathscr{A}$ such that $a \leq \alpha$ and $b \leq \alpha$,*

$$\pi^\alpha(\hat{a} \cap \hat{b}) = \pi^\alpha(\hat{a})\pi^\alpha(\hat{b}) \tag{5.4.28}$$

then for any $\alpha_1 \leq \alpha$, and any $a, b \leq \alpha_1$ one has that,

$$\pi^{\alpha_1}(\hat{a} \cap \hat{b}) = \pi^{\alpha_1}(\hat{a})\pi^{\alpha_1}(\hat{b}) \tag{5.4.29}$$

5.4.6 Predecomposition and natural transformations

Definition 5.4.5. Let $\mathscr{A}$ be a poset and G be a functor from $\mathscr{A}$ to **IHilb**; for any $(\alpha, a) \in \mathscr{A}_1$, when $c \leq a$, let,

$$S_c(\alpha, a) = L(\alpha, c) \cap \bigcap_{d \lneq c} L(\alpha, d)^{\perp} \tag{5.4.30}$$

otherwise

$$S_c(\alpha, a) = 0 \tag{5.4.31}$$

Futhermore let $(\alpha, a), (\beta, b) \in \mathscr{A}_1$ be such that $(\alpha, a) \geq (\beta, b)$, for any $v \in S_c(\beta, b)$, let

$$S_{c\alpha a}^{\beta b}(v) = G_{\alpha}^{\beta}(v) \tag{5.4.32}$$

This reduces to $S_{c\alpha a}^{\beta b} = 0$ when $c \not\leq b$.

Let for any $(\alpha, a) \in \mathscr{A}_1$, $s_c(\alpha, a) : L(\alpha, a) \to S_c(\alpha, a)$ be the orhogonal projection on $S_c(\alpha, a)$. Furthermore we shall denote V_c the restriction of S_c to $\mathscr{A}$, i.e. for any $a \in \mathscr{A}$,

$$V_c(a) = S_c(a, a) \tag{5.4.33}$$

and for $b \in \mathscr{A}$ such that $b \leq a$,

$$V_{ca}^{\ b} = S_{caa}^{\ bb} \tag{5.4.34}$$

Proof. Let $\alpha, \beta, a \in \mathscr{A}$ such that $\alpha \geq \beta \geq a$, let $b \in \mathscr{A}$ such that $b \lneq a$, let $v \in S_b(\beta, a)$, let $w \in L(\alpha, b)$ then as $L_{\alpha b}^{\beta b}$ is an isomorphism of **IHilb** (Proposition 5.4.6) one has that there is $w_1 \in L(\beta, b)$ such that $w = G_{\alpha}^{\beta}(v)$ and,

$$\langle G_{\alpha}^{\beta}(v), w \rangle = \langle G_{\alpha}^{\beta}(v), G_{\alpha}^{\beta}(w_1) \rangle = \langle s_a(v), w_1 \rangle = 0 \tag{5.4.35}$$

This show that $S_{c\alpha a}^{\beta a}$ is well defined and therefore the S_c is well defined.

□

Proposition 5.4.11. *Let $\mathscr{A}$ be any poset and $G : \mathscr{A} \to$ **IHilb** be a functor, then for any $c \in \mathscr{A}$, S_c is a functor from $\mathscr{A}_1$ to **IHilb**. Furthermore for any $c \in \mathscr{A}$, $s_c : L \to S_c$ is a natural transformation of funtors of **Hilb**, i.e. for any $(\alpha, a), (\beta, b) \in \mathscr{A}_1$ such that $(\alpha, a) \geq (\beta, b)$,*

$$S_{c\alpha a}^{\beta b} s_c(\beta, b) = s_c(\alpha, a) L_{\alpha a}^{\beta b} \tag{5.4.36}$$

Proof. Let $(\alpha, a), (\beta, b) \in \mathscr{A}$ be such that $(\alpha, a) \geq (\beta, b)$, as G_α^β is an isometry so is $S_{c\alpha a}^{\beta b}$; for $(\gamma, g) \in \mathscr{A}_1$ such that $(\gamma, g) \leq (\beta, b)$ and for $v \in S_d(\gamma, g)$ one has that,

$$S_{c\alpha a}^{\beta b} S_{c\beta b}^{\gamma g}(v) = G_\alpha^\beta G_\beta^\gamma(v) = G_\alpha^\gamma(v) \tag{5.4.37}$$

$$S_{c\alpha a}^{\beta b} S_{c\beta b}^{\gamma g}(v) = S_{c\alpha a}^{\gamma g} \tag{5.4.38}$$

Let $(\alpha, a), (\beta, a) \in \mathscr{A}_1$ be such that $\alpha \geq \beta$, let $v \in L(\beta, a)$ and $w \in S_c(\beta, a)$; let us recall that $L_\alpha^{\beta a \dagger}$ is an isomorphism of **IHilb** (Proposition 5.4.6), therefore,

$$\langle L_{\alpha a}^{\beta a \dagger} s_c(\alpha, a) L_{\alpha a}^{\beta a}(v), w\rangle = \langle s_c(\alpha, a) L_{\alpha a}^{\beta a}(v), L_{\alpha a}^{\beta a}(w)\rangle \tag{5.4.39}$$

As $s_c(\alpha, a)$ is the orthogonal projection on $S_c(\alpha, a) \subseteq L(\alpha, a)$ one has that,

$$\langle L_{\alpha a}^{\beta a \dagger} s_c(\alpha, a) L_{\alpha a}^{\beta a}(v), w\rangle = \langle L_{\alpha a}^{\beta a}(v), L_{\alpha a}^{\beta a}(w)\rangle = \langle v, w\rangle \tag{5.4.40}$$

Therefore by definition of $s_c(\beta, a)$,

$$s_c(\beta, a)(v) = L_{\alpha a}^{\beta a \dagger} s_c(\alpha, a) L_{\alpha a}^{\beta a}(v) \tag{5.4.41}$$

and,

$$S_{c\alpha a}^{\beta a} s_c(\beta, a) = s_c(\alpha, a) L_{\alpha a}^{\beta a} \tag{5.4.42}$$

Furthermore by definition one has that for $(\alpha, a), (\alpha, b) \in \mathscr{A}_1$ such that $a \geq b$,

$$S_{c\alpha a}^{\alpha b} s_c(\alpha, b) = s_c(\alpha, a) L_{\alpha a}^{\alpha b} \tag{5.4.43}$$

and so for any $(\alpha, a), (\beta, b) \in \mathscr{A}_1$ such that $(\alpha, a) \geq (\beta, b)$,

$$S_{c\alpha a}^{\beta b} s_c(\beta, b) = S_{c\alpha a}^{\beta a} S_{c\beta a}^{\beta b} s_c(\beta, b) = s_c(\alpha, a) L_{\alpha a}^{\beta a} L_{\beta a}^{\beta b} \tag{5.4.44}$$

$$S_{c\alpha a}^{\beta b} s_c(\beta, b) = s_c(\alpha, a) L_{\alpha a}^{\beta b} \tag{5.4.45}$$

□

5.4.7 Main Theorem

Theorem 5.4.1. *Let $\mathscr{A}$ be a well founded poset and G be a functor from $\mathscr{A}$ to **IHilb**, G is decomposable if and only if it statifies the intersection property, i.e.*

$$\forall \alpha, a, b \in \mathscr{A} \ s.t. \ a \leq \alpha, b \leq \alpha, \quad G_\alpha^a G_\alpha^{a\,\dagger} G_\alpha^b G_\alpha^{b\,\dagger} = \hat{G}_{\alpha\hat{\alpha}}^{\alpha\hat{a}\cap\hat{b}} \hat{G}_{\alpha\hat{\alpha}}^{\alpha\hat{a}\cap\hat{b}\dagger} \tag{5.4.46}$$

Lemma 5.4.2. *Let $\mathscr{A}$ be a well-founded poset then any subposet of $\mathscr{A}$ is well founded.*

Proof. Proposition 3.5.3

□

Proof. Let $\alpha \in \mathscr{A}$, by Lemma 5.4.2 and by Theorem 5.3.1 one has that $\bigoplus_{d\in\mathscr{A}} s_d(\alpha,.) : G(\alpha,.) \to \bigoplus_{d\in\mathscr{A}} S_d(\alpha,.)$ is an isomorphims and an isometry. By definition for any $a, b, c \in \mathscr{A}$ such that $\alpha \geq a \geq b \geq c$, $S_{c\alpha a}^{\alpha b}$ is the identity application. Furthermore for any $a \in \mathscr{A}$ such that $a \leq \alpha$, $\phi_a : G(a) \to G(\alpha, a)$ is an isomorphism and therefore by Lemma 5.4.11 $\bigoplus_{d\in\mathscr{A}} S_{d\alpha a}^{\alpha a}$ is an isomorphism. Then for any $d \in \mathscr{A}$, $S_{d\alpha a}^{\alpha a}$ is an isomorphism and so as,

$$S_{d\alpha a}^{\alpha\alpha} S_{d\alpha a}^{\alpha a} = S_{d\alpha a}^{\alpha\alpha} = S_{d a}^{\alpha} \tag{5.4.47}$$

$S_{d a}^{\alpha}$ is an isomophism which shows that the intersection property implies decomposable.

The necessary condition is a consequence of Proposition 5.4.3 and Theorem 5.3.1.

□

5.5 How to relate the interaction decomposition for Hilbert spaces over a finite poset to the interaction decomposition for presheaves?

In this section we will limit our attention to finite posets$\mathscr{A}$, and to subspaces, $(H_a, a \in \mathscr{A})$, of a given Hilbert space H; we want to give a brief overview of how and when to build a canonical decomposition for the collection of orthogonal projectors $(\pi_a, a \in \mathscr{A})$. It is a different point of view from the one presented in Section 2 and we shall show that both of the points of view are equivalent when $(H_a, a \in \mathscr{A})$ is decomposable or $(\pi_a, a \in \mathscr{A})$ is decomposable, but that if not hey aren't equivalent.

5.5.1 How to relate both interaction decompositions

Proposition 5.5.1. *Let H be a Hilbert spaces, and $\mathscr{A}$ a finite poset; if $(H_a \in \boldsymbol{Gr}\, H)$ is decomposable then $(\pi_a, a \in \mathscr{A})$ is decomposable and for any $a \in \mathscr{A}$,*

$$s_a = s_a^{\perp} \tag{5.5.1}$$

Proof. Let $(H_a, a \in \mathscr{A})$ be decomposable therefore by definition, for any $a \in \mathscr{A}$,

$$\pi_a = \sum_{b \leq a} s_a^{\perp} \tag{5.5.2}$$

One also has by definition that,

$$\pi_a = \sum_{b \leq a} s_a \tag{5.5.3}$$

As $\mu_{\mathscr{A}}$ is injective, $s_a = s_a^{\perp}$ and therefore $(\pi_a, a \in \mathscr{A})$ is also decomposable. □

Proposition 5.5.2. *Let $\mathscr{A}$ be a finite poset, let H be a Hilbert space, let $(H_a, a \in \mathscr{A})$ be a collection of Hilbert subspace of H. Suppose that $(\pi_a, a \in \mathscr{A})$ is decomposable, where π_a is the orthogonal projection on H_a, then for any $a \in \mathscr{A}$,*

$$s_a = s_a^{\perp} \tag{5.5.4}$$

and $(H_a \in \boldsymbol{Gr}\, H, a \in \mathscr{A})$ is decomposable.

Proof. Let us note $\mu_{\mathscr{A}}$ as μ; let us recall that by definition of $(S_a, a \in \mathscr{A})$ for any $a \in \mathscr{A}$,

$$\pi_a s_b^{\perp} = 1[b \leq a] s_b^{\perp} \tag{5.5.5}$$

Let $a \in \mathscr{A}$, let $v \in H$ and $w \in S_a$,

$$\langle s_a(v), w \rangle = \sum_{b \leq a} \mu(a,b) \langle \pi_b s_a(v), w \rangle \tag{5.5.6}$$

As for any $b \in \mathscr{A}$, π_b is a projector,

$$\langle s_a(v), w \rangle = \sum_{b \leq a} \mu(a,b) \langle s_a(v), \pi_b w \rangle = \sum_{b \leq a} \mu(a,b) \delta_b(a) \langle s_a(v), \pi_b w \tag{5.5.7}$$

and so as $\mu(a,a) = 1$ for any $a \in \mathscr{A}$,

$$\langle v, w \rangle = \langle s_a(v), w \rangle \tag{5.5.8}$$

Therefore $s_a = s_a^{\perp}$ which implies that $(H_a \in \mathbf{Gr}\, H, a \in \mathscr{A})$ is decomposable. □

Remark 5.5.1. Proposition 5.5.2 implies that in this case for any $a \in \mathscr{A}$, $\operatorname{im} s_a$ are Hilbert spaces.

The previous propositions (Propositions 5.5.1,5.5.2) and Theorems 5.3.1,4.3.1 that for subspace of a given Hilbert space it is equivalent to prove that the collection of orthogonal projections $(\pi_a, a \in \mathscr{A})$ is decomposable or that $(H_a, a \in \mathscr{A})$ is decomposable; therefore one can use the intersection property for collection of projector or for collections of Hilbert spaces to verify if the presheaf given by the projectors is decomposable. Furthermore when $(\pi_a, a \in \mathscr{A})$ is decomposable for any $\mathscr{B} \in \mathscr{U}(\mathscr{A})$,

$$\pi(\mathscr{B}) = \sum_{b \in \mathscr{B}} s_b = \sum_{b \in \mathscr{B}} s_b^{\perp} \tag{5.5.9}$$

However if $(H_a, a \in \mathscr{A})$ is not decomposable, $(s_a^{\perp}, a \in \mathscr{A})$ and $(s_a, a \in \mathscr{A})$ are usually different from one another. Indeed let $\mathscr{A} = \{0, 0^{'}, 2\}$ such that $0 \leq 1$, $0^{'} \leq 2$; let $H_1 = H = \mathbb{R} \bigoplus \mathbb{R}$ with for any $(\lambda_1, \lambda_2), (\mu_1, \mu_2) \in H$,

$$\langle (\lambda_1, \lambda_2), (\mu_1, \mu_2) \rangle = \sum_{i=0,1} \lambda_i \mu_i \tag{5.5.10}$$

Let $H_0 = \mathbb{R} \bigoplus 0$, let $H_{0'} = \mathbb{R} e_1 \oplus e_2$. One remarks that by construction,

$$\sum_{a \in \mathscr{A}} s_a = \mathrm{id} \tag{5.5.11}$$

but that,

$$\sum_{a \in \mathscr{A}} s_a^{\perp} = s_0^{\perp} + s_{0'}^{\perp} \neq \mathrm{id} \tag{5.5.12}$$

as H_0 and $H_{0'}$ aren't othogonal to one another.

Appendix

5.A Results for the category of Pre-Hilbert spaces and Hilbert spaces

5.A.1 Completion

Definition 5.A.1. We shall consider several categories; the category that has as objects Hilbert spaces and as morphism continuous linear applications will be noted **Hilb**, when the morphism are isometries it is **IHilb**; when the objects are pre-Hilbert and the morphisms are continuous linear applications the corresponding category is **PHilb**; finally the category that has as object pre-Hilbert spaces and as morphism isometries shall be denoted as **IPhilb** and the category that has as objects vector spaces and morphisms linear applications shall be noted as **Vect**.

Proposition 5.A.1. *Let H be a pre-Hilbert space, there is a Hilbert space that we shall note as $\mathscr{C}\, H$ and a continuous injective morphism $\eta_H : H \to \mathscr{C}\, H$ such that for any Hilbert spaces H_1 and any continuous linear application (of* ***PHilb****), $\phi : H \to H_1$, there is a unique continuous linear application $\mathscr{C}\, \phi : \mathscr{C}\, H \to H_1$ such that*

$$\mathscr{C}\, \phi \circ \eta_H = \phi \tag{5.A.1}$$

i.e. such that the following diagram commutes,

$$\begin{tikzcd} H \arrow[r, "\phi"] \arrow[dr, "\eta_H"'] & H_1 \\ & \mathscr{C}\, H \arrow[u, "\mathscr{C}\, \phi"'] \end{tikzcd} \tag{5.A.2}$$

$\mathscr{C}\, H$ is called the completion of H.

5.A.2 Epic and monic

Proposition 5.A.2. *Monomorphism of* ***Philb*** *are injective continuous linear applications and epimorphism are surjective continuous linear applications.*

Monomorphisms of ***Hilb*** *are injective applications and epimorphism are maps that have a dense image.*

Monomorphisms of ***IHilb*** *are injective applications and epimorphism are surjective applications.*

Proof. Let H, H_1 be pre-Hilbert spaces, let $\phi : H \to H_1$ be monic, let $v, v_1 \in H$ be such that $\phi(v) = \phi(v_1)$. Let,

$$\begin{array}{rcccc} i & : & \mathbb{K} & \to & H \\ & & \lambda & \mapsto & \lambda v \end{array} \tag{5.A.3}$$

and,

$$\begin{array}{rcccc} i_1 & : & \mathbb{K} & \to & H \\ & & \lambda & \mapsto & \lambda v_1 \end{array} \tag{5.A.4}$$

Then,

$$\phi i = \phi i_1 \tag{5.A.5}$$

and so,

$$i = i_1 \tag{5.A.6}$$

Therefore $v = i(1) = i_1(1) = v_1$.

The previous proof also holds in **Hilb** as any finite dimentional normed vector space is complete, i.e. i and i_1 are morphisms of **Hilb** when H is a Hilbert space.

To show that injective application are monomorphisms, let H, H_1, H_2 be Hilbert or pre-Hilbert spaces and let $\phi : H_1 \to H_2$ be an injective application, let $\psi : H \to H_1$ and $\psi_1 : H \to H_1$ be such that,

$$\phi\psi = \phi\psi_1 \tag{5.A.7}$$

then for any $v \in H$, $\phi(\psi(v)) = \phi(\psi_1(v))$ and as ϕ is injective, $\psi_1(v) = \psi(v)$.

Let us now show that if $\phi : H \to H_1$ is an epimorphism of pre-Hilbert spaces its image is dense in H_2. Let us note $K = \operatorname{im}\phi$; we shall distinguish between the closure of K in H_1, that we shall note as $\overline{K} \subseteq H_1$, and the closure of K in $\mathscr{C}\, H_1$, that we shall note as $\overline{K} \subseteq \mathscr{C}\, H_1$. Let us note p the projection onto $\overline{K} \subseteq \mathscr{C}\, H_1$; then for any $v \in H_1$ and $w \in K$,

$$\langle v, w\rangle = \langle p(v), w\rangle \tag{5.A.8}$$

therefore $\langle v, .\rangle = \langle p(v), .\rangle$, i.e.,

$$\langle v, \phi\rangle = \langle p(v), \phi\rangle \tag{5.A.9}$$

and so $v = p(v)$ in $\mathscr{C}\, H_1$, but as $v \in H_1$ and $p(v) \in \overline{K} \subseteq \mathscr{C}\, H_1$ then $v \in \overline{K} \subseteq H_1$.

When H and H_1 are Hilbert spaces, $\langle v, .\rangle$ and $\langle p(v), .\rangle$ are morphism of h therefore, epimorphism have a dense image.

Let $\phi : H \to H_1$ have a dense image in H_1, let $\psi, \psi_1 : H_1 \to H_2$ be two morphims such that,

$$\psi\phi = \psi_1\phi \tag{5.A.10}$$

Let $v \in H_1$, there is a sequence $(w_n \in H, n \in \mathbb{N})$ such that $\lim_{n\to\infty} \phi(w_n)v$, therefore,

$$\psi(v) = \lim_{n\to\infty} \psi\phi(w_n) = \lim_{n\to\infty} \psi_1\phi(w_n) = \psi_1(v) \tag{5.A.11}$$

This argument still holds for morphism of **Hilb** and therefore epimorphisms of **Hilb** are exactly the applications that have a dense image.

Following the same argument than for **Philb** one can show that epimorphisms of **IHilb** are injective maps.

Let $\phi : H \to H_1$ be an epimorphism of **IHilb**, then ϕ has a dense image; but as H is a Hilbert space and as ϕ is an isometry, $\operatorname{im}\phi$ is a Hilbert space. In particular $\operatorname{im}\phi$ is closed and therefore $\operatorname{im}\phi = H_1$. Now when $\phi : H \to H_1$ is surjective it is a monomorphism.

□

Remark 5.A.1. In general, for **Hilb** and **Philb**, epic and monic does not imply isomorphism. However in **IHilb** it does.

Proposition 5.A.3. *Let ϕ be a morphism of **Hilb** such that ϕ is injective and surjective, then ϕ is an isomorphism.*

Proof. Let ϕ be injective and surjective, therefore by Banach-Schauder theorem it is an isomorphism as its inverse is continuous.

□

Remark 5.A.2. There are morphisms that are epic and monic of **Philb** but that are not isomorphisms; let $H = \bigoplus_{n\in\mathbb{N}} \mathbb{K}$, and let, $s : H \to H$ be such that for any $n \in \mathbb{N}$,

$$s(e_n) = \frac{1}{n} e_n \tag{5.A.12}$$

then s is injective and surjective but is not an isomorphism as its inverse $d : H \to H$, defined for any $n \in \mathbb{N}$ as,

$$d(e_n) = n e_n \tag{5.A.13}$$

is not bounded.

5.A.3 Forgetful Functors and left adjoints

Definition 5.A.2. Let U : **Hilb** $\to$ **Philb** be the forgetful functor that forgets the fact that a Hilbert space is complete, i.e. any Hilbert space is a pre-Hilbert space and any morphism of Hilbert spaces is a morphism of pre-Hilbert spaces.

Proposition 5.A.4. *The left adjoint to U is $\mathscr{C}$. Furthermore, $U\mathscr{C}U\mathscr{C} = U\mathscr{C}$; U is fully faithfull and $\mathscr{C}$ is faithfull.*

Proof. $\eta : \mathrm{id}_{\mathbf{PHilb}} \to U\,\mathscr{C}$ is a natural transformation. Let for any Hilbert space, H, $\epsilon_H = \mathrm{id}$ then ϵ is a natural transformation from $\mathrm{id}_{\mathbf{Hilb}} \to \mathscr{C}\,U$ where $\mathscr{C}\,U$ cen be identified to $\mathrm{id}_{\mathbf{Hilb}}$, here we choose $\mathscr{C}\,U = \mathrm{id}_{\mathbf{Hilb}}$. One has that,

$$U \star \epsilon\eta \star U = id_U \tag{5.A.14}$$

and,

$$\epsilon \star \mathscr{C}\,\mathscr{C} \star \eta = id_{\mathscr{C}} \tag{5.A.15}$$

Therefore $\mathscr{C}$ is left adjoint to U (remark under Proposition A.5.7 Appendix A [35]). Furthermore as $\mathscr{C}\,U = id$, $U\,\mathscr{C}\,U\,\mathscr{C} = U\,\mathscr{C}$ and as ϵ is an isomorphism and η is a monomorphism, one has that U is fully faithfull and $\mathscr{C}$ is faithfull (Proposition A.5.9 [35]). □

Corollary 5.A.1. *The left adjoint to* $U|^{\mathbf{PIhilb}}_{\mathbf{IHilb}}$ *is* $\mathscr{C}\,|^{\mathbf{IHilb}}_{\mathbf{PIhilb}}$.

Proof. η and ϵ have their morphism in **IPhilb** and **IHilb**, i.e. they are isometries. □

Remark 5.A.3. The forgethful functor U_1 : **IPhilb** $\to$ **Philb** does not have left adjoint functor as, 0 is final in **PHilb** but 0 is not final in **IPhilb** (there is no isometry form $\mathbb{K} \oplus \mathbb{K}$ to 0).

Furthermore if U_1 has a right adjoint, for any object H of **PHilb**, object H_1 of **IPhilb** and morphism of **Philb**, $\phi : U_1 H_1 \to H$, if there were $G(H)$ an object of **IPhilb** and $\epsilon_H : U_1 GH \to H$ such that there was a unique isometry $\psi : H_1 \to GH$ for which $\phi = \epsilon_H \psi$ then,

$$\|\phi\| \leq \|\epsilon\| \tag{5.A.16}$$

The last equation leads to a contradiction when choosing $\|\phi\| > \|\epsilon\|$.

Proposition 5.A.5. *Let H be a Hilbert space and H_1 a pre-Hilbert subspace of H;*

$$\overline{H_1} \cong \mathscr{C}\,H_1 \tag{5.A.17}$$

Proof. Let H_2 be a Hilbert space and $\phi : H_1 \to H_2$ be a continuous linear application. Let $(x_n \in H_1, n \in \mathbb{N})$ be a sequence such that

$$\lim_{n\to\infty} x_n = x \tag{5.A.18}$$

with $x \in H$; then $(x_n, n \in \mathbb{N})$ is a Cauchy sequence, therefore $(\phi(x_n), n \in \mathbb{N})$ is a Cauchy sequence and as H_2 is complete there is $y \in H_2$ such that,

$$\lim_{n \to n\infty} \phi(x_n) = y \tag{5.A.19}$$

We shall say that xRy.

For any other sequence $(x_{1n}, n \in N)$ such that,

$$\lim_{n \to \infty} x_{1n} = x \tag{5.A.20}$$

and for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\|y - y_1\| \leq \|y - \phi(x_n)\| + \|y_1 - \phi(x_{1n})\| + \|\phi\|\|x_N - x_{1N}\| < \epsilon \tag{5.A.21}$$

Therefore the relation R is a functional relation and it extends ϕ and it is the unique map to do so.

□

5.A.4 Direct sum

We shall now recall and extend some results on the Direct sum in the catergory of Hilbert space (see for example [38]).

Definition 5.A.3. Let $U_2 : \mathbf{Philb} \to \mathbf{Vect}$ be the forgethful functor that forgets the scalar product of a pre-Hilbert space, i.e. any pre-Hilbert space is a vector space and any morphism of pre-Hilbert spaces is a vector space morphism.

Definition 5.A.4. Let I be any set and $(H_i, \langle , \rangle_i, i \in I)$ be a collection of pre-Hilbert spaces. Let $\bigoplus_{i \in I} U_2 H_i$ be the direct sum in **Vect**. For any $u, v \in \bigoplus_{i \in I} U_2 H_i$ let,

$$\langle u, v \rangle = \sum_{i \in I} \langle u_i, v_i \rangle_i \tag{5.A.22}$$

$(\bigoplus_{i \in I} U_2 H_i, \langle , \rangle)$ is a pre-Hilbert space and we shall note also note is as $\bigoplus_{i \in I} H_i$. We shall note its inclusions as ψ_i and the projections as π_i.

Remark 5.A.4. Let $(H_i, i \in I)$ be a collection of pre-Hilbert spaces, the inclusion $\psi_i : H_i \to \bigoplus_{i \in I} H_i$ is an isometry.

Proposition 5.A.6. ***Philb***, ***Hilb***, ***IHilb***, ***IPhilb*** *are not cocomplete for non finite diagrams, i.e. if D is a functor from any category $\boldsymbol{C}$ to one of these four categories then* $\operatorname{colim} D$ *does not necessarily exist.*

Proof. Let $\mathbf{C} = \mathbb{N}$ seen as a set and not as a poset. Let D be a functor from $\mathbb{N}$ to **Hilb**.

We shall show that there can't be a Hilbert space $(\operatorname{colim} D, (\eta_n, n \in \mathbb{N}))$ such that for any H Hilbert space and $(\phi_n : D(n) \to H, n \in \mathbb{N})$ collection of continuous linear applications, these morphisms factor through η_n, i.e. there is ϕ such that for any $n \in \mathbb{N}$,

$$\phi_n = \phi\eta_n \tag{5.A.23}$$

Let $D(n) = \mathbb{K}$, let us assume that $\operatorname{colim} D$ exists. Let us first show that for any $n \in \mathbb{N}$, η_n is injective. Let $H = \mathscr{C} \bigoplus_{n\in\mathbb{N}} \mathbb{K}$, then as the inclusions ψ_n are injective, one has that,

$$\|\eta_n\| \neq 0 \tag{5.A.24}$$

Let for any $n \in \mathbb{N}$,

$$\begin{array}{rcccc} \phi_n & : & \mathbb{K} & \to & \mathbb{K} \\ & & \lambda & \mapsto & n\lambda/\|\eta_n\| \end{array} \tag{5.A.25}$$

Then for any $n \in \mathbb{N}$,

$$\|\phi\| \geq n \tag{5.A.26}$$

which contradictory. Therefore for **Hilb**, colimits over any category do not necessarily exist.

As $\mathscr{C}$ is left adjoint to U the forgetful functor from **Hilb** to **Philb**, one has that,

$$\operatorname{colim} D = \operatorname{colim} \mathscr{C}\, UD = \mathscr{C} \operatorname{colim} UD \tag{5.A.27}$$

As $\operatorname{colim} D$ does not necessarily exist, $\operatorname{colim} UD$ does not necessarily exist which shows that colimits do not necessarily exist in **Philb**.

Let us now show that the same statement holds in **IHilb**. Let D be defined as previously and let us assume that $\operatorname{colim} D$ exists. For any $n \in \mathbb{N}$, $\psi_n : D(n) \to \mathscr{C} \bigoplus_{n\in\mathbb{N}} \mathbb{K}$ factor through η_n, i.e there is $\psi : \operatorname{colim} D \to \bigoplus_{n\in\mathbb{N}} \mathbb{K}$ such that for any $n \in \mathbb{N}$,

$$\psi_n = \psi\eta_n \tag{5.A.28}$$

Therefore for any $\lambda \in \bigoplus \mathbb{K}$,

$$\psi(\sum_{n\in\mathbb{N}} \lambda_n \eta_n(1)) = \bigoplus_{n\in\mathbb{N}} \lambda_n \tag{5.A.29}$$

And ψ is an isometry, therefore,

$$\operatorname{colim} D \cong \mathscr{C} \bigoplus_{n\in\mathbb{N}} \mathbb{K} \tag{5.A.30}$$

Let us now consider for any $n \in \mathbb{N}$,$\phi_n = \mathrm{id}$. Then ϕ can't be an isometry as $\phi(e_n \times n) = \phi(e_m \times m)$ for any $n, m \in \mathbb{N}$.

One concludes that in **IHilb** not every diagram D has a colimit and, using the same argument that enabled us to conclude for **Philb** from h, we conclude that **IPhilb** does not have all colimits.

□

Remark 5.A.5. In **IHilb**, following the same lines than in Proposition 5.A.6 but for two copies of $\mathbb{K}$, enables us to show there can be no finite sum, i.e. **IHilb** is not (finitely) cocomplete.

5.A.5 Quotient of a pre-Hilbert space by a subspace

Remark 5.A.6. Let H be a pre-Hilbert space; the data of its scalar product is the same as the data of its norm by polarization identity, i.e. if $\mathbb{K} = \mathbb{R}$, for any $u, v \in H$,

$$\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right) \tag{5.A.31}$$

and if $\mathbb{K} = \mathscr{C}$,

$$\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\|u-iv\|^2 - i\|u+iv\|^2\right) \tag{5.A.32}$$

We will note $(H, \langle,\rangle)$ when we will refer to its scalar product and $(H, \|.\|)$ when we refer to its norm.

Proposition 5.A.7. *Let $(H, \|.\|)$ be a pre-Hilbert space and H_1 be a closed subspace of H_1; let for any $v \in H$,*

$$\|v\|_0 = \inf_{y\in H_1} \|v - y\| \tag{5.A.33}$$

Then $(H/H_1, \|.\|_0)$ is a pre-Hilbert space. Furthermore $\pi : H \to H/H_1$, the quotient map, is continuous.

Notation 5.A.1. Let H be a Hilbert space and H_1 a closed Hilbert subspace of H, when we will refer to H/H_1 as being a pre-Hilbert space we will be refering to its canonical norm $\|.\|_0$ (Proposition 5.A.7).

Lemma 5.A.1 (Fréchet-Von Neumann-Jordan)**.** *Let* $(B, \|.\|)$ *be a* $\mathbb{K}$*-normed space, whose norm satifies the parallelogram identity, i.e for any* $u, v \in B$*,*

$$\|u+v\| + \|u-v\| = 2(\|u\| + \|v\|) \tag{5.A.34}$$

then B *is a pre-Hilbert space.*

Proof. Refer to [39] □

Lemma 5.A.2. *Let* H *be a pre-Hilbert space and* $H_1 \subseteq H$ *a pre-Hilbert subspace; let* $x, y \in H$*, let* $\epsilon > 0$*, there is* $\delta > 0$ *such that one can choose* $v_1, v_2 \in H_1$ *such that,*

$$\inf_{u \in H_1} \|x-u\| + \delta > \|x - v_1\| \tag{5.A.35}$$

$$\inf_{u \in H_1} \|y-u\| + \delta > \|y - v_2\| \tag{5.A.36}$$

And

$$\inf_{u \in H_1} \|x+y-u\| + \epsilon > \|x+y-v_1-v_2\| \tag{5.A.37}$$

Proof. We shall consider for this proof that the Hilbert space is a real Hilbert space, the proof for a complex Hilbert space follows the same line.

Let us consider $x \in H$, $u \in H_1$; let us note $k = u - v$. Let $\epsilon > 0$ and let u be such that,

$$\|x-u\| \leq \min_{v \in H_1} \|x-v\| + \epsilon \tag{5.A.38}$$

then, for any $w \in H_1$,

$$\|x-u\| \leq \|x-v\| = \epsilon \tag{5.A.39}$$

One has that,

$$\|x-v\|^2 = \|x-u\|^2 + 2\langle x-u, k\rangle + \|k\|^2 \tag{5.A.40}$$

Therefore,

$$2\langle x-u, k\rangle \leq \|k|^2 + \epsilon \tag{5.A.41}$$

The previous computations enable us to conclude that

$$\|x-u\| \leq \min_{v \in H_1} \|x-v\| + \epsilon \tag{5.A.42}$$

if and only if, for any $k \in H_1$,

$$2\langle x-u, k\rangle \leq \|k\|^2 + \epsilon \tag{5.A.43}$$

Furthermore if $u \in H_1$ is such that for any $k \in H_1$,

$$2\langle x-u, k\rangle \leq \|k\|^2 + \epsilon \tag{5.A.44}$$

then, for any $k \in H_1$ such that $k \neq 0$,

$$2\langle x-u, \frac{k}{\|k\|}\rangle \leq \|k\| + \epsilon \tag{5.A.45}$$

and so for any $k \in H_1$ such that $\|k\| = 1$,

$$2\langle x-u, k\rangle \leq \epsilon \tag{5.A.46}$$

Furthermore if for any $k \in H_1$ such that $\|k\| = 1$,

$$2\langle x-u, k\rangle \leq \epsilon \tag{5.A.47}$$

then for any $k \in H_1$,

$$2\langle x-u, \frac{k}{\|k\|}\rangle \leq \|k\|\epsilon \tag{5.A.48}$$

Assume that $\epsilon \leq 1$, then for $\|k\| \geq \epsilon$,

$$2\langle x-u, \frac{k}{\|k\|}\rangle \leq \|k\|^2 \leq \|k\|^2 + \epsilon \tag{5.A.49}$$

and when $\|k\| \leq \epsilon$,

$$2\langle x-u, \frac{k}{\|k\|}\rangle \leq \epsilon^2 \leq \|k\|^2 + \epsilon \tag{5.A.50}$$

Therefore,

$$\|x-u\| \leq \min_{v \in H_1} \|x-v\| + \epsilon \tag{5.A.51}$$

if and only if for any $k \in H_1$ such that $\|k\| = 1$,

$$2\langle x-u, \frac{k}{\|k\|}\rangle \leq \epsilon \tag{5.A.52}$$

Let $1 \geq \epsilon > 0$, let $\delta = \epsilon/2$, let $y \in H$ and let $u, v \in H_1$ be such that,

$$\|x-u\| \leq \min_{v \in H_1} \|x-v\| + \delta \tag{5.A.53}$$

and,

$$\|x-u\| \leq \min_{v\in H_1} \|x-v\| + \delta \tag{5.A.54}$$

Then for any $k \in H_1$ such that

$$2\langle x+y-u-v, k\rangle \leq \epsilon \tag{5.A.55}$$

and,

$$\|x+y-u-v\| \leq \min_{v\in H_1} \|x+y-v\| + \epsilon \tag{5.A.56}$$

□

Let us now prove Proposition 5.A.7,

Proof. Let H/H_1 be the quotient vector space of H with respect to H_1, let for any $x \in H$,

$$\|x\|_0 = \min_{u\in H_1} \|x-u\| \tag{5.A.57}$$

For any $u \in H_1$, $\|x+u\|_0 = \|x\|_0$; therefore $\|.\|_0$ factorizes through the quotient map $[.] : H \to H/H_1$, i.e.for any $x \in H$,

$$\|[x]\|_0 = \min_{v\in H_1} \|x-v\| \tag{5.A.58}$$

is well defined.

Assume that $\|[x]\|_0 = 0$ then there is a sequence $(x_n \in H_1, n \in \mathbb{N})$ such that $\lim_{n\to\infty} x_n = x$ and as H_1 is closed in H, one has that $x \in H$. Therefore for any $x \in H$,

$$\|[x]\|_0 = 0 \implies [x] = 0 \tag{5.A.59}$$

The norm defined by a scalar product is rigid in the sense that any sequence that optimises Equation 5.A.33 is unique with respect to the Cauchy equivalence (Lemma 5.A.2); therefore one can show that the parallelogram identity also holds for the quotient norm. In other words, Lemma 5.A.2 unables us to assert that for any $x, y \in H$ and $\epsilon > 0$, there is $u, v \in H_1$ such that,

$$|\|[x+y]\|_0+\|[x-y]\|_0-2(\|[x]\|_0+\|[y]\|_0)| \leq |\|x-u+y-v\|+\|x-u-(y-v)\|-2(\|x-u\|+\|y-v\|)|+\epsilon \tag{5.A.60}$$

and therefore for any $\epsilon > 0$,

$$|\|[x+y]\| + \|[x-y]\| - 2(\|[x]\| + \|[y]\|)| \leq \epsilon \tag{5.A.61}$$

Lemma 5.A.1 shows that $\|.\|_0$ is the norm associated to a scalar product, which end the proof.

Finaly, for any $x \in H$,

$$\|[x]\|_0 \leq \|x\| \tag{5.A.62}$$

Therefore π is continuous. □

Remark 5.A.7. An other way to prove Proposition 5.A.7 is to remark that if one completes H and H_1, then if one notes p the orthogonal projection on $H_1^\perp \subseteq \mathscr{C}\, H$, one has that for any $v \in H \subseteq \mathscr{C}\, H$,

$$\|p(v)\| = \min_{u \in \mathscr{C}\, H_1} \|v - u\| \tag{5.A.63}$$

Furthermore as $\|.\|$ is by definition continuous, one has that,

$$\inf_{u \in \mathscr{C}\, H_1} \|v - u\| = \inf_{u \in H_1} \|v - u\| \tag{5.A.64}$$

Therefore $\|[v]\|^2 = \langle p(v), p(v)\rangle$ which by polarisation defines a positive bilinear form (or sesquilinear form) on H/H_1; as H_1 is closed in H it is also a definite bilinear form as for any $v \in H$, if $\langle (v), p(v)\rangle = 0$ then v in the closure of H_1 in $\mathscr{C}\, H$, but as $v \in H$ it must be in the closure of H_1 in H which is H. Therefore it is a scalar product.

Proposition 5.A.8. *Let $(H, \|.\|)$ be a pre-Hilbert space and H_1 be a closed pre-Hilbert subspace of H; we shall note $i : H_1 \to H$ the inclusion. The coequalizer* $\mathrm{Coeq}(i, 0)$ *exists and is equal to H/H_1.*

Proof. Let $\phi : H \to H_2$ be a morphism such that,

$$\phi \circ i = 0 \tag{5.A.65}$$

Then ϕ factors uniquely through, $\pi : H \to H/H_1$, i.e. there is a unique $\phi_1 : H/H_1 \to H_2$ such that,

$$\phi_1 \pi_= \phi \tag{5.A.66}$$

Let us now show that ϕ_1 is continuous. For any $v \in H$,

$$\|\phi_1([v])\| \leq \|\phi\| \|v\| \tag{5.A.67}$$

therefore for any $u \in H_1$,

$$\|\phi_1([v])\| \leq \|\phi\| \|v + u\| \tag{5.A.68}$$

and,

$$\|\phi_1([v])\| \leq \|\phi\| \inf_{u \in H_1} \|v + u\| \tag{5.A.69}$$

Which show that ϕ_1 is continuous and ends the proof. □

Proposition 5.A.9. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a left adjoint functor to $G : \boldsymbol{D} \to \boldsymbol{C}$; let $\mathbb{D} : D \to \boldsymbol{C}$ be a diagram in $\boldsymbol{C}$, $F(\operatorname{colim} \mathbb{D}) = \operatorname{colim} F\mathbb{D}$.*

Corollary 5.A.2. *Let H be a pre-Hilbert space and H_1 be a closed subspace of H, then*

$$\mathscr{C}(H/H_1) = \mathscr{C}\, H / \mathscr{C}\, H_1 \tag{5.A.70}$$

Proof. By Proposition 5.A.8 and Proposition 5.A.9.

□

5.A.6 For IHilb, the poset of subobjects is a complet join-lattice

Remark 5.A.8. Let $\mathbf{C}$ be any category and let $\phi_1 : A \to B$, $\phi_2 : C \to B$ be two monomorphism and $\phi : A \to B$ be a morphism. If there is $\psi : D \to B$ a monomorphism and $\psi_1 : A \to D$ and $\psi_2 : B \to D$ two morphisms such that,

$$\psi\psi_1 = \phi_1 \tag{5.A.71}$$

$$\psi\psi_2 = \phi_2 \tag{5.A.72}$$

then,

$$\psi_1 = \psi_2\phi \tag{5.A.73}$$

Indeed as,

$$\psi\psi_2\phi = \phi_2\phi = \phi = \psi\psi_1 \tag{5.A.74}$$

and as ψ is a monomorphism,

$$\psi_1 = \psi_2\phi \tag{5.A.75}$$

This justifies that in what follows we only talk about collections of monomorphism with a given codomain and not about functors.

Proposition 5.A.10. *Let $(H_i \to H, i \in I)$ be a collection of monomorphism of H for the category **IHilb**; let us refer to $H_i \to H$ simply as H_i.*

*There is a cocone over the functor $(H_i, i \in I)$ in the category of monomorphism with codomain H (of **Philb**), $\bigvee_{i\in I} H_i$, that is initial. In other words there is a monomorphism $\bigvee_{i\in I} H_i \to H$, and a collection of morphisms $(\eta_i : H_i \to \bigvee_{i\in I} H_i, i \in I)$, satisfying for any $i \in I$,*

$$\bigvee_{i\in I} H_i \eta_i = H_i \tag{5.A.76}$$

such that if a monomorphism $W \to H$ and a collection $(\phi_i : H_i \to W, i \in I)$ of morphisms satisfies,

$$W\phi_i = H_i \tag{5.A.77}$$

then there is a unique morphisms $\bigvee_{i\in I} \phi_i$ such that,

$$\bigvee_{i\in I} \phi_i \eta_i = \phi_i \tag{5.A.78}$$

i.e the following diagram commute,

$$\begin{tikzcd}
H_i \arrow[r, "H_i"] \arrow[d, "\eta_i"'] \arrow[dr, "\phi_i"] & H \\
\bigvee H_i \arrow[r, "\bigvee \phi_i"'] \arrow[urr, bend right, "\bigvee H_i"'] & W \arrow[u, "W"']
\end{tikzcd} \tag{5.A.79}$$

Proof. Let $W \to H$ be a monomorphism and let $(\phi_i : H_i \to W, i \in I)$ be a collecton of morphisms that satisfies,

$$W\phi_i = H_i \tag{5.A.80}$$

Let $\overline{\sum_{i\in I} H_i}$ be the closure of the pre-Hilbert space generated by the $(\operatorname{im} H_i, i \in I)$. Let,

$$\bigvee_{i\in I} H_i = \overline{\sum_{i\in I} H_i} \to H \tag{5.A.81}$$

let for $i \in I$, $\eta_i : H_i \to \bigvee_{i\in I} H_i$ be H_i with codomain restricted to $\overline{\sum_{i\in I} H_i}$; and similarly let $\bigvee_{i\in I} H_i \to H$ be the inclusion of $\overline{\sum_{i\in I} H_i}$ into H. Then there is a unique linear application $\bigvee_{i\in I} \phi$ such that,

$$\bigvee_{i\in I} \phi_i\eta_i = \phi_i \tag{5.A.82}$$

Let us note $\bigvee_{i\in I} \phi_i$ as Φ, then for any $v \in \bigoplus_{i\in I} H_i$,

$$\Phi(\sum_{i\in I} H_i(v_i)) = \sum_{i\in I} \phi_i(v_i) \tag{5.A.83}$$

and the image of W is $\overline{\sum_{i\in I} H_i}$ and Φ is the inverse of $W|^{\bigvee_{i\in I} H_i}$. Therefore Φ is also an isomety which ends the proof.

□

Definition 5.A.5. Let **C** be a category and A and object of **C**. A subobject of A is an isomorphism class of monomorphisms, where two monomorphims $i : B \to A$ and $j : B_1 \to A$ are equivalent if and only if there is an isomorphims $\phi : B \to B_1$ such that $j\phi = i$.

One can define a preorder on monomorphisms as follows,

$$i \leq j \quad \iff \quad \exists \phi : B \to B_1, j\phi = i \tag{5.A.84}$$

The preoder defines an order between classes of equivalence sets; we refer the collection of these classes, equipped with the order, as the subobject poset $\mathrm{Sub}(A)$.

Remark 5.A.9. Let **C** be any category, let $A \to C$, $B \to C$ be two monomorphism such that $[A] \leq [B]$. Then there is $\phi : A \to B$ such that,

$$B\phi = A \tag{5.A.85}$$

Let $\psi, \psi_1 : D \to A$ be two morphisms such that,

$$\phi\psi = \phi\psi_1 \tag{5.A.86}$$

then $B\phi\psi = B\phi\psi_1$ and, $A\psi = \psi_1$. So $\psi = \psi_1$, and this show that ϕ is a monomorphism.

Corollary 5.A.3. *Let H be a Hilbert space, the "poset" Sub(H) of **IHilb** is a complete join-lattice, i.e. for any set I and collection of objects $([H_i], i \in I)$ of Sub(H) there is a subobject $\vee_{i\in I}[H_i]$ such that for any object J of Sub(H) such that for any $i \in I$, $[H_i] \leq J$, one has that,*

$$\vee_{i\in I} [H_i] \leq J \tag{5.A.87}$$

Proof. Let $J = [W]$ be a subobject of H then, applying Proposition 5.A.10, one gets that,

$$[\vee_{i\in I} H_i] \leq [W] \tag{5.A.88}$$

therefore, $\vee_{i\in I}[H_i] = [\vee_{i\in I} H_i]$. $\square$

Chapter 6

Extra-fine sheaves and interaction decompositions

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The work presented in this chapter is the result of a collaboration with D. Bennequin, O. Peltre, JP Vigneaux. and I would like to thank them again for our collaboration.

Abstract

We introduce an original notion of extra-fine sheaf on a topological space, and a variant (hyper-extra-fine) for which Čech cohomology in strictly positive degree vanishes. We provide a characterization of such sheaves when the topological space is a partially ordered set (poset) equipped with the Alexandrov topology. Then we further specialize our results to some sheaves of vector spaces and injective maps, where extra-fineness is (essentially) equivalent to the decomposition of the sheaf into a direct sum of subfunctors, known as interaction decomposition, and can be expressed by a sum-intersection condition. We use these results to compute the dimension of the space of global sections when the presheaves are freely generated over a functor of sets, generalizing classical counting formulae for the number of solutions of the linearized marginal problem (Kellerer and Matúš). We finish with a comparison theorem between the Čech cohomology associated to a covering and the topos cohomology of the poset with coefficients in the presheaf, which is also the cohomology of a cosimplicial local system over the nerve of the poset. For that, we give a detailed treatment of cosimplicial local systems on simplicial sets. The appendixes present presheaves, sheaves and Čech cohomology, and their application to the marginal problem.

6.1 Introduction

This article develops cohomological tools to study collections of data associated to hypergraphs, or to more general partially ordered sets (posets). The kind of data we will consider is organized in families of sets indexed by the elements of the poset, forming covariant and contravariant functors with respect to the partial ordering, which are called respectively copresheaves and presheaves over the poset. Such functors have been applied to several problems at the crossroad of data analysis, information theory, coding theory, logic, computation, and bayesian learning. We will mention below some of these problems and develop several applications of

the cohomological approach.

In this work, we see a partially ordered set (poset) $\mathscr{A}$ as a small category such that:

1. there is at most one morphism between two objects;
2. if $a \to b$ and $b \to a$, then $a = b$.

An hypergraph is a particular case of poset, whose objects are some finite subsets of an index set I, and there exist a morphism $S \to S'$ whenever $S' \subseteq S$. An abstract simplicial complex is an hypergraph $\mathscr{K}$ that satisfies an additional property: if S belongs to $\mathscr{K}$, then every subset of S belongs to $\mathscr{K}$ too.

A presheaf on a category $\mathscr{A}$ is a contravariant functor F from $\mathscr{A}$ to the category of sets $\mathscr{S}$, in other terms it is a covariant functor on the opposite category $F : \mathscr{A}^{op} \to \mathscr{S}$. A copresheaf is just a covariant functor $F : \mathscr{A} \to \mathscr{S}$. The presheaves of classical sheaf theory on topological spaces [40] are obtained when $\mathscr{A}$ is the category of open sets of some topological space, which is an example of poset.

Abstract simplicial complexes play a prominent role in *persistent homology* [41,42], a technique to extract topological features that is a cornerstone of applied algebraic topology. The basic idea is to replace a sequence of data points in a metric space by an abstract simplicial complex induced by a proximity parameter (e.g. the Čech complex or the Vietoris-Rips complex). Then homological tools (spectral sequences) are applied to an increasing family of complexes for defining invariant quantities of the data.

Curry's dissertation [43] showed that persistent homology can be extended in several directions involving sheaves on posets of parameters.

Curry [43] also gave a systematic treatment of sheaves defined on another kind of complexes, the cellular complexes (giving cellular sheaves and cosheaves), which he traces back to Zeeman's Ph.D. thesis [44]. A spectral theory of such sheaves was later developed by Hansen and Ghrist [45]. Those works list several situations that can be modeled by cellular (co)sheaves, which include network coding, sensor networks, distributed consensus, flocking, synchronization and opinion dynamics, among other things.

Along similar lines, a series of works by Robinson and collaborators [46–48] argued that sheaves are a canonical model for the integration of information provided by interconnected sensors. In those works, the vertices of an abstract simplicial complex represent heterogeneous data sources and the abstract simplexes some sort of interaction between these sources. It is claimed that sheaves constitute a

canonical data structure if one requires sufficient generality to represent all sensors of interest and the ability to summarize information faithfully. A similar approach is taken by Mansourbeigi in his doctoral dissertation [49].

Independently, Abramsky and his collaborators (see e.g. [50, 51]) have used sheaves and cosheaves on simplicial complexes to study *contextuality*. In this situation, the vertices represent observables, the simplices represent joint measurements (measurement contexts) and the maximal faces of the complex are called *maximal contexts*. The functor associates to each context a set of possible outcomes or a set of probabilities on those outcomes. *Contextuality* refers to the fact that it can happen that some sections of the probability functor (i.e. coherent collections of "local" probabilities) are not compatible with a globally defined probability law. In this article, we refer to this problem as the *probabilistic marginal problem*. There are also linearized versions of this problem, as well as "possibilistic" versions.

In all these examples, homology and cohomology is used to determine the "shape" of the simplicial complex or the relevant geometrical invariants of the associated sheaves.

Simplicial complexes are particularly convenient because they have a geometric realization as CW-complexes, so they can be studied using standard tools in algebraic topology e.g. standard homology and cohomology theories. Hypergraphs were introduced in combinatorics, not in geometry, hence their geometrical study is less straightforward. There have been several proposals to define (co)homological invariants of hypergraphs. A recent paper by Bressan, Li, Ren and Wu [52] defines the *embedded homology* of an hypergraph $\mathscr{H}$, which equals the homology of the smallest abstract simplicial complex that contains $\mathscr{H}$. A specific cohomology of k-regular hypergraphs (i.e. containing only subsets of cardinality k) was introduced by Chung and Graham [53] motivated by some problems in combinatorics.

The present article develops an alternative approach, based on sheaf theory and simplicial methods. We equip the poset $\mathscr{A}$ with the lower or upper Alexandrov topology (see Section 6.3), obtaining the topological space $X_{\mathscr{A}}$ or $X^{\mathscr{A}}$, respectively. There is a well-known equivalence of categories between covariant (resp. contravariant) set-valued functors on $\mathscr{A}$ and sheaves on $X_{\mathscr{A}}$ (resp. $X^{\mathscr{A}}$) i.e.

$$[\mathscr{A}, \mathscr{S}] \cong \mathrm{Sh}(X_{\mathscr{A}}), \qquad [\mathscr{A}^{op}, \mathscr{S}] \cong \mathrm{Sh}(X^{\mathscr{A}}); \tag{6.1.1}$$

in both cases the morphisms are the natural transformations. In other words, we can see a (co)presheaf on $\mathscr{A}$ as a usual sheaf on a topological space, with which Čech cohomology can be used. This cohomology is convenient from a computational viewpoint and well adapted to study the global sections of the sheaf.

Here, we are particularly interested in the following setting, which is adapted to a wide variety of problems, as mentioned above. One introduces an hypergraph $\mathscr{A}$

with vertex set I. The elements of I represent elementary observables or sources, and the elements α of $\mathscr{A}$ represent interactions or joint measurements. To take into account the internal degrees of freedom of each object of $\mathscr{A}$, one introduces a covariant set-valued functor $E : \mathscr{A} \to \mathscr{S}$ of possible outcomes, associating to each object α of $\mathscr{A}$ a set E_α, and to each arrow $\alpha \to \beta$ a surjective map $E_\alpha \to E_\beta$. The local probabilities on each E_α or the functions over each E_α give rise to other important functors, that can be covariant or contravariant.

In particular, the study of the special case of real-valued functions of the probability laws on finite sets E_α over a simplicial complex $\mathscr{A}$ gives a natural interpretation in terms of topos theory and cohomology [54] of the information quantities defined by Shannon and Kullback, or by Von Neumann in the quantum case, cf. [11, 55]. These results were later extended to presheaves of functions of statistical frequencies, and to gaussian laws in Euclidean space [12]. The cohomologies which were used here are not of the type of Čech, they are based on the action of variables on probabilities by conditioning, expressed as non-trivial modules in the topos of presheaves over $\mathscr{A}$. A conjecture is that computing cohomology in degrees higher than one will give entirely new information quantities.

The vector spaces $\{V_\alpha\}_{\alpha\in\mathscr{A}}$ of numerical functions on the sets E_α, and the inclusions $j_{\alpha\beta} : V_\beta \to V_\alpha$ induced by the projections $\pi_{\beta\alpha}$, whenever $\alpha \to \beta$, form a contravariant functor, which is an example of *linear injective presheaf*, that is a presheaf made of vector spaces and injective maps between them, see Section 6.4. For each $\alpha \in \mathscr{A}$, one can introduce the *boundary observables* $V'_\alpha = \sum_{\beta:\beta\subsetneq\alpha} V_\beta$ and then define the *interaction subspace* as any supplement of V'_α, in such a way that $V_\alpha = V'_\alpha \oplus S_\alpha$. We are going to prove (Theorem 6.5.1) that, if $\mathscr{A}$ is closed under all intersections, V_α equals $\bigoplus_{\beta:\beta\subset\alpha} S_\alpha$; equivalently, the functor V can be written as a direct sum of subfunctors, $V = \bigoplus_{\gamma\in\mathscr{A}} S^\gamma$ such that $S^\gamma(\alpha) = j_{\alpha\beta}(S_\beta)$. This constitutes an example of an *interaction decomposition*.

Provided that a general poset $\mathscr{A}$ satisfies an adequate finiteness condition, an interaction decomposition $V = \bigoplus_{\gamma\in\mathscr{A}} S^\gamma$ of an injective presheaf over an arbitrary poset $\mathscr{A}$ is equivalent to a sum-intersection property discovered by G. Sergeant-Perthuis, that we call *condition G*:

$$\forall \alpha, \beta \in \mathscr{A} \text{ such that } \alpha \to \beta, \quad V_{\alpha\beta} \cap \left(\sum_{\substack{\gamma:\alpha \xrightarrow{\neq} \gamma \\ \gamma \not\to \beta}} V_{\alpha\gamma} \right) \subset \sum_{\substack{\gamma:\alpha \xrightarrow{\neq} \gamma \\ \beta \xrightarrow{\neq} \gamma}} V_{\alpha\gamma}. \tag{6.1.2}$$

Here $V_{\alpha\beta} = j_{\alpha\beta} V_\beta$, whenever $\alpha \to \beta$, and $\gamma \xrightarrow{\neq} \delta$ means that $\gamma \to \delta$ and $\gamma \neq \delta$.

In turn, an interaction decomposition can be rephrased in purely topological terms, through the concept of *extra-fine* sheaf on a topological space, reminiscent

of the classical notion of fine sheaf, and a variant called hyper-extra-fine. Then we prove, as it was the case for fine presheaves on paracompact spaces, that hyper-extra-fine implies acyclic for the Čech cohomology (Theorem 1) on any topological space. In Section 6.3 we characterize extra-fine sheaves on the Alexandrov spaces $X_{\mathscr{A}}$ or $X^{\mathscr{A}}$ by the property of *interaction decomposition.*

Suppose that I is a finite set. Let $\{E_i\}_{i\in I}$ be a collection of finite sets and define $E_\beta = \prod_{i\in\beta} E_i$, for each subset β of I. The functor of probabilities P (on an hypergraph $\mathscr{A}$ or the full abstract simplex $\Delta(I)$) associates to each β the probability measures on E_β and to each arrow $\alpha \to \beta$ the marginalization map $P(\alpha) \to P(\beta)$ that is precomposition by $\pi_{\beta\alpha}^{-1}$. A section of P over $\mathscr{A}$ is a "coherent" collection of local probabilities, also called *pseudomarginal* in the literature. The (discrete) *probabilistic marginal problem* is the following: given a pseudomarginal p in $\Gamma_{\mathscr{A}}(P)$, when is it possible to find $q \in \Gamma_{\Delta(I)}(P)$ that restricts to p? The *linearized* version of the problem asks for measures that may be signed, subject to a normalization condition. One can see them as elements of the copresheaf $\overline{F}$ on $\mathscr{A}$ that associates to each subset β the space of functions $f : E_\beta \to \mathbb{R}$ that sum 1, and to each arrow the marginalization map. It is the *predual* of the sheaf $\overline{V}$, which is V quotiented by the constant functions. We will see that the interaction decomposition of $\overline{V}$ induces an interaction decomposition of $\overline{F}$, which is useful to answer the linearized version of the marginal problem. In particular, Theorem 6.5.3 proves that there is a surjection $\Gamma_{\Delta(I)}(P) \to \Gamma_{\mathscr{A}}(P)$, and the index formula of Theorem 6.5.4 gives an explicit quantification of the dimension of these spaces in terms of the dimensions of the interaction subspaces and the Euler characteristic of the poset. This constitutes a novel topological reformulation of the classical results by Kellerer [13] and Matúš [21]; we hope it will lead to new ideas to tackle the probabilistic version of the problem.

We expect that sheaf-theoretic constructions on hypergraphs will give a better understanding of certain algorithms in Statistics or Machine learning. In this direction, Olivier Peltre (cf. [27] and his thesis [26]) has developed a cohomological understanding of the Belief Propagation Algorithm (in the generalized version of [16]); the algorithm appears as a non-linear dispersion flow. Higher dimensional analogs are promising tools. Grégoire Sergeant-Perthuis (cf. [31, 34, 56, 57] and his forthcoming thesis) focused on defining the thermodynamical limit in the category of Markov Kernels, extending several constructions of statistics and statistical physics such as the decomposition into interaction subspaces, first introduced for factor spaces [14, 19], the space of Hamiltonians, infinite-volume Gibbs state, and the renormalisation group.

In both these works, the same result appears: the vanishing of sheaf cohomology (in the toposic form, or in Čech form respectively) in degree larger than one

(i.e. acyclicity, without contractility) for the case of an injective presheaf V over $\mathscr{A}$, under a certain condition relating the intersections and the sums of the subspaces given by the faces: the condition G in Section 6.4. The main goal of this article is to enunciate and prove this result, and to place it in a topological context.

Section 6.2 defines an original notion of *extra-fine* presheaf, and a technical variant *hyper-extra-fine* presheaf, over a topological space X, that is reminiscent of the classical notion of fine sheaf. Then we prove, as it was the case for fine presheaves on paracompact spaces, that hyper-extra-fine implies acyclic for the Čech cohomology (Theorem 1) on any topological space. In Section 6.3 we characterize extra-fine sheaves on the Alexandrov spaces $X_{\mathscr{A}}$ or $X^{\mathscr{A}}$ by the property of *interaction decomposition*. Hyper-extra-fine corresponds to the case of posets conditionally stable under finite products (for a definition, see Section 6.3).

In Section 6.4 we consider presheaves V of $\mathbb{K}$ vector spaces (over any field $\mathbb{K}$) and injective maps. For such presheaves, we give an alternative characterization of extra-fineness through the *sum-intersection condition* G; this is the first main result of the article (Theorem 6.4.2). We do that without any finiteness condition on the vector spaces, and only weak finiteness conditions for the poset. Then we study duality, proving the acyclicity theorem for the predual cosheaves.

Section 6.5 contains the definition of *free presheaves* generated by a covariant set-valued functor E over a commutative field $\mathbb{K}$ (the usual case in data analysis over hypergraphs). We establish the condition G for the injective presheaf V of functions from E to $\mathbb{K}$, when the poset $\mathscr{A}$ has conditional coproducts (meaning stable by non-empty intersections in the case of hypergraphs). Then the acyclicity is deduced for the sheaf induced by V on $X^{\mathscr{A}}$, when $\mathscr{A}$ is stable by all the finite coproducts, including the empty one (Theorem 6.5.1). Then we compute the cohomology of this sheaf when $\mathscr{A}$ is stable by all the finite coproducts, excluding the empty one (Theorem 6.5.2): it is the sum of the ordinary cohomology of $\mathscr{A}$ in all degrees, and of the cohomology of degree zero of a restricted sheaf of functions (where the sum of coordinates is zero). This is the second main result of this article. We also prove a version of the marginal theorem (surjection in Čech cohomology, Theorem 6.5.3), which seems to be new in this generality. We deduce an index theorem for the Euler characteristic of the marginal sheaves (Theorem 6.5.4).

Finally, Section 6.6 comes back to a general topological space X and preshaves of abelian groups, to provide the homotopy equivalence of the Čech cochain complex of an open covering of a presheaf with the cochain complex of the *nerve of the category* generated by the covering; this is done for a general notion of cosimplicial coefficients (Theorem 6.6.2). This answers a natural question in our framework, and permits to identify the Cech cohomology on $X_{\mathscr{A}}$ and the topos cohomology induced by the global sections functor on the poset $\mathscr{A}$, provided $\mathscr{A}$ is conditionally

closed under coproducts. The proof is surprisingly cumbersome, which is reminiscent of the known fact that there exists an homotopy equivalence between a finite simplicial complex and its barycentric subdivision but non-canonically.

In all the above sections we take care of morphisms between presheaves and naturality behaviors, or functoriality.

Three appendices are added at the end, where we summarize the main objects and constructions involved in the article: sheaves, Čech cohomology, and Möbius functions, among other things. The first two are written for people that are not familiar at all with topology.

The main results and illustrations are in Sections 6.4 and 6.5, about *interaction decomposition*, *intersection condition G*, and *free sheaves*, in a context of infinite sets. The other sections are of more expository nature.

Acknowledgments: The authors thank warmly an anonymous referee for pointing several imprecisions in a preceding version.

6.2 Fine, extra-fine, super-local and acyclic

In this section, we consider presheaves of abelian groups over a topological space X. See Appendix 6.A for some basic topological definitions and notations. We use Čech cohomology as presented in any standard reference, e.g. [58], but all relevant definitions can also be found in Sections 6.6.1 and 6.6.2 under the formalism of cosimplicial local systems.

Definition 6.2.1 (*Fine presheaf*, cf. [58, Sec. 6.8])**.** A presheaf F of abelian groups over a topological space X is said to be *fine* if for every locally finite [1] open covering $\mathscr{U}$ of X, there exists a family $\{e_V\}_{V\in\mathscr{U}}$ of endomorphisms of F (i.e. natural transformations $e_V : F \to F$, whose components $e_V(W)$ we denote by $e_{V|W}$), such that:

(i) For all $V \in \mathscr{U}$ and every open set W, one has $e_{V|W}|(W \setminus \bar{V}) = 0$.

(ii) For every open W that encounters only a finite number of closures $\overline{V}$ of elements V of $\mathscr{U}$, and every $x \in F(W)$, we have $x = \sum_{V\in\mathscr{U}} e_{V|W}(x)$.

More generally, we say that a family $\{e_V\}_{V\in\mathscr{U}}$ of endomorphisms of F, indexed by elements of an open cover, is a a *partition of unity* (or *partition of identity*) adapted to $\mathscr{U}$, when it satisfies (i) and the following condition:

[1] By definition, an open cover $\mathscr{U}$ of X is locally finite when every point of X has a neighborhood which meets a finite number of elements of $\mathscr{U}$.

(ii') For every open W, and every element x of $F(W)$, there exists only a finite number of elements V of $\mathscr{U}$, such that $e_{V|W}(x) \neq 0$, and we have $x = \sum_{V \in \mathscr{U}} e_{V|W}(x)$.

Fine presheaves are part of the classical literature on sheaf theory, see also [40, Sec. 3.7] and [59, p. 42], although the classical definitions require that the space X is paracompact. Positive dimensional cohomology of a paracompact topological space with coefficients in a fine presheaf vanishes [58, Thm. 6.8.4], and this fact has important implications, for instance in the comparison of Alexander and Čech cohomology. We propose here a variant of this notion that plays a fundamental role in our investigations.

Definition 6.2.2 (*Extra-fine presheaf*)**.** A presheaf F of abelian groups over the topological space X is said to be *extra-fine* if for every open covering $\mathscr{V}$ of X, there exists a finer open covering $\mathscr{U}$ and a partition of unity $\{e_V\}_{V \in \mathscr{U}}$ adapted to $\mathscr{U}$ (i.e. 6.2.1-(ii') is satisfied), such that

(i') for all $V \in U$ and $W \in U$, $e_{V|W} \neq 0$ implies $W \subseteq V$;

(iii) for all $V, W \in \mathscr{U}$ such that $V \neq W$, $\quad e_V \circ e_W = e_W \circ e_V = 0$.

Remark that in this definition, the covering $\mathscr{U}$ is not required to be locally finite.

Definition 6.2.3 (*Hyper-extra-fine presheaf*)**.** F is *hyper-extra-fine*, if for every open covering $\mathscr{V}$ of X, there exists a finer open covering $\mathscr{U}$ that satisfies (ii'), (i'), (iii), and that is closed by non-empty finite intersections, i.e. for every collection of elements $U_1, ..., U_n$ in $\mathscr{U}$ that have a non-empty intersection $U_1 \cap ... \cap U_n$, this intersection belongs to $\mathscr{U}$.

Lemma 6.2.1. *If a partition of unity satisfies condition (iii), then for all $V \in \mathscr{U}$ the equality $e_V \circ e_V = e_V$ holds.*

Proof. For any open set W, and any section $s \in F(W)$, we have a finite decomposition $s = \sum_{U \in \mathscr{U}} e_U(s)$, then

$$e_V(s) = \sum_U e_V \circ e_U(s) = e_V \circ e_V(s). \tag{6.2.1}$$

□

Thus a partition of unity $\{e_V\}_{V \in \mathscr{U}}$ that satisfies (iii) is a family of projections, decomposing the presheaf F is a direct sum; we refer to this as a *local orthogonal decomposition* of the functor. If (i') is also satisfied, we speak of a *super-local orthogonal decomposition.*

The condition (i') for a partition of unity is named *super-locality*; it is certainly exceptional for usual topologies, but useful for the particular topologies we are interested in in this text.

Remark 6.2.1 (Lack of functoriality). Let $f : X \to Y$ be a continuous map and $\mathscr{F}$ a fine presheaf of abelian groups over X, the presheaf $\mathscr{G} = f_*\mathscr{F}$ on Y is fine [58, Thm. 6.8.3], but it can happen that $\mathscr{F}$ is extra-fine on X and that $\mathscr{G} = f_*\mathscr{F}$ is not extra-fine. The problematic property is super-locality. For the inverse image of a presheaf $\mathscr{G}$ over Y, both fine and extra-fine fail to be transmitted from $\mathscr{G}$ to $f^{-1}\mathscr{G}$.

We shall see that positive dimensional Čech cohomology of an hyper extra-fine presheaf vanishes. To fix some notations, we summarize here the construction of Čech cohomology; more details can be found in Sections 6.6.1 and 6.6.2.

Let $\mathscr{U}$ be an open covering of a topological space X, and for each $n \in \mathbb{N}$, let $K_n(\mathscr{U})$ denote the set of sequences of length $n+1$, $u = (U_0, ..., U_n)$, of elements of $\mathscr{U}$ such that the intersection $U_u = U_0 \cap ... \cap U_n$ is non-empty. For $n \in \mathbb{N}$, a *Čech cochain* of F of degree n with respect to $\mathscr{U}$ is a element $\{c(u)\}_{u \in K_n(\mathscr{U})}$ of $\prod_{u \in K_n(\mathscr{U})} F(U_u)$. The set of n-cochains is denoted $C^n(\mathscr{U}; F)$; it is an abelian group.

A coboundary operator $\delta : C^n(\mathscr{U}; F) \to C^{n+1}(\mathscr{U}; F)$ is then introduced, as a linear map such that

$$(\delta c)(U_0, ..., U_{n+1}) = \sum_{i=0}^{n+1} (-1)^i c(U_0, ..., \widehat{U_i}, ..., U_{n+1}) | U_0 \cap ... \cap U_{n+1}, \tag{6.2.2}$$

where $\widehat{U_i}$ means that U_i is omitted. When we want to be more precise we write $\delta = \delta_n^{n+1}$ at degree n.

It is well known that $\delta \circ \delta = 0$, which allows one to define the Čech cohomology of F over $\mathscr{U}$ in degree n as the quotient abelian group $H^n(\mathscr{U}; F) = \ker(\delta_n^{n+1}) / \operatorname{im}(\delta_{n-1}^n)$. As explained in Appendix 6.B, the set of open coverings of X with the relation of refinement is a *directed set*. And the Čech cohomology of F over X is defined as

$$\forall n \in \mathbb{N}, \quad H^n(X; F) = \varinjlim H^n(\mathscr{U}; F). \tag{6.2.3}$$

See [40, Ch. 5], [58, Sec. 6.7.11] or [59, Sec. 0.3].

From the definition of δ_0^1, it is clear that the group $H^0(\mathscr{U}; F)$ can be identified with the group of global sections of F over X, for any open covering $\mathscr{U}$. Hence $H^0(X; F)$ coincides with every $H^0(\mathscr{U}; F)$ and also corresponds to global sections.

A presheaf is called *acyclic* if its cohomology is zero for every degree $n \geq 1$.

Theorem 6.2.1. *A presheaf F of abelian groups that is hyper-extra fine is also acyclic. More precisely, for every open covering $\mathscr{V}$, and every integer $n \geq 1$, there*

exists an open covering $\mathscr{U}$ finer than $\mathscr{V}$ such that the cohomology group $H^n(\mathscr{U}; F)$ is zero.

Proof. We adapt the argument given by Spanier in the case of paracompact spaces [58, Thm. 6.8.4].

Given $\mathscr{V}$, let $\mathscr{U}$ be an open covering finer than $\mathscr{V}$ that satisfies (i'), (ii'), and (iii), and is closed under finite non-empty intersections.

Consider a cochain ψ for $\mathscr{U}$ and F of degree $q \geq 1$, which is a cocycle i.e. $\delta\psi = 0$. Then for every collection $U_0, U_1, ..., U_q, U_{q+1}$ of elements of $\mathscr{U}$, we have

$$\psi(U_1, ..., U_{q+1})|U_0 \cap ... \cap U_{q+1} \\ = \sum_{k=1}^{q+1} (-1)^{k+1} \psi(U_0, ..., \widehat{U_k}, ..., U_{q+1})|U_0 \cap ... \cap U_{q+1}. \quad (6.2.4)$$

Set $U_0 = U$. Remark that $U_1 \cap \cdots U_{q+1}$ is also non-empty and an element of $\mathscr{U}$. We deduce from (6.2.4) that when U contains $U_1 \cap ... \cap U_{q+1}$,

$$e_U\psi(U_1, ..., U_{q+1})|U_1 \cap ... \cap U_{q+1} \\ = \sum_{k=1}^{q+1} (-1)^{k+1} e_U\psi(U, U_1, ..., \widehat{U_k}, ..., U_{q+1})|U_1 \cap ... \cap U_{q+1}, \quad (6.2.5)$$

and when U does not contain $U_1 \cap ... \cap U_{q+1}$, the super-locality implies that

$$e_U(\psi(U_1, ..., U_{q+1})) = 0. \quad (6.2.6)$$

For any $U \in \mathscr{U}$, we define a $(q-1)$-cochain ϕ_U for F and the covering $\mathscr{U}$ as follows: given $V_0, ..., V_{q-1} \in \mathscr{U}$, if $V_0 \cap ... \cap V_{q-1} \subseteq U$ then

$$\phi_U(V_0, ..., V_{q-1}) = e_U(\psi(U, V_0, ..., V_{q-1})|V_0 \cap ... \cap V_{q-1}), \quad (6.2.7)$$

and if $V_0 \cap ... \cap V_{q-1} \nsubseteq U$ then

$$\phi_U(V_0, ..., V_{q-1}) = 0. \quad (6.2.8)$$

By definition of the coboundary operator, in both cases we have

$$(\delta\phi_U)(U_1, ..., U_{q+1}) = \sum_{k=1}^{q+1} (-1)^{k+1} \phi_U(U_1, ..., \widehat{U_k}, ..., U_{q+1})|U_1 \cap ... \cap U_{q+1}; \quad (6.2.9)$$

which gives, when U contains $U_1 \cap ... \cap U_{q+1}$,

$$(\delta\phi_U)(U_1, ..., U_{q+1}) = \sum_{k=1}^{q+1} (-1)^{k+1} e_U(\psi(U, U_1, ..., \widehat{U_k}, ..., U_{q+1})|U_1 \cap ... \cap U_{q+1}), \quad (6.2.10)$$

and, when U does not contain $U_1 \cap ... \cap U_{q+1}$, gives $(\delta\phi_U)(U_1, ..., U_{q+1}) = 0$. Consequently, in any case we get

$$\delta\phi_U(U_1, ..., U_{q+1}) = e_U(\psi(U_1, ..., U_{q+1})). \tag{6.2.11}$$

Then we define ϕ by summing over the open sets U in $\mathscr{U}$, and using (ii'), we obtain $\delta\phi = \psi$, which proves the theorem. □

6.3 Alexandrov topologies and sheaves

6.3.1 Basic definitions

A *partially ordered set* (poset) is a set with a binary relation $\leq$ that is reflexive, antisymmetric and transitive. Equivalently, it is a small category $\mathscr{A}$ such that:

1. for any pair of objects α, β, there is at most one morphism from α to β, and
2. if there is a morphism from α to β and a morphism from β to α, then $\alpha = \beta$.

Starting with a partially ordered set $\mathrm{Ob}\,\mathscr{A}$, there exists an arrow $\alpha \to \beta$ if and only if $\beta \leq \alpha$.

The categorical coproduct between two objects α and α' of $\mathscr{A}$ is an object β such that $\alpha \to \beta$ and $\alpha' \to \beta$, that additionally satisfies the following property: for any $\omega \in \mathscr{A}$, if $\alpha \to \omega$ and $\alpha' \to \omega$, then $\beta \to \omega$. Such β is denoted $\alpha \vee \alpha'$ and called *coproduct* (or *sup*) of α and α'; it is unique. We shall not suppose that our categories have all finite coproducts, but sometimes we impose the following *conditional existence of coproducts*: for any $\alpha, \alpha' \in \mathscr{A}$, if there exists $\omega \in \mathscr{A}$ such that $\alpha \to \omega$ and $\alpha' \to \omega$, then $\alpha \vee \alpha'$ exists.

The dual notion is the product $\alpha \wedge \alpha'$ of α and α', called *meet*. In [55], Vigneaux introduced posets subject to conditional existence of meets under the name of *conditional meet semilattices*; they are the fundamental ingredient to introduce *information cohomology*.

Example 6.3.1. Let $\mathscr{K}$ be an abstract simplicial complex i.e. a family of subsets of a given set I such that if $\alpha \in \mathscr{K}$, then every subset of α is also in $\mathscr{K}$. In this structure all coproducts exist, $\alpha \vee \beta = \alpha \cap \beta$, but meets only exists conditionally.

P. S. Alexandrov introduced a natural topology on the set of objects of a poset $\mathscr{A}$, given by a basis of open sets $U_\alpha = \{\beta \mid \alpha \to \beta\}$, indexed $\alpha \in \mathscr{A}$.[2] We will

[2] To justify the definition, one must verify that an intersection $U_\alpha \cap U_{\alpha'}$ is a union of sets $U_\beta, \beta \in B$; but if $\alpha \to \beta$ and $\alpha' \to \beta$, we have $U_\beta \subseteq U_\alpha \cap U_{\alpha'}$, then $U_\alpha \cap U_{\alpha'} = \bigcup_{\beta \in U_\alpha \cap U_{\alpha'}} U_\beta$. The same argument shows that the intersection of every family of open sets is an open set.

name this topology the lower Alexandrov topology (A-topology) of $\mathscr{A}$, and denote $X_{\mathscr{A}}$ the topological space obtained in this way.

Dually, the upper sets $U^\beta = \{\alpha \mid \alpha \to \beta\}$, indexed by objects $\beta \in \mathscr{A}$, form the basis of a topology that we call upper A-topology of $\mathscr{A}$. The corresponding topological space is denoted $X^{\mathscr{A}}$. Clearly, it is the lower A-topology of the opposite category $\mathscr{A}^{op}$.

Remark that if $\alpha \to \beta$ then $U_\alpha \supseteq U_\beta$ and $U^\alpha \subseteq U^\beta$. Also, whenever $\mathscr{A}$ possesses conditional coproducts and $U_\alpha \cap U_{\alpha'}$ is non-empty, one has $U_\alpha \cap U_{\alpha'} = U_{\alpha \vee \alpha'}$; the element in $U_\alpha \cap U_{\alpha'}$ is a common upper bound of α and α'.

A general reference for Alexandrov spaces, finite topological spaces, and their relations to simplical complexes is [60]. For instance, the reader can find there the following result.

Lemma 6.3.1 ([60, Prop. 1.2.1])**.** *Let $\mathscr{A}, \mathscr{B}$ be posets. A map $f : \operatorname{Ob}\mathscr{A} \to \operatorname{Ob}\mathscr{B}$ is order preserving (equivalently, defines a covariant functor from $\mathscr{A}$ to $\mathscr{B}$) if and only if f is continuous for the lower (or upper) A-topology.*

It is a classical result, corollary of a more general result of Grothendieck [54], that the functors on a poset $\mathscr{A}$ can be seen as classical sheaves on the associated topological space. This equivalence is an easy consequence of the Comparison lemma, cf. [61, Thm. 1.1.8], a restatement of the original [62, Thm. III.4.1]. However, we give just below a simple explicit proof.

Proposition 6.3.1. *Every covariant functor F from $\mathscr{A}$ to the category of sets, can be extended to a sheaf on $X_{\mathscr{A}}$, and this extension is unique.*

Proof. Let F be a covariant functor on $\mathscr{A}$. Suppose that F extends to a sheaf F on $X_{\mathscr{A}}$. For any open set $U = \bigcup_{\alpha \in U} U_\alpha$, we must have $F(U) = \varprojlim_{\alpha \in U} F(\alpha)$, that we identify with the set of collections $(s_\alpha)_{\alpha \in U}$, with $s_\alpha \in F(\alpha)$, such that for any pair α, α' in U and any element β in $U_\alpha \cap U_{\alpha'}$, the images of s_α and $s_{\alpha'}$ in $F(\beta)$ coincide ("coherent collection"). This proves the uniqueness of the extension. And in any case, this formula defines a presheaf F on $X_{\mathscr{A}}$ i.e. for the lower A-topology. Let us verify that F is a sheaf. First, let $\mathscr{U}$ be a covering of an open U, and s, s' two elements of $F(U)$ such that $s|_V = s'|_V$ for all $V \in \mathscr{U}$; in this case, for each $\alpha \in U$, the components s_α and s'_α (in $F(\alpha)$) of s and s' are necessarily the same, so $s = s'$. Concerning the second axiom of a sheaf, suppose that a collection s_V is defined for $V \in \mathscr{U}$, and that $s_V|_{V \cap W} = s_W|_{V \cap W}$ whenever $V, W \in \mathscr{U}$ have nonempty intersection, then by restriction to the U_α for $\alpha \in U$ we get a coherent section over U. This proves the existence of the extension. $\square$

Neither the (conditional) existence of coproducts or products nor any finiteness hypothesis are used in the previous proof. A similar proposition holds for the upper topology, but in this case the sheaves are in correspondence with contravariant

functors on $\mathscr{A}$ (i.e. presheaves on $\mathscr{A}$).

Remark 6.3.1. The construction in the above proposition establishes an equivalence of the category $\mathrm{PSh}(\mathscr{A})$ of presheaves over $\mathscr{A}$, having for morphisms the natural transformations, with the category of sheaves over $X_{\mathscr{A}}$, having for morphisms the natural transformations. In fact these two categories are isomorphic toposes cf. [63, Prop. 4.1], [62], [64].

Remark 6.3.2 (Functoriality). In the case of posets and their associated Alexandrov topologies, the direct images and inverse images of sheaves (or presheaves) are easy to handle.

Let $f : \mathscr{A} \to \mathscr{B}$ be a morphism of posets, i.e. an increasing map; f is continuous for the lower and the upper topologies.

If G is a sheaf of sets on $\mathscr{B}$ for the lower A-topology, its inverse image is defined at the level of germs of sections by the formula $(f^*G)(\alpha) = G(f(\alpha))$, which gives the stack in α.

If F is a sheaf of sets on $\mathscr{A}$ for the lower A-topology; its direct image is defined by $(f_*F)(\beta) = (f_*F)(U_\beta) = (f^{-1}F)(U_\beta) = F(f^{-1}(U_\beta))$, where the open set $f^{-1}(U_\beta)$ is the set of elements $\alpha \in \mathscr{A}$ such that $f(\alpha) \subseteq \beta$.

If β does not meet the image of f, this is the empty set. For sheaves of abelian groups, and β non-intersecting $f(\mathscr{A})$, we have $f_*F(\beta) = 0_\beta$. For instance, if $\mathscr{A}$ is a sub-poset of $\mathscr{B}$, and J the injection: J_*F coincides with F on $\mathscr{A}$ and is zero in its complement.

Analog results hold true for the upper topology and for the contravariant functors on $\mathscr{A}$ and $\mathscr{B}$.

6.3.2 Extra-fine presheaves on posets

Consider a poset $\mathscr{A}$, and the induced topological space $X_{\mathscr{A}}$ whose underlying set is $\mathrm{Ob}\,\mathscr{A}$, equipped with the lower A-topology. Let us denote by $\mathscr{U}_{\mathscr{A}}$ the covering of $X_{\mathscr{A}}$ by the open sets $U_\alpha, \alpha \in \mathscr{A}$.

By definition of the lower A-topology, $\mathscr{U}_{\mathscr{A}}$ refines any other open covering. So by taking the injective limit (i.e. the colimit) on the category of coverings preordered by refinement (cf. Appendix 6.B), Theorem 2.5 implies that if F is an hyper-extra-fine sheaf on $X_{\mathscr{A}}$, for any $n \geq 1$ we have $H^n(X; F) = 0$.

Due to the maximality of the open covering $\mathscr{U}_{\mathscr{A}}$, the existence of a super-local orthogonal decomposition for F subordinated to $\mathscr{U}_{\mathscr{A}}$ implies that F is extra-fine.

However, in general, $\mathscr{U}_{\mathscr{A}}$ is not the only finest open covering of $\mathscr{A}$. The relation of refinement only defines a pre-order.

Therefore it can easily happen that a sheaf F is extra-fine, but that F is not super-local for $\mathscr{U}_{\mathscr{A}}$.

In the applications we have in mind, described in the introduction and Section 6.5, the covering $\mathscr{U}_{\mathscr{A}}$ is super-local for the sheaf F; in this case we say that F over $X_{\mathscr{A}}$ is *canonically extra-fine*. This property implies that F is extra-fine, because every open covering is less fine than $\mathscr{U}_{\mathscr{A}}$.

Remark 6.3.3. When $\mathscr{A}$ is stable by conditional finite coproducts, every finite non-empty intersection of elements of $\mathscr{U}_{\mathscr{A}}$ is itself an element of $\mathscr{U}_{\mathscr{A}}$. Therefore, if a sheaf F over $X_{\mathscr{A}}$ is canonically extra-fine, then it is hyper-extra-fine, by Definition 6.2.3.

If F is canonically extra-fine, there is a super-local orthogonal decomposition $\{e_\alpha\}_{\alpha\in\mathscr{A}}$ associated to the covering $\mathscr{U}_A$. From the axioms, the images $S_\alpha = \operatorname{im} e_\alpha$ define sub-sheaves of F such that $F = \bigoplus_{\alpha\in\mathscr{A}} S_\alpha$.
Moreover, for any open set U of $X_{\mathscr{A}}$, $e_\alpha(U)$ is the projection on $S_\alpha(U)$ parallel to $\bigoplus_{\beta:\beta\neq\gamma} S_\beta(U)$.
We will see the relation with the *interaction decomposition* in the next section.

This implies the following result.

Proposition 6.3.2. *Let F be a canonically extra-fine sheaf over $X_{\mathscr{A}}$, where $X_{\mathscr{A}}$ denotes the topological space defined by a poset $\mathscr{A}$ equipped with its lower A-topology. Then,*

$$H^0(X_{\mathscr{A}}; F) = \bigoplus_{\alpha\in\mathscr{A}} H^0(\mathscr{U}_{\mathscr{A}}; S_\alpha).$$

Proof. Recall that $H^0(X; F) = H^0(\mathscr{U}; F)$ for any open covering. The naturality of the e_α implies that $S_\alpha(\beta)$ is mapped to $S_\alpha(\gamma)$ by the map $F\iota$ induced by $\iota : \beta \to \gamma$. Hence the set of global sections of F can be computed as the direct sum of sections of S_α. □

Remark 6.3.4. In the above results, the groups $F(\alpha) = F(U_\alpha)$, for $\alpha \in \mathscr{A}$, or $F(U)$, for $U \in \mathscr{U}$, are not supposed finitely generated. This is a good point because, starting with a covariant functor F of finitely generated abelian groups over $\mathscr{A}$, it can happen that the sheaf extending F to $X_{\mathscr{A}}$ in Proposition 1 is not made of finitely generated abelian groups.

6.3.3 Finiteness conditions

In what follows we will sometimes consider posets $\mathscr{A}$ that satisfy some finiteness condition.

We say that $\mathscr{A}$ is *locally finite*, if for every arrow $\alpha \to \beta$, the intersection $U_\alpha^\beta = U_\alpha \cap U^\beta$ is finite. In other terms, there exist only a finite number of chains without repetition beginning at α and ending at β.

A stronger property is (lower) *closure finite*, meaning that every U_α is finite. This is the case for the poset associated to a CW complex (cf. [65]).

A more convenient condition for us will be the hypothesis of *locally finite dimension*: for every object α of $\mathscr{A}$, the non-degenerate chains $\alpha \xrightarrow{\neq} \beta_1 \xrightarrow{\neq} \beta_2 \xrightarrow{\neq} \cdots$ are all finite and their length is uniformly bounded; the least upper bound is called the *dimension* of α.

Note that the conditions of local finiteness is self-dual, i.e. it holds for $\mathscr{A}$ if and only its hold for $\mathscr{A}^{op}$. This is not the case for closure finiteness or locally finite dimension. The posets $\mathscr{A}$ and $\mathscr{A}^{op}$ are both closure finite if and only if they are finite. The posets $\mathscr{A}$ and $\mathscr{A}^{op}$ are both of locally finite dimension if and only if there exists a number d such that any sequence $\alpha \to ... \to \beta$ of length bigger than $d+1$ has a repetition; in this case we say that $\mathscr{A}$ has finite dimension, or finite depth.

The most elegant finiteness condition is Lower Well Foundedness: there exists no infinite chain without repetition (cf. [31]).

In the case of finite posets and sheaves of finitely generated abelian groups, we can assert that the cohomology is finitely generated.

6.4 Interaction decomposition

6.4.1 Condition G and the equivalence theorem

Let $\mathscr{A}$ be an arbitrary poset, and let V be a contravariant functor on $\mathscr{A}$, valued in the category of vector spaces over a commutative field $\mathbb{K}$. We suppose that for each $\rho : \alpha \to \beta$ in $\mathscr{A}$, the map $j_{\alpha\beta} = V(\rho) : V(\beta) \to V(\alpha)$ is injective. We call V a *linear injective presheaf*, or simply an injective presheaf.

To get a sheaf on a topological space from V, we must consider the upper A-topology and not the lower one, because $U^\beta \supseteq U^\alpha$ whenever $\alpha \to \beta$. In what follows we denote by $X^{\mathscr{A}}$ the set $\operatorname{Ob}\mathscr{A}$ equipped with the upper A-topology.

We write $V_{\alpha\beta}$ instead of $j_{\alpha\beta}(V_\beta)$. For a partition of unity associated to V, if it exists, $e_{\alpha|\beta}$ is an endomorphism of V_β.

Definition 6.4.1. An *interaction decomposition* of an injective presheaf V is a family of vector sub-spaces S_γ of V_γ, indexed by $\gamma \in \mathscr{A}$, such that

$$\forall \alpha \in \mathscr{A}, \quad V_\alpha = \bigoplus_{\alpha \to \beta} j_{\alpha\beta} S_\beta. \tag{6.4.1}$$

Let us introduce, for every $\gamma \in \mathscr{A}$, the vector space

$$\forall \alpha \in \mathscr{A}, \quad S^\gamma(\alpha) = \begin{cases} S_{\alpha\gamma} = j_{\alpha\gamma} S_\gamma & \text{if } \alpha \to \gamma \\ 0 & \text{if } \alpha \nrightarrow \gamma \end{cases}, \tag{6.4.2}$$

this defines a presheaf $\mathscr{S}^\gamma$ on $\mathscr{A}$. The interaction decomposition corresponds to a decomposition of the presheaf V:

$$V = \bigoplus_{\gamma \in \mathscr{A}} \mathscr{S}^\gamma. \tag{6.4.3}$$

The name *interaction decomposition* comes from Statistical Physics, where the spaces $\{V_\alpha\}_{\alpha \in \mathscr{A}}$ are spaces of functions depending on local variables over a lattice. An important old example corresponds to Wick's theorem, used in remormalization theory and Wiener analysis; a particular case is the decomposition of functions in sum of Bernoulli polynomials or Hermite polynomials, cf. Sinai's *Theory of Phase Transition: rigorous results* [36]. The notion of interaction decomposition also plays a fundamental role in other domains of Probability and Statistics, cf. [14].

Suppose that F is a canonically extra-fine sheaf on $X^{\mathscr{A}}$ in $\mathbb{K}$-vector spaces and that for every $\alpha \to \beta$ in $\mathscr{A}$, the linear map $F_\beta \to F_\alpha$ is injective; then the beginning of Section 6.3.2 proves that the induced presheaf on $\mathscr{A}$ has a natural interaction decomposition. Under a fairly strong finiteness hypothesis, the reciprocal statement is true, as shown by the following lemma.

Definition 6.4.2. We say that $\mathscr{A}$ is *lower finitely covered*, or satisfies the condition (a), if it has a finite subset $\mathscr{A}_{in}$ such that, for any object $\beta \in \mathscr{A}$, there exists an object $\alpha \in \mathscr{A}_{in}$ with $\alpha \to \beta$.

This condition is verified, for instance, if $\mathscr{A}$ possesses an initial object or if it is a finite poset.

Lemma 6.4.1. *Assume that $\mathscr{A}$ satisfies (a). Let $\{S_\gamma\}_{\gamma \in \mathscr{A}}$ be an interaction decomposition of an injective presheaf V, and let F_V be the associated sheaf over $X^{\mathscr{A}}$; then F_V is canonically extra-fine. More precisely, F_V is isomorphic to the direct sum of the sheaves F_{S^α} for $\alpha \in \mathscr{A}$. The family of projections onto $S_{\alpha\beta}$ parallel to $\bigoplus_{\beta' : \beta \neq \beta'} S_{\alpha\beta'}$, extends to the partition of unity $\{e_\beta\}_{\beta \in \mathscr{A}}$, which over any open set W is the projection on $F_{S^\beta}(W)$ parallel to $\bigoplus_{\beta' : \beta \neq \beta'} F_{S^{\beta'}}(W)$.*

Proof. By the maximality of the covering $U^{\mathscr{A}}$ of $X^{\mathscr{A}}$ by the $\{U^\alpha\}_{\alpha \in \mathscr{A}}$, it is sufficient to verify the axioms (i'), (ii') and (iii) for this covering $U^{\mathscr{A}}$. For (i'), if $U^\alpha \not\subseteq U^\beta$, then $\alpha \nrightarrow \beta$, which in turn implies that $S_{\alpha\beta} = 0$ so $e_{\beta|\alpha} = 0$. Remark: as every point outside a U^α belongs to an open set $U^{\alpha'}$ with $\alpha \neq \alpha'$, this implies the condition (i).

For (ii'), let us consider $x \in V_\alpha$; to have $e_{\gamma|\alpha}(x) \neq 0$, we must have $\gamma \to \alpha$, but the definition of interaction decomposition tells that x belongs to the direct sum of the spaces $S_{\alpha\beta}$, thus only a finite number of the $e_{\gamma|\alpha}(x)$ are different from zero. Now consider any open set W in $X^{\mathscr{A}}$: it is an arbitrary union of open sets U^ω, for $\omega \in W$. Let us denote by $W^\uparrow$ the set of γ such that $\omega \to \gamma$ for some $\omega \in W$. We claim that

$$F_V(W) \cong \bigoplus_{\gamma \in W^\uparrow} S_\gamma. \tag{6.4.4}$$

In fact, one has

$$\begin{aligned} F_V(W) &= \varprojlim_{\omega \in W} F_V(\omega) \\ &\cong \left\{ (s_\omega) \in \prod_{\omega \in W} \bigoplus_{\omega \to \gamma} S_{\omega\gamma} \mid \text{for every arrow } \alpha \to \beta,\ j_{\alpha\beta}(s_\beta) = s_\alpha \right\}. \end{aligned}$$

It is evident that any element of $\bigoplus_{\gamma \in W^\uparrow} S_\gamma$ determines an element of this limit. In the other direction, remark that an element of $F_V(W)$ is uniquely determined by an element $(t_\gamma)_\gamma$ of $\prod_{\gamma \in W^\uparrow} S_\gamma$. But only a finite number of the t_γ can be nonzero, since $F_V(W)$ also injects into $\prod_{\alpha \in \mathscr{A}_{in}} \bigoplus_{\alpha \to \gamma} S^\gamma(\alpha)$, where the product is finite.

For (iii) consider $\beta \neq \beta'$ lower than $\alpha \in \mathscr{A}$, then by definition of the projector $e_{\beta|\alpha}$ the space $S_{\alpha|\beta'}$ belongs to its kernel. □

Together with the discussion before Proposition 6.3.2, the lemma implies the following result.

Theorem 6.4.1. *For any injective presheaf of vector spaces V,*

1. *if the associated sheaf on $X^{\mathscr{A}}$ is canonically extra-fine then V admits an interaction decomposition, and*

2. *if V has an interaction decomposition, and $\mathscr{A}$ satisfies (a), then the associated sheaf on $X^{\mathscr{A}}$ is canonically extra-fine.*

We will see that an interaction decomposition implies the following statement, that we call *condition G*:

$$\forall \alpha, \beta \in \mathscr{A} \text{ such that } \alpha \to \beta, \quad V_{\alpha\beta} \cap \left(\sum_{\substack{\gamma : \alpha \xrightarrow{\neq} \gamma \\ \gamma \not\to \beta}} V_{\alpha\gamma} \right) \subset \sum_{\substack{\gamma : \alpha \xrightarrow{\neq} \gamma \\ \beta \xrightarrow{\neq} \gamma}} V_{\alpha\gamma}, \tag{G}$$

where $\beta \xrightarrow{\neq} \gamma$ means that $\beta \to \gamma$ and $\beta \neq \gamma$.

The condition G implies in turn the existence of an interaction decomposition, if one supposes that the poset $\mathscr{A}$ is of local finite (lower) dimension. As told in the preceding section, this means that for every object α of $\mathscr{A}$, the non-degenerate chains $\alpha \xrightarrow{\neq} \beta_1 \xrightarrow{\neq} \beta_2 \xrightarrow{\neq} \cdots$ are all finite and their length is uniformly bounded; the least upper bound is called the *dimension* of α.

Theorem 6.4.2. *Let V be a linear injective presheaf on a poset $\mathscr{A}$.*

1. *If the the condition G is satisfied and $\mathscr{A}$ is of locally finite dimension (in the lower direction), the presheaf V on $\mathscr{A}$ admits an interaction decomposition;*

2. *If the presheaf V on $\mathscr{A}$ admits an interaction decomposition, the condition G is satisfied.*

Proof. For each $\alpha \in \mathscr{A}$, we define the *boundary sum* $V'_\alpha = \sum_{\beta:\alpha\to\beta,\alpha\neq\beta} V_{\alpha\beta}$, and we choose any supplementary space S_α of it. Hence it remains to prove that V'_α is the direct sum $\bigoplus_{\beta:\alpha\to\beta,\beta\neq\alpha} S_{\alpha\beta}$. We prove this by recurrence in the dimension of α.

First, if α has dimension zero, then it is maximal, which means that $\alpha \to \beta$ implies $\alpha = \beta$. So $V'_\alpha = 0$ and the claim is then trivially true.

Let us suppose now that the recurrence hypothesis hods true in dimension smaller or equal than $r-1$, for some $r \geq 1$, and consider α of dimension r.

Let B be the set of maximal cells β such that $\alpha \to \beta$ and $\alpha \neq \beta$. And for $\beta \in B$, consider $x \in V_{\alpha\beta}$, and suppose it also belongs to the algebraic sum $\sum_{\gamma:\alpha\to\gamma,\beta\neq\gamma,\gamma\neq\alpha} V_{\alpha\gamma}$. As β is maximal, we have

$$x \in \sum_{\substack{\gamma:\alpha\xrightarrow{\neq}\gamma \\ \gamma\not\to\beta}} V_{\alpha\gamma}. \tag{6.4.5}$$

Then, applying the condition G to x, we deduce that x belongs to the sum of $V_{\alpha\gamma}$ over the $\gamma \in \mathscr{A}$ such that $\alpha \xrightarrow{\neq} \gamma$ and $\beta \xrightarrow{\neq} \gamma$, which coincides with $V'_{\alpha\beta} = j_{\alpha\beta}V'_\beta$, consequently

$$V'_\alpha = S_{\alpha\beta} \oplus (V'_{\alpha\beta} + \sum_{\beta'\neq\beta,\beta'\in B} V_{\alpha\beta'}). \tag{6.4.6}$$

For $\beta' \in B$, $\beta' \neq \beta$, consider $x' \in V_{\alpha\beta'}$ and suppose it also belongs to the algebraic sum $\sum_{\beta"\in B,\beta"\neq\beta,\beta"\neq\beta'} V_{\alpha\gamma}$. As β' is maximal, we have

$$x \in \sum_{\substack{\gamma:\alpha\xrightarrow{\neq}\gamma \\ \gamma\not\to\beta'}} V_{\alpha\gamma} \tag{6.4.7}$$

Then, applying the condition G to x', we deduce that x' belongs to the sum of $V_{\alpha\gamma}$ over the $\gamma \in \mathscr{A}$ such that $\alpha \xrightarrow{\neq} \gamma$ and $\beta' \xrightarrow{\neq} \gamma$, which is the space $V'_{\alpha\beta'} = j_{\alpha\beta'}V'_\beta$,

consequently

$$V'_\alpha = S_{\alpha\beta} \oplus S_{\alpha\beta'} \oplus (V'_{\alpha\beta} + V'_{\alpha\beta'} + \sum_{\beta" \in B \setminus \{\beta,\beta'\}} V_{\alpha\beta"}). \tag{6.4.8}$$

By (possibly transfinite) induction, we get

$$V'_\alpha = \bigoplus_{\beta \in B} S_{\alpha\beta} \oplus \sum_{\beta \in B} V'_{\alpha\beta}. \tag{6.4.9}$$

Then we conclude by applying the recurrence hypothesis to the spaces V'_β, and the transitivity, $j_{\alpha\gamma} = j_{\alpha\beta} \circ j_{\beta\gamma}$.

We prove now the second claim. Suppose that there exists an interaction decomposition of V, as in (6.4.1). Let us fix $\alpha \in \mathscr{A}$, look at $\alpha \to \beta$, and consider a vector x in $S_{\alpha\beta}$. Suppose that this vector is equal to a finite sum $y_1 + ... + y_m$ of elements of $V_{\alpha\beta_1}, ..., V_{\alpha\beta_m}$ respectively, with $\alpha \to \beta_i$ but $\beta \not\to \beta_i$. Applying to each of these vectors y_i the projector $e_{\beta|\alpha}$, we find zero: however $e_{\beta|\alpha}(x) = x$ by definition of $S_{\alpha\beta}$, thus $x = 0$. Now consider any vector z in $V_{\alpha\beta}$; by interaction decomposition, z is (in a unique way) the sum of a vector $x \in S_{\alpha\beta}$, and a vector y in the space $V'_{\alpha\beta}$, equal to the sum of the $S_{\alpha\gamma}$ for $\beta \xrightarrow{\neq} \gamma$, which is included in the sum of all the $V_{\alpha\gamma}$ for $\beta \xrightarrow{\neq} \gamma$. Then the condition G is proved. □

In this generality, the theorem above appears for the first time in [31], by G. Sergeant-Perthuis. It holds true if we replace the property of local finite dimension by the noetherian property of well-foundedness.

We also refer to condition G as *sum-intersection property.* Before the work of G. Sergeant-Perthuis, a particular case of this property appeared in the book of Lauritzen (see Proposition B.5 in the Appendix B of [14]), as a corollary of the interaction decomposition. Lauritzen considers a finite poset $\mathscr{A}$ and a presheaf of finite dimensional vector spaces $\{V_a\}_{a \in \mathscr{A}}$ that admits an interaction decomposition; the property is stated for two open subsets U, V of the lower space $X_{\mathscr{A}}$ (named generating classes, the topology was not mentioned) in the following form:

$$\sum_{c \in R(U \cap V)} V_c = \left(\sum_{a \in R(U)} V_a \right) \cap \left(\sum_{b \in R(V)} V_b \right), \tag{6.4.10}$$

where $R(U)$ denotes the set of all elements a of $\mathscr{A}$ such that U_a is included in U.

In [31] it is assumed that all the $V(\alpha)$, $\alpha \in \mathscr{A}$, belong to a fixed vector space W, this is not a restriction, because it is always possible to inject all of them in the colimit of V seen as a functor over $\mathscr{A}^{op}$.

Let us recall that the connected composents of a poset $\mathscr{A}$ are the equivalent classes of the equivalence relation generated by the preorder.

Theorem 6.4.3. *Let V be an injective presheaf with an interaction decomposition, F_V the sheaf on $X^{\mathscr{A}}$ associated to it, and F_{S^α} the sheaf on $X^{\mathscr{A}}$ induced by S^α.*

1. *For $\gamma \in \mathscr{A}$, if γ is final in its connected component, then $H^0(X^{\mathscr{A}}, F_{S^\gamma}) \cong S_\gamma$, where S_γ denotes the stalk at γ, and $H^0(X^{\mathscr{A}}, F_{S^\gamma}) = 0$ otherwise.*

2. *When the finite products exist conditionally in $\mathscr{A}$ and $\mathscr{A}$ is lower finitely covered, the sheaf F_V on $X^{\mathscr{A}}$ associated to V is acyclic and*

$$H^0(X^{\mathscr{A}}, F_V) \cong \bigoplus_{\alpha \in \mathscr{A}} H^0(X^{\mathscr{A}}, F_{S^\alpha}) \cong \bigoplus_{\alpha \in \mathscr{A}^f} S_\alpha,$$

where $\mathscr{A}^f$ are the final elements of each connected component when they exist.

Proof. To compute $H^0(X^{\mathscr{A}}, F_{S^\gamma})$ it is sufficient to consider the canonical covering $\mathscr{U}^{\mathscr{A}}$. The group $H^0(\mathscr{U}^{\mathscr{A}}, F_{S^\gamma})$ is the set of global sections of S^γ. The presheaf S^γ is zero outside U^γ and is $(j_{\alpha\gamma} S_\gamma)$ for $\alpha \in U^\gamma$. For this sheaf any arrow is zero or an isomorphism. A section is given by a collection $s_\alpha \in j_{\alpha\gamma} S^\gamma$, for $\alpha \in \mathscr{A}$, such that for any arrow $\alpha \to \beta$, we have $j_{\alpha\beta}(s_\beta) = s_\alpha$. Therefore, if a zero appears, the global section is zero; the only case where no zero map appears is for γ in $\mathscr{A}^{out}$.

Since V admits an interaction decomposition and (a) is verified, the sheaf F_V is canonically extra-fine in virtue of Theorem 6.4.1: F_V is decomposed in a direct sum of the F_{S^α}. The Čech cohomology of a direct sum of sheaves is the direct sum of their cohomology; this follows by projection of the cochain and naturality of the coboundary operator, hence $H^0(X^{\mathscr{A}}, F_V) \cong \bigoplus_{\alpha \in \mathscr{A}} H^0(X^{\mathscr{A}}, F_{S^\alpha})$. Cf. Proposition 6.3.2.

Finally, given the conditional existence of products, canonically extra-fine implies hyper-extra-fine, so the acyclicity of F_V results from the Theorem 6.2.1. □

Remark 6.4.1. As we will see in Section 6.6, the vanishing of $H^1(X^{\mathscr{A}}, F_{S^\alpha})$ and $H^2(X^{\mathscr{A}}, F_{S^\alpha})$ follows from the injectivity in the comparison between Čech and topos cohomology, proved by Grothendieck in Tohoku. This does not require the hypothesis on finite coproducts, but an hypothesis is needed for the argument of direct sum.

6.4.2 Duality

We suppose that the presheaf V on $\mathscr{A}$ admits an interaction decomposition. For each $\alpha \in \mathscr{A}$, let V_α^* be the (algebraic) dual vector space of the vector space $V_\alpha = V(\alpha)$; the transpose maps ${}^t j_{\alpha\beta}; \alpha \to \beta$ define a covariant functor on $\mathscr{A}$, then a presheaf for the lower topology on $\mathscr{A}$. But the transposed maps ${}^t e_\alpha; \alpha \in \mathscr{A}$ give a decomposition of this presheaf into the product of the presheaves $S_\alpha^*; \alpha \in \mathscr{A}$,

not into a direct sum. Therefore we need to follow another way to dualize the Theorem 6.4.3.

This can be done adding a further hypothesis. Let us suppose that there exists a covariant functor F on $\mathscr{A}$ (equivalently, a topological sheaf on $X_{\mathscr{A}}$), with surjective arrows $\pi^{\beta\alpha}$ for $\alpha \to \beta$, such that $V_\alpha = F_\alpha^*$ and $j_{\alpha\beta} = {}^t\pi^{\beta\alpha}$ for all pairs α, β with $\alpha \to \beta$. Then for every $\alpha \in \mathscr{A}$, the space F_α embeds naturally in V_α^*, in such a manner that, for every pair α, β with $\alpha \to \beta$, $j_{\alpha\beta}$ induces the map $\pi^{\beta\alpha}$. Let us denote by e_α^* the restriction of ${}^te_\alpha$ to F. Given the following lemma, this gives a family of orthogonal projectors from F to the dual copresheaf V^* (by the same argument given in the proof of Lemma 6.2.1). We do not ask that ${}^te_\alpha$ preserves F.

Lemma 6.4.2. $\mathrm{id}_F = \sum_{\alpha\in\mathscr{A}} e_\alpha^*$, *in the sense of finite sum when applied to a given vector* $g \in F_\beta$ *for any* $\beta \in \mathscr{A}$.

Proof. Consider $\gamma \in \mathscr{A}$ and a basis $\{f_j \mid j \in J\}$ of F_γ as a vector space over $\mathbb{K}$, the space $V_\gamma = F_\gamma^*$ is isomorphic to the product $\mathbb{K}^J$, in such a manner that the duality is given by the natural evaluation. The space F_γ itself is isomorphic to the space of scalar functions on J which are zero outside a finite subset.

For $j \in J$, note $x_j = f_j^*$ the element of V_γ corresponding to f_j (in the dual basis). The set A_j of elements $\beta \in \mathscr{A}$ such that $e_\beta(x_j) \neq 0$ is finite, and we have

$$x_j = \sum_{\beta\in A_j} e_{\beta|\gamma}(x_j). \tag{6.4.11}$$

Choose $g \in F_\gamma$. For any $\alpha \in \mathscr{A}$, we have

$$\langle x_j, e_{\alpha|\gamma}^*(g)\rangle = \langle e_{\alpha|\gamma}(x_j), g\rangle, \tag{6.4.12}$$

where the bracket denotes the form of incidence from $V_\gamma^* \times V_\gamma$ to $\mathbb{K}$. Then

$$\langle x_j, g\rangle = \sum_{\beta\in A_j} \langle e_{\beta|\gamma}(x_j), g\rangle = \langle x_j, \sum_{\beta\in A_j} e_{\beta|\gamma}^*(g)\rangle. \tag{6.4.13}$$

If α does not belong to A_j, we have

$$0 = \langle e_{\alpha|\gamma}(x_j), g\rangle = \langle x_j, e_{\alpha|\gamma}^*(g)\rangle, \tag{6.4.14}$$

i.e. x_j vanishes at $e_{\alpha|\gamma}^*(g)$. Therefore, for every $j \in J$ and $g \in F$,

$$\langle x_j, g\rangle = \langle x_j, \sum_{\beta\in\mathscr{A}} e_{\beta|\gamma}^*(g)\rangle. \tag{6.4.15}$$

Let us denote by B_g a finite set of indexes $k \in J$ such that

$$g = \sum_{k\in B_g} g^k f_k. \tag{6.4.16}$$

Equivalently, if j does not belong to B_g, we have

$$0 = \langle x_j, g\rangle = \langle x_j, \sum_{\beta\in\mathscr{A}} e^*_{\beta|\gamma}(g)\rangle, \tag{6.4.17}$$

and if $j \in B_g$,

$$g^j = \langle x_j, g\rangle = \langle x_j, \sum_{\beta\in\mathscr{A}} e^*_{\beta|\gamma}(g)\rangle. \tag{6.4.18}$$

Now, consider any element $x \in V_\gamma = F^*_\gamma$, it is identified with the numerical function that assigns $x(j) \in \mathbb{K}$ to $j \in J$. Then, using the above equations (6.4.17) and (6.4.18), we get

$$\begin{aligned}\langle x, g\rangle &= \langle x, \sum_{k\in B_g} g^k f_k\rangle \\ &= \sum_{k\in B_g} g^k x(k) = \sum_{k\in B_g} \langle x(k)x_k, \sum_{\beta\in\mathscr{A}} e^*_{\beta|\gamma}(g)\rangle \\ &= \langle x, \sum_{\beta\in\mathscr{A}} e^*_{\beta|\gamma}(g)\rangle,\end{aligned}$$

which implies the desired result. □

We noted that for F, in general the interaction decomposition does not hold, but something else holds true, which is sufficient in many applications.

The images of $e^*_{\alpha|\beta}$ for β describing $\mathscr{A}$ define a sub-sheaf of V^*, that we denote T^α. And we denote by T_α its stalk at $\alpha \in \mathscr{A}$. The above lemma tells that over $\mathscr{A}$, the cosheaf F is isomorphic to the sum of the cosheaves T_α.

In what follos we denote by $\widetilde{F}$, or simply F when there is no risk of confusion, the associated sheaf over $X_{\mathscr{A}}$.

To obtain a decomposition of the associated sheaf $\widetilde{F}$ over $X_{\mathscr{A}}$, we need an hypothesis, dual of the condition (a) in Definition 6.4.2, i.e. we assume that A^{op} is lower finitely covered, and we say that $\mathscr{A}$ is *upper finitely covered*, or satisfied the condition (a^*).

Corollary 6.4.1. *When the finite coproducts exist conditionally in $\mathscr{A}$, and when $\mathscr{A}$ satisfies the condition (a^*), the sheaf induced by F on $X_{\mathscr{A}}$ is acyclic and $H^0(X_{\mathscr{A}}, F) \cong \bigoplus_{\alpha\in\mathscr{A}} H^0(X_{\mathscr{A}}, T^\alpha) = \bigoplus_{\alpha\in\mathscr{A}} T_\alpha$.*

Proof. The Čech cohomology of a direct sum of sheaves is the direct sum of their cohomology; this follows by projection of the cochain and naturality of the coboundary operator.

Moreover, for any $\gamma \in \mathscr{A}$, the presheaf T^γ is zero in the complementary set of U^γ and is $(j_{\alpha\gamma}S_\gamma)^*$ for $\alpha \in U^\gamma$. Therefore its space of global section can be identified with the stalk at γ.

Therefore the corollary results from the following lemma.

□

Lemma 6.4.3. *Let T be a presheaf on $\mathscr{A}$, equipped with the* lower *A-topology, that is supported on a set U^γ for $\gamma \in \mathscr{A}$. If for every $\alpha, \beta \in U^\gamma$ such that $\alpha \to \beta$ the morphism $\pi^{\beta\alpha}$ is an isomorphism, then T is acyclic and $H^0(X_{\mathscr{A}}, T) = T_\gamma$.*

Proof. It is sufficient to prove the acyclicity for the covering by the $\{U_\alpha\}_{\alpha\in\mathscr{A}}$. Every space T_α is zero except if $\alpha \to \gamma$, then we can consider that every cochain takes its value in T_γ, whatever being its degree. Considering a cochain c of degree n, if it is a cocycle, for any family $\alpha_1, ..., \alpha_{n+1}$ in $\mathscr{A}$, we have, in T_γ:

$$c(\alpha_1, ..., \alpha_{n+1}) = \sum_{k=1}^{n+1} (-1)^{k+1} c(\gamma, \alpha_1, ..., \widehat{\alpha_k}, ..., \alpha_{n+1}), \tag{6.4.19}$$

which tells that c is equal to $\delta\phi$, where ϕ is the $(n-1)$-cochain defined by

$$\forall \beta_1, ..., \beta_n \in \mathscr{A}, \quad \phi(\beta_1, ..., \beta_n) = c(\gamma, \beta_1, ..., \beta_n). \tag{6.4.20}$$

This establishes the lemma. □

Remark 6.4.1 also holds for this result.

In the following section we will need a variant of the lemma (6.4.3), concerning the relative cohomology. Suppose that $\mathscr{A}$ is a sub-poset of $\mathscr{B}$, and that we have a presheaf T on $X_{\mathscr{B}}$ (i.e. for the lower A-topology on $\mathscr{B}$), which is supported on a set U^γ for $\gamma \in \mathscr{B}$, such that every morphism $\pi^{\beta\alpha}$ with $\alpha \to \beta \to \gamma$ is an isomorphism. Then we consider the sheaf S over $\mathscr{A}$, obtained by restriction.
We assume that both $\mathscr{A}$ and $\mathscr{B}$ are closed by finite coproducts.

Lemma 6.4.4. *Under the above hypotheses, $\forall n \geq 1, H^n(\mathscr{B}, \mathscr{A}; T, S) = 0$.*

See Appendix 6.B for the definition of relative cohomology.

Proof. It is sufficient to prove the result for the cohomology of the covering by the open sets $\{U_\beta\}_{\beta\in\mathscr{B}}$, and their traces on $\mathscr{A}$. By definition, a relative cochain $c \in C^n(\mathscr{B}, \mathscr{A}; T, S)$ takes the value 0 on every family of $n+1$ elements of $\mathscr{A}$. If it is a cocycle, for any family $\alpha, \alpha_1, ..., \alpha_{n+1}$ of $n+2$ elements in $\mathscr{B}$, we have, in T_γ:

$$c(\alpha_1, ..., \alpha_{n+1}) = \sum_{k=1}^{n+1} (-1)^{k+1} c(\alpha, \alpha_1, ..., \widehat{\alpha_k}, ..., \alpha_{n+1}), \tag{6.4.21}$$

which tells that c is equal to $\delta\phi$, where ϕ is the $(n-1)$-cochain defined by

$$\forall \beta_1, ..., \beta_n \in \mathscr{B}, \quad \phi(\beta_1, ..., \beta_n) = c(\alpha, \beta_1, ..., \beta_n). \tag{6.4.22}$$

If U^γ has empty intersection with $\mathscr{A}$, taking $\alpha = \gamma$, we have $\phi \in C^{n-1}(\mathscr{B}, \mathscr{A}; T, S)$, and $c = d\phi$. And if α belongs to $\mathscr{A} \cap U^\gamma$, the cochain ϕ belongs to $C^{n-1}(\mathscr{B}, \mathscr{A}; T, S)$, and $c = d\phi$. This establishes the lemma. □

6.5 Factorization of free sheaves

6.5.1 Free presheaves and intersection properties

In many applications to Statistical Physics or Bayesian Learning, the presheaves that appear are free modules, generated by subsets of a fixed set.

A set I is given (non-necessarily finite) and the poset $\mathscr{A}$ is a sub-poset (i.e. a subcategory) of the poset $(\mathscr{P}_f(I), \to)$ of finite subsets of I, ordered in such a way that $A \to B$ iff $B \subseteq A$. The poset $\mathscr{A}$ is automatically of locally finite dimension. The pair $(\mathscr{A}, I)$ is named an *hypergraph*. We consider a covariant functor (a.k.a. copresheaf) of sets E on $\mathscr{A}$, such that, for every $\alpha \in \mathscr{A}$, the set $E_\alpha = E(\alpha)$ can be identified with the cartesian product $\prod_{i\in\alpha} E_i$ by surjective maps $\pi^{i\alpha} = E(\alpha \to i)$. By naturality, all the maps $\pi^{\beta\alpha} : E_\alpha \to E_\beta$ are surjective. If the empty set $\emptyset$ belongs to $\mathscr{A}$, the set $E_\emptyset$ is a singleton $* = \{\emptyset\}$. In this case, for every element $\alpha \in \mathscr{A}$, there exists a unique map $\pi^{\emptyset\alpha} : E_\alpha \to E_\emptyset$.

Note that E is a sheaf of sets for the lower A-topology on $\mathscr{A}$, and for every arrow $\alpha \to \beta$, the map $\pi^{\beta\alpha}$ is the restriction of sections from the open set U_α to the open set U_β.

A commutative field $\mathbb{K}$ of any characteristic is given. For every $\alpha \in \mathscr{A}$, we define V_α as the space of all functions from E_α to $\mathbb{K}$. We say that V is the *free presheaf* generated by E. If $\emptyset \in \mathscr{A}$, the space $V_\emptyset$ is canonically isomorphic to $\mathbb{K}$. If $\alpha \to \beta$, i.e. $\beta \subseteq \alpha$, we get a natural application $j_{\alpha\beta} : V_\beta \to V_\alpha$, which is linear and injective. Therefore V is a particular case of injective presheaf over $\mathscr{A}$. As before, $V_{\alpha\beta}$ designates the image of $j_{\alpha\beta}$ in V_α. Using the projection $\pi^{\beta\alpha}$ we can identify $V_{\alpha\beta}$ with the space of numerical functions of x_α that depend only on the variables x_β, these functions are named the *cylindrical functions with respect to* $\pi^{\beta\alpha}$.

Definition 6.5.1 (Reduced functor)**.** The sub-functor of constants $\mathbb{K}_{\mathscr{A}}$ maps each $\alpha \in \mathscr{A}$ to the one dimensional vector subspace $\mathbb{K}_\alpha$ of constant functions, embedded in V_α. The *reduced functor* (or reduced free presheaf) $\overline{V}_\alpha; \alpha \in \mathscr{A}$ is made of the quotient vector spaces $V_\alpha/\mathbb{K}_\alpha$.

If $\emptyset \in \mathscr{A}$, for every $\alpha \in \mathscr{A}$, we have $\mathbb{K}_\alpha = V_{\alpha\emptyset}$.

Definition 6.5.2 (Intersection property)**.** The hypergraph $(\mathscr{A}, I)$ satisfies the *strong (resp. weak) intersection proprety*, if, for every pair (α, α') in $\mathscr{A}$ (resp. every pair having non-empty intersection in $\mathscr{P}(I)$), the intersection $\alpha \cap \alpha'$ belongs to $\mathscr{A}$.

Remark 6.5.1. If $\mathscr{A}$ satisfies the strong intersection property, all the coproducts $\alpha \vee \alpha'$ exist; if $\mathscr{A}$ satisfies the weak intersection property, the coproducts exist conditionally, i.e. $\alpha \vee \alpha'$ exists as soon as α and α' have a common majorant (under the relation $\to$).

If $\mathscr{A}$ has non-intersecting elements, the strong intersection property implies that the empty set $\emptyset$ belongs to $\mathscr{A}$, then $\mathscr{A}$ possesses a unique final element, that is $\emptyset$. If $\mathscr{A}$ satisfies the weak intersection property, it possesses conditional coproducts (here intersections) in the categorical sense of Section 6.3.

Proposition 6.5.1. *If $\mathscr{A}$ has the strong intersection property, the condition G is satisfied by the free presheaf V.*

Proof. Consider $\alpha \to \beta$ in $\mathscr{A}$ (i.e. $\beta \subseteq \alpha$), and a vector v in $V_{\alpha\beta}$ that satisfies

$$v = \sum_{\substack{\gamma:\alpha\to\gamma,\gamma\neq\alpha,\\ \gamma\not\to\beta}} v_\gamma, \tag{6.5.1}$$

for some $v_\gamma \in V_{\alpha\gamma}$.

The above decomposition tells that for every $x_\beta \in E_\beta$, and for any collection of elements $\{y_j\}_{j\in\alpha\setminus\beta}$, where $y_j \in E_j$, we have

$$v(x_\beta, y_{\alpha\setminus\beta}) = \sum_{\substack{\gamma:\alpha\to\gamma,\gamma\neq\alpha,\\ \gamma\not\to\beta}} v_\gamma(x_\gamma); \tag{6.5.2}$$

where on the right, the components of x_γ are x_i with $i \in \beta \cap \gamma$ and y_j with $j \in (\alpha \setminus \beta) \cap \gamma$.

For each index $k \in \alpha \setminus \beta$, we choose a fixed y_k^0, and replace everywhere in the formula the variable x_k by this value. The formula continues to hold true. In the expression $v_\gamma(x_\gamma)$, the variables x_i that do not belong to $\beta \cap \gamma$, are constants $y_k^0; k \in \alpha\setminus\beta$. Moreover the intersection of β and γ is a strict subset of β, because β is assumed to be not included in γ.
This gives

$$v(x_\beta, y^0_{\alpha\setminus\beta}) = \sum_{\substack{\gamma:\alpha\to\gamma,\gamma\neq\alpha,\\ \beta\to\beta\cap\gamma,\beta\neq\beta\cap\gamma}} v^0_{\beta\cap\gamma}(x_{\beta\cap\gamma}). \tag{6.5.3}$$

And, for all possible $\omega \in \mathscr{A}$, $\omega \subset \beta$, $\beta \neq \omega$, if we bring together the γ such that $\alpha \to \gamma, \gamma \neq \alpha, \beta \cap \gamma = \omega$, this gives

$$v(x_\beta, y^0_{\alpha\setminus\beta}) = \sum_{\substack{\omega:\alpha\to\omega,\omega\neq\alpha,\\ \beta\to\omega,\beta\neq\omega}} w_\omega(x_\omega). \tag{6.5.4}$$

Which is the expected result. □

Remark 6.5.2. Without the strong intersection property the result is false. Take for instance, $I = \{i, j\}$, $\mathscr{A} = \{i; j; \alpha = (i, j)\}$, a non-zero constant function belongs to $V_{\alpha i}$, but cannot belong to the image of a strict subset of $\{j\}$.

Proposition 6.5.2. *If $\mathscr{A}$ has the weak intersection property, the condition G is satisfied by the reduced functor $\overline{V}$.*

Proof. Repeat the proof of Proposition 6.5.1, but distinguish the cases where $\beta \cap \gamma$ is empty or not. When it is empty the respective function v_β of $(x_\gamma, y_{\gamma'})$ belongs to the constants. □

Now remind that, by construction, the poset $\mathscr{A}$ is of locally finite dimension (it is even locally finite), then the following proposition results directly from the prop. 6.5.1 (resp. 6.5.2) and the Theorem 6.4.2.

Theorem 6.5.1. *If $\mathscr{A}$ has the strong (resp. weak) intersection property, the free presheaf V (resp. $\overline{V}$) has an interaction decomposition.*

Theorem 6.5.1 generalizes a theorem of existence of an interaction decomposition for factor spaces that, under different forms, has been known for long time in probability theory, but only for finite posets and finite dimensional vector spaces, cf. [13, 14, 19, 21].

As in the preceding section, denote by V'_α the sum of the $V_{\alpha\beta}$ over $\beta \subsetneq \alpha$, (resp. $\overline{V}'_\alpha$ the sum of the $\overline{V}_{\alpha\beta}$ over $\beta \subsetneq \alpha$) and take a supplementary subspace S_α of V'_α in V_α (resp. $\overline{S}_\alpha$ of $\overline{V}'_\alpha$ in $\overline{V}_\alpha$). The *interaction decomposition* gives

$$\forall \alpha \in \mathscr{A}, \quad V_\alpha = \bigoplus_{\beta \subseteq \alpha} S_\beta, \tag{6.5.5}$$

resp.

$$\forall \alpha \in \mathscr{A}, \quad \overline{V}_\alpha = \bigoplus_{\beta \subseteq \alpha} \overline{S}_\beta. \tag{6.5.6}$$

6.5.2 Duality: Free copresheaves

Note $F_\alpha = \mathbb{K}^{(E_\alpha)}$ the space of functions with finite supports, which can be seen as the vector spaces freely generated by the set E_α over the field $\mathbb{K}$. Its dual space is $V_\alpha = \mathbb{K}^{E_\alpha}$ and the transpose of the natural map $\pi^{\beta\alpha} : F_\alpha \to F_\beta$ is $j_{\alpha\beta}$. The vector spaces F_α and the maps $\pi^{\beta\alpha}$ define a covariant functor (i.e. a copresheaf) over $\mathscr{A}$ (resp. a sheaf on $X_{\mathscr{A}}$) named the *free copresheaf* (resp. the *free sheaf*) generated by E.

We can apply Corollary 6.4.1 in the preceding section to get the following result.

Proposition 6.5.3. *When $\mathscr{A}$ satisfies the strong intersection property and the finiteness condition (a^*), F is acyclic and $H^0(X_{\mathscr{A}}, F) \cong \bigoplus_{\alpha \in \mathscr{A}} S^*_\alpha$.*

Respectively, denote by $\overline{F}_\alpha$ the subspace of $F_\alpha = \mathbb{K}^{(E_\alpha)}$ defined by annihilating the sum of the coordinates in the canonical basis. Its dual space is $\overline{V}_\alpha = \mathbb{K}^{E_\alpha}/\mathbb{K}_\alpha$. The transpose of the natural map $\pi^{\beta\alpha} : \overline{F}_\alpha \to \overline{F}_\beta$ is again $j_{\alpha\beta}$. This forms a sheaf over $X_{\mathscr{A}}$, named the *restricted free sheaf* generated by E. As before, we obtain the following.

Proposition 6.5.4. *When $\mathscr{A}$ satisfies the weak intersection property and the finiteness condition (a^*), the sheaf $\overline{F}$ over $X_{\mathscr{A}}$ is acyclic and $H^0(X_{\mathscr{A}}, \overline{F}) \cong \bigoplus_{\alpha\in\mathscr{A}} \overline{S}^*_\alpha$.*

In the case of finite sets $\{E_i\}_{i\in I}$ and I finite, this result was established in H.G. Kellerer [13]. See also [21] and Appendix 6.C below.

Theorem 6.5.2. *If the hypergraph $(\mathscr{A}, I)$ satisfies the weak intersection hypothesis, and the finiteness condition (a^*), for any covariant functor of sets E on the category $\mathscr{A}$, the Čech cohomology $H^*(X_{\mathscr{A}}; F)$ of the induced free sheaf F is naturally isomorphic to the sum of $H^\bullet(X_{\mathscr{A}}; \overline{F})$ which is concentrated in degree zero, and of the full Čech cohomology (with trivial coefficients $\mathbb{K}$) of the topological space $X_{\mathscr{A}}$ (i.e. the poset $\mathscr{A}$ equipped with the lower Alexandrov topology).*

Proof. The sheaf F over $\mathscr{A}$ is decomposed into the sum of the sheaf $\overline{F}$ and the constant sheaf $\mathbb{K}_A$; this induces a decomposition in direct sum of the cochain complexes. One of them gives gives $H^\bullet(X_{\mathscr{A}}; \overline{F})$, which is concentrated in degree zero as just said by the preceding proposition, whereas the other one gives the standard Čech cohomology of $\mathscr{A}$. □

When $\mathscr{A}$ is the poset of a finite simplicial complex, it satisfies the weak intersection property, and the standard Čech cohomology (with constant coefficients) on $X_{\mathscr{A}}$ is isomorphic to the singular or simplicial cohomology with coefficients in $\mathbb{K}$. See the introduction to Section 6.6.

6.5.3 Relative cohomology and marginal theorem

In addition to $\mathscr{A} \subseteq \mathscr{P}_f(I)$, consider another poset $\mathscr{B}$ satisfying the same kind of hypotheses, with respect to a set J, i.e. $\mathscr{B} \subseteq \mathscr{P}_f(J)$.

Definition 6.5.3. A *strict morphism* from $(\mathscr{A}, I)$ to $(\mathscr{B}, J)$ is the pair (f, f_I) of a functor (i.e. an increasing map) $f : \mathscr{A} \to \mathscr{B}$, and a map $f_I : I \to J$, such that $\forall i \in I$ and all $\alpha \in \mathscr{A}$ such that $i \in \alpha$, one has $f_I(i) \in f(\alpha) \subseteq J$. For simplicity, we will denote $f_I = f$.

As before, let E be the sheaf of sets over $\mathscr{A}$ given by products of the sets $\{E_i\}_{i\in I}$ i.e. such that $\alpha \mapsto E_\alpha \cong \prod_{i\in\alpha} E_i$; we call the E_i *basic sets*. Consider a strict morphism $f : \mathscr{A} \to \mathscr{B}$. For every $j \in J$, let us define E'_j as the product of the E_i for $i \in I$ such that $f(i) = j$.

Proposition 6.5.5. *The direct image f_*E over $\mathscr{B}$ is given by the products of the basic sets $\{E'_j\}_{j\in J}$.*

Proof. For $\beta \in \mathscr{B}$, the set $f_*E(U_\beta)$ (in the lower A-topology) is the subset of the product of the E_α over $\alpha \in f^{-1}(\beta)$ formed by the families $s_\alpha; f(\alpha) \subseteq \beta$, that are compatible on the intersections $U_\alpha \cap U_{\alpha'}$. Each set E_α is the product of the sets $E_i; i \in \alpha$, and the compatibility condition tells that for any pair α, α' with $f(\alpha) \subseteq \beta$ and $f(\alpha') \subseteq \beta$, the restriction of s_α and $s_{\alpha'}$ to their common terminal points coincide. This implies that $E(f^{-1}(U_\beta))$ is the product of the $E_i; i \in I$ such that $f(i) \in \beta$, then it is the product of the E'_j for $j \in \beta$. □

In particular, E'_j coincides with the set $(f_*E)_j$ which corresponds to the direct image of sheaves.

Definition 6.5.4. A *simplicial morphism* from $\mathscr{A}$ to $\mathscr{B}$ is a strict morphism $f : \mathscr{A} \to \mathscr{B}$, such that $\forall \alpha \in \mathscr{A}$, the restriction of f_I to the set $\alpha \in \mathscr{P}_f(I)$ is surjective onto the set $f(\alpha) \in \mathscr{P}_f(J)$.

Proposition 6.5.6. *Let $f : \mathscr{A} \to \mathscr{B}$ be injective and simplicial, and let F' be a sheaf on $\mathscr{B}$, given by products of the basic sets $E'_j; j \in J$. The inverse image f^*F' over $\mathscr{A}$ is given by the products of the basic sets $\{E_i = E'_{f(i)}\}_{i\in I}$.*

Proof. For $\alpha \in \mathscr{A}$, by definition of $f^{-1}F'$ (which coincide with f^*F' in the case of posets), $(f^*E')_\alpha = E'_{f(\alpha)}$ is the product of the sets $E'_j; j \in f(\alpha)$, and this product coincide with the product of the sets $E'_{f(i)}$ for $i \in \alpha$ because f is simplicial and injective. □

In the following result, we consider the *restricted subsheaves* $\overline{F}$ and $\overline{F}'$, and we assume that both A and B verify the weak intersection property.

Theorem 6.5.3. *Let $J : \mathscr{A} \to \mathscr{B}$ be an inclusion of posets, strict and simplicial. If $\overline{F}'$ is a restricted free copresheaf over $\mathscr{B}$, then the inverse image $\overline{F} = J^*\overline{F}'$ over $\mathscr{A}$ is restricted, and we have a natural surjection from $\overline{F}'$ to $J_*\overline{F}$, and the induced natural map in cohomology $J^* : H^0(\mathscr{B}; \overline{F}') \to H^0(\mathscr{A}; \overline{F})$ is surjective.*

Proof. Along $\mathscr{A}$, the stalk of $\overline{F}'$ and $J_*\overline{F}$ coincide. From Theorem 6.4.2 and the long exact sequence in Čech cohomology (Appendix 6.B), we get the following exact sequence:

$$0 \to H^0(\mathscr{B}, \mathscr{A}; \overline{F}', \overline{F}) \to H^0(\mathscr{B}; \overline{F}') \to H^0(\mathscr{A}; \overline{F}) \to H^1(\mathscr{B}, \mathscr{A}; \overline{F}', \overline{F}) \to 0. \tag{6.5.7}$$

Then the theorem is equivalent to the vanishing of $H^1(\mathscr{B}, \mathscr{A}; \overline{F}', \overline{F}) = 0$. To prove the latter, we proceed as in the proof of Corollary 6.4.1: we decompose $\overline{F}'$ over $\overline{B}$ and then $\overline{F}$ accordingly over $\overline{A}$ in direct sums of sheaves $T^\beta; \beta \in \overline{\mathscr{B}}$ and $S^\alpha; \alpha \in \overline{\mathscr{A}}$

respectively, which satisfy the hypotheses of Lemma 6.4.4. Then we conclude by applying Lemma 6.4.4 and the natural isomorphism in cohomology between A (resp. B) and $\overline{A}$ (resp. $\overline{B}$) for the restricted sheaves. $\square$

In the context of finite probabilities, quotienting F to obtain $\overline{F}$ corresponds to the tangent equation of the probability restriction of sum 1, and the surjection of Theorem 6.5.3 is equivalent to a result of H.G. Kellerer [13].

In the Appendix 6.C, based on the preceding sections, we prove the following index formula, which generalizes the result of Kellerer [13] and Matúš [21].

Theorem 6.5.4. *If the poset $\mathscr{A}$ is finite and satisfies the weak intersection property, and if the $\{E_i\}_{i\in I}$ are finite sets, then*

$$\chi(\mathscr{A};F)=\sum_{k=0}^{\infty}(-1)^k\dim_{\mathbb{K}}H^k(\mathscr{U}_{\mathscr{A}};F)=\sum_{\alpha,\beta\in\mathscr{A}}\mu_{\alpha\beta}N_\beta; \tag{6.5.8}$$

where $\mu_{\alpha,\beta}$ is the Möbius function of $\mathscr{A}$, and, for each $\alpha\in\mathscr{A}$, N_α denotes the cardinality of E_α.

We will also prove that

$$\chi(\mathscr{A};F)=\dim_{\mathbb{K}}H^0(\mathscr{U}_{\mathscr{A}};\overline{F})+\chi(\mathscr{A}); \tag{6.5.9}$$

where $\chi(\mathscr{A})$ denotes the Euler characteristic of $\mathscr{A}$, in every possible sense: as a metric subspace of the simplex $\mathscr{P}(I)$, as the lower or upper Hausdorff topological space in Čech cohomology, or as an abstract poset; this is also the Euler characteristic of the nerve of the category $\mathscr{A}$.

6.6 Nerves of categories and nerves of coverings

Any *contravariant* functor G of abelian groups on a poset $\mathscr{A}$ produces a sheaf, also denoted by G, on the topological space $X^{\mathscr{A}}$, whose underlying set is $\operatorname{Ob}\mathscr{A}$, equipped with the upper A-topology. This is equivalent with the dual statement for covariant functors on $\mathscr{A}^{op}$. But G is also an abelian object in the topos $\mathrm{PSh}(\mathscr{A})$, cf. [62], [64]. And in the context of topos theory, it is customary to study another cohomology, that is the graded derived functor $H^\bullet(\mathscr{A},-)$ of

$$\Gamma_{\mathscr{A}}(-)=\mathrm{Hom}_{\mathrm{Ab}(\mathscr{A})}(\mathbb{Z},-)\cong\mathrm{Hom}_{\mathrm{PSh}(\mathscr{A})}(*,-); \tag{6.6.1}$$

cf. [66]. In the following lines, we give a more explicit and topological definition of this functor, according to [62], [64].

The nerve of a small category $\mathscr{C}$ is the simplicial set whose n simplices are sequences $c_0 \to \cdots \to c_n$ of composable arrows in $\mathscr{C}$, and whose face operators are

$$d_i(c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} c_n) = \begin{cases} c_1 \to \cdots c_n & \text{if } i = 0 \\ c_0 \to \cdots c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \to \cdots \to c_n & \text{if } 0 < i < n \\ c_0 \to \cdots \to c_{n-1} & \text{if } i = n \end{cases} . \quad (6.6.2)$$

For background and details, see Section 6.6.1 below. This permits to define a *canonical cochain complex* $(C^n(A, G), d)$ whose cohomology is precisely $H^\bullet(\mathscr{A}, G)$, cf. [64, Prop. 6.1]. This complex comes from a canonical projective resolution of the constant presheaf $\mathbb{Z}$ [62, Ex. V.2.3.6].

The n-cochains are

$$C^n(\mathscr{A}, G) = \prod_{a_n \to \cdots \to a_0 \text{ in } \mathscr{A}} G(a_n) = \prod_{a_0 \to \cdots \to a_n \text{ in } \mathscr{A}^{op}} G(a_n) \quad (6.6.3)$$

and the coboundary $\delta : C^{n-1}(\mathscr{A}, G) \to C^n(\mathscr{A}, G)$ is given by

$$(\delta g)_{a_0 \to \cdots \to a_n} = \sum_{i=0}^{n-1} (-1)^i g_{d_i(a_0 \to \cdots a_n)} + (-1)^n G(\varphi_n) g_{d_n(a_0 \to \cdots a_n)}, \quad (6.6.4)$$

where φ_n is the $\mathscr{A}$-morphism from a_n to a_{n-1} in the sequence $a_n \to \cdots \to a_0$.

Remark 6.6.1. This complex and its analog for a covariant homology were rediscovered by O. Peltre in the context of his doctoral work [26], which gives a homological interpretation of the generalized Belief Propagation algorithm [16], which is applied in statistical physics, bayesian learning and decoding processes. One of the initial motivations behind the present article was to understand better the connections of it with Čech cohomology and sheaf cohomology.

Example 6.6.1. Consider a sheaf S^γ for $\gamma \in \mathscr{A}$ of abelian groups over $\mathscr{A}$, which is zero outside U^γ and such that for any arrow $\alpha \to \beta$ in U^γ the morphism $j_{\alpha\beta}$ is an isomorphism of group. This was the situation for the factors of an interaction decomposition. Then the above cohomology is null in degree ≥ 1, and equal to zero or to $S^\gamma(\gamma)$, according to the fact that γ has or not a successor in $\mathscr{A}$. The proof is the exact copy of the Lemma 6.4.3.

The analogue for the sheaf T_γ over the opposite category $\mathscr{A}^{op}$ tells that the cohomology is zero in degree ≥ 1, but it is always isomorphic to $T_\gamma(\gamma)$ in degree zero.

In this section, our aim is to compare this cohomology with the topological Čech cohomology of the sheaf G on $X^{\mathscr{A}}$ that we have studied in the previous sections. We will prove (Corollary 6.6.2) that, when $\mathscr{A}$ possesses conditional

products (i.e. the covering $\mathscr{U}_{\mathscr{A}}$ is closed by finite non-empty intersections), these two cohomologies are naturally isomorphic.

The equivalence of category between sheaves over $X^{\mathscr{A}}$ and $\mathrm{PSh}(\mathscr{A})$, reminded in (6.3.1) does not imply in general the isomorphism of the two cohomology theories.

For any topological space, the isomorphism between Čech and topos cohomology in degree 0 and 1 is a general result, due to [67, Sec. 3.8]. The fact that in every degree, the topos cohomology of sheaves over a topological space coincides with the topos cohomology of any equivalent topos is proved in [62, Ch. V] see also [68], cohomology of sheaves. Grothendieck also established an injection of Čech cohomology group of degree 2 into the sheaf cohomology group of degree 2. This injectivity is sufficient to establish the vanishing of H^2 for the sheaves S^γ or T_γ without appealing to the existence of conditional products or coproducts in $\mathscr{A}$.

The methods of comparison based on spectral sequences [69], developed in Tohoku [67, Ch. III] and the SGA4 [62], might apply in our situation, but our aim in this last section is to describe an explicit natural isomorphism between the two cohomolgy theories, when it exists. We will achieve this goal by putting the two theories in a same more general context of cosimplicial sheaves over simplicial sets. The explicit comparison of the cohomology theories is is the content of the theorem 6.6.2 below.

When G is the constant sheaf $\mathbb{Z}$, $H^\bullet(C(N(\mathscr{A}),\mathbb{Z}),d)$ corresponds to the simplicial cohomology of $|N(\mathscr{A})|$, the geometric realization of the nerve $N(\mathscr{A})$ (see Remark 6.6.2); it is known to be naturally isomorphic to the singular cohomology of $|N(\mathscr{A})|$ (cf. [70]). In turn, for paracompact spaces, as are the finite dimensional CW complexes, the Čech cohomology $\check{H}^\bullet(X^{\mathscr{A}},\mathbb{Z})$ is isomorphic to the singular cohomology of $X^{\mathscr{A}}$. Hence the isomorphism between $H^\bullet(C(N(\mathscr{A}),\mathbb{Z}),d) \cong \check{H}^\bullet(X^{\mathscr{A}},\mathbb{Z})$ is implied by the homotopy equivalence between $|N(\mathscr{A})|$ and $X^{\mathscr{A}}$, see May [71]. We are looking here for an extension of this result in the context of sheaves, and without paracompacity.

For that purpose, we introduce a general framework of *cosimplicial local systems* on simplicial sets. We will remind below the definition of simplicial sets and simplicial objects in a category. The nerve $K_\bullet(\mathscr{U})$ of a covering $\mathscr{U}$ introduced in Section 6.2 and the nerve $N(\mathscr{C})$ of a category $\mathscr{C}$ are examples of simplicial sets. Cosimplicial local systems are functorial assignments of local data to the simplexes and morphisms of a simplicial set. It appears that both Čech cohomology and the cohomology introduced by (6.6.3)-(6.6.4) become particular cases of this general construction and can be compared in this framework.

6.6.1 Simplicial sets and nerves of coverings

Simplicial sets can be traced back to Eilenberg and Zilber [72]—under the name "complete semi-simplicial sets". They became ubiquitous in algebraic topology, due to the works of Segal, Grothendieck, Kan, Quillen, May and many others. The subject was treated in great detail by May in [71]; also [66, Ch. 8] is a good introduction.

Let Δ be the category whose objects are the finite ordered sets $[n] = \{0 < 1 < ... < n\}$, for each $n \in \mathbb{N}$, and whose morphisms are nondecreasing monotone functions. Given any category $\mathscr{C}$, a simplicial object S in $\mathscr{C}$ is a contravariant functor from Δ to $\mathscr{C}$ i.e. $S : \Delta^{op} \to \mathscr{C}$. When $\mathscr{C}$ is the category of sets, S is called a *simplicial set*. One can define analogously simplicial groups, modules, etc.

Although Δ has many morphisms, which seem complicated at first sight, they can be conveniently expressed in terms of certain morphisms known as *face and degeneracy maps*. For each $n \in \mathbb{N}$ and $i \in [n]$, the face map $d_i^n : [n] \to [n+1]$ is given by

$$d_i^n(j) = j \quad \text{if} \quad j < i, \qquad d_i^n(j) = j+1 \quad \text{if} \quad j \geq i. \tag{6.6.5}$$

Similarly, for each $i \in [n+1]$, the degeneracy map $s_i^{n+1} : [n+1] \to [n]$ is

$$s_i^{n+1}(j) = j \quad \text{if} \quad j \leq i, \qquad s_i^{n+1}(j) = j-1 \quad \text{if} \quad j > i. \tag{6.6.6}$$

Normally the super-index is dropped.

Given a morphism $\varphi : [m] \to [n]$ of Δ, let $i_1, ..., i_s$ be the elements of $[n]$ not in $\varphi([m])$, in reverse order, and let $j_1, ..., j_t$, in order, be the elements of $[m]$ such that $\varphi(j) = \varphi(j+1)$. Then

$$\varphi = d_{i_1} \cdots d_{i_s} s_{j_1} \cdots s_{j_t}. \tag{6.6.7}$$

Remark that $m - t + s = n$. This factorization is unique [71, Sec. I.2].

The face and degeneracy maps satisfy some relations:

$$\forall n \in \mathbb{N}, 0 \leq j < k \leq n, \quad s_j^{n+1} \circ d_k^n = d_{k-1}^n \circ s_j^{n+1}, \tag{6.6.8}$$

$$\forall n \in \mathbb{N}, 0 \leq j \leq n+1, \quad s_j^{n+1} \circ d_j^n = s_{j+1}^{n+1} \circ d_j^n = \mathrm{id}_{n+1}, \tag{6.6.9}$$

$$\forall n \in \mathbb{N}, n+1 \geq j > k+1 \geq 1, \quad s_j^{n+1} \circ d_k^n = d_k^n \circ s_{j-1}^{n+1}. \tag{6.6.10}$$

A (simplicial) morphism from a simplicial set S to a simplicial set S' is a natural transformation of functors: a collection of maps $\{f_n : S([n]) \to S'([n])\}_{n \in \mathbb{N}}$ such that, for each morphism $\varphi : [m] \to [n]$ in Δ, the diagram

$$\begin{array}{ccc} S([n]) & \xrightarrow{S\varphi} & S([m]) \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_m} \\ S'([n]) & \xrightarrow{S\varphi} & S'([m]) \end{array} \tag{6.6.11}$$

commutes.

Example 6.6.2 (Simplex). A basic example of simplicial set is the *k-simplex* Δ^k [71, Def. I.5.4], which is the presheaf represented by $[k]$. This means that $\Delta^k_n = \Delta^k([n])$ equals $\mathrm{Hom}([n],[k])$, and the map $\Delta^k\varphi : \mathrm{Hom}([m],[k]) \to \mathrm{Hom}([n],[k])$ induced by $\varphi : [m] \to [n]$ is given by precomposition with φ.

Example 6.6.3 (Nerve of a covering). Let X be a topological space and $\mathscr{U}$ an open covering of X. The *nerve of the covering* $\mathscr{U}$ is the set $K(\mathscr{U})$ of finite sequences of elements of $\mathscr{U}$ having a non-empty intersection. It has a natural structure of simplicial set: $K_n = K([n])$ is the set of sequences of length $n+1$, denoted $(U_0, ..., U_n)$, and for any nondecreasing function $\varphi_{m,n}$ from m to n, there is a map $\varphi^*_{m,n} : K_n \to K_m$ given by

$$\varphi^*_{m,n}(V_0, ..., V_n) = (V_{\varphi(0)}, ..., V_{\varphi(m)}). \tag{6.6.12}$$

In other terms, $K_n(\mathscr{U})$ is the set of maps $u : [n] \to \mathscr{U}$ such that the intersection of the images are non-empty, and if $\varphi : [m] \to [n]$ is a morphism and $v \in K_n(\mathscr{U})$, then $\varphi^*_{m,n}(v) = v \circ \varphi$.

Hence the map $s^*_i = K(s^{n+1}_i)$ is given by

$$s^*_i(U_0, ..., U_n) = (U_0, ..., U_{i-1}, U_i, U_i, U_{i+1}, ..., U_n); \tag{6.6.13}$$

is also called *degeneracy map*, whereas $d^*_i = K(d^{n+1}_i)$ is given by

$$d^*_i(U_0, ..., U_{n+1}) = (U_0, ..., U_{i-1}, \widehat{U_i}, U_{i+1}, ..., U_{n+1}), \tag{6.6.14}$$

and called *face map*.

For each $u = (U_0, ..., U_n)$, we denote by U_u the intersection $U_0 \cap ... \cap U_n$. It is easily verified that, for every morphism $\varphi_{m,n} : [m] \to [n]$ and every $v \in K_n(\mathscr{U})$, one has $U_v \subseteq U_{\varphi^*(v)}$. In particular, $U_{\varphi^*(v)}$ is non-empty if U_v is non-empty.

Example 6.6.4 (Nerve of a category). To any small category $\mathscr{C}$ is naturally associated a simplicial set $N(\mathscr{C})$, named its nerve: the elements of $N_n(\mathscr{C})$ are the covariant functors from the poset $[n]$ to $\mathscr{C}$, and the morphisms are obtained by right composition.

Concretely an element of degree n is a sequence

$$a = \alpha_0 \to \alpha_1 \to ... \to \alpha_n. \tag{6.6.15}$$

The action s^*_i of s_i is the repetition of the object α_i via the insertion of an identity id_{α_i}; the action d^*_i of d_i is the deletion of α_i via the composition of $\alpha_{i-1} \to \alpha_i$ and $\alpha_i \to \alpha_{i+1}$.

More generally if

$$b = \beta_0 \to \beta_1 \to ... \to \beta_m \tag{6.6.16}$$

belongs to $N_m(\mathscr{C})$, and $\varphi : n \to m$ is non-decreasing, then

$$\varphi^*(b) = \beta_{\varphi(0)} \to \beta_{\varphi(1)} \to ... \to \beta_{\varphi(n)}. \tag{6.6.17}$$

Example 6.6.5 (Barycentric subdivision of the nerve of a covering). Consider the category $\mathscr{A}(\mathscr{U})$ which has for objects the non-empty intersections of the elements of $\mathscr{U}$, and for morphisms the inclusions, then the nerve $N(\mathscr{A}(\mathscr{U}))$ is the barycentric subdivision of the simplicial set $K(\mathscr{U})$. This was remarked by Segal [69], interpreting [72].

Remark 6.6.2 (Geometric realization). It is reassuring to know that any simplicial set gives rise to a CW-complex, even if this is not directly used in the present text. The *geometric realization* $|K|$ of the simplicial set K is a topological space obtained as the quotient of the disjoint union of the products $K_n \times \Delta(n)$, where $K_n = K([n])$ and $\Delta(n) \subset \mathbb{R}^{n+1}$ is the geometric standard simplex, by the equivalence relation that identifies $(x, \varphi_*(y))$ and $(\varphi^*(x), y)$ for every nondecreasing map $\varphi : [m] \to [n]$, every $x \in K_n$ and every $y \in \Delta(m)$; here f^* is $K(f)$ and f_* is the unique linear map from $\Delta(n)$ to $\Delta(m)$ that maps the canonical vector e_i to $e_{f(i)}$. For every $n \in \mathbb{N}$, K_n is equipped with the discrete topology and $\Delta(n)$ with its usual compact topology, the topology on the union over $n \in \mathbb{N}$ is the weak topology, i.e. a subset is closed if and only if its intersection with each closed simplex is closed, and the realization is equipped with the quotient topology. In particular, even it is not evident at first sight, the realization of the simplicial set Δ^k is the standard simplex $\Delta(k)$. See [71, Ch. III].

The cartesian product of two simplicial sets K and L is taken as it must be for functors to $\mathscr{E}$, that is term by term: $(K \times L)([n]) = K([n]) \times L([n])$ at the level of objects, and similarly for the maps.

Definition 6.6.1. Let $f : K \to L$ and $g : K \to L$ be two simplicial maps, a *simplicial homotopy* from f to g is a simplicial map $h : K \times \Delta^1 \to L$, such that $f = h \circ (\mathrm{id}_K \times d_0)$ and $g = h \circ (\mathrm{id}_K \times d_1)$.

Simplicial homotopy is an equivalence relation, compatible with composition of simplicial maps.

Example 6.6.6 (Homotopy induced by a projection of coverings). A covering $\mathscr{U}$ is called a *refinement* of another covering $\mathscr{U}'$ when every set of $\mathscr{U}$ is contained in some set of $\mathscr{U}'$. In that case, there exists a map $\lambda : \mathscr{U} \to \mathscr{U}'$, called *projection*, such that for every $U \in \mathscr{U}$ one has $U \subseteq \lambda(U)$. It is also said that $\mathscr{U}$ is *finer* than $\mathscr{U}'$ [70].

A projection map $\lambda : \mathscr{U} \to \mathscr{U}'$ induces a simplicial morphism λ_* from the simplicial set $K(\mathscr{U})$ to the simplcial set $K(\mathscr{U}')$:

$$\lambda_*(u) = \lambda \circ u; \tag{6.6.18}$$

Proposition 6.6.1. *If $\mathscr{U}$ is a refinement of $\mathscr{U}'$, two projections λ, μ from $\mathscr{U}$ to $\mathscr{U}'$ induce homotopic simplicial maps λ_*, μ_* from $K(\mathscr{U})$ to $K(\mathscr{U}')$.*

Proof. Let $u = (U_0, ..., U_n)$ be an element of $K_n(\mathscr{U})$, and $\varphi_i = (0, .., 0, 1, ..., 1) \in \Delta^1_n$ with the first 1 at place i between 0 and $n+1$, we put

$$h(u, \varphi_i) = (\lambda(U_0), ..., \lambda(U_{i-1}), \mu(U_i), ..., \mu(U_n)). \tag{6.6.19}$$

□

6.6.2 Cosimplicial local systems and their cohomology

We present here a general definition of cohomology for cosimplicial local systems on simplicial sets.

Definition 6.6.2. A *cosimplicial local system* of sets F over the simplicial set K is a family F_u indexed by the elements u of K, and, for any morphism $\varphi : [m] \to [n]$ and any $v \in K_n$, a given application $F(\varphi, v) : F_u \to F_v$, where $u = \varphi^*_{m,n}(v)$, such that $F(\psi, w) \circ F(\varphi, v) = F(\psi \circ \varphi, w)$, for $\varphi : [m] \to [n]$, $\psi : [n] \to [p]$, $w \in K_p$, $v = \psi^*_{n,p}(w)$, $u = \varphi^*_{m,n}(v)$.

Remark 6.6.3. A definition of simplicial local systems appeared in the work of Halperin [73]. In his case the arrows are in the reverse direction, i.e. for $\varphi : [m] \to [n]$, $v \in K_n$, a map $\varphi^*_v : F_v \to F_{\varphi^*(v)}$.

Example 6.6.7 (Čech system). Take a presheaf F over the topological space X and consider an open covering $\mathscr{U}$ of X. Then for $u \in K(\mathscr{U})$, define $F_u = F(U_u)$, and for $\varphi : [m] \to [n]$, $v \in K_n$, take for $F(\varphi, v)$ the restriction from $F(U_{\varphi^*_v})$ to $F(U_v)$. This defines a cosimplicial system over $K(\mathscr{U})$.

Example 6.6.8 (Upper and lower systems associated to a functor). Let F be a contravariant functor from $\mathscr{C}$ to the category of sets $\mathscr{E}$. We can define a cosimplicial local system F^* over the nerve $N(\mathscr{C})$ (see Example 6.6.4 for the definition and the notation), named the *upper* system, by taking $F^*(a) = F(\alpha_n)$, and for a morphism $\psi : n \to p$, and an element $b = \beta_0 \to ... \to \beta_p$ in $N_p(\mathscr{C})$, denoting by α_n the last element of $a = \psi^*(b)$, we have $\alpha_n = \beta_{\varphi(n)}$, and this comes with a canonical arrow f going to β_p, then we take $\psi^*_b = f^*$, going from $F_b = F(\beta_p)$ to $F_a = F(\alpha_n)$.

In the dual manner, if F is covariant, we can define the lower cosimplicial system F_*, by taking $F_*(a) = F(\alpha_0)$. Taking again the element b and the morphism ψ, we use now the fact that the first element of $a = \psi^*(b)$ is $\alpha_0 = \beta_{\varphi(0)}$, which comes with a canonical arrow g in $\mathscr{C}$ from b_0 to it, and we can take $\varphi^*_b = g_*$ from $F_*(b)$ to $F_*(a)$.

Replacing $\mathscr{C}$ by the opposite (or dual) category $\mathscr{C}^{op}$, we exchange contravariant functors with covariant ones, and lower systems with upper ones.

Remark 6.6.4. Introduce the category $\mathscr{S}(K)$, having for objects the elements of K, and for arrows between two elements $v \in K_n$ and $u \in K_m$ the elements φ of

$\Delta(m,n)$ such that $\varphi^*(v) = u$. Then a cosimplicial local system F over K is a contravariant functor (i.e. a presheaf) from $\mathscr{S}(K)$ to the category of sets.

Definition 6.6.3. Let F be a cosimplicial local system over the simplicial set K; for each $n \in \mathbb{N}$, a *simplicial cochain* of F of degree n is an element $(c_u)_{u \in K_n}$ of the product $C^n(K;F) = \prod_{u \in K_n} F_u$.

When F is a local system of abelian groups, $C^n(K;F)$ has a natural structure of abelian group. In what follows, we stay in this abelian context.

The *coboundary operator* $\delta : C^n(K;F) \to C^{n+1}(K;F)$ is defined by

$$(\delta c)(v) = \sum_{i=0}^{n+1} (-1)^i F(d_i, v)(c(d_i^*(v))), \tag{6.6.20}$$

for any element v of $C^{n+1}(K;F)$. In the expression, d_i^* is $K(d_i)$; remark that the sum takes place in F_v. When we want to be more precise, we write $\delta = \delta_n^{n+1}$ at degree n. The operator δ is also named the differential of the cochain complex $C^n(K;F), n \in \mathbb{N}$.

Proposition 6.6.2. *For all $n \in \mathbb{N}$, the equality $\delta_{n+1}^{n+2} \circ \delta_n^{n+1} = 0$ holds. In short, $\delta \circ \delta = 0$.*

Proof. The expression of $\delta \circ \delta(c)(w)$ is the sum of elementary terms of the form $(-1)^k F(d_j^{n+1} \circ d_i^n, w)(c(d_i^* \circ d_j^* w))$, with $i \neq j$ and $k = i + j$ if $j < i$, $k = i + j + 1$ if $j > i$.

It is easy to verify that the maps d satisfy the relation $d_j d_i = d_i d_{j-1}$ if $i < j$ [66, Ex. 8.1.1]. It follows that the terms in the sum cancel two by two.[3] □

A sequence $\{C^n\}_{n \in \mathbb{N}}$ of abelian groups with an operator δ of degree $+1$ and square zero, is named a differential complex, or a cochain complex.

Definition 6.6.4 (Cohomology of a cosimplicial local system)**.** The cohomology group in degree $n \in \mathbb{N}$ of the local system F over the simplicial set K is the quotient abelian group

$$H^n(K;F) = \ker(\delta_n^{n+1}) / \operatorname{im}(\delta_{n-1}^n).$$

By convention $\delta_{-1}^0 = 0$.

Equivalently, the cohomology $H^\bullet(K;F)$ of F over K, seen as graded vector space, is the cohomology of the complex of simplicial cochains $(C^\bullet(K,F), \delta)$.

[3] This is a short argument with big consequences. When did it appear for the first time? Who came up with it? Euler, Poincaré, Noether, Lefschetz, Alexander?

As usual, the elements of $\ker \delta^n$ are called n-cocycles, and those in the image of δ^{n-1} are the n-coboundaries.

For example, a 0-cochain is a collection $(c_u)_{u\in K_0}$, and it a 0-cocycle if for any $v \in K_2$,

$$0 = F(d_0, v)(c_{d_0^* v}) - F(d_1, v)(c_{d_1^* v}). \tag{6.6.21}$$

In the particular case of an open covering $\mathscr{U}$ of a topological space X, and a presheaf of abelian groups F on X, the group $H^0(K_{\mathscr{U}}; F)$ (for the associated local system on the nerve of the covering $\mathscr{U}$) coincide with the set of global sections of F on X, i.e. the families $(c_U) \in \prod_{U\in\mathscr{U}} F(U)$ of local sections of F whose restriction coincide on the non-empty intersections $U \cap V$, $U, V \in \mathscr{U}$.

Remark 6.6.5. Let G be a contravariant functor over a poset $\mathscr{A}$. The simplicial cochain complex $(C^n(N(\mathscr{A}^{op}), G^*), \delta)$ associated to the upper local system of $G : \mathscr{A}^{op} \to \mathscr{E}$, in the sense of the preceding definitions, is precisely the cochain complex introduced by (6.6.3)-(6.6.4). One could also compute this cohomology from the complex $(C^n(N(\mathscr{A}), G_*), \delta)$: the cochains are the same, and the differential only differs by a sign when n is odd.

Similarly, if G is covariant over $\mathscr{A}$, then one can see it as a presheaf on $\mathscr{A}^{op}$; its sheaf cohomology can be computed as the cohomology of $(C^n(N(\mathscr{A}), G^*), \delta)$ or $(C^n(N(\mathscr{A}^{op}), G_*), \delta)$

As mentioned before, these complexes were rediscovered by O. Peltre [27] for understanding geometrically the generalized belief propagation algorithm of [16].

Given two cochain complexes of abelian groups $C^\bullet$ and $D^\bullet$, whose differential operators have degree $+1$ (i.e. ascending complexes) a cochain map (or cochain morphism) is a collection $\{f^n : C^n \to D^n\}_{n\in\mathbb{Z}}$ of morphism of groups that commute with the differentials. In other words, it is a morphism of graded abelian groups of degree zero that commute with the differentials.

A chain map between two cochain complexes sends coboundaries to coboundaries and cocycles to cocycles, thus it induces a map at the level of cohomology.

Example 6.6.9. Let $\mathscr{U}$ be a refinement of $\mathscr{U}'$, and $\lambda : \mathscr{U} \to \mathscr{U}'$ an adapted projection. The simplicial morphism λ_* of the simplicial set $K(\mathscr{U})$ to the simplicial set $K(\mathscr{U}')$ in the last section induces, for each integer n, a map λ^* from $C^n(\mathscr{U}'; F)$ to $C^n(\mathscr{U}; F)$ defined by $\lambda^*(c') = c' \circ \lambda_*$. More concretely,

$$(\lambda^* c')(U_0, ..., U_n) = c'(\lambda(U_0), ..., \lambda(U_n)). \tag{6.6.22}$$

This map commutes with the Čech differential, then it induces a map in cohomology

$$\lambda^* : H^n(\mathscr{U}'; F) \to H^n(\mathscr{U}; F). \tag{6.6.23}$$

Given two cochain maps $f^\bullet, g^\bullet$, a cochain homotopy from $f^\bullet$ to $g^\bullet$ is a morphism of graded groups h from $C^\bullet$ to $D^\bullet$ of degree -1 such that

$$d \circ h + h \circ d = g - f. \tag{6.6.24}$$

This defines an equivalence relation on cochain maps (of degree zero) which is compatible with the composition of maps.

Proposition 6.6.3 ([70, Thm. 4.4])**.** *If two cochain morphisms are homotopic, they give the same application in cohomology.*

Proof. If c is a cocycle of C, and if h is an homotopy from f to g, then $dh(c) = g(c) - f(c)$, thus $g(c)$ and $f(c)$ have the same classes in cohomology. □

Definition 6.6.5 (Lift of simplicial map to a local system)**.** let F (resp. G) be a cosimplicial local system over the simplicial set K (resp. L), and $f : K \to L$ a simplicial map, a *lift* $\widetilde{f}$ of f from G to F is family of maps $\widetilde{f}_u : G_{f(u)} \to F_u$, for each $u \in K$, such that, for any morphism $\varphi : m \to n$, any $v \in K_n$, $u = \varphi^* v \in K_m$,

$$F(\varphi, v) \circ \widetilde{f}_u = \widetilde{f}_v \circ G(\varphi, f(v)). \tag{6.6.25}$$

An example is given by a morphism (f, φ) between a presheaf $\mathscr{F}$ over X and a presheaf $\mathscr{G}$ over Y, when we consider two open coverings $\mathscr{U}$ and $\mathscr{V}$ of X and Y respectively, such that $\mathscr{U}$ is finer than the open covering $f^{-1}(\mathscr{V})$. In this case, we choose a projection λ from $\mathscr{U}$ to $\mathscr{V}$ i.e. $\forall U \in \mathscr{U}$, $U \subseteq f^{-1}\lambda(U)$. As we have seen, this defines a simplicial map from $K(\mathscr{U})$ to $K(\mathscr{V})$, that we write f_λ; then, for $u = (U_0, ..., U_n)$, the group $G(f_\lambda(u))$ can be identified with $G(\bigcap_{i=0}^n(\lambda(U_i))$, and for every element g in this group, we pose

$$\widetilde{f_\lambda}_u(g) = \varphi(g) \in F(\bigcap_{i=0}^{n}(U_i)); \tag{6.6.26}$$

because φ can be seen as a morphism of $f^{-1}\mathscr{G}$ to $\mathscr{F}$.

Definition 6.6.6. Two pairs $(f, \widetilde{f})$, $(g, \widetilde{g})$ of morphisms and lifts are *simplicially homotopic* if there exists a simplicial homotopy $h : K \times \Delta^1 \to L$ from f to g, and a family of maps $\widetilde{h}_{u,s}, u \in K, s \in \Delta^1$ from $G_{h(u,s)}$ to F_u, such that $\widetilde{f} = \widetilde{j_0} \circ \widetilde{h}$ and $\widetilde{g} = \widetilde{j_1} \circ \widetilde{h}$, where $j_0 = \mathrm{id}_K \times s_0$ and $j_1 = \mathrm{id}_K \times s_1$.

As a consequence of Proposition 6.6.1, two choices of projections in the construction of the map of local systems associated to a morphism of presheaves gives two homotopic morphisms in the simplicial sense.

Suppose given two local systems F, G over K, L respectively, and a lift $\widetilde{f}$ of $f : K \to L$. The following formula defines a natural morphism $\widetilde{f}^*$ from $C^\bullet(L; G)$ to $C^\bullet(K; F)$:

$$\widetilde{f}^*(c_L)(u) = \widetilde{f}_u(c_L(f(u)). \tag{6.6.27}$$

Lemma 6.6.1. *$\widetilde{f}^*$ commutes with the differentials.*

Definition 6.6.7. Let G be a cosimplicial local system over a simplicial set L, and f a simplicial map from K to L, the family $F_u = G_{f(u)}$, for $u \in K$, with the maps $F(\varphi, v) = G(\varphi, f(v))$ is a cosimplicial local system over K, named the pull-back of G, and denoted by $f^*(G)$.

Example 6.6.10. Start with a cosimplicial local system F over a simplicial set K; then, over the product $K \times \Delta^1$, we define a local system $\pi^* F$ by taking, for $u \in K_n$ and $s \in \Delta^1$, $\pi^* F(u, s) = F(u)$. Then consider the two injections $j_0 = Id_K \times s_0$ and $j_1 = Id_K \times s_1$ from $K = K \times \Delta^0$ to $K \times \Delta^1$; the two pull-back $j_0^* \pi^* F$ and $j_1^* \pi^* F$ coincide with F. Thus we have evident lifts $\widetilde{j_0}$ and $\widetilde{j_1}$ from F to $\pi^* F$. They are homotopic in the simplicial sense, the map h from $K \times \Delta^1$ to itself being the identity, and the lift being the natural identification.

From $C^\bullet(K \times \Delta^1; \pi^* F)$ to $C^\bullet(K; F)$, the two chain maps $\widetilde{j_0}^*$ and $\widetilde{j_1}^*$ are given by the following formulas, for c be a n-cochain of $K \times \Delta^1$ with value in $\pi^* F$), and u an element of K_n

$$\widetilde{j_0}^* c(u) = c(u, (0, ..., 0)), \tag{6.6.28}$$

$$\widetilde{j_1}^* c(u) = c(u, (1, ..., 1)). \tag{6.6.29}$$

Lemma 6.6.2. *$\widetilde{j_0}^*$ and $\widetilde{j_1}^*$ are homotopic as chain maps.*

Proof. Let C be a $(n+1)$-cochain of $K \times \Delta^1$ with value in $\pi^* F$, and u an element of K_n, we pose

$$H(C)(u) = \sum_{j=0}^{n+1} (-1)^j F(s_j, u)(C(s_j^*(u), 1_j^{n+1})), \tag{6.6.30}$$

where, for $n \in \mathbb{N}$ and $j \in [n+2]$, 1_j^{n+1} denotes the element $(0, ..., 0, 1, ..., 1)$ of Δ^1_{n+1}, where the first 1 is at the place j. That gives $1_{n+2}^{n+1} = (0, ..., 0) = s_0(0)$ and $1_0^{n+1} = (1, ..., 1) = s_1(0)$.
This defines an endomorphism H of degree -1 of $C^\bullet(K; F)$. Now we compute, for $c \in C^n(K \times \Delta^1; \pi^* F)$:

$$H(\delta c)(u) = \sum_{j=0}^{n+1} \sum_{k=0}^{n+1} (-1)^{j+k} F(s_j, u) \circ F(d_k, s_j^* u) c(d_k^* s_j^* u, d_k^* 1_j^{n+1}), \tag{6.6.31}$$

and

$$\delta(H(c))(u) = \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{j+k} F(d_k, u) \circ F(s_j, d_k^* u) c(s_j^* d_k^* u, 1_j^{n+1}). \tag{6.6.32}$$

Let us add $H(\delta c)(u)$ and $\delta(H(c))(u)$, in virtue of the relations (6.6.8), most of the terms annihilate. The only ones that survive correspond to the terms $(s_j d_j)^*$ and $(s_j d_{j-1})^*$. Note that $d_j^* 1_j^{n+1} = 1_j^n$ and that $d_{j-1}^* 1_j^{n+1} = 1_{j-1}^n$, then, due to the signs, they annihilate two by two, except the extreme terms, for $j = 0$ and $j = n+1$, giving

$$\begin{aligned}\delta(H(c))(u) + H(\delta c)(u) &= c(u, 1_0^n) - c(u, 1_{n+1}^n) \\ &= c(u, (1, ..., 1) - c(u, (0, ..., 0)). \quad (6.6.33)\end{aligned}$$

Then H is a chain homotopy operator from $\widetilde{j_0}^*$ to $\widetilde{j_1}^*$, as we desired. □

Proposition 6.6.4. *Let f, g be two simplicial maps from the simplicial set K to the simplicial set L, let F, G be cosimplicial systems over K and L respectively, and $\widetilde{f}, \widetilde{g}$ two lifts over K; suppose that the pais $(f, \widetilde{f})$, $(g, \widetilde{g})$ are simplicially homotopic, then the induced maps of cochains complexes are homotopic.*

Proof. The map $\widetilde{h}$ is a pullback of the simplicial map h to the local systems $\pi^* F$ on $K \times \Delta^1$ and G on L. Thus wee get a chain-map

$$\widetilde{h}^* : C^\bullet(L; G) \to C^\bullet(K \times \Delta^1; \pi^* F). \quad (6.6.34)$$

On the other side, we have two natural cochain maps, for $k = 0, 1$,

$$\widetilde{j_k}^* : C^\bullet(K \times \Delta^1; \pi^* F) \to C^\bullet(K; F). \quad (6.6.35)$$

Applying the Lemma 6.6.2, there exists an homotopy from $\widetilde{j_0}^*$ to $\widetilde{j_1}^*$:

$$H : C^\bullet(K; F) \to C^{\bullet - 1}(K; F); \quad (6.6.36)$$

therefore, applying the Lemma 6.6.1:

$$\begin{aligned}\widetilde{g}^* - \widetilde{f}^* &= \widetilde{h}^* \circ (\widetilde{j_1}^* - \widetilde{j_0}^*) \\ &= \widetilde{h}^* \circ (d \circ H + H \circ d) = d \circ (\widetilde{h}^* \circ H) + (\widetilde{h}^* \circ H). \quad (6.6.37)\end{aligned}$$

□

Corollary 6.6.1. *The induced morphisms in cohomology are the same.*

Theorem 6.6.1 (cf. [70, Ch. IX])**.** *If $\mathscr{U}$ is a refinement of $\mathscr{U}'$, two projections λ, μ from $\mathscr{U}$ to $\mathscr{U}'$ give the same application in cohomology.*

Proof. The simplicial maps λ_* and μ_* from $K(\mathscr{U})$ to $K(\mathscr{U}')$ are homotopic in the simplicial sense, then the maps λ^* and μ^* are homotopic in the sense of maps of differential complexes, cf. last section). □

6.6.3 Comparison theorems

Given a covering $\mathscr{U}$ of a topological space X, let $\mathscr{A}(\mathscr{U})$ denote the poset whose objects are the non-empty finite intersections of elements of $\mathscr{U}$, ordered by inclusion (thus the morphisms go from intersections to partial intersections), and $N(\mathscr{U}) = N(\mathscr{A}(\mathscr{U}))$ denotes the nerve of the category $\mathscr{A}(\mathscr{U})$, cf. Example 6.6.5. Note that the set of open sets in $\mathscr{U}$ giving an object of $\mathscr{A}(\mathscr{U})$ is not an element of the structure.

The objects of $\mathscr{A}(\mathscr{U})$ make an open covering of X which is finer than $\mathscr{U}$. By choosing for each object a of $\mathscr{A}(\mathscr{U})$ an element of $\mathscr{U}$ which contains a, we obtain a map from $\mathscr{A}(\mathscr{U})$ to $\mathscr{U}$, that we denote π; it is a projection in the sense of Eilenberg-Steenrod, see Example 6.6.6. In what follows we will always assume that for $U \in \mathscr{U}$, $\pi(U) = U$. The map π induces a simplicial map π_* from the simplicial set $N(\mathscr{U})$ to the simplicial set $K(\mathscr{U})$ (which is the usual nerve of the covering $\mathscr{U}$ in the sense of Example 6.6.3); it maps the sequence $V_0 \to \cdots \to V_n$ of elements of $\mathscr{A}(\mathscr{U})$ to the sequence $(\pi(V_0), ..., \pi(V_n))$ of elements of $\mathscr{U}$.

Given a presheaf F of abelian groups over X, we have defined the cosimplicial local system of Čech $F^\vee$ over $K(\mathscr{U})$ (cf. Example 6.6.7). To define a local system over $N(\mathscr{U})$, we restrict F to a presheaf on $\mathscr{A}(\mathscr{U})$ and we take the *lower* cosimplicial local system F_* over $N(\mathscr{U})$, as in Example 6.6.8. Given an element $v = (V_0 \to \cdots \to V_n)$ of $N_n(\mathscr{U})$, remark that $V_0 = \bigcap_{i=0}^n V_i \subseteq \bigcap_{i=0}^n \pi(V_i)$, hence there is a well-defined restriction map from $F^\vee(\pi_* v)$ to $F_*(v)$. This defines a lift $\tilde{\pi}$ of π_* from $F^\vee$ to F_* in the sense of Definition 6.6.5, hence a morphism

$$\pi^* : C^\bullet(K(\mathscr{U}); F^\vee) \to C^\bullet(N(\mathscr{U}); F_*) \tag{6.6.38}$$

Theorem 6.6.2. *The map π^* is an homotopy equivalence between $C^\bullet(K(\mathscr{U}))$ and $C^\bullet(N(\mathscr{U}))$.*

Before presenting the proof, let us see how this implies the isomorphism between topos cohomology and Čech cohomology in the case of abelian presheaves on a poset, provided it is a conditional meet semilattice i.e. that products exists conditionally. Let G be a contravariant functor of abelian groups on a poset $\mathscr{A}$, and let $\widehat{G}$ be the induced sheaf on the upper A-space $X^{\mathscr{A}}$. We have seen that the topos cohomology of $G \in \mathrm{PSh}(\mathscr{A})$ is isomorphic to the cohomology of the cochain complex $(C^\bullet(N(\mathscr{A}), G_*), \delta)$, whereas the Čech cohomology of $\widehat{G}$ is the cohomology of the complex $(C^\bullet(K(\mathscr{U}^{\mathscr{A}}), G^\vee), \delta)$.

The space $X^{\mathscr{A}}$ has a finest canonical open covering $\mathscr{U}^{\mathscr{A}}$ made by the upper sets $U^\alpha; \alpha \in \mathscr{A}$. An inclusion $U^\alpha \subseteq U^\beta$ corresponds to an arrow $\alpha \to \beta$, then the natural inclusion $\mathscr{A} \hookrightarrow \mathscr{A}(\mathscr{U}^{\mathscr{A}})$ is a covariant functor, and induces an injective simplicial covariant functor $\iota : N(\mathscr{A}) \hookrightarrow N(\mathscr{U}^{\mathscr{A}})$.

We have a diagram of simplicial sets

$$N(\mathscr{A}) \xrightarrow{\iota} N(\mathscr{U}^{\mathscr{A}}) \xrightarrow{\pi_*} K(\mathscr{U}^{\mathscr{A}}) \tag{6.6.39}$$

where the last arrow is induced by any projection $\pi : \mathscr{A}(\mathscr{U}^{\mathscr{A}}) \to \mathscr{U}^{\mathscr{A}}$ that is the identity on $\mathscr{U}^{\mathscr{A}}$. Let us denote by j the simplicial map $\pi_* \circ \iota$. Given $a = (\alpha_0 \to \cdots \to \alpha_n)$ in $N(\mathscr{A})$, we have

$$G(\alpha_0) = G_*(a) = G^\vee(j(a)) = \widehat{G}(\cap_{i=0}^n U^{\alpha_i}) = \widehat{G}(U^{\alpha_0}), \tag{6.6.40}$$

then there is a lift of j from $G^\vee$ to G_* (cf. Definition 6.6.5) given by identities. Thus we deduce a morphism of chain complexes

$$j^* : C^\bullet(K(\mathscr{A}); G^\vee) \to C^\bullet(N(\mathscr{A}); G_*) \tag{6.6.41}$$

that induces a morphism in cohomology.

Corollary 6.6.2. *If products exist conditionally in $\mathscr{A}$, the chain map j^* is a chain equivalence up to homotopy, thus induces an isomorphism in cohomology.*

Proof. Under the hypothesis, one has $\mathscr{A}(\mathscr{U}^{\mathscr{A}}) = \mathscr{U}^{\mathscr{A}}$, since every intersection $U^{\alpha_0} \cap \cdots \cap U^{\alpha_n}$ equals $U^{\alpha_0 \wedge \cdots \wedge \alpha_n}$. Hence $N(\mathscr{A}) \cong N(\mathscr{U}^{\mathscr{A}})$. The map π_* is induced by $\pi = \mathrm{id}_{\mathscr{U}_{\mathscr{A}}}$. The claim then follows from Theorem 6.6.2. □

The equivalence above is natural in the category of posets with presheaves up to homotopy.

We close this section with the proof of Theorem 6.6.2, inspired by a classical argument of Eilenberg and Steenrod: starting with a simpicial complex K, they associated to it the poset $\mathscr{A}(K)$, whose elements are the faces (simplicies) of K; the nerve N of this poset is naturally isomorphic to the barycentric subdivision of K (cf. [69]). In [72, pp. 177-178], the authors proved that there exists an homotopy equivalence between N and K. The following proof is an adaptation of their argument to this more general setting.

Proof of Theorem 6.6.2. 1) first we construct by recurrence over n a linear application Sd^n from $C^n(N(\mathscr{U}))$ to $C^n(K(\mathscr{U}))$, having the two following properties:
(i) (locality) for any $c \in C^n(N(\mathscr{U}))$ and any collection $u = (U_0, ..., U_n)$, the value $(Sd^n c)(u)$ in $F(U_u)$ depends only of the values of c on the descendent of the open sets $U_i; i = 0, ..., n$, i.e. the values $c(v) \in F(V_n)$ for the sequences $v = (V_0, .., V_n)$, where each $V_i; i = 0, ..., n$ is included in a $U_j; j = 0, ..., n$;
(ii) (morphism of cochain complex) $d \circ Sd^* = Sd^* \circ d$.

For $n = 0$, and $U_0 \in \mathscr{U} = K_0(\mathscr{U})$, we take $Sd^0(c)(U_0) = c(U_0)$, this is allowed because U_0 is also an element of $N_0(\mathscr{U})$. The condition (i) is evidently satisfied, and (ii) is empty in this degree.

For $n = 1$, $c \in C^1(N(\mathscr{U}))$ and $u = (U_0, U_1)$, we pose $Sd^1c(U_0, U_1) = c(U_0, U_u) - c(U_1, U_u)$, where $U_u = U_0 \cap U_1$. This is local, and for $c_0 \in C^0(N(\mathscr{U}))$:

$$\begin{aligned} Sd^1(dc_0)(U_0, U_1) &= (c_0(U_0) - c_0(U_u)) - (c_0(U_1) - c_0(U_u)) \\ &= dc_0(U_0, U_1) = d \circ Sd^0(c)(U_0, U_1). \end{aligned} \quad (6.6.42)$$

Then take $n \geq 2$, and suppose that a map Sd^q is constructed for every $q \leq n-1$, satisfying (i) and (ii). Take a cochain c in $C^n(N(\mathscr{U}))$, and consider an element $u = (U_0, ..., U_n)$ of $K_n(\mathscr{U})$; remind we note U_u the intersection (necessary non-empty) of the $U_i, i = 0, ..., n$. We define an element $c_u \in C^{n-1}(N(\mathscr{U}))$ by taking on every decreasing sequence $v = (V_0, ..., V_{n-1})$,

$$c_u(v) = c(V_0, ..., V_{n-1}, U_u), \quad (6.6.43)$$

if V_{n-1} contains U_u and taking $c_u(V_0, ..., V_{n-1}) = 0$ in the opposite case.
Then we define

$$Sd^n(c)(u) = \sum_{i=0}^{n} (-1)^{n-i} Sd^{n-1}(c_u)(U_0, ..., \widehat{U_i}, ..., U_n)|U_u. \quad (6.6.44)$$

The locality (i) follows from the recurrence hypothesis: the definition of c_u depends only of U_u which is a descendent of u, and this is the same for the restriction to U_u, moreover the value of $Sd^{n-1}(c_u)$ on $(U_0, ..., \widehat{U_i}, ..., U_n)$ depends only of the values of c_u on the sequences of descendent of $(U_0, ..., \widehat{U_i}, ..., U_n)$.

For (ii), we have to compute $Sd^n(dc)(u)$ for a cochain $c \in C^{n-1}(N(\mathscr{U}); F)$. For a decreasing sequence $V_0, ..., V_{n-1}$, then, by writing $V_n = U_u$, we have

$$\begin{aligned} (dc)_u(V_0, ..., V_{n-1}) &= dc(V_0, ..., V_{n-1}, U_u) \\ &= \sum_{j=0}^{n} (-1)^j c(V_0, ..., \widehat{V_j}, ..., V_n)|U_u \\ &= d(c_u)(V_0, ..., V_{n-1})|U_u + (-1)^n c(V_0, ..., V_{n-1})|U_u; \end{aligned}$$

where c_u is also defined by $c_u(V_0, ..., V_{n-2}) = c(V_0, ..., V_{n-2}, U_u)$ if V_{n-2} contains U_u and $c_u(V_0, ..., V_{n-2}) = 0$ in the opposite case. Which gives for reference, when c belongs to $C^{n-1}(N(\mathscr{U}); F)$:

$$d(c_u) = (dc)_u + (-1)^{n-1} c. \quad (6.6.45)$$

It follows from the recurrence hypothesis that

$$\begin{aligned}
Sd^n(dc)(u) &= \sum_{i=0}^{n}(-1)^{n-i}Sd^{n-1}((dc)_u)(U_0,...,\widehat{U_i},...,U_n)|U_u \\
&= \sum_{i=0}^{n}(-1)^{n-i}(Sd^{n-1}d(c_u))(U_0,...,\widehat{U_i},...,U_n)|U_u \\
&+ \sum_{i=0}^{n}(-1)^{i}(Sd^{n-1}c)(U_0,...,\widehat{U_i},...,U_n)|U_u \\
&= \sum_{i=0}^{n}(-1)^{n-i}(d\circ Sd^{n-1}(c_u))(U_0,...,\widehat{U_i},...,U_n)|U_u \\
&+ \sum_{i=0}^{n}(-1)^{i}(Sd^{n-1}c)(U_0,...,\widehat{U_i},...,U_n)|U_u \\
&= (-1)^n(d\circ d)Sd^n(c_u)(u) + d(Sd^{n-1}(c))(u) \\
&= d\circ Sd^{n-1}(c)(u).
\end{aligned}$$

Therefore Sd^n verifies (ii).
2) let us prove that the composition $Sd^*\circ\pi^*$ is homotopic to the identity of $C^\bullet(K(\mathscr{U});F)$.

For that purpose we construct a sequence of homomorphisms,

$$D_K^{n+1} : C^{n+1}(K(\mathscr{U});F) \to C^n(K(\mathscr{U});F), \tag{6.6.46}$$

for $n\geq 0$, by recurrence over the integer n, such that

$$Id - Sd^n\circ\pi^n = d\circ D_K^n + D_K^{n+1}\circ d. \tag{6.6.47}$$

For $n=0$, and $c\in C^1(K(\mathscr{U});F)$, we simply take $D_K^1(c)(U)=0$. This works because, if c is a 0-cochain of $K(\mathscr{U}),F$,

$$(Sd^0\circ\pi^0c)(U_0) = c(U_0). \tag{6.6.48}$$

For $n=1$, and $c\in C^2(K(\mathscr{U});F)$, take

$$D_K^2c(U_0,U_1) = c(U_0,U_1,\pi(U_0\cap U_1)|U_0\cap U_1. \tag{6.6.49}$$

This gives for $c'\in C^2(K(\mathscr{U});F)$:

$$D_K^2(dc')(U_0,U_1) = c'(U_1,\pi(U_u))|U_u - c'(U_0,\pi(U_u))|U_u + c'(U_0,U_1); \tag{6.6.50}$$

where as usual we have denoted $U_0\cap U_1$ by the symbol U_u. On the other side,

$$\begin{aligned}
(Sd^1\circ\pi^1c')(U_0,U_1) &= -(\pi^1c')_u(U_1)|U_u + (\pi^1c')_u(U_0)|U_u \\
&= -(\pi^1c')(U_1,U_u)|U_u + (\pi^1c')(U_0,U_u)|U_u \\
&= -c'(U_1,\pi(U_u))|U_u + c'(U_0,\pi(U_u))|U_u.
\end{aligned}$$

Then $Id - Sd^1 \circ \pi^1 = D_K^2 \circ d + d \circ D_K^1$, as we expected.

More generally, for any n, consider consider a n-cochain c of $K(\mathscr{U})$ with respect to the local system F. For a sequence $u' = (U'_0, ..., U'_{n-1})$ in $\mathscr{U}$, let us define

$$c_u^\pi(U'_0, ..., U'_{n-1}) = c(U'_0, ..., U'_{n-1}, \pi(U_u)) \tag{6.6.51}$$

if $U_{u'} \supseteq \pi(U_u)$, and $c_u^\pi(U'_0, ..., U'_{n-1}) = 0$ if not.

Then consider a decreasing sequence $v = (V_0, ..., V_{n-1})$ in $\mathscr{A}_{\mathscr{U}}$. If $U_u \subseteq U_{\pi(v)} = \bigcap_{i=0}^{n-1} \pi(V_i)$,

$$\begin{aligned}(\pi^n c)_u(V_0, ..., V_{n-1}) &= \pi^n c(V_0, ..., V_{n-1}, U_u) \\ &= c(\pi(V_0), ..., \pi(V_{n-1}), \pi(U_u)) \\ &= c_u^\pi(\pi(V_0), ..., \pi(V_{n-1})) \\ &= \pi^{n-1}(c_u^\pi)(V_0, ..., V_{n-1}).\end{aligned}$$

If $U_u \nsubseteq U_{\pi(v)}$, we have $(\pi^n c_n)_u(v) = 0 = (\pi^{n-1}(c_u^\pi))(v)$. Therefore, in all cases

$$(\pi^n c)_u(v) = \pi^{n-1}(c_u^\pi)(v). \tag{6.6.52}$$

Now assume that D_K^{q+1} is defined for $q \leq n$, satisfying the homotopy relation for $Id - Sd^q \circ \pi^q$, and consider a n-cochain c of $K(\mathscr{U})$ with respect to the local system F; for every sequence $u = (U_0, ..., U_n)$ in $\mathscr{U}$, the chosen definition of Sd^n gives

$$(Sd^n \circ \pi^n(c))(U_0, ..., U_n) = \sum_{i=0}^{n} (-1)^{n-i} (Sd^{n-1}(\pi^n c)_u)(U_0, ..., \widehat{U_i}, ..., U_n)|U_u. \tag{6.6.53}$$

Thus, applying (6.6.52) we get

$$(Sd^n \circ \pi^n(c))(U_0, ..., U_n)) = \sum_{i=0}^{n} (-1)^{n-i} Sd^{n-1} \circ \pi^{n-1}(c_u^\pi)(U_0, ..., \widehat{U_i}, ..., U_n)|U_u. \tag{6.6.54}$$

By applying the hypothesis of recurrence, we get

$$\begin{aligned}(Sd^n \circ \pi^n(c))(U_0, ..., U_n)) = (-1)^n \sum_{i=0}^{n} (-1)^i c_u^\pi(U_0, ..., \widehat{U_i}, ..., U_n)|U_u \\ + \sum_{i=0}^{n} (-1)^{n+1-i} D_K^n \circ d(c_u^\pi)(U_0, ..., \widehat{U_i}, ..., U_n)|U_u \\ + \sum_{i=0}^{n} (-1)^{n+1-i} d(D_K^{n-1}(c_u^\pi))(U_0, ..., \widehat{U_i}, ..., U_n)|U_u.\end{aligned} \tag{6.6.55}$$

The last sum is zero due to $d \circ d = 0$, and the first sum is $(-1)^n d(c_u^\pi)$, therefore

$$(Sd^n \circ \pi^n(c))(U_0, ..., U_n) = (-1)^n d(c_u^\pi)(U_0, ..., U_n) \\ + \sum_{i=0}^{n} (-1)^{n+1-i} D_K^n \circ d(c_u^\pi)(U_0, ..., \widehat{U_i}, ..., U_n)|U_u \quad (6.6.56)$$

As we obtained a formula for $d(c_u)$, we obtain a formula for $d(c_u^\pi)|U_u$. In fact, writing $U_{n+1} = \pi(U_u)$,

$$\begin{aligned}(dc)_u^\pi(U_0, ..., U_n)|U_u &= dc(U_0, ..., U_n, \pi(U_u))|U_u \\ &= \sum_{j=0}^{n+1} (-1)^j c(U_0, ..., \widehat{U_j}, ..., U_n)|U_u \\ &= d(c_u^\pi)(U_0, ..., U_n)|U_u + (-1)^{n+1} c(U_0, ..., U_n)|U_u;\end{aligned}$$

Then, replacing $d(c_u^\pi)|U_u$ by $(dc)_u^\pi + (-1)^n c$ in the formula (6.6.55), we get

$$(Sd^n \circ \pi^n(c))(U_0, ..., U_n) = c(U_0, ..., U_n) + (-1)^n (dc)_u^\pi(U_0, ..., U_n) \\ + (-1)^{n+1} d \circ D_K^n \circ (dc)_u^\pi)(U_0, ..., \widehat{U_i}, ..., U_n)|U_u - d \circ D_k^n(c)(U_0, ..., U_n). \quad (6.6.57)$$

Assuming that we have defined D_K^n on $C^n(K(\mathscr{U}); F)$, we define D_K^{n+1} on $C^{n+1}(K(\mathscr{U}); F)$ by the following formula:

$$D_K^{n+1}(c') = (-1)^{n+1}(c')_u^\pi + (-1)^{n+1} d D_K^n(c')_u^\pi. \quad (6.6.58)$$

This gives the awaited result.
3) To finish the proof of the theorem, we have to demonstrate that the composition $\pi^* \circ Sd^*$ is homotopic to the identity of $C^\bullet(N(\mathscr{U}); F)$. For that, we construct a sequence of homomorphisms,

$$D_N^{n+1} : C^{n+1}(N(\mathscr{U}); F) \to C^n(N(\mathscr{U}); F), \quad (6.6.59)$$

by recurrence over the integer $n \geq 0$, such that

$$Id - \pi^n \circ Sd^n = d \circ D_N^n + D_N^{n+1} \circ d. \quad (6.6.60)$$

For $n = 0$, and $c \in C^1(N(\mathscr{U}); F)$, we define $D_N^1(c)(V_0) = c(\pi(V_0), V_0)$. Remember that if c is a zero cochain for $N(\mathscr{U})$, $Sd^0 c(U_0) = c(U_0)$. Then

$$\pi^0 Sd^0 c(V_0) = c(\pi(V_0)) = c(V_0) + (c\pi(V_0)) - c(V_0)) = c(V_0) - D_N^1(dc))(V_0); \quad (6.6.61)$$

which gives $c - \pi^0 Sd^0 c = D_N^1(dc))$ as desired.

Now assume the recurrence hypothesis, that there exist operators D_K^{q+1} for $q \leq n$, satisfying the homotopy relation for $Id - \pi^q \circ Sd^q$, and consider a n-cochain c of $N(\mathscr{U})$ with respect to the local system F; for every decreasing sequence $v = (V_0, ..., V_n)$ in $\mathscr{A}_{\mathscr{U}}$, we have

$$\begin{aligned}(\pi^n \circ Sd^n(c))(v) &= (Sd^n c)(\pi(V_0), ..., \pi(V_n))|V_n \\ &= \sum_{i=0}^{n} (-1)^{n-i} Sd^{n-1}(c)_{\pi(v)}(\pi(V_0), ..., \widehat{\pi(V_i)}, ..., \pi(V_n))|V_n \\ &= \sum_{i=0}^{n} (-1)^{n-i} \pi^{n-1}(Sd^{n-1}(c)_{\pi(v)})(V_0, ..., \widehat{V_i}, ..., V_n)|V_n;\end{aligned}$$

which gives by applying the hypothesis of recurrence:

$$\begin{aligned}(\pi^n \circ Sd^n(c))(V_0, ..., V_n) &= \sum_{i=0}^{n} (-1)^{n-i} c_{\pi(v)}(V_0, ..., \widehat{V_i}, ..., V_n)|V_n \\ &+ \sum_{i=0}^{n} (-1)^{n+1-i} D_N^n(dc_{\pi(v)})(V_0, ..., \widehat{V_i}, ..., V_n)|V_n \\ &+ \sum_{i=0}^{n} (-1)^{n+1-i} d \circ D_N^{n-1}(c_{\pi(v)})(V_0, ..., \widehat{V_i}, ..., V_n)|V_n. \quad (6.6.62)\end{aligned}$$

The last sum is zero due to $d \circ d = 0$, the first one is equal to $(-1)^n d(c_{\pi(v)})(v)$, and the second one to $(-1)^{n+1} d(D_N^n(dc_{\pi(v)})(v)$, that is

$$(\pi^n \circ Sd^n(c))(v) = (-1)^n d(c_{\pi(v)})(v) + (-1)^{n+1} d(D_N^n(dc_{\pi(v)})(v). \quad (6.6.63)$$

But the relation (6.6.45) tells that

$$d(c_{\pi(v)})(v) = (dc)_{\pi(v)}(v) + (-1)^n c(v). \quad (6.6.64)$$

Thus by substituting, we get

$$\begin{aligned}(\pi^n \circ Sd^n(c))(v) = c(v) &+ (-1)^n (dc)_{\pi(v)})(v) \\ &- d(D_N^n(c)(v) - (-1)^n d(D_N^n(d(c_{\pi(v)}))(v); \quad (6.6.65)\end{aligned}$$

which gives the expected result,

$$c(v) - (\pi^n \circ Sd^n(c))(v) = d(D_N^n(c))(v) + D^{n+1}(dc)(v)_{\pi(v)})(v); \quad (6.6.66)$$

if we define, for any $c' \in C^{n+1}(N(\mathscr{U}); F)$ and any v in $N_n(\mathscr{U})$:

$$D_N^{n+1}(c')(v) = (-1)^{n+1} c'_{\pi(v)}(v) + (-1)^n dD_N^n(c'_{\pi(v)}(v). \quad (6.6.67)$$

This ends the proof. □

The constructions made in the proof show that the homotopy equivalence is natural in the category of open covering of topological spaces and morphisms of local systems.

Appendix

6.A Topology and sheaves

Remind that a *topological space* is a set X, equipped with a subset $\mathscr{T}$ of the set of parts $\mathscr{P}(X)$—named its *topology*—that is supposed to contain X and the empty set $\emptyset$, and to be closed under union and finite intersection. A map $f : X \to Y$ between topological spaces is said *continuous* if the inverse image of an open set is an open set. A topology $\mathscr{T}$ is said *finer* than a topology $\mathscr{T}'$ if the identity is continuous from $X_{\mathscr{T}}$ to $X_{\mathscr{T}'}$. It is equivalent to ask that $\mathscr{T}' \subseteq \mathscr{T}$ as elements of $\mathscr{P}(\mathscr{P}(X))$.

An *open covering* of an open set $V \in \mathscr{T}$ is a subset $\mathscr{U} \subseteq \mathscr{T}$ such that $V = \bigcup_{U \in \mathscr{U}} U$.

A topology $\mathscr{T}$ can be seen as a category, whose objects are the open sets of X (i.e. the elements of $\mathscr{T}$); whenever $U \subseteq V$, there is one $U \to V$. The resulting category $\mathscr{T}$ is a poset (see Section 6.3).

A *presheaf* $\mathscr{F}$ over a topological space X is a contravariant functor from $\mathscr{T}$ to the category of sets $\mathscr{E}$, i.e. a family of sets $\{F(U) = F_U\}_{U \in \mathscr{T}}$, and maps $\{\pi_{VU}\}_{(U \to V) \in \mathscr{T}}$ such that $\pi_{UU} = Id_{F(U)}$ and $\pi_{WV} \circ \pi_{VU} = \pi_{WU}$ when $W \subseteq V \subseteq U$. Frequently we will note $\pi_{VU}(s) = s|V$, as a restriction. Sometimes, the elements s of F_U are named sections of F over U.

A *sheaf* is a presheaf which satisfies the two following axioms:

1. For every $V \in \mathscr{T}$ and every open covering $\mathscr{U} \subseteq \mathscr{T}$ of V, if s, t are two elements of F_V such that for any $U \in \mathscr{U}$ we have $s|U = t|U$, then $t = s$.

2. For every $V \in \mathscr{T}$ and every open covering $\mathscr{U} \subseteq \mathscr{T}$ of V, if a family $(s_U)_{U \in \mathscr{U}} \in \prod_{U \in \mathscr{U}} F_U$ is such that for all $U, U' \in \mathscr{U}$, $s_U|(U \cap U') = s_{U'}|(U \cap U')$, then there exists $s \in F_V$ such that for all $U \in \mathscr{U}$, $s|U = s_U$.

The notion of presheaf extends to any category $\mathscr{C}$ in place of $\mathscr{E}$: just take a contravariant functor from T to C. However the definition of sheaf requires a priori that $\mathscr{C}$ is a sub-category of $\mathscr{E}$.

One of the main theorems in sheaf theory is the existence of a canonical sheaf $\mathscr{F}^{\sim}$ associated to a presheaf $\mathscr{F}$ on $(X, \mathscr{T})$, built as follows [74, Sec. II.5]. One says $s \in FU$ and $t \in FV$ have the same germ at x if there exists $W \subseteq U \cap V$ such that $s|W = t|W$. Having the same germ at x is an equivalence relation and one denotes $\mathrm{germ}_x\, s$ the corresponding equivalence class. More precisely, one can describe the set of all germs as a colimit $\varinjlim_{x \in U} FU$ over all open neighborhoods of U; the resulting set F_x is called the *stalk* of F at x. Set $\Lambda_F = \prod_{x \in X} F_x$, and introduce the obvious projection $p : \Lambda_F \to X$. Any $s \in FU$ determines a map $\dot{s} : U \to \Lambda_P,\ x \mapsto (x, \mathrm{germ}_x\, s)$, which is a section of p. The set Λ_F is topologized

introducing $\{\dot{s}(U) \mid U \in \mathscr{T}, s \in FU\}$ as a basis of open sets. Then $F^\sim$ is defined as the sheaf of (continuous) sections of Λ_P over the opens of X. This means that an element of $F^\sim(U)$ is a family $(s_x) \in \prod_{x \in U} \mathscr{F}_x$ which is locally a germ of $\mathscr{F}$: for all $y \in U$, there exist $V \in \mathscr{T}$ and $t \in FV$ such that $y \in V \subseteq U$ and for all $x \in V$, $\operatorname{germ}_x t = s_x$. The map $s \mapsto \dot{s}$ defines a natural transformation $F \to F^\sim$, which is an isomorphism when F is a sheaf.

We consider now the functoriality of sheaves. Let $f : X \to Y$ be a continuous map; it induces a functor $f^{-1} : \mathscr{T}_Y \to \mathscr{T}_X$ between the topologies (seen as categories) of Y and X, respectively.

1. If $\mathscr{F}$ is a presheaf over X, the *direct image* $f_*\mathscr{F}$ is defined on Y by the formula: $f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$. If $\mathscr{F}$ is a sheaf, this is also the case for $f_*\mathscr{F}$ [74]. (In fact, f_*F is also the pullback $(f^{-1})^*F$ of F under the functor f^{-1} according to [54, Sec. I.5].)

2. If $\mathscr{G}$ is a presheaf over Y, the *inverse image* $f^{-1}\mathscr{G}$ is defined on X by the formula: $f^{-1}\mathscr{G}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(V)$, where the limit is taken over the directed family of opens subsets V of Y which contain $f(U)$. Even if $\mathscr{F}$ is a sheaf, in general $f^{-1}\mathscr{G}$ is not a sheaf. We make use of the sheafification, and define the *pullback* of G by $f^*\mathscr{G} = (f^{-1}\mathscr{G})^\sim$.

The functors f_*, f^{-1} between the corresponding categories of presheaves are adjoint i.e. for any presheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y, there exist natural bijections

$$\operatorname{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})) \cong \operatorname{Hom}_Y(\mathscr{G}, f_*\mathscr{F}). \tag{6.A.1}$$

Similarly, f^* is left adjoint to f_* in the categories of sheaves.

Definition 6.A.1. A map of presheaves (resp. sheaves) from $(X, \mathscr{F})$ to $Y, \mathscr{G})$ is a pair (f, φ), where $f : X \to Y$ is continuous, and φ is a morphism from $\mathscr{G}$ to $f_*\mathscr{F}$, or equivalently a morphism φ^* from $f^{-1}\mathscr{G}$ (resp. $f^*\mathscr{G}$) to $\mathscr{F}$.

6.B Čech cohomology

We summarize some facts concerning Čech cohomology.

6.B.1 Limit over coverings

A *preorder* is a set P with a binary relation that is transitive and reflexive. Equivalently, is a small category $\mathscr{P}$ where there exists at most one arrow between two objects. The preorder $\mathscr{P}$ is called directed if for any objects a and b of $\mathscr{P}$, there exists an object c such that $a \to c$ and $b \to c$.

As we saw in Section 6.6.2, a covering $\mathscr{U}$ is called a *refinement* of another covering $\mathscr{U}'$ when every set of $\mathscr{U}$ is contained in some set of $\mathscr{U}'$. In that case, there exists a map $\lambda : \mathscr{U} \to \mathscr{U}'$, called *projection*, such that for every $U \in \mathscr{U}$ one has $U \subseteq \lambda(U)$. It is also said that $\mathscr{U}$ is *finer* than $\mathscr{U}'$ [70].

This notion of refinement does not give in general a partial ordering among coverings, but only a pre-order. So it is unlike the notion of finer topology, which corresponds to the natural partial ordering by inclusion of subsets. This can be illustrated with two coverings of $\mathbb{R}$, such that $\mathscr{U} = \{]n, \infty[\mid n \text{ even}\}$ and $\mathscr{U}' = \{]n, \infty[\mid n \text{ odd}\}$.

Lemma 6.B.1. *The category of open coverings of X, such that $\mathscr{U} \to \mathscr{U}'$ if $\mathscr{U}'$ refines $\mathscr{U}$, is a* directed set.

Proof. If $\mathscr{U}$ and $\mathscr{V}$ are open coverings of X, the set of non-empty intersections $U \cap V$, for $U \in \mathscr{U}$ and $V \in \mathscr{V}$ is a refinement of both $\mathscr{U}$ and $\mathscr{V}$. □

Given a directed set $\mathscr{P}$, a *directed system of sets* (associated to $\mathscr{P}$) is a covariant functor from $\mathscr{P}$ to a category $\mathscr{C}$, i.e. a family of objects E_a for $a \in \mathscr{P}$, and a family f_{ab} of morphisms $E_a \to E_b$, associated to ordered pairs $a \preceq b$, such that $\forall a, \quad f_{aa} = 1_{E_a}$ and $\forall a, b, c, \quad a \preceq b \preceq c \Rightarrow f_{ac} = f_{bc} \circ f_{ab}$.

By definition a *direct limit* of such direct system in the category $\mathscr{C}$ is an object E with a set of morphisms $E_a \to E, a \in \mathscr{P}$, such that for any $a \preceq b$ $\varphi_b \circ f_{ab} = \varphi_a$, which is initial, i.e. for any object Y and set $\psi_a : E_a \to Y$ verifying the same rule there exist a morphism $h : E \to Y$ making all evident diagrams commutative. If such a limit exists it is unique up to unique isomorphism, and denoted $\varinjlim E_a$.

When $\mathscr{C}$ is the category of sets $\mathscr{E}$, the direct limit always exists, it is a the quotient of the union of the disjoint sets $\widehat{E}_a = E_a \times \{a\}$ by the equivalence relation $e_a \approx e_b$ if there exists $c \in C$, with $a \preceq c$, $b \preceq c$ and $f_{ac}(e_a) = f_{bc}(e_b)$, i.e. asymptotic equality. If the category $\mathscr{C}$ is the subcategory of $\mathscr{E}$ made by abelian groups and their morphisms, the direct limit is an abelian group.

Definition 6.B.1. For all $n \in \mathbb{N}$, $H^n(X; F) = \varinjlim H^n(\mathscr{U}; F)$, the direct limit being associated to the directed set of open coverings of X.

6.B.2 Functoriality

Suppose given a map of presheaves $(f, \varphi) : (X, \mathscr{F}) \to (Y, \mathscr{G})$, and two open coverings $\mathscr{U}$, $\mathscr{V}$ of X and Y respectively, such that $\mathscr{U}$ is a refinement of $f^{-1}(\mathscr{V})$.
We can choose a projection map λ from $\mathscr{U}$ to $\mathscr{V}$, i.e. $\forall U \in \mathscr{U}$, $U \subseteq f^{-1}(\lambda(V))$. From Proposition 6.6.1, two such maps are homotopic in the simplicial sense. This induces a natural application of chain complexes:

$$(f, \varphi, \lambda)^* : C^\bullet(\mathscr{V}; \mathscr{G}) \to C^\bullet(\mathscr{U}; \mathscr{F}), \tag{6.B.1}$$

which commutes with the coboundary operators.

Consider the particular case of an inclusion $J : X \hookrightarrow Y$. A covering $\mathscr{V}$ of Y induce a covering $\mathscr{U}$ of X, made of the (non-empty) intersections $V \cap X$ for $V \in \mathscr{V}$; there is an evident projection λ from $\mathscr{U}$ to $\mathscr{V}$.

Hypothesis: the map φ is surjective, i.e. for any open set V in Y the map $\varphi_V : \mathscr{G}(V) \to \mathscr{F}(V \cap X)$ is surjective.

In particular this happens if $\mathscr{G} = J_*(\mathscr{F})$ over Y.

If $c \in C^n(\mathscr{U}; \mathscr{F})$, there exists $\tilde{c} \in C^n(\mathscr{V}; \mathscr{G})$ such that, for any family $V_0, ..., V_n$ of elements of $\mathscr{V}$, we have

$$\varphi(\tilde{c}(V_0, ..., V_n)) = c(V_0 \cap X, ..., V_n \cap X) \in \mathscr{F}(\bigcap_{i=0}^{n}(V_i \cap X) = \varphi(\mathscr{G}(\bigcap_{i=0}^{n}(V_i)). \quad (6.B.2)$$

This gives $\widetilde{f_\lambda}^*(\tilde{c}) = c$, then the map $\widetilde{f_\lambda}^* = (f, \varphi, \lambda)^*$ is surjective.

Let us define

$$C^\bullet(\mathscr{V}, \mathscr{U}; \mathscr{G}, \mathscr{F}) = \ker((f, \varphi, \lambda)^*). \quad (6.B.3)$$

By the snake's lemma, we obtain a natural long exact sequence in cohomology:

$$\begin{aligned} ... \to H^q(\mathscr{V}; \mathscr{G}) \to H^q(\mathscr{U}; \mathscr{F}) \to H^{q+1}(\mathscr{V}, \mathscr{U}; \mathscr{G}, \mathscr{F}) \\ \to H^{q+1}(\mathscr{V}; \mathscr{G}) \to H^{q+1}(\mathscr{U}; \mathscr{F}) \to ... \quad (6.B.4) \end{aligned}$$

This sequence survive to the direct limits over coverings and gives an exact of Čech cohomology of the pair $(\mathscr{F}, \mathscr{G})$ over the pair (X, Y).

6.C Möbius inversion and Index formula

This is a continuation of Section 6.5 on free sheaves. Our aim is to give a proof of the Theorem 6.5.4.

We introduce now the hypothesis that the $\{E_i\}_{i \in I}$ are finite sets, of respective cardinality N_i.

If N_α denotes the cardinality of E_α, we have $N_\alpha = \prod_{i \in \alpha} N_i$.

If we suppose that $\mathscr{A}$ is finite, and that it satisfies the strong intersection property, the sheaf F has a decomposition in direct sum:

$$\forall \alpha \in \mathscr{A}, \quad F_\alpha = \bigoplus_{\beta \subseteq \alpha} T_\beta, \quad (6.C.1)$$

Let us denote by D_α the dimension of T_α, for $\alpha \in \mathscr{A}$. We have $N_\alpha = \sum_{\alpha \to \beta} D_\beta$. Then the Möbius inversion formula gives

$$\forall \alpha \in \mathscr{A}, \quad D_\alpha = \sum_{\alpha \to \beta} \mu_{\alpha,\beta} N_\beta; \quad (6.C.2)$$

where the integral numbers $\mu_{\alpha\beta}$ are the Möbius coefficents of $\mathscr{A}$.

Let us remind what are these coefficients [25]. For any locally finite poset $\mathscr{A}$, they are defined by the two following equations:

$$\forall \alpha, \gamma, \quad \delta_{\alpha=\gamma} = \sum_{\beta|\alpha\to\beta\to\gamma} \mu_{\alpha,\beta} = \sum_{\alpha\to\beta\to\gamma} \mu_{\beta,\gamma}. \tag{6.C.3}$$

$$\forall \alpha, \beta, \quad \alpha \nrightarrow \beta \Rightarrow \mu_{\alpha,\beta} = 0. \tag{6.C.4}$$

This gives a function from $\mathscr{A} \times \mathscr{A}$ to $\mathbb{Z}$, which is named the Möbius function of the poset. The Möbius function of $\mathscr{A}^{op}$ is given by $\mu^*_{\beta,\alpha} = \mu_{\alpha,\beta}$.

For example, if $\mathscr{A}$ is the full set of parts of a finite set I, including the empty set or not, we have, for $\beta \subseteq \alpha$:

$$\mu_{\alpha,\beta} = (-1)^{|\alpha|-|\beta|}, \tag{6.C.5}$$

where $|\alpha|$ denotes the cardinality of α, for any $\alpha \in \mathscr{A}$. This formula is called the inclusion-exclusion principle. When $\beta = \emptyset$, the above formula holds true if we pose $|\emptyset| = -1$.

If $\alpha \supseteq \omega$ are two elements of $\mathscr{A}$, and if $\mathscr{A}(\alpha, \omega)$ is the sub-poset of $\mathscr{A}$ made by the elements β such that $\alpha \to \beta \to \omega$, the restriction of the Möbius function of $\mathscr{A}$ to $\mathscr{A}(\alpha, \omega)$ coincides with the Möbius function of $\mathscr{A}(\alpha, \omega)$.

The formula (6.C.5) extends to the poset associated to any simplicial complex. This follows from the preceding assertion, because in the case of a manifold, for every pair of elements α, ω of $\mathscr{A}$ such that $\omega \subseteq \alpha$, the elements β between α and ω are the same in $\mathscr{A}$ or in the simplex defined by α.

If $\mathscr{A}$ verifies the strong intersection property, for each $\alpha \in \mathscr{A}$, the dimension of $H^0(\mathscr{U}_{\mathscr{A}}; T_\alpha)$ is D_α, then Theorem 6.5.1 and Proposition 6.5.3 imply:

Proposition 6.C.1. *If the poset $\mathscr{A}$ is finite and satisfies the strong intersection property, and if the $\{E_i\}_{i\in I}$ are finite sets,*

$$\dim_{\mathbb{K}} H^0(\mathscr{U}_{\mathscr{A}}; F) = \sum_{\alpha,\beta\in\mathscr{A}} \mu_{\alpha\beta} N_\beta. \tag{6.C.6}$$

In particular for the full simplex $\Delta(n-1) = \mathscr{P}(J)$, if J has cardinality n, and if $N_i = N$ for any vertex, the dimension of $H^0(\mathscr{A}; V)$ is N^n.

Proof. Since we include the empty set, with $V_\emptyset = S_\emptyset$ of dimension 1, we get:

$$\begin{aligned}
\sum_{\alpha,\beta\in\mathscr{A}} \mu_{\alpha\beta} N_\beta &= \sum_{k=0}^{n} C_n^k \sum_{l=0}^{k} C_k^l (-1)^{k-l} N^l \\
&= \sum_{k=0}^{n} (-1)^k C_n^k (1-N)^k = \sum_{k=0}^{n} C_n^k (N-1)^k \\
&= (N-1+1)^n = N^n.
\end{aligned}$$

$\square$

Remark 6.C.1. In this case, if we remove the empty set, and compute the expression we get the same result

$$\begin{aligned}\sum_{\alpha,\beta\in\mathscr{A}} \mu_{\alpha\beta}N_\beta &= \sum_{k=1}^{n} C_n^k \sum_{l=1}^{k} C_k^l(-1)^{k-l}N^l = \sum_{k=1}^{n}(-1)^k C_n^k((1-N)^k-1)\\ &= \sum_{k=1}^{n} C_n^k(N-1)^k - \sum_{k=1}^{n} C_n^k(-1)^k\\ &= ((N-1+1)^n-1)-((1-1)^n-1) = N^n.\end{aligned}$$

Let us now delete the maximal face $\alpha = I$, then the poset $\mathscr{A}$ becomes the boundary $\partial\Delta(n-1)$ of the $(n-1)$-simplex. If we include the empty set in $\mathscr{A}$, and compute the dimension of H^0; we obtain

$$\begin{aligned}\sum_{\alpha,\beta\in\mathscr{A}^{\times+}} \mu_{\alpha\beta}N_\beta &= \sum_{k=0}^{n-1} C_n^k \sum_{l=0}^{k} C_k^l(-1)^{k-l}N^l\\ &= \sum_{k=0}^{n-1}(-1)^k C_n^k(1-N)^k = \sum_{k=0}^{n-1} C_n^k(N-1)^k\\ &= (N-1+1)^n-(N-1)^n\\ &= N^n-(N-1)^n.\end{aligned}$$

Remark 6.C.2. Now the expression $\sum_{\alpha,\beta\in\mathscr{A}} \mu_{\alpha\beta}N_\beta$ is not the same if we exclude $\emptyset$, because in this case, we have

$$\begin{aligned}\sum_{\alpha,\beta\in\mathscr{A}^{\times}} \mu_{\alpha\beta}N_\beta &= \sum_{k=1}^{n-1} C_n^k \sum_{l=1}^{k} C_k^l(-1)^{k-l}N^l\\ &= \sum_{k=1}^{n-1}(-1)^k C_n^k((1-N)^k-1) = \sum_{k=1}^{n-1} C_n^k(N-1)^k - \sum_{k=1}^{n-1} C_n^k(-1)^k\\ &= ((N-1+1)^n-1-(N-1)^n)-((1-1)^n-1-(-1)^n)\\ &= N^n-(N-1)^n+(-1)^n.\end{aligned}$$

We will see just below why there is a difference for the boundary $\partial\Delta(n-1)$ and not for the simplex $\Delta(n-1)$!

If $\mathscr{A}$ satisfies the weak intersection property, then

$$\forall\alpha\in\mathscr{A},\quad \overline{F}_\alpha = \bigoplus_{\beta\subseteq\alpha}\overline{T}_\beta. \tag{6.C.7}$$

Proposition 6.C.2. *If the poset $\mathscr{A}$ is finite and satisfies the weak intersection property, and if the $\{E_i\}_{i\in I}$ are finite sets,*

$$\dim_{\mathbb{K}} H^0(\mathscr{U}_{\mathscr{A}}; \overline{F}) = \sum_{\alpha,\beta\in\mathscr{A}} \mu_{\alpha\beta}(N_\beta - 1). \tag{6.C.8}$$

Proof. We apply Proposition 6.5.4 as we applied Proposition 6.5.3 to prove Proposition 6.C.1. □

Definition 6.C.1. Let $\mathscr{A}$ be a finite poset, the *Euler characteristic* of $\mathscr{A}$ is defined by

$$\chi(\mathscr{A}) = \sum_{\alpha,\beta\in\mathscr{A}} \mu_{\alpha\beta}. \tag{6.C.9}$$

In fact, the Euler characteristic was defined by Rota [25], when $\mathscr{A}$ contains a maximal element I and a minimal element $\emptyset$, by the formula

$$E(\mathscr{A}) = 1 + \mu_{I,\emptyset}. \tag{6.C.10}$$

But take any finite poset $\mathscr{A}$, and add formally to $\mathscr{A}$ a maximal element 1 and a minimal element 0, obtaining a poset $\mathscr{A}^+$. Then, for any $\alpha \in \mathscr{A}$,

$$0 = \mu(\alpha, 0) + \sum_{\beta\in\mathscr{A},\beta\subseteq\alpha} \mu(\alpha, \beta), \tag{6.C.11}$$

and

$$0 = \mu(1, 0) + \mu(0, 0) + \sum_{\alpha\in\mathscr{A}} \mu(\alpha, 0). \tag{6.C.12}$$

Consequently

$$\chi(\mathscr{A}) = \mu(1, 0) + \mu(0, 0) = E(\mathscr{A}^+). \tag{6.C.13}$$

Therefore the two definitions accord. See also the categorical extension of these ideas by Tom Leinster [75].

The *Hall formula* (cf. [25]), tells that

$$E(\mathscr{A}) = r_0 - r_1 + r_2 - ...; \tag{6.C.14}$$

where each r_k is the number of non degenerate chains of length k in $\mathscr{A}$. This number is the Euler characteristic of the nerve $N(\mathscr{A})$ of the category $\mathscr{A}$, therefore $\chi(\mathscr{A})$ coincides with the Euler characteristics of $N(\mathscr{A})$. But we have seen in Section 6.6 that the Čech cohomology of the (lower) Hausdorff space $\mathscr{A}$ with coefficients in $\mathbb{Z}$, is isomorphic to the simplicial cohomology of $N(\mathscr{A})$. Then $\chi(\mathscr{A})$ also coincides with the Euler-Čech characteristic of the (lower) Hausdorff space $\mathscr{A}$. By duality of the Möbius function, this is also true for the upper topology.

From the inclusion-exclusion formula, it is easy to show that for the poset of a simplicial complex, the number $\chi(\mathscr{A})$ is the alternate sum of the numbers of faces of each dimension:

$$\chi(\mathscr{A}) = a_0 - a_1 + ... \tag{6.C.15}$$

as in the original definition by Euler.

Now consider $\mathscr{A}$ (finite) as a topological subspace $\mathscr{A}_t$ of the simplex $\mathscr{P}(I)$; its closure $\overline{\mathscr{A}}$ is a simplicial complex. Moreover, if $\mathscr{A}$ satisfies the weak intersection property, the inclusion of $\mathscr{A}_t$ in $\overline{\mathscr{A}}$ is an equivalence of homotopy; therefore, in this case, $\chi(\mathscr{A})$ is also the usual Euler characteristic of the metric space $\mathscr{A}$.

Consequently, Proposition 6.C.2 can be rephrased by the following formula

$$\dim_{\mathbb{K}} H^0(\mathscr{U}_{\mathscr{A}}; \overline{F}) + \chi(\mathscr{A}) = \sum_{\alpha,\beta\in\mathscr{A}} \mu_{\alpha\beta} N_\beta. \tag{6.C.16}$$

Applying Theorem 6.5.1, we get the following result:

Theorem 6.5.4. If the poset $\mathscr{A}$ is finite and satisfies the weak intersection property, and if the $\{E_i\}_{i\in I}$ are finite sets, then

$$\chi(\mathscr{A}; F) = \sum_{k=0}^{\infty} (-1)^k \dim_{\mathbb{K}} H^k(\mathscr{U}_{\mathscr{A}}; F) = \sum_{\alpha,\beta\in\mathscr{A}} \mu_{\alpha\beta} N_\beta. \tag{6.C.17}$$

Remark that we also have

$$\chi(\mathscr{A}; F) = \dim_{\mathbb{K}} H^0(\mathscr{U}_{\mathscr{A}}; \overline{F}) + \chi(\mathscr{A}). \tag{6.C.18}$$

In the example of $\Delta(n-1)$, we have $\chi(\mathscr{A}) = 1$, and when $\mathscr{A} = \partial\Delta(n-1)$ we have $\chi(\mathscr{A}) = 1 + (-1)^n$, therefore, with all the N_i equals to N, this explains the results obtained in the previous remarks.

The copresheaf $\overline{F}$ corresponds to signed measures of sum 1. One can deduce, for instance, that compatible measures of sum 1 over the poset $\partial\Delta(n-1)$ come from a global measure, that always exists (Marginal Theorem) and depends on $(N-1)^n$ degrees of freedom. However, in general, none of these measures is positive.

Chapter 7

A note on (co-)homology and interaction decomposition

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7.1 Introduction

In this chapter we remark that the decomposable functors and presheaves over a poset $\mathscr{A}$ are respectively projective and injective objects in the category $(k-)\mathbf{Vect}^{\mathscr{A}}$, $\mathbf{Vect}^{\mathscr{A}^{op}}$; this gives an other proof of their acyclicity and completes the results of the previous chapter.

7.2 Decomposable functor and presheaves are respectively projective and injective objects

Let us recall that the right derived functor, $R\lim$, for $\lim : \mathbf{Vect}^{\mathscr{A}} \to \mathbf{Vect}$ exists and similarly the left derived functor, $L\,\mathrm{colim}$, for $\mathrm{colim} : \mathbf{Vect}^{\mathscr{A}} \to \mathbf{Vect}$ ([76]). We shall show that a functor from $(G, F) : \mathscr{A} \to \mathbf{Split}$ that is decomposable is acyclic, where $\mathbf{Split}$ is a subcategory of $\mathbf{Vect} \times \mathbf{Vect}^{op}$ for a reference field $\mathbb{K}$.

More precisely that for any $i \in \mathbb{N}^*$,

$$L^i \operatorname{colim} G = 0 \tag{7.2.1}$$

$$R^i \lim F = 0 \tag{7.2.2}$$

Let us first make explicit what we mean by decomposable presheaves.

Definition 7.2.1 (Decomposable presheaf)**.** A presheaf $F : \mathscr{A} \to \mathbf{Mod}$ is decomposable if and only if there is a collection of functors $(S_a : \mathscr{A} \to \operatorname{core}(\mathbf{Mod}), a \in \mathscr{A})$ such that,

$$F \cong \bigoplus_{a \in \mathscr{A}} S_a 1[a \leq .] \tag{7.2.3}$$

Remark 7.2.1. Let us remark that if a functor $G : \mathscr{A} \to \mathbf{Mod}$ is decomposable with decomposition $(S_a 1[a \leq .], a \in \mathscr{A})$ then one can replace S_a by the constant functor $\tilde{S}_a = \operatorname{colim} S_a$ and still get,

$$G \cong \bigoplus_{a \in \mathscr{A}} \tilde{S}_a 1[a \leq .] \tag{7.2.4}$$

One can also do the same for a decomposable presheaf F. In other words this means an equivalent definition of decomposability, respectively for a functor G and a cofuntor F, is that there is a collection of modules $(S_a, a \in \mathscr{A})$ such that,

$$G \cong \bigoplus_{a \in \mathscr{A}} S_a 1[a \leq] \tag{7.2.5}$$

$$F \cong \bigoplus_{a \in \mathscr{A}} S_a 1[a \leq] \tag{7.2.6}$$

$$\tag{7.2.7}$$

Proposition 7.2.1. *Let $G : \mathscr{A} \to$ **Vect** be a decomposable functor and $F : \mathscr{A} \to$ **Vect** be a decomposable cofuntor, then G is a projective object and F is an injective object.*

Lemma 7.2.1. *Let $\mathscr{A}$ be a poset, let $a \in \mathscr{A}$, let S be a vector space, the functor $S1[a \leq .] : \mathscr{A} \to$ **Vect** is a projective object and the cofunctor $S1[a \leq .] : \mathscr{A}^{op} \to$ **Vect** is an injective object.*

Proof. Let A, B two functors from $\mathscr{A}$ to **Vect**, let S be a vector space, let $\phi : S1[a \leq .] \to A$ be a natural transformation and $p : B \to A$ an epimorphism. Let $\widetilde{\phi_a} : V \to B(a)$ be such that $p_a \widetilde{\phi_a} = \phi_a$; for any $a \leq b$, let $\tilde{\phi}_b = B^a_b \widetilde{\phi_a}$ and for $a \not\leq b$, $\tilde{\phi}_b = 0$. Let $a \leq b$ then,

$$p_b\tilde{\phi}_b = p_b B^a_b \tilde{\phi}_a = A^a_b p_a \tilde{\phi}_a \tag{7.2.8}$$

Therefore,

$$p_b\tilde{\phi}_b = A^a_b \phi_a = \phi_b \tag{7.2.9}$$

If $a \not\leq b$, $0 = p_b 0$ and so or any $a \in \mathscr{A}$, $p_a\tilde{\phi}_a = \phi_a$.

Let us now show that $\tilde{\phi}$ is a natural transformations. Let $b, c \in \mathscr{A}$ such that $a \leq b \leq c$, then, $B^b_c\tilde{\phi}_b = B^b_c B^a_b \tilde{\phi}_a = \tilde{\phi}_c$, if $a \not\leq b$ and $b \leq c$, $0 = B^b_c 0$, and therefore $\tilde{\phi}$ is a natural transformation such that, $p\tilde{\phi} = \phi$.

Following similar steps one shows that the cofunctor $S1[a \leq] : \mathscr{A}^{op} \to \mathbf{Vect}$ is an injective object.

□

Remark 7.2.2. Let us remark that in general when a functor G is decomposable it is not an injectif object and when a cofuntor F is decomposable it is not a projective object.

Proof of Proposition 7.2.1:

Proof. A direct sum of projective objects is a projective object and a direct sum of injective objects is an injective object.

□

7.2.1 Acyclicity

Theorem 7.2.1 (Acyclicity of decomposable functors/cofunctors)**.** *Let* $(G, F) : \mathscr{A} \to$ ***Split***$(\mathbb{K} -$ ***Vect***$)$*, if* (G, F) *is decomposable, then for any* $i \in \mathbb{N}^*$*,*

$$L^i \operatorname{colim} G = 0 \tag{7.2.10}$$

$$R^i \lim F = 0 \tag{7.2.11}$$

Proof. Injective and projective objects are acyclic for respectively right and left derived functors.

□

Chapter 8

Gibbs states in the category of Markov Kernels

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8.1 Introduction

Giry [1] and Lawvere [2], have initiated the study of the categories behind measurable applications and probability kernels which unified several constructions in probability. As an example one can reinterpret the Kolmogorov extension theorem as a statement on the existence of the limit of a presheaf in the category of the probability kernels (see [1]).

Proposition (Kolmogorov's extension theorem)**.** *Let $I = \mathbb{N}$, let $E = \prod_{i\in I} E_i$ be product of finite set, let $E_{[n]} = \prod_{k\leq n} E_k$. Any sequence $(P_n \in \mathbb{P}(E_{[n]}), n \in \mathbb{N}^*)$ of probability measures respectively over $E_{[n]}$ for $n \in \mathbb{N}^*$, such that for any $n \in \mathbb{N}^*$,*

$$\pi^{[n+1]}_{[n],*} P_{n+1} = P_n \tag{8.1.1}$$

has a unique extension into a probability measure $P \in \mathbb{P}(E)$ over E such that,

$$\pi_{[n]_*} P = P_{[n]} \tag{8.1.2}$$

More generally Giry in [1] gives new conditions for the existence of limits in this category, generalizing the Kolmogorov's extension theorem and generalizing the fact that one can reconstruct a probability distribution, $P \in \mathbb{P}(E)$, from a sequence of its conditional expectations $(P(x_{n+1}|x_k, k \leq n), n \in \mathbb{N})$ (see for example Proposition 2.38 [77] for a Markov process). We start by recalling somes basic definitions for these categories.

The tools that we have developed in the previous sections were so because we hope that they can help in characterizing, using algebraic topology, infinite volume Gibbs measures (see Georgii's classical presentation of the subject [24]). We will recall what specifications and Gibbs measures are and show what the respective concept in the category of Markov kernels is. We will relate specifications to functors that take values in **Split** and this will allow us to single out decomposable specifications for which one can characterize the limit and therefore their associated Gibbs measures.

What we want to convey in this chapter is that instead of seing Gibbs measures as true measures on a probability spaces, which most of the time fails because the category of probability kernels does not have enough limits, one should study the associated diagram over a poset and send it in a category with enough algebraic structure, for example the category of vector spaces, but it could also be of Banach spaces or C^*-algebras. Doing so one can define invariants thanks to the cohomology theories introduced in Chapters 6,7.

8.2 Mes, Kern categories

8.2.1 Mes

Definition 8.2.1. The category that has as objects measurable spaces and as morphism measurable maps will be denoted as **Mes**.

Definition 8.2.2. Let $(E, \mathscr{E})$ be a measurable space, let us denote $\mathbb{P}(E)$ the space of probability measure on E with σ-algebra the smallest algebra that makes the evaluations maps measurable, i.e. for any $A \in \mathscr{E}$,

$$ev_A : \mathbb{P}(E) \to [0,1] \tag{8.2.1}$$
$$\mathbb{P} \to \mathbb{P}(A) \tag{8.2.2}$$

is measurable.

8.2.2 Kern

Definition 8.2.3 (Probability Kernel Category)**.** Let $(E, \mathscr{E})$, $(E, \mathscr{E}_1)$ be two measurable spaces, a kernel from $(E, \mathscr{E})$ to $(E, \mathscr{E}_1)$, denoted as $E \to E_1$, is a function $\pi : \mathscr{E}_1 \times E \to [0,1]$ such that:

1. $\pi(.|\omega)$ is a probability measure on $(E_1, \mathscr{E}_1)$ for all $\omega \in E$
2. $\pi(A|.)$ is $\mathscr{E}$-measurable for any $A \in \mathscr{E}_1$.

One can compose probability kernels as follows: let $\pi : E \to E_1$, $\pi_1 : E_1 \to E_2$, for any $A \in \mathscr{E}_2$ and $\omega \in E$,

$$\pi_1 \circ \pi(A|\omega) = \int \pi_1(A|\omega_1)\pi(d\omega_1|\omega) \tag{8.2.3}$$

Proposition 8.2.1. *The collection that has as objects measurable spaces and as morphisms probability kernels with compositions given by Equation 8.2.3 is a category, that we shall call the category of probability kernels and that we denote as* ***Kern****.*

Proof. It is a category and is equivalent to the Kleisli category of **Mes** (see [1]). □

Proposition 8.2.2. ***Mes*** *is a subcategory of* ***Kern****.*

Proof. Any measurable function $f : (E, \mathscr{E}) \to (E_1, \mathscr{E}_1)$ can be seen as the following kernel, for any $A \in \mathscr{E}_1$ and $\omega \in E$,

$$\pi_f(A|\omega) = 1[f(\omega) \in A] \tag{8.2.4}$$

□

Remark 8.2.1. The identity kernel $\mathrm{id}_E : (E, \mathscr{E}) \to (E, \mathscr{E})$ is the kernel associated to the identity map from $(E, \mathscr{E})$ to $(E, \mathscr{E})$.

We will note $\mathbf{Kern}\left((E, \mathscr{E}), (E, \mathscr{E}_1)\right)$ the set of probability kernels from E to E_1.

Proposition 8.2.3. *Let for any $\pi \in$ **Kern**(E, E_1) and any $P \in \mathbb{P}(E)$,*

$$\mathbb{P}(\pi)(P) = \pi P \tag{8.2.5}$$

*Then $\mathbb{P}$ is a functor from **Kern** to **Mes**.*

Proof. For any $A \in \mathscr{E}_1$ and $\pi \in \mathbf{Kern}(E, E_1)$, $ev_A \mathbb{P}(\pi)$ is the limit of measurable application, therefore it is measurable. Functoriality is a consequence of associativity of the composition law for kernels: for any $P \in \mathbb{P}(E)$, $\pi \in \mathbf{Kern}(E, E_1)$, $\pi_1 \in \mathbf{Kern}(E_1, E_2)$,

$$\pi_1(\pi P) = (\pi_1 \pi) P \tag{8.2.6}$$

□

Notation 8.2.1. For π a probability kernel from E to E_1 and $P \in \mathbb{P}(E)$ we will denote $\mathbb{P}(\pi)P$ simply as $\pi(P)$.

Let us denote $*$ the measurable set constituted of one point. For any measurable space E,

$$\mathbf{Kern}(*, E) \cong \mathbb{P}(E) \tag{8.2.7}$$

$\mathbf{Kern}(\star, .)$ is an embedding, i.e. is injective on objects and faithfull, of **Kern** into **Mes**. Let $\mathscr{A}$ be a poset and $G : \mathscr{A} \to \mathbf{Kern}$ be a functor, we shall call the sections of this functor the compatible measures:

$$\mathbf{Kern}^{\mathscr{A}}(*, G) := \{(P_a \in \mathbb{P}(G(a)), a \in \mathscr{A}) | \quad \forall b \leq a, G_a^b P_b = P_a\} \tag{8.2.8}$$

However these sections can't necessarily be represented as measures over a fixed measurable space, in otherwords limits are not necessarily representable; finding some condition on functors to be representable is also the subject [1].

Remark 8.2.2. One can justify the definition, Equation 8.2.8, of sections of a functor from a poset $\mathscr{A}$ to **Kern** by noting that,

$$\mathbf{Kern}^{\mathscr{A}}(i_* *, G) \cong \lim_{a \in \mathscr{A}} \mathbf{Kern}(*, G(a)) = \lim_a \mathbb{P}(G(a)) \tag{8.2.9}$$

where the equality is in **Set** and in the last equality the explicit mention of the forgethfull functor from **Mes** to **Set** is ommited.

Remark 8.2.3. Let $E = \prod_{k\in\mathbb{N}} E_n$, let us recall that for any $a, b \in \mathscr{P}_f(\mathbb{N})$ such that $b \subseteq a$, p^a_b denote the projection from E_b to E_a but also the marginalization $\mathbb{P}(p^a_b) : \mathbb{P}(E_a) \to \mathbb{P}(E_b)$; these marginalization define a cofunctor that we will denote as p. The Kolmogorov extension theorem is equivalent to saying that,

$$\lim_{a\in\mathscr{P}_f(\mathbb{N})} \mathbb{P}(p(a)) \simeq \mathbb{P}(E) \tag{8.2.10}$$

8.3 Specifications, generalized Gibbs states

In Chapter 6 we studied the linearized marginal problem and showed how decomposability generalizes Kellerer's result; an analogous problem to the maginal problem is finding the Gibbs measures of a specification. We will recall what specifications and Gibbs measures are as defined in statistical physics [24]; we extend this definition to some functors from a poset to **Kern** and show how decomposability helps to characterize Gibbs measures (Theorem 8.5.1).

8.3.1 L^∞ functor

We saw in the previous section that in **Kern** limits are not necessarily representable, to get around this problem we want to send each measurable space to its space of bounded measurable functions; doing so will enable us to apply the previous equivalence results on decomposability to **Kern** and characterize the sections of decomposable functors.

Definition 8.3.1 (Category of Banach space)**.** We shall call the category of Banach spaces the category that has as objects Banach spaces and as morphisms linear contractions, i.e linear maps $a : B \to B_1$ such that $\|a\| \leq 1$.

Definition 8.3.2 (L^∞ cofunctor)**.** Let E be measurable space, $L^\infty(E)$ is the set of bounded measurable functions from X to $\mathbb{R}$. Let $\pi \in \mathbf{Kern}(E, E_1)$,

$$\begin{array}{rcccc} L^\infty(\pi) & : & L^\infty(E_1) & \to & L^\infty(E) \\ & & f & \mapsto & \pi(f) = \int f(\omega)\pi(d\omega|.) \end{array} \tag{8.3.1}$$

$L^\infty : \mathbf{Kern} \to \mathbf{Ban}$ is a functor.

Remark 8.3.1. L^∞ is an embedding of **Kern** into **Ban**. **Ban** is in particular a subcategory of **Vect**, let us note the forgetful functor as i : **Ban** $\to$ **Vect**. The category of vector spaces **Vect** is complete and co-complete (the diagram can be non finite), therefore any functor $G : \mathscr{A} \to$ **Kern**, from any poset $\mathscr{A}$, has a limit, $\lim iL^\infty G$, in **Vect**.

Let us remark that for any measurable space E, $\mathbb{P}(E)$ can be identified to a subset of the topological dual of $L^\infty(E)$ with the operator norm, dual that we denote as $L^\infty_*(E)$. $_*$: **Vect** $\to$ **Vect** sends a vector space to its dual and is a coendofunctor. For any poset $\mathscr{A}$ and any functor $G : \mathscr{A} \to$ **Kern**, $\mathbb{P}(G)$ is a subfunctor (in **Set**) of $L^\infty_*(G)$, in otherwords for any $a, b \in \mathscr{A}$, such that $b \leq a$,

$$L^\infty_*(G^b_a)\mathbb{P}(E_b) \subseteq \mathbb{P}(E_a) \tag{8.3.2}$$

8.3.2 Specifications and Gibbs measures

Let us first recall what specifications are in Statistical Physics [24].

Definition 8.3.3 (Proper Kernel)**.** Let $\mathscr{E} \subseteq \mathscr{E}_1$ be two σ-algebras of a set E, a kernel $\pi \in$ **Kern** $((E, \mathscr{E}_1), (E, \mathscr{E}))$ is proper if and only if, for any $A \in \mathscr{E}$, any $B \in \mathscr{E}_1$ and any $\omega \in E$,

$$\pi(A \cap B|.) = \pi(A|.)1_B \tag{8.3.3}$$

Remark 8.3.2. Let $\mathscr{E}_1 \subseteq \mathscr{E}$ be two σ-algebras of E. Let us denote $i : (E, \mathscr{E}) \to (E, \mathscr{E}_1)$ the identity map from $(E, \mathscr{E})$ to $(E, \mathscr{E}_1)$, note that it is not the identity morphism in **Kern** but rather an inclusion as it appears clearly when composing by L^∞. A kernel $\pi : (E, \mathscr{E}_1) \to (E, \mathscr{E})$ is proper if and only if,

$$i \circ \pi = \mathrm{id} \tag{8.3.4}$$

Equation 8.3.4 is a restatement of the standard result that a kernel π is proper if and only if for any $B \in \mathscr{E}_1$,

$$\pi(B|.) = 1_B \tag{8.3.5}$$

For more details one can refer to 1.1 [24].

Let us now give the usual definition of specification. Let I be any set, let $(E_i, i \in I)$ be a collection of measurable spaces and E is still the product $\prod_{i \in I} E_i$ with the product σ-algebra; let us recall that we denote $E_a = \prod_{i \in a} E_i$ and $\pi_a : E \to E_a$ the associated projection. Let $\mathscr{E}$ be the whole σ-algebra on E and for any $a \in \mathscr{P}(I)$ let $\mathscr{E}_a$ be the smallest sigma algebra that makes π_a measurable.

Notation 8.3.1. Let us recall that we denote $\mathscr{P}_f(I)$ the set of finite parts of I.

Definition 8.3.4 (Standard specification)**.** A specification with parameter set I and state spaces $(E, \mathscr{E})$ is a collection $(\gamma_a, a \in \mathscr{P}_f(I))$ such that for any $a \in \mathscr{P}_f(I)$, $\gamma_a \in \mathbf{Kern}\,((E, \mathscr{E}_{\overline{a}}), (E, \mathscr{E}))$ and which satisfies that for any $b \subseteq a$, i.e $\overline{a} \subseteq \overline{b}$, any $A \in \mathscr{E}$,

$$\gamma_b i^a \gamma_a(A|.) = \gamma_a[\gamma_b(A|.)] = \gamma_a(A|.) \tag{8.3.6}$$

Remark 8.3.3. The previous definition corresponds to Definition (1.23) [24] when E is a product over the same measurable space X over a countably infinite set.

Definition 8.3.5 (Gibbs measures)**.** Let γ be a specification with state space E, the set of probability measures,

$$\mathscr{G}(\gamma) = \{\mu \in \mathbb{P}(E) \quad : \quad \mathbb{E}_\mu(A|\mathscr{E}_{\overline{a}}) = \gamma_a(A|.)\ \mu \text{ a.s.}\} \tag{8.3.7}$$

is called the set of Gibbs measures of γ.

In the standard definition of specification (Definition 8.3.4) the markov kernels encodes border conditions for an experiments that takes place on a finite number of random variables ($a \in \mathscr{P}_f(I)$); in order to insure that changing the border conditions "at infinity" does not influence the experiment one asks that the specification is quasi-local which is a regularity condition we will not detail here.

8.3.3 Specifications and Gibbs states in Kern

Let us now give an some properties for Gibbs measures that will motivate the extension we propose in this chapter. Let I be any set and $E = \prod_{i \in I} E$, for any $a, b \in \mathscr{P}(I)$ we will note the projection p^a_b as i^a_b. What we have in mind is the fact that $\mathscr{E}_b$ measurable function are immerged in $\mathscr{E}_a$ measurable functions. From this collection of morphism of $\mathbf{Mes} \subseteq \mathbf{Kern}$, one can define the following functor G. For any $a, b \in \mathscr{P}_f(I)$ such that $b \leq a$ let,

$$G(a) = E_{\overline{a}} \tag{8.3.8}$$

$$G^b_a = i^{\overline{b}}_{\overline{a}} \tag{8.3.9}$$

Remark 8.3.4. $\emptyset$ is initial in $\mathscr{A} = \mathscr{P}_f(I)$ and for any $a, b \in \mathscr{A}$ such that $b \subseteq a$,

$$G^b_a G^\emptyset_b = G^\emptyset_a \tag{8.3.10}$$

This remark will be significant for the further development.

Proposition 8.3.1. *Let $(\gamma^a, a \in \mathscr{P}_f(I))$ be a specification on the state space E, for any $a, b \in \mathscr{P}_f(I)$ such that $b \subseteq a$, let F^a_b be the unique kernel in $\boldsymbol{Kern}(E_{\overline{a}}, E_{\overline{b}})$ such that the following diagram commutes,*

$$\begin{tikzcd} E_{\overline{a}} \arrow[r, "\gamma^a"] \arrow[d, "F^a_b"'] & E \\ E_{\overline{b}} \arrow[ur, "\gamma^b"'] & \end{tikzcd} \tag{8.3.11}$$

then F is a cofunctor of $\boldsymbol{Kern}^{\mathscr{P}_f(I)}$.

Proof. Let $b \subseteq a$, let F^a_b satisfy Equation 8.3.11, then,

$$i_{\overline{b}}\gamma^b F^a_b = i_{\overline{b}}\gamma^a \tag{8.3.12}$$

therefore,

$$F^a_b = i_{\overline{b}}\gamma^a \tag{8.3.13}$$

For any $a, b \in \mathscr{A}$ such that $b \subseteq a$ let $F^a_b = i_{\overline{b}}\gamma^a$ then for $c \subseteq b \subseteq a$,

$$F^b_c F^a_b = i_{\overline{c}}\gamma^b i_{\overline{b}}\gamma^a \tag{8.3.14}$$

Equation 8.3.6 can be rewritten as, for $b \subseteq a$,

$$\gamma^b i_{\overline{b}}\gamma^a = \gamma^a \tag{8.3.15}$$

therefore,

$$F^b_c F^a_b = F^a_c \tag{8.3.16}$$

□

The Gibbs measures are by definitions sections of the cocone $(E_{\overline{a}} \to E, a \in \mathscr{P}_f(I))$, for any $P \in \mathscr{G}(\gamma)$, let $P_a = i_{\overline{a}}P$ then, for any $a, b \in \mathscr{P}_f(I)$ such that $b \subseteq a$,

$$\gamma^b(F^a_b P_a) = P \tag{8.3.17}$$

therefore, $F^a_b P_a = i_{\overline{b}}P$ and

$$F^a_b P_a = P_b \tag{8.3.18}$$

We just showed that Gibbs measures of γ are sections of F. The aim of the previous discussion is to emphasize the fact that we don't need to represent the

compatible measures $\mathbf{Kern}^{\mathscr{A}}(*, F)$ as measures over E; in fact in practice there are stricly less Gibbs measures than sections of F. Our aim in the rest of this chapter is to give a definition of specifications that does not refer to an ambiant probability space E.

One could also think of defining specifications on any subset of I, in otherwords to consider $\gamma^a : E_a \to E$ for any $a \in \mathscr{P}(I)$ with compatibility conditions. The set of sections of the associated functor, F_1, over $\mathscr{P}(I)$ are strictly smaller than the set of sections over the ultrafilter of cofinite sets: the Gibbs measures $P \in \mathscr{G}(\gamma)$ would be completely characterized by $P = \gamma^{\emptyset}$, however the section of $F_1|\mathscr{P}_f(I)^c$ can contain more that one probability measure.

We therefore propose the following definition of specification in **Kern**.

Definition 8.3.6 (Specification)**.** Let $\mathscr{A}$ be a poset, a specification is a couple (G, F) of cofunctor-functor where $G : \mathscr{A}^{op} \to \mathbf{Mes}$ and $F : \mathscr{A} \to \mathbf{Kern}$ such that for any $a, b \in \mathscr{A}$ with $b \leq a$,

$$G^a_b F^b_a = \mathrm{id} \tag{8.3.19}$$

Definition 8.3.7 (Gibbs measures for specifications)**.** Let $\gamma = (G, F)$ be a specification over $\mathscr{A}$, we shall call the Gibbs measures of γ the sections of F,

$$\mathscr{G}(\gamma) = \mathbf{Kern}^{\mathscr{A}}(*, F) \tag{8.3.20}$$

Proposition 8.3.2. *Let $\gamma = (G, F)$ be a specification over $\mathscr{A}$,*

$$\mathscr{G}(\gamma) \subseteq \boldsymbol{Kern}^{\mathscr{A}^{op}}(*, G) \tag{8.3.21}$$

Proof. For any $b, a \in \mathscr{A}$ such that $b \leq a$, and $P \in \mathscr{G}(\gamma)$,

$$F^b_a(P_b) = P_a \tag{8.3.22}$$

And therefore,

$$P_b = G^a_b F^b_a(P_b) = G^a_b P_a \tag{8.3.23}$$

□

Proposition 8.3.3. *Let $\gamma = (G, F)$ be a specification over $\mathscr{A}$, let,*

$$\epsilon : \lim_{a \in \mathscr{A}} \boldsymbol{Kern}(*, F(a)) \to \boldsymbol{Kern}^{\mathscr{A}}(*, F)$$

be the co-unit map. For any $a, b \in \mathscr{A}$ such that $b \leq a$, $G^a_b|^{\mathrm{im}\,\epsilon_b}_{\mathrm{im}\,\epsilon_a}$ and $F^b_a|^{\mathrm{im}\,\epsilon_a}_{\mathrm{im}\,\epsilon_b}$ are isomorphisms.

Proof. Let $a, b \in \mathscr{A}$ suh that $b \leq a$; let $\mu \in \mathbf{Kern}(*, F(b))$ be such that there is $\nu \in \mathbf{Kern}^{\mathscr{A}}(*, F)$ such that $\mu = \nu_b$, then

$$F_b^a G_a^b \mu = F_b^a G_a^b \nu_b = F_b^a \nu_a = \nu_b = \mu \tag{8.3.24}$$

and so,

$$F_b^a G_a^b|_{\operatorname{im} \epsilon_b} = \mathrm{id} \tag{8.3.25}$$

By definition the following always holds:

$$G_a^b F_b^a = \mathrm{id} \tag{8.3.26}$$

□

8.4 Category of specifications and decomposability

Let us recall some properties and definitions of Chapter 4 that we shall adapt to **Kern**.

Proposition 8.4.1 (Category of $\mathscr{A}$-Specifications)**.** *The collection that has as objects $\mathscr{A}$-specification and as natural transformations couples of natural transformations $(\phi, \psi) : \gamma_1 \to \gamma_2$ from $\gamma_1 = (G_1, F_1)$ to $\gamma_2 = (G_2, F_2)$, with $(\phi_a, a \in \mathscr{A}) \in \boldsymbol{Mes}^{\mathscr{A}^{op}}(G_2, G_1)$ and $(\psi_a, a \in \mathscr{A}) \in \boldsymbol{Kern}^{\mathscr{A}}(F_1, F_2)$ such that, for any $a \in \mathscr{A}$,*

$$\phi_a \psi_a = \mathrm{id} \tag{8.4.1}$$

is a category that we shall denote as $\boldsymbol{Sp}(\mathscr{A})$.

Proof. $\mathbf{Sp}(\mathscr{A})$ is the category of functors from $\mathscr{A}$ to the subcategory of $\mathbf{Mes}^{op} \times \mathbf{Kern}$ that has as objects any (E, E), where E is a measurable space, and as morphisms $(\phi, \psi) \in \mathbf{Mes}^{op}(E_1, E) \times \mathbf{Kern}(E, E_1)$ such that,

$$\phi\psi = \mathrm{id} \tag{8.4.2}$$

□

Example 8.4.1. The simplest example of category of specifications is the categroy over a point $\mathbf{Sp}(*)$; a morphisms is a couple of a random variable $f \in \mathbf{Mes}(X, Y)$ and a kernel $\pi \in \mathbf{Kern}(Y, X)$ such that for any $A \in \mathscr{E}_Y$ and $\omega \in Y$,

$$\pi(f^{-1}(A)|\omega) = 1[\omega \in f^{-1}(A)] \tag{8.4.3}$$

Let μ be a positive measures of X with bounded total value, i.e. $\mu(X) < +\infty$, let $g \in L^\infty(X)$, the conditional expectation with respect to μ and f of g is the following Radon-Nickodym derivative,

$$E_\mu[g|f] = \frac{df_*(g\mu)}{df_*\mu} \tag{8.4.4}$$

Take any probability measure $P \in \mathscr{G}(f, \pi)$, Equation 8.4.3 implies that for any $\mathscr{A} \in \mathscr{E}_X$ and $\omega \in Y$,

$$\pi(A|\omega) = E_P[A|f](\omega) \quad f_*P \text{ a .s.} \tag{8.4.5}$$

In this chapter **Split** is a subcategry of $\mathbb{R} - \mathbf{Vect} \times \mathbb{R} - \mathbf{Vect}^{op}$, in other word we consider only $\mathbb{R}-$vector spaces.

Proposition 8.4.2. *Let $\gamma = (G, F)$ be a specification over $\mathscr{A}$, L^∞ extends into a functor from $\boldsymbol{Sp}(\mathscr{A})$ to $\boldsymbol{Split}^{\mathscr{A}}$ by composition on the left on functors and cofunctors, i.e. $L^\infty\gamma = (L^\infty G, L^\infty F)$.*

Proof. $L^\infty : \mathbf{Kern} \to \mathbf{Vect}$ is a cofunctor therefore $L^\infty G$ is a functor and $L^\infty F$ is a cofuntor, furthermore, for $a, b \in \mathscr{A}$ such that $b \leq a$,

$$L^\infty(G^a_b F^b_a) = L^\infty(F^b_a) L^\infty(G^a_b) \tag{8.4.6}$$

□

Definition 8.4.1 (Decomposability)**.** We shall say that a specification γ is decomposable if $L^\infty\gamma$ is decomposable.

Definition 8.4.2 (Conditional product)**.** Let $\mathscr{A}$ be a poset, we shall say that its has conditional products if and only if for any $a \in \mathscr{A}$, $\hat{a}$ has products; for $b \leq a$, $c \leq a$, we shall denote the conditional product as, $b \cap_a c$.

Theorem 8.4.1 (Intersection property and decomposability)**.** *Let $\mathscr{A} \in \hat{\mathscr{P}}_f$ be a poset with conditional products, let γ be a specification such that for any $a, b, c \in \mathscr{A}$, such that $a \leq c$ and $b \leq c$,*

$$F^b_a G^a_b F^c_a G^a_c = F^{b \cap_a c}_a G^a_{b \cap_a c} \tag{8.4.7}$$

Then γ is decomposable.

Proof. $(L^\infty G, L^\infty F)$ satisfies the Intersection property (Definition 4.7 Chapter 4) therefore it is decomposable (Corollary 4.4 Chapter 4).

□

8.5 Gibbs measures of decomposable specifications

To conclude this chapter let us characterize Gibbs states of decomposable specifications.

For any poset $\mathscr{A}$, symmetrizing the order defines the following equivalence relation,

$$\forall a, b \in \mathscr{A},\ a \sim b \iff a \leq b \text{ or } b \leq a \tag{8.5.1}$$

The equivalence classes of this equivalence relation are the connected components of $\mathscr{A}$ that we will denote as $\mathscr{C}(\mathscr{A})$. To each element of $a \in \mathscr{A}$ one can associate its connected component $\mathscr{C}(a)$. If each connected components has a minimum element, in other words if for any $C \in \mathscr{C}(\mathscr{A})$, and any $b \in C$, there is $c \in C$ such that, $c \leq b$, then we shall denote, $\mathscr{C}_*(\mathscr{A})$ as the collection of these minimuù elements; if not $\mathscr{C}_*(\mathscr{A}) = \emptyset$.

Theorem 8.5.1. *Let γ be a specification of $\boldsymbol{Sp}(\mathscr{A})$. When γ is decomposable and if at least one of the connected component of $\mathscr{A}$ does not have a minimum element, i.e. when,*

$$\mathscr{C}_*(\mathscr{A}) = \emptyset \tag{8.5.2}$$

then,

$$\mathscr{G}(\gamma) = \emptyset \tag{8.5.3}$$

if not,

$$\mathscr{G}(\gamma) = \prod_{a \in \mathscr{C}_*(\mathscr{A})} \mathbb{P}(\gamma(a)) \tag{8.5.4}$$

Proof. Let us denote $L^\infty G$ as i and, $L^\infty F$ as π; (i, π) is decomposable, let $(S_a, a \in \mathscr{A})$ be its decomposition. For any $a, b \in \mathscr{A}$ such that $b \leq a$ and $\mu \in \mathscr{G}(\gamma)$, let us denote $L^\infty \mu$ as ν.

$$\nu_b \pi_b^a (\sum_{c \leq a} S_c(a)(v)) = \nu_a (\sum_{c \leq a} S_c(a)(v)) \tag{8.5.5}$$

therefore,

$$\nu_b (\sum_{c \leq b} S_{cb}^{\ a}(v)) = \nu_a i_a^b (\sum_{c \leq b} S_{cb}^{\ a}(v)) = \nu_a (\sum_{c \leq a} S_c(a)(v) \tag{8.5.6}$$

and so,

$$\mu_a(\sum_{\substack{c\leq a\\ c\not\leq b}} S_c(a)) = 0 \tag{8.5.7}$$

Therefore for any $a \notin \mathscr{C}_*(\mathscr{A})$, $\nu_a|_{S_a(a)} = 0$. However we note that as $\nu \in \lim \hom_{\mathbf{Vect}}(i, \mathbb{R})$, and as,

$$\operatorname{colim} i \simeq \bigoplus_{a\in\mathscr{A}} S_a(a) \tag{8.5.8}$$

ν is uniquely determined by $(\nu_a|_{S_a(a)}, a \in \mathscr{A})$; if there is a connected component $C \in \mathscr{C}(\mathscr{A})$ that does not have a minimal element, for any $a \in C$,

$$\nu_a|_{S_a} = 0 \tag{8.5.9}$$

Therefore for any $a \in C$, $\nu_a = 0$; this is contradictory with the fact that $\mu_a \in \mathbb{P}(\gamma(a))$ and so,

$$\mathscr{G}(\gamma) = \emptyset \tag{8.5.10}$$

When $\mathscr{C}_*(\mathscr{A})$ is non empty for any functor, H, from $\mathscr{A}$ to **Set**,

$$\lim H \simeq \prod_{a\in\mathscr{C}_*(\mathscr{A})} H(a) \tag{8.5.11}$$

therefore,

$$\mathscr{G}(\gamma) = \prod_{a\in\mathscr{C}_*(\mathscr{A})} \mathbb{P}(\gamma(a)) \tag{8.5.12}$$

□

Let us conclude this chapter remarking that decomposable specifications are the simplest general specifications that one can consider but that must of the time they do not have Gibbs measure; this would lead us to believe that Gibbs measures are not really informative for general specifications. On the other hand for a $\mathscr{A}$-specification decomposability is strongly correlated to finitness conditions on $\mathscr{A}$, typically that for any $a \in \mathscr{A}$, $|\hat{a}|$ is finite; this property does not hold for $\mathscr{P}_f(I)^{op}$ and even worse $\mathscr{P}_f(I)^{op}$ is not well founded. We are aware that this chapter raises more questions than it answers and we hope that pursuing in this direction will give new insight on the thermodynamic limit, specially that the approach followed in this chapter will enable us to use tools from agebraic geometry to study phase transition.

Chapter 9

Regionalized optimization

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9.1 Introduction

Yedidia, Freeman, Weiss have shown in their reference article, *Constructing Free Energy Approximations and Generalized Belief Propagation Algorithms* [16], that there is a variational principle underlying the General Belief Propagation, by introducing a region-based free energy approximation of the MaxEnt free energy, that we will call the Generalized Bethe free energy. They sketched a proof that fixed points of the General Belief Propagation are critical points of this free energy, this proof was completed in the thesis of Peltre [26] who also replaced this algorithm in a new topological setting.

Definition 9.1.1 (MaxEnt)**.** Let E be a finite set, let $H \in \mathbb{R}^E$ and $U \in \mathbb{R}$, maximizing the entropy under the contraint of mean energy U is the following optimization problem,

$$\sup_{\substack{p\in\mathbb{P}(E)\\ \mathbb{E}_p[H]=U}} -\sum_{x\in E} p(x)\ln p(x) \tag{9.1.1}$$

where $S(p) = -\sum_{x\in E} p(x)\ln p(x)$ is the entropy of p.

MaxEnt can be re-expressed using Lagrange multipliers,

$$\inf_{p\in\mathbb{P}(E)} \mathbb{E}_p[\beta H] - S(p) \tag{9.1.2}$$

$\mathbb{E}_p[H] - \frac{1}{\beta}S(p)$ is called the Free energy .

Let I and $E = \prod_{i\in I} E_i$ be two finite sets. Let $\mathscr{A} \subseteq \mathscr{P}(I)$ be a poset and let $\mathbb{P}_{>0}(E_a)$ be the set of strictly positive probability laws on E_a. Let us denote F the maginalization cofunctor restricted to $\mathscr{A}$, i.e. for $a, b \in \mathscr{A}$ such that $b \leq a$ and $P \in \mathbb{P}_{>0}(E_a)$, $F^a_b(P) = \pi^a_{b\,*}P$.

Definition 9.1.2 (Generalized Bethe Free Energy)**.** Let I and $E = \prod_{i\in I} E_i$ be finite sets, let $(H_a \in \mathbb{R}^{E_a}, a \in \mathscr{A})$ be a collection of Hamiltonians and let for any $p \in \lim F$,

$$F_{\text{Bethe}}(p) = \sum_{a\in\mathscr{A}} \sum_{b\geq a} \mu(b,a)\left(\mathbb{E}_{p_a}[H_a] - S_a(p_a)\right) \tag{9.1.3}$$

The Generalized Bethe Free Energy is a free energy associated to a region, encoded by the poset $\mathscr{A}$; it has been introduced as an approximation for the free energy of probability laws that factor according to $\mathscr{A}$, i.e. for $P \in \mathscr{F}_{\mathscr{A}}$ and this approximation is exact over directed acyclic graphs (see Section 9.3.2). We like to see it as a 'global' cost function, over $\lim F$, reconstructed from the local ones $(\mathbb{E}_{p_a}[H_a] - S_a(p_a))$. In this chapter we extend these ideas in two directions: firstly for other functionals than entropy, which will lead us to the first application we will present which is what we call a local PCA, and, secondly, to allow more general compatibility conditions than the marginalization compatibility condition. For this second direction we have an application in mind which is to take the limit over diagrams in **Kern** as we shall briefly present in Section 9.3.2; this offers new possibilities for the use of the General Belief Propagations.

Peltre in his PhD thesis gave a meaningful characterization of the critical points of the Generalized Bethe Free Energy (Theorem 4.22 [26]) in order to complete the proof of the equivalence between critical point of the Generalized Bethe Free energy and the fix points of the General Belief Propagation of [16].

Theorem. *Let I and $E = \prod_{i\in I} E_i$ be finite sets, let $(H_a \in \mathbb{R}^{E_a}, a \in \mathscr{A})$ be a collection of Hamiltonians. The critical points of F_{Bethe}, $(p^*_a, a \in \mathscr{A})$, are such*

that there is a collection $(m_{a\to b} \in \bigoplus_{\substack{a,b:\\ b\leq a}} \mathbb{R}^{E_b})$ *and a collection* $(\lambda_a \in \mathbb{R}, a \in \mathscr{A})$ *such that for any* $a \in \mathscr{A}$,

$$-\ln(p_a^*) = H_a + \lambda_a + \sum_{b\leq a}\left(\sum_{c\leq b} j_{ac} m_{b\to c} - \sum_{c\geq b} j_{ab} m_{c\to b}\right) \tag{9.1.4}$$

where $j_{a,b}$ *is the inclusion of* $\mathbb{R}^{E_b}$ *into* $\mathbb{R}^{E_a}$.

Let us now present the extension we propose in this chapter. Let F be a cofunctor from a finite poset $\mathscr{A}$ to $\mathbf{Vect}_f$, the category of finite vector spaces, and let $(f_a : F(a) \to \mathbb{R}, a \in \mathscr{A})$ be a collection of local cost functions, we propose the followin global cost function,

$$\begin{array}{cccc} f & : & \lim F & \to & \mathbb{R} \\ & & (x_a, a \in \mathscr{A}) & \mapsto & \sum_{a\in\mathscr{A}} \sum_{b\leq a} \mu(a,b) f_b(x_b) \end{array} \tag{9.1.5}$$

and we will motivate our choice in Section 9.2.1. We will call this function the global cost function with respect to F and $(f_a, a \in \mathscr{A})$; we will call the associated optimization problem,

$$\sup_{x\in\lim F} f(x) \tag{9.1.6}$$

the regionalized optimization with respect to F and $(f_a, a \in \mathscr{A})$. One can think of f as the less redundant reconstruction of a global cost function from the local cost function $(f_a, a \in \mathscr{A})$.

Notation 9.1.1. Let G be a functor from $\mathscr{A}$ to $R-\mathbf{Mod}$, let us denote the cofunctor $\hom_{R-\mathbf{Mod}}(G, R)$ as G^*; similarly for any morphism of modules, $\phi : M \to M_1$, we shall denote $\hom(\phi, R)$ as ϕ^*.

When $(f_a : F(a) \to \mathbb{R}, a \in \mathscr{A})$ is a collection of differentiable applications, this framework gives the right algebraic setting so that one can generalize Proposition 9.1 and characterize the critical points of the global cost function.

Theorem 9.2.1. An element $x \in \lim F$ is a critical point of the global cost function f with respect to a presheaf F and a collection of differentiable applications $(f_a : F(a) \to \mathbb{R}, a \in \mathscr{A})$ if and only if there is $(l_{a\to b} \in \bigoplus_{\substack{a,b:\\ b\leq a}} F(b)^*)$ such that for any $a \in \mathscr{A}$,

$$d_x f(a) = \sum_{b\leq a} F_b^{a*}\left(\sum_{c\leq b} F_c^{b*} l_{b\to c} - \sum_{c\geq b} l_{c\to b}\right) \tag{9.1.7}$$

The critical points of the global cost function are simply the critical points of the local ones up to messages. When the cofunctor is decomposable whe show the following characterization of the critical points that is even simpler.

Theorem 9.2.3. An element $x \in \lim F$ is a critical point of the global cost function f with respect to a decomposable presheaf F and a collection of differentiable applications $(f_a : F(a) \to \mathbb{R}, a \in \mathscr{A})$ if and only if for any $a \in \mathscr{A}$,

$$s_a^*(a)(d_x f_a) = 0 \tag{9.1.8}$$

where $(s_a, a \in \mathscr{A})$ are the projections on the $S_a, a \in \mathscr{A}$, of the decomposition of F.

Assume furthermore that each $F(a)$, with $a \in \mathscr{A}$, is a Hilbert with a scalar product that we denote as $\langle , \rangle$. Let for any $a \in \mathscr{A}$,

$$d_x f_a = \langle \partial_x f_a | \tag{9.1.9}$$

and if $G = F^\dagger$ is decomposable, let $(S_a, a \in \mathscr{A})$ be its canonical decomposition, then critical points are as such that for any $a \in \mathscr{A}$,

$$s_a(a)\partial_x f_a = 0 \tag{9.1.10}$$

We then give two examples of applications of local optimization. The first one is a local version of the PCA that is particularly adapted to times series and more generally to data with a natural filtration. The second is a generalization of the Bethe Free Energy, that we call the local MaxEnt, when considering any cofunctor in **Kern** and not just the one given by marginalizations. We briefly discuss how to implement this optimization problem using projected gradient descent, in particular when the cofunctor is decomposable.

9.2 Regionalized Optimization

9.2.1 Global Cost Function and critical points

Definition of mobius inversion for funtors and presheaves

Definition 9.2.1 (Mobius inversion associated to a functor)**.** Let $G : \mathscr{A} \to$ **Mod** be a functor from a finite poset to the category of (R-)modules; let $\eta_G : \bigoplus_{a \in \mathscr{A}} G(a) \to \bigoplus_{a \in \mathscr{A}} G(a)$ be such that for any $a \in \mathscr{A}$ and $v \in \bigoplus_{a \in \mathscr{A}} G(a)$,

$$\mu_G(v)(a) = \sum_{b \leq a} \mu(a, b) G_a^b(v_b) \tag{9.2.1}$$

Proposition 9.2.1. *Let $G : \mathscr{A} \to$ **Mod** be a functor from a finite poset to the category of modules, μ_G is invertible and its inverse, denoted ζ_G, is defined as follows, for any $a \in \mathscr{A}$ and $v \in \bigoplus_{a\in\mathscr{A}} G(a)$,*

$$\zeta_G(v)(a) = \sum_{b\leq a} G_a^b(v_b) \tag{9.2.2}$$

Proof. Let $v \in \bigoplus_{a\in\mathscr{A}} G(a)$ and $a \in \mathscr{A}$,

$$\zeta_G\mu_G(v)(a) = \sum_{b\leq a}\sum_{c\leq b} \mu(b,c)G_a^b G_b^c(v_c) \tag{9.2.3}$$

therefore,

$$\zeta_G\mu_G(v)(a) = \sum_{c\leq a}\left(\sum_{b:\ c\leq b\leq a} \mu(b,c)\right) G_a^c(v_c) = G_a^a(v_a) \tag{9.2.4}$$

Furthermore,

$$\mu_G\zeta_G(v)(a) = \sum_{b\leq a}\mu(a,b)\sum_{c\leq b} G_a^c(v_c) = v_a \tag{9.2.5}$$

□

Remark 9.2.1. Let us remark for any poset $(\mathscr{A}, \leq)$ one can reverse the relations, in other words, for any $a, b \in \mathscr{A}$,

$$a \leq_{op} b \iff b \leq a \tag{9.2.6}$$

We sall also denote $\leq_{op}$ as $\geq$ and the corresponding poset as $\mathscr{A}^{op}$ or $(\mathscr{A}, \geq)$. One has that, for any $a, b \in \mathscr{A}$ such that $a \geq b$,

$$\eta_{\mathscr{A}^{op}}(b,a) = \eta_{\mathscr{A}}(a,b) \tag{9.2.7}$$

$$\mu_{\mathscr{A}^{op}}(b,a) = \mu_{\mathscr{A}}(a,b) \tag{9.2.8}$$

In particular for any $G : \mathscr{A} \to$ **Mod** functor from a finite poset to the category of modules,

$$\mu_{G^*} = (\mu_G)^* \tag{9.2.9}$$

as for any $(l_a \in G(a)^*, a \in \mathscr{A})$,

$$\sum_{a\in\mathscr{A}}\sum_{b\leq a}\mu(a,b)l_a G_a^b = \sum_{b\in\mathscr{A}}\sum_{a\geq b}\mu(a,b){G^*}_b^a(l_a) \tag{9.2.10}$$

Regionalized Cost Function

Definition 9.2.2 (Regionalized optimization Cost Function)**.** Let $F : \mathscr{A} \to \mathbf{Vect}_f$ be a cofunctor over a finite poset, $\mathscr{A}$, to $\mathbb{R} - \mathbf{Vect}_f$, the category of finite dimensional ($\mathbb{R}$-)vector spaces, let $(f_a \in \hom_{\mathbf{Set}}(F(a), \mathbb{R}), a \in \mathscr{A})$ be a collection of functions that we will call local cost functions. We shall call the regionalized optimization with respect to F and $(f_a, a \in \mathscr{A})$, the following optimization problem,

$$\sup_{x \in \lim F} \sum_{a \in \mathscr{A}} c(a) f_a(x_a) \tag{9.2.11}$$

where for any $a \in \mathscr{A}$,

$$c(a) = \sum_{b \geq a} \mu(b, a) \tag{9.2.12}$$

$f = \sum_{a \in \mathscr{A}} c(a) f_a$ is the global cost function.

Remark 9.2.2. In the previous definition, we considered cofuntors over a given poset because in the applications it is this way that it appears; however, as explained in Remark 9.2.1 we could also state the previous definition for functors $F : \mathscr{A}^{op} \to \mathbf{Vect}_f$ and in this case,

$$c(a) = \sum_{b \leq_{op} a} \mu_{op}(a, b) \tag{9.2.13}$$

Let us now give one justification of why the proposed optimization problem is a good candidate for defining a global optimization problem from the local cost functions.

Proposition 9.2.2. *Let $\mathscr{A}$ be a poset that has a maximum element, that we shall note as* 1*, let $F : \mathscr{A} \to \boldsymbol{Vect}_f$ be a cofuntor and let $(f_a \in \hom_{\boldsymbol{Set}}(F(a), \mathbb{R}), a \in \mathscr{A})$ be a collection of differentiable cost functions, then for any $x \in \lim F$,*

$$\sum_{a \in \mathscr{A}} c(a) f_a(x_a) = f_1(x_1) \tag{9.2.14}$$

and,

$$\sup_{x \in \lim F} \sum_{a \in \mathscr{A}} c(a) f_a(x_a) = \sup_{x_1 \in F(1)} f_1(x_1) \tag{9.2.15}$$

Proof.

$$\sum_{a \in \mathscr{A}} \sum_{b \leq a} \mu(a, b) f_b(x_b) = \sum_{b \in \mathscr{A}} \sum_{b \leq a \leq 1} \mu(a, b) f_b(x_b) \tag{9.2.16}$$

And so,

$$\sum_{a\in\mathscr{A}}\sum_{b\leq a}\mu(a,b)f_b(x_b)=\sum_{b\in\mathscr{A}}\delta_b(1)f_b(x_b)=f_1(x_1) \tag{9.2.17}$$

□

Let us now focus on how to handle local constraints; let us first recall that when one considers an optimization problem $f\in\hom_{\mathbf{Set}}(X,\mathbb{R})$ with constraint $r\in\hom_{\mathbf{Set}}(X,\mathbb{R})$, in order to solve for some $\mu\in\mathbb{R}$ the following problem,

$$\sup_{r(x)=\mu} f(x) \tag{9.2.18}$$

one can also consider the other optimization problem using a Lagrange multipliers,

$$\sup_{x\in X} f(x)+\lambda r(x) \tag{9.2.19}$$

If there are multiple constrainsts there is a Lagrange multiplier for each constraint. When looking at critical points of Equation 9.2.18, if the cost and constraints are differentiable, one finds that there is a relation between μ and λ defined as,

$$\mu R\ \lambda \iff \exists x\in\{r=\mu\},\ d_xf=\lambda d_xr \tag{9.2.20}$$

In the context of regionalized optimization we have not explicitly discussed the case of constraints in Definition 9.2.2. The reason behind this omission is that we implicitly assume that the constraints on each x_a are in f_a, in other words if we are interested in a collection of cost functions $(f_a\in\hom_{\mathbf{Set}}(F(a),\mathbb{R}),a\in\mathscr{A})$ with constraints $(r_a\in\hom_{\mathbf{Set}}(F(a),\mathbb{R}),a\in\mathscr{A})$ we consider the modified cost function, for $a\in\mathscr{A}$

$$f_a^{'}=f_a+\lambda_a r_a \tag{9.2.21}$$

Each λ_a as explained above (Equation 9.2.20) is a place holder for a constraint of the kind $r_a(x_a)=\mu_a$.

We can't expect that optimizing,

$$\sup_{\substack{x\in\lim F\\ \forall a\in\mathscr{A},\ r_a(x_a)=\mu_a}}\sum_{a\in\mathscr{A}}c(a)f_a(x_a) \tag{9.2.22}$$

is in general the same than optimizing,

$$\sup_{x\in\lim F}\sum_{a\in\mathscr{A}}c(a)\left(f_a(x_a)+\lambda_a r_a(x_a)\right) \tag{9.2.23}$$

Indeed $\sum_{a\in\mathscr{A}} c(a)\lambda_a r_a(x_a)$ is a reconstruction of a global constraint from a local constraint. However when $r \in \lim_{a\in\mathscr{A}} \hom_{\mathbf{Set}}(F(a),\mathbb{R})$ there is a correspondance between imposing the constraints locally, Equation 9.2.22, and the global reconstructed contraint, Equation 9.2.23.

Proposition 9.2.3. *Let $r \in \lim_a \hom_{\boldsymbol{Set}}(F(a),\mathbb{R})$, for any $x \in \lim F$ and $(\lambda_a \in \mathbb{R}, a \in \mathscr{A})$ then,*

$$\sum_{a\in\mathscr{A}} c(a)\lambda_a r_a(x_a) = \sum_{a\in\mathscr{A}} \lambda_a^{'} r_a(x_a) \tag{9.2.24}$$

with,

$$\lambda^{'} = \mu_{\mathscr{A}}\lambda \tag{9.2.25}$$

Proof. For any $x \in \lim F$,

$$\sum_{a\in\mathscr{A}} c(a)\lambda_a r_a(x_a) = \sum_{b\in\mathscr{A}}\sum_{a\geq b} \mu(a,b)\lambda_b r_b(x_b) \tag{9.2.26}$$

But,

$$r_b(x_b) = r_b(F_b^a x_a) = r_a(x_a) \tag{9.2.27}$$

therefore,

$$\sum_{a\in\mathscr{A}} c(a)\lambda_a r_a(x_a) = \sum_{a\in\mathscr{A}} \left(\sum_{b\leq a} \mu(a,b)\lambda_a\right) r_a(x_a) \tag{9.2.28}$$

$\lambda^{'} = \mu_{\mathscr{A}}\lambda$ satisfies Equation 9.2.24 and the correspondence between λ and $\lambda^{'}$ is one to one as $\mu_{\mathscr{A}}$ is invertible.

□

Proposition 9.2.3 tells us that in the case of compatible constraints, the difference between Equation 9.2.22 and Equation 9.2.23 is a question of reparametrization of the Lagrange multipliers. The only example of this kind that we shall consider is the regionalized optimization of MaxEnt; as in this context the values of the constraints matter, in other words we are give the μ's of the relation in Equation 9.2.20, we shall consider a problem of the first kind (Equation 9.2.22) and in this context one has the following proposition.

Proposition 9.2.4. *Let F be a presheaf from $\mathscr{A}$ to $\boldsymbol{Vect}_f$, let $(f_a, a \in \mathscr{A})$ be a collection of cost functions, let $r \in \lim_a \hom_{\boldsymbol{Set}}(F(a),\mathbb{R})$. Let $\mu \in \mathbb{R}$, then $(\overline{F}(a) = \{r_a = \mu\}, a \in \mathscr{A}$ is stable by $(F_b^a, b \leq a)$, in otherwords for $a, b \in \mathscr{A}$ such that $b \leq a$,*

$$\overline{F}^a_b = F^a_b|^{\overline{F}(a)}_{\overline{F}(b)} \tag{9.2.29}$$

defines a functor and is a subobject of F.

Proof. Let $a \in \mathscr{A}$ and $x_a \in \overline{F}(a)$ then,

$$r_b(F^a_b(x_a)) = r_a(x_a)) = \mu \tag{9.2.30}$$

□

If furthermore the constraints are linear, Proposition 9.2.4 tells us that under such constraints, one can refer to the development for the presheaf $\overline{F}$.

Critical points of regionalized optimization

Theorem 9.2.1. *An element $x \in \lim F$ is a critical point of the regionalized optimization with respect to a cofuntor F from a finite poset $\mathscr{A}$ to $\boldsymbol{Vect}_f$ and a collection $(f_a, a \in \mathscr{A})$ if and only if,*

$$\mu_{F^*} d_x f|_{\lim F} = 0 \tag{9.2.31}$$

where for any $a \in \mathscr{A}$ and $u \in \bigoplus_{a\in\mathscr{A}} F(a)$, $d_x f(u) = \sum_{a\in\mathscr{A}} d_x f_a(u_a)$.

Proof. Let us note $c(u) = \sum_{a\in\mathscr{A}} c(a) f_a(u_a)$ for $u \in \lim F$, then any critical point $x \in \lim F$ is defined by,

$$d_x c|_{\lim F} = 0 \tag{9.2.32}$$

Then for any $u \in \lim F$,

$$\sum_{a\in\mathscr{A}} \sum_{b\leq a} \mu(a,b) d_x f_b(u_b) = 0 \tag{9.2.33}$$

Furthermore for any $b \leq a$, $F^a_b(u_a) = u_b$ therefore,

$$\sum_{a\in\mathscr{A}} \sum_{b\leq a} \mu(a,b) d_x f_b F^a_b(u_a) = 0 \tag{9.2.34}$$

in other words,

$$\sum_{a\in\mathscr{A}} \mu^*_F(d_x f)(u_a) = \sum_{a\in\mathscr{A}} \mu_{F^*}(d_x f)(u_a) = 0 \tag{9.2.35}$$

which can be restated as,

$$\mu_{F^*} d_x f|_{\lim F} = 0 \tag{9.2.36}$$

□

Let us remark that the following exact sequence holds,

$$0 \to \lim F \to \bigoplus_{a \in \mathscr{A}} F(a) \xrightarrow{\delta} \bigoplus_{\substack{a,b \in \mathscr{A} \\ a \geq b}} F(b) \tag{9.2.37}$$

where for any $v \in \bigoplus_{\substack{a,b \in \mathscr{A} \\ a \geq b}} F(b)$ and $a, b \in \mathscr{A}$ such that $b \leq a$,

$$\delta(v)(a,b) = F_b^a(v_a) - v_b \tag{9.2.38}$$

As in **Vect** any object is injective in particular $\hom_{\mathbf{Vect}}(., \mathbb{R})$ is exact and by dualizing, one gets the following exact sequence,

$$0 \leftarrow (\lim F)^* \leftarrow \bigoplus_{a \in \mathscr{A}} F(a)^* \xleftarrow{d} \bigoplus_{\substack{a,b \in \mathscr{A} \\ a \geq b}} F(b)^* \tag{9.2.39}$$

with $d = \delta^*$; therefore $\lim F \cong \operatorname{colim} F^\star$ and explicitly for any $l_{a \to b} \in \bigoplus_{\substack{a,b \in \mathscr{A} \\ a \geq b}} F(b)^*$ and $a \in \mathscr{A}$,

$$\mathrm{d}m(a) = \sum_{a \geq b} F_b^{a*}(l_{a \to b}) - \sum_{b \geq a} l_{b \to a} \tag{9.2.40}$$

So in particular on can restate Equation 9.2.31 as,

$$\mu_F^* d_x f \in \operatorname{im} \mathrm{d} \tag{9.2.41}$$

which can be rewritten as the fact that there is $(l_{a \to b} \in F(b)^* | a, b \in \mathscr{A}, b \leq a)$ such that,

$$d_x f = \zeta_{F^*} \mathrm{d} l \tag{9.2.42}$$

9.2.2 Local Cost Function for a functor

Let us extend what we said in the previous section to an analogous for functors. In Definition 9.2.2 the requirement is that F is a presheaf over $\mathscr{A}$, if F where a functor on $\mathscr{A}$ Theorem 9.2.1 would not hold. To remedy to this one needs to consider a functor (G, F) to $\mathbf{Split}(\mathbb{R} - \mathbf{Vect}_f)$, the subcategory of $\mathbb{R} - \mathbf{Vect} \times \mathbb{R} - \mathbf{Vect}^{op}$ as defined in Chapter 4, that we will simply denote as **Split**.

Definition 9.2.3. Let (G, F) be a functor from a finite poset $\mathscr{A}$ to **Split**, the regionalized optimization with respect to (G, F) and a collection $(f_a \in \hom_{\mathbf{Set}}(G(a), \mathbb{R}), a \in \mathscr{A})$ is,

$$\sup_{u \in \lim F} \sum_{a \in \mathscr{A}} c_{op}(a) f_a(u_a) \tag{9.2.43}$$

where for any $a \in \mathscr{A}$,

$$c_{op}(a) = \sum_{b:a \geq b} \mu_{op}(b,a) = \sum_{b:b \leq a} \mu(a,b) \tag{9.2.44}$$

Theorem 9.2.2. *Let (G, F) be a functor from a finite poset $\mathscr{A}$ to **Split**, let $f_a \in \hom_{\boldsymbol{Set}}(G(a), \mathbb{R}), a \in \mathscr{A})$ be a collection of differentiable functions, the critcal points with respect to the associated regionalized optimization problem are the $x \in \lim F$ such that,*

$$\mu_{G^*} d_x f|_{\lim F} = 0 \tag{9.2.45}$$

Proof. Let us remark that for any $u \in \lim G$,

$$F^a_b u_a = F^a_b G^b_a u_b = u_b \tag{9.2.46}$$

Therefore,

$$\lim G \subseteq \lim F \tag{9.2.47}$$

For any $x, u \in \lim F$

$$\mathrm{d} \sum_{b \in \mathscr{A}} \sum_{a:b \leq a} \mu(a,b) \mathrm{d} f_a(u_a) = \mathrm{d} \sum_{a \in \mathscr{A}} \sum_{b \leq a} \mu(a,b) \mathrm{d} f_a(G^b_a u_b) = \sum_{b \in \mathscr{A}} \mu_{G^*} \mathrm{d} f(b) \tag{9.2.48}$$

Therefore,

$$\mu_{G^*} \mathrm{d}_x f|_{\lim F} = 0 \tag{9.2.49}$$

□

9.2.3 Regionalized optimization for decomposable presheaves

Theorem 9.2.3. *Let $\mathscr{A}$ be a finite poset and $F : \mathscr{A} \to \boldsymbol{Vect}_f$ be a cofunctor such that F^* is decomposable, let $(f_a, a \in \mathscr{A})$ be a collection of differentiable cost functions; let $(S_a, a \in \mathscr{A})$ be a decomposition of F^* and $(s_a : F^* \to S_a, a \in \mathscr{A})$ the respective projections, then the critical points $x \in \lim F$ of the global cost function are such that, for any $a \in \mathscr{A}$,*

$$s_a(a)(d_x f_a) = 0 \tag{9.2.50}$$

Proof. $\bigoplus_{a\in\mathscr{A}} S_a(a) \cong \operatorname{colim} F^*$ and,

$$\begin{array}{rcccc} \psi & : & \bigoplus_{a\in\mathscr{A}} F(a)^* & \to & \bigoplus_{a\in\mathscr{A}} S_a(a) \\ & & (l_a, a\in\mathscr{A}) & \mapsto & (s_a(a)(l_a), a\in\mathscr{A}) \end{array} \tag{9.2.51}$$

is such that $\operatorname{coim}\psi = \operatorname{colim} F^*$, therefore $\mu_{F^*} d_x f|_{\lim F} = 0$ if an only if for any $a\in\mathscr{A}$, $s_a(a)\mu_{F^*} d_x f = 0$, so,

$$\sum_{b\leq a} s_a(a) F_b^{a*} d_x f_b = 0 \tag{9.2.52}$$

But by definition $(s_a, a\in\mathscr{A})$ are natural transformation therefore,

$$s_a(a) F_b^{a*} = S_{a\,a}^{\;\;b} s_a(b) \tag{9.2.53}$$

and $s_a(b) = 1[a\leq b]s_a(a)$; therefore,

$$0 = \sum_{b\leq a} s_a(a) F_b^{a*} d_x f_b = s_a(a) d_x f_a \tag{9.2.54}$$

□

Remark 9.2.3. In Theorem 9.2.3 we can chose a decomposition such that for any $a, b\in\mathscr{A}$, $S_b(a)\subseteq F^*(a)$. We could also have assumed that S_a is a constant functor and that $F^* = \bigoplus_{a\in\mathscr{A}} S_a 1[a\leq .]$, Equation 9.2.3 would then be rewritten as, for any $a\in\mathscr{A}$,

$$s_a(d_x f_a) = 0 \tag{9.2.55}$$

In practice the problems and algorithms we will consider are not given up to isomorphism but they come with a specific chart, the spaces $(F(a), a\in\mathscr{A})$ have a precise meaning so we consider that $S_b(a)\subseteq F^*(a)$.

When for any $a\in\mathscr{A}$, $(F(a), \langle,\rangle_a)$ is a Hilbert space then $F^*\cong F^\dagger$ and when F^* is decomposable so is $F^\dagger$ as a functor from $\mathscr{A}$ to $\mathbf{Vect}_f$. Let us consider a decomposition of $F^\dagger$ that is such that for any $a, b\in\mathscr{A}$, $S_a(b)\subseteq F(b)$, of course,

$$\lim F \cong \operatorname{colim} F^\dagger \cong \bigoplus_{a\in\mathscr{A}} S_a(a) \tag{9.2.56}$$

Notation 9.2.1. Let us denote $\mathbf{IHilb}_f$ the category of finite dimensional Hilbert spaces which has as morphisms isometries.

$F^\dagger$ is not necessarily decomposable as a functor from $\mathscr{A}$ to $\mathbf{IHilb}_f$ as $\psi : \bigoplus_a F(a) \to \bigoplus_a S_a(a) \subseteq \bigoplus_a F(a)$ is not necessarily an isometry (the projectors S_a are not orthogonal projector). When computing the gradient descent in the next section we will want an explicit expression of the projection on $\lim F$, if

$i : \lim F \to \bigoplus_{a\in\mathscr{A}} F(a)$ is an isometry then its ajoint $\pi : \bigoplus_{a\in\mathscr{A}} F(a) \to \lim F$, which is isomorphic to the restriction $|_{\lim F} : \bigoplus_{a\in\mathscr{A}} F(a)^* \to (\lim F)^*$, induces the othogonal projection onto $\lim F$, $i\pi$, as

$$\bigoplus_{a\in\mathscr{A}} F(a) = \ker \pi \overset{\perp}{\bigoplus} \lim F \tag{9.2.57}$$

however we don't have its explicit expression. When $F^\dagger$ is decomposable, we do have the explicit expression of this projection,

$$p_{\lim F} = \bigoplus_{a\in\mathscr{A}} s_a(a) \tag{9.2.58}$$

where as explained in Chapter 5,

$$s_a(a) = \sum_{b\leq a} \mu(a,b)\pi^a(\hat{b}) = \sum_{b\leq a} \mu(a,b) F_b^{a\dagger} F_b^a \tag{9.2.59}$$

Notation 9.2.2. Let H be a Hilbert space and H_1 be a closed subspace of H, we shall note p_H the orthogonal projection on H_1.

Corollary 9.2.1. *Let $\mathscr{A}$ be a finite poset and let $G : \mathscr{A} \to \boldsymbol{IHilb}_f$ be a decomposable functor from $\mathscr{A}$ to finite dimensional Hilbert spaces, let $(S_a, a \in \mathscr{A})$ be its canonical decomposition. Let $(f_a, a \in \mathscr{A})$ be a collection of differentiable cost functions, then the critical points $x \in \lim F$ of the global cost function are such that, for any $a \in \mathscr{A}$,*

$$\sum_{b\leq a} \mu(a,b) G_a^b G_a^{b\dagger}(\partial_x f_a) = 0 \tag{9.2.60}$$

Remark 9.2.4. When $\mathscr{A}$ is finite and $F : \mathscr{A} \to \mathbf{Vect}_f$ is a decomposable cofuntor, there is a collection of scalar products on each $F(a)$, $a \in \mathscr{A}$ such that F is a decomposable cofunctor from $\mathscr{A}$ to $\mathbf{IHilb}_f$ as explained in Chapter 5.

9.2.4 A word about implementation

For an unconstrained problem a general method to find critical points of a cost function is the gradient descent. Let $f : H \to \mathbb{R}$ be a differentiable function from a finite dimensional Hilbert space to $\mathbb{R}$, the gradient descent is a sequence $(x_t, t \in \mathbb{N})$ such that,

$$\forall t \in \mathbb{N}\ x_{t+1} = x_t - \partial_{x_t} f \tag{9.2.61}$$

where x_0 is an initial condition.

When the optimization problem is constrained, i.e. the optimisation is done over $M \subseteq H$, one needs to project at each step x_{t+1} on the constraint space [78];

$$\forall t \in \mathbb{N}\ x_{t+1} = p_M\left(x_t - \partial_{x_t} f\right) \tag{9.2.62}$$

and $x_0 \in M$. By definition for any $x \in H$,

$$p_M(x) = \mathbf{argmin}_{y \in M} \|x - y\|^2 \tag{9.2.63}$$

In general one does not have an algebraic definition of p_M. In the case of a local optimization for a decomposable functor $G : \mathscr{A} \to \mathbf{IHilb}_f$ over a finite poset and a collection of cost functions $(f_a, a \in \mathscr{A})$, one has an explicit expression of the projection on $\lim G^\dagger$, for any $a \in \mathscr{A}$,

$$p_{\lim G^\dagger} = \bigoplus_{a \in \mathscr{A}} \sum_{b \leq a} \mu(a,b) G^b_a {G^\dagger}^a_b \tag{9.2.64}$$

9.3 Applications

9.3.1 Local PCA

We would like to see if the knowledge of the interaction decomposition can help us to decompose the covariance matrix of a process over a filtration in components that only take into account the new information added at each step of the filtration. For example when considering a time series, the natural filtration is given by $(\mathscr{F}_{\leq t}, t \in [0,T])$, where $\mathscr{F}_{\leq t}$ are all the measurable sets that only depend on the t first times periods, and $T \in \mathbb{N}$. If one would like to do a PCA on $(X_t, t \in [0,T])$ then the principal components will most of the time depend on all the times between $[0,T]$, i.e. they will be in $\mathscr{F}_{\leq T}$ and we will not have taken into consideration that physically the information arrives in an ordered manner; worse, on can exchange the variables with a permutation $\sigma : [0,T] \to [0,T]$ and the principal components won't change. We want to take into consideration the supplementary information that the filtration gives us on the process when doing a PCA. We shall see that the algebraic point of view does not allow many improvement however the geometric one does.

Algebraic point of view

Notation 9.3.1. We shall reserve capital letters X for random varaibles and x for one of its ralisation.

Let $(X_{i,k}, i \in [n], k \in [d])$ be a random sample of size n of a random vector of dimension d, i.e. each $X_{i,k}$ is a random variables that takes values in $\mathbb{R}$.

Notation 9.3.2. In this section $[n]$ will denote an integer interval $[1, n] \subseteq \mathbb{N}$.

Definition 9.3.1 (Empirical Covariance)**.** Let $(X_{i,k}, i \in [n], k \in [d])$ be a random sample of size n of a random vector of dimension d. The empirical mean is the vector consituted of,

$$\overline{X}_k = \frac{1}{n} \sum_{i \in [n]} X_{i,k} \tag{9.3.1}$$

with $k \in [d]$. The empirical covariance matrix is defined as,

$$\widehat{\mathrm{Cov}}(X_k, X_l) = \frac{1}{n} \sum_{i \in [n]} (X_{i,k} - \overline{X}_k)(X_{i,l} - \overline{X}_l) \tag{9.3.2}$$

with $k, l \in [d]$,

Proposition 9.3.1. *Let $(x_{i,k}, i \in [n], k \in [d])$ be a sample of size n of a random vector of dimension d. Let for any $\lambda, \mu \in \mathbb{R}^d$,*

$$\mathrm{g_c}(\lambda, \mu) = \sum_{k,l \in [d]} \lambda_k \mu_l \, \widehat{\mathrm{Cov}}(x_k, x_l) \tag{9.3.3}$$

Then $\mathrm{g_c}$ is a positive bilinear form on $\mathbb{R}^d$.

Proof. Let $\lambda, \mu \in \mathbb{R}^d$, let for any $i \in [n]$,

$$Y_i = \sum_{k \in [d]} \lambda_k X_{i,k} \tag{9.3.4}$$

$$Z_i = \sum_{k \in [d]} \mu_k X_{i,k} \tag{9.3.5}$$

$$\overline{Y} = \frac{1}{n} \sum_{i \in [n]} Y_i \tag{9.3.6}$$

$$\overline{Z} = \frac{1}{n} \sum_{i \in [n]} Z_i \tag{9.3.7}$$

Then,

$$\mathrm{g_c}(\lambda, \mu) = \widehat{\mathrm{Cov}}(Y, Z) = \frac{1}{n} \sum_{i \in [n]} (Y_i - \overline{Y})(Z_i - \overline{Z}) \tag{9.3.8}$$

And,

$$\mathrm{g_c}(\lambda, \lambda) \geq 0 \tag{9.3.9}$$

□

On the other hand there is a canonical sclar product on $\mathbb{R}^d$ which is such that for any $\lambda, \mu \in \mathbb{R}^d$,

$$g(\lambda, \mu) = \sum_{k \in [d]} \lambda_k \mu_k \tag{9.3.10}$$

For b any bilinear form on $\mathbb{R}^{[d]}$ there is a unique endomorphism $m(b) \in \mathscr{L}(\mathbb{R}^d)$, such that for any $\lambda, \mu \in \mathbb{R}^d$,

$$\mathrm{g_c}(\lambda, \mu) = g(m(b)\lambda, \mu) \tag{9.3.11}$$

If b is symmetric then,

$$m(b)^\dagger = m(b) \tag{9.3.12}$$

Of course, in the case of $\mathrm{g_c}$ for a given sample $(x_{i,k}, i \in [n], k \in [d])$,

$$m(\mathrm{g_c})_{k,l} = \widehat{\mathrm{Cov}}(x_k, x_l) \tag{9.3.13}$$

Let us now recall that the Spectral theorem states that any auto-ajoint matrix is diagonalizable; this will allow us to order the eigenvectors of $m(\mathrm{g_c})$ with respect to its eigenvalues and project on the $d_1 \leq d$ first eignevectors.

Proposition 9.3.2. *Let $(x_{i,k}, i \in [n], k \in [d])$ be a sample of size n of a random vector of dimension d, the covariance matrix is diagonalizable in a orthonormal basis, all its eigenvalues are positive.*

Let us order the eigenvalues of $m(\mathrm{g_c})$ in a decreasing order; let $(u_1, ...u_d)$ be the associated eigenvectors of norm 1, i.e. $g(u_k, u_k) = 1$ for any $k \in [d]$. Let us note U the matrix of coordinates of the $(u_k, k \in [d])$, i.e $Ue_k = u_k$ for any $k \in [d]$. We then have that for $k, l \in [d]$,

$$g(u_k, u_l) = \delta_k(l)\lambda_k \tag{9.3.14}$$

and for any $k, l \in [d]$ such that $k \leq l$,

$$\lambda_k \geq \lambda_l \tag{9.3.15}$$

In particular u_1 is an eigenvector of norm 1 that has the maximal empirical covariance as we shall see in the geometric point of view of the PCA.

The PCA consists in keeping only the covariance induced by the first $d_1 \leq d$ eigenvectors, i.e. for any $\lambda, \mu \in \mathbb{R}^d$,

$$g_{PCA}(\lambda, \mu) = \sum_{k,k_1 \in [d_1]} g\left(\sum_{l \in [d]} U^{\dagger}_{k,l} \lambda_l u_k, \sum_{l \in [d]} U^{\dagger}_{k_1,l} \lambda_l u_{k_1}\right) \tag{9.3.16}$$

To summarize, we consider a finite dimensional Hilbert space (H, g) and a positive bilinear form $\mathrm{g_c}$ on H that we diagonalize. Let us remark that it works the same way if instead of considering the empirical covariance, i.e. $\widehat{\mathrm{Cov}}(Y, Z)$ with $Y = \sum_{k \in [d]} \lambda_k X_k$, $Z = \sum_{k \in [d]} \lambda_k X_k$, we consider the true covariance. To be more explicit, let $\langle X_i, i \in [d] \rangle \subseteq L^2(H, \mathbb{P})$ with $(X_i \in H, i \in [d])$ a collection of random variables, $\mathbb{P} \in \mathbb{P}(H)$ a probability measure and $L^2(H, \mathbb{P})$ the Hilbert space of square integrable functions on H with respect to $\mathbb{P}$. Let us consider the following bilinear form, for $u, v \in \langle X_i, i \in [d] \rangle$,

$$\mathrm{g_c}(u, v) = \mathbb{E}_{\mathbb{P}}[(u - \mathbb{E}[u])(v - \mathbb{E}[v])] \tag{9.3.17}$$

The choice of the collection of random variables $(X_k, k \in [d])$ prescribes the way we shall do the PCA; indeed the following application,

$$\phi : \mathbb{R}^d \to \langle X_i, i \in [d] \rangle \tag{9.3.18}$$

$$\lambda \to \sum_{i \in [d]} \lambda_i X_i \tag{9.3.19}$$

defines that way we pullback $\mathrm{g_c}$ on $\mathbb{R}^d$ where $\mathbb{R}^d$ is seen as a Hilbert space with respect to the canonical scalar product g.

Generalization of the algebraic point of view

The generalization of the reference scalar product g is a functor $G : \mathscr{A} \to \mathbf{IHilb}_f$ from a finite poset $\mathscr{A}$ to $\mathbf{IHilb}_f$; it encodes the filtration with respect to $a \in \mathscr{A}$, in other words it is as if we were given a collection $(X^a_k | k \in [m_a], a \in \mathscr{A})$ of random variables in a certain $L^2(\Omega_a, \mathbb{P}_a)$ and a way to relate the $(X^b_k, k \in [m_b])$ and $(X^a_k, k \in [m_a])$ for $b \leq a$.

Notation 9.3.3. For H a finite dimensional vector space let us denote $B_{\geq 0}(H)$ the set of positive bilinear forms, i.e for any $v \in H$, and $\mathrm{g_c} \in B_{\geq 0}(H)$,

$$\mathrm{g_c}(v, v) \geq 0 \tag{9.3.20}$$

The generalization of $\mathrm{g_c}$ is a collection of $({\mathrm{g_c}}^a \in B_{\geq 0}(G(a)), a \in \mathscr{A})$ of positive bilinear forms. We would like to be able to do a PCA that takes into consideration the local structure of the random variables.

To do so, we choose $G : \mathscr{A} \to \mathbf{IHilb}_f$ such that it is decomposable, which is a property on the underlying functor in **Vect** that is not very restrictive. We consider $(S_a, a \in \mathscr{A})$ its decomposition. We want to find a collection of integers $(n_a \in \mathbb{N}, a \in \mathscr{A})$ and a collections of orthogonal basis $((u_i^a \in S_a(a), i \in [n_a]), a \in \mathscr{A})$, i.e. for any $a \in \mathscr{A}, k, l \in [n_a]$,

$$g_a(u_k^a, u_l^a) = \delta_k(l)\|u_k^a\| \tag{9.3.21}$$

where g_a is the scalar product of $G(a)$, such that for $a \in \mathscr{A}, k \in [n_a]$, there is $\lambda_k^a \in \mathbb{R}$ such that,

$$m({\mathrm{g_c}}^a)u_k^a = \lambda_k^a u_k^a \tag{9.3.22}$$

Remark 9.3.1. We do not ask that the basis of S_a are orthonormal as it is possible that S_a is reduced to 0; this is not problematic as we are interested in the principal directions when doing a PCA, or more precisely on the projection on these directions therefore one does not need, in the algebraic point of view, to specify that the eigenvectors are chosen to be of norm 1.

The previous requirements (Equation 9.3.22) is equivalent to asking that for any $a, b, b_1 \in \mathscr{A}$ such that $b, b_1 \leq a$,

$${\mathrm{g_c}}^a|_{S_b(a)\times S_{b_1}(a)} = \delta_b(b_1)\,{\mathrm{g_c}}^a|_{S_b(a)\times S_{b_1}(a)} \tag{9.3.23}$$

In this case one can order the eigenvector in each $S_a(a)$ with respect to their eigenvalues and keep only the $k \in \mathbb{N}$ first eigenvector in each S_a, if $k > |S_a(a)|$ this mean the associated projections $\pi_a : S_a \to S_a$ is the identity map. We do not detail more this procedure as it is very restrictive and we would not expect it to happen even in the simplest cases of a process filtered by time $(X_t, t \in [0, T])$. However the geometric point of view, as we will see, offers more perspectives and when G is decomposable it is a compromise between interaction decomposition and diagonalisation of the covariance matrix.

Geometric point of view

In the geometrical approach to PCA, one wants to solve the following optimization problem,

$$\sup_{\substack{Y=\sum_{k\in[d]}\lambda_k X_k \\ \|\lambda\|=\mu}} \widehat{\mathrm{Cov}}(Y,Y) = \sup_{\substack{\lambda\in\mathbb{R}^d \\ \|\lambda\|=\mu}} \sum_{i\in[n]}\sum_{k\in[d]}\sum_{l\in[d]} \lambda_k\lambda_l X_{i,k}X_{i,l} \tag{9.3.24}$$

In other words in order to get the first principal component one must maximize, for some $\gamma \in \mathbb{R}$,

$$\sup_{\lambda\in\mathbb{R}^n} \mathrm{g_c}(\lambda,\lambda) + \gamma\|\lambda\|^2 \tag{9.3.25}$$

As said previously for the local version of this problem we need to consider a collection of compatible transformations, over a poset, between (finite dimensional) Hilbert spaces, in other words a functor G from a finite poset $\mathscr{A}$ to $\mathbf{IHilb}_f$. We are given a collection of positive bilinear forms on each $G(a), a \in \mathscr{A}$, and we want to find the vector that maximizes the associated global quadratic form under the "global" constraint on the norms of the projections of this vector.

Definition 9.3.2 (Local PCA)**.** Let $\mathscr{A}$ be a finite poset, let $G : \mathscr{A} \to \mathbf{IHilb}_f$ be a functor from $\mathscr{A}$ to the category of finite dimensional Hilbert spaces. Let $({\mathrm{g_c}}^a \in B_{\geq 0}(G(a)), a \in \mathscr{A})$ be a collection of positive bilinear forms over each $G(a)$ (not necessarily compatible); the (family of) local PCA optimization problem with respect to G and $({\mathrm{g_c}}^a, a \in \mathscr{A})$ is,

$$\sup_{v\in\lim G^\dagger} \sum_{a\in\mathscr{A}} c(a)\left({\mathrm{g_c}}^a(v_a,v_a) + \gamma_a g^a(v_a,v_a)\right) \tag{9.3.26}$$

For example when one is given a collection of samples $(X^a_{i,k} \in G(a) | i \in \coprod_{a\in\mathscr{A}} I_a, k \in [d])$, one can look for critical points of the local PCA with respect to,

$$\mathrm{g}_{\mathrm{c}a}(v,u) = \frac{1}{|I_a|}\sum_{i\in I_a}\sum_{k\in[d]}\sum_{l\in[d]} v_k X^a_{i,k} X^a_{i,l} u_l \tag{9.3.27}$$

Proposition 9.3.3 (Critical points of local PCA in the decomposable case)**.** *Let $\mathscr{A}$ be a finite poset and let $G : \mathscr{A} \to \mathbf{IHilb}_f$ be decomposable and $(S_a, a \in \mathscr{A})$ be its decomposition. Let $(\mathrm{g}_{\mathrm{c}a}, a \in \mathscr{A})$ be a collection of positive bilinear forms over each $G(a)$ and $(m_a \in \mathscr{L}(G(a)), a \in \mathscr{A})$ the associated endomorphisms, then the critical points, $v \in \lim G^\dagger$, of the local PCA are such that for any $a \in \mathscr{A}$*

$$s_a(a)m_a(v_a) = -\gamma_a s_a(a)(v_a) \tag{9.3.28}$$

Proof. For any $u \in \bigoplus_{a\in\mathscr{A}} G(a)$ and $a \in \mathscr{A}$,

$$\partial_u f_a = m_a u_a + \gamma_a u_a \tag{9.3.29}$$

One ends the proof by applying Corollary 9.2.1. □

If furthermore the $(m(\mathrm{g_c}^a), a \in \mathscr{A})$ satisfy some stability conditions, if for any $a \in \mathscr{A}$,

$$m_a \sum_{b<a} S_b(a) \subseteq \sum_{b<a} S_b(a) \tag{9.3.30}$$

then the critical points $u \in \lim G^\dagger$ are such that,

$$S_a m_a S_a(u_a) = -\gamma_a u_a \tag{9.3.31}$$

and in this context local PCA consist in ordering the eigenvalues of each $S_a m_a S_a$, $a \in \mathscr{A}$, and keeping only the k first eigenvectors of each $S_a(a)$. The regionalized optimization with respect to the geometric point of view of PCA is strictly more general that the extension of the algebraic point of view but keeps the same spirit.

Examples

Let us now give examples of when local PCA can be of use. We mentioned the case of a time series $(X_t \in V, t \in [1,T])$, where V is a finite dimensional vector space. One can for example consider that at each time one is given a reference basis, an experimentally meaningful basis, $(u_k, k \in [1,d])$, and identifies V with $\mathbb{R}^d$. Let us denote U_k^t the coordinate,

$$U_k^t : \prod_{t\in[T]} V \to \mathbb{R} \tag{9.3.32}$$

$$(\sum_{k\in[d]} \lambda_k^t u_k, t \in [T]) \to \lambda_k^t \tag{9.3.33}$$

Let $P \in \mathbb{P}(\prod_{t\in[T]} V)$ be such that any $U_k^t \in L^2(\prod_{t\in[T]} V, P)$; let us consider the following functor, let $\mathscr{A} = [T]$ and for any $t \in \mathscr{A}$,

$$G(t) = \langle U_k^\tau |\ \tau \in [t], k \in [d]\rangle \subseteq L^2(\prod_{t\in[T]} V, P) \tag{9.3.34}$$

where for any $\sum_{\tau\leq t}\sum_{k\in[d]} \lambda_k^\tau U_k^\tau, \sum_{\tau\leq t}\sum_{k\in[d]} \rho_k^\tau U_k^\tau \in G(t)$,

$$g_t(\sum_{\tau\leq t}\sum_{k\in[d]} \lambda_k^\tau U_k^\tau, \sum_{\tau\leq t}\sum_{k\in[d]} \rho_k^\tau U_k^\tau) = \sum_{\tau\leq t}\sum_{k\in[d]} \lambda_k^\tau \rho_k^\tau \tag{9.3.35}$$

For $t_1 \leq t$, $G_t^{t_1}$ is the inclusion $G(t_1) \subseteq G(t)$ and finally, for any $t \in [T]$ and $u, v \in G(t)$,

$$\mathrm{g_c}^t(u,v) = \mathbb{E}_{\mathbb{P}}\left((u-\mathbb{E}[u])(v-\mathbb{E}[v])\right) \tag{9.3.36}$$

For experimental samples $(x_{t,i}|\ t \in [T], i \in [n])$, $\mathrm{g_c}^t$ is replaced by,

$$\mathrm{g_c}^t(u,v) = \widehat{\mathrm{Cov}}(u,v) \tag{9.3.37}$$

One could also want to change basis at every time step which would correspond to a moving frame; what we just said in the previous setting still holds but as the basis of V, $(u_k^t, k \in [d])$, now depends on t in Equation 9.3.35 U_k^t is the coordinate with respect to u_k^t.

An other example is the one where the random variables can't be all observed at the same times. To be more precise we consider a collection of random variables $(X_l \in V, l \in I)$, in a finite vector space V, and observables are in $G(a) = \langle X_{k,l} | l \in a, k \in [d]\rangle$, where $a \in \mathscr{A} \subseteq \mathscr{P}(I)$; these are the analogous to $a-$factor spaces; if one considers a fixed basis of V the local PCA in this situation is a restatement of what we said for the first example of time series with $t \in [T]$ replaced by $a \in \mathscr{A}$.

9.3.2 Local Optimisation for Entropy

Generalized Bethe free energy as regionalized optimization

The Generalized Bethe Free energy is the local version of the maximum of entropy under energy constraint for the presheaf given by marginalizations. Let I be a finite set and let $E = \prod_{i \in I} E_i$ be a product of finite sets. Let $\mathscr{A} \subseteq \mathscr{P}(I)$ be a subposet of the parts of I and for any $a \in \mathscr{A}$ let $F(a) = \mathbb{P}_{>0}(E_a)$. For any $a, b \in \mathscr{A}$ such that $b \subseteq a$, we shall note the marginalisation, $\pi^a_{b\,*} : \mathbb{P}_{>0}(E_a) \to \mathbb{P}_{>0}(E_b)$ as F^a_b. π is a cofunctor from $\mathscr{A}$ to **Mes** $\subseteq$ **Kern**. $F = \mathbb{P}_{>0}(\pi)$ is a cofunctor from $\mathscr{A}$ to **Mes** and we can extend it to a cofuntor $\tilde{F}$ from $\mathscr{A}$ to **Vect** by defining for any $a \in \mathscr{A}$, $\tilde{F}(a) = \mathbb{R}^{E_a}$ and for $b \in \mathscr{A}$ such that $b \leq a$ and $f \in F(a)$, any $x \in E_b$,

$$\tilde{F}^a_b(x) = \sum_{y \in E_{a \setminus b}} f(x,y) \tag{9.3.38}$$

Let us note $1_a \in \mathbb{R}^{E_a}$ the constant function that equals to 1 and let us denote $\langle 1_a|$ the associated linear form for the canonical scalar product on $\mathbb{R}^{E_a}$. The fact that F is a suboject of $\tilde{F}$ can be restated as saying that for any $a, b \in \mathscr{A}$ such that $a \leq b$, $\langle 1_b | \tilde{F}^a_b = \langle 1_a|$. In other words the constraint that $p \in F(a)$ is compatible with $\tilde{F}$ and, as discussed in Section 9.2.1 considering the global reconstruction of the constraints, with the lagrange multiplier, is the same than considering the local one, here we choose to keep these constraints local as we restrict our attention to

probability distributions. Let us recall what the Generalized Bethe Free Energy with respect to local opitimization framework we developed in the previous section.

Definition (Generalized Bethe Free Energy)**.** Let $(H_a \in \mathbb{R}^{E_a}, a \in \mathscr{A})$ be a collection of Hamiltonians respectively in the a-factor spaces, the Generalized Bethe Free energy with respect to $\mathscr{A}$ is for any $p \in \lim F$,

$$F(p) = \sum_{a \in \mathscr{A}} c(a) \left(\mathbb{E}_{p_a}[H_a] - S_a(p_a) \right) \tag{9.3.39}$$

The associated optimization problem is,

$$\inf_{p \in \lim F} F(p) \tag{9.3.40}$$

which can be rewritten as,

$$\inf_{\substack{p \in \lim \tilde{F} \\ \forall a \in \mathscr{A}\, p_a > 0 \\ \sum_{x_a \in E_a} p_a(x_a) = 1}} \sum_{a \in \mathscr{A}} c(a) \left(\sum_{x_a \in E_a} p(x_a) H_a(x_a) + \sum_{x_a \in E_a} p_a(x_a) \ln p_a(x_a) \right) \tag{9.3.41}$$

Let us now show that the Generalized Bethe free energy is the usual free energy when the poset is a Directed Acyclic Graph.

Bethe free energy and DAG

In Markov decision process one is interested in minimizing the relative entropy between two distribution, P, Q over, for example, a finite set E with the requirement that P factors according to a Directed Acyclic Graph (DAG). In orther words one want to minimize,

$$\mathrm{DKL}(P \| Q) = \sum_{x \in E} P(x)[-\ln Q(x)] - \left(- \sum_{x \in E} P(x) \ln P(x) \right) \tag{9.3.42}$$

with $P \in \Theta \subseteq \mathbb{P}(E)$, and Θ is the set of probability distributions that factor according a DAG as we shall explain now.

Definition 9.3.3 (DAG)**.** A finite directed acyclic graph is a finite directed graph that has no directed cycles, in other words there is no sequence of directed edges that starts at a vertex and ends at the same vertex.

Remark 9.3.2. Let us remark that as Graphs, G, are collection of vertices I and of edges A, one can see it as a poset, $\mathscr{A}(G)$, with relation the inclusion, in other words, for any $v \in I$ and $e \in A$,

$$v \leq e \iff v \in e \tag{9.3.43}$$

Notation 9.3.4. Let $G = (I, A)$ be a finite graph, for $e \in A$ we shall note $d(e)$ its degree.

Let us recall the following propositions (see [19], [16], [79]),

Proposition 9.3.4 (Factorization on Dags)**.** *Let I be a finite set and let $E = \prod_{i \in I} E_i$ be a product of finite sets, let $G = (I, A)$ be a finite DAG; $p \in \mathbb{P}_{>0}(E)$ factors accordingly to $\mathscr{A}(G)$, i.e. $p \in G_{\mathscr{A}(G)}$ if and only if for any $x \in E$,*

$$p(x) = \frac{\prod_{e \in A} \pi_{e*} p(x_e)}{\prod_{v \in W} \pi_{v*} p^{d(v)-1}(x_v)} \tag{9.3.44}$$

Proposition 9.3.5. *I be a finite set and let $E = \prod_{i \in I} E_i$ be a product of finite sets, let $G = (I, A)$ be a finite DAG, let $(H_a \in \mathbb{R}^{E_a}, a \in \mathscr{A})$ be a collection of Hamiltonians respectively in the a-factor spaces; the following application,*

$$\begin{array}{rccc} \phi & : & \lim F & \to & G_{\mathscr{A}(G)} \\ & & (p_a, a \in \mathscr{A}(G)) & \mapsto & \frac{\prod_{e \in A} p_e}{\prod_{v \in W} p_v^{d(v)-1}} \end{array} \tag{9.3.45}$$

is a bijection and,

$$F\left(p_a, a \in \mathscr{A}(G)\right)) = \mathbb{E}_{\phi(p)}[\sum_{e \in A} H_e + \sum_{v \in I} H_i] - S(\phi(p)) \tag{9.3.46}$$

Proof. Let for any $p \in \mathbb{P}_{>0}(E) \cap G_{\mathscr{A}(G)}$,

$$\begin{array}{rccc} \psi & : & \mathbb{P}(E) \cap G_{\mathscr{A}(G)} & \to & \lim F \\ & & P & \mapsto & (\pi_{a,*} P, a \in \mathscr{A}(G)) \end{array} \tag{9.3.47}$$

Then $\phi\psi = \mathrm{id}$ and $\psi\phi = \mathrm{id}$, furthermore for any $P \in \mathbb{P}(E) \cap G_{\mathscr{A}(G)}$,

$$\mathbb{E}_P[\sum_{a \in \mathscr{A}} H_a] = \mathbb{E}_P[\sum_{a \in \mathscr{A}} \sum_{b \leq a} \mu(a, b) H_a] = \sum_{a \in \mathscr{A}} \sum_{b \leq a} \mu(a, b) \mathbb{E}_{\pi_{a,*}}[H_a] \tag{9.3.48}$$

and by Proposition 9.3.4,

$$S(p) = \sum_{e \in A} S(\pi_{e,*} P) - \sum_{v \in I} (d(v) - 1) S(\pi_{v,*} P) = \sum_{a \in \mathscr{A}(G)} c(a) S(\pi_{a,*} P) \tag{9.3.49}$$

which ends the proof.

□

Proposition 9.3.5 shows that the Generalized Bethe Free Energy is strictly more general in its approach and enables to approach factorisation on hypergraphs (see Chapter 2 and factorisation over posets of factor spaces $\mathscr{F}_{\mathscr{A}}$).

Local Optimisation for MaxEnt

The regionalized optimization (Definition 9.2.2) of the entropy with energy constraints is in fact more general than the Generalized Bethe Free energy (Definition 9.3.2). Let us consider presheaves f from $\mathscr{A}$ to **Kern** that takes values in finite sets, i.e. for any $a \in \mathscr{A}$, $F(a)$ is a finite set and such that for any $a, b \in \mathscr{A}$ such that $b \leq a$ and any $\omega \in F(a), \omega_1 \in F(b)$,

$$f(\omega|\omega_1) > 0 \tag{9.3.50}$$

In particular, the underlying cofuntor one considers for the Generalized Bethe Free Energy is the collection of projections $(\pi^a_b | a, b \subseteq I, b \subseteq a)$.

Let for $a, b \in \mathscr{A}$ such that $b \leq a$,

$$\begin{array}{rcccc} F^a_b & : & \mathbb{P}_{>0}\left(f(a)\right) & \to & \mathbb{P}_{>0}\left(f(b)\right) \\ & & p & \mapsto & F^a_b p \end{array} \tag{9.3.51}$$

And F^a_b can be extended to a cofuntor from $\mathscr{A}$ to $\mathbf{Vect}_f$ as follows,

$$\begin{array}{rcccc} \tilde{F}^a_b & : & \mathbb{R}^{f(a)} & \to & \mathbb{R}^{f(b)} \\ & & f & \mapsto & \left(\sum_{\omega \in F(a)} f(\omega) F^a_b(\omega_1|\omega), \omega_1 \in f(b)\right) \end{array} \tag{9.3.52}$$

For any $a, b \in \mathscr{A}$ such that $a \leq b$, $\langle 1_b | \tilde{F}^a_b = \langle 1_a |$ as F^a_b is a probability kernel, therefore F is a subobject of $\tilde{F}$. In other words the constraint that $p \in \mathbb{P}_{>0}(f(a))$ is compatible with $\tilde{F}$.

Let us remark that in this case the cost functions are not defined on all $\mathbb{R}^{E_a}$ but only on $\mathbb{R}^{E_a}_{>0}$, however $\prod_{a \in \mathscr{A}} \mathbb{R}^{E_a}_{>0} \cap \lim \tilde{F}$ is an open subset of $\tilde{F}$ therefore the characterization of the critical points stated in Section 9.2.1, Theorem 9.2.1 still holds.

Chapter 10

Evolution of ribosomal protein network architectures

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The work presented in this chapter is the result of a collaboration with Y. Timsit and D. Bennequin; I am very grateful for our very stimulating collaboration and humbled by the depth of their vision on nature.

Abstract

To perform an accurate protein synthesis, ribosomes accomplish complex tasks involving the long-range communication between its functional centres such as the peptidyl transfer centre, the tRNA bindings sites and the peptide exit tunnel. How information is transmitted between these sites remains one of the major challenges in current ribosome research. Many experimental studies have revealed that some r-proteins play essential roles in remote communication and the possible involvement of r-protein networks in these processes have been recently proposed. Our phylogenetic, structural and mathematical study reveals that of the three kingdom's r-protein networks converged towards non-random graphs where r-proteins collectively coevolved to optimize interconnection between functional centres. The massive acquisition of conserved aromatic residues at the interfaces and along the extensions of the newly connected eukaryotic r-proteins also highlights that a strong selective pressure acts on their sequences probably for the formation of new allosteric pathways in the network.

10.1 Introduction

Ribosome structures [80–83] have evolved by the accretion of rRNA and ribosomal (r)-proteins around a universal core considered as a relic of ancient translation systems that co-evolved with the genetic code [84–88]. They followed distinct evolutionary pathways to form the bacterial, archaeal and eukaryotic ribosomes [89–93] whose overall structures are well conserved within kingdoms [94–105]. A concomitant increase in the complexity of ribosome assembly processes, ribosome structures, efficiency and fidelity of protein production is observed from prokaryotes to eukaryotes [106–108]. While the past decade studies have provided a detailed mechanistic understanding of almost all of the translation steps, one of the major challenges in ribosome research is how information is transmitted and processed between remote functional sites such as the tRNA binding sites, the peptidyl-transfer centre (PTC) and the peptide exit tunnel, during protein synthesis.

Growing experimental evidences have shown that distant ribosomal functional sites not only continuously "sense" incoming molecular signals but also "transmit"

them to each other. For example, long-range signalling between the decoding centre monitors the correct geometry of the codon-anticodon and other distant sites such as the Sarcin Ricin Loop (SRL) or the E-tRNA site [109]. The r-protein uL3 also plays a key role in the allosteric coordination of the peptidyl transferase centre (PTC) [110] and the A-site [111]. Similarly, the r-proteins that sense the nascent peptide within the exit tunnel participate in the regulation of co-translational folding and communicate with remote functional sites such as the PTC [112, 113]. Communication processes also coordinate the complex ribosomal movements during translation, such as the ratchet-like motion between the two subunits [114–116].

In addition to their roles in rRNA folding and ribosome assembly [117–122], extensions systematically form complex r-protein networks through tiny interactions, in the three kingdom ribosomes [81, 82, 123]. Unlike most protein networks that occur transiently in the cells [124], r-protein networks are particular in that they are woven by tiny interactions in mature ribosomes, once the stages of their biogenesis is complete [107]. It has been suggested that they could contribute to information transfer and processing during the course of protein synthesis [123, 125].

Understanding how r-protein networks have evolved is an indispensable step to get further insights about their biological significance. Several questions remain to be answered: are the tiny interfaces structurally and phylogenetically conserved in the three domains of life? How has r-protein network connectivity evolved over time and does graph theory provide information about their evolution and functionality? Here, we present a global study of the evolution of r-protein networks, through a phylogenetic, structural and mathematical analysis of their architectures.

10.2 Results

10.2.1 r-protein network conservation and expansion during evolution

Beyond their fascinating diversity, the interactions between ribosomal proteins (PPi) constitute variations around a common theme: how to maintain the tiniest interfaces between long filamentous extensions (ext) or globular domains (G) of a wide variety of r-protein structures (Supplementary Fig.1). All of these interactions, which may also contact functional sites (mRNA, tRNAs, PTC or peptide exit tunnel), form complex interactomes that have been analysed from an evolutionary and a functional perspective.

Comparing the three kingdoms' networks makes it possible to identify the evolutionary status of all their components (Supplementary Tables 1 and 2) and to trace how they have developed over time. Network archaeology has first revealed the existence of a universal (ABE) network, that consists of 49 strictly conserved connections probably present before the radiation of the bacteria and archaea [91] (Supplementary Fig.2). From this common core, the Bacterial (B), Archaeal (A) and Eukaryotic (E) networks have gradually developed through the addition of new proteins and/or new connections. Each kingdom ribosome displays a network with distinct but well-conserved architecture (Supplementary Fig.3). Although tiny, PPi interfaces are both structurally and phylogenetically well conserved within each kingdom (Supplementary Fig.4–8). Noticeably, conservation mapping shows that interfaces are significantly better conserved in eukaryotes, especially in the small subunit.

Except some minor variations, the interactomes of each kingdom are very well conserved even in distant clades. In some bacterial species, the acquisitions of new extensions are generally associated with the formation of new inter-protein contacts. For example, the "new" C-mix extension of M. smegmatis bS16 interacts with uS4 [103] and the E. coli uS14 internal loop (15-54; E. coli numbering) interacts with uS19 [100, 101] (Supplementary Fig.9). In these two cases, rRNA structures of the different specie's ribosomes are conserved and cannot simply justify changes in protein network. In another hand, variations in network connectivity may be also associated with changes in rRNA structures. In the M. smegmatis ribosome, the path of bL17 extension differs from that of the other bacterial species: instead of contacting uL3, the bL17 N-seg returns to its own globular domain and stabilizes a specie's specific extra-helical adenine.

The architecture of the eukaryotic network is also well conserved in the three distant clades opistokhonts, chromoalveota and excavata analysed (Supplementary Fig.10A,B). A particularly interesting feature of plasmodium ribosome network [98] is the direct connection between uL4 with uL16 (Supplementary Fig.10C). In the human ribosome [99], eL6 has a different trajectory that complicates the interaction network compared to yeast (Supplementary Fig.10D). In the leishmania ribosome, the new extensions that have been also reported for the r-proteins uL13 and eL33 are thought to stabilize the fragmented rRNAs specific to this particular clade [102] (Supplementary Figs. 10E,F). Thus, while the networks are very well preserved even in very distant taxa within each kingdom [91,126], they also display minor variations in some species where the new connections may reflect particular adaptations of the translation systems.

Another way of understanding the networks is their functional organization. As a first approximation, r-proteins can be distinguished according to the contacts they form with the different categories of functional centres (Supplementary Fig.11). While some proteins cluster in modules around the main functional centers of the ribosome: mRNAs, tRNAs, PTCs and the peptide tunnel, others build bridges between these modules or between ribosome subunits. Mainly developed in the small subunit (SSU), the ABE network contains a high proportion of connections between r-proteins and functional sites. On the contrary, bridges develop in the later phases of ribosome evolution. Different categories of bridges can be distinguished according to the number and type of functional sites they connect. For example, proteins that bridge two different functional sites are common in bacteria and archaea. In contrast, proteins such as eL15, eL20 and eL21 that link together three functional sites are only found in eukaryotes (Supplementary Figs. 11D-E).

10.2.2 Molecular mechanisms of network expansions.

How new incoming r-proteins or newly acquired extensions contribute to network expansions? Systematic comparison of the networks made it possible to identify, one by one, the events that marked the expansion of the networks at each evolutionary transition. The networks are progressively woven by a combination of interactions between protein components (G or ext) of different evolutionary statuses (ABE, B, A or E) (Fig.1). The new interactions either reinforce existing links between r-proteins, or connect nodes that were not previously connected. Each subunit starts its evolutionary history from different network architectures: an already densely connected SSU and a poorly connected LSU in the ABE network. Whereas the number of connections of the two subunits evolved roughly symmetrically in archaea, bacteria and eukarya underwent a spectacular development of the LSU connectivity (Supplementary Fig.12 and Supplementary Table 3). Network expansion is also characterized by an increase in multiple connections between protein pairs (Supplementary Table 4). Noticeably, the r-proteins that display the highest number of multiple connections correspond to mRNA binders (uS3, uS5).

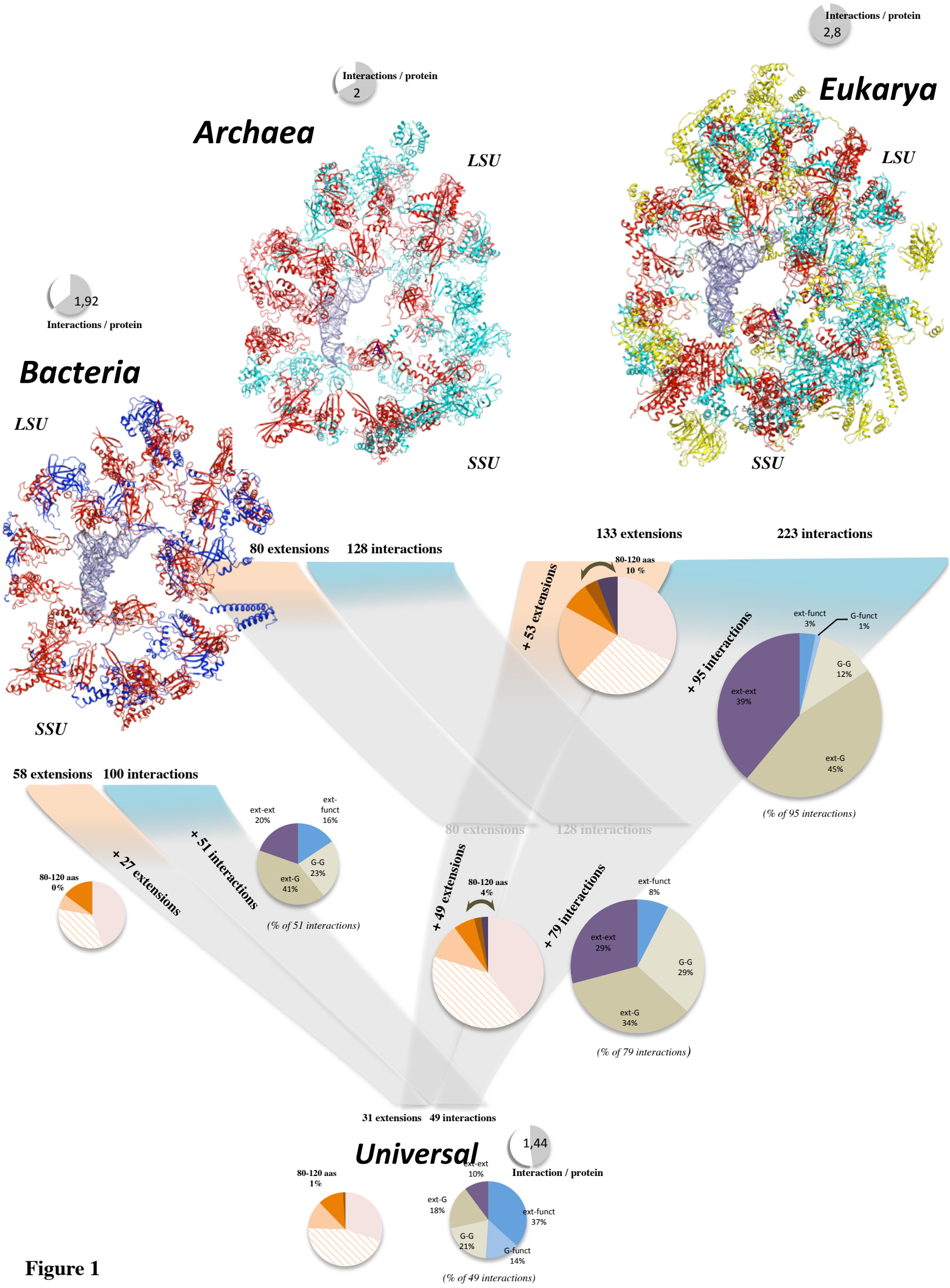

Interactions / protein
2,8
Eukarya
LSU
SSU
Interactions / protein
2
Archaea
LSU
SSU
1,92
Interactions / protein
Bacteria
LSU
SSU
80 extensions
128 interactions
133 extensions
223 interactions
+ 53 extensions
80-120 aas
10 %
+ 95 interactions
ext-funct
3%
G-funct
1%
G-G
12%
ext-ext
39%
ext-G
45%
(% of 95 interactions)
58 extensions
100 interactions
+ 27 extensions
80-120 aas
0 %
+ 51 interactions
ext-ext
20%
ext-
funct
16%
G-G
23%
ext-G
41%
(% of 51 interactions)
80 extensions
128 interactions
+ 49 extensions
80-120 aas
4%
+ 79 interactions
ext-funct
8%
ext-ext
29%
G-G
29%
ext-G
34%
(% of 79 interactions)
31 extensions
49 interactions
Universal
1,44
Interaction / protein
80-120 aas
1%
ext-ext
10%
ext-G
18%
ext-funct
37%
G-G
21%
G-funct
14%
(% of 49 interactions)

Figure 1

Figure 1: Evolution of the r-protein interactomes. In the three kingdoms' networks, the r-protein components are represented according to their evolutionary status with the following colour code: red: universal; blue: bacteria; cyan: archaea; yellow: eukarya. Pie charts report the proportion of extension sizes and of contact types that have been acquired at each transition, respectively. Abbreviations: ext-ext: extension-extension; ext-G: extension-globular domain; ext-funct: extension-functional site; G-G: globular domain-globular domain; G-funct: globular domain-functional site.

Early (ABE→ B, ABE→ A) and late (A→ E) network expansions display specific molecular evolutionary features (Fig.1; Supplementary Fig.12). Whereas a massive incorporation of new interacting r-proteins char- acterizes the early evolutionary stages, it becomes a minority in the A→ E transition where ancient r-proteins become more connected to each other. During the ABE→ B transition, most of the new interactions of the bacterial network (58% U-B interactions) occur between the universal proteins (U) (G or ext) and the newly acquired bacterial r-proteins (B). Interactions between incoming bacterial proteins (B-B) only represent 19% of the interactions and a minority of contacts involve bacterial acquired extensions of universal proteins (Ub) with universal r-protein (12% Ub-U) or with bacterial proteins (9% Ub-B). Thus, in the bacterial ribosome, extensions acquired in bacteria (Ub) only play a minor role in networking. During the ABE→ A transition, the majority of new contacts are also formed between the universal proteins (U) and the newly acquired archaeal proteins (A) (47% U-A interactions). Similar to bacteria, 19% of the interactions are observed between the newly acquired proteins (A-A). However, a greater contribution (35%) of the archaeal specific extensions (Ua) contributes to the formation of the A network. The A→ E transition displays new evolutionary traits. Although it is the most connected, the eukaryotic network incorporates only 11 new proteins, the smallest number of new incoming r-proteins and most of the new contacts (57%) are mediated by the newly acquired extensions in universal and archael r-proteins (Ue and Ae). The A→ E transition therefore displays a sharp increase of connections between ancient and previously unconnected r-proteins (universal or archaeal) that have acquired new extensions.

As a result, each evolutionary transition is distinguishable by different proportions of contact types. For example, while virtually all interactions between r-proteins and functional sites have already been established in the universal network (45% of the total interactions), the connections to functional centres sharply decrease from 14% in B, to 4% and 2% in A and E networks, respectively. Another remarkable evolutionary feature is everincreasing number of contact involving extensions during evolution. For example, the eukaryotic networks have the lowest proportion of G-G interactions and the highest G-ext (45%) and ext-ext (39%)

interactions. Among them, the majority of new contacts in E involve C-terminal extensions. The eukaryotic LSU network evolution is unique in that it displays the highest number of extension-extension interactions.

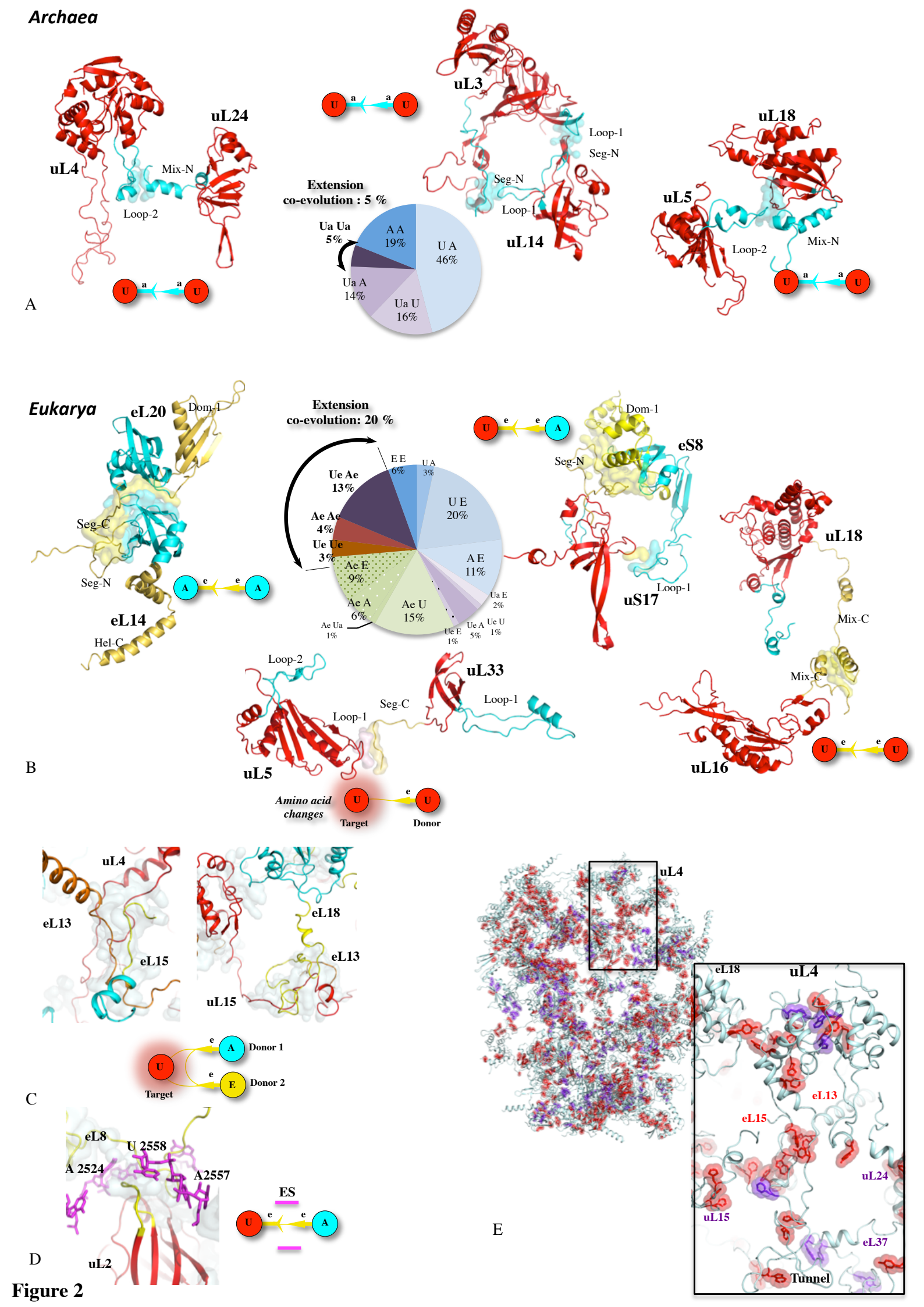
Thus, networking gradually depends on the emergence of new extensions (Supplementary Fig.13) and a majority of r-proteins develop extensions that are systematically involved in network interactions (Supplementary Table 5). A majority of r-proteins (uL3, uL4, uL18, uL22, uL23 and uL33) that have acquired new extensions at all evolutionary stages belongs to the LSU. The categories of newly acquired extensions also vary according to evolving transitions (Supplementary Fig.14). While acquiring internal extensions (loops and β-hairpins) is an early evolutionary trait, a strong bias towards C-terminal extensions is observed during the A$\rightarrow$ E transition. The increase of the proportion of "mix" shows that the unstructured segments (seg) acquire gradually more β-helical regions. A net increase of extension sizes is also observed during evolution (Supplementary Fig.14). The percentage of extensions greater than 80 amino acids increases from 1 to 10% from the ABE to E networks. The longer extensions interconnect more distant r-proteins in the networks: the distances between the interconnected proteins gradually increase in course of the evolution (Supplementary Fig.15). Extensions can therefore be considered as an evolutionary solution for inter-connecting remote network nodes while preserving the ribosome or protein overall structures and sizes.

Unlike the growing roles of extensions in protein networks, extension-rRNA interactions only slightly increase from prokaryotes to eukaryotes (Supplementary Fig.16A,B) and are very moderate compared to the jump in inter-protein connectivity. Moreover, the eukaryotic ribosome reveals that eukaryotic specific extensions rather interact with other r-proteins than with eukaryotic specific RNA expansion segments (Supplementary Fig.16C). Also, extensions acquired in eukaryotes contribute little to RNA interactions (22–26%, for SSU and LSU, respectively) relatively to their archaeal or universal counterparts (78%). This demonstrates that the acquisition of new extensions during evolution are mainly linked to their roles in protein–protein interactions.

10.2.3 Co-evolution fr networking

The expansion of networks also relies on a curious phenomenon of coordinated acquisition of extensions or r-proteins that establish new contacts (extension-protein: Ub-B, Ua-A, Ue-E; extension-extension: Ua-Ua, Ue-Ue, Ae-Ae) (Fig.2; Supplementary Table 3). At each evolutionary transition, extensions have been concurrently acquired, or coevolved [127] for connecting remote r-proteins. In the

bacterial ribosome, the bacterial extensions acquired by the universal r-proteins (uL5-mix-N, uL13-seg-N, uL23-β-HP) coincide with the incorporation of new bacterial proteins (bL31, bL20-bL21, bL34, respectively) to which they are connected. In the ABE→ A transition, several coordinated acquisitions of new extensions such as uL4-loop-2 and uL24-mix-N that interact together are observed (Fig.2A). This phenomenon is significantly amplified in the A→ E transition where many new E specific extensions have been acquired in both universal and archaeal r-proteins to inter-connect them (Fig.2B). Coevolutions between multiple r-proteins or rRNA partners have been detected in eukaryotic ribosomes. Tripartite assembly of co-occurred extensions on a third target r-protein (eL13-eL15 → uL4 and of eL13-eL18 → uL15) or with rRNA ES have been observed (Fig.2C). A remarkable structural motif is formed by the interaction of the extensions of uL2 and eL8 whose interface is literally caged inside a sphere formed by unpaired bases of ES (Fig.2D). Most of the coevolved extensions display highly conserved residues along themselves and in the interfaces they form with their partners (Supplementary Table 6).


Archaea
uL4
uL24
Mix-N
Loop-2
U a a U
uL3
Loop-1
Seg-N
Seg-N
Loop-1
uL14
Extension
co-evolution : 5 %
Ua Ua
5%
A A
19%
U A
46%
Ua A
14%
Ua U
16%
uL18
uL5
Loop-2
Mix-N
A
Eukarya
eL20
Dom-1
Seg-C
Seg-N
eL14
Hel-C
A e e A
Extension
co-evolution: 20 %
E E
6%
U A
3%
Ue Ae
13%
U E
20%
Ae Ae
4%
Ue Ue
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A E
11%
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Ae A
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15%
Ua E
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Ae Ua
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Seg-N
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uL18
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Loop-2
uL33
Loop-1
Seg-C
Loop-1
uL5
Mix-C
uL16
U e e U
B
Amino acid
changes
U e U
Target
Donor
uL4
eL13
eL15
eL18
eL13
uL15
U e A Donor 1
e E Donor 2
Target
C
eL8
U 2558
A 2524
A2557
uL2
ES
U e e A
D
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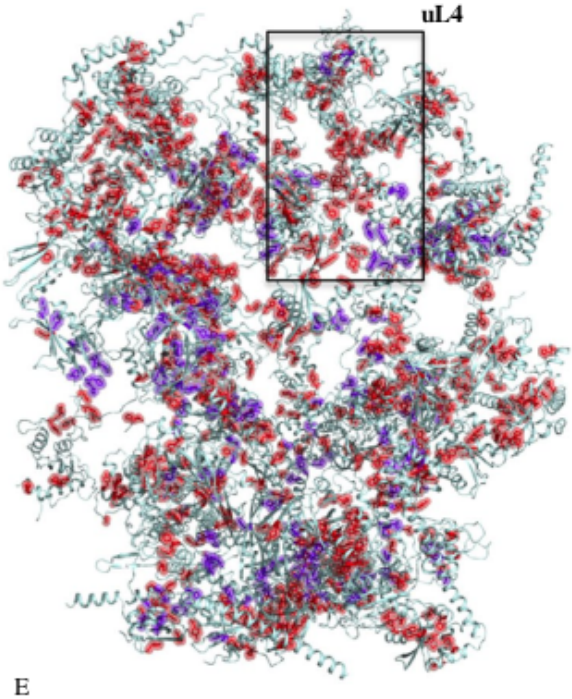
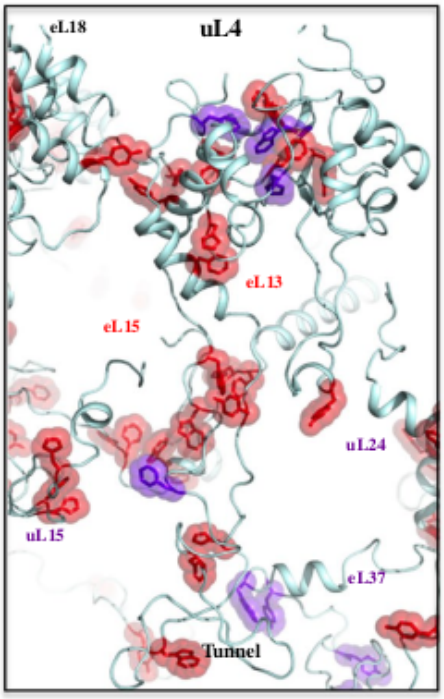
eL18
uL4
eL13
eL15
uL24
uL15
eL37
Tunnel
E

Figure 2


uL4

E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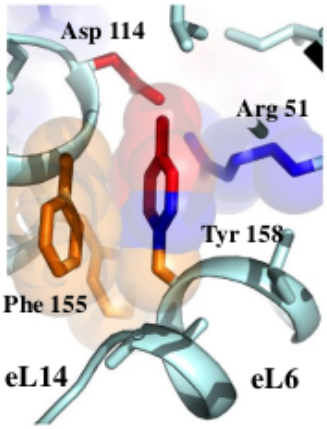
uL4
eL13
eL15
uL24
uL15
eL37
Tunnel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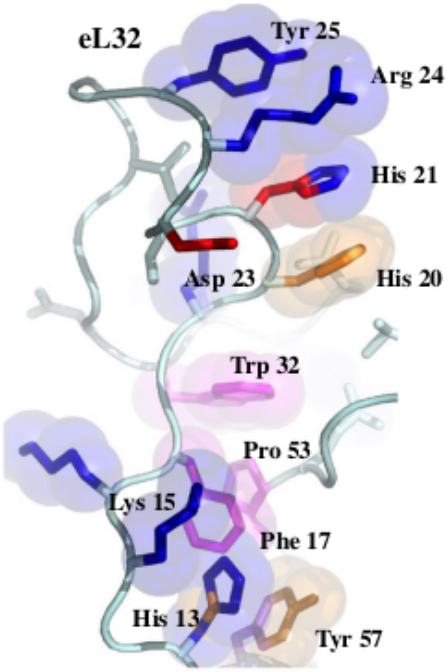
Arg 51
Tyr 158
Phe 155
eL14
eL6

F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Arg 24
His 21
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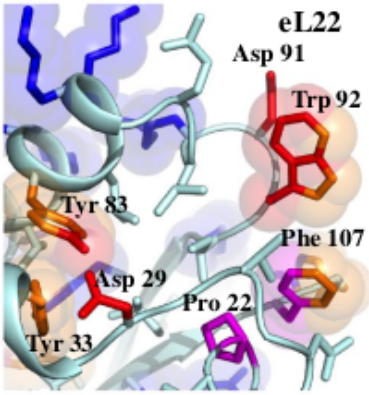
His 20
Trp 32
Pro 53
Lys 15
Phe 17
His 13
Tyr 57

H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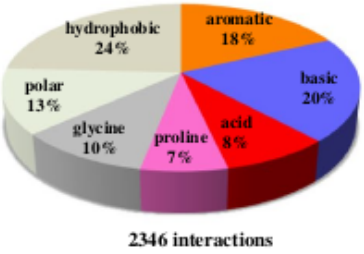
Asp 91
Trp 92
Tyr 83
Phe 107
Asp 29
Pro 22
Tyr 33

G


hydrophobic 24%
aromatic 18%
basic 20%
polar 13%
glycine 10%
proline 7%
acid 8%
2346 interactions

I

Figure 2: Coevolutions for networking. Extension coevolutions in archaea (A) and eukarya (B). The extensions and r-proteins are coloured according to their evolutionary status. Pie charts report the proportions of the connections specifically acquired in the A and E networks according to their evolutionary status. (C) Multiple extension coevolutions toward a common "ancient" target. (D) Coevolution of a eukaryotic PPi interface (uL2–eL8) with unpaired bases of ES. Abbreviations: U, A, B and E correspond to universal, archaeal, bacterial and eukaryotic r-proteins. Ub: Universal protein with an extension acquired in (B) bacteria. Ua: Universal protein with an extension acquired in (A) archaea. Ue: Universal protein with an extension acquired in (E) eukarya. Ae: Archaeal protein with an extension acquired in (E) eukarya. (E) Massive acquisition of new conserved aromatic residues in the eukaryotic ribosome. New conserved and "ancient" (already conserved in archaea) aromatic residues are represented with red and purple blue spheres, respectively. The rectangle represents a close-up around the uL4 r-protein. (F) Interactions between conserved aromatic residues and other conserved amino acids at the interface eL6–eL14. (G) Intra molecular anion-π and proline-π interactions between conserved aromatic residues in the globular domain of r-protein eL22. (H) Cluster of conserved aromatic residues in the extension of eL32 and their interactions with various amino acids. From F to H, the atoms of the aromatic residues (orange) are coloured in function of their interacting partners: blue: basic (lys, arg); red: acidic (glu, asp) and magenta: proline. (I) Proportions of conserved aromatic residues involved in intra or intermolecular interactions with other types of conserved amino acids.

Co-evolution is also observed at the amino acid level (Fig.2E). The A $\rightarrow$ E transition provides the unique opportunity to follow the evolution of r-proteins common to both kingdoms, at the amino acid level, according to their change in connectivity in the network. We have tracked changes in all target proteins that receive new contacts in eukaryotes while remaining in a similar rRNA context (Supplementary Table 3). This reveals that the formation of new contacts systematically correlate with the apparition of new conserved residues, mainly aromatic ones, not only at the new interfaces but also regularly spread along the r-proteins (Supplementary Fig.17). As aresult, a massive acquisition of new conserved aromatic residues parallels the spectacular jump in connectivity of the eukaryotic network and likely accounts for the higher conservation rate of the eukaryotic r-protein and PPi (Fig.2E; Supplementary Table 7). A strong selective pressure is therefore exerted on the sequence of the newly interconnected proteins. Firstly, conserved amino acids that were associated with contacts already present in archaea are not only still conserved, but very often change from "similar" (conservation by amino acid type: aromatic, basic, acidic, polar or hydrophobic) to strictly conserved. Second, the new aromatic amino acids most often form clusters and structural motifs where they participate in combined π–π, cation-π, anion-π or proline-π interactions [128–131] (Fig.2F-I). Thirdly, and surprisingly, the newly conserved amino acids most often appear, not only at the new interfaces, but also

evenly distributed along the newly connected proteins (Fig.2E-I). A similarcorrelation between the appearance of new aromatic residues and the formation of new contacts has been also observed in intra-kingdom evolutions as for example in uS4 of M. smegmatis (Supplementary Fig.9B) or uS19 of E. coli (Supplementary Fig.9F). It is therefore likely that these new conserved amino acids have been selected to play a role both in the formation of interfaces but also in the constitution of new allosteric pathways. The complexity and particular electrostatic properties of these conserved structural motifs merit further study and reinforces the previous hypotheses that they may play a particular role in the transfer of information between the proteins in the network [123, 125].

A special case is the "distant" approach of the extensions of uL4 and uL15, without forming direct contact in both archaea and eukaryotes (Supplementary Fig.18). Yet, despite the fact that there is no direct contact between them, aromatic residues are retained, at the level of their closer approach (10 Å), on both partners. But although the bacteria have developed a longer extension on uL15 that establishes true contact with uL4, they have also retained this structural motif and the aromatic residues (Supplementary Fig.18). This suggests that even aromatic residues that are not in direct contact could participate to allosteric communication. Thus, this structural motif could help to establish an important functional interconnection between the tRNA-E and tunnel modules (Supplementary Fig.11).

10.2.4 Graph theory highlights node functionalities.

Graph theory [124] has been used to further characterize the evolution of r-protein networks. Previous studies in which rRNA was considered as nucleotide network revealed interesting correlations between graph properties and ribosome - functionality [132]. Here, the r-protein networks have been modelled as undirected graphs where the nodes correspond to the centres of mass of the r-protein globular domains or the functional centres and the edges represent their interactions (a single connection between a protein pair) (Fig.3A; Supplementary Fig.19). In the three kingdom's networks, the node interconnectivity and their centrality values are significantly distinct from that of random graphs (Supplementary Table 8). Secondly, interesting correspondences between the graph theory and the ribosome biology have been found. For example, betweenness centrality (BC) that defines the number of shortest paths passing through a node is a key determinant of network functionality in that it measures the extent to which a node influences the information spread on the network. Remarkably, without knowing ribosome biology, mathematics indicates that in the three kingdom's networks, the BC maxima correspond to the PTC, which catalyses the peptidyl-transfer reaction during

protein synthesis (Fig.4; Supplementary Fig.19). This reveals that r-protein network architectures have evolved convergently, so that the PTC becomes a major player in information spread, in perfect agreement with its central role in ribosome catalysis [110] and allostery [132, 133]. In eukaryotic ribosomes, network control is also distributed to other actors including uL16, tRNA-A, eL8 and uL4. Interestingly, uL16 has been also highlighted by biochemical studies as a key player in information transmission of the eukaryotic ribosome [115] and disease [134] . In a similar way, the eigenvector centrality (EV) maxima that illustrate the importance of a node (by its property of being itself connected to important nodes) correspond to the key proteins that establish bridges between functional modules in eukaryotes (Fig.3A; Fig.4; Supplementary Fig.19A).

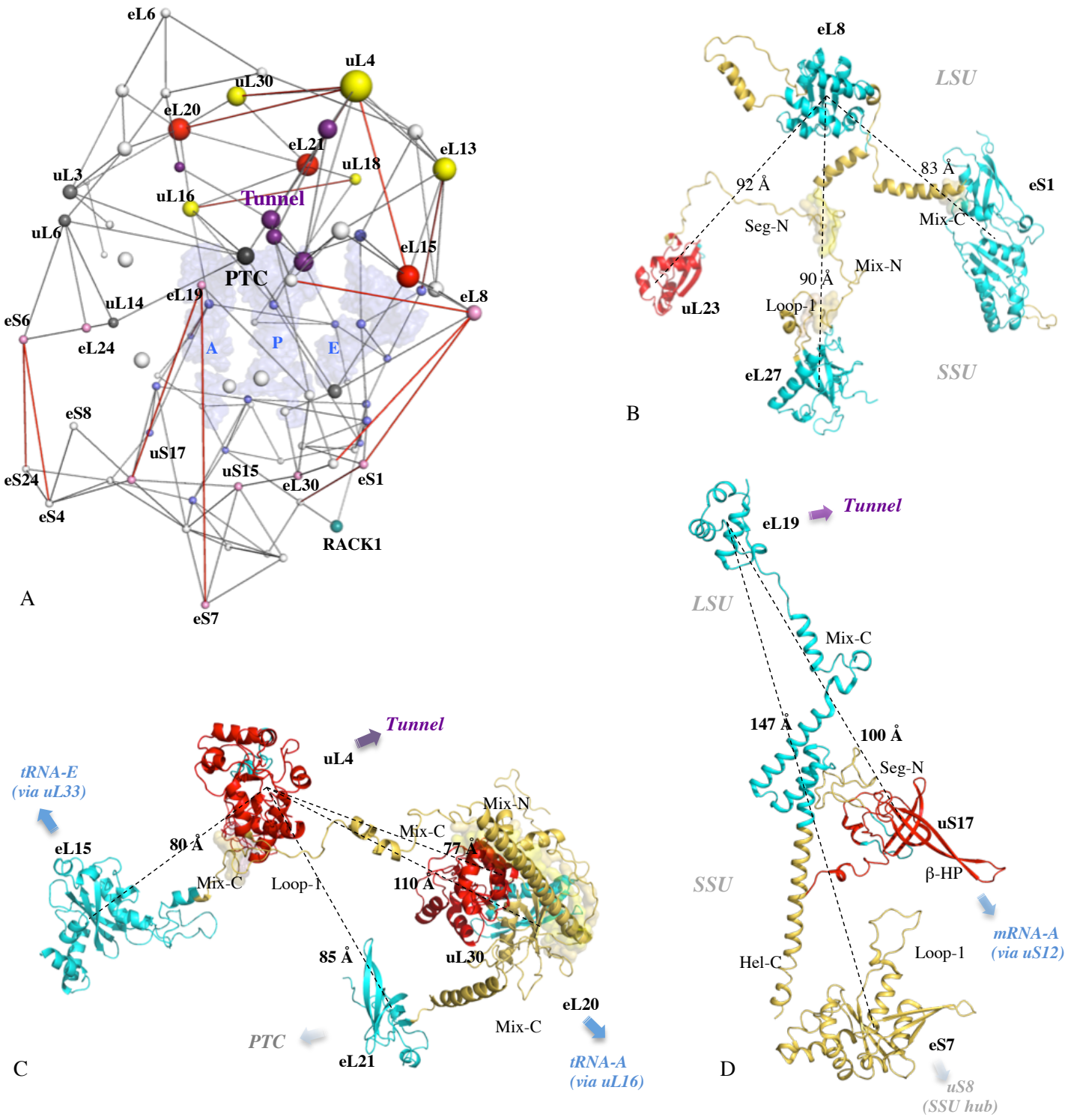
Tracking the fate of the network nodes during the evolutionary transitions also highlights that those undergoing the most important jumps in centrality values, correspond to the proteins that play a critical role in bridging functional modules (Supplementary Fig.19B,C). Furthermore, the distances between these particular nodes are among the largest in the network and are the result of the co-evolution of their extensions (Fig.3). This observation therefore provides a functional meaning to the increase in the size of the extensions and the progressive increase in the distances of the interconnected r-proteins over the course of evolution (Supplementary Table 9): they gradually interconnect distant functional centres. The intra- and inter-connectivity of the functional modules grows indeed steadily over the course of evolution (Fig.4). Within functional clusters, some r-proteins such as uL4 or uL16 have a special fate in that they are selected by the evolution to establish a bridge with other clusters. But another phenomenon consists in the appearance of new bridges between functional modules by proteins that develop proteins between them. Eukaryotes are the only ones to have proteins such as eL15, eL20, eL21 that bridge three distinct categories of functional sites such as PTC, the tunnel and a tRNA binding site (Fig.4).

Overall, graph theory shows that the interconnectivity of the network is not the result of chance, but rather the result of selective pressure that has led to a functional optimization of its architecture for bridging and spreading information between functional modules. Thus, to better understand the evolution of r-proteins, it is necessary to take into account both the increase in their connectivity but also their changing mathematical and functional status in the network.

10.2.5 Discussion

The role of r-proteins and their extensions has long been enigmatic. It has been suggested that due to their charged and dynamic nature, extensions could con-

tribute to ribosome assembly in facilitating the rRNA folding [117–122]. Mutagenic studies have shown, for example, that the N-terminal extensions of uS12, eL8, uL29 and uL30 play a role in the assembly of the bacterial SSU [135] or the eukaryotic LSU [136]. However, other studies demonstrated that extensions participate in both translation and ribosomal assembly [137–140] and many studies from Dinman's group have revealed that they play essential roles in long-range communication between functional sites [111, 141–143]. Several studies have also highlighted the role of r-protein communication for efficient translation and its link to yeast signalling pathways [144]. The dramatic expansion of r-protein networks during evolution fits well with these studies and opens new perspectives towards more complex extension functions than rRNA assembly or stabilization.


eL6
uL4
uL30
eL20
eL21
eL13
uL18
uL3
uL16
Tunnel
uL6
eL15
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eL19
eL8
uL14
eS6
eL24
A
P
E
eS8
uS17
uS15
eS24
eL30
eS1
eS4
RACK1
eS7
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eL8
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B
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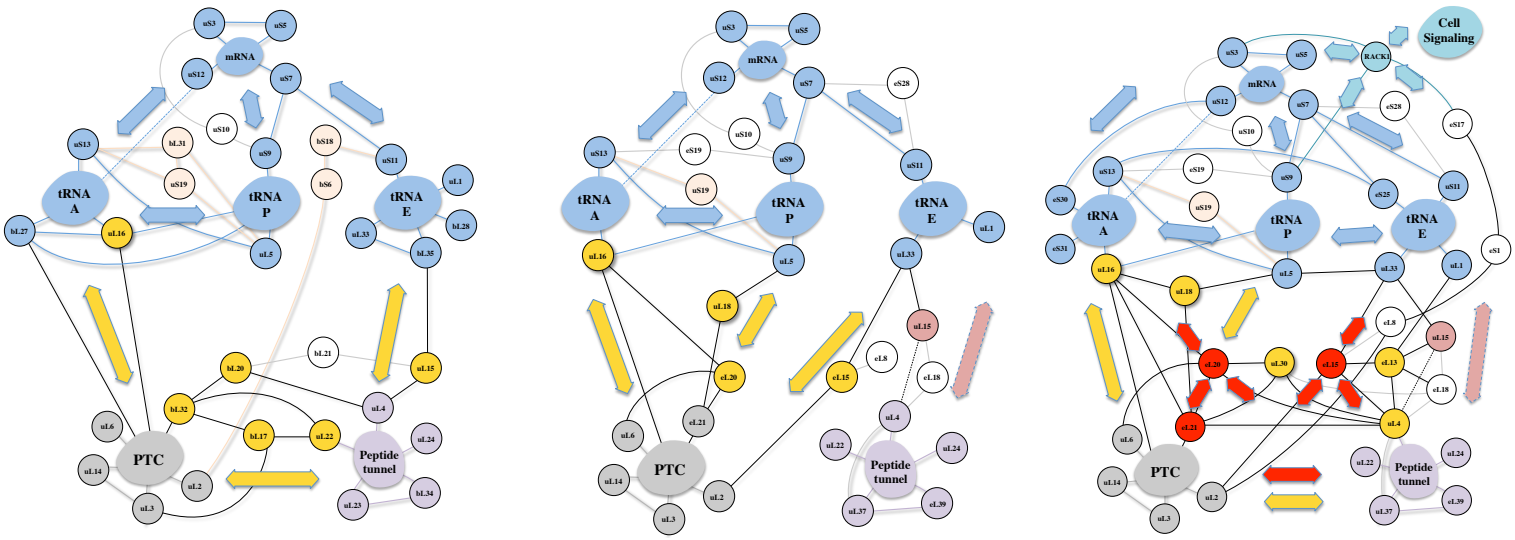
Tunnel
LSU
Mix-C
147 Å
100 Å
Seg-N
uS17
β-HP
SSU
mRNA-A
(via uS12)
Hel-C
Loop-1
eS7
uS8
(SSU hub)
D
Tunnel
uL4
tRNA-E
(via uL33)
Mix-N
eL15
80 Å
Mix-C
77 Å
Mix-C
Loop-1
110 Å
85 Å
uL30
eL20
Mix-C
PTC
eL21
tRNA-A
(via uL16)
C

Figure 3

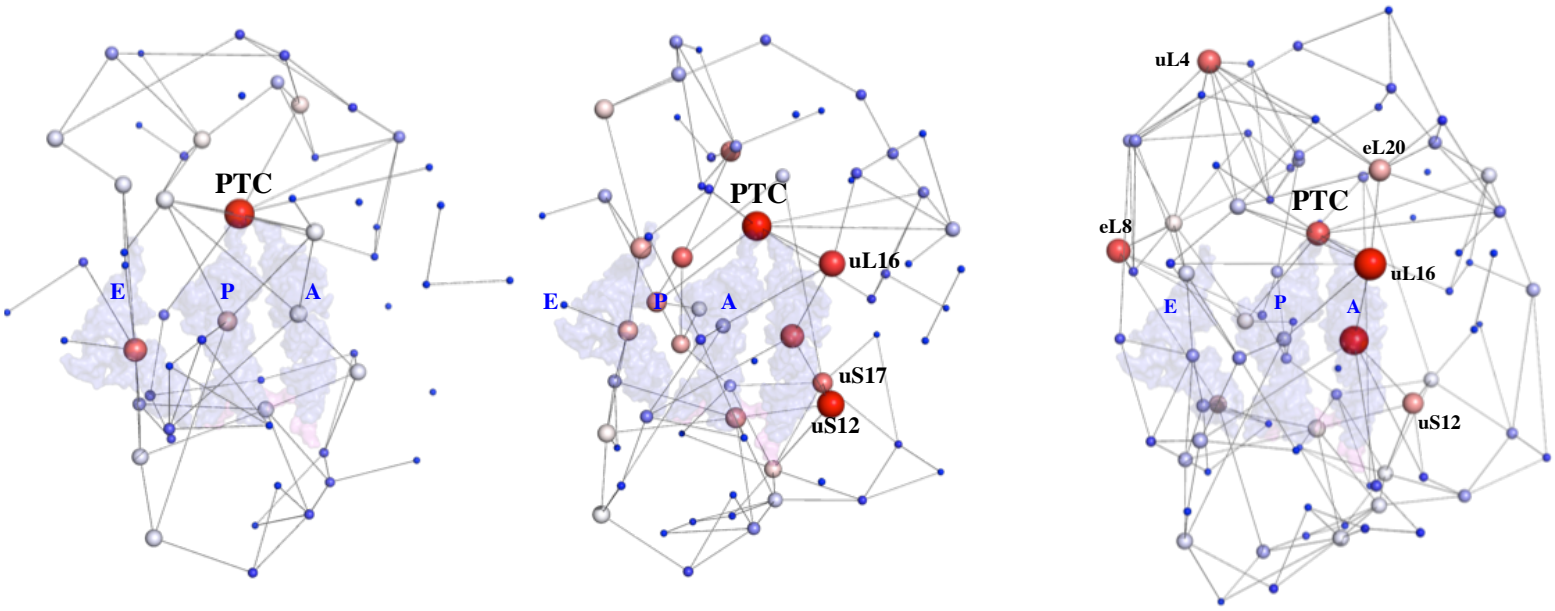
Figure 3: Extensions coevolve to connect remote functional modules. (A) Graph of the eukaryotic network with the largest distances between interconnected centres of mass are represented in red. The sphere radius are proportional to the values of the eigenvector centrality and are coloured according to their functional status (Supplementary Fig. 11): light blue: mRNA or tRNA functional modules; gray: PTC functional module; violet: tunnel functional module; light pink: subunit brides; yellow: node that bridges two distinct functional modules; red: node that bridges three distinct functional modules. The three tRNAs are represented by pale blue transparent surfaces. (B-D) Interconnections of the most distant r-proteins by co-evolution of their extensions. R-protein components are represented according to their evolutionary status.

Our study reveals that network expansion is produced by the collective evolution of r-proteins. The asymmetrical evolution of the two subunits, the heterogeneous landscape of the r-protein evolutionary history and graph theory show that the evolution of network connectivity did not occur at random. On the contrary, the concerted acquisition of new extensions and connections gradually relates the functional modules, progres- sively differentiate the functional status of the nodes and places the functional centres in central positions of the network. Moreover, in the course of evolution, certain connections involving key r-proteins belonging to func- tional modules are also strengthened and give rise to multiple binding. The strong selective pressure that is also expressed at the amino acid establishes an interesting link between the network architectures and the r-protein phylogeny (Supplementary Table 7) and suggests that the networks have gradually evolved sophisticated allosteric pathways within and between r-proteins. The congruence between independent evolutionary traits indicates that the network architectures evolved to relate and optimize the information spread between functional modules.

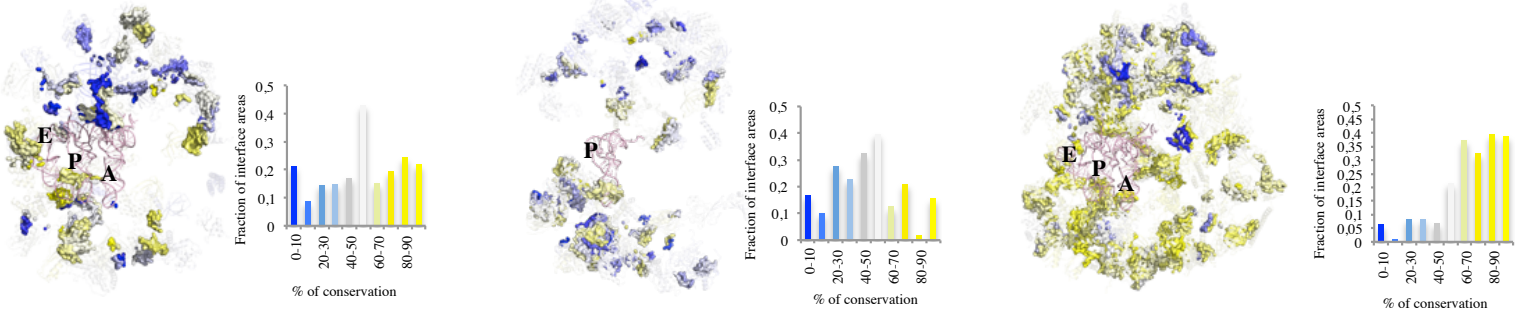
Pathways between functional centres



Betweenness Centrality (shortest paths)



Interface conservations



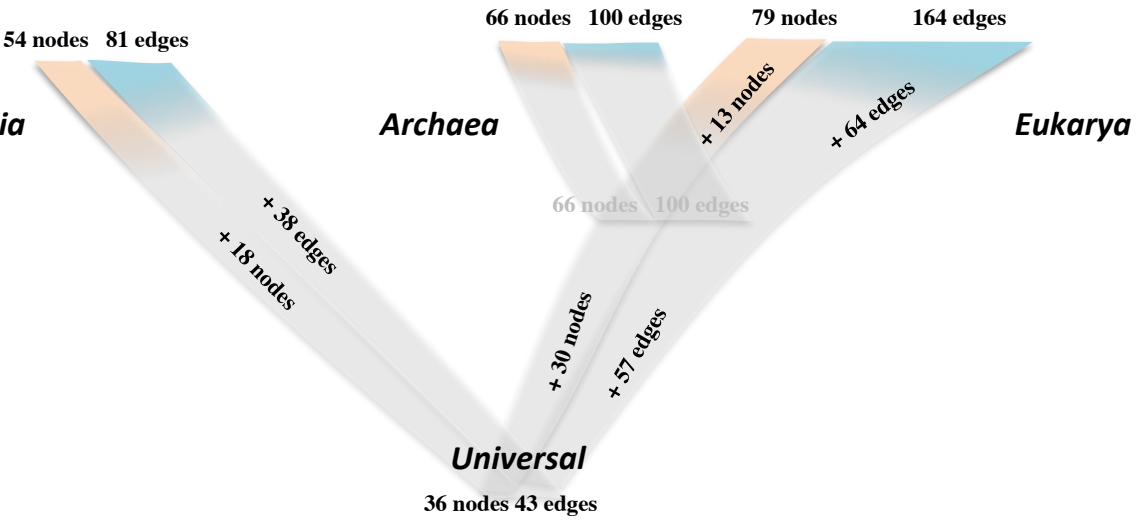


Figure 4

Figure 4: Congruence of evolutionary traits in network expansions. The three kingdoms' networks are represented according to (from top to bottom) their functional, mathematical and phylogenetic properties. The colour codes corresponding to the functional status of the nodes is the same as in Fig. 3. The graphs are represented by connected spheres whose radius and colour gradient (white > red) are proportional to the betweenness centrality values. Largest red sphere correspond to maxima of betweenness centrality. The three tRNAs A, P and E and the mRNA are represented by transparent surfaces. The mapping of the interfaces and the distribution of the conserved surfaces are represented by a colour gradient from blue (0 conservation) to yellow (100% conservation).

Thus, r-proteins may have evolved towards multiple functions. While they likely retained their primary functions devoted to the assembly and stabilization of the rRNA, they may have acquired new functions related to the increasing complexity of ribosome tasks in the course of evolution. Previous studies have revealed that rRNA can also participate in remote communication between functional sites [132]. rRNA and r-protein networks may have also co-evolved for optimizing not only assembly processes but also r-protein synthesis [125]. The expansion of network connectivity indeed follows the increasing accuracy and complexity of the ribosome's tasks from prokaryotes to eukaryotes [106, 108]. The burst of LSU connectivity parallels its gradual specialization in regulating multiple sub-functions that co-emerged with the growth of cellular complexity.

Our study predicts that any changes in the r-proteins that may affect network connectivity should modify the translation efficiency and accuracy and provides new hints to understand both ribosome heterogeneity that plays a key role in translation regulation [145] and diseases associated with r-protein mutations [134]. In addition, the network shortcuts in parasitic or virulent species such as the uL4-uL16 interaction that connect directly the tRNA-A to the tunnel modules in plasmodium or the new bS16-uS4 connection in mycobacterium opens a conceptual framework for therapeutic perspectives. Thus, networking can circumvent the physical constraints associated with growth and the addition of new constituents in an evolving system. Our study suggests that accretion and networking probably co-existed since the earliest life forms.

10.3 Methods

10.3.1 Selection of ribosome structures in the PDB.

The highest resolution X-ray and cryo-EM ribosome structures representative of the three kingdoms (B: Bacteria; A: Archaea; E: Eukarya) have been selected from

the Protein Data Bank (PDB) [146] and used to analyse the r-protein networks and the interactions between the r-protein components: the globular domains (G), the extensions (Ext) or the functional sites (PTC, tunnel, mRNA, tRNAs) [95–100] (Supplementary Table 10). The ribosomes in the three kingdoms of life were structurally aligned using the program pymol [147]. The r-proteins have been named according the new nomenclature [148].

10.3.2 Analysis of r-protein extension evolution.

Protein extensions are distinct parts of the protein structure that protrude from globular domains [92, 93, 122, 123]. They have been divided into five categories according to their structural properties: segments, mix (segments and α-helices), α-helices, loops, and β-hairpins. Internal extensions such as loops or β-hairpins result from insertions within the peptidic chain. External extensions have been added to the N- or C-terminal ends. Equivalent proteins may also differ by extra-domains (dom) across kingdoms. Insertion and deletions (Indels) events in proteins are often used as taxon specific markers [149] . The comparison of the superimposed r-proteins of A, B and E ribosomes has made it possible to follow the evolutionary history of their extensions and domains. The gradual acquisition of extensions during the major evolutionary transitions (ABE $\rightarrow$ A, ABE $\rightarrow$ B and A $\rightarrow$ E) has been then inferred. As previously observed [120], this procedure indicated that bL33 and eL42 that co-locate close to the E-site have the same globular domain but developed different extensions in A, B and E. They were renamed "universal protein uL33". The fact that bL33 belongs to a set of bacterial specific genes located in the clusters containing the universal protein genes [150] supports this hypothesis.

The r-proteins, their extensions and their inter-connections have been compared in the three kingdom's ribosomes and have been coloured according to their evolutionary status (Supplementary Table 2). The same colour code is used throughout the whole manuscript: red: Universal (ABE); blue: Bacteria (B); cyan: common between Archaea and Eukarya (A); yellow: Eukarya (E). The statistics of extension types and sizes in the three kingdoms are based on extensions that are visible in the high-resolution ribosome structures (Supplementary Table 10). Although some extension size variability has been observed in phylogenetic studies [89, 92, 93, 151], the extensions identified in our structural analysis correspond to a representative extension set of each kingdom [91].

10.3.3 Phylogenetic analysis

Due to their complex evolutionary history [92, 93] r-proteins are difficult to align across kingdoms. Thus, r-proteins in their entirety (G + ext) were aligned in each kingdom separately, to characterize their conservation profiles. Eubacterial and archaeal r-proteins sequence alignments have been kindly provided by U. Wolf [151]. A representative dataset of eukaryotic r-proteins covering most of the eukaryotic taxons [126] has been selected from NR and aligned with the program Muscle [152]. The aligned sequences have been then visualized, inspected and analysed with the program Jalview [153]. The quantitative values of the conservation profiles have been retrieved from the LOGOs to calculate the proportions of conserved residues at the PPi interfaces. Residues that occurred in the same position of more than 80% of the sequence were considered as conserved.

10.3.4 Analysis of the inter-molecular contacts.

The networks include all the protein–protein and protein-functional sites (PTC, tunnel, tRNA-sites, mRNA) interactions [83,125,154]. The intermolecular interactions constituting the networks have been calculated in the 3 kingdoms' ribosome structures (Supplementary Table 10). The selected high-resolution structures correspond ribosomes trapped in the elongation cycle. Due to possible X-ray diffraction or cryo-electron-microscopy experimental errors, parts of the structures may be missing or not well defined in the PDB models. Hence, the networks determined here may have omitted a few interactions that have not been detected in the ribosome structures. Scripts systematically calculated the contacts using areaimol from CCP4 [155] with the default parameters (diffmode: "compare" and probe radius 1.4 Å) (Supplementary Fig.4A). This provided an exhaustive list of protein–protein interactions (PPi) for each r-protein network. The distributions and the average values of the interface areas of the whole ribosome, large subunit (LSU), small subunit (SSU) networks shows that they remain globally tiny throughout the evolution (Supplementary Fig.4B,C).

Most of the network contacts are "permanent" within the mature ribosome structure. However transient contacts occur during the elongation cycle. For example, interactions between the two-subunit bridges are formed and broken during the ratchet like motion (uL5-uS13 form different contacts in the two rotational states). In the current models, transiently bound translation factors (such as EF-Tu and EF-G in bacteria) are not included. The universal network described here is a minimal network that only contains the r-proteins conserved in the three kingdoms. However, there are several examples of r-proteins that occupy equivalent sites across kingdom (Supplementary Fig.3). It is therefore possible that a

precursor of such proteins also existed in the ancestral network.

10.3.5 Network representations

To highlight the evolution of connectivity, the networks have been represented in two-dimensional cartoons where the evolutionary statuses of r-proteins, extensions and connections are indicated by different colours and representation codes (Supplementary Fig.3). The interactions between r-protein and functional sites have been taken from the literature and from the present analysis [81–83]. All of the kingdom specific interactions have been listed (Supplementary Table 3). Simplified 3D network models have been used to describe the expansion of the network architecture during evolution. The centres of mass (COM) of the globular domains of interconnected proteins have been linked. The COM of r-proteins devoid of globular domains was calculated using the entire protein coordinates. The distances between the COM of each protein pairs of the network have been systematically calculated. Connections between the three tRNAs and the mRNA are drawn but are not reported in the statistics of r-protein connectivity.

10.3.6 Graph theory analysis

To characterize the networks, several notions of centralities have been defined; centrality indices are a way to capture the centrality of a node in a graph and there are different definitions of centralities (degree, closeness, betweenness, eigenvector...) as there are different notions of "centrality" to capture. These quantities may be an indicator of the graph functionality as for example, whether a graph can transfer and treat information. In order to accept a hypothesis or claim, one needs to have strong evidence against the opposite claim. We therefore assumed that the networks we observed were nothing but random, in other words a typical realisation of a random graph (following an Erdős–Rényi model) [156, 157]; it is our null hypothesis. By studying the law of the centralities under this hypothesis of randomness, we built a test statistic (and a test) that enabled us to decide if it is plausible that the networks we are studying are purely random or not. The answer is that one can reject the hypothesis that the networks are generated randomly (Supplementary Table 8). As one does not have access to the expression of the probability of the centralities under the null hypothesis, one needs to sample with respect to an Erdős–Rényi model the law of the test statistic and from there build a test. This leads to a new class of statistical hypothesis testing and the natural notion of coherence (all the details can be found in the Supplementary Graph Theory).

10.4 Acknowledgements

We thank Guillaume Blanc for his help in the phylogenetic studies of eukaryotic r-proteins and Eric Westhof, Dominique Burnouf, Julie Thompson, Cedric Notredame and Claudio Scazzocchio for valuable discussions.

10.5 Author contributions

Y.T. wrote the manuscript and G.S.P. and D.B. performed the mathematical analysis.

10.6 Supplementary document 1: Mathematical methods, Statistics of centralities

10.6.1 Centralities

In our study of the properties of the ribosomal network we have considered several centralities, let us recall their definition.

Notations: We will note a graph $G = (V, E)$ where V is its collection of vertices and E its collection of edges, in other words subsets of V of cardinal 2; in particular there are at most one edge between two vertices. For any finite W, we note $|W|$ its cardinal and in what follows $n = |V|$.

For $n \in \mathbb{N}$, we will denote the interval $[1, n] \subseteq \mathbb{N}$ as $[n]$.

Definition 10.6.1 (Degree centrality)**.** For any $v \in V$, the degree of v, noted as deg(v), is

$$deg(v) = |\{e \in E| \quad v \in e\}|$$

For any two $u, v \in V$, the length of a shortest path between u and v will be noted $d(u, v)$, a path being a sequence of edges and the length of a path being the size of the sequence. The number of shortest paths between $u, v \in V$ will be denoted $\sigma_{u,v}$.

Definition 10.6.2 (Closeness centrality)**.** The closeness centrality of a node u is given by,

$$C_c(u) = \frac{|V| - 1}{\sum\limits_{v:v \neq u} d(u,v)}$$

A vertex has a high closeness centrality when it is close to many vertices.

Remark 10.6.1. This definition is made according to the python package *Networkx*.

Definition 10.6.3 (Betweenness centrality)**.** For $v, v_1, u \in V$ the number of shortest path between v, v_1 that pass through u is denoted $\sigma_{v,v_1}(u)$.

The betweenness centrality of a vertex u is,

$$C_B(u) = \sum_{\substack{v,v_1: \\ v \neq u \neq v_1}} \frac{\sigma_{v,v_1}(u)}{\sigma_{v,v_1}}$$

A vertex has a high betweenness centrality when many shortest paths pass through it.

Definition 10.6.4 (Eigenvector centrality)**.** Let A be the adjacency matrix of G, and x the eigenvector (normalized for the l^2 norm) of its largest eigenvalue,

$$C_E(u) = x(u)$$

A vertex has high eigenvector centrality when it is specially involved in the heat propagation on the graph.

Remark 10.6.2. The adjacency matrix of a graph $G = (V, E)$ is defined as, for $u, v \in V$, $m_{u,v} = 1[\{u, v\} \in E]$ and there exist a unique normalized eigenvector for the maximal eigenvalue; all its components are strictly positive (Perron-Frobenius).

10.6.2 Sampling and coherence

In general when one wants to build a statistical test one needs to know the law of the test statistic; we show that in fact one can pass by this constraint but at the cost of defining a notion of coherence. Note that this argument is an improvement with respect to the litterature we know.

Sampling a probability law

Let Ω be a universe of events, i.e. a measurable space, P be a probability law on Ω. For a random variable $Y : \Omega \to \Omega'$, i.e. a measurable function between two measurable spaces, we will note $P(Y \in A) = P(\{\omega : \quad Y(\omega) \in A\})$. For example

$P(Id_\Omega \in \Omega) = 1$. We will note $cd(Y)$ the codomain of Y, i.e. Ω'.

Let us consider a random variable X from Ω to $[n]$ and $X_i, i \in [N]$ i.i.d random variables distributed as X, that we will denote as $X_{[N]}$. Then the empirical frequency count is

$$\forall k \in [n], \quad \widehat{XP}(k) := \frac{1}{N} \sum_{i \in [N]} 1[X_i = k]$$

where $1[X_i = k]$ is the indicatrix function of the set $\{X_i = k\}$. Then $\widehat{XP}$ is a random variable from $\Omega \times [n]$ to $[0, 1] \subseteq \mathbb{R}$.

For $\omega \in \Omega$, $\widehat{XP}(\omega)$ is, most of the time, a good approximation of X_*P, i.e. the probability law of X; we will note X_*P as XP; this is a way to bypass the curse of dimensionality by going from exponential to polynomial complexity.

For any subset S of $cd(X)$,

$$P(||(\widehat{XP} - XP)_{|S}||_\infty \geq M) \leq \sum_{k \in S} P(|\widehat{XP}(k) - XP(k)| \geq M)$$
$$\leq \frac{1}{NM^2} \sum_{k \in S} V_P(1[X = k])$$

Where $V_P(1[X = k])$ is the variance of the random variable $1[X = k]$ which is given by $V(1[X = k]) = XP(k)(1 - XP(k))$. So the right hand side can be simplified into, $\frac{1}{NM^2}(\sum_{k \in S} XP(k) - \sum_{k \in S} XP(k)^2)$, which is always smaller than $\frac{1}{NM^2}(1 - \sum_{k \in S} XP(k)^2) \leq \frac{1}{NM^2}$.

So to conclude,

$$P(||(\widehat{XP} - XP)_{|S}||_\infty \geq M) \leq \frac{1}{NM^2} \tag{10.6.1}$$

Therefore, if one wants to sample the marginal probability of X with an error of $\epsilon > 0$ at risk α, we need to have a sample of size, $N = \frac{1}{\alpha \epsilon^2}$. For $M = \epsilon = 0.1$, and $\alpha = 0.05$ one gets $N = 2000$.

For any $S \subseteq cd(X)$, let us denote $\widehat{XP}(X \in S) := \frac{1}{N} \sum_{i \in [N]} 1[X_i \in S]$. Let $Y = 1[X \in S]$, $\widehat{YP}(1) = \frac{1}{N} \sum_{i \in [N]} 1[X_i \in S] = \widehat{XP}(X \in S)$, we get from Equation (10.6.1),

$$P(\widehat{XP}(X \in S) - P(X \in S)) \leq \frac{1}{NM^2} \tag{10.6.2}$$

Conditional test and coherence

We consider a variable T which will be the test statistic. We will assume that T goes from a measurable space Γ to $[n]$. Let us recall the usual decisions rule,

Definition 10.6.5. Let t be a realisation of T on the data; if under H_0, $P(T \geq t|H_0) \leq \alpha$, one rejects the null hypothesis, i.e. H_0.

Remark 10.6.3. $P(T \geq t|H_0)$ is the $p-$value.

Remark 10.6.4. A decision rule is in fact a way to force a binary logic on a non binary object. The usual decision rule 10.6.5 can be written as an indicatix function $1[P(T \geq .) \leq \alpha] : \Omega \to \{0, 1\}$, where Ω is the universe and α is a given risk. So for any risk one has a truth value which corresponds to the rejection of the test and looking at the p-value is looking at all the $\omega \in \Omega$ such that H_0 can be rejected. In particular one concludes that for two tests, $1[P(T \geq .) \leq \alpha] \leq 1[P(T_1 \geq .) \leq \alpha]$, is the same than saying that the rejection for T implies the rejection for T_1.

$\Omega = [m]$, $\Gamma = \Omega$, hypothesis H_0 is that $X = Id_\Omega$ is distributed according to P. Let us choose a sample $x_i, i \in [N]$, in $\Omega^{[N]}$, for $t \in [n]$,

$$\widehat{TP}_{x_{[N]}}(t) = \frac{1}{N} \sum_{i \in [N]} 1[T(x_i) = t]$$

Definition 10.6.6 (Conditional test)**.** Let t be a realisation of T on the data. The conditional test is defined as rejecting the null hypothesis if $\widehat{TP}_{x_{[N]}}(T \geq t) \leq \alpha$[1].

Note that

$$\widehat{TP}_{x_{[N]}}(T \geq t) := \frac{1}{N} \sum_{i \in [N]} 1[T(x_i) \geq t]$$

If $x_{[N]}$ and t are such that

$$|\widehat{TP}_{x_{[N]}}(T \geq t) - P(T \geq t)| > \epsilon \tag{10.6.3}$$

then $\widehat{TP}_{x_{[N]}}(T \geq t) \leq (\alpha - \epsilon)$ implies that $P(T \geq t) \leq \alpha$. This serves as motivation for the conditional test. And one can see that the testing problem is divided

[1]The inequality is not strict as T takes values in a finite set: for example consider T a binary random variable and assume $T = 0$ almost surely, then astrict inequality would lead to an incoherent decision rule

in two steps, the first one is to verifiy that $x_{[N]}$ verifies the inequality 10.6.3 and the second beeing the conditional test.

If we would like to refine the conditional test (Definition 10.6.6), we could ask that the rejection process is taken into consideration only when under H_0, $P(\widehat{TP}_{X_{[n]}}(Id_{cd(T)} \geq T) \leq \alpha) \leq \alpha_1$ for given α_1.

Under H_0 the law of $X_{[N]}$, i.i.d with respect to X, is $\bigotimes_{k\in[N]} P$. Therefore the law of $(X_{[N]}, X)$ is $\bigotimes_{[N+1]} P$ (as X independent of $X_{[N]}$ by hypothesis).

Definition 10.6.7 (Coherent conditional test)**.** For a given couple α, α_1, one rejects the null hypothesis if $\widehat{TP}_{x_{[N]}}(T \geq t) \leq \alpha$ and $P(\widehat{TP}_{X_{[n]}}(Id_{cd(T)} \geq T) \leq \alpha) \leq \alpha_1$.

As we said before it is difficult to compute TP and it was the whole point of the procedure to avoid computing it, however we can get a bound on the probability that appears in the second condition.

Theorem 10.6.1. *If* $\alpha_1 \geq \alpha + 3/(2^{2/3}N^{\frac{1}{3}})$ *then* $P(\widehat{TP}_{X_{[N]}}(Id_{cd(T)} \geq T) \leq \alpha) \leq \alpha_1$

Proof. For any $\epsilon > 0$

$$P(\widehat{TP}_{X_{[N]}}(Id_{cd(T)} \geq T) \leq \alpha) = \int \bigotimes_{[N+1]} P(x_{[N+1]}) 1[\widehat{TP}_{x_{[N]}}(Id_{cd(T)} \geq T(x_{N+1})) \leq \alpha]$$

$$= \int \bigotimes_{[N]} dP(x_{[N]}) \sum d[TP](t) 1[\widehat{TP}_{x_{[N]}}(T \geq t) \leq \alpha)] 1[|\widehat{TP}_{x_{[N]}}(T \geq t) - P(T \geq t)| \leq \epsilon]$$

$$+ \int \bigotimes_{[N]} dP(x_{[N]}) \sum d[TP](t) 1[\widehat{TP}_{x_{[N]}}(T \geq t) \leq \alpha)] 1[|\widehat{TP}_{x_{[N]}}(T \geq t) - P(T \geq t)| > \epsilon]$$

If $\widehat{TP}_{X_{[N]}}[(T \geq t) \leq \alpha]$ and $|\widehat{TP}_{X_{[N]}}(T \geq t) - P(T \geq t)| < \epsilon$ then $P(T \geq t) \leq \alpha + \epsilon$ and the first part of the sum is lower than $\alpha + \epsilon$. The second part of the sum is lower than $\int \bigotimes_{[n]} dP(x_{[N]}) \sum d[TP](t) 1[|\widehat{TP}_{x_{[N]}}(T \geq t) - P(T \geq t)| > \epsilon] \leq 1/N\epsilon^2$ (by Equation (10.6.2). So one gets that for any $\epsilon > 0$,

$$P(\widehat{TP}_{X_{[N]}}(T \geq t) \leq \alpha) \leq \alpha + \epsilon + 1/(N\epsilon^2)$$

One would like to find at least one $\epsilon > 0$ such that $\alpha + \epsilon + 1/(N\epsilon^2) \leq \alpha_1$.

This inequality has solutions if and only if $\alpha_1 \geq \alpha + 3/(2^{2/3}N^{\frac{1}{3}})$.

□

Remark 10.6.5. In the usual test, the consistency of the test is justified by the fact that, under H_0, when N goes to ∞, T is almost surely constant.

In the case of the conditional test and coherent conditional test, there are two possible forms of consistency. When $N \to \infty$, $\widehat{TP}_{X_{[N]}}(T \geq t)$ is almost surely $P(T \geq t)$ under H_0 and this form is satisfied by our method. The second one is to insure that the test converges when both N and n tend to infinity. However n is dependent on the experimental data and not of the method; then this form of consistency makes no sense for the ribosomal networks.

Application to centralities

Our data is the graph of connections inside of a ribosome $g = (V, E)$ and $n = |V|$. We will start with the hypothesis H_0 that the ribosomal networks are random in each kingdom and refute this hypothesis thanks to a test. The model for randomness that we choose, in other word the hypothesis H_0, is the Erdös-Rényi random graph model $G(n, p)$, where n is the number of nodes of the ribosomal network and p is the proportion of edges in the observed graph with respect to the complete graph, i.e. $p = \frac{2|E|}{n(n-1)}$.

Most of the centralities, that we will generically denote as C, take values in $\mathbb{R}$ and we want to give ourselves bins on the values of these centralities; in other words we consider a collection of sets $R_k, k \in [n]$ and consider the random variable for all $x \in V$, $X(x) = \sum k 1[C(G)(x) \in R_k]$. In practive we choose as R_k a disjoint union of sets on the range of values of the centralities of the graph g we observe, the one on wich we want to do a test. X is a finite random variable (as $cd(X)$ is finite).

We now consider the histogram of X, in other words:

$$Y(k) = \frac{1}{n} \sum_{x \in V} 1[C(G)(x) \in \underset{j \leq k}{\cup} R_j]$$

Y is a finite random variable. The test statistic we would like to consider is $T = ||Y - E[Y]||_\infty$, where $E[Y]$ is the expectation of Y under H_0. However we don't have access to $E[Y]$, therefore we sample $E[Y]$ and we get $\widehat{E}[Y]$. The test statistic we consider is $T = ||Y - \widehat{E}[Y]||_\infty$, which is a finite random variable, i.e. $cd(T)$ is finite; the conditional p-values are given in the table of p-values.

In our case $N = 2000$ and so α_1 has to be greater than $\alpha + 0.15$ by Theorem.(10.6.1), for $N = 4000$, α_1 must be greater or equal than $\alpha + 0.119$. We then conclude that H_0 is rejected.

Bibliography

[1] M. Giry, "A categorical approach to probability theory," in *Categorical aspects of topology and analysis*, vol. 915 of *Lecture notes in Mathematics*, pp. 68–85, Springer, 1982.

[2] B. Lawvere, "The category of probabilistic mappings," 1962.

[3] T. Fritz, "A synthetic approach to markov kernels, conditional independence and theorems on sufficient statistics," *Advances in Mathematics*, 2020.

[4] T. Fritz and P. Perrone, "A probability monad as the colimit of spaces of finite samples," *Theory and Applications of Categories*, 2019.

[5] T. Fritz and D. I. Spivak, "Internal probability valuations." Workshop Categorical Probability and Statistics, 2020.

[6] N. Čensov, *Statistical Decision Rules*. Translations of Mathematical Monographs, 1980.

[7] J. C. Baez, T. Fritz, and T. Leinster, "A characterization of entropy in terms of information loss," *Entropy*, 2011.

[8] M. Gromov, "In a search for a structure, part 1: On entropy." `https://www.ihes.fr/~gromov/`, 2013.

[9] M. Marcolli, "Gamma spaces and information," *Journal of Geometry and Physics*, 2019.

[10] Y. I. Manin and M. Marcolli, "Homotopy theoretic and categorical models of neural information networks." arXiv:2006.15136, 2020.

[11] P. Baudot and D. Bennequin, "The homological nature of entropy," *Entropy*, vol. 17, no. 5, pp. 3253–3318, 2015.

[12] J. P. Vigneaux, *Topology of Statistical Systems: A Cohomological Approach to Information Theory*. PhD thesis, Université de Paris, 2019.

[13] H. G. Kellerer, "Masstheoretische Marginalprobleme," *Mathematische Annalen*, vol. 153, no. 3, pp. 168–198, 1964.

[14] S. L. Lauritzen, *Graphical Models.* Oxford Science Publications, 1996.

[15] S. L. Lauritzen, "Interaction models," in *Selected Works of Terry Speed*, ch. 10, pp. 91–94, Springer, 2012.

[16] J. S. Yedidia, W. T. Freeman, and Y. Weiss, "Constructing free-energy approximations and generalized belief propagation algorithms," *IEEE Transactions on information theory*, vol. 51, no. 7, pp. 2282–2312, 2005.

[17] Y. Timsit, G. Sergeant-Perthuis, and D. Bennequin, "Evolution of ribosomal protein network architecture," *Scientific Reports*, 2021.

[18] D. Rudrauf, G. Sergeant-Perthuis, O. Belli, Y. Tisserand, and G. Di Marzo Serugendo, "The role of consciousness in biological cybernetics: emergent adaptive and maladaptive behaviours in artificial agents governed by the projective consciousness model." arXiv:2012.12963, 2020.

[19] T. P. Speed, "A note on nearest-neighbour gibbs and markov probabilities," *Sankhyā: The Indian Journal of Statistics, Series A*, 1979.

[20] T. H. Chan and R. W. Yeung, "Probabilistic inference using function factorization and divergence minimization," in *Towards an Information Theory of Complex Networks: Statistical Methods and Applications* (M. D. et al., ed.), ch. 3, pp. 47–74, Springer, 2011.

[21] F. Matúš, "Discrete marginal problem for complex measures," *Kybernetika*, vol. 24, no. 1, pp. 36–46, 1988.

[22] R. Davidson, "Determination of confounding," in *Stochastic Analysis* (D. Kendall and E. Harding, eds.), John Wiley and Sons, 1973.

[23] S. J. Haberman, "Direct products and linear models for complete factorial tables," *Annals of Statistics*, 1975.

[24] H.-O. Georgii, *Gibbs Measures and Phase Transitions.* de Gruyter, 1988.

[25] G.-C. Rota, "On the foundations of combinatorial theory I. Theory of Möbius functions," *Probability theory and related fields*, vol. 2, no. 4, pp. 340–368, 1964.

[26] O. Peltre, "Homology of Message-Passing Algorithms." `http://opeltre.github.io`, 2020. Ph.D. thesis (preprint).

[27] O. Peltre, "A homological approach to belief propagation and Bethe approximations," in *International Conference on Geometric Science of Information*, pp. 218–227, Springer, 2019.

[28] J. Pearl, *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. Morgan Kaufmann Publishers, 1988.

[29] R. W. Yeung, *A First Course in Information Theory*. Springer, 2002.

[30] A. P. Dawid, "Separoids: A mathematical framework for conditional independence and irrelevance," *Annals of Mathematics and Artificial Intelligence 3*, 2001.

[31] G. Sergeant-Perthuis, "Intersection property and interaction decomposition." arXiv:1904.09017v1, 2019.

[32] N. Bourbaki, *Théorie des ensembles*. Springer, 1939.

[33] H. Langseth, "The Hammersley-Clifford Theorem and its impact on modern statistics."

[34] G. Sergeant-Perthuis, "Bayesian/graphoid intersection property for factorisation models." arXiv:1903.06026v1, 2019.

[35] F. Loregian, "Coend calculus." arXiv:1501.02503, 2019.

[36] Y. G. S. Sinai, *Theory of phase transitions: Rigourous results*. Pergamon Press, 1982.

[37] T. Lévy, "Mesures gaussiennes et espaces de fock," 2003.

[38] C. Heunen, "On the functor l^2," in *Computation, Logic, Games, and Quantum Foundations* (G. Goos, J. Hartmanis, and J. v. Leeuwen, eds.), pp. 107–121, Springer, 2013.

[39] P. Jordan and J. Von Neumann, "On inner products in linear, metric spaces," *Annals of Mathematics*, 1935.

[40] R. Godement, *Topologie algébrique et théorie des faisceaux*. No. v. 1 in Actualités scientifiques et industrielles, Hermann, 1964.

[41] E. Carlsson, G. Carlsson, and V. De Silva, "An algebraic topological method for feature identification," *International Journal of Computational Geometry & Applications*, vol. 16, no. 04, pp. 291–314, 2006.

[42] R. Ghrist, "Barcodes: the persistent topology of data," *Bulletin of the American Mathematical Society*, vol. 45, no. 1, pp. 61–75, 2008.

[43] J. Curry, *Sheaves, cosheaves and applications.* PhD thesis, The University of Pennsylvania, 2013. arXiv:1303.3255.

[44] E. Zeeman, "Dihomology: I. relations between homology theories," *Proceedings of the London Mathematical Society*, vol. 3, no. 1, pp. 609–638, 1962.

[45] J. Hansen and R. Ghrist, "Toward a spectral theory of cellular sheaves," 2018.

[46] M. Robinson, "Understanding networks and their behaviors using sheaf theory," in *2013 IEEE Global Conference on Signal and Information Processing*, pp. 911–914, IEEE, 2013.

[47] M. Robinson, C. Joslyn, E. Hogan, and C. Capraro, "Conglomeration of heterogeneous content using local topology," tech. rep., American University, Mar, 2015.

[48] M. Robinson, "Sheaves are the canonical data structure for sensor integration," *Information Fusion*, vol. 36, pp. 208–224, 2017.

[49] S. M.-H. Mansourbeigi, *Sheaf Theory as a Foundation for Heterogeneous Data Fusion.* PhD thesis, Utah State University, 2010.

[50] S. Abramsky and A. Brandenburger, "The sheaf-theoretic structure of non-locality and contextuality," *New Journal of Physics*, vol. 13, no. 11, p. 113036, 2011.

[51] S. Abramsky, R. S. Barbosa, K. Kishida, R. Lal, and S. Mansfield, "Contextuality, Cohomology and Paradox," in *24th EACSL Annual Conference on Computer Science Logic (CSL 2015)* (S. Kreutzer, ed.), vol. 41 of *Leibniz International Proceedings in Informatics (LIPIcs)*, (Dagstuhl, Germany), pp. 211–228, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2015.

[52] S. Bressan, J. Li, S. Ren, and J. Wu, "The embedded homology of hypergraphs and applications," *Asian Journal of Mathematics*, vol. 23, no. 3, pp. 479–500, 2019.

[53] F. Chung and R. Graham, "Cohomological aspects of hypergraphs," *Transactions of the American Mathematical Society*, vol. 334, no. 1, pp. 365–388, 1992.

[54] M. Artin, A. Grothendieck, and J.-L. Verdier, *Théorie des Topos et Cohomologie Étale des Schémas: Séminaire de Géométrie Algébrique du Bois-Marie 1963/64 - SGA 4. Tome 1.* Lecture notes in mathematics, Springer-Verlag, 1972.

[55] J. P. Vigneaux, "Information structures and their cohomology," *Theory and Applications of Categories*, vol. 35, no. 38, pp. 1476–1529, 2020.

[56] G. Sergeant-Perthuis, "Around intersection property and interactiond decomposition," 2020.

[57] G. Sergeant-Perthuis, "Interaction decomposition for presheaves." arXiv:2008.09029, 2020.

[58] E. H. Spanier, *Algebraic Topology.* McGraw-Hill, 1966.

[59] P. Griffiths and J. Harris, *Principles of Algebraic Geometry.* Wiley Classics Library, John Wiley & Sons, 1978.

[60] J. Barmak, *Algebraic Topology of Finite Topological Spaces and Applications.* Lecture Notes in Mathematics, Springer Berlin Heidelberg, 2011.

[61] O. Caramello, *Theories, Sites, Toposes: Relating and Studying Mathematical Theories Through Topos-theoretic 'bridges'.* Oxford University Press, 2018.

[62] M. Artin, A. Grothendieck, and J.-L. Verdier, *Théorie des Topos et Cohomologie Étale des Schémas: Séminaire de Géométrie Algébrique du Bois-Marie 1963/64 - SGA 4. Tome 2.* Lecture notes in mathematics, Springer-Verlag, 1972.

[63] O. Caramello, "A topos-theoretic approach to Stone-type dualities." arXiv:1103.3493, 2011.

[64] I. Moerdijk, *Classifying Spaces and Classifying Topoi.* Lecture Notes in Mathematics, Springer Berlin Heidelberg, 2006.

[65] J. H. Whitehead, "Combinatorial homotopy. I," *Bulletin of the American Mathematical Society*, vol. 55, no. 3, pp. 213–245, 1949.

[66] C. A. Weibel, *An introduction to homological algebra.* Printed in USA: Cambridge University Press, 1994.

[67] A. Grothendieck, "Sur quelques points d'algèbre homologique, I," *Tohoku Mathematical Journal*, vol. 9, no. 2, pp. 119–221, 1957.

[68] T. Stacks Project Authors, "*Stacks Project*." `https://stacks.math.columbia.edu`, 2018.

[69] G. Segal, "Classifying spaces and spectral sequences," *Publications Mathématiques de l'IHÉS*, vol. 34, pp. 105–112, 1968.

[70] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*. Princeton University Press, 1952.

[71] J. May, *Simplicial Objects in Algebraic Topology*. Chicago Lectures in Mathematics, University of Chicago Press, 1992.

[72] S. Eilenberg and J. A. Zilber, "Semi-simplicial complexes and singular homology," *Annals of Mathematics*, pp. 499–513, 1950.

[73] S. Halperin, *Lectures on minimal models*, vol. 9. Gauthier-Villars, 1983.

[74] S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Mathematical Sciences Research Institute Publications, New York, USA: Springer-Verlag, 1992.

[75] T. Leinster, "The Euler characteristic of a category," *Documenta Mathematica*, vol. 13, pp. 21–49, 2008.

[76] J. Gallier and J. Quaintance, "Homology, cohomology, and sheaf cohomology." `https://www.seas.upenn.edu/~jean/sheaves-cohomology.pdf`, 2020.

[77] M. Hairer, "Ergodic properties of markov processes." `http://www.hairer.org/notes/Markov.pdf`, 2018.

[78] A. Ang, "Projected gradient algorithm."

[79] D. Bennequin, O. Peltre, G. Sergeant-Perthuis, and J. P. Vigneaux, "Extra-fine sheaves." arXiv:2009.12646, 2020.

[80] T. M. Schmeing and V. Ramakrishnan, "What recent ribosome structures have revealed about the mechanism of translation," *Nature*, 2009.

[81] S. Klinge, F. Voigts-Hoffmann, M. Leibundgut, and N. Ban, "Atomic structures of the eukaryotic ribosome," *Trends Biochem. Sci.*, 2012.

[82] S. e. a. Melnikov, "One core, two shells: bacterial and eukaryotic ribosomes," *Nat. Struct. Mol. Biol.*, 2012.

[83] D. N. Wilson and J. H. Doudna Cate, "The structure and function of the eukaryotic ribosome," *Cold Spring Harb Perspect Biol*, 2012.

[84] M. J. e. a. Belousoff, "Ancient machinery embedded in the contemporary ribosome," *Biochem. Soc. Trans.*, 2010.

[85] H. Grosjean and E. Westhof, "An integrated, structure- and energy-based view of the genetic code," *Nucleic Acids Res.*, 2016.

[86] A. S. e. a. Petrov, "History of the ribosome and the origin of translation," *Natl. Acad. Sci. U.S.A.*, 2015.

[87] N. A. Kovacs, A. S. Petrov, K. A. Lanier, and L. D. Williams, "Frozen in time: The history of proteins," *Mol. Biol. Evol.*, 2017.

[88] V. Alva, J. Söding, and A. N. Lupas, "A vocabulary of ancient peptides at the origin of folded proteins," *Elife*, 2015.

[89] O. Lecompte, R. Ripp, J.-C. Thierry, D. Moras, and O. Poch, "Comparative analysis of ribosomal proteins in complete genomes: an example of reductive evolution at the domain scale," *Nucleic Acids Res.*, 2002.

[90] H. Hartman, P. Favaretto, and T. F. Smith, "The archaeal origins of the eukaryotic translational system," *Archaea*, 2006.

[91] P. Forterre, "The universal tree of life: an update," *Front Microbiol*, 2015.

[92] S. Melnikov, K. Manakongtreecheep, and D. Söll, "Revising the structural diversity of ribosomal proteins across the three domains of life," *Mol. Biol. Evol.*, 2018.

[93] P. Vishwanath, P. Favaretto, H. Hartman, S. C. Mohr, and T. F. Smith, "Ribosomal protein-sequence block structure suggests complex prokaryotic evolution with implications for the origin of eukaryotes," *Mol. Phylogenet. Evol.*, 2004.

[94] N. Ban, P. Nissen, J. Hansen, P. B. Moore, and T. A. Steitz, "The complete atomic structure of the large ribosomal subunit at 2.4 a resolution," *Science*, 2000.

[95] A. e. a. Ben-Shem, "The structure of the eukaryotic ribosome at 3.0 Å resolution," *Science*, 2011.

[96] J.-P. e. a. Armache, "Promiscuous behaviour of archaeal ribosomal proteins: implications for eukaryotic ribosome evolution," *Nucleic Acids Res.*, 2013.

[97] R. M. Voorhees, I. S. Fernández, S. H. W. Scheres, and R. S. Hegde, "Structure of the mammalian ribosome-sec61 complex to 3.4 Å resolution," *Cell*, 2014.

[98] W. e. a. Wong, "Cryo-em structure of the plasmodium falciparum 80s ribosome bound to the anti-protozoan drug emetine," *Elife*, 2014.

[99] H. Khatter, A. G. Myasnikov, S. K. Natchiar, and B. P. Klaholz, "Structure of the human 80s ribosome," *Nature*, 2015.

[100] J. e. a. Noeske, "High-resolution structure of the escherichia coli ribosome," *Nat. Struct. Mol. Biol.*, 2015.

[101] Y. S. Polikanov, S. V. Melnikov, D. Söll, and T. A. Steitz, "Structural insights into the role of rrna modifications in protein synthesis and ribosome assembly," *Nat. Struct. Mol. Biol.*, 2015.

[102] X. e. a. Zhang, "Structures and stabilization of kinetoplastid-specific split rrnas revealed by comparing leishmanial and human ribosomes," *Nat Commun*, 2016.

[103] J. e. a. Hentschel, "The complete structure of the mycobacterium smegmatis 70s ribosome," *Cell Rep*, 2017.

[104] Z. e. a. Li, "Cryo-em structures of the 80s ribosomes from human parasites trichomonas vaginalis and toxoplasma gondii," *Cell Res.*, 2017.

[105] P.-D. e. a. Coureux, "Cryo-em study of start codon selection during archaeal translation initiation," *Nat Commun*, 2016.

[106] K. Joshi, L. Cao, and P. J. Farabaugh, "The problem of genetic code misreading during protein synthesis," *Yeast*, 2019.

[107] C. Peña, E. Hurt, and V. G. Panse, "Eukaryotic ribosome assembly, transport and quality control," *Nat. Struct. Mol. Biol.*, 2017.

[108] D. Balchin, M. Hayer-Hartl, and F. U. Hartl, "In vivo aspects of protein folding and quality control," *Science*, 2016.

[109] H. S. Zaher and R. Green, "Fidelity at the molecular level: lessons from protein synthesis," *Cell*, 2009.

[110] N. Polacek and A. S. Mankin, "The ribosomal peptidyl transferase center: structure, function, evolution, inhibition," *Crit. Rev. Biochem. Mol. Biol.*, 2005.

[111] A. Meskauskas and J. D. Dinman, "A molecular clamp ensures allosteric coordination of peptidyltransfer and ligand binding to the ribosomal a-site," *Nucleic Acids Res.*, 2010.

[112] S. Pechmann, F. Willmund, and J. Frydman, "The ribosome as a hub for protein quality control," *Mol. Cell*, 2013.

[113] D. N. Wilson, S. Arenz, and R. Beckmann, "Translation regulation via nascent polypeptide-mediated ribosome stalling," *Curr. Opin. Struct. Biol.*, 2016.

[114] A. Korostelev, D. N. Ermolenko, and H. F. Noller, "Structural dynamics of the ribosome," *Curr Opin Chem Biol*, 2008.

[115] S. O. e. a. Sulima, "Eukaryotic rpl10 drives ribosomal rotation," *Nucleic Acids Res.*, 2014.

[116] S. P. e. a. Gulay, "Tracking fluctuation hotspots on the yeast ribosome through the elongation cycle," *Nucleic Acids Res.*, 2017.

[117] Y. Timsit, Z. Acosta, F. Allemand, C. Chiaruttini, and M. Springer, "The role of disordered ribosomal protein extensions in the early steps of eubacterial 50 s ribosomal subunit assembly," *Int J Mol Sci*, 2009.

[118] Y. Timsit, F. Allemand, C. Chiaruttini, and M. Springer, "Coexistence of two protein folding states in the crystal structure of ribosomal protein l20," *EMBO Rep.*, 2006.

[119] D. N. Wilson and K. H. Nierhaus, "Ribosomal proteins in the spotlight," *Crit. Rev. Biochem. Mol. Biol.*, 2005.

[120] D. J. Klein, P. B. Moore, and T. A. Steitz, "The roles of ribosomal proteins in the structure assembly, and evolution of the large ribosomal subunit," *J. Mol. Biol.*, 2004.

[121] D. E. Brodersen, W. M. Clemons, A. P. Carter, B. T. Wimberly, and V. Ramakrishnan, "Crystal structure of the 30 s ribosomal subunit from thermus thermophilus: structure of the proteins and their interactions with 16 s rna," *J. Mol. Biol.*, 2002.

[122] Z. e. a. Peng, "A creature with a hundred waggly tails: intrinsically disordered proteins in the ribosome," *Cell. Mol. Life Sci.*, 2014.

[123] O. Poirot and Y. Timsit, "Neuron-like networks between ribosomal proteins within the ribosome," *Sci Rep*, 2016.

[124] A.-L. Barabási and Z. N. Oltvai, "Network biology: understanding the cell's functional organization," *Nat. Rev. Genet.*, 2004.

[125] Y. Timsit and D. Bennequin, "Nervous-like circuits in the ribosome facts, hypotheses and perspectives," *Int J Mol Sci*, 2019.

[126] F. Burki, A. J. Roger, M. W. Brown, and A. G. B. Simpson, "The new tree of eukaryotes," *Trends Ecol. Evol. (Amst.)*, 2020.

[127] S. C. Lovell and D. L. Robertson, "An integrated view of molecular coevolution in protein-protein interactions," *Mol. Biol. Evol.*, 2010.

[128] S. K. Burley and G. A. Petsko, "Aromatic-aromatic interaction: A mechanism of protein structure stabilization.," *Science*, 1985.

[129] D. A. Dougherty, "The cation-π interaction," *Acc Chem Res*, 2013.

[130] B. L. Schottel, H. T. Chifotides, and K. R. Dunbar, "Anion–π interactions," *Chem. Soc. Rev.*, 2007.

[131] N. J. Zondlo, "Aromatic–proline interactions: Electronically tunable ch/π interactions.," *Acc. Chem. Res.*, 2013.

[132] H. David-Eden and Y. Mandel-Gutfreund, "Revealing unique properties of the ribosome using a network based analysis," *Nucleic Acids Res.*, 2008.

[133] R. Rakauskaite and J. D. Dinman, "rrna mutants in the yeast peptidyltransferase center reveal allosteric information networks and mechanisms of drug resistance," *Nucleic Acids Res.*, 2008.

[134] S. O. Sulima, I. J. F. Hofman, K. De Keersmaecker, and J. D. Dinman, "How ribosomes translate cancer," *Cancer Discov*, 2017.

[135] D. Calidas, H. Lyon, and G. M. Culver, "The n-terminal extension of s12 influences small ribosomal subunit assembly in escherichia coli," *RNA*, 2014.

[136] B. Tutuncuoglu, J. Jakovljevic, S. Wu, N. Gao, and J. L. Woolford, "The n-terminal extension of yeast ribosomal protein l8 is involved in two major remodeling events during late nuclear stages of 60s ribosomal subunit assembly," *RNA*, 2016.

[137] O. e. a. Galkin, "Roles of the negatively charged n-terminal extension of saccharomyces cerevisiae ribosomal protein s5 revealed by characterization of a yeast strain containing human ribosomal protein s5," *RNA*, 2007.

[138] R. M. Voorhees, A. Weixlbaumer, D. Loakes, A. C. Kelley, and V. Ramakrishnan, "Insights into substrate stabilization from snapshots of the peptidyl transferase center of the intact 70s ribosome," *Nat. Struct. Mol. Biol.*, 2009.

[139] A. e. a. Fernández-Pevida, "The eukaryote-specific n-terminal extension of ribosomal protein s31 contributes to the assembly and function of 40s ribosomal subunits," *Nucleic Acids Res.*, 2016.

[140] M. G. e. a. Lawrence, "The extended loops of ribosomal proteins ul4 and ul22 of escherichia coli contribute to ribosome assembly and protein translation," *Nucleic Acids Res.*, 2016.

[141] M. H. J. Rhodin and J. D. Dinman, "A flexible loop in yeast ribosomal protein l11 coordinates p-site trna binding," *Nucleic Acids Res.*, 2010.

[142] I. e. a. Kisly, "The functional role of el19 and eb12 intersubunit bridge in the eukaryotic ribosome," *J. Mol. Biol.*, 2016.

[143] J. e. a. Mailliot, "Crystal structures of the ul3 mutant ribosome: Illustration of the importance of ribosomal proteins for translation efficiency," *J. Mol. Biol.*, 2016.

[144] N. Singh, S. Jindal, A. Ghosh, and A. A. Komar, "Communication between rack1/asc1 and us3 (rps3) is essential for rack1/asc1 function in yeast saccharomyces cerevisiae," *Gene*, 2019.

[145] N. R. Genuth and M. Barna, "The discovery of ribosome heterogeneity and its implications for gene regulation and organismal life," *Mol. Cell.*, 2018.

[146] H. M. e. a. Berman, "The protein data bank," *Nucleic Acids Res*, 2000.

[147] "The pymol molecular graphics system, version 2.0 schrödinger, llc. 69. ban, n. et al. a new system for naming ribosomal proteins. curr. opin."

[148] N. e. a. Ban, "A new system for naming ribosomal proteins," *Curr. Opin. Struct. Biol.*, 2014.

[149] P. Ajawatanawong and S. L. Baldauf, "Evolution of protein indels in plants, animals and fungi," *BMC Evol. Biol.*, 2013.

[150] E. Roberts, A. Sethi, J. Montoya, C. R. Woese, and Z. Luthey-Schulten, "Molecular signatures of ribosomal evolution," *Proc. Natl. Acad. Sci. U.S.A.*, 2008.

[151] N. Yutin, P. Puigbò, E. V. Koonin, and Y. I. Wolf, "Phylogenomics of prokaryotic ribosomal proteins," *PLOS One*, 2012.

[152] R. C. Edgar, "Muscle: multiple sequence alignment with high accuracy and high throughput," *Nucleic Acids Res.*, 2004.

[153] P. V. Troshin, J. B. Procter, and G. J. Barton, "Java bioinformatics analysis web services for multiple sequence alignment–jabaws:msa," *Bioinformatics*, 2011.

[154] D. Graifer and G. Karpova, "Roles of ribosomal proteins in the functioning of translational machinery of eukaryotes," *Biochimie*, 2015.

[155] M. D. e. a. Winn, "Overview of the ccp4 suite and current developments," *Acta Crystallogr. D Biol. Crystallogr.*, 2011.

[156] D. R. Cox and D. V. Hinkley, *Theoretical Statistics.* CRC Press, 1974.

[157] B. Bollobas, *Random graphs.* Cambrigde University Press, 2001.

[158] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology.* Princeton Mathematical Series, 1952.

[159] T. Tao, "245b, notes 7: Well-ordered sets, ordinals, and zorn's lemma (optional)," January 2009.

[160] S. Mac Lane, *Categories for the Working Mathematician.* Springer, 1978.

[161] E. Mendelson, *Introduction to Mathematical Logic.* Chapman & Hall, 1997.

[162] J. E. Besag, "Spatial interaction and the statistical analysis of lattice systems," *J. Roy. Statist. Soc., Series B*, 1974.

[163] M. Averintsev, "A method of describing discrete parameter random fields," *Problemy Peredaci Informatsii*, 1970.

[164] R. Dobrushin, "The description of a random field by means of its conditional probabilities and conditions of its regularity," *Theory Probability and Appl.*, 1968.

[165] G. R. Grimmet, "A theorem about random fields," *Bull. Lond. Math. Soc.*, 1973.

[166] J. Moussouris, "Gibbs and markov random systems with constraints," *J. Statistical Physics*, 1974.

[167] C. Preston, "Generalised gibbs states and markov random fields," *Advances in Applied Probability*, 1973.

[168] C. Preston, "Gibbs states on countable sets," *Cambridge University Press*, 1974.

[169] C. Preston, "Random fields," *Springer-Verlag, Lectures notes in Mathematics*, 1976.

[170] S. Sherman, "Markov random fields and gibbs random fields," *Israel J. Math*, 1978.

[171] F. Spitzer, "Random fields and interacting particle systems." Notes on lectures given at the MAA Summer Seminar.

[172] P. Suomela, "Factorrings of finite-dimensional distributions," *Commentationes Physico-Mathematics*, 1972.

[173] P. Suomela, "Construction of nearest-neighbour systems," *Annales Acad. Scientiarum Fenn. Ser. A. Math. Diss.*, 1976.

[174] P. Suomela, "Dependence stuctures of dicrete random fields," *Report on Computer Science, Economiscs and Statistisc*, 1978.

[175] N. N. Vorob'ev, "Consistent families of measures and their extensions," *Theory Probability and Appl.*, 1962.

[176] N. N. Vorob'ev, "Markov measures and markov extensions," *Theory Probability and Appl.*, 1963.

[177] N. Vorob'ev, "Consistent families of measures and their extensions," *Theory of Probability & Its Applications*, vol. 7, no. 2, pp. 147–163, 1962.

[178] J. P. Vigneaux, "Generalized information structures and their cohomology." arXiv:1709.07807, 2017.

[179] D. Dugger and D. C. Isaksen, "Topological hypercovers and 1-realizations," *Mathematische Zeitschrift*, vol. 246, no. 4, pp. 667–689, 2004.

[180] A. C. van Enter, R. Fernández, and A. D. Sokal, "Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of gibbsian theory," *Journal of Statistical Physics*, 1993.