Mean-field stochastic control with constraints in law

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Chapter 1

Introduction

Mean-field stochastic control problems are stochastic control problems where the state dynamic as well as the criteria to optimize depend, in a non trivial way, on the statistical distribution of the system. They arise from two directions. On the one hand they are a convenient extension to the classical theory of stochastic control. They allow for cost criteria which are non-linear functionals of the probability distribution of the system. On the other hand, they arise as "mean-field" limit for control problems of large number of interacting agents. The theory was largely developed in the last ten years, in connection with the sibling theory of mean-field games. Mean-field games were introduced independently by Lasry and Lions in [100, 101, 102] and Huang, Caines, Malhamé in [29] and [30]. It aims at describing stochastic differential games when the number of players tends to infinity. We refer to the textbooks [14, 36, 43, 44], as well as the lecture notes [5, 34, 40, 56, 119] and the video lectures [104] for a full account of the theory.

From a theoretical point of view, the main questions in mean-field control theory fall into two categories. On the one hand, one wants to solve the mean-field stochastic control problem. This means finding the optimal value that can be achieved and, if possible, determining an optimal control. On the other hand, one would like to properly understand the connection between the mean-field problem and control problems for large number of interacting agents.

The goal of this introduction is threefold. First, we want to introduce the main questions of the theory and some available results from the literature. We do not aim for exhaustiveness and this is rather an opportunity to introduce basic concepts that will appear throughout the manuscript. Secondly, we want to address specifically mean-field control problems with constraints in law which are the main subject of this dissertation. Finally we present the contributions of this thesis.

1.1 Mean-field stochastic control

In their simplest version, stochastic mean-field control problems take the following form. The dynamic of the system is described by a stochastic differential equation of McKean-Vlasov type

$$dX_t = b(X_t, \mathcal{L}(X_t), \alpha_t)dt + \sqrt{2}dB_t. \tag{1.1}$$

The state of the system X_t is a random variable taking values in $\mathbb{R}^d$. We denote by $\mathcal{L}(X_t)$ the probability distribution of X_t, sometimes referred to as its law. In the above equation, $(\alpha_t)_{0\leqslant t\leqslant T}$ is the control process, taking values in the control set A. Moreover the stochastic differential equation is driven by a standard Brownian motion $(B_t)_{0\leqslant t\leqslant T}$. The goal is to choose the control in order to minimize over the time interval $[0, T]$ a cost in expectation form

$$J(\alpha) = \mathbb{E}\left[\int_0^T L(X_t, \alpha_t)dt + \int_0^T \mathcal{F}(\mathcal{L}(X_t))dt + \mathcal{G}(\mathcal{L}(X_T))\right]. \tag{1.2}$$

In the above expression, L is a running cost which depends on the state of the system and the control. The running cost $\mathcal{F}$ and the terminal cost $\mathcal{G}$ are of "mean-field" type. This means that they are functions of the probability distribution $\mathcal{L}(X_t)$ of the process X_t. They are defined over $\mathcal{P}(\mathbb{R}^d)$, the infinite dimensional space of Borel probability measures over $\mathbb{R}^d$.

This problem exceeds the scope of the standard theory of stochastic control (see [68]) because of this dependence of b, $\mathcal{F}$ and $\mathcal{G}$ on $\mathcal{L}(X_t)$. Consequently, knowing the state of the system X_t at time t is not sufficient to act optimally. One needs as well to know the full probability distribution of X_t. This leads to an intrinsically infinite dimensional problem and requires analysis as well as differential calculus tools over $\mathcal{P}(\mathbb{R}^d)$. For the same reasons, the dynamic programming approach requires new techniques to be adapted to the mean-field setting. The resulting Hamilton-Jacobi-Bellman (HJB) equation has to be stated over $\mathcal{P}(\mathbb{R}^d)$ and the standard techniques to study HJB equations break-down in infinite dimension.

The study of mean-field control problems is partly motivated because they arise as limit of control problems for large numbers of interacting agents. The pre-limit problem involves a huge number of systems and, presumably the "mean-field" limit can be easier to handle. The N-agent problem can be described as follows. Consider $N \geqslant 1$ agents —or players, or particles— denoted $X_t^{1,N}, \ldots, X_t^{N,N}$ evolving according to the stochastic differential equations

$$dX_t^{i,N} = b(X_t^{i,N}, \widehat{\mu}_t^{N,x}, \alpha_t^{i,N})dt + \sqrt{2}dB_t^{i,N}, \tag{1.3}$$

where $\widehat{\mu}_t^{N,x} := \dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{X_t^{i,N}}$ is the empirical distribution of the agents. The Brownian motions are independent and they represent the private noises to which each agent is subject. We assume that a central planner chooses the controls $(\alpha_t^{i,N})_{0\leqslant t\leqslant T}$ for each agent, in order to minimize the cost

$$J^N(((\alpha_t^{i,N})_{0\leqslant t\leqslant T})_{1\leqslant i\leqslant N}) := \mathbb{E}\left[\int_0^T \frac{1}{N}\sum_{i=1}^N L(X_t^{i,N}, \alpha_t^{i,N})dt + \int_0^T \mathcal{F}(\widehat{\mu}_t^{N,x})dt + \mathcal{G}(\widehat{\mu}_T^{N,x})\right]. \tag{1.4}$$

To understand, at least formally, the connection between the N-agent problem and the mean-field problem we can look for controls of the form

$$\alpha_t^{i,N} = \alpha(t, X_t^{i,N}, \widehat{\mu}_t^{N,x}),$$

for some function $\alpha : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to A$. This means that we expect that the optimal strategy for each player is a (common) function of its state $X_t^{i,N}$, time $t \in [0,T]$ and the distribution of the population $\widehat{\mu}_t^{N,x}$ at time t. Provided α is regular enough with respect to its arguments and the agents are initialized from independent and identically distributed (iid) initial positions, propagation of chaos dictates that the players become increasingly independent as their number goes to infinity, see [79, 114, 126]. Moreover the dynamic of a typical player is given by

$$\begin{cases} dX_t = b(X_t, \mathcal{L}(X_t), \alpha(t, X_t, \mu(t))dt + \sqrt{2}dB_t \\ \mu(t) = \mathcal{L}(X_t) \end{cases} \tag{1.5}$$

and the (stochastic) empirical distribution $\widehat{\mu}_t^{N,x}$ of the players converges to the (deterministic) measure $\mu(t)$. This means that a typical player does not interact anymore with the rest of the population but rather with its own statistical distribution. At the limit, the cost functional takes the form (1.2). We have reduced the complexity of the problem, from N to one player. The price to pay is that the dynamic of the system and the cost to minimize now depend on $\mathcal{L}(X_t)$. The rigorous connection between Problem (1.13) and Problem (1.10) is unfortunately not obvious. Indeed it is not clear that passing to the mean-field limit in the system and optimizing over the controls are commutative operations.

In the rest of this introduction we present the main approaches developed in the literature in order to study the mean-field problem and its connection with the N-agent problem.

1.1.1 Optimal control of SDEs of McKean-Vlasov type and the Pontryagin maximum principle

Depending on the objective to achieve, it is convenient to formulate the mean-field control problem alternatively as an optimal control problem for stochastic differential equations, as an optimal control problem for a parabolic equation, the Fokker-Planck equation and finally, as an optimization problem over a set of probability measures satisfying martingale constraints.

The first and most natural formulation of the control problem goes as follows. We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ endowed with a d-dimensional Brownian motion $B = (B_t)_{0 \leqslant t \leqslant T}$. An admissible control is a measurable process taking values in a control space A and satisfying the integrability condition $\mathbb{E}\left[\int_0^T |\alpha_t|^2 dt\right] < +\infty$. If we assume that b satisfies suitable Lipschitz continuity conditions, for any initial position $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$, and any admissible $(\alpha_t)_{0 \leqslant t \leqslant T}$, there is a unique solution to the McKean-Vlasov stochastic differential equation

$$X_t = X_0 + \int_0^t b(X_t, \mathcal{L}(X_t), \alpha_t)dt + \sqrt{2}B_t, \qquad \forall t \in [0,T]. \tag{1.6}$$

We seek to minimize the cost

$$J((\alpha_t)_{0 \leqslant t \leqslant T}) := \mathbb{E}\left[\int_0^T L(X_t, \alpha_t)dt + \int_0^T \mathcal{F}(\mathcal{L}(X_t))dt + \mathcal{G}(X_T)\right],$$

over all the admissible controls.

This problem can be solved, under specific regularity and convexity assumptions, using the Pontryagin maximum principle. To this end, we introduce, for each admissible control $(\alpha_t)_{0\leqslant t\leqslant T}$ and associated process $(X_t)_{0\leqslant t\leqslant T}$, an adjoint state $(Y_t, Z_t)_{0\leqslant t\leqslant T}$ as the solution of the backward stochastic differential equation,

$$\begin{cases} dY_t = -\left[\partial_x b(X_t, \mathcal{L}(X_t), \alpha_t)Y_t + \partial_x L(X_t, \alpha_t) + D_m\mathcal{F}(\mathcal{L}(X_t), X_t)\right]dt + Z_t dB_t, \\ Y_T = D_m\mathcal{G}(\mathcal{L}(X_T), X_T), \end{cases} \tag{1.7}$$

where "D_m" denotes a suitable derivative with respect to the measure variable. See Section 1.3 of this introduction for the exact definition. Under specific regularity and growth assumptions on the data $b, L, \mathcal{F}$ and $\mathcal{G}$ which guarantee among other things, for each admissible control $(\alpha_t)_{0\leqslant t\leqslant T}$ and associated process $(X_t)_{0\leqslant t\leqslant T}$, the well-posedness of (1.7), the Pontryagin maximum principle can be stated as follows.

Theorem 1.1. *Assume that the control set A is convex and*

$$a \mapsto b(x, \mu, a).y + L(x, a)$$

is convex over A for all $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$. Then we have the following necessary condition for optimality: if $(\alpha_t)_{0\leqslant t\leqslant T}$ is an optimal control and $(X_t)_{0\leqslant t\leqslant T}$ is the associated optimally controlled process, then, for all $a \in A$,

$$b(X_t, \mathcal{L}(X_t), \alpha_t).Y_t + L(X_t, \alpha_t) \leqslant b(X_t, \mathcal{L}(X_t), a).Y_t + L(X_t, a),$$

$dt \otimes \mathbb{P}$-almost surely, where $(X_t, Y_t, Z_t)_{0\leqslant t\leqslant T}$ is solution to the forward-backward system of stochastic differential equations

$$\begin{cases} dX_t = b(X_t, \mathcal{L}(X_t), \alpha_t)dt + \sqrt{2}dB_t, \\ dY_t = -\left[\partial_x b(X_t, \mathcal{L}(X_t), \alpha_t)Y_t + \partial_x L(X_t, \alpha_t) + D_m\mathcal{F}(\mathcal{L}(X_t), X_t)\right]dt + Z_t dB_t, \\ Y_T = D_m\mathcal{G}(\mathcal{L}(X_T), X_T). \end{cases} \tag{1.8}$$

Conversely, let $(\alpha_t)_{0\leqslant t\leqslant T}$ be an admissible control with associated process $(X_t)_{0\leqslant t\leqslant T}$ and adjoint process $(Y_t, Z_t)_{0\leqslant t\leqslant T}$. We further assume that $\mathcal{G}$ is (displacement) convex and that

$$(x, \mu, a) \mapsto b(x, \mu, a).Y_t + L(x, a) + \mathcal{F}(\mu)$$

is, $dt \otimes \mathbb{P}$ almost-everywhere, convex. If

$$b(X_t, \mathcal{L}(X_t), \alpha_t).Y_t + L(X_t, \alpha_t) = \inf_{a\in A}\left\{b(X_t, \mathcal{L}(X_t), a).Y_t + L(X_t, a)\right\},$$

$dt \otimes \mathbb{P}$ almost-everywhere, then $(\alpha_t)_{0\leqslant t\leqslant T}$ is an optimal control.

This theorem is an extension of the standard stochastic Pontryagin maximum principle which can be found in [135]. It was proved by Carmona and Delarue (in a more general setting) in [42] where the authors prove as well the well-posedness of the system of forward-backward stochastic differential equations. We refer to the first book of Carmona and Delarue

[43] for a detailed presentation of this result, including a discussion about the notion of convexity required for this theorem to hold.

The Pontryagin maximum principle is, by definition, limited to situations where strong optimal solutions are expected to exist. In the theory of stochastic control, this is more the exception than the rule. Therefore we need to introduce weak formulations to the optimal control problem. This is usually done by introducing measure-valued controls and controlled martingales.

1.1.2 Compactness methods and the martingale problem

The control problem in its weak formulation is described as follows. We denote by $\mathcal{C}^d := \mathcal{C}([0,T],\mathbb{R}^d)$ the path space and $\mathcal{V}$ the control space consisting of measures q over $[0,T]\times A$ with the Lebesgue measure as first marginal and such that $\int_0^T\int_A |a|^2 dq(t,a) < +\infty$. A control $q \in \mathcal{V}$ can be identified as a flow $(q_t)_{t\in[0,T]}$ of probability measures over A by disintegration with respect to its first marginal.

We let (X,Λ) be the identity processes over $(\mathcal{C}^d\times\mathcal{V})$. That is $(X_t,\Lambda_t)(x,q) = (x(t),q_t)$ for any $(x,q)\in\mathcal{C}^d\times\mathcal{V}$ and all $t\in[0,T]$. We look for probability measures m over $\mathcal{C}^d\times\mathcal{V}$ such that X_0 is distributed according to μ_0 under m,

$$\varphi(X_t) - \int_0^t\int_A \mathcal{L}\varphi(X_s, m_s^x, a)d\Lambda_s(a)ds$$

is a martingale under m for all smooth compactly supported $\varphi:\mathbb{R}^d\to\mathbb{R}$, where we use the notations $\mathcal{L}\varphi(t,x,\mu,a) = b(x,\mu,a).D\varphi(x) + \Delta\varphi(x)$ for the generator of the diffusion and $m_s^x := X_s\#m$ for the push-forward of m by X_s. We denote by $\mathcal{R}$ the set of such measures and we look for $m\in\mathcal{R}$ which minimizes the cost function

$$\Gamma(m) := \mathbb{E}^m\left[\int_0^T\int_{\mathbb{R}^d} L(X_t,a)d\Lambda_t(a)dt\right] + \int_0^T \mathcal{F}(m_t^x)dt + \mathcal{G}(m_T^x).$$

It is important to notice that a "strong" control $(\alpha_t)_{0\leqslant t\leqslant T}$ defined on some probability space $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$ with associated controlled process $(X_t)_{0\leqslant t\leqslant T}$ solution to the McKean-Vlasov SDE (1.6) induces a relaxed control $(X, dt\delta_{\alpha_t}(da)))\#\mathbb{P}\in\mathcal{P}(\mathcal{C}^d\times\mathcal{V})$. We denote by $\mathcal{R}_s$ the set of such strong controls.

Under appropriate assumptions on b to ensure the well-posedness of (1.6) for constant controls and appropriate assumptions on $L,\mathcal{F},\mathcal{G}$ and μ_0 to ensure the lower semi-continuity and coercivity of Γ, Lacker proved -in a more general and difficult setting where the dynamics have controlled, possibly degenerate volatility- the following result which can be found in [96].

Theorem 1.2. *Optimal relaxed controls exist. Moreover, the infimum over the relaxed controls is the same as the infimum over the strong controls*

$$\inf_{m\in\mathcal{R}}\Gamma(m) = \inf_{m\in\mathcal{R}_s}\Gamma(m).$$

Under additional convexity assumptions, one can prove the existence of Markovian controls, that is controls $m \in \mathcal{R}$ such that $m(\Lambda_t = \delta_{\alpha(t,X_t)}),$ for a.e. $t \in [0,T]) = 1$ for some measurable function $\alpha : [0,T] \times \mathbb{R}^d \to A$. Assume that, for each $(x,\mu) \in \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ the set

$$K(x,\mu) := \{(b(x,\mu,a), z) : a \in A, z \leqslant L(x,a) + \mathcal{F}(\mu)\} \subset \mathbb{R}^d \times \mathbb{R}$$

is convex. Under this additional assumption, the following result is proved in [96].

Theorem 1.3. *For all $m \in \mathcal{R}$, there exists a Markovian control $\widetilde{m} \in \mathcal{R}$ satisfying $\widetilde{m}^x_t = m^x_t$ for all $t \in [0,T]$, as well as $\Gamma(\widetilde{m}) \leqslant \Gamma(m)$. In particular there exists an optimal Markovian control.*

To prove the existence of optimal solutions, one usually relies on Aldou's criteria to prove compactness of solutions to stochastic differential equations with appropriate uniform time regularity. Uniform time regularity for candidates $m \in \mathcal{R}$ such that $\Gamma(m) \leqslant C$ for some $C > 0$ follows from the martingale constraint and energy estimates obtained thanks to the coercivity of Γ. This is achieved, for instance in Proposition 3.5 of [96]. The approximation of relaxed controls by strong controls is particularly technical when the volatility coefficient depends on the control, a situation that we do not consider here.

The deep connection between solutions to stochastic differential equations and solutions to martingale problems is demonstrated in the seminal work of Stroock and Varadhan [125]. It was broadly used in the context of stochastic control, see [94], and notably for applications in large deviations theory [27, 60, 64, 65]. In the theory of stochastic control, two milestones are the papers [62] and more recently [96]. In the first one, the authors investigate the existence of optimal Markovian control under very broad assumptions, using the martingale problem approach. In [96], Lacker generalizes these results to mean-field systems and rigorously connects the mean-field problem and problems for large number of interacting particles. More recently these results were further extended to controlled McKean-Vlasov SDEs with a common noise in [59].

1.1.3 Optimal control of the Fokker-Planck equation and the mean-field game system of partial differential equations

Mean-field stochastic control problems are control problems for stochastic processes. However it is convenient —especially for the type of constraints presented in Section 1.2 of this introduction— to directly formulate them as control problems for the law of this processes. If α_t has the form $\alpha_t = \alpha(t, X_t)$ then the law $\mu(t) := \mathcal{L}(X_t)$ of X_t solves, by Itô's lemma, the equation

$$\begin{cases} \partial_t \mu + \operatorname{div}(b(x,\mu(t),\alpha(t,x))\mu) - \Delta\mu = 0, \quad \text{in } (0,T)\times\mathbb{R}^d, \\ \mu(0) = \mathcal{L}(X_0). \end{cases} \tag{1.9}$$

This is a non-linear parabolic partial differential equation (pde) named the Fokker-Planck equation. We refer to [16] for its general theory, and to [63, 129] for its in connections with stochastic differential equations. The cost functional (1.2) can be rewritten solely as a

function of μ and α and we are led to consider the new problem

$$\inf_{(\alpha,\mu)} \int_0^T \int_{\mathbb{R}^d} L(x, \alpha(t,x)) d\mu(t)(x)dt + \int_0^T \mathcal{F}(\mu(t))dt + \mathcal{G}(\mu(T)), \tag{1.10}$$

where the minimum is taken over couples (α, μ) satisfying the Fokker-Planck equation (1.9). We are now facing a deterministic control problem for a (non-linear) parabolic pde.

In the special case where we control directly the drift of the diffusion, that is $b(x, \mu, \alpha) = \alpha$, Problem (1.10) has been widely studied in the literature in the context of potential mean-field games. We denote by $H(x,p) := \sup_{q\in\mathbb{R}^d} \{-p.q - L(x,q)\}$ the Hamiltonian of the system and assume that L is convex and coercive with respect to the control variable.

Theorem 1.4. *Under these conditions, optimal solutions (α, μ) exist and satisfy*

$$\alpha = -\partial_p H(x, Du),$$

for some solution (u, μ) to the mean-field game system of partial differential equations

$$\begin{cases} -\partial_t u + H(x, Du) - \Delta u = \dfrac{\delta \mathcal{F}}{\delta m}(\mu(t), x) & \text{in } (0,T)\times\mathbb{R}^d, \\ \partial_t \mu - \operatorname{div}(D_p H(x, Du)\mu) - \Delta\mu = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ \mu(0) = \mu_0, u(T,x) = \dfrac{\delta \mathcal{G}}{\delta m}(\mu(T), x) & \text{in } \mathbb{R}^d. \end{cases} \tag{1.11}$$

The first equation is a backward Hamilton-Jacobi-Bellman equation satisfied by an adjoint state u and from which we derive the optimal control α. The second equation is a forward Fokker-Planck equation satisfied by the optimal solution. The existence of solutions and the derivation of the optimality conditions can first be found in the seminal work [102], see also [24] and Chapters 2 and 3 of this dissertation. Usually the necessary conditions are obtained through convex duality techniques, using generally the Fenchel-Rockafellar theorem as in [24, 37] or the Von-Neumann theorem as in [115]. The form of the optimality conditions as well as the techniques to derive them are reminiscent of the the Benamou-Brenier formulation of optimal transport, see [12].

This system of partial differential equations is known as the mean-field game system. In mean-field game theory, u is the value function of an infinitesimal player. The other unknown μ represents the density of the players at equilibrium.

The mean-field game system was introduced by Lasry and Lions in [102]. Since then, its analysis has generated numerous works and we refer to [14, 36, 43, 44, 81, 100, 101, 102] for existence and uniqueness results in various contexts. System (1.11) has a particular form among general mean-field game systems. Indeed, the source term and the terminal condition for the HJB equation are derivatives of the costs $\mathcal{F}$ and $\mathcal{G}$. We refer to these games as potential mean-field games and they exhibit some particular structural conditions. They allow for instance for the manipulation of weak solutions to the pde system, as in [37, 39]. They also permit to discuss the stability of solutions as in [24]. The study of their long time behavior was also pursued in [38, 108]. System (1.11) was also analyzed for its connections with mean-field control problems in the early works [2, 3, 4].

1.1.4 Connection with control problems for large number of interacting agents.

In this section we explain how Problem (1.10) arises as limit problem for large number of interacting agents. To this end, consider $N \geqslant 1$ agents (or players, or particles) $(X_t^{1,N}, \dots, X_t^{N,N})$, evolving according to the stochastic differential equations

$$\begin{cases} dX_t^{i,N} = b(X_t^{i,N}, \alpha_t^{i,N}, \widehat{\mu}_t^{N,x})dt + \sqrt{2}dB_t^{i,N}, \\ \widehat{\mu}_t^{N,x} = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \delta_{X_t^{i,N}}, \end{cases} \tag{1.12}$$

starting from independent initial positions $X_0^{1,N}, \dots, X_0^{N,N}$, identically distributed according to $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ and driven by independent Brownian motions. A central planner chooses the controls $(\alpha_t^{i,N})_{0 \leqslant t \leqslant T}$ for each agent, in order to minimize the cost

$$\mathbb{E}\left[\int_0^T \frac{1}{N}\sum_{i=1}^{N} L(X_t^{i,N}, \alpha_t^{i,N})dt + \int_0^T \mathcal{F}(\widehat{\mu}_t^{N,x})dt + \mathcal{G}(\widehat{\mu}_T^{N,x})\right] \tag{1.13}$$

where $(X_t^{i,N})_{0 \leqslant t \leqslant T}$ satisfies the dynamic (1.12) for all $1 \leqslant i \leqslant N$.

We distinguish two main approaches to the convergence problem. On the one hand, we can argue by compactness methods. This is successfully achieved in various contexts in [27, 59, 65, 96]. One first introduces suitable weak formulations of the control problems. We already presented the martingale problem formulation of the mean-field problem. For the N-particle system we proceed similarly. We denote by $(X^{i,N}, \Lambda^{i,N})$ the canonical process on $(\mathcal{C}^d \times \mathcal{V})^N$ and define the empirical measures

$$\widehat{\mu}^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{(X^{i,N}, \Lambda^{i,N})}, \qquad \widehat{\mu}_t^{N,x} := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}.$$

We define $\mathcal{R}^N$ as the set of probability $P_N \in \mathcal{P}((\mathcal{C}^d \times \mathcal{V})^N)$ under which $(X_0^{i,N})_{i=1,\dots,N}$ are iid with law μ_0 and

$$\varphi(X_t^{1,N}, \dots, X_t^{N,N}) - \sum_{i=1}^{N} \int_0^t \int_{\mathbb{R}^d} \mathcal{L}_i^N \varphi(X_s^{1,N}, \dots, X_s^{N,N}, a) d\Lambda_s^{i,N}(a) ds$$

is a martingale under P_N, for all smooth, compactly supported φ with

$$\mathcal{L}_i^N \varphi(x_1, \dots, x_N, \mu, a) := D_{x_i}\varphi(x_1, \dots, x_N).b(x_i, \mu, a) + \Delta_{x_i}\varphi(x_1, \dots, x_N),$$

for all $(x_1, \dots, x_N, \mu, a) \in (\mathbb{R}^d)^N \times \mathcal{P}(\mathbb{R}^d) \times A$. The N-state problem in its weak formulation is therefore to minimize over $P_N \in \mathcal{R}^N$ the cost functional

$$\mathbb{E}^{P_N}\left[\int_0^T \left(\int_{\mathbb{R}^d} \frac{1}{N}\sum_{i=1}^{N} L(X_t^{i,N}, a) d\Lambda_t^{i,N}(a) + \mathcal{F}(\widehat{\mu}_{t,x}^N)\right) dt + \mathcal{G}(\widehat{\mu}_T^{N,x})\right]$$

where $\mathbb{E}^{P_N}$ is the expectation under P_N. In the setting of Theorem (1.2), Lacker proved in [96] the following result, where $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ is endowed with a suitable topology.

Theorem 1.5. *For each $N \geqslant 1$, let $P_N \in \mathcal{R}^N$ be an ϵ_N-optimal control for the N-particle problem, for some sequence $\epsilon_N \to 0$. Then $P_N \# \widehat{\mu}^N$ is precompact in $\mathcal{P}(\mathcal{P}(\mathcal{C}^d \times \mathcal{V}))$ and every limit point is supported on the set $\mathcal{R}^*$ of optimal solutions to the mean-field problem.*

Conversely, if $P \in \mathcal{P}(\mathcal{P}(\mathcal{C}^d \times \mathcal{V}))$ is supported on $\mathcal{R}^*$, then there exists a sequence $\epsilon_N \to 0$ and a sequence of relaxed ϵ_N-optimal controls $P_N \in \mathcal{R}^N$ such that $P_N \# \widehat{\mu}^N \to P$.

There are two key steps to prove the above result. The first one is to prove the compactness of the sequence $(P_N \# \widehat{\mu}^N)_{N \in \mathbb{N}}$ and to identify weak-limit points. This is achieved thanks to uniform time regularity inherited from the martingale constraint and energy estimates derived from the coercivity of Γ. Then we can pass to the limit in the martingale problems. The second step is to show that any relaxed control for the mean-field problem can be approximated by sequences of relaxed controls for the N-particle problems. This means that, for any $m \in \mathcal{R}$, one can find, for all $N \in \mathbb{N}$, $P_N \in \mathcal{R}^N$ such that $P_N \# \widehat{\mu}^N \to \delta_m$ in $\mathcal{P}(\mathcal{P}(\mathcal{C}^d \times \mathcal{V}))$ and $\mathbb{E}^{P_N}\left[\Gamma(\widehat{\mu}^N)\right] \to \Gamma(m)$. This is achieved thanks to coupling techniques from the theory of propagation of chaos.

The other way to prove the convergence and obtain rates of convergence along the way, is to rely on the regularity of the value function for the mean-field problem. We discuss this point in the special case where $b(x, \mu, a) = a$ and $A = \mathbb{R}^d$. The value function $\overline{\mathcal{U}} : [0, T] \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is then defined, for all $(t_0, \mu_0) \in [0, T] \times \mathcal{P}(\mathbb{R}^d)$ by

$$\overline{\mathcal{U}}(t_0, \mu_0) := \inf_{(\mu, \alpha)} \int_{t_0}^T \int_{\mathbb{R}^d} L(x, \alpha(t, x)) d\mu(t)(x) dt + \int_{t_0}^T \mathcal{F}(\mu(t)) dt + \mathcal{G}(\mu(T))$$

where the infimum is taken over the couples (μ, α) in $\mathcal{C}([t_0, T], \mathcal{P}(\mathbb{R}^d)) \times L^2_{dt \otimes \mu(t)}([t_0, T] \times \mathbb{R}^d, \mathbb{R}^d)$ satisfying

$$\begin{cases} \partial_t \mu + \operatorname{div}(\alpha \mu) - \Delta \mu = 0 \quad \text{in } (t_0, T) \times \mathbb{R}^d, \\ \mu(t_0) = \mu_0. \end{cases} \tag{1.14}$$

If $\overline{\mathcal{U}}$ is smooth enough, we can prove that it satisfies the dynamic programming equation

$$\begin{cases} -\partial_t \mathcal{U} + \displaystyle\int_{\mathbb{R}^d} H(x, D_m \mathcal{U}, m) dm(x) - \int_{\mathbb{R}^d} \operatorname{div}_x D_m \mathcal{U}(m, x) dm(x) \\ \qquad\qquad = \mathcal{F}(m) \qquad \text{in } (0, T) \times \mathcal{P}(\mathbb{R}^d), \\ \mathcal{U}(T, m) = \mathcal{G}(m) \qquad \text{in } \mathcal{P}(\mathbb{R}^d). \end{cases} \tag{1.15}$$

Now, let us define $\mathcal{V}^N(t_0, \mathbf{x}^N)$ for $t \in [0, T]$ and $\mathbf{x}^N = (x^{1,N}, \dots x^{N,N}) \in (\mathbb{R}^d)^N$ as the value function for the N-agent problem when the players start at t_0 with $X^{i,N}_{t_0} = x^{i,N}$ for all $1 \leqslant i \leqslant N$. We are faced with a standard stochastic control problem and we can show under appropriate regularity assumptions on the data, that $\mathcal{V}^N$ is a strong solution to the HJB equation

$$\begin{cases} -\partial_t \mathcal{V}^N(t,\mathbf{x}^N) + \dfrac{1}{N}\displaystyle\sum_{i=1}^N H(x^{i,N}, ND_{x^{i,N}}\mathcal{V}^N(t,\mathbf{x}^N)) - \sum_{i=1}^N \Delta_{x^{i,N}}\mathcal{V}^N(t,\mathbf{x}^N) \\ \qquad\qquad = \mathcal{F}(\dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{x^{i,N}}) \qquad \text{in } (0,T)\times(\mathbb{R}^d)^N, \\ \mathcal{V}^N(T,\mathbf{x}^N) = \mathcal{G}(\dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{x^{i,N}}) \qquad \text{in } (\mathbb{R}^d)^N. \end{cases} \tag{1.16}$$

For all $N \geqslant 1$ we can project $\overline{\mathcal{U}}$ onto $[0,T]\times(\mathbb{R}^d)^N$ and define the function $\overline{\mathcal{V}}^N$ for all $t \in [0,T]$ and $\mathbf{x}^N = (x^{1,N}, \dots x^{N,N}) \in (\mathbb{R}^d)^N$ by

$$\overline{\mathcal{V}}^N(t,\mathbf{x}^N) = \overline{\mathcal{U}}(t, \frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}}).$$

Using equation (1.15) we can prove that $\overline{\mathcal{V}}^N$ solves Equation (1.16) up to a small error term, that is

$$\begin{cases} -\partial_t \overline{\mathcal{V}}^N(t,\mathbf{x}^N) + \dfrac{1}{N}\displaystyle\sum_{i=1}^N H(x^{i,N}, ND_{x^{i,N}}\overline{\mathcal{V}}^N(t,\mathbf{x}^N)) - \sum_{i=1}^N \Delta_{x^{i,N}}\overline{\mathcal{V}}^N(t,\mathbf{x}^N) \\ \qquad\qquad = \mathcal{F}(\dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{x^{i,N}}) + E_N(t,\mathbf{x}^N) \qquad \text{in } (0,T)\times(\mathbb{R}^d)^N, \\ \overline{\mathcal{V}}^N(T,\mathbf{x}^N) = \mathcal{G}(\dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{x^{i,N}}) \qquad \text{in } (\mathbb{R}^d)^N, \end{cases} \tag{1.17}$$

with

$$E_N(t,\mathbf{x}^N) := \frac{-1}{N^2}\sum_{j=1}^N \mathrm{Tr}(D^2_{mm}\overline{\mathcal{U}}(t, \frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}}, x^{j,N}, x^{j,N}).$$

See Section 1.3 for the definition of the second order derivative D^2_{mm}. If $D^2_{mm}\overline{\mathcal{U}}$ is bounded, we have that $|E_N| \leqslant \frac{1}{N}$ and by a standard comparison principle for parabolic equations, we can deduce the estimate

$$|\overline{\mathcal{U}}(t_0, \frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}}) - \mathcal{V}^N(t_0,\mathbf{x}^N)| \leqslant \frac{C}{N}$$

for some $C > 0$ independent from $N \geqslant 1$, $t_0 \in [0,T]$ and $\mathbf{x}^N \in (\mathbb{R}^d)^N$.

Moreover, still assuming that $\overline{\mathcal{U}}$ is a smooth function, it provides us, for each initial positions (t,μ), with a Lipschitz continuous optimal feedback control

$$\alpha^*(t,\mu,x) := -\partial_p H(x, D_m\overline{\mathcal{U}}(t,\mu,x))$$

for the mean-field problem. We can test this control for the population of particles and deduce proper propagation of chaos results for the optimal trajectories of the N-particle problem, see [80]. Therefore the difficulty comes down to proving the desired regularity for the value function $\overline{\mathcal{U}}$ over $[0,T] \times \mathcal{P}(\mathbb{R}^d)$. This is not merely a technical challenge or a by-product of the regularity of the data and this cannot be achieved without structural conditions which guarantee, at least, the uniqueness of solutions to the control problem for each initial position. In [36] Chapter 3.7 such regularity is proved (in a compact setting) under suitable convexity assumptions on the costs $\mathcal{F}$ and $\mathcal{G}$, by analyzing the mean-field game system (1.11) and suitable linearizations of this system around the initial condition μ_0. A similar approach is also carried on in [42].

Finally we mention other approaches to the convergence problem. For problems with finite state space or for problems where all the players face the same noise, one can use techniques from the theory of viscosity solutions to HJB equations to prove the convergence of the value functions. This respectively done in [78] and [47]. For the deterministic setting (namely when problem (1.10) is considered without diffusion), we refer to [46, 71, 72, 78].

1.2 Mean-field stochastic control with constraints in law

Chapters 2, 3 and 4 of this manuscript are dedicated to the optimization problem (1.10) when the solution μ is constrained to satisfy, either the terminal constraint

$$\Psi(\mu(T)) \leqslant 0, \tag{1.18}$$

in Chapter 2, or, in Chapters 3 and 4, the state constraint

$$\Psi(\mu(t)) \leqslant 0, \qquad \forall t \in [0,T], \tag{1.19}$$

where $\Psi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ satisfies similar regularity conditions as $\mathcal{F}$ and $\mathcal{G}$.

In this section we give some motivations to study these constrained problems.

Constraint in law. Constraints in law arise naturally in applications in economy and finance, as a way to control the risk associated with a given strategy. The most striking examples are the Mean-Variance portfolio selection problem, see [107], and models with value at risk constraints as in [92]. Probability constraints are also a convenient way to relax almost-sure constraints. In the classical super-hedging problem, the controlled process must match almost surely a given random variable at the terminal time. This might be too hard to replicate. To tackle this issue, Föllmer and Leukert introduced in [70] the method of quantile hedging which can be seen as a stochastic control problem with constraints in law.

Mean-field constraint. For mean-field control problems, the constraint can also arise as a limit "mean-field" constraint. We refer for instance to the works of Seguret [121] in connection with the problem of smart charging of electric vehicles. Mean-field constraint also arise when the density of a population of agents is constrained to remain below a threshold.

This problem leads to local constraints (the constraint is a function of the density of the law of state process) and it was addressed in [39, 57, 109, 110, 120].

Large deviations for interacting diffusion processes. Finally we explain an application to large deviations theory. Consider N interacting (non-controlled) particles $X_t^{i,N}$ subject to the dynamics

$$\begin{cases} dX_t^{i,N} = b(X_t^{i,N}, \widehat{\mu}_t^{N,x})dt + dB_t^{i,N} \quad \text{for } t \in [0,T], \\ X_0^{i,N} = x_0^{i,N} \in \mathbb{R}^d, \end{cases} \tag{1.20}$$

where $(B_t^{1,N}), \dots, (B_t^{N,N})$ are N independent Brownian motions, $\widehat{\mu}_t^{N,x} = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$ is the empirical distribution of the particles and $x_0^{1,N}, \dots, x_0^{N,N}$ are deterministic initial positions in $\mathbb{R}^d$ such that $\frac{1}{N}\sum_{i=1}^N \delta_{x_0^{i,N}}$ converges to some μ_0 in $\mathcal{P}(\mathbb{R}^d)$. We introduce the rate function $I : \mathcal{C}([0,T], \mathcal{P}(\mathbb{R}^d)) \to \mathbb{R}$ defined by

$$I(\mu) := \inf_{(\alpha)} \int_0^T \int_{\mathbb{R}^d} \frac{1}{2}|\alpha(t,x)|^2 d\mu(t)(x)dt \tag{1.21}$$

where the infimum is taken over the $\alpha \in L^2_{\mu(t)\otimes dt}([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ such that the time marginals $(\mu(t))_{0\leqslant t\leqslant T}$ of μ satisfy the Fokker-Planck equation

$$\begin{cases} \partial_t \mu + \operatorname{div}([b(x,\mu(t)) + \alpha(t,x)]\,\mu) - \frac{1}{2}\Delta\mu = 0 \quad \text{in } (0,T)\times\mathbb{R}^d, \\ \mu(0) = \mu_0. \end{cases} \tag{1.22}$$

In their seminal work [55], Dawson and Gärtner proved that the family of probability measures $(\widehat{\mu}^{N,x})_{N\geqslant 1}$ satisfies a large deviation principle with rate function I (in [55] the rate function is not exactly (1.21) but the two formulations are equivalent, see for instance [13]). This means that, for suitable subsets $\mathcal{K} \subset \mathcal{C}([0,T], \mathcal{P}(\mathbb{R}^d))$, it holds

$$\lim_{N\to+\infty} \frac{1}{N} \log \mathbb{P}\left[\widehat{\mu}^N \in \mathcal{K}\right] = -\inf_{\mu\in\mathcal{K}} I(\mu).$$

Therefore, if one is interested in the asymptotic of the probabilities $\mathbb{P}\left[\Psi(\widehat{\mu}_T^{N,x}) \leqslant 0\right]$ as N tends to infinity, one is led to minimize (1.21) over couples $(\mu,\alpha) \in \mathcal{C}([0,T], \mathcal{P}(\mathbb{R}^d)) \times L^2_{m(t)\otimes dt}([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ satisfying (1.22) in the sense of distributions as well as the terminal constraint $\Psi(\mu(T)) \leqslant 0$. If one is interested in the asymptotic of the probabilities $\mathbb{P}\left[\Psi(\widehat{\mu}_t^{N,x}) \leqslant 0, \forall t \in [0,T]\right]$, one is led to solve the same problem with, this time, the state constraint $\Psi(\mu(t)) \leqslant 0$, $\forall t \in [0,T]$.

We refer to the works [26, 27, 64] for the deep connections between mean-field control and the theory of large deviations for weakly interacting diffusions and how the weak convergence methods introduced in Section 1.1.2 can be used to prove the large deviations principle.

1.3 The space of probability measures

In this section we recall some basic facts about the space of probability measures $\mathcal{P}(\mathbb{R}^d)$, that will be used throughout the manuscript.

Metric aspect. In this dissertation, we mostly focus on subsets $\mathcal{P}_p(\mathbb{R}^d)$ of $\mathcal{P}(\mathbb{R}^d)$, for $p \geqslant 1$. The space $\mathcal{P}_p(\mathbb{R}^d)$ is the set of Borel probability measures μ over $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |x|^p d\mu(x) < +\infty.$$

It is endowed with the p-th Wasserstein distance d_p from optimal transport, defined, for $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$ by

$$d_p^p(\mu,\nu) = \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x-y|^p d\pi(x,y), \tag{1.23}$$

where Γ is the set of transport plans between μ and ν, that is the set of Borel probability measures γ over $\mathbb{R}^d \times \mathbb{R}^d$ satisfying, for all Borel subset A of $\mathbb{R}^d$, $\gamma(A \times \mathbb{R}^d) = \mu(A)$ and $\gamma(\mathbb{R}^d \times A) = \nu(A)$. With this metric, $\mathcal{P}_p(\mathbb{R}^d)$ is a complete separable metric space. Moreover, for a given a sequence $(\mu_n) \in (\mathcal{P}_p(\mathbb{R}^d))^{\mathbb{N}}$,

$$\lim_{n\to+\infty} d_p(\mu_n,\mu) = 0 \Longleftrightarrow \begin{cases} \mu_n \text{ narrowly converges to } \mu, \\ (\mu_n) \text{ has uniformly integrable } p\text{-moments.} \end{cases} \tag{1.24}$$

We recall that (μ_n) narrowly converges to μ if

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} f(x) d\mu_n(x) = \int_{\mathbb{R}^d} f(x) d\mu(x),$$

for every continuous and bounded function $f : \mathbb{R}^d \to \mathbb{R}$. We refer to the books [8] and [130] for the problem of optimal transport and the properties of the Wassertein distance.

Differential calculus. We are going to solve optimization problems defined over the space $\mathcal{P}_2(\mathbb{R}^d)$ and, to this end, we need a suitable notion of differentiability with respect to probability measures. We say that a map $\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is $\mathcal{C}^1$ over $\mathcal{P}_2(\mathbb{R}^d)$ if there is a jointly continuous map $\dfrac{\delta \mathcal{F}}{\delta m} : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ satisfying the integrability condition

$$\sup_{\mu \in \mathcal{K}} \frac{1}{1+|x|^2} \left| \frac{\delta \mathcal{F}}{\delta m}(\mu, x) \right| < +\infty, \qquad \text{for all bounded subset} \quad \mathcal{K} \subset \mathcal{P}_2(\mathbb{R}^d),$$

and such that, for all $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, one has the following:

$$\int_0^1 \int_{\mathbb{R}^d} \frac{\delta \mathcal{F}}{\delta m}((1-h)\mu + h\nu, x) d(\nu - \mu)(x) dh = \mathcal{F}(\nu) - \mathcal{F}(\mu).$$

The derivative is defined up to an additive constant and we always assume that, for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \frac{\delta \mathcal{F}}{\delta m}(\mu, x) d\mu(x) = 0.$$

In the terminology of [43] it means that $\mathcal{F}$ admits a linear functional derivative. When $\dfrac{\delta \mathcal{F}}{\delta m}$ is differentiable with respect to the space variable, we define the intrinsic derivative

$$D_m\mathcal{F}(\mu,x) := D_x\frac{\delta\mathcal{F}}{\delta m}(m,x).$$

If, for every $x\in\mathbb{R}^d$, the map $\mu\mapsto\frac{\delta\mathcal{F}}{\delta m}(\mu,x)$ is $\mathcal{C}^1$ we say that $\mathcal{F}$ is $\mathcal{C}^2$ and we denote by $\frac{\delta^2\mathcal{F}}{\delta m^2}$ its derivative, defined, for all $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ and all $x,y\in\mathbb{R}^d$ by

$$\frac{\delta^2\mathcal{F}}{\delta m^2}(\mu,x,y)=\frac{\delta}{\delta m}\left(\frac{\delta\mathcal{F}}{\delta m}(\mu,x)\right)(y).$$

Finally, if $\mathcal{F}$ is $\mathcal{C}^2$ and if $\frac{\delta^2\mathcal{F}}{\delta m^2}(\mu,x,y)$ is twice differentiable in (x,y), we let

$$D^2_{mm}\mathcal{F}(\mu,x,y) := D^2_{x,y}\frac{\delta^2\mathcal{F}}{\delta m^2}(\mu,x,y).$$

Equivalently

$$D^2_{mm}\mathcal{F}(\mu,x,y)=D_m\left(D_m\mathcal{F}(.,x)\right)(\mu,y).$$

We refer to the books [43] and [36] for details on the notion(s) of derivative over the Wasserstein space. The reader, not familiar with differential calculus in the space of probability measures, can assume that functionals like $\mathcal{F}:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ are linear with respect to the measure variable. This means that there is a measurable function $f:\mathbb{R}^d\to\mathbb{R}$ with at most quadratic growth, such that, for all $\mu\in\mathcal{P}_2(\mathbb{R}^d)$,

$$\mathcal{F}(\mu)=\int_{\mathbb{R}^d}f(x)d\mu(x).$$

In this case $\mathcal{F}$ admits a linear derivative, given for all $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, by

$$\frac{\delta\mathcal{F}}{\delta m}(\mu,x)=f(x)-\int_{\mathbb{R}^d}f(y)d\mu(y).$$

If f is a differentiable function, then $\mathcal{F}$ admits an intrinsic derivative $D_m\mathcal{F}$ given, for all $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ by

$$D_m\mathcal{F}(\mu,x)=D_xf(x).$$

1.4 Organization of the manuscript and summary of the main results

This dissertation contains four chapters, in addition to this introduction. In Chapter 2, which constitutes the first section, we investigate a stochastic control problem with terminal constraint in law. We prove the existence of optimal solutions and characterize them with a second-order mean-field game system of partial differential equations associated with an exclusion condition.

The second section of this manuscript is dedicated to optimal control problems where the state is a flow of probability measures constrained to stay, at all time, in some region of the

Wasserstein space of probability measures. In Chapter 3 we prove the existence of optimal solutions and exhibit some optimality conditions, once again in the form of a mean-field game system of pde's. The state constraint gives rise to delicate questions of time regularity for the optimal solutions. We address this questions and propose a geometric condition on the constraint under which optimal controls are Lipschitz continuous in time. Chapter 4 is an extension of Chapter 3. We show how mean-field control problems with constraints in law arise as limit of control problems for interacting agents with symmetric, almost-sure constraints. We also discuss some connections of this result with large deviations principles for (uncontrolled) weakly interacting particle systems. Although the results of Chapter 4 rely on the main results of Chapter 3, the former can be read independently.

Finally, in Chapter 5 we investigate the convergence problem in mean-field control without constraint. In this joint work with Pierre Cardaliaguet, Joe Jackson and Panagiotis Souganidis, we find an algebraic rate of convergence for the convergence of the value functions of the N-particle problems toward the value function of the mean-field problem, in a setting where uniqueness of optimal controls for the mean-field problem is not expected and therefore the value function is not expected to be differentiable.

1.4.1 Chapter 2: Stochastic control with terminal constraint in law

This chapter is devoted to a stochastic control problem with terminal constraint in law. This problem was originally motivated by applications in economy and finance, where it is natural to impose terminal constraints on the statistical distribution of the controlled state, see [70, 83, 84, 92, 107]. From a control theoretic perspective, optimality conditions were investigated in [75, 117] for linear constraints and in the paper of Pfeiffer [116] for a problem with non-linear terminal constraint[1]. A dynamic programming principle for stochastic control problems with linear constraints can also be found in [50]. Our contribution is to prove the existence of strong optimal Markovian controls for a problem with non-linear costs and constraint, and characterize these controls through a (second-order, fully non-linear) mean-field game system of pde's. Along the way we put forward a qualification assumption adapted to the infinite dimensional setting, as well as new weak formulation of the problem as an optimal control problem for the Fokker-Planck equation. We also exhibit some regularity and growth conditions on the Hamiltonian of the system to ensure simultaneously the (approximate) controllability of the Fokker-Planck equation and the regularity of the backward HJB equation which appear in the system of optimality conditions.

The problem takes the following form, where $\mathcal{A}$ is an appropriate class of controls,

$$\inf_{\alpha_t \in \mathcal{A}} \mathbb{E}\left[\int_0^T (f_1(t, X_t, \alpha_t) + f_2(t, \mathcal{L}(X_t)))dt + g(\mathcal{L}(X_T))\right]$$

under the constraint $\Psi(\mathcal{L}(X_T)) \leqslant 0$ for the diffusion

[1] A word of caution on the terminology: throughout this dissertation, linearity and convexity are usually understood with respect to the measure variable. For instance a functional $\mathcal{F} : m \mapsto \int_{\mathbb{R}^d} f(x)dm(x)$ is linear and therefore convex but it does not bear any meaning on the linearity or convexity of $f : \mathbb{R}^d \to \mathbb{R}$.

$$dX_t = b(t, X_t, \alpha_t)dt + \sqrt{2}\sigma(t, X_t, \alpha_t)dB_t,$$

with the initial condition given by $\mathcal{L}(X_0) = m_0 \in \mathcal{P}(\mathbb{R}^d)$. The constraint Ψ satisfies, among other regularity assumptions, the qualification condition

$$\begin{cases} \exists m_f \in \mathcal{P}(\mathbb{R}^d), \Psi(m_f) < 0, \\ \dfrac{\delta\Psi}{\delta m}(m, .) \neq 0, \text{ whenever } \Psi(m) = 0. \end{cases} \tag{1.25}$$

We define the Hamiltonian of the system

$$H(t, x, p, M) := \sup_{a\in A} \left\{-b(t, x, a).p - \sigma^t\sigma(t, x, a).M - f_1(t, x, a)\right\}.$$

Under specific assumptions on f_1, f_2, g, b and σ which guarantee the regularity and the controllability of the system we prove, in Theorem (2.2), that optimal feedback controls $\alpha : [0, T] \times \mathbb{R}^d \to A$ exist and satisfy

$$\begin{aligned} H(t, x, D\phi(t, x), D^2\phi(t, x)) = &- b(t, x, \alpha(t, x)).D\phi(t, x) - \sigma^t\sigma(t, x, \alpha(s, x)).D^2\phi(t, x) \\ &- f_1(t, x, \alpha(t, x)) \end{aligned} \tag{1.26}$$

for some solution (λ, ϕ, m) of the system of optimality conditions:

$$\begin{cases} -\partial_t u(t, x) + H(t, x, Du(t, x), D^2u(t, x)) = \dfrac{\delta f_2}{\delta m}(t, m(t), x) & \text{in } (0, T) \times \mathbb{R}^d \\ \partial_t m - \operatorname{div}(\partial_p H(t, x, Du(t, x), D^2u(t, x))m) \\ \qquad + \displaystyle\sum_{i,j} \partial^2_{ij}((\partial_M H(t, x, Du(t, x), D^2u(t, x)))_{ij}m) = 0 & \text{in } (0, T) \times \mathbb{R}^d \\ u(T, x) = \lambda\dfrac{\delta\Psi}{\delta m}(m(T), x) + \dfrac{\delta g}{\delta m}(m(T), x) \text{ in } \mathbb{R}^d, m(0) = m_0 \\ \lambda\Psi(m(T)) = 0, \ \Psi(m(T)) \leqslant 0, \ \lambda \geqslant 0, \end{cases}$$

Our strategy consists first in introducing a suitable relaxed problem (RP) and then showing that optimal solutions for the relaxed problem yield optimal solutions for the original problem. The relaxed problem has the following form,

$$\inf_{(m,\omega,W)\in\mathbb{K}} J_{RP}(m, \omega, W), \tag{1.27}$$

where $\mathbb{K}$ is the set of triples $(m, \omega, W) \in \mathcal{C}([0, T], \mathcal{P}(\mathbb{R}^d)) \times \mathcal{M}([0, T] \times \mathbb{R}^d, \mathbb{R}^d) \times \mathcal{M}([0, T] \times \mathbb{R}^d, \mathbb{S}_d(\mathbb{R}))$ such that ω and W are absolutely continuous with respect to $m(t) \otimes dt$,

$$\partial_t m + \operatorname{div}\omega - \sum_{i,j} \partial^2_{ij} W_{ij} = 0 \tag{1.28}$$

holds in the sense of distributions, $m(0) = m_0$ and $\Psi(m(T)) \leqslant 0$. Above $\mathbb{S}_d(\mathbb{R})$ is the space of symmetric square matrices of size d and we denoted by $\mathcal{M}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$ (respectively

by $\mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{S}_d(\mathbb{R})\ )$ the space of finite $\mathbb{R}^d$-valued (respectively $\mathbb{S}^d(\mathbb{R})$-valued) Radon measures over $[0,T]\times\mathbb{R}^d$. The cost J_{RP} is defined on $\mathbb{K}$ by

$$J_{RP}(m,\omega,W) := \int_0^T \int_{\mathbb{R}^d} L\left(t,x,\frac{d\omega}{dt\otimes dm}(t,x),\frac{dW}{dt\otimes dm}(t,x)\right) dm(t)(x)dt + \int_0^T f_2(t,m(t))dt + g(m(T)),$$

with

$$L(t,x,q,N) := \sup_{(p,M)\in\mathbb{R}^d\times\mathbb{S}_d(\mathbb{R})} \{-p.q - M.N - H(t,x,p,M)\} = H^*(t,x,-q,-N).$$

By standard compactness methods, using the controllability of the system, we prove the existence of relaxed solutions $(\widetilde{m},\widetilde{\omega},\widetilde{W})$ in Lemma 2.3. In Proposition 2.3 we proceed to show that solutions $(\widetilde{m},\widetilde{\omega},\widetilde{W})$ to the relaxed problem are also minimizers of the linearized functional

$$J^l_{RP}(m,\omega,W) := \int_0^T \int_{\mathbb{R}^d} L\left(t,x,\frac{d\omega}{dt\otimes dm}(t,x),\frac{dW}{dt\otimes dm}(t,x)\right) dm(t)(x)dt + \int_0^T \int_{\mathbb{R}^d} \frac{\delta f_2}{\delta m}(t,\widetilde{m}(t),x)dm(t)(x)dt + \int_{\mathbb{R}^d} \frac{\delta g}{\delta m}(\widetilde{m}(T),x)dm(T)(x) \tag{1.29}$$

among triples (m,ω,W) that satisfy the Fokker-Planck equation with $m(0)=m_0$ and with m satisfying the linearized constraint

$$\int_{\mathbb{R}^d} \frac{\delta\Psi}{\delta m}(\widetilde{m}(T),x)dm(T)(x) \leqslant 0. \tag{1.30}$$

By a minmax theorem we prove in Theorem 2.5 that the infimum of the linearized problem is equal to

$$\sup_{(\lambda,\phi)} \int_{\mathbb{R}^d} \phi(0,x)m_0(dx), \tag{1.31}$$

where the supremum is taken over the couples $(\lambda,\phi)\in\mathbb{R}^+\times\mathcal{C}^{1,2}_b([0,T]\times\mathbb{R}^d)$ such that

$$\begin{cases} -\partial_t\phi(t,x) + H(t,x,D\phi(t,x),D^2\phi(t,x)) \leqslant \dfrac{\delta f_2}{\delta m}(t,\widetilde{m}(t),x) \text{ in } [0,T]\times\mathbb{R}^d, \\ \phi(T,x) \leqslant \lambda\dfrac{\delta\Psi}{\delta m}(\widetilde{m}(T),x) + \dfrac{\delta g}{\delta m}(\widetilde{m}(T),x) \text{ in } \mathbb{R}^d. \end{cases} \tag{1.32}$$

To solve the dual problem, we use the controllability of the system to find an *a priori* upper bound on λ in Lemma 2.5 and we show that, for any $\lambda \geqslant 0$, there is a unique solution to the HJB equation in $\mathcal{C}^{1,2}_b([0,T]\times\mathbb{R}^d)$ in Theorem 2.1. This last step is, by far, the most technical since H is not Lipschitz continuous with respect to the gradient variable,

and therefore the standard regularity results on fully non-linear, uniformly parabolic HJB equations do not apply. Using the weak Bernstein method we prove that viscosity solutions to the HJB equation are Lipschitz continuous which is, by standard arguments, enough to conclude.

Once we have found optimal solutions $(\widetilde{\lambda}, \widetilde{\phi})$ for the dual problem, we can express optimal controls to the relaxed problem in terms of $(\widetilde{\lambda}, \widetilde{\phi})$. Arguing by verification, we show that these optimal controls are also optimal for the original problem and deduce the optimality conditions.

1.4.2 Chapter 3: Optimal control of the Fokker-Planck equation under state constraint: optimality conditions

This chapter is devoted to an optimal control problem similar to the one of Chapter 2 but, this time, the constraint is imposed throughout the whole time horizon. For deterministic control problems in finite dimension, state constraints have been widely studied and we refer to the survey of Frankowska [74] and the lecture notes of Bonnans [17] for an overview. As for unconstrained problems, the results usually take two forms. The dynamic programming principle leads to the characterization of the value function as the unique "constrained" viscosity solution to the corresponding HJB equation. This result was proved in the seminal work of Soner [123]. The other strategy is to show optimality conditions, usually thanks to the Pontryagin maximum principle. The effect of the constraint is then captured by the presence, in the system, of an additional Lagrange multiplier. This approach is convenient to address the delicate question of time regularity of the optimal controls in the presence of state constraints. This is achieved in [77, 85]. First order control problems with state constraints in the Wasserstein space were investigated in [18, 19]. First order mean-field game systems with state constraints were also studied in a series of papers [31, 32] and the structure of the present paper is closely inspired from [31] (although our techniques are completely different). Finally let us mention the reference [76] where the authors prove first and second order optimality conditions for stochastic control problems with linear state constraints. In this work the authors put forward a stochastic inward pointing condition to ensure the normality of the optimality conditions. This condition is the finite dimensional analog of our qualification condition 1.42, introduced below.

In this chapter, we study an optimal control problem for the Fokker-Planck equation where the state is constrained to stay in some region of the Wasserstein space of probability measures. We prove the existence of optimal feedback controls and characterize them with a second order mean-field game system of pdes associated with an exclusion condition. Our main contributions are to give the system of optimality conditions when the constraint and the cost are non-linear functionals of the measure variable and to prove the Lipschitz continuity of optimal controls under an appropriate geometric condition on the constraint.

More precisely the problem takes the following form. We seek to minimize a cost

$$J(\alpha, m) := \int_0^T \int_{\mathbb{R}^d} L(x, \alpha(t,x)) dm(t)(x) dt + \int_0^T \mathcal{F}(m(t)) dt + \mathcal{G}(m(T))$$

over pairs (α, m) with $m \in \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d))$ and $\alpha \in L^2_{m(t)\otimes dt}\left([0,T] \times \mathbb{R}^d, \mathbb{R}^d\right)$ satisfying in

the sense of distributions the Fokker-Planck equation:

$$\partial_t m + \mathrm{div}(\alpha m) - \Delta m = 0 \tag{1.33}$$

with the initial condition $m(0) = m_0 \in \mathcal{P}_2(\mathbb{R}^d)$. The flow of probability measures m is constrained to satisfy the inequality

$$\Psi(m(t)) \leqslant 0, \qquad \forall t \in [0,T]$$

for some function $\Psi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$.

We assume, in addition to specific regularity assumptions, that the constraint Ψ satisfies

$$\begin{cases} \Psi(m_0) < 0, \\ \Psi \text{ is convex.} \end{cases} \tag{1.34}$$

This is a stronger requirement than the qualification condition (1.25) of Chapter 2. Our first main contribution in this chapter is to derive the system of optimality conditions for the problem with state constraint. Assuming as well some coercivity and regularity conditions for

$$H(x,p) := \sup_{q \in \mathbb{R}^d} \{-p.q - L(x,p)\}$$

and some regularity conditions for the mean-field costs $\mathcal{F}$ and $\mathcal{G}$, we prove, in Theorem 3.4, that optimal solutions (α, m) exist and satisfy $\alpha = -\partial_p H(., Du)$ for some solution (u, m, ν, η) of the system of optimality conditions

$$\begin{cases} -\partial_t u(t,x) + H(x, Du(t,x)) - \Delta u(t,x) \\ \qquad\qquad = \nu(t)\dfrac{\delta \Psi}{\delta m}(m(t),x) + \dfrac{\delta \mathcal{F}}{\delta m}(m(t),x) & \text{in } (0,T)\times\mathbb{R}^d, \\ \partial_t m - \mathrm{div}(\partial_p H(x, Du(t,x))m) - \Delta m = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ u(T,x) = \eta \dfrac{\delta \Psi}{\delta m}(m(T),x) + \dfrac{\delta \mathcal{G}}{\delta m}(m(T),x) & \text{in } \mathbb{R}^d, \\ m(0) = m_0, \end{cases} \tag{1.35}$$

associated with the exclusion conditions

$$\Psi(m(t)) = 0, \nu\text{-almost-everywhere in } [0,T] \tag{1.36}$$

$$\eta \Psi(m(T)) = 0, \tag{1.37}$$

where $\nu \in \mathcal{M}^+([0,T])$ and $\eta \in \mathbb{R}^+$ are Lagrange multipliers associated to the constraint. We discuss in details the well-posedness of the backward HJB equation. It is *a priori* non standard since the multiplier ν is a measure. As a consequence u and the optimal control α are discontinuous in time. Yet we show that u is still smooth in the space variable. To prove this result we need to introduce, for small parameters $\epsilon, \delta > 0$ solutions to the penalized problems

$$\inf_{(m,\alpha)} J_{\epsilon,\delta}(\alpha, m) \tag{1.38}$$

where the infimum runs over all (m, α) satisfying (1.33) (but not necessarily the state constraint) and $J_{\epsilon,\delta}$ is defined by

$$J_{\epsilon,\delta}(\alpha, m) := \int_0^T \int_{\mathbb{R}^d} L(x, \alpha(t,x)) dm(t)(x) dt + \int_0^T \mathcal{F}(m(t)) dt + \frac{1}{\epsilon} \int_0^T \Psi^+(m(t)) dt + \mathcal{G}(m(T)) + \frac{1}{\delta} \Psi^+(m(T)),$$

where we used the notation $\Psi^+(m) = \max(\Psi(m), 0)$. By a linearization procedure akin to what is done in Chapter 2, we prove, in Theorem 3.5 that optimal solutions (α, m) exist and satisfy $\alpha = -\partial_p H(., Du)$ for some solution (u, m, λ, β) of

$$\begin{cases} -\partial_t u(t,x) + H(x, Du(t,x)) - \Delta u(t,x) \\ \qquad = \dfrac{\lambda(t)}{\epsilon} \dfrac{\delta \Psi}{\delta m}(m(t), x) + \dfrac{\delta \mathcal{F}}{\delta m}(m(t), x) & \text{in } (0,T) \times \mathbb{R}^d, \\ \partial_t m - \operatorname{div}(\partial_p H(x, Du(t,x)) m) - \Delta m = 0 & \text{in } (0,T) \times \mathbb{R}^d, \\ u(T,x) = \dfrac{\beta}{\delta} \dfrac{\delta \Psi}{\delta m}(m(T), x) + \dfrac{\delta \mathcal{G}}{\delta m}(m(T), x) \quad \text{in } \mathbb{R}^d, \qquad m(0) = m_0. \end{cases} \tag{1.39}$$

where, $\lambda \in L^\infty([0,T])$ and $\beta \in \mathbb{R}^+$ satisfy

$$\lambda(t) \begin{cases} = 0 & \text{if } \Psi(m(t)) < 0 \\ \in [0,1] & \text{if } \Psi(m(t)) = 0 \\ = 1 & \text{if } \Psi(m(t)) > 0 \end{cases} \tag{1.40}$$

$$\beta \begin{cases} = 0 & \text{if } \Psi(m(T)) < 0 \\ \in [0,1] & \text{if } \Psi(m(T)) = 0 \\ = 1 & \text{if } \Psi(m(T)) > 0. \end{cases} \tag{1.41}$$

To pass to the limit, as ϵ and δ go to 0, we first use condition (1.34) to build, in Lemma 3.4, admissible candidates which remain uniformly inside the constraint at all times and deduce, in Lemma 3.5 some bounds on λ and β of the form

$$\int_0^T \frac{\lambda(t)}{\epsilon} dt + \frac{\beta}{\delta} < C,$$

for some $C > 0$ independent of ϵ and δ. By parabolic regularity, we show, in Theorem 3.6 that this is enough to bound uniformly u and its space derivatives in (1.39) and then we can pass to the limit as ϵ and δ go to 0 to obtain the optimality conditions for the constrained problem.

Our second main contribution in this chapter (although presented first in the manuscript) is to provide an additional geometric assumption on the constraint under which the Lagrange multiplier ν is in fact a bounded function over $[0,T]$. This condition reads as follows

$$\int_{\mathbb{R}^d} |D_m \Psi(m,x)|^2 dm(x) \neq 0, \qquad \text{whenever } \Psi(m) = 0. \tag{1.42}$$

Notice that $\displaystyle\int_{\mathbb{R}^d} |D_m \Psi(m,x)|^2 dm(x)$ is nothing but the squared norm of the Wasserstein gradient of Ψ in $L^2(dm)$ at the point m. To prove this additional regularity, we apply a method already used in [33] and [31] for control problems in finite dimension. We look at the second order derivative of $t \mapsto \Psi(m^{\epsilon,\delta}(t))$ when $m^{\epsilon,\delta}$ is a solution to the penalized problem. We show, in Proposition 3.5 that it can be written, whenever $\Psi(m^{\epsilon,\delta}(t)) \neq 0$ as

$$\frac{d^2}{dt^2}\Psi(m^{\epsilon,\delta}(t)) = \frac{\lambda(t)}{\epsilon}\int_{\mathbb{R}^d} D_m\Psi(m^{\epsilon,\delta}(t),x).D^2_{pp}H(x,Du^{\epsilon,\delta}(t,x))D_m\Psi(m^{\epsilon,\delta}(t),x)dm^{\epsilon,\delta}(t)(x) + O(1)$$

where $O(1)$ is bounded independently from ϵ and δ and can be expressed in terms of $m^{\epsilon,\delta}$ as well as $u^{\epsilon,\delta}$ and its space derivatives. Using conditions (1.40) and (1.42) as well as the convexity of the Hamiltonian with respect to the momentum variable, we prove that $\frac{d^2}{dt^2}\Psi(m^{\epsilon,\delta}(t))$ cannot be negative when ϵ is small enough. As a consequence, maxima of $t \mapsto \Psi(m^{\epsilon,\delta}(t))$ must occur when $\Psi(m^{\epsilon,\delta}(t)) \leqslant 0$ which means that optimal solutions to the penalized problem remain inside the constraint when the penalization is strong enough. As a direct by-product, solutions to the constrained problem are also solutions to the penalized problem with a strong enough penalization and therefore enjoy the same regularity. In particular, we prove, in Theorem 3.3 that ν belongs to $L^\infty([0,T])$ and that optimal controls are Lipschitz continuous in time.

1.4.3 Chapter 4: Optimal control of the Fokker-Planck equation under state constraint: the mean-field limit

In this chapter we apply the results of Chapter 3, in particular the existence of bounded, Lipschitz continuous optimal controls for the mean-field problem, in order to study the mean-field limit for problems with constraints. We show that the mean-field problem arises as limit of control problems for finitely many interacting agents subject to symmetric, almost-sure constraints. The pre-limit and limit problems are very different in nature. For the N-particle system, the constraint has to be satisfied almost-surely and therefore optimal controls blow-up near the boundary to compensate the effect of the non-degenerate noise. At the limit, the problem is deterministic, the constraint is of mean-field type and optimal controls are bounded and Lipschitz continuous. The mean-field limit is now rather well understood with the different approaches introduced in Section 1.1.4, however it seems that our result is the first of his kind for second order problems with constraints. This being said, making logarithmic transforms reminiscent of [67, 68] in the special case of a purely quadratic Hamiltonian, we see that our result has well-known counterparts in the theory of large deviations for weakly interacting (uncontrolled) particles, see [26, 27, 55, 64].

More precisely, the goal of this chapter is to investigate the connection between the optimization problem:

$$\inf_{(\alpha,\mu)} \int_0^T \int_{\mathbb{R}^d} L(x,\alpha(t,x))d\mu(t)(x)dt + \int_0^T \mathcal{F}(\mu(t))dt + \mathcal{G}(\mu(T)) \tag{mfP}$$

subject to

$$\begin{cases} \mu \in \mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R}^d)), \alpha \in L^2_{dt\otimes\mu(t)}([0,T]\times\mathbb{R}^d,\mathbb{R}^d), \\ \partial_t\mu + \operatorname{div}(\alpha\mu) - \Delta\mu = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ \mu(0) = \mu_0 \in \mathcal{P}_2(\mathbb{R}^d), \\ \Psi(\mu(t)) \leqslant 0 & \forall t \in [0,T], \end{cases}$$

and a control problem for a large number N of interacting particles:

$$\inf_{(\alpha_t^{i,N})_{1\leqslant i\leqslant N}} \mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\int_0^T \frac{1}{N}\sum_{i=1}^N L(X_t^{i,N},\alpha_t^{i,N})dt + \int_0^T \mathcal{F}(\widehat{\mu}_t^{N,x})dt + \mathcal{G}(\widehat{\mu}_T^{N,x})\right] \tag{NP}$$

subject to

$$\begin{cases} dX_t^{i,N} = \alpha_t^{i,N}dt + \sqrt{2}dB_t^{i,N}, \\ (X_0^{1,N},\dots,X_0^{N,N}) \sim \mu_0^{\otimes N} \text{ under } \mathbb{P}, \\ \Psi(\widehat{\mu}_t^{N,x}) \leqslant 0 \text{ for all } t\in[0,T] \qquad \mathbb{P}^{\gamma_N}-\text{almost-surely.} \end{cases}$$

In the latter problem, $(B^{i,N})_{1\leqslant i\leqslant N}$ are N independent standard Brownian motions supported on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. We denoted by

$$\widehat{\mu}_t^{N,x} := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

the empirical measures and finally $\mathbb{P}^{\gamma_N} := \mathbb{P}\left[\cdot|\Psi(\widehat{\mu}_0^{N,x}) \leqslant -\gamma_N\right]$ is the conditional probability with respect to the event $\left\{\Psi(\widehat{\mu}_0^{N,x}) \leqslant -\gamma_N\right\}$ for some suitable rate $\gamma_N > 0$ such that $\gamma_N \to 0$ as $N\to+\infty$. The conditioning being necessary to ensure that the particles start from inside the constraint. If we denote by $\overline{\mathcal{U}}(\mu_0)$, the value of Problem (mfP) and by $\mathcal{U}_N(\mu_0)$, the value of Problem (NP) our main result, Theorem 4.1, is to prove the convergence of $\mathcal{U}_N(\mu_0)$ toward $\overline{\mathcal{U}}(\mu_0)$, when $N\to+\infty$.

We proceed in two steps. First we prove that $\limsup_{N\to+\infty}\mathcal{U}_N(\mu_0) \leqslant \overline{\mathcal{U}}(\mu_0)$ in Theorem 4.2. To this end, we need to find a way to transform an admissible control for the mean-field problem into an admissible control for the N-particle problem. The difficulty is that, if the particles each follow a bounded control, admissible for the mean-field problem, then the empirical measure of the system will almost-surely leave the constraint. To overcome this issue, we build explicit feedback controls which allow the particles to stay strictly inside a little ball of $(\mathbb{R}^d)^N$ for as long as needed without paying to big a cost. Therefore we can virtually stop the particle system when the empirical measures get close to the boundary of the constraint.

In Theorem 4.3, we prove that $\limsup_{N\to+\infty}\overline{\mathcal{U}}(\mu_0) \leqslant \mathcal{U}^N(\mu_0)$ relying on the compactness methods introduced in Section 1.1.2. This boils down to finding weak limit points of sequences of nearly optimal weak solutions to the N-particle problem. Once we know that $\mathcal{U}_N(\mu_0)$ is bounded independently from N, this follows from the line of arguments of [96] for problems without constraint.

In Section 4.4 we discuss connections of these results with the problem of large deviations for weakly interacting particle systems.

1.4.4 Chapter 5: A rate of convergence for the optimal control of McKean-Vlasov dynamics

Chapter 5 is a joint work with Pierre Cardaliaguet, Joe Jackson and Panagiotis Souganidis. The goal is to investigate quantitatively the validity of the mean-field approximation, in

situations where uniqueness is not expected for the mean-field problem and, therefore, the value function is *a priori* not differentiable with respect to the measure argument. The motivation is to fill the gap between the two extreme regimes described in Section 1.1.4 of this introduction. We recall that the convergence of the value functions is well understood under mild regularity and coercivity assumptions since the works of Lacker [96], without common noise and Djete, Possamaï and Tan [59] for problems with a common noise. On the other hand, in the special case where the diffusion is non-degenerate and the cost functionals are convex with respect to the measure variable, we can prove that the value function for the mean-field problem is a smooth function over the space of probability measures, see [36]. As a consequence we can prove propagation of chaos for the optimal trajectories and quantitative rates of convergence for the value functions. Let us mention that several other papers have studied the question of the mean field limit of optimal control problems, for example [46] and [71] investigate the problem without noise by $\Gamma-$convergence techniques. The recent contribution [78] studies the mean field limit without idiosyncratic but with common noise using partial differential equations (PDE for short) techniques.

The main contribution of this chapter is to provide an algebraic rate of convergence for the value functions in a setting where the value function of the limit problem is not expected to be smooth.

We use the notations of Section 1.1.4 in this section to describe our results. Our main theorem asserts that there exists $\beta \in (0,1]$ (depending on the dimension only) and $C > 0$ (depending on the data of the problem) such that, for all $t \in [0,T]$ and all $\mathbf{x}^N = (x^{1,N}, \dots, x^{N,N}) \in \times(\mathbb{R}^d)^N$ it holds

$$|\overline{\mathcal{U}}(t_0, \frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}}) - \mathcal{V}^N(t_0, \mathbf{x}^N)| \leqslant \frac{C}{N^\beta}(1 + M_2^{1/2}(\frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}})) \tag{1.43}$$

where $M_2(\frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}}) = \frac{1}{N}\sum_{i=1}^N (x^{i,N})^2$.

The first step is to prove, in Lemma 5.1 and Lemma 5.3, uniform in N, L^∞, Lipschitz and semi-concavity estimates for $\mathcal{V}^N$. This is possible under suitable regularity assumptions on the data, since $\mathcal{V}^N$ solves, by dynamic programming, the uniformly parabolic backward HJB equation

$$\begin{cases} -\partial_t \mathcal{V}^N(t,\mathbf{x}^N) + \frac{1}{N}\sum_{i=1}^N H(x^{i,N}, N D_{x^{i,N}} \mathcal{V}^N(t,\mathbf{x}^N)) - \sum_{i=1}^N \Delta_{x^{i,N}} \mathcal{V}^N(t,\mathbf{x}^N) \\ \qquad\qquad = \mathcal{F}(\frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}}) \qquad \text{in } (0,T)\times(\mathbb{R}^d)^N, \\ \mathcal{V}^N(T,\mathbf{x}^N) = \mathcal{G}(\frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}}) \qquad \text{in } (\mathbb{R}^d)^N. \end{cases} \tag{1.44}$$

Then we proceed, in Proposition 5.3 to show the inequality

$$\mathcal{V}^N(t_0,\mathbf{x}^N) \leqslant \overline{\mathcal{U}}(t_0, \frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}}) + \frac{C}{N^\beta}(1 + M_2^{1/2}(\frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}})).$$

This not too difficult because we can check that the function

$$\widehat{\mathcal{V}}^N(t,\mu) := \int_{(\mathbb{R}^d)^N} \mathcal{V}^N(t,x^{1,N},\dots,x^{N,N})d\mu^{\otimes N}(x^{1,N},\dots,x^{N,N})$$

is a smooth subsolution to the dynamic programming equation of the mean-field problem, up to a small error that we can quantify using the results of [73]. Indeed, it holds that

$$\begin{cases} -\partial_t\widehat{\mathcal{V}}^N(t,\mu) - \displaystyle\int_{\mathbb{R}^d} \mathrm{div}(D_m\widehat{\mathcal{V}}^N(t,\mu,x))d\mu(x) \\ + \displaystyle\int_{\mathbb{R}^d} H(y,D_m\widehat{\mathcal{V}}^N(t,\mu,x))d\mu(x) \leqslant \widehat{\mathcal{F}}^N(\mu) \ \text{ in } \ (0,T)\times\mathcal{P}(\mathbb{R}^d), \\ \widehat{\mathcal{V}}^N(T,\mu) = \widehat{\mathcal{G}}^N(\mu) \ \text{ in } \ \mathcal{P}(\mathbb{R}^d), \end{cases}$$

where

$$\widehat{\mathcal{F}}^N(\mu) = \int_{(\mathbb{R}^d)^N} \mathcal{F}(\frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}})d\mu^{\otimes N}(\mathbf{x}^N) \ \text{ and } \ \widehat{\mathcal{G}}^N(m) = \int_{(\mathbb{R}^d)^N} \mathcal{G}(\frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}})d\mu^{\otimes N}(\mathbf{x}^N).$$

The results of [73] allow us to estimate the differences between $\widehat{\mathcal{V}}^N$ and $\mathcal{V}^N$, between $\widehat{\mathcal{F}}^N$ and $\mathcal{F}$ and between $\widehat{\mathcal{G}}^N$ and $\mathcal{G}$ thanks to the regularity of $\mathcal{V}^N$, $\mathcal{F}$ and $\mathcal{G}$.

The reverse inequality is more intricate. Indeed it is not clear how to transform an optimal control for $\mathcal{V}^N$, which depends on the position of each particles (two particles which are very close can go into opposite directions) into an admissible control for the mean-field problem. To overcome this issue, we break the particles into subgroups, so that, in each subgroup, the particles have approximately the same drift. This is possible because we proved a -priori that the controls for each particle are bounded independently from N. Now we can prove, using a concentration inequality, that the empirical measure of each sub-group converges toward the solution of a Fokker-Planck equation when the number of particles gets larger. We can interpolate between these solutions arising from each subgroup, to find a candidate for the mean-field problem which is quantitatively close, for a small time at least, to the empirical measure of the whole system of particles. We combine this construction in the proof of Proposition 5.4 with our Lipschitz and semi-concavity estimates as well as a convenient doubling of variables argument inspired by techniques from the theory of viscosity solutions for Hamilton-Jacobi equations.

Our strategy relies mostly on the regularity of the value function for the N-particle system and we only use the Lipschitz continuity of the value function and dynamic programming principle for the mean-field problem. Therefore, we can extend our result to problems with a common noise, as long as the private noises remain non-degenerate. This is possible thanks to the recent results of [59, 58, 97] on mean-field control problems in the presence of a common noise. In this more general setting, we define

$$\mathcal{V}^N(t_0,\mathbf{x}_0^N) := \inf_{\alpha\in\mathcal{A}^N} \mathbb{E}\left[\int_{t_0}^T (\frac{1}{N}\sum_{k=1}^N L(X_t^k,\alpha_t^k) + \mathcal{F}(\widehat{\mu}_t^{N,x})dt + \mathcal{G}(\widehat{\mu}_T^{N,x})\right],$$

where $\mathcal{A}^N$ is an appropriate class of controls, $\mathbf{x}_0^N = (x_0^{1,N}, \dots, x_0^{N,N})$ is the initial position of the particles and

$$X_t^k = x_0^{k,N} + \int_{t_0}^t \alpha_s^k ds + \sqrt{2}(B_t^k - B_{t_0}^k) + \sqrt{2a_0}(B_t^0 - B_{t_0}^0) \qquad t \in [t_0, T].$$

The $(B^k)_{k\geqslant 0}$ are independent d-dimensional Brownian motions defined on a fixed filtered probability space and $a_0 \geqslant 0$ is the intensity of the common noise. We also define the value function for the mean-field control problem with common noise

$$\mathcal{U}(t_0, \mu_0) := \inf_{\alpha} \mathbb{E}[\int_{t_0}^T \big(L(X_t, \alpha_t) + \mathcal{F}(\mathcal{L}(X_t|\mathcal{F}_t^{B^0}))\big) + \mathcal{G}(\mathcal{L}(X_T|\mathcal{F}_T^{B^0}))],$$

where the infimum is taken, once again, over an appropriate set of admissible controls, $\mathbb{F}^{B^0} = (\mathcal{F}_t^{B^0})_{0\leqslant t\leqslant T}$ denotes the filtration generated by B^0, $\mathcal{L}(X_t|\mathcal{F}_t^{B^0})$ is the law of X_t conditioned upon $\mathcal{F}_t^{B^0}$, and

$$X_t = \overline{X}_{t_0} + \int_{t_0}^t \alpha_s(X_s) ds + \sqrt{2}(B_t - B_{t_0}) + \sqrt{2a_0}(B_t^0 - B_{t_0}^0),$$

with B another Brownian motion, $\overline{X}_{t_0}$ a random initial condition with law μ_0 and B^0, B and $\overline{X}_{t_0}$ mutually independent. Under appropriate assumptions on the data $L, \mathcal{F}$ and $\mathcal{G}$ we prove in Section 5.4 that the estimate (1.43) still holds in the presence of a common noise.

Part I

Stochastic Control With Terminal Constraint in Law

Chapter 2

Stochastic Control With Terminal Constraint in Law

This chapter was accepted for publication in the Journal of Optimization Theory and Applications.

Introduction

This paper is devoted to the study of stochastic optimal control problems with constraints on the law $\mathcal{L}(X_T)$ of the controlled process at the terminal time. Our problem takes the following form :

$$\inf_{\alpha_t \in \mathcal{A}} \mathbb{E}\left[\int_0^T (f_1(t, X_t, \alpha_t) + f_2(t, \mathcal{L}(X_t)))dt + g(\mathcal{L}(X_T))\right]$$

under the constraint $\Psi(\mathcal{L}(X_T)) \leqslant 0$ for the diffusion:

$$dX_t = b(t, X_t, \alpha_t)dt + \sqrt{2}\sigma(t, X_t, \alpha_t)dB_t$$

with the initial condition given by $\mathcal{L}(X_0) = m_0$ for some m_0 in $\mathcal{P}_2(\mathbb{R}^d)$, the space of probability measures over $\mathbb{R}^d$ with finite second order moment. Here, $f_1 : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}$ and $f_2 : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ are the instantaneous costs, $g : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is the terminal cost, $\Psi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is the final constraint, $b : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \times A \to \mathbb{S}_d(\mathbb{R})$ are respectively the drift and the volatility of the controlled process X and α and is the control process valued in the control space A. We look in particular for optimal Markov policies, that is control processes (α_t) which are optimal among all admissible controls and for which there exists some measurable function $\alpha : [0,T] \times \mathbb{R}^d \to A$ such that, for all $t \in [0,T]$, $\alpha_t = \alpha(t, X_t)$.

We are going to show that optimal Markov policies are related to the solutions of the following system of partial differential equations, where the unknown (λ, ϕ, m) belongs to $\mathbb{R}^+ \times \mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^d) \times \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d))$:

$$\begin{cases} -\partial_t u(t,x) + H(t,x,Du(t,x),D^2u(t,x)) = \dfrac{\delta f_2}{\delta m}(t,m(t),x) & \text{in } [0,T]\times\mathbb{R}^d \quad (2.1a) \\ \partial_t m - \operatorname{div}(\partial_p H(t,x,Du(t,x),D^2u(t,x))m) \\ \qquad + \displaystyle\sum_{i,j} \partial^2_{ij}((\partial_M H(t,x,Du(t,x),D^2u(t,x)))_{ij}m) = 0 & \text{in } [0,T]\times\mathbb{R}^d \quad (2.1b) \\ u(T,x) = \lambda\dfrac{\delta \Psi}{\delta m}(m(T),x) + \dfrac{\delta g}{\delta m}(m(T),x) \text{ in } \mathbb{R}^d, m(0) = m_0 & (2.1c) \\ \lambda\Psi(m(T)) = 0,\ \Psi(m(T)) \leqslant 0,\ \lambda \geqslant 0, & (2.1d) \end{cases}$$

where $H(t,x,p,M) := \sup_{a\in A}\{-b(t,x,a).p - \sigma^t\sigma(t,x,a).M - f_1(t,x,a)\}$ is the Hamiltonian of the system. The forward equation, Equation 2.1b is a Fokker-Planck equation which describes the evolution of the probability distribution m of the optimally controlled process. The backward equation, Equation 2.1a is an Hamilton-Jacobi-Bellman equation satisfied by the adjoint state u. The nonnegative parameter λ is the Lagrange multiplier associated to the terminal constraint. The forward and backward equations are coupled through the source term for the HJB equation, the terminal condition for the HJB equation and the exclusion condition $\lambda\Psi(m(T)) = 0$.

Our main result, Theorem 2.2 states that, under suitable growth and regularity assumptions, optimal Markov policies $\alpha \in L^0([0,T]\times\mathbb{R}^d, A)$ exist and satisfy :

$$\alpha(t,x) \in \operatorname{argmax}_{a\in A}\left\{-b(t,x,a).Du(t,x) - \sigma^t\sigma(t,x,a).D^2u(t,x) - f_1(t,x,a)\right\}$$

for some solution (λ, u, m) of the above system of PDEs. Notice that we do not a priori require Ψ to be a convex function. When $\Psi(m) = \displaystyle\int_{\mathbb{R}^d} h(x)m(dx)$ for some function $h : \mathbb{R}^d \to \mathbb{R}$ and for all $m \in \mathcal{P}_2(\mathbb{R}^d)$, we say that the constraint is linear. When the costs f_2 and g are linear as well we recover the problem of stochastic optimal control under expectation constraint (as in [22], [50], [117]).

Such problems arise in economy and finance when an agent tries to minimize a cost (maximize a utility function) under constraints on the probability distribution of the final output. These types of constraints can take into account the risk given by the dispersion of the cost. There has recently been a surge of interest for this kind of problems. For instance [84] and [83] use similar formulations to study respectively the problem of calibration of local-stochastic volatility models and the problem of portfolio allocation with prescribed terminal wealth distribution. Probability constraints of the form $\mathbb{P}[h(X_T) \leqslant 0] \leqslant 1-\epsilon$ also fall into our analysis since they can be written as functions of the law $\mathcal{L}(X_T)$ of X_T. In state constrained problems, the constraint is directly imposed on the process X_T and must be satisfied almost-surely. Such constraints might be too stringent or even impossible to satisfy and probability constraints might allow to find controls with a better reward and a controlled probability of failure/success.

Stochastic control problems with terminal constraints have been extensively studied in the literature. Optimal control problems under stochastic target constraints have been studied in Bouchard, Elie and Imbert [21] using the geometric dynamic programming principle proposed

in Soner and Touzi [124] . In Föllmer and Leukert [70], the authors introduce the notion of quantile hedging to relax almost-sure constraints into probability constraints. In Yong and Zhou [135] Chapter 3, necessary optimality conditions are proved in the form of a system of forward/backward stochastic differential equations. More recently the problem with constraints on the law of the process has been studied in Pfeiffer [116] and in Pfeiffer, Tan and Zhou [117]. In these works, the authors prove that the problem can be reduced to a "standard" problem (without terminal constraint) by adding a term involving $\lambda^* h$ — in the case where the constraint has the form $\mathbb{E}\left[h(X_T)\right] \leqslant 0$— to the final cost for some optimal Lagrange multiplier λ^*. A dual problem over the Lagrange multipliers associated to the constraints is exhibited using abstract duality results. In Pfeiffer, Tan and Zhou [117], the authors provide necessary and sufficient optimality conditions for problems with multiple equality and inequality expectation constraints with much less restrictions on the data than we do and in a path dependent framework. However [117] needs to assume some controllability condition (Assumption 3.1.*ii*) and works with a compact control set. In our framework, the corresponding controllability condition would be to assume a priori that there exist some control α such that $\mathbb{E}(h(X_T^\alpha)) < 0$. In our analysis, we are able to prove such controllability condition when H satisfies suitable assumptions.

The novelty of the present work is to provide a framework in which both controllability and existence of strong regular solutions for the Stochastic Control problem can be proved. We also believe that our necessary conditions for optimality can lead to efficient numerical methods using techniques already developed for similar kind of coupled PDE systems as in Achdou and Capuzzo Dolcetta [1]. We are also able to handle costs of mean-field type.

Our strategy is to study a relaxed problem which is an optimal control problem for the Fokker-Planck equation and then rely on the regularity of the data to show that optimal controls for the relaxed problem yield optimal controls for the original problem. The relaxed problem is the following :

$$\inf_{(m,\omega,W)} \int_0^T \int_{\mathbb{R}^d} L(t,x,\frac{d\omega}{dt\otimes dm}(t,x),\frac{dW}{dt\otimes dm}(t,x))dm(t)(x)dt + \int_0^T f_2(t,m(t))dt + g(m(T)),$$

where

$$L(t,x,q,N) := \sup_{(p,M)\in\mathbb{R}^d\times\mathbb{S}_d(\mathbb{R})} \{-p.q - M.N - H(t,x,p,M)\} = H^*(t,x,-q,-N)$$

and the infimum is taken over the triples $(m,\omega,W) \in \mathcal{C}([0,T],\mathcal{P}_1(\mathbb{R}^d))\times\mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)\times\mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{S}_d(\mathbb{R}))$ for which ω and W are absolutely continuous with respect to $m(t)\otimes dt$ and (m,ω,W) satisfy in the sense of distribution the Fokker-Planck equation:

$$\partial_t m + \mathrm{div}\omega - \sum_{i,j}\partial^2_{ij}W_{ij} = 0$$

together with the initial condition $m(0) = m_0$ and the terminal constraint $\Psi(m(T)) \leqslant 0$. Notice that here and in the following, we denote by $\mathbb{S}_d(\mathbb{R})$ the space of symmetric matrices of size d, endowed with the inner product $M.N := \mathrm{Tr}(MN)$ and by $\mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ (respectively by $\mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{S}_d(\mathbb{R}))$) the space of $\mathbb{R}^d$-valued (respectively $\mathbb{S}_d(\mathbb{R})$-valued) Borel measures on $[0,T]\times\mathbb{R}^d$ with finite total variation.

In order to study the relaxed problem, we rely on duality techniques that originated in the theory of Optimal Transport (see [118], [130], [131] and [12]) and were further developed in the theory of Mean Field Games. Indeed, when the game has a potential structure — see for instance Lasry, Lions [102], Cardaliaguet, Graber, Porretta and Tonon [37], Briani and Cardaliaguet [24] and Orrieri, Porretta and Savaré [115] — the system of partial differential equations which describes the distribution of the players and the value function of a typical infinitesimal player can be obtained as optimality conditions for an optimal control problem for the Fokker-Planck equation. In this framework, the necessary conditions are obtained through convex duality techniques, using generally the Fenchel-Rockafellar theorem as in [37], [24] or the Von-Neumann theorem as in [115]. We follow this path and — when the final constraint as well as the costs f_2 and g are linear — we are able to exhibit a dual problem, which is an optimal control problem for the HJB equation involving the Lagrange multiplier $\lambda \in \mathbb{R}^+$ associated to the terminal constraint. It takes the following form :

$$\sup_{(\lambda,\phi)} \int_{\mathbb{R}^d} \phi(0,x) dm_0(x),$$

where the supremum runs over the couples $(\lambda, \phi) \in \mathbb{R}^+ \times \mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^d)$ satisfying

$$\begin{cases} -\partial_t \phi(t,x) + H(t,x,D\phi(t,x), D^2\phi(t,x)) \leqslant f_2'(t,x) \text{ in } [0,T] \times \mathbb{R}^d \\ \phi(T,x) \leqslant \lambda h(x) + g'(x) \text{ in } \mathbb{R}^d, \end{cases}$$

and where $f_2' : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $g' : \mathbb{R}^d \to \mathbb{R}$ are such that $f_2(t,m) = \displaystyle\int_{\mathbb{R}^d} f_2'(t,x) dm(x)$ and $g(m) = \displaystyle\int_{\mathbb{R}^d} g'(x) dm(x)$.

The necessary conditions for optimality then follow from the lack of duality gap between the relaxed and the dual problems. We can then address more general constraints $\Psi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and costs $f_2 : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, $g : \mathcal{P}_2(\mathbb{R}^d)$ by "linearizing" the costs and the constraint around solutions of the relaxed problem.

Using convex duality techniques to solve optimal control problems for diffusion processes is of course not new. It can be traced back at least to Fleming and Vermes [69], where the philosophy is very close to ours. In Tan and Touzi, [127] the authors extend the usual Monge-Kantorovitch optimal transportation problem to a stochastic framework. The mass is transported along a continuous semimartingale and the initial and terminal distributions are prescribed. Studying optimal control problems for the Fokker-Planck equation in order to understand the stochastic control problem is less common and it seems adapted to problems where the constraints only act on the law of the process. We refer to the works of Blaquière [15] and more recently, Mikami [111] and Mikami and Thieullen [112] where similar approaches are developed in connection with the so-called Schrödinger problem. This approach has been followed recently by Guo, Loeper and Wang [84] and Guo, Loeper, Langrené and Ning [83] for problems with various expectation constraints. In both papers, the authors show that their original problem is in duality with a problem of optimal control of sub-solutions of an HJB equation. This dual problem is solved numerically. Our relaxation is in the spirit of classical works in convex analysis (see [61]) but usually probabilists prefer to study another relaxation of the initial problem through the martingale problem (see Stroock

and Varadhan [125]), as in El Karoui, Jeanblanc-Picqué and Nguyen [62] or Lacker [95]. These different ways to relax the initial problem are, of course, connected and the correspondences between the diffusion processes, the martingale problem and the Fokker-Planck equation are now well established starting from the seminal work of [125] and more recently Figalli [63] and Trevisan [129].

Under very general assumptions, as in [69], one is usually able to see that the original problem is in duality with a problem of optimal control of the HJB equation. However, existence of solutions for this dual problem is much harder to come by and requires particular structural conditions. Essentially, the dual problem has a solution if the Hamilton-Jacobi-Bellman equation admits a regular solution. This is of course rather difficult to obtain. Regularity results for the Hamilton-Jacobi-Bellman equation where the control appears in the volatility as in Fleming and Soner [68] Chapter $IV.4$, usually rely upon three things : the regularity of the coefficients of the diffusion and of the costs functionals, the compactness of the control set and finally the uniform parabolicity of the equation. The last point means that there must be some $\Lambda^- > 0$ such that the volatility coefficient satisfies (uniformly in the time/state/control variables) $\sigma^t\sigma \geqslant \Lambda^- I_d$.

In studying terminal constraints, compact control sets are not satisfactory since we would not be able to show, in full generality, that the constraint can indeed be reached with a finite cost. Part of the challenge of the paper is to find a framework in which the process is sufficiently "controllable" but the HJB equation is still solvable. For that we need to impose restrictions on the coefficients.

In particular, we require some growth assumptions on the Hamiltonians and its derivatives. This allows us to use the weak Bernstein method as in Ishii and Lions [88], Barles [10], Lions and Souganidis [105] and Armstrong and Cardaliaguet [9] (among others) to prove that the viscosity solution of the HJB equation is Lipschitz in time and space.

As it is well-known, controllability for such systems is related to the coercivity of the Hamiltonian H in the momentum variable. As we will show, imposing a strictly super-linear polynomial growth (in p) for $H(t,x,p,0) := \sup_{a\in A} -b(t,x,a).p - f_1(t,x,a)$ allows to show that the agent can take (with a relaxed control) any instantaneous drift without paying too big a cost.

The rest of the paper is organized as follows : in Section 2.1 we present our assumptions and the precise statement of the problem. We also give our main results there. In Section 2.2 we introduce and study the problem of optimal control of the Fokker-Planck equation. Our main results, Theorems 2.2 and 2.3 are then proved in Section 2.3. Finally we give in Section 2.4 a detailed study of the Hamilton-Jacobi-Bellman equation which is crucial to our analysis.

2.1 Main Results

In this section we first present our notations and our standing assumptions. Then we briefly discuss some properties of the Lagrangian L and finally we state our main results.

2.1.1 Notations and Functional Spaces

The d-dimensional euclidean space is denoted by $\mathbb{R}^d$ and the space of real matrices of size d by $\mathbb{M}_d(\mathbb{R})$. The space of symmetric matrices of size $d \times d$ is denoted by $\mathbb{S}_d(\mathbb{R})$. The subset of $\mathbb{S}_d(\mathbb{R})$ consisting of positive symmetric matrices is denoted by $\mathbb{S}_d^+(\mathbb{R})$ and $\mathbb{S}_d^{++}(\mathbb{R})$ is the subset of $\mathbb{S}_d(\mathbb{R})$ consisting of definite-positive symmetric matrices. Recall that $\mathbb{S}_d^{++}(\mathbb{R})$ is endowed with a smooth (analytic) square root : $\sqrt{.} : \mathbb{S}_d^{++}(\mathbb{R}) \to \mathbb{S}_d^{++}(\mathbb{R})$ (see for instance [125] Lemma 5.2.1). Sometimes we will use $\mathrm{S_p}(A)$ to denote the set of eigenvalues of a square matrix A. The euclidean space $\mathbb{R}^d$ is endowed with its canonical scalar product : $x.y := \sum_{i=1}^d x_i y_i$ and the associated norm $|x|^2 := \sum_{i=1}^d x_i^2$. The space $\mathbb{M}_d(\mathbb{R})$ is endowed with its canonical scalar product : $M.N := \mathrm{Tr}({}^tMN)$ and the associated norm $|M|^2 := \mathrm{Tr}({}^tMM)$, where $\mathrm{Tr}(M)$ is the trace of M and ${}^tM$ is the transpose of M. Sometimes we will use the operator norm on $\mathbb{M}_d(\mathbb{R})$: $|||M||| := \sup_{x\in\mathbb{R}^d} \frac{|Mx|}{|x|}$. For two real numbers r_1 and r_2, $r_1 \wedge r_2$ is the minimum of r_1 and r_2 and $r_1 \vee r_2$ is the maximum of r_1 and r_2. If η is a σ-finite positive measure on a measurable space $(\Omega, \mathcal{F})$, μ is a σ-finite vector measure on $(\Omega, \mathcal{F})$ and μ is absolutely continuous with respect to η we write $\frac{d\mu}{d\eta} \in L^1(\eta)$ for the Radon-Nikodym derivative of μ with respect to η. If E is a locally compact, complete, separable metric space and $l \geqslant 1$ is an integer, $\mathcal{C}_0(E, \mathbb{R}^l)$ is the space of $\mathbb{R}^l$-valued continuous functions on X, vanishing at infinity. It is endowed with the topology of uniform convergence. Its topological dual $\big(\mathcal{C}_0(E, \mathbb{R}^l)\big)^*$ can be identified thanks to Riesz theorem as the space $\mathcal{M}(E, \mathbb{R}^l)$ of $\mathbb{R}^l$-valued Borel measures with finite total variation on E, normed by total variation. We will often consider the weak-* topology on $\mathcal{M}(E, \mathbb{R}^l)$. When $l = 1$ we simply note $\mathcal{C}_0(E)$ and $\mathcal{M}(E)$. $\mathcal{M}^+(E) \subset \mathcal{M}(E)$ is the cone of finite non-negative measures. The set of Borel probability measures over E is denoted by $\mathcal{P}(E)$. If $r \geqslant 1$, $\mathcal{P}_r(E)$ is the set of Borel probability measures over E with finite moment of order r. It is endowed with the topology given by the Wasserstein distance d_r of order r. If X is a random variable taking values into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, its law is denoted by $\mathcal{L}(X) \in \mathcal{P}(\mathbb{R}^d)$. We say that $U : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ if $\mathcal{C}^1$ if there is a bounded continuous function $\frac{\delta U}{\delta m} : \mathcal{P}_1(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ such that, for any $m_1, m_2 \in \mathcal{P}_1(\mathbb{R}^d)$,

$$U(m_1) - U(m_2) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-t)m_2 + tm_1, x)(m_1 - m_2)(dx)dt.$$

This derivative is defined up to an additive constant and we use the standard normalization convention : $\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m, x)m(dx) = 0$. See [43] for details on the notion(s) of derivatives in the space of measures.

We consider a finite, fixed horizon $T > 0$. The set of continuous functions from $[0, T]$ to $\mathcal{P}(\mathbb{R}^d)$ and from $[0, T]$ to $\mathcal{P}_r(\mathbb{R}^d)$ for $r \geqslant 1$ are respectively denoted by $\mathcal{C}([0, T], \mathcal{P}(\mathbb{R}^d))$ and by $\mathcal{C}([0, T], \mathcal{P}_r(\mathbb{R}^d))$. The space of measurable functions defined on $[0, T] \times \mathbb{R}^d$ with values into the measurable space Y is denoted by $L^0([0, T] \times \mathbb{R}^d, Y)$. If $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is sufficiently smooth, $Du : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $D^2u : [0, T] \times \mathbb{R}^d \to \mathbb{S}^d(\mathbb{R})$ denote respectively the differential and the Hessian of u with respect to the space variable x. The space of continuous functions u on $[0, T] \times \mathbb{R}^d$ for which $\partial_t u$, Du and D^2u exist and are continuous is denoted by $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}^d)$ and $\mathcal{C}_b^{1,2}([0, T] \times \mathbb{R}^d)$ is the subspace of $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}^d)$ consisting of functions u for which u, $\partial_t u$, Du and D^2u are bounded. If $n \in \mathbb{N}^*$ and $\alpha \in (0, 1)$, $\mathcal{C}_b^{n+\alpha}(\mathbb{R}^d)$ is the space of bounded continuous real functions on $\mathbb{R}^d$ for which the first n-derivatives

are continuous and bounded and the n-th derivative is α-Hölder continuous. We say that $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is in $\mathcal{C}_b^{\frac{n+\alpha}{2},n+\alpha}([0,T] \times \mathbb{R}^d)$ if ϕ is continuous in both variables together with all derivatives $D_t^r D_x^s \phi$ with $2r + s \leqslant n$. Moreover, $\|\phi\|_{\frac{n+\alpha}{2},n+\alpha}$ is bounded, where

$$\|\phi\|_{\frac{n+\alpha}{2},n+\alpha} := \sum_{2r+s\leq n} \|D_t^r D_x^s \phi\|_\infty + \sum_{2r+s=n} \sup_{t\in[0,T]} \|D_t^r D_x^s \phi(t,.)\|_\alpha + \sum_{0<n+\alpha-2r-s<2} \sup_{x\in\mathbb{R}^d} \|D_t^r D_x^s \phi(.,x)\|_{\frac{n+\alpha-2r-s}{2}}.$$

2.1.2 Assumptions

In all the following, A is a closed subset of an euclidean space, $T > 0$ is a finite horizon and $r_2 \geqslant r_1 > 1$ are two parameters. The conjugate exponents of r_1 and r_2 are respectively denoted by r_1^* and r_2^*. The data are:

$$\begin{aligned}(b,\sigma,f_1) : [0,T] \times \mathbb{R}^d \times A &\to \mathbb{R}^d \times \mathbb{S}_+^d(\mathbb{R}) \times \mathbb{R},\\ f_2 : [0,T] \times \mathcal{P}_1(\mathbb{R}^d) &\to \mathbb{R},\\ g : \mathcal{P}_1(\mathbb{R}^d) &\to \mathbb{R},\\ \Psi : \mathcal{P}_1(\mathbb{R}^d) &\to \mathbb{R},\\ m_0 &\in \mathcal{P}_{r_1^*}(\mathbb{R}^d).\end{aligned}$$

We define the Hamiltonian of the system, for all $(t,x,p,M) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d(\mathbb{R})$:

$$H(t,x,p,M) = \sup_{a\in A} \left\{-b(t,x,a).p - \sigma(t,x,a)^t \sigma(t,x,a).M - f_1(t,x,a)\right\}$$

1. **Assumptions on b, σ, f_1, f_2 and g**

   (a) For all $R > 0$, b, σ and f_1 as well as the partial derivatives $\partial_x b$, $\partial_t b$, $\partial_{xx}^2 b$, $\partial_x \sigma$, $\partial_t \sigma$, $\partial_{xx}^2 \sigma$, $\partial_x f_1$, $\partial_t f_1$, $\partial_{xx}^2 f_1$, are continuous and bounded on $[0,T] \times \mathbb{R}^d \times (A \cap B(0,R))$; $\partial_x b$, $\partial_x \sigma$ and $\partial_x f_1$ are globally bounded.

   (b) b has at most a linear growth and σ satisfies $\Lambda^- I_d \leqslant \sigma^t \sigma(t,x,a) \leqslant \Lambda^+ I_d$ for some $\Lambda^+ \geqslant \Lambda^- > 0$ uniformly in (t,x,a).

   (c) f_1 is continuous and coercive with respect to a: there is $\delta > 0$ and $C_1, C_2 > 0$ such that, for all (t,x,a), $f_1(t,x,a) \geqslant C_1|a|^{1+\delta} - C_2$.

   (d) f_2 is continuous, bounded and has one linear derivative in m. The first order functional derivative $\dfrac{\delta f_2}{\delta m} : [0,T] \times \mathcal{P}_1(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ is globally Lipschitz continuous, bounded and $x \to \dfrac{\delta f_2}{\delta m}(t,m,x)$ belongs to $\mathcal{C}_b^{3+\alpha}(\mathbb{R}^d)$ with bounds uniform in (t,m).

   (e) g is continuous, bounded and has one functional derivative in m such that $x \to \dfrac{\delta g}{\delta m}(m,x)$ belongs to $\mathcal{C}_b^{3+\alpha}(\mathbb{R}^d)$ with bounds uniform in m.

2. **Assumptions on the Hamiltonian**

   (a) H is C^1 in (t,x,p,M). The partial derivatives $\partial_x H$, $\partial_p H$ and $\partial_M H$ are Lipschitz in $[0,T]\times\mathbb{R}^d\times B(0,R)\times B(0,R)$ for all $R>0$.

   (b) There is some $\alpha_1,\alpha_2>0$ and $C_H>0$ such that, for all $(t,x,p)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$,

$$\alpha_1|p|^{r_1}-C_H\leqslant H(t,x,p,0)\leqslant\alpha_2|p|^{r_2}+C_H.$$

   (c) $\partial_t H(t,x,p,M)$ is bounded over $[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}_d(\mathbb{R})$.

   (d) There is some positive constant $C_{\partial_p H}$ and an exponent $\nu\geqslant 1$ such that, for all $(t,x,p,M)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}_d(\mathbb{R})$

$$|\partial_p H(t,x,p,M)|\leqslant C_{\partial_p H}(1+|p|^{\nu}).$$

   (e) $\partial_x H$ is uniformly in $(t,x,p)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ Lipschitz continuous in M.

   (f) i. **Either** $f_2=0$ and the limit $\lim_{|p|\to+\infty}\dfrac{|p|^2-\partial_x H(t,x,p,0).p}{H^2(t,x,p,0)}=0$ holds uniformly in $(t,x)\in[0,T]\times\mathbb{R}^d$

   ii. **or** $f_2\neq 0$ and there is some $C_{\partial_x H}>0$ such that $|\partial_x H(t,x,p,0)|\leqslant C_{\partial_x H}(1+|p|)$.

3. **Assumptions on the constraint Ψ**

   (a) Ψ is continuous and admits a functional derivative such that $x\to\dfrac{\delta\Psi}{\delta m}(m,x)$ belongs to $\mathcal{C}_b^{3+\alpha}(\mathbb{R}^d)$ with bounds uniform in m.

   (b) There is at least one $m\in\mathcal{P}_1(\mathbb{R}^d)$ such that $\Psi(m)<0$.

   (c) For all $m\in\mathcal{P}_1(\mathbb{R}^d)$ such that $\Psi(m)=0$ there exists $x_0\in\mathbb{R}^d$ such that $\dfrac{\delta\Psi}{\delta m}(m,x_0)<0$.

Remark 2.1. Assumption 1 is sufficient to uniquely define the controlled process X^α for any control $\alpha\in\mathcal{A}$ (see below for the definitions). If A were compact with $f_2=0$, we would be in the setting of [68] Chapter IV.4 and these assumptions would guarantee the existence of a smooth value function (in $\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)$).

Remark 2.2. The upper bound in Assumption 2b is a coercivity assumption on the cost f_1 relatively to the drift b. Taking the definition of H, we see that it is equivalent to ask that, for all $(t,x,a)\in[0,T]\times\mathbb{R}^d\times A$, $f_1(t,x,a)\geqslant\alpha_2'|b(t,x,a)|^{r_2^*}-C_H$, for some $\alpha_2'>0$. It will be a source of compactness throughout the paper. The lower bound in Assumption 2b is a "weak"-controllability condition and we will discuss it further in Lemma 2.2.

Remark 2.3. Using the Envelope theorem (see for instance [113]) we see that H being $\mathcal{C}^1$ —Assumption 2a— in the p, M-variables implies that, for any $a(t,x,p,M) \in A$ such that $H(t,x,p,M) = -b(t,x,a(t,x,p,M)).p - \sigma^t\sigma(t,x,a(t,x,p,M)) - f_1(t,x,a(t,x,p,M))$ we get $\partial_p H(t,x,p,M) = -b(t,x,a(t,x,p,M))$ and $\partial_M H(t,x,p,M) = -\sigma^t\sigma(t,x,a(t,x,p,M))$.

Consequently, drift and volatility must agree on potentially different optimal controls with common values $-\partial_p H(t,x,p,M)$ and $\sqrt{-\partial_M H(t,x,p,M)}$ respectively. Notice that the growth conditions on the cost f_1 and the drift b ensure that for any $(t,x,p,M) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_d(\mathbb{R})$, there exists at least one such $a(t,x,p,M)$ in A.

Remark 2.4. Using the envelope theorem and the uniform ellipticity condition in Assumption 1b we see that for all (t,x,p,M), $\Lambda^- I_d \leqslant -\partial_M H(t,x,p,M) \leqslant \Lambda^+ I_d$, a fact that we will repeatedly used throughout the paper.

Remark 2.5. We use (the restrictive) Assumptions 2c, 2d, 2e, 2f in order to find Lipschitz estimates for the solution of the Hamilton-Jacobi-Bellman equation and to deduce that it is well-posed in $\mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^d)$. Assumptions 2a is then sufficient to show that the solution is actually in $\mathcal{C}_b^{\frac{3+\alpha}{2},3+\alpha}([0,T] \times \mathbb{R}^d)$. When Assumption 2b hold, Assumption 2(f)ii is stronger than Assumption 2(f)i but we use it to find Lipschitz estimates which are independent from the time regularity of the source term of the HJB equation.

Remark 2.6. Assumption 3c is a tranversality condition. When Ψ is convex, this assumption is equivalent to the existence of some probability measure $m \in \mathcal{P}_1(\mathbb{R}^d)$ such that $\Psi(m) < 0$.

The following observations will be useful in order to translate the properties of the Hamiltonian H into properties of the Lagrangian L defined for all $(t,x,q,N) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_d(\mathbb{R})$ by

$$L(t,x,q,M) := \sup_{(p,M) \in \mathbb{R}^d \times \mathbb{S}_d(\mathbb{R})} \{-p.q - M.N - H(t,x,p,M)\}.$$

Taking convex conjugates in 2b we see that this assumption can be reformulated in terms of L: for all $(t,x,q) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\alpha_2' |q|^{r_2^*} - C_H \leqslant L(t,x,q,0) \leqslant \alpha_1' |q|^{r_1^*} + C_H, \tag{2.2}$$

where, for $i = 1,2$, $\alpha_i' = \alpha_i^{-\frac{1}{r_i - 1}} \left(r_i - 1 \right) r_i^{\frac{-r_i}{r_i - 1}}$ and $r_i^* = \frac{r_i}{r_i - 1}$ is the conjugate exponent of r_i.

Throughout the article, the following dual representation for L will be useful.

Lemma 2.1. *Under Assumption 1 above, for all $(t,x,q,N) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_d(\mathbb{R})$, $L(t,x,q,N) < +\infty$ if and only if there is $\mathbf{q}_A \in \mathcal{P}_1(A)$ such that $\int_A b(t,x,a) d\mathbf{q}_A(a) = q$ and $\int_A \sigma^t\sigma(t,x,a) d\mathbf{q}_A(a) = N$ and in this case*

$$L(t,x,q,N) = \min_{\mathbf{q}_A} \int_A f_1(t,x,a) d\mathbf{q}_A(a),$$

where the minimum is taken over the $\mathbf{q}_A \in \mathcal{P}_1(A)$ such that $\int_A b(t,x,a) d\mathbf{q}_A(a) = q$ and $\int_A \sigma^t\sigma(t,x,a) d\mathbf{q}_A(a) = N$.

Proof. It is elementary to show that for all $(t,x,p,M) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_d(\mathbb{R})$,

$$H(t,x,p,M) = \sup_{\mathbf{q}_A \in \mathcal{P}_1(A)} \left\{ \int_A (-b(t,x,a).p - \sigma^t\sigma(t,x,a) - f_1(t,x,a)) d\mathbf{q}_A(a) \right\}$$

and therefore L reads as follows for all $(t,x,q,N) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_d(\mathbb{R})$,

$$L(t,x,q,N) = \sup_{p,M} \left\{ \inf_{\mathbf{q}_A \in \mathcal{P}_1(A)} \left\{ -p.q - M.N + \int_A (f_1(t,x,a) + b(t,x,a).p + \sigma^t\sigma(t,x,a).M) d\mathbf{q}_A(a) \right\} \right\}.$$

The result follows by exchanging the "sup" and the "inf". To this end we use Von Neumann Theorem 2.6 in the Appendix. The coercivity of f_1 as well as results of [8] (Proposition 7.1.5) about the lower semicontinuity of functions defined on the space of probability measures allow to ensure that the use of the minmax theorem is licit. □

From this dual representation we can see that the lower-bound on $H(t,x,p,0)$ —or equivalently the upper-bound on $L(t,x,q,0)$— is a "weak"-controllability condition. It ensures that the agent can take any drift with a relaxed (i.e measure-valued) control without paying more than the r_1^*-power of the drift :

Lemma 2.2. *Fix $(t,x) \in [0,T] \times \mathbb{R}^d$. It holds that $H(t,x,p,0) \geqslant \alpha_1|p|^{r_1} - C_H$ for all $p \in \mathbb{R}^d$ if and only if, for all $q \in \mathbb{R}^d$ there exists $\mathbf{q}_A \in \mathcal{P}_1(A)$ such that $q = \int_A b(t,x,a) d\mathbf{q}_A(a)$ and $\int_A f_1(t,x,a) d\mathbf{q}_A(a) \leqslant \alpha_1'|q|^{r_1^*} + C_H$.*

For example, the growth condition on H is satisfied if $Conv(Im(b(t,x,.)) = \mathbb{R}^d$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and for all $(t,x,a) \in [0,T] \times \mathbb{R}^d \times A$, $\alpha_2'|b(t,x,a)|^{r_2^*} - C_H \leqslant f_1(t,x,a) \leqslant \alpha_1'|b(t,x,a)|^{r_1^*} + C_H$.

2.1.3 Main Results

Throughout the article, we consider a fixed filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ satisfying the usual conditions and supporting an adapted, standard d-dimensional Brownian motion $(B_t)_{t \geqslant 0}$. We fix a $\mathcal{F}_0$-measurable random variable X_0, independent of (B_t) and such that X_0 belongs to $L^{r_1^*}(\mathbb{P})$. The control process $\alpha = (\alpha_t)_{t \geqslant 0}$ is a progressively measurable process valued in A with finite $\mathbb{L}^2(\Omega \times [0,T))$-norm. We denote by $\mathcal{A}$ the set of control processes. From the Cauchy-Lipschitz theorem, we know that for every $\alpha \in \mathcal{A}$, there exists a unique $\mathbb{F}$-adapted process X^α satisfying :

$$dX_t = b(t, X_t, \alpha_t)dt + \sqrt{2}\sigma(t, X_t, \alpha_t)dB_t$$

with the initial condition $X_0^\alpha = X_0$. A particular class of controls which is of interest is the one of Markovian controls (or Markov policies). A control process α is a Markovian control if there is a measurable function $\alpha : [0,T] \times \mathbb{R}^d \to$ such that, for all $t \in [0,T]$, $\alpha_t = \alpha(t, X_t^\alpha)$. We now introduce the cost functional $J_{SP} : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$

$$J_{SP}(\alpha) := \mathbb{E}\left[\int_0^T \left(f_1(s, X_s^\alpha, \alpha_s) + f_2(s, \mathcal{L}(X_s^\alpha))\right) ds + g(\mathcal{L}(X_T^\alpha))\right].$$

The optimal control problem we are interested in is to minimize $J_{SP}(\alpha)$ over $\alpha \in \mathcal{A}$ under the constraint $\Psi(\mathcal{L}(X_T)) \leqslant 0$.
If there exists a continuous function $h : \mathbb{R}^d \to \mathbb{R}$ such that, for all $m \in \mathcal{P}_1(\mathbb{R}^d)$, $\Psi(m) = \int_{\mathbb{R}^d} h(x)dm(x)$ then will say that the final constraint is **linear**. We define the set of admissible controls $\mathcal{U}_{ad}$

$$\mathcal{U}_{ad} := \{\alpha \in \mathcal{A} : \Psi(\mathcal{L}(X_T^\alpha)) \leqslant 0 \text{ and } J_{SP}(\alpha) < +\infty\}.$$

The problem in strong formulation is thus :

$$\inf_{\alpha \in \mathcal{U}_{ad}} J_{SP}(\alpha). \tag{SP}$$

The fact that $\mathcal{U}_{ad}$ is not empty is not trivial in itself but in our setting we will show that there are indeed admissible controls. Our results are the following :

Theorem 2.1 (HJB equation)**.** *Take $g' \in \mathcal{C}_b^{3+\alpha}(\mathbb{R}^d)$ and $f_2' \in \mathcal{C}_b([0,T], \mathcal{C}_b^{3+\alpha}(\mathbb{R}^d))$ such that $t \to f_2'(t,x) \in \mathcal{C}^\alpha([0,T])$ for all $x \in \mathbb{R}^d$ with bounds uniform in x. Assume further that Assumptions 1 and 2 hold with 2(f)i in force if $f_2' = 0$ and 2(f)ii in force if $f_2' \neq 0$. Then the Hamilton-Jacobi-Bellman equation*

$$\begin{cases} -\partial_t \phi(t,x) + H(t,x,D\phi(t,x), D^2\phi(t,x)) = f_2'(t,x) & \text{in } [0,T] \times \mathbb{R}^d \\ \phi(T,x) = g'(x) & \text{in } \mathbb{R}^d \end{cases}$$

admits a unique strong solution $\phi \in \mathcal{C}_b^{\frac{3+\alpha}{2}, 3+\alpha}([0,T] \times \mathbb{R}^d)$.

Theorem 2.2 (General Constraint)**.** *Under Assumptions 1, 2 and 3, there exist optimal Markov policies. Moreover, if $(\alpha_t) \in \mathcal{A}$ is an optimal Markov policy, then there exists $(\lambda, \phi, m) \in \mathbb{R}^+ \times \mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^d)) \times \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d))$ such that, for $m(t) \otimes dt$-almost all (t,x) in $[0,T] \times \mathbb{R}^d$*

$$\begin{aligned} H(t,x,D\phi(t,x), D^2\phi(t,x)) = &- b(t,x,\alpha(t,x)).D\phi(t,x) - \sigma^t\sigma(t,x,\alpha(s,x)).D^2\phi(t,x) \\ &- f_1(t,x,\alpha(t,x)) \end{aligned} \tag{2.3}$$

and (λ, ϕ, m) satisfies the system of optimality conditions :

$$\begin{cases} -\partial_t \phi(t,x) + H(t,x,D\phi(t,x), D^2\phi(t,x)) = \dfrac{\delta f_2}{\delta m}(t, m(t), x) & \text{in } [0,T] \times \mathbb{R}^d \\ \partial_t m - \operatorname{div}(\partial_p H(t,x,D\phi(t,x), D^2\phi(t,x))m) \\ \qquad + \displaystyle\sum_{i,j} \partial_{ij}^2((\partial_M H(t,x,D\phi(t,x), D^2\phi(t,x)))_{ij} m) = 0 & \text{in } [0,T] \times \mathbb{R}^d \\ \phi(T,x) = \lambda \dfrac{\delta \Psi}{\delta m}(m(T), x) + \dfrac{\delta g}{\delta m}(m(T), x) \text{ in } \mathbb{R}^d,\ m(0) = m_0, \\ \lambda \Psi(m(T)) = 0,\ \Psi(m(T)) \leqslant 0,\ \lambda \geqslant 0. \end{cases} \tag{OC}$$

Furthermore, $m(t)$ is actually the law of the optimally controlled process X_t^α and the value of the problem -denoted by $V_{SP}(X_0)$- is given by

$$V_{SP}(X_0) := \inf_{\alpha \in \mathcal{U}_{ad}} J_{SP}(\alpha) = \int_{\mathbb{R}^d} \phi(0,x) dm_0(x) + \int_0^T f_2(t, m(t)) dt + g(m(T)).$$

When the constraint and the costs f_2 and g are convex in the measure variable, we are able to show that the conditions are also sufficient :

Theorem 2.3 (Convex constraint and convex costs)**.** *If Ψ, f_2 and g are convex in the measure argument and Assumptions 1, 2 and 3 hold, then the conditions of Theorem 2.2 are also sufficient conditions: if $\alpha \in L^0([0,T] \times \mathbb{R}^d, A)$ satisfies 2.3 for some (ϕ, m, λ) satisfying OC then the SDE*

$$dX_t = b(t, X_t, \alpha(t, X_t))dt + \sqrt{2}\sigma(t, X_t, \alpha(t, X_t))dB_t$$

starting from X_0 has unique strong solution X_t, it holds that $m(t) = \mathcal{L}(X_t)$ and $\alpha_t := \alpha(t, X_t)$ is a Markovian solution to SP.

Remark 2.7. Using standard parabolic PDE techniques and the regularity of ϕ, we can show that, provided m_0 admits a density in $\mathcal{C}_b^{2+\alpha}(\mathbb{R}^d)$, $m(t)$ in Theorem 2.2 admits a density $m(t,x)$ with respect to the Lebesgue measure such that $m \in \mathcal{C}_b^{\frac{2+\alpha}{2}, 2+\alpha}([0,T] \times \mathbb{R}^d)$.

Remark 2.8. In Theorems 2.2 and 2.3, the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and the Brownian motion (B_t) introduced at the beginning of this section are a priori fixed. In the terminology of stochastic control it means that we deal with strong solutions to the stochastic control problem.

Remark 2.9. In the spirit of the Karush-Kuhn-Tucker theorem, multiple inequality constraints $\Psi_i(m(T)) \leqslant 0 \ \forall i \in [\![1,n]\!]$ can be considered provided they satisfy some qualification condition. We would say that the constraint is qualified at $\widetilde{m} \in \mathcal{P}_1(\mathbb{R}^d)$ provided there exists some $m \in \mathcal{P}_1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \frac{\delta \Psi_i}{\delta m}(\widetilde{m}, x) dm(x) < 0$ for all $i \in [\![1,n]\!]$ such that $\Psi_i(\widetilde{m}) = 0$. If $n = 2$ a sufficient condition would be $\frac{\delta \Psi_i}{\delta m}(\widetilde{m}, .) \in L^2(\mathbb{R}^d)$ for $i = 1, 2$ and $\int_{\mathbb{R}^d} \frac{\delta \Psi_1}{\delta m}(\widetilde{m}, x) \frac{\delta \Psi_2}{\delta m}(\widetilde{m}, x) dx > 0$. For $n \geqslant 2$ the condition would be satisfied everywhere if the constraints Ψ_i are convex, satisfy Assumption 3 and if there is some $m \in \mathcal{P}_1(\mathbb{R}^d)$ such that $\Psi_i(m) < 0$ for all $i \in [\![1,n]\!]$.

2.2 A relaxed Problem: Optimal Control of the Fokker-Planck Equation

Definition 2.1. The relaxed problem is

$$\inf_{(m,\omega,W) \in \mathbb{K}} J_{RP}(m, \omega, W), \tag{RP}$$

where $\mathbb{K}$ is the set of triples $(m,\omega,W) \in \mathcal{C}([0,T],\mathcal{P}_1(\mathbb{R}^d)) \times \mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{R}^d) \times \mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{S}_d(\mathbb{R}))$ such that ω and W are absolutely continuous with respect to $m(t)\otimes dt$,

$$\partial_t m + \mathrm{div}\omega - \sum_{i,j} \partial^2_{ij} W_{ij} = 0 \tag{2.4}$$

holds in the sense of distributions, $m(0) = m_0$ and $\Psi(m(T)) \leqslant 0$. The cost J_{RP} is defined on $\mathbb{K}$ by

$$\begin{aligned} J_{RP}(m,\omega,W) := &\int_0^T \int_{\mathbb{R}^d} L\left(t,x,\frac{d\omega}{dt\otimes dm}(t,x),\frac{dW}{dt\otimes dm}(t,x)\right) dm(t)(x)dt \\ &+ \int_0^T f_2(t,m(t))dt + g(m(T)). \end{aligned}$$

Notice that the first term in the objective function J_{RP} is convex in the variables (m,ω,W) and that the Fokker-Planck equation and the initial condition are linear in (m,ω,W). Therefore the problem is linear/convex when the final constraint as well as the costs f_2 and g are convex.
We say that (m,ω,W) in $\mathcal{C}([0,T],\mathcal{P}_1(\mathbb{R}^d)) \times \mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{R}^d) \times \mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{S}_d(\mathbb{R}))$ satisfies the Fokker-Planck equation (FPE) 2.4 with initital condition $m(0) = m_0$ if and only if, for all $\varphi \in \mathcal{C}(\mathbb{R}^d)$ with compact support and all $\phi \in \mathcal{C}^{1,2}((0,T)\times\mathbb{R}^d)$ with compact support we have

$$\int_0^T \int_{\mathbb{R}^d} \partial_t\phi(t,x)dm(t)(x)dt + \int_0^T \int_{\mathbb{R}^d} D\phi(t,x).d\omega(t,x) + \int_0^T \int_{\mathbb{R}^d} D^2\phi(t,x).dW(t,x) = 0$$

and the initial condition $\displaystyle\int_{\mathbb{R}^d} \varphi(x)dm(0)(x) = \int_{\mathbb{R}^d} \varphi(x)dm_0(x)$.
Moreover, if ω and W are absolutely continuous with respect to $m(t)\otimes dt$ the above relations hold if φ and ϕ are respectively taken in $\mathcal{C}_b(\mathbb{R}^d)$ and $\mathcal{C}^{1,2}_b((0,T)\times\mathbb{R}^d)$ (see [129] Remark 2.3). In this case, we have for all $\phi \in \mathcal{C}^{1,2}_b([0,T]\times\mathbb{R}^d)$ and for all $t_1,t_2 \in [0,T]$

$$\begin{aligned} &\int_{\mathbb{R}^d} \phi(t_2,x)dm(t_2)(x) = \int_{\mathbb{R}^d} \phi(t_1,x)dm(t_1)(x) \\ &+ \int_{t_1}^{t_2} \left[\partial_t\phi(t,x) + D\phi(t,x).\frac{d\omega}{dm\otimes dt}(t,x) + \frac{dW}{dm\otimes dt}(t,x).D^2\phi(t,x)\right] dm(t)(x)dt. \end{aligned}$$

Let us recall some known results about the link between solutions of the FPE and solutions to the SDE.

Proposition 2.1.

1. *Suppose that m is a solution to the Fokker-Planck equation*

$$\begin{cases} \partial_t m + \mathrm{div}(b(t,x)m) - \sum_{i,j} \partial^2_{i,j}\left((\sigma^t\sigma(t,x))_{ij}m\right) = 0 \\ m(0) = m_0. \end{cases} \tag{2.5}$$

with coefficients $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{M}_d(\mathbb{R})$, *Borel functions satisfying*

$$\int_0^T \int_{\mathbb{R}^d} \left(|b(t,x)| + |\sigma(t,x)|^2\right) dm(t)(x)dt < +\infty.$$

Then there is a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, *an adapted Brownian motion* $(B_t)_{t\geqslant 0}$ *and an adapted process* $(X_t)_{0\leqslant t\leqslant T}$ *such that*

$$\mathcal{L}(X_0) = m_0,$$
$$dX_t = b(t,X_t)dt + \sqrt{2}\sigma(t,X_t)dB_t.$$

Moreover, for all $t \in [0,T]$, $\mathcal{L}(X_t) = m(t)$.

2. *Conversely, suppose that* $(X_s)_{s\geqslant 0}$ *is a strong solution of the stochastic differential equation*

$$\begin{cases} dX_s = b(s,X_s)ds + \sqrt{2}\sigma(s,X_s)dB_s \\ X|_{t=0} = X_0 \end{cases}$$

on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *endowed with an adapted Brownian motion* (B_t) *with* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *and* $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{M}_d(\mathbb{R})$ *Borel-measurable functions such that*

$$\mathbb{P}\left[\int_0^T \left(|b(s,X_s)| + |\sigma(s,X_s)|^2\right) ds < +\infty\right] = 1$$

and let $m(t) := \mathcal{L}(X_t) = X_t\#\mathbb{P}$, *then* m *satisfies the Fokker-Planck equation 2.5.*

Proof. The second part follows from Itô's lemma and is standard. For the first part we need to combine the argument of [90] and [129]. From [129] Theorem 2.5 we know that this statement is equivalent to the existence of a solution to the so-called martingale problem and from [90] Chapter 4, we know that existence of a solution to the martingale problem is equivalent to the existence of a weak solution to the SDE. □

Let $V_{RP}(m_0)$ be the value of the relaxed problem. The link with the usual compactification / convexification (see [62] and [95]) method in stochastic optimal control is the following :

Proposition 2.2.

$$V_{RP}(m_0) = \inf_{\mathbf{q}_A, m} \{\int_0^T \int_{\mathbb{R}^d} \int_A f_1(t,x,a)d\mathbf{q}_A(t,x)(a)dm(t)(x)dt + \int_0^T f_2(t,m(t))dt + g(m(T))\},$$

where the infimum is taken over the couples $(\mathbf{q}_A, m) \in L^0([0,T]\times\mathbb{R}^d, \mathcal{P}_1(A))\times\mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d))$ *that satisfy in the sense of distributions the Fokker-Planck equation*

$$\partial_t m + \operatorname{div}(\int_A b(t,x,a)d\mathbf{q}_A(t,x)(a)m) - \sum_{i,j} \partial^2_{ij}(\left(\int_A \sigma^t\sigma(t,x,a)d\mathbf{q}_A(t,x)(a)\right)_{ij} m) = 0$$

together with the initial condition $m(0) = m_0$ *and the terminal constraint* $\Psi(m(T)) \leqslant 0$.

Proof. The proof follows from the dual representation of L in Lemma 2.1 and a measurable selection argument as in [125] Theorem 12.1.10. For every competitor (m, ω, W) such that $L(t, x, \frac{d\omega}{dt \otimes dm}(t, x), \frac{dW}{dt \otimes dm}(t, x)) < +\infty$ one can find a measurable function $\mathbf{q}_A : [0, T] \times \mathbb{R}^d \to \mathcal{P}_1(A)$ such that, for every $(t, x) \in [0, T] \times \mathbb{R}^d$ one has

$$L(t, x, \frac{d\omega}{dt \otimes dm}(t, x), \frac{dW}{dt \otimes dt}(t, x)) = \int_A f_1(t, x, a) d\mathbf{q}_A(t, x)(a),$$

and

$$\left(\frac{d\omega}{dt \otimes dm}(t, x), \frac{dW}{dt \otimes dm}(t, x) \right) = \left(\int_A b(t, x, a) d\mathbf{q}_A(t, x)(a), \int_A \sigma^t \sigma(t, x, a) d\mathbf{q}_A(t, x)(a) \right).$$

□

2.2.1 Analysis of the Relaxed Problem

We will need the following facts :

Lemma 2.3. *There exists $(m, \omega, W) \in \mathbb{K}$ such that $J_{RP}(m, \omega, W) < +\infty$.*

Proof. We have to check that we can indeed reach the final constraint with a finite cost. By continuity of Ψ we can find $x_0, ..., x_n \in \mathbb{R}^d$ such that $\Psi(\frac{1}{n} \sum_{i=1}^n \delta_{x_i}) < 0$. Fix some $\delta > 0$. Let i be in $[\![0, n]\!]$. For all $c > 0$ we can find $\mathbf{q}^c \in L^0([0, T] \times \mathbb{R}^d, \mathcal{P}_1(A))$ such that $\int_A b(t, x, a) d\mathbf{q}^c(t, x)(a) = c(x_i - x)$ and $\int_A f_1(t, x, a) d\mathbf{q}^c(t, x)(a) \leqslant \alpha_1' c^{r_1^*} |x_i - x|^{r_1^*} + C_H$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$ (see Lemma 2.2). We define the measurable function $\widetilde{\sigma}_c(t, x) := \left(\int_A \sigma^t \sigma(t, x, a) d\mathbf{q}^c(t, x)(a) \right)^{\frac{1}{2}}$. Notice that $\Lambda^- I_d \leqslant \widetilde{\sigma}_c^t \widetilde{\sigma}_c(t, x) \leqslant \Lambda^+ I_d$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$. We can use the result of Krylov in part 2.6 of [93] (existence of weak solutions to stochastic differential equations with bounded measurable coefficients and uniformly non-degenerate volatility) and find a filtered probability space $(\Omega^1, \mathcal{F}^1, \mathbb{F}^1, \mathbb{P}^1)$ satisfying the usual conditions, an adapted Brownian motion (B_t), a $\mathcal{F}_0^1$ measurable random variable X_0 with law m_0 and a solution Y_t^c of the stochastic differential equation

$$dY_t^c = ce^{ct} x_i dt + \sqrt{2} \widetilde{\sigma}_c'(t, Y_t^c) dB_t$$

starting from X_0 with $\widetilde{\sigma}_c'(t, y) = e^{ct} \widetilde{\sigma}_c(t, e^{-ct} y)$. By Ito's lemma, $X_t^c := e^{-ct} Y_t^c$ solves the SDE

$$dX_t^c = c(x_i - X_t^c) dt + \sqrt{2} \widetilde{\sigma}_c(t, X_t^c) dB_t$$

starting from X_0 and we have, for all $t \in [0, T]$

$$X_t^c = x_i + (X_0 - x_i) e^{-ct} + \sqrt{2} e^{-ct} \int_0^t \widetilde{\sigma}_c(s, X_s^c) e^{cs} dB_s.$$

Using the Burkholder-Davis-Gundy inequality and the upper bound on $\sigma^t \sigma$ we get

$$\begin{aligned}\mathbb{E}^1(|X_t^c - x_i|^{r_1^*}) &\leqslant 2^{r_1^*-1}e^{-r_1^* ct}\mathbb{E}^1(|X_0 - x_i|^{r_1^*}) + 2^{\frac{3r_1^*-2}{2}}e^{-r_1^* ct}\mathbb{E}^1\left(|\int_0^t \widetilde{\sigma}_c(s, X_s^c)e^{cs}dB_s|^{r_1^*}\right)\\ &\leqslant 2^{r_1^*-1}e^{-r_1^* ct}\mathbb{E}^1(|X_0 - x_i|^{r_1^*}) + 2^{\frac{3r_1^*-2}{2}}e^{-r_1^* ct}\mathbb{E}^1\left((\int_0^t \mathrm{Tr}(\widetilde{\sigma}^t\widetilde{\sigma}_c(s, X_s^c)e^{2cs}ds)^{\frac{r_1^*}{2}}\right)\\ &\leqslant 2^{r_1^*-1}e^{-r_1^* ct}\mathbb{E}^1(|X_0 - x_i|^{r_1^*}) + 2^{\frac{3r_1^*-2}{2}}(d\Lambda^+)^{\frac{r_1^*}{2}}\left(\frac{e^{-2ct}-1}{2c}\right)^{\frac{r_1^*}{2}},\end{aligned}$$

where $\mathbb{E}^1$ is the expectation under $\mathbb{P}^1$. In particular, taking $t = T$ we see that, for c sufficiently large we have $d_{r_1^*}(\mathcal{L}(X_T^c), \delta_{x_i}) \leqslant \delta$. Now, for such a c, we let $m^i(t) = \mathcal{L}(X_t^c)$, $\omega^i = c(x_i - x)m^i$, $W^i = \widetilde{\sigma}_c^t\widetilde{\sigma}_c(t, x)m^i$. Since f_2 and g are bounded functions, and thanks to the upper bound on f_1 we have that

$$J_{RP}(m^i, \omega^i, W^i) \leqslant C\left(1 + \int_0^T \mathbb{E}^1(|X_t^c - x_i|^{r_1^*})dt\right) < +\infty.$$

Now we do the same for all $i \in [\![0, n]\!]$ and we let $(m, \omega, W) := \dfrac{1}{n}\displaystyle\sum_{i=1}^n (m^i, \omega^i, W^i)$. The triple (m, ω, W) solves the Fokker-Planck equation starting from m_0. Now by convexity of

$$(m, \omega, W) \to \int_0^T \int_{\mathbb{R}^d} L\left(t, x, \frac{d\omega}{dm \otimes dt}(t, x), \frac{dW}{dm \otimes dt}(t, x)\right) dm(t)(x)dt$$

and using the fact that f_2 and g are bounded we get that $J_{RP}(m, \omega, W) < +\infty$. Finally

$$d_{r_1^*}\left(\frac{1}{n}\sum_{i=1}^n m^i(T), \frac{1}{n}\sum_{i=1}^n \delta_{x_i}\right) \leqslant C(n)\delta$$

for some non negative constant $C(n)$. For δ small enough we get that $J_{RP}(m, \omega, W) < +\infty$ and $\Psi(m(T)) < 0$ which concludes the proof.

□

Lemma 2.4. *1. Any point $(m, \omega, W) \in \mathbb{K}$ with $J_{RP}(m, \omega, W) < +\infty$, satisfies the following estimate for some constant C_{r_2} depending only on r_2: for any $0 < s \leqslant t < T$,*

$$\mathbf{d}_{r_2^*}(m(s), m(t))^{r_2^*} \leqslant C_{r_2}(t-s)^{r_2^*-1}\int_{\mathbb{R}^d}\int_0^T \left|\frac{d\omega}{dt \otimes dm}(u, x)\right|^{r_2^*} dm(u)(x)du + C_{r_2}\Lambda^+(t-s)^{\frac{r_2^*}{2}}. \tag{2.6}$$

2. There exists some $M > 0$ such that

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^d} |x|^{r_2^*} dm(t)(x) + |\omega|([0, T] \times \mathbb{R}^d) + |W|([0, T] \times \mathbb{R}^d) \leqslant M \tag{2.7}$$

whenever $J_{RP}(m, \omega, W) \leqslant \inf J_{RP} + 1$.

Proof. First observe that, since $J_{RP}(m,\omega,W) < +\infty$, by the dual formula for L of Lemma 2.1, we know that $m(t)\otimes dt$-almost-everywhere : $\Lambda^- I_d \leqslant \dfrac{dW}{dt\otimes dm} \leqslant \Lambda^+ I_d$. Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},(X,B))$ be a weak solution to the SDE

$$dX_t = \frac{d\omega}{dt\otimes dm}(t,X_t)dt + \sqrt{2\frac{dW}{dt\otimes dm}}(t,X_t)dB_t$$

with $\mathcal{L}(X_t) = m(t)$ for all $t\in[0,T]$. The existence of such a solution is ensured by the fact that m solves the FPE with coefficients $\dfrac{d\omega}{dt\otimes dm}, \dfrac{dW}{dt\otimes dm}$ (see Proposition 2.1). Now, for all $0\leqslant s<t\leqslant T$, with M_{r_2} and C_{r_2} positive constants depending only on r_2 we have

$$\begin{aligned}
\mathbf{d}_{r_2^*}(m(s),m(t))^{r_2^*} &\leqslant \mathbb{E}\left[|X_t-X_s|^{r_2^*}\right]\\
&\leqslant 2^{r_2^*-1}\mathbb{E}\left[\Big|\int_s^t \frac{d\omega}{dt\otimes dm}(u,X_u)du\Big|^{r_2^*}\right] + 2^{r_2^*-1}\mathbb{E}\left[\Big|\int_s^t \sqrt{2\frac{dW}{dt\otimes dm}}(u,X_u)dB_u\Big|^{r_2^*}\right]\\
&\leqslant (2(t-s))^{r_2^*-1}\mathbb{E}\left[\int_s^t \left|\frac{d\omega}{dt\otimes dm}(u,X_u)\right|^{r_2^*} du\right]\\
&+ 2^{r_2^*}M_{r_2}\mathbb{E}\left(\left[\int_s^t \mathrm{Tr}(\frac{dW}{dt\otimes dm})(u,X_u)du\right]^{\frac{r_2^*}{2}}\right)\\
&\leqslant C_{r_2}(t-s)^{r_2^*-1}\int_{\mathbb{R}^d}\int_0^T \left|\frac{d\omega}{dt\otimes dm}(u,x)\right|^{r_2^*} dm(u)(x)du + C_{r_2}\Lambda^+(t-s)^{\frac{r_2^*}{2}},
\end{aligned}$$

where we used Jensen inequality for the term involving ω and Burkholder-Davis-Gundy inequality for the other one.

For the second part of the lemma, let us take $(m,\omega,W)\in\mathbb{K}$ such that $J_{RP}(m,\omega,W)\leqslant \inf J_{RP}+1$. From the growth assumptions on L, there exists $M_1>0$ (which does not depend on the particular (m,ω,W)) such that

$$\int_{\mathbb{R}^d}\int_0^T \left|\frac{d\omega}{dt\otimes dm}(u,x)\right|^{r_2^*} dm(u)(x)du \leqslant M_1.$$

Using the estimate proven in the first part of the lemma, we see that for all $t,s\in[0,T]$, $\mathbf{d}_{r_2^*}(m(s),m(t))\leqslant M_1'$ for some $M_1'>0$ which, once again, does not depend on the particular choice of (m,ω,W). This yields the uniform estimate on $\displaystyle\int_{\mathbb{R}^d}|x|^{r_2^*}dm(t)(x) < M_1''$ for some new $M_1''>0$. The uniform estimate on $|\omega|$ follows by Hölder's inequality

$$\begin{aligned}
|\omega|([0,T]\times\mathbb{R}^d) &\leqslant \left(\int_0^T\int_{\mathbb{R}^d} dm(t)(x)dt\right)^{1/r_2}\left(\int_0^T\int_{\mathbb{R}^d}\left|\frac{d\omega}{dt\otimes dm}(t,x)\right|^{r_2^*} dm(t)(x)dt\right)^{1/r_2^*}\\
&\leqslant T^{1/r_2}M_1^{1/r_2^*}.
\end{aligned}$$

Finally, $m(t) \otimes dt$-almost everywhere $\mathrm{S_p}\left(\frac{dW}{dt \otimes dm}(t,x)\right) \in [\Lambda^-, \Lambda^+]$ which means that $|W|([0,T] \times \mathbb{R}^d) \leqslant \sqrt{d}\Lambda^+$. The claim follows taking $M = M_1'' + T^{1/r_2} M_1^{1/r_2^*} + \sqrt{d}\Lambda^+$. □

From this we can conclude with:

Theorem 2.4. *J_{RP} achieves its minimum at some point $(\widetilde{m}, \widetilde{\omega}, \widetilde{W})$ in $\mathbb{K}$.*

Proof. This follows from the direct method of calculus of variations. Let (m_n, ω_n, W_n) be a minimizing sequence such that, for all $n \in \mathbb{N}$, $J_{RP}(m_n, \omega_n, W_n) \leqslant \inf J_{RP} + 1$. Using the Estimate 2.7 in Lemma 2.4 we can use Arzela-Ascoli theorem on the one hand and Banach-Alaoglu theorem on the other hand to extract a subsequence (still denoted (m_n, ω_n, W_n)) converging to $(\widetilde{m}, \widetilde{\omega}, \widetilde{W}) \in \mathcal{C}([0,T], \mathcal{P}_{r_2^*}(\mathbb{R}^d)) \times \mathcal{M}([0,T] \times \mathbb{R}^d, \mathbb{R}^d) \times \mathcal{M}([0,T] \times \mathbb{R}^d, \mathbb{S}_d(\mathbb{R}))$ in $\mathcal{C}([0,T], \mathcal{P}_\delta(\mathbb{R}^d)) \times \mathcal{M}([0,T] \times \mathbb{R}^d, \mathbb{R}^d) \times \mathcal{M}([0,T] \times \mathbb{R}^d, \mathbb{S}_d(\mathbb{R}))$ for any $\delta \in (1, r_2^*)$. It remains to show that $(\widetilde{m}, \widetilde{\omega}, \widetilde{W})$ belongs to $\mathbb{K}$ and is indeed a minimum. The Fokker-Planck equation and the initial and final conditions are easily deduced from the weak-$*$ convergence of measures. To conclude we can use Theorem 2.34 of [7] to show that absolute continuity of ω_n and W_n with respect to $m_n(t) \otimes dt$ is preserved when we take limits and that $J_{RP}(\widetilde{m}, \widetilde{\omega}, \widetilde{W}) \leqslant \liminf_n J_{RP}(m_n, \omega_n, W_n)$. So $(\widetilde{m}, \widetilde{\omega}, \widetilde{W})$ is indeed a minimum of J_{RP} in $\mathbb{K}$. □

2.2.2 Necessary Conditions for the Linear Case

In this section we suppose that Ψ is linear: there is a function $h : \mathbb{R}^d \to \mathbb{R}$ such that, for all $m \in \mathcal{P}_1(\mathbb{R}^d)$, $\Psi(m) = \int_{\mathbb{R}^d} h(x) m(dx)$. We also suppose that h belongs to $\mathcal{C}_b^{3+\alpha}(\mathbb{R}^d)$ for some $\alpha \in (0,1)$ and that there exists $x_T \in \mathbb{R}^d$ such that $h(x_T) < 0$. Under these assumptions, Ψ satisfies Assumption 3. We also suppose that f_2 and g are linear in m with $f_2(t,m) = \int_{\mathbb{R}^d} f_2'(t,x) dm(x)$ and $g(m) = \int_{\mathbb{R}^d} g'(x) dm(x)$ with $g' \in \mathcal{C}_b^{3+\alpha}(\mathbb{R}^d)$ and f_2' satisfying the assumptions of Theorem 2.1. Let us introduce a dual problem for RP.

Definition 2.2 (Dual Problem)**.** The dual problem is :

$$\sup_{(\lambda,\phi) \in \mathbb{R}^+ \times \mathbb{A}, \phi \in \mathrm{HJ}^-(\lambda h + g)} \int_{\mathbb{R}^d} \phi(0,x) m_0(dx), \tag{DP}$$

where $\mathbb{A} = \mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^d)$ and, for all $(\lambda, \phi) \in \mathbb{R}^+ \times \mathbb{A}$, ϕ belongs to $\mathrm{HJ}^-(\lambda h + g')$ if and only if :

$$\begin{cases} -\partial_t \phi(t,x) + H(t,x,D\phi(t,x), D^2\phi(t,x)) \leqslant f_2'(t,x) \text{ in } [0,T] \times \mathbb{R}^d \\ \phi(T,x) \leqslant \lambda h(x) + g'(x) \text{ in } \mathbb{R}^d \end{cases} \tag{2.8}$$

The main theorem of this part is a duality result between RP and DP :

Theorem 2.5.

$$\min_{(m,\omega,W) \in \mathbb{K}} J_{RP}(m,\omega,W) = \sup_{(\lambda,\phi) \in \mathbb{R}^+ \times \mathbb{A}, \phi \in HJ^-(\lambda h + g)} \int_{\mathbb{R}^d} \phi(0,x) m_0(dx).$$

To prove Theorem 2.5 the idea is to write the relaxed problem RP as a min/max problem and use the Von Neumann theorem to conclude. The statement of the Von-Neumann theorem is given in Appendix 2.5.1.

Proof of Theorem 2.5. **Step 1: Further Relaxation**

First we need to enlarge the space of test functions $\mathbb{A}$ to allow for functions with linear growth. More precisely, we define $\mathbb{A}'$ as the subset of $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^d)$ consisting of functions ϕ such that

$$\|(\partial_t\phi)^-\|_\infty + \|\phi^+\|_\infty + \|D\phi\|_\infty + \|D^2\phi\|_\infty + \left\|\frac{|\phi|+|\partial_t\phi|}{1+|x|}\right\|_\infty < +\infty,$$

where $(\partial_t\phi)^- = \min(\partial_t\phi, 0)$ and $\phi^+ = \max(\phi, 0)$.

Owing to the estimates of Lemma 2.4 and using an approximation argument similar to [129] Remark 2.3 we see that any minimizer of the relaxed problem RP satisfies the Fokker-Planck equation against any function $\phi \in \mathbb{A}'$. Now we define $\mathbb{B}$ to be the set of tuples (m,ω,W,n) in $\mathcal{M}^+([0,T]\times\mathbb{R}^d)\times\mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)\times\mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{S}_d(\mathbb{R}))\times\mathcal{M}^+(\mathbb{R}^d)$ such that ω and W are absolutely continuous with respect to m. The cost J'_{RP} is defined on $\mathbb{B}$ by

$$J'_{RP}(m,\omega,W,n) = \int_0^T\int_{\mathbb{R}^d}\left[L\left(t,x,\frac{d\omega}{dm}(t,x),\frac{dW}{dm}(t,x)\right)+f'_2(t,x)\right]dm(t,x)+\int_{\mathbb{R}^d}g'(x)dn(x).$$

If $(\widetilde{m},\widetilde{\omega},\widetilde{W})$ is a solution of the relaxed problem we claim that

$$J_{RP}(\widetilde{m},\widetilde{\omega},\widetilde{W}) = J'_{RP}(\widetilde{m},\widetilde{\omega},\widetilde{W},\widetilde{m}(T,dx)) = \inf_{(m,\omega,W,n)} J'_{RP}(m,\omega,W,n),$$

where the infimum is taken over the (m,ω,W,n) in $\mathbb{B}$ satisfying,

$$\forall\phi\in\mathbb{A}',\ \int_0^T\int_{\mathbb{R}^d}(\partial_t\phi m + D\phi.\omega + D^2\phi.W)+\int_{\mathbb{R}^d}\phi(0,x)dm_0(x)-\int_{\mathbb{R}^d}\phi(T,x)dn(x)=0, \quad (2.9)$$

$$\int_{\mathbb{R}^d}h(x)dn(x)\leqslant 0. \quad (2.10)$$

Indeed, since $(\widetilde{m},\widetilde{\omega},\widetilde{W},\widetilde{m}(T))$ belongs to $\mathbb{B}$ and satisfies the Fokker-Planck equation, it is clear that $J'_{RP}(\widetilde{m},\widetilde{\omega},\widetilde{W},\widetilde{m}(T))\geqslant\inf_{(m,\omega,W,n)}J'_{RP}(m,\omega,W,n)$. Now let us take $(m,\omega,W,n)\in\mathbb{B}$ satisfying (2.9) for every $\phi\in\mathbb{A}'$ and such that $J'_{RP}(m,\omega,W,n)<+\infty$.

Testing (2.9) for constants functions we get that $n(\mathbb{R}^d)=1$ and for any $f\in\mathcal{C}([0,T])$, taking $\phi(t)=\displaystyle\int_0^t f(s)ds$ as a test function in (2.9) we get

$$\int_0^T\int_{\mathbb{R}^d}f(t)dm(t,x)=\int_0^T f(t)dt.$$

This means that the time marginal of m is the Lebesgue measure over $[0,T]$. If $m=dm_t(x)dt$ is a disintegration of m with respect to its time marginal, we deduce that

$$\int_0^T f(t)m_t(\mathbb{R}^d)dt = \int_0^T f(t)dt$$

for all $f \in \mathcal{C}([0,T])$ and therefore $m_t(\mathbb{R}^d) = 1$, for dt-almost-all $t \in [0,T]$.

Now we can follow Lemma 2.6 and the discussion in the proof of Theorem 2.4 to deduce that m admits a continuous representative $m' \in \mathcal{C}([0,T], \mathcal{P}_{r_2^*}(\mathbb{R}^d))$. We then get $n = m'(T)$ from (2.9). Therefore (m', ω, W) belongs to $\mathbb{K}$,

$$J'_{RP}(m,\omega,W,n) = J_{RP}(m',\omega,W) \geqslant J_{RP}(\widetilde{m},\widetilde{\omega},\widetilde{W})$$

and the claim is proved. Now, observe that, for any point (m,ω,W,n) in $\mathbb{B}$

$$\sup_{(\lambda,\phi)\in\mathbb{R}^+\times\mathbb{A}'}\left[\int_0^T\int_{\mathbb{R}^d}(\partial_t\phi m + D\phi.\omega + D^2\phi.W) + \int_{\mathbb{R}^d}\phi(0,x)dm_0(x) + \int_{\mathbb{R}^d}(\lambda h(x) - \phi(T,x))\,dn(x)\right]$$
$$= \begin{cases} 0 & \text{if } (m,\omega,W,n) \text{ satisfies (2.9) and (2.10),} \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore we deduce that

$$V_{RP}(m_0) := \min_{(m,\omega,W)\in\mathbb{K}} J_{RP}(m,\omega,W) = \inf_{(m,\omega,W,n)\in\mathbb{B}}\sup_{(\lambda,\phi)\in\mathbb{R}^+\times\mathbb{A}'}\mathcal{L}((\lambda,\phi),(m,\omega,W,n)),$$

where $\mathcal{L} : \mathbb{R}^+ \times \mathbb{A}' \times \mathbb{B} \to \mathbb{R}$ is defined by:

$$\begin{aligned}\mathcal{L}((\lambda,\phi),(m,\omega,W,n)) = &\int_0^T\int_{\mathbb{R}^d}(L\left(t,x,\frac{d\omega}{dm}(t,x),\frac{dW}{dm}(t,x)\right) + f_2'(t,x))dm(t,x) + \int_{\mathbb{R}^d}g'(x)dn(x)\\ &+\int_0^T\int_{\mathbb{R}^d}\partial_t\phi(t,x)dm(t,x) + D\phi(t,x).d\omega(t,x) + D^2\phi(t,x).dW(t,x)\\ &+\int_{\mathbb{R}^d}\phi(0,x)dm_0(x) - \int_{\mathbb{R}^d}\phi(T,x)dn(x) + \lambda\int_{\mathbb{R}^d}h(x)dn(x).\end{aligned}$$

Step 2: Analysis of the Lagrangian

We immediately check that for all $(\lambda,\phi) \in \mathbb{R}^+ \times \mathbb{A}'$, $(m,\omega,W,n) \to \mathcal{L}((\lambda,\phi),(m,\omega,W,n))$ is convex and for all $(m,\omega,W,n) \in \mathbb{B}$, $(\lambda,\phi) \to \mathcal{L}((\lambda,\phi),(m,\omega,W,n))$ is concave. Now $\mathcal{L}$ can be rewritten as the sum of four terms, $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \int_{\mathbb{R}^d}\phi(0,x)m_0(dx)$ where,

$$\begin{aligned}\mathcal{L}_1((\lambda,\phi),m) &:= \int_0^T\int_{\mathbb{R}^d}\left[\partial_t\phi(t,x) - H(t,x,D\phi(t,x),D^2\phi(t,x)) + f_2'(t,x)\right]dm(t,x),\\ \mathcal{L}_2((\lambda,\phi),(m,\omega,W)) &:= \int_0^T\int_{\mathbb{R}^d}f^{(\lambda,\phi)}\left(t,x,\frac{d\omega}{dm}(t,x),\frac{dW}{dm}(t,x)\right)dm(t,x),\\ \mathcal{L}_3((\lambda,\phi),n) &:= \int_{\mathbb{R}^d}\left[\lambda h(x) + g'(x) - \phi(T,x)\right]dn(x),\end{aligned}$$

with $f^{(\lambda,\phi)} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_d(\mathbb{R}) \to \mathbb{R}$ defined by,

$$f^{(\lambda,\phi)}(t,x,q,N) = L\,(t,x,q,N) + H(t,x,D\phi(t,x),D^2\phi(t,x)) + q.D\phi(t,x) + N.D^2\phi(t,x).$$

Now suppose that $(m_k,\omega_k,W_k,n_k)_{k\in\mathbb{N}}$ weakly-$*$ converges to some point (m,ω,W,n) and satisfies the uniform estimate

$$\begin{aligned}\max\{ &\int_0^T \int_{\mathbb{R}^d} (1+|x|)dm_k(t,x), \int_{\mathbb{R}^d} (1+|x|)dn_k(x), \int_0^T \int_{\mathbb{R}^d} \left|\frac{d\omega_k}{dm_k}(t,x)\right|^{r_2^*} dm_k(t,x),\\ &\int_0^T \int_{\mathbb{R}^d} \chi_{[0,\Lambda^+]}\left(\frac{dW_k}{dm_k}(t,x)\right) dm_k(t,x)\} \leqslant M \end{aligned} \tag{2.11}$$

for some $M > 0$ and for all $k \in \mathbb{N}$. For $\mathcal{L}_1$ and $\mathcal{L}_3$ we can proceed similarly and prove that $\mathcal{L}_1((\lambda,\phi),m) \leqslant \liminf_{k\to+\infty} \mathcal{L}_1((\lambda,\phi),m_k)$ and $\mathcal{L}_3((\lambda,\phi),n) \leqslant \liminf_{k\to+\infty} \mathcal{L}_3((\lambda,\phi),n_k)$, for every $(\lambda,\phi) \in \mathbb{R}^d \times \mathbb{A}'$. Let us detail this point for $\mathcal{L}_3$ since the same argument works for $\mathcal{L}_1$. If (λ,ϕ) belongs to $\mathbb{R}^+ \times \mathbb{A}'$ then

$$\begin{aligned}\liminf_{k\to+\infty} \mathcal{L}_3((\lambda,\phi),n_k) &\geqslant \sup_{\phi' \in \mathcal{C}_0(\mathbb{R}^d),\, \phi' \leqslant \lambda h + g' - \phi(T)} \liminf_{k\to+\infty} \int_{\mathbb{R}^d} \phi'(x) dn_k(x)\\ &= \sup_{\phi' \in \mathcal{C}_0(\mathbb{R}^d),\, \phi' \leqslant \lambda h + g' - \phi(T)} \int_{\mathbb{R}^d} \phi'(x) dn(x)\\ &= \int_{\mathbb{R}^d} \left[\lambda h(x) + g'(x) - \phi(T,x)\right] dn(x)\\ &= \mathcal{L}_3((\lambda,\phi),n)\end{aligned}$$

where we can argue by approximation, using Lebesgue dominated convergence and (2.11) to prove the second inequality.

For $\mathcal{L}_2$ we need to proceed differently. Being $f^{(\lambda,\phi)}$ is nonnegative, lower-semicontinuous and for all $(t,x) \in [0,T] \times \mathbb{R}^d$, $(q,N) \to f^{(\lambda,\phi)}(t,x,q,N)$ being convex, we can follow [7] Theorem 2.34 and Example 2.36 to prove that ω and W are absolutely continuous with respect to m and $\mathcal{L}_2((\lambda,\phi),(m,\omega,W)) \leqslant \liminf_{k\to+\infty} \mathcal{L}_2((\lambda,\phi),(m_k,\omega_k,W_k))$. Finally, we have that

$$\mathcal{L}((\lambda,\phi),(m,\omega,W,n)) \leqslant \limsup_{k\to+\infty} \mathcal{L}((\lambda,\phi),(m_k,\omega_k,W_k,n_k)). \tag{2.12}$$

Step 3: Min/Max argument

Now we are going to use the Von Neumann Theorem 2.6 to show that

$$\inf_{(m,\omega,W,n)\in\mathbb{B}} \sup_{(\lambda,\phi)\in\mathbb{R}^+\times\mathbb{A}'} \mathcal{L}((\phi,\lambda),(m,\omega,W,n)) = \sup_{(\lambda,\phi)\in\mathbb{R}^+\times\mathbb{A}'} \inf_{(m,\omega,W,n)\in\mathbb{B}} \mathcal{L}((\lambda,\phi),(m,\omega,W,n)).$$

To check that the hypothesis of the theorem are satisfied, we define $\varphi^*(t,x) := \sqrt{1+|x|^2}(t - T - 1)$ and $\phi^*(t,x) := \left(\sqrt{1+|x|^2} + C_1\right)(t - T - 1) + C_2$, where

$$C_1 = \|H\left(.,.,D\varphi^*(.,.),D^2\varphi^*(.,.)\right) - f_2'(.,.)\|_\infty + 1$$

and $C_2 = -||g'||_\infty + C_1 - 1$. Then we let

$$C^* := \sup_{(\lambda,\phi)\in\mathbb{R}^+\times\mathbb{A}'} \inf_{(m,\omega,W,n)\in\mathbb{B}} \mathcal{L}(\lambda,\phi),(m,\omega,W,n)) + 1$$

and we check that

$$\mathbb{B}^* := \{(m,n,\omega,W)\in\mathbb{B} \text{ such that } \mathcal{L}((0,\phi^*),(m,\omega,W,n)) \leqslant C^*\}$$

is not empty and that there exists some $M > 0$ such that any $(m,\omega,W,n) \in \mathbb{B}^*$ satisfies Estimate (2.11). We deduce that $\mathbb{B}^*$ is (strongly) bounded and using (2.12) we see that $\mathbb{B}^*$ is weakly-$*$ compact. Now we can use (2.11) and (2.12) once again to show that for all $C > 0$ and all $(\lambda,\phi) \in \mathbb{R}^+ \times \mathbb{A}'$,

$$\mathbb{B}^* \cap \{(m,n,\omega,W)\in\mathbb{B} \text{ such that } \mathcal{L}((\lambda,\phi),(m,\omega,W,n)) \leqslant C\}$$

is (possibly empty and) compact. Therefore we can apply the Von Neumann theorem, Theorem 2.6 to show that

$$\inf_{(m,\omega,W,n)\in\mathbb{B}} \sup_{(\lambda,\phi)\in\mathbb{R}^+\times\mathbb{A}'} \mathcal{L}((\phi,\lambda),(m,\omega,W,n)) = \sup_{(\lambda,\phi)\in\mathbb{R}^+\times\mathbb{A}'} \inf_{(m,\omega,W,n)\in\mathbb{B}} \mathcal{L}((\lambda,\phi),(m,\omega,W,n)).$$

Step 4: Computation of the dual problem

Let $(\lambda,\phi) \in \mathbb{R}^+ \times \mathbb{A}'$ be fixed and consider the problem

$$\inf_{(m,\omega,W,n)\in\mathbb{B}} \mathcal{L}((\lambda,\phi),(m,\omega,W,n)).$$

Recall the definitions of $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ in Step 2 of the proof and observe first that, for fixed (m,n),

$$\inf_{(\omega,W)} \mathcal{L}_2((\lambda,\phi),(m,\omega,W)) = 0$$

with the infimum being achieved if and only if,

$$\begin{cases} \omega = -\partial_p H(t,x,D\phi(t,x),D^2\phi(t,x))m, \\ W = -\partial_M H(t,x,D\phi(t,x),D^2\phi(t,x))m. \end{cases}$$

Therefore it holds that

$$\begin{aligned}\inf_{(m,\omega,W,n)\in\mathbb{B}} \mathcal{L}((\lambda,\phi),(m,\omega,W,n)) = &\inf_{m\in\mathcal{M}^+([0,T]\times\mathbb{R}^d)} \mathcal{L}_1((\lambda,\phi),m) + \inf_{n\in\mathcal{M}^+(\mathbb{R}^d)} \mathcal{L}_3((\lambda,\phi),n) \\ &+ \int_{\mathbb{R}^d} \phi(0,x)dm_0(x)\end{aligned}$$

but we have

$$\inf_{m\in\mathcal{M}^+([0,T]\times\mathbb{R}^d)} \mathcal{L}_1((\lambda,\phi),m) = \begin{cases} 0 & \text{if } -\partial_t\phi + H(t,x,D\phi,D^2\phi) \leqslant f_2'(t,x) \text{ in } [0,T]\times\mathbb{R}^d, \\ -\infty & \text{otherwise} \end{cases}$$

and

$$\inf_{n\in\mathcal{M}^+(\mathbb{R}^d)} \mathcal{L}_3((\lambda,\phi),n) = \begin{cases} 0 & \text{if } \phi(T,x) \leqslant \lambda h(x) + g'(x) \text{ in } \times\mathbb{R}^d, \\ -\infty & \text{otherwise.} \end{cases}$$

so we can conclude that

$$\begin{aligned}\inf_{(m,\omega,W,n)\in\mathbb{B}} \sup_{(\lambda,\phi)\in\mathbb{R}^+\times\mathbb{A}'} \mathcal{L}((m,\omega,W,n),(\lambda,\phi)) &= \sup_{(\lambda,\phi)\in\mathbb{R}^+\times\mathbb{A}'} \inf_{(m,\omega,W,n)\in\mathbb{B}} \mathcal{L}((m,\omega,W,n),(\lambda,\phi)) \\ &= \sup_{(\lambda,\phi)\in\mathbb{R}^+\times\mathbb{A}',\phi\in \mathrm{HJ}^-(\lambda h+g')} \int_{\mathbb{R}^d} \phi(0,x) m_0(dx),\end{aligned}$$

where $\phi \in \mathrm{HJ}^-(\lambda h + g')$ for some $(\lambda,\phi) \in \mathbb{R}^+ \times \mathbb{A}'$ if and only if

$$\begin{cases} -\partial_t\phi(t,x) + H(t,x,D\phi(t,x),D^2\phi(t,x)) \leqslant f_2'(t,x) \text{ in } [0,T]\times\mathbb{R}^d, \\ \phi(T,x) \leqslant \lambda h(x) + g'(x) \text{ in } \mathbb{R}^d. \end{cases}$$

Finally, we get $\min_{(m,\omega,W)\in\mathbb{K}} J_{RP}(m,\omega,W) = \sup_{(\lambda,\phi)\in\mathbb{R}^+\times\mathbb{A}',\phi\in \mathrm{HJ}^-(\lambda h+g')} \int_{\mathbb{R}^d} \phi(0,x) m_0(dx)$. □

Notice that this duality is not surprising and holds under very general conditions (see for instance [69]). In particular the volatility σ can be degenerate. However the existence of solutions to the dual problem requires stronger assumptions. In particular we need strong solutions to the HJB equation and that is why we need Theorem 2.1.

Lemma 2.5. *The dual problem has a finite value which is achieved at some point* $(\widetilde{\lambda},\widetilde{\phi}) \in \mathbb{R}^+ \times \mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)$ *such that :*

$$\begin{cases} -\partial_t\widetilde{\phi} + H(t,x,D\widetilde{\phi}(t,x),D^2\widetilde{\phi}(t,x)) = f_2'(t,x) \text{ in } [0,T]\times\mathbb{R}^d \\ \widetilde{\phi}(T,x) = \widetilde{\lambda}h(x) + g'(x) \text{ in } \mathbb{R}^d. \end{cases}$$

Proof. The finiteness follows from the fact that

$$\sup_{(\lambda,\phi)\in\mathbb{R}^+\times\mathbb{A}',\phi\in \mathrm{HJ}^-(\lambda h+g)} \int_{\mathbb{R}^d} \phi(0,x) dm_0(x) = \min_{(m,\omega,W)\in\mathbb{K}} J_{RP}(m,\omega,W) < +\infty.$$

Using the continuity of h and the fact that $h(x_T) < 0$ for some $x_T \in \mathbb{R}^d$, we can follow Lemma 2.3 to build $(\overline{m},\overline{\omega},\overline{W}) \in \mathbb{K}$ such that $J_{RP}(\overline{m},\overline{\omega},\overline{W}) < +\infty$ and $\int_{\mathbb{R}^d} h(x) d\overline{m}(T)(x) < 0$. Consider (λ,ϕ) a candidate for the dual problem. Since $(\overline{m},\overline{\omega},\overline{W})$ satisfies the Fokker-Planck equation we have, taking ϕ as a test function

$$\begin{aligned}&\int_{\mathbb{R}^d} \phi(T,x) d\overline{m}(T)(x) = \int_{\mathbb{R}^d} \phi(0,x) dm_0(x) \\ &+ \int_0^T \int_{\mathbb{R}^d} \left[\partial_t\phi(t,x) + \frac{d\overline{\omega}}{d\overline{m}\otimes dt}(t,x).D\phi(t,x) + \frac{d\overline{W}}{d\overline{m}\otimes dt}(t,x).D^2\phi(t,x) \right] d\overline{m}(t)(x)dt.\end{aligned}$$

Using the inequations satisfied by ϕ and the definition of L we get after reorganizing the terms

$$\lambda\left(-\int_{\mathbb{R}^d} h(x)d\overline{m}(T)(x)\right) \leqslant J_{RP}(\overline{m},\overline{\omega},\overline{W}) - \int_{\mathbb{R}^d}\phi(0,x)dm_0(x). \tag{2.13}$$

Now if we take $(\phi_n,\lambda_n)_{n\in\mathbb{N}}$ a maximizing sequence, the above inequality shows that $(\lambda_n)_{n\in\mathbb{N}}$ is bounded. Taking a subsequence we can suppose that $(\lambda_n)_{n\in\mathbb{N}}$ converges to some $\widetilde{\lambda}\geqslant 0$. By comparison, $(\widetilde{\phi},\widetilde{\lambda})$ is a solution of the dual problem, where $\widetilde{\phi}\in\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)$ is solution to

$$\begin{cases} -\partial_t\phi(t,x) + H(t,x,D\phi(t,x),D^2\phi(t,x)) = f_2'(t,x) \text{ in } [0,T]\times\mathbb{R}^d \\ \phi(T,x) = \widetilde{\lambda}h(x) + g'(x) \text{ in } \mathbb{R}^d. \end{cases} \tag{2.14}$$

The existence of $\widetilde{\phi}$ and the well-posedness of (2.14) in $\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)$ is guaranteed by Theorem 2.1.

□

Remark 2.10. In the proof of the previous lemma, we showed as a by product that λ is bounded independently from ϕ, m. In particular using inequality (2.13) for a maximizing sequence and using the duality result of Theorem 2.5 we get that $\widetilde{\lambda}$ satisfies

$$\widetilde{\lambda} \leqslant \frac{J_{RP}(\overline{m},\overline{\omega},\overline{W}) - V_{RP}(m_0)}{-\displaystyle\int_{\mathbb{R}^d} h(x)d\overline{m}(T)(x)}$$

for any candidate $(\overline{m},\overline{\omega},\overline{W})$ such that $\displaystyle\int_{\mathbb{R}^d} h(x)d\overline{m}(T)(x) < 0$.

Corollary 2.1. *If $(\widetilde{m},\widetilde{\omega},\widetilde{W})$ and $(\widetilde{\lambda},\widetilde{\phi})$ are points where respectively the primal and the dual problems are achieved, then*

$$\widetilde{\omega} = -\partial_p H(t,x,D\widetilde{\phi},D^2\widetilde{\phi}(t,x))\widetilde{m}(t)\otimes dt,$$

$$\widetilde{W} = -\partial_M H(t,x,D\widetilde{\phi},D^2\widetilde{\phi}(t,x))\widetilde{m}(t)\otimes dt$$

and $(\widetilde{\lambda},\widetilde{\phi},\widetilde{m})$ satisfies the optimality conditions

$$\begin{cases} -\partial_t\widetilde{\phi}(t,x) + H(t,x,D\widetilde{\phi}(t,x),D^2\widetilde{\phi}(t,x)) = f_2'(t,x) & \text{in } [0,T]\times\mathbb{R}^d \\ \partial_t\widetilde{m} - \operatorname{div}(\partial_p H(t,x,D\widetilde{\phi}(t,x),D^2\widetilde{\phi}(t,x))\widetilde{m}) & \\ \qquad + \displaystyle\sum_{i,j}\partial_{ij}^2((\partial_M H(t,x,D\widetilde{\phi}(t,x),D^2\widetilde{\phi}(t,x)))_{ij}\widetilde{m}) = 0 & \text{in } [0,T]\times\mathbb{R}^d \\ \widetilde{\phi}(T,x) = \widetilde{\lambda}h(x) + g'(x) \text{ in } \mathbb{R}^d,\ \widetilde{m}(0) = m_0 & \\ \widetilde{\lambda}\displaystyle\int_{\mathbb{R}^d} h(x)d\widetilde{m}(T)(x) = 0,\ \int_{\mathbb{R}^d} h(x)d\widetilde{m}(T)(x)\leqslant 0,\ \widetilde{\lambda}\geqslant 0. & \end{cases} \tag{2.15}$$

Proof of Corollary 2.1. Let $(\phi, \lambda) \in \mathbb{A}$ and $(m, \omega, W) \in \mathbb{K}$ points where the primal and the dual problems are achieved. One has $\int_{\mathbb{R}^d} \phi(0,x) dm_0(x) = J_{RP}(m, \omega, W)$. Given the constraint on ϕ and the fact that m is non-negative we get

$$\begin{aligned}\int_{\mathbb{R}^d} \phi(0,x) dm_0(x) - \int_0^T \int_{\mathbb{R}^d} (-\partial_t \phi(t,x) + H(t,x,D\phi(t,x), D^2\phi(t,x))) dm(t)(x)dt \\ \geqslant \int_0^T \int_{\mathbb{R}^d} \left[L\left(t, x, \frac{d\omega}{dt \otimes dm}(t,x), \frac{dW}{dt \otimes dm}(t,x)\right)\right] dm(t)(x)dt \\ + \int_{\mathbb{R}^d} g'(x) dm(T)(x).\end{aligned}$$

Yet, (m, ω, W) solves the Fokker-Planck equation and $\phi(T,x) \leqslant \lambda h(x) + g'(x)$ for all $x \in \mathbb{R}^d$ so

$$\begin{aligned}&\lambda \int_{\mathbb{R}^d} h(x) dm(T)(x) - \int_0^T \int_{\mathbb{R}^d} D\phi(t,x).d\omega(t,x) - \int_0^T \int_{\mathbb{R}^d} D^2\phi(t,x).dW(t,x) \\ &\geqslant \int_0^T \int_{\mathbb{R}^d} \left[L\left(t, x, \frac{d\omega}{dt \otimes dm}(t,x), \frac{dW}{dt \otimes dm}(t,x)\right) + H(t,x,D\phi(t,x), D^2\phi(t,x))\right] dm(t)(x)dt.\end{aligned}$$

Remember that $\int_{\mathbb{R}^d} h(x) dm(T)(x) \leqslant 0$ and $\lambda \geqslant 0$ so

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^d} [L\left(t, x, \frac{d\omega}{dt \otimes dm}(t,x), \frac{dW}{dt \otimes dm}(t,x)\right) + H(t,x,D\phi(t,x), D^2\phi(t,x)) \\ &- D\phi(t,x).\frac{d\omega}{dt \otimes dm}(t,x) - D^2\phi(t,x).\frac{dW}{dt \otimes dm}(t,x))] dm(t)(x)dt \\ &\leqslant \lambda \int_{\mathbb{R}^d} h(x) dm(T)(x) \leqslant 0.\end{aligned}$$

But, by definition of L, the integrand is always nonnegative. So, $m(t) \otimes dt$-ae we have

$$\begin{aligned}&- D\phi(t,x).\frac{d\omega}{dt \otimes dm}(t,x) - D^2\phi(t,x).\frac{dW}{dt \otimes dm}(t,x) \\ &= L\left(t, x, \frac{d\omega}{dt \otimes dm}(t,x), \frac{dW}{dt \otimes dm}(t,x)\right) + H(t,x,D\phi(t,x), D^2\phi(t,x))\end{aligned}$$

and since H is differentiable, $m(t) \otimes dt$-ae it holds

$$\begin{cases} \dfrac{d\omega}{dt \otimes dm}(t,x) = -\partial_p H(t,x,D\phi(t,x), D^2\phi(t,x)) \\ \dfrac{dW}{dt \otimes dm}(t,x) = -\partial_M H(t,x,D\phi(t,x), D^2\phi(t,x)). \end{cases}$$

Finally, since all the inequalities at the beginning of this proof are actually equalities, we get the necessary conditions for optimality. □

2.3 Proof of the Main Results

2.3.1 Linearization

Let us fix $(\widetilde{m}, \widetilde{\omega}, \widetilde{W})$ a solution of the relaxed problem. The linearized problem is to minimize

$$J^l_{RP}(m,\omega,W) := \int_0^T \int_{\mathbb{R}^d} L\left(t, x, \frac{d\omega}{dt\otimes dm}(t,x), \frac{dW}{dt\otimes dm}(t,x)\right) dm(t)(x)dt + \int_0^T \int_{\mathbb{R}^d} \frac{\delta f_2}{\delta m}(t,\widetilde{m}(t),x)dm(t)(x)dt + \int_{\mathbb{R}^d} \frac{\delta g}{\delta m}(\widetilde{m}(T),x)dm(T)(x) \tag{2.16}$$

among triples (m,ω,W) that satisfy the Fokker-Planck equation with $m(0)=m_0$ and with m satisfying the linearized constraint

$$\int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(\widetilde{m}(T),x)dm(T)(x) \leqslant 0. \tag{2.17}$$

Notice that we are in the setting of Section 2.2.2 with $f_2'(t,x) = \frac{\delta f_2}{\delta m}(t,\widetilde{m}(t),x)$, $g'(x) = \frac{\delta g}{\delta m}(\widetilde{m}(T),x)$ and $h(x) = \frac{\delta \Psi}{\delta m}(\widetilde{m}(T),x)$.

Proposition 2.3. *Let $(\widetilde{m}, \widetilde{\omega}, \widetilde{W})$ be a fixed solution to the relaxed problem. If $\Psi(\widetilde{m}(T)) = 0$ then $(\widetilde{m}, \widetilde{\omega}, \widetilde{W})$ is a solution of the linearized problem (2.16). If $\Psi(\widetilde{m}(T)) < 0$ then $(\widetilde{m}, \widetilde{\omega}, \widetilde{W})$ is a solution of the linearized problem (2.16) without the final constraint.*

Proof. Suppose that $\Psi(\widetilde{m}(T)) = 0$. By Assumption (3c) there is some $x_0 \in \mathbb{R}^d$ such that $\frac{\delta \Psi}{\delta m}(\widetilde{m}(T), x_0) < 0$ and we can proceed as in Lemma 2.3 (the constraint being then the linear one: $\widetilde{\Psi}(m) = \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(\widetilde{m}(T),x)dm(x)$) and find (m',ω',W') such that

$$\begin{cases} m'(0) = m_0 \\ \partial_t m' + \operatorname{div}(\omega') - \sum_{i,j} \partial^2_{ij} W'_{ij} = 0 \\ \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(\widetilde{m}(T),x)dm'(T)(x) < 0 \\ J^l_{RP}(m',\omega',W') < +\infty. \end{cases}$$

Now let (m,ω,W) be any candidate for the linearized problem (in particular (m,ω,W) satisfies the linearized constraint (2.17)). Let $\epsilon \in (0,1)$ and define $(m^\epsilon,\omega^\epsilon,W^\epsilon) := (1-\epsilon)(m,\omega,W) + \epsilon(m',\omega',W')$ (we perturb (m,ω,W) a little bit so that it satisfies strictly the linearized constraint). Let $\lambda \in (0,1)$ and define

$$(m^\epsilon_\lambda, \omega^\epsilon_\lambda, W^\epsilon_\lambda) := (1-\lambda)(\widetilde{m}, \widetilde{\omega}, \widetilde{W}) + \lambda(m^\epsilon, \omega^\epsilon, W^\epsilon).$$

We have that

$$\Psi(m_\lambda^\epsilon(T)) = \Psi(\widetilde{m}(T)) + \lambda \int_{\mathbb{R}^d} \frac{\delta\Psi}{\delta m}(\widetilde{m}(T), x) dm^\epsilon(T)(x) + \circ(\lambda), \tag{2.18}$$

yet it holds

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{\delta\Psi}{\delta m}(\widetilde{m}(T), x) dm^\epsilon(T)(x) &= (1-\epsilon) \int_{\mathbb{R}^d} \frac{\delta\Psi}{\delta m}(\widetilde{m}(T), x) dm(T)(x) \\ &\quad + \epsilon \int_{\mathbb{R}^d} \frac{\delta\Psi}{\delta m}(\widetilde{m}(T), x) dm'(T)(x) < 0\end{aligned}$$

and therefore $\Psi(m_\lambda^\epsilon(T)) \leqslant 0$ for small enough λ. Now, by convexity of

$$(m, \omega, W) \to \Gamma(m, \omega, W) := \int_0^T \int_{\mathbb{R}^d} L\left(t, x, \frac{d\omega}{dt \otimes dm}(t, x), \frac{dW}{dt \otimes dm}(t, x)\right) dm(t)(x) dt$$

and optimality of $(\widetilde{m}, \widetilde{\omega}, \widetilde{W})$ for the relaxed problem we have

$$\begin{aligned}\Gamma(\widetilde{m}, \widetilde{\omega}, \widetilde{W}) &\leqslant \Gamma(m_\lambda^\epsilon, \omega_\lambda^\epsilon, W_\lambda^\epsilon) + \int_0^T \left[f_2(t, m_\lambda^\epsilon(t)) - f_2(t, \widetilde{m}(t))\right] dt + g(m_\lambda^\epsilon(T)) - g(\widetilde{m}(T)) \\ &\leqslant (1-\lambda)\Gamma(\widetilde{m}, \widetilde{\omega}, \widetilde{W}) + \lambda\Gamma(m^\epsilon, \omega^\epsilon, W^\epsilon) + \int_0^T \left[f_2(t, m_\lambda^\epsilon(t)) - f_2(t, \widetilde{m}(t))\right] dt \\ &\quad + g(m_\lambda^\epsilon(T)) - g(\widetilde{m}(T))\end{aligned}$$

which gives

$$\Gamma(\widetilde{m}, \widetilde{\omega}, \widetilde{W}) \leqslant \Gamma(m^\epsilon, \omega^\epsilon, W^\epsilon) + \frac{1}{\lambda} \int_0^T \left[f_2(t, m_\lambda^\epsilon(t)) - f_2(t, \widetilde{m}(t))\right] dt + \frac{1}{\lambda} \left[g(m_\lambda^\epsilon(T)) - g(\widetilde{m}(T))\right].$$

Now we let λ go to 0 and use once again the convexity of Γ to get

$$\begin{aligned}J_{RP}^l(\widetilde{m}, \widetilde{\omega}, \widetilde{W}) &= \Gamma(\widetilde{m}, \widetilde{\omega}, \widetilde{W}) \\ &\leqslant \Gamma(m^\epsilon, \omega^\epsilon, W^\epsilon) + \int_0^T \int_{\mathbb{R}^d} \frac{\delta f_2}{\delta m}(t, \widetilde{m}(t), x) dm^\epsilon(t)(x) dt \\ &\quad + \int_{\mathbb{R}^d} \frac{\delta g}{\delta m}(\widetilde{m}(T), x) dm^\epsilon(T)(x) \\ &\leqslant J_{RP}^l(m, \omega, W) + \epsilon \left(J_{RP}^l(m', \omega', W') - J_{RP}^l(m, \omega, W)\right),\end{aligned}$$

where the first equality comes from the normalization condition in the definition of the linear derivative. We get the result letting $\epsilon \to 0$. When $\Psi(\widetilde{m}(T)) < 0$ there is no need to perturb (m, ω, W) since (2.18) shows that $\Psi(m_\lambda^0(T)) \leqslant 0$ for small enough λ independently from the sign of $\displaystyle\int_{\mathbb{R}^d} \frac{\delta\Psi}{\delta m}(\widetilde{m}(T), x) dm(T)(x)$ and we can take $\epsilon = 0$ in the rest of the proof. □

2.3.2 General Constraint

Proof of Theorem 2.2. Recall that, on the one hand we want to prove the existence of optimal Markovian controls for (SP) and on the other hand we want to prove that optimal controls, if Markovian, satisfy some necessary conditions. Let $(\widetilde{m}, \widetilde{\omega}, \widetilde{W})$ be a solution of the relaxed problem. We can apply Proposition 2.3 and Corollary 2.1 to find some $(\widetilde{\lambda}, \widetilde{\phi})$ in $\mathbb{R}^+ \times \mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)$ such that $(\widetilde{m}, \widetilde{\lambda}, \widetilde{\phi})$ satisfies the system of optimality conditions (2.15) with $f_2'(t,x) = \dfrac{\delta f_2}{\delta m}(t, \widetilde{m}(t), x)$, $g'(x) = \dfrac{\delta g}{\delta m}(\widetilde{m}(T), x)$ and $h(x) = \dfrac{\delta \Psi}{\delta m}(\widetilde{m}(T), x)$. Notice that, when $\Psi(\widetilde{m}(T)) < 0$ we can take $\lambda = 0$ since $(\widetilde{m}, \widetilde{\omega}, \widetilde{W})$ is a solution of the linearized problem without constraint in this case. In general, let $\widetilde{\alpha}$ be a measurable function such that, for all $(t,x) \in [0,T]\times\mathbb{R}^d$

$$\begin{aligned} H(t,x,D\widetilde{\phi}(t,x), D^2\widetilde{\phi}(t,x)) = &- b(t,x,\widetilde{\alpha}(t,x)).D\widetilde{\phi}(t,x) - \sigma^t\sigma(t,x,\widetilde{\alpha}(t,x)).D^2\widetilde{\phi}(t,x) \\ &- f_1(t,x,\widetilde{\alpha}(t,x)). \end{aligned}$$

We use the assumption that H is continuously differentiable in (p,M). Indeed, in this case one has, thanks to the Envelope theorem (see [113]),

$$\begin{cases} \partial_p H(t,x,D\widetilde{\phi}(t,x), D^2\widetilde{\phi}(t,x)) = -b(t,x,\widetilde{\alpha}(t,x)) \\ \partial_M H(t,x,D\widetilde{\phi}(t,x), D^2\widetilde{\phi}(t,x)) = -\sigma^t\sigma(t,x,\widetilde{\alpha}(t,x)). \end{cases} \tag{2.19}$$

Since $\partial_p H$ and $\partial_M H$ are supposed to be locally Lipschitz continuous respectively in p and M and uniformly in x and since $|\partial_M H|$ is bounded from below by $\sqrt{d}\Lambda^- > 0$, using the fact (Theorem 2.1) that $\widetilde{\phi}$ belongs to $\mathcal{C}_b^{\frac{3+\alpha}{2},3+\alpha}([0,T]\times\mathbb{R}^d)$ we see that the coefficients of the functions, $(t,x) \to \partial_p H(t,x,D\widetilde{\phi}(t,x), D^2\widetilde{\phi}(t,x))$ and $(t,x) \to \partial_M H(t,x,D\widetilde{\phi}(t,x), D^2\widetilde{\phi}(t,x))$ are Lipschitz in x, uniformly in t. Thus there is a unique strong solution of the SDE

$$d\widetilde{X}_t = b(t,\widetilde{X}_t,\widetilde{\alpha}(t,\widetilde{X}_t))dt + \sqrt{2}\sigma(t,\widetilde{X}_t,\widetilde{\alpha}(t,\widetilde{X}_t))dB_t$$

starting from X_0. Therefore $\mathcal{L}(\widetilde{X}_t) = \widetilde{m}(t)$ for all $t \in [0,T]$ and, in particular, $\Psi(\mathcal{L}(\widetilde{X}_T)) \leqslant 0$. This means that $\widetilde{\alpha}_t := \widetilde{\alpha}(t,\widetilde{X}_t)$ is admissible for the strong problem. Since H is $\mathcal{C}^1$ we know that for all $(t,x,p,M) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}_d(\mathbb{R})$,

$$\begin{aligned} H(t,x,p,M) = &\, p.\partial_p H(t,x,p,M) + M.\partial_M H(t,x,p,M) \\ &- L(t,x,-\partial_p H(t,x,p,M), -\partial_M H(t,x,p,M)) \end{aligned} \tag{2.20}$$

and therefore, (2.19) implies that

$$L(t,x,-\partial_p H(t,x,D\widetilde{\phi}(t,x), D^2\phi(t,x)), -\partial_M H(t,x,D\phi(t,x), D^2\phi(t,x))) = f_1(t,x,\widetilde{\alpha}(t,x))$$

and thus $J_{SP}(\widetilde{\alpha}) = J_{RP}(\widetilde{m},\widetilde{\omega},\widetilde{W}) = V_{SP}(X_0)$ from which it comes that $V_{RP}(m_0) \geqslant V_{SP}(X_0)$. The reverse inequality being clear, we get $V_{RP}(m_0) = V_{SP}(X_0)$ and $\widetilde{\alpha}$ is a solution to the

strong problem. This shows in particular that optimal controls for the strong problem (SP) do exist. Now take α a Markovian solution to the strong problem. If X is the corresponding process, we take $(m, \omega, W) = (m, b(t,x,\alpha)m, \sigma^t\sigma(t,x,\alpha)m)$. Then, (m,ω,W) is admissible for the relaxed problem and we have $J_{RP}(m,\omega,W) \leqslant J_{SP}(\alpha) = V_{SP}(X_0)$. And thus, $J_{RP}(m,\omega,W) = V_{RP}(m_0)$. Finally, (m,ω,W) is optimal for the relaxed problem and we can apply Proposition 2.3 and Corollary 2.1 to conclude. Now if we use $\widetilde{\phi}$ in (OC) as a test function for the Fokker-Planck equation, recalling (2.20) as well as the normalization convention for the linear functional derivative we get that

$$\begin{aligned}&\int_{\mathbb{R}^d} \widetilde{\phi}(0) dm_0(x)\\ &= \int_0^T \int_{\mathbb{R}^d} L\left(t,x,-\partial_p H(t,x,D\widetilde{\phi}(t,x),D^2\widetilde{\phi}(t,x)), -\partial_M H(t,x,D\widetilde{\phi}(t,x),D^2\widetilde{\phi}(t,x))\right) d\widetilde{m}(t)(x)dt\end{aligned}$$

and therefore $V_{SP}(X_0) = \int_{\mathbb{R}^d} \widetilde{\phi}(0)dm_0(x) + \int_{\mathbb{R}^d} f_2(t,\widetilde{m}(t))dt + g(\widetilde{m}(T))$.

□

2.3.3 Convex Constraint and Convex Costs

Now we show that the conditions are also sufficient when Ψ, f_2 and g are convex functions in the measure variable. Notice that this case covers in particular the problem with expectation constraint and costs in expectation form when Ψ, f_2 and g are linear.

Proof of Theorem 2.3. Let $(\widetilde{\lambda}, \widetilde{\phi}, \widetilde{m})$ be a solution to the system of optimality conditions (OC) and let $\widetilde{X}_t$ be the solution to

$$\begin{cases} d\widetilde{X}_t = b(t, X_t, \widetilde{\alpha}(t,\widetilde{X}_t))dt + \sqrt{2}\sigma(t,\widetilde{X}_t,\widetilde{\alpha}(t,\widetilde{X}_t))dB_t \\ \widetilde{X}_0 = X_0. \end{cases}$$

for some measurable function $\widetilde{\alpha} : [0,T]\times\mathbb{R}^d \to A$ such that, for any $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$\begin{aligned}H(t,x,D\widetilde{\phi}(t,x),D^2\widetilde{\phi}(t,x)) = &-b(t,x,\widetilde{\alpha}(t,x)).D\widetilde{\phi}(t,x) - \sigma^t\sigma(t,x,\widetilde{\alpha}(t,x)).D^2\widetilde{\phi}(t,x)\\ &- f_1(t,x,\widetilde{\alpha}(t,x)).\end{aligned}$$

Since

$$\begin{aligned}b(t,x,\widetilde{\alpha}(t,x)) &= -\partial_p H(t,x,D\widetilde{\phi}(t,x),D^2\widetilde{\phi}(t,x)),\\ \sigma^t\sigma(t,x,\widetilde{\alpha}(t,x)) &= -\partial_M H(t,x,D\widetilde{\phi}(t,x),D^2\widetilde{\phi}(t,x))\end{aligned}$$

and ϕ belongs to $\mathcal{C}_b^{3+\alpha,\frac{3+\alpha}{2}}([0,T]\times\mathbb{R}^d)$ the SDE admits a unique strong solution.
We are going to show that $\widetilde{\alpha}_t := \widetilde{\alpha}(t,\widetilde{X}_t)$ is a solution to the optimal control problem. The law of $\widetilde{X}_t$ is $\widetilde{m}(t)$ and we deduce that $\Psi(\mathcal{L}(\widetilde{X}_T)) \leqslant 0$ and $\widetilde{\alpha}_t$ is admissible. Now we show that $\widetilde{\alpha}_t$ is indeed optimal among the admissible strategies. Let α_t be an admissible control, X_t the corresponding process and $m(t) := \mathcal{L}(X_t)$. Let also J'_{SP} be defined on $\mathcal{U}_{ad}$ as follows

$$J'_{SP}(\alpha_t) := \mathbb{E}\left(\int_0^T \left(f_1(t, X_t, \alpha_t) + \frac{\delta f_2}{\delta m}(t, \widetilde{m}(t), X_t)\right) dt + \frac{\delta g}{\delta m}(\widetilde{m}(t), X_T) + \widetilde{\lambda}\frac{\delta \Psi}{\delta m}(\widetilde{m}(T), X_T)\right).$$

Using a classical verification argument and the fact that $\widetilde{\phi}$ solves the HJB equation, we get that $J'_{SP}(\widetilde{\alpha}_t) \leqslant J'_{SP}(\alpha_t)$. Now by convexity of Ψ, f_2 and g we get

$$\mathbb{E}\left[\int_0^T f_2(t, \widetilde{m}(t))dt - \int_0^T f_2(t, m(t))dt + \int_0^T \frac{\delta f_2}{\delta m}(t, \widetilde{m}(t), X_t)dt\right] \leqslant 0,$$

$$\mathbb{E}\left[g(\widetilde{m}(T)) - g(m(T)) + \frac{\delta g}{\delta m}(\widetilde{m}(T), X_T)\right] \leqslant 0$$

and

$$\widetilde{\lambda}\mathbb{E}\left[\frac{\delta \Psi}{\delta m}(\widetilde{m}(T), X_T)\right] = \widetilde{\lambda}\left(\Psi(\widetilde{m}(T)) + \mathbb{E}\left[\frac{\delta \Psi}{\delta m}(\widetilde{m}(T), X_T)\right]\right) \leqslant \widetilde{\lambda}\Psi(m(T)) \leqslant 0.$$

Therefore we get that $J_{SP}(\widetilde{\alpha}) \leqslant J_{SP}(\alpha)$ and $\widetilde{\alpha}$ is optimal for the strong problem. □

2.4 The HJB Equation

The aim of this section is to show that the HJB equation

$$\begin{cases} -\partial_t u(t,x) + H(t, x, Du(t,x), D^2u(t,x)) = f'_2(t,x) & \text{in } [0,T]\times\mathbb{R}^d \\ u(T,x) = g'(x) & \text{in } \mathbb{R}^d \end{cases} \tag{2.21}$$

admits a unique strong solution $u \in \mathcal{C}_b^{\frac{3+\alpha}{2},3+\alpha}([0,T]\times\mathbb{R}^d)$ when H satisfies Assumptions (1), (2), g' belongs to $\mathcal{C}_b^{3+\alpha}(\mathbb{R}^d)$ and $f'_2 \in \mathcal{C}_b([0,T], \mathcal{C}_b^{3+\alpha}(\mathbb{R}^d))$ is Hölder continuous in time, uniformly in x.

We start by observing that it is enough to prove that (2.21) admits a bounded Lipschitz continuous viscosity solution. Indeed, if u is such a solution, it follows from the following line of arguments that u belongs to $\mathcal{C}_b^{\frac{3+\alpha}{2},3+\alpha}([0,T]\times\mathbb{R}^d)$. If u is Lipschitz continuous in space we can use Theorem VII.3 in [88] to deduce that u is semi-concave with a modulus of semi-concavity uniform in (t,x). Now, using the uniform parabolicity of the equation, the fact that u is Lipschitz and semi-concave we can prove that u is also semi-convex (see [87] Theorem 4 with the help of [6]) and therefore Du is continuous and Lipschitz in space. At this point, using the uniform parabolicity of the equation and the Hölder regularity in time of f'_2, we can use the results of [133] and [134] (see also the last section of [23]) to deduce that u belongs to $\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)$. Finally, by differentiating the equation we can use results on uniformly parabolic linear PDEs (Theorem IV.5.1 of [98]) to conclude that u belongs to $\mathcal{C}_b^{\frac{3+\alpha}{2},3+\alpha}([0,T]\times\mathbb{R}^d)$.

We first show the existence of bounded, Lipschitz continuous viscosity solutions to (2.21) when f'_2 is also globally Lipschitz continuous (and not just Hölder continuous in time) and then we use an approximation argument.

Lemma 2.6. *Suppose that Assumptions* (1) *and* (2) *hold. Take* $g' \in \mathcal{C}_b^2(\mathbb{R}^d)$ *and suppose that* $f_2' \in \mathcal{C}_b([0,T]\times\mathbb{R}^d)$ *is globally Lipschitz continuous in* (t,x) *and* $\mathcal{C}^1$ *in* x*. Then* (2.21) *admits a unique bounded, globally Lischitz continuous viscosity solution.*

Proof. We proceed as follows. For $K \geqslant \|Dg'\|_\infty$ we let u be the unique viscosity solution to

$$\begin{cases} \max\{-\partial_t u + H(t,x,Du,D^2u) - f_2'(t,x), |Du| - 2K\} = 0 & \text{in } [0,T]\times\mathbb{R}^d \\ u(T,x) = g'(x) & \text{in } \mathbb{R}^d. \end{cases} \tag{2.22}$$

The existence of u is guaranteed by Proposition 2.4 in Appendix 2.5.2. In particular, u is $2K$-Lipschitz continuous in x and L-Lipschitz continuous in t for some $L > 0$ independent from K. We are going to show that, for K large enough, u is K-Lipschitz continuous in x. As a direct by-product, u will be the unique viscosity solution to (2.21) and it will satisfy the Lipschitz regularity given by Proposition 2.4.

Now we proceed to show that u is K-Lipschitz continuous in x. Let $K \geqslant 2\|Dg'\|_\infty$ such that $H(t,x,p,0) > L + \|f_2'\|_\infty$ for all $(t,x,p) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ such that $|p| \geqslant K-1$, where L is an upper bound for the time-Lipschitz constant of u. We are going to show that for all $(t,x,y) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$, $u(t,x) - u(t,y) \leqslant K|x-y|$ when K is large enough. Suppose on the contrary that $\delta := \sup_{(t,x,y)\in\{[0,T]\times\mathbb{R}^d\times\mathbb{R}^d}\{u(t,x) - u(t,y) - K|x-y|\}$ is positive. Let β be a small positive parameter and define

$$\phi_\beta(t,x,y) := u(t,x) - u(t,y) - K|x-y| - \beta|y|^2 - \beta\frac{1}{t}.$$

The function ϕ_β reaches its maximum at some point $(\overline{t},\overline{x},\overline{y}) \in (0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ and there is $\beta_0 > 0$ such that for $0 < \beta \leqslant \beta_0$

$$\phi_\beta(\overline{t},\overline{x},\overline{y}) \geqslant \frac{\delta}{2}. \tag{2.23}$$

Suppose that $\beta \leqslant \beta_0$ and $\overline{t} = T$, then

$$\begin{aligned} \frac{\delta}{2} \leqslant \phi_\beta(T,\overline{x},\overline{y}) &= u(T,\overline{x}) - u(T,\overline{y}) - K|\overline{x}-\overline{y}| - \beta|\overline{y}|^2 - \frac{\beta}{T} \\ &\leqslant (\|Dg'\|_\infty - K)|\overline{x}-\overline{y}|. \end{aligned}$$

But this is impossible since $K > \|Dg'\|_\infty$ and $\delta > 0$. Thus for all $\beta \leqslant \beta_0$, $\overline{t} \neq T$. From (2.23) we deduce that $\beta|\overline{y}|^2 \leqslant 2\|u\|_\infty$ and therefore $\beta|\overline{y}| \to 0$ as $\beta \to 0$. We also deduce that

$$\begin{aligned} \frac{\delta}{2} &\leqslant u(\overline{t},\overline{x}) - u(\overline{t},\overline{y}) - K|\overline{x}-\overline{y}| \\ &\leqslant 2K|\overline{x}-\overline{y}| - K|\overline{x}-\overline{y}| \leqslant K|\overline{x}-\overline{y}|, \end{aligned}$$

which leads to the lower bound

$$|\overline{x}-\overline{y}| \geqslant \frac{\delta}{2K}, \tag{2.24}$$

and, in particular, $\overline{x} \neq \overline{y}$. Since $\overline{t} \neq T$ for $\beta \leqslant \beta_0$, we can apply the maximum principle for semi-continuous functions from [53]. Let $\varphi_\beta(t,x,y) = K|x-y| + \beta\frac{1}{t}$. Computing the various derivatives for $|x-y| > 0$ gives

$$\left\{\begin{array}{l} \partial_t\varphi_\beta(t,x,y) = -\frac{\beta}{t^2} \\ \partial_x\varphi_\beta(t,x,y) = \frac{K}{|x-y|}(x-y) \\ \partial_y\varphi_\beta(t,x,y) = \frac{K}{|x-y|}(y-x) \\ D^2\varphi_\beta(t,x,y) = \frac{K}{|x-y|}\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} - \frac{K}{|x-y|^3}\begin{pmatrix} (x-y)\otimes(x-y) & -(x-y)\otimes(x-y) \\ -(x-y)\otimes(x-y) & (x-y)\otimes(x-y) \end{pmatrix}. \end{array}\right.$$

In particular, if $N := (x-y)\otimes(x-y)$, then $N \geqslant 0$ (rank one symetric matrice with positive trace) and thus it is elementary to show that $\begin{pmatrix} N & -N \\ -N & N \end{pmatrix} \geqslant 0$. Therefore, it holds

$$D^2\varphi_\beta(t,x,y) \leqslant \frac{K}{|x-y|}\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}.$$

Now, from the maximum principle, we get $\nu \in \mathbb{R}$, $X, Y \in \mathbb{S}_d(\mathbb{R})$ such that

$$\left\{\begin{array}{l} \left(\nu, K\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|}, X\right) \in \overline{\mathcal{P}}^{2,+}u(\overline{t},\overline{x}) \\ \left(\nu + \frac{\beta}{\overline{t}^2}, K\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|} - 2\beta\overline{y}, Y\right) \in \overline{\mathcal{P}}^{2,-}u(\overline{t},\overline{y}) \end{array}\right.$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -(Y+2\beta I_d) \end{pmatrix} \leqslant 3\frac{K}{|\overline{x}-\overline{y}|}\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}.$$

Observe that $|\nu|$ is bounded by L_T the time -Lipschitz constant of u and thus $|\nu|$ is bounded independently of K. Now we use the equation satisfied by u to get

$$H\left(\overline{t},\overline{x},K\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|},X\right) - f_2'(\overline{t},\overline{x}) \leqslant \nu \leqslant H\left(\overline{t},\overline{y},K\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|} - 2\beta\overline{y},Y\right) - f_2'(\overline{t},\overline{y}).$$

From now on, we let $\overline{\xi} := K\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|}$ and $\gamma := \frac{3K}{|\overline{x}-\overline{y}|}$. We are going to show that the information

$$\left\{\begin{array}{l} |\nu| \leqslant L_T \\ |\overline{\xi}| = K \\ \begin{pmatrix} X & 0 \\ 0 & -(Y+2\beta I_d) \end{pmatrix} \leqslant \gamma\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} \\ H\left(\overline{t},\overline{x},\overline{\xi},X\right) - f_2'(\overline{t},\overline{x}) \leqslant \nu \leqslant H\left(\overline{t},\overline{y},\overline{\xi}-2\beta\overline{y},Y\right) - f_2'(\overline{t},\overline{y}) \end{array}\right. \tag{2.25}$$

is inconsistent whenever K is sufficiently large. Let $\overline{\eta} := \overline{\xi} - 2\beta\overline{y}$ and for any $\lambda \in [0,1]$, $x_\lambda := (1-\lambda)\overline{x} + \lambda\overline{y}$ and $\xi_\lambda := (1-\lambda)\overline{\xi} + \lambda\overline{\eta} = \overline{\xi} - 2\lambda\beta\overline{y}$. From [9], Lemma A.2, there exists a $\mathcal{C}^1$ map, $\lambda \to Z_\lambda$ from $[0,1] \to \mathbb{S}_d(\mathbb{R})$ such that

$$\left\{\begin{array}{l} \frac{d}{d\lambda}Z_\lambda = \gamma^{-1}Z_\lambda^2, \\ Z_0 = X \\ \forall\lambda \in [0,1],\ X \leqslant Z_\lambda \leqslant Y + 2\beta I_d. \end{array}\right.$$

Let us define $l : [0,1] \to \mathbb{R}$ by $l(\lambda) = H(\bar{t}, x_\lambda, \xi_\lambda, Z_\lambda) - f_2'(\bar{t}, x_\lambda)$, so that $l(0) = H\left(\bar{t}, \overline{x}, \overline{\xi}, X\right) - f_2'(\bar{t}, \overline{x}) \leqslant \nu$ and (using $Z_1 \leqslant Y + 2\beta I_d$ and the boundness of $\partial_M H$)

$$l(1) = H(\bar{t}, \overline{y}, \overline{\eta}, Z_1) \geqslant H(\bar{t}, \overline{y}, \overline{\eta}, Y + 2\beta I_d) - f_2'(\bar{t}, \overline{y}) \geqslant H(\bar{t}, \overline{y}, \overline{\eta}, Y) - C\beta - f_2'(\bar{t}, \overline{y}) \geqslant \nu - C\beta,$$

where $C = 2\Lambda^+\sqrt{d}$. Thus, l being $\mathcal{C}^1$, there exists $\lambda \in [0,1]$ such that

$$l(\lambda) = \nu - C\lambda\beta, \tag{2.26}$$
$$l'(\lambda) \geqslant -C\beta. \tag{2.27}$$

From inequality (2.26) we are going to obtain a lower bound on $|Z_\lambda| = \sqrt{\mathrm{Tr}(Z_\lambda^2)}$ and from inequality (2.27) we are going to obtain an upper bound on $|Z_\lambda|$. Combining the two bounds will get a contradiction for K large enough. First we exploit (2.26). It gives us

$$\begin{aligned} H(\bar{t}, x_\lambda, \xi_\lambda, 0) - f_2'(\bar{t}, x_\lambda) - \nu + C\lambda\beta &= H(\bar{t}, x_\lambda, \xi_\lambda, 0) - H(\bar{t}, x_\lambda, \xi_\lambda, Z_\lambda) \\ &\leqslant -\partial_M H(\bar{t}, x_\lambda, \xi_\lambda, Z_\lambda).Z_\lambda \\ &\leqslant \sqrt{d}\Lambda^+|Z_\lambda|, \end{aligned}$$

where we used Cauchy-Schwarz inequality at the last step. Therefore we have

$$|Z_\lambda| \geqslant \frac{H(\bar{t}, x_\lambda, \xi_\lambda, 0) - f_2'(\bar{t}, x_\lambda) - \nu + C\lambda\beta}{\sqrt{d}\Lambda^+}. \tag{2.28}$$

Now we use (2.27). Computing the derivative of l gives

$$\begin{aligned} l'(\lambda) = {} & \partial_x H(\bar{t}, x_\lambda, \xi_\lambda, Z_\lambda).(\overline{y} - \overline{x}) - \partial_x f_2'(\bar{t}, x_\lambda).(\overline{y} - \overline{x}) - 2\beta\partial_p H(\bar{t}, x_\lambda, \xi_\lambda, Z_\lambda).\overline{y} \\ & + \gamma^{-1}\partial_M H(\bar{t}, x_\lambda, \xi_\lambda, Z_\lambda).Z_\lambda^2 \geqslant -C\beta \end{aligned}$$

and, since $-\partial_M H \geqslant \Lambda^- I_d$ and $\partial_x H$ satisfies Assumption (2e), we get

$$\begin{aligned} |Z_\lambda|^2 &\leqslant \frac{1}{\Lambda^-}\left[\gamma C\beta + \gamma\partial_x H(\bar{t}, x_\lambda, \xi_\lambda, Z_\lambda).(\overline{y} - \overline{x}) - \gamma\partial_x f_2'(\bar{t}, x_\lambda).(\overline{y} - \overline{x}) - 2\beta\gamma\partial_p H(\bar{t}, x_\lambda, \xi_\lambda, Z_\lambda).\overline{y}\right] \\ &\leqslant \frac{1}{\Lambda^-}\left[\gamma C\beta + \gamma\partial_x H(\bar{t}, x_\lambda, \xi_\lambda, 0).(\overline{y} - \overline{x}) - \gamma\partial_x f_2'(\bar{t}, x_\lambda).(\overline{y} - \overline{x}) - 2\beta\gamma\partial_p H(\bar{t}, x_\lambda, \xi_\lambda, Z_\lambda).\overline{y}\right] \\ &\quad + \frac{C}{\Lambda^-}\gamma(\overline{y} - \overline{x}).Z_\lambda, \end{aligned}$$

and therefore, for another $C > 0$, we find

$$\begin{aligned} |Z_\lambda|^2 \leqslant {} & C(\gamma\partial_x H(\bar{t}, x_\lambda, \xi_\lambda, 0).(\overline{y} - \overline{x}) - \gamma\partial_x f_2'(\bar{t}, x_\lambda).(\overline{y} - \overline{x}) + \gamma^2|\overline{x} - \overline{y}|^2 \\ & + \gamma\beta - 2\beta\gamma\partial_p H(\bar{t}, x_\lambda, \xi_\lambda, Z_\lambda).\overline{y}). \end{aligned}$$

On the one hand, being $|\xi| = K$, $\gamma = \frac{3K}{|\overline{x}-\overline{y}|} \leqslant \frac{6K^2}{\delta}$ (thanks to (2.24)), $\lim_{\beta\to 0} \beta|\overline{y}| = 0$ and recalling Assumption (2d) on $\partial_p H$ we find that

$$|\gamma\beta| + |2\beta\gamma\partial_p H(\bar{t}, x_\lambda, \xi_\lambda, Z_\lambda).\overline{y})| \leqslant 1$$

for β (depending on K) small enough. Finally, being $\partial_x f_2'$ bounded and $\gamma(\overline{x} - \overline{y}) = 3\overline{\xi} = 3\xi_\lambda + 6\lambda\beta\overline{y}$ we find that

$$|Z_\lambda|^2 \leqslant C(1 + |\xi_\lambda|^2 - \partial_x H(\bar{t}, x_\lambda, \xi_\lambda, 0).\xi_\lambda) \tag{2.29}$$

for some $C > 0$, as soon as β is small enough. Combining (2.28) and (2.29), we get, for some new $C > 0$ independent from $(K, \delta, \beta, \lambda)$,

$$H(\bar{t}, x_\lambda, \xi_\lambda, 0)^2 \leqslant C\left(1 + \nu^2 + |\xi_\lambda|^2 - \xi_\lambda.\partial_x H(\bar{t}, x_\lambda, \xi_\lambda, 0)\right).$$

We get a contradiction letting $\beta \to 0$ as soon as K is big enough since $|\xi| = K$, $H(t, x, p, 0) \geqslant \alpha_1|p|^{r_1} - C_H$ with $r_1 > 1$ for all (t, x, p) and $\partial_x H$ satisfies Assumption (2f) (either (2(f)i) or (2(f)ii) is enough to conclude here).

□

To conclude the proof of Theorem 2.1 we need to show Lipschitz estimates which hold when f_2' is merely Hölder continuous in time. This requires more space regularity for f_2' and also Assumption (2(f)ii) instead of Assumption (2(f)i).

Proof of Theorem 2.1. When $f_2' = 0$, the previous lemma and the discussion at the beginning of this section are enough to conclude. In the general case, take a smooth kernel ρ with support in $[-1, 1]$ and define for all $n \in \mathbb{N}^*$, $\rho_n(r) := n\rho(nr)$ and $f_2'^{(n)}(t, x) := \int_{-1}^{1} f_2'(s, x)\rho_n(t - s)ds$, where we extended f_2' to $[-1, T + 1] \times \mathbb{R}^d$ by $f_2'(t, x) = f_2'(0, x)$ for $t \in [-1, 0]$ and $f_2'(t, x) = f_2'(T, x)$ for $t \in [T, T + 1]$. We also define u_n to be the viscosity solution to

$$\begin{cases} -\partial_t u_n(t, x) + H(t, x, Du_n(t, x), D^2u_n(t, x)) = f_2'^{(n)}(t, x) & \text{in } [0, T] \times \mathbb{R}^d \\ u(T, x) = g'(x) & \text{in } \mathbb{R}^d. \end{cases}$$

Thanks to the previous lemma and the discussion at the beginning of this section we know that u_n actually belongs to $\mathcal{C}_b^{\frac{3+\alpha}{2}, 3+\alpha}([0, T] \times \mathbb{R}^d)$. Now we use Bernstein method. We define $w_n := \frac{1}{2}e^{\mu t}\left|Du_n\right|^2$ for some $\mu > 0$ and we get, after differentiating the HJB equation and taking scalar product with $e^{\mu t}Du_n$:

$$\begin{aligned}
&- \partial_t w_n(t, x) + \partial_p H(t, x, Du_n(t, x), D^2u_n(t, x)).Dw_n(t, x) \\
&+ \partial_M H(t, x, Du_n(t, x), D^2u_n(t, x)).D^2w_n(t, x) \\
&= Df_n(t, x).Du_n(t, x)e^{\mu t} - \partial_x H(t, x, Du_n(t, x), D^2u_n(t, x)).Du_n(t, x)e^{\mu t} \\
&+ e^{\mu t}\partial_M H(t, x, Du_n(t, x), D^2u_n(t, x)).(D^2u_n(t, x))^2 - \frac{1}{2}\mu e^{\mu t}|Du_n(t, x)|^2 \\
&\leqslant Df_2'^{(n)}(t, x).Du_n(, x)e^{\mu t} + C_{\partial_x H}(1 + |Du_n(t, x)| + |D^2u_n(t, x)|)e^{\mu t}|Du_n(t, x)| \\
&- e^{\mu t}\Lambda^-|D^2u_n(t, x)|^2 - \frac{1}{2}\mu e^{\mu t}|Du_n(t, x)|^2,
\end{aligned}$$

where we used the growth assumption (2(f)ii) on $\partial_x H$, Assumption (2e) and the uniform ellipticity of H. Now we can choose $\mu = \mu(\|Df\|_\infty, C_{\partial_x H}, \Lambda^-) > 0$ such that the right-hand side of the above expression is bounded by above and, by the maximum principle for parabolic equations we get that $\|Du_n\|_\infty \leqslant C$ for some $C = C(\|Dg'\|_\infty, \|Df_2'\|_\infty, C_{D_x H}, \Lambda^-) > 0$.

Now we let $v_n := \partial_t u_n$. By differentiating the HJB equation with respect to time we get that v_n solves

$$\begin{cases} -\partial_t v_n(t,x) + \partial_p H(t,x,Du_n(t,x),D^2u_n(t,x)).Dv_n(t,x) \\ +\partial_M H(t,x,Du_n(t,x),D^2u_n(t,x)).D^2v_n(t,x) = -\partial_t H(t,x,Du_n(t,x),D^2u_n(t,x)) + \partial_t f_2^{\prime(n)}(t,x) \\ v_n(T,x) = H(T,x,Dg'(x),D^2g'(x)) - f_2^{\prime(n)}(T,x). \end{cases}$$

Fix $(t_0,x_0) \in [0,T]\times\mathbb{R}^d$ and consider a weak solution $m_n \in \mathcal{C}([t_0,T],\mathcal{P}_2(\mathbb{R}^d))$ to the adjoint equation

$$\begin{cases} \partial_t m_n - \operatorname{div}(\partial_p H(t,x,Du_n(t,x),D^2u_n(t,x))m_n) \\ + \sum_{i,j=1}^d \partial^2_{i,j}((\partial_M H(t,x,Du_n(t,x),D^2u_n(t,x)))_{i,j}m) = 0 \quad \text{in } [t_0,T]\times\mathbb{R}^d \\ m_n(t_0) = \delta_{x_0} \end{cases}$$

Integrating v_n against m_n gives, after integration by part and reorganizing the terms:

$$\begin{aligned} v_n(t_0,x_0) = &\int_{\mathbb{R}^d}\Big[H(T,x,Dg'(x),D^2g'(x)) - f_2^{\prime(n)}(T,x)\Big]dm^n(T)(x) \\ &- \int_{t_0}^T\int_{\mathbb{R}^d}\partial_t H(t,x,Du_n(t,x),D^2u_n(t,x))dm^n(t)(x)dt \\ &+ \int_{t_0}^T\int_{\mathbb{R}^d}\partial_t f_2^{\prime(n)}(t,x)dm^n(t)(x)dt \end{aligned}$$

But, again by integration by part, we have

$$\begin{aligned} \int_{t_0}^T\int_{\mathbb{R}^d}\partial_t f_2^{\prime(n)}(t,x)dm^n(t)(x)dt = &\int_{\mathbb{R}^d} f_2^{\prime(n)}(T,x)dm(T)(x) - f_2^{\prime(n)}(t_0,x_0) \\ &+ \int_{t_0}^T\int_{\mathbb{R}^d}\partial_p H(t,x,Du_n(t,x),D^2u_n(t,x)).Df_2^{\prime(n)}(t,x)dm^n(t)(x)dt \\ &+ \int_{t_0}^T\int_{\mathbb{R}^d}\partial_M H(t,x,Du_n(t,x),D^2u_n(t,x)).D^2f_2^{\prime(n)}(t,x)dm^n(t)(x)dt \end{aligned}$$

and we can conclude, using the growth assumption on $\partial_p H$, Assumption (2d), and the boundness of $\partial_M H$ and $\partial_t H$, that $\|\partial_t u_n\|_\infty \leqslant C$ for some $C > 0$ depending only on $\|Du_n\|_\infty$, $\|f_2'\|_\infty$, $\|Df_2'\|_\infty$, $\|D^2f_2'\|_\infty$, $\|Dg'\|_\infty$, $\|D^2g'\|_\infty$ but not on $\|\partial_t f_2^{\prime(n)}\|_\infty$.

Combining the two above estimates, we can use the stability of viscosity solutions to show that u_n converges locally uniformly to u, the unique viscosity solution to (2.21) and that u is therefore a globally Lipschitz function. Following the discussion at the beginning of this section this is enough to conclude that u belongs to $\mathcal{C}_b^{\frac{3+\alpha}{2},3+\alpha}([0,T]\times\mathbb{R}^d)$.

□

Conclusion In this paper we investigated a stochastic control problem with constraints on the probability distribution of the output. By reformulating the problem as a control problem for the PDE satisfied by the time marginals of the process we were able to prove the existence of solutions and characterize them. The optimal trajectories and associated controls are given by a coupled system of PDEs associated with an exclusion condition. We proved the sufficiency of these conditions under suitable convexity assumptions.

2.5 Appendix

2.5.1 Von-Neumann Theorem

Since it appears twice in our article and in particular in the proof of Theorem 2.5 we recall the statement of the Von-Neumann theorem we are using. The statement and proof can be found in the Appendix of [115] and in a slightly different setting, in [122].

Theorem 2.6. *(Von Neumann) Let $\mathbb{A}$ and $\mathbb{B}$ be convex sets of some vector spaces and suppose that $\mathbb{B}$ is endowed with some Hausdorff topology. Let $\mathcal{L}$ be a function satisfying :*

$$a \to \mathcal{L}(a,b) \text{ is concave in } \mathbb{A} \text{ for every } b \in \mathbb{B},$$

$$b \to \mathcal{L}(a,b) \text{ is convex in } \mathbb{B} \text{ for every } a \in \mathbb{A}.$$

Suppose also that there exists $a_* \in \mathbb{A}$ and $C_* > \sup_{a\in\mathbb{A}} \inf_{b\in\mathbb{B}} \mathcal{L}(a,b)$ such that :

$$\mathbb{B}_* := \{b \in \mathbb{B}, \mathcal{L}(a_*,b) \leqslant C_*\} \text{ is not empty and compact in } \mathbb{B},$$

$$b \to \mathcal{L}(a,b) \text{ is lower semicontinuous in } \mathbb{B}_* \text{ for every } a \in \mathbb{A}.$$

Then,

$$\min_{b\in\mathbb{B}} \sup_{a\in\mathbb{A}} \mathcal{L}(a,b) = \sup_{a\in\mathbb{A}} \inf_{b\in\mathbb{B}} \mathcal{L}(a,b).$$

Remark 2.11. The fact that the infimum in the "inf sup" problem is in fact a minimum is part of the theorem.

2.5.2 A comparison principle

In this section we prove comparison for viscosity solutions of the following equation

$$\max\{-\partial_t u + H(t,x,Du,D^2u) - f_2'(t,x), |Du| - K\} = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d \tag{2.30}$$

where $K > 0$.

Proposition 2.4. *Suppose that Assumptions* (1) *and* (2) *hold and suppose that $f_2' \in \mathcal{C}_b([0,T]\times\mathbb{R}^d)$ is Lipschitz continuous in $x \in \mathbb{R}^d$ uniformly in $t \in [0,T]$.*

1. *(Comparison). Let u be a bounded continuous viscosity sub-solution to* (2.30) *and v be a bounded continuous viscosity super-solution to* (2.30) *such that, $x \to u(T,x)$ is uniformly continuous and such that, for all $x \in \mathbb{R}^d$: $u(T,x) \leqslant v(T,x)$. Then for all $(t,x) \in [0,T] \times \mathbb{R}^d$, $u(t,x) \leqslant v(t,x)$.*

2. *(Existence of solutions). Assume further that g' belongs to $C_b^2(\mathbb{R}^d)$. Then, for all $K \geqslant 2\|Dg'\|_\infty$ there exists a unique bounded, continuous viscosity solution u to* (2.30) *such that $u(T,x) = g'(x)$ over $\mathbb{R}^d$.*

3. *(Regularity). Assume as well that f_2' is Lipschitz continuous in the time variable. Then, there is $L > 0$ (independent from K) such that*

$$|u(t,x) - u(s,y)| \leqslant L|t-s| + K|x-y|, \qquad \text{for all } (t,x),(s,y) \text{ in } [0,T] \times \mathbb{R}^d.$$

Proof. **(Comparison)** Let $\epsilon > 0$ be a fixed parameter, and define $u_\epsilon(t,x) = u(t,x) - \epsilon(T-t)$. We are going to show that $u_\epsilon(t,x) \leqslant v(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$. Suppose on the contrary that $\delta := \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} u_\epsilon(t,x) - v(t,x) > 0$. Let $\alpha > 0$ and $\beta > 0$ be small parameters and

$$\phi_{\alpha,\beta}(t,x,y) := u_\epsilon(t,x) - (1+\alpha^2)v(t,y) - \frac{|x-y|^2}{\alpha} - \beta|y|^2 - \frac{\beta}{t}.$$

Since u and v are bounded, $\phi_{\alpha,\beta}$ achieves its maximum at some point $(\bar{t},\overline{x},\overline{y}) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d$. Moreover, there exists $\epsilon_0 \in (0,1)$ such that, for $|\alpha| + |\beta| \leqslant \epsilon_0$, $\phi_{\alpha,\beta}(\bar{t},\overline{x},\overline{y}) \geqslant \frac{\delta}{2}$. In particular, it holds that $\beta|\overline{y}|^2 \leqslant \|u\|_\infty + \|v\|_\infty + \frac{\delta}{2} =: C_1$ and therefore $\beta|\overline{y}| \leqslant \sqrt{\beta C_1}$.

First we suppose that there exists $\epsilon_1 \in (0,\epsilon_0)$ such that $\bar{t} \neq T$ as soon as $|\alpha|+|\beta| \leqslant \epsilon_1$. We fix $\alpha \in (0, \frac{\epsilon_1}{2})$ and we let $\beta \in (0, \frac{\epsilon_1}{2})$. We can apply the maximum principle for semicontinuous functions from [53] and find that there exist $\nu \in \mathbb{R}$, $X, Y \in \mathbb{S}_d(\mathbb{R})$ such that

$$\begin{cases} (\nu, \frac{2}{\alpha}(\overline{x}-\overline{y}), X) \in \overline{\mathcal{P}}^{2,+} u_\epsilon(\bar{t},\overline{x}) \\ (\nu + \frac{\beta}{t^2}, \frac{2}{\alpha}(\overline{x}-\overline{y}), Y) \in \overline{\mathcal{P}}^{2,-}((1+\alpha^2)v(\bar{t},\overline{y}) + \beta|\overline{y}|^2) \end{cases}$$

and

$$-\frac{6}{\alpha} I_{2d} \leqslant \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \frac{6}{\alpha} \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}. \tag{2.31}$$

Yet we have the equivalence

$$\left(\nu, \frac{2}{\alpha}(\overline{x}-\overline{y}), X\right) \in \overline{\mathcal{P}}^{2,+} u_\epsilon(\bar{t},\overline{x}) \Leftrightarrow \left(\nu - \epsilon, \frac{2}{\alpha}(\overline{x}-\overline{y}), X\right) \in \overline{\mathcal{P}}^{2,+} u(\bar{t},\overline{x})$$

and, letting $\mu := 1 + \alpha^2$, it holds as well

$$\begin{aligned} &\left(\nu + \frac{\beta}{t^2}, \frac{2}{\alpha}(\overline{x}-\overline{y}), Y\right) \in \overline{\mathcal{P}}^{2,-}\left(\mu v(\bar{t},\overline{y}) + \beta|\overline{y}|^2\right) \\ &\Leftrightarrow \left(\frac{\nu}{\mu} + \frac{\beta}{\mu t^2}, \frac{2}{\alpha\mu}(\overline{x}-\overline{y}) - 2\frac{\beta}{\mu}\overline{y}, \frac{Y}{\mu} - \frac{2\beta}{\mu} I_d\right) \in \overline{\mathcal{P}}^{2,-} v(\bar{t},\overline{y}). \end{aligned}$$

Now we use the inequations satisfied by u and v to get

$$\begin{cases} \max\{-\nu+\epsilon+H(\bar{t},\overline{x},\dfrac{2}{\alpha}(\overline{x}-\overline{y}),X)-f_2'(\bar{t},\overline{x}),\dfrac{2}{\alpha}|\overline{x}-\overline{y}|-K\}\leqslant 0 \\ \max\{-\dfrac{\nu}{\mu}-\dfrac{\beta}{\mu t^2}+H(\bar{t},\overline{y},\dfrac{2}{\alpha\mu}(\overline{x}-\overline{y})-2\dfrac{\beta}{\mu}\overline{y}),\dfrac{Y}{\mu}-\dfrac{2\beta}{\mu}I_d)-f_2'(\bar{t},\overline{y}),|\dfrac{2}{\alpha\mu}(\overline{x}-\overline{y})-2\dfrac{\beta}{\mu}\overline{y}|-K\}\geqslant 0. \end{cases} \tag{2.32}$$

Notice in particular that $\dfrac{2}{\alpha}|\overline{x}-\overline{y}|\leqslant K$. Since $\mu>1$, we get for β sufficiently small (depending on α) that $|\frac{2}{\alpha\mu}(\overline{x}-\overline{y})-2\frac{\beta}{\mu}\overline{y}|\leqslant K$ and therefore we deduce from (2.32) the two inequalities

$$\begin{cases} H\left(\bar{t},\overline{x},\frac{2}{\alpha}(\overline{x}-\overline{y}),X\right)-f_2'(\bar{t},\overline{x})\leqslant\nu-\epsilon \\ H\left(\bar{t},\overline{y},\frac{2}{\alpha\mu}(\overline{x}-\overline{y})-2\frac{\beta}{\mu}\overline{y},\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d\right)-f_2'(\bar{t},\overline{y})\geqslant\frac{\nu}{\mu}. \end{cases}$$

Multiplying the second inequality by μ and substracting the two inequalities leads to

$$\begin{aligned} \epsilon\leqslant{}&\mu H\left(\bar{t},\overline{y},\frac{2}{\alpha\mu}(\overline{x}-\overline{y})-2\frac{\beta}{\mu}\overline{y},\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d\right)-H\left(\bar{t},\overline{x},\frac{2}{\alpha}(\overline{x}-\overline{y}),X\right)\\ &-\mu f_2'(\bar{t},\overline{y})+f_2'(\bar{t},\overline{x}). \end{aligned} \tag{2.33}$$

Recalling that $\mu=1+\alpha^2$ we rewrite (2.33) as follows

$$\begin{aligned} \epsilon\leqslant{}&H(\bar{t},\overline{y},\frac{2}{\alpha}(\overline{x}-\overline{y}),Y)-H(\bar{t},\overline{x},\frac{2}{\alpha}(\overline{x}-\overline{y}),X)+f_2'(\bar{t},\overline{x})-f_2'(\bar{t},\overline{y})\\ &+\mu H\left(\bar{t},\overline{y},\frac{2}{\alpha\mu}(\overline{x}-\overline{y})-2\frac{\beta}{\mu}\overline{y},\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d\right)-H(\bar{t},\overline{y},\frac{2}{\alpha}(\overline{x}-\overline{y}),Y)\\ &+f_2'(\bar{t},\overline{y})-\mu f_2'(\bar{t},\overline{y}) \end{aligned}$$

Recalling the definition of H and the Lipschitz regularity of b and f_1 with respect to the space variable we get

$$\begin{aligned} &H(\bar{t},\overline{y},\frac{2}{\alpha}(\overline{x}-\overline{y}),Y)-H(\bar{t},\overline{x},\frac{2}{\alpha}(\overline{x}-\overline{y}),X)\\ &\qquad=\sup_{a\in A}\left\{-b(\bar{t},\overline{y},a).\frac{2}{\alpha}(\overline{x}-\overline{y})-\sigma^t\sigma(\bar{t},\overline{y},a).Y-f_1(\bar{t},\overline{y},a)\right\}\\ &\qquad-\sup_{a\in A}\left\{-b(\bar{t},\overline{x},a).\frac{2}{\alpha}(\overline{x}-\overline{y})-\sigma^t\sigma(\bar{t},\overline{x},a).X-f_1(\bar{t},\overline{x},a)\right\}\\ &\qquad\leqslant\sup_{a\in A}\left\{\left(b(\bar{t},\overline{x},a)-b(\bar{t},\overline{y},a)\right).\frac{2}{\alpha}(\overline{x}-\overline{y})\right\}+\sup_{a\in A}\left\{f_1(\bar{t},\overline{x},a)-f_1(\bar{t},\overline{y},a)\right\}\\ &\qquad+\sup_{a\in A}\left\{\sigma^t\sigma(\bar{t},\overline{x},a).X-\sigma^t\sigma(\bar{t},\overline{y},a).Y\right\}\\ &\qquad\leqslant C\left(\frac{2}{\alpha}|\overline{x}-\overline{y}|^2+|\overline{x}-\overline{y}|\right)+\sup_{a\in A}\left\{\sigma^t\sigma(\bar{t},\overline{x},a).X-\sigma^t\sigma(\bar{t},\overline{y},a).Y\right\}. \end{aligned}$$

Now we use estimate (2.31) and the Lipschitz regularity of σ with respect to the space variable to deduce that, for all $a \in A$,

$$\begin{aligned}
\sigma^t\sigma(\bar{t},\overline{x},a).X - \sigma^t\sigma(\bar{t},\overline{y},a).Y &= \operatorname{Tr}\begin{pmatrix}\sigma^t\sigma(\bar{t},\overline{x},a) & \sigma(\bar{t},\overline{x},a)^t\sigma(\bar{t},\overline{y},a)\\ \sigma^t\sigma(\bar{t},\overline{y},a) & \sigma(\bar{t},\overline{y},a)^t\sigma(\bar{t},\overline{x},a)\end{pmatrix}\begin{pmatrix}X & 0\\ 0 & -Y\end{pmatrix}\\
&\leqslant \frac{6}{\alpha}\operatorname{Tr}\begin{pmatrix}\sigma^t\sigma(\bar{t},\overline{x},a) & \sigma(\bar{t},\overline{x},a)^t\sigma(\bar{t},\overline{y},a)\\ \sigma^t\sigma(\bar{t},\overline{y},a) & \sigma(\bar{t},\overline{y},a)^t\sigma(\bar{t},\overline{x},a)\end{pmatrix}\begin{pmatrix}I_d & -I_d\\ -I_d & I_d\end{pmatrix}\\
&\leqslant \frac{6}{\alpha}|\sigma(\bar{t},\overline{x},a)-\sigma(\bar{t},\overline{y},a)|^2\\
&\leqslant \frac{C}{\alpha}|\overline{x}-\overline{y}|^2,
\end{aligned}$$

and therefore, for some constant $C>0$, it holds

$$H(\bar{t},\overline{y},\frac{2}{\alpha}(\overline{x}-\overline{y}),Y) - H(\bar{t},\overline{x},\frac{2}{\alpha}(\overline{x}-\overline{y}),X) \leqslant C\left(\frac{2}{\alpha}|\overline{x}-\overline{y}|^2 + |\overline{x}-\overline{y}|\right). \tag{2.34}$$

It remains to estimate

$$\begin{aligned}
&\mu H\left(\bar{t},\overline{y},\frac{2}{\alpha\mu}(\overline{x}-\overline{y}) - 2\frac{\beta}{\mu}\overline{y},\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d\right) - H(\bar{t},\overline{y},\frac{2}{\alpha}(\overline{x}-\overline{y}),Y)\\
&= \alpha^2 H\left(\bar{t},\overline{y},\frac{2}{\alpha\mu}(\overline{x}-\overline{y}) - 2\frac{\beta}{\mu}\overline{y},\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d\right)\\
&+ H\left(\bar{t},\overline{y},\frac{2}{\alpha\mu}(\overline{x}-\overline{y}) - 2\frac{\beta}{\mu}\overline{y},\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d\right) - H(\bar{t},\overline{y},\frac{2}{\alpha}(\overline{x}-\overline{y}),Y).
\end{aligned}$$

On the one hand, using the boundness of $\partial_M H$ (see Remark (2.4)) and the growth condition (2b) on $H(t,x,p,0)$, we get the following, where $C_K>0$ depends from now on $K>0$ but not on α and β and may change from line to line,

$$\begin{aligned}
H\left(\bar{t},\overline{y},\frac{2}{\alpha\mu}(\overline{x}-\overline{y}) - 2\frac{\beta}{\mu}\overline{y},\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d\right) &\leqslant H\left(\bar{t},\overline{y},\frac{2}{\alpha\mu}(\overline{x}-\overline{y}) - 2\frac{\beta}{\mu}\overline{y},0\right) + C\left|\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d\right|\\
&\leqslant C_K\left(1+|Y|\right),
\end{aligned}$$

where we use the fact that $\left|\frac{2}{\alpha\mu}(\overline{x}-\overline{y}) - 2\frac{\beta}{\mu}\overline{y}\right| \leqslant K$ which comes from (2.32). To estimate $|Y|$ we use (both sides of) the matrix inequality (2.31) to deduce that $-\frac{6}{\alpha}I_d \leqslant Y \leqslant \frac{6}{\alpha}I_d$ and therefore $|||Y||| \leqslant \frac{6}{\alpha}$. As a consequence we get

$$\alpha^2 H\left(\bar{t},\overline{y},\frac{2}{\alpha\mu}(\overline{x}-\overline{y}) - 2\frac{\beta}{\mu}\overline{y},\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d\right) \leqslant C_K(\alpha^2+\alpha). \tag{2.35}$$

On the other hand, using again the boundness of $\partial_M H$ as well as Assumption (2d) on $\partial_p H$ we find

$$
\begin{aligned}
&H\left(\bar{t},\overline{y},\frac{2}{\alpha\mu}(\overline{x}-\overline{y})-2\frac{\beta}{\mu}\overline{y},\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d\right)-H(\bar{t},\overline{y},\frac{2}{\alpha}(\overline{x}-\overline{y}),Y)\\
&=H\left(\bar{t},\overline{y},\frac{2}{\alpha\mu}(\overline{x}-\overline{y})-2\frac{\beta}{\mu}\overline{y},\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d\right)-H\left(\bar{t},\overline{y},\frac{2}{\alpha}(\overline{x}-\overline{y}),\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d\right)\\
&+H\left(\bar{t},\overline{y},\frac{2}{\alpha}(\overline{x}-\overline{y}),\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d\right)-H(\bar{t},\overline{y},\frac{2}{\alpha}(\overline{x}-\overline{y}),Y)\\
&\leqslant C_K\left(\left|\frac{2}{\alpha\mu}(\overline{x}-\overline{y})-2\frac{\beta}{\mu}\overline{y}-\frac{2}{\alpha}(\overline{x}-\overline{y})\right|+\left|\frac{Y}{\mu}-\frac{2\beta}{\mu}I_d-Y\right|\right)\\
&\leqslant C_K\left(\alpha|\overline{x}-\overline{y}|+\beta|\overline{y}|+\beta+\alpha^2|Y|\right)\\
&\leqslant C_K\left(\alpha|\overline{x}-\overline{y}|+\beta|\overline{y}|+\beta+\alpha\right) \qquad (2.36)
\end{aligned}
$$

Finally, using the boundness and the Lipschitz continuity with respect to x of f_2' we obtain from (2.33), (2.34), (2.35) and (2.36)

$$
\epsilon\leqslant C_K\left(\frac{1}{\alpha}|\overline{x}-\overline{y}|^2+|\overline{x}-\overline{y}|+\alpha^2+\alpha+\alpha|\overline{x}-\overline{y}|+\beta|\overline{y}|+\beta\right).
$$

Recalling that $\beta|\overline{y}|\leqslant\sqrt{C_1\beta}$ and that $\frac{2}{\alpha}|\overline{x}-\overline{y}|\leqslant K$ we get a contradiction letting first $\beta\to 0$ and then $\alpha\to 0$.

Remember that we assumed the existence of $\epsilon_1\in(0,\epsilon_0)$ such that $\bar{t}\neq T$ whenever $|\alpha|+|\beta|\leqslant\epsilon_1$. Now we suppose that for all $\epsilon_1>0$ there exist α,β with $|\alpha|+|\beta|\leqslant\epsilon_1$ and $\bar{t}=T$. Then we can construct a sequence (α_n,β_n) such that for all $n\in\mathbb{N}^*$, $|\alpha_n|+|\beta_n|\leqslant\frac{1}{n}$ and $\bar{t}=T$. Let ω be a modulus of continuity for $x\to u(T,x)$. We have

$$
\begin{aligned}
\frac{\delta}{2}\leqslant\phi_{\alpha_n,\beta_n}(T,\overline{x},\overline{y})&\leqslant u(T,\overline{x})-(1+\alpha_n^2)v(T,\overline{y})\\
&\leqslant u(T,\overline{y})-v(T,\overline{y})+\omega(|\overline{x}-\overline{y}|)-\alpha_n^2v(T,\overline{y})\\
&\leqslant\omega(|\overline{x}-\overline{y}|)-\alpha_n^2v(T,\overline{y})
\end{aligned}
$$

and we get a contradiction letting $n\to+\infty$. Thus we get that, for all $\epsilon>0$, $u_\epsilon\leqslant v$ on $[0,T]\times\mathbb{R}^d$. Letting ϵ go to 0 gives, for all $(t,x)\in[0,T]\times\mathbb{R}^d$, $u(t,x)\leqslant v(t,x)$.

(Existence of viscosity solutions) For the second point of the proposition, we observe that, being g' in $\mathcal{C}_b^2(\mathbb{R}^d)$, $g'(x)-C_{g'}(T-t)$ and $g'(x)+C_{g'}(T-t)$ are respectively viscosity sub-solution and super-solution to (2.30) with terminal condition $u(T,x)=g'(x)$ over $\mathbb{R}^d$, as soon as $K\geqslant 2\|Dg'\|_\infty$. Using Perron's method we get, for all $K\geqslant 2\|Dg\|_\infty$, the existence and uniqueness of a bounded, continuous viscosity solution u to

$$
\begin{cases}
\max\{-\partial_t u+H(t,x,Du,D^2u)-f_2'(t,x),|Du|-K\}=0 & \text{in } (0,T)\times\mathbb{R}^d,\\
u(T,x)=g'(x) & \text{in } \mathbb{R}^d
\end{cases}
\qquad (2.37)
$$

(Regularity) Being u a solution to (2.37), it is straightforward that u is K-Lipschitz continuous with respect to the space variable. We now prove the time regularity. As previously observed, for

$$
C_{g'}\geqslant\sup_{(t,x)}|H(t,x,Dg'(t,x),D^2g'(t,x))-f_2'(t,x)|,
$$

$g'(x) - C_{g'}(T-s)$ and $g'(x) + C_{g'}(T-s)$ are respectively viscosity super-solution and sub-solution to (2.37). By comparison we have that $|u(T-s,x) - g'(x)| \leqslant C_{g'}s$ for all $s \in [0,T]$. If we fix $s \in [0,T]$ and define for all $(t,x) \in [s,T] \times \mathbb{R}^d$, $v(t,x) = u(t-s,x)$ it is plain to check that $v^+(t,x) := v(t,x) - C'st$ and $v^-(t,x) := v(t,x) + C'st$ are respectively sub and super-solutions to (2.37) over $[s,T] \times \mathbb{R}^d$, where C' is such that

$$|H(t-s,x,p,M) - H(t,x,p,M)| + |f_2'(t-s,x) - f_2'(t,x)| \leqslant C's$$

for all $s \in [0,T]$, all $t \in [s,T]$ and all $(x,p,M) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_d(\mathbb{R})$. By comparison we find that for all $s \in [0,T]$, all $t \in [s,T]$ and all $x \in \mathbb{R}^d$,

$$\begin{aligned} u(t,x) - v^+(t,x) &\leqslant \sup_{x \in \mathbb{R}^d} u(T,x) - v^+(T,x) \\ &\leqslant \sup_{x \in \mathbb{R}^d} g'(x) - u(T-s,x) + C'Ts \leqslant C_{g'}s + C'Ts. \end{aligned}$$

Doing the same with v^- we get that $|u(t,x) - u(t-s,x)| \leqslant (C_{g'} + C'T)s$ for all $s \in [0,T]$, all $t \in [s,T]$ and all $x \in \mathbb{R}^d$. As a consequence, we find $L > 0$, independent of $K > 0$ such that, for all $(t,x),(s,y) \in [0,T] \times \mathbb{R}^d$ it holds

$$|u(t,x) - u(t,y)| \leqslant L|t-s| + K|x-y|.$$

This concludes the proof of the proposition. □

Part II

Optimal control of the Fokker-Planck equation with state constraints in the Wasserstein space

Chapter 3

Optimality Conditions

Introduction

This paper is devoted to the study of an optimal control problem of the Fokker-Planck equation under state constraints on the space of probability measures. The formulation of the problem is the following. We seek to minimize a cost

$$J(\alpha, m) := \int_0^T \int_{\mathbb{R}^d} L(x, \alpha(t,x)) dm(t)(x) dt + \int_0^T \mathcal{F}(m(t)) dt + \mathcal{G}(m(T))$$

over pairs (α, m) with $m \in \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d))$ and $\alpha \in L^2_{m(t)\otimes dt}\left([0,T] \times \mathbb{R}^d, \mathbb{R}^d\right)$ (the control) satisfying in the sense of distributions the Fokker-Planck equation:

$$\partial_t m + \operatorname{div}(\alpha m) - \Delta m = 0 \tag{3.1}$$

with the initial condition $m(0) = m_0 \in \mathcal{P}_2(\mathbb{R}^d)$. The flow of probability measures m is also constrained to satisfy the inequality

$$\Psi(m(t)) \leqslant 0, \qquad \forall t \in [0,T]$$

for some function $\Psi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ satisfying additional conditions. Here $\mathcal{P}_2(\mathbb{R}^d)$ is the set of probability measures over $\mathbb{R}^d$ with finite second order moment. The functions $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ are the running costs and $g : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is the final cost.

Our main motivation comes from the theory of stochastic control. The corresponding problem is to minimize:

$$\mathbb{E}\left[\int_0^T L(X_t, \alpha_t) dt + \int_0^T \mathcal{F}(\mathcal{L}(X_t)) dt + \mathcal{G}(\mathcal{L}(X_T))\right]$$

over solutions of the stochastic differential equation

$$dX_t = \alpha_t dt + \sqrt{2} dB_t,$$

where the controller controls their drift α_t starting from a random position X_0 such that $\mathcal{L}(X_0) = m_0$ and under the constraint that $\Psi(\mathcal{L}(X_t)) \leqslant 0$ for all $t \in [0,T]$. In this context,

it is well-known that $\mathcal{L}(X_t)$ solves Equation (3.1) in the sense of distributions and therefore the stochastic control problem reduces to a problem of optimal control of the Fokker-Planck equation (see Chapter 2 and the references therein). Stochastic optimal control problems with constraints on the probability distribution of the output have raised some interest in the past few years in connection with quantile hedging in [70], stochastic target problems with [21, 22] and stochastic control problems with expectation constraints -see [50, 82, 84, 116, 117] - to name a few.

Given the type of constraints we are studying, here it is convenient to state our problem directly as an optimal control problem in the Wasserstein space. Such problems have been studied recently but mostly for control problems for the continuity equation (namely without diffusion term). Different approaches have been considered. In [89, 106] the authors use the dynamic programming approach and prove that the value function is the viscosity (in a sense adapted to the infinite dimensional setting) of an HJB equation. Whereas in [18, 19, 20] the authors prove some adapted forms of the Pontryagin maximum principle. Notice that optimal control problems for the Fokker-Planck equation were previously considered in [45, 66] but without constraint. Here we emphasize that the constraint is a smooth function defined on the Wasserstein space. In particular, our results do not cover the case of local constraints where the constraint acts on the density (when it exists) of m. This latter problem was addressed in [39, 57, 109, 110, 120].

Here we follow the path initiated in Chapter 2 of this thesis for a problem with terminal constraint and prove some optimality conditions in the form of a coupled system of partial differential equations associated with an exclusion condition. One of the equations is a Fokker-Planck equation satisfied by the solution of the problem. The other equation is a Hamilton-Jacobi-Bellman equation which is satisfied by an adjoint state, and from which we derive an optimal control. Besides these two equations, the exclusion condition reflects the effect of the constraint on the system. Our strategy is to proceed by penalization. We solve the penalized problem in a way that is closely related to Mean Field Game theory. Indeed, when the game has a potential structure - see for instance [24, 37, 102, 115] - the system of partial differential equations which describes the value function of a typical infinitesimal player and the distribution of the players can be obtained as optimality conditions for an optimal control problem for the Fokker-Planck equation. With this optimality conditions at hand we proceed to show that solutions to the penalized problem – when the penalization term is large enough– stay inside the constraint at all times and are therefore solutions to the constrained problem. This second step is inspired by ideas in finite dimensional optimal control theory (see [74]). In particular we follow a method used in [33, 31]. The idea is to look at local maximum points of the function $t \mapsto \Psi(m(t))$ for some solution m of the penalized problem and prove that they cannot satisfy $\Psi(m(t)) > 0$ when the penalization is strong enough. To this end we compute the second order derivative of $t \mapsto \Psi(m(t))$ thanks to the optimality conditions previously proved. An interplay between the convexity of the Hamiltonian of the system, a tranversality assumption on the constraint and various estimates on the solutions of the optimality conditions of the penalized problem allows us to conclude. As a by-product of this method we can show that the solutions of the constrained problem enjoy the same regularity as the solutions of the penalized problem. In particular optimal controls are proved to be Lipschitz continuous. This result might seem surprising since the presence of state constraints generally leads to optimal controls which behave badly

in time (see [74] and the references therein). However it is reminiscent of classical results in finite dimensional optimal control theory in the presence of suitable regularity, growth and convexity assumptions as in see [77, 85].

The rest of the paper is organized as follows. In Section 3.1 we introduce the notations and state some useful preliminary results on the Fokker-Planck equation and the HJB equation on the one hand, and on the differentiability of maps defined on the space of measures on the other hand. We also state a form of Itô's lemma for flows of probability measures. In Section 3.2 we state the standing assumptions and our main results. In Section 3.3 we obtain optimality conditions for the penalized problem. In Section 3.4 we prove our main theorem. In section 3.5 we extend our results to a more general setting. Finally, we postpone to Section 3.6 some technical results for the Hamilton-Jacobi equation satisfied by the adjoint state, that we use throughout the paper.

Notations For a map u defined on $[0,T]\times\mathbb{R}^d$ we will frequently use the notation $u(t)$ to denote the function $x\mapsto u(t,x)$. Notice that $u(t)$ is therefore a function defined on $\mathbb{R}^d$. If a function u defined on $[0,T]\times\mathbb{R}^d$ is sufficiently smooth, we denote by $\partial_t u$ the partial derivative with respect to t and by $Du, \Delta u := \mathrm{div} Du, D^2u$ (if u is a scalar function) or $Du, \vec{\Delta}u := \vec{\mathrm{div}}Du$ if u is vector-valued, the derivatives with respect to x. The Wasserstein space of Borel probability measures over $\mathbb{R}^d$ with finite moment of order $r\geqslant 1$ is denoted by $\mathcal{P}_r(\mathbb{R}^d)$. It is endowed with the r-Wasserstein distance d_r. The space of n-times differentiable bounded real functions over $\mathbb{R}^d$ with continuous and bounded derivatives is denoted by $\mathcal{C}^n_b(\mathbb{R}^d)$. Given $m\in\mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R}^d))$ we denote by $L^2_{m(t)\otimes dt}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ the space of $\mathbb{R}^d$-valued, $m(t)\otimes dt$-square-integrable functions over $[0,T]\times\mathbb{R}^d$. The space of finite Radon measures over $[0,T]$ is denoted by $\mathcal{M}([0,T])$, the subset of non-negative measures by $\mathcal{M}^+([0,T])$ and the set of $\mathbb{R}^d$-valued Borel measures over $[0,T]\times\mathbb{R}^d$ with finite total variation by $\mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$. The space of symmetric matrices of size d is denoted by $\mathbb{S}_d(\mathbb{R})$. We denote by $C^{1,2}_b([0,T]\times\mathbb{R}^d)$ the space of bounded functions with one bounded continuous derivative in time and two bounded continuous derivatives in space. Finally we denote by $W^{1,\infty}([0,T]\times\mathbb{R}^d)$ the subspace of $L^\infty([0,T]\times\mathbb{R}^d)$ consisting of functions which have one bounded distributional derivative in space and one bounded distributional derivative in time. For $n\geqslant 1$ we denote by E_n the subspace of $\mathcal{C}^n(\mathbb{R}^d)$ consisting of functions u such that

$$\|u\|_n := \sup_{x\in\mathbb{R}^d}\frac{|u(x)|}{1+|x|}+\sum_{k=1}^n\sup_{x\in\mathbb{R}^d}\left|D^k u(x)\right|<+\infty.$$

Similarly we define $E_{n+\alpha}$ for $n\geqslant 1$ and $\alpha\in(0,1)$ to be the subset of E_n consisting of functions u satisfying

$$\|u\|_{n+\alpha} := \|u\|_n+\sup_{x\neq y}\frac{|D^n u(x)-D^n u(y)|}{|x-y|^\alpha}<+\infty.$$

For $\alpha\in(0,1)$ we say that $u\in\mathcal{C}([0,T]\times\mathbb{R}^d)$ belongs to the parabolic Hölder space $\mathcal{C}^{(1+\alpha)/2,1+\alpha}([0,T]\times\mathbb{R}^d)$ if u is differentiable in x and

$$\|u\|_{\frac{1+\alpha}{2},1+\alpha} := \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u(t,x)| + \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |Du(t,x)| + \sup_{t\in[0,T]}\sup_{x\neq y} \frac{|Du(t,x)-Du(t,y)|}{|x-y|^\alpha}$$
$$+ \sup_{x\in\mathbb{R}^d}\sup_{t\neq s} \frac{|u(t,x)-u(s,x)|}{|t-s|^{(1+\alpha)/2}} + \sup_{x\in\mathbb{R}^d}\sup_{t\neq s} \frac{|Du(t,x)-Du(s,x)|}{|t-s|^{\alpha/2}}$$

is finite. Finally we will use the heat kernel P_t associated to $-\Delta$ defined, when it makes sense, by

$$P_t f(x) := \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}} f(y)dy.$$

3.1 Preliminaries

We start by introducing the main protagonists of this paper. The first one is the Fokker-Planck equation.

The Fokker-Planck equation. Given $m \in \mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R}^d))$ and $\alpha \in L^2_{m(t)\otimes dt}\left([0,T]\times\mathbb{R}^d,\mathbb{R}^d\right)$, we say that (m,α) satisfies the Fokker-Planck equation:

$$\partial_t m + \operatorname{div}(\alpha m) - \Delta m = 0 \tag{3.2}$$

if for all $\varphi \in \mathcal{C}_c^\infty((0,T)\times\mathbb{R}^d)$ we have

$$\int_0^T \int_{\mathbb{R}^d} \left[\partial_t\varphi(t,x) + D\varphi(t,x).\alpha(t,x) + \Delta\varphi(t,x)\right] dm(t)(x)dt = 0. \tag{3.3}$$

Using an approximation argument similar to [129] Remark 2.3, we can extend the class of test functions to $\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)$ and for all $\varphi \in \mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)$ and all $t_1,t_2\in[0,T]$ it holds

$$\int_{\mathbb{R}^d} \varphi(t_2,x)dm(t_2)(x) = \int_{\mathbb{R}^d} \varphi(t_1,x)dm(t_1)(x)$$
$$+ \int_{t_1}^{t_2}\int_{\mathbb{R}^d} \left[\partial_t\varphi(t,x) + D\varphi(t,x).\alpha(t,x) + \Delta\varphi(t,x)\right] dm(t)(x)dt.$$

Throughout the paper, we will repeatedly use the following properties of solutions to the Fokker-Planck equation. The proofs are given in the appendix.

Proposition 3.1. *Assume that $m \in \mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R}^d))$ and $\alpha \in L^2_{m(t)\otimes dt}\left([0,T]\times\mathbb{R}^d,\mathbb{R}^d\right)$ satisfy the Fokker-Planck equation (3.2), starting from the initial position $m_0 \in \mathcal{P}_2(\mathbb{R}^d)$ then,*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x|^2 dm(t)(x) + \sup_{t\neq s} \frac{d_2^2(m(t),m(s))}{|t-s|} \leqslant C$$

for some $C = C(\int_{\mathbb{R}^d} |x|^2 dm_0(x), \int_0^T\int_{\mathbb{R}^d} |\alpha(t,x)|^2 dm(t)(x)dt) > 0$.

We also have the following compactness result.

Proposition 3.2. *Assume that, for all $k \geqslant 1$, (m_k, ω_k) solve the Fokker-Planck equation starting from m_0 and satisfies the uniform energy estimate*

$$\int_0^T \int_{\mathbb{R}^d} |\alpha_k(t,x)|^2 dm_k(t)(x)dt \leqslant C,$$

for some $C > 0$ independent of k. Then, up to taking a sub-sequence, $(m_k, \alpha_k m_k)$ converges in $\mathcal{C}^{\frac{1-\delta}{2}}([0,T], \mathcal{P}_{2-\delta}(\mathbb{R}^d)) \times \mathcal{M}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ for any $\delta \in (0,1)$ toward some (m, ω). The curve m belongs to $\mathcal{C}^{1/2}([0,T], \mathcal{P}_2(\mathbb{R}^d))$, ω is absolutely continuous with respect to $m(t) \otimes dt$, it holds that

$$\int_0^T \int_{\mathbb{R}^d} \left| \frac{d\omega}{dm(t) \otimes dt}(t,x) \right|^2 dm(t)(x)dt \leqslant \liminf_{k \to +\infty} \int_0^T \int_{\mathbb{R}^d} \left| \frac{d\omega_k}{dm_k(t) \otimes dt}(t,x) \right|^2 dm_k(t)(x)dt$$

and, finally, $(m, \frac{d\omega}{dt \otimes dm})$ solves the Fokker-Planck equation (3.2) starting from m_0.

The HJB equation The second protagonist of this paper is the following Hamilton-Jacobi-Bellman equation. It involves the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ of the system. For the following definition to make sense and the next theorem to hold, H is assumed to satisfy Assumption (AH), introduced in the next section.

Definition 3.1. Let $f \in L^1([0,T], E_n)$ and $g \in E_{n+\alpha}$ for some $n \geqslant 2$. We say that $u \in L^1([0,T], E_n)$ is a solution to

$$\begin{cases} -\partial_t u + H(x, Du) - \Delta u = f & \text{in } [0,T] \times \mathbb{R}^d, \\ u(T,x) = g & \text{in } \mathbb{R}^d, \end{cases} \tag{3.4}$$

if, for dt-almost all $t \in [0,T]$ it holds, for all $x \in \mathbb{R}^d$

$$u(t,x) = P_{T-t} g(x) + \int_t^T P_{s-t} f(s)(x) ds - \int_t^T P_{s-t} \left[H(., Du(s,.)) \right](x) ds.$$

Let us point out that a solution $u \in \mathcal{C}([0,T], E_n)$ for $n \geqslant 3$ is differentiable in time whenever f is continuous and, at these times, the HJB equation is satisfied in the usual sense.

We introduce this notion to handle solutions which are smooth in x at each time but not necessarily regular in the time variable.

The following theorem is proved in Section 3.6.1.

Theorem 3.1. *Take $n \geqslant 2$. Assume that f belongs to $L^1([0,T], E_n)$, g belongs to $E_{n+\alpha}$ and H satisfies Assumption* (AH) *then,*

- *The HJB equation* (3.4) *admits a unique solution u in $\mathcal{C}([0,T], E_n)$ in the sense of definition 3.1 and it satisfies the estimate*

$$\sup_{t \in [0,T]} \|u(t)\|_n \leqslant C\Big(\int_0^T \|f(t)\|_n dt, \|g\|_n \Big).$$

- *Assume that* (f_m, g_m) *belongs to* $L^1([0,T], E_n) \times E_{n+\alpha}$ *for all* $m \geqslant 1$ *and that* f_m *converges to* f *in* $L^1([0,T], E_n)$ *and* g_m *converges to* g *in* $E_{n+\alpha}$. *Let* u_m *be the solution to* (3.4) *with data* (f_m, g_m), *then* u_m *converges to* u *in* $L^\infty([0,T], E_n)$.

Differentiability on the Wasserstein space and Itô's formula for flows of probability measures. We say that a map $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^m$ is $\mathcal{C}^1$ if there exists a jointly continuous map $\dfrac{\delta U}{\delta m} : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^m$ such that, for any bounded subset $\mathcal{K} \subset \mathcal{P}_2(\mathbb{R}^d)$, $x \to \dfrac{\delta U}{\delta m}(m,x)$ has at most quadratic growth in x uniformly in $m \in \mathcal{K}$ and such that, for all $m, m' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-h)m + hm', x) d(m' - m)(x) dh.$$

The function $\dfrac{\delta U}{\delta m}$ is defined up to an additive constant and we adopt the normalization convention

$$\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m,x) dm(x) = 0.$$

In the terminology of [43] it means that U admits a linear functional derivative. When the map $x \mapsto \dfrac{\delta U}{\delta m}(m,x)$ is differentiable we define the intrinsic derivative of U

$$D_m U(m,x) := D_x \frac{\delta U}{\delta m}(m,x).$$

The following form of Itô's lemma -formulated in terms of SDEs- is proved (under more general assumptions) in [43] Theorem 5.99.

Proposition 3.3. *Take* $m \in \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d))$ *and* $\alpha \in L^2_{m(t) \otimes dt}\left([0,T] \times \mathbb{R}^d, \mathbb{R}^d\right)$ *such that* (m, α) *is a solution of the Fokker-Planck equation* (3.2) *and suppose that* $U : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ *is* $\mathcal{C}^1$ *with* $\dfrac{\delta U}{\delta m}$ *satisfying*

$$\forall m \in \mathcal{P}_2(\mathbb{R}^d), \qquad x \mapsto \frac{\delta U}{\delta m}(m,x) \in \mathcal{C}^2(\mathbb{R}^d),$$

with $(m,x) \mapsto D_m U(m,x)$ *and* $(m,x) \mapsto D_x D_m U(m,x)$ *being bounded on* $\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$ *and jointly continuous. Then, for all* $t \in [0,T]$, *it holds that*

$$\begin{aligned} U(m(t)) = U(m(0)) &+ \int_0^t \int_{\mathbb{R}^d} D_m U(m(s), x).\alpha(s,x) dm(s)(x) ds \\ &+ \int_0^t \int_{\mathbb{R}^d} \mathrm{div}_x D_m U(m(s), x) dm(s)(x) ds. \end{aligned}$$

Notice that Proposition 5.48 of [43] ensures that U satisfies the assumptions of Theorem 5.99.

3.2 Main results and assumptions

First, consider the unconstrained problem

$$\inf_{(\alpha,m)} J(\alpha, m), \tag{uP}$$

where

$$J(\alpha, m) := \int_0^T \int_{\mathbb{R}^d} L(x, \alpha(t,x)) dm(t)(x) dt + \int_0^T \mathcal{F}(m(t)) dt + \mathcal{G}(m(T))$$

is the total cost and the infimum runs over all (α, m) such that

$$\begin{cases} m \in \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d)), \\ \alpha \in L^2_{dt\otimes m(t)}([0,T]\times\mathbb{R}^d, \mathbb{R}^d), \\ \partial_t m + \operatorname{div}(\alpha m) - \Delta m = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d, \\ m(0) = m_0, \end{cases} \tag{3.5}$$

where the Fokker-Planck equation is understood in the sense of distributions. Here, the Lagrangian L is defined by

$$L(x,q) := \sup_{p\in\mathbb{R}^d} \{-p.q - H(x,p)\}$$

and the data are the finite horizon $T > 0$, the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, the mean-field costs $\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $\mathcal{G} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and the initial measure $m_0 \in \mathcal{P}_2(\mathbb{R}^d)$. The above data are supposed to satisfy the following conditions for some fixed integer $n \geqslant 3$.

For $\mathcal{U} = \mathcal{F}, \mathcal{G}$, the map $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ satisfies

$$\begin{cases} \mathcal{U} \text{ is a bounded from below, } \mathcal{C}^1 \text{ map} \\ \text{and } \dfrac{\delta \mathcal{U}}{\delta m} \text{ belongs to } \mathcal{C}(\mathcal{P}_2(\mathbb{R}^d), E_{n+\alpha}). \end{cases} \tag{Ureg}$$

For some positive constant $C_0 > 0$ it holds

$$\begin{cases} \text{for all } (x,p) \in \mathbb{R}^d \times \mathbb{R}^d, \\ C_0^{-1}|p|^2 - C_0 \leqslant H(x,p) \leqslant C_0|p|^2 + C_0. \end{cases} \tag{AH2}$$

$$H \text{ belongs to } \mathcal{C}^n(\mathbb{R}^d \times \mathbb{R}^d). \tag{AH3}$$

$$\begin{cases} H \text{ and its derivatives are bounded on sets of the form} \\ \mathbb{R}^d \times B(0,R) \text{ for all } R > 0. \end{cases} \tag{AH4}$$

$$\text{For all } (x,p) \in \mathbb{R}^d \times \mathbb{R}^d,\ |D_x H(x,p)| \leqslant C_0(1+|p|). \tag{AH5}$$

$$\begin{cases} \text{There is some } \mu > 0 \text{ such that, for all } (x,p) \in \mathbb{R}^d \times \mathbb{R}^d, \\ \dfrac{1}{\mu} I_d \leqslant D_{pp} H(x,p) \leqslant \mu I_d. \end{cases} \tag{AH6}$$

For convenience we put all the assumptions concerning H into

$$\text{Assume } AH2, AH3, AH4, AH5, AH6. \tag{AH}$$

Notice that, taking convex conjugates, we see that L satisfies a coercivity condition similar to Assumption (AH2): for all $(x, q) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$\frac{1}{4C_0}|q|^2 - C_0 \leqslant L(x, q) \leqslant \frac{C_0}{4}|q|^2 + C_0,$$

and the first term in the total cost J looks very much like a kinetic energy.

A typical example of functions satisfying the condition (Ureg) is the class of cylindrical functions of the form

$$\mathcal{F}(m) = F\left(\int_{\mathbb{R}^d} f_1(x)dm(x), \dots, \int_{\mathbb{R}^d} f_k(x)dm(x)\right),$$

where F and the f_i, $1 \leqslant i \leqslant k$ are smooth with bounded derivatives. Assumption (Ureg) also implies that $(m, x) \to D_m\mathcal{U}(m, x)$ is uniformly bounded in $\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$ and therefore, a simple application of Kantorovitch-Rubinstein duality for d_1 proves that $\mathcal{U}$ is Lipschitz continuous with respect to this distance.

Under the above assumptions on $\mathcal{F}$, $\mathcal{G}$ and H it is well-known (see [24] or Chapter 2 of this thesis), that solutions (m, α) of Problem (uP) exist and satisfy $\alpha(t, x) = -D_pH(x, Du(t, x))$ with (m, u) solution to the Mean-Field Game (MFG) system of partial differential equations

$$\begin{cases} -\partial_t u(t,x) + H(x, Du(t,x)) - \Delta u(t,x) = \dfrac{\delta \mathcal{F}}{\delta m}(m(t), x) & \text{in } (0,T) \times \mathbb{R}^d, \\ \partial_t m - \operatorname{div}(D_pH(x, Du(t,x))m) - \Delta m = 0 & \text{in } (0,T) \times \mathbb{R}^d, \\ u(T,x) = \dfrac{\delta \mathcal{G}}{\delta m}(m(T), x) \quad \text{in } \mathbb{R}^d, \qquad m(0) = m_0, \end{cases} \tag{3.6}$$

where the unknown (u, m) belong to $\mathcal{C}^{1,2}((0, T) \times \mathbb{R}^d)$.

The purpose of the present work is to investigate the effect of a state constraint

$$\Psi(m(t)) \leqslant 0, \forall t \in [0, T],$$

on the problem above. Here $\Psi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ satisfies the regularity assumption (Ureg) and is convex for the linear structure of $\mathcal{P}_2(\mathbb{R}^d)$

$$\begin{cases} \text{For all } m_1, m_2 \in \mathcal{P}_2(\mathbb{R}^d) \text{ and all } \lambda \in [0,1], \\ \Psi((1-\lambda)m_1 + \lambda m_2) \leqslant (1-\lambda)\Psi(m_1) + \lambda\Psi(m_2). \end{cases} \tag{APsiConv}$$

We also need to assume that the problem is initialized at a point m_0 in the interior of the constraint that is

$$\Psi(m_0) < 0. \tag{APsiInside}$$

In addition to the previous assumptions we will ask for second-order differentiability with respect to the measure variable for Ψ.

$$\begin{cases} \text{For all } x \in \mathbb{R}^d,\ m \mapsto \dfrac{\delta\Psi}{\delta m}(m,x) \text{ is } \mathcal{C}^1 \text{ with } (x,y) \mapsto \dfrac{\delta^2\Psi}{\delta m^2}(m,x,y) := \dfrac{\delta^2\Psi}{\delta m^2}(m,x)(y) \\ \text{in } \mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d) \text{ for all } m \in \mathcal{P}_2(\mathbb{R}^d) \text{ and } \dfrac{\delta^2\Psi}{\delta m^2}(m,x,y) \text{ and its derivatives being} \\ \text{jointly continuous and bounded in } \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{APsiC2}$$

Notice that Assumption (APsiC2) implies in particular (see for instance [43] Remark 5.27) that the map $(m,x) \mapsto D_m\Psi(m,x)$ is uniformly Lipschitz continuous over $\mathcal{P}_1(\mathbb{R}^d) \times \mathbb{R}^d$.

Finally we require the following geometric assumption on the constraint.

$$\int_{\mathbb{R}^d} |D_m\Psi(m,x)|^2\, dm(x) \neq 0 \text{ whenever } \Psi(m) = 0. \tag{APsiTrans}$$

The transversality assumption (APsiTrans) is not necessary to get the optimality conditions however it is the key assumption to obtain the time regularity of optimal controls.

An example of constraint $\Psi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ satisfying Assumptions (Ureg), APsiConv and APsiC2 is

$$\Psi(m) := \int_{\mathbb{R}^d} \psi(x) dm(x)$$

where ψ is any function in E_n. If if holds as well that $|D\psi(x)| \neq 0$ whenever $\psi(x) \geqslant 0$ then Ψ satisfies Assumption (APsiTrans). Indeed if $\displaystyle\int_{\mathbb{R}^d} |D\psi(x)|^2 dm(x) = 0$ then m must be concentrated on the set of points in $\mathbb{R}^d$ where $\psi(x) < 0$ and therefore it cannot be that $\displaystyle\int_{\mathbb{R}^d} \psi(x) dm(x) = 0$.

Example 3.1. A typical example which satisfies Assumptions (Ureg), (APsiConv), (APsiC2) and (APsiTrans) that we have in mind is

$$\Psi(m) = \int_{\mathbb{R}^d} \left(\sqrt{|x - x_0|^2 + \delta^2} - \delta\right) dm(x) - \kappa$$

with $x_0 \in \mathbb{R}^d$, $\delta > 0$ and $\kappa > 0$. We can compute the derivatives, for all $m \in \mathcal{P}_2(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$:

$$\frac{\delta\Psi}{\delta m}(m,x) = \sqrt{|x - x_0|^2 + \delta^2} - \delta - \int_{\mathbb{R}^d} \left(\sqrt{|y - x_0|^2 + \delta^2} - \delta\right) dm(y),$$

$$D_m\Psi(m,x) = \frac{x - x_0}{\sqrt{|x - x_0|^2 + \delta^2}},$$

$$\mathrm{div}_x D_m\Psi(m,x) = \frac{d}{\sqrt{|x - x_0|^2 + \delta^2}} - \frac{|x - x_0|^2}{(|x - x_0|^2 + \delta^2)^{3/2}},$$

$$\frac{\delta^2\Psi}{\delta m^2}(m,x_1,x_2) = 0.$$

In particular, it holds that

$$\int_{\mathbb{R}^d} |D_m\Psi(m,x)|^2 dm(x) = \int_{\mathbb{R}^d} \frac{|x-x_0|^2}{|x-x_0|^2+\delta^2} dm(x),$$

and it is easily seen that Ψ satisfies the tranversality condition APsiTrans.

We can finally state the main problem of interest in this paper:

$$\inf_{(\alpha,m)} \int_0^T \int_{\mathbb{R}^d} L(x,\alpha(t,x))dm(t)(x)dt + \int_0^T \mathcal{F}(m(t))dt + \mathcal{G}(m(T)) \tag{P}$$

where the infimum runs over the pairs (m,α) satisfying (3.5) and the state constraint

$$\forall t \in [0,T], \qquad \Psi(m(t)) \leqslant 0.$$

Over the course of the paper we will introduce several auxiliary problems. The main one is the following. For $\epsilon,\delta > 0$ the penalized problem $(P_{\epsilon,\delta})$ is

$$\inf_{(m,\alpha)} J_{\epsilon,\delta}(\alpha,m) \tag{$P_{\epsilon,\delta}$}$$

where the infimum runs over all (m,α) satisfying (3.5) (but not necessarily the state constraint) and $J_{\epsilon,\delta}$ is defined by

$$\begin{aligned} J_{\epsilon,\delta}(\alpha,m) := &\int_0^T \int_{\mathbb{R}^d} L(x,\alpha(t,x))dm(t)(x)dt + \int_0^T \mathcal{F}(m(t))dt + \frac{1}{\epsilon}\int_0^T \Psi^+(m(t))dt \\ &+ \mathcal{G}(m(T)) + \frac{1}{\delta}\Psi^+(m(T)) \\ &= J(\alpha,m) + \frac{1}{\epsilon}\int_0^T \Psi^+(m(t))dt + \frac{1}{\delta}\Psi^+(m(T)). \end{aligned}$$

Here and in the following, $\Psi^+(m) = \Psi(m) \vee 0 = \max(\Psi(m),0)$. Notice that Problem $(P_{\epsilon,\delta})$ is very similar to Problem (uP) although we have to deal with the non-differentiability at 0 of the map $r \mapsto \max(r,0)$.

We now state our main results. The first one is not expected without Assumption (APsiTrans). Roughly speaking, it asserts that optimal solutions to the penalized problems $P_{\epsilon,\delta}$ stay inside the constraint when the penalization is strong enough.

Theorem 3.2. *Take $n \geqslant 3$. Assume that* (AH) *holds for H,* (Ureg) *holds for $\mathcal{F}$ and $\mathcal{G}$. Assume further that Ψ satisfies Assumptions* (Ureg), (APsiConv), (APsiInside), (APsiC2) *and* (APsiTrans). *Then there exist $\epsilon_0,\delta_0 > 0$ depending on m_0 only through the value $\Psi(m_0)$ such that, for all (ϵ,δ) in $(0,\epsilon_0)\times(0,\delta_0)$ Problems $(P_{\epsilon,\delta})$ and* (P) *have the same solutions.*

As a consequence we find the following optimality conditions for the optimal control problem with constraint.

Theorem 3.3. *Under the same assumptions as Theorem 3.2, Problem* (P) *admits at least one solution and, for any solution (α,m) there exist $u \in \mathcal{C}([0,T],E_n)$, $\nu \in L^\infty([0,T])$ and $\eta \in \mathbb{R}^+$ such that*

$$\alpha = -D_pH(x,Du) \tag{3.7}$$

and

$$\begin{cases} -\partial_t u(t,x) + H(x, Du(t,x)) - \Delta u(t,x) \\ \qquad\qquad = \nu(t)\dfrac{\delta\Psi}{\delta m}(m(t),x) + \dfrac{\delta\mathcal{F}}{\delta m}(m(t),x) & \text{in } (0,T)\times\mathbb{R}^d, \\ \partial_t m - \operatorname{div}(D_pH(x,Du(t,x))m) - \Delta m = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ u(T,x) = \eta\dfrac{\delta\Psi}{\delta m}(m(T),x) + \dfrac{\delta\mathcal{G}}{\delta m}(m(T),x) & \text{in } \mathbb{R}^d, \\ m(0) = m_0, \end{cases} \tag{3.8}$$

where the Fokker-Planck equation is understood in the sense of distributions and u solves the HJB equation in the sense of Definition 3.1.

The Lagrange multipliers ν and η satisfy

$$\nu(t) = \begin{cases} 0 & \text{if } \Psi(m(t)) < 0 \\ \nu(t) \in \mathbb{R}^+ & \text{if } \Psi(m(t)) = 0, \end{cases} \qquad \eta = \begin{cases} 0 & \text{if } \Psi(m(T)) < 0 \\ \eta \in \mathbb{R}^+ & \text{if } \Psi(m(T)) = 0. \end{cases} \tag{3.10}$$

(3.9)

If we also assume that $\mathcal{F}$ and $\mathcal{G}$ are convex in the measure variable, then the above conditions are sufficient conditions: if (m,α) satisfies $\Psi(m(t)) \leqslant 0$ for all $t \in [0,T]$ and if there exists (u,ν,η) such that (3.7)*,* (3.8)*,* (3.9) *and* (3.10) *hold then (α, m) is a solution to* (P)*.*

The strength of the above result relies on the regularity of the Lagrange multiplier ν associated to the constraint that for all $t \in [0,T]$, $\Psi(m(t)) \leqslant 0$. Indeed we would a priori expect ν to be a finite Radon measure over $[0,T]$ but here we find that ν belongs to $L^\infty([0,T])$. As a consequence – and as explained in Remark 3.3 below– optimal controls are Lipschitz continuous in time.

We complete this section with a few comments.

Remark 3.1. Arguing as in [31], in the proof of Theorem 3.1, we can use the expression of $\frac{d^2}{dt^2}\Psi(m(t))$ given by Proposition 3.5 to express $\nu(t)$ as a (non-local) feedback function of $Du(t)$, $D^2u(t)$ and $m(t)$.

Remark 3.2. Computing the cost of an optimal control we see that the value of the problem denoted by $\overline{\mathcal{U}}(m_0)$ is given by

$$\overline{\mathcal{U}}(m_0) = \int_{\mathbb{R}^d} u(0,x)dm_0(x) + \int_0^T \mathcal{F}(m(t))dt + \mathcal{G}(m(T))$$

for any solution $(m, -D_pH(x,Du))$ of (P).

Remark 3.3. Differentiating the HJB equation with respect to x shows that Du actually belongs to $W^{1,\infty}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ and since Du is also continuous and D_pH is locally Lipschitz we get that α is Lipschitz continuous. In particular the Stochastic Differential Equation

$$X_t = X_0 + \int_0^t \alpha(s, X_s)ds + \sqrt{2}B_t$$

where $X_0 \sim m_0$, admits a unique strong solution and we can proceed as in Chapter 2 in the proof of Theorem 2.2 to find strong solutions to the stochastic analog of Problem (P) (as stated in the introduction).

Remark 3.4. Ideally we would like to consider constraints of the form $\Psi(m) = \int_{\mathbb{R}^d} |x|^2 dm(x) - \kappa$ (which does not satisfy the growth conditions of Assumptions (Ureg) and (APsiC2)) for some $\kappa > 0$. However this would significantly increase the technicality of the paper and we leave this case for future research. Among other difficulties we would have to solve the backward HJB equation in (3.8) when the source term has a quadratic growth in the space variable.

Remark 3.5. Our results could be naturally extended to multiple (possibly time dependent) equality or inequality constraints under suitable qualification conditions but we focus on this case of just one inequality constraint for the sake of clarity in an already technical paper.

Optimality conditions without Assumptions (APsiC2) **and** (APsiTrans)**.** When Assumptions (APsiC2) and (APsiTrans) are not satisfied we do not expect the conclusions of Theorem 3.2 to hold. However, we can pass to the limit as ϵ, δ go to 0 in the Penalized problem $(P_{\epsilon,\delta})$ and find the optimality conditions for the constrained problem.

Theorem 3.4. *Assume that AH holds for H, Ureg holds for $\mathcal{F}$ and $\mathcal{G}$. Assume further that Ψ satisfies Assumptions* (Ureg)*,* (APsiConv) *and* (APsiInside)*. Then the conclusions of Theorem* (3.3) *hold true with $\nu \in \mathcal{M}^+([0,T])$, and $u \in L^\infty([0,T], E_n)$. The exclusion condition for ν now reads $\Psi(m(t)) = 0$, for ν-almost all $t \in [0,T]$. Finally the optimal control α belongs to $BV_{loc}([0,T] \times \mathbb{R}^d, \mathbb{R}^d) \bigcap L^\infty([0,T], \mathcal{C}_b^{n-1}(\mathbb{R}^d, \mathbb{R}^d))$.*

In this (slightly more) general case, we lose the time regularity of the optimal controls. This is due to the shocks that can occur when the optimal curve $t \to m(t)$ touches the constraint. Indeed, the set of times where the optimal control is not continuous, is contained into the support of the singular part of the Lagrange multiplier ν. However, the space regularity of the backward component u of the system and of the optimal control $-D_pH(x, Du)$ remains.

The proof of Theorem 3.4 is the aim of section 3.5 where we discuss in particular the well-posedness of the HJB equation when the Lagrange multiplier ν belongs to $\mathcal{M}^+([0,T])$.

3.3 The penalized problem

In this section we analyze the penalized problem $(P_{\epsilon,\delta})$. The main result is the following.

Theorem 3.5. *Problem $(P_{\epsilon,\delta})$ admits at least one solution and, for any solution (α, m) of $(P_{\epsilon,\delta})$ there exist $u \in \mathcal{C}([0,T], E_n)$, $\lambda \in L^\infty([0,T])$ and $\beta \in [0,1]$ such that*

$$\alpha = -D_pH(x, Du)$$

and

$$\begin{cases} -\partial_t u(t,x) + H(x, Du(t,x)) - \Delta u(t,x) \\ \qquad\qquad = \dfrac{\lambda(t)}{\epsilon}\dfrac{\delta\Psi}{\delta m}(m(t),x) + \dfrac{\delta\mathcal{F}}{\delta m}(m(t),x) & \text{in } (0,T)\times\mathbb{R}^d, \\ \partial_t m - \operatorname{div}(D_p H(x, Du(t,x))m) - \Delta m = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ u(T,x) = \dfrac{\beta}{\delta}\dfrac{\delta\Psi}{\delta m}(m(T),x) + \dfrac{\delta\mathcal{G}}{\delta m}(m(T),x) \quad \text{in } \mathbb{R}^d, \qquad m(0) = m_0. \end{cases} \tag{3.11}$$

Moreover, λ and β satisfy

$$\lambda(t)\begin{cases} = 0 & \text{if } \Psi(m(t)) < 0 \\ \in [0,1] & \text{if } \Psi(m(t)) = 0 \\ = 1 & \text{if } \Psi(m(t)) > 0, \end{cases} \tag{3.12}$$

$$\beta\begin{cases} = 0 & \text{if } \Psi(m(T)) < 0 \\ \in [0,1] & \text{if } \Psi(m(T)) = 0 \\ = 1 & \text{if } \Psi(m(T)) > 0. \end{cases} \tag{3.13}$$

The proof of Theorem 3.5 will be divided into three steps. First we are going to prove the existence of (relaxed) solutions to the problem. This is Lemma 3.1. In the second step, we will show that these relaxed solutions are actually solutions of a suitable linearized problem. This is Lemma 3.2. Finally, we will conclude the proof of Theorem 3.5 by computing the optimality conditions for this linearized problem. The three steps above are very similar to what is done in [24] Lemma 3.1 and in Chapter 2, Section 2.2 of this thesis. Here, however we have to deal with the lack of differentiability at 0 of the function $r \mapsto \max(0, r)$. We also proceed differently at the end of the proof of Theorem 3.5, where we argue by verification to avoid the unnecessary use of a min/max argument.

We start with the existence of relaxed solutions. A relaxed candidate is a pair (m, ω) such that

$$\begin{cases} m \in \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d)), \\ \omega \in \mathcal{M}([0,T]\times\mathbb{R}^d, \mathbb{R}^d), \\ \partial_t m + \operatorname{div}(\omega) - \Delta m = 0 \quad \text{in } (0,T)\times\mathbb{R}^d, \\ m(0) = m_0, \end{cases} \tag{3.14}$$

where the Fokker-Planck equation is once again understood in the sense of distributions.

A relaxed solution is a minimizer over all the relaxed candidates of the following functional still denoted (with a slight abuse of notations) by $J_{\epsilon,\delta}$

$$\begin{aligned} J_{\epsilon,\delta}(m,\omega) := \int_0^T \int_{\mathbb{R}^d} L(x, \frac{d\omega}{dt\otimes dm(t)}(t,x))dm(t)(x)dt + \int_0^T \mathcal{F}(m(t))dt + \frac{1}{\epsilon}\int_0^T \Psi^+(m(t))dt \\ + \mathcal{G}(m(T)) + \frac{1}{\delta}\Psi^+(m(T)), \end{aligned}$$

where we set $J_{\epsilon,\delta}(m,\omega) = +\infty$ if ω is not absolutely continuous with respect to $dt \otimes m(t)$.

Lemma 3.1. *Problem $(P_{\epsilon,\delta})$ admits at least one relaxed solution.*

The existence of relaxed solutions is standard (see [24] or Chapter 2 Section 2.2) but we give the proof in Appendix 3.7.1 for the sake of completeness and because we will use the same line of arguments at different points in our analysis.

Notice that it would not be more difficult to obtain weak solutions directly for the constrained problem. However, for the constrained problem, we don't know how to directly compute the optimality conditions and more importantly they would not give us the regularity of the Lagrange multipliers that we get thanks to our penalization procedure.

Now we fix a solution $(\widetilde{m}, \widetilde{\omega})$ of the penalized problem and we proceed to show that $(\widetilde{m}, \widetilde{\omega})$ is solution to a suitable linearized problem for which it will be easier to compute the optimality conditions. In the proof of the following lemma we will use a smooth distance-like function. To this end we consider a family $(\varphi_i)_{i\in\mathbb{N}}$ of functions in $\mathcal{C}^2_b(\mathbb{R}^d)$ such that for $m_1, m_2 \in \mathcal{P}_2(\mathbb{R}^d)$ we have

$$m_1 = m_2 \Leftrightarrow \forall i \in \mathbb{N} \quad \int_{\mathbb{R}^d} \varphi_i(x) d(m_1 - m_2)(x) = 0,$$

and we define $q : \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ by

$$q(m_1, m_2) := \sum_{i=0}^{+\infty} \frac{\left| \int_{\mathbb{R}^d} \varphi_i d(m_1 - m_2) \right|^2}{2^i (1 + \|\varphi_i\|^2_\infty + \|D\varphi_i\|^2_\infty)}.$$

Notice that q satisfies

$$\begin{cases} q(m_1, m_2) \geqslant 0 & \forall m_1, m_2 \in \mathcal{P}_2(\mathbb{R}^d) \\ q(m_1, m_2) = 0 \text{ if and only if } m_1 = m_2. \end{cases} \tag{3.15}$$

It is straightforward to verify that q is $\mathcal{C}^1$ with respect to both of its arguments and that

$$\frac{\delta q}{\delta m_1}(m_1, m_2)(x) = \sum_{i=0}^{+\infty} \frac{2 \int_{\mathbb{R}^d} \varphi_i d(m_1 - m_2)}{2^i (1 + \|\varphi_i\|^2_\infty + \|D\varphi_i\|^2_\infty)} (\varphi_i(x) - \int_{\mathbb{R}^d} \varphi_i dm_1).$$

In particular we have

$$\begin{cases} \int_{\mathbb{R}^d} \dfrac{\delta q}{\delta m_1}(m_1, m_2)(y) dm_1(y) = 0 & \forall m_1, m_2 \in \mathcal{P}_2(\mathbb{R}^d), \\ \dfrac{\delta q}{\delta m_1}(m_1, m_1)(x) = 0 & \forall m_1 \in \mathcal{P}_2(\mathbb{R}^d) \text{ and } \forall x \in \mathbb{R}^d. \end{cases} \tag{3.16}$$

Lemma 3.2. *Let $(\widetilde{m}, \widetilde{\omega})$ be a fixed solution to Problem $(P_{\epsilon,\delta})$. Then there exist $\lambda \in L^\infty([0,T])$ and $\beta \in \mathbb{R}^+$ satisfying*

$$\lambda(t) = \begin{cases} 0 & \text{if } \Psi(\widetilde{m}(t)) < 0, \\ \lambda(t) \in [0,1] & \text{if } \Psi(\widetilde{m}(t)) = 0, \\ 1 & \text{if } \Psi(\widetilde{m}(t)) > 0, \end{cases} \qquad \beta \begin{cases} = 0 & \text{if } \Psi(\widetilde{m}(T)) < 0, \\ \in [0,1] & \text{if } \Psi(\widetilde{m}(T)) = 0, \\ = 1 & \text{if } \Psi(\widetilde{m}(T)) > 0, \end{cases} \tag{3.17, 3.18}$$

such that $(\widetilde{m}, \widetilde{\omega})$ minimizes

$$J^l_{\epsilon,\delta}(\omega,m) := \int_0^T \int_{\mathbb{R}^d} L(x, \frac{d\omega}{dt\otimes dm(t)}(t,x))dm(t)(x)dt + \int_0^T \int_{\mathbb{R}^d} \left[\frac{\lambda(t)}{\epsilon}\frac{\delta\Psi}{\delta m}(\widetilde{m}(t),x) + \frac{\delta\mathcal{F}}{\delta m}(\widetilde{m}(t),x)\right] dm(t)(x)dt + \int_{\mathbb{R}^d} \left[\frac{\beta}{\delta}\frac{\delta\Psi}{\delta m}(\widetilde{m}(T),x) + \frac{\delta\mathcal{G}}{\delta m}(\widetilde{m}(T),x),\right] dm(T)(x)$$

over the pairs (m,ω) *satisfying* (3.14). *Once again, we set* $J^l_{\epsilon,\delta}(m,\omega) = +\infty$ *if* ω *is not absolutely continuous with respect to* $dt\otimes m(t)$.

Proof. To avoid uniqueness issues we modify the cost function $J_{\epsilon,\delta}$ so that the new problem reads

$$\inf\left[J_{\epsilon,\delta}(m,\omega) + \int_0^T q(m(t),\widetilde{m}(t))dt\right]. \tag{3.19}$$

If $(\widetilde{m}',\widetilde{\omega}')$ is a solution of the above problem, then $\widetilde{m}' = \widetilde{m}$. This is a direct consequence of (3.15) and the fact that $(\widetilde{m},\widetilde{\omega})$ is a solution of the penalized problem. We use this function q (and not the Wasserstein distance for instance) because it is smooth and therefore we can differentiate it to get optimality conditions and also because $\frac{\delta q}{\delta m}(\widetilde{m},\widetilde{m},x) = 0$ for all $x\in\mathbb{R}^d$ (see (3.16)): therefore q will not appear in the optimality conditions for $(\widetilde{m},\widetilde{\omega})$. Now, we introduce a suitable regularization of the function $r\mapsto \max(0,r)$. For all $h>0$, let $\gamma_h:\mathbb{R}\to\mathbb{R}^+$ be functions satisfying

$$\begin{cases} \gamma_h\in\mathcal{C}^2(\mathbb{R}), \gamma_h\geqslant 0,\\ \gamma_h(r) = \max(0,r) \text{ in } \mathbb{R}\backslash[-h,h],\\ \sup_{r\in\mathbb{R}} |\gamma_h'(r)| \leqslant 1,\\ \sup_{r\in\mathbb{R}} |\gamma_h(r) - \max(0,r)| \to 0 \text{ as } h\to 0. \end{cases}$$

We consider the regularized, penalized cost functionals

$$J_{\epsilon,\delta,h}(m,\omega) := \int_0^T \int_{\mathbb{R}^d} L(x, \frac{d\omega}{dt\otimes dm(t)}(t,x))dm(t)(x)dt + \int_0^T \mathcal{F}(m(t))dt + \frac{1}{\epsilon}\int_0^T \Psi_h(m(t))dt + \mathcal{G}(m(T)) + \frac{1}{\delta}\Psi_h(m(T))$$

where Ψ_h is defined for all $m\in\mathcal{P}_2(\mathbb{R}^d)$ by $\Psi_h(m) = \gamma_h(\Psi(m))$. Now we argue as in the proof of Lemma 3.1 (see Appendix 3.7.1) and find for all $h\in(0,1)$ a solution (m_h,ω_h) of

$$\inf\left[J_{\epsilon,\delta,h}(m,\omega) + \int_0^T q(m(t),\widetilde{m}(t))dt\right]. \tag{3.20}$$

Taking for granted that we can find a candidate $(\overline{m},\overline{\omega})$ such that $J(\overline{m},\overline{\omega}) < +\infty$ and $\Psi(\overline{m}(t)) \leqslant 0$ for all $t\in[0,T]$ (we explicitly construct such a candidate in Lemma 3.4 in

Section 3.4.1 below) we find that $J_{\epsilon,\delta,h}(m_h,\omega_h)$ is bounded from above by $J(\overline{m},\overline{\omega})$ independently of ϵ,δ and h. By coercivity of L we deduce that

$$\int_0^T \int_{\mathbb{R}^d} \left| \frac{d\omega_h}{dt \otimes dm_h(t)}(t,x) \right|^2 dm_h(t)(x)dt \leqslant C$$

for some $C > 0$ independent of ϵ,δ and h. Following the proof of Lemma 3.1 in Appendix 3.7.1, we deduce that (m_h,ω_h) converges, up to a sub-sequence, in $\mathcal{C}([0,T],\mathcal{P}_r(\mathbb{R}^d)) \times \mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ for some $r \in (1,2)$ to an element (m',ω') of $\mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R}^d)) \times \mathcal{M}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ satisfying (3.14) with ω' absolutely continuous with respect to m'. Let us prove that (m',ω') is a minimizer of (3.19) and therefore, by uniqueness –that is why we added the q-term in the cost functional–, $m' = \widetilde{m}$. We just need to show that

$$J_{\epsilon,\delta}(m',\omega') + \int_0^T q(m'(t),\widetilde{m}(t))dt \leqslant J_{\epsilon,\delta}(\widetilde{m},\widetilde{\omega}).$$

However, for any $h \in (0,1)$, using the minimality of (m_h,ω_h) for Problem (3.20) it holds,

$$\begin{aligned}
&J_{\epsilon,\delta}(m',\omega') + \int_0^T q(m'(t),\widetilde{m}(t))dt - J_{\epsilon,\delta}(\widetilde{m},\widetilde{\omega}) \\
&= J_{\epsilon,\delta,h}(m_h,\omega_h) + \int_0^T q(m_h(t),\widetilde{m}(t))dt - J_{\epsilon,\delta,h}(\widetilde{m},\widetilde{\omega}) \\
&+ J_{\epsilon,\delta}(m',\omega') - J_{\epsilon,\delta,h}(m_h,\omega_h) + \int_0^T q(m'(t),\widetilde{m}(t))dt - \int_0^T q(m_h(t),\widetilde{m}(t))dt \\
&+ J_{\epsilon,\delta,h}(\widetilde{m},\widetilde{\omega}) - J_{\epsilon,\delta}(\widetilde{m},\widetilde{\omega}) \\
&\leqslant J_{\epsilon,\delta}(m',\omega') - J_{\epsilon,\delta,h}(m_h,\omega_h) + \int_0^T q(m'(t),\widetilde{m}(t))dt - \int_0^T q(m_h(t),\widetilde{m}(t))dt \\
&+ J_{\epsilon,\delta,h}(\widetilde{m},\widetilde{\omega}) - J_{\epsilon,\delta}(\widetilde{m},\widetilde{\omega}).
\end{aligned}$$

Since $\displaystyle\int_0^T q(m'(t),\widetilde{m}(t))dt - \int_0^T q(m_h(t),\widetilde{m}(t))dt$ and $J_{\epsilon,\delta,h}(\widetilde{m},\widetilde{\omega}) - J_{\epsilon,\delta}(\widetilde{m},\widetilde{\omega})$ converge to 0 as h converges to 0, it is sufficient to prove that $J_{\epsilon,\delta}(m',\omega') \leqslant \liminf_{h\to 0} J_{\epsilon,\delta,h}(m_h,\omega_h)$. For all $h > 0$ we can rewrite

$$J_{\epsilon,\delta,h}(m_h,\omega_h) = J_{\epsilon,\delta}(m_h,\omega_h) + \frac{1}{\epsilon}\int_0^T \left[\Psi_h(m_h(t)) - \Psi^+(m_h(t))\right] dt + \frac{1}{\delta}\left[\Psi(m_h(T)) - \Psi^+(m_h(T))\right]$$

but

$$\lim_{h\to 0} \frac{1}{\epsilon}\int_0^T \left[\Psi_h(m_h(t)) - \Psi^+(m_h(t))\right] dt + \frac{1}{\delta}\left[\Psi_h(m_h(T)) - \Psi^+(m_h(T))\right] = 0$$

and therefore $\liminf_{h\to 0} J_{\epsilon,\delta,h}(m_h,\omega_h) = \liminf_{h\to 0} J_{\epsilon,\delta}(m_h,\omega_h)$. Finally we can conclude by lower semi-continuity of $J_{\epsilon,\delta}$ that $\liminf_{h\to 0} J_{\epsilon,\delta}(m_h,\omega_h) \leqslant J_{\epsilon,\delta}(m',\omega')$. The lower semi-continuity of $J_{\epsilon,\delta}$ can be proved following Theorem 2.34 of [7].

Now we argue as in Chapter 3.7 Section 2.3.1 to show that, for all $h > 0$, (m_h,ω_h) is actually an infimum of the linearized problem

$$\inf J^l_{\epsilon,\delta,h}(m,\omega) + \int_0^T \int_{\mathbb{R}^d} \frac{\delta q}{\delta m_1}(m_h(t), \widetilde{m}(t), x) dm(t)(x) dt \tag{3.21}$$

where the infimum is still taken over relaxed candidates (m,ω) satisfying (3.14) with the linearized cost functional $J^l_{\epsilon,\delta,h}$ defined by

$$\begin{aligned} J^l_{\epsilon,\delta,h}(\omega, m) = &\int_0^T \int_{\mathbb{R}^d} L(x, \frac{d\omega}{dt \otimes dm(t)}(t,x)) dm(t)(x) dt \\ &+ \int_0^T \int_{\mathbb{R}^d} \left[\frac{1}{\epsilon} \frac{\delta \Psi_h}{\delta m}(m_h(t), x) + \frac{\delta \mathcal{F}}{\delta m}(m_h(t), x) \right] dm(t)(x) dt \\ &+ \int_{\mathbb{R}^d} \left[\frac{1}{\delta} \frac{\delta \Psi_h}{\delta m}(m_h(T), x) + \frac{\delta \mathcal{G}}{\delta m}(m_h(T), x) \right] dm(T)(x), \end{aligned}$$

with, once again $J^l_{\epsilon,\delta,h}(\omega, m) = +\infty$ if ω is not absolutely continuous with respect to $m(t) \otimes dt$.

Indeed, take a candidate (m,ω) with finite cost, take $l \in (0,1)$ and define $(m_l, \omega_l) := (1-l)(m_h, \omega_h) + l(m,\omega)$. By minimality of (m_h, ω_h) we have, for all $l \in (0,1)$

$$\frac{1}{l} \left[J_{\epsilon,\delta,h}(m_h, \omega_h) + \int_0^T q(m_h(t), \widetilde{m}(t)) dt - J_{\epsilon,\delta,h}(m_l, \omega_l) - \int_0^T q(m_l(t), \widetilde{m}(t)) dt \right] \leqslant 0.$$

The statement is proved letting $l \to 0$ in the expression above and using, on the one hand, the convexity of $(m,\omega) \mapsto \int_0^T \int_{\mathbb{R}^d} L(x, \frac{d\omega}{dt \otimes dm(t)}(t,x)) dm(t)(x)$ and, on the other hand, the differentiability of the mean-field costs.

Now we are going to pass to the limit in the linearized problems when $h \to 0$.

On the one hand, being the family of functions $t \mapsto \gamma_h'(\Psi(m_h(t))$ bounded in $L^\infty([0,T])$, it converges –up to a sub-sequence– for the weak-$*$ topology $\sigma(L^\infty, L^1)$ of $L^\infty([0,T])$ to a function λ in $L^\infty([0,T])$. It is easily seen that λ satisfies (3.17). On the other hand the functions $t \mapsto \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(m_h(t), x) dm(t)(x)$ converge uniformly to $t \mapsto \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(\widetilde{m}(t), x) dm(t)(x)$ as h goes to 0. Therefore we can conclude that, up to a sub-sequence,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} \frac{\delta \Psi_h}{\delta m}(m_h(t), x) dm(t)(x) dt &= \int_0^T \gamma_h'(\Psi(m_h(t)) \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(m_h(t), x) dm(t)(x) dt \\ &\to \int_0^T \lambda(t) \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(\widetilde{m}(t), x) dm(t)(x) dt \end{aligned}$$

as h goes to 0. A similar statement holds for $\frac{1}{\delta} \int_{\mathbb{R}^d} \frac{\delta \Psi_h}{\delta m}(m_h(T), x) dm(T)(x)$ and we can conclude that, up to a sub-sequence, $J^l_{\epsilon,\delta,h}(m,\omega)$ converges to $J^l_{\epsilon,\delta}(m,\omega)$ for any relaxed candidate (m,ω), where $J^l_{\epsilon,\delta}$ is defined in the statement of the lemma for some λ, β satisfying the conditions (3.17) and (3.18). We deduce that $(\widetilde{m}, \omega')$ is an infimum of $J^l_{\epsilon,\delta}$. Notice that the term involving $\frac{\delta q}{\delta m_1}$ in (3.21) disappeared since $\frac{\delta q}{\delta m_1}(\widetilde{m}(t), \widetilde{m}(t), x) = 0$ for all $x \in \mathbb{R}^d$.

To conclude that $(\widetilde{m}, \widetilde{\omega})$ is a solution to the linearized problem, it suffices to notice that, $(\widetilde{m}, \widetilde{\omega})$ being a solution to the penalized problem it must hold that

$$\int_0^T \int_{\mathbb{R}^d} L(x, \frac{d\widetilde{\omega}}{dt \otimes d\widetilde{m}(t)}(t,x)) d\widetilde{m}(t)(x)dt \leqslant \int_0^T \int_{\mathbb{R}^d} L(x, \frac{d\omega'}{dt \otimes d\widetilde{m}(t)}(t,x)) d\widetilde{m}(t)(x)dt$$

(all the other terms in the $J_{\epsilon,\delta}$ only involve $\widetilde{m}$) and therefore $J^l_{\epsilon,\delta}(\widetilde{m}, \widetilde{\omega}) \leqslant J^l_{\epsilon,\delta}(\widetilde{m}, \omega')$. This concludes the proof of the lemma.

□

Before we can prove Theorem 3.5 we need the following duality formula.

Lemma 3.3. *Assume that $(m, \alpha) \in \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d)) \times L^2_{dt \otimes dm(t)}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ solves the Fokker-Planck equation (3.2) in the sense of distributions. Assume that $u \in \mathcal{C}([0,T], E_n)$ is a solution to the HJB equation (3.4) in the sense of Definition 3.4 with inputs $(f, g) \in L^1([0,T], E_n) \times E_{n+\alpha}$. Then, for all $t_1, t_2 \in [0,T]$ it holds*

$$\begin{aligned} \int_{\mathbb{R}^d} u(t_2, x) dm(t_2)(x) = &\int_{\mathbb{R}^d} u(t_1, x) dm(t_2)(x) - \int_{t_1}^{t_2} \int_{\mathbb{R}^d} f(t,x) dm(t)(x) dt \\ &+ \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left[H(x, Du(t,x)) + \alpha(t,x).Du(t,x) \right] dm(t)(x) dt. \end{aligned} \tag{3.22}$$

Proof. We take a sequence of functions $f_m \in \mathcal{C}([0,T], E_n)$ converging to f in $L^1([0,T], E_n)$ and we let u_m be the corresponding solutions to the HJB equation with data (f_m, g). Being f_m in $\mathcal{C}([0,T], E_n)$, it is straightforward from the definition of solution 3.4 that u_m is differentiable in time, $\partial_t u_m$ belongs to $L^\infty([0,T], E_{n-2})$ and the HJB equation is satisfied in the strong sense. The curve $m(t)$ being bounded in $\mathcal{P}_2(\mathbb{R}^d)$, an approximation argument similar to [129] Remark 2.3 shows that the integration by part formula (3.3) holds for u_m and therefore, we get

$$\begin{aligned} \int_{\mathbb{R}^d} u_m(t_2, x) dm(t_2)(x) - &\int_{\mathbb{R}^d} u_m(t_1, x) dm(t_1)(x) \\ &= \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left[\partial_t u_m(t,x) + \alpha(t,x).Du_m(t,x) + \Delta u_m(t,x) \right] dm(t)(x) dt \\ &= \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left[\alpha(t,x).Du_m(t,x) + H(x, Du_m(t,x)) - f_m(t,x) \right] dm(t)(x) dt \end{aligned}$$

where we used the equation satisfied by u_m at the last line. Now we can use the stability result of Theorem 3.1 to pass to the limit as $m \to +\infty$ and conclude the proof of the proposition. □

Finally we can conclude the proof of Theorem 3.5.

Proof of Theorem 3.5. We consider $\widetilde{u} \in \mathcal{C}([0,T], E_n)$ solution to

$$\begin{cases} -\partial_t \widetilde{u}(t,x) + H(x, D\widetilde{u}(t,x)) - \Delta \widetilde{u}(t,x) \\ \qquad\qquad = \dfrac{\lambda(t)}{\epsilon} \dfrac{\delta \Psi}{\delta m}(\widetilde{m}(t), x) + \dfrac{\delta \mathcal{F}}{\delta m}(\widetilde{m}(t), x) \qquad \text{in } (0,T) \times \mathbb{R}^d, \\ \widetilde{u}(T,x) = \dfrac{\beta}{\delta} \dfrac{\delta \Psi}{\delta m}(\widetilde{m}(T), x) + \dfrac{\delta \mathcal{G}}{\delta m}(\widetilde{m}(T), x) \qquad \text{in } \mathbb{R}^d, \end{cases} \tag{3.23}$$

—the existence of such a solution is guaranteed by Theorem 3.1— and we proceed by verification. We use Lemma 3.3 to get

$$\int_{\mathbb{R}^d} \widetilde{u}(0,x) dm_0(x) = -\int_0^T \int_{\mathbb{R}^d} \left[H(x, D\widetilde{u}(t,x)) + \frac{d\widetilde{\omega}}{dt \otimes d\widetilde{m}}(t,x).D\widetilde{u}(t,x) \right] d\widetilde{m}(t) dt.$$

Here we used the equation satisfied by $\widetilde{u}$ and the convention $\displaystyle\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m,x) dm(x) = 0$ for all $m \in \mathcal{P}_2(\mathbb{R}^d)$ and all $\mathcal{C}^1$ map U. But the inequality

$$-H(x, D\widetilde{u}(t,x)) - \frac{d\widetilde{\omega}}{dt \otimes d\widetilde{m}(t)}(t,x).D\widetilde{u}(t,x) \leqslant L(x, \frac{d\widetilde{\omega}}{dt \otimes d\widetilde{m}}(t,x))$$

holds, with equality if and only if

$$\frac{d\widetilde{\omega}}{dt \otimes d\widetilde{m}}(t,x) = -D_p H(x, D\widetilde{u}(t,x)).$$

Therefore,

$$\int_{\mathbb{R}^d} \widetilde{u}(0,x) dm_0(x) \leqslant J^l_{\epsilon,\delta}(\widetilde{m}, \widetilde{\omega})$$

with equality if and only if $\dfrac{d\widetilde{\omega}}{dt \otimes d\widetilde{m}}(t,x) = -D_p H(x, D\widetilde{u}(t,x))$, $dt \otimes \widetilde{m}(t)$-almost everywhere. Now if we consider $\widetilde{m}'$ solution to

$$\partial_t \widetilde{m}' - \operatorname{div}(D_p H(x, D\widetilde{u}(t,x)) \widetilde{m}') + \Delta \widetilde{m}' = 0$$

with $\widetilde{m}'(0) = m_0$, a similar computation shows that

$$\int_{\mathbb{R}^d} \widetilde{u}(0,x) dm_0(x) = J^l_{\epsilon,\delta}(-D_p H(x, D\widetilde{u}(t,x)) \widetilde{m}', \widetilde{m}')$$

which means that the cost $\displaystyle\int_{\mathbb{R}^d} \widetilde{u}(0,x) dm_0(x)$ can indeed be reached and, by minamility of $(\widetilde{\omega}, \widetilde{m})$ we get

$$\int_{\mathbb{R}^d} \widetilde{u}(0,x) dm_0(x) = \inf_{(\omega,m)} J^l_{\epsilon,\delta} \tag{3.24}$$

and

$$\widetilde{\omega} = -D_p H(x, D\widetilde{u}(t,x)) \widetilde{m}(t) \otimes dt.$$

Combining the Fokker-Planck equation in (3.14) where $\widetilde{\omega}$ is replaced by $-D_pH(x, D\widetilde{u}(t,x))\widetilde{m}(t)\otimes dt$ with the HJB equation (3.23) and recalling that λ and β satisfy the conditions of Lemma 3.2 concludes the proof of the theorem. □

3.4 From the penalized problems to the constrained one

The first goal of this section is to find estimates on the system of optimality conditions (3.11) which are independent from ϵ and δ. This is Section 3.4.1. Next we prove the regularity and find suitable expressions for the first two derivatives of the map $t \mapsto \Psi(m(t))$ when (m, α) is a solution to the penalized problem. This is Section 3.4.2. Finally we prove Theorems 3.2 and 3.3 in Section 3.4.3.

3.4.1 Uniform (in epsilon, delta) estimates

First we construct a candidate $(\overline{m}, \overline{\alpha})$ which stays uniformly inside the constraint at all time with a finite cost.

Lemma 3.4. *Provided $\Psi(m_0) < 0$, we can build a trajectory $(\overline{m}, \overline{\alpha})$ in $\mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d)) \times L^2_{dt\otimes m(t)}([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ such that $J(\overline{\alpha}, \overline{m}) < +\infty$ and $\Psi(\overline{m}(t)) \leqslant -\theta$ for all t in $[0,T]$, for some $\theta > 0$.*

Proof. First we introduce a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions and supporting a standard, adapted Brownian motion (B_t). Take $c > 0$ and consider a solution to the SDE

$$dX_t = -c(X_t - X_0)dt + \sqrt{2}dB_t$$

where X_0 is $\mathcal{F}_0$-measurable, with law m_0 and independent from the Brownian motion. A simple application of Itô's lemma proves that X_t can be rewritten as

$$X_t = X_0 + \sqrt{2}\int_0^t e^{-c(t-s)}dB_s \tag{3.25}$$

and therefore

$$\mathbb{E}\left[|X_t - X_0|^2\right] = 2\int_0^t e^{-2c(t-s)}ds = \frac{1}{c}(1 - e^{-2ct}).$$

Now let $\overline{m}(t)$ be the law of X_t. The above computation shows that

$$d_2^2(\overline{m}(t), m_0) \leqslant \frac{1}{c}$$

for all $t \in [0,T]$. With an abstract mimicking argument as in [25] we can find a measurable drift $\overline{\alpha} : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ such that

$$\partial_t\overline{m} + \operatorname{div}(\overline{\alpha}\overline{m}) - \Delta\overline{m} = 0$$

and

$$\int_0^T \int_{\mathbb{R}^d} |\overline{\alpha}(t,x)|^2 d\overline{m}(t)(x)dt \leqslant \mathbb{E}\left[\int_0^T c^2 \left|X_t - X_0\right|^2 dt\right] \leqslant cT.$$

However a direct computation, using Jensen's inequality, shows that it is enough to take, for all $(t,x) \in (0,T] \times \mathbb{R}^d$,

$$\overline{\alpha}(t,x) := \frac{c}{\overline{m}(t,x)} \int_{\mathbb{R}^d} (x-y) m^y(t,x) dm_0(y)$$

where $m^y(t)$ is the solution to

$$\begin{cases} \partial_t m^y - c\mathrm{div}((x-y)m^y) - \Delta m^y = 0 \\ m^y(0) = \delta_y. \end{cases}$$

Notice that X_0 being independent from the Brownian motion, we easily deduce from (3.25) that $\overline{m}(t,x) > 0$ for all $(t,x) \in (0,T] \times \mathbb{R}^d$.

Being Ψ Lipschitz continuous and $\Psi(m_0) < 0$ we can choose c large enough so that $\Psi(\overline{m}(t)) \leqslant \frac{\Psi(m_0)}{2}$ for all $t \in [0,T]$ and this concludes the proof of the lemma.

□

Using this particular candidate and the convexity of the constraint we can obtain the following estimate which is crucial to find compactness in the problem.

Although the notations do not make it clear, from now on (m,u,λ,β) will generally denote a solution to the optimality conditions (3.11) for the penalized problem $(P_{\epsilon,\delta})$ and therefore depend upon a particular (ϵ,δ).

Lemma 3.5. *There is a constant $M = M(\Psi(m_0)) > 0$ such that, for all $\epsilon, \delta > 0$ and for all tuple (u,m,λ,β) satisfying the conditions of Theorem 3.5 it holds*

$$\frac{1}{\epsilon}\int_0^T \lambda(t)dt + \frac{\beta}{\delta} \leqslant M.$$

Proof. By Lemma 3.4 we can build a solution of the Fokker-Planck equation $(\overline{\alpha}, \overline{m})$ such that $J(\overline{\alpha}, \overline{m}) < +\infty$ and, for all $t \in [0,T]$, $\Psi(\overline{m}(t)) \leqslant -\theta$ for some $\theta > 0$ independent of t. Using the fact that $(\overline{m}, \overline{\alpha})$ solves the Fokker-Planck equation, we can apply Lemma 3.3 to get

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} \left[\overline{\alpha}(t,x).Du(t,x) + H(x,Du(t,x)) - \frac{\lambda(t)}{\epsilon}\frac{\delta\Psi}{\delta m}(m(t),x) - \frac{\delta\mathcal{F}}{\delta m}(m(t),x)\right] d\overline{m}(t)(x)dt \\ = \int_{\mathbb{R}^d} \left[\frac{\beta}{\delta}\frac{\delta\Psi}{\delta m}(m(T),x) + \frac{\delta\mathcal{G}}{\delta m}(m(T),x)\right] d\overline{m}(T)(x) - \int_{\mathbb{R}^d} u(0,x)dm_0(x). \end{aligned}$$

Now, reorganizing the terms and using the fact that, by definition of L, we have for all (t,x) in $[0,T] \times \mathbb{R}^d$

$$\overline{\alpha}(t,x).Du(t,x) + H(x, Du(t,x)) \geqslant -L(x, \overline{\alpha}(t,x)),$$

we get

$$\begin{aligned} -\int_0^T \int_{\mathbb{R}^d} \frac{\lambda(t)}{\epsilon} \frac{\delta\Psi}{\delta m}(m(t),x) d\overline{m}(t)(x)dt - \int_{\mathbb{R}^d} \frac{\beta}{\delta} \frac{\delta\Psi}{\delta m}(m(T),x) d\overline{m}(T)(x) \\ \leqslant \int_0^T \int_{\mathbb{R}^d} \left[L(x, \overline{\alpha}(t,x)) + \frac{\delta\mathcal{F}}{\delta m}(m(t),x) \right] d\overline{m}(t)(x)dt \\ + \int_{\mathbb{R}^d} \frac{\delta\mathcal{G}}{\delta m}(m(T),x) d\overline{m}(T)(x) - \int_{\mathbb{R}^d} u(0,x) dm_0(x). \end{aligned} \tag{3.26}$$

On the one hand -using (3.24) in the proof of Theorem 3.5 and the notations therein- we have that $\int_{\mathbb{R}^d} u(0,x)dm_0(x) = J_l^{\epsilon,\delta}(\widetilde{m},\widetilde{\omega})$. But the linearized costs cancel when applied to $(\widetilde{m},\widetilde{\omega})$ and therefore $J_l^{\epsilon,\delta}(\widetilde{m},\widetilde{\omega}) = J(\widetilde{m},\widetilde{\omega})$. And since L, $\mathcal{F}$ and $\mathcal{G}$ are bounded from below we get a lower bound on $\int_{\mathbb{R}^d} u(0,x)dm_0(x)$ independent of ϵ and δ. The other terms in the right-hand side of (3.26) are also bounded from above since $J(\overline{\alpha},\overline{m}) < +\infty$ and since $x \mapsto \frac{\delta\mathcal{F}}{\delta m}(m,x)$ and $x \mapsto \frac{\delta\mathcal{G}}{\delta m}(m,x)$ are bounded in E_n with bounds uniform in m and $\overline{m}(t)$ belongs to $\mathcal{P}_2(\mathbb{R}^d)$ for all $t \in [0,T]$. On the other hand, by convexity of Ψ we get for all $t \in [0,T]$,

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{\delta\Psi}{\delta m}(m(t),x) d\overline{m}(t)(x) &\leqslant \Psi(\overline{m}(t)) - \Psi(m(t)) \\ &\leqslant -\theta - \Psi(m(t)) \end{aligned}$$

and by definition of λ and β we have $\lambda(t)\Psi(m(t)) \geqslant 0$ for all $t \in [0,T]$ and $\beta\Psi(m(T)) \geqslant 0$ and thus, if $C > 0$ is an upper bound for the right-hand side of (3.26) we get

$$\int_0^T \frac{\lambda(t)}{\epsilon} dt + \frac{\beta}{\delta} \leqslant \frac{C}{\theta},$$

which concludes the proof of the Lemma.

□

Remark 3.6. Notice that this estimate, together with the construction of Lemma (3.4) are the only steps which require the convexity of Ψ, Assumption (APsiConv) as well as the condition that $\Psi(m_0)$ must be strictly negative, Assumption (APsiInside).

We can combine this Lemma with Theorem 3.1 to find uniform in ϵ, δ estimates for the system of Optimality Conditions (3.11).

Proposition 3.4. *There is some $C > 0$ such that, for any $\epsilon, \delta > 0$ and any solution (m, u, λ, β) of* (3.11) *satisfying* (3.12) *and* (3.13) *it holds*

$$\sup_{t\in[0,T]} \|u(t)\|_n \leqslant C.$$

At this stage, the above estimates would be sufficient to pass to the limit when ϵ and δ go to zero in the penalized problem $(P_{\epsilon,\delta})$. We would find, at the limit, solutions of the constrained problem (P) and passing to the limit in the optimality conditions we would find that the solutions to the constrained problem satisfy similar conditions with $\frac{\lambda}{\epsilon}$ replaced by a non-negative Radon measure $\nu \in \mathcal{M}^+([0,T])$. This would lead to a priori discontinuous (in time) optimal controls. However, we refrain from following such approach for now. Instead we are going to exhibit a special behavior of the optimal solutions of the penalized problem. Indeed we are going to show in the next section that solutions of the penalized problem stay inside the constraint when the penalization is strong enough. Consequently it is sufficient to take ϵ and δ small to get solutions to the constrained problem and optimal controls for the constrained problem are still continuous.

3.4.2 Second order analysis

The special behavior (described just above) of the solutions will be a simple consequence of the fact that we cannot have simultaneously $\Psi(m(t)) > 0$ and $\frac{d^2}{dt^2}\Psi(m(t)) \leqslant 0$ (here m is a solution to $(P_{\epsilon,\delta})$) when the penalization is strong enough. The purpose of this section is to prove the regularity and a suitable expansion of the map $t \mapsto \Psi(m(t))$.

Recall that $\mathbb{S}_d(\mathbb{R})$ is the space of symmetric matrices of size d.

Proposition 3.5. *Suppose that (m, u, λ, β) is a solution of (3.11) for some $\epsilon, \delta > 0$. Then the map $t \mapsto \Psi(m(t))$ is $\mathcal{C}^1$ in $[0,T]$ and $\mathcal{C}^2$ in $[0,T] \bigcap \{t : \Psi(m(t)) \neq 0\}$ with derivatives given by*

$$\frac{d}{dt}\Psi(m(t)) = -\int_{\mathbb{R}^d} D_m\Psi(m(t),x).D_pH(x,Du(t,x))dm(t)(x) + \int_{\mathbb{R}^d} \mathrm{div}_x D_m\Psi(m(t),x)dm(t)(x)$$

and

$$\frac{d^2}{dt^2}\Psi(m(t)) = \frac{\lambda(t)}{\epsilon}\int_{\mathbb{R}^d} D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))D_m\Psi(m(t),x)dm(t)(x) + F(Du(t), D^2u(t), D\Delta u(t), m(t))$$

for some functional $F : \mathcal{C}_b(\mathbb{R}^d,\mathbb{R}^d) \times \mathcal{C}_b(\mathbb{R}^d,\mathbb{S}^d(\mathbb{R})) \times \mathcal{C}_b(\mathbb{R}^d,\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ independent of ϵ and δ and bounded in sets of the form $\mathcal{A} \times \mathcal{P}_2(\mathbb{R}^d)$ for bounded subsets $\mathcal{A}$ of $\mathcal{C}_b(\mathbb{R}^d,\mathbb{R}^d) \times \mathcal{C}_b(\mathbb{R}^d,\mathbb{S}^d(\mathbb{R})) \times \mathcal{C}_b(\mathbb{R}^d,\mathbb{R}^d)$.

Proof. Since Ψ is supposed to satisfy Assumption (Ureg), we can use Proposition 3.3 and, for all $t \in [0,T]$ we get

$$\Psi(m(t)) = \Psi(m_0) - \int_0^t\int_{\mathbb{R}^d} D_m\Psi(m(s),x).D_pH(x,Du(s,x))dm(s)(x)ds + \int_0^t\int_{\mathbb{R}^d} \mathrm{div}_x D_m\Psi(m(s),x)dm(s)(x)ds.$$

Being u in $\mathcal{C}([0,T],E_n)$ and m in $\mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R}^d))$ we get that $t \mapsto \Psi(m(t))$ is $\mathcal{C}^1$ with

$$\begin{aligned}\frac{d}{dt}\Psi(m(t)) = &-\int_{\mathbb{R}^d} D_m\Psi(m(t),x).D_pH(x,Du(t,x))dm(t)(x)\\ &+\int_{\mathbb{R}^d} \mathrm{div}_x D_m\Psi(m(t),x)dm(t)(x).\end{aligned}$$

Now we assume that $\Psi(m(t)) \neq 0$. We denote by $l(t,x)$ the integrand

$$l(t,x) := -D_m\Psi(m(t),x).D_pH(x,Du(t,x)) + \mathrm{div}_x D_m\Psi(m(t),x)$$

The parameter λ is constant (equal to 0 or 1) in a neighborhood (t_1,t_2) of t because of the exclusion condition (3.12) and u solves the HJB equation according to Definition 3.1 so we have that u belongs to $\mathcal{C}^{1,2}((t_1,t_2)\times\mathbb{R}^d)$. Moreover,

$$\partial_t u(t,x) = H(x,Du(t,x)) - \Delta u(t,x) - \frac{\lambda(t)}{\epsilon}\frac{\delta\Psi}{\delta m}(m(t),x) - \frac{\delta\mathcal{F}}{\delta m}(m(t),x)$$

and u belongs to $\mathcal{C}([0,T],E_n)$ with $n \geqslant 3$. This means that $\partial_t u$ is differentiable with respect to x with

$$\begin{aligned}-\partial_t Du(t,x) + D_xH(x,Du(t,x)) + D^2u(t,x)D_pH(x,Du(t,x)) - D\Delta u(t,x)\\ = \frac{\lambda(t)}{\epsilon}D_m\Psi(m(t),x) + D_m\mathcal{F}(m(t),x).\end{aligned}$$

But m solves the Fokker-Planck equation, Ψ satisfies Assumptions (Ureg) and (APsiC2) so we can apply Proposition 3.3 to $D_m\Psi(m(t),x)$ and $\mathrm{div}_x D_m\Psi(m(t),x)$ and deduce that l belongs to $\mathcal{C}_b^{1,2}((t_1,t_2)\times\mathbb{R}^d)$ and therefore $t \mapsto \frac{d}{dt}\Psi(m(t))$ is differentiable at t with

$$\frac{d^2}{dt^2}\Psi(m(t)) = \int_{\mathbb{R}^d}\left[\partial_t l(t,x) - D_pH(x,Du(t,x)).Dl(t,x) + \Delta l(t,x)\right]dm(t)(x).$$

Computing $\partial_t l$ leads to

$$\begin{aligned}\partial_t l(t,x) = &-\frac{d}{dt}D_m\Psi(m(t),x).D_pH(x,Du(t,x)) + \frac{d}{dt}\mathrm{div}_x D_m\Psi(m(t),x)\\ &- D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))\partial_t Du(t,x)\\ &= -\frac{d}{dt}D_m\Psi(m(t),x).D_pH(x,Du(t,x)) + \frac{d}{dt}\mathrm{div}D_m\Psi(m(t),x)\\ &- D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))D^2u(t,x)D_pH(x,Du(t,x))\\ &- D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))D_xH(x,Du(t,x))\\ &+ D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))D\Delta u(t,x)\\ &+ \frac{\lambda(t)}{\epsilon}D_m\Psi(m(t),x)D^2_{pp}H(x,Du(t,x)).D_m\Psi(m(t),x)\\ &+ D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))D_m\mathcal{F}(m(t),x),\end{aligned}$$

and therefore

$$\frac{d^2}{dt^2}\Psi(m(t)) = \frac{\lambda(t)}{\epsilon}\int_{\mathbb{R}^d} D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))D_m\Psi(m(t),x)dm(t)(x) + F(Du(t),D^2u(t),D\Delta u(t),m(t))$$

with

$$\begin{aligned}F(Du(t),D^2u(t),D\Delta u(t),m(t)) = &\int_{\mathbb{R}^d}\left[-D_pH(x,Du(t,x)).Dl(t,x)+\Delta l(t,x)\right]dm(t)(x)\\ &-\int_{\mathbb{R}^d}\frac{d}{dt}D_m\Psi(m(t),x).D_pH(x,Du(t,x))dm(t)(x)\\ &+\int_{\mathbb{R}^d}\frac{d}{dt}\mathrm{div}_xD_m\Psi(m(t),x)dm(t)(x)\\ &-\int_{\mathbb{R}^d}D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))D^2u(t,x)D_pH(x,Du(t,x))dm(t)(x)\\ &-\int_{\mathbb{R}^d}D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))D_xH(x,Du(t,x))dm(t)(x)\\ &+\int_{\mathbb{R}^d}D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))D\Delta u(t,x)dm(t)(x)\\ &+\int_{\mathbb{R}^d}D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))D_m\mathcal{F}(m(t),x)dm(t)(x).\end{aligned}$$

□

Remark 3.7. An explicit formula for Dl, Δl or F is not necessary for our purpose however a tedious but straightforward computation leads to

$$
\begin{aligned}
\frac{d^2}{dt^2}\Psi(m(t)) = \frac{\lambda(t)}{\epsilon}\int_{\mathbb{R}^d} & D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))D_m\Psi(m(t),x)dm(t)(x)\\
&+\int_{\mathbb{R}^d}\Delta_x\mathrm{div}_x D_m\Psi(x,m(t))dm(t)(x)\\
&+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathrm{div}_x\mathrm{div}_y D^2_{mm}\Psi(m(t),x,y)dm(t)(x)dm(t)(y)\\
&-2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathrm{div}_y D^2_{mm}\Psi(m(t),x,y).D_pH(x,Du(t,x))dm(t)(x)dm(t)(y)\\
&-2\int_{\mathbb{R}^d}\overrightarrow{\Delta}_x D_m\Psi(m(t),x).D_pH(x,Du(t,x))dm(t)(x)\\
&+\int_{\mathbb{R}^d} D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))D_m\mathcal{F}(m(t),x)dm(t)(x)\\
&+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D^2_{mm}\Psi(m(t),x,y)D_pH(x,Du(t,x).D_pH(y,Du(t,y))dm(t)(x)dm(t)(y)\\
&+\int_{\mathbb{R}^d} D_xD_m\Psi(m(t),x)D_pH(x,Du(t,x)).D_pH(x,Du(t,x))dm(t)(x)\\
&-2\int_{\mathbb{R}^d} D_xD_m\Psi(m(t),x).D^2u(t,x)D^2_{pp}H(x,Du(t,x))dm(t)(x)\\
&-\sum_{i=1}^n\int_{\mathbb{R}^d} D_{x_i}\frac{\delta\Psi}{\delta m}(m(t),x)D^2u(t,x).D^2u(t,x)D^2_{pp}\partial_{p_i}H(x,Du(t,x))dm(t)(x)\\
&-\int_{\mathbb{R}^d} D_m\Psi(m(t),x).\overrightarrow{\Delta}_x D_pH(x,Du(t,x))dm(t)(x)\\
&-2\int_{\mathbb{R}^d} D_xD_m\Psi(m(t),x)D^2_{xp}H(x,Du(t,x))dm(t)(x)\\
&-\int_{\mathbb{R}^d} D_m\Psi(m(t),x).D^2_{pp}H(x,Du(t,x))D_xH(x,Du(t,x))dm(t)(x)\\
&+\int_{\mathbb{R}^d} D^2_{xp}H(x,Du(t,x))D_m\Psi(m(t),x).D_pH(x,Du(t,x))dm(t)(x)\\
&-2\sum_{i=1}^n\int_{\mathbb{R}^d}\partial_{x_i}\frac{\delta\Psi}{\delta m}(m(t),x)D^2_{xp}\partial_{p_i}H(x,Du(t,x)).D^2u(t,x)dm(t)(x).
\end{aligned}
$$

The formula above shows in particular that the terms in $D\Delta u$ cancel out and thus F depends only on the derivatives of u up to order two.

3.4.3 Proof of the main theorems

Proposition 3.6. *There is some $\epsilon_0, \delta_0 > 0$ such that any solution (m, α) of Problem $(P_{\epsilon,\delta})$ for some $(\epsilon, \delta) \in (0, \epsilon_0] \times (0, \delta_0]$ stays inside the constraint at all time:*

$$\forall t \in [0,T], \quad \Psi(m(t)) \leqslant 0.$$

Proof. The proof follows closely the methodology of [31] Lemma 3.7. Toward a contradiction we suppose that there exist a sequence $(\epsilon_k, \delta_k)_{k\in\mathbb{N}} \in ((0,1)\times(0,1))^{\mathbb{N}}$ converging to $(0,0)$, corresponding solutions $(m_k, -D_pH(x, Du_k(t,x)))_{k\in\mathbb{N}}$ satisfying the conditions of Theorem 3.5 with corresponding multipliers (λ_k, β_k) and times $(t_k)_{k\in\mathbb{N}} \in (0,T]$ which are local maximum points of $t \mapsto \Psi(m_k(t))$ and such that $\Psi(m_k(t_k)) > 0$. The couples (m_k, ω_k) are uniformly bounded in $\mathcal{C}^{1/2}([0,T], \mathcal{P}_2(\mathbb{R}^d)) \times \mathcal{M}([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ and we can assume that they converge in $\mathcal{C}^{1/2-\delta}([0,T], \mathcal{P}_{2-\delta}(\mathbb{R}^d)) \times \mathcal{M}([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$, for some $\delta \in (0,1)$, toward some solution $(\widetilde{m}, \widetilde{\omega})$ to the constrained problem. In particular, $\Psi(\widetilde{m}(t)) \leqslant 0$ for all $t \in [0,T]$.

We first notice that, thanks to Proposition 3.4, for large enough k, $\beta_k < 1$ and therefore $\Psi(m_k(T)) \leqslant 0$ and $t_k \neq T$.

Using Proposition 3.5 yields that $t \mapsto \Psi(m_k(t))$ is C^2 in a neighborhood of t_k and,

$$\begin{aligned}
\frac{d^2}{dt^2}\Psi(m_k(t))|_{t=t_k} &= \frac{1}{\epsilon_k}\int_{\mathbb{R}^d} D_m\Psi(m_k(t_k),x).D^2_{pp}H(x, Du_k(t_k,x))D_m\Psi(m_k(t_k),x)dm_k(t_k)(x)\\
&\quad + F(Du_k(t_k), D^2u_k(t_k), D\Delta u_k(t_k), m_k(t_k))\\
&\geqslant \frac{1}{\mu\epsilon_k}\int_{\mathbb{R}^d} |D_m\Psi(m_k(t_k),x)|^2 dm_k(t_k)(x)\\
&\quad + F(Du_k(t_k), D^2u_k(t_k), D\Delta u_k(t_k), m_k(t_k)),
\end{aligned}$$

where we used the convexity assumption (AH6) on the Hamiltonian H. On the one hand, using the estimates of Proposition 3.4 we have that $F(Du_k(t), D^2u_k(t), D\Delta u_k(t), m_k(t))$ is bounded independently from k. On the other hand, using the regularity assumption (APsiC2) and up to taking a subsequence we can assume that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^d} |D_m\Psi(m_k(t_k),x)|^2 dm_k(t_k)(x) = \int_{\mathbb{R}^d} |D_m\Psi(\widetilde{m}(\widetilde{t}),x)|^2 d\widetilde{m}(\widetilde{t})(x)$$

for some $\widetilde{t} \in [0,T]$ such that $\Psi(\widetilde{m}(\widetilde{t})) = 0$. This is where Assumption (APsiTrans) comes into play. Since $\Psi(\widetilde{m}(\widetilde{t})) = 0$, we have that

$$\int_{\mathbb{R}^d} |D_m\Psi(\widetilde{m}(\widetilde{t}),x)|^2 d\widetilde{m}(\widetilde{t})(x) > 0,$$

and we deduce that, $\dfrac{d^2}{dt^2}\Psi(m_k(t))|_{t=t_k} > 0$ for k large enough. This leads to a contradiction since t_k is assumed to be a local maximum point of $t \to \Psi(m_k(t))$. □

Theorem 3.2 is a direct consequence of the above proposition.

Proof of Theorem 3.2. Denote by $\overline{\mathcal{U}}_{\epsilon,\delta}$ the value of Problem $(P_{\epsilon,\delta})$ and by $\overline{\mathcal{U}}$ the value of the constrained problem (P). We assume that (ϵ,δ) belongs to $(0,\epsilon_0)\times(0,\delta_0)$ with (ϵ_0,δ_0) the parameters from Proposition 3.6.

We have that $\overline{\mathcal{U}}_{\epsilon,\delta} = \overline{\mathcal{U}}$ and the minimizers for problems $(P_{\epsilon,\delta})$ and (P) coincide.

Indeed, it is straightforward that $\overline{\mathcal{U}}_{\epsilon,\delta} \leqslant \overline{\mathcal{U}}$. Now if (m_1,α_1) is a solution to Problem $(P_{\epsilon,\delta})$, by Proposition 3.6, (m_1,α_1) is admissible for Problem (P). This means that $\overline{\mathcal{U}}_{\epsilon,\delta} = J_{\epsilon,\delta}(m_1,\alpha_1) = J(m_1,\alpha_1) \geqslant \overline{\mathcal{U}}$ and, therefore $\overline{\mathcal{U}}_{\epsilon,\delta} = \overline{\mathcal{U}}$ and (m_1,α_1) is a solution to (P).

Conversely, if (m_2, α_2) is a solution to (P) then $J_{\epsilon,\delta}(m_2, \alpha_2) = J(m_2, \alpha_2) = \overline{\mathcal{U}} = \overline{\mathcal{U}}_{\epsilon,\delta}$ and (m_2, α_2) is a solution to $(P_{\epsilon,\delta})$.

Looking carefully at the proof of Proposition 3.6, using Theorem (3.1) with the estimates given by Proposition 3.4 and Lemma 3.4 we see that the threshold (ϵ_0, δ_0) depends on m_0 only through the value $\Psi(m_0)$.

□

Now we are finally able to conclude the proof of Theorem 3.3.

Proof of Theorem 3.3. We use Theorem 3.2 and the optimality conditions for the penalized problem: If (m, α) is any solution to Problem (P), we can find $(\epsilon, \delta) \in (0, \epsilon_0) \times (0, \delta_0)$, $\lambda \in L^\infty([0,T])$, $\beta \geqslant 0$, $u \in \mathcal{C}([0,T], \mathcal{C}_b^n(\mathbb{R}^d))$ such that $\alpha(t,x) = -D_p H(x, Du(t,x))$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and (m, u, λ, β) satisfies the conditions of Theorem 3.5. Taking $\nu(t) := \frac{\lambda(t)}{\epsilon}$ and $\eta := \frac{\beta}{\delta}$ concludes the proof of the first part of the theorem.

Now, if we suppose that $\mathcal{F}$ and $\mathcal{G}$ are convex in the measure variable we can proceed as in Chapter 2 Section 2.3.3 and easily show that the conditions are sufficient. □

3.5 The general case

The goal of this section is to prove Theorem 3.4 . We first need to extend the results of Theorem 3.1 to HJB equations with right hand-side of the form $\nu\psi_1 + \varphi_1$ where ν belongs to $\mathcal{M}^+([0,T])$ and ψ_1, φ_1 belong to $\mathcal{C}([0,T], E_n)$.

3.5.1 The HJB equation

Definition 3.2. Suppose that $n \geqslant 3$. Let ψ_1, φ_1 be in $C([0,T], E_n)$ and ψ_2 be in $E_{n+\alpha}$. Let also ν be in $\mathcal{M}^+([0,T])$. We say that $u \in L^1([0,T], E_n)$ is a solution to

$$\begin{cases} -\partial_t u + H(x, Du) - \Delta u = \nu\psi_1 + \varphi_1, & \text{in } [0,T] \times \mathbb{R}^d \\ u(T,x) = \psi_2, & \text{in } \mathbb{R}^d, \end{cases} \tag{3.27}$$

if, for almost all $t \in [0,T]$, for all $x \in \mathbb{R}^d$,

$$\begin{aligned} u(t,x) = P_{T-t}\psi_2(x) + \int_0^T \mathbb{1}_{(t,T]} P_{s-t}\psi_1(s)(x) d\nu(s) + \int_t^T P_{s-t}\varphi_1(s)(x) ds \\ - \int_t^T P_{s-t}\left[H(., Du(s,.))\right](x) ds. \end{aligned}$$

Theorem 3.6. *Suppose that $n \geqslant 3$. Let ψ_1, φ_1 be in $C([0,T], E_n)$ and ψ_2 be in E_n. Let also ν be in $\mathcal{M}^+([0,T])$. Under these conditions, there is a unique solution $u \in L^\infty([0,T], E_n)$ to (3.27) in the sense of Definition 3.2. Moreover it satisfies*

$$\text{essup}_{t\in[0,T]} \|u(t)\|_n \leqslant C(|\nu|, \sup_{t\in[0,T]} \|\psi_1(t)\|_n, \sup_{t\in[0,T]} \|\varphi_1(t)\|_n, \|\psi_2\|_n),$$

where $|\nu|$ is the total variation norm of ν.

In the following proposition, we abuse notations and we let $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be the unique element in its equivalence class of $L^\infty([0,T], E_n)$ satisfying, for all $(t,x) \in [0,T] \times \mathbb{R}^d$

$$u(t,x) = P_{T-t}\psi_2(x) + \int_0^T \mathbb{1}_{(t,T]}(s) P_{s-t}\psi_1(s)(x) d\nu(s) + \int_t^T P_{s-t}\varphi_1(s)(x) ds - \int_t^T P_{s-t}\left[H(.,Du(s,.))\right](x) ds.$$

We have the following stability result.

Proposition 3.7. *Assume that $(\nu_m)_{m\geqslant 1} \in L^\infty([0,T])$ converges in $\mathcal{M}^+([0,T])$ toward ν. Let $u_m \in \mathcal{C}([0,T], E_n)$ be the solution to the HJB equation (3.27) with data $(\nu_m, \psi_1, \varphi_1, \psi_2)$ with $\psi_1, \varphi_1 \in \mathcal{C}([0,T], E_n)$ and $\psi_2 \in E_{n+\alpha}$. Then, for all $(t,x) \in [0,T] \times \mathbb{R}^d$ such that $\nu(\{t\}) = 0$, it holds:*

$$\lim_{m\to+\infty} u_m(t,x) = u(t,x),$$
$$\lim_{m\to+\infty} Du_m(t,x) = Du(t,x).$$

Once again, these technical results are postponed to Section 3.6.2.

3.5.2 Optimality conditions in the general case

We first prove a lemma similar to Lemma 3.2.

Lemma 3.6. *Let $(\widetilde{m}, \widetilde{\omega})$ be a solution to the constrained Problem (P). Then there exist $\nu \in \mathcal{M}^+([0,T])$ and $\eta \in \mathbb{R}^+$ satisfying*

$$\Psi(\widetilde{m}(t)) = 0, \nu - ae \qquad (3.28) \qquad\qquad \eta\Psi(\widetilde{m}(T)) = 0, \qquad (3.29)$$

and such that $(\widetilde{m}, \widetilde{\omega})$ minimizes

$$J^l(\omega, m) := \int_0^T \int_{\mathbb{R}^d} L(x, \frac{d\omega}{dt \otimes dm(t)}(t,x)) dm(t)(x) dt + \int_0^T \int_{\mathbb{R}^d} \frac{\delta\Psi}{\delta m}(\widetilde{m}(t), x) dm(t)(x) d\nu(t) + \int_0^T \int_{\mathbb{R}^d} \frac{\delta\mathcal{F}}{\delta m}(\widetilde{m}(t), x) dm(t)(x) dt + \int_{\mathbb{R}^d} \left[\eta \frac{\delta\Psi}{\delta m}(\widetilde{m}(T), x) + \frac{\delta\mathcal{G}}{\delta m}(\widetilde{m}(T), x)\right] dm(T)(x), \qquad (3.30)$$

over the pairs (m, ω) satisfying (3.14) and where we set, $J^l(m,\omega) = +\infty$ if ω is not absolutely continuous with respect to $dt \otimes m(t)$.

Proof. We take $\epsilon, \delta > 0$ and $(m^{\epsilon,\delta}, \omega^{\epsilon,\delta})$ solutions to the penalized problems $P_{\epsilon,\delta}$. As $\epsilon, \delta \to 0$, $(m^{\epsilon,\delta}, \omega^{\epsilon,\delta})$ converges in $\mathcal{C}([0,T], \mathcal{P}_{2-\delta}(\mathbb{R}^d)) \times \mathcal{M}([0,T] \times \mathbb{R}^d, \times\mathbb{R}^d)$ for $\delta \in (0,1)$ to a solution to the constrained problem that we can assume, without loss of generality, to be $(\widetilde{m}, \widetilde{\omega})$. Now

$(m^{\epsilon,\delta}, \omega^{\epsilon,\delta})$ is also a solutions to the linearized problems of Lemma 3.2 for some $\lambda^{\epsilon,\delta}, \beta^{\epsilon,\delta} \in L^\infty([0,T]) \times \mathbb{R}^+$ satisfying the exclusion conditions

$$\lambda^{\epsilon,\delta}(t) \begin{cases} = 0 & \text{if } \Psi(m^{\epsilon,\delta}(t)) < 0 \\ \in [0,1] & \text{if } \Psi(m^{\epsilon,\delta}(t)) = 0 \\ = 1 & \text{if } \Psi(m^{\epsilon,\delta}(t)) > 0, \end{cases} \qquad \beta^{\epsilon,\delta} \begin{cases} = 0 & \text{if } \Psi(m^{\epsilon,\delta}(T)) < 0 \\ \in [0,1] & \text{if } \Psi(m^{\epsilon,\delta}(T)) = 0 \\ = 1 & \text{if } \Psi(m^{\epsilon,\delta}(T)) > 0. \end{cases}$$

Using the controllability lemma 3.4 and arguing as in Lemma 3.5 we can infer that $\frac{\lambda^{\epsilon,\delta}}{\epsilon}$ is bounded in $L^1([0,T])$ independently from $(\epsilon,\delta) > 0$ and $\frac{\beta^{\epsilon,\delta}}{\delta}$ is also bounded in $\mathbb{R}^+$. Let us take $\nu \in \mathcal{M}^+([0,T])$ to be a limit point of $\frac{\lambda^{\epsilon,\delta}}{\epsilon}$ and η a limit point of $\frac{\beta^{\epsilon,\delta}}{\delta}$ as $\epsilon, \delta \to 0$. It is plain to check that $\Psi(\widetilde{m}(t)) = 0$ for ν-almost all $t \in [0,T]$ and $\eta\Psi(\widetilde{m}(T)) = 0$. Now we can argue as in the proof of Lemma 3.2, passing to the limit in the linearized problems to conclude that $(\widetilde{m}, \widetilde{\omega})$ is indeed a minimum of (3.30). □

We now take $u \in L^\infty([0,T], E_n)$ to be the solution to

$$\begin{cases} -\partial_t u + H(x, Du) - \Delta u = \nu \dfrac{\delta\Psi}{\delta m}(\widetilde{m}(t), x) + \dfrac{\delta\mathcal{F}}{\delta m}(\widetilde{m}(t), x) & \text{in } [0,T] \times \mathbb{R}^d, \\ u(T,x) = \eta \dfrac{\delta\Psi}{\delta m}(\widetilde{m}(T), x) + \dfrac{\delta\mathcal{G}}{\delta m}(\widetilde{m}(T), x) & \text{in } \mathbb{R}^d. \end{cases} \tag{3.31}$$

We also assume that u is defined for all $(t,x) \in [0,T] \times \mathbb{R}^d$ (and not just dt-almost everywhere) by

$$\begin{aligned} u(t,x) = \eta P_{T-t} \frac{\delta\Psi}{\delta m}(\widetilde{m}(T))(x) + P_{T-t} \frac{\delta\mathcal{G}}{\delta m}(\widetilde{m}(T))(x) + \int_0^T \mathbb{1}_{(t,T]}(s) P_{s-t} \frac{\delta\Psi}{\delta m}(\widetilde{m}(s))(x) d\nu(s) \\ + \int_t^T P_{s-t} \frac{\delta\mathcal{F}}{\delta m}(\widetilde{m}(s))(x) ds - \int_t^T P_{s-t} \left[H(., Du(s,.)) \right](x) ds. \end{aligned} \tag{3.32}$$

Using an approximation argument and Proposition 3.7, we have the following duality relation:

Proposition 3.8. *Let $u \in L^\infty([0,T], E_n)$ be a solution to (3.31) satisfying (3.32) for all $(t,x) \in [0,T] \times \mathbb{R}^d$. Let also $(m,\alpha) \in \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d)) \times L^2_{dt \otimes dm(t)}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ be a solution in the sense of distributions to*

$$\begin{cases} \partial_t m + \operatorname{div}(\alpha m) - \Delta m = 0, & \text{in } (0,T) \times \mathbb{R}^d, \\ m(0) = m_0. \end{cases}$$

Then the following duality formula holds for any $t_1 \in [0,T]$ such that $\nu(\{t_1\}) = 0$,

$$\begin{aligned} \int_{\mathbb{R}^d} u(t_1,x) dm(t_1)(x) = \eta \int_{\mathbb{R}^d} \frac{\delta\Psi}{\delta m}(\widetilde{m}(T), x) dm(T)(x) + \int_{\mathbb{R}^d} \frac{\delta\mathcal{G}}{\delta m}(\widetilde{m}(T), x) dm(T)(x) \\ - \int_{t_1}^T \int_{\mathbb{R}^d} \left[H(x, Du(t,x)) + \alpha(t,x).Du(t,x) \right] dm(t)(x) dt \\ + \int_{t_1}^T \int_{\mathbb{R}^d} \frac{\delta\Psi}{\delta m}(\widetilde{m}(t), x) dm(t)(x) d\nu(t) + \int_{t_1}^T \int_{\mathbb{R}^d} \frac{\delta\mathcal{F}}{\delta m}(\widetilde{m}(t), x) dm(t)(x) dt. \end{aligned} \tag{3.33}$$

We can conclude with the proof of Theorem 3.4.

Proof of Theorem 3.4. We proceed similarly to the proof of Theorem 3.5. Take $(\widetilde{m}, \widetilde{\omega})$ a relaxed solution to the constrained problem P. Let also $u \in L^\infty([0,T], E_n)$ be the solution to (3.31) satisfying (3.32) with ν and η satisfying respectively (3.28) and (3.29).

Recall that the linearized cost J^l is defined in Lemma 3.6. On the one hand, by definition of L, it holds that

$$\begin{aligned} J^l(\widetilde{m}, \widetilde{\omega}) &= \int_0^T \int_{\mathbb{R}^d} L(x, \frac{d\widetilde{\omega}}{dt \otimes d\widetilde{m}(t)}(t,x)) d\widetilde{m}(t)(x) dt \\ &\geqslant -\int_0^T \int_{\mathbb{R}^d} \left[\frac{d\widetilde{\omega}}{dt \otimes d\widetilde{m}(t)}(t,x).Du(t,x) + H(x, Du(t,x)) \right] d\widetilde{m}(t)(x) dt \end{aligned}$$

with equality if and only if

$$\frac{d\widetilde{\omega}}{dt \otimes d\widetilde{m}(t)} = -D_p H(x, Du), dt \otimes d\widetilde{m}(t) - ae. \tag{3.34}$$

Being $\Psi(m_0) < 0$, it holds that $\nu(\{0\}) = 0$ because of the exclusion condition (3.28) and we can use the duality relation (3.33) with $t_1 = 0$ and $\alpha = \dfrac{d\widetilde{\omega}}{dt \otimes d\widetilde{m}(t)}$ to conclude that

$$J^l(\widetilde{m}, \widetilde{\omega}) \geqslant \int_{\mathbb{R}^d} u(0,x) dm_0(x).$$

On the other hand, we can apply relation (3.33) to the candidate $(m', -D_p H(x, Du(t,x))m')$ where m' is solution to

$$\begin{cases} \partial_t m' - \operatorname{div}(D_p H(x, Du(t,x))m') - \Delta m' = 0, \quad \text{in } (0,T) \times \mathbb{R}^d \\ m'(0) = m_0. \end{cases}$$

We get $J^l(m', -D_p H(x, Du(t,x))m') = \displaystyle\int_{\mathbb{R}^d} u(0,x) dm_0(x)$ and we can conclude that the infimum of the linearized problem is indeed $\displaystyle\int_{\mathbb{R}^d} u(0,x) dm_0(x)$, it is achieved at $(\widetilde{m}, \widetilde{\omega})$ and (3.34) holds true. Collecting the equations satisfied by u and $\widetilde{m}$, relation (3.34) as well as the exclusion conditions of Lemma 3.6, we get the optimality conditions for the constrained problem.

□

3.6 Technical Results about the HJB equation

We start with a (slightly unusual) version of Grönwall lemma.

Lemma 3.7. *Assume that $l : [0,T] \to \mathbb{R}^+$ is a bounded measurable map which satisfies, for some $C_1, C_2 > 0$ and almost all $t \in [0,T]$,*

$$l(t) \leqslant C_1 + C_2 \int_t^T \frac{l(s)}{\sqrt{s-t}} ds. \tag{3.35}$$

Then, for almost all $t \in [0,T]$,

$$l(t) \leqslant C_1(1 + C_2\sqrt{\pi}\sqrt{T-t})e^{C_2^2\pi(T-t)}.$$

Proof. Arguing by induction, using (3.35) we find that, for all $t \in [0,T]$ and all $n \in \mathbb{N}^*$, it holds

$$\begin{aligned} l(t) &\leqslant C_1\left(1 + \sum_{k=1}^{n} C_2^k I_k(t)\right) + C_2^{n+1}\int_t^T\int_{t_1}^T \dots \int_{t_n}^T \frac{l(t_{n+1})}{\sqrt{t_1 - t}\dots\sqrt{t_{n+1}-t_n}} dt_1 \dots dt_{n+1} \\ &\leqslant C_1\left(1 + \sum_{k=1}^{n} C_2^k I_k(t)\right) + \|l\|_\infty C_2^{n+1} I_{n+1}(t), \end{aligned} \tag{3.36}$$

where $I_k : [0,T] \to \mathbb{R}$ is defined for all $k \in \mathbb{N}^*$ by

$$I_k(t) = \int_t^T\int_{t_1}^T \dots \int_{t_{k-1}}^T \frac{1}{\sqrt{t_1 - t}\dots\sqrt{t_k - t_{k-1}}} dt_1 \dots t_k.$$

Noticing that $I_k(t) = \displaystyle\int_t^T \frac{1}{\sqrt{t_1 - t}} I_{k-1}(t_1)dt_1$ for all $k \geqslant 2$ and that $I_1(t) = 2\sqrt{T-t}$, we find by induction that, for all $n \geqslant 2$,

$$I_n(t) = 2B_1 \dots B_{n-1}(T-t)^{n/2}$$

where the B_k are given, for all $k \geqslant 1$ by,

$$\begin{aligned} B_k &= \int_0^1 (1-u)^{k/2} u^{-1/2} du \\ &= \beta(1/2, k/2+1) \\ &= \frac{\Gamma(1/2)\Gamma(k/2+1)}{\Gamma(k/2+3/2)} \end{aligned}$$

where β and Γ are Euler's functions. As a consequence, we get, for all $k \geqslant 1$ and all $t \in [0,T]$,

$$I_k(t) = \frac{\pi^{k/2}}{\Gamma(k/2+1)}(T-t)^{k/2}.$$

In particular, $\lim_{n\to+\infty} C_2^{n+1} I_{n+1}(t) = 0$ for all t in $[0,T]$. Now we can compute

$$\begin{aligned} \sum_{k=1}^{N} C_2^k \frac{\pi^{k/2}}{\Gamma(k/2+1)}(T-t)^{k/2} &= \sum_{k=1}^{+\infty} C_2^{2k} \frac{\pi^k}{\Gamma(k+1)}(T-t)^k + \sum_{k=0}^{+\infty} C_2^{2k+1} \frac{\pi^{k+1/2}}{\Gamma(k+1+1/2)}(T-t)^{k+1/2} \\ &\leqslant \sum_{k=1}^{+\infty} C_2^{2k} \frac{\pi^k}{k!}(T-t)^k + \sum_{k=0}^{+\infty} C_2^{2k+1} \frac{\pi^{k+1/2}}{k!}(T-t)^{k+1/2} \\ &\leqslant e^{C_2^2\pi(T-t)} - 1 + C_2\sqrt{\pi}\sqrt{T-t}e^{C_2^2\pi(T-t)} \\ &\leqslant (1 + C_2\sqrt{\pi}\sqrt{T-t})e^{C_2^2\pi(T-t)} - 1. \end{aligned}$$

Finally, we deduce from (3.36) that, for all $t \in [0,T]$,

$$l(t) \leqslant C_1(1 + C_2\sqrt{\pi}\sqrt{T-t})e^{C_2^2\pi(T-t)}.$$

□

3.6.1 Proof of Theorem 3.1

Lemma 3.8. *Assume that $u \in \mathcal{C}([0,T], E_n)$ is a solution to the HJB equation (3.4) with $f \in \mathcal{C}([0,T], E_n)$ and $g \in E_n$. Then*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |Du(t,x)| \leqslant C(\int_0^T \|f(t)\|_1 dt, \|g\|_1).$$

Proof. We use the classical Bernstein method. Let $\mu > 0$ and $w(t,x) := \frac{1}{2}e^{\mu t}|Du(t,x)|^2$. Being f in $\mathcal{C}([0,T], E_n)$, u is smooth in space and satisfies the HJB equation in the strong sense. Differentiating the equation with respect to x and taking the scalar product with $e^{\mu t}Du(t,x)$ gives

$$\begin{aligned}
&-\partial_t w(t,x) + Dw(t,x).D_pH(x,Du(t,x)) - \Delta w(t,x) \\
&= -\mu w(t,x) - D_xH(x,Du(t,x)).e^{\mu t}Du(t,x) + Df(t,x).e^{\mu t}Du(t,x) - e^{\mu t}|D^2u(t,x)|^2.
\end{aligned}$$

Now, by assumption on H, $|D_xH(x,Du(t,x))| \leqslant C_0(1 + |Du(t,x)|)$ and therefore, for $\mu = 2C_0$,

$$\begin{aligned}
-\partial_t w(t,x) + Dw(t,x).D_pH(x,Du(t,x)) - \Delta w(t,x) &\leqslant C_0 e^{\mu t}|Du(t,x)| + Df(t,x).e^{\mu t}Du(t,x) \\
&\leqslant \sqrt{2}e^{C_0T}\left(C_0 + \|f(t)\|_1\right) \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \sqrt{w(s,y)}.
\end{aligned}$$

By comparison between w and the obvious super-solution

$$(t,x) \mapsto \frac{1}{2}e^{2C_0T}\|g\|_1^2 + \sqrt{2}e^{C_0T} \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \sqrt{w(s,y)} \int_t^T \left(C_0 + \|f(s)\|_1\right) ds$$

we deduce that, for all $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$w(t,x) \leqslant C(1 + \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \sqrt{w(s,y)})$$

for some $C = C(\int_0^T \|f(t)\|_1 dt, \|g\|_1) > 0$. And therefore, $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |Dw(t,x)| \leqslant C$ for another constant $C = C(\int_0^T \|f(t)\|_1 dt, \|g\|_1) > 0$.

□

Lemma 3.9. *Assume that $u \in \mathcal{C}([0,T], E_n)$ is a solution to the HJB equation with data $f \in L^1([0,T], E_n)$ and $g \in E_n$ and assume that u satisfies the estimate of the previous lemma then*

$$\sup_{t\in[0,T]} \|u(t)\|_n \leqslant C(\int_0^T \|f(t)\|_n, \|g\|_n).$$

Proof. For all $(t,x) \in [0,T] \times \mathbb{R}^d$, it holds that

$$\begin{aligned}|u(t,x)| &\leqslant |P_{T-t}g(x)| + \int_t^T |P_{s-t}f(s)(x)|ds + \int_t^T |P_{s-t}\left[H(.,Du(s,.))\right](x)|ds \\ &\leqslant 2\sqrt{T}\|g\|_0(1+|x|) + 2\sqrt{T}\int_t^T \|f(s)\|_0(1+|x|)ds + C(1 + \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |Du(s,x)|)\end{aligned}$$

for some $C = C(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d)} |Du(t,x)|) > 0$. Above with use the fact that $\sup_{x\in\mathbb{R}^d} |P_t g(x)| \leqslant \sup_{x\in\mathbb{R}^d} |g(x)|$ for a bounded function g and $\sup_{x\in\mathbb{R}^d} \frac{|P_t g(x)|}{1+|x|} \leqslant 2\sqrt{T}\sup_{x\in\mathbb{R}^d} \frac{|g(x)|}{1+|x|}$ for a function g with linear growth. Since u is assumed to satisfy the Lipschitz estimate of the previous lemma 3.8, it holds that

$$\sup_{t\in[0,T]} \|u(t)\|_1 \leqslant C(\int_0^T \|f(t)\|_1 dt, \|g(t)\|_1).$$

Now we proceed with higher order derivatives and we argue by induction. Take $k \geqslant 2$ and assume that we have shown that

$$\sup_{t\in[0,T]} \|u(t)\|_{k-1} \leqslant C(\int_0^T \|f(t)\|_{k-1} dt, \|g(t)\|_{k-1}).$$

Using the inequality $\sup_{x\in\mathbb{R}^d} |DP_t g(x)| \leqslant \frac{C}{\sqrt{t}} \sup_{x\in\mathbb{R}^d} |g(x)|$ we get

$$\begin{aligned}|D^k u(t,x)| &\leqslant |P_{T-t}D^k g(x)| + \int_t^T |P_{s-t}D^k f(s)(x)|ds + \int_t^T |DP_{s-t}D^{k-1}\left[H(.,Du(s,.))\right](x)|ds \\ &\leqslant \|g\|_k + \int_t^T \|f(s)\|_k ds + C\int_t^T \frac{\sup_{x\in\mathbb{R}^d} |D^{k-1}\left[H(x,Du(s,x))\right]|}{\sqrt{s-t}} ds.\end{aligned}$$

But we can find a constant $C = C(\sup_{t\in[0,T]} \|u(t)\|_{k-1})$ such that

$$\sup_{x\in\mathbb{R}^d} |D^{k-1}H(x,Du(s,x))| \leqslant C(1 + \sup_{x\in\mathbb{R}^d} |D^k u(s,x)|)$$

and therefore, by Grönwall's lemma 3.7,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |D^k u(t,x)| \leqslant C(\|g\|_k, \int_0^T \|f(t)\|_k dt, \sup_{t\in[0,T]} \|u(t)\|_{k-1})$$

and we conclude by induction.

□

Following similar computations we can prove the following stability result.

Lemma 3.10. *Assume that $f_1, f_2 \in L_1([0,T], E_n)$ and $g_1, g_2 \in E_n$. Suppose that, $u_1, u_2 \in \mathcal{C}([0,T], E_n)$ are the respective solutions the HJB equation with data $(f_1, g_1), (f_2, g_2)$ and satisfy the estimate of Lemma 3.9. Then*

$$\sup_{t\in[0,T]} \|u_1(t) - u_2(t)\|_n \leqslant C(\int_0^T \|f_1(t) - f_2(t)\|_n dt + \|g_1 - g_2\|_n).$$

for some $C = C(\int_0^T \|f_1(t)\|_n dt, \int_0^T \|f_2(t)\|_n dt, \|g_1\|_n, \|g_2\|_n) > 0.$

Proof. For all $(s,x) \in [0,T] \times \mathbb{R}^d$ we can write

$$\begin{aligned} &H(x, Du_1(s,x)) - H(x, Du_2(s,x)) \\ &\qquad = (Du_1(s,x) - Du_2(s,x)). \int_0^1 D_p H(x, rDu_1(s,x) + (1-r)Du_2(s,x))dr \end{aligned}$$

and deduce that, for all $k \geqslant 1$,

$$\sup_{x\in\mathbb{R}^d} |D^{k-1}\left[H(x, Du_1(s,x)) - H(x, Du_2(s,x))\right]| \leqslant C\|u_1(s) - u_2(s)\|_k$$

for some $C = C(\|u_1(s)\|_k, \|u_2(s)\|_k) > 0$. The proof of the lemma follows from this observation and the same computations as the proof of Lemma 3.9. □

Lemma 3.11. *Assume that $u \in L^\infty([0,T], E_n)$ solves the HJB equation with data $(f,g) \in \mathcal{C}([0,T], E_n) \times E_{n+\alpha}$ then u belongs to $\mathcal{C}([0,T], E_n)$.*

Proof. Let us take $k \in [\![1, n]\!]$. We fix $h > 0$. For $t \in [0, T-h]$ it holds

$$\begin{aligned} D^k u(t+h,x) - D^k u(t,x) &= P_{T-t-h} D^k g(x) - P_{T-t} D^k g(x) \\ &+ \int_{t+h}^T P_{s-t-h} D^k f(s)(x) ds - \int_t^T P_{s-t} D^k f(s)(x) ds \\ &+ \int_{t+h}^T DP_{s-t-h} D^{k-1} H(., Du(s,.))(x) ds - \int_t^T DP_{s-t} D^{k-1} H(., Du(s,.))(x) ds \\ &= \Delta_1 + \Delta_2 + \Delta_3. \end{aligned}$$

We estimate the three differences as follows:

$$|\Delta_1| = |P_{T-t-h} D^k g(x) - P_{T-t} D^k g(x)| \leqslant |D^k g(x) - P_h D^k g(x)| \leqslant h^{\alpha/2} ||g||_{k+\alpha}.$$

Now for the term involving f:

$$\begin{aligned}
|\Delta_2| &= |\int_{t+h}^{T} P_{s-t-h}D^k f(s)(x)ds - \int_t^T P_{s-t}D^k f(s)(x)ds| \\
&= |\int_t^{T-h} P_{s-t}D^k f(s+h)(x)ds - \int_t^T P_{s-t}D^k f(s)(x)ds| \\
&= |\int_t^{T-h} P_{s-t}(D^k f(s+h) - D^k f(s))(x)ds - \int_{T-h}^T P_{s-t}D^k f(s)(x)ds| \\
&\leqslant \int_0^{T-h} \|f(s+h) - f(s)\|_k ds + C\sqrt{h} \sup_{t\in[0,T]} \|f(t)\|_{k-1}.
\end{aligned}$$

Finally for the term involving the Hamiltonian

$$\begin{aligned}
|\Delta_3| &= |\int_{t+h}^{T} DP_{s-t-h}D^{k-1}H(.,Du(s,.))(x)ds - \int_t^T DP_{s-t}D^{k-1}H(.,Du(s,.))(x)ds| \\
&= |\int_t^{T-h} DP_{s-t}D^{k-1}\left[H(.,Du(s+h,.) - H(.,Du(s,.))\right](x)ds \\
&- \int_{T-h}^T DP_{s-t}D^{k-1}\left[H(.,Du(s,.))\right](x)ds| \\
&\leqslant C(\text{essup}_{t\in[0,T]} \|u(t)\|_k) \int_t^{T-h} \frac{\sup_{x\in\mathbb{R}^d} |D^k u(s+h,x) - D^k u(s,x)|}{\sqrt{s-t}} ds \\
&+ C(\text{essup}_{t\in[0,T]} \|u(t)\|_k)\sqrt{h} \\
&\leqslant C(\text{essup}_{t\in[0,T]} \|u(t)\|_k)(\sqrt{h} + \int_t^{T-h} \frac{\|u(s+h) - u(s)\|_k}{\sqrt{s-t}} ds).
\end{aligned}$$

Using again Grönwall Lemma 3.7, we get, for all $t \in [0,T]$,

$$\begin{aligned}
\|u(t+h) - u(t)\|_n \leqslant C(\text{essup}_{t\in[0,T]} \|u(t)\|_n)(h^{\alpha/2}\|g\|_{n+\alpha} + \int_0^{T-h} \|f(s+h) - f(s)\|_n ds \\
+ \sqrt{h} \sup_{t\in[0,T]} \|f(t)\|_{n-1}).
\end{aligned}$$

Being f in $\mathcal{C}([0,T], E_n)$, the right-hand side converges to 0 when h goes to 0 and therefore

$$\lim_{h\to 0} \sup_{t\in[0,T-h]} \|u(t+h) - u(t)\|_n = 0$$

which concludes that u belongs to $\mathcal{C}([0,T], E_n)$.

□

As a consequence, we get the existence of solutions from the classical case.

Proposition 3.9. *Take $f \in L^1([0,T], E_n)$ and $g \in E_{n+\alpha}$. Then there exists a unique solution in $u \in \mathcal{C}([0,T], E_n)$ to the HJB equation with data (f,g) and it satisfies the estimate of Lemma 3.9.*

Proof of Theorem 3.1. We take a sequence of smooth functions $f_m : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $g_m : \mathbb{R}^d \to \mathbb{R}$ converging respectively to f in $L^1([0,T], E_n)$ and to g in $E_{n+\alpha}$. For each m, the existence of a strong solution $u_m \in \mathcal{C}([0,T], E_n)$ follows from Schauder theory and our a priori Lipschitz estimate. Thanks to the previous lemma, we know that u_m is a Cauchy sequence in $L^\infty([0,T], E_n)$ and therefore it converges in this space to some u. The subspace $\mathcal{C}([0,T], E_n)$ being closed in $L^\infty([0,T], E_n)$ we have that u belongs to $\mathcal{C}([0,T], E_n)$. We can also pass to the limit in the equation

$$u_m(t,x) = P_{T-t} g_m(x) + \int_t^T P_{s-t} f_m(s)(x) ds - \int_t^T P_{t-s} \left[H(., Du_m(s,.)) \right](x) ds$$

to conclude that u is a solution.

The uniqueness of solutions is a straightforward consequence of the stability estimate of the previous lemma. □

3.6.2 Proof of Theorem 3.6.

Proof. Proof of Theorem 3.6 We can remark that u is a solution (3.27) if and only if $v := u - z$ is a solution to

$$\begin{cases} -\partial_t v + H(x, Dv + Dz) - \Delta v = 0 & \text{in } [0,T] \times \mathbb{R}^d, \\ v(T,x) = 0 & \text{in } \mathbb{R}^d. \end{cases}$$

where

$$z(t,x) := P_{T-t}\psi_2(x) + \int_t^T P_{s-t}\psi_1(s,x) d\nu(s) + \int_t^T P_{s-t}\varphi_1(s)(x) ds.$$

Proceeding exactly as in the proof of Theorem 3.1, we find that there exists a unique solution $v \in L^\infty([0,T], E_n)$ and it satisfies

$$\text{essup}_{t\in[0,T]} \|v(t)\|_n \leqslant C\left(\int_0^T \|z(t)\|_n dt\right).$$

We can also proceed directly (without relying on the existence of solutions for smooth z) and follow Chapter 4.3.3. in [36]. We recall the main steps here. Up to finding Lipschitz a priori estimates on the solutions using Bernstein method, as in Lemma 3.8 , we can pretend that H is Lipschitz continuous.

Take $u' \in L^\infty([0,T], E_n)$ and define $u = \Phi(u')$ by

$$\begin{aligned} u(t,x) = P_{T-t}\psi_2(x) + \int_t^T P_{s-t}\psi_2(s)(x) d\nu(s) + \int_t^T P_{s-t}\varphi_1(s)(x) ds \\ - \int_t^T P_{s-t} \left[H(., Du'(s,.)) \right](x) ds. \end{aligned}$$

Following the same computations as in Lemma 3.9 we can prove that, for all $1 \leqslant k \leqslant n$, it holds

$$\text{essup}_{(s,x)\in[t,T]\times\mathbb{R}^d} |D^k u(s,x)| \leqslant C(1 + \int_t^T \frac{\|u'(s)\|_k}{\sqrt{s-t}} ds)$$

for some $C = C(\|\nu\|, \sup_{t\in[0,T]} \|\psi_1(t)\|_n, \sup_{t\in[0,T]} \|\varphi_1(t)\|_n, \|\psi_1\|_n, \text{essup}_{t\in[0,T]} \|u'(t)\|_{k-1}) > 0$. We use this to construct by induction $(\Lambda_0, (\lambda_1, \Lambda_1), \dots, (\lambda_n, \Lambda_N))$ depending only on the data $(|\nu|, \sup_{t\in[0,T]} \|\psi_1(t)\|_n, \sup_{t\in[0,T]} \|\varphi_1(t)\|_n, \|\psi_1\|_n)$, such that

$$\text{essup}_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{|u(t,x)|}{1+|x|} \leqslant \Lambda_0,$$

$$\text{essup}_{(s,x)\in[t,T]\times\mathbb{R}^d} |D^k u(s,x)| \leqslant \lambda_k e^{\Lambda_k (T-t)}$$

for all $1 \leqslant k \leqslant n$, whenever u' satisfies the same estimates. We let $\mathcal{B}$ be the subset of functions in $L^\infty([0,T], E_n)$ satisfying these estimates. By construction $\mathcal{B}$ is stable under the action of Φ. Now if we take two functions u_1', u_2' in $\mathcal{B}$ and if we define $u_1 = \Phi(u_1')$, $u_2 = \Phi(u_2')$, $w = u_1 - u_2$ and $w' = u_1' - u_2'$ we find, following yet again the same computations, that

$$\text{essup}_{s\in[t,T]} \|w(t)\|_n \leqslant C \int_t^T \frac{\|w'(s)\|_n}{\sqrt{s-t}} ds \tag{3.37}$$

for some $C > 0$ depending only on the constants defining $\mathcal{B}$. Thanks to estimate (3.37) we find that Φ is a contraction in $L^1_{e^{\mu t}dt}([0,T], E_n)$ for μ large enough. Finally we take any $u_1 \in \mathcal{B}$ and we build by induction $u_{m+1} = \Phi(u_m)$. We know that u_m converges in $L^1_{e^{\mu t}dt}([0,T], E_n)$ to some u. However we can use inequality (3.37) once again to prove that u_m actually converges in $L^\infty([0,T], E_n)$. The set $\mathcal{B}$ being closed in $L^\infty([0,T], E_n)$ we can conclude that the limit u belongs to $\mathcal{B}$ and is a solution to the HJB equation.

The uniqueness of solutions is proved as in the proof of Theorem 3.1. □

We let $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be the unique element in its equivalence class of $L^\infty([0,T], E_n)$ satisfying, for all $(t,x) \in [0,T] \times \mathbb{R}^d$

$$\begin{aligned} u(t,x) = P_{T-t}\psi_2(x) + \int_0^T \mathbb{1}_{(t,T]}(s) P_{s-t}\psi_1(s)(x) d\nu(s) + \int_t^T P_{s-t}\varphi_1(s)(x) ds \\ - \int_t^T P_{s-t}\left[H(., Du(s,.))\right](x) ds. \end{aligned}$$

We also define, for all $(t,x) \in [0,T] \times \mathbb{R}^d$

$$z(t,x) := P_{T-t}\psi_2(x) + \int_0^T \mathbb{1}_{(t,T]}(s) P_{s-t}\psi_1(s,x) d\nu(s) + \int_t^T P_{s-t}\varphi_1(s)(x) ds \tag{3.38}$$

and

$$v(t,x) := -\int_t^T P_{s-t}\left[H(., Du(s,.))\right](x) ds. \tag{3.39}$$

Lemma 3.12. *The function v belongs to $L^\infty([0,T],E_n)\bigcap \mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\mathbb{R}^d)$ for any $\alpha\in(0,1)$ and satisfies the following estimate*

$$\begin{aligned}\text{essup}_{(t,x)\in[0,T]\times\mathbb{R}^d}|v(t,x)| + \text{essup}_{t\in[0,T]}\|v(t)\|_n \\ + \text{essup}_{s\neq t}\frac{\|v(t)-v(s)\|_{n-2}}{\sqrt{t-s}} + \text{essup}_{t\neq s}\frac{\|v(t)-v(s)\|_{n-3}}{|t-s|}\leqslant C,\end{aligned}$$

for some $C=C(|\nu|,\|\psi_2\|_n,\sup_{t\in[0,T]}\|\psi_1(t)\|_n,\sup_{t\in[0,T]}\|\varphi_1(t)\|_n)>0$.

Proof. The estimate on $\text{essup}_{t\in[0,T]}\|v(t)\|_n$ follows from the fact that $v=u-z$ and the analog estimate for u. The L^∞ estimate is also straightforward since

$$|v(t,x)|\leqslant\int_t^T\sup_{x\in\mathbb{R}^d}|H(x,Du(s,x)|ds\leqslant C(\sup_{t\in[0,T]}\|u(t)\|_1).$$

Notice that, on the contrary to u, the function v is bounded on $[0,T]\times\mathbb{R}^d$.

We turn our attention to the time regularity. We take $k\geqslant 0$ and $t_1,t_2\in[0,T]$ such that $t_2\geqslant t_1$. We can write

$$\begin{aligned}D^kv(t_1,x)-D^kv(t_2,x) &= \int_{t_1}^{t_2}D^k\left[P_{s-t_1}\left[H(.,Du(s,.))\right](x)\right]ds \\ &+\int_{t_2}^T D^k\left[P_{s-t_1}\left[H(.,Du(s,.))\right](x)-P_{s-t_2}\left[H(.,Du(s,.))\right](x)\right]ds. \qquad (3.40)\end{aligned}$$

On the one hand, for all $(s,x)\in[t_1,t_2]\times\mathbb{R}^d$,

$$|D^k\left[P_{s-t_1}\left[H(.,Du(s,.))\right](x)\right]|\leqslant C(\text{essup}_{t\in[0,T]}\|u(t)\|_{k+1}).$$

On the other hand, for all $s\in[t_2,T]$,

$$\begin{aligned}P_{s-t_1}\left[H(.,Du(s,.))\right](x)-P_{s-t_2}\left[H(.,Du(s,.))\right](x) &= -\int_{t_1}^{t_2}\frac{d}{dt}P_{s-t}\left[H(.,Du(s,.))\right](x)dt \\ &= \int_{t_1}^{t_2}\Delta P_{s-t}\left[H(.,Du(s,.))\right](x)dt = \int_{t_1}^{t_2}P_{s-t}\Delta\left[H(.,Du(s,.))\right](x)dt,\end{aligned}$$

and therefore,

$$\begin{aligned}D^k[P_{s-t_1}\left[H(.,Du(s,.))\right](x)-P_{s-t_2}\left[H(.,Du(s,.))\right](x)] \\ = -\int_{t_1}^{t_2}DP_{s-t}D^{k-1}\Delta\left[H(.,Du(s,.))\right](x)dt \qquad (3.41)\end{aligned}$$

from which it follows that,

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}|D^k\int_{t_2}^T[P_{s-t_1}\left[H(.,Du(s,.))\right](x)-P_{s-t_2}\left[H(.,Du(s,.))\right](x)ds]| \\ \leqslant C\int_{t_2}^T\int_{t_1}^{t_2}\frac{1}{\sqrt{s-t}}dtds\leqslant 2C(T-t_2)\sqrt{t_2-t_1},\end{aligned}$$

for some $C = C(\sup_{t\in[0,T]} \|u(t)\|_{k+2}) > 0$. We conclude from (3.40) that

$$\begin{aligned}\operatorname{essup}_{x\in\mathbb{R}^d} |D^k v(t_1,x) - D^k v(t_2,x)| &\leqslant C(\operatorname{essup}_{t\in[0,T]} \|u(t)\|_{k+1})(t_2-t_1) \\ &\quad + C(\operatorname{essup}_{t\in[0,T]} \|u(t)\|_{k+2})\sqrt{t_2-t_1} \\ &\leqslant C\sqrt{t_2-t_1}\end{aligned}$$

with $C = C(\operatorname{essup}_{t\in[0,T]} \|u(t)\|_{k+2}) > 0$. If we don't do the integration by parts in step (3.41) we find that

$$\|v(t_1) - v(t_2)\|_k \leqslant C(t_2 - t_1)$$

for some $C = C(\operatorname{essup}_{t\in[0,T]} \|u(t)\|_{k+3}) > 0$. Being $n \geqslant 3$ we get that v is bounded in $\mathcal{C}^{(1+\alpha)/2,1+\alpha}([0,T]\times\mathbb{R}^d)$ for all $\alpha \in (0,1)$. This concludes the proof of the lemma. □

Proof. Proof of Proposition 3.7. We take a sequence of non-negative functions $\nu_m : [0,T] \to \mathbb{R}$ in $L^\infty([0,T])$ such that ν_m converges to ν is $\mathcal{M}^+([0,T])$. We also let u_m be the solution to the associated HJB equation and define z_m and v_m according to 3.38 and 3.39 respectively with u replaced by u_m. Being ν_m in $L^\infty([0,T])$, z_m, v_m and u_m are continuous in time and space. Clearly it is enough to prove the convergence of z_m toward z, Dz_m toward Dz and v_m toward v. Assume that $\nu(\{t\}) = 0$. In this case, we have, for all $x \in \mathbb{R}^d$

$$z_m(t,x) \to z(t,x),$$

$$Dz_m(t,x) \to Dz(t,x),$$

as $m \to +\infty$.

Using the previous lemma, we also have that the sequence (v_m) is uniformly bounded in $\mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\mathbb{R}^d)$ for any $\alpha \in (0,1)$. Up to a sub-sequence it converges (locally) to some $\widetilde{v}$ in $\mathcal{C}^{\frac{1+\beta}{2},1+\beta}([0,T]\times\mathbb{R}^d)$ for some $\beta \in (0,\alpha)$. Using Lebesgue dominated convergence theorem we can pass to the limit as $m \to +\infty$ in the equality

$$v_m(t,x) = -\int_t^T P_{s-t}\left[H(.,Dv_m(s,.) + Dz_m(s,.))\right](x)ds,$$

we conclude that, for all $(t,x) \in [0,T]\times\mathbb{R}^d$

$$\widetilde{v}(t,x) = -\int_t^T P_{s-t}\left[H(.,D\widetilde{v}(s,.) + Dz(s,.))\right](x)ds.$$

If we let $\widetilde{u} := \widetilde{v} + z$, we have that $\widetilde{u}$ solves the HJB equation and, by uniqueness, $\widetilde{u} = u$ in $L^\infty([0,T],E_n)$. Therefore $v(t,x) = \widetilde{v}(t,x)$ for all $(t,x) \in [0,T]\times\mathbb{R}^d$ and we conclude that v_m converges locally uniformly to v in $\mathcal{C}^{\frac{1+\beta}{2},1+\beta}([0,T]\times\mathbb{R}^d)$.

□

3.7 Appendix

3.7.1 Existence of relaxed solutions

Proof of Proposition 3.1. Consider a weak solution of

$$\begin{cases} dX_t = \alpha(t, X_t)dt + \sqrt{2}dB_t, \\ X_{t=0} = X_0 \sim m_0 \end{cases}$$

such that $\mathcal{L}(X_t) = m(t)$, $\forall t \in [0, T]$. The existence of such a solution is guaranteed by the fact that (α, m) solves the Fokker-Planck equation (see [129] and also Proposition 2.1 in Chapter 2). Using Jensen inequality, we get for $t, s \in [0, T]$ with $s < t$

$$\begin{aligned} \mathbb{E}(|X_t - X_s|^2) &\leqslant 2\mathbb{E}\left[\left|\int_s^t \alpha(u, X_u)du\right|^2\right] + 4\mathbb{E}\left[|B_t - B_s|^2\right] \\ &\leqslant 2(t-s)^2\mathbb{E}\left[\int_s^t |\alpha(u, X_u)|^2 \frac{du}{t-s}\right] + 4(t-s) \\ &\leqslant 2(t-s)\int_0^T \int_{\mathbb{R}^d} |\alpha(t,x)|^2 dm(t)(x)dt + 4(t-s) \end{aligned}$$

and therefore

$$d_2(m(s), m(t)) \leqslant C\sqrt{t-s}$$

for some $C = C(\int_0^T \int_{\mathbb{R}^d} |\alpha(t,x)|^2 dm(t)(x)dt) > 0$ since $d_2(m(s), m(t)) \leqslant \mathbb{E}(|X_t - X_s|^2)^{1/2}$. Taking $s = 0$ in the above computation also shows that

$$\int_{\mathbb{R}^d} |x|^2 dm(t) \leqslant 2\mathbb{E}(|X_t - X_s|^2) + 2\int_{\mathbb{R}^d} |x|^2 dm_0(x) \leqslant C$$

for another $C = C(\int_{\mathbb{R}^d} |x|^2 dm_0(x), \int_0^T \int_{\mathbb{R}^d} |\alpha(t,x)|^2 dm(t)(x)dt) > 0$.

□

Proof of Proposition 3.2. We set $\omega^n = \alpha^n m^n$. By Cauchy-Schwarz inequality we find that the total variation $|\omega^n|$ of ω^n is uniformly bounded. Indeed we have

$$\begin{aligned} |\omega^n| &= \int_0^T \int_{\mathbb{R}^d} \left|\frac{d\omega^n}{dt \otimes dm^n(t)}(t,x)\right| dm^n(t)(x)dt \\ &\leqslant \sqrt{T}\left(\int_0^T \int_{\mathbb{R}^d} \left|\frac{d\omega^n}{dt \otimes dm^n(t)}(t,x)\right|^2 dm^n(t)(x)dt\right)^{1/2}. \end{aligned}$$

This estimate together with Proposition 3.1 allow us to use Banach-Alaoglu theorem on the one hand and Ascoli theorem on the other hand and deduce that for all $r \in (1, 2)$, up to a subsequence, $(m^n, \omega^n)_{n\in\mathbb{N}}$ converges in $\mathcal{C}([0,T], \mathcal{P}^r(\mathbb{R}^d)) \times \mathcal{M}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ to some element $(\widetilde{m}, \widetilde{\omega})$ of $\mathcal{C}([0,T], \mathcal{P}^r(\mathbb{R}^d)) \times \mathcal{M}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$. It is straightforward that $\widetilde{m}(0) = m_0$ and

the fact that $(\widetilde{m}, \widetilde{\omega})$ satisfies the Fokker-Planck equation is a consequence of the weak-$*$ convergence of measures. Using Theorem 2.34 of [7] (see also Exemple 2.36) in [7]) we find that ω is absolutely continuous with respect to $m(t) \otimes dt$ and

$$\int_0^T \int_{\mathbb{R}^d} \left| \frac{d\omega}{dt \otimes dm(t)}(t,x) \right|^2 dm(t)(x)dt \leqslant \liminf_{n \to +\infty} \int_0^T \int_{\mathbb{R}^d} |\alpha_n(t,x)|^2 dm^n(t)(x)dt.$$

By Proposition 3.1 again, this shows that m belongs to $\mathcal{C}^{1/2}([0,T], \mathcal{P}_2(\mathbb{R}^d))$.

□

Now we give the proof of Lemma 3.1.

Proof of Lemma 3.1. This is precisely Theorem 2.4 in Chapter 2 and the result follows from Proposition 3.1 and Proposition 3.2. We consider a minimizing sequence (m^n, ω^n) satisfying (3.14) and such that, for all $n \in \mathbb{N}$, $J_{\epsilon,\delta}(m^n, \omega^n) \leqslant \inf J_{\epsilon,\delta}(m^n, \omega^n) + 1$. By coercivity of H and therefore -by taking convex conjugates- of L we find that there is $C_1 > 0$ such that, for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^d} \int_0^T \left| \frac{d\omega^n}{dt \otimes dm^n(t)}(t,x) \right|^2 dm^n(t)(x)dt \leqslant C_1. \tag{3.42}$$

Using that (m^n, ω^n) satisfies the Fokker-Planck equation and m_0 belongs to $\mathcal{P}_2(\mathbb{R}^d)$ we deduce from Proposition (3.2) that, for all $r \in (1,2)$, up to a subsequence, $(m^n, \omega^n)_{n \in \mathbb{N}}$ converges in $\mathcal{C}([0,T], \mathcal{P}^r(\mathbb{R}^d)) \times \mathcal{M}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ to some element $(\widetilde{m}, \widetilde{\omega})$ of $\mathcal{C}([0,T], \mathcal{P}^2(\mathbb{R}^d)) \times \mathcal{M}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ which satisfies the Fokker-Planck equation with initial position $m(0) = m_0$. To conclude we use Theorem 2.34 of [7] to prove that

$$J_{\epsilon,\delta}(\widetilde{m}, \widetilde{\omega}) \leqslant \liminf_{n \to +\infty} J_{\epsilon,\delta}(m_n, \omega_n).$$

Therefore $(\widetilde{m}, \widetilde{\omega})$ is indeed a minimum of $J_{\epsilon,\delta}$.

□

Chapter 4

Mean-field limit

Introduction

The goal of this chapter is to investigate the connection between the optimization problem:

$$\inf_{(\alpha,\mu)} \int_0^T \int_{\mathbb{R}^d} L(x,\alpha(t,x))d\mu(t)(x)dt + \int_0^T \mathcal{F}(\mu(t))dt + \mathcal{G}(\mu(T)) \tag{mfP}$$

subject to

$$\begin{cases} \mu \in \mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R}^d)), \alpha \in L^2_{dt\otimes\mu(t)}([0,T]\times\mathbb{R}^d,\mathbb{R}^d) & \\ \partial_t\mu + \mathrm{div}(\alpha\mu) - \Delta\mu = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ \mu(0) = \mu_0 \in \mathcal{P}_2(\mathbb{R}^d) & \\ \Psi(\mu(t)) \leqslant 0 & \forall t \in [0,T] \end{cases}$$

and a control problem for a large number N of interacting particles:

$$\inf_{(\alpha_t^{i,N})_{1\leqslant i\leqslant N}} \mathbb{E}^{\mathbb{P}^{\gamma_N}} \left[\int_0^T \frac{1}{N}\sum_{i=1}^N L(X_t^{i,N},\alpha_t^{i,N})dt + \int_0^T \mathcal{F}(\widehat{\mu}_t^{N,x})dt + \mathcal{G}(\widehat{\mu}_T^{N,x}) \right] \tag{NP}$$

subject to

$$\begin{cases} dX_t^{i,N} = \alpha_t^{i,N}dt + \sqrt{2}dB_t^{i,N}, & \\ (X_0^{1,N},\dots,X_0^{N,N}) \sim \mu_0^{\otimes N} \text{ under } \mathbb{P}, & \\ \Psi(\widehat{\mu}_t^{N,x}) \leqslant 0 \text{ for all } t\in[0,T] & \mathbb{P}^{\gamma_N} - \text{almost-surely.} \end{cases}$$

In the latter problem, $(B^{i,N})_{1\leqslant i\leqslant N}$ are N independent standard Brownian motions supported on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. We denoted by

$$\widehat{\mu}_t^{N,x} := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

the empirical measures and finally $\mathbb{P}^{\gamma_N} := \mathbb{P}\left[.|\Psi(\widehat{\mu}_0^{N,x}) \leqslant -\gamma_N\right]$ is the conditional probability with respect to the event $\left\{\Psi(\widehat{\mu}_0^{N,x}) \leqslant -\gamma_N\right\}$ for some suitable rate $\gamma_N > 0$ such that $\gamma_N \to 0$

as $N \to +\infty$. The conditioning being necessary to ensure that the particles start from inside the constraint.

The former problem was thoroughly analyzed in Chapter 3 where we showed, in particular the existence of bounded, Lipschitz continuous optimal controls, for any initial position $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\Psi(\mu_0) < 0$. The latter problem (NP) is, however very different in nature. Indeed, the state is a process $(X_t^{1,N}, \dots X_t^{N,N})_{0 \leqslant t \leqslant T}$ valued in $(\mathbb{R}^d)^N$, the empirical measures $\widehat{\mu}_t^{N,x}$ are, by definition, random and the constraint has to be satisfied almost-surely. This type of constraint leads to new difficulties. Indeed, to dominate the effect of the diffusion, the controls cannot remain bounded and the value function associated to this problem is expected to blow-up near the boundary.

Without constraint, the connection between Problem (mfP) and Problems (NP) is by now well understood. Under more general structure conditions, Lacker proved in [96] that the law of the empirical measures of weak solutions to the N-particle system converge to probability measures supported on the set of optimal solutions to the mean-field problem and therefore convergence of the value functions hold. Taking advantage of the regularizing effect of the diffusion and uniform in N Lipschitz and semi-concavity estimates for the value functions of the N-particles system, it was shown in [35] that convergence actually holds with a rate (see Chapter 5). In the same setting, Cardaliaguet and Souganidis later proved in [41] a propagation of chaos around "stable" solutions of the mean-field problem. Finally we mention that, under convexity assumptions on the mean-field costs $\mathcal{F}$ and $\mathcal{G}$ it is shown in [36] that the value function associated to the mean-field control problem is a smooth (enough) function in the Wasserstein space. In this setting it is not difficult to prove that the convergence of the value functions holds with an optimal rate and we have quantitative propagation estimates for the optimal trajectories to the N-particles system toward the solution to the mean-field control problem.

We also mention that recent progresses were made in order to characterize the value function for the mean-field problem, in the general situation where it is not expected to be smooth. Similarly to the finite dimensional case, we expect the value function to be the unique viscosity solution (in some sense) to the dynamic programming equation. Different approaches have been taken in [28, 48, 51, 52]. The most general result, so far, being [52], where the authors rely on the approximation of the mean-field control problem by control problems for finite numbers of interacting particles.

Stochastic control problems with state constraint and non-degenerate diffusions were addressed in the seminal work [99] of Lasry and Lions. They showed that the blow-up behavior of the value function is directly related to the growth of the Hamiltonian and provided rates of divergence. This problem was later revisited by Leonori and Porretta in [103] where the authors also prove the rate of divergence of the gradient of the value function.

In this chapter we prove the convergence of the value functions for the problems with almost-sure constraints toward the value function for the mean-field problem. Let us denote by $\overline{\mathcal{U}}(\mu_0)$, the value of Problem (mfP) and $\mathcal{U}_N(\mu_0)$, the value of Problem (NP) which are rigorously defined in Sections 4.1 and 4.2 respectively. Similarly to [35] we proceed in two steps. On the one hand we prove that

$$\overline{\mathcal{U}}(\mu_0) \leqslant \liminf_{N \to +\infty} \mathcal{U}_N(\mu_0).$$

This boils down to finding weak limit points of sequences of nearly optimal weak solutions

to the N-particle problem. Once we know that $\mathcal{U}_N(\mu_0)$ is bounded independently from N, this follows from the line of arguments of [96] for problems without constraint. On the other hand, proving that

$$\limsup_{N\to+\infty} \mathcal{U}_N(\mu_0) \leqslant \overline{\mathcal{U}}(\mu_0)$$

requires more care. Indeed an admissible control for the mean-field problem is, in general, not admissible for the particle system because of the almost-sure constraint. We also mention that, in contrast with [35], the value function $\mathcal{U}_N(\mu_0)$ is certainly not smooth as a function of μ_0 (at least because we introduced conditional expectations and because of the almost-sure constraint) and therefore we cannot see it as a smooth sub-solution to the dynamic programming equation satisfied by $\overline{\mathcal{U}}$. Our strategy can be described as follows. Given an admissible control α for the mean-field control problem, we consider the particle system starting from an initial position $(X_0^{1,N},\dots,X_N^{N,N}) \sim^{\mathbb{P}} \mu_0^{\otimes N}$,

$$X_t^{i,N} = X_0^{i,N} + \int_0^{t\wedge\tau_N} \alpha(s, X_s^{i,N})ds + \int_{t\wedge\tau_N}^t \beta_t^{i,N}dt + \sqrt{2}B_t^{i,N}$$

where $\tau_N := \inf\{t \geqslant 0, \Psi(\widehat{\mu}_t^{N,x}) \geqslant -\frac{\gamma_N}{2}\}$ and $\beta_t^{i,N}$ is a feedback control designed so that, $\mathbb{P}^{\gamma_N}$-almost-surely

$$\frac{1}{N}\sum_{i=1}^N \left|X_t^{i,N} - X_{\tau_N}^{i,N}\right|^2 \leqslant r_N^2 \qquad \forall t \geqslant \tau_N,$$

where r_N is a small radius depending on γ_N which guarantees that, $\mathbb{P}^{\gamma_N}$-almost-surely,

$$\Psi(\widehat{\mu}_t^{N,x}) \leqslant 0, \qquad \forall t \in [0,T].$$

If α is bounded, Lipschitz continuous and taken so that the corresponding solution μ to

$$\begin{cases} \partial_t\mu + \mathrm{div}(\alpha\mu) - \Delta\mu = 0 \quad \text{in } (0,T)\times\mathbb{R}^d, \\ \mu(0) = \mu_0 \end{cases} \tag{4.1}$$

satisfies $\Psi(\mu(t)) \leqslant -\delta$, for all $t \in [0,T]$, for some $\delta > 0$, we expect a strong convergence of $\widehat{\mu}_t^{N,x}$ toward $\mu(t)$ for $t \in [0,\tau_N]$ and therefore $\tau_N \wedge T$ must converge to T. The key step is to build $(\beta_t^{i,N})_{1\leqslant i\leqslant N}$ so that its contribution to the cost for the N-particle problem, vanishes as $N \to +\infty$. This can be done only if $\Psi(\mu_0) < 0$ and γ_N does not converge too fast to 0. We also need to prove that it is enough to approximate admissible candidates (α,μ) such that α is bounded, Lipschitz continuous with respect to the space variable and $\Psi(\mu(t)) \leqslant -\delta$ for all $t \in [0,T]$, for some $\delta > 0$. Overall, our main result is the following.

Theorem 4.1. *Let Assumption* (A) *(introduced below) hold. Assume further that* $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ *satisfies* $\Psi(\mu_0) < 0$ *and* $\int_{\mathbb{R}^d} |x|^{d+5} d\mu_0(x) < +\infty$. *Then*

$$\lim_{N\to+\infty} \mathcal{U}_N(\mu_0) = \overline{\mathcal{U}}(\mu_0),$$

whenever $\gamma(N) = N^{-\theta}$, *with* $\theta < \frac{1}{d+8}$.

The rate γ_N is certainly not optimal but it is sufficient to ensure the convergence of $\mathcal{U}_N(\mu_0)$ toward $\overline{\mathcal{U}}(\mu_0)$. Let us also notice that $\mathbb{P}(\left\{\Psi(\widehat{\mu}_0^{N,x}) \leqslant -\gamma_N\right\}) \to 1$ as $N \to +\infty$ as soon as $\gamma_N \to 0$ since $\Psi(\mu_0) < 0$ and $(X_0^{1,N}, \dots, X_0^{N,N})$ is distributed according to $\mu_0^{\otimes N}$ under $\mathbb{P}$.

Connection with large deviations for non-interacting diffusions. Our results are closely related to the large deviations principle for (non-interacting) diffusions. Indeed if we consider the probability

$$v^N(t, \mathbf{x}^N) := \mathbb{P}(\forall s \in [0,t], \Psi\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}+\sqrt{2}B_t^{i,N}}\right) < 0)$$

where $\mathbf{x}^N = (x^{1,N}, \dots, x^{N,N})$ and $(B_t^{1,N}, \dots, B_t^{N,N})$ are N independent d-dimensional standard Brownian motions supported on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then v^N satisfies the heat equation

$$\begin{cases} \partial_t v^N - \Delta v^N = 0 & \text{in } (0,T) \times \Omega_N, \\ v^N = 0 & \text{in } (0,T) \times \partial\Omega_N, \\ v^N = 1 & \text{in } \{0\} \times \Omega_N, \end{cases}$$

where $\Omega_N := \left\{(x^{1,N}, \dots, x^{N,N}) \in \mathbb{R}^{dN}, \Psi(\frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}}) < 0\right\}$. If we make a logarithmic transformation and let

$$u^N(t, \mathbf{x}^N) = -\frac{2}{N}\log v^N(T-t, \mathbf{x}^N),$$

we obtain that u^N solves

$$\begin{cases} -\partial_t u^N + \frac{N}{2}|Du^N|^2 - \Delta u^N = 0, & \text{in } (0,T) \times \Omega_N, \\ u^N = +\infty, & \text{in } (0,T) \times \partial\Omega_N, \\ u^N = 0 & \text{in } \{T\} \times \Omega_N, \end{cases}$$

which is the dynamic programming equation for Problem (NP) when $\mathcal{F} = \mathcal{G} = 0$ and $L(x,q) = \frac{1}{2}|q|^2$ for all $(x,q) \in \mathbb{R}^d \times \mathbb{R}^d$. In section 4.4 we discuss this rigorously. Notice that this method to obtain estimates on the probability v^N by making a logarithmic transformation and studying the stochastic control problem corresponding to the resulting Hamilton-Jacobi-Bellman equation is reminiscent of [67].

The rest of the chapter is organized as follows. First we introduce, in Section 4.1 the N-particle problem and explain what boundary behavior can be expected. Then we briefly recall the main features of the mean-field problem and provide a stability result with respect to the constraint in Section 4.2. We then prove the convergence results in Section 4.3. We start with Theorem 4.2 which gives the more difficult $\limsup_{N\to+\infty} \mathcal{U}_N(\mu_0) \leqslant \overline{\mathcal{U}}(\mu_0)$ and finish with the reverse $\overline{\mathcal{U}}(\mu_0) \leqslant \liminf_{N\to+\infty} \mathcal{U}_N(\mu_0)$ in Theorem 4.3. Finally in Section 4.4 we discuss an application of our result to large deviations theory.

4.1 The system of particles with almost-sure constraints

4.1.1 Assumptions

We first give the assumptions satisfied by L, $\mathcal{F}$, $\mathcal{G}$ and Ψ. For $\mathcal{U} = \mathcal{F}, \mathcal{G}, \Psi$, the map $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ satisfies

$$\mathcal{U} \text{ is a bounded from below, } \mathcal{C}^1 \text{ map and } \frac{\delta \mathcal{U}}{\delta m} \text{ belongs to } \mathcal{C}(\mathcal{P}_2(\mathbb{R}^d), E_{3+\alpha}), \tag{4.2}$$

where $E_{3+\alpha}$ is the subset of $\mathcal{C}^3(\mathbb{R}^d)$ consisting of functions u such that

$$\sup_{x\in\mathbb{R}^d} \frac{|u(x)|}{1+|x|} + \sup_{x\in\mathbb{R}^d} |Du(x)| + \sup_{x\in\mathbb{R}^d} |D^3u(x)| + \sup_{x\neq y\in\mathbb{R}^d} \frac{|D^3u(x) - D^3u(y)|}{|x-y|^\alpha} < +\infty.$$

The Lagrangian L verifies $L(x,q) = \sup_{p\in\mathbb{R}^d} \{-p.q - H(x,p)\}$ for all $(x,q) \in \mathbb{R}^d \times \mathbb{R}^d$ where H satisfies the following conditions for some $C_0 > 0$.

- For all $(x,p) \in \mathbb{R}^d \times \mathbb{R}^d$, $C_0^{-1}|p|^2 - C_0 \leqslant H(x,p) \leqslant C_0|p|^2 + C_0$.
- H belongs to $\mathcal{C}^3(\mathbb{R}^d \times \mathbb{R}^d)$.
- H and its derivatives are bounded on sets of the form $\mathbb{R}^d \times B(0,R)$ for all $R > 0$.
- For all $(x,p) \in \mathbb{R}^d \times \mathbb{R}^d$, $|D_x H(x,p)| \leqslant C_0(1+|p|)$.
- For all $(x,p) \in \mathbb{R}^d \times \mathbb{R}^d$, $\frac{1}{C_0} I_d \leqslant D_{pp} H(x,p) \leqslant C_0 I_d$.

For the constraint, we also assume that Ψ is convex, that it satisfies the regularity condition

$$\left\{ \begin{array}{c} \text{For all } x \in \mathbb{R}^d,\ m \mapsto \dfrac{\delta \Psi}{\delta m}(m,x) \text{ is } \mathcal{C}^1 \text{ with } (x,y) \mapsto \dfrac{\delta^2 \Psi}{\delta m^2}(m,x,y) \\ \text{in } \mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d) \text{ for all } m \in \mathcal{P}_2(\mathbb{R}^d) \text{ and } \dfrac{\delta^2 \Psi}{\delta m^2}(m,x,y) \text{ and its derivatives being} \\ \text{jointly continuous and bounded in } \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d. \end{array} \right. \tag{4.3}$$

and the transversality condition

$$\int_{\mathbb{R}^d} |D_m \Psi(m,x)|^2 dm(x) \neq 0, \text{ whenever } \Psi(m) = 0 \tag{TransCondPsi}$$

We also assume that there is at least one $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\Psi(\mu) < 0$.

For convenience we put all of the above assumptions into

$$\text{Assume the above assumptions.} \tag{A}$$

4.1.2 Statement of the problem

Throughout this section we fix some $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\Psi(\mu_0) < 0$. In its strong formulation, the N-state control problem is described as follows. We fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions and endowed with N independent adapted Brownian motions $(B_t^{i,N})_{i=1,\dots,N}$. We also assume that there are N independent $\mathcal{F}_0$-measurable initial positions $(X_0^{i,N})_{i=1,\dots,N}$ distributed according to μ_0 and independent from the Brownian motions.

For some rate $\gamma_N > 0$ which goes to 0 as $N \to +\infty$ we denote $\mathbb{P}^{\gamma_N}$ the conditional probability with respect to the event $\left\{\Psi(\widehat{\mu}_0^{N,x}) \leqslant -\gamma_N\right\}$.

The controller's problem is to minimize over controls $(\alpha_t^{i,N})_{i=1,\dots,N}$ adapted to the filtration generated by the Brownian motions and the initial positions

$$J^N((\alpha_t^{i,N})_{1\leqslant i\leqslant N}) := \mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\int_0^T \left(\frac{1}{N}\sum_{i=1}^N L(X_t^{i,N}, \alpha_t^{i,N}) + \mathcal{F}(\widehat{\mu}_t^{N,x})\right) dt + \mathcal{G}(\widehat{\mu}_T^{N,x})\right]$$

under the dynamics

$$X_t^{i,N} = X_0^{i,N} + \int_0^t \alpha_s^{i,N} ds + \sqrt{2} B_s^{i,N}$$

where $\widehat{\mu}_t^{N,x} := \dfrac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$ and the particles are constrained to satisfy $\mathbb{P}^{\gamma_N}$-almost surely the inequality

$$\Psi(\widehat{\mu}_t^{N,x}) < 0, \qquad \text{for all } t \in [0,T].$$

We denote by $\mathcal{U}_N(\mu_0)$ the value of the above problem. Notice that the initial positions are i.i.d under $\mathbb{P}$ but not under $\mathbb{P}^{\gamma_N}$.

We also define

$$\Omega_N := \left\{(x^1, \dots, x^N) \in \mathbb{R}^{dN}, \Psi(\frac{1}{N}\sum_{i=1}^N \delta_{x^i}) < 0\right\}.$$

The condition (TransCondPsi) on the Wasserstein gradient of Ψ at the boundary ensures that the closure $\overline{\Omega_N}$ of Ω_N in $(\mathbb{R}^d)^N$ is

$$\overline{\Omega_N} = \left\{(x^1, \dots, x^N) \in (\mathbb{R}^d)^N, \Psi(\frac{1}{N}\sum_{i=1}^N \delta_{x^i}) \leqslant 0\right\}.$$

Similarly we define $\Omega_\infty := \left\{\mu \in \mathcal{P}_2(\mathbb{R}^d), \Psi(\mu) < 0\right\}$ and we have that

$$\overline{\Omega_\infty} := \left\{\mu \in \mathcal{P}_2(\mathbb{R}^d), \Psi(\mu) \leqslant 0\right\}.$$

Remark 4.1. Notice that it could very well happen that $\Omega_N = \varnothing$ for small values of N. However we neglect this detail since we always assume that there is some $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\Psi(\mu_0) < 0$. Approximating μ_0 by atomic measures, we find that Ω_N is not empty for N large enough.

Heuristic concerning the boundary behavior. If we denote by $u^N(t, x^{1,N}, \dots, x^{N,N})$ the value of the same problem but this time initialized at a deterministic position $(x^{1,N}, \dots, x^{N,N}) \in \Omega_N$ at time $t \in [0,T]$, we expect u^N to satisfy

$$\begin{cases} -\partial_t u^N + \frac{1}{N}\sum_{i=1}^N H(x^{i,N}, N D_{x^{i,N}} u^N) - \Delta u^N = \mathcal{F}(\frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}}) & \text{in } [0,T)\times\Omega_N, \\ u^N = +\infty & \text{in } [0,T)\times\partial\Omega_N, \\ u^N(T, x^{1,N}, \dots, x^{N,N}) = \mathcal{G}(\frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}}) & \text{in } \overline{\Omega_N}, \end{cases}$$

Well-posedness of the above equation with such a boundary condition as well as its rigorous connection with the control problem is a challenging question.

In [99], Lasry and Lions studied explosive solutions to

$$\begin{cases} v + \frac{1}{p}|Dv|^p - \Delta v = f & \text{in } \Omega, \\ v = +\infty & \text{in } \partial\Omega, \end{cases}$$

where Ω is the interior of a compact domain of $\mathbb{R}^d$ with a smooth boundary and $p > 1$. In case f is smooth and bounded, they proved that $W^{2,r}(\Omega), \forall r > 1$ explosive solutions exist if and only if $p \leqslant 2$. They also proved that, in this case, there is a unique explosive solution and that it behaves near the boundary like $d(x)^{\frac{p-2}{p-1}}$ if $p < 2$ and like $|\log(d(x))|$ if $p = 2$, where $d(x)$ is the distance to the boundary. When $p < 2$ they proved moreover that this solution is indeed the value function for the related control problem

$$\inf_{(\alpha_t)} \mathbb{E}\left[\int_0^{+\infty} e^{-t}(p-1)p^{-p/(p-1)}|\alpha_t|^p dt + \int_0^{+\infty} f(X_t)dt\right]$$

under the dynamic

$$\begin{cases} dX_t = \alpha_t dt + dB_t, \\ X_0 = x \in \Omega \end{cases} \tag{4.4}$$

and the constraint

$$X_t \in \Omega \qquad \forall t \geqslant 0, \qquad \mathbb{P} - \text{almost-surely.}$$

Leonori and Porretta further proved, in [103], refinements of these results together with the blow-up behavior of the gradient of the solution. In particular, for the quadratic case which is of interest to us, they proved that $v - \log(d(x))$ is bounded and Lipschitz over $\overline{\Omega}$. If we translate these results to our setting, keeping track of the dimension and using Ψ to measure the distance to the boundary, this suggests (at least in the infinite horizon case) that

$$u^N(x^{1,N}, \dots, x^{N,N}) + \frac{1}{N}\log(-\Psi(\frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N}})) \in W^{1,\infty}(\overline{\Omega_N}).$$

Even in the infinite horizon setting we don't try to replicate the aforementioned results because the connection between the solution to the HJB equation and the value function in the quadratic case is not clear. However they give us a good idea of the expected behavior of the system near the boundary.

4.2 The mean-field problem

For some $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\Psi(\mu_0) < 0$, the problem is to minimize

$$J(\alpha, \mu) := \int_0^T \int_{\mathbb{R}^d} L(x, \alpha(t,x)) d\mu(t)(x)dt + \int_0^T \mathcal{F}(\mu(t))dt + \mathcal{G}(\mu(T))$$

over couples $(\mu, \alpha) \in \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d)) \times L^2_{dt\otimes\mu(t)}([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ satisfying in the sense of distributions the Fokker-Planck equation

$$\begin{cases} \partial_t \mu + \operatorname{div}(\alpha\mu) - \Delta\mu = 0 \quad \text{in } (0,T)\times\mathbb{R}^d \\ \mu(0) = \mu_0, \end{cases}$$

under the constraint that $\Psi(\mu(t)) \leqslant 0$ for all $t \in [0,T]$.

In Chapter 3, we showed that optimal controls for this problems exist and are bounded and Lipschitz continuous provided $\Psi(\mu_0) < 0$. We complete the analysis with a stability result with respect to the constraint. To this end we introduce, for all $\delta > 0$ small, $\overline{\mathcal{U}}^\delta(\mu_0)$ the value of the same problem associated to the constraint $\Psi(\mu(t)) \leqslant -\delta$ for all $t \in [0,T]$.

Proposition 4.1. *Assume Assumption* (A) *and assume as well that* $\Psi(\mu_0) < 0$. *Then it holds*

$$\lim_{\delta\to 0} \overline{\mathcal{U}}^\delta(\mu_0) = \overline{\mathcal{U}}(\mu_0).$$

Proof. Using the controllability result of Lemma 3.4, in Chapter 3, we know that $\overline{\mathcal{U}}^\delta(\mu_0)$ is uniformly bounded for $\delta \in [0, -\frac{\Psi(\mu_0)}{2}]$. By standard estimates, using the coercivity of L and the fact that μ_0 belongs to $\mathcal{P}_2(\mathbb{R}^d)$ we find some $R > 0$ such that, for all $\delta \in [0, -\frac{\Psi(\mu_0)}{2}]$, for all $t \in [0,T]$,

$$\int_{\mathbb{R}^d} |x|^2 d\mu^\delta(t)dt < R^2$$

whenever $(\mu^\delta(t))_{0\leqslant t\leqslant T}$ is optimal for $\overline{\mathcal{U}}^\delta(\mu_0)$. Now we use Assumption (TransCondPsi) as well as the continuity of $\mu \mapsto \Psi(\mu)$ and $\mu \mapsto \int_{\mathbb{R}^d} |D_m\Psi(\mu)|^2 d\mu(x)$ with respect to d_1, to conclude that there exists some $\eta_R > 0$ such that $\int_{\mathbb{R}^d} |D_m\Psi(\mu,x)|^2 d\mu(x) \geqslant \eta_R$, whenever $|\Psi(\mu)| \leqslant \eta_R$ and $\int_{\mathbb{R}^d} |x|^2 d\mu(x) < 2R^2$. If we define $\delta^0 := \frac{\eta_R}{4}$, it is plain to check that $\int_{\mathbb{R}^d} |D_m\Psi(\mu,x)|^2 d\mu(x) \geqslant \eta_R$ whenever $\delta \in [0,\delta^0]$, $|\Psi(\mu) + \delta| \leqslant \frac{\eta_R}{4}$ and $\int_{\mathbb{R}^d} |x|^2 d\mu(x) \leqslant 2R^2$. As a consequence, we can apply Theorem 3.2 in Chapter 3 to $\overline{\mathcal{U}}^\delta(\mu_0)$ and conclude that there is some $\epsilon_0 > 0$ such that, for all $\delta \in [0,\delta^0]$

$$\overline{\mathcal{U}}^{\delta}(\mu_0) = \inf_{(\alpha,\mu)} \int_0^T \int_{\mathbb{R}^d} L(x,\alpha(t,x))d\mu(t)(x)dt + \int_0^T \mathcal{F}(\mu(t))dt + \mathcal{G}(\mu(T)) + \frac{1}{\epsilon_0}\left[\int_0^T (\Psi(\mu(t)+\delta)^+)dt + (\Psi(\mu(T))+\delta)^+\right] \tag{4.5}$$

where the infimum is taken over the couples (α,μ) in $\mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R}^d)) \times L^2_{dt\otimes\mu(t)}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ satisfying

$$\begin{cases} \partial_t\mu + \operatorname{div}(\alpha\mu) - \Delta\mu = 0 \quad \text{in } (0,T)\times\mathbb{R}^d, \\ \mu(0) = \mu_0, \end{cases}$$

but not necessarily the state constraint. Now consider $(\widetilde{\mu},\widetilde{\alpha})$ an optimal solution for $\overline{\mathcal{U}}(\mu_0)$. On the one hand it is obvious that

$$\overline{\mathcal{U}}(\mu_0) \leqslant \overline{\mathcal{U}}^{\delta}(\mu_0)$$

for any $\delta \geqslant 0$. On the other hand, for $\delta \in [0,\delta^0]$, using $(\widetilde{\mu},\widetilde{\alpha})$ as a candidate in (4.5), it comes

$$\begin{aligned} \overline{\mathcal{U}}^{\delta}(\mu_0) - \overline{\mathcal{U}}(\mu_0) &\leqslant \frac{1}{\epsilon_0}\left[\int_0^T (\Psi(\widetilde{\mu}(t))+\delta)^+)dt + (\Psi(\widetilde{\mu}(T))+\delta)^+\right] \\ &\leqslant \frac{(T+1)\delta}{\epsilon_0}. \end{aligned}$$

As a consequence, we have the limit

$$\lim_{\delta\to 0} \overline{\mathcal{U}}^{\delta}(\mu_0) = \overline{\mathcal{U}}(\mu_0),$$

which concludes the proof of the proposition.

□

4.3 Mean field limit

The main result of this section is to prove Theorem (4.1), that is the convergence of $\mathcal{U}^N(\mu_0)$ to $\overline{\mathcal{U}}(\mu_0)$ as $N \to +\infty$.

4.3.1 From mean-field to almost-sure constraint

In this section we prove the first inequality in Theorem (4.1).

Theorem 4.2. *Let Assumption* (A) *hold. Assume further that* μ_0 *satisfies* $\Psi(\mu_0) < 0$ *and* $\int_{\mathbb{R}^d} |x|^{d+5} d\mu_0(x) < +\infty$. *Then it holds that*

$$\limsup_{N\to+\infty} \mathcal{U}^N(\mu_0) \leqslant \overline{\mathcal{U}}(\mu_0),$$

whenever $\gamma_N = N^{-\theta}$ *with* $\theta < \frac{1}{d+8}$.

Proof. To prove Theorem (4.2) we proceed as follows. First we fix $\delta > 0$ and we take $\alpha : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ to be an optimal control for $\overline{\mathcal{U}}^{\delta}(\mu_0)$. Using Theorem 3.4 in Chapter 3, we know that α is bounded and Lipschitz continuous in the space variable uniformly in time. We let μ be the corresponding trajectory, solution to

$$\begin{cases} \partial_t \mu + \operatorname{div}(\alpha\mu) - \Delta\mu = 0 \quad \text{in } (0,T) \times \mathbb{R}^d, \\ \mu(0) = \mu_0. \end{cases}$$

In particular, $\Psi(\mu(t)) \leqslant -\delta$ for all $t \in [0,T]$. We let $(X_t^{1,N}, \dots, X_t^{N,N})_{0 \leqslant t \leqslant T}$ be the solution to

$$X_t^{i,N} = X_0^{i,N} + \int_0^{t \wedge \tau_N} \alpha(s, X_s^{i,N}) dt + \int_{t \wedge \tau_N}^t \beta_t^{i,N} dt + \sqrt{2} B_t^{i,N}$$

where $\tau_N := \inf\{t \geqslant 0, \Psi(\widehat{\mu}_t^{N,x}) \geqslant -\frac{\gamma_N}{2}\}$, with the convention $\inf\{\varnothing\} = +\infty$, and $\beta_t^{i,N}$ is the feedback control, defined for all $t \geqslant \tau_N \wedge T$ by

$$\beta_t^{i,N} = \frac{4(X_t^{i,N} - X_{\tau_N \wedge T}^{i,N})}{\sum_{i=1}^N |X_t^{i,N} - X_{\tau_N \wedge T}^{i,N}|^2 - r_N^2 N} - 2\frac{d}{r_N^2}(X_t^{i,N} - X_{\tau_N \wedge T}^{i,N}),$$

with $r_N = \frac{\gamma_N}{4C_\Psi}$ and C_Ψ a Lipschitz constant for Ψ with respect to d_2.

Lemma 4.1. *$\mathbb{P}^{\gamma_N}$-almost-surely, it holds that,*

$$\frac{1}{N}\sum_{i=1}^N \left| X_t^{i,N} - X_{\tau_N \wedge T}^{i,N} \right|^2 \leqslant r_N^2, \qquad \forall t \geqslant \tau_N \wedge T.$$

Moreover, the following estimate holds

$$\mathbb{E}^{\mathbb{P}^{\gamma_N}} \left[\int_{\tau_N \wedge T}^T \frac{1}{N} \sum_{i=1}^N \left| \beta_t^{i,N} \right|^2 dt \right] \leqslant \frac{16d}{r_N^2 N} \mathbb{E}^{\mathbb{P}^{\gamma_N}} \left[e^{T - T \wedge \tau_N} \right] + \frac{8d^2}{r_N^2} \mathbb{E}^{\mathbb{P}^{\gamma_N}} \left[T - T \wedge \tau_N \right].$$

We continue with the ongoing proof. We have taken r_N and β_N is such a way that $\mathbb{P}^{\gamma^N}$-almost-surely

$$\Psi(\widehat{\mu}_t^{N,x}) \leqslant \frac{-\gamma_N}{4} \qquad \forall t \in [0,T].$$

Indeed, by definition of τ_N, $\mathbb{P}^{\gamma_N}$-almost-surely,

$$\Psi(\widehat{\mu}_t^{N,x}) \leqslant -\frac{\gamma_N}{2} \qquad \text{for all } t \leqslant \tau_N,$$

and, $\mathbb{P}^{\gamma_N}$-almost-surely, by definition of r_N and Lemma 4.1, it holds, whenever $t \geqslant \tau_N$

$$\begin{aligned} \left| \Psi(\widehat{\mu}_t^{N,x}) - \Psi(\widehat{\mu}_{\tau_N}^{N,x}) \right| &\leqslant C_\Psi d_2(\widehat{\mu}_t^{N,x}, \widehat{\mu}_{\tau_N}^{N,x}) \\ &\leqslant \frac{C_\Psi}{\sqrt{N}} \left(\sum_{i=1}^N |X_t^{i,N} - X_{\tau_N}^{i,N}|^2 \right)^{1/2} \\ &\leqslant C_\Psi \times r_N \leqslant \frac{\gamma_N}{4} \end{aligned}$$

and, as a consequence, being $\Psi(\widehat{\mu}^{N,x}_{\tau_N}) = -\frac{\gamma_N}{2}$, it holds that $\Psi(\widehat{\mu}^{N,x}_t) \leqslant -\frac{\gamma_N}{4}$. Therefore, we have an admissible control for $\mathcal{U}^N(\mu_0)$.

Now, by standard propagation of chaos estimates (see [86]) it holds that

$$\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_N]} d_2^2(\mu(t),\widehat{\mu}^{N,x}_t)\right] \leqslant CN^{-\frac{2}{d+8}}.$$

As a consequence, we get

$$\begin{aligned}
\mathbb{P}^{\gamma_N}\left[\tau_N < T\right] &= \mathbb{P}^{\gamma_N}\left[\exists t < T, \Psi(\widehat{\mu}^{N,x}_t) \geqslant \frac{-\gamma_N}{2}\right] \\
&\leqslant \mathbb{P}^{\gamma_N}\left[\exists t < \tau_N, \Psi(\widehat{\mu}^{N,x}_t) \geqslant \frac{-3\gamma_N}{4}\right] \\
&\leqslant \mathbb{P}^{\gamma_N}\left[\sup_{t\in[0,T\wedge\tau_N]} d_2^2(\widehat{\mu}^{N,x}_t,\mu(t)) \geqslant \frac{1}{C_\Psi^2}(\delta - \frac{3\gamma_N}{4})^2\right],
\end{aligned}$$

where we use the facts that $\Psi(\mu(t)) \leqslant -\delta$, for all $t \in [0,T]$ as well as

$$|\Psi(\widehat{\mu}^{N,x}_t) - \Psi(\mu(t))| \leqslant C_\Psi d_2(\widehat{\mu}^{N,x}_t,\mu(t)).$$

Using Markov's inequality we conclude that,

$$\begin{aligned}
\mathbb{P}^{\gamma_N}\left[\tau_N < T\right] &\leqslant \frac{C_\Psi^2 \mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\sup_{t\in[0,T\wedge\tau_N]} d_2^2(\widehat{\mu}^{N,x}_t,\mu(t))\right]}{(\delta - \frac{3\gamma_N}{4})^2} \\
&\leqslant \frac{C_\Psi^2 \mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_N]} d_2^2(\widehat{\mu}^{N,x}_t,\mu(t))\right]}{(\delta - \frac{3\gamma_N}{4})^2 \mathbb{P}\left[\Psi(\widehat{\mu}^{N,x}_0) \leqslant -\gamma_N\right]} \\
&\leqslant CN^{-\frac{2}{d+8}},
\end{aligned}$$

for some $C > 0$ independent of N. We deduce immediately that

$$\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[T - T\wedge\tau_N\right] \leqslant T\mathbb{P}^{\gamma_N}\left[\tau_N < T\right] \leqslant CN^{-\frac{2}{d+8}}.$$

Using Lemma (4.1), we get

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\int_{\tau_N\wedge T}^T \frac{1}{N}\sum_{i=1}^N |\beta_t^{i,N}|^2\right] &\leqslant \frac{C}{r_N^2 N} + C\frac{\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[T - T\wedge\tau_N\right]}{r_N^2} \\
&\leqslant C\frac{N^{-\frac{2}{d+8}}}{\gamma_N^2}.
\end{aligned}$$

But we have chosen γ_N so that $\lim_{N\to+\infty} \frac{N^{-\frac{2}{d+8}}}{\gamma_N^2} = 0$. Therefore it holds that

$$\lim_{N\to+\infty} \mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\int_{\tau_N\wedge T}^{T} \frac{1}{N}\sum_{i=1}^{N} |\beta_t^{i,N}|^2\right] = 0.$$

We easily deduce that,

$$\lim_{N\to+\infty} \mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\sup_{t\in[0,T]} d_2^2(\mu(t), \widehat{\mu}_t^{N,x})\right] = 0.$$

As a consequence, α being bounded and $\mathcal{F}$ and $\mathcal{G}$ Lipschitz continuous with respect to d_1,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\int_0^{T\wedge\tau_N} \frac{1}{N}\sum_{i=1}^{N} L(X_t^{i,N}, \alpha(t, X_t^{i,N}))dt + \int_0^{T\wedge\tau_N} \mathcal{F}(\widehat{\mu}_t^{N,x})dt\right] \\
&+ \mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\int_{T\wedge\tau_N}^{T} \frac{1}{N}\sum_{i=1}^{N} L(X_t^{i,N}, \beta_t^{i,N}))dt + \int_{T\wedge\tau_N}^{T} \mathcal{F}(\widehat{\mu}_t^{N,x})dt\right] \\
&+ \mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\mathcal{G}(\widehat{\mu}_T^{N,x})\right] = \int_0^T \int_{\mathbb{R}^d} L(x, \alpha(t,x))d\mu(t)(x)dt + \int_0^T \mathcal{F}(\mu(t))dt + \mathcal{G}(\mu(T)) + o(1)
\end{aligned}$$

as $N \to +\infty$. Finally, being α optimal for $\overline{\mathcal{U}}^{\delta}(\mu_0)$ we have that

$$\limsup_{N\to+\infty} \mathcal{U}^N(\mu_0) \leqslant \overline{\mathcal{U}}^{\delta}(\mu_0).$$

Yet, we have proved, in Proposition (4.1) that $\lim_{\delta\to 0} \overline{\mathcal{U}}^{\delta}(\mu_0) = \overline{\mathcal{U}}(\mu_0)$ and therefore,

$$\limsup_{N\to+\infty} \mathcal{U}^N(\mu_0) \leqslant \overline{\mathcal{U}}(\mu_0),$$

which concludes the proof of the theorem.

□

Proof of Lemma (4.1) . For $\eta \geqslant 0$ small, we introduce the stopping time

$$\tau^{\eta} := \inf\{t \geqslant \tau_N \wedge T, \frac{1}{N}\sum_{i=1}^{N} |X_t^{i,N} - X_{\tau_N\wedge T}^{i,N}|^2 \geqslant r_N^2 - \eta\},$$

with the convention that $\inf\{\varnothing\} = +\infty$.

For $\eta > 0$ and $T' > T$, we write $\mathbf{B}_t =^t (B_t^{1,N}, \ldots, B_t^{N,N})$ and $\mathbf{Y}_t =^t (X_t^{i,N}, \ldots, X_t^{N,N})$ and

apply Itô's lemma to get

$$
\begin{aligned}
-e^{-\tau^\eta\wedge T'}\log(r_N^2-\frac{|\mathbf{Y}_{\tau^\eta\wedge T'}-\mathbf{Y}_{\tau_N\wedge T}|^2}{N}) &= -e^{-\tau_N\wedge T}\log(r_N^2)+\int_{\tau_N\wedge T}^{\tau^\eta\wedge T'}e^{-t}\log(r_N^2-\frac{|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}{N})dt\\
&+\int_{\tau_N\wedge T}^{\tau^\eta\wedge T'}e^{-t}\left[\frac{4(\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T})}{|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2-r_N^2N}.\frac{2(\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T})}{Nr_N^2-|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}-\frac{2d}{r_N^2}\frac{2|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}{Nr_N^2-|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}\right]dt\\
&+\int_{\tau_N\wedge T}^{\tau^\eta\wedge T'}e^{-t}\left[\frac{2dN}{Nr_N^2-|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}+\frac{4|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}{(Nr_N^2-|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2)^2}\right]dt\\
&+\sqrt{2}\int_{\tau_N\wedge T}^{\tau^\eta\wedge T'}\frac{2(\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T})}{Nr_N^2-|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}.d\mathbf{B}_t\\
&\leqslant -4\int_{\tau_N\wedge T}^{\tau^\eta\wedge T'}e^{-t}\frac{|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}{(|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2-r_N^2N)^2}dt+\int_{\tau_N\wedge T}^{\tau^\eta\wedge T'}e^{-t}\left[-\frac{4d}{r_N^2}\frac{|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2-dN}{Nr_N^2-|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}\right]dt\\
&+\sqrt{2}\int_{\tau_N\wedge T}^{\tau^\eta\wedge T'}\frac{2(\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T})}{Nr_N^2-|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}.d\mathbf{B}_t
\end{aligned}
$$

However, an elementary analysis reveals that

$$
-\frac{4d}{r_N^2}\frac{|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2-dN}{Nr_N^2-|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}\leqslant\frac{2d}{r_N^2}
$$

whenever $0\leqslant|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2<Nr_N^2$. Therefore, we get, multiplying by $e^{\tau_N\wedge T}$ and taking expectations,

$$
\begin{aligned}
-\mathbb{E}^{\mathbb{P}^{\gamma_N}}&\left[e^{\tau_N\wedge T-\tau^\eta\wedge T'}\log(r_N^2-\frac{|\mathbf{Y}_{\tau^\eta\wedge T'}-\mathbf{Y}_{\tau_N\wedge T}|^2}{N})\right]\\
&+4\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\int_{\tau_N\wedge T}^{\tau^\eta\wedge T'}e^{\tau_N\wedge T-t}\frac{|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}{(|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2-r_N^2N)^2}dt\right]\leqslant\frac{2d}{r_N^2}.
\end{aligned}
$$

Letting $T'\to+\infty$, using the definition of τ^η and Lebesgue dominated convergence theorem leads to

$$
-\log(\eta)\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[e^{\tau_n\wedge T-\tau^\eta}\mathbb{1}_{\{\tau^\eta<+\infty\}}\right]+4\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\int_{\tau_N\wedge T}^{\tau^\eta}e^{\tau_N\wedge T-t}\frac{|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}{(|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2-r_N^2N)^2}dt\right]\leqslant\frac{2d}{r_N^2}. \tag{4.6}
$$

Notice that both terms in the left-hand side of 4.6 are non-negative for $\eta\leqslant 1$. Letting $\eta\to 0$, we get, on the one hand that $\tau^0=+\infty$, $\mathbb{P}^{\gamma_N}$-almost surely and, on the other hand, we obtain

$$
4\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\int_{\tau_N\wedge T}^{+\infty}e^{\tau_N\wedge T-t}\frac{|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2}{(|\mathbf{Y}_t-\mathbf{Y}_{\tau_N\wedge T}|^2-r_N^2N)^2}dt\right]\leqslant\frac{2d}{r_N^2}.
$$

It follows that,

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\int_{\tau_N\wedge T}^{T}\frac{1}{N}\left|\frac{4(\mathbf{Y}_t-\mathbf{Y}_{\tau_N})}{|\mathbf{Y}_t-\mathbf{Y}_{\tau_N}|^2-r_N^2N}-2\frac{d}{r_N^2}(\mathbf{Y}_t-\mathbf{Y}_{\tau_N})\right|^2dt\right]\\
&\quad\leqslant 2\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\int_{\tau_N\wedge T}^{T}\frac{1}{N}\left|\frac{4(\mathbf{Y}_t-\mathbf{Y}_{\tau_N})}{|\mathbf{Y}_t-\mathbf{Y}_{\tau_N}|^2-r_N^2N}\right|^2dt\right]+2\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\int_{\tau_N\wedge T}^{T}\frac{1}{N}\left|2\frac{d}{r_N^2}(\mathbf{Y}_t-\mathbf{Y}_{\tau_N})\right|^2dt\right]\\
&\quad\leqslant \frac{16d^2}{r_N^2N}\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[e^{T-T\wedge\tau_N}\right]+\frac{8d^2}{r_N^4}\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[\int_{\tau_N\wedge T}^{T}\frac{1}{N}|\mathbf{Y}_t-\mathbf{Y}_{\tau_N}|^2dt\right]\\
&\quad\leqslant \frac{16d^2}{r_N^2N}\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[e^{T-T\wedge\tau_N}\right]+\frac{8d^2}{r_N^2}\mathbb{E}^{\mathbb{P}^{\gamma_N}}\left[T-T\wedge\tau_N\right],
\end{aligned}
$$

where we used, for the last inequality, the fact that, $\mathbb{P}^{\gamma_N}$-almost-surely, for all $t\geqslant\tau_N$,

$$|\mathbf{Y}_t-\mathbf{Y}_{\tau_N}|^2\leqslant Nr_N^2.$$

This concludes the proof of the lemma.

□

4.3.2 From almost-sure constraint to mean-field constraint

To prove the second inequality we rely on compactness methods developed, in the context of Large Deviations by Budhiraja, Dupuis and Fischer [27] and, in the context of mean-field control, by Lacker [96] and Djete, Possamaï and Tan [59]. To this end we need to introduce suitable weak formulations of the control problems.

Let us first introduce some notations. We denote by $\mathcal{C}^d:=\mathcal{C}([0,T],\mathbb{R}^d)$ the path space. The control space $\mathcal{V}$ is defined as the set of non-negative measures q over $[0,T]\times\mathbb{R}^d$ with the Lebesgue measure as time marginal and such that

$$\int_{[0,T]\times\mathbb{R}^d}|a|^2dq(t,a)<+\infty.$$

We denote by $(X^{i,N},\Lambda^{i,N})$ the canonical process on $(\mathcal{C}^d\times\mathcal{V})^N$ and define the empirical measures

$$\widehat{\mu}^N:=\frac{1}{N}\sum_{i=1}^N\delta_{(X^{i,N},\Lambda^{i,N})},\qquad \widehat{\mu}_t^{N,x}:=\frac{1}{N}\sum_{i=1}^N\delta_{X_t^{i,N}}.$$

We define $\mathcal{R}^N$ as the set of probabilities $P_N\in\mathcal{P}_2((\mathcal{C}^d\times\mathcal{V})^N)$ under which $(X_0^{i,N})_{i=1,\dots,N}$ are iid with law μ_0 and

$$\varphi(X_t^{1,N},\dots,X_t^{N,N})-\sum_{i=1}^N\int_0^t\int_{\mathbb{R}^d}\mathcal{L}_i^N\varphi(X_s^{1,N},\dots,X_s^{N,N},a)d\Lambda_s^{i,N}(a)ds$$

is a martingale under P_N, for all smooth, compactly supported φ with

$$\mathcal{L}_i^N\varphi(x_1,\dots,x_N,a):=D_{x_i}\varphi(x_1,\dots,x_N).a+\Delta_{x_i}\varphi(x_1,\dots,x_N).$$

The control rule P_N is also assumed to satisfy

$$P_N^{\gamma_N}(\Psi(\widehat{\mu}_{t,x}^N) \leqslant 0, \forall t \geqslant 0) = 1.$$

where we wrote, for simplicity $P_N^{\gamma_N} := P_N\left[.|\Psi(\widehat{\mu}_0^{N,x}) \leqslant -\gamma_N\right]$. The N-state problem in its weak formulation is therefore to minimize over $P_N \in \mathcal{R}^N$

$$\mathbb{E}^{P_N^{\gamma_N}}\left[\int_0^T \left(\int_{\mathbb{R}^d} \frac{1}{N}\sum_{i=1}^N L(X_t^{i,N}, a) d\Lambda_t^{i,N}(a) + \mathcal{F}(\widehat{\mu}_{t,x}^N)\right) dt + \mathcal{G}(\widehat{\mu}_T^{N,x})\right]$$

where $\mathbb{E}^{P_N^{\gamma_N}}$ is the expectation under $P_N^{\gamma_N}$.

Similarly, for the mean-field problem we introduce the controlled martingale formulation. The control problem is described as follows. We let (X, Λ) be the identity processes over $(\mathcal{C}^d \times \mathcal{V})$ and we look for probabilities m over $\mathcal{C}^d \times \mathcal{V}$ such that X_0 is distributed according to μ_0 under m,

$$\varphi(X_t) - \int_0^t \int_{\mathbb{R}^d} \mathcal{L}\varphi(X_s, a) d\Lambda_s(a) ds$$

is a martingale under m for all smooth compactly supported $\varphi : \mathbb{R}^d \to \mathbb{R}$, with $\mathcal{L}\varphi(x, a) = D\varphi(x).a + \Delta\varphi(x)$. The measure m is also assumed to satisfy the constraint

$$\Psi(X_t \# m) \leqslant 0 \ \forall t \geqslant 0.$$

We denote by $\mathcal{R}$ the set of such measures and we look for $m \in \mathcal{R}$ which minimizes the cost function

$$\Gamma(m) := \mathbb{E}^m\left[\int_0^T \int_{\mathbb{R}^d} L(X_t, a) d\Lambda_t(a) dt\right] + \int_0^T \mathcal{F}(X_t \# m) dt + \mathcal{G}(X_T \# m).$$

Before going on with the main result of this section, we make two remarks. The first one is that the value of the relaxed problem is no greater than $\mathcal{U}_N(\mu_0)$. Indeed any "strong" control induces a weak control with a lower cost. On the other hand, it is easier to show that the value of the relaxed mean-field problem is equal to $\overline{\mathcal{U}}(\mu_0)$. This follows for instance from the fact that $\overline{\mathcal{U}}(\mu_0)$ is the value of an optimal control problem without constraint but with a strong enough penalization. And for problems without constraint, the equivalence between the different formulations is well known, see [96].

We are ready to prove the desired inequality:

Theorem 4.3. *Take P_N a sequence of ϵ_N-optimal solutions to the relaxed N-particles problem, for some sequence $\epsilon_N \to 0$. Then the sequence $\mu^N \# P_N^{\gamma_N}$ is relatively compact in $\mathcal{P}_p(\mathcal{P}_p(C^d \times \mathcal{V}))$ for every $p \in (1, 2)$. Every limit point is supported on the set of solutions to the relaxed mean-field problem and it holds that*

$$\overline{\mathcal{U}}(\mu_0) \leqslant \liminf_{N \to +\infty} \mathcal{U}^N(\mu_0).$$

Proof. We will closely follow the steps of [96] and therefore we only highlight the differences due to the constraint. In light of [95] Corollary B.2, to prove the pre-compactness of $\mu^N \# P_N^{\gamma_N}$, it suffices to prove that the mean measures $\frac{1}{N}\sum_{i=1}^N (X^{i,N}, \Lambda^{i,N}) \# P_N^{\gamma_N}$ are tight and to prove that

$$\sup_N \mathbb{E}^{P_N^{\gamma_N}} \frac{1}{N}\sum_{i=1}^N \left[\sup_{t\in[0,T]} |X_t^{i,N}|^2 + \int_0^T \int_{\mathbb{R}^d} |a|^2 d\Lambda_t^{i,N}(a) dt \right] < +\infty. \tag{4.7}$$

The tightness of the mean measures actually follows from (4.7) thanks to the compactness result of Proposition 3.5 in [96] noticing that a martingale under P_N remains a martingale under $P_N^{\gamma_N}$. By standard estimates, it is enough to prove

$$\sup_N \mathbb{E}^{P_N^{\gamma_N}} \left[\frac{1}{N}\sum_{i=1}^N |X_0^{i,N}|^2 \right] < +\infty$$

as well as

$$\sup_N \mathbb{E}^{P_N^{\gamma_N}} \left[\int_0^T \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^d} |a|^2 d\Lambda_t^{i,N}(a) dt \right] < +\infty$$

in order to get (4.7). The former follows from

$$\mathbb{E}^{P_N^{\gamma_N}} \left[\frac{1}{N}\sum_{i=1}^N |X_0^{i,N}|^2 \right] \leqslant \frac{\mathbb{E}^{P_N}\left[|X_0^{1,N}|^2 \right]}{P_N\left[\Psi(\widehat{\mu}_0^{N,x}) \leqslant -\gamma_N \right]} \leqslant 2\int_{\mathbb{R}^d} |x|^2 d\mu_0(x)$$

which holds for N large enough since $P_N\left[\Psi(\widehat{\mu}_0^{N,x}) \leqslant -\gamma_N \right] \to 1$ as $N \to +\infty$. The latter follows from the coercivity of L, the boundness of $\mathcal{F}$, $\mathcal{G}$ and the fact that we took the P_N as ϵ_N-optimal solutions for the N-particle problem whose values are bounded independently from N (as proved in Theorem (4.2)). Now we take a limit point $P \in \mathcal{P}_p(\mathcal{P}_p(\mathcal{C}^d \times \mathcal{V}))$ and prove that P is supported on the set of solutions to the mean field relaxed problem. First we have that $\widehat{\mu}_0^{N,x} \# P_N^{\gamma_N} \to \delta_\mu$ in $\mathcal{P}_p(\mathcal{P}_p(\mathbb{R}^d))$. This follows from Glivenko-Cantelli law of large numbers since $\frac{1}{N}\sum_{i=1}^N \mathbb{E}^{P_N^{\gamma_N}}\left[|X_0^{i,N}|^2 \right]$ is bounded independently from N and, for all $f \in \mathcal{C}_b(\mathcal{P}_p(\mathbb{R}^d))$, as $N \to +\infty$

$$\left| \mathbb{E}^{P_N^{\gamma_N}}\left[f(\widehat{\mu}_0^{N,x}) \right] - \mathbb{E}^{P_N}\left[f(\widehat{\mu}_0^{N,x}) \right] \right| \leqslant 2\|f\|_\infty (1 - P_N\left[\Psi(\widehat{\mu}_0^{N,x}) \leqslant -\gamma_N \right]) \to 0.$$

Following [96] Proposition 5.2 we have that P is supported on the set of measures solution to the martingale problem. It remains to show that the constraint is satisfied P-almost surely at the limit. By continuity of Ψ, for all $t \in [0,T]$ it holds that

$$P\left(\left\{ m \in \mathcal{P}_p(\mathcal{C}^d \times \mathcal{V}), \Psi(X_t \# m) \leqslant 0 \right\} \right) \geqslant \limsup_{N\to+\infty} P_N^{\gamma_N}(\Psi(\widehat{\mu}_t^{N,x}) \leqslant 0) = 1.$$

Since P-almost surely m satisfies the martingale problem, we have that P-almost surely $t \to X_t \# m$ is continuous and therefore we have that

$$P\left(\left\{ m \in \mathcal{P}_p(\mathcal{C}^d \times \mathcal{V}), \quad \Psi(X_t \# m) \leqslant 0 \quad \forall t \in [0,T] \right\} \right) = 1.$$

The fact that P is supported on the set of optimal solutions of the mean-field relaxed problem follows from the lower semi-continuity of the cost functional as proved in [96] Lemma 4.1 and Theorem (4.2). Indeed it holds that

$$\begin{aligned}\int_{\mathcal{P}_p(\mathcal{C}^d\times\mathcal{V})}\Gamma(\mu)dP(\mu) &\leqslant \liminf_{N\to+\infty}\mathbb{E}^{P_N^{\gamma_N}}\left[\Gamma(\widehat{\mu}^N)\right]\\ &\leqslant \limsup_{N\to+\infty}\mathcal{U}_N(\mu_0)\leqslant\overline{\mathcal{U}}(\mu_0)\end{aligned}$$

and we already pointed out, before stating Theorem (4.3), that $\overline{\mathcal{U}}(\mu_0)=\inf_{m\in\mathcal{R}}\Gamma(m)$. Finally we have that

$$\overline{\mathcal{U}}(\mu_0)\leqslant\liminf_{N\to+\infty}\mathbb{E}^{P_N^{\gamma_N}}\left[\Gamma(\widehat{\mu}^N)\right]\leqslant\liminf_{N\to+\infty}\mathcal{U}_N(\mu_0)$$

which concludes the proof of the theorem. □

4.4 Application to Large Deviations

Throughout this section we take $\mathcal{F}=\mathcal{G}=0$ and $L(x,q)=\frac{1}{2}|q|^2$ for all $(x,q)\in\mathbb{R}^d\times\mathbb{R}^d$. In this setting, $\overline{\mathcal{U}}(\mu_0)$ reads

$$\overline{\mathcal{U}}(\mu_0)=\inf_{(\alpha,\mu)}\int_0^T\int_{\mathbb{R}^d}\frac{1}{2}|\alpha(t,x)|^2d\mu(t)(x)dt \tag{4.8}$$

where the infimum is taken over the couples (α,μ) satisfying

$$\begin{cases}\mu\in\mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R}^d)),\alpha\in L^2_{dt\otimes\mu(t)}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)\\ \partial_t\mu+\mathrm{div}(\alpha\mu)-\Delta\mu=0 & \text{in } (0,T)\times\mathbb{R}^d,\\ \mu(0)=\mu_0\in\mathcal{P}_2(\mathbb{R}^d),\\ \Psi(\mu(t))\leqslant 0 & \forall t\in[0,T].\end{cases}$$

For all $N\geqslant 1$, we fix a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ supporting N d-dimensional independent standard Brownian motions $(B_t^{1,N},\dots,B_t^{N,N})$. For $\mathbf{x}^N=(x^{1,N},\dots,x^{N,N})$ in Ω^N, we define $u^N(0,\mathbf{x}^N)$ as follows

$$u^N(0,\mathbf{x}^N):=\inf_{(\alpha_t^i)_{1\leqslant i\leqslant N}}\mathbb{E}\left[\int_0^T\frac{1}{N}\sum_{i=1}^N\frac{1}{2}|\alpha_t^{i,N}|^2dt\right]$$

where the infimum is taken over controls $(\alpha^{i,N})_{i=1,\dots,N}$ adapted to the filtration generated by the Brownian motions, such that the dynamics

$$X_t^{i,N}=x^{i,N}+\int_0^t\alpha_s^{i,N}ds+\sqrt{2}B_t^{i,N},$$

satisfy, $\mathbb{P}$-almost surely the inequality

$$\Psi(\widehat{\mu}_t^{N,x})\leqslant 0,\qquad\text{for all } t\in[0,T],$$

where $\widehat{\mu}_t^{N,x} := \dfrac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}$.

We assume that

$$\text{there is some } x_0 \in \mathbb{R}^d \text{ such that } \Psi(\delta_{x_0}) < 0. \tag{4.9}$$

As a special case of Theorem (4.1) we have the following result.

Proposition 4.2. *For all $N \geqslant 1$, take $\boldsymbol{x}_0^N = (x_0, \dots, x_0)$ such that $\Psi(\delta_{x_0}) < 0$. Then it holds that*

$$\lim_{N \to +\infty} u^N(0, \boldsymbol{x}_0^N) = \overline{\mathcal{U}}(\delta_{x_0}).$$

Notice that, the initial position being deterministic and strictly inside the constraint, there is no need for conditioning as we did in the previous sections. Equivalently, $\mathbb{P}^{\gamma_N} = \mathbb{P}$ for N large enough.

In the rest of this section we assume, for simplicity, that

$$\begin{gathered}\text{The constraint } \{\Psi \leqslant 0\} \text{ is contained inside the ball} \\ \left\{m \in \mathcal{P}_2(\mathbb{R}^d), d_1(m, m_0) \leqslant R\right\} \text{ for some } R > 0.\end{gathered} \tag{4.10}$$

As a consequence, the constraints $\Omega_N := \left\{(x_1, \dots, x_N) \in \mathbb{R}^{dN}, \Psi(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}) < 0\right\}$ are bounded for all $N \geqslant 1$. We also assume that

$$\begin{gathered}\frac{\delta^2 \Psi}{\delta m^2} \text{ has a linear derivative,} \\ \text{with bounded and jointly continuous first order derivatives in the euclidean variables.}\end{gathered} \tag{4.11}$$

Under Assumption (A) as well as these additional assumptions, for all $N \geqslant 1$, the constraint Ω_N is open, bounded and $\partial\Omega_N$ is a manifold of class $\mathcal{C}^3$.

For $(t, \mathbf{x}^N = (x^{1,N}, \dots, x^{N,N})) \in [0,T] \times (\mathbb{R}^d)^N$ we introduce the probability

$$v^N(t, \mathbf{x}^N) := \mathbb{P}(\forall s \in [0,t], \Psi\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^{i,N} + \sqrt{2}B_t^{i,N}}\right) < 0)$$

where $(B_t^{1,N}, \dots, B_t^{N,N})$ are N independent d-dimensional standard Brownian motions supported on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We are precisely in the framework of [68] section VI.6 and we can conclude that v^N is $\mathcal{C}^{1,2}$ in $(0,T] \times \Omega_N$ and satisfies

$$\begin{cases} \partial_t v^N - \Delta v^N = 0, & \text{in } (0,T) \times \Omega_N \\ v^N = 0, & \text{in } (0,T) \times \partial\Omega_N \\ v^N = 1 & \text{in } \{0\} \times \Omega_N. \end{cases}$$

Moreover, $v^N(t, \mathbf{x}) > 0$ for all $(t, \mathbf{x}^N) \in (0,T] \times \Omega_N$.

Proposition 4.3. *For all $\boldsymbol{x}^N \in \Omega_N$ it holds*

$$u^N(0, \boldsymbol{x}^N) = -\frac{2}{N} \log v^N(T, \boldsymbol{x}^N),$$

where u^N is the value function defined in (4.8)*.*

Proof. For all $(t, \mathbf{x}^N) \in [0, T) \times \Omega_N$ we define $w^N(t, \mathbf{x}^N) := -\frac{2}{N} \log v^N(T - t, \mathbf{x}^N)$. We are going to proceed by verification to show that $w^N(0, \mathbf{x}^N) = u^N(0, \mathbf{x}^N)$ in Ω_N. For $\mathbf{x}^N = (x^{1,N}, \dots, x^{N,N}) \in \Omega_N$, we define the following particle system

$$\begin{aligned}\mathbf{X}_t^N &:= \mathbf{x}^N - \int_0^{t\wedge\tau} N D w^N(t, \mathbf{X}_t^N) dt + \sqrt{2} \int_0^{t\wedge\tau} d\mathbf{B}_t^N \\ &= \mathbf{x}^N + \int_0^{t\wedge\tau} 2 \frac{D v^N(T - t, \mathbf{X}_t^N)}{v^N(T - t, \mathbf{X}_t^N)} dt + \sqrt{2} \int_0^{t\wedge\tau} d\mathbf{B}_t^N,\end{aligned}$$

where $\mathbf{B}_t^N :=^t (B_t^{1,N}, \dots, B_t^{N,N})$ and τ is the first exit time from Ω_N:

$$\tau := \inf\{t \geqslant 0, \mathbf{X}_t^N \notin \Omega_N\}.$$

For $\eta \geqslant 0$ small, we introduce the stopping time

$$\tau^\eta := \inf\{t \geqslant 0, v^N(T - t, \mathbf{X}_t^N) \leqslant \eta\}.$$

Notice that, by definition of v^N, it holds that $\tau^0 = \tau$. Applying Itô's formula to $\log v^N(T - t, \mathbf{X}_t^N)$ yields, for $\eta > 0$,

$$\begin{aligned}\log v^N(T - \tau^\eta \wedge T, \mathbf{X}^N_{\tau^\eta \wedge T}) &= \log v^N(T, \mathbf{x}^N) \\ &+ \int_0^{\tau^\eta \wedge T} \left[\frac{-\partial_t v^N}{v^N} + 2 \left| \frac{D v^N}{v^N} \right|^2 + \frac{\Delta v^N}{v^N} - \left| \frac{D v^N}{v^N} \right|^2 \right] (T - t, \mathbf{X}_t^N) dt \\ &+ \int_0^{\tau^\eta \wedge T} 2 \frac{D v^N(T - t, \mathbf{X}_t^N)}{v^N(T - t, \mathbf{X}_t^N)} . d\mathbf{B}_t^N \\ &= \log v^N(T, \mathbf{x}^N) + \int_0^{\tau^\eta \wedge T} \left| \frac{D v^N(T - t, \mathbf{X}_t^N)}{v^N(T - t, \mathbf{X}_t^N)} \right|^2 dt \\ &+ \int_0^{\tau^\eta \wedge T} 2 \frac{D v^N(T - t, \mathbf{X}_t^N)}{v^N(T - t, \mathbf{X}_t^N)} . d\mathbf{B}_t^N.\end{aligned}$$

Taking expectations and recalling the definition of τ^η we get

$$\log(\eta) \mathbb{P}(\tau^\eta \leqslant T) + \mathbb{P}(\tau^\eta > T) \geqslant \log v^N(T, \mathbf{x}^N).$$

As a consequence,

$$\lim_{\eta \to 0} \mathbb{P}(\tau^\eta \leqslant T) = 0$$

and the control $-NDw^N(T-t, \mathbf{X}_t^N)$ is admissible. Let us show that it is optimal. Recalling the equation satisfied by v^N, it holds that

$$\begin{cases} -\partial_t w^N + \frac{N}{2}|Dw^N|^2 - \Delta w^N = 0, & \text{in } (0,T)\times\Omega_N \\ w^N = +\infty, & \text{in } (0,T)\times\partial\Omega_N \\ w^N = 0 & \text{in } \{T\}\times\Omega_N. \end{cases}$$

Let us take another admissible control α with the associated solution $\mathbf{Y}^N$ to the SDE:

$$\mathbf{Y}_t^N := \mathbf{x}^N + \int_0^t \alpha_s ds + \sqrt{2}\int_0^t d\mathbf{B}_s^N.$$

Being α admissible, it holds that $\mathbf{Y}_t^N$ belongs to Ω_N for all $t \in [0,T]$ almost surely. We can apply Itô's lemma to w^N and get

$$\begin{aligned} 0 = \mathbb{E}\left[w^N(T, \mathbf{Y}_T^N)\right] &= w^N(0,\mathbf{x}^N) + \mathbb{E}\left[\int_0^T \partial_t w^N(t,\mathbf{Y}_t^N) + \alpha_t.Dw^N(t,\mathbf{Y}_t^N) + \Delta w^N(t,\mathbf{Y}_t^N)\right] dt \\ &= w^N(0,\mathbf{x}^N) + \mathbb{E}\left[\int_0^T \left(\alpha_t.Dw^N(t,\mathbf{Y}_t^N) + \frac{N}{2}|Dw^N(t,\mathbf{Y}_t^N)|^2\right) dt\right] \\ &\geqslant w^N(0,\mathbf{x}^N) - \mathbb{E}\left[\int_0^T \frac{1}{2N}|\alpha_t|^2 dt\right] \end{aligned}$$

with equality if and only if $\alpha_t = -NDw^N(t,\mathbf{Y}_t^N)$. This means that the control $-NDw^N(t,\mathbf{Y}_t^N)$ is optimal and the optimal value is given by $w^N(0,\mathbf{x}^N)$ which concludes the proof of the proposition. □

Finally we obtain the following convergence.

Corollary 4.1. *Let Assumption* (A) *as well as Assumptions* (4.9), (4.10) *and* (4.11) *hold. Assume that* $\Psi(\delta_{x_0}) < 0$ *and write* $\boldsymbol{x}_0^N = (x_0,\dots,x_0)$. *Then it holds*

$$\lim_{N\to+\infty} \frac{2}{N}\log v^N(T,\boldsymbol{x}_0^N) = -\overline{\mathcal{U}}(\delta_{x_0}).$$

This is a very special case of the general result of Dawson of Gärtner, [55]. However the optimality conditions of Chapter 3 give a new way to compute the limit $-\overline{\mathcal{U}}(\delta_{x_0})$.

Part III

A rate of convergence for the optimal control of McKean-Vlasov dynamics

Chapter 5

A rate of convergence for the mean-field limit

This chapter is based on a joint work with Pierre Cardaliaguet, Joe Jackson and Panagiotis Souganidis.

5.1 Introduction

We consider an optimal control problem with a large number of particles. The value function for this optimization problem reads

$$\mathcal{V}^N(t_0, \mathbf{x}_0) := \inf_{\alpha \in \mathcal{A}^N} \mathbb{E}\left[\int_{t_0}^T \left(\frac{1}{N}\sum_{k=1}^N L(X_t^k, \alpha_t^k) + \mathcal{F}(m^N_{\mathbf{X}_t})\right) dt + \mathcal{G}(m^N_{\mathbf{X}_T}) \right], \tag{5.1}$$

where $T > 0$ is a finite horizon, $t_0 \in [0, T]$ is the initial time, and $\mathbf{x}_0 = (x_0^1, \dots, x_0^N) \in (\mathbb{R}^d)^N$ is the initial position of the N particles. The infimum is taken over the set $\mathcal{A}^N$ of progressively measurable $(\mathbb{R}^d)^N$-valued processes $\alpha = (\alpha^k)_{k=1}^N$ in $L^2([0,T] \times \overline{\Omega}; (\mathbb{R}^d)^N)$ and $\mathbf{X} = (X^1, \dots, X^N)$ solves, for each $k \in \{1, \dots, N\}$,

$$X_t^k = x_0^k + \int_{t_0}^t \alpha_s^k ds + \sqrt{2} B_t^k + \sqrt{2a_0} B_t^0 \qquad t \in [t_0, T]. \tag{5.2}$$

The $(B^k)_{k \geqslant 0}$ are independent d-dimensional Brownian motions defined on the fixed filtered probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{F}}, \overline{\mathbb{P}})$ satisfying the usual conditions, and $L^2([0,T] \times \overline{\Omega}; (\mathbb{R}^d)^N)$ denotes the set of square-integrable and progressively measurable processes taking values in $(\mathbb{R}^d)^N$. Here δ_x is the Dirac mass at x, and the empirical measure $m^N_{\mathbf{X}_t}$ is given by

$$m^N_{\mathbf{X}_t} := \frac{1}{N}\sum_{k=1}^N \delta_{X_t^k}. \tag{5.3}$$

The cost function $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is supposed to be convex in the second variable and smooth while the maps $\mathcal{F}, \mathcal{G} : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ are assumed to be smooth and bounded over the space $\mathcal{P}_1(\mathbb{R}^d)$ of Borel measures on $\mathbb{R}^d$ with a finite first moment (precise assumptions will be given in Section 5.2). The constant $a_0 \geqslant 0$ is the level of the common noise, and the $(B^k)_{k \geqslant 1}$ are viewed as independent or idiosyncratic noises.

5.1.1 Our results

To describe our result we need to introduce the map $\mathcal{U} : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, where $\mathcal{P}_2(\mathbb{R}^d)$ is the space of Borel measures on $\mathbb{R}^d$ with a finite second moment, given, for $(t_0, m_0) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, by

$$\mathcal{U}(t_0, m_0) := \inf_{\alpha} \mathbb{E}[\int_{t_0}^{T} \big(L(X_t, \alpha_t) + \mathcal{F}(\mathcal{L}(X_t|\mathcal{F}_t^{B^0}))\big) + \mathcal{G}(\mathcal{L}(X_T|\mathcal{F}_T^{B^0}))], \tag{5.4}$$

where the infimum is taken over an appropriate set of admissible controls (this will be made precise later), $\mathbb{F}^{B^0} = (\mathcal{F}_t^{B^0})_{0 \leqslant t \leqslant T}$ denotes the filtration generated by B^0, $\mathcal{L}(X_t|\mathcal{F}_t^{B^0})$ is the law of X_t conditioned upon $\mathcal{F}_t^{B^0}$, and

$$X_t = \overline{X}_{t_0} + \int_{t_0}^{t} \alpha_s(X_s)ds + \sqrt{2}(B_t - B_{t_0}) + \sqrt{2a_0}(B_t^0 - B_{t_0}^0), \tag{5.5}$$

with B another Brownian motion, $\overline{X}_{t_0}$ a random initial condition with law m_0 and B^0, B and $\overline{X}_{t_0}$ mutually independent.

Although it is known (more about this later in the introduction) that, as N tends to infinity, $\mathcal{V}^N$ converges to $\mathcal{U}$, the existing convergence results come without any rate. Our main result is the following algebraic convergence rate: there exists $\beta \in (0,1]$ (depending on dimension only) and $C > 0$ (depending on the data of the problem) such that, for any $(t, \mathbf{x}) \in [0,T] \times (\mathbb{R}^d)^N$ one has:

$$\big|\mathcal{V}^N(t, m_{\mathbf{x}}^N) - \mathcal{U}(t, m_{\mathbf{x}}^N)\big| \leqslant CN^{-\beta}(1 + M_2^{1/2}(m_{\mathbf{x}}^N)), \tag{5.6}$$

where $M_2(m_{\mathbf{x}}^N) = N^{-1}\sum_{i=1}^N |x^i|^2$ is the second-order moment of the measure $m_{\mathbf{x}}^N$. Although the exact value of β could be traced back through the computation, it is clearly not optimal. In particular, it is very far from the one obtained in the standard particle system. In the same way, even if some dependence with respect to a moment of the measure is expected, the dependence given here is probably far from sharp.

5.1.2 Background and related literature

The convergence of $\mathcal{V}^N$ to $\mathcal{U}$ was shown by Lacker [96] in a very general framework and for suitable initial data but without common noise, that is $a_0 = 0$ in (5.2). Very recently, the results of [96] have been extended in Djete, Possamaï and Tan [59] to problems with a common noise and interaction through the controls. Beside [59, 96] several other papers have studied the question of the mean field limit of optimal control problems, for example Cavagnari, Lisini, Orrieri and Savaré [46] and Fornasier, Lisini, Orrieri and Savaré [71] investigate the problem without noise by $\Gamma-$convergence techniques. The recent contribution of Gangbo, Mayorga and Swiech [78] studies the mean field limit without idiosyncratic but with common noise using partial differential equations (PDE for short) techniques. This is

possible thanks to the fact that $\mathcal{V}^N$ solves the Hamilton-Jacobi equation

$$\begin{cases} -\partial_t \mathcal{V}^N(t,\mathbf{x}) - \sum_{j=1}^N \Delta_{x^j} \mathcal{V}^N(t,\mathbf{x}) - a_0 \sum_{i,j=1}^N \mathrm{tr}(D^2_{ij}\mathcal{V}^N(t,\mathbf{x})) + \frac{1}{N}\sum_{j=1}^N H(x^j, N D_{x^j}\mathcal{V}^N(t,\mathbf{x})) \\ \qquad\qquad = \mathcal{F}(m^N_{\mathbf{x}}) \qquad \text{in } (0,T)\times(\mathbb{R}^d)^N \\ \mathcal{V}^N(T,\mathbf{x}) = \mathcal{G}(m^N_{\mathbf{x}}) \qquad \text{in } (\mathbb{R}^d)^N \end{cases} \tag{5.7}$$

where

$$H(x,p) = \sup_{\alpha\in\mathbb{R}^d}[-p\cdot\alpha - L(x,\alpha)],$$

while $\mathcal{U}$ is expected to solve (in some sense) the infinite dimensional Hamilton-Jacobi equation

$$\begin{cases} -\partial_t \mathcal{U}(t,m) - (1+a_0)\int_{\mathbb{R}^d} \mathrm{div}_y(D_m\mathcal{U}(t,m,y))m(dy) \\ \qquad -a_0\int_{\mathbb{R}^{2d}} \mathrm{tr}(D^2_{mm}\mathcal{U}(t,x,m,y,y'))m(dy)m(dy') \\ \qquad +\int_{\mathbb{R}^d} H(y, D_m\mathcal{U}(t,m,y))m(dy) = \mathcal{F}(m) \qquad \text{in } (0,T)\times\mathcal{P}_2(\mathbb{R}^d) \\ \mathcal{U}(T,m) = \mathcal{G}(m) \qquad \text{in } \mathcal{P}_2(\mathbb{R}^d) \end{cases} \tag{5.8}$$

For the definition of the derivatives $D_m\mathcal{U}$ and $D^2_{mm}\mathcal{U}$ we refer to books of Cardaliaguet, Delarue, Lasry and Lions [36] and Carmona and Delarue [43].

One of the reasons for introducing the value functions is that they provide optimal feedbacks for the optimization problems. For the particle system, this optimal feedback is given (rigorously) by

$$\alpha^*_i(t,\mathbf{x}) = -D_pH(x_i, N D_{x_i}\mathcal{V}^N(t,\mathbf{x})),$$

while for the limit system it takes the form (at least formally)

$$\alpha^*_t(x,m) = -D_pH(x, D_m\mathcal{U}(t,m,x)).$$

The difficulty in the PDE analysis of [78] is that, in the absence of the idiosyncratic noise, the value functions $\mathcal{V}^N$ are not smooth in general. Thus in [78] the equation (5.7) has to be interpreted in the viscosity sense. A suitable notion of viscosity solution for the infinite dimensional Hamilton-Jacobi equation (5.8) without idiosyncratic noise is introduced in [78] , and then is proven that $\mathcal{V}^N$ converges to this viscosity solution. In the presence of idiosyncratic noise the notion of viscosity solution to (5.8) is not understood yet and we will not try to use this approach.

5.1.3 Our results continued

While the existing results mentioned above demonstrate the convergence of $\mathcal{V}^N$ to $\mathcal{U}$ under many different technical hypotheses and using a variety of techniques, none provides a rate

of convergence. Our main result fills this gap in the literature, by providing a rate of convergence of $\mathcal{V}^N$ to $\mathcal{U}$ in the presence of both idiosyncratic and common noise.

The primary challenge we face is related to the (lack of) regularity of $\mathcal{U}$. Indeed, if $\mathcal{U}$ is a smooth solution solution to (5.8), then the projections $\mathcal{U}^N : [0,T] \times (\mathbb{R}^d)^N \to \mathbb{R}$ given by $\mathcal{U}^N(t,\mathbf{x}) = \mathcal{U}(t, m^N_{\mathbf{x}})$ are smooth solutions of the Hamilton-Jacobi equation

$$\begin{cases} -\partial_t \mathcal{U}^N(t,\mathbf{x}) - \sum_{j=1}^N \Delta_{x^j} \mathcal{U}^N(t,\mathbf{x}) - a_0 \sum_{i,j=1}^N \operatorname{tr}(D^2_{ij} \mathcal{V}^N(t,\mathbf{x})) + \frac{1}{N} \sum_{j=1}^N H(x^j, N D_{x^j} \mathcal{U}^N(t,\mathbf{x})) \\ \qquad\qquad = \mathcal{F}(m^N_{\mathbf{x}}) + E_N(t,\mathbf{x}) \qquad \text{in } (0,T) \times (\mathbb{R}^d)^N \\ \mathcal{U}^N(T,\mathbf{x}) = \mathcal{G}(m^N_{\mathbf{x}}) \qquad \text{in } (\mathbb{R}^d)^N, \end{cases} \tag{5.9}$$

with

$$E_N(t,\mathbf{x}) = -\frac{1}{N^2} \sum_{j=1}^N \operatorname{tr}(D_{mm} \mathcal{U}(t, m^N_{\mathbf{x}}, x_i, x_i)).$$

If $D_{mm}\mathcal{U}$ is bounded, then we see immediately that $|E_n| = O(1/N)$. Thus $\mathcal{U}^N$ solves the same equation as $\mathcal{V}^N$, up to a term of order $O(1/N)$. By a comparison argument, we conclude that $|\mathcal{U} - \mathcal{V}| = O(1/N)$, that is, there exists a constant C such that, for all $t \in [0,T]$ and $\mathbf{x} \in (\mathbb{R}^d)^N$,

$$|\mathcal{V}^N(t,\mathbf{x}) - \mathcal{U}(t, m^N_{\mathbf{x}})| \leqslant \frac{C}{N}.$$

See also [80] for more on what convergence results can be obtained once (5.8) has a sufficiently smooth solution. This argument is similar to the approach taken in [36, 44] to study the convergence problem in the context of mean field games (see Lasry and Lions [102]) in situations where a classical solution to the so-called master equation is known to exist (see Bayraktar and Cohen [11] and Cecchin and Pelino [49] for related results). In this setting, convergence is related to the propagation of chaos for the optimal trajectories of the game.

Of course, the simple argument outlined above works only when the value function $\mathcal{U}$ is smooth. For instance this would be is the case if the maps $\mathcal{F}$ and $\mathcal{G}$ were convex and sufficiently smooth (see the discussion in Chap. 3.7 of [36]). However, we will not assume such a convexity property and the map $\mathcal{U}$ is expected to present discontinuities in its first order derivative; for instance, an example can be found in Briani and Cardaliaguet [24]. Because of this, the techniques in [36, 43, 44] break down.

When the value function is not smooth, the convergence rate has been studied primarily in the case of finite state space (see Kolokoltsov [91] and Cechin [47]). In this finite state space setting, the convergence rate is of order $1/\sqrt{N}$. Indeed, as explained in [47], the particle system is then a kind of discretization of the continuous McKean-Vlasov equation.

The situation is different and much more difficult in the continuous state space. This might come as a surprise since the convergence rate for particle systems is very well understood; see, for instance, Fournier and Guillin, [73]). The main difficulty we face is that, even though the optimal feedback in the particle system remains bounded independently of N (see Lemma 5.1), it cannot be expected to be uniformly continuous as a function of the empirical measure. Indeed this uniform continuity would imply the C^1 regularity of the limit $\mathcal{U}$, which does not hold in general. So we have to find a way to show that, despite the fact that the controls played by each particle might be very different, a kind of concentration of measure takes place.

5.1.4 Strategy of the proof

A few words on the method of proof are now in order. Let us first point out that we do not rely on a propagation of chaos, which we cannot prove at this stage. Indeed, as for a given initial condition there might be several optimal trajectories for the limit problem, a propagation of chaos is not expected to hold without additional assumptions on the initial data. The main ingredients for the proof are, uniform in N, Lipschitz and semiconcavity estimates for $\mathcal{V}^N$, and a concentration inequality. To bound from above $\mathcal{V}^N$ by $\mathcal{U}$ is relatively easy, because $\mathcal{V}^N$ can be transformed into an approximate subsolution for the Hamilton-Jacobi equation (5.8). The opposite inequality is much trickier, because it seems impossible to transform an optimal control for the $\mathcal{V}^N$ (in which the control depend on each particle) into a feedback for $\mathcal{U}$. We overcome this difficulty by dividing the particles into subgroups in such a way that the optimal controls for the particles in each subgroup are close and show a propagation of chaos, based on a concentration inequality, for each subgroup. The proof being technical, we first show the result when there is no common noise ($a_0 = 0$) and, in a second step, extend the result to problems with a common noise.

5.1.5 Organization of the paper

The paper is organized as follows. In the rest of the introduction we fix notation. We state the assumptions and the main result in section 5.2. As the proof of the convergence rate is technical, we start in section 5.3 with the problem without common noise. Indeed this case contains the main ideas without the extra technicalities due to the common noise. We first give some estimates on $\mathcal{V}^N$ and $\mathcal{U}$ (subsection 5.3.1), then show a first and relatively easy bound from above for $\mathcal{V}^N$ in subsection 5.3.2. The main part of the proof, that is, the bound from below, which is the aim of subsection 5.3.3 requires a concentration inequality proved in subsection 5.3.4. We finally explain the adaptation of the proof to the case with common noise in section 5.4.

5.1.6 Notation

For $\mathbf{x} = (x^1, ..., x^N) \in (\mathbb{R}^d)^N$, $m^N_{\mathbf{x}} \in \mathcal{P}(\mathbb{R}^d)$ denotes the empirical measure of $\mathbf{x}$, that is, $m^N_{\mathbf{x}} = \frac{1}{N}\sum_{i=1}^N \delta_{x^i}$. We write I_d for the identity matrix in $\mathbb{R}^d$, and B_R for the ball in $\mathbb{R}^d$ centered at the origin with radius R. If $\varphi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is smooth enough, we denote by $D\varphi$, $\Delta\varphi$ and $D^2\varphi$ the derivatives with respect to space and by $\partial_t\varphi$ and $\partial_{tt}\varphi$ the derivatives

with respect to time. Similarly, for $\mathcal{V} = \mathcal{V}(t, x^1, ..., x^N) : [0,T] \times (\mathbb{R}^d)^N \to \mathbb{R}$, we define the derivatives $D_{x^k}\mathcal{V}$, $\Delta_{x^k}\mathcal{V}$, $\partial_t\mathcal{V}$. We denote by $\mathcal{P}(\mathbb{R}^d)$ the set of Borel probability measures on $\mathbb{R}^d$ and note that, if $m \in \mathcal{P}(\mathbb{R}^d)$ has a density, for simplicity of notation, m is also be used to denote the density. Given $m \in \mathcal{P}(\mathbb{R}^d)$ and $p \geqslant 1$, we denote by $M_p(m)$ the p^{th} moment of m, that is, $M_p(m) = \int_{\mathbb{R}^d} |x|^p dm$, and by $\mathcal{P}_p(\mathbb{R}^d)$ the set of $m \in \mathcal{P}(\mathbb{R}^d)$ such that $M_p(\mathbb{R}^d) < \infty$. We endow $\mathcal{P}_p(\mathbb{R}^d)$ with the Wasserstein metric $\mathbf{d}_p$, defined by

$$\mathbf{d}_p^p(m, m') := \inf_{\pi \in \Pi(m,m')} \int_{\mathbb{R}^d} |x-y|^p d\pi(x,y),$$

where $\Pi(m, m')$ is the set of all $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ with marginals m and m'. We recall the duality formula

$$\mathbf{d}_1(m, m') = \sup_{\phi \in \mathbf{L}} \int_{\mathbb{R}^d} \phi d(m - m'),$$

where $\mathbf{L}$ is the set of all 1-Lipschitz functions from $\mathbb{R}^d$ to $\mathbb{R}$. We write $\mathbf{L}_R$ for the set of all 1-Lipschitz functions $\phi : B_R \subset \mathbb{R}^d \to [-R, R]$. For any $\phi \in \mathbf{L}_R$, we denote by $\widetilde{\phi}$ the extension $\widetilde{\phi} : \mathbb{R}^d \to [-R, R]$ (note that $\widetilde{\phi}$ is also 1-Lipschitz) given by

$$\widetilde{\phi}(x) = \begin{cases} \phi(x) & |x| \leqslant R, \\ \frac{2R-|x|}{R}\phi(\frac{R}{|x|}x) & R < |x| < 2R, \\ 0 & |x| \geqslant 2R. \end{cases}$$

Finally, for $\mathcal{U} : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ is smooth enough, $\dfrac{\delta U}{\delta m} : \mathcal{P}_1(\mathbb{R}^d) \times \mathbb{R} \to \mathbb{R}$ denotes the linear functional derivative, which satisfies, for all $m, m' \in \mathcal{P}_1(\mathbb{R}^d)$ and all $h \in (0,1)$,

$$\mathcal{U}(m') - \mathcal{U}(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-h)m + hm', x)(m' - m)(dx)dh.$$

We use the standard convention $\displaystyle\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m, x)m(dx) = 0$ for all $m \in \mathcal{P}_1(\mathbb{R}^d)$. If $\dfrac{\delta \mathcal{U}}{\delta m}$ is differentiable with respect to the space variable, we define the L-derivative of $\mathcal{U}$ by $D_m U(m, x) = D_x \dfrac{\delta U}{\delta m}(m, x)$. Higher order derivatives are defined similarly. We refer to [36, 43] for the properties of the L-derivatives.

5.2 Assumptions and main result

5.2.1 Assumptions

We now state our standing assumptions on the maps H, F and G, which constitute the data of our problem. We keep in mind that $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Legendre transform of H with respect to the last variable:

$$L(x, a) = \sup_{p \in \mathbb{R}^d} [-a \cdot p - H(x, p)].$$

We assume that

$$\begin{cases} \text{There exist constants } c,\, C > 0 \text{ such that} \\ -C + c|p|^2 \leqslant H(x,p) \leqslant C + \frac{1}{c}|p|^2 \quad \text{for all} \quad (x,p) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{5.10}$$

$$H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \text{ is of class } C^2, \tag{5.11}$$

$$\begin{cases} H \text{ is locally strictly convex with respect to the last variable,} \\ \text{that is, for any } R > 0, \text{ there exists } c_R > 0 \text{ such that} \\ D^2_{pp} H(x,p) \geqslant c_R I_d \quad \text{for all} \quad (x,p) \in \mathbb{R}^d \times \overline{B_R}, \end{cases} \tag{5.12}$$

$$\begin{cases} \text{there exists a constant } C > 0 \text{ such that} \\ |D_x H(x,p)| \leqslant C(|p| + 1) \quad \text{for all } (x,p) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{5.13}$$

$$\begin{cases} \text{for any } R > 0, \text{ there exists } C_R > 0 \text{ such that} \\ |D^2_{xx} H(x,p)| + |D^2_{xp} H(x,p)| \leqslant C_R \quad \text{for all} \quad (x,p) \in \mathbb{R}^d \times \overline{B_R}, \end{cases} \tag{5.14}$$

$$\mathcal{F} : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R} \text{ is of class } C^2 \text{ with } \mathcal{F},\, D_m\mathcal{F},\, D^2_{ym}\mathcal{F} \text{ and } D^2_{mm}\mathcal{F} \text{ uniformly bounded,} \tag{5.15}$$

and, finally,

$$\mathcal{G} : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R} \text{ is of class } C^4 \text{ with all derivatives up to order 4 uniformly bounded.} \tag{5.16}$$

For simplicity, in what follows we put together all the assumptions above in

$$\text{assume that (5.10), (5.11), (5.12), (5.13), (5.14), (5.15) and (5.16) hold,} \tag{5.17}$$

Remark 5.1. We make the following comments regarding (5.17).

(i) The strict convexity of H with respect to the gradient variable is standard in optimal control. In particular, it implies that L has the same regularity as H.

(ii) Although (5.13), which is used to obtain, independent of N, Lipschitz estimates on the value function $\mathcal{V}^N$ (see Lemma 5.1), is more restrictive, but we do not know it is possible to avoid it. It is satisfied, for instance, by a Hamiltonian of the form $H(x,p) = |p|^2 + V(x) \cdot p$ for some smooth and globally Lipschitz continuous vector field $V : \mathbb{R}^d \to \mathbb{R}^d$.

(iii) The fact that the "full" Hamiltonian $(x,p,m) \to H(x,p) - \mathcal{F}(m)$ has a separate form is not completely necessary. Some (small) extensions are possible, but we have decided to keep it in a separate form in order to avoid unnecessary technicalities.

(iv) The uniform bounds on $D_m\mathcal{F}$ and $D_m\mathcal{G}$ imply that both maps are Lipschitz continuous in $\mathcal{P}_1(\mathbb{R}^d)$. The additional smoothness is used to obtain, independent of N, semiconcavity estimates on the value function $\mathcal{V}^N$ (see Lemma 5.3).

(v) As L is the Legendre transform of H, Assumptions (5.10) and (5.11) imply that

$$\begin{cases} \text{for any } R > 0, \text{ there exists } C_R > 0 \text{ such that} \\ |D_a L(x,a)| \leqslant C_R \quad \text{for all} \quad (x,a) \in \mathbb{R}^d \times \overline{B_R} \end{cases} \tag{5.18}$$

Indeed, let $|a| \leqslant R$, $x \in \mathbb{R}^d$ and $p = D_a L(x,a)$. Then, by (5.11), we have $L(x,a) = -a \cdot p - H(x,p)$. By (5.10), we have

$$L(x,a) \geqslant -R|p| - C + (1/c)|p|^2$$

while

$$L(x,a) \leqslant \sup_{p'} \{-a \cdot p' + C - c|p'|^2\} \leqslant C + \frac{R^2}{4c},$$

which implies (5.18).

5.2.2 Formulation of the problem

For concreteness, we fix throughout the paper a filtered probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{F}} = (\overline{\mathcal{F}})_{t \geqslant 0}, \overline{\mathbb{P}})$ satisfying the usual conditions and hosting independent d-dimensional Brownian motions B^0 and $(B^k)_{k \in \mathbb{N}}$.

Definition of $\mathcal{V}^N$

The definition of $\mathcal{V}^N$ and the relevant quantities/functions were given and explained in the introduction–see (5.1), (5.2) and (5.3).

As explained in the introduction, it is well known that under (5.17), $\mathcal{V}^N$ is the unique classical solution to the Hamilton-Jacobi equation (5.7) and that the infimum in (5.1) is achieved (in feedback form) by the function $\alpha = (\alpha^k)_{k=1}^N : [0,T] \times (\mathbb{R}^d)^N \to \mathbb{R}^N$ given by

$$\alpha_k(t, \mathbf{x}) = -D_p H(x^k, N D_{x^k} \mathcal{V}^N(t, \mathbf{x})). \tag{5.19}$$

The definition of $\mathcal{U}$ without common noise

Suppose now that $a_0 = 0$. To define $\mathcal{U}$, we find it more intuitive to work with closed-loop controls, and to view the problem in terms of deterministic control of the associated Fokker-Planck equation. For fixed $(t_0, m_0) \in [t_0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, let $\mathcal{A}(t_0, m_0)$ to be the set of pairs (m, α) with $m = (m_t)_{t \in [t_0,T]} = (m(t,\cdot))_{t \in [t_0,T]} \in C^0([t_0,T]; \mathcal{P}_2(\mathbb{R}^d))$, $\alpha : [t_0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ measurable and such that

1. m solves (in the sense of distributions) the Fokker-Planck equation

$$\partial_t m = \Delta m - \operatorname{div}(m\alpha) \ \text{ in } \ (t_0,T] \times \mathbb{R}^d \quad \text{and} \quad m_{t_0} = m_0,$$

2. $\int_{t_0}^T \int_{\mathbb{R}^d} |\alpha(t,x)|^2 m(t,dx)dt < \infty$.

Then we define $\mathcal{U} : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ by

$$\mathcal{U}(t_0, m_0) = \inf_{(m,\alpha) \in \mathcal{A}(t_0,m_0)} \Big\{ \int_{t_0}^T \Big(\int_{\mathbb{R}^d} L(x, \alpha(t,x)) m(t,dx) + \mathcal{F}(m_t) \Big) dt + \mathcal{G}(m_T) \Big\} \tag{5.20}$$

One advantage to using this deterministic formulation of the McKean-Vlasov control problem is that, at least in the absence of common noise, the dynamic programming principle is straightforward. In particular, we can assert the following dynamic programming principle, which will be useful in what follows.

Proposition 5.1. *Assume* (5.17). *Then, for any* $0 \leqslant t_0 \leqslant t_1 \leqslant T$,

$$\mathcal{U}(t_0, m_0) = \inf_{(m,\alpha)\in\mathcal{A}(t_0,m_0)} \Big\{ \int_{t_0}^{t_1} \big(\int_{\mathbb{R}^d} L(x, \alpha(t,x)) m_t(dx) + \mathcal{F}(m_t) \big) dt + \mathcal{U}(t_1, m_{t_1}) \Big\}.$$

The definition of $\mathcal{U}$ with common noise

To define $\mathcal{U}$ when $a_0 > 0$, we once again use a form of closed-loop formulation, but this time the relevant Fokker-Planck equation becomes stochastic and we work with a notion of weak solution.

For fixed $(t_0, m_0) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, we define a control rule $\mathcal{R} \in \mathcal{A}(t_0, m_0)$ to be a tuple

$$\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, m, \alpha),$$

where

1. $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P})$ is a filtered probability space supporting the d-dimensional Brownian motion W,
2. $\alpha = (\alpha_t)_{t_0\leqslant t\leqslant T}$ is a $\mathbb{F}$-progressively measurable taking values in $L^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and such that α is uniformly bounded, in the sense that
$$\Big\| \sup_{t\in[t_0,T]} \|\alpha_t\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)} \Big\|_{L^\infty(\Omega)} < \infty. \tag{5.21}$$
3. m satisfies the stochastic McKean-Vlasov equation
$$\begin{cases} dm_t(x) = \left[(1+a_0)\Delta m_t(x) - \operatorname{div}(m_t\alpha_t(x))\right] dt + \sqrt{2a_0} Dm_t(x) \cdot dW_t \ \text{ in } \ (t_0,T]\times\mathbb{R}^d \\ m_{t_0} = m_0 \ \text{ in } \ \mathbb{R}^d. \end{cases} \tag{5.22}$$

The last condition means that, $\mathbb{P}-$a.s., for any smooth test function $\phi \in C_c^\infty([0,T]\times\mathbb{R}^d)$ with a compact support and for any $t \in [t_0, T]$ one has,

$$\int_{\mathbb{R}^d} \phi_t(x) m_t(dx) = \int_{\mathbb{R}^d} \phi_0(x)\overline{m}_0(dx) + \int_{t_0}^t \int_{\mathbb{R}^d} (\partial_t\phi_s(x) + \alpha_s(x)\cdot D\phi_s(x) + (1+a_0)\Delta\phi_s(x)) m_s(dx)ds$$
$$+ \int_{t_0}^t \sqrt{2a^0} \int_{\mathbb{R}^d} D\phi_s(x) m_s(dx) \cdot dW_s.$$

Now we define

$$\mathcal{U}(t_0, m_0) = \inf_{\mathcal{R}\in\mathcal{A}(t_0,m_0)} \mathbb{E}^{\mathbb{P}}\Big[\int_{t_0}^T \big(\int_{\mathbb{R}^d} L(x, \alpha_t(x)) m_t(dx) + \mathcal{F}(m_t) \big) dt + \mathcal{G}(m_T) \Big]. \tag{5.23}$$

The connection to the informal description (5.4) of $\mathcal{U}$ is that, if α is a bounded $L^\infty(\mathbb{R}^d;\mathbb{R}^d)$-valued process defined on some filtered probability space probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P})$

supporting independent Brownian motions B and W, α is a adapted to the filtration of W and X is a strong solution to the McKean-Vlasov equation

$$X_t = \overline{X}_{t_0} + \int_{t_0}^t \alpha_s(X_s)ds + \sqrt{2}(B_t^0 - B_{t_0}^0) + \sqrt{2a_0}(W_t^0 - W_{t_0}^0), \tag{5.24}$$

then $(\Omega, \mathcal{F}, \mathbb{F}^W, W, m, \alpha) \in \mathcal{A}(t_0, m_0)$, where $m_t = \mathcal{L}(X_t|W)$, that is, m is the conditional law of X given the filtration of the Brownian motion W.

As in the case $a_0 = 0$, we have the following dynamic programming principle.

Proposition 5.2. *Assume* (5.17). *Then, for any* $0 \leqslant t_0 < t_1 \leqslant T$, *for* $\mathcal{U}$ *defined by* (5.23), *we have*

$$\mathcal{U}(t_0, m_0) = \inf_{(m,\alpha)\in\mathcal{A}(t_0,m_0)} \mathbb{E}^{\mathbb{P}}\left[\int_{t_0}^{t_1}(\int_{\mathbb{R}^d} L(x, \alpha_t(x))m_t(dx) + \mathcal{F}(m_t))dt + \mathcal{U}(t_1, m_{t_1})\right].$$

Unlike in the case without common noise, where the control problem is deterministic and thus the dynamic programming principle is straightforward, in the common noise case we will need to use some machinery from Djete, Possamaï and Tan [59] and Lacker, Shkolnikov and Zhang [97] to verify that the dynamic programming principle holds in this setting. To streamline the presentation, we present the proof of Proposition 5.2 as well as of some other technical results from [58, 59, 97] in the Appendix.

Remark 5.2. We could have defined $\mathcal{U}$ using (5.23) when $a_0 = 0$ as well, and, in the end, it would be possible (thanks in part to Lemma 5.2 below) to prove that this is equivalent to the definition (5.20). We chose to define things separately with and without common noise mostly to avoid some unnecessary technicalities and to simplify the presentation for the reader interested in the case without common noise. The only mathematical reason for splitting up the definitions is that for technical reasons it is convenient to work with L^∞ feedback controls in the case of common noise, whereas without common noise we have no difficulty working with square-integrable controls.

5.2.3 The main result

With $\mathcal{V}^N$ defined by (5.1), $\mathcal{U}$ defined by (5.20), if $a_0 = 0$, or (5.23), if $a_0 > 0$, we have the following result.

Theorem 5.1. *Suppose that Assumption* (5.17) *holds. Then there exists* $\beta \in (0, 1]$, *which depends only on* d *and* $C > 0$ *depending on the data such that, for any* $(t, \mathbf{x}) \in [0, T] \times (\mathbb{R}^d)^N$,

$$\left|\mathcal{V}^N(t, \mathbf{x}) - \mathcal{U}(t, m_{\mathbf{x}}^N)\right| \leqslant CN^{-\beta}(1 + M_2(m_{\mathbf{x}}^N)).$$

For the convenience of the reader we repeat here the strategy of the proof. We detail in section 5.3 the proof of Theorem 5.1, the adaptation to the case $a_0 > 0$ being the aim of section 5.4. The proof of Theorem 5.1 requires several steps: We first provide uniform regularity estimates on $\mathcal{V}^N$ (Lipschitz and semiconcavity estimates, see Lemma 5.1 and 5.3). Then we show how to bound from above $\mathcal{V}^N$ by $\mathcal{U}$ plus an error term (Proposition 5.3). This

estimate is relatively easy and boils down to transforming the map $\mathcal{V}^N$ into a subsolution of the HJ equation (5.8). The converse estimate is more involve and is the aim of Proposition 5.4. The technical reason is that we found no way to embed $\mathcal{U}$ into the equation for $\mathcal{V}^N$ as a subsolution. Actually, since $\mathcal{U}$ is semiconcave, it is naturally a supersolution of that equation and the remaining term is a priori large. We overcome this issue by using locally optimal feedback of the $N-$problem for the continuous one, the main difficulty being to compare the empirical measure in the $N-$problem to the solution of the Fokker-Planck equation. This step is by no means trivial and relies on a key concentration result (Lemma 5.7).

5.3 The proof of Theorem 5.1 without common noise

Throughout this section we assume that $a_0 = 0$.

5.3.1 Some regularity estimates

Throughout the proof we use systematically the fact that $\mathcal{V}^N$ is the unique solution of the uniformly parabolic backward PDE (5.7) and, therefore, is smooth. We first start with regularity estimates for $\mathcal{V}^N$.

Lemma 5.1. *Assume* (5.17). *There exists a constant $C > 0$ such that, for any $N \geqslant 1$,*

$$\|\mathcal{V}^N\|_\infty + N \sup_j \|D_{x^j}\mathcal{V}^N\|_\infty + \|\partial_t \mathcal{V}^N\|_\infty \leqslant C.$$

Remark 5.3. The estimate on $D_{x^j}\mathcal{V}^N$ implies that the optimal feedback of the problem, given by $\alpha^k(t,x) = -D_pH(x^i, ND_{x^j}\mathcal{V}^N(t,\mathbf{x}))$ remains uniformly bounded.

Proof. The bound on $\mathcal{V}^N$ is obvious. We note that $w^i = D_{x^i}\mathcal{V}^N$ satisfies

$$\begin{cases} -\partial_t w^i(t,\mathbf{x}) - \displaystyle\sum_{k=1}^N \Delta_{x^k} w^i(t,\mathbf{x}) + \frac{1}{N} D_x H(x^i, N w^i(t,\mathbf{x})) \\ + \displaystyle\sum_{k=1}^N D_p H(x^k, N D_{x^k}\mathcal{V}^N(t,\mathbf{x})) \cdot D_{x^k} w^i(t,\mathbf{x}) = \frac{1}{N} D_m \mathcal{F}(m^N_{\mathbf{x}}, x^i) \quad \text{in } \ (0,T)\times(\mathbb{R}^d)^N, \\ w^i(T,\mathbf{x}) = \frac{1}{N} D_m \mathcal{G}(m^N_{\mathbf{x}}, x^i) \ \text{ in } \ (\mathbb{R}^d)^N. \end{cases} \tag{5.25}$$

It follows from the maximum principle that $N\|D_{x^j}\mathcal{V}^N\|_\infty$ is bounded uniformly in N and j.

In the same way, $w^t = \partial_t \mathcal{V}^N$ satisfies

$$\begin{cases} -\partial_t w^t(t,\mathbf{x}) - \sum_{k=1}^N \Delta_{x^k} w^t(t,\mathbf{x}) + \\ \qquad + \sum_{k=1}^N D_p H(x^k, N D_{x^k}\mathcal{V}^N(t,\mathbf{x})) \cdot D_{x^k} w^t(t,\mathbf{x}) = 0 \ \text{ in } \ (0,T)\times(\mathbb{R}^d)^N, \\ w^t(T,\mathbf{x}) = -\frac{1}{N}\sum_{k=1}^N \operatorname{tr}\left[D^2_{y,m}\mathcal{G}(m^N_{\mathbf{x}}, x^k) + \frac{1}{N}[D^2_{m,m}\mathcal{G}(m^N_{\mathbf{x}}, x^k, x^k)\right] \\ \qquad + \frac{1}{N}\sum_k H(x^k, D_m\mathcal{G}(m^N_{\mathbf{x}}, x^k) - \mathcal{F}(m^N_{\mathbf{x}}) \ \text{ in } \ (\mathbb{R}^d)^N, \end{cases} \tag{5.26}$$

and the uniform bound on $\|\partial_t \mathcal{V}^N\|_\infty$ follows again from the maximum principle. □

Lemma 5.2. *Assume* (5.17). *There is a constant C such that $\mathcal{U}$ satisfies*

$$|\mathcal{U}(t_0,m_0) - \mathcal{U}(s_0,\overline{m}_0)| \leqslant C\Big(|t_0-s_0|^{1/2} + \mathbf{d}_1(m,\overline{m})\Big), \quad \forall\ t_0,s_0\in[0,T],\ m_0,\overline{m}_0\in\mathcal{P}_2(\mathbb{R}^d).$$

Moreover, there exists a constant $C>0$ such that, if $(t_0,m_0)\in[0,T]\times\mathcal{P}_1(\mathbb{R}^d)$ and (m,α) is optimal in the definition of $\mathcal{U}(t_0,m_0)$ in (5.4), *then $\|\alpha\|_\infty \leqslant C$.*

Proof. The result is standard so we only sketch the argument and refer to [24] for more details. Fix $(t_0,\overline{m}_0)\in[0,T]\times\mathcal{P}_1(\mathbb{R}^d)$. It follows from (5.17) that there exists at least a pair (m,α) optimal in the definition of $\mathcal{U}(t_0,\overline{m}_0)$. Moreover, for such optimal pair (m,α), there exists a map $u:[t_0,T]\times\mathbb{R}^d\to\mathbb{R}$ with $\alpha_t(x) = -D_pH(x,Du(t,x))$ and such that (u,m) solves the system

$$\begin{cases} -\partial_t u(t,x) - \Delta u(t,x) + H(x,Du(t,x)) = \dfrac{\delta\mathcal{F}}{\delta m}(m_t,x) \ \text{ in } \ (t_0,T)\times\mathbb{R}^d \\ \partial_t m_t(x) - \Delta m_t(x) - \operatorname{div}(D_pH(x,Du(t,x))m_t(s)) = 0 \ \text{ in } \ (t_0,T)\times\mathbb{R}^d, \\ m_{t_0} = \overline{m}_0,\ u(T,x) = \dfrac{\delta\mathcal{G}}{\delta m}(m_T,x) \ \text{ in } \ \mathbb{R}^d. \end{cases}$$

Arguing as for the Lipschitz estimate in Lemma 5.1, one can check that there exists a constant $C>0$ such that $\|Du\|_\infty\leqslant C$, and, since $\alpha = -D_pH(x,Du)$, $\|\alpha\|_\infty\leqslant C$. By the standard parabolic regularity this implies that $\|D\alpha\|_\infty = \|D[D_pH(\cdot,Du(\cdot,\cdot))]\|_\infty\leqslant C$.

Fix $\overline{m}_1\in\mathcal{P}_1(\mathbb{R}^d)$ and let μ be the solution to

$$\partial_t\mu - \Delta\mu + \operatorname{div}(\mu\alpha) = 0 \ \text{ in } \ (t_0,T)\times\mathbb{R}^d \ \text{ with } \ \mu(t_0) = \overline{m}_1.$$

It is easy to check that there exists C depending on $\|D\alpha\|_\infty$ and on T such that

$$\sup_{t\in[t_0,T]} \mathbf{d}_1(\mu(t),m(t)) \leqslant C\mathbf{d}_1(\overline{m}_1,\overline{m}_0).$$

Thus, for some constant C depending on T, on the regularity of L, $\mathcal{F}$ and $\mathcal{G}$ and on $\|D\alpha\|_\infty$,

$$\begin{aligned}\mathcal{U}(t_0,\overline{m}_1) &\leqslant \int_{t_0}^T \Big(\int_{\mathbb{R}^d} L(x,\alpha_t(x))\mu(t,dx) + \mathcal{F}(\mu(t))\Big)dt + \mathcal{G}(\mu(T)) \\ &\leqslant \int_{t_0}^T \Big(\int_{\mathbb{R}^d} L(x,\alpha_t(x))m(t,dx) + \mathcal{F}(m(t))\Big)dt + \mathcal{G}(m(T)) + C\sup_{t\in[t_0,T]} \mathbf{d}_1(\mu(t),m(t)) \\ &\leqslant \mathcal{U}(t_0,\overline{m}_0) + C\mathbf{d}_1(\overline{m}_1,\overline{m}_0).\end{aligned}$$

This establishes the estimate

$$|\mathcal{U}(t_0,m_0) - \mathcal{U}(t_0,\overline{m}_0)| \leqslant C\mathbf{d}_1(m_0,\overline{m}_0). \tag{5.27}$$

Finally, we fix $s_0 < t_0$, and we choose (m,α) optimal in the definition of $\mathcal{U}(s_0,m_0)$. By dynamic programming (Proposition 5.1), we have

$$\mathcal{U}(s_0,m_0) = \int_{s_0}^{t_0} \Big(\int_{\mathbb{R}^d} L(x,\alpha(t,x))m_t(dx) + \mathcal{F}(m_t)\Big)dt + \mathcal{U}(t_0,m_{t_0}),$$

and so

$$\begin{aligned}|\mathcal{U}(s_0,m_0) - \mathcal{U}(t_0,m_0)| &\leqslant \Big|\int_{s_0}^{t_0} \Big(\int_{\mathbb{R}^d} L(x,\alpha(t,x))m_t(dx) + \mathcal{F}(m_t)\Big)dt\Big| + |\mathcal{U}(t_0,m_{t_0}) - \mathcal{U}(t_0,m_0)| \\ &\leqslant C(t_0-s_0) + C\mathbf{d_1}(m_{t_0},m_0) \leqslant C(t_0-s_0) + C(t_0-s_0)^{1/2} \leqslant C(t_0-s_0)^{1/2},\end{aligned}$$

where we have used (5.27) and the boundedness of α, together with the fact that Assumption 5.10 implies a similar inequality for L. This completes the proof. □

The key estimate on $\mathcal{V}^N$ is given in the following Lemma.

Lemma 5.3. *Assume* (5.17). *There exists an independent of N constant C,such that, for any $N \geqslant 1$ and any $\xi = (\xi^i) \in (\mathbb{R}^d)^N$ and $\xi^0 \in \mathbb{R}$,*

$$\sum_{i,j=1}^N D^2_{x^ix^j}\mathcal{V}^N(t,\mathbf{x})\xi^i\cdot\xi^j + 2\sum_{i=1}^N D^2_{x^it}\mathcal{V}^N(t,\mathbf{x})\cdot\xi^i\xi^0 + D^2_{tt}\mathcal{V}^N(t,\mathbf{x})(\xi^0)^2 \leqslant \frac{C}{N}\sum_{i=1}^N |\xi^i|^2 + C(\xi^0)^2. \tag{5.28}$$

Remark 5.4. Inequality (5.28) plays a crucial role in the proof of Lemma 5.9 below. Since $\mathcal{V}^N$ converges to $\mathcal{U}$, it follows that (5.28) implies the semi-concavity of the extension $\widetilde{\mathcal{U}} : [0,T]\times L^2((\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}});\mathbb{R}^d) \to \mathbb{R}$ defined, for $X \in L^2(\widetilde{\Omega},\mathbb{R}^d)$, by

$$\widetilde{\mathcal{U}}(t,X) := \mathcal{U}(t,\mathcal{L}(X)),$$

where $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ is a fixed atomless probability space and $\mathcal{L}(X)$ is the law of the random variable X.

Proof. For $1 \leqslant i, j, k \leqslant N$, let

$$\begin{cases} \omega^i = D_{x^i}\mathcal{V}^N \cdot \xi^i, \quad \omega^{i,j} = D^2_{x^i x^j}\mathcal{V}^N \xi^i \cdot \xi^j, \quad \omega^0 = \partial_t \mathcal{V}^N \xi_0, \quad \omega^{0,i} = \omega^{i,0} = \partial_t D_{x^i}\mathcal{V}^N \cdot \xi^0 \xi^i \\ \omega^{0,0} = \partial_{tt}\mathcal{V}^N (\xi^0)^2, \quad \widetilde{\omega} = \sum_{i,j=0}^N \omega^{i,j} \quad \text{and} \quad \sigma_k = \sum_{i=0}^N D_{x^k}\omega^i. \end{cases}$$

A straightforward computation gives

$$\begin{aligned} &- \partial_t \widetilde{\omega} - \sum_{k=1}^N \Delta_{x^k}\widetilde{\omega} + \sum_{k=1}^N D_{x^k}\widetilde{\omega}.D_p H(x^k, N D_{x^k}\mathcal{V}^N(t,\mathbf{x})) \\ =& - N\sum_{k=1}^N D^2_{pp}H(x^k, N D_{x^k}\mathcal{V}^N(t,\mathbf{x}))\sigma_k \cdot \sigma_k - 2\sum_{k=1}^N D^2_{xp}H(x^k, N D_{x^k}\mathcal{V}^N(t,\mathbf{x}))\xi^k.\sigma_k \\ &- \frac{1}{N}\sum_{i=1}^N D^2_{xx}H(x^i, n D_{x^i}\mathcal{V}^N(t,\mathbf{x}))\xi^i.\xi^i \\ &+ \frac{1}{N^2}\sum_{i,j=1}^N D^2_{mm}\mathcal{F}(m^N_{\mathbf{x}}, x^i, x^j)\xi^i.\xi^j + \frac{1}{N}\sum_{i=1}^N D_y D_m \mathcal{F}(m^N_{\mathbf{x}}, x^i)\xi^i.\xi^i \end{aligned}$$

Denote by γ the right-hand-side of the equality above. Recalling that H is strongly convex in the p variable and that $N\partial_{x^k}\mathcal{V}^N$ is bounded, we have, for all $1 \leqslant k \leqslant N$,

$$- N D^2_{pp}H\sigma_k \cdot \sigma_k - 2D^2_{xp}H\xi^k.\sigma_k \leqslant \frac{C}{N}|\xi^k|^2.$$

We can use again the Lipschitz bounds on $\mathcal{V}^N$ and (5.14) to deduce that

$$\gamma(t,\mathbf{x}) \leqslant \frac{C}{N}\sum_{k=1}^N |\xi^i|^2.$$

Next, fix $(t_0, \mathbf{x}_0)$ and consider the weak solution m^N to

$$\begin{cases} \partial_t m^N(t,\mathbf{x}) - \sum_{k=1}^N \Delta_{x^k} m^N(t,\mathbf{x}) - \sum_{k=1}^N \operatorname{div}(D_p H(x^k, N D_{x^k}\mathcal{V}^N(t,\mathbf{x}))m^N) = 0 \ \text{ in } \ (t_0,T)\times(\mathbb{R}^d)^N, \\ m^N(t_0,\cdot) = \delta_{\mathbf{x}_0} \ \text{ in } \ (\mathbb{R}^d)^N. \end{cases}$$

Integrating the equation satisfied by $\widetilde{\omega}$ against m^N we deduce that, for all $(t_0, \mathbf{x}_0) \in [0,T] \times (\mathbb{R}^d)^N$,

$$\widetilde{\omega}(t_0, \mathbf{x_0}) \leqslant \sup_{\mathbf{x}} \|\widetilde{\omega}(T,\mathbf{x})\|_\infty + \frac{C}{N}\sum_{k=1}^N |\xi^k|^2.$$

In order to bound the right-hand side of the inequality above, we first note that, by the equation satisfied by $\mathcal{V}^N$, we have

$$\partial_t \mathcal{V}^N(T,\mathbf{x}) = -\sum_{k=1}^N \Delta_{x^k}\mathcal{G}^N(\mathbf{x}) + \frac{1}{N}\sum_{k=1}^N H(x^k, N D_{x^k}\mathcal{G}^N(\mathbf{x})) - \mathcal{F}^N(\mathbf{x}),$$

where $\mathcal{F}^N(\mathbf{x}) := \mathcal{F}(m^N_{\mathbf{x}})$ and $\mathcal{G}^N(\mathbf{x}) := \mathcal{G}(m^N_{\mathbf{x}})$, and, similarly,

$$\partial^2_{tt}\mathcal{V}^N(T,\mathbf{x}) = -\sum_{k=1}^N \Delta_{x^k}\partial_t\mathcal{V}^N(T,\mathbf{x}) + \sum_{k=1}^N D_pH(x^k, ND_{x^k}\mathcal{G}^N(\mathbf{x}))\cdot D_{x^k}\partial_t\mathcal{V}^N(T,\mathbf{x}).$$

Recalling the expressions of the derivatives of $\mathcal{F}^N$ and $\mathcal{G}^N$ in function of the derivatives of $\mathcal{F}$ and $\mathcal{G}$ in Proposition 5.35 of [43], we obtain, after a tedious but straightforward computation that, under our standing assumptions on $\mathcal{F}$ and $\mathcal{G}$, the quantity $\sup_{\mathbf{x}}\|\widetilde{\omega}(T,\mathbf{x})\|_\infty$ is bounded by $\frac{C}{N}\sum_{i=1}^N|\xi^i|^2 + C(\xi^0)^2$.

□

5.3.2 The easy estimate

The second step in the proof of Theorem 5.1 is an upper bound of $\mathcal{V}^N$ in terms of $\mathcal{U}$. Our strategy will be to first compare $\mathcal{U}$ to $\widehat{\mathcal{V}}^N$, where

$$\widehat{\mathcal{V}}^N(t,m) := \int_{(\mathbb{R}^d)^N}\mathcal{V}^N(t,\mathbf{x})\prod_{j=1}^N m(dx^j). \tag{5.29}$$

We start with a Lemma, whose proof is a straightforward computation which is essentially the same as the one carried out in the proof of Proposition 3.1 in Cardaliaguet and Masoero [38]. Hence, we omit the details.

Lemma 5.4. *Let $\widehat{\mathcal{V}}^N$ be given by (5.29). Then $\widehat{\mathcal{V}}^N$ is smooth and satisfies the inequality*

$$\begin{cases} -\partial_t\widehat{\mathcal{V}}^N(t,m) - \displaystyle\int_{\mathbb{R}^d}\mathrm{div}(D_m\widehat{\mathcal{V}}^N(t,m,y))m(dy) \\ \quad + \displaystyle\int_{\mathbb{R}^d}H(y,D_m\widehat{\mathcal{V}}^N(t,m,y))m(dy) \leqslant \widehat{\mathcal{F}}^N(m) \quad \text{in} \quad (0,T)\times\mathcal{P}_1(\mathbb{R}^d), \\ \widehat{\mathcal{V}}^N(T,m) = \widehat{\mathcal{G}}^N(m) \quad \text{in} \quad \mathcal{P}_1(\mathbb{R}^d), \end{cases}$$

where

$$\widehat{\mathcal{F}}^N(m) = \int_{(\mathbb{R}^d)^N}\mathcal{F}(m^N_{\mathbf{x}})\prod_{j=1}^N m(dx^j) \quad \textit{and} \quad \widehat{\mathcal{G}}^N(m) = \int_{(\mathbb{R}^d)^N}\mathcal{G}(m^N_{\mathbf{x}})\prod_{j=1}^N m(dx^j).$$

Next we prove the easier inequality in Theorem (5.1).

Proposition 5.3. *There exist constants C (depending on the data) and β (depending only on d) such that for any $(t,\mathbf{x}_0)\in[0,T]\times(\mathbb{R}^d)^N$,*

$$\mathcal{V}^N(t,m^N_{\mathbf{x}_0}) \leqslant \mathcal{U}(t,m^N_{\mathbf{x}_0}) + C(1+M_2^{1/2}(m^N_{\mathbf{x}_0}))N^{-\beta}. \tag{5.30}$$

Proof. Theorem 1 in [73] gives constants C and β depending only on d such that, for any $m\in\mathcal{P}_2(\mathbb{R}^d)$ and for all $N\in\mathbb{N}$,

$$\int_{(\mathbb{R}^d)^N}\mathbf{d}_1(m^N_{\mathbf{x}},m)\prod_{i=1}^N m(dx^i) \leqslant CM_2^{1/2}(m)N^{-\beta}.$$

Fix $(t_0, m_0) \in [0,T) \times \mathcal{P}_2(\mathbb{R}^d)$ and let α^* be optimal in the definition of $\mathcal{U}(t_0, m_0)$. Using Lemma (5.4) together with a standard verification argument, for example, using Itô's formula in [43], we see that

$$\widehat{\mathcal{V}}^N(t_0, m_0) \leqslant \inf_{\alpha \in \mathcal{A}(t_0, m_0)} \Big\{ \int_{t_0}^T \Big(\int_{\mathbb{R}^d} L(x, \alpha(t,x)) m_t(dx) + \widehat{\mathcal{F}}^N(m_t) \Big) dt + \widehat{\mathcal{G}}^N(m_T)] \Big\}$$

and, hence,

$$\widehat{\mathcal{V}}^N(t_0, m_0) \leqslant \int_{t_0}^T \Big(\int_{\mathbb{R}^d} L(x, \alpha^*(t,x)) + \widehat{\mathcal{F}}^N(m_t) \Big) dt + \widehat{\mathcal{G}}^N(m_T). \tag{5.31}$$

Since, in view of Lemma 5.2, α^* is uniformly bounded by a constant independent of N, an easy computation shows that the corresponding state process satisfies

$$\sup_{t \in [t_0, T]} \int_{\mathbb{R}^d} |x|^2 m(t, dx) \leqslant (1 + CT) \int_{\mathbb{R}^d} |x|^2 m_0(dx) + CT.$$

It then follows from the Lipschitz continuity of $\mathcal{F}$ with respect to $\mathbf{d_1}$ that

$$\widehat{\mathcal{F}}^N(m(t)) \leqslant \mathcal{F}(m(t)) + C \int_{(\mathbb{R}^d)^N} \mathbf{d}_1(m^N_{\mathbf{x}}, m(t)) \prod_{j=1}^N m(t, dx^j) \leqslant \mathcal{F}(m(t)) + C(1 + M_2^{1/2}(m_0)) N^{-\beta}$$

and, in the same way,

$$\widehat{\mathcal{G}}^N(m(T)) \leqslant \mathcal{G}(m(t)) + C(1 + M_2^{1/2}(m_0)) N^{-\beta}.$$

So, by the optimality of α^*, we can use (5.31) together with the estimates of $\widehat{\mathcal{F}}^N$ and $\widehat{\mathcal{G}}$ to obtain

$$\begin{aligned} \widehat{\mathcal{V}}^N(t_0, m_0) &\leqslant \mathbb{E}[\int_{t_0}^T \big(L(X_t, \alpha_t^*) + \mathcal{F}(\mathcal{L}(X_t)) \big) dt + \mathcal{G}(\mathcal{L}(X_T))] + C(1 + M_2^{1/2}(m_0)) N^{-\beta} \\ &\leqslant \mathcal{U}(t_0, m_0) + C(1 + M_2^{1/2}(m_0)) N^{-\beta}. \end{aligned}$$

Fix now $\mathbf{x}_0 \in (\mathbb{R}^d)^N$. Then the Lipschitz estimate on $\mathcal{V}^N$ and the same argument as above yield

$$\Big| \mathcal{V}^N(t_0, \mathbf{x}_0) - \widehat{\mathcal{V}}^N(t_0, m^N_{\mathbf{x}_0}) \Big| \leqslant C(1 + M_2^{1/2}(m^N_{\mathbf{x}_0})) N^{-\beta}.$$

Putting together the last two estimates gives (5.30). □

5.3.3 The main estimate

The main step of the proof is to show the opposite inequality.

Proposition 5.4. *There exists a constant $\beta \in (0,1]$, which depends only on the dimension, and a constant $C > 0$, which depends on the data, such that, for any $N \geqslant 1$ and any $(t, \mathbf{x}) \in [0,T] \times (\mathbb{R}^d)^N$,*

$$\mathcal{U}(t, m^N_{\mathbf{x}}) - \mathcal{V}^N(t, \mathbf{x}) \leqslant C N^{-\beta} (1 + \frac{1}{N} \sum_{i=1}^N |x^i|^2). \tag{5.32}$$

Proof. Following a viscosity solutions-type argument, we double the variables and, for $\theta, \lambda \in (0,1)$, we set

$$M := \max_{(t,\mathbf{x}),(s,\mathbf{y})\in[0,T]\times(\mathbb{R}^d)^N} e^s(\mathcal{U}(s,m^N_{\mathbf{y}}) - \mathcal{V}^N(t,\mathbf{x})) - \frac{1}{2\theta N}\sum_{i=1}^N |x^i - y^i|^2 - \frac{1}{2\theta}|s-t|^2 - \frac{\lambda}{2N}\sum_{i=1}^N |y^i|^2.$$

We denote by $((t_0,\mathbf{x}_0),(s_0,\mathbf{y}_0))$ a maximum point in the expression above and we continue estimating in the next lemma the error related to the penalization

Lemma 5.5. *There exists a constant C such that, for any $i \in \{1,\dots,N\}$,*

$$\frac{1}{N}\sum_{i=1}^N |x_0^i - y_0^i|^2 + |s_0 - t_0|^2 \leqslant C\theta^2 \quad \textit{and} \quad \frac{1}{N}\sum_{i=1}^N |y_0^i|^2 \leqslant C\lambda^{-1}.$$

We postpone the proof of the lemma and continue with the ongoing proof.

As pointed out in the introduction, the main difficulty is that it does not seem possible, at least to us, how to transform an optimal control for the $\mathcal{V}^N$ (in which the control depend on each particle) into a feedback for $\mathcal{U}$. We overcome this difficulty by dividing the players into subgroups in such a way that the optimal controls for the agents in each subgroup are close and showing a propagation of chaos-type result for each subgroup using a a concentration inequality.

We begin explaining how to create the subgroups based on an appropriate partition of $\{1,\dots,N\}$. Since we will use it again in the next section, state and prove the following lemma.

Lemma 5.6. *For each $\delta > 0$ there exist a constant C depending only on the data, and a partition $(C^j)_{j\in\{1,\dots,J\}}$ of $\{1,\dots,N\}$ and $\overline{\alpha}^j \in \mathbb{R}^d$ for $j = 1,\dots,J$ such that $J \leqslant C\delta^{-d}$ and, for all $k \in C^j$,*

$$\left|H(x_0^k, ND_{x^k}\mathcal{V}^N(t_0,\mathbf{x}_0)) + \overline{\alpha}^j \cdot (ND_{x^k}\mathcal{V}^N(t_0,\mathbf{x}_0)) + L(x_0^k,\overline{\alpha}^j)\right| \leqslant C\delta. \tag{5.33}$$

Proof. Let $\widehat{\alpha}^k(t,\mathbf{x}) = -D_pH(x^k, ND_{x^k}\mathcal{V}^N(t,\mathbf{x}))$be the optimal feedback for particle k, and recall (see Remark 5.3),that there exists R depending only on the data such that $|\widehat{\alpha}^k(t,\mathbf{x})| \leqslant R$.

Given $\delta > 0$, we can find a δ-covering of $B_R \subset \mathbb{R}^d$ consisting of $J \leqslant C\delta^{-d}$ balls of radius δ entered at $(\overline{\alpha}_j)_{j\in\{1,\dots,J\}} \subset B_R$.

Then, we choose the partition $(C^j)_{j\in 1,\dots J}$ such that, for each $k \in C^j$, $|\widehat{\alpha}^k(t,\mathbf{x}) - \overline{\alpha}^j| \leqslant \delta$. It follows that, for each $k \in C^j$,

$$\begin{aligned}
&|H(x_0^k, ND_{x^k}\mathcal{V}^N(t_0,\mathbf{x}_0)) + \overline{\alpha}^j \cdot (ND_{x^k}\mathcal{V}^N(t_0,\mathbf{x}_0)) + L(x_0^k,\overline{\alpha}^j)| \\
&= |\big(\alpha^j - \widehat{\alpha}^k(t_0,\mathbf{x}_0)\big) \cdot (ND_{x^k}\mathcal{V}^N(t_0,\mathbf{x}_0)) + L(x_0^k,\overline{\alpha}^j) - L(x_0^k,\widehat{\alpha}(t_0,\mathbf{x}_0))| \\
&\leqslant \big(ND_{x^k}\mathcal{V}^N(t_0,\mathbf{x}_0)) + \|D_aL\|_{L^\infty(\mathbb{R}^d\times B_R)}\big)|\widehat{\alpha}^k(t_0,\mathbf{x}_0) - \overline{\alpha}^j| \leqslant C\delta,
\end{aligned}$$

where we have used 5.18. □

Fix $j \in \{1, \dots, J\}$, set $\alpha^k = \overline{\alpha}^j$ if $k \in C^j$ and let

$$X^k_{t_0+\tau} = x^k_0 + \tau\alpha^k + \sqrt{2}B^k_\tau \ \text{ and } \ Y^k_{s_0+\tau} = y^k_0 + \tau\alpha^k + \sqrt{2}B^k_\tau,$$

$$m^j_{\mathbf{Y}_{s_0+\tau}} = \frac{1}{n^j}\sum_{k\in C^j} \delta_{Y^k_{s_0+\tau}} \ \text{ and } \ m^j_{\mathbf{X}_{t_0+\tau}} = \frac{1}{n^j}\sum_{k\in C^j} \delta_{X^k_{t_0+\tau}},$$

consider the solution m^j to

$$\begin{cases} \partial_t m^j - \Delta m^j + \overline{\alpha}^j \cdot Dm^j = 0 \ \text{ in } \ (s_0, T)\times\mathbb{R}^d, \\ m^j(s_0,\cdot) = m^j_{\mathbf{y}_0} \ \text{ in } \ \mathbb{R}^d, \end{cases}$$

and, finally, set $m(s) = \frac{1}{N}\sum_{j\in J} n^j m^j(s)$.

We state next the concentration inequality we need for the proof.

Lemma 5.7. *There exist positive constants $\beta \in (0, 1/2)$. which depending on d, and C, which depends only on $\sup_j |\overline{\alpha}^j|$, d and T, such that, for all $h \geqslant 0$,*

$$\mathbb{E}\left[\mathbf{d}_1(m^j(s_0+h), m^j_{\mathbf{Y}_{s_0+h}})\right] \leqslant C(1 + M_2^{1/2}(m^j(s_0)))\frac{h^\beta}{(n^j)^\beta}, \tag{5.34}$$

and

$$\mathbb{E}\left[\mathbf{d}_1(m^j(s_0+h), m^j_{\mathbf{X}_{t_0+h}})\right] \leqslant \frac{1}{n^j}\sum_{k\in C^j} |x^k_0 - y^k_0| + C(1 + M_2^{1/2}(m^j(s_0)))\frac{h^\beta}{(n^j)^\beta} \qquad \forall h \geqslant 0,$$

and, as a consequence,

$$\mathbb{E}\left[\mathbf{d}_1(m(s_0+h), m^N_{\mathbf{Y}_{s_0+h}})\right] \leqslant C\delta^{-d\beta}(1 + \lambda^{-\frac{1}{2}})\frac{h^\beta}{N^\beta}$$

and

$$\mathbb{E}\left[\mathbf{d}_1(m(s_0+h), m^N_{\mathbf{X}_{t_0+h}})\right] \leqslant C\theta + C\delta^{-d\beta}(1 + \lambda^{-\frac{1}{2}})\frac{h^\beta}{N^\beta}.$$

We postpone the proof of Lemma 5.7 to subsection 5.3.4 except for the one of the third inequality, since it contains an argument is needed for the ongoing proof.

Proof of the third inequality in Lemma 5.7. Using the first two inequalities of Lemma 5.7 as well as the Cauchy-Schwarz inequality, the concavity of the maps $n \to n^{1-\beta}$ and $n \to n^{1-2\beta}$, the fact that $\sum_j n^j = N$, and the assumption that $\beta \in (0, 1/2)$, and recalling that $|J| = C\delta^{-d}$ and the estimate of $M_2(m(s_0))$ in Lemma 5.5 we obtain the following string of inequalities which prove the claim.

$$\begin{aligned}
\mathbb{E}\left[\mathbf{d}_1(m(s_0+h), m^N_{\mathbf{Y}_{s_0+h}})\right] &\leqslant \sum_{j\in J}\frac{n^j}{N}\mathbb{E}\left[\mathbf{d}_1(m^j(s_0+h), m^j_{\mathbf{Y}_{s_0+h}})\right] \\
&\leqslant C\sum_{j\in J}\frac{n^j}{N}(1+M_2^{1/2}(m^j(s_0)))\frac{h^\beta}{(n^j)^\beta} \\
&\leqslant Ch^\beta\sum_{j\in J}\frac{(n^j)^{1-\beta}}{N} + Ch^\beta(\sum_{j\in J}\frac{n^j}{N}M_2(m^j(s_0)))^{1/2}(\sum_{j\in J}\frac{n^j}{N(n^j)^{2\beta}})^{1/2} \\
&\leqslant Ch^\beta\frac{|J|}{N}\left(\sum_{j\in J}\frac{n^j}{|J|}\right)^{1-\beta} + CM_2^{1/2}(m(s_0))h^\beta\sqrt{\frac{|J|}{N}}(\sum_{j\in J}\frac{1}{|J|}(n^j)^{1-2\beta})^{1/2} \\
&\leqslant C|J|^\beta N^{-\beta}h^\beta + CM_2^{1/2}(m(s_0))|J|^\beta N^{-\beta}h^\beta.
\end{aligned}$$

□

We proceed with the ongoing proof.

The Lipschitz regularity of $\mathcal{U}$ in Lemma 5.2 and the definition of $\mathbf{X}_t$ and $\mathbf{Y}_t$ give

$$\begin{aligned}
M &\geqslant \mathbb{E}\Big[e^{s_0+h}(\mathcal{U}(s_0+h, m^N_{\mathbf{Y}_{s_0+h}}) - \mathcal{V}^N(t_0+h, \mathbf{X}_{t_0+h})) \\
&\qquad - \frac{1}{2\theta}\left(\frac{1}{N}\sum_{k=1}^N |Y^k_{s_0+h} - X^k_{t_0+h}|^2 + (t_0-s_0)^2\right) - \frac{\lambda}{2N}\sum_{i=1}^N |Y^i_{s_0+h}|^2\Big] \\
&\geqslant \mathbb{E}\Big[e^{s_0+h}(\mathcal{U}(s_0+h, m(s_0+h)) - \mathcal{V}^N(t_0+h, \mathbf{X}_{t_0+h}))\Big] - C\delta^{-d\beta}(1+\lambda^{-\frac{1}{2}})\frac{h^\beta}{N^\beta} \\
&\qquad - \frac{1}{2\theta}\left(\frac{1}{N}\sum_{k=1}^N |y^k_0 - x^k_0|^2 + (s_0-t_0)^2\right) - \frac{\lambda}{2N}\sum_{i=1}^N(|y^i_0| + Ch^{1/2})^2.
\end{aligned}$$

To proceed we need a dynamic programming-type argument, which is stated next. Its proof is postponed for later in the paper.

Lemma 5.8. *With the notation above, we have*

$$\begin{aligned}
\mathcal{U}(s_0+h, m(s_0+h)) &\geqslant \mathcal{U}(s_0, m_{\mathbf{y}_0^N}) \\
&\qquad - \int_{s_0}^{s_0+h}(\sum_{j=1}^J\int_{\mathbb{R}^d}\frac{1}{N}n^jL(x,\overline{\alpha}^j)m^j(s,x)dx + \mathcal{F}(m(s)))ds.
\end{aligned}$$

Using Itô's formula for $\mathcal{V}^N$ we find

$$\begin{aligned}
M &\geqslant e^{s_0+h}\mathcal{U}(s_0, m^N_{\mathbf{y}_0}) - e^{s_0+h}\int_{s_0}^{s_0+h}(\int_{\mathbb{R}^d}\sum_{j=1}^J\frac{1}{N}n^jL(x,\overline{\alpha}^j)m^j(s,x)dx + \mathcal{F}(m(s)))ds \\
&\quad - e^{s_0+h}\mathbb{E}\left[\mathcal{V}^N(t_0,\mathbf{x}_0) + \int_{t_0}^{t_0+h}(\partial_t\mathcal{V}^N(t,\mathbf{X}_t) + \sum_{k=1}^N(\Delta_{x^k}\mathcal{V}^N(t,\mathbf{X}_t) + \alpha^k\cdot D_{x^k}\mathcal{V}(t,\mathbf{X}_t)))dt\right] \\
&\quad - C\delta^{-d\beta}(1+\lambda^{-\frac{1}{2}})\frac{h^\beta}{N^\beta} - \frac{1}{2\theta}\left(\frac{1}{N}\sum_{k=1}^N |y^k_0 - x^k_0|^2 + (s_0-t_0)^2\right) - \frac{\lambda}{2N}\sum_{i=1}^N(|y^i_0| + Ch^{1/2})^2.
\end{aligned}$$

Since the $\overline{\alpha}^j$ are uniformly bounded, the map $L(\cdot, \overline{\alpha}^j)$ is uniformly Lipschitz independently of j. Hence, using Lemma 5.7 and Lemma 5.5, we find

$$\begin{aligned}
&\int_{s_0}^{s_0+h} \int_{\mathbb{R}^d} \sum_{j=1}^{J} \frac{1}{N} n^j L(x, \overline{\alpha}^j) m^j(s,x) dx ds \\
&\leqslant \mathbb{E}\left[\int_{s_0}^{s_0+h} \sum_{j=1}^{J} \Big(\sum_{k \in C^j} \frac{1}{N} L(X^k_{t_0 - s_0 + s}, \overline{\alpha}^j) + C \frac{1}{N} n^j \mathbf{d}_1(m^j(s), m^j_{\mathbf{X}_{t_0 - s_0 + s}}) \Big) ds \right] \\
&\leqslant \mathbb{E}\left[\int_{t_0}^{t_0+h} \sum_{k=1}^{N} \frac{1}{N} L(X^k_s, \alpha^k) ds \right] + C\theta h + C \sum_{j=1}^{J} \frac{1}{N} n^j (1 + M_2^{1/2}(m^j_{s_0})) \frac{h^\beta}{(n^j)^\beta} \\
&\leqslant \mathbb{E}\left[\int_{t_0}^{t_0+h} \sum_{k=1}^{N} \frac{1}{N} L(X^k_s, \alpha^k) ds \right] + C\theta h + C\delta^{-d\beta}(1 + \lambda^{-\frac{1}{2}}) \frac{h^\beta}{N^\beta}.
\end{aligned}$$

Note that the last inequality we used exactly the same argument as for the proof given above of the third inequality of Lemma 5.7.

Hence, recalling the optimality of $(\mathbf{x}_0, \mathbf{y}_0)$ and using the equation for $\mathcal{V}^N$, we get

$$\begin{aligned}
0 \geqslant\ & (e^{s_0+h} - e^{s_0})(\mathcal{U}(s_0, m^N_{\mathbf{y}_0}) - \mathcal{V}^N(t_0, \mathbf{x}_0)) - C\delta^{-d\beta}(1 + \lambda^{-\frac{1}{2}}) \frac{h^\beta}{N^\beta} - C\lambda h^{1/2} N^{-1} \sum_{i=1}^{N} |y^i_0| - C\theta h \\
& - e^{s_0+h} \mathbb{E}\left[\int_{s_0}^{s_0+h} (\mathcal{F}(m(s)) - \mathcal{F}(m^N_{\mathbf{X}_{s_0 - t_0 + s}})) ds \right] \\
& - e^{s_0+h} \mathbb{E}\left[\frac{1}{N} \int_{t_0}^{t_0+h} \sum_{k=1}^{N} (L(X^k_s, \alpha^k) + \alpha^k \cdot (N D_{x^k} \mathcal{V}(s, \mathbf{X}_s)) + H(X^k_s, N D_{x^k} \mathcal{V}(s, \mathbf{X}_s))) ds \right].
\end{aligned}$$

Using the Lipschitz regularity of $\mathcal{F}$ and Lemma 5.7 to deal with the difference of the $\mathcal{F}$ and Lemma 5.5 to deal with the term in $\sum_i |y^i_0|$, we find

$$\begin{aligned}
0 \geqslant\ & e^{s_0} h (\mathcal{U}(s_0, m^N_{\mathbf{y}_0}) - \mathcal{V}^N(t_0, \mathbf{x}_0)) - C\delta^{-d\beta}(1 + \lambda^{-\frac{1}{2}}) \frac{h^\beta}{N^\beta} - C\lambda^{1/2} h^{1/2} - C\theta h - Ch^2 \\
& - e^{s_0+h} \mathbb{E}\left[\frac{1}{N} \int_{t_0}^{t_0+h} \sum_{k=1}^{N} (L(X^k_s, \alpha^k) + \alpha^k \cdot (N D_{x^k} \mathcal{V}(s, \mathbf{X}_s)) + H(X^k_s, N D_{x^k} \mathcal{V}(s, \mathbf{X}_s)) ds) ds \right].
\end{aligned}$$

The regularity of L and H and the uniform boundedness of the α^k and of $N D_{x^k} \mathcal{V}^N$ allow to infer that

$$\begin{aligned}
0 \geqslant\ & e^{s_0} h (\mathcal{U}(s_0, m^N_{\mathbf{y}_0}) - \mathcal{V}^N(t_0, \mathbf{x}_0)) - C\delta^{-d\beta}(1 + \lambda^{-\frac{1}{2}}) \frac{h^\beta}{N^\beta} - C\lambda^{1/2} h^{1/2} - C\theta h - Ch^2 \\
& - e^{s_0+h} \mathbb{E}\left[\frac{1}{N} \int_{t_0}^{t_0+h} \sum_{k=1}^{N} (L(x^k_0, \alpha^k) ds + \alpha^k \cdot (N D_{x^k} \mathcal{V}(s, \mathbf{X}_s)) + H(x^k_0, N D_{x^k} \mathcal{V}(s, \mathbf{X}_s))) ds \right] - Ch^{3/2}.
\end{aligned}$$

and, in view of (5.33),

$$\begin{cases} 0 \geqslant e^{s_0} h(\mathcal{U}(s_0, m^N_{\mathbf{y}_0}) - \mathcal{V}^N(t_0, \mathbf{x}_0)) - C\delta^{-d\beta}(1+\lambda^{-\frac{1}{2}})\frac{h^\beta}{N^\beta} - C\lambda^{1/2}h^{1/2} - C\theta h - Ch^{3/2} \\ -C\mathbb{E}\left[\frac{1}{N}\int_{t_0}^{t_0+h}\sum_{k=1}^N |ND_{x^k}\mathcal{V}^N(s,\mathbf{X}_s)) - ND_{x^k}\mathcal{V}^N(s,\mathbf{x}_0))|ds\right] - Ch\delta. \end{cases} \tag{5.35}$$

The semiconcavity of $\mathcal{V}^N$ and the penalization by the term in θ give the next lemma. The proof is postponed to end of the section.

Lemma 5.9. *For any* $(t,\mathbf{x}) \in [0,T]\times(\mathbb{R}^d)^N$,

$$\begin{aligned} &\sum_{k=1}^N |D_{x^k}\mathcal{V}^N(t,\mathbf{x}) - D_{x^k}\mathcal{V}^N(t_0,\mathbf{x}_0)| \\ &\qquad \leqslant \frac{C}{N}\sum_{k=1}^N |x^k - x_0^k| + \left(\frac{C}{N\theta}\sum_{k=1}^N (|x^k-x_0^k| + |x^k-x_0^k|^2)\right)^{1/2} + \frac{C}{\theta^{1/2}}|t-t_0|^{1/2}. \end{aligned}$$

We continue with the ongoing proof. Inserting the estimate of Lemma 5.9 in (5.35), we obtain

$$\begin{aligned} 0 &\geqslant e^{s_0} h(\mathcal{U}(s_0, m^N_{\mathbf{y}_0}) - \mathcal{V}^N(t_0, \mathbf{x}_0)) - C\delta^{-d\beta}(1+\lambda^{-\frac{1}{2}})\frac{h^\beta}{N^\beta} - C\lambda^{1/2}h^{1/2} - C(\theta+\delta)h - Ch^{3/2} \\ &\quad - C\mathbb{E}\left[\int_{t_0}^t \Big(\frac{C}{N}\sum_{k=1}^N |X_s^k - x_0^k| + \left(\frac{C}{N\theta}\sum_{k=1}^N (|X_s^k-x_0^k| + |X_s^k-x_0^k|^2)\right)^{1/2} + \frac{1}{\theta^{1/2}}|s-t_0|^{1/2}\Big)ds\right] \\ &\geqslant e^{s_0} h(\mathcal{U}(s_0, m^N_{\mathbf{y}_0}) - \mathcal{V}^N(t_0, \mathbf{x}_0)) - C\delta^{-d\beta}(1+\lambda^{-\frac{1}{2}})\frac{h^\beta}{N^\beta} \\ &\qquad - C(\theta+\delta)h - C\lambda^{1/2}h^{1/2} - C\theta^{-1/2}h(h^{1/2}+h)^{1/2}. \end{aligned}$$

Dividing by h we find, for each choice of $\theta,\lambda,\delta>0$ and $0<h\leqslant (T-s_0)\wedge(T-t_0)$, the estimate

$$e^{s_0}(\mathcal{U}(s_0, m^N_{\mathbf{y}_0}) - \mathcal{V}^N(t_0, \mathbf{x}_0)) \leqslant C\frac{h^{\beta-1}}{N^\beta\delta^{d\beta}}(1+\lambda^{-1/2}) + C(\theta+\delta) + C\lambda^{1/2}h^{-1/2} + Ch^{1/4}\theta^{-1/2}.$$

We take $\theta = h^{\alpha_1}$, $\delta = (\frac{\lambda^{-1/2}h^{\beta-1}}{N^\beta})^{\alpha_2}$, $\lambda = N^{-\alpha_3}$ and $h = N^{-\alpha_4}$. By fixing appropriate choices of α_1, α_2, α_3 and finally α_4 we deduce that the estimate

$$e^{s_0}(\mathcal{U}(s_0, m^N_{\mathbf{y}_0}) - \mathcal{V}^N(t_0, \mathbf{x}_0)) \leqslant CN^{-\widetilde{\beta}} \tag{5.36}$$

holds for some $\widetilde{\beta} = \widetilde{\beta}(\beta) \in (0,1/2)$ and for all values of N such that $h = N^{-\alpha_4} \leqslant (T-s_0)\wedge(T-t_0)$. For those values of N such that $h = N^{-\alpha_4} \geqslant (T-s_0)\wedge(T-t_0)$, we have by Lemma 5.5 that $(T-s_0)\vee(T-s_0) \leqslant h + C\theta$, and so using Lemmas 5.1 and 5.2, we have

$$\begin{aligned} |\mathcal{U}(s_0, m^N_{\mathbf{y}_0}) - \mathcal{V}^N(t_0, \mathbf{x}_0)| &\leqslant |\mathcal{U}(s_0, m^N_{\mathbf{y}_0}) - \mathcal{G}(m^N_{\mathbf{y}_0})| + |\mathcal{G}(m^N_{\mathbf{y}_0}) - \mathcal{G}(m^N_{\mathbf{x}_0})| + |\mathcal{G}(m^N_{\mathbf{x}_0}) - \mathcal{V}^N(t_0, \mathbf{x}_0)| \\ &\leqslant C(h+\theta)^{1/2} + C\theta + C(h+\theta) \leqslant CN^{-\widetilde{\beta}}, \end{aligned}$$

where in the last line we choose $\widetilde{\beta}$ even smaller if necessary. With this choice of $\widetilde{\beta}$, we have now established that (5.36) holds for all values of N.

Finally, we conclude that, for all $(t, \mathbf{x}) \in [0,T] \times (\mathbb{R}^d)^N$,

$$e^t(\mathcal{U}(t, m_{\mathbf{x}}^N) - \mathcal{V}^N(t, \mathbf{x})) \leqslant e^{s_0}(\mathcal{U}(s_0, m_{\mathbf{y}_0}^N) - \mathcal{V}^N(t_0, \mathbf{x}_0)) + \frac{\lambda}{2N}\sum_{i=1}^N |x^i|^2$$
$$\leqslant CN^{-\min(\widetilde{\beta}, \alpha_3)}(1 + \frac{1}{N}\sum_{i=1}^N |x^i|^2).$$

□

Before proving the various lemmas used in the proof of Proposition 5.4, we complete the proof of the main result.

Proof of Theorem 5.1. Combining Proposition 5.3 and Proposition 5.4 we know that there exist $\beta \in (0,1]$ (depending on dimension) and $C > 0$ such that, for any $(t, \mathbf{x}) \in [0,T] \times (\mathbb{R}^d)^N$,

$$\left|\mathcal{U}(t, m_{\mathbf{x}}^N) - \mathcal{V}^N(t, \mathbf{x}))\right| \leqslant CN^{-\beta}(1 + M_2^{1/2}(m_{\mathbf{x}}^N) + M_2(m_{\mathbf{x}}^N)) \leqslant CN^{-\beta}(1 + M_2(m_{\mathbf{x}}^N)).$$

□

We continue with the proofs of the several auxiliary results sated earlier.

Proof of Lemma 5.5. The proof of the first statement is an immediate consequence of the uniform bound on $\mathcal{U}$ and $\mathcal{V}^N$ and of the Lipschitz estimate for $\mathcal{V}^N$. □

Proof of Lemma 5.8. For $K \in \mathbb{N}$ and any nonnegative integrable functions $m_0^1, \dots, m_0^K$ on $\mathbb{R}^d$ such that $\sum_{k=1}^K m_0^k \in \mathcal{P}(\mathbb{R}^d)$, let

$$\mathcal{U}^K(t_0, m_0^1, \dots, m_0^K) := \inf_{(m^1,\beta^1),\dots,(m^K,\beta^K)} \int_{t_0}^T (\int_{\mathbb{R}^d} \sum_{k=1}^K L(x, \frac{\beta^k(t,x)}{m^k(t,dx)}) m^k(t,x)dx + \mathcal{F}(\sum_{k=1}^K m^k(t)))dt$$
$$+ \mathcal{G}(\sum_{k=1}^K m^k(T)),$$

where the infimum is taken over the tuple of measures (m^k, β^k) (the β^k being a vector measure) with $\beta^k << m^k$ such that (m^k, β^k) solve in the sense of distributions

$$\partial_t m^k - \Delta m^k + \operatorname{div}(\beta^k) = 0 \ \text{ in } \ (t_0, T] \times \mathbb{R}^d \ \text{ and } \ m^k(t_0) = m_0^k \ \text{ in } \ \mathbb{R}^d.$$

We establish next that

$$\mathcal{U}^K(t_0, m_0^1, \dots, m_0^K) = \mathcal{U}(t_0, m_0^1 + \cdots + m_0^K),$$

and the result will then follow from Proposition 5.1.

Since obviously $\mathcal{U}^K(t_0, m_0^1, \dots, m_0^K) \leqslant \mathcal{U}(t_0, m_0^1 + \cdots + m_0^K)$, next we concentrate on the reverse inequality.

Fix $\varepsilon > 0$, let $(m^1, \beta^1, \dots, m^K, \beta^K)$ be $\varepsilon-$optimal for $\mathcal{U}^K(t_0, m_0^1, \dots, m_0^K)$, and set $\beta = \sum_{k=1}^N \beta^K$ and $m(t) = \sum_{k=1}^N m^k(t)$. Then (m, β) solves

$$\partial_t m - \Delta m + \operatorname{div}(\beta) = 0 \ \text{ in } \ (t_0, T] \times \mathbb{R}^d \ \text{ and } \ m(t_0) = m_0 \ \text{ in } \ \mathbb{R}^d.$$

and we have

$$\begin{aligned}
&\varepsilon + \mathcal{U}^K(t_0, m_0^1, \dots, m_0^K) \\
&\quad \geqslant \int_{t_0}^T \Big(\int_{\mathbb{R}^d} \sum_{k=1}^K L\big(x, \frac{\beta^k(t,x)}{m^k(t,x)}\big) \frac{m^k(t,x)}{m(t,x)} m(t,x)dx + \mathcal{F}\big(\sum_{k=1}^K m^k(t)\big)\Big)dt + \mathcal{G}\big(\sum_{k=1}^K m^k(T)\big) \\
&\quad \geqslant \int_{t_0}^T \Big(\int_{\mathbb{R}^d} L\big(x, \frac{\sum_{k=1}^K \beta^k(t,x)}{m(t,x)}\big) m(t,x)dx + \mathcal{F}(m(t))\Big)dt + \mathcal{G}(m(T)) \\
&\quad \geqslant \mathcal{U}(t_0, m_0),
\end{aligned}$$

where the second inequality follows from the convexity of the map $(\beta, m) \to mL(x, \frac{\beta}{m})$ and the third one by the definition of $\mathcal{U}$. □

Proof of Lemma 5.9. Set $p^k = D_{x^k}\mathcal{V}(t_0, \mathbf{x}_0)$ and $p^t = \partial_t \mathcal{V}(t_0, \mathbf{x}_0)$. Then, in view of Lemma 5.3, we have, for any $(t, \mathbf{x}), (t_0, \mathbf{x}_0) \in [0, T] \times (\mathbb{R}^d)^N$,

$$\mathcal{V}^N(t, \mathbf{x}) - \mathcal{V}^N(t_0, \mathbf{x}_0) - \sum_{k=1}^N p^k \cdot (x^k - x_0^k) - p^t(t - t_0) \leqslant \frac{C}{N} \sum_{k=1}^N |x^k - x_0^k|^2 + C(t - t_0)^2.$$

The optimality of $(t_0, \mathbf{x}_0, s_0, \mathbf{y}_0)$ also gives, for any $(t, \mathbf{x})$,

$$\frac{1}{2\theta N} \sum_{i=1}^N |x^i - y_0^i|^2 + \frac{1}{2\theta}(t - s_0)^2 + \mathcal{V}^N(t, \mathbf{x}) \geqslant \frac{1}{2\theta N} \sum_{i=1}^N |x_0^i - y_0^i|^2 + \frac{1}{2\theta}(t_0 - s_0)^2 + \mathcal{V}^N(t_0, \mathbf{x}_0). \quad (5.37)$$

From (5.37), we conclude that

$$p^k = \frac{y_0^k - x_0^k}{\theta N} \ \text{ and } \ p^t = \frac{s_0 - t_0}{\theta}.$$

Furthermore, rearranging (5.37) yields

$$\begin{aligned}
\mathcal{V}^N(t, x) - \mathcal{V}^N(t_0, x_0) &\geqslant \frac{1}{2\theta N} \sum_{k=1}^N |x_0^k - y_0^k|^2 - \frac{1}{2\theta N} \sum_{k=1}^N |x^k - y_0^k|^2 + \frac{1}{2\theta N}|t_0 - s_0|^2 - \frac{1}{2\theta N}|t - s_0|^2 \\
= \frac{1}{2\theta N} \sum_{k=1}^N |x_0^k - y_0^k|^2 &- \frac{1}{2\theta N} \sum_{k=1}^N |(x^k - x_0^k) + (x_0^k - y_0^k)|^2 + \frac{1}{2\theta N}|t_0 - s_0|^2 - \frac{1}{2\theta N}|(t - t_0) + (t_0 - s_0)|^2 \\
&= \sum_{k=1}^N p^k \cdot (x^k - x_0^k) + p^t(t - t_0) - \sum_{k=1}^N \frac{1}{2\theta N}|x^k - x_0^k|^2 - \frac{1}{2\theta}(t - t_0)^2.
\end{aligned}$$

and, after some elementary manipulations,

$$\mathcal{V}^N(t,\mathbf{x}) - \mathcal{V}^N(t_0,\mathbf{x}_0) - \sum_{k=1}^N p^k \cdot (x^k - x_0^k) - p^t(t-t_0) \geqslant -\frac{1}{2\theta N}\sum_{k=1}^N |x^k - x_0^k|^2 - \frac{1}{2\theta}(t-t_0)^2.$$

Assuming that $\theta \leqslant (2C)^{-1}$, tt follows that

$$w(t,\mathbf{x}) = \mathcal{V}^N(t_0,\mathbf{x}_0) - \mathcal{V}^N(t,\mathbf{x}) + \sum_{k=1}^N p^k \cdot (x^k - x_0^k) + p^t(t-t_0) + \frac{C}{N}\sum_{k=1}^N |x^k - x_0^k|^2 + C(t-t_0)^2$$

is convex and satisfies

$$0 \leqslant w(t,\mathbf{x}) \leqslant \frac{1}{\theta N}\sum_{k=1}^N |x^k - x_0^k|^2 + \frac{1}{\theta}(t-t_0)^2.$$

Thus, for any $(t,\mathbf{x})$ and any $(s,\mathbf{y})$, we have

$$\sum_{k=1}^N D_{x^k} w(t,\mathbf{x}) \cdot (y^k - x^k) + \partial_t w(t,\mathbf{x})(s-t) \leqslant w(t,\mathbf{x}) + \sum_{k=1}^N D_{x^k} w(t,\mathbf{x}) \cdot (y^k - x^k) + \partial_t w(t,\mathbf{x})(s-t)$$
$$\leqslant w(s,\mathbf{y}) \leqslant \frac{1}{\theta N}\sum_{k=1}^N |y^k - x_0^k|^2 + \frac{1}{\theta}(s-t_0)^2.$$

Letting $y^k = x_0^k + \frac{1}{2}\theta N D_{x^k} w(t,\mathbf{x})$ and $s = t_0 + \frac{1}{2}\theta \partial_t w(t,x)$ in the inequality above, we obtain

$$\frac{\theta N}{4}\sum_{k=1}^N |D_{x^k} w(t,\mathbf{x})|^2 \leqslant \sum_{k=1}^N D_{x^k} w(t,\mathbf{x}) \cdot (x^k - x_0^k) + \partial_t w(t,x)(t-t_0), \tag{5.38}$$

and, after using the Cauchy-Schwarz inequality,

$$\sum_{k=1}^N |D_{x^k} w(t,\mathbf{x})| \leqslant N^{1/2}\left(\sum_{k=1}^N |D_{x^k} w(t,\mathbf{x})|^2\right)^{1/2} \tag{5.39}$$
$$\leqslant N^{1/2}\left(\frac{4}{N\theta}\sum_{k=1}^N |x_0^k - x^k||D_{x^k} w(t,\mathbf{x})| + \frac{4}{N\theta}|\partial_t w(t,x)||t-t_0|\right)^{1/2}.$$

Recalling the definition of w and that $|D_{x^k}\mathcal{V}^N| \leqslant C/N$ and $|\partial_t \mathcal{V}^N| \leqslant C$, we find

$$|D_{x^k} w(t,\mathbf{x})| = |-D_{x^k}\mathcal{V}^N(t,\mathbf{x}) + p^k + \frac{2C}{N}(x^k - x_0^k)| \leqslant CN^{-1} + \frac{2C}{N}|x^k - x_0^k|$$

and

$$|\partial_t w(t,x)| = |-\partial_t \mathcal{V}^N(t_0,\mathbf{x}_0) + 2C(t-t_0)| \leqslant C.$$

Returning to (5.39), we have

$$\sum_{k=1}^N |-D_{x^k}\mathcal{V}^N(t,\mathbf{x}) + p^k + \frac{2C}{N}(x^k - x_0^k)| \leqslant \left(\frac{C}{N\theta}\sum_{k=1}^N |x_0^k - x^k| + \frac{C}{N\theta}\sum_{k=1}^N |x_0^k - x^k|^2 + \frac{C}{\theta}|t-t_0|\right)^{1/2},$$

from which we deduce the result by the definition of p^k. □

5.3.4 Proof of the concentration inequality

To complete the proof of Proposition 5.4, we are need to show Lemma 5.7. For this, it is convenient to introduce a few more facts.

Let $\mathcal{L}(\epsilon, R)$ be the ϵ-covering number of $\mathbf{L}_R$ with respect to the L^∞-distance, that is,

$$\mathcal{L}(\epsilon, R) = \inf\{k \in \mathbb{N} : \text{ there exist } \phi_1, ..., \phi_k \in \mathbf{L}_R \text{ such that for all } \phi \in \mathbf{L}_R, \|\phi - \phi_j\|_{L^\infty} < \epsilon \text{ for some } j\}.$$

It is known (see, for example, [128]) that

$$\mathcal{L}(\epsilon, 1) \leqslant \exp\{C\epsilon^{-d}\}, \tag{5.40}$$

and, after a rescaling argument,

$$\mathcal{L}(\epsilon, R) \leqslant \exp\{C\big(\frac{R}{\epsilon}\big)^d\}. \tag{5.41}$$

Indeed, if $\{\phi_1, ..., \phi_n\} \in \mathbf{L}$ is ϵ/R-dense in $\mathbf{L}$, then $\{\widetilde{\phi}_1, ..., \widetilde{\phi}_n\}$ is ϵ-dense in $\mathbf{L}_R$, where $\widetilde{\phi}_i(x) = R\phi(\frac{x}{R})$. Thus (5.41) follows from (5.40).

We need two preliminary estimates and note that, without loss of generality, we can take $t_0 = 0$ in what follows. Finally, we recall the notation after Lemma 5.5.

Lemma 5.10. *There exists a constant $C > 0$ such that, for any $\phi \in \mathbf{L}$ and any $j \in \{1, ..., J\}$, we have*

$$\mathbb{P}[\int_{\mathbb{R}^d} \phi(m^j(h) - m^j_{Y_h}) > x] \leqslant \exp\left\{-\frac{n^j x^2}{Ch}\right\}.$$

Proof. Let u the solution of

$$\begin{cases} -\partial_t u - \Delta u - \alpha^j \cdot Du = 0 \ \text{ in } \ (0,h) \times \mathbb{R}^d, \\ u(h) = \phi \ \text{ in } \ \mathbb{R}^d; \end{cases}$$

note that, since $\|D\phi\| \leqslant 1$, $\|Du\|_\infty \leqslant 1$.

Using Itô's formula and the equation for m^j, we get

$$\int_{\mathbb{R}^d} \phi(m^j(h) - m^j_{Y_h}) = -\sqrt{2}\frac{1}{n^j}\sum_{k \in C^j}\int_0^h Du(s, Y^k_s).dB^k_s$$

The random variables $h^{-1/2}\int_0^h Du(s, Y^k_s)dB^k_s$ are independent and sub-Gaussian, uniformly in k. Indeed, viewing $h^{-1/2}\int_0^h Du(\cdot, Y^k)dB^k$ as a time-changed Brownian motion, we have that $h^{-1/2}\int_0^h Du(t, Y^k_t)dB^k_t \overset{(d)}{=} B_\tau$, where B is a standard Brownian motion and $\tau \leqslant 1$ is a stopping time (we use here that $\|Du\|_\infty \leqslant 1$). In particular,

$$\mathbb{P}[\int_0^h Du(s, Y^k_s)dB^k_s > x] \leqslant \mathbb{P}[\sup_{0 \leqslant t \leqslant 1} |B_t| > h^{1/2}x],$$

from which the claim follows easily.

We may now apply Hoeffding's inequality (see, for example, Proposition 2.5 in [132]) to complete the proof. □

Lemma 5.11. *There exists a constant C such that, for any $j \in \{1, ..., , J\}$ and $R > 0$,*

$$\mathbb{E}[\sup_{\phi \in \mathbf{L}_R} \int_{\mathbb{R}^d} \widetilde{\phi}\big(m^j(h) - m^j_{Y_h}\big)] \leqslant C(1 + R^{\frac{d}{d+2}})(n^j)^{\frac{-1}{d+2}} h^{\frac{1}{d+2}}.$$

Proof. We fix $\epsilon > 0$ and use the estimate on $\mathcal{L}(\epsilon, R)$ to choose $K \leqslant \exp\{C\big(\frac{R}{\epsilon}\big)^d\}$ and $\phi_1, ..., \phi_K$ in $\mathbf{L}_R$ such that, for each $\phi \in \mathbf{L}_R$, there exists $k \in \{1, ..., K\}$ such that $\|\phi - \phi_k\|_{L^\infty(B_R)} < \epsilon$, and hence $\left\|\widetilde{\phi} - \widetilde{\phi}_k\right\|_{L^\infty(\mathbb{R}^d)} \leqslant \epsilon$.

Then, using Lemma 5.10 and the upper bound on K, for any $x > \epsilon$, we have

$$\begin{aligned}\mathbb{P}[\sup_{\phi \in \mathbf{L}_R} \int_{\mathbb{R}^d} \widetilde{\phi}(m^j(h) - m^j_{Y_h}) > x] &\leqslant \mathbb{P}[\,\exists k \ \text{ such that } \int_{\mathbb{R}^d} \widetilde{\phi}_k(m^j(h) - m^j_{Y_h}) > x - \epsilon] \\ &\leqslant \sum_{k=1}^{K} \mathbb{P}[\int_{\mathbb{R}^d} \widetilde{\phi}_k(m^j(h) - m^j_{Y_h}) > x - \epsilon] \leqslant \exp\left\{C\big(\frac{R}{\epsilon}\big)^d - \frac{n^j(x-\epsilon)^2}{Ch}\right\}. \quad (5.42)\end{aligned}$$

We fix a small positive parameter γ, and note that, if

$$\epsilon = \gamma^{-\frac{1}{d}} R h^{1/d} x^{-2/d} (n^j)^{-1/d},$$

then

$$R \exp\left\{C\big(\frac{R}{\epsilon}\big)^d - \frac{n^j x^2}{Ch}\right\} = R \exp\left\{C\gamma \frac{n^j x^2}{h} - \frac{n^j(x-\epsilon)^2}{Ch}\right\}. \quad (5.43)$$

Further computations reveal that there is a constant C such that $x > 2\epsilon$ as soon as

$$x \geqslant C \frac{R^{\frac{d}{d+2}} h^{\frac{1}{d+2}}}{\gamma^{\frac{1}{d+2}} (n^j)^{\frac{1}{d+2}}}. \quad (5.44)$$

By choosing γ even smaller, and deduce, in view of (5.42) and (5.43), that, for some constant C and all R, x as in (5.44),

$$P[\sup_{\phi \in \mathbf{L}_R} \int_{\mathbb{R}^d} \widetilde{\phi}(m^j(h) - m^j_{Y_h}) > x] \leqslant \exp\left\{-\frac{n^j x^2}{Ch}\right\}.$$

It follows that

$$\begin{aligned}\mathbb{E}[\sup_{\phi \in \mathbf{L}_R} \int_{\mathbb{R}^d} \widetilde{\phi}(m^j(h) - m^j_{Y_h})] &\leqslant \int_0^{C(\frac{R^d h}{n^j})^{\frac{1}{d+2}}} 1dx + \int_{C(\frac{R^d h}{n^j})^{\frac{1}{d+2}}}^{\infty} \exp\{\frac{-n^j x^2}{Ch}\} dx \\ &\leqslant C(1 + R^{\frac{d}{d+2}})(n^j)^{\frac{-1}{d+2}} h^{\frac{1}{d+2}}.\end{aligned}$$

□

Finally, we give the proof of the concentration inequality.

Proof of Lemma 5.7. Throughout this argument, C denotes a positive constant which, although changing from line to line, depends only on d, T, and $\sup_j |\alpha^j|$.

Next, we prove (5.34) in the case $t_0 = 0$. We fix $R > 0$, and note that, any $\psi \in \mathbf{L}$ normalized with $\psi(0) = 0$, can be written as $\psi = \widetilde{\phi} + \varphi$, with $\phi \in \mathbf{L}_R$ and $|\varphi| \leqslant |x|\mathbf{1}_{B_R^c}$.

Thus, for any $h \in (0, 1]$, we get

$$\begin{aligned}
\mathbb{E}[\mathbf{d_1}(m^j(h), m^j_{Y_h})] &= \mathbb{E}[\sup_{\phi\in\mathbf{L}} \int_{\mathbb{R}^d} \phi(m^j(h) - m^j_{Y_h})] \\
&\leqslant \mathbb{E}[\sup_{\phi\in\mathbf{L}_R} \int_{\mathbb{R}^d} \widetilde{\phi}(m^j(h) - m^j_{Y_h})] + \int_{\mathbb{R}^d} |x|\mathbf{1}_{B_R^c} m^j(h) + \mathbb{E}[\int_{\mathbb{R}^d} |x|\mathbf{1}_{B_R^c} m^j_{Y_h}] \\
&\leqslant \mathbb{E}[\sup_{\phi\in\mathbf{L}_R} \int_{\mathbb{R}^d} \widetilde{\phi}(m^j(h) - m^j_{Y_h})] + \frac{M_2(m^j(h))}{R} + \frac{\mathbb{E}[M_2(m^j_{Y_h})]}{R} \\
&\leqslant \mathbb{E}[\sup_{\phi\in\mathbf{L}_R} \int_{\mathbb{R}^d} \widetilde{\phi}(m^j(h) - m^j_{Y_h})] + C\frac{(1 + M_2(m^j(0)))}{R}. \qquad (5.45)
\end{aligned}$$

Using Lemma 5.11, we find that

$$\begin{aligned}
\mathbb{E}[\mathbf{d_1}(m^j(h), m^j_{Y_h})] &\leqslant C(1 + R^{\frac{d}{d+2}})(n^j)^{\frac{-1}{d+2}} h^{\frac{1}{d+2}} + C\frac{(1 + M_2(m^j(0)))}{R} \\
&\leqslant C(1 + R)(n^j)^{\frac{-1}{d+2}} h^{\frac{1}{d+2}} + C\frac{(1 + M_2(m^j(0)))}{R}.
\end{aligned}$$

Optimizing in R, that is, taking $R = (n^j)^{\frac{1}{2d+4}} h^{-\frac{1}{2d+4}} \sqrt{1 + M_2(m^j(0))}$, gives the result with $\beta = \frac{1}{2d+4}$. □

5.4 The proof of Theorem 5.1 with a common noise

We now show that the method developed above can be adapted to problems with a common noise, that is, when a_0 is positive. Recall that $\mathcal{V}^N$ and $\mathcal{U}$ are defined by (5.1) and (5.23) respectively.

Proof of Theorem 5.1 when $a_0 > 0$. Since the proof follows closely the one in the case $a^0 = 0$, here we emphasize and explain the main differences.

We first note that the estimates of Lemma 5.1 and 5.3 remain valid (with the same proof), that is, there exists $C > 0$ such that

$$\|\mathcal{V}^N\|_\infty + N \sup_j \|D_{x^j}\mathcal{V}^N\|_\infty + \|\partial_t \mathcal{V}^N\|_\infty \leqslant C,$$

and, for any $(t, \mathbf{x}) \in [0, T] \times (\mathbb{R}^d)^N$, $(\xi^i)_{i=1,\dots,N} \in (\mathbb{R}^d)^N$ and $\xi^0 \in \mathbb{R}$,

$$\sum_{i,j=1}^N D^2_{x^i x^j}\mathcal{V}^N(t, \mathbf{x})\xi^i \cdot \xi^j + 2\sum_{i=1}^N D^2_{x^i t}\mathcal{V}^N(t, \mathbf{x})\xi^i\xi^0 + D^2_{tt}\mathcal{V}^N(t, \mathbf{x})(\xi^0)^2 \leqslant \frac{C}{N}\sum_{i=1}^N |\xi^i|^2 + C(\xi^0)^2.$$

We note for later use that this implies that the conclusion of Lemma 5.9 still holds, because it relies only on the above estimates.

However, the proof of Lemma 5.2 does not adapt to the case $a_0 > 0$. Hence, we need a new argument which relies on some results of [59].

In particular, we have the following analogue of Lemma 5.2.

Lemma 5.12. *Assume* (5.17). *There exists a constant $C > 0$ depending only on the data such that, for all $s, t \in [0, T]$ and all $m, m' \in \mathcal{P}_4(\mathbb{R}^d)$*

$$|\mathcal{U}(s, m) - \mathcal{U}(t, m')| \leqslant C\Big(\mathbf{d}_1(m, m') + |t - s|\Big).$$

Moreover, there exists a constant $C > 0$ such that, for any $\epsilon > 0$ and any $(t_0, m_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, there exists an ϵ-optimal control rule $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{F}, W, m, \alpha) \in \mathcal{A}(t_0, m_0)$ for $\mathcal{U}(t_0, m_0)$ such that

$$\|\alpha\|_\infty \leqslant C.$$

Proof. Fix $R > 0$ and let $\mathcal{V}^{N,R}$ and $\mathcal{U}^R$ denote the values of the problems defining $\mathcal{V}^N$ and $\mathcal{U}$ when controls are restricted to the ball $B_R \subset \mathbb{R}^d$.

More precisely, define $\mathcal{A}^{N,R}$ to be the set of $\alpha = (\alpha^k)_{k=1}^N$ such that $|\alpha^k| \leqslant R$ for each R, and $\mathcal{A}^R(t_0, m_0)$ to be the set of $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, m, \alpha) \in \mathcal{A}(t_0, m_0)$ such that $|\alpha| \leqslant R$. Then define $\mathcal{V}^{N,R}$ exactly as in (5.1) but with $\mathcal{A}^{N,R}$ replacing $\mathcal{A}$ and define $\mathcal{U}^R$ exactly as in (5.23) but with $\mathcal{A}^R(t_0, m_0)$ replacing $\mathcal{A}^R$.

Then Proposition 5.5 and Theorem 3.6 of [59] give

$$\lim_{N\to\infty} \mathcal{V}^{N,R}(t, \mathbf{x}^N) = \mathcal{U}^R(t, m)$$

where $\mathbf{x}^N = (x^1, ..., x^N)$, $m \in \mathcal{P}_4(\mathbb{R}^d)$ and $x^1, x^2, ... \in \mathbb{R}^d$ are such that

$$\sup_N \frac{1}{N}\sum_{i=1}^N |x^i|^4 < \infty \quad \text{and} \quad \frac{1}{N}\sum_{i=1}^N \delta_{x^i} \to m \in \mathcal{P}_2(\mathbb{R}^d).$$

It follows from Lemma 5.1 and Lemma 5.13, that there is $R_0 > 0$ such that $\mathcal{V}^{N,R_0} = \mathcal{V}^N$ and $\mathcal{U}^{R_0} = \mathcal{U}$, and so we infer that, for all x_i and m as above,

$$\lim_{N\to\infty} \mathcal{V}^N(t, \mathbf{x}^N) = \mathcal{U}(t, m).$$

Hence, the uniform regularity on $\mathcal{V}^N$ established in (5.2), which, as noted above, holds equally well when $a_0 > 0$, is enough to conclude that, for some $C > 0$,

$$|\mathcal{U}(t, m) - \mathcal{U}(t, m')| \leqslant C d_1(m, m') \quad \text{for all } m, m' \in \mathcal{P}_4(\mathbb{R}^d).$$

Finally, for any $\epsilon > 0$ and any (t_0, m_0), we can choose an ϵ-optimal pair (m, α) for $\mathcal{U}^{R_0}$, and that this control is also ϵ-optimal for $\mathcal{U}$. This completes the proof. □

Let $\widehat{\mathcal{V}}^N$ be defined in Lemma 5.3. Then it is easily checked that $\widehat{\mathcal{V}}^N$ is smooth and satisfies, with $\widehat{\mathcal{F}}^N$ and $\widehat{\mathcal{G}}^N$ as in Lemma 5.3,

$$\begin{cases} -\partial_t \widehat{\mathcal{V}}^N(t,m) - (1+a^0)\displaystyle\int_{\mathbb{R}^d} \mathrm{div}_y(D_m \widehat{\mathcal{V}}^N(t,m,y))m(dy) \\ \qquad -a_0 \displaystyle\int_{\mathbb{R}^{2d}} \mathrm{tr}(D^2_{mm}\widehat{\mathcal{V}}^N(t,x,m,y,y'))m(dy)m(dy') \\ \qquad + \displaystyle\int_{\mathbb{R}^d} H(y, D_m \widehat{\mathcal{V}}^N(t,m,y))m(dy) \leqslant \widehat{\mathcal{F}}(m) \ \text{ in } \ (0,T)\times \mathcal{P}_1(\mathbb{R}^d), \\ \widehat{\mathcal{V}}^N(T,m) = \widehat{\mathcal{G}}(m) \ \text{ in } \ \mathcal{P}_1(\mathbb{R}^d), \end{cases}$$

Then, as in the proof of Lemma 5.3, it is possible to use Itô's formula for conditional measures (see, for example, [44]) to infer that, for any solution (m,α) to (5.22),

$$\widehat{\mathcal{V}}^N(t_0,m_0) \leqslant \mathbb{E}\left[\int_{t_0}^T (\int_{\mathbb{R}^d} L(x,\alpha_t(x))m_t(x)dx + \widehat{\mathcal{F}}^N(m_t))dt + \widehat{\mathcal{G}}^N(m_T)\right].$$

Using the same argument as in the proof of Lemma 5.3 with Lemma 5.12 replacing Lemma 5.7, we arrive at

$$\mathcal{V}^N(t_0, m^N_{\mathbf{x}_0}) \leqslant \mathcal{U}(t_0, m^N_{\mathbf{x}_0}) + C(1+M_2^{1/1}(m^N_{\mathbf{x}_0}))N^{-\beta}.$$

We now turn to the opposite inequality. As before, for $\theta,\lambda \in (0,1)$,

$$M := \max_{(t,\mathbf{x}),(s,\mathbf{y})\in[0,T]\times(\mathbb{R}^d)^N} e^s(\mathcal{U}(s,m^N_{\mathbf{y}}) - \mathcal{V}^N(t,\mathbf{x})) - \frac{1}{2\theta N}\sum_i |x^i-y^i|^2 - \frac{1}{2\theta}|s-t|^2 - \frac{\lambda}{2N}\sum_{i=1}^N |y^i|^2,$$

and denote by $((t_0,\mathbf{x}_0),(s_0,\mathbf{y}_0))$ a maximum point in the expression above.

As in Lemma 5.5 we have

$$\frac{1}{N}\sum_{i=1}^N |x^i_0 - y^i_0|^2 + |t_0-s_0|^2 \leqslant C\theta^2 \qquad \frac{1}{N}\sum_{i=1}^N |y^i_0|^2 \leqslant C\lambda^{-1}.$$

Next, for $\delta>0$, we use the partition $(C^j)_{j\in\{1,\dots,J\}}$ be a partition of $\{1,\dots,N\}$ constructed in Lemma 5.6.

We set $\alpha^k = \overline{\alpha}^j$ if $k\in C^j$, and let

$$X^k_{t_0+\tau} = x^k_0 + \tau\alpha^k + \sqrt{2}B^k_\tau + \sqrt{2a^0}B^0_\tau, \quad Y^k_{s_0+\tau} = y^k_0 + \tau\alpha^k + \sqrt{2}B^k_\tau + \sqrt{2a^0}B^0_\tau,$$
$$\text{and } \ m^j_{\mathbf{Y}_{s_0+\tau}} = \frac{1}{n^j}\sum_{k\in C^j} \delta_{Y^k_{s_0+\tau}},$$

and m^j be the solution to

$$dm^j_t = \left[(1+a^0)\Delta m^j_t - \overline{\alpha}^j\cdot Dm^j_t\right] + \sqrt{2a_0}Dm^j_t\cdot dB^0_t \ \text{ in } \ (t_0,T]\times\mathbb{R}^d \ \text{ and } \ m^j_{t_0} = m^j_{\mathbf{y}_0} \ \text{ in } \ \mathbb{R}^d.$$

Finally, we set $m_s = N^{-1}\sum_{j\in J} n^j m^j_s$, and claim that, for all $h\geqslant 0$ and $j\in\{1,\dots,J\}$,

$$\mathbb{E}\left[\mathbf{d}_1(m^j_{s_0+h}, m^j_{\mathbf{Y}_{s_0+h}})\right] \leqslant C(1+M_2^{1/2}(m^j_{s_0}))\frac{h^\beta}{(n^j)^\beta}, \tag{5.46}$$

and

$$\mathbb{E}\left[\mathbf{d}_1(m_{s_0+h}, m^N_{\mathbf{X}_{t_0+h}})\right] \leqslant C\theta + C\delta^{-d\beta}(1+\lambda^{-\frac{1}{2}})\frac{h^\beta}{N^\beta}. \tag{5.47}$$

The proof follows from Lemma 5.7. Indeed, to establish (5.46), we first note that the process $(m_t)_{t\in[s-0,T]}$ solves (5.22) in the sense of distribution (with B^0 replacing W) if and only if the process $\widetilde{m}_t = (Id - \sqrt{2a^0}(B^0_t - B_{t_0}))\sharp m_t$ solves $\mathbb{P}-$a.s. in the (classical) sense of distributions, with $\widetilde{\alpha}_t(x) = \alpha_t(x + \sqrt{2a^0}(B^0_t - B^0_{t_0})$, the equation

$$d\widetilde{m}_t(x) = [\Delta\widetilde{m}_t(x) - \mathrm{div}(\widetilde{m}_t\widetilde{\alpha}_t(x))]\,dt \;\text{ in }\; [t_0,T]\times\mathbb{R}^d \quad \widetilde{m}_{t_0} = m_0 \;\text{ in }\; \mathbb{R}^d, \tag{5.48}$$

Next, we consider

$$\widetilde{m}^j_{t_0+\tau} = (Id - \sqrt{2a^0}B^0_\tau)\sharp m^j_{t_0+\tau} \;\text{ and }\; \widetilde{\mathbf{Y}}^k_{t_0+\tau} = \mathbf{Y}^k_{t_0+\tau} - \sqrt{2a^0}B^0_\tau,$$

and notice that $\widetilde{m}^j$ and $\widetilde{\mathbf{Y}}^k$ solve the same equations as in Lemma 5.7, and, hence, (5.46) holds with $\widetilde{m}^j_{t_0+h}$ replacing $m^j_{t_0+h}$ and $m^j_{\widetilde{\mathbf{Y}}_{t_0+h}}$ replacing $m^j_{\mathbf{Y_{t_0+h}}}$.

Since

$$m^j_{t_0+r} = \widetilde{m}^j_{t_0+\tau} \star \delta_{\sqrt{2\alpha^0}B_\tau},$$

and

$$m^j_{\mathbf{Y_{t_0}}+\tau} = \frac{1}{n^j}\sum_{k\in C^j}\delta_{\widetilde{\mathbf{Y}}^k_{t_0+\tau}+\sqrt{2a^o}B^0_\tau} = \Big(\frac{1}{n^j}\sum_{k\in C^j}\delta_{\widetilde{\mathbf{Y}}^k_{s_0+\tau}}\Big)\star\delta_{\sqrt{2a^o}B^0_\tau} = \widetilde{m}^j_{\mathbf{Y_{s_0}}+\tau}\star\delta_{\sqrt{2a^o}B^0_\tau},$$

we can conclude that

$$\mathbb{E}\left[\mathbf{d}_1(m^j_{s_0+h}, m^j_{\mathbf{Y}_{s_0+h}})\right] = \mathbb{E}\left[\mathbf{d}_1(m^j_{s_0+h} * \delta_{\sqrt{2\alpha^0}B^0_h},\, m^j_{\mathbf{Y}_{s_0+h}} * \delta_{\sqrt{2\alpha^0}B^0_h})\right] = \mathbb{E}\left[\mathbf{d}_1(\widetilde{m}^j_{s_0+h}, m^j_{\widetilde{\mathbf{Y}}_{s_0+h}})\right],$$

and so (5.46) holds. The proof for (5.47) is similar.

We proceed with the proof by noticing that the Dynamic Programming in Lemma 5.8 still holds but with an expectation, since now the measures are random, and with Proposition 5.2 replacing Proposition 5.1.

Moreover, since the conclusion of Lemma 5.9 also holds as already pointed out, we can argue as in the proof of Proposition 5.4 (the time-regularity provided by Lemma 5.12 replacing that in Lemma 5.2) that

$$\mathcal{U}(t, m^N_{\mathbf{x}}) - \mathcal{V}^N(t,\mathbf{x})) \leqslant C\frac{1}{N^\beta}(1 + \frac{1}{N}\sum_{i=1}^N |x^i|^2).$$

The conclusion then follows as in the proof of Theorem 5.1. □

5.5 Appendix

The purpose of this appendix is to adapt some technical results from [58] and [59] to our setting. Most importantly, we need to infer the dynamic programming principle (Proposition 5.2) in our setting from the dynamic programming principle which is stated in Theorem 3.1 of [59]. Most of the arguments here are straightforward adaptations of the superposition and mimicking results achieved in [97], and so the proofs are only sketched.

Following Definition 2.1 in [58] and Definition 2.3 [59] we define, for each $(t_0, m_0) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, the set of weak controls $\mathcal{A}_w(t_0, m_0)$ to be the set of tuples

$$\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbb{G} = (\mathcal{G})_{0 \leqslant t \leqslant T}, X, B, W, m, \alpha)$$

such that

1. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space equipped with filtrations $\mathbb{G}$, $\mathbb{F}$ such that, for all $0 \leqslant t \leqslant T$, $\mathcal{G}_t \subset \mathcal{F}_t$ and $\mathcal{F}_t \vee \mathcal{F}_T^B \perp \mathcal{G}_T | \mathcal{G}_t$.
2. $X = (X_t)_{0 \leqslant t \leqslant T}$ is a continuous, $\mathbb{F}$-adapted $\mathbb{R}^d$ valued process.
3. $\alpha = (\alpha_t)_{t_0 \leqslant t \leqslant T}$ is a bounded, $\mathbb{F}$-predictable process taking values in $\mathbb{R}^d$.
4. (B, W) is a $\mathbb{R}^d \times \mathbb{R}^d$-valued standard $\mathbb{F}$ Brownian motion, W is $\mathbb{G}$-adapted, and $\mathcal{F}_t \vee \sigma(B) \perp \mathcal{G}_T$.
5. $m = (m_t)_{t_0 \leqslant t \leqslant T}$ is a $\mathbb{G}$-predictable process taking values in $\mathcal{P}_2(\mathbb{R}^d)$ and such that $m_t = \mathcal{L}(X_t | \mathcal{G}_t)$ for $d\mathbb{P} \otimes ds$-a.e. $(s, \omega) \in [t, T] \times \Omega$.
6. the state equation

$$X_t = X_{t_0} + \int_{t_0}^t \alpha_s ds + \sqrt{2}(B_t - B_{t_0}) + \sqrt{2a_0}(W_t - W_{t_0}), \quad \mathcal{L}(X_{t_0}) = m_0$$

holds for all $t_0 \leqslant t \leqslant T$.

We also define

$$\mathcal{U}_w(t_0, m_0) := \inf_{\mathcal{R} \in \mathcal{A}_w(t_0, m_0)} \mathbb{E}^{\mathbb{P}}[\int_{t_0}^T (L(X_t, \alpha_t) + \mathcal{F}(m_t))dt + \mathcal{G}(m_T)]$$

In our context, a superposition principle is a result asserting the following: given a control rule $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, m, \alpha) \in \mathcal{A} \in \mathcal{R}(t_0, m_0)$, we can find an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathbb{G})$ of $(\Omega, \mathcal{F}, \mathbb{F})$ hosting another Brownian motion B independent of $\mathbb{F}$ and a process X such that $dX_t = \alpha_t(X_t)dt + \sqrt{2}dB_t + \sqrt{2a_0}dW_t$ such that $m_t = \mathcal{L}(X_t | \mathcal{F}_t)$. We refer to [97] for details. The superposition results of [97] are useful to us because we need to apply some technical results from [58, 59], and the superposition results allow us to check that our formulation is equivalent to the one used in [58, 59].

In what follows, for technical reasons, that is, to have the coercivity condition on the cost appearing in Assumption 2.1 of [59], we will work with a truncated version of the weak

formulation defined here. Namely, we define $\mathcal{A}_W^R(t_0, m_0)$ just as $\mathcal{A}_w(t_0, m_0)$, but with the controls α required to take values in $B_R \subset \mathbb{R}^d$. Then, we use

$$\mathcal{U}_w^R(t_0, m_0) := \inf_{\mathcal{R} \in \mathcal{A}_w^R(t_0, m_0)} \mathbb{E}^{\mathbb{P}}[\int_{t_0}^T (L(X_t, \alpha_t) + \mathcal{F}(m_t))dt + \mathcal{G}(m_T)]$$

We also truncate the original form of the problem, by defining $\mathcal{U}^R$ just like $\mathcal{U}$, but with controls α required to take values in $B_R \subset \mathbb{R}^d$.

The following can be obtained using the superposition and following results of [97], as in the proof of Theorem 8.3 of [97].

Proposition 5.5. *For each R, $\mathcal{U}_w^R = \mathcal{U}^R$.*

It is also useful to note that our regularity result Lemma 5.1, which holds also in the case $a_0 > 0$, can be used to infer that $\mathcal{U}^R = \mathcal{U}$ for all $R \geqslant R_0$.

Lemma 5.13. *There exists R_0 depending on the data such that, for each $R \geqslant R_0$, $\mathcal{U}^R = \mathcal{U}$.*

Proof. Theorems 3.1 and 3.6 in [59] together with Proposition 5.5 allow to conclude that, for each $R > 0$, we have the following form of convergence of $\mathcal{V}^{N,R}$ to $\mathcal{U}^R$.

For all $t \in [0, T]$, $\mu \in \mathcal{P}_4(\mathbb{R}^d)$ and $x_i \in \mathbb{R}^d$ such that

$$\sup_N \frac{1}{N} \sum_{i=1}^N |x_i|^4 < \infty \quad \text{and} \quad \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \underset{N \to \infty}{\to} m \quad \text{in} \quad \mathcal{P}_2(\mathbb{R}^d)$$

we have, for $\mathbf{x}^N = (x_1, ..., x_N) \in (\mathbb{R}^d)^N$,

$$\lim_{N \to \infty} \mathcal{V}^{R,N}(t, \mathbf{x}^N) = \mathcal{U}^R(t, m). \tag{5.49}$$

Next, notice that, by (5.1) (see Remark 5.3), there is R_0 depending only on the data such that, for all $R \geqslant R_0$, $\mathcal{V}^{N,R} = \mathcal{V}^N$. Thus (5.49) actually gives, for all $R \geqslant R_0$,

$$\lim_{N \to \infty} \mathcal{V}^N(t, \mathbf{x}^N) = \mathcal{U}^R(t, m)$$

It follows that

$$\mathcal{U} = \mathcal{U}^{R_0}.$$

Indeed, clearly $\mathcal{U} \leqslant \mathcal{U}^{R_0}$.

For the other inequality, for any (t_0, m_0), we can choose $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, m, \alpha)$ to be ϵ-optimal in the definition of $\mathcal{U}(t_0, m_0)$. Since α is bounded by hypothesis, there exist $R \geqslant R_0$ such that $\mathcal{R} \in \mathcal{A}^R(t_0, m_0)$, and, hence,

$$\mathcal{U}^{R_0}(t_0, m_0) = \mathcal{U}^R(t_0, m_0) \leqslant \mathcal{U}(t_0, m_0) + \epsilon.$$

Letting $\epsilon \to 0$ gives $\mathcal{U}(t_0, m_0) = \mathcal{U}^{R_0}(t_0, m_0)$. □

Now, we turn to the dynamic programming principle, that is, Proposition 5.2.

Proof of Proposition 5.2. We combine Theorem 3.1 of [58] with Proposition 5.5 to conclude that, for all $0 \leqslant t_0 \leqslant t_1 \leqslant T$ and any $R \geqslant R_0$,

$$\begin{aligned}\mathcal{U}(t_0,m_0) = \mathcal{U}^R(t_0,m_0) = \mathcal{U}^R_W(t_0,m_0) &= \inf_{\mathcal{R}\in\mathcal{A}^R_W(t_0,m_0)} \mathbb{E}^{\mathbb{P}}[\int_{t_0}^{t_1}(L(X_t,\alpha_t)+\mathcal{F}(m_t))dt + \mathcal{U}^R_W(t_1,m_{t_1})] \\ &= \inf_{\mathcal{R}\in\mathcal{A}^R_W(t_0,m_0)} \mathbb{E}^{\mathbb{P}}[\int_{t_0}^{t_1}(L(X_t,\alpha_t)+\mathcal{F}(m_t))dt + \mathcal{U}^R(t_1,m_{t_1})] \\ &= \inf_{\mathcal{R}\in\mathcal{A}^R_W(t_0,m_0)} \mathbb{E}^{\mathbb{P}}[\int_{t_0}^{t_1}(L(X_t,\alpha_t)+\mathcal{F}(m_t))dt + \mathcal{U}(t_1,m_{t_1})].\end{aligned}$$

Since R can be arbitrarily large, it is easy to see that the above imply

$$\mathcal{U}(t_0,m_0) = \inf_{\mathcal{R}\in\mathcal{A}_w(t_0,m_0)} \mathbb{E}^{\mathbb{P}}[\int_{t_0}^{t_1}(L(X_t,\alpha_t)+\mathcal{F}(m_t))dt + \mathcal{U}(t_1,m_{t_1})].$$

To get from here to

$$\mathcal{U}(t_0,m_0) = \inf_{\mathcal{R}\in\mathcal{A}(t_0,m_0)} \mathbb{E}^{\mathbb{P}}\Big[\int_{t_0}^{t_1}\Big(\int_{\mathbb{R}^d} L(x,\alpha_t(x))m_t(dx) + \mathcal{F}(m_t)\Big)dt + \mathcal{U}(t_1,m_{t_1})\Big],$$

we again use the superposition and adapt arguments results from [97]. □

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RÉSUMÉ

On étudie des problèmes de contrôle stochastique de type champ-moyen avec des contraintes sur la loi du processus. Le but est de caractériser les stratégies optimales grâce à un système couplé d'équations aux dérivées partielles qui est celui de la théorie des jeux à champ-moyen. On étudie tout d'abord un problème avec contrainte terminale puis un problème avec contrainte à chaque instant. Pour ce second problème on étudie la régularité en temps des contrôles optimaux sous des conditions géométriques sur la contrainte. On montre également comment ce problème apparaît comme limite de problèmes de contrôle où un grand nombre d'agents sont soumis à des contraintes symétriques et presque-sûres. Dans une dernière partie, on étudie quantitativement, en obtenant un taux de convergence, la limite champ-moyen pour des problèmes de contrôle stochastique sans contrainte.

MOTS CLÉS

Contrôle Stochastique; Théorie des Jeux à Champ Moyen; Optimisation sous Contraintes; Limite de Champ Moyen

ABSTRACT

We analyse mean-field stochastic control problems under constraints in law. The goal is to characterize optimal solutions thanks to a mean-field game system of partial differential equations. We start with a problem with terminal constraint and go on with a problem with constraint at all time. For this second problem we prove the time regularity of optimal controls under suitable geometric assumptions on the constraint. We also explain how this problem arises as limit of control problems for a large number of interacting agents with symmetric, almost-sure constraints. Finally we investigate quantitatively the mean-field limit for stochastic control problems without constraint and prove that this convergence holds with a rate.

KEYWORDS

Stochastic Control; Mean-Field Control; Mean-Field Games; Mean-Field Limit; Constrained Optimization