Analyse et analyse numérique d'EDP issues de la thermomécanique des fluides

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THÈSE

Pour obtenir le diplôme de doctorat

Spécialité **MATHEMATIQUES**

Préparée au sein de l'**Université de Rouen Normandie**

Analyse et analyse numérique d'EDP issues de la thermomécanique des fluides

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Thèse soutenue le 20/12/2023

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Remerciements

D'abord et avant tout je voudrais exprimer toute ma gratitude à Olivier Guibé, qui a dirigé ma thèse. Depuis mon mémoire de master et tout au long de mon parcours par la suite il a su se montrer présent et m'a beaucoup apporté tant sur le plan scientifique que sur le plan humain. Sa qualité de travail, sa rigueur disciplinaire mais aussi son humour et sa bonne humeur quotidienne m'ont donnés un cadre épanouissant dont je lui suis reconnaissante.

Je voudrais ensuite remercier Roger Lewandowski et Alain Miranville pour avoir accepté d'être les rapporteurs de ma thèse et pour l'application qu'ils ont mis à cette tâche. Ce sont des membres éminents dans mon domaine de recherche et je suis très honorée de les compter parmi mon jury de thèse.

Je souhaiterais aussi remercier Carole Rosier et Hasnaa Zidani d'avoir bien voulu prendre sur leur temps pour faire partie du jury.

Des remerciements particuliers vont à Boris Andreinaov qui a accepté de prendre place non seulement parmi le jury de la thèse mais aussi au sein de mon CSI. Je voudrais en profiter aussi pour remercier Nicolas Forcadel qui a accepté d'être le deuxième membre de mon CSI.

Mes remerciements vont à Enrique Fernández-Cara pour son enrichissante collaboration à mon travail de recherche ainsi que son accueil chaleureux à l'Université de Séville.

J'aimerais ensuite remercier tous les membres du LMRS avec une attention particulière pour Adel pour la bienveillance dont il a fait preuve avec moi.

Je voudrais adresser un immense merci à tous les doctorants qui m'ont accompagnée : Adrien, Anas, Angèle, Audrey, Cécile, Gauthier, Nessim, Samuel, Séverin, Silvio et Victor. Vous m'avez apportez la joie au milieu de tout ce travail, grâce à nos discussions les midis, nos restaurants les jeudi, votre humour et votre bonne humeur quotidienne. Merci Anas pour le chocolat noir quotidien dans le bureau (sans oublier la chasse aux insectes). Merci Cécile, Gauthier, Séverin et Victor de m'avoir supportée dès le début et pour toutes les belles aventures ensemble. Un grand merci à Séverin et Victor de m'avoir fait découvrir le Badminton. Merci Adrien, maintenant on peut rejouer à Catane. Que je joins aux doctorants malgré lui, merci Nicolas pour ton support avec tous les petits écoliers.

Pour tous mes amis libanais : qu'ils soient en Europe avec moi : Perla, Rafka, Yara, Ghady et tout particulièrement Cynthia et Karen. Merci d'avoir partager avec moi cette expérience, d'être venus me voir et de m'avoir encouragée sans relâches. Ceux que j'ai rencontrés en arrivant à Rouen : Mickeal, Nissrine et Rania. Il était important pour moi de savoir que je pouvais compter sur vous. Enfin ceux qui sont au Liban : Claudia, Daniel, Ghiwa, Hadi, Patrick, Rami et Romario. Merci pour vos jolis accueils lorsque je revenais vous voir.

Enfin, je suis profondément reconnaissante envers mes parents, ma soeur et mon frère d'avoir été une source d'encouragement et du support sans faille tout au long de ces années. Mon dernier remerciement s'adresse à François. Merci d'avoir été là à chaque étape du chemin, de croire en moi et de me pousser à persévérer (les cookies seront bientôt de retour).

Résumé

Dans cette thèse nous nous intéressons à des systèmes non linéaires d'évolution issus d'un modèle de solidification non isotherme avec prise en compte de la convection. Ces systèmes consistent en un couplage non linéaire de trois équations aux dérivés partielles : la première est l'équation de la phase, la deuxième est l'équation de conservation de l'énergie et la troisième est l'équation de Navier-Stokes incompressible. Plus précisément, nous nous intéressons à établir des résultats sur l'existence de solutions pour ce type de systèmes en dimension 2 et 3 ainsi que sur la convergence de solutions approchées par la méthode des volumes finis. Une des particularités de ce type de système est la présence d'un terme naturellement dans L^1 dans l'équation de conservation de l'énergie, ce qui demande un traitement particulier.

Cette thèse est divisée en deux parties.

La première partie comporte deux chapitres et est consacrée à l'étude des problèmes avec données L^1 et des conditions aux limites de type Neumann. Pour traiter ces problèmes et ces données peu régulières, nous nous plaçons dans le cadre des solutions renormalisées. Nous établissons dans un premier chapitre un résultat de convergence des solution approchées par la méthode des volumes finis vers l'unique solution renormalisée à médiane nulle dans le cas d'une équation de convection-diffusion elliptique. Dans le second chapitre nous nous intéressons à un problème parabolique non-linéaire, avec des conditions de Neumann non homogène et un terme de convection. Pour ce problème nous donnons une définition de solution renormalisée et nous montrons l'existence et l'unicité d'une telle solution.

La deuxième partie comporte deux chapitres et est consacrée à l'étude du système en dimension 2 et 3. Dans un premier chapitre nous traitons le cas de la dimension 2 et nous définissons une notion de solution faibles–renormalisées. Avec notamment l'aide des résultats d'existence et de stabilité établis dans la première partie pour l'équation de la conservation de l'énergie, nous démontrons un résultat d'existence de solution. Le dernier chapitre aborde le cas plus délicat de la dimension 3. L'absence de résultat général de stabilité et d'unicité pour l'équation de Navier-Sokes en dimension 3 nous impose tout d'abord à transformer le système en un système formellement équivalent. Par approximation et passage à la limite nous démontrons un résultat d'existence de solution dans un sens faible.

Mots clés : Volumes finis ; Solutions renormalisées ; Problème de Convection-diffusion ; Donnée L^1, Existence de solutions ; Phase-field ; Équations de Navier-Stokes ; Problèmes Parabolique ; Problèmes Elliptique ; Systèmes non linéaires.

Abstract

In this thesis, we focus on nonlinear evolutionary systems derived from a non-isothermal solidification problem with melt convection. These systems consist of three partial differential equations. The first is the phase-field equation coupled with the heat equation and the incompressible Navier-Stokes equation. More precisely, we are interested in the existence of solutions for these types of systems in the two-dimensional and the three-dimensional cases, and in the convergence of a finite volume approximation. One of the particularities of this type of system is the presence of a term naturally in L^1 in the energy conservation equation, which requires special treatment.

This thesis is divided into two parts.

The first part is divided into two chapters and is devoted to the study of problems with L^1 data and Neumann-type boundary conditions. To deal with these problems, and with data that are not very regular, we use the framework of renormalized solutions. In the first chapter, we establish a convergence result for solutions approximated by the finite volume method to the unique renormalized solution with zero median in the case of an elliptic convection-diffusion equation. In the second chapter, we focus on a non-linear parabolic problem with non-homogeneous Neumann conditions and a convection term. For this problem, we provide a definition of a renormalized solution and we show the existence and uniqueness of such a solution.

The second part is devoted to the study of the system in dimensions 2 and 3. The first chapter deals with the dimension 2 and defines the notion of weak–renormalized solutions. With the help of the existence and stability results established in the first part for the conservation of energy equation, we prove the existence of a weak–renormalized solution. The final chapter considers the trickier case of dimension 3. The absence of a general stability and uniqueness result for the 3-dimensional Navier-Sokes equation requires us to transform the system into a formally equivalent one. By approximation and passage to the limit, we prove the existence of a solution in a weak sense.

Key words: Finite volume; Renormalized solutions; Convection-diffusion problem, L^1 data, existence of solutions; Phase-field; Navier-Stokes equations; Parabolic problems; Elleptic problems; Nonlinear systems.

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Introduction

Dans cette thèse, nous étudions des systèmes non linéaires d'évolution dérivant de modèles de thermomécanique des fluides. Plus précisément, nous nous concentrons sur l'étude de problèmes de solidification non isotherme avec prise en compte de la convection. Ce type de problème se présente sous la forme d'un système constitué de trois équations aux dérivées partielles

— l'équation de la phase, dans laquelle la température intervient

— l'équation de conservation de l'énergie, dans laquelle interviennent la vélocité du fluide et la phase

— l'équation de Navier-Stokes pour la conservation du mouvement de fluide.

Dans le cadre de la version simplifiée étudiée ici, nous travaillons avec des EDP couplées du type

$$\left\{\begin{array}{l} \varphi_t + u\cdot\nabla\varphi - \xi\Delta\varphi = \varphi(\varphi-1)(1-2\varphi) + F(\theta) \\ \theta_t + l\varphi_t + u\cdot\nabla\theta - \operatorname{div}(\kappa\nabla\theta) = \nu D(u) : D(u) + g \\ u_t + (u\cdot\nabla)u - \operatorname{div}(\nu D(u)) + \nabla p = f, \quad \operatorname{div} u = 0 \end{array}\right. \tag{1}$$

avec $\kappa = \kappa(\varphi,\theta)$ et $\nu = \nu(\varphi,\theta)$. Ces équations sont satisfaites dans $\Omega \times (0,T)$ où Ω est un domaine borné de $\mathbb{R}^2$ ou $\mathbb{R}^3$.

La première équation est celle de la phase φ dans laquelle interviennent naturellement la température via $F(\theta)$ et le déplacement de la phase via $u\cdot\nabla\varphi$. Le champ de phase φ représente un état intermédiaire qui varie entre l'état liquide (φ=1) et l'état solide (φ=0). La deuxième équation représente la conservation de l'énergie. Elle inclut des termes tels que $\nu D(u) : D(u)$ (lié au mouvement du fluide et à la source de chaleur) et $u\cdot\nabla\theta$ (associé au transport de la chaleur par advection). Enfin, la troisième équation est l'équation de Navier-Stokes, décrivant le mouvement du fluide (considéré comme incompressible). Elle fait intervenir la vitesse u (un champ de vecteurs dans $\mathbb{R}^2$ ou $\mathbb{R}^3$) et la pression p.

En ce qui concerne l'analyse mathématique et numérique du système (1), nous nous intéressons principalement dans cette thèse aux questions d'existence de solutions en dimension 2 et 3 du système (1) ainsi qu'aux questions de convergence des solutions des schémas aux volumes finis. Ces résultats sont difficiles à obtenir parce qu'il s'agit d'un système non linéaire fortement couplé et que le membre de droite de l'équation de la chaleur, $\nu D(u) : D(u)$, est naturellement un terme dans L^1. La littérature existante sur l'analyse mathématique de ce type de problème concerne essentiellement des modèles linéarisés en dimension 2 et dimension 3 (pas de terme en $\nu D(u) : D(u)$), ou encore des systèmes de type Boussinesq avec conditions aux limites de Dirichlet, qui ne font pas intervenir l'équation de la phase. Concernant l'analyse numérique, la littérature existante se concentre principalement sur les équations avec second membre L^2 ou second membre L^1 avec des conditions aux bords de Dirichlet. La complexité et la difficulté pour étudier un tel système d'EDP viennent du fait de ne pas négliger les effets du déplacement sur la température, l'évolution de la température, etc. De plus, en dimension 3, le couplage de l'équation de Navier-Stokes avec l'équation de la chaleur nécessite des conditions et des transformations supplémentaires pour le système.

L'originalité du travail présenté ici réside dans la prise en compte du terme non linéaire $\nu D(u) : D(u)$ et du terme $l\varphi_t$ dans l'équation de conservation de l'énergie, ainsi que dans l'intégration d'une viscosité et d'une conductivité thermique dépendantes à la fois de la température et de la phase. De plus, nous supposons qu'il n'y a pas d'échange de chaleur aux bords et considérons par conséquent dans notre analyse les conditions aux bords de Neumann pour l'équation de la conservation de l'énergie. En ce qui concerne l'équation de la phase, les conditions aux bords de Neumann sont également considérées. Enfin, pour l'équation de Navier-Stokes, nous utilisons les conditions de Dirichlet en dimension 2 et les conditions de glissement (Slip boundary condtions) en dimension 3.

Afin de mieux comprendre le système et parvenir à démontrer l'existence de solutions en dimension 2 et dimension 3, une étude détaillée de l'équation de la chaleur avec second membre L^1 et conditions aux bords de Neumann est essentielle. De même, pour l'analyse numérique, nous sommes amenés dans un premier temps à étudier la convergence de solutions des schémas aux volumes finis pour une équation elliptique de type convection-diffusion avec second membre L^1 et conditions aux bords de Neumann. Étant donné que nous travaillons avec des données L^1, il est indispensable d'adopter une notion de solution appropriée pour cette faible régularité. Nous utiliserons dans cette thèse la notion de solutions normalisées.

Cette thèse est donc divisées en deux parties. Nous allons dans un premier temps donner une idée générale du contenu de chacune de ces partie.

La première partie porte sur l'étude des problèmes avec des données L^1 et des conditions aux limites de type Neumann. Elle est scindée en deux chapitres.

Dans le Chapitre 1, nous étudions la convergence de la solution approchée par des schémas aux volumes finis de type « cell-centered », pour une équation elliptique de type convection-diffusion :

$$\begin{aligned} -\operatorname{div}(\lambda(u)\nabla u - \boldsymbol{v}u) &= f \quad \text{dans } \Omega, \\ (\lambda(u)\nabla u - \boldsymbol{v}u)\cdot \boldsymbol{n} &= 0 \quad \text{sur } \partial\Omega, \end{aligned} \tag{2}$$

où Ω est un ouvert borné de $\mathbb{R}^N$, $N \geq 2$ et $\boldsymbol{n}$ est le vecteur normal unitaire. La fonction $\boldsymbol{v} \in L^p(\Omega)^N$, $2 < p < +\infty$ si $N = 2$ et $p = N$ si $N \geq 3$, λ est une fonction continue telle que $\lambda_\infty \geq \lambda(u) \geq \mu > 0$ avec λ_∞ et μ deux nombres réels et $f \in L^1(\Omega)$.
Pour gérer le caractère non coercif de l'équation et la donnée L^1, nous combinons à la fois les techniques développées pour les solutions renormalisées et celles de la méthode des volumes finis. Pour traiter les conditions aux limites de Neumann, nous choisissons des solutions avec une médiane nulle et prouvons un résultat de convergence.

Dans le chapitre 2, nous nous intéressons à l'équation parabolique suivante

$$\begin{aligned} &\frac{\partial \theta}{\partial t} - \operatorname{div}\left(A(x,t,\theta)\nabla\theta\right) + u\cdot\nabla\theta = f - \operatorname{div} g && \text{dans } Q_T\ ,\\ &\theta(t=0) = \theta_0 && \text{dans } \Omega\ ,\\ &\Big(A(x,t,\theta)\nabla\theta - g\Big)\cdot \boldsymbol{n} = h && \text{sur } \partial\Omega\times(0,T)\ . \end{aligned} \tag{3}$$

où $Q_T = \Omega\times(0,T)$, Ω est un ouvert bornée de $\mathbb{R}^N$, $T>0$, $N\geq 2$ et $\boldsymbol{n}$ est le vecteur unitaire normal au bord $\partial\Omega$. Ici, f respectivement h sont dans $L^1(Q_T)$, respectivement $L^1(0,T;L^1(\partial\Omega))$, g est un élément de $L^2(Q_T)^N$ et u est un vecteur satisfaisant des conditions spécifiques.
Dans un premier temps, nous démontrons l'existence et la stabilité d'une solution renormalisée de l'équation (3) puis nous prouvons son unicité sous une condition supplémentaire concernant la matrice A.

La deuxième partie de cette thèse est dédiée à l'étude du système (1) en dimension 2 et dimension 3. Elle est elle-même scindée en deux chapitres.

L'objectif principal du chapitre 3 est de prouver l'existence de solutions faible-renormalisées, dans le cadre bidimensionnel et sans aucune condition de petitesse sur l, du problème de changement de phase décrit ci-dessous :

$$\varphi_t + u\cdot\nabla\varphi - \xi\Delta\varphi = -\varphi(1-\varphi)(1-2\varphi+c\theta) \qquad \text{dans } Q_T, \tag{4}$$
$$\theta_t - \operatorname{div}\left(\kappa(\varphi,\theta)\nabla\theta\right) + \operatorname{div}(u\theta) = \nu(\varphi,\theta)D(u) : D(u) + g + l\varphi_t \qquad \text{dans } Q_T, \tag{5}$$
$$u_t + (u\cdot\nabla)u - \operatorname{div}\left(\nu(\varphi,\theta)D(u)\right) + \nabla p = f, \quad \operatorname{div} u = 0 \qquad \text{dans } Q_T, \tag{6}$$
$$\nabla\varphi\cdot\boldsymbol{n} = 0, \quad \kappa(\varphi,\theta)\nabla\theta\cdot\boldsymbol{n} = 0, \quad u = 0 \qquad \text{sur}\,\Sigma, \tag{7}$$
$$\varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0(x), \quad u(x,0) = u_0(x) \qquad \text{dans } \Omega, \tag{8}$$

où $Q_T = \Omega\times(0,T)$, $\Sigma = \partial\Omega\times(0,T)$ Ω un ouvert borné de $\mathbb{R}^N$ dont le bord $\partial\Omega$ est Lipschitz, $N=2$, $T>0$, et $\boldsymbol{n}$ est le vecteur normal unitaire au bord $\partial\Omega$. La fonction g est dans $L^1(Q_T)^2$, f est dans $L^2(Q_T)^2$, les fonctions κ et ν sont $C^0(\mathbf{R}^2)$ avec $0<\kappa_0\leq\kappa\leq\kappa_1$, $0<\nu_0\leq\nu\leq\nu_1$ et $0\leq\varphi_0\leq 1$ avec $l>0$, $c>0$ et $\xi>0$.
Le résultat d'existence est établi en utilisant la stabilité des solutions renormalisées décrit dans le chapitre 2 ainsi que le couplage entre les trois équations de ce système.

Enfin, dans le chapitre 4, nous étudions le cas tridimensionnel. Plus précisément, avec les conditions de Navier-Stokes spécifiques à cette partie, le système se réécrit :

$$\varphi_t + u\cdot\nabla\varphi - \xi\Delta\varphi = -\varphi(1-\varphi)(1-2\varphi+c\theta) \qquad \text{dans } Q_T, \tag{9}$$
$$\theta_t - \operatorname{div}\left(\kappa(\varphi,\theta)\nabla\theta\right) + \operatorname{div}(u\theta) = \nu(\varphi,\theta)D(u) : D(u) + g + l\varphi_t \qquad \text{dans } Q_T, \tag{10}$$
$$u_t + (u\cdot\nabla)u - \operatorname{div}\left(\nu(\varphi,\theta)D(u)\right) + \nabla p = f, \quad \operatorname{div} u = 0 \qquad \text{dans } Q_T, \tag{11}$$
$$\nabla\varphi\cdot\boldsymbol{n} = 0, \quad \kappa(\varphi,\theta)\nabla\theta\cdot\boldsymbol{n} = 0, \qquad \text{sur } \Sigma, \tag{12}$$
$$u\cdot\boldsymbol{n} = 0, \quad \alpha u_\tau + (\nu(\theta,\varphi)D(u)\boldsymbol{n})_\tau = 0, \qquad \text{sur } \Sigma, \tag{13}$$
$$\varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0(x), \quad u(x,0) = u_0(x) \qquad \text{dans } \Omega, \tag{14}$$

où $Q_T = \Omega\times(0,T)$, $\Sigma = \partial\Omega\times(0,T)$, $\Omega\in C^{1,1}$ un ouvert de $\mathbb{R}^N$ dont le bord $\partial\Omega$ est Lipschitz, $N=3$, $T>0$, et $\boldsymbol{n}$ est le vecteur normal unitaire au bord $\partial\Omega$. La fonction g est dans $L^1(Q_T)^2$, f est dans $L^2(Q_T)^2$, les fonctions κ et ν sont $C^0(\mathbf{R}^2)$ avec $0<\kappa_0\leq\kappa\leq\kappa_1$, $0<\nu_0\leq\nu\leq\nu_1$ et $0\leq\varphi_0\leq 1$ avec $l>0$, $c>0$ et $\xi>0$.

En dimension 3 la difficulté majeure est la question de l'unicité et la stabilité de la solution de l'équation de Navier-Stokes : en particulier, la convergence des énergies n'est pas établie en général. Il ne semble pas alors possible de considérer l'équation (10) avec le terme de dissipation $\nu(\varphi, \theta) D(u) : D(u)$. Dans ce cas, l'idée est de transformer le système (9)–(11) en un système formellement équivalent. Avec des conditions de type « slip boundary » et pour le système formellement équivalent, on établit un résultat d'existence de solution faible sans condition de petitesse sur l.

Partie I : Problèmes avec des données L^1 et des conditions aux bords de type Neumann

Les deux chapitres de cette section se concentrent sur deux équations ayant un principal point commun : la donnée L^1. Pour ces deux équations, l'une elliptique et l'autre parabolique, nous utiliserons le cadre des solutions renormalisées.

Dans le cadre des données L^1, la solution des problèmes elliptiques ou paraboliques n'appartient pas à $H^1(\Omega)$ ou $L^2(0, T; H^1(\Omega))$. Boccardo et Gallouët ont montré dans [26] l'existence de solutions au sens des distributions pour une classe générale d'opérateurs non linéaires avec conditions aux limites de type Dirichlet. Pour les conditions aux limites non homogènes, Prignet dans [93] a montré l'existence de solutions au sens de distribution pour des problèmes elliptiques avec second membre mesure. Cependant, dans les deux cas, les solutions obtenus ne sont pas uniques (voir l'exemple de Serrin [98]). Par conséquent, il est essentiel de définir une notion de solution qui garantit l'unicité et la stabilité. Dans les années 90, les notions de SOLA (solutions obtenues comme limites d'approximation) (voir [48]), solutions entropiques (voir [46]) et solutions renormalisées (voir [84, 46]) ont été développées. Ces solutions assurent l'existence, l'unicité et la stabilité et sont équivalentes dans le cas d'une donnée L^1.

Comme mentionné précédemment, nous nous intéressons ici à la notion de solutions renormalisées. Cette notion a été introduite par DiPerna et Lions dans le cadre des équations de Boltzmann dans [51] et développées ensuite pour les cas elliptiques dans [85, 47] et les cas paraboliques dans [22, 23, 80]. La littérature sur ce sujet est abondante, la notion de solution renormalisée permettant d'aborder des questions d'existence, stabilité et unicité là où la notion de solution faible ne le permet pas.

En raison de la faible régularité de la solution de ces problèmes, la technique des solutions renormalisées impose la régularité sur les tronquées de cette solution. Il est donc nécessaire de définir correctement une notion de gradient. Pour une fonction u finie presque partout, $T_k(u)$ est dans $H^1(\Omega)$: ainsi ∇u peut être défini dans le sens où $\nabla T_k(u) = \mathbb{1}_{\{|u|<k\}} \nabla u$ avec $T_k(r) = \min(k, \max(r, -k))$. L'équation au sens renormalisé revient formellement à utiliser des fonctions tests de type $h(u)\varphi$ où h est une fonction régulière à support compact : l'équation est « vérifiée » là où u est tronquée. Pour compenser le manque d'information quand u tend vers l'infini, une condition dite « décroissance de l'énergie des tronquées » est ajoutée.

Le deuxième outil utilisé, notamment dans le premier chapitre, est la méthode des volumes finis. Cette approche de discrétisation est utilisée pour résoudre les équations aux dérivées partielles décrivant une variété de phénomènes physiques tels que la mécanique des fluides, la diffusion de la chaleur, etc. La méthode des volumes finis consiste à diviser le domaine d'étude en un maillage discret de petits volumes de contrôle K, puis à appliquer les lois de conservation à chaque volume de contrôle et enfin à résoudre les équations en les intégrant sur chacun de ses volumes. Grâce à sa capacité à diviser le domaine en différentes cellules de contrôle, cette méthode peut être appliquée à des domaines de forme géométrique complexe, ce qui en fait une méthode flexible pour des simulations réalistes tout en garantissant la conservation locale des quantités physiques.

Eymard, Gallouët et Herbin présentent dans leur livre Finite volume methods [60], les principales techniques de résolution de problèmes elliptiques, paraboliques, hyperboliques avec les volumes finis. Dans ce livre, des versions discrètes des inégalités de Poincaré et des injections de Sobolev ont été démontrées, ainsi que des lemmes de compacités permettant le passage à la limite dans ces schémas. La principal difficulté vient du gradient au sens des volumes finis, qui ne peut converger que faiblement, étant donné que l'approximation par les volumes finis est constante sur chaque volume de contrôle, et par conséquent n'appartient pas à un Sobolev.

En ce qui concerne l'approximation par volumes finis des équations elliptiques avec des conditions aux limites de Neumann, une étude a été établit dans [41] dans le cadre des données L^2 pour des solutions à moyenne nulle. Dans le cadre des données L^1 et conditions aux bords de Dirichlet, la plupart des articles montrent que la solution du schéma converge vers une solution au sens des distributions du problème, nous citons [69] et [56] dans cette direction.

La combinaison des techniques de solutions renormalisées et de l'approximation des volumes finis a récemment été réalisée pour une équation non coercitive avec des données L^1 et des conditions aux limites de Dirichlet. Cependant ce type d'étude est peu présente dans la littérature.

Volumes finis et solutions renormalisées

Dans ce chapitre nous étudions l'équation de convection-diffusion

$$\begin{aligned} -\operatorname{div}(\lambda(u)\nabla u - \boldsymbol{v}u) &= f \quad \text{dans } \Omega, \\ (\lambda(u)\nabla u - \boldsymbol{v}u)\cdot\boldsymbol{n} &= 0 \quad \text{sur } \partial\Omega. \end{aligned} \tag{15}$$

Ici Ω est un ouvert polygonal de $\mathbb{R}^d$, $d \geq 2$, $\boldsymbol{n}$ le vecteur normale unitaire $\partial\Omega$ et λ une fonction continue telle que $\lambda_\infty \geq \lambda(u) \geq \mu > 0$ avec λ_∞ et μ deux réelles. La fonction $\boldsymbol{v}$ appartient à $L^p(\Omega)^d$ avec $2 < p < +\infty$ si $d = 2$, $p = d$ si $d \geq 3$, et f appartient à $L^1(\Omega)$ et vérifie $\int_\Omega f = 0$.

Comme nous l'avons expliqué, l'étude des équations elliptiques à données L^1 nécessite une notion appropriée de solution qui fournit des résultats d'existence, de stabilité et d'unicité. Dans le cas de conditions aux bords de type Neumann, en raison du manque de régularité de la solution, la moyenne peut ne pas exister pour les problèmes non linéaires avec données L^1, ce qui implique des difficultés supplémentaires dans la dérivation des estimations et dans la définition d'une notion appropriée de solution renormalisée. Dans [2] et [15], les auteurs ont choisit la médiane à la place de la moyenne ce qui leur a permis d'établir l'existence et l'unicité de solutions renormalisées à médiane nulle pour ce type d'équations dans le cas continu. L'un des principaux outils est l'inégalité de Poincaré-Wirtinger avec la médiane ([106]).

Nous nous intéressons alors à la convergence de la solution approchée par la méthode des volumes finis vers la solution renormalisée du problème (15). La principale originalité de ce travail est de considérer l'équation non coercitive (15) avec des données L^1 et de combiner les techniques développées dans [15] et la méthode des volumes finis afin de prouver le résultat de convergence. Nous choisissons dans cette étude de travailler avec la médiane. De ce fait, nous montrons tout d'abord une inégalité discrète de type Poincaré-Wirtinger avec médiane en s'inspirant du travail effectué dans [14] (où l'inégalité discrète de Poincaré-Wirtinger avec moyenne était étudié).

Proposition (Inégalité discrète de Poincaré-Wirtinger avec médiane)*.* Soit Ω un ouvert borné connexe polyédrique de $\mathbb{R}^N$ et soit $\mathcal{M}$ un maillage admissible. Alors, $\forall\, 1 \leq p < +\infty$ il existe une constante $C > 0$ qui dépend uniquement de Ω, d et p telle que

$$\|u - c\|_{0,p} \leq \frac{C}{\xi^{(p-1)/p}} |u|_{1,p,\mathcal{M}}, \qquad \forall u \in X(\mathcal{T}) \tag{16}$$

où c appartient à $\mathrm{med}(u)$.

Ensuite, nous considérons le schéma aux volumes finis suivant : Soit $\mathcal{M}$ maillage admissible. Pour $K \in \mathcal{T}$ et $\sigma \in \mathcal{E}(K)$, on définit $\boldsymbol{v}_{K,\sigma} = \frac{1}{|D_\sigma|} \int_{D_\sigma} \boldsymbol{v} \cdot \mathbf{n}_{K,\sigma} \, \mathrm{d}x$,

$$\forall K \in \mathcal{T}, \sum_{\sigma \in \mathcal{E}_{int}(K)} \frac{|\sigma|}{d_\sigma} \lambda(u)_\sigma (u_K - u_L) + \sum_{\sigma \in \mathcal{E}_{int}(K)} |\sigma| \boldsymbol{v}_{K,\sigma} u_{\sigma,+} = \int_K f \, \mathrm{d}x, \tag{17}$$

$$\forall \sigma = K|L \in \mathcal{E}_{int},\ u_{\sigma,+} = u_K \text{ si } \boldsymbol{v}_{K,\sigma} \geq 0,\ u_{\sigma,+} = u_L \text{ sinon.} \tag{18}$$

tel que $u_{\sigma,-}$ satisfait $\{u_{\sigma,+}, u_{\sigma,-}\} = \{u_K, u_L\}$.
Pour $\sigma \in \mathcal{E}_{int}$,

$$\forall \sigma = K|L \in \mathcal{E}_{int},\ \min[\lambda(u_K), \lambda(u_L)] \leq \lambda(u)_\sigma \leq \max[\lambda(u_K), \lambda(u_L)], \tag{19}$$

avec $\lambda(u)_\sigma$ est par exemple la valeur moyenne de $\lambda(u_K)$ et $\lambda(u_L)$ si $\sigma \in \mathcal{E}_{int}$.

Nous démontrons l'existence de solutions approchées en utilisant un argument de point fixe et en adaptant les arguments développés dans [41] pour un problème linéaire avec des conditions de Neumann, un second membre régulier et une solution à moyenne nulle.

Théorème*.* Soit $\mathcal{M}$ un maillage admissible. Alors, il existe une solution $u_\mathcal{T} = (u_K)_{K \in \mathcal{M}}$ aux schémas (17) tel que $\underline{\mathrm{med}}(u_\mathcal{T}) = 0$ où $\underline{\mathrm{med}}(u)$ est défini par

$$\underline{\mathrm{med}}(u) = \inf \left\{ t \in \mathbb{R} : \mathrm{meas}\{x \in \Omega : u(x) > t\} \leq \frac{\mathrm{meas}(\Omega)}{2} \right\}, \tag{20}$$

Enfin, après avoir montré une version discrète de la décroissance à l'infini de l'énergie tronquée, nous établissons le principal résultat de convergence énoncé ci-dessous. Le fait que la formulation renormalisée soit non linéaire et contienne un terme de type $|\nabla T_k(u)|^2$ donne des difficultés supplémentaires par rapport à une équation linéaire (comme dans [41] par exemple). Pour cela nous mélangeons les techniques des volumes finis et celles des solutions renormalisées avec une stratégie similaire à [77] qui considère une équation à donnée L^1 avec des conditions aux limites de Dirichlet.

Théorème. Soit $(\mathcal{M}_m)_{m\geq 1}$ une suite de maillages admissibles avec $h_{\mathcal{M}_m} \to 0$ quand $m \to \infty$. Soit $u_m = (u_K^m)_{K\in\mathcal{T}_m} \in X(\mathcal{T}_m)$ une solution du schéma (17) telle que $\underline{\mathrm{med}}(u_m) = 0$. Alors, u_m converge vers l'unique solution renormalisée u à médiane nulle du problème (15), lorsque $m \to \infty$, dans le sens

$$\begin{gathered} u_m \text{ converge vers } u \text{ p.p. dans } \Omega, \\ \forall n \in \mathbb{N},\ \nabla_{\mathcal{M}_m} T_n(u_m) \text{ converge vers } \nabla T_n(u), \text{ faiblement dans } (L^2(\Omega))^d. \end{gathered}$$

Problèmes paraboliques quasi-linéaires

Nous nous intéressons dans ce chapitre aux questions d'existence et d'unicité de solutions renormalisées pour le problème parabolique

$$\begin{aligned} &\frac{\partial\theta}{\partial t} - \operatorname{div}\left(A(x,t,\theta)\nabla\theta\right) + u\cdot\nabla\theta = f - \operatorname{div} g && \text{dans } Q_T\ , \\ &\theta(t=0) = \theta_0 && \text{dans } \Omega\ , \\ &\Big(A(x,t,\theta)\nabla\theta - g\Big)\cdot \boldsymbol{n} = h && \text{sur } \partial\Omega\times(0,T)\ , \end{aligned} \tag{21}$$

où $Q_T = \Omega\times(0,T)$, Ω est un ouvert bornée de $\mathbb{R}^N$, $T > 0$, $N \geq 2$ et $\boldsymbol{n}$ est le vecteur unitaire normal au bord $\partial\Omega$. Ici, f respectivement h sont dans $L^1(Q_T)$, respectivement dans $L^1(0,T;L^1(\partial\Omega))$, g est un élément de $L^2(Q_T)^N$ et u est un champ de vecteurs tel que $u = 0$ sur $\partial\Omega$ et $\operatorname{div} u = 0$ dans Q_T (typiquement u est la solution d'une équation de Navier-Stokes). Enfin, la matrice A est une fonction de Carathéodory.

Dans le cas des équations paraboliques non linéaires, la question d'existence de solutions renormalisées a été abordée initialement par Blanchard et Murat dans [22] dans le cas de donnée L^1 avec un opérateur de type Leray-Lions. Des résultats d'existence sont également montrés par Blanchard et Redwane, dans [25], pour les problèmes non linéaires d'évolution et par Blanchard, Murat et Redwane, dans [23] pour un second membre dans $L^1 + W^{-1,p'}$ et un opérateur de type $a(t,x,u,Du)$. Une nouvelle formulation des solutions renormalisées équivalente à celle originale a été introduite ensuite par Blanchard et Poretta dans [24] dans le cas du problème de Stefan. Nous adaptons cette approche à notre cas avec les conditions de limites non homogènes de type Neumann.

Le cas parabolique présente en général plus de difficultés que le cas elliptique par la présence et le manque de régularité de la dérivée en temps. Pour surmonter cette difficulté supplémentaire nous utilisons l'approche développée dans [24] avec les moyennes de Steklov et des lemmes d'intégrations par parties appropriés. Le point crucial est d'obtenir une convergence forte des troncatures de la solution approchée.

Concernant l'unicité de la solution renormalisée, la matrice A dépend de la solution et le terme $\operatorname{div}(g)$ présent dans l'équation (3) ne nous permet pas d'appliquer un principe de contraction L^1. Ainsi, la principale difficulté est de trouver une fonction test qui permet de contrôler la dérivée en temps, les termes provenant de l'opérateur et les termes provenant de $-\operatorname{div}(g)$. En adaptant les techniques développées dans [23] et dans [50] dans le cadre des problèmes avec conditions aux bords de Dirichlet, nous montrons l'unicité de la solution renormalisée.

En utilisant la formulation renormalisée de Blanchard et Porreta dans [24] pour le problème avec des conditions aux limites de type Dirichlet, nous énonçons la définition de solution renormalisée du problème (3) ainsi que les deux principaux résultats d'existence et d'unicité de ce chapitre.

Définition. Soit θ une fonction mesurable sur Q_T. Alors, θ est une solution renormalisée de (21) si

$$\theta \in L^\infty(0,T;L^1(\Omega)), \tag{22}$$
$$T_k(\theta) \in L^2(0,T;H^1(\Omega)),\ \forall\ k>0, \tag{23}$$
$$\lim_{n\to+\infty}\frac{1}{n}\int_{\{(x,t)\in Q_T:\, n<|\theta(x,t)|<2n\}} A(x,t,\theta)\nabla\theta\cdot\nabla\theta\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{24}$$

et pour tout $S\in W^{2,\infty}(\mathbb{R})$ avec S' à support compact et $\forall\ \varphi\in C_c^\infty([0,T)\times\bar{\Omega})$, θ satisfait

$$\begin{aligned}-\int_{Q_T}\varphi_t S(\theta)\,\mathrm{d}x\,\mathrm{d}t &- \int_\Omega \varphi(0)S(\theta_0)\,\mathrm{d}x + \int_{Q_T} A(x,t,\theta)\nabla\theta\cdot\nabla\theta S''(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_{Q_T} S'(\theta)A(x,t,\theta)\nabla\theta\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} u\cdot\nabla\theta S'(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_{Q_T} fS'(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\nabla(S'(\theta)\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} hS'(\theta)\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t.\end{aligned} \tag{25}$$

Le premier résultat est l'existence d'une solution.

Théorème. Il existe une solution renormalisée de (21) dans le sens de la définition ci-dessus.

Comme dans la plupart des problèmes à donnée L^1, la preuve consiste à considérer un problème approché pour lequel on sait, d'après [79, 96], qu'une solution existe. On obtient alors une série d'estimations a priori et les théorèmes de compacité de type Aubin-Lions sont alors utilisés pour extraire une sous-suite et définir une limite. Par rapport aux équations elliptiques à donnée L^1 la présence conjointe de la dérivée en temps θ_t ou $S(\theta)_t$ (qui n'est pas assez régulière) et du terme en $S''(\theta)A(x,t,\theta)\nabla\theta\cdot\nabla\theta$ est le principal obstacle pour passer à la limite et obtenir que cette limite est une solution renormalisée. Une première méthode a été développée par Blanchard, Murat et Redwane dans [34], qui utilise la régularisée de Landes. Nous adoptons ici la méthode développée par Blanchard et Porrerra dans [24], plus courte, qui utilise les moyennes de Steklov et des inégalités d'intégration par parties. Pour se faire et comme nous considérons ici des conditions de Neumann non homogènes nous donnons un lemme d'intégration par parties adapté à notre problème.

Sous une hypothèse supplémentaire et naturellee sur la matrice A, nous obtenons l'unicité de la solution renormalisée.

Théorème. Nous supposons de plus que A est localement Lipschitzienne. Alors, la solution renormalisée définie précédemment du problème (3) est unique.

Système de changement de phase modélisant les solidifications non isothermes

Dans cette partie, nous nous intéressons à l'étude mathématique des problèmes de solidification non isotherme dans le cas de la dimension 2 et 3.

Nous choisissons ici la méthode du champ de phase pour modéliser la solidification et la fusion des matériaux. Cette méthode se caractérise par une modélisation efficace du processus due aux avantages significatifs qu'elle apporte.
Rappelons tout d'abord qu'un matériau occupant une région Ω peut se présenter sous deux phases : solide ou liquide. La méthodologie classique de Stefan, également connue sous le nom « sharp interface methodology » pour la solidification, néglige l'épaisseur de l'interface entre un solide et un liquide, de sorte que les régions de différentes phases sont séparées par des surfaces régulières (de largeur nulle). Cette hypothèse de largeur nulle de l'interface n'est pas réaliste dans plusieurs situations importantes. En revanche, dans la méthodologie du champ de phase, les zones de mélange et les couches de transition sont naturelles et sont déterminées par les valeurs de certaines variables spécifiques appelées champs de phase. Nous notons également que dans la méthodologie de champ de phase, l'intégration de plusieurs phénomènes physiques importants se fait de manière complètement systématique et claire. En effet, en incluant ces phénomènes dans une fonction mathématique spécifique (la fonctionnelle d'énergie libre, aussi appelée fonctionnelle d'entropie), il devient possible de dériver les équations de champ de phase correspondantes et qui sera alors standards.

Fix, dans [65], fut parmi les premiers auteurs à adopter cette approche de champ de phase pour la modélisation de la solidification de matériaux simples. Les aspects mathématiques de cette méthode ainsi que sa relation avec l'approche classique des modèles de Stefan ont ensuite été développés par plusieurs auteurs, notamment par Caginalp dans [36], par Caginalp et Jones dans [37], par Morosanu et Motreanu dans [83], par Penrose et Fife dans [88], et par McFadden, Wheeler et Anderson dans [82].
Nous pouvons également citer quelques articles connus qui adoptent cette méthode et en font l'analyse théorique et numérique, [95] par Rappaz et Scheid, [30] par Boldrini et Planas, [28] par Boldrini, Caretta et Fernández-Cara ou [44] par Cherfils, Gatti and Miranville.
Étant donné que la possibilité d'un écoulement dans des parties non solidifiées du matériau est très probable, la méthode de champ de phase a été ensuite développé pour inclure un terme de convection, notons par exemple les travaux [3, 12, 63, 92] dans cette direction.

L'équation du champ de phase considérée dans ce travail dérive des modèles présentés dans [37] et [12]. Elle se présente sous la forme

$$\varphi_t + u{\cdot}\nabla\varphi - \xi\Delta\varphi = -\varphi(1-\varphi)(1-2\varphi+c\theta),$$

où ξ et c sont deux constantes positives, θ est la température et u est la vélocité du fluide. La fonction de champ de phase φ est utilisée pour distinguer les phases du matériau à l'instant t. Elle varie entre l'état liquide $\varphi = 1$ et l'état solide $\varphi = 0$. Le terme $u \cdot \theta$ représente la convection et la fonction non-linéaire $F(\varphi,\theta) = -\varphi(1-\varphi)(1-2\varphi+c\theta)$, issue de la théorie de Landau-Ginzburg, correspond au potentiel à double puits.

L'équation de la phase est accompagnée de l'équation de conservation de l'énergie qui s'écrit

$$\theta_t - \operatorname{div}(\kappa(\varphi,\theta)\nabla\theta) + \operatorname{div}(u\theta) = \nu(\varphi,\theta)D(u) : D(u) + g + l\varphi_t,$$

où θ représente la température à l'instant t, $l > 0$ est associé à la chaleur latente, κ et ν sont deux fonctions positives dépendant de φ et θ qui représentent respectivement la conductivité thermique et la viscosité du fluide u, g est une force extérieure et $D(u)$ est le tenseur des taux de déformation symétrique du champ de vitesse du fluide.

Nous ajoutons à ces deux équations l'équation de Navier-Stokes qui est essentielle pour modéliser le mouvement du fluide dans les régions non-solides. Nous considérons le cas du fluide incompressible

$$\begin{aligned} &u_t + (u\cdot\nabla)u - \mathrm{div}\,(\nu(\varphi,\theta)D(u)) + \nabla p = f \\ &\mathrm{div}\ u = 0, \end{aligned}$$

où u représente le champ de vecteurs de vitesse du fluide, p représente la pression, f décrit les force appliquées au fluide et ν est la viscosité qui dépend de φ et θ.

Le système étudié résulte donc d'un couplage non linéaire de ces trois équations et se présente sous la forme

$$\varphi_t + u\cdot\nabla\varphi - \xi\Delta\varphi = -\varphi(1-\varphi)(1-2\varphi+c\theta) \tag{26}$$
$$\theta_t - \mathrm{div}\,(\kappa(\varphi,\theta)\nabla\theta) + \mathrm{div}\,(u\theta) = \nu(\varphi,\theta)D(u) : D(u) + g + l\varphi_t \tag{27}$$
$$u_t + (u\cdot\nabla)u - \mathrm{div}\,(\nu(\varphi,\theta)D(u)) + \nabla p = f \tag{28}$$
$$\mathrm{div}\ u = 0, \tag{29}$$

avec

1. $\xi > 0$ et $c > 0$
2. $l > 0$ $\kappa, \nu \in C^0(\mathbb{R}^2)$ avec $0 < \kappa_0 \leq \kappa \leq \kappa_1$ et $0 < \nu_0 \leq \nu \leq \nu_1$
3. $g \in L^1(Q_T)^2$ and $f \in L^2(Q_T)^2$.

Dans cette partie nous étudions l'existence de solution du système (26)-(29) en dimension 2 et 3.

Dans le cas de la dimension 2, des résultats d'existence de solutions sont établis pour des modèles de changement de phase dans [30], [29], [28]. Cependant, le terme de dissipation était considéré nul dans la plupart des travaux. Dans [11] et [18], pour des systèmes de type Boussinesq avec terme de dissipation non nul, les auteurs utilisent la notion des solutions renormalisées et montrent des résultats d'existence de solutions faibles renormalisées. Dans le cadre des modèles de solidification non isotherme avec terme de dissipation non nul, Fernández-Cara et Vaz ont montré dans [63] et [102] l'existence d'une solution faible-renormalisée. Nous nous inspirons de ces travaux pour établir notre résultat d'existence en dimension 2. D'autres types de modèles issus de la thermo-mécanique des fluides, comme les modèles NSKTE qui couplent l'équation de Navier-Stokes et une équation cinétique de la turbulence (voir par exemple [40, 78]) donnent des systèmes d'EDP avec des difficultés similaires.

Dans le cas plus délicat de la dimension 3, l'analyse a été faite pour des modèles de changement de phase sous l'effet d'un champ magnétique dans [58] et dans [31] pour des modèles de changement de phase d'un alliage avec convection. Dans [80], [32] et [33] les auteurs étudient l'existence en dimension 3 pour des systèmes de type Boussinesq ou Navier-Stokes Fourrier. L'approche adoptée dans ces trois travaux consiste à établir une transformation dans le système étudié afin d'obtenir les résultats d'existence souhaités. Nous utiliserons la méthode de [32] pour étudier notre système.

Chapitre 3 : Cas de la Dimension 2

Ce chapitre de la thèse est dédié à l'analyse mathématique du système (26)-(29) décrit ci-dessus en dimension 2. Pour cette étude, nous considérons les conditions aux bords et les conditions initiales suivante :

$$\nabla\varphi \cdot \boldsymbol{n} = 0, \quad \kappa(\varphi,\theta)\nabla\theta \cdot \boldsymbol{n} = 0, \quad u = 0 \qquad \text{sur } \Sigma, \tag{30}$$

$$\varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0(x), \quad u(x,0) = u_0(x) \qquad \text{dans } \Omega. \tag{31}$$

Les données u_0, θ_0 et φ_0 sont respectivement dans $L^2(\Omega)^2$, $L^1(\Omega)$ et $L^\infty(\Omega)$ avec $u_0 \cdot \boldsymbol{n} = 0$ sur $\partial\Omega$ et $\operatorname{div} u_0 = 0$ dans Ω. De plus, nous supposons que $0 \leq \varphi_0 \leq 1$.

Dans le cas de la dimension 2, nous avons l'existence et l'unicité de la solution faible $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ des équations de Navier-Stokes. La principale difficulté résulte, comme expliqué précédemment, de l'énergie de dissipation $|Du|^2$ qui est alors un élément de $L^1(Q_T)$ ainsi que de la présence de θ dans le second membre de l'équation de la phase et sans oublier les termes de convection dans chacune des équations du système. Pour palier cette faible régularité, nous utilisons dans ce chapitre la notion de solution renormalisée pour l'équation de conservation de l'énergie. L'existence et l'unicité de la solution renormalisée pour ce type d'équation, avec les conditions aux bords de Neumann, a été démontrés dans le chapitre 2 de cette thèse.

Des résultats sur l'existence d'une solution faible-renormalisée pour un système non linéaire de type Boussinesq ont été présentés dans [11] et [18]. Une étude mathématique pour un modèle de solidification similaire au nôtre a été réalisée par C-Vaz et E-Fernández-Cara dans [63] et un résultat d'existence d'une solution faible–normalisée a été établi. Cependant, dans [63], les auteurs ont supposé que l était négligeable et ont imposé des conditions de Dirichlet la température.

La nouveauté de ce chapitre est que nous considérons les conditions aux bords de Neumann pour l'équation de la conservation d'énergie et nous prouvons l'existence de solutions faibles renornalisées sans aucune conditions de petitesses sur l. Pour cela, nous utilisons le couplage d'équations et la stabilité des solutions renormalisées montrée dans le chapitre 2.

Le résultat principal du chapitre est énoncé ci-dessous.

Théorème. Soit Ω un ouvert bornée de $\mathbb{R}^N$, avec $N = 2$. On suppose que les hypothèses (1)-(3) sont satisfaites. Alors, il existe au moins une solution faible-renormalisée (φ, θ, u) définie sur $\Omega \times (0,T)$ du système (26)-(29) telle que

$$u \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}), \quad u_t \in L^2(0,T;\mathcal{V}); \tag{32}$$

$$\varphi \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \quad \varphi_t \in L^2(0,T;H^1(\Omega)'); \tag{33}$$

$$0 \leq \varphi \leq 1; \tag{34}$$

$$\varphi(t=0) = \varphi_0, \quad u(t=0) = u_0; \tag{35}$$

$$\theta \in L^\infty(0,T;L^1(\Omega)),\ T_k(\theta) \in L^2(0,T;H^1(\Omega)),\ \forall k > 0; \tag{36}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{(x,t)\in Q_T:\, n<|\theta(x,t)|<2n\}} \kappa(\varphi,\theta)|\nabla\theta|^2 \,\mathrm{d}x\,\mathrm{d}t = 0; \tag{37}$$

et

$$\begin{aligned} \int_0^T \langle u_t, v\rangle_{\mathcal{V}',\mathcal{V}}\,\mathrm{d}t + \int_{Q_T} \nu(\varphi,\theta)\nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} (u\cdot\nabla)u\cdot v\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{Q_T} f v\,\mathrm{d}x\,\mathrm{d}t \qquad \forall v \in L^2(0,T;\mathcal{V}), \end{aligned} \tag{38}$$

$$\begin{aligned}\int_0^T \langle \varphi_t, \psi\rangle \,\mathrm{d}t + \int_{Q_T} \xi \nabla\varphi \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} (u\cdot\nabla\varphi)\cdot\psi\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{Q_T} -\varphi(1-\varphi)(1-2\varphi+c\theta)\psi\,\mathrm{d}x\,\mathrm{d}t \qquad \forall\psi\in L^2(0,T;H^1(\Omega));\end{aligned} \tag{39}$$

et pour tout $\beta \in W^{2,\infty}(\mathbb{R})$ avec β' à support compact et pour tout $\eta \in C_c^\infty([0,T)\times\Omega)$, on a

$$\begin{aligned}-\int_{Q_T} \eta_t\beta(\theta)\,\mathrm{d}x\mathrm{d}t + \int_{Q_T}\kappa(\varphi,\theta)\nabla\beta(\theta)\cdot\nabla\eta\,\mathrm{d}x\mathrm{d}t + \int_{Q_T}\kappa(\varphi,\theta)\nabla\theta\cdot\nabla\beta'(\theta)\eta\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{Q_T} u\cdot\nabla\theta\beta'(\theta)\eta\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}\beta'(\theta)\nu(\varphi,\theta)D(u):D(u)\eta\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\beta'(\theta)\eta\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T \langle l\varphi_t, \beta'(\theta)\eta\rangle\,\mathrm{d}t + \int_\Omega \eta(0)\beta(\theta_0)\,\mathrm{d}x.\end{aligned} \tag{40}$$

Dans cette définition les espaces $\mathcal{H}$ et $\mathcal{V}$ sont les espaces standards (voir [101]) pour résoudre les équations de Navier-Sokes :

$$\mathcal{H} = \{v\in L^2(\Omega)^N : \operatorname{div} v = 0 \text{ dans } \Omega,\ v\cdot\boldsymbol{n} = 0 \text{ sur } \partial\Omega\},$$
$$\mathcal{V} = \{v\in H_0^1(\Omega)^N : \operatorname{div} v = 0 \text{ dans } \Omega\}.$$

Pour démontrer ce théorème d'existence on considère tout d'abord un problème approché, pour lequel le théorème de point fixe de Schauder permet d'obtenir une solution. La principale difficulté est alors d'obtenir des estimations a priori : en prenant les 3 équations indépendamment les unes des autres il semble nécessaire de supposer que l est suffisamment petit. Pour ne pas avoir la contrainte de petitesse sur l nous utilisons le couplage des deux équations (26) et (27) et un choix approprié de fonctions tests. L'estimation ainsi obtenue est cruciale d'une part pour démontrer des estimations a priori sur la phase et par la suite de passer à la limite dans le système approché.

Chapitre 4 : Cas de la Dimension 3

Le chapitre 4 est consacré à l'étude du système (26)-(29) en dimension 3. Étant donné que la viscosité du fluide et la conductivité thermique dépendent de la température et de la phase, un changement sur les conditions aux bords de la vitesse du fluide est nécessaire afin d'obtenir des estimations sur la pression en dimension 3. En adoptant l'approche développée dans [32], nous imposons les conditions aux bords suivantes :

$$\nabla\varphi\cdot\boldsymbol{n} = 0, \quad \kappa(\varphi,\theta)\nabla\theta\cdot\boldsymbol{n} = 0, \qquad \text{sur } \Sigma, \tag{41}$$
$$u\cdot\boldsymbol{n} = 0, \qquad \alpha u_\tau + (\nu(\theta,\varphi)D(u)\boldsymbol{n})_\tau = 0 \qquad \text{sur } \Sigma, \tag{42}$$
$$\varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0(x), \quad u(x,0) = u_0(x) \qquad \text{dans } \Omega. \tag{43}$$

Les données u_0, θ_0 et φ_0 sont respectivement dans $L^2(\Omega)^2$, $L^1(\Omega)$ et $L^\infty(\Omega)$ avec $u_0\cdot\boldsymbol{n} = 0$ sur $\partial\Omega$ et $\operatorname{div} u_0 = 0$ dans Ω. De plus, nous supposons que $0 \le \varphi_0 \le 1$.

Le cas de la dimension 3 est beaucoup plus problématique que celui de la dimension 2. En effet, la fonction u n'est pas assez régulière pour être prise comme fonction test en présence du terme dans L^1, et la convergence des énergies n'est par conséquent plus garantie. Ainsi, il est plus approprié de transformer le système (26)-(29) en un système formellement équivalent qui garantit que tous les termes passeront à la limite. Comme dans [80] pour les équations de Rayleigh-Benard on écrit une équation pour u et une équation pour l'énergie totale $\theta + \frac{|u|^2}{2}$. Formellement cela consiste à multiplier l'équation de Navier-Stokes par u et à l'ajouter à l'équation en θ, le terme $\nu(\varphi,\theta)D(u) : D(u)$ « disparaît » remplacé par plusieurs autres termes.

Nous étudions donc l'existence de solutions du système suivant

$$\varphi_t + \mathrm{div}\,(u\varphi) - \xi\Delta\varphi = F(\varphi,\theta) \qquad \text{dans } Q_T, \tag{44}$$

$$(\theta + \frac{|u|^2}{2})_{,t} - \mathrm{div}\,(\kappa(\varphi,\theta)\nabla\theta) + \mathrm{div}\,(u(\frac{|u|^2}{2} + \theta + p)) = \mathrm{div}(\nu(\varphi,\theta)D(u)u) + g + l\varphi_t + f\cdot u \qquad \text{dans } Q_T, \tag{45}$$

$$\theta_t - \mathrm{div}\,(\kappa(\varphi,\theta)\nabla\theta) + \mathrm{div}(\theta u) \geq \nu(\varphi,\theta)D(u) : D(u) + g + l\varphi_t \qquad \text{dans } Q_T, \tag{46}$$

$$u_t + (u\cdot\nabla)u - \mathrm{div}\,(\nu(\varphi,\theta)D(u)) + \nabla p = f, \quad \mathrm{div}\ u = 0 \qquad \text{dans } Q_T, \tag{47}$$

$$\nabla\varphi\cdot\boldsymbol{n} = 0, \quad \kappa(\varphi,\theta)\nabla\theta\cdot\boldsymbol{n} = 0, \qquad \text{sur } \Sigma \tag{48}$$

$$u\cdot\boldsymbol{n} = 0, \quad \alpha u_\tau + (\nu(\theta,\varphi)D(u)\boldsymbol{n})_\tau = 0 \qquad \text{sur } \Sigma, \tag{49}$$

$$\varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0(x), \quad u(x,0) = u_0(x) \qquad \text{dans } \Omega. \tag{50}$$

Dans le cas où la viscosité du fluide et la conductivité thermique sont constantes, P.L.Lions, dans [80], a proposé deux approches pour l'étude du système du type Navier-Stokes-Fourier afin d'établir l'existence de solutions. Cependant, nous ne pouvons pas adapter cette approche dans notre cas où la viscosité du fluide et la conductivité thermique dépendent de la température. Feireisl et Málek, dans [61], ont étudié l'équation de Navier-Stokes avec des coefficients qui dépendent de la température. Cette approche a été développée par la suite dans [32] pour le système de Navier-Stokes-Fourier. En combinant cette technique avec le couplage entre les trois équations, nous parvenons à montrer l'existence de solutions dites « suitable weak-solution » ([35]) du système (44)-(50) sans aucune condition sur l.

Le résultat principal du chapitre est énoncé ci-dessous

Théorème. Soit $\Omega \in C^{1,1}$ un ouvert de $\mathbb{R}^3$. On suppose que les hypothèses (1)-(3) sont satisfaites. Alors, il existe une solution faible (φ,θ,u,p) du système (26)-(29) telle que

$$u \in C(0,T;L^2_{weak}(\Omega)^3) \cap L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}), \quad u_t \in L^{\frac{5}{3}}(0,T;W_{\boldsymbol{n}}^{-1,\frac{5}{3}}); \tag{51}$$

$$p \in L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}}(\Omega)); \tag{52}$$

$$\varphi \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)), \quad \varphi_t \in L^2(0,T;W^{-1,2}(\Omega)); \tag{53}$$

$$0 \leq \varphi \leq 1; \tag{54}$$

$$\theta \in L^\infty(0,T;L^1(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega)),\ \forall\, 1 \leq p < 5/4; \tag{55}$$

$$\theta \in L^1(0,T;L^r(\Omega)),\ \forall 1 \leq r < \frac{2^*}{2}; \tag{56}$$

$$E(t,\phi) \in C([0,T]), \quad \lim_{t\to 0+} E(t,\phi) = E_0(\phi) \quad \forall\phi \in C(\bar{\Omega}); \tag{57}$$

$$\lim_{t\to 0+} \|u(t) - u_0\|_{L^2(\Omega)} = 0,\ \varphi(t=0) = \varphi_0; \tag{58}$$

et

$$\begin{aligned}
&\int_0^T \langle u_t, v\rangle \,\mathrm{d}t + \int_{Q_T} \nu(\varphi,\theta) D(u)\cdot D(v)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega} \alpha u v\,\mathrm{d}\omega\,\mathrm{d}t \\
&\quad + \int_{Q_T} (u\cdot\nabla)u\cdot v\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} p \operatorname{div} v\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} f v\,\mathrm{d}x\,\mathrm{d}t \qquad \forall v\in L^\infty(0,T; W^{1,\infty}_{\boldsymbol{n}});
\end{aligned} \tag{59}$$

$$\begin{aligned}
&\int_0^T \langle \varphi_t, \psi\rangle \,\mathrm{d}t + \int_{Q_T} \xi\nabla\varphi\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u\varphi\nabla\psi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{Q_T} -\varphi(1-\varphi)(1-2\varphi+c\theta)\psi\,\mathrm{d}x\,\mathrm{d}t \qquad \forall\psi\in L^2(0,T;W^{1,2}(\Omega));
\end{aligned} \tag{60}$$

$$\begin{aligned}
&-\int_{Q_T} (\frac{|u|^2}{2}+\theta)\phi_t\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u(\frac{|u|^2}{2}+\theta+p)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \kappa(\varphi,\theta)\nabla\theta\cdot\nabla\phi\,\mathrm{d}t\,\mathrm{d}x \\
&\qquad + \int_0^T\int_{\partial\Omega} \alpha|u|^2\phi\,\mathrm{d}x\,\mathrm{d}\omega - \int_{Q_T} \nu(\varphi,\theta)D(u)u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q_T} g\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \langle l\varphi_t, \phi\rangle \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} f u\phi\,\mathrm{d}x\,\mathrm{d}t + E_0(\phi) \qquad \forall\phi\in C_c^\infty([0,T)\times\bar\Omega).
\end{aligned} \tag{61}$$

De plus,

$$\begin{aligned}
&-\int_{Q_T} \theta\phi_t\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u\theta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \kappa(\varphi,\theta)\nabla\theta\cdot\nabla\phi\,\mathrm{d}t\,\mathrm{d}x - \int_{Q_T} \nu(\varphi,\theta)|D(u)|^2\phi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \geq \int_{Q_T} g\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \langle l\varphi_t, \phi\rangle \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega} \theta_0\phi_0\,\mathrm{d}x\,\mathrm{d}t \qquad \forall\phi\in C_c^\infty([0,T)\times\bar\Omega), \phi\geq 0,
\end{aligned} \tag{62}$$

où la fonction $E(t,\cdot)$ est définie par

$$E(t,\phi) := \int_\Omega (\theta(t,x) + \frac{|u(t,x)|^2}{2})\phi(x)\,\mathrm{d}x \quad \forall\phi\in C(\bar\Omega)$$

et $E_0(\phi)$ par

$$E_0(\phi) := \int_\Omega (\theta_0 + \frac{|u_0|^2}{2})\phi\,\mathrm{d}x \quad \forall\phi\in C(\bar\Omega).$$

Nous utilisons ici l'approche développée dans [34]. Une méthode de Galerkin permet tout d'abord de considérer un problème approché mais pour un fluide quasi-compressible. Plus précisément la condition $\varepsilon\Delta p = -\operatorname{div} u$, avec ε un paramètre qui tend vers 0, remplace la condition d'incompressibilité du système original. Le terme $\nu(\varphi,\theta)D(u) : D(u)$ est tronqué et le terme $(u\cdot\nabla)u$ est régularisé dans ce système approché. Cela permet, à ce stade d'approximation, de pouvoir passer du système original à celui « formellement » équivalent. En mélangeant les techniques du cas bi-dimensionnel et la stratégie de [34] nous pouvons alors établir une série d'estimations a priori. Le fait que l'ouvert Ω soit régulier est primordial pour contrôler la pression dans des espaces convenables. Ces estimations a priori sont suffisantes pour appliquer les lemmes de compacité de type Aubin-Lions et construire une solution limite du système.

Part I

Problems with L^1 data and Neumann boundary conditions

Chapter 1

Finite volume scheme and renormalized solutions for nonlinear elliptic Neumann problem with L^1 data

1.1 Introduction

In the present chapter, we are interested in the discretization by the cell-centered finite volume method of the following convection-diffusion equation with Neumann boundary conditions and L^1 data:

$$\begin{aligned} -\operatorname{div}(\lambda(u)\nabla u - \boldsymbol{v}u) = f \quad &\text{in } \Omega, \\ (\lambda(u)\nabla u - \boldsymbol{v}u) \cdot \vec{n} = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here Ω is a bounded polygonal connected open subset of $\mathbb{R}^d$, $d \geq 2$, $\vec{n}$ is the outer unit normal to $\partial\Omega$ and λ is a continuous function such that $\lambda_\infty \geq \lambda(u) \geq \mu > 0$ with λ_∞ and μ two real numbers. The function $\boldsymbol{v}$ lies in $L^p(\Omega)^d$ with $2 < p < +\infty$ if $d = 2$, $p = d$ if $d \geq 3$, and f belongs to $L^1(\Omega)$ and satisfies the compatibility condition $\int_\Omega f = 0$.

Considering elliptic equations with L^1 data requires a precise meaning of the solution. Indeed we cannot expect in general to obtain a usual weak solution that belongs to $H_0^1(\Omega)$ for Dirichlet boundary conditions or to $H^1(\Omega)$ for Neumann boundary conditions. Elliptic equations with L^1 data and Dirichlet boundary conditions are widely studied in the literature. In [26] Boccardo and Gallouët have obtained the existence of a solution in the sense of distributions for a fairly class of monotone operator with measure data. However, it is known that this solution is not unique in general (see the counter example of Serrin [98]). To overcome the lack of uniqueness results, it is possible to use in the linear case the duality method (see [100]) or, for general nonlinear operators, the notion of entropy solution (see [46]), the notion of solution obtained as limit of approximation (SOLA) (see [48]) or the notion of renormalized solution (see [84, 46]). The previous three notions of solution are equivalent in the L^1 case and provide existence, stability, and uniqueness results for a large class of elliptic equations. As far as the approximation of elliptic equations with Dirichlet boundary conditions and L^1 data is concerned, the method of finite volume (see [60]) allows us to consider such equations. In [69] the authors have studied equation (1.1) with $\boldsymbol{v} = 0$ and with a measure data (and Dirichlet boundary conditions). In [56] the authors have considered a linear noncoercive equation (similar to (1.1)) with measure data (and Dirichlet boundary conditions). In both papers [69, 56] the authors have established the convergence of the finite volume approximation to a solution in the sense of distributions. More precisely for the equation $-\Delta u + \operatorname{div}(\boldsymbol{v}u) = f$ in Ω with Dirichlet boundary conditions, the limit u of the finite volume scheme verifies

$$\begin{cases} u \in \displaystyle\bigcap_{q<d/(d-1)} W_0^{1,q}(\Omega) \\ \displaystyle\int_\Omega \nabla u \nabla \varphi \,\mathrm{d}x - \int_\Omega u\boldsymbol{v}\nabla\varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x, \quad \forall \varphi \in \bigcup_{s>d} W_0^{1,s}(\Omega). \end{cases}$$

Recently mixing the techniques of renormalized solution and the finite volume approximation has been performed in [77] for a noncoercive equation with L^1 data and Dirichlet boundary conditions: the author proves that the limit of the finite volume scheme is the renormalized solution of the equation. Concerning the finite elements approximation the model case of the equation $-\operatorname{div}(A\nabla u) = f$ with Dirichlet boundary conditions is dealt in [39].

In the present chapter, we have to face a noncoercive equation, to an L^1 data and to Neumann boundary conditions. To our knowledge such a situation is less studied in the literature both in the continuous case and the discrete case. One of the difficulties in the variational and linear case is that the kernel is nontrivial and that we have to impose an additional condition on the solution to ensure uniqueness result, which is in general $\int_\Omega u dx = 0$. In [57] by using the Fredholm theory the authors have studied the operator associated with the linear version of (1.1). They prove that the linear version of (1.1) with $(H^1)'$ data verifying a compatibility condition admits a unique weak solution. Moreover, they deduce existence and uniqueness results for elliptic and coercive equations of the type $-\operatorname{div}(A(x,u)\nabla u)) = \mu$ with Neumann boundary condition, where μ is a bounded Radon measure. The finite volume approximation of (1.1) with f belonging to $L^2(\Omega)$ (with zero mean value) is studied in [41]. As in the continuous case, the finite volume approximation requires the study of the kernel and for different approximations of the convective terms the authors prove in [41] that the finite volume approximation converges to a weak solution of (1.1). For the class of nonlinear elliptic equations $-\Delta_p u = f$ with Neumann boundary conditions, L^1 data and for small value of p it is well known that the solution is not in general a summable function so that the mean value has no meaning. To overcome this obstacle, in [2, 15] the authors have chosen the median value which is well-defined instead of the mean value. In [15] an appropriate definition of renormalized solutions is given, which gives an existence result (see also [17] for the uniqueness question). The main originality of the present paper is to consider noncoercive equation (1.1) with L^1 data and to mix the techniques developed in [15] and the finite volume method. We choose here the median value instead of the mean value as in [41]. Since Poincaré-Wirtinger inequality is crucial in general we state in Proposition 1.2.8 an appropriate discrete Poincaré-Wirtinger inequality involving the median value (see Appendix for the proof, in the spirit of [14]). In Theorem 1.3.2 we prove that the finite volume approximation of (1.1) converges to the renormalized solution with a null median.

The Chapter is organized as follows. In Section 2 we recall some definitions, in particular the median of a measurable function. Moreover, we present in Section 2 the continuous case and the notion of renormalized solution of (1.1) and, at last, the finite volume tools and the scheme. The main results are stated in Section 3. Section 4 is devoted to derive the a priori estimates for the solutions of the scheme. Using Section 4 we prove the existence of a solution to the scheme in Section 5 while the convergence analysis is performed in Section 6. Finally we give in Appendix the proof of the discrete Poincaré-Wirtinger inequality involving the median (instead of the mean value).

1.2 Assumptions and definitions

Let Ω be a connected open bounded polygonal subset of $\mathbb{R}^d$, $d \geq 2$. We consider the following nonlinear elliptic problem with Neumann boundary conditions:

$$\begin{cases} -\operatorname{div}(\lambda(u)\nabla u - \boldsymbol{v}u) = f & \text{in } \Omega, \\ (\lambda(u)\nabla u - \boldsymbol{v}u)\cdot\vec{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\vec{n}$ is the outer unit normal to $\partial\Omega$. We assume that

$$\boldsymbol{v} \in L^p(\Omega)^d \text{ with } 2 < p < +\infty \text{ if } d = 2, p = d \text{ if } d \geq 3, \tag{1.3}$$

$$\lambda \text{ is a continuous function such that } \lambda_\infty \geq \lambda(r) \geq \mu > 0, \forall r \in \mathbb{R}, \tag{1.4}$$

with λ_∞ and μ two real numbers. Moreover, we assume that

$$f \in L^1(\Omega), \tag{1.5}$$

and it satisfies the compatibility condition

$$\int_\Omega f \,\mathrm{d}x = 0. \tag{1.6}$$

As explained in the Introduction we deal with solutions whose median is equal to zero. Let us recall that if u is a measurable function, we define the median of u (with respect to the Lebesgue measure), denoted by $\operatorname{med}(u)$ as the set of real numbers t such that

$$\begin{aligned} \operatorname{meas}\{x \in \Omega : u(x) > t\} &\leq \frac{\operatorname{meas}(\Omega)}{2} \\ \operatorname{meas}\{x \in \Omega : u(x) < t\} &\leq \frac{\operatorname{meas}(\Omega)}{2}. \end{aligned}$$

It is known that $\operatorname{med}(u)$ is non-empty compact interval (see [106]). Let us explicitly observe that if $0 \in \operatorname{med}(u)$ then

$$\begin{aligned} \operatorname{meas}\{x \in \Omega : u(x) > 0\} &\leq \frac{\operatorname{meas}(\Omega)}{2} \\ \operatorname{meas}\{x \in \Omega : u(x) < 0\} &\leq \frac{\operatorname{meas}(\Omega)}{2}. \end{aligned}$$

We denote $\underline{\operatorname{med}}(u)$ by

$$\underline{\operatorname{med}}(u) = \inf\left\{t \in \mathbb{R} : \operatorname{meas}\{x \in \Omega : u(x) > t\} \leq \frac{\operatorname{meas}(\Omega)}{2}\right\}, \tag{1.7}$$

and $\overline{\operatorname{med}}(u)$ by

$$\overline{\operatorname{med}}(u) = \sup\left\{t \in \mathbb{R} : \operatorname{meas}\{x \in \Omega : u(x) > t\} \geq \frac{\operatorname{meas}(\Omega)}{2}\right\}. \tag{1.8}$$

We observe that if u is an element of $H^1(\Omega)$ (Ω being a connected domain), the median of u is uniquely determined; $\operatorname{med}(u) = \underline{\operatorname{med}} u = \overline{\operatorname{med}}(u)$. However, it is not the case for the finite volume approximation of (1.19) which is a piecewise-constant function; the median is then the compact interval of $\mathbb{R}$ $[\underline{\operatorname{med}}(u), \overline{\operatorname{med}}(u)]$.

In the whole paper, T_n, $n \geq 0$, denotes the truncation at height n that is

$$T_n(s) = \min(n, \max(s, -n)), \quad \forall s \in \mathbb{R}.$$

1.2.1 Continuous Case

In this subsection we precise the notion of solution of equation (1.2). Indeed as explained in the Introduction, considering elliptic equations with L^1 data requires an appropriate notion of solution which provides existence, stability, and uniqueness results. There is a wide literature on the Dirichlet case. In the Neumann case, due to the lack of regularity of the solution, the mean value may not exist for nonlinear problems with L^1 data, which gives additional difficulties in deriving estimates and in defining an appropriate notion of a renormalized solution. We refer mainly to [53] for linear problems using the duality method and to [93], [2] and [15] for nonlinear problems. In [2] and [15] the authors have chosen the median instead of the mean value (which may not exist if the solution is not integrable) and one of the main tools is the following Poincaré-Wirtinger inequality, see [106].

Proposition 1.2.1. *If $u \in W^{1,p}(\Omega)$, then*

$$\|u - \mathrm{med}(u)\|_{L^p(\Omega)} \leq C\|\nabla u\|_{(L^p(\Omega))^d} \tag{1.9}$$

where C is a constant depending on p, d, Ω.

In [15] the authors prove the existence of a renormalized solution for a class of nonlinear problems and prove in [17] uniqueness results under additional assumptions. In the present paper, we use the framework of renormalized solutions. In the particular case of equation (1.2), let us recall the following definition (see [15]).

Definition 1.2.2. *A real function u defined in Ω is a renormalized solution to (1.2) if*

$$u \text{ is measurable and finite almost everywhere in } \Omega, \tag{1.10}$$

$$T_n(u) \in H^1(\Omega), \text{ for any } n > 0, \tag{1.11}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{x\in\Omega, |u(x)|\leq n\}} \lambda(u)|\nabla u|^2 \,\mathrm{d}x = 0, \tag{1.12}$$

and the following equation holds

$$\begin{aligned} &\int_\Omega S(u)\lambda(u)\nabla u \cdot \nabla\varphi \,\mathrm{d}x + \int_\Omega S'(u)\lambda(u)\varphi\nabla u \cdot \nabla u \,\mathrm{d}x \\ &- \int_\Omega uS(u)\boldsymbol{v} \cdot \nabla\varphi \,\mathrm{d}x - \int_\Omega uS'(u)\varphi\boldsymbol{v} \cdot \nabla u \,\mathrm{d}x = \int_\Omega f\varphi S(u) \,\mathrm{d}x, \end{aligned} \tag{1.13}$$

for every $S \in W^{1,\infty}(\mathbb{R})$ having compact support and for every $\varphi \in L^\infty(\Omega) \cap H^1(\Omega)$.

By combining [15] and [17] we have the following existence and uniqueness result.

Theorem 1.2.3. *Let us assume that (1.3)–(1.6) hold true. Then there exists a unique renormalized solution u of (1.2) such that $\mathrm{med}(u) = 0$.*

Remark 1.2.4. *As far as the uniqueness is concerned equation (1.2) is not directly in the scope of [17]. Indeed uniqueness results are mainly obtained for equations whose prototype is $-\mathrm{div}(a(x, \nabla u) + \Phi(x, u)) = f$ with Neumann boundary conditions. The operator $a(x, \nabla u)$ does not depend on u. Due to the presence of $\lambda(u)$ in equation (1.2) the quasilinear character allows one to obtain the uniqueness by a changement of unknown. Since $\lambda(r)$ is a continuous function* *such that $\lambda_\infty \geq \lambda(r) \geq \mu > 0$, by defining $\widetilde{\lambda}(r) = \int_0^r \lambda(s)\mathrm{d}s$ and $w = \widetilde{\lambda}(u)$, we can verify that the function w has a null median and that w is a renormalized solution of*

$$\begin{cases} -\operatorname{div}(\nabla w - \boldsymbol{v}\widetilde{\lambda}^{-1}(w)) = f & \text{in } \Omega, \\ (\nabla w - \boldsymbol{v}\widetilde{\lambda}^{-1}(w)) \cdot \vec{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.14}$$

At last since the function $\widetilde{\lambda}^{-1}$ is Lipschitz continuous, Theorem 4.2 of [17] allows one to conclude that w is unique so that u is unique.

1.2.2 Finite Volume

We now introduce the discrete settings. Let us first recall the notion of admissible discretization of Ω, the definitions of the discrete norms, and the space of piecewise functions associated with an admissible mesh following [60].

Definition 1.2.5 (Admissible mesh)**.** *An admissible mesh $\mathcal{M}$ of Ω is given by a finite family $\mathcal{T}$ of disjoint open convex polygonal subsets of Ω, a finite family $\mathcal{E}$ of disjoint subsets of $\bar{\Omega}$ (the edges) consisting in non-empty open convex subsets of affine hyperplanes and a family $\mathcal{P} = (x_K)_{K\in\mathcal{T}}$ of points in Ω such that*

- $\bar{\Omega} = \cup_{K\in\mathcal{T}}\bar{K}$,
- *each $\sigma \in \mathcal{E}$ is a non-empty open subset of ∂K for some $K \in \mathcal{T}$,*
- *by denoting $\mathcal{E}_K = \{\sigma \in \mathcal{E}, \sigma \subset \partial K\}$, $\partial K = \cup_{\sigma\in\mathcal{E}_K}\sigma$ for all $K \in \mathcal{T}$,*
- *for all $K \neq L$ in $\mathcal{T}$, either the $(d-1)-$dimentional measure of $\bar{K}\cap\bar{L}$ is zero or $\bar{K}\cap\bar{L} = \bar{\sigma}$ for some $\sigma \in \mathcal{E}$, which is then denoted $\sigma = K|L$,*
- *for all $K \in \mathcal{T}$, $x_K \in K$,*
- *for all $\sigma = K|L \in \mathcal{E}$, the straight line (x_K, x_L) intersects and is orthogonal to σ,*
- *for all $\sigma \in \mathcal{E}$ such that $\sigma \subset \partial\Omega \cap \partial K$, the line which is orthogonal to σ and goes through x_K intersects σ.*

In the whole of the present paper, we use the following notations associated with an admissible discretization. In the set of edges $\mathcal{E}$, we distinguish the set of interior edges $\mathcal{E}_{int}$ and the set of boundary edges $\mathcal{E}_{ext}$. We denote by $\mathrm{m}(K)$ the d-dimensional measure of a control volume K and $\mathrm{m}(\sigma)$ the $(d-1)$-dimensional measure of σ. For all $\sigma \in \mathcal{E}_K$, $\mathbf{n}_{K,\sigma}$ is the unit normal to σ outwards K. If $\sigma = K|L \in \mathcal{E}_{int}$, we denote by d_σ the Euclidian distance between x_K and x_L, $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$ and $d_\sigma = d_{K,\sigma}$ if $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K$.
The size of the mesh is defined by

$$h_{\mathcal{M}} = \sup_{K\in\mathcal{T}}\mathrm{diam}(K).$$

We assume that the mesh satisfies the following assumption

$$\exists\ \xi > 0 \text{ such that } d(x_K, \sigma) \geq \xi d_\sigma, \quad \forall\mathcal{T} \in \mathcal{E}, \forall\sigma \in \mathcal{E}_K. \tag{1.15}$$

An example of admissible mesh in the sense of the above definition is shown in Figure 1.1.

The space of piecewise functions associated to an admissible mesh, denoted by $X(\mathcal{M})$, is defined as the set of functions from Ω to $\mathbb{R}$ which are constant over each control volume of the mesh.

Definition 1.2.6 (Discrete $W^{1,p}$ norm)**.** *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d \geq 2$, and let $\mathcal{M}$ be an admissible mesh. For $u = (u_K)_{K\in\mathcal{T}} \in X(\mathcal{T})$ and $p \in [1, +\infty[$, the discrete $W^{1,p}$-semi-norm is defined by*

$$|u|_{1,p,\mathcal{M}} = \left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} \frac{\mathrm{m}(\sigma)}{d_\sigma^{p-1}} |u_K - u_L|^p \right)^{\frac{1}{p}}, \qquad \forall u \in X(\mathcal{T})$$

and the discrete $W^{1,p}$-norm is defined by

$$\|u\|_{1,p,\mathcal{M}} = \|u\|_{0,p} + |u|_{1,p,\mathcal{M}}, \qquad \forall u \in X(\mathcal{T})$$

where $\|u\|_{0,p}$ is the L^p norm for piecewise constant functions, $\forall p \in [1, +\infty[$,

$$\|u\|_{0,p} = \left(\int_\Omega |u(x)|^p \mathrm{d}x \right)^{\frac{1}{p}} = \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)|u_K|^p \right)^{\frac{1}{p}}, \ \forall u \in X(\mathcal{T}).$$

We present now discrete functional analysis results. We refer the reader to [[41], Lemma 6.1] for a proof of the following discrete Sobolev inequality.

Proposition 1.2.7 (Discrete Sobolev inequality)**.** *Let Ω be a bounded polygonal open subset of $\mathbb{R}^d$ and let $\mathcal{M}$ be an admissible mesh satisfying* (1.15)*. Let $q < +\infty$ if $d = 2$ and $q = \frac{2d}{d-2}$ if $d \geq 3$. Then there exists $C = C(\Omega, \xi, q)$ such that, for all $u = (u_K)_{K\in\mathcal{T}} \in X(\mathcal{T})$,*

$$\|u\|_{0,q} \leq C \left(|u|_{1,2,\mathcal{M}} + \|u\|_{0,2} \right). \tag{1.16}$$

In the already cited references discrete Poincaré and Poincaré-Wirtinger inequalities are related to the discrete space $W_0^{1,p}(\Omega)$ and zero boundary condition or the discrete space $W^{1,p}(\Omega)$ with discrete mean value. We derive here a discrete Poincaré-Wirtinger inequality involving the median. The proof is given in the appendix.

Proposition 1.2.8 (Discrete Poincaré-Wirtinger median inequality)**.** *Let Ω be an open bounded connected polyhedral domain of $\mathbb{R}^d$ and let $\mathcal{M}$ be an admissible mesh satisfying* (1.15)*. Then for $1 \leq p < +\infty$ there exists a constant $C > 0$ only depending on Ω, d and p such that*

$$\|u - c\|_{0,p} \leq \frac{C}{\xi^{(p-1)/p}} |u|_{1,p,\mathcal{M}}, \qquad \forall u \in X(\mathcal{T}) \tag{1.17}$$

where c belongs to $\mathrm{med}(u)$*.*

Theorem 1.2.9 (Discrete Rellich's theorem)**.** *Let $(\mathcal{M}_m)_{m\geq 1}$ be a sequence of admissible meshes satisfying* (1.15) *and such that $h_{\mathcal{M}_m} \to 0$ as $m \to \infty$. If $v_m \in X(\mathcal{T}_m)$ is such that $(|v_m|_{1,2,\mathcal{M}} + \|v_m\|_{0,2})$ is bounded, then $(v_m)_{m\in\mathbb{N}}$ is relatively compact in $L^2(\Omega)$. Furthermore, any limit in $L^2(\Omega)$ of a subsequence of $(v_m)_{m\in\mathbb{N}}$ belongs to $H^1(\Omega)$.*

Let us now define a discrete finite volume gradient introduced equivalently in [[42], Lemma 4.4], [[54], Lemma 6.5] or [[59], Definition 2].

Definition 1.2.10 (Discrete finite volume gradient)**.** *For $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}(K)$, we define the volume $D_{K,\sigma}$ as the cone of basis σ and of opposite vertex x_K. Then, we define the "diamond-cell" D_σ (see Figure 1.1) by*

$$\begin{aligned} D_\sigma &= D_{K,\sigma} \cup D_{L,\sigma} && \text{if } \sigma = K|L \in \mathcal{E}_{int}, \\ D_\sigma &= D_{K,\sigma} && \text{if } \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K, \end{aligned}$$

and

$$\mathrm{m}(D_\sigma) = \frac{1}{d} d_\sigma \mathrm{m}(\sigma).$$

The approximate gradient $\nabla_{\mathcal{M}} u$ of a function $u \in X(\mathcal{T})$ is defined as a piece-wise constant function over each diamond cell and given by

$$\begin{aligned} &\forall \sigma \in \mathcal{E}_{int},\ \sigma = K|L, && \nabla_{\mathcal{M}} u(\boldsymbol{x}) = d \frac{u_L - u_K}{d_\sigma} \mathbf{n}_{K,\sigma}, && \forall \boldsymbol{x} \in D_\sigma, \\ &\forall \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K, && \nabla_{\mathcal{M}} u(\boldsymbol{x}) = 0, && \forall \boldsymbol{x} \in D_\sigma. \end{aligned}$$

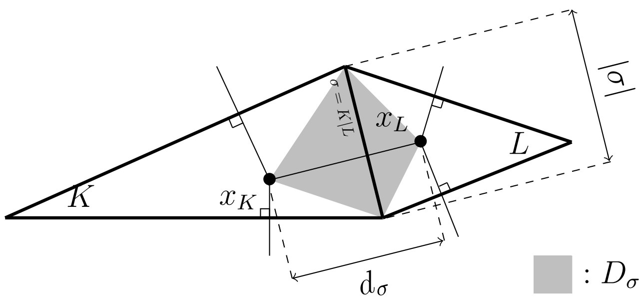

Figure 1.1 – The diamond D_σ

Let us then give the convergence property of the discrete gradient (see e.g., in the case of Dirichlet boundary condition, [42] and [59] in L^2 context, and [76] in the L^1 context).

Lemma 1.2.11 (Weak convergence of the finite volume gradient)**.** *Let $(\mathcal{M}_m)_{m\geq 1}$ be a sequence of admissible meshes satisfying* (1.15) *and such that $h_{\mathcal{M}_m} \to 0$ as $m \to \infty$. Let $v_m \in X(\mathcal{T}_m)$ and let us assume that there exists $\alpha \in [1, +\infty[$ and $C > 0$ such that $\|v_m\|_{1,\alpha,\mathcal{M}_m} \leq C$, and that v_m converges in $L^1(\Omega)$ to $v \in W^{1,\alpha}(\Omega)$. Then $\nabla_{\mathcal{M}_m} v_m$ converges to ∇v weakly in $L^\alpha(\Omega)^d$.*

We now define the finite volume scheme. Let $\mathcal{M}$ be an admissible mesh in the sense of definition 1.2.5. For $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, we define $\boldsymbol{v}_{K,\sigma}$ by

$$\boldsymbol{v}_{K,\sigma} = \frac{1}{\mathrm{m}(D_\sigma)} \int_{D_\sigma} \boldsymbol{v} \cdot \mathbf{n}_{K,\sigma} \, \mathrm{d}x. \tag{1.18}$$

We consider the following finite volume scheme for (1.1)

$$\forall K \in \mathcal{T}, \sum_{\sigma \in \mathcal{E}_{int}(K)} \frac{\mathrm{m}(\sigma)}{d_\sigma} \lambda(u)_\sigma (u_K - u_L) + \sum_{\sigma \in \mathcal{E}_{int}(K)} \mathrm{m}(\sigma) \boldsymbol{v}_{K,\sigma} u_{\sigma,+} = \int_K f \, \mathrm{d}x, \tag{1.19}$$

and

$$\forall \sigma = K|L \in \mathcal{E}_{int},\ u_{\sigma,+} = \begin{cases} u_K & \text{if } \boldsymbol{v}_{K,\sigma} \geq 0, \\ u_L & \text{otherwise.} \end{cases} \tag{1.20}$$

We denote $u_{\sigma,-}$ the downstream choice of u, i.e. $u_{\sigma,-}$ is such that $\{u_{\sigma,+}, u_{\sigma,-}\} = \{u_K, u_L\}$, $\forall \sigma \in \mathcal{E}_{int}$.

Finally,

$$\forall \sigma = K|L \in \mathcal{E}_{int},\ \min[\lambda(u_K), \lambda(u_L)] \leq \lambda(u)_\sigma \leq \max[\lambda(u_K), \lambda(u_L)], \tag{1.21}$$

where $\lambda(u)_\sigma$ is for example the mean value of $\lambda(u_K)$ and $\lambda(u_L)$ if $\sigma \in \mathcal{E}_{int}$.

1.3 Main results

Our main results on the finite volume scheme are the following. The first one states that there exists at least one solution to the scheme. It is a generalization of Theorem 2.5 in [41] in the context of a quasilinear problem with a median value constraint instead of a mean value constraint. The second one gives the convergence of this solution to the unique renormalized solution of the continuous problem with a null median, as the size of the mesh tends to 0.

Theorem 1.3.1 (Existence of the solution of the scheme)**.** *Let us assume that* (1.3)–(1.6) *hold. Let $\mathcal{M}$ be an admissible mesh in the sense of Definition 1.2.5 satisfying* (1.15)*. Then there exists a solution $u_{\mathcal{T}} = (u_K)_{K\in\mathcal{M}}$ to* (1.19)–(1.21) *having* $\underline{\mathrm{med}}(u_{\mathcal{T}}) = 0$.

Theorem 1.3.2 (Convergence of the solution of the scheme)**.** *Let $(\mathcal{M}_m)_{m\geq 1}$ be a sequence of admissible meshes in the sense of Definition 1.2.5, which satisfy* (1.15) *and such that $h_{\mathcal{M}_m}$ goes to 0 as $m \to \infty$. Let $u_m = (u_K^m)_{K\in\mathcal{T}_m} \in X(\mathcal{T}_m)$ be a solution of* (1.19) *such that* $\underline{\mathrm{med}}(u_m) = 0$. *Then u_m converges to the unique renormalized solution u of* (1.1) *having* $\mathrm{med}(u) = 0$, *in the sense that*

$$u_m \textit{ converges to } u \textit{ a.e. in } \Omega,$$
$$\forall n \in \mathbb{N},\ \nabla_{\mathcal{M}_m} T_n(u_m) \textit{ converges to } \nabla T_n(u), \textit{ weakly in } (L^2(\Omega))^d,$$

as $m \to \infty$.

Remark 1.3.3. *As explained in the Introduction we choose in the present paper a constraint on the median value instead of the mean value. It allows one to mix the techniques developed in [15] and the finite volume. Observe that the median is an appropriate choice in [2, 15] to deal with nonlinear elliptic equations with L^1 data and Neumann boundary conditions since we cannot expect to have a solution u (in the sense of distribution or the renormalized sense) of $-\Delta_p u = f$ with p closed to 1 such that the solution belongs to $L^1(\Omega)$. However under the restriction $p > 2 - 1/N$ and using the Boccardo-Gallouët estimates it is possible to solve $-\Delta_p u = f$ with f in L^1 and Neumann boundary conditions in the sense of distributions with a mean value equal to zero, see [93]. As far as equation* (1.1) *is concerned a natural question is to solve it and to approximate it with $\int_\Omega u dx = 0$ and not* $\mathrm{med}(u) = 0$. *To our knowledge, the continuous case is not dealt with in the literature. Starting from an approximate problem the difficulties are similar in passing to the limit in the continuous case and the discrete case: the crucial steps are the a priori estimates stated in Section 4. Since we cannot give all the details of a possible proof we refer to [10].*

1.4 A priori estimates

This section is devoted to derive *a priori* estimates of the solution of the scheme (1.19), which are crucial to extract subsequences using compactness results and then to pass to the limit in the scheme. Let us observe that we adapt the strategy developed in [15] for the continuous problem (1.2) with Neumann boundary conditions to the discrete case and that we use the techniques developed in [41] for the approximation of the solution to problem (1.2) with a more regular data and Neumann boundary conditions and the ones of [56] and [77] which study the approximation of equations with L^1 (or measure data) with Dirichlet boundary conditions.

Proposition 1.4.1 (Estimate on $\ln(1+|u_{\mathcal{M}}|)$ with $\underline{\text{med}}(u_{\mathcal{M}})=0$)**.** *Let $\mathcal{M}$ be an admissible mesh satisfying* (1.15)*. If $u_{\mathcal{M}}=(u_K)_{K\in\mathcal{T}}$ is a solution to (1.19), such that $\underline{\text{med}}(u_{\mathcal{M}})=0$, then*

$$\|\ln(1+|u_{\mathcal{M}}|)\|^2_{1,2,\mathcal{M}} \leq C\left(2\|f\|_{L^1(\Omega)} + d|\Omega|^{\frac{p-2}{p}}\,\|\boldsymbol{v}\|^2_{(L^p(\Omega))^d}\right), \tag{1.22}$$

where $C=C(\Omega,\mu,\beta,p,\xi)$ is a positive constant.

Proof. A *log*-estimate was obtained in [Proposition 3.1, [56]] in the case of Dirichlet boundary conditions and $\boldsymbol{v}\in(C(\bar{\Omega}))^d$. Since we deal with Neumann boundary conditions, $\boldsymbol{v}\in(L^p(\Omega))^d$ and the specific choice of $\underline{\text{med}}(u_{\mathcal{M}})=0$, we will adapt the proof derived in [56] and explain the modifications. As in [56], let $\varphi(s)=\displaystyle\int_0^s \frac{dt}{(1+|t|)^2}$. Taking $\varphi(u_K)$ as a test function in the scheme (1.19) and reordering the sums yield

$$\sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u)_\sigma(u_K-u_L)(\varphi(u_K)-\varphi(u_L)) \leq \|f\|_{L^1(\Omega)} + \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)|\boldsymbol{v}_{K,\sigma}|u_{\sigma,+}(\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})). \tag{1.23}$$

To control the second term of the right-hand side of (1.23) we introduce the set of edges $\mathcal{A}$ (see [56]) by

$$\mathcal{A}=\{\sigma\in\mathcal{E}_{int}\,;\; u_{\sigma,+}\geq u_{\sigma,-},\; u_{\sigma,+}<0\}\cup\{\sigma\in\mathcal{E}_{int}\,;\; u_{\sigma,+}<u_{\sigma,-},\; u_{\sigma,+}\geq 0\}, \tag{1.24}$$

Since φ is non-decreasing, as in [56] we obtain

$$\sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)|\boldsymbol{v}_{K,\sigma}|u_{\sigma,+}(\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})) \leq \sum_{\sigma\in\mathcal{A}} \mathrm{m}(\sigma)|\boldsymbol{v}_{K,\sigma}|u_{\sigma,+}(\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})). \tag{1.25}$$

Now using Cauchy-Schwarz and Hölder inequalities, and the following inequality (see Lemma 3.1, [56]), $\forall\sigma\in\mathcal{A}$, $|u_{\sigma,+}|^2|\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})|^2\leq|u_{\sigma,-}-u_{\sigma,+}||\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})|$, we obtain

$$\begin{aligned}
\sum_{\sigma\in\mathcal{A}} \mathrm{m}(\sigma)|\boldsymbol{v}_{K,\sigma}||u_{\sigma,+}|(\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})) &\leq \left(\sum_{\sigma\in\mathcal{A}} \mathrm{m}(\sigma)d_\sigma|\boldsymbol{v}_{K,\sigma}|^2\right)^{\frac{1}{2}} \\
&\quad\times\left(\sum_{\sigma\in\mathcal{A}} \frac{\mathrm{m}(\sigma)}{d_\sigma}|u_{\sigma,+}|^2|\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{\sigma\in\mathcal{A}} \mathrm{m}(\sigma)d_\sigma\right)^{\frac{p-2}{p}} \left(\sum_{\sigma\in\mathcal{A}} \mathrm{m}(\sigma)d_\sigma|\boldsymbol{v}_{K,\sigma}|^p\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left(\sum_{\sigma\in\mathcal{A}}\frac{\mathrm{m}(\sigma)}{d_\sigma}|u_{\sigma,+}|^2|\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})|^2\right)^{\frac{1}{2}}$$
$$\leq\left(\sum_{\sigma\in\mathcal{A}}\mathrm{m}(\sigma)d_\sigma\right)^{\frac{p-2}{p}}\left(\sum_{\sigma\in\mathcal{A}}\mathrm{m}(\sigma)d_\sigma|\boldsymbol{v}_{K,\sigma}|^p\right)^{\frac{1}{p}}$$
$$\times\left(\sum_{\sigma\in\mathcal{A}}\frac{\mathrm{m}(\sigma)}{d_\sigma}|u_k-u_L||\varphi(u_K)-\varphi(u_L)|\right)^{\frac{1}{2}}. \quad (1.26)$$

Recalling that $\sum_{\sigma\in\mathcal{A}}\mathrm{m}(\sigma)d_\sigma\leq\sum_{\sigma\in\mathcal{E}_{int}}\mathrm{m}(\sigma)d_\sigma=d\mathrm{m}(\Omega)$ and since the term $\left(\sum_{\sigma\in\mathcal{E}_{int}}\mathrm{m}(\sigma)d_\sigma|\boldsymbol{v}_{K,\sigma}|^p\right)^{\frac{1}{p}}$ is bounded by $d^{\frac{1}{p}}\|\boldsymbol{v}\|_{(L^p(\Omega))^d}$, by Young's inequality we get

$$\sum_{\sigma\in\mathcal{A}}\mathrm{m}(\sigma)|\boldsymbol{v}_{K,\sigma}||u_{\sigma,+}|(\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+}))\leq\frac{1}{2\beta}d\mathrm{m}(\Omega)^{\frac{p-2}{p}}\|\boldsymbol{v}\|^2_{(L^p(\Omega))^d}$$
$$+\frac{\beta}{2}\mathrm{m}(\Omega)^{\frac{p-2}{p}}\sum_{\sigma\in\mathcal{E}_{int}}\frac{\mathrm{m}(\sigma)}{d_\sigma}(u_K-u_L)(\varphi(u_K)-\varphi(u_L)), \quad (1.27)$$

where $\beta>0$. Since $0<\mu\leq\lambda(u)$, an appropriate choice of β gives

$$\sum_{\sigma\in\mathcal{E}_{int}}\frac{\mathrm{m}(\sigma)}{d_\sigma}(u_K-u_L)(\varphi(u_K)-\varphi(u_L))\leq C(\Omega,\mu,\beta,p)\left(2\|f\|_{L^1(\Omega)}+d|\Omega|^{\frac{p-2}{p}}\|\boldsymbol{v}\|^2_{(L^p(\Omega))^d}\right). \quad (1.28)$$

Moreover we have, for all $(x,y)\in\mathbb{R}^2$, $(\ln(1+|x|)-\ln(1+|y|))^2\leq(x-y)(\varphi(x)-\varphi(y))$. It follows that

$$\sum_{\sigma\in\mathcal{E}_{int}}\frac{\mathrm{m}(\sigma)}{d_\sigma}\left(\ln(1+|u_K|)-\ln(1+|u_L|)\right)^2\leq C(\Omega,\mu,\beta,p)\left(2\|f\|_{L^1(\Omega)}+d|\Omega|^{\frac{p-2}{p}}\|\boldsymbol{v}\|^2_{(L^p(\Omega))^d}\right). \quad (1.29)$$

Since $\underline{\mathrm{med}}(\ln(1+u_{\mathcal{M}}))=0$, the discrete Poincaré-Wirtinger inequality (1.17) implies that

$$\|\ln(1+|u_{\mathcal{M}}|)\|^2_{1,2,\mathcal{M}}\leq C\left(2\|f\|_{L^1(\Omega)}+d|\Omega|^{\frac{p-2}{p}}\parallel\boldsymbol{v}\parallel^2_{(L^p(\Omega))^d}\right).$$

□

Let us state a corollary which is a consequence of Proposition 1.4.1 and is necessary for the proof of the estimate of Proposition 1.4.3 and for Proposition 1.4.4. It may be found in [56] and is recalled here with its proof, for the sake of completeness.

Corollary 1.4.2. *Let $\mathcal{M}$ be an admissible mesh satisfying* (1.15). *If $u_{\mathcal{M}}=(u_K)_{K\in\mathcal{T}}\in X(\mathcal{T})$ is a solution to (1.19) and, for $n>0$, $E_n=\{|u_{\mathcal{M}}|>n\}$, then there exists $C>0$ only depending on $(\Omega,\boldsymbol{v},f,d,p,\xi)$ such that*

$$\mathrm{meas}(E_n)\leq\frac{C}{(\ln(1+n))^2}. \quad (1.30)$$

Proof. On the one hand, using Proposition 1.4.1 we have

$$\|\ln(1+|u_{\mathcal{M}}|)\|^2_{1,2,\mathcal{M}}\leq C\left(2\|f\|_{L^1(\Omega)}+d|\Omega|^{\frac{p-2}{p}}\parallel\boldsymbol{v}\parallel^2_{(L^p(\Omega))^d}\right). \quad (1.31)$$

On the other hand, since $\underline{\mathrm{med}}(\ln(1+|u_{\mathcal{M}}|)) = 0$, by the discrete Poincaré-Wirtinger median inequality (1.17), we have that there exists $C > 0$ only depending on (Ω, d, ξ) such that

$$\|\ln(1+|u_{\mathcal{M}}|)\|_{0,2} \leq C|u_{\mathcal{M}}|_{1,2,\mathcal{M}}. \tag{1.32}$$

Therefore, using (1.31) and (1.32), there exists $C > 0$ only depending on $(\Omega, \boldsymbol{v}, f, d, p, \xi)$ such that

$$\|\ln(1+|u_{\mathcal{M}}|)\|_{0,2}^2 \leq C. \tag{1.33}$$

Finally, due to the fact that $\mathrm{meas}(E_n) = \mathrm{meas}\left(\{\ln(1+|u_{\mathcal{M}}|) \geq \ln(1+n)\}\right)$, the Chebyshev inequality and (1.33) lead to the result. □

Proposition 1.4.3 (Estimate on $T_n(u_{\mathcal{M}})$)**.** *Let $\mathcal{M}$ be an admissible mesh satisfying* (1.15)*. If $u_{\mathcal{M}} = (u_K)_{K\in\mathcal{T}} \in X(\mathcal{T})$ is a solution to (1.19) having $\underline{\mathrm{med}}(u_{\mathcal{M}}) = 0$, then for any $n \geq 0$, there exists $C > 0$ only depending on $(\Omega, \boldsymbol{v}, f, n, d, \xi)$ such that*

$$\|T_n(u_{\mathcal{M}})\|_{1,2,\mathcal{M}} \leq C. \tag{1.34}$$

Let $(\mathcal{M}_m)_{m\geq 1}$ be a sequence of admissible meshes satisfying (1.15) *and such that $h_{\mathcal{M}_m}$ goes to zero as $m \to \infty$ and let $u_m = (u_K^m)_{K\in\mathcal{T}_m} \in X(\mathcal{T}_m)$ be a solution to (1.19) having $\underline{\mathrm{med}}(u_m) = 0$. Then there exists a measurable function u finite a.e. in Ω such that, up to a subsequence (still indexed by m),*

$$T_n(u) \in H^1(\Omega), \text{ for any } n > 0, \tag{1.35}$$

$$\mathrm{med}(u) = 0, \tag{1.36}$$

$$T_n(u_m) \to T_n(u) \text{ strongly in } L^2(\Omega) \text{ and a.e }, \tag{1.37}$$

$$\nabla_{\mathcal{M}_m} T_n(u_m) \rightharpoonup \nabla T_n(u) \text{ in } (L^2(\Omega))^d, \ \forall n > 0. \tag{1.38}$$

Proof. The proof is divided into 2 steps. First, we prove that $T_n(u_{\mathcal{M}})$ satisfies the a priori estimate (1.34). In the second step, considering a sequence of admissible meshes $\mathcal{M}_m$, we prove that the solution u_m to the scheme (1.19) converges to a function u as m goes to infinity and that (1.35)–(1.38) hold true.

Step 1. *Estimate on $T_n(u_{\mathcal{M}})$.*

After multiplying each equation of the scheme by $T_n(u_K)$, summing over each control volume and reordering the sums, we obtain $T_1 + T_2 = T_3$ with

$$T_1 = \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u)_\sigma(u_K - u_L)(T_n(u_K) - T_n(u_L)),$$
$$T_2 = \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma}u_{\sigma,+}(T_n(u_K) - T_n(u_L)),$$
$$T_3 = \sum_{K\in\mathcal{T}} \int_K f\, T_n(u_K)\,\mathrm{d}x.$$

Since T_n is bounded by n, we obtain that $|T_3| \leq n\|f\|_{L^1(\Omega)}$. Then

$$T_1 \leq n\|f\|_{L^1(\Omega)} - T_2. \tag{1.39}$$

Let $\sigma \in \mathcal{E}$. By the definition (1.20) of $u_{\sigma,+}$ and recalling that $u_{\sigma,-}$ is the downstream choice of u, if $\boldsymbol{v}_{K,\sigma} \geq 0$ it gives

$$\boldsymbol{v}_{K,\sigma}(T_n(u_K) - T_n(u_L)) = \boldsymbol{v}_{K,\sigma}(T_n(u_{\sigma,+}) - T_n(u_{\sigma,-})),$$

and if $\boldsymbol{v}_{K,\sigma} < 0$ it gives

$$\boldsymbol{v}_{K,\sigma}(T_n(u_K) - T_n(u_L)) = -\boldsymbol{v}_{K,\sigma}(T_n(u_{\sigma,+}) - T_n(u_{\sigma,-})).$$

In consequence, T_2 can be written as

$$-T_2 = \frac{1}{n}\sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)\,|\boldsymbol{v}_{K,\sigma}|\,u_{\sigma,+}(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})). \tag{1.40}$$

As in the proof of estimate (1.22), we use the set of edges $\mathcal{A} = \{\sigma \in \mathcal{E}_{int}\,;\ u_{\sigma,+} \geq u_{\sigma,-},\ u_{\sigma,+} < 0\} \cup \{\sigma \in \mathcal{E}_{int}\,;\ u_{\sigma,+} < u_{\sigma,-},\ u_{\sigma,+} \geq 0\}$. Since T_n is non decreasing we have

$$-T_2 \leq \frac{1}{n}\sum_{\sigma\in\mathcal{A}} \mathrm{m}(\sigma)\,|\boldsymbol{v}_{K,\sigma}|\,u_{\sigma,+}(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})). \tag{1.41}$$

Due to the fact that $\forall\sigma \in \mathcal{A}$, if $|u_{\sigma,+}| \geq n$ then $|u_{\sigma,-}| \geq n$ we deduce that

$$u_{\sigma,+}(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})) = T_n(u_{\sigma,+})(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})),\ \forall\sigma \in \mathcal{A}.$$

It follows that

$$-T_2 \leq \sum_{\sigma\in\mathcal{A}} \mathrm{m}(\sigma)\,|\boldsymbol{v}_{K,\sigma}|\,T_n(u_{\sigma,+})\ \left(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})\right).$$

Moreover, using Cauchy-Schwarz and Young inequalities, taking into account that $\left(\sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)d_\sigma|\boldsymbol{v}_{K,\sigma}|^2\right)^{\frac{1}{2}}$ is bounded by $d^{\frac{1}{2}}\|\boldsymbol{v}\|_{(L^2(\Omega))^d}$, we obtain

$$\begin{aligned}
-T_2 &\leq \sum_{\sigma\in\mathcal{A}} \mathrm{m}(\sigma)\,|\boldsymbol{v}_{K,\sigma}|\,T_n(u_{\sigma,+})\ (T_n(u_{\sigma,-}) - T_n(u_{\sigma,+}))\\
&\leq \left(\sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)d_\sigma|\boldsymbol{v}_{K,\sigma}|^2\right)^{\frac{1}{2}} \left(\sum_{\sigma\in\mathcal{A}} \frac{\mathrm{m}(\sigma)}{d_\sigma} T_n(u_{\sigma,+})^2\,(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+}))^2\right)^{\frac{1}{2}}\\
&\leq n\,d^{\frac{1}{2}}\|\boldsymbol{v}\|_{L^2(\Omega)^d}\left(\sum_{\sigma\in\mathcal{A}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\,(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+}))^2\right)^{\frac{1}{2}}\\
&\leq \frac{1}{2\beta}n^2\,d\|\boldsymbol{v}\|^2_{L^2(\Omega)^d} + \frac{\beta}{2}\sum_{\sigma\in\mathcal{A}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+}))^2\\
&\leq \frac{1}{2\beta}n^2\,d\|\boldsymbol{v}\|^2_{L^2(\Omega)^d} + \frac{\beta}{2}\sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(T_n(u_K) - T_n(u_L))^2,
\end{aligned}$$

where $\beta > 0$. Since $T_n(u_K) - T_n(u_L) \leq u_K - u_L$, we have

$$-T_2 \leq \frac{1}{2\beta}n^2\,d\|\boldsymbol{v}\|^2_{(L^2(\Omega))^d} + \frac{\beta}{2}\sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u_K - u_L)\left(T_n(u_K) - T_n(u_L)\right),$$

and we can deduce that

$$\begin{aligned}
\sum_{\sigma\in\mathcal{E}_{int}} \lambda(u)_\sigma\frac{\mathrm{m}(\sigma)}{d_\sigma}(u_K - u_L)\left(T_n(u_K) - T_n(u_L)\right) &\leq n\|f\|_{L^1(\Omega)} + \frac{1}{2\beta}n^2\,d\|\boldsymbol{v}\|^2_{(L^2(\Omega))^d}\\
&\quad + \frac{\beta}{2}\sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u_K - u_L)\left(T_n(u_K) - T_n(u_L)\right).
\end{aligned}$$

Recalling that $\underline{\mathrm{med}}(T_n(u_{\mathcal{M}})) = 0$ and $0 \le \mu \le \lambda(u)$, an appropriate choice of β and the Poincaré-Wirtinger inequality (1.17) lead to the result.

Step 2. *In this step we consider sequence of admissible meshes $(\mathcal{M}_m)_{m\ge1}$ satisfying* (1.15) *and such that $h_{\mathcal{M}_m}$ goes to zero as $m \to \infty$. If $u_m = (u_K^m)_{K\in\mathcal{T}_m} \in X(\mathcal{T}_m)$ denotes a solution to (1.19) having $\underline{\mathrm{med}}(u_m) = 0$, we show that there exists a measurable function u finite a.e. in Ω such that* (1.35)–(1.38) *hold true.*

The method is widely used for elliptic equations with L^1 (or measure data) (see e.g. [46]) and consists in proving that, up to subsequence, u_m is a Cauchy sequence in measure. For the convenience of the reader we give the complete arguments. For any n, in view of Step 1 we know that the sequence $\left(\|T_n(u_m)\|_{1,2,\mathcal{M}_m}\right)_{m\ge1}$ is bounded (uniformly with respect to m). By Theorem 1.2.9 and a diagonal process (n being a natural number), up to a subsequence still indexed by m, we deduce that, for any $n \in \mathbb{N}$, there exists v_n belonging to $H^1(\Omega)$ such that

$$T_n(u_m) \to v_n, \text{ a.e. in } \Omega, \text{ as } m \to \infty. \tag{1.42}$$

We now prove that u_m is a Cauchy sequence in measure. Let $\omega > 0$. For all $n > 0$, and all $m, p \ge 0$, we have

$$\{|u_m - u_p| > \omega\} \subset \{|u_m| > n\} \cup \{|u_p| > n\} \cup \{|T_n(u_m) - T_n(u_p)| > \omega\}.$$

Let $\varepsilon > 0$ fixed. By Corollary 1.4.2, let $n > 0$ such that, for all $m, p \ge 0$,

$$\mathrm{meas}(\{|u_m| > n\}) + \mathrm{meas}(\{|u_p| > n\}) < \frac{\varepsilon}{2}.$$

Once n is chosen, since $T_n(u_m)$ converges almost everywhere to v_n as m goes to infinity we obtain

$$\exists m_0 > 0\,;\ \forall m, p \ge m_0 \quad \mathrm{meas}(\{|T_n(u_m) - T_n(u_p)| > \omega\}) \le \frac{\varepsilon}{2}.$$

Therefore, we deduce that $\forall m, p \ge m_0$

$$\mathrm{meas}\{|u_m - u_p| > \omega\} < \varepsilon.$$

Hence $(u_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in measure. Consequently, up to a subsequence still indexed by m, there exists a measurable function u such that

$$u_m \to u \text{ a.e. in } \Omega. \tag{1.43}$$

It follows from Corollary 1.4.2 that u is finite a.e. in Ω.

Moreover by the pointwise convergence (1.42) of $T_n(u_m)$ for any $n \in \mathbb{N}$ we deduce that $T_n(u) = v_n \in H^1(\Omega)$. Applying Theorem 1.2.11 we obtain that

$$\nabla_{\mathcal{M}_m} T_n(u_m) \rightharpoonup \nabla T_n(u) \text{ in } \left(L^2(\Omega)\right)^d, \text{ as } m \to \infty.$$

It remains to prove that $\mathrm{med}(u) = 0$. Due to the point-wise convergence of u_m to u, the sequence $\mathbb{1}_{\{u_m>0\}}\mathbb{1}_{\{u>0\}}$ converges to $\mathbb{1}_{\{u>0\}}$ a.e. as $h_{\mathcal{M}}$ goes to zero. Recalling that $\underline{\mathrm{med}}(u_m) = 0$ Fatou's lemma leads to

$$\begin{aligned}
\mathrm{meas}\{u(x) > 0\} &\le \liminf \int_\Omega \mathbb{1}_{\{u_m>0\}}\mathbb{1}_{\{u>0\}}\mathrm{d}x \\
&\le \liminf \mathrm{meas}\{u_m(x) > 0\} \\
&\le \frac{\mathrm{meas}(\Omega)}{2}.
\end{aligned}$$

Analogously from the convergence of $\mathbb{1}_{\{u_m<0\}}\mathbb{1}_{\{u<0\}}$ to $\mathbb{1}_{\{u<0\}}$ a.e. as $m\to\infty$

$$\text{meas}\{u(x)<0\}\le\frac{\text{meas}(\Omega)}{2}.$$

It follows that $0\in\text{med}(u)$. Since we have for n large enough $\text{med}(T_n(u))=\text{med}(u)$ and since $T_n(u)$ belongs to $H^1(\Omega)$, the median of u is unique and it is equal to 0. □

Let us recall that in the renormalized framework the decay of the energy (1.12) plays an important role to derive stability or uniqueness results. In the following proposition, we show a discrete version of the decay of the energy (uniformly with respect to the sequence of the admissible meshes). Having (1.44) ans (1.45) is crucial to pass to the limit in the scheme.

Proposition 1.4.4 (Discrete estimate on the energy)**.** *Let $(\mathcal{M}_m)_{m\ge1}$ be a sequence of admissible meshes satisfying* (1.15) *and such that $h_{\mathcal{M}_m}\to0$ as $m\to\infty$. For any $m\ge0$, let us consider $u_m=(u_K^m)_{K\in\mathcal{T}_m}\in X(\mathcal{T}_m)$ a solution to (1.19) and let u be a measurable function finite a.e. in Ω such that, up to a subsequence still indexed by m, the second part of Proposition 1.4.3 holds. Then we have*

$$\lim_{n\to+\infty}\overline{\lim_{h_{\mathcal{M}_m}\to0}}\frac{1}{n}\sum_{\sigma\in\mathcal{E}_{int}}\frac{\text{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma(u_K^m-u_L^m)(T_n(u_K^m)-T_n(u_L^m))=0, \tag{1.44}$$

where $u_L=0$ if $\sigma\in\mathcal{E}_{ext}$, and

$$\lim_{n\to+\infty}\overline{\lim_{h_{\mathcal{M}_m}\to0}}\frac{1}{n}\sum_{\sigma\in\mathcal{E}_{int}}\text{m}(\sigma)\,|\boldsymbol{v}_{K,\sigma}|\,|u_{\sigma,+}^m|\,|T_n(u_{\sigma,+}^m)-T_n(u_{\sigma,-}^m)|=0. \tag{1.45}$$

Proof. Let $m\ge1$ and $u_m=\left(u_K^m\right)_{K\in\mathcal{T}_m}$ be a solution of (1.19). Multiplying each equation of the scheme by $\frac{T_n(u_K^m)}{n}$, summing over $K\in\mathcal{M}$ and gathering by edges we find

$$T_1+T_2=T_3$$

with

$$T_1=\frac{1}{n}\sum_{\sigma\in\mathcal{E}_{int}}\frac{\text{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma(u_K^m-u_L^m)(T_n(u_K^m)-T_n(u_L^m)), \tag{1.46}$$

$$T_2=\frac{1}{n}\sum_{\sigma\in\mathcal{E}_{int}}\text{m}(\sigma)\boldsymbol{v}_{K,\sigma}u_{\sigma,+}^m(T_n(u_K^m)-T_n(u_L^m)), \tag{1.47}$$

$$T_3=\frac{1}{n}\sum_{K\in\mathcal{M}}\int_K f\,T_n(u_K^m)\,\mathrm{d}x. \tag{1.48}$$

According to the definition of u_m, we have

$$T_3=\int_\Omega f\frac{T_n(u_m)}{n}\,\mathrm{d}x.$$

Due to (1.43), $T_n(u_m)$ converges to $T_n(u)$ as m goes to infinity in $L^\infty(\Omega)$ weak-$\star$ and a.e. Since f belongs to $L^1(\Omega)$ it follows that

$$\lim_{h_{\mathcal{M}_m}\to0}T_3=\int_\Omega f\frac{T_n(u)}{n}\,\mathrm{d}x.$$

Recalling that u is finite almost everywhere in Ω, $\frac{T_n(u)}{n}$ converges to 0 a.e. in Ω as n goes to infinity. Therefore, since f belongs to $L^1(\Omega)$ and $\left|\frac{T_n}{n}\right|$ is bounded by one, the Lebesgue dominated theorem allows one to conclude that

$$\lim_{n\to+\infty} \lim_{h_{\mathcal{M}_m}\to 0} T_3 = 0. \tag{1.49}$$

We now study the term T_2. We know from (1.40) that it can be written as

$$T_2 = \frac{1}{n} \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)\,|\boldsymbol{v}_{K,\sigma}|\, u^m_{\sigma,+}(T_n(u^m_{\sigma,+}) - T_n(u^m_{\sigma,-})). \tag{1.50}$$

Recalling the definition of the subset of edges $\mathcal{A}$

$$\mathcal{A} = \{\sigma\in\mathcal{E}_{int}\,;\ u^m_{\sigma,+} \geq u^m_{\sigma,-},\ u^m_{\sigma,+} < 0\} \cup \{\sigma\in\mathcal{E}_{int}\,;\ u^m_{\sigma,+} < u^m_{\sigma,-},\ u^m_{\sigma,+} \geq 0\}, \tag{1.51}$$

we denote by $\mathcal{B}$ the subset of edges such that

$$\mathcal{B} = \{\sigma\in\mathcal{E}_{int}\,;\ u^m_{\sigma,+} \geq u^m_{\sigma,-},\ u^m_{\sigma,+} \geq 0\} \cup \{\sigma\in\mathcal{E}_{int}\,;\ u^m_{\sigma,+} < u^m_{\sigma,-},\ u^m_{\sigma,+} < 0\}. \tag{1.52}$$

Then T_2 can be written as

$$\begin{aligned} T_2 =& \frac{1}{n} \sum_{\sigma\in\mathcal{A}} \mathrm{m}(\sigma)\,|\boldsymbol{v}_{K,\sigma}|\, u^m_{\sigma,+}(T_n(u^m_{\sigma,+}) - T_n(u^m_{\sigma,-})) \\ &+ \frac{1}{n} \sum_{\sigma\in\mathcal{B}} \mathrm{m}(\sigma)\,|\boldsymbol{v}_{K,\sigma}|\, u^m_{\sigma,+}(T_n(u^m_{\sigma,+}) - T_n(u^m_{\sigma,-})) \\ &= T_{2,1} + T_{2,2}. \end{aligned} \tag{1.53}$$

Using the fact that T_n is non decreasing and $T_n(0) = 0$, we notice that

$$|T_{2,1}| = -T_{2,1} = -\frac{1}{n} \sum_{\sigma\in\mathcal{A}} \mathrm{m}(\sigma)\,|\boldsymbol{v}_{K,\sigma}|\, u^m_{\sigma,+}(T_n(u^m_{\sigma,+}) - T_n(u^m_{\sigma,-})),$$
$$|T_{2,2}| = T_{2,2} = \frac{1}{n} \sum_{\sigma\in\mathcal{B}} \mathrm{m}(\sigma)\,|\boldsymbol{v}_{K,\sigma}|\, u^m_{\sigma,+}(T_n(u^m_{\sigma,+}) - T_n(u^m_{\sigma,-})),$$

and $|T_2| = -T_{2,1} + T_{2,2}$.
As far as $-T_{2,1}$ is concerned, we observe that for any $\sigma\in\mathcal{A}$, $|u^m_{\sigma,+}| \geq n$ implies $|u^m_{\sigma,-}| \geq n$. To deal with this term, as in [77], we split the sum on $\{|u^m_{\sigma,+}| \leq r\}$ and on $\{r \leq |u^m_{\sigma,+}| \leq n\}$ where r is a positive real number which will be chosen later. We have

$$-T_{2,1} = I_1 + I_2$$

with

$$I_1 = \frac{1}{n} \sum_{\substack{\sigma\in\mathcal{A} \\ |u^m_{\sigma,+}|\leq r}} \mathrm{m}(\sigma)\,|\boldsymbol{v}_{K,\sigma}|\, u^m_{\sigma,+}(T_n(u^m_{\sigma,-}) - T_n(u^m_{\sigma,+})), \tag{1.54}$$

$$I_2 = \frac{1}{n} \sum_{\substack{\sigma\in\mathcal{A} \\ r\leq|u^m_{\sigma,+}|\leq n}} \mathrm{m}(\sigma)\,|\boldsymbol{v}_{K,\sigma}|\, u^m_{\sigma,+}(T_n(u^m_{\sigma,-}) - T_n(u^m_{\sigma,+})). \tag{1.55}$$

Recalling that $\left(\sum\limits_{\sigma\in\mathcal{A}} \mathrm{m}(\sigma) d_\sigma |\boldsymbol{v}_{K,\sigma}|^2\right)^{\frac{1}{2}}$ is bounded by $d^{\frac{1}{2}}\|\boldsymbol{v}\|_{(L^2(\Omega))^d}$, the Cauchy-Schwarz inequality and the Young inequality yields that

$$
\begin{aligned}
|I_1| &= \frac{1}{n} \sum_{\substack{\sigma\in\mathcal{A}\\ |u^m_{\sigma,+}|\le r}} \mathrm{m}(\sigma)\, |\boldsymbol{v}_{K,\sigma}|\, u^m_{\sigma,+}(T_n(u^m_{\sigma,-}) - T_n(u^m_{\sigma,+}))\\
&\le \frac{1}{n}\left(\sum_{\substack{\sigma\in\mathcal{A}\\ |u^m_{\sigma,+}|\le r}} \mathrm{m}(\sigma) d_\sigma |\boldsymbol{v}_{K,\sigma}|^2\right)^{\frac{1}{2}} \left(\sum_{\substack{\sigma\in\mathcal{A}\\ |u^m_{\sigma,+}|\le r}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u^m_{\sigma,+})^2 (T_n(u^m_{\sigma,-}) - T_n(u^m_{\sigma,+}))^2\right)^{\frac{1}{2}}\\
&\le \frac{r}{n}\left(\sum_{\substack{\sigma\in\mathcal{A}\\ |u^m_{\sigma,+}|\le r}} \mathrm{m}(\sigma) d_\sigma |\boldsymbol{v}_{K,\sigma}|^2\right)^{\frac{1}{2}} \left(\sum_{\substack{\sigma\in\mathcal{A}\\ |u^m_{\sigma,+}|\le r}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u^m_K - u^m_L)(T_n(u^m_K) - T_n(u^m_L))\right)^{\frac{1}{2}}\\
&\le \frac{1}{n}\left[\frac{r^2 d \,\|\, \boldsymbol{v}\,\|^2_{(L^2(\Omega))^d}}{2\beta} + \frac{\beta}{2}\sum_{\substack{\sigma\in\mathcal{A}\\ |u^m_{\sigma,+}|\le r}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u^m_K - u^m_L)(T_n(u^m_K) - T_n(u^m_L))\right],
\end{aligned}
\tag{1.56}
$$

where $\beta > 0$ (to be chosen later). To control the second term I_2, we distinguish the case $d \ge 3$ and $d = 2$. We know that for any $\sigma \in \mathcal{A}$, $|u^m_{\sigma,+}| \ge r$ implies $|u^m_{\sigma,-}| \ge r$; if $d \ge 3$ the equality

$$
\frac{1}{d} + \frac{d-2}{2d} + \frac{1}{2} = 1,
$$

and the Hölder inequality gives

$$
\begin{aligned}
|I_2| &\le \frac{1}{n} \sum_{\substack{\sigma\in\mathcal{A}\\ r\le|u^m_{\sigma,+}|\le n\\ r\le|u^m_{\sigma,-}|}} \mathrm{m}(\sigma)\, |\boldsymbol{v}_{K,\sigma}|\, T_n(u^m_{\sigma,+})\, (T_n(u^m_{\sigma,-}) - T_n(u^m_{\sigma,+}))\\
&\le \frac{1}{n}\left(\sum_{\substack{\sigma\in\mathcal{A}\\ r\le|u^m_{\sigma,+}|\le n\\ r\le|u^m_{\sigma,-}|}} \mathrm{m}(\sigma)\, d_\sigma\, |\boldsymbol{v}_{K,\sigma}|^d\right)^{\frac{1}{d}} \left(\sum_{\substack{\sigma\in\mathcal{A}\\ r\le|u^m_{\sigma,+}|\le n\\ r\le|u^m_{\sigma,-}|}} \mathrm{m}(\sigma)\, d_\sigma |T_n(u^m_{\sigma,+})|^{\frac{2d}{d-2}}\right)^{\frac{d-2}{2d}}\\
&\quad\times \left(\sum_{\substack{\sigma\in\mathcal{A}\\ r\le|u^m_{\sigma,+}|\le n\\ r\le|u^m_{\sigma,-}|}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(T_n(u^m_{\sigma,-}) - T_n(u^m_{\sigma,+}))^2\right)^{\frac{1}{2}}\\
&\le \frac{1}{n}\left(\sum_{\substack{\sigma\in\mathcal{A}\\ r\le|u_{\sigma,+}|\le n\\ r\le|u^m_{\sigma,-}|}} \mathrm{m}(\sigma)\, d_\sigma\, |\boldsymbol{v}_{K,\sigma}|^d\right)^{\frac{1}{d}} \left(\sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)\, d_\sigma |T_n(u^m_{\sigma,+})|^{\frac{2d}{d-2}}\right)^{\frac{d-2}{2d}}\\
&\quad\times \left(\sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(T_n(u^m_{\sigma,-}) - T_n(u^m_{\sigma,+}))^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Recalling that $\underline{\text{med}}(T_n(u_m)) = 0$, the discrete Poincaré-Wirtinger inequality (1.17) and the discrete Sobolev inequality (1.16) lead to

$$|I_2| \leq C_1 \frac{d^{\frac{1}{2}} \|\boldsymbol{v}\|_{(L^d(E_r))^d}}{n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma} (T_n(u_K^m) - T_n(u_L^m))^2, \tag{1.57}$$

where E_r is the set where $|u_{\sigma,+}^m| \geq r$ and $C_1 > 0$ is a constant independent of n and $\mathcal{M}$. If $d = 2$, similar arguments lead to

$$|I_2| \leq C_2 \frac{d^{\frac{1}{2}} \|\boldsymbol{v}\|_{(L^p(E_r))^d}}{n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma} (T_n(u_K^m) - T_n(u_L^m))^2, \tag{1.58}$$

where $C_2 > 0$ is a constant independent of n and $\mathcal{M}$.
In view of Corollary 1.4.2 and since $\boldsymbol{v} \in (L^p(\Omega))^d$ ($2 < p < +\infty$ if $d = 2$, $p = d$ if $d \geq 3$), the absolute continuity of the integral implies that there exists $r > 0$ (independent of m) such that for all m

$$d^{\frac{1}{2}} \|\boldsymbol{v}\|_{(L^p(E_r))^d} \leq \frac{1}{2}. \tag{1.59}$$

Then from (1.57), (1.58) and (1.59) we obtain

$$|I_2| \leq \frac{C_3}{2n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma} (u_K^m - u_L^m)(T_n(u_K^m) - T_n(u_L^m)). \tag{1.60}$$

Recalling that $-T_2 \leq -T_{2,1}$, the inequalities (1.60) and (1.56) lead to

$$\begin{aligned} -T_2 &\leq |I_1 + I_2| \\ &\leq \frac{1}{n} \frac{r^2 d \parallel \boldsymbol{v} \parallel^2_{(L^2(\Omega))^d}}{2\beta} + \frac{C_4}{2n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma} (u_K^m - u_L^m)(T_n(u_K^m) - T_n(u_L^m)), \end{aligned} \tag{1.61}$$

where C_4 is a positive constant depending on β and C_3. Since $0 < \mu \leq \lambda(u_m)$, we choose $\beta > 0$ such that the second term of the right-hand side of (1.61) is $\leq \dfrac{T_1}{2}$. It follows that

$$\begin{aligned} -T_2 &\leq \frac{1}{n} \frac{r^2 d \parallel \boldsymbol{v} \parallel^2_{(L^2(\Omega))^d}}{2\beta} + \frac{T_1}{2} \\ &\leq R(n, h_{\mathcal{T}_m}) + \frac{T_1}{2}, \end{aligned} \tag{1.62}$$

with R verifying $\lim\limits_{n \to +\infty} \overline{\lim\limits_{h_{\mathcal{T}_m} \to 0}} R(n, h_{\mathcal{T}_m}) = 0$.
Since $T_1 + T_2 = T_3$, (1.62) allows one to conclude that

$$\lim_{n \to +\infty} \overline{\lim_{h_{\mathcal{M}_m} \to 0}} \frac{1}{n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma} \lambda(u_m)_\sigma (u_K^m - u_L^m)(T_n(u_K^m) - T_n(u_L^m)) = 0, \tag{1.63}$$

which gives (1.44). We are now in a position to prove (1.45). Recalling that $T_1 + T_2 = T_3$, we have $T_1 + T_{2,2} \leq |T_{2,1}| + T_3$. Since $T_{2,2}$ is non negative, using (1.49), (1.62) we get

$$T_1 \leq \frac{C_5}{2} T_1 + R + T_3 \text{ with}$$

$$\lim_{n\to+\infty}\overline{\lim_{h_{\mathcal{T}_m}\to 0}}\, R = 0,$$
$$\lim_{n\to+\infty}\overline{\lim_{h_{\mathcal{T}_m}\to 0}}\, T_3 = 0.$$

As a consequence we obtain

$$\lim_{n\to+\infty}\overline{\lim_{h_{\mathcal{T}_m}\to 0}}\, |T_{2,1}| = 0. \tag{1.64}$$

Moreover, writing again $T_1 + T_{2,2} \le |T_{2,1}| + T_3$, (1.64), (1.49) and (1.44) imply that

$$\lim_{n\to+\infty}\overline{\lim_{h_{\mathcal{T}_m}\to 0}}\, T_{2,2} = 0. \tag{1.65}$$

Therefore from (1.64) and (1.65) we deduce (1.45). □

1.5 Existence of a solution to the scheme

In this section, we prove that there exists at least one solution to the discrete scheme. Since the scheme is nonlinear we use a fixed-point argument together with the study of the linear version of our problem for which we adapt the arguments developed in [41] for a linear problem with Neumann boundary conditions and mean value.

Proof of Theorem 1.3.1. The proof is divided into 2 steps. In Step 1 with the help of [41] we construct a map in view of the fixed point argument. In Step 2 using estimates in Proposition 1.4.1 we conclude with the Brouwer fixed point theorem the existence of a solution.
Step 1. Let $\tilde{u} = (\tilde{u}_K)_{K\in\mathcal{T}} \in X(\mathcal{T})$ and let us consider the linear scheme

$$\forall K \in \mathcal{M}, \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(\tilde{u})_\sigma(u_K - u_L) + \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma}u_{\sigma,+} = \int_K f\,\mathrm{d}x, \tag{1.66}$$

where $u = (u_K)_{K\in\mathcal{T}} \in X(\mathcal{T})$ is the unknown. Following [41], it can be rewritten as the linear system

$$\mathbb{A}U = F \tag{1.67}$$

where $U = (u_K)_{K\in\mathcal{T}}$, $F = (\int_K f dx)_{K\in\mathcal{T}}$ and $\mathbb{A}$ is the square matrix of size $\mathrm{card}(\mathcal{T}) \times \mathrm{card}(\mathcal{T})$ with

$$\begin{cases} \mathbb{A}_{K,K} = \displaystyle\sum_{\sigma\in\mathcal{E}_{K,int}} \mathrm{m}(\sigma)\Big(\frac{\lambda(\tilde{u}_\sigma)}{d_\sigma} + \boldsymbol{v}^+_{K,\sigma}\Big), \quad \forall K \in \mathcal{T}, \\ \mathbb{A}_{K,L} = \mathrm{m}(\sigma)\Big(-\dfrac{\lambda(\tilde{u}_\sigma)}{d_\sigma} - \boldsymbol{v}^-_{K,\sigma}\Big), \quad \forall K \in \mathcal{T}, \forall L \in N(K), \text{ with } \sigma = K|L, \\ \mathbb{A}_{K,L} = 0, \quad \forall K \in \mathcal{T}, \forall L \notin N(K). \end{cases}$$

At this step having f belonging to $L^1(\Omega)$ or $f \in L^2(\Omega)$ does not play any role. From Proposition 3.1 in [41] (see also Remark 2.4 in [41] when $-\Delta u$ is replaced by $-\operatorname{div}(a(x)\nabla u)$, which is the isotropic case) it follows that

— $\dim(\ker(\mathbb{A})) = 1$ and any (non zero) element U belonging to $\ker(\mathbb{A})$ verifies either $u_K > 0$ for all $K \in \mathcal{T}$ or $u_K < 0$ for all $K \in \mathcal{T}$.

— $\ker(\mathbb{A}^\top) = \mathbb{R}(1, 1, \cdots, 1)^\top$ and thus

$$\mathrm{Im}(\mathbb{A}) = \Big\{(F_K)_{K\in\mathcal{T}}\,;\ \sum_{K\in\mathcal{T}} F_K = 0\Big\}.$$

Since $\int_\Omega f \,\mathrm{d}x = 0$, F belongs to $\mathrm{Im}(\mathbb{A})$ and then there exists at least $\overline{U}$ solution of $\mathbb{A}\overline{U} = F$. If $V = (v_K)_{K\in\mathcal{T}}$ denotes an element of $\ker(\mathbb{A})$ such that $v_k > 0$ for any $K \in \mathcal{T}$, for any $\lambda \in \mathbb{R}$, the vector $\overline{U} + \lambda V$ is a solution of (1.67). Since $v_K > 0$, $\forall K \in \mathcal{T}$, the function $\lambda \mapsto \underline{\mathrm{med}}(\overline{U} + \lambda V)$ is continuous and increasing while $\lim_{\lambda\to+\infty} \underline{\mathrm{med}}(\overline{U} + \lambda V) = +\infty$ and $\lim_{\lambda\to-\infty} \underline{\mathrm{med}}(\overline{U} + \lambda V) = -\infty$. It follows that there exists at least one solution $u \in X(\mathcal{T})$ verifying the scheme (1.66) and $\underline{\mathrm{med}}(u) = 0$.

The uniqueness is a consequence of the characterization of $\ker(\mathbb{A})$. As a conclusion we can define the map Γ from $X(\mathcal{T})$ into $X(\mathcal{T})$ by

$$\forall \tilde{u} \in X(\mathcal{T}),\ \Gamma\Big(\tilde{u}\Big) = u$$

where $u \in X(\mathcal{T})$ is the unique solution of (1.66) such that $\underline{\mathrm{med}}(u) = 0$.

Step 2. In this step we prove that

$$\exists C > 0, \forall \tilde{u} \in X(\mathcal{T}), \quad \|\Gamma(\tilde{u})\|_{L^\infty} \le C, \tag{1.68}$$

$$\Gamma \text{ is a continuous map} \tag{1.69}$$

in order to apply the Brouwer fixed point theorem.

The boundedness of Γ relies on Proposition 1.4.1. Indeed by replacing $\lambda(u)_\sigma$ by $\lambda(\tilde{u})_\sigma$ in the proof of Proposition 1.4.1 it can be shown that there exists $C > 0$ (not depending on $\tilde{u}$) such that

$$\|\ln(1 + |\Gamma(\tilde{u})|)\|^2_{1,2,\mathcal{M}} \le C\left(2\|f\|_{L^1(\Omega)} + d|\Omega|^{\frac{p-2}{p}}\,\|\boldsymbol{v}\|^2_{(L^p(\Omega))^d}\right), \tag{1.70}$$

Since $\Gamma(\tilde{u})$ lies in a finite dimension vector space we obtain that (1.68) holds true.

We now prove that Γ is a continuous map. Let $(\tilde{u}_n)_{n\in\mathbb{N}}$ and $\tilde{u}$ belonging to $X(\mathcal{T})$ such that $\tilde{u}_n$ goes to $\tilde{u}$ as n goes to infinity. In view of (1.68) up to a subsequence, still indexed by n, there exists $w \in X(\mathcal{T})$ such that $\Gamma(\tilde{u}_n)$ tends to w as n goes to infinity. Recalling that the coefficient of the matrix $\mathbb{A}$ is continuous with respect to $\tilde{u}$ we obtain that w is a solution to the scheme

$$\forall K \in \mathcal{M},\ \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(\tilde{u})_\sigma(w_K - w_L) + \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma}w_{\sigma,+} = \int_K f\,\mathrm{d}x. \tag{1.71}$$

The mesh $\mathcal{T}$ being fixed since we consider constant piecewise functions the fact that $\underline{\mathrm{med}}\Big(\Gamma(\tilde{u}_n)\Big) = 0$ for any $n \in \mathbb{N}$ implies that $\underline{\mathrm{med}}(w) = 0$. Recalling that $u = \Gamma(\tilde{u})$ is the unique solution of (1.66) with $\underline{\mathrm{med}}(u) = 0$ we conclude that $u = w$. Since the limit does not depend on the subsequence we obtain that Γ is a continuous map.

In view of (1.68) and (1.69) the Brouwer fixed point theorem allows one to conclude the proof of Theorem 1.3.1. □

1.6 Convergence of the scheme

In this section, we prove using Section 4 that the function u obtained in Proposition 1.4.3 is a renormalized solution to equation (1.2). The main difficulty is the equation (1.13): in short the test functions in (1.13) are of the kind $\varphi S(u)$ which is nonlinear in u. The strategy is to mix renormalized techniques and finite volume with the use of a discrete version of $\varphi S_n(u_{\mathcal{M}_m})$ (see below the definition of S_n) where φ belongs to $\mathcal{D}(\overline{\Omega})$ (see [77] in the case of Dirichlet boundary condition). Before proving Theorem 1.3.2 we give in Lemma 1.6.1 a convergence result concerning $S_n(u_{\mathcal{M}_m})$ (see e.g. [76, 77]) and in Corollary 1.6.2 the asymptotic behavior of an extra term (with respect to the continuous case) which appears when we pass to the limit in the discrete equation.

First, let us define the function S_n, $n \geq 1$, by

$$S_n(s) = \begin{cases} 0 & \text{if } s \leq -2n, \\ \dfrac{s}{n} + 2 & \text{if } -2n \leq s \leq -n, \\ 1 & \text{if } -n \leq s \leq n, \\ \dfrac{-s}{n} + 2 & \text{if } n \leq s \leq 2n, \\ 0 & \text{if } s \geq 2n. \end{cases} \tag{1.72}$$

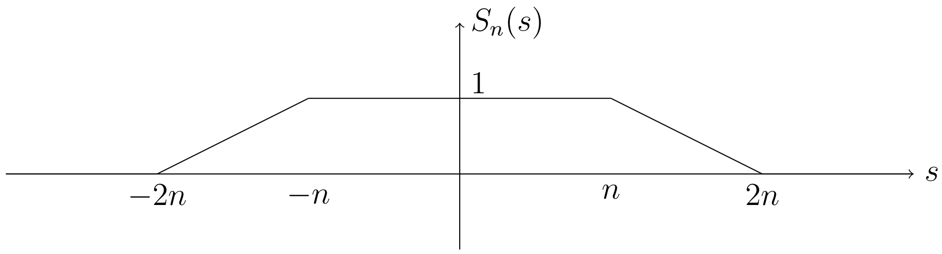

Figure 1.2 – The function S_n

Lemma 1.6.1. *Let $(\mathcal{M}_m)_{m\geq 1}$ be a sequence of admissible meshes satisfying* (1.15) *such that $h_{\mathcal{M}_m} \to 0$ as $m \to \infty$ and let $u_m = (u_K^m)_{K\in\mathcal{T}_m} \in X(\mathcal{T}_m)$ be a sequence of solution of (1.19) such that the conclusions of Proposition 1.4.3 hold true. We define the function $\overline{S}_n^m$ over the diamonds by*

$$\forall \sigma = K|L \in \mathcal{E}_{int}, \forall x \in D_\sigma, \quad \overline{S}_n^m(x) = \frac{S_n(u_K^m) + S_n(u_L^m)}{2},$$
$$\forall \sigma \in \mathcal{E}_{ext}, \forall x \in D_\sigma, \quad \overline{S}_n^m(x) = S_n(u_K^m),$$

and the function $\overline{\lambda}^m$ by

$$\forall \sigma \in \mathcal{E}, \forall x \in D_\sigma \quad \overline{\lambda}^m(x) = \lambda(u_m)_\sigma.$$

Then the functions $\overline{S}_n^m$ and $\overline{\lambda}$ converge respectively to $S_n(u)$ and $\lambda(u)$ in $L^q(\Omega)$ $\forall q \in [1, +\infty[$ and in $L^\infty(\Omega)$ weak-$*$, as $h_{\mathcal{M}_m} \to 0$, where u is the limit of $u_{\mathcal{M}_m}$.

Proof. Since the functions $\overline{S}_n^m$ are bounded with respect to m it is sufficient to prove that the convergence holds true in $L^2(\Omega)$. By Proposition 1.4.3 u_m converges to u a.e. in Ω. Since S_n is a continuous and bounded function, the Lebesgue-dominated convergence theorem gives that $S_n(u_m) \to S_n(u)$ in $L^2(\Omega)$, so that it is sufficient to study the behavior of $\overline{S}_n^m - S_n(u_m)$. Recalling that S_n is Lipschitz continuous and has a compact support, we have

$$\begin{aligned}
\|\overline{S}_n^m - S_n(u_m)\|^2_{L^2(\Omega)} &= \int_\Omega |\overline{S}_n^m - S_n(u_m(x))|^2 \,\mathrm{d}\, x \\
&= \sum_{\sigma\in\mathcal{E}} \int_{D_\sigma} |\overline{S}_n^m(x) - S_n(u_m(x))|^2 \,\mathrm{d}\, x \\
&= \sum_{\sigma\in\mathcal{E}_{int}} \int_{D_\sigma} \left| \frac{S_n(u_K^m) + S_n(u_L^m)}{2} - S_n(u_m(x)) \right|^2 \mathrm{d}\, x \\
&= \frac{1}{4} \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(D_\sigma)\, |S_n(u_K^m) - S_n(u_L^m)|^2 \\
&\le \frac{1}{4n^2} \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(D_\sigma) |T_{2n}(u_K^m) - T_{2n}(u_L^m)|^2 \\
&= \frac{1}{4dn^2} \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma) d_\sigma \left| \frac{T_{2n}(u_K^m) - T_{2n}(u_L^m)}{d_\sigma} \right|^2 (d_\sigma)^2 \\
&\le \frac{1}{4dn^2} |T_{2n}(u_m)|^2_{1,2,\mathcal{M}} \,(h_{\mathcal{M}_m})^2,
\end{aligned}$$

so that $\|\overline{S}_n^m - S_n(u_m)\|^2_{L^2(\Omega)}$ goes to zero as $h_{\mathcal{M}_m} \to 0$.

As far as $\overline{\lambda}^m$ is concerned let us first define the function $\overline{u}^m$ by

$$\begin{aligned}
\forall\sigma = K|L \in \mathcal{E}_{int}, \forall x \in D_\sigma, \quad \overline{u}^m(x) &= \frac{u_K^m + u_L^m}{2}, \\
\forall\sigma \in \mathcal{E}_{ext}, \forall x \in D_\sigma, \quad \overline{u}^m(x) &= u_K^m.
\end{aligned}$$

In view of the definition of $\lambda(u)_\sigma$ we have in Ω

$$\min(\lambda(u_m), \lambda(2\overline{u}^m - u_m)) \le \overline{\lambda}^m \le \max(\lambda(u_m), \lambda(2\overline{u}^m - u_m)). \tag{1.73}$$

In view of already used arguments for $\overline{S}^m$, since the function T_n is Lipschitz continuous, $T_n(\overline{u}^m)$ converges to $T_n(u)$ in $L^q(\Omega)$, $\forall q \in [1, +\infty[$ and in L^∞ weak-$*$. By the diagonal process, up to a subsequence still index by m, $\overline{u}^m$ goes to u a.e. in Ω as $h_{\mathcal{M}_m}$ goes to zero. Since λ is a bounded continuous function we obtain, up to a subsequence, that $\overline{\lambda}^m$ converges to $\lambda(u)$ in $L^q(\Omega)$, $\forall q \in [1, +\infty[$ and in L^∞ weak-$*$ as $h_{\mathcal{M}_m}$ goes to zero. To conclude it is sufficient to observe that the limit of $\overline{\lambda}^m$ is independent of the subsequence. □

Corollary 1.6.2. *Let $(\mathcal{M}_m)_{m\ge1}$ be a sequence of admissible meshes satisfying* (1.15) *such that $h_{\mathcal{M}_m} \to 0$ as $m \to \infty$. Let $u_m = (u_K^m)_{K\in\mathcal{T}_m} \in X(\mathcal{T}_m)$ be a sequence of solution of (1.19) such that the conclusions of Proposition 1.4.3 hold true. Then we have*

$$\lim_{n\to+\infty} \overline{\lim_{h_{\mathcal{M}_m}\to 0}} \sum_{\substack{\sigma\in\mathcal{E}_{int} \\ |u_K^m|\le 2n \\ |u_L^m|>4n}} \lambda(u)_\sigma \frac{\mathrm{m}(\sigma)}{d_\sigma} |u_L^m| = 0. \tag{1.74}$$

Proof. For $m \in \mathbb{N}$, let us consider $K \in \mathcal{T}_m$ and $n \in \mathbb{N}$. On one hand if $|u_K^m| \le 2n$ and $u_L^m > 4n$ then

$$(u_K^m - u_L^m)(T_{4n}(u_K^m) - T_{4n}(u_L^m)) \ge \frac{u_L^m}{2} 2n \ge 0.$$

On the other hand, if $|u_K^m| \le 2n$ and $u_L^m < -4n$ then

$$(u_K^m - u_L^m)(T_{4n}(u_K^m) - T_{4n}(u_L^m)) \ge \frac{-u_L^m}{2} 2n \ge 0.$$

It follows that

$$\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ |u_K^m|\le 2n\\ |u_L^m|>4n}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma |u_L^m| \le \frac{1}{4n}\sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma (u_L^m - u_K^m)(T_{4n}(u_K^m) - T_{4n}(u_L^m)). \tag{1.75}$$

Using the discrete estimate on energy (1.44), we have

$$\lim_{n\to+\infty}\lim_{h_{\mathcal{M}_m}\to 0}\frac{1}{4n}\sum_{\sigma\in\mathcal{E}_{int}}\lambda(u_m)_\sigma \frac{\mathrm{m}(\sigma)}{d_\sigma}(u_L^m - u_K^m)(T_{4n}(u_K^m) - T_{4n}(u_L^m)) = 0, \tag{1.76}$$

so that (1.75) and (1.76) give (1.74). □

We are now in a position to prove Theorem 1.3.2.

Proof of Theorem 1.3.2 Let $m \ge 1$ and let us consider $u_m = (u_K^m)_{K\in\mathcal{T}_m}$ be a solution of the scheme (1.19).

For φ a function belonging to $\mathcal{D}(\overline{\Omega})$ we denote by φ_m the function defined by $\varphi_K = \varphi(x_K)$ for all $K \in \mathcal{T}_m$. For $n \in \mathbb{N}$, multiplying each equation of the scheme by $\varphi(x_K)S_n(u_K^m)$ (which is a discrete version of the test function used in the renormalized formulation), summing over the control volumes and gathering by edges, we get

$$T_1 + T_2 = T_3$$

with

$$\begin{aligned}
T_1 &= \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma (u_K^m - u_L^m)(\varphi(x_K)S_n(u_K^m) - \varphi(x_L)S_n(u_L^m)),\\
T_2 &= \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma}u_{\sigma,+}^m(\varphi(x_K)S_n(u_K^m) - \varphi(x_L)S_n(u_L^m)),\\
T_3 &= \sum_{K\in\mathcal{T}}\int_K f\,\varphi(x_K)S_n(u_K^m).
\end{aligned}$$

Since $S_n(u_m) \to S_n(u)$ a.e. and L^∞ weak $\star$, by the regularity of φ, $\varphi_m \to \varphi$ uniformly and $|f\,\varphi_m\,S_n(u_m)| \le C_\varphi |f| \in L^1(\Omega)$, the Lebesgue theorem ensures that

$$T_3 = \int_\Omega f\,\varphi_m\,S_n(u_m)\,\mathrm{d}\,x \xrightarrow[h_{\mathcal{M}_m}\to 0]{} \int_\Omega f\,\varphi\,S_n(u)\,\mathrm{d}\,x. \tag{1.77}$$

We now study the convergence of the diffusion term. We write

$$\begin{aligned}
T_1 &= \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma (u_K^m - u_L^m)(\varphi(x_K)S_n(u_K^m) - \varphi(x_L)S_n(u_L^m))\\
&= T_{1,1} + T_{1,2}
\end{aligned}$$

with

$$T_{1,1} = \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma S_n(u_K^m)\,(u_K^m - u_L^m)\,(\varphi(x_K)-\varphi(x_L)),$$
$$T_{1,2} = \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma \varphi(x_L)\,(u_K^m - u_L^m)\,(S_n(u_K^m)-S_n(u_L^m)).$$

According to the definition of S_n we have

$$|T_{1,2}| \le \frac{1}{n}\sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma \varphi(x_L)\,(u_K^m - u_L^m)\,(T_{2n}(u_K^m)-T_{2n}(u_L^m)),$$

so that (1.44) give

$$\lim_{n\to+\infty}\overline{\lim_{h_{\mathcal{M}_m}\to 0}}\, T_{1,2} = 0. \tag{1.78}$$

The main difference with respect to the continuous case is that u_K^m is truncated while u_L is not in $T_{1,1}$. To control this term we have to write

$$\begin{aligned} T_{1,1} &= \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma S_n(u_K^m)\,(T_{2n}(u_K^m) - u_L^m)\,(\varphi(x_K)-\varphi(x_L)),\\ &= I + II + III \end{aligned}$$

with

$$I = \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma \frac{S_n(u_K^m)+S_n(u_L^m)}{2}\,(T_{4n}(u_K^m)-T_{4n}(u_L^m))\,(\varphi(x_K)-\varphi(x_L)),$$
$$II = \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma S_n(u_K^m)\,(T_{4n}(u_L^m)-u_L^m)\,(\varphi(x_K)-\varphi(x_L)),$$
$$III = \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma \frac{S_n(u_K^m)-S_n(u_L^m)}{2}\,(T_{4n}(u_K^m)-T_{4n}(u_L^m))\,(\varphi(x_K)-\varphi(x_L))$$

We first study the asymptotic behavior of II and III as the parameter $h_{\mathcal{M}_n}$ goes to zero. Since

$$\begin{aligned} |II| &\le \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u)_\sigma |S_n(u_K^m)|\,|T_{4n}(u_L^m)-u_L^m|\,|\varphi(x_K)-\varphi(x_L)|\\ &\le 2\|\varphi\|_{L^\infty(\Omega)} \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ |u_K^m|\le 2n\\ |u_L^m|>4n}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma |u_L^m|, \end{aligned}$$

and due to Corollary 1.6.2 we obtain that

$$\overline{\lim_{h_{\mathcal{M}}\to 0}}\, |II| \le \omega(n)\|\varphi\|_{L^\infty(\Omega)}, \tag{1.79}$$

where $\omega(n)$ tends to zero as n goes to infinity. In the sequel of the present proof $\omega(n)$, is a generic positive quantity such that $\lim_{n\to+\infty} w(n) = 0$. According to the definition of S_n, we have

$$|III| \le \frac{\|\varphi\|_{L^\infty(\Omega)}}{n} \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}\lambda(u_m)_\sigma |u_K^m - u_L^m|\,|T_n(u_K^m)-T_n(u_L^m)|,$$

recalling (1.44) of Proposition 1.4.4 we obtain that

$$\overline{\lim_{h_{\mathcal{T}}\to 0}} |III| \leq \omega(n)\|\varphi\|_{L^\infty(\Omega)}. \tag{1.80}$$

We now turn to I. By rewriting I as integral over the diamonds D_σ (see e.g. [42, 59, 54]) we have

$$\begin{aligned}
I &= \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma) d_\sigma \lambda(u_m)_\sigma \frac{S_n(u_K^m)+S_n(u_L^m)}{2}\,\frac{T_{4n}(u_K^m)-T_{4n}(u_L^m)}{d_\sigma}\,\frac{\varphi(x_K)-\varphi(x_L)}{d_\sigma}\\
&= \sum_{\sigma\in\mathcal{E}_{int}} \int_{D_\sigma} \overline{\lambda}^m(x)\overline{S}_n^m(x)\nabla_{\mathcal{M}_m}T_{4n}(u_m)\cdot\nabla\varphi\,\mathrm{d}\,x\\
&\quad+\sum_{\sigma\in\mathcal{E}_{int}} d\mathrm{m}(D_\sigma)\lambda(u_m)_\sigma \frac{S_n(u_K^m)+S_n(u_L^m)}{2}\,\frac{T_{4n}(u_K^m)-T_{4n}(u_L^m)}{d_\sigma}\\
&\qquad\qquad\times\left[\frac{\varphi(x_K)-\varphi(x_L)}{d_\sigma}+\frac{1}{\mathrm{m}(D_\sigma)}\int_{D_\sigma}\nabla\varphi\cdot\eta_{K,\sigma}\,\mathrm{d}\,x\right]\\
&= I_1+I_2.
\end{aligned}$$

By Lemma 1.6.1 $\overline{S}_n^m \to S_n(u)$ in $L^q(\Omega)$ for all $q\in[1,+\infty[$ and $\overline{\lambda}^m\to\lambda(u)$ in L^∞ weak $\star$, while $\nabla_{\mathcal{M}_m}T_{4n}(u_m)$ tends to $\nabla T_{4n}(u)$ weakly in $(L^2(\Omega))^d$. Since φ belongs to $C_c^\infty(\Omega)$ we conclude that

$$\lim_{h_{\mathcal{M}_m}\to 0} I_1 = \int_\Omega \lambda(u)S_n(u)\nabla T_{4n}(u)\cdot\nabla\varphi\,\mathrm{d}x.$$

By the regularity of φ we see that

$$\begin{aligned}
|I_2| &\leq \sum_{\sigma\in\mathcal{E}_{int}} d\mathrm{m}(D_\sigma)\lambda(u_m)_\sigma\left|\frac{S_n(u_K^m)+S_n(u_L^m)}{2}\right|\frac{|T_{4n}(u_K^m)-T_{4n}(u_L^m)|}{d_\sigma}\\
&\qquad\qquad\times\left|\frac{\varphi(x_K)-\varphi(x_L)}{d_\sigma}+\frac{1}{\mathrm{m}(D_\sigma)}\int_{D_\sigma}\nabla\varphi\cdot\eta_{K,\sigma}\,d\,x\right|\\
&\leq \lambda_\infty\|\varphi\|_{W^{2,\infty}(\Omega)}h_{\mathcal{M}}\|T_{4n}(u_m)\|_{1,1,\mathcal{M}},
\end{aligned}$$

thus

$$\lim_{h_{\mathcal{M}_m}\to 0} I_2 = 0. \tag{1.81}$$

We now study the convergence of the convection term T_2. We have

$$\begin{aligned}
T_2 &= \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma}u_{\sigma,+}^m(\varphi(x_K)S_n(u_K^m)-\varphi(x_L)S_n(u_L^m))\\
&= \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \boldsymbol{v}_{K,\sigma}\geq 0}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma}u_{\sigma,+}^m(\varphi(x_K)S_n(u_{\sigma,+}^m)-\varphi(x_L)S_n(u_{\sigma,-}^m))\\
&\quad+\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \boldsymbol{v}_{K,\sigma}<0}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma}u_{\sigma,+}^m(\varphi(x_K)S_n(u_{\sigma,-}^m)-\varphi(x_L)S_n(u_{\sigma,+}^m))\\
&= \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \boldsymbol{v}_{K,\sigma}\geq 0}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma}u_{\sigma,+}^m S_n(u_{\sigma,+}^m)(\varphi(x_K)-\varphi(x_L))
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \boldsymbol{v}_{K,\sigma}\geq 0}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma} u^m_{\sigma,+}\varphi(x_L)(S_n(u^m_{\sigma,+}) - S_n(u^m_{\sigma,-}))\\
&- \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \boldsymbol{v}_{K,\sigma}< 0}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma} u^m_{\sigma,+}S_n(u^m_{\sigma,+})(\varphi(x_L) - \varphi(x_K))\\
&- \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \boldsymbol{v}_{K,\sigma}< 0}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma} u^m_{\sigma,+}\varphi(x_K)(S_n(u^m_{\sigma,+}) - S_n(u^m_{\sigma,-}))\\
= {}& T_{2,1} + T_{2,2} + T_{2,3}
\end{aligned}
$$

with

$$
\begin{aligned}
T_{2,1} &= \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma} u^m_{\sigma,+}S_n(u^m_{\sigma,+})(\varphi(x_K) - \varphi(x_L))\\
T_{2,2} &= \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \boldsymbol{v}_{K,\sigma}\geq 0}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma} u^m_{\sigma,+}\varphi(x_L)(S_n(u^m_{\sigma,+}) - S_n(u^m_{\sigma,-}))\\
T_{2,3} &= -\sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \boldsymbol{v}_{K,\sigma}< 0}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma} u^m_{\sigma,+}\varphi(x_K)(S_n(u^m_{\sigma,+}) - S_n(u^m_{\sigma,-}))
\end{aligned}
$$

Since

$$|T_{2,2} + T_{2,3}| \leq \frac{\|\varphi\|_{L^\infty(\Omega)}}{n} \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)\,|\boldsymbol{v}_{K,\sigma}|\,|u^m_{\sigma,+}|\,|T_{2n}(u^m_{\sigma,+}) - T_{2n}(u^m_{\sigma,-})|,$$

we deduce from (1.45) that

$$\lim_{n\to+\infty} \overline{\lim_{h_{\mathcal{M}_m}\to 0}} T_{2,2} + T_{2,3} = 0. \tag{1.82}$$

For the term $T_{2,1}$, we have

$$
\begin{aligned}
T_{2,1} &= \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)\boldsymbol{v}_{K,\sigma} u^m_{\sigma,+}S_n(u^m_{\sigma,+})(\varphi(x_K) - \varphi(x_L))\\
&= \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)d_\sigma}{d\,\mathrm{m}(D_\sigma)}\, u^m_{\sigma,+}\, S_n(u^m_{\sigma,+})\, d\frac{\varphi(x_K) - \varphi(x_L)}{d_\sigma} \int_{D_\sigma} \boldsymbol{v}\cdot\eta_{K,\sigma}\,\mathrm{d}x\\
&= -\sum_{\sigma\in\mathcal{E}_{int}} \int_{D_\sigma} T_{2n}(u^m_{\sigma,+})S_n(u^m_{\sigma,+})\boldsymbol{v}\cdot\nabla_{\mathcal{M}}\varphi_{\mathcal{M}}\,\mathrm{d}x.
\end{aligned}
$$

We define the function $\overline{G}^m_n$ defined over the diamonds by

$$\forall\sigma = K|L \in \boldsymbol{\mathcal{E}}_{int}, \forall x\in D_\sigma, \quad \overline{G}^m_n(x) = T_{2n}(u^m_{\sigma,+})S_n(u^m_{\sigma,+})$$

Then $T_{2,1}$ reads as

$$T_{2,1} = -\int_\Omega \overline{G}^m_n\boldsymbol{v}\cdot\nabla_{\mathcal{M}_m}\varphi_m\,\mathrm{d}x.$$

Since the function $r\mapsto T_{2n}(r)S_n(r)$ is Lipschitz continuous and bounded, with the help of arguments already used in the proof of Lemma 1.6.1, we can show that $\overline{G}^m_n$ converges to $T_{2n}(u)S_n(u)$ in L^∞ weak $\star$, as $h_{\mathcal{M}_m}\to 0$. Recalling that $\nabla_{\mathcal{M}_m}\varphi_m$ converges weakly in $(L^2(\Omega))^d$ we obtain

$$\lim_{h_{\mathcal{M}_m}\to 0} T_{2,1} = -\int_\Omega T_{2n}(u)S_n(u)\boldsymbol{v}\cdot\nabla\varphi\,\mathrm{d}x. \tag{1.83}$$

We now pass to the limit in the scheme first as $h_{\mathcal{M}_m}$ goes to zero and then as n goes to infinity. Gathering equations (1.77) to (1.83), allows one to conclude that

$$\int_\Omega \lambda(u)S_n(u)\,\nabla u\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega u\,S_n(u)\,\boldsymbol{v}\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega f\,\varphi\,S_n(u)\,\mathrm{d}x = \lim_{h_{\mathcal{M}_m}\to 0} T(n,\varphi) \quad (1.84)$$

where $\lim\limits_{h_{\mathcal{M}_m}\to 0} |T(n,\varphi)| \le \|\varphi\|_{L^\infty(\Omega)}\omega(n)$ with $\omega(n)\to 0$ as $n\to+\infty$. Since $S_n(u)\lambda(u)\nabla u$, $uS_n(u)\boldsymbol{v}$ and $fS_n(u)$ belongs respectively to $(L^2(\Omega))^d$, $L^2(\Omega)$ and $L^1(\Omega)$ a density argument gives that (1.84) holds true for any φ lying in $H^1(\Omega)\cap L^\infty(\Omega)$.

Let S be a function in $W^{1,\infty}(\mathbb{R})$ with compact support, contained in the interval $[-k,k]$, $k>0$ and let $\psi\in H^1(\Omega)\cap L^\infty(\Omega)$. Using the function $S(u)\psi$ in (1.84), we deduce that

$$\Bigg| \int_\Omega \lambda(u)\nabla u\,S_n(u)S(u)\nabla\psi\,\mathrm{d}x + \int_\Omega \lambda(u)\nabla u\,S_n(u)\psi\nabla u\,S'(u)\,\mathrm{d}x \\ - \int_\Omega u\,S_n(u)\,S(u)\,\boldsymbol{v}\cdot\nabla\psi\,\mathrm{d}x - \int_\Omega u\,S_n(u)\,S'(u)\,\psi\,\boldsymbol{v}\cdot\nabla u\,\mathrm{d}x \\ - \int_\Omega \psi S(u)S_n(u) f\,\mathrm{d}x\Bigg| \le \|\varphi\|_{L^\infty(\Omega)}\omega(n).$$

By observing that $S_n(u)S(u)=S(u)$ and $S_n(u)S'(u)=S'(u)$ a.e. in Ω for n sufficiently large, by passing to the limit as n goes to infinity we obtain the condition (1.13) of Definition 1.2.2, that is

$$\int_\Omega \lambda(u)\nabla u\,S(u)\nabla\psi\,\mathrm{d}x + \int_\Omega \lambda(u)\nabla u\,\psi\nabla u\,S'(u)\,\mathrm{d}x \\ - \int_\Omega u\,S(u)\,\boldsymbol{v}\cdot\nabla\psi\,\mathrm{d}x - \int_\Omega u\,S'(u)\,\psi\,\boldsymbol{v}\cdot\nabla u\,\mathrm{d}x = \int_\Omega \psi S(u)\,f\,\mathrm{d}x.$$

We now turn to the decay of the energy. As a consequence of (1.44), we get

$$\lim_{n\to\infty}\overline{\lim_{h_{\mathcal{M}_m}\to 0}}\frac{1}{n}\sum_{\sigma\in\mathcal{E}_{int}} \lambda(u)\frac{\mathrm{m}(\sigma)}{d_\sigma}(T_{2n}(u_K^m)-T_{2n}(u_L^m))^2 = 0,$$

and

$$\sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(T_{2n}(u_K^m)-T_{2n}(u_L^m))^2 = \sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma)d_\sigma\left(\frac{T_{2n}(u_K^m)-T_{2n}(u_L^m)}{d_\sigma}\right)^2 \\ = \sum_{\sigma\in\mathcal{E}_{int}} d\mathrm{m}(D_\sigma)\left(\frac{T_{2n}(u_K^m)-T_{2n}(u_L^m)}{d_\sigma}\right)^2 \\ = \frac{1}{d}\int_\Omega |\nabla_{\mathcal{M}_m}T_{2n}(u_m)|^2\,\mathrm{d}x.$$

so that $\lim\limits_{n\to\infty}\overline{\lim\limits_{h_{\mathcal{M}_m}\to 0}}\frac{1}{n}\int_\Omega |\nabla_{\mathcal{M}_m}T_{2n}(u_m)|^2\,\mathrm{d}x = 0$. Since $\nabla_{\mathcal{M}_m}T_{2n}(u_m)$ converges weakly in $(L^2(\Omega))^d$, we have also

$$\frac{1}{n}\int_\Omega |\nabla T_{2n}(u)|^2\,\mathrm{d}x \le \liminf_{h_{\mathcal{M}}\to 0}\frac{1}{n}\int_\Omega |\nabla_{\mathcal{M}}T_{2n}(u_{\mathcal{M}})|^2\,\mathrm{d}x,$$

which leads to

$$\lim_{n\to\infty} \frac{1}{n} \int_\Omega \lambda(u)|\nabla T_{2n}(u)|^2 \,\mathrm{d}x = 0.$$

Since u is finite almost everywhere in Ω and since $T_n(u) \in H^1(\Omega)$ for any $n > 0$ we can conclude $u_{\mathcal{M}_m}$ converges to u which is is the unique renormalized solution with a null median. $\square$

1.7 Appendix

Discrete functional inequalities are useful for the study of finite volume schemes. Discrete Sobolev inequalities are proved in [45] for Dirichlet boundary conditions and in [41] for non-Dirichlet boundary conditions. In [14] discrete Gagliardo-Nirenberg-Sobolev and Poincaré-Sobolev inequalities for some finite volume schemes are proved. There does not seem to be any proof of Discrete Poincaré-Wirtinger median inequality. The authors in [14] use the continuous embedding of the space $BV(\Omega)$ into $L^{\frac{d}{d-1}}(\Omega)$ for a Lipschitz domain $\Omega \subset \mathbb{R}^d$, with $d \geq 2$ to establish discrete inequalities. We will use this method to prove the Discrete Poincaré-Wirtinger median inequality (1.17).

Let us first recall some results concerning functions of bounded variation (more details about these functions can be found in [106]). Let Ω be an open set of $\mathbb{R}^d$ and $u \in L^1(\Omega)$. The total variation of u in Ω, denoted by $TV_\Omega(u)$, is defined by

$$TV_\Omega(u) = \sup\left\{\int_\Omega u(x)\mathrm{div}(\phi(x)) \,\mathrm{d}x,\ \phi \in C_c^1(\Omega),\ |\phi(x)| \leq 1,\ \forall x \in \Omega\right\}.$$

The function $u \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if $TV_\Omega(u) < +\infty$. The space $BV(\Omega)$ is endowed with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + TV_\Omega(u).$$

The space $BV(\Omega)$ is a natural space to study finite volume approximations. Indeed, for $u = (u_K)_{K\in\mathcal{T}} \in X(\mathcal{T})$, we have

$$TV_\Omega(u) = \sum_{\substack{\sigma\in\mathcal{E}_{int}\\ \sigma=K|L}} \mathrm{m}(\sigma)|u_L - u_K| = |u|_{1,1,\mathcal{M}} < +\infty.$$

The discrete space $X(\mathcal{T})$ is included in $L^1(\Omega) \cap BV(\Omega)$ and we have

$$\|u\|_{BV(\Omega)} = \|u\|_{1,1,\mathcal{M}}.$$

Our starting point for the discrete Poincaré-Wirtinger median inequality is the continuous embedding of $BV(\Omega)$ into $L^{\frac{d}{d-1}}(\Omega)$ for Lipschitz bounded connected domain Ω of $\mathbb{R}^d$, $d \geq 2$, written in the following theorem (see [106] for more details).

Theorem 1.7.1 ([106]). *There exists a constant $C(\Omega) > 0$ only depending on Ω such that, for all $u \in BV(\Omega)$,*

$$\left(\int_\Omega |u - m|^{\frac{d}{d-1}} \,\mathrm{d}x\right)^{\frac{d-1}{d}} \leq C(\Omega)\, TV_\Omega(u), \tag{1.85}$$

where $m \in \mathrm{med}(u)$.

In the spirit of [14] (which studied Discrete Poincaré-Wirtinger mean inequality), let us prove now the following proposition

Proposition 1.7.2 (Discrete Poincaré-Wirtinger median inequality)**.** *Let Ω be an open bounded connected polyhedral domain of $\mathbb{R}^d$ and let $\mathcal{M}$ be an admissible mesh satisfying* (1.15)*. Then for $1 \leq p < +\infty$ there exists a constant $C > 0$ only depending on Ω, d and p such that*

$$\|u - c\|_{0,p} \leq \frac{C}{\xi^{(p-1)/p}} |u|_{1,p,\mathcal{M}}, \qquad \forall u \in X(\mathcal{T}) \tag{1.86}$$

where c is in $\mathrm{med}(u)$*.*

Proof of Proposition 1.7.2. Let $u = (u_K)_{K\in\mathcal{T}}$ be a function of $X(\mathcal{T})$ and let m be an element of $\mathrm{med}(u)$. We define $v \in X(\mathcal{T})$ by $v_K = (u_k - m)|u_k - m|^{p-1}$ for all $K \in \mathcal{T}$. Since $m \in \mathrm{med}(u)$, we have $0 \in \mathrm{med}(v)$, using inequality (1.85), we obtain

$$\|v\|_{0,\frac{d}{d-1}} \leq C(\Omega)|v|_{1,1,\mathcal{M}}, \tag{1.87}$$

and using the inclusion of $L^{\frac{d}{d-1}}(\Omega)$ into $L^1(\Omega)$, we get

$$\|v\|_{0,1} \leq C(\Omega, d)|v|_{1,1,\mathcal{M}}, \tag{1.88}$$

where the constant C depends on Ω and d.
Moreover, for all $K, L \in \mathcal{T}$, we have

$$\begin{aligned} |v_K - v_L| &= |u_K - u_L||v'(w_{LK})|, \qquad \forall w_{LK} \in [u_k, u_L], \\ &\leq p|u_K - u_L||w_{LK} - m|^{p-1} \\ &\leq p|u_K - u_L| \left(|u_K - m|^{p-1} + |u_L - m|^{p-1}\right). \end{aligned} \tag{1.89}$$

Therefore, gathering (1.88) and (1.89), we obtain

$$\begin{aligned} \|v\|_{0,1} = \||u - m|^p\|_{0,1} = \|u - m\|_{0,p}^p \\ \leq C \sum_{\substack{\sigma\in\mathcal{E}_{int} \\ \sigma=K|L}} \mathrm{m}(\sigma) p|u_k - u_L| \left(|u_K - m|^{p-1} + |u_L - m|^{p-1}\right). \end{aligned} \tag{1.90}$$

Using Hölder's inequality we get,

$$\begin{aligned} \|u - m\|_{0,p}^p \leq pC &\left(\sum_{\substack{\sigma\in\mathcal{E}_{int} \\ \sigma=K|L}} \frac{\mathrm{m}(\sigma)}{d_\sigma^{p-1}} |u_K - u_L|^p \right)^{\frac{1}{p}} \\ &\times \left(\sum_{\substack{\sigma\in\mathcal{E}_{int} \\ \sigma=K|L}} \mathrm{m}(\sigma) \left(d_\sigma^{\frac{p-1}{p}}\right)^{\frac{p}{p-1}} \left(|u_K - m|^{p-1} + |u_L - m|^{p-1}\right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\leq pC|u - m|_{1,p,\mathcal{M}} \left(\sum_{\substack{\sigma\in\mathcal{E}_{int} \\ \sigma=K|L}} \mathrm{m}(\sigma) d_\sigma \frac{p}{p-1} \left(|u_K - m|^p + |u_L - m|^p\right) \right)^{\frac{p-1}{p}}. \end{aligned} \tag{1.91}$$

The regularity constraint (1.15) on the mesh ensures that

$$\sum_{\sigma\in\mathcal{E}_{int}} \mathrm{m}(\sigma) d_\sigma \le \frac{1}{\xi} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) d(x_K,\sigma) = \frac{N}{\xi} \sum_{K\in\mathcal{T}} \mathrm{m}(K). \tag{1.92}$$

Then applying the previous inequality (1.92) and a discrete integration by parts, we get

$$\begin{aligned} \|u-m\|_{0,p}^p &\le pC(\Omega)|u-m|_{1,p,\mathcal{M}} \left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) d_\sigma |u_K - m|^p\right)^{\frac{p-1}{p}} \\ &\le \frac{C(\Omega,p,d)}{\xi^{\frac{p-1}{p}}} |u-m|_{1,p,\mathcal{M}} \|u-m\|_{0,p}^{p-1}. \end{aligned} \tag{1.93}$$

Then we obtain the general result

$$\|u-m\|_{0,p} \le \frac{C}{\xi^{(p-1)/p}} |u|_{1,p,\mathcal{M}}, \qquad \forall u \in X(\mathcal{T}).$$

□

Chapter 2

Existence and Uniqueness of renormalized solutions for parabolic Neumann problem with L^1 data

2.1 Introduction

In the present paper, we study a class of nonlinear parabolic equations with L^1 data and non-homogeneous Neumann boundary conditions whose prototype is

$$\begin{cases} \dfrac{\partial \theta}{\partial t} - \operatorname{div}\left(A(x,t,\theta)\nabla\theta\right) + u\cdot\nabla\theta = f - \operatorname{div} g & \text{in } Q_T\ , \\ \theta(t=0) = \theta_0 & \text{in } \Omega\ , \\ \Big(A(x,t,\theta)\nabla\theta - g\Big)\cdot\vec{n} = h & \text{on } \partial\Omega\times(0,T)\ , \end{cases} \tag{2.1}$$

where Q_T is the cylinder $\Omega\times(0,T)$, Ω is a bounded open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$, $T>0$, $N\geq 2$ and $\vec{n}$ is the unit outward normal to $\partial\Omega$. Here f, resp. h belong to $L^1(Q_T)$, resp. $L^1(0,T;L^1(\partial\Omega))$, while g is an element of $L^2(Q_T)^N$ and u is a vector field verifying specific conditions.

We are interested in proving the existence and uniqueness results of Equation (2.1). The motivations for this study come from fluid thermo-mechanics, namely Navier-Stokes-Boussinesq type system, which couples the Navier-Stokes equations and the energy conservation equation (2.1) in which appears the term $u\cdot\nabla\theta$ and the nonlinear dissipation that is in general related to $|Du|^2$ (u is the velocity of the fluid) and which belongs to $L^1(Q_T)$.

The main difficulties to handle Equation (2.1) are the Neumann boundary conditions, the presence of the term $u\cdot\nabla\theta$ and the lack of regularity of the right-hand side (f and h belong to L^1). It is well known that parabolic or elliptic equations with L^1 data need an appropriate notion of solution, since there is no uniqueness result concerning the solutions in the sense of distributions (see [98] in the context of steady-state equation). In the present paper, we choose to use the notion of renormalized solutions. The notion of a renormalized solution was introduced in [51, 52] for first-order equations and has been developed for elliptic problems with L^1 data in [85] (see also [86]) and has been extended to the case of bounded measure data in [47]. Concerning parabolic equations with L^1 data it was was adapted in [22, 23, 80] (see also [89] for a definition of renormalized solution to parabolic equation with general measure data and [24] for Stefan problems). The notion of entropy solution (which is equivalent to the notion of a renormalized solution in the L^1 case) developed in [13] may also be used for parabolic equations of type (2.1) (see [94]). There is a wide literature (see among others [26, 85, 86, 13, 47, 89, 22, 23, 90, 24, 49]) on elliptic or parabolic equations with L^1 data and with Dirichlet boundary conditions but comparatively few dealing with Neumann boundary conditions (see e.g. [80, 8, 7, 16]).

In the present paper, we propose a definition of renormalized solution to (2.1). In particular, since the regularity of the solution is only given through the truncates of the solution (see Definition 2.2.3) we need to define a notion of trace (see Definition 2.2.1), in the spirit of [8], to deal with the nonhomogeneous Neumann boundary conditions. We establish the existence of a renormalized solution in Theorem 2.4.1. We mainly use the approach of [24] with Steklov average and appropriate by-part integration Lemmas. The crucial point is to obtain the strong convergence of the truncates of the approximated solution. Under more restrictive assumptions on the matrix field A, namely a local Lipschitz condition, we prove in Theorem 2.6.1 the uniqueness of the renormalized solution. It is worth noting that we cannot have an L^1 contraction principle since the operator $\theta \longmapsto -\operatorname{div}(A(x,t,\theta)\nabla\theta)$ is pseudo-monotone in θ and since the term $\operatorname{div}(g)$ is present. It leads to additional difficulties which are overcome by a specific choice of test functions, a by-part integration formula, and a technical Lemma proved in [50].

The paper is organized as follows: in Section 2 we specify the assumptions on the data and we give the definition of a renormalized solution to Equation (2.1). Section 3 is devoted to the time-derivative, and by-part integration Lemmas, which play a crucial role in this context. In Section 4 we state and prove the existence of a solution (see Theorem 2.4.1). We give some properties of the renormalized solution in Section 5, in particular Boccardo-Gallouët type estimates of the solution in $L^1(0,T;L^r(\Omega))$. In Section 5 we establish the uniqueness of the renormalized solution (Theorem 2.6.1).

2.2 Assumptions and definitions

Let us consider the following nonlinear parabolic equation with non homogeneous Neumann boundaries conditions

$$\begin{cases} \dfrac{\partial\theta}{\partial t} - \operatorname{div}\left(A(x,t,\theta)\nabla\theta\right) + u\cdot\nabla\theta = f - \operatorname{div} g & \text{in } Q_T\ , \\ \theta(t=0) = \theta_0 & \text{in } \Omega\ , \\ \Big(A(x,t,\theta)\nabla\theta - g\Big)\cdot\vec{n} = h & \text{on } \partial\Omega\times(0,T)\ , \end{cases} \tag{2.2}$$

where Q_T is the cylinder $\Omega\times(0,T)$, Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $T>0$, $N\geq 2$ and $\vec{n}$ is the unit outward normal to $\partial\Omega$.

We assume that the matrix A is a Carathéodory function and it satisfies

$$A(x,t,s)\xi\cdot\xi \geq \alpha|\xi|^2,\ \ \forall s\in\mathbb{R}, \forall\xi\in\mathbb{R}^N,\ \text{ for a.e } (x,t) \text{ in } Q_T, \tag{2.3}$$

where $\alpha>0$ is given real number and there exists $\beta>0$ such that

$$|A(x,t,s)|\leq\beta,\ \ \forall s\in\mathbb{R},\ \text{ for a.e } (x,t) \text{ in } Q_T. \tag{2.4}$$

As far as the vector field u is concerned, recalling that u is a solution of the Navier-Stokes equation, we assume that

$$u \in L^2(0,T;H_0^1(\Omega)^N) \cap L^\infty(0,T;L^2(\Omega)^N), \text{ and } \operatorname{div}(u) = 0. \tag{2.5}$$

Finally, we assume that

$$f \in L^1(Q_T), \tag{2.6}$$
$$g \in L^2(Q_T)^N, \tag{2.7}$$
$$h \in L^1(\partial\Omega \times (0,T)), \tag{2.8}$$
$$\theta_0 \in L^1(\Omega). \tag{2.9}$$

In order to deal with L^1 data we denote by T_k, for any $k > 0$, the truncation function at height $\pm k$ (see Figure 2.1) ; $T_k(r) = \max(\min(r,K) - K)$, for any $r \in \mathbb{R}$. We also define $\widetilde{T}_k$ the antiderivative of T_k, vanishing in 0, that is the $W^{2,\infty}(\mathbb{R})$-function $\widetilde{T}_k$ defined by

$$\widetilde{T}_k(r) = \int_0^r T_k(s)\,\mathrm{d}s.$$

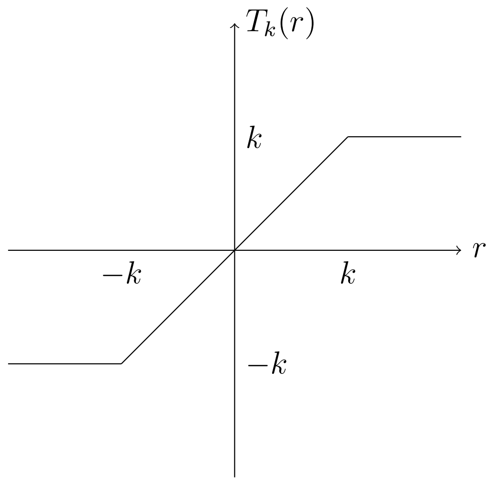

Figure 2.1 – The function T_k

Since the regularity of the renormalized solution is given through the truncated of the solution, we now recall the definitions of the gradient and the trace of functions whose truncated verify appropriate conditions. We follow [13] for the definition of the gradient and [8, 9, 75] for the definition of the trace.

Definition 2.2.1. *Let $\theta : \Omega \times (0,T) \to \mathbb{R}$ be a measurable function such that $T_k(\theta) \in L^2(0,T;H^1(\Omega))$ for every $k > 0$.*

1. *Then there exists a unique measurable function $\psi : Q_T \to \mathbb{R}^N$ such that*

$$\nabla T_K(\theta) = \psi\,\chi_{\{|\theta|<K\}} \quad a.e. \text{ in } Q_T, \tag{2.10}$$

where $\chi_{\{|\theta|<K\}}$ denotes the characteristic function of $\{(x,t) \in Q_T : |\theta(x,t)| < K\}$. We define ψ as the gradient of θ and write $\psi = \nabla\theta$.

2. *If moreover*

$$\sup_{k\geq 1} \frac{1}{k} \|T_k(\theta)\|^2_{L^2(0,T;H^1(\Omega))} < +\infty \tag{2.11}$$

then there exists a unique measurable function $w : \partial\Omega \times (0,T) \to \mathbb{R}$ *such that*

$$\tau\left(T_K(\theta)\right) = T_K(w) \quad \textit{a.e. in } \partial\Omega \times (0,T), \tag{2.12}$$

where $\tau : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ *is the trace operator. We define the function* w *as the trace of* θ *on* $\partial\Omega \times (0,T)$ *and set, formally,*

$$\tau(\theta) = w.$$

Remark 2.2.2. *The gradient* $\nabla\theta$ *is well defined in view of the regularity of* $T_k(\theta)$ *and of the fact that* θ *is finite almost everywhere in* Q_T*:* $Q_T = \cup_{n\in\mathbb{N}}\{|\theta| < n\}$ *(up to a set of zero measure) and* $T_n(T_{n+1}(\theta)) = T_n(\theta)$ *are the main ingredients to justify the existence and the uniqueness of* $\nabla\theta$ *(see [13]).*

To define the trace, assumption (2.11) *and the continuous embedding of* $H^1(\Omega)$ *into* $L^2(\partial\Omega)$ *leads to*

$$k \operatorname{meas}\{(x,t) \in \partial\Omega \times (0,T); |\tau(T_k(\theta))| \geq k\} \leq C,$$

where $C > 0$ *is a constant independent of* k*. Then, up to a set of zero measure,*

$$\partial\Omega \times (0,T) = \bigcup_{n\in\mathbb{N}} \{(x,t) \in \partial\Omega \times (0,T); |\tau(T_k(\theta))| < n\}$$

and the equality $\tau(T_n(T_{n+1}(\theta))) = \tau(T_n(\theta))$ *almost everywhere in* $\partial\Omega \times (0,T)$ *allow one to construct the function* w*.*

As far as the trace is concerned, as noticed in [9, 75] in the stationary case, having only $T_k(\theta) \in L^2(0,T;H^1(\Omega))$ *for any* $k > 0$ *is not sufficient to define a notion of trace for* θ *which is finite almost everywhere in* $\partial\Omega \times (0,T)$*. Indeed if we consider the function* $\theta(x,t) = \ln(x)$ *defined on* $(0,1) \times (0,T)$*, we have* $T_k(\theta) \in L^2(0,T;H^1(0,1))$ *but it is not possible to define a (finite) trace in* $\{0\} \times (0,T)$*. Notice that* $\theta \in L^\infty(0,T;L^1((0,1))$ *(which is the expected regularity of the renormalized solution, see* (2.13)*) and, for any* $k > 1$*,*

$$\int_0^T \int_0^1 |\nabla T_k(\theta)|^2 \, \mathrm{d}x \, \mathrm{d}t \geq (1 - \exp(-k)) \exp(k).$$

Thus imposing a control on $T_k(\theta)$ *is crucial to obtain a finite trace.*

Following [24] we now give the definition of a renormalized solution to Problem (2.2).

Definition 2.2.3. *Let* θ *be a measurable function on* Q_T*. Then* θ *is a renormalized solution of* (2.2) *if*

$$\theta \in L^\infty(0,T;L^1(\Omega)), \tag{2.13}$$

$$T_k(\theta) \in L^2(0,T;H^1(\Omega)), \ \textit{for any } k > 0, \tag{2.14}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{(x,t)\in Q_T:\, n<|\theta(x,t)|<2n\}} A(x,t,\theta)\nabla\theta \cdot \nabla\theta \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{2.15}$$

and for any $S \in W^{2,\infty}(\mathbb{R})$ *such that* S' *has a compact support and any* $\varphi \in C_c^\infty([0,T)\times\bar\Omega)$, θ *satisfies*

$$\begin{aligned}
&-\int_{Q_T} \varphi_t S(\theta)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varphi(0)S(\theta_0)\,\mathrm{d}x + \int_{Q_T} A(x,t,\theta)\nabla\theta\cdot\nabla\theta S''(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{Q_T} S'(\theta)A(x,t,\theta)\nabla\theta\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} u\cdot\nabla\theta S'(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_{Q_T} fS'(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\nabla(S'(\theta)\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} hS'(\theta)\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t
\end{aligned}\tag{2.16}$$

Remark 2.2.4. *Concerning the decay of the truncated energy, notice that using a Cesaro argument (see [73])* (2.15) *is equivalent to*

$$\lim_{n\to+\infty}\frac{1}{n}\int_{\{(x,t)\in Q_T:\,|\theta(x,t)|<n\}} A(x,t,\theta)\nabla\theta\cdot\nabla\theta\,\mathrm{d}x\,\mathrm{d}t = 0.$$

As usual conditions (2.13) *and* (2.14) *allow one to define* $\nabla\theta$ *in the sense of Definition 2.2.1. As far as the trace is concerned it is necessary to combine* (2.13), (2.14), (2.15) *and the Poincaré-Wirtinger inequality to deduce that*

$$\lim_{k\to+\infty}\frac{1}{k}\|T_k(\theta)\|^2_{L^2(0,T;H^1(\Omega))} = 0.$$

It follows that Definition 2.2.1 applies for the trace of θ.

Concerning (2.16) *since* S' *and* S'' *have a compact support we have, for a sufficiently large* k *and in view of assumption* (2.4), (2.5), (2.7) *and* (2.8),

$$\begin{gathered}
A(x,t,\theta)\nabla\theta\cdot\nabla\theta S''(\theta)\varphi = A(x,t,\theta)\nabla T_k(\theta)\cdot\nabla T_k(\theta)S''(\theta)\varphi \in L^1(Q_T),\\
S'(\theta)A(x,t,\theta)\nabla\theta\cdot\nabla\varphi = S'(\theta)A(x,t,\theta)\nabla T_k(\theta)\cdot\nabla\varphi \in L^2(Q_T),\\
u\cdot\nabla\theta S'(\theta)\varphi = u\cdot\nabla T_k(\theta)S'(\theta)\varphi \in L^1(Q_T),\\
g\nabla(S'(\theta)\varphi) = S''(\theta)g\nabla T_k(\theta)\varphi + S'(\theta)g\nabla\varphi T_k(\theta) \in L^1(Q_T),\\
hS'(\theta)\varphi = hS'(\nu(\theta))\varphi \in L^1(\partial\Omega\times(0,T)).
\end{gathered}$$

Since the functions $\varphi_t S(\theta)$, $\varphi(0)S(\theta_0)$ *and* $fS'(\theta)\varphi$ *are integrable, we deduce that all the terms in* (2.16) *are well defined. It is worth noting that the formal Neumann boundary conditions in* (2.2) *are taken into account in* (2.16). *Indeed, for any* $S\in W^{2,\infty}(\mathbb{R})$ *such that* S' *has a compact support and any* $\varphi\in C_c^\infty((0,T)\times\bar\Omega)$, *we have*

$$\begin{aligned}
&\int_{Q_T} S'(\theta)\Big(A(x,t,\theta)\nabla\theta - g\Big)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_{\partial\Omega\times(0,T)} S'(\theta)h\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t\mathrm{d}t \\
&\quad = \int_{Q_T}\varphi_t S(\theta)\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u\cdot\nabla\theta S'(\theta)\varphi\,\mathrm{d}x - \int_{Q_T} A(x,t,\theta)\nabla\theta\cdot\nabla\theta S''(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad + \int_{Q_T} fS'(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} S''(\theta)g\nabla\theta\varphi\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{2.17}$$

which is formally equivalent to $S'(\theta)\Big(A(x,t,\theta)\nabla\theta - g\Big)\cdot\vec n = S'(\theta)h$ *on* $\partial\Omega\times(0,T)$. *Since the trace of* θ *is finite almost everywhere we obtain, formally, the prescribed Neumann boundary conditions.*

As far as the initial condition is concerned, notice that the equation (2.17) *implies that* $S(\theta)_t$ *belongs to* $L^1(0,T;(W^{1,q}(\Omega))')$ *for any* $q>N$. *It follows that* $S(\theta)$ *belongs to* $C^0([0,T];(W^{1,q}(\Omega))')$ *and it gives a sense to the initial condition through* $S(\theta)(t=0)=S(\theta_0)$.

We explicitly remark that is enough to have $\varphi\in W^{1,\infty}(Q_T)$ *with* $\varphi(T)=0$ *in* (2.3.1).

2.3 Time derivative tools

In the whole paper, since we consider Neumann boundary conditions we must be careful with the notion of the time derivative θ_t (or $\partial_t\theta$, or $\frac{\partial\theta}{\partial t}$): here it is not the derivative in the sense of distributions in Q_T but the weak derivative, i.e. if θ belongs to $L^2(0,T;E)$, where E is a Banach space, θ_t is a continuous linear operator acting from $\mathcal{D}(0,T)$ with value into a Banach space E : $\theta_t \in \mathcal{D}'((0,T);E)$. We refer to [105, 96, 68]. More precisely in the present paper we are mainly dealing with two cases.

The first case is the classical framework for equations of type (2.2) with more regular data, that is $u \equiv 0$, $f \in L^2(Q_T)$ and $h \in L^2(0,T;H^{1/2}(\partial\Omega))$. If we define $W(0,T) = \{v \in L^2(0,T;H^1(\Omega), \partial_t v \in L^2(0,T;(H^1(\Omega))')\}$, we are in the case of Gelfand's triple, $H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))'$. Here $L^2(\Omega)$ is the pivot space and, as usual, $L^2(\Omega)$ is identified to its dual space $L^2(\Omega)'$. Gelfdand's tripe then provides time regularity

$$W(0,T) \subset C(0,T;L^2(\Omega))$$

and by-part integration formula: for any $v, w \in W(0,T)$ and for any $0 < t_1 < t_2 < T$ we have

$$\int_\Omega v(x,t_2)w(x,t_2)\,\mathrm{d}x - \int_\Omega v(x,t_1)w(x,t_1)\,\mathrm{d}x = \int_{t_1}^{t_2} \langle \partial_t v, w\rangle_{H',H}\,\mathrm{d}t + \int_{t_1}^{t_2} \langle \partial_t w, v\rangle_{H',H}\,\mathrm{d}t.$$

We will also use the following by-part integration formula:

Lemma 2.3.1. *Let Ω be an open subset of $\mathbb{R}^N$ and Q_T is the cylinder $\Omega\times(0,T)$. Assume that $G:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function such that $\widetilde{G}(0)=0$, that is $\widetilde{G}$ is defined by*

$$\widetilde{G}(r) = \int_0^r G(s)\,\mathrm{d}s.$$

Let $w \in L^2(0,T;H^1(\Omega))$ and $w_0 \in L^\infty(\Omega)$ such that $w_t \in L^2(0,T;(H^1(\Omega))')$ and $w(0)=w_0$ in $(H^1(\Omega))'$. Then

$$-\int_0^T \langle w_t, G(w)\xi\rangle\,\mathrm{d}t = \int_{Q_T} \xi_t \widetilde{G}(w)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \xi(0)\widetilde{G}(w_0)\,\mathrm{d}x,$$

for any $\xi \in W^{1,\infty}(Q_T)$ such that $\xi(T)=0$.

When we consider the parabolic equation (2.2) with L^1 data, the situation is more intricate. The regularity of the solution is given through the truncated and, if S is a $W^{2,\infty}(\mathbb{R})$ function such that $S(0)=0$ and the support of S' is compact, the time derivative $S(\theta)_t$ belongs to $L^1(Q_T) + L^2(0,T;(H^1(\Omega))') + L^1(\partial\Omega\times(0,T))$, in the sense that there exists $f_1 \in L^1(Q_T)$, $g_1 \in (L^2(Q_T))^N$ and $h_1 \in L^1(\partial\Omega\times(0,T))$ such that, for any $\varphi \in \mathcal{C}_c^\infty((0,T)\times\bar{\Omega})$ we have

$$-\int_{Q_T} S(\theta)\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} f_1\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g_1\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} h_1\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t.$$

Notice that the previous formula makes sense for less regular φ, for example $\varphi \in \mathcal{C}_c^1((0,T)\times\bar{\Omega})$. Moreover, the pivot space being $L^2(\Omega)$, concerning the continuity with respect to the time variable we have $S(\theta) \in L^2(0,T;(W^{1,q}(\Omega))')$ and $\partial_t S(\theta) \in L^2(0,T;(W^{1,q}(\Omega))')$ for $q > N$, so that $S(\theta) \in C([0,T];(W^{1,q}(\Omega))')$.

One of the main obstacles in dealing with Equation (2.2) is the lack of regularity of the solution with respect to the time variable. Obtaining compactness and passing to the limit in the renormalized formulation is then more complicated than in the elliptic case. For the analysis of our parabolic problem, we use the approach developed by Blanchard-Poretta in [24] (see also [90]) for a class of Stefan problem with L^1 data and Dirichlet boundary conditions. They give in [24] (see also [9, 7]) weak integration by parts Lemma, see below Lemma 2.3.2 which concerns forward translation while Lemma 2.3.3 concerns backward translation. Coupled with the Steklov average it plays a crucial role in our proofs and gives a shorter proof that the method developed earlier in [23, 25] which uses Landes approximation (in the Dirichlet case).

Lemma 2.3.2 ([24, Lem. 2.1]). *Let Ω be an open subset of $\mathbb{R}^N$ and Q_T is the cylinder $\Omega\times(0,T)$. Assume that $G:\mathbb{R}\to\mathbb{R}$ is a Lipschitz, continuous and nondecreasing functions such that $\widetilde{G}(0)=0$, that is $\widetilde{G}$ is defined by*

$$\widetilde{G}(r)=\int_0^r G(s)\,\mathrm{d}s.$$

Let $w\in L^1(Q_T)$ and $w_0\in L^1(\Omega)$. If $G(w)\in L^\infty(Q_T)$, $\widetilde{G}(w)\in L^1(Q_T)$, $G(w_0)\in L^\infty(\Omega)$ and $\widetilde{G}(w_0)\in L^1(\Omega)$, then

$$\begin{aligned}\int_{Q_T}(w-w_0)\frac{\partial}{\partial t}\left(\xi\frac{1}{h}\int_t^{t+h}G(w(x,\tau))\,\mathrm{d}\tau\right)\mathrm{d}x\,\mathrm{d}t\le\int_{Q_T}\xi_t\bigg(w\frac{1}{h}\int_t^{t+h}G(w)\,\mathrm{d}\tau\\-\frac{1}{h}\int_t^{t+h}G(w)w\,\mathrm{d}\tau+\frac{1}{h}\int_t^{t+h}\widetilde{G}(w)\,\mathrm{d}\tau\bigg)\mathrm{d}x\,\mathrm{d}t+\int_\Omega\xi(0)\widetilde{G}(w_0)\,\mathrm{d}x,\end{aligned}$$

for any $\xi\in W^{1,\infty}(Q_T)$ such that $\xi\geqslant 0$ and $\xi(T)=0$.

Lemma 2.3.3 ([24, Lem. 2.3]). *Let Ω be an open subset of $\mathbb{R}^N$ and Q_T is the cylinder $\Omega\times(0,T)$. Assume that $G:\mathbb{R}\to\mathbb{R}$ is a Lipschitz and non-decreasing function such that $\widetilde{G}(0)=0$, that is $\widetilde{G}$ is defined by*

$$\widetilde{G}(r)=\int_0^r G(s)\,\mathrm{d}s.$$

Let $w\in L^1(Q_T)$ and $w_0\in L^1(\Omega)$, we define $w(x,t)=w_0$ for $t<0$. If $G(w)\in L^\infty(Q_T)$, $\widetilde{G}(w)\in L^1(Q_T)$, $G(w_0)\in L^\infty(\Omega)$, $\widetilde{G}(w_0)\in L^1(\Omega)$, then

$$\begin{aligned}\int_{Q_T}(w-w_0)\frac{\partial}{\partial t}\left(\xi\frac{1}{h}\int_{t-h}^{t}G(w(x,\tau))\,\mathrm{d}\tau\right)\mathrm{d}x\,\mathrm{d}t\ge\int_{Q_T}\xi_t\bigg(w\frac{1}{h}\int_{t-h}^{t}G(w)\,\mathrm{d}\tau\\-\frac{1}{h}\int_{t-h}^{t}G(w)w\,\mathrm{d}\tau+\frac{1}{h}\int_{t-h}^{t}\widetilde{G}(w)\,\mathrm{d}\tau\bigg)\mathrm{d}x\,\mathrm{d}t+\int_\Omega\xi(0)\widetilde{G}(w_0)\,\mathrm{d}x,\end{aligned}$$

for any $\xi\in W^{1,\infty}(Q_T)$ such that $\xi\geqslant 0$ and $\xi(T)=0$.

For our needs, we explicitly state the following by-part integration formula which is an extension of Lemma 2.3.1 under weaken regularity. Such a by-part integration formula has various forms (see, e.g., [1, 38, 24, 9, 7, 64]) and it is crucial for establishing the stability of the solution and the uniqueness Theorem 2.6.1. For the sake of completeness we give the proof.

Lemma 2.3.4. *Let Ω be an open subset of $\mathbb{R}^N$ and Q_T is the cylinder $\Omega\times(0,T)$. Assume that $G:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function such that $\widetilde{G}(0)=0$, that is $\widetilde{G}$ is defined by*

$$\widetilde{G}(r)=\int_0^r G(s)\,\mathrm{d}s.$$

Let $w \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T)$ and $w_0 \in L^\infty(\Omega)$. Assume that there exists $f_1 \in L^1(Q_T)$, $g_1 \in (L^2(Q_T))^N$ and $h_1 \in L^1(\partial\Omega \times (0,T))$ such that, for any $\varphi \in \mathcal{C}_c^\infty([0,T) \times \bar{\Omega})$ we have

$$\begin{aligned} -\int_{Q_T} w\partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varphi(0)w_0 dx \\ = \int_{Q_T} f_1\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g_1 \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} h_1\varphi \,\mathrm{d}\sigma(x)\,\mathrm{d}t. \end{aligned} \tag{2.18}$$

Then

$$\begin{aligned} -\int_{Q_T} \xi_t\widetilde{G}(w)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \xi(0)\widetilde{G}(w_0)\,\mathrm{d}x = \int_{Q_T} f_1\xi G(w)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{Q_T} g_1 \cdot \nabla\Big(\xi G(w)\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} h_1\xi G(w)\,\mathrm{d}\sigma(x)\,\mathrm{d}t. \end{aligned} \tag{2.19}$$

for any $\xi \in W^{1,\infty}(Q_T)$ such that $\xi(T) = 0$.

Proof. Let us assume that G is non decreasing and let us consider $\xi \in C_c^1([0,T) \times \bar{\Omega})$, with $\xi \geq 0$.

The forward Steklov average, denoted by W_h, is defined for $h > 0$ by

$$(x,t) \in \Omega \times [0,T) \mapsto W_h(x,t) = \frac{1}{h}\int_t^{t+h} G(w(x,\tau))\,\mathrm{d}\tau).$$

Since the support of ξ is compact in $[0,T) \times \Omega$, for h small enough, the function $\xi W_h$is well defined, belongs to $H^1(0,T;H^1(\Omega))$ and is such that $(\xi W_h)(T) = 0$. From (2.18) it follows that

$$\begin{aligned} -\int_{Q_T} w\partial_t(\xi W_h)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \xi(0)W_h(0)w_0 dx \\ = \int_{Q_T} f_1\xi W_h\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g_1 \cdot \nabla(\xi W_h)\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} h_1\xi W_h\,\mathrm{d}\sigma(x)\,\mathrm{d}t. \end{aligned} \tag{2.20}$$

By apply Lemma 2.3.2 we also have

$$\begin{aligned} \int_{Q_T} (w - w_0)\frac{\partial}{\partial t}\Big(\xi W_h\Big)\,\mathrm{d}x\,\mathrm{d}t \leq \int_{Q_T} \xi_t\Bigg(wW_h - \frac{1}{h}\int_t^{t+h} G(w)w\,\mathrm{d}\tau \\ + \frac{1}{h}\int_t^{t+h} \widetilde{G}(w)\,\mathrm{d}\tau\Bigg)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \xi(0)\widetilde{G}(w_0)\,\mathrm{d}x. \end{aligned}$$

Since

$$\int_{Q_T} (w - w_0)\frac{\partial}{\partial t}\Big(\xi W_h\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} w\partial_t(\xi W_h)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \xi(0)W_h(0)w_0 dx$$

we obtain that, for any $h > 0$ small enough,

$$\begin{aligned} \int_{Q_T} -f_1\xi W_h\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} g_1 \cdot \nabla(\xi W_h)\,\mathrm{d}x\,\mathrm{d}t - \int_{\partial\Omega\times(0,T)} h_1\xi W_h\,\mathrm{d}\sigma(x)\,\mathrm{d}t \\ \leq \int_{Q_T} \xi_t\Bigg(wW_h - \frac{1}{h}\int_t^{t+h} G(w)w\,\mathrm{d}\tau \\ + \frac{1}{h}\int_t^{t+h} \widetilde{G}(w)\,\mathrm{d}\tau\Bigg)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \xi(0)\widetilde{G}(w_0)\,\mathrm{d}x. \end{aligned} \tag{2.21}$$

We now pass to the limit as h goes to zero. By the properties of the Steklov average we have, as h goes to zero, for any $\eta > 0$

$$W_h \to G(w) \text{ strongly in } L^2(0,T-\eta;H^1(\Omega)),$$
$$W_h \to G(w) \text{ strongly in } L^2(0,T-\eta;H^{1/2}(\partial\Omega)),$$

and since w is bounded

$$W_h \to G(w) \text{ weak-}* \text{ in } L^1(\Omega \times (0,T-\eta)),$$
$$W_h \to G(w) \text{ weak-}* \text{ in } L^1(\partial\Omega \times (0,T-\eta)).$$

Similarly we have

$$\frac{1}{h}\int_t^{t+h} G(w)w \to G(w)w\,\mathrm{d}\tau \text{ weak-}* \text{ in } L^1(\Omega \times (0,T-\eta)),$$
$$\frac{1}{h}\int_t^{t+h} \widetilde{G}(w)\mathrm{d}\tau \to \widetilde{G}(w)\,\mathrm{d}\tau \text{ weak-}* \text{ in } L^1(\Omega \times (0,T-\eta)).$$

Since the support of ξ is compact in $[0,T)\times\bar{\Omega}$ we can pass to the limit in (2.21) and we deduce that

$$\begin{aligned} -\int_{Q_T} f_1\xi G(w)\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} g_1\cdot\nabla\Big(\xi G(w)\Big)\,\mathrm{d}x\,\mathrm{d}t - \int_{\partial\Omega\times(0,T)} h_1\xi G(w)\,\mathrm{d}\sigma(x)\,\mathrm{d}t \\ \le \int_{Q_T}\xi_t\widetilde{G}(w)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \xi(0)\widetilde{G}(w_0)\,\mathrm{d}x. \end{aligned} \tag{2.22}$$

To prove the reverse inequality we have to combine (2.20), the backward Steklov average and Lemma 2.3.3. However defining W_{-h} by

$$(x,t)\in\Omega\times[0,T)\mapsto W_{-h}(x,t) = \frac{1}{h}\int_t^{t-h} G(w(x,\tau))\,\mathrm{d}\tau,$$

needs to extend w for $t<0$. To this end we consider $(w_{0j})_{j\in\mathbb{N}}$ a sequence of $\mathcal{C}_c^\infty(\Omega)$ which converges to w_0 a.e. in Ω and in $L^\infty(\Omega)$ weak-$*$ and we set $w(t) = w_{0j}$ for $t<0$. With similar methods, combining Lemma 2.3.3 and (2.20) allows one to obtain, for $h>0$ small enough,

$$\begin{aligned} &\int_\Omega \xi(0)W_{-h}(0)w_0dx - \int_\Omega \xi(0)W_{-h}(0)w_{0,j}dx \\ &\qquad\int_{Q_T} -f_1\xi W_{-h}\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} g_1\cdot\nabla(\xi W_{-h})\,\mathrm{d}x\,\mathrm{d}t - \int_{\partial\Omega\times(0,T)} h_1\xi W_{-h}\,\mathrm{d}\sigma(x)\,\mathrm{d}t \\ &\qquad\qquad\ge \int_{Q_T}\xi_t\Bigg(wW_{-h} - \frac{1}{h}\int_{t-h}^t G(w)w\,\mathrm{d}\tau \\ &\qquad\qquad\qquad + \frac{1}{h}\int_{t-h}^t \widetilde{G}(w)\,\mathrm{d}\tau\Bigg)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \xi(0)\widetilde{G}(w_{0j})\,\mathrm{d}x. \end{aligned} \tag{2.23}$$

Since

$$\left|\int_\Omega \xi(0)W_{-h}(0)(w_0 - w_{0,j})dx\right| \le C\int_\Omega |w_0 - w_{0,j}|dx$$

we can pass to the limit in (2.23), first as h goes to zero and then as j goes to infinity and we obtain the reverse inequality of (2.22).

It follows that (2.19) holds for any nondecreasing function G and any non negative $\xi \in C_c^1([0,T) \times \bar{\Omega})$. Then (2.19) holds for any $\xi \in C_c^1([0,T) \times \bar{\Omega})$. By density and using the decomposition $G = G_1 + G_2$ (where G_1 is nondecreasing and G_2 is nonincreasing) we deduce that (2.19) holds true for any Lipschitz continous function G and any $\xi \in W^{1,\infty}(Q_T)$ with $\xi(T) = 0$. □

2.4 Existence Results

In this section, we present the proof for the existence of a renormalized solution of (2.2).

Theorem 2.4.1. *Under the assumptions* (2.3)-(2.9)*, there exists a renormalized solution of* (2.2) *in the sense of Definition 2.2.3.*

Proof. The proof is divided into 5 steps. In a standard way, we begin by introducing an approximate problem whose data are smooth enough and converge in some sense to the data. Using the fixed point argument described in Step 1, we prove the existence of a weak solution θ_ε to our approximate problem (2.24). Then, in Step 2, we establish a priori estimates on the weak solution θ_ε and on the truncate $T_k(\theta_\varepsilon)$. These estimates and compactness tools make it possible to show in Step 3, the convergence of θ_ε to a measurable function θ and in Step 4, the strong convergence of $T_k(\theta_\varepsilon)$ to $T_k(\theta)$. Finally, in Step 5, by passing to the limit in the approximate problem we prove that θ is a renormalized solution to (2.2).

Step 1: Approximate problem

For $\varepsilon > 0$. We define the approximate problem

$$\begin{cases} \dfrac{\partial \theta_\varepsilon}{\partial t} - \operatorname{div}\left(A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon\right) + u \cdot \nabla T_{1/\varepsilon}(\theta_\varepsilon) = f^\varepsilon - \operatorname{div}(g) & \text{in } Q_T \ , \\ \theta_\varepsilon(t=0) = \theta_0^\varepsilon & \text{in } \Omega \ , \\ \big(A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon - g\big) \cdot \vec{n} = h^\varepsilon & \text{on } \partial\Omega \times (0,T) \ , \end{cases} \tag{2.24}$$

with $f^\varepsilon = T_{1/\varepsilon}(f)$ and $\theta_0^\varepsilon = T_{1/\varepsilon}(\theta_0)$, and $h^\varepsilon = T_{1/\varepsilon}(h)$.

We then have the following existence result.

Theorem 2.4.2. *Assume that* (2.3)-(2.9) *hold true. Then there exists at least one weak solution* θ_ε *to problem* (2.24)*.*

Proof. The proof relies on a fixed point argument.

Let $\hat{\theta} \in L^2(Q_T)$. Classical arguments (see.e.g [79, 96]) allow one to deduce that there exists a unique θ_ε solution to the problem

$$\begin{cases} \dfrac{\partial \theta_\varepsilon}{\partial t} - \operatorname{div}(A(x,t,\hat{\theta})\nabla\theta_\varepsilon) + \operatorname{div}(uT_{1/\varepsilon}(\hat{\theta})) = f^\varepsilon - \operatorname{div}(g) & \text{in } Q_T \ , \\ \theta_\varepsilon(t=0) = \theta_0^\varepsilon & \text{in } \Omega \ , \\ (A(x,t,\hat{\theta})\nabla\theta_\varepsilon - g) \cdot \vec{n} = h^\varepsilon & \text{on } \partial\Omega \times (0,T) \ , \end{cases} \tag{2.25}$$

such that

$$\theta_\varepsilon \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \tag{2.26}$$

$$\partial_t \theta_\varepsilon \in L^2(0,T;(H^1(\Omega))'), \tag{2.27}$$

$$\theta_\varepsilon(t=0) = \theta_0^\varepsilon. \tag{2.28}$$

and

$$\begin{aligned}\int_0^T \left\langle \frac{\partial \theta_\varepsilon}{\partial t}, \varphi \right\rangle \mathrm{d}t + \int_{Q_T} A(x,t,\hat{\theta}) \nabla\theta_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u T_{1/\varepsilon}(\hat{\theta}) \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ = \int_{Q_T} f^\varepsilon \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} h^\varepsilon \varphi \,\mathrm{d}\sigma(x)\,\mathrm{d}t \qquad \forall \varphi \in L^2(0,T;H^1(\Omega)).\end{aligned} \tag{2.29}$$

It follows that we can consider the mapping $\Gamma^\varepsilon : L^2(Q_T) \to L^2(Q_T)$ defined by

$$\Gamma^\varepsilon(\hat{\theta}) = \theta_\varepsilon, \qquad \forall \hat{\theta} \in L^2(Q_T),$$

where θ_ε is the unique solution of (2.25), verifying (2.26), (2.27) and (2.29). In the sequel, we will show that Γ^ε is compact, continuous and there exists a ball B of $L^2(Q_T)$ such that $\Gamma^\varepsilon(B) \subset B$.

Γ^ε **is compact:** let us consider a bounded sequence $\hat{\theta}_n$ in $L^2(Q_T)$ and let $\theta_n = \Gamma^\varepsilon(\hat{\theta}_n)$ (we omit the dependence in ε). By the definition of Γ^ε, for a fixed $n \geq 1$, the function θ_n is the unique solution of the problem (2.25) with θ_n in place of θ_ε such that (2.29) holds true. Choosing $\varphi = \theta_n$ as a test function in (2.29), we obtain from assumption (2.3) and by-part integration lemma 2.3.1 that, for a.e. $t \in (0,T)$,

$$\begin{aligned}\frac{1}{2}\int_\Omega |\theta_n(t)|^2 \,\mathrm{d}x + \alpha \int_0^t \int_\Omega |\nabla\theta_n|^2 \,\mathrm{d}x\,\mathrm{d}s \leq \int_0^t \int_\Omega |f^\varepsilon \theta_n| \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega |u T_{1/\varepsilon}(\hat{\theta}_n) \nabla\theta_n| \,\mathrm{d}x\,\mathrm{d}s \\ + \int_0^t \int_\Omega |g\nabla\theta_n| \,\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_\Omega |\theta_n(0)|^2 \,\mathrm{d}x + \int_0^t \int_{\partial\Omega} |h^\varepsilon \theta_n| \,\mathrm{d}\sigma(x)\,\mathrm{d}s.\end{aligned}$$

Now using Hölder and Young inequalities we get

$$\begin{aligned}\frac{1}{2}\int_\Omega |\theta_n(t)|^2 \,\mathrm{d}x + \alpha \int_0^t \int_\Omega |\nabla\theta_n|^2 \,\mathrm{d}x\,\mathrm{d}s \leq \frac{1}{2}\|f^\varepsilon\|^2_{L^2(Q_T)} + \frac{1}{2}\|\theta_n\|^2_{L^2(Q_T)} + \frac{1}{2\beta}\|u T_{1/\varepsilon}(\hat{\theta}_n)\|^2_{L^2(Q_T)} \\ + \frac{\beta}{2}\|\nabla\theta_n\|^2_{L^2(Q_T)} + \frac{1}{2}\|\theta_n^\varepsilon(0)\|^2_{L^2(\Omega)} + \frac{1}{2\beta}\|g\|^2_{L^2(Q_T)} + \frac{\beta}{2}\|\theta_n\|^2_{L^2(0,T;H^1(\Omega))} \\ + \frac{1}{2\beta}\|h^\varepsilon\|_{L^2(\partial\Omega\times(0,T))} + \frac{C(\Omega)\beta}{2}\|\theta_n\|^2_{L^2(0,T;H^1(\Omega))}\end{aligned}$$

where $\beta > 0$ and where $C(\Omega) > 0$ is such that

$$\forall v \in H^1(\Omega), \quad \int_{\partial\Omega} v^2 \,\mathrm{d}\sigma(x) \leq C(\Omega)\|v\|^2_{H^1(\Omega)}.$$

In view of the definition of f^ε, θ_0^ε and h^ε and recalling that $u \in L^2(Q_T)^2$, choosing β such that $\alpha - \frac{\beta}{2}(2 + C(\Omega)) > 0$ implies that for a.e. $t \in (0,T)$

$$\int_\Omega |\theta_n(t)|^2 \,\mathrm{d}x + \left(\alpha - \frac{\beta}{2}(2 + C(\Omega))\right) \int_0^t \int_\Omega |\nabla\theta_n|^2 \,\mathrm{d}x\,\mathrm{d}s \leq C_1 + C_2 \int_0^t \int_\Omega |\theta_n|^2 \,\mathrm{d}x\,\mathrm{d}s, \tag{2.30}$$

where C_1 and C_2 are independent of n. By the Gronwall lemma and since the second term of the left-hand-side of (2.30) is positive, we deduce that there exists $C_3 > 0$ independent of n, such that for a.e. $t \in (0,T)$,

$$\int_\Omega |\theta_n(t)|^2 \,\mathrm{d}x \le C_3. \tag{2.31}$$

It follows from (2.31) and (2.30) that

$$\theta_n \text{ is bounded in } L^2(0,T;H^1(\Omega)). \tag{2.32}$$

On the other hand in view of equation (2.29) we deduce that

$$\frac{\partial \theta_n}{\partial t} \text{ is bounded in } L^2(0,T;H^1(\Omega)'). \tag{2.33}$$

Then by (2.32), (2.33) and by using Aubin-Simon Lemma (see e.g. [99]) we get that θ_n belongs to a compact set of $L^2(Q_T)$.

Γ^ε **is continuous:** let $\hat{\theta}_n$ and $\hat{\theta} \in L^2(Q_T)$ such that $\hat{\theta}_n \to \hat{\theta}$ strongly in $L^2(Q_T)$. Let θ_n belongs to $L^2(Q_T)$ such that $\theta_n = \Gamma^\varepsilon(\hat{\theta}_n)$. Proceeding as in the compactness of Γ^ε, we get that θ_n is bounded in $L^2(0,T;H^1(\Omega))$ and $\dfrac{\partial \theta_n}{\partial t}$ is bounded in $L^2(0,T;H^1(\Omega)')$, as a consequence, there exists a subsequence (still denoted by θ_n), a measurable function θ_ε such that, as n goes to infinity,

$$\theta_n \rightharpoonup \theta_\varepsilon \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{2.34}$$

$$\frac{\partial \theta_n}{\partial t} \rightharpoonup \frac{\partial \theta_\varepsilon}{\partial t} \quad \text{weakly in } L^2(0,T;H^1(\Omega)'), \tag{2.35}$$

$$\theta_n \to \theta_\varepsilon \quad \text{strongly in } L^2(Q_T), \tag{2.36}$$

$$\theta_n \to \theta_\varepsilon \quad \text{a.e. in } Q_T. \tag{2.37}$$

To get the continuity of Γ^ε we need to show that $\theta_\varepsilon = \Gamma^\varepsilon(\hat{\theta})$. Indeed, since up to a subsequence (still denoted by $\hat{\theta}_n$) $\hat{\theta}_n \to \hat{\theta}$ a.e. in Q_T and due to the assumptions on A, we have by the Lebesgue dominated theorem that

$$A(x,t,\hat{\theta}_n) \to A(x,t,\hat{\theta})$$

a.e in Q_T and in $L^\infty(Q_T)$ weak-$*$, as $n \to +\infty$. From (2.34) it follows that

$$\int_0^t \int_\Omega A(x,t,\hat{\theta}_n)\nabla\theta_n \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}s \to \int_0^t \int_\Omega A(x,t,\hat{\theta})\nabla\theta_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}s,$$

as $n \to +\infty$, for all φ in $L^2(0,T;H^1(\Omega))$. Moreover, taking into account that $T_{1/\varepsilon}(\hat{\theta}_n) \to T_{1/\varepsilon}(\hat{\theta})$ a.e in Q_T and $L^\infty(Q_T)$ weak-$*$, we obtain that

$$\int_0^t \int_\Omega u T_{1/\varepsilon}(\hat{\theta}_n)\nabla\varphi \,\mathrm{d}x\,\mathrm{d}s \to \int_0^t \int_\Omega u T_{1/\varepsilon}(\hat{\theta})\nabla\varphi \,\mathrm{d}x\,\mathrm{d}s, \qquad \forall \varphi \in L^2(0,T;H^1(\Omega)).$$

Then, passing to the limit in (2.29) (with θ_n in place of θ_ε and $\hat{\theta}_n$ in place of $\hat{\theta}$) we get that θ_ε satisfies (2.29). Since there exists a unique weak solution to (2.29) we obtain that the whole sequence θ_n converges to θ_ε in $L^2(Q_T)$ such that $\theta_\varepsilon = \Gamma^\varepsilon(\hat{\theta})$. It follows that Γ^ε is continuous.

There exists a ball B of $L^2(Q_T)$ such that $\Gamma^\varepsilon(B) \subset B$: let $\hat{\theta} \in B_{L^2(Q_T)}(0,R)$. From (2.31) we have

$$\|\Gamma^\varepsilon(\hat{\theta})\|_{L^\infty(0,T;L^2(\Omega))} \leq C_4$$

where $C_4 > 0$ is independent of $\hat{\theta}$. Thus clearly, Γ preserves the ball of $L^2(Q_T)$ of center 0 and radius TC_4.
For a fixed $\varepsilon > 0$, the Leray-Schauder fixed point theorem allows one to conclude that there exists at least one weak solution to (2.24). □

Step 2: a priori estimates

In this step we prove that there exists $C_5 > 0$, independent of ε and k, such that

$$\|\theta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C_5 \tag{2.38}$$

$$\|T_k(\theta_\varepsilon)\|^2_{L^2(0,T;H^1(\Omega))} \leq C_5(k+1), \quad \forall k > 0. \tag{2.39}$$

Moreover, we prove that θ_ε verifies

$$\lim_{n\to+\infty} \sup_\varepsilon \frac{1}{n}\int_{\{n\leq|\theta_\varepsilon|\leq 2n\}} |\nabla\theta_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.40}$$

We firstly prove (2.38) by a standard method: taking $T_1(\theta_\varepsilon)$ as a test function in (2.29) and by integrating on $(0,t)$ we have for a.e. $t \in (0,T)$

$$\begin{aligned}\int_\Omega \widetilde{T}_1(\theta_\varepsilon)(t)\,\mathrm{d}x + \int_0^t\int_\Omega A(x,s,\theta_\varepsilon)\nabla T_1(\theta_\varepsilon)\cdot\nabla T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s \\ + \int_0^t\int_\Omega u\cdot\nabla T_{1/\varepsilon}(\theta_\varepsilon)T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega \widetilde{T}_1(\theta_0^\varepsilon)\,\mathrm{d}x \\ + \int_0^t\int_\Omega f^\varepsilon T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega g\nabla T_1(\theta_\varepsilon)\,\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\partial\Omega} h^\varepsilon T_1(\theta_\varepsilon)\,\mathrm{d}\sigma(x)\,\mathrm{d}s.\end{aligned} \tag{2.41}$$

Since $\mathrm{div}(u) = 0$ and $u \in L^2(0,T;H_0^1(\Omega)^N)$ we have

$$\int_0^t\int_\Omega u\cdot\nabla T_{1/\varepsilon}(\theta_\varepsilon)T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_\Omega u\cdot\nabla(\widetilde{H}(\theta_\varepsilon))\,\mathrm{d}x\,\mathrm{d}s = 0, \tag{2.42}$$

where $\widetilde{H}(r) = \int_0^r \mathbb{1}_{\{|s|<1/\varepsilon\}}T_1(s)\,\mathrm{d}s$. Using the ellipticity assumption (2.3) on the matrix A and Young inequality we obtain, for a.e. $t \in (0,T)$,

$$\int_\Omega \widetilde{T}_1(\theta_\varepsilon)(t)\,\mathrm{d}x \leq C\Big(\|\theta_0^\varepsilon\|_{L^1(\Omega)} + \|f^\varepsilon\|_{L^1(Q_T)} + \|g\|^2_{L^2(Q_T)} + \|h^\varepsilon\|_{L^1(\partial\Omega\times(0,T))}\Big) \tag{2.43}$$

where $C > 0$ is a positive constant independent on ε. Since $|r| \leq \frac{1}{2} + \widetilde{T}_1(r)$ for any $r \in \mathbb{R}$, in view of the definition of f^ε, θ_0^ε and h^ε we deduce that (2.38) holds.

We now turn to estimate (2.39). Similarly using the test function $T_k(\theta_\varepsilon)$ instead of $T_1(\theta_varepsilon)$ we obtain

$$\|\nabla T_k(\theta_\varepsilon)\|^2_{L^2(Q_T)} \leq C(k+1)$$

where $C > 0$ is independent of ε and k. To obtain (2.39) is sufficient to remark that

$$\|T_k(\theta_\varepsilon)\|^2_{L^2(Q_T)} \leq k\|T_k(\theta_\varepsilon)\|_{L^1(Q_T)} \leq kT\|\theta_\varepsilon\|_{L^\infty(L^1(\Omega))}.$$

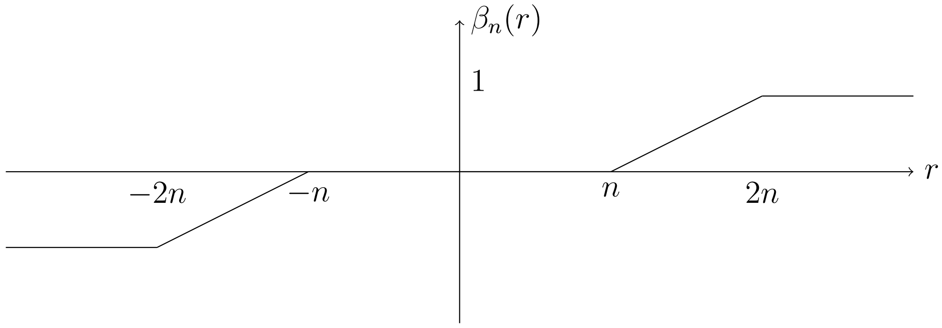

Figure 2.2 – The function β_n

We now turn to (2.40). Let us define $\beta_n(r) = \dfrac{1}{n}(T_{2n}(r) - T_n(r))$ and $\widetilde{\beta}_n(r) = \int_0^r \beta(s)\,\mathrm{d}s$.

As for (2.38) and (2.39), we deduce by taking $\beta_n(\theta_\varepsilon)$ as a test function in (2.25) and by using property (2.42), that, for a.e. $t \in (0,T)$,

$$\int_\Omega \widetilde{\beta}_n(\theta_\varepsilon)(t)\,\mathrm{d}x + \int_0^t\int_\Omega A(x,s,\theta_\varepsilon)\nabla\beta_n(\theta_\varepsilon)\cdot\nabla\beta_n(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega \widetilde{\beta}_n(\theta_0^\varepsilon)\,\mathrm{d}x + \int_0^t\int_\Omega f^\varepsilon\beta_n(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega g\nabla\beta_n(\theta_\varepsilon)\,\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\partial\Omega} h^\varepsilon\beta_n(\theta_\varepsilon)\,\mathrm{d}\sigma(x)\,\mathrm{d}s. \tag{2.44}$$

In view of the definition of β_n since $\tilde{\beta}_n \geq 0$ and using the ellipticity assumption (2.3) on the matrix A and Young inequality we obtain by integrating on $(0,T)$

$$\frac{1}{n}\int_{\{n\le|\theta_\varepsilon|\le 2n\}} |\nabla\theta_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \le C\Big(\int_{\{|\theta_0^\varepsilon|\geqslant n\}}|\theta_0^\varepsilon|\,\mathrm{d}x + \int_{\{|\theta_\varepsilon|\geqslant n\}}|f^\varepsilon|\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{n}\int_{Q_T}|g|^2\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)}|h^\varepsilon|\mathbb{1}_{\{|\theta_\varepsilon|\ge n\}}\,\mathrm{d}\sigma(x)\,\mathrm{d}s\Big) \tag{2.45}$$

According to the definition of θ_0^ε we have $|\theta_0^\varepsilon| \leqslant |\theta_0|$. Since θ_0 lies in $L^1(\Omega)$ we obtain

$$\begin{aligned} 0 \le \lim_{n\to+\infty}\sup_\varepsilon \operatorname{meas}\{x\in\Omega; |\theta_0^\varepsilon(x)| > n\} &\le \lim_{n\to+\infty}\sup_\varepsilon \operatorname{meas}\{x\in\Omega; |\theta_0| > n\} \\ &\le \lim_{n\to+\infty}\sup_\varepsilon \frac{1}{n}\int_\Omega |\theta_0|\,\mathrm{d}x = 0. \end{aligned} \tag{2.46}$$

On the other hand, θ_ε is in $L^\infty(0,T;L^1(\Omega))$, then

$$\lim_{n\to+\infty}\sup_\varepsilon \operatorname{meas}\{(x,t)\in Q_T; |\theta_\varepsilon(x,t)| > n\} = 0. \tag{2.47}$$

Hence, by the equi-integrability of f^ε, we get

$$\lim_{n\to+\infty}\sup_\varepsilon \int_0^T\int_{\{|\theta_\varepsilon|\geqslant n\}} |f^\varepsilon|\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.48}$$

Concerning the integral on $\partial\Omega\times(0,T)$, estimate (2.39) gives, for any $k > 0$

$$\|\tau(T_k(\theta_\varepsilon))\|^2_{L^2(0,T;H^{1/2}(\partial\Omega))} \le C_5(k+1)$$

and then, with the continuous embedding $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$, for any $k > 0$

$$\|\tau(T_k(\theta_\varepsilon))\|^2_{L^2(\partial\Omega\times(0,T))} \leq C_5(k+1).$$

It follows that

$$\lim_{n\to+\infty} \sup_\varepsilon \operatorname{meas}\{(x,t) \in \partial\Omega \times (0,T); |\tau(\theta_\varepsilon)| \geq n\} = 0.$$

Hence, by the equi-integrability of h^ε we get

$$\lim_{n\to+\infty} \sup_\varepsilon \int_{\partial\Omega\times(0,T)} |h^\varepsilon| \mathbb{1}_{\{|\theta_\varepsilon|\geq n\}} \,\mathrm{d}\sigma(x)\,\mathrm{d}s = 0.$$

Since it is clear that

$$\lim_{n\to+\infty} \frac{1}{n} \int_{Q_T} |g|^2 \,\mathrm{d}x\,\mathrm{d}t$$

we can conclude that

$$\lim_{n\to+\infty} \sup_\varepsilon \frac{1}{n} \int_{\{n\leq|\theta_\varepsilon|\leq 2n\}} |\nabla\theta_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.49}$$

Remark 2.4.3. *We explicitly remark that the weak solution θ_ε of* (2.25) *is also a solution in the renormalized sense. The regularities* (2.13)–(2.15) *of θ_ε hold true and we have for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support and any $\varphi \in C_c^\infty([0,T) \times \bar{\Omega})$*

$$\begin{aligned}
&- \int_{Q_T} \varphi_t S(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varphi(0) S(\theta_0^\varepsilon)\,\mathrm{d}x + \int_{Q_T} A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon \cdot \nabla\theta_\varepsilon S''(\theta_\varepsilon)\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{Q_T} S'(\theta_\varepsilon) A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon \cdot \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} u \cdot \nabla T_{1/\varepsilon}(\theta_\varepsilon) S'(\theta_\varepsilon)\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q_T} f^\varepsilon S'(\theta_\varepsilon)\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\nabla(S'(\theta_\varepsilon)\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} h^\varepsilon S'(\theta_\varepsilon)\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t.
\end{aligned} \tag{2.50}$$

With the help of Lemma 2.3.4 and using, for any $t \in (0,T)$, a approximation of $\mathbb{1}_{[0,t]}$, it is possible to recover (2.38), (2.39) *and* (2.40). *It is important to obtain stability result for renormalized solutions of equation* (2.2) *(see Remark 2.4.4).*

Step 3: Convergence of θ_ε

We show in this step, using a method developed in [19, 25] (see e.g. [19, Lemma 2]), that there exists a measurable function θ finite a.e. in Q_T such that, up to a subsequence still indexed by ε,

$$T_k(\theta_\varepsilon) \rightharpoonup T_k(\theta) \text{ in } L^2(0,T;H^1(\Omega)),\ \forall k > 0, \tag{2.51}$$

$$\theta_\varepsilon \to \theta \text{ a.e. in } Q_T. \tag{2.52}$$

Let H_k be a smooth approximation of the function T_k, that is an increasing function of $C^\infty(\mathbb{R})$ defined by

$$H_k(r) = \begin{cases} r & \text{for} \quad |r| \leq \dfrac{k}{2}, \\ k\operatorname{sgn}(r) & \text{for} \quad |r| \geq k. \end{cases} \tag{2.53}$$

For a fixed $k > 0$, the sequence $H_k(\theta_\varepsilon)$ satisfies independently of ε,

$$H_k(\theta_\varepsilon) \text{ is bounded in } L^2(0,T;H^1(\Omega)), \tag{2.54}$$

Concerning the time derivative we have for any $\varphi \in C_c^\infty([0,T) \times \bar{\Omega})$

$$-\int_{Q_T} \varphi_t H_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varphi(0) H_k(\theta_0^\varepsilon)\,\mathrm{d}x = -\int_{Q_T} A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon \cdot \nabla\theta_\varepsilon H_k''(\theta_\varepsilon)\varphi\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_{Q_T} H_k'(\theta) A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon \cdot \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u \cdot \nabla T_{1/\varepsilon}(\theta_\varepsilon) H_k'(\theta_\varepsilon)\varphi\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{Q_T} f H_k'(\theta_\varepsilon)\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\nabla(H_k'(\theta_\varepsilon)\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} h H_k'(\theta_\varepsilon)\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t. \quad (2.55)$$

Indeed, we note that $DH_k(\theta_\varepsilon) = H_k'(\theta_\varepsilon) DT_k(\theta_\varepsilon)$, from (2.38) and (2.39) we get that $H_k(\theta_\varepsilon)$ is bounded in $L^2(0,T;H^1(\Omega))$. For (2.55), we multiply the equation in θ_ε of (2.24) by $H_k'(\theta^\varepsilon)$.

From all the previous estimates and (2.55) we deduce that $\partial_t H_k(\theta_\varepsilon)$ lies in a bounded subset of $L^1(0,T;W^{1,s}(\Omega)')$ for $s > N$. From (2.54) and the compact embedding $H^1(\Omega) \Subset L^2(\Omega)$, using the fact that L^2 is the pivot space we are in the situation $H^1(\Omega) \Subset L^2(\Omega) \subset W^{1,s}(\Omega)'$. We can use Aubin Simon's compactness results (see [99], Corollary 4) that is: for any $k > 0$, $H_k(\theta_\varepsilon)$ lies in a compact set of $L^2(Q_T)$. Thus, by a diagonal process (k being a natural number) there exists a measurable function η_k defined on Q_T for any $k > 0$ such that, up to a subsequence still indexed by ε, $H_k(\theta_\varepsilon) \to \eta_k$ a.e. in Q_T as $\varepsilon \to 0$. Due to the specific choice of the function H_k, we deduce that for any $k > 0$ (k being a natural number)

$$T_k(\theta_\varepsilon) \to T_k(\eta_k) \text{ a.e. in } Q_T, \text{ as } \varepsilon \to 0. \quad (2.56)$$

Since if $k' < k$, $T_{k'}(T_k(\theta_\varepsilon)) = T_{k'}(\theta_\varepsilon)$ we have $\eta_{k'} = T_{k'}(\eta_k)$, so that $\eta_{k'} = \eta_k$ a.e. in $\{|\eta_k| \le k'\}$. It allows to construct a measurable function, denoted by θ, defined in Q_T with value in $\bar{\mathbb{R}}$ such that, for any $k > 0$

$$\theta_\varepsilon \to \theta \quad \text{a.e. in } Q_T, \text{ as } \varepsilon \to 0. \quad (2.57)$$

By remarking that θ_ε is bounded in $L^\infty(0,T;L^1(\Omega))$ we obtain by Fatou lemma that θ is finite almost everywhere in Q_T and the convergence holds for any $k > 0$ (and not only for natural numbers).

We now turn to the convergence of the trace of θ_ε. We firstly remark that due to estimates (2.38) and (2.39) we know that θ is finite a.e. in Q_T and that

$$\frac{1}{k}\|T_k(\theta)\|_{L^2(0,T;H^1(\Omega))} \le C,$$

where $C > 0$ is independent of k. Then the trace of θ is well-defined in the sense of Definition 2.2.1.

We then prove that the trace of $T_k(\theta_\varepsilon)$ is a Cauchy sequence in $L^2(\partial\Omega \times (0,T))$. Recalling that the injection from $H^1(\Omega)$ into $L^2(\partial\Omega)$ is compact, Lions Lemma implies that, for any $\delta > 0$, there exists $C(\delta)$ verifying for any $v \in L^2(H^1(\Omega))$

$$\|\tau(v)\|_{L^2(\partial\Omega)} \le \delta\|\nabla v\|_{L^2(\Omega)} + C(\delta)\|v\|_{L^2(\Omega)}.$$

Taking $v = T_k(\theta_\varepsilon) - T_k(\theta_{\varepsilon'})$ which belongs to $L^2(0,T;H^1(\Omega))$ and integrating the previous inequality on $(0,T)$ and it follows that, for any $\delta > 0$, there exists $C(\delta)$ verifying for any ε and ε'

$$\|\tau(T_k(\theta_\varepsilon)) - \tau(T_k(\theta_{\varepsilon'}))\|_{L^2(\partial\Omega\times(0,T))} \le \delta\|\nabla T_k(\theta_\varepsilon) - \nabla T_k(\theta_{\varepsilon'})\|_{L^2(Q_T)}$$
$$+ C(\delta)\|T_k(\theta_\varepsilon) - T_k(\theta_{\varepsilon'})\|_{L^2(Q_T)}.$$

In view of estimates (2.39) and the point-wise convergence of θ_ε in Q_T we obtain that $\tau(T_k(\theta_\varepsilon))$ is a Cauchy sequence in $L^2(\partial\Omega \times (0,T))$. The weak convergence of $T_k(\theta_\varepsilon)$ in $L^2(0,T;H^1(\Omega))$ then allows one to deduce that $\tau(T_k(\theta_\varepsilon))$ converges to $\tau(T_k(\theta))$ in $L^2(\partial\Omega \times (0,T))$. Up to the expense of extracting subsequences (and with the diagonal process) we obtain that

$$\tau(\theta_\varepsilon) \to \tau(\theta) \quad \text{a.e. in } \partial\Omega \times (0,T), \text{ as } \varepsilon \to 0. \tag{2.58}$$

Step 4: Convergence of truncated energies

This step aims to show that for every $k > 0$,

$$T_k(\theta_\varepsilon) \to T_k(\theta) \text{ strongly in } L^2(0,\tau;H^1(\Omega)) \text{ for any } 0 < \tau < T. \tag{2.59}$$

The main tool we use in the proof of (2.59) is the Steklov time-average for functions $z : (0,T) \times \Omega \to \mathbb{R}$. More precisely, for $\eta \in (0,T)$ we define for every $h \in (0,\eta)$ the functions $z_h : (0,T-\eta) \times \Omega \to \mathbb{R}$ and $z_{-h} : [\eta,T] \times \Omega \to \mathbb{R}$ by

$$z_h(t,x) = \frac{1}{h}\int_t^{t+h} z(s,x)\,\mathrm{d}s \text{ and } z_{-h}(t,x) = \frac{1}{h}\int_{t-h}^{t} z(s,x)\,\mathrm{d}s. \tag{2.60}$$

Note that for every $z \in L^2(0,T;H^1(\Omega))$ we have $z_h \in H^1(0,T-\eta;H^1(\Omega))$. In addition, for all $\varphi \in C_c^\infty([0,T))$

$$\varphi z_h \to \varphi z \text{ strongly in } L^2(0,T;H^1(\Omega)) \text{ as } h \to 0. \tag{2.61}$$

At the expense of defining an extension of $z(t)$ for $t < 0$, by a function belonging to $H^1(\Omega)$, the function z_{-h} is defined on Q_T and we have similar results:

$$\varphi z_{-h} \to \varphi z \text{ strongly in } L^2(0,T;H^1(\Omega)) \text{ as } h \to 0. \tag{2.62}$$

Mixing Steklov average and renormalized solution for parabolic equations with L^1 data was introduced by Blanchard-Porretta in [24] (see also [90]). Coupling to the weak by-part integration formulas of Lemmas 2.3.2 and 2.3.3 it allows to prove the strong convergence (2.59).

First, let us take $\varphi = \xi$ such that $\xi \in C_c^\infty[0,T)$. By using $T_k(\theta_\varepsilon)\varphi$ in (2.29) and by-part integration Lemma 2.3.4 we get

$$\begin{aligned}
&-\int_{Q_T} \xi_t \widetilde{T}_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \xi(0)\widetilde{T}_k(\theta_0^\varepsilon)\,\mathrm{d}x + \int_{Q_T} A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon \cdot \nabla T_k(\theta_\varepsilon)\xi\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{Q_T} T_k(\theta_\varepsilon)A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon \cdot \nabla\xi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} u \cdot \nabla T_{1/\varepsilon}(\theta_\varepsilon)T_k(\theta_\varepsilon)\xi\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q_T} f^\varepsilon T_k(\theta_\varepsilon)\xi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\nabla(T_k(\theta_\varepsilon)\xi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} h^\varepsilon T_k(\theta_\varepsilon)\xi\,\mathrm{d}\sigma(x)\,\mathrm{d}t.
\end{aligned} \tag{2.63}$$

Since ξ is independent of x and $\mathrm{div}(u) = 0$, the fourth and the fifth term of (2.63) are equal to zero. Now we pass to the limit in (2.63) as ε goes to zero. Due to the Boccardo-Gallouët estimates (see [26] and Lemma 2.5.2) and the almost everywhere convergence of θ_ε to θ in Q_T, we have as a consequence of Vitali's Theorem that

$$\theta_\varepsilon \to \theta \text{ stronly in } L^p(Q_T), \tag{2.64}$$

for $1 \leq p < 1 + \frac{2}{N}$ as ε goes to zero. In addition since θ_0^ε converges to θ_0 strongly in $L^1(\Omega)$, then

$$\lim_{\varepsilon\to 0} -\int_{Q_T} \xi_t \widetilde{T}_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \xi(0)\widetilde{T}_k(\theta_0^\varepsilon)\,\mathrm{d}x = -\int_{Q_T} \xi_t \widetilde{T}_k(\theta)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \xi(0)\widetilde{T}_k(\theta_0)\,\mathrm{d}x \qquad (2.65)$$

Moreover because f is in $L^1(Q_T)$ the Lebesgue dominated Theorem implies that

$$\lim_{\varepsilon\to 0} \int_{Q_T} f^\varepsilon T_k(\theta_\varepsilon)\xi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} f T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t. \qquad (2.66)$$

Similarly, by the point-wise convergence of the trace of θ_ε and the definition of h^ε we have

$$\lim_{\varepsilon\to 0} \int_{\partial\Omega\times(0,T)} h^\varepsilon T_k(\theta_\varepsilon)\xi\,\mathrm{d}\sigma(x)\,\mathrm{d}t = \int_{\partial\Omega\times(0,T)} h T_k(\theta)\xi\,\mathrm{d}\sigma(x)\,\mathrm{d}t. \qquad (2.67)$$

At last the weak convergence (2.51) of $T_k(\theta_\varepsilon)$ gives that

$$\lim_{\varepsilon\to 0} \int_{Q_T} g\nabla(T_k\theta_\varepsilon)\xi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} g\nabla T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t \qquad (2.68)$$

It follows from (2.51), (2.65), (2.66), (2.67) and (2.68) that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{Q_T} A(x,t,\theta_\varepsilon)\nabla T_k(\theta_\varepsilon)\cdot\nabla T_k(\theta_\varepsilon)\xi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{Q_T} \xi_t \widetilde{T}_k(\theta)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \xi(0)\widetilde{T}_k(\theta_0)\,\mathrm{d}x + \int_{Q_T} f T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t \qquad (2.69)\\
&\quad\quad + \int_{Q_T} g\nabla T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} h T_k(\theta)\xi\,\mathrm{d}\sigma(x)\,\mathrm{d}t.
\end{aligned}$$

We now prove by passing to the limit in the equation and with the help of Steklov average that

$$\begin{aligned}
&\int_{Q_T} A(x,t,\theta)\nabla T_k(\theta)\cdot\nabla T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \geq \int_{Q_T} \xi_t \widetilde{T}_k(\theta)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \xi(0)\widetilde{T}_k(\theta_0)\,\mathrm{d}x + \int_{Q_T} f T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad + \int_{Q_T} g\nabla T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} h T_k(\theta)\xi\,\mathrm{d}\sigma(x)\,\mathrm{d}t.
\end{aligned}$$

We first define the functions S_n, for all $r \in \mathbb{R}$, by

$$S_n'(r) = 1 - \frac{1}{n}|T_{2n}(r) - T_n(r)|, \qquad S_n(r) = \int_0^r S_n'(s)\,\mathrm{d}s.$$

By taking $S(r) = S_n(r)$ in (2.50) and by using $\mathrm{div}(u) = 0$ we obtain, for any $\varphi \in C_c^\infty([0,T)\times\Omega)$,

$$\begin{aligned}
-\|\varphi\|_{L^\infty(Q_T)}&\frac{1}{n}\int_0^T \int_{\{n\leq|\theta_\varepsilon|\leq 2n\}} |A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\theta_\varepsilon|\,\mathrm{d}x\,\mathrm{d}t \\
&\leq -\int_{Q_T} \varphi_t S_n(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varphi(0)S_n(\theta_0^\varepsilon)\,\mathrm{d}x \\
&\quad + \int_{Q_T} S_n'(\theta_\varepsilon)A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} u\cdot\widetilde{N}^\varepsilon(\theta_\varepsilon)\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \qquad (2.70)\\
&\quad - \int_{Q_T} f^\varepsilon S_n'(\theta_\varepsilon)\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} g\nabla(S_n'(\theta_\varepsilon)\varphi)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{\partial\Omega\times(0,T)} h^\varepsilon S_n'(\theta_\varepsilon)\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t.
\end{aligned}$$

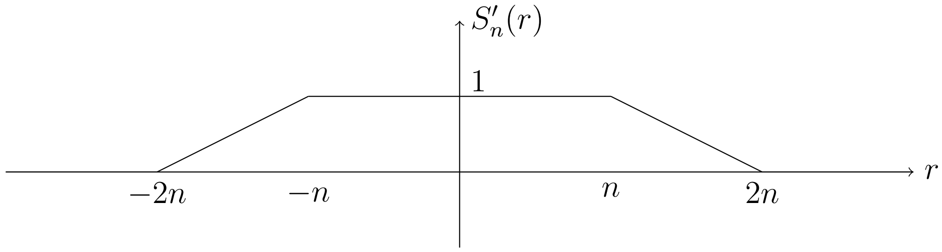

Figure 2.3 – The function S_n'

where $\widetilde{N}^\varepsilon(s) = \int_0^s \mathbb{1}_{\{|r|<1/\varepsilon\}} h_n(r)\,\mathrm{d}r$.

We now pass to the limit as ε goes to zero in (2.70). According to the definition of S_n, we have $|S_n'(r)| \leq 1$, since f^ε converges to f in $L^1(Q_T)$ and since θ_ε converges a.e. to θ we get by the Lebesgue-dominated Theorem

$$\lim_{\varepsilon\to 0} \int_{Q_T} f^\varepsilon S_n'(\theta_\varepsilon)\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} f S_n'(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t. \tag{2.71}$$

Similarly, we have

$$\lim_{\varepsilon\to 0} \int_{\partial\Omega\times(0,T)} h^\varepsilon S_n'(\theta_\varepsilon)\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t = -\int_{\partial\Omega\times(0,T)} h S_n'(\theta)\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t. \tag{2.72}$$

Moreover, writing $g\nabla(S_n'(\theta_\varepsilon)\varphi) = S_n''(\theta_\varepsilon)g\nabla T_{2n}(\theta_\varepsilon)\varphi + S_n'(\theta_\varepsilon)g\nabla\varphi$ together the weak convergence of $\nabla T_{2n}(\theta_\varepsilon)$ and already used arguments lead to

$$\lim_{\varepsilon\to 0} \int_{Q_T} g\nabla(S_n'(\theta_\varepsilon)\varphi)\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} g\nabla(S_n'(\theta)\varphi)\,\mathrm{d}x\,\mathrm{d}t. \tag{2.73}$$

Note that S_n is a bounded Lipschtiz function and $\varphi(T) = 0$. Due to the convergence of $\theta_\varepsilon \to \theta$ in $L^1(Q_T)$ and $\theta_0^\varepsilon \to \theta_0$ in $L^1(\Omega)$, as $\varepsilon \to 0$ we have

$$\begin{aligned}\lim_{\varepsilon\to 0} -\int_{Q_T} \varphi_t S_n(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varphi(0)S_n(\theta_0^\varepsilon)\,\mathrm{d}x &= -\int_{Q_T} \varphi_t S_n(\theta)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varphi(0)S_n(\theta_0)\,\mathrm{d}x \\ &= -\int_{Q_T} \varphi_t(S_n(\theta) - S_n(\theta_0))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.74}$$

For the third term of the right-hand-side of (2.70) we observe that by the assumption (2.4) on A and the almost everywhere convergence of θ_ε

$$S_n'(\theta_\varepsilon)A(x,t,T_{2n}(\theta_\varepsilon)) \to S_n'(\theta)A(x,t,T_{2n}(\theta)), \tag{2.75}$$

a.e. in Q_t and in $L^\infty(Q_T)$ weak-$*$ as $\varepsilon \to 0$. This and (2.51) imply that

$$\lim_{\varepsilon\to 0} \int_{Q_T} S_n'(\theta_\varepsilon)A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon \cdot \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} S_n'(\theta)A(x,t,\theta)\nabla\theta \cdot \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t. \tag{2.76}$$

As far as the fourth term of the right-hand-side of (2.70) is concerned, in view of the definition of $\widetilde{N}^\varepsilon$ and the point-wise convergence of θ we have

$$\widetilde{N}^\varepsilon(\theta_\varepsilon) \to \int_0^\theta h_n(s)ds = S_n(\theta)$$

a.e. in Q_T and in L^∞ weak-$*$ as ε goes to zero. It follows that

$$\lim_{\varepsilon\to 0}\int_{Q_T} u\cdot \widetilde{N}^\varepsilon(\theta_\varepsilon)\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} u\cdot S_n(\theta)\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{2.77}$$

Let

$$\omega(n) := \limsup_{\varepsilon}\frac{1}{n}\int_0^T\int_{\{n\le|\theta_\varepsilon|\le 2n\}} A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\theta_\varepsilon\,\mathrm{d}x\,\mathrm{d}t. \tag{2.78}$$

As a consequence of the convergences (2.71), (2.74), (2.76), (2.77), (2.73) together with (2.70) we obtain for any $\varphi\in C_c^\infty([0,T)\times\Omega)$

$$\begin{aligned}
-\|\varphi\|_{L^\infty(Q_T)}\omega(n) \le &-\int_{Q_T}\varphi_t(S_n(\theta)-S_n(\theta_0))\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{Q_T} S_n'(\theta)A(x,t,\theta)\nabla\theta\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}u\cdot S_n(\theta)\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_{Q_T} fS_n'(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_{Q_T} g\nabla(S_n'(\theta)\varphi)\,\mathrm{d}x\,\mathrm{d}t-\int_{\partial\Omega\times(0,T)} hS_n'(\theta)\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t.
\end{aligned}\tag{2.79}$$

We now take $\varphi=\xi W_{-h}$ where $W_{-h}=(T_k(\theta))_{-h}=\frac{1}{h}\int_{t-h}^{h}T_k(\theta(\tau))\,\mathrm{d}\tau$ in (2.79), with $\xi\in C_c^\infty([0,T))$, $0\leqslant\xi\leqslant 1$. We define $\theta(t)=\Theta_{0j}$ for $t<0$, such that $\Theta_{0,j}$ is a sequence of $C_c^\infty(\Omega)$ functions which converges to θ_0 as j goes to infinity. By noting that $\|\xi W_{-h}\|_{L^\infty(Q_T)}\leqslant k$ we get

$$\begin{aligned}
-k\omega(n)\le\ &-\int_{Q_T}\frac{\partial(\xi W_{-h})}{\partial t}(S_n(\theta)-S_n(\theta_0))\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{Q_T} S_n'(\theta)A(x,t,\theta)\nabla\theta\cdot\nabla(\xi W_{-h})\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}u\cdot S_n(\theta)\nabla(\xi W_{-h})\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_{Q_T} fS_n'(\theta)\xi W_{-h}\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_{Q_T} g\nabla(S_n'(\theta)\xi W_{-h})\,\mathrm{d}x\,\mathrm{d}t-\int_{\partial\Omega\times(0,T)} hS_n'(\theta)\xi W_{-h}\,\mathrm{d}\sigma(x)\,\mathrm{d}t.
\end{aligned}\tag{2.80}$$

From the Integration by parts Lemma 2.3.3 we have that

$$\begin{aligned}
-\int_{Q_T}(S_n(\theta)-S_n(\Theta_{0,j}))\frac{\partial}{\partial t}\left(\xi\frac{1}{h}\int_{t-h}^{t}T_k(\theta(\tau))\,\mathrm{d}\tau\right)\,\mathrm{d}x\,\mathrm{d}t \le -\int_{Q_T}\xi_t\Bigg(S_n(\theta)\frac{1}{h}\int_{t-h}^{t}T_k(\theta(\tau))\,\mathrm{d}\tau\\
-\frac{1}{h}\int_{t-h}^{t}S_n(\theta)T_k(\theta)\,\mathrm{d}\tau+\frac{1}{h}\int_{t-h}^{t}\widetilde{T}_k(S_n(\theta))\,\mathrm{d}\tau\Bigg)\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\xi(0)\widetilde{T}_k(S_n(\Theta_{0,j}))\,\mathrm{d}x.
\end{aligned}\tag{2.81}$$

Due to the properties of the Steklov average, we recall that, as h goes to zero, for any $\eta>0$

$$\begin{aligned}
W_{-h}\to T_k(\theta)\quad&\text{in } L^2(0,T-\eta;H^1(\Omega)),\\
\frac{1}{h}\int_{t-h}^{t}S_n(\theta)T_k(\theta)\,\mathrm{d}\tau\to S_n(\theta)T_k(\theta)\quad&\text{in } L^2(0,T-\eta;H^1(\Omega)),\\
\frac{1}{h}\int_{t-h}^{t}\widetilde{T}_k(S_n(\theta))\,\mathrm{d}\tau\to\widetilde{T}_k(S_n(\theta)\quad&\text{in } L^2(0,T-\eta;H^1(\Omega)).
\end{aligned}$$

Since ξ has a compact support in $[0,T)\times\Omega$ we can pass to the limit in (2.80) and (2.81), as h goes to zero. We obtain, for any $j\geq 1$,

$$\begin{aligned}-k\omega(n)\leq\ &-\int_{Q_T}\xi_t\widetilde{T}_k(S_n(\theta))\,\mathrm{d}x\,\mathrm{d}t\\&-\int_\Omega \xi(0)(\widetilde{T}_k(S_n(\Theta_{0,j}))-S(\theta_0)T_k(\Theta_{0,j})+S_n(\Theta_{0,j})T_k(\Theta_{0,j}))\,\mathrm{d}x\\&+\int_{Q_T}\xi S_n'(\theta)A(x,t,\theta)\nabla\theta\cdot\nabla T_k(\theta)\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}u\cdot S_n(\theta)\nabla T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t\\&-\int_{Q_T}fS_n'(\theta)\xi T_k(\theta)\,\mathrm{d}x\,\mathrm{d}t\\&-\int_{Q_T}g\nabla(S_n'(\theta)\xi T_k(\theta))\,\mathrm{d}x\,\mathrm{d}t-\int_{\partial\Omega\times(0,T)}hS_n'(\theta)\xi T_k(\theta)\,\mathrm{d}\sigma(x)\,\mathrm{d}t.\end{aligned}$$

Due to the definition of $\Theta_{0,j}$ then letting as j go to infinity we deduce

$$\begin{aligned}-k\omega(n)\leq\ &-\int_{Q_T}\xi_t\widetilde{T}_k(S_n(\theta))\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\xi(0)\widetilde{T}_k(S_n(\theta_0))\mathrm{d}x\\&+\int_{Q_T}\xi S_n'(\theta)A(x,t,\theta)\nabla\theta\cdot\nabla T_k(\theta)\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}u\cdot S_n(\theta)\nabla T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t\\&-\int_{Q_T}fS_n'(\theta)\xi T_k(\theta)\,\mathrm{d}x\,\mathrm{d}t-\\&-\int_{Q_T}g\nabla(S_n'(\theta)\xi T_k(\theta))\,\mathrm{d}x\,\mathrm{d}t-\int_{\partial\Omega\times(0,T)}hS_n'(\theta)\xi T_k(\theta)\,\mathrm{d}\sigma(x)\,\mathrm{d}t.\end{aligned}\tag{2.82}$$

Finally, we pass to the limit in (2.82) as n goes to infinity. From (2.40), the left-hand-side of (2.82) goes to 0. On the other hand in view of the definition of S_n we know that $S_n(\theta)$ (resp. $S_n(\theta_0)$) converges almost everywhere to θ (resp. θ_0). Since θ belongs to $L^\infty(0,T;L^1(\Omega))$, since θ_0 belongs to $L^1(\Omega)$, the inequality $\widetilde{T}_k(S_n(r))\leq k|r|$ for all $r\in\mathbb{R}$ and the Lebesgue dominated Theorem allow one to deduce

$$\begin{aligned}&\lim_{n\to\infty}-\int_{Q_T}\xi_t\widetilde{T}_k(S_n(\theta))\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\xi(0)\widetilde{T}_k(S_n(\theta_0))\,\mathrm{d}x\\&=-\int_{Q_T}\xi_t\widetilde{T}_k(\theta)\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\xi(0)\widetilde{T}_k(\theta_0)\,\mathrm{d}x.\end{aligned}\tag{2.83}$$

Moreover, the term $S_n(\theta)\nabla T_k(\theta)$ is identified with $T_k(\theta)\nabla T_k(\theta)$ for $2n\geq k$ and ξ is independent of x, hence by the Stoke's formula and since $\mathrm{div}(u)=0$ we have

$$\int_{Q_T}u\cdot S_n(\theta)\nabla T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t=0.$$

For the third term of the right-hand-side of (2.82), we note that $S_n'(\theta)A(x,t,\theta)\nabla\theta\cdot\nabla T_k(\theta)$ is equal to $S_n'(\theta)A(x,t,\theta)\nabla T_k(\theta)\cdot\nabla T_k(\theta)$ for $2n\geq k$. Due to the definition of S_n, for $n>k$ we have $S_n'(\theta)\nabla T_k(\theta)=\nabla T_k(\theta)$, so that

$$\lim_{n\to\infty}\int_{Q_T}\xi S_n'(\theta)A(x,t,\theta)\nabla\theta\cdot\nabla T_k(\theta)\,\mathrm{d}x\,\mathrm{d}t=\int_{Q_T}\xi A(x,t,\theta)\nabla\theta\cdot\nabla T_k(\theta)\,\mathrm{d}x\,\mathrm{d}t.\tag{2.84}$$

Additionally, with the fact that $f\in L^1(Q_T)$ and $h\in L^1(\partial\Omega\times(0,T))$ the Lebesgue dominated theorem yields that

$$\lim_{n\to\infty}\int_{Q_T}fS_n'(\theta)\xi T_k(\theta)\,\mathrm{d}x\,\mathrm{d}t=\int_{Q_T}f\xi T_k(\theta)\,\mathrm{d}x\,\mathrm{d}t,\tag{2.85}$$

$$\lim_{n\to\infty}\int_{\partial\Omega\times(0,T)}hS_n'(\theta)\xi T_k(\theta)\,\mathrm{d}\sigma(x)\,\mathrm{d}t=\int_{\partial\Omega\times(0,T)}h\xi T_k(\theta)\,\mathrm{d}\sigma(x)\,\mathrm{d}t.\tag{2.86}$$

Recalling that $g \in L^2(Q_T)$, it follows from the above convergences that

$$
\begin{aligned}
0 \leq &-\int_{Q_T} \xi_t \widetilde{T}_k(S(\theta))\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \xi(0)\widetilde{T}_k(S(\theta_0))\,\mathrm{d}x + \int_{Q_T} \xi A(x,t,\theta)\nabla\theta\cdot\nabla T_k(\theta)\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_{Q_T} f\xi T_k(\theta)\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} \xi g\nabla T_k(\theta)\,\mathrm{d}x\,\mathrm{d}t - \int_{\partial\Omega\times(0,T)} h\xi T_k(\theta)\,\mathrm{d}\sigma(x)\,\mathrm{d}t
\end{aligned}
\tag{2.87}
$$

Thus, by comparing (2.69) and (2.87) we deduce that

$$
\lim_{\varepsilon\to 0}\int_{Q_T} A(x,t,\theta_\varepsilon)\nabla T_k(\theta_\varepsilon)\cdot\nabla T_k(\theta_\varepsilon)\xi\,\mathrm{d}x\,\mathrm{d}t \leq \int_{Q_T} A(x,t,\theta)\nabla T_k(\theta)\cdot\nabla T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t. \tag{2.88}
$$

Now, recalling the ellipticity assumption (2.3) on A we know that

$$
\begin{aligned}
\alpha &\int_{Q_T} \xi|\nabla T_k(\theta_\varepsilon) - \nabla T_k(\theta)|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \leq \int_{Q_T} \xi A(x,t,\theta_\varepsilon)(\nabla T_K(\theta_\varepsilon) - \nabla T_k(\theta))\cdot(\nabla T_k(\theta_\varepsilon) - \nabla T_k(\theta))\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{2.89}
$$

Since $A(x,t,\theta_\varepsilon)$ converges to $A(x,t,\theta)$ in L^∞ weak-$*$ and a.e. in Q_T we obtain that $\xi|\nabla T_k(\theta_\varepsilon) - \nabla T_k(\theta)|^2$ converges to zero strongly in $L^1(Q_T)$, for any $\xi \in C_c^\infty([0,1))$ with $0 \leq \xi \leq 1$. Hence, for any $0 < \tau < T$

$$
T_k(\theta_\varepsilon) \to T_k(\theta) \text{ strongly in } L^2(0,\tau;H^1(\Omega)). \tag{2.90}
$$

Step 5: θ is a renormalized solution to (2.2)

To conclude that θ is a renormalized solution of (2.2) it remains to prove that (2.15) holds true and to pass to the limit in (2.50) as ε goes to zero.

By (2.40) we know that

$$
\lim_{n\to+\infty}\limsup_{\varepsilon}\frac{1}{n}\int_{\{n<|\theta_\varepsilon|<2n\}} A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\theta_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = 0.
$$

Since $A(x,t,\theta_\varepsilon)$ converges to $A(x,t,\theta)$ a.e. in Q_T and in L^∞ weak-$*$, and since $T_{2n}(\theta_\varepsilon) - T_n(\theta_\varepsilon)$ converges weakly to $T_{2n}(\theta) - T_n(\theta)$ in $L^2(0,T;H^1(\Omega))$, assumption (2.3) and a lower semi continuity argument give (2.15).

We now pass to the limit in (2.50). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let K be a positive real number such that $\operatorname{supp} S' \subset [-K,K]$ and let $\varphi \in C_c^\infty([0,T)\times\Omega)$. We have, for any $\varepsilon > 0$

$$
\begin{aligned}
&-\int_{Q_T} \varphi_t S(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varphi(0)S(\theta_0^\varepsilon)\,\mathrm{d}x + \int_{Q_T} A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\theta_\varepsilon S''(\theta_\varepsilon)\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{Q_T} S'(\theta_\varepsilon)A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} u\cdot\nabla T_{1/\varepsilon}(\theta_\varepsilon)S'(\theta_\varepsilon)\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q_T} f^\varepsilon S'(\theta_\varepsilon)\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\nabla(S'(\theta_\varepsilon)\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} h^\varepsilon S'(\theta_\varepsilon)\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t.
\end{aligned}
\tag{2.91}
$$

The only difference with passing to the limit in (2.70) in Step 4 is that we now have the strong convergence result (2.59).

By observing that $A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\theta_\varepsilon S''(\theta_\varepsilon)=A(x,t,T_K(\theta_\varepsilon))\nabla T_K(\theta_\varepsilon)\cdot\nabla S'(T_K(\theta_\varepsilon))$ a.e. in Q_T, using the strong convergence of the truncation (2.59) and the fact that the support of φ is compact in $[0,T)\times\Omega$ lead to

$$\lim_{\varepsilon\to 0}\int_{Q_T} A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\theta_\varepsilon S''(\theta_\varepsilon)\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_{Q_T} A(x,t,\theta)\nabla\theta\cdot\nabla\theta S''(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t. \tag{2.92}$$

The convergences derived in Step 4 and (2.92) permits us to pass to the limit and to obtain that θ satisfies

$$\begin{aligned}&-\int_{Q_T}\varphi_t S(\theta)\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\varphi(0)S(\theta_0)\,\mathrm{d}x+\int_{Q_T}A(x,t,\theta)\nabla\theta\cdot\nabla\theta S''(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t\\&+\int_{Q_T}S'(\theta)A(x,t,\theta)\nabla\theta\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}u\cdot\nabla\theta S'(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t\\&=\int_{Q_T}S'(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}g\nabla(S'(\theta)\varphi)\,\mathrm{d}x\,\mathrm{d}t+\int_{\partial\Omega\times(0,T)}h^\varepsilon S'(\theta)\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t,\end{aligned}\tag{2.93}$$

and then we conclude that θ is a renormalized solution of (2.2). □

Remark 2.4.4 (Stability of the renormalized solution.). *For $\varepsilon>0$ let f_ε belonging to $L^1(Q_T)$, g_ε belonging to $L^2(Q_T^N)$, $h_\varepsilon\in L^1(\partial\Omega\times(0,T))$, $\theta_0^\varepsilon\in L^1(\Omega)$ and $u\in L^2(0,T;H_0^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ such that $\operatorname{div}(u)=0$ in Q_T and $u=0$ on $\partial\Omega\times(0,T)$. For any $\varepsilon>0$, let us consider θ_ε a renormalized solution of*

$$\begin{cases}\dfrac{\partial\theta_\varepsilon}{\partial t}-\operatorname{div}\left(A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon\right)+u\cdot\nabla\theta_\varepsilon=f_\varepsilon-\operatorname{div}(g_\varepsilon) & \text{in } Q_T\ ,\\ \theta_\varepsilon(t=0)=\theta_0^\varepsilon & \text{in } \Omega\ ,\\ (A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon-g_\varepsilon)\cdot\vec{n}=h_\varepsilon & \text{on } \partial\Omega\times(0,T)\ ,\end{cases}\tag{2.94}$$

where A is a Carathéodory function verifying (2.3) *and* (2.4).

Moreover, assume that

$$\theta_0^\varepsilon\to\theta_0\ \text{ strongly in } L^1(\Omega), \tag{2.95}$$
$$f^\varepsilon\to f\ \text{ strongly in } L^1(Q_T), \tag{2.96}$$
$$g_\varepsilon\to g\ \text{ strongly in } L^2(Q_T)^N \tag{2.97}$$
$$h_\varepsilon\to h\ \textit{strongly in } L^1(\partial\Omega\times(0,T)). \tag{2.98}$$

Then, up to a subsequence (still indexed by ε) θ_ε converges to θ a.e. in Q_T as ε goes to zero where θ is a renormalized solution of

$$\begin{cases}\dfrac{\partial\theta}{\partial t}-\operatorname{div}\left(A(x,t,\theta)\nabla\theta\right)+u\cdot\nabla\theta=f-\operatorname{div}(g) & \text{in } Q_T\ ,\\ \theta(t=0)=\theta_0 & \text{in } \Omega\ ,\\ (A(x,t,\theta)\nabla\theta-g)\cdot\vec{n}=h & \text{on } \partial\Omega\times(0,T)\ .\end{cases}\tag{2.99}$$

Since in Steps 3 and 4 of the proof of Theorem 2.4.1 we use mainly the renormalized formulation the only difference is the way to obtain a priori estimates in Step 2. Indeed, for any $\xi\in C_c^\infty([0,T))$ with $0\le\xi\le 1$, Lemma (2.3.4) gives for any $k>0$,

$$\begin{aligned}&-\int_{Q_T}\xi_t\widetilde{T}_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\xi(0)\widetilde{T}_k(\theta_0^\varepsilon)\,\mathrm{d}x+\int_{Q_T}A(x,t,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla T_k(\theta_\varepsilon)\xi\,\mathrm{d}x\,\mathrm{d}t\\&=\int_{Q_T}f^\varepsilon T_k(\theta_\varepsilon)\xi\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}g_\varepsilon\nabla(T_k(\theta_\varepsilon)\xi)\,\mathrm{d}x\,\mathrm{d}t+\int_{\partial\Omega\times(0,T)}h^\varepsilon T_k(\theta_\varepsilon)\xi\,\mathrm{d}\sigma(x)\,\mathrm{d}t.\end{aligned}\tag{2.100}$$

Taking $k=1$ and $\xi(t)=\int_t^T \varphi(s)ds$ where $\varphi \in C_c^\infty(0,T)$ verifies $\varphi \geq 0$, the ellipticity condition on A and Young inequality allows one to obtain

$$\int_0^T \varphi(t)\int_\Omega \widetilde{T}_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \leq M\|\varphi\|_{L^1(0,T)},$$

where $M>0$ does not depend on ε. It follows that $\int_\Omega \widetilde{T}_1(\theta_\varepsilon)\,\mathrm{d}x$ (which is non negative) is bounded in $L^\infty(0,T)$, that is θ_ε bounded in $L^\infty(0,T;L^1(\Omega))$.

To obtain the estimates on $T_k(\theta_\varepsilon)$, it is sufficient to use (2.100) *and $\xi = \min(\frac{(T-\delta-1)^+}{\delta},1)$ and then to let δ go to zero. The uniform decay of the truncated energies is obtained with a similar process.*

Since Steps 3 and 4 remain unchanged we can derive the stability, up to a subsequence, of the renormalized solution.

2.5 Properties of the renormalized solution

In this section, we give two results on the renormalized solutions of problem (2.2). The first proposition, which is classical, states that $T_k(\theta)$ is formally an admissible test function. It allows one to recover the $L^\infty(L^1)$ bound and is useful to derive a stability result. The second result concerns the Boccardo–Gallouët estimates with some precisions on the contribution on the bound in $L^\infty(L^1)$ and on the bound on $T_k(\theta)$ in $L^2(H^1)$.

Proposition 2.5.1. *Assume that* (2.3)–(2.9) *hold true. Let θ a renormalized solution of* (2.2)*. Then, for any $k>0$ and for any $\xi \in C_c^\infty([0,T[)$ we have*

$$\begin{aligned}-\int_{Q_T}\xi_t\widetilde{T}_k(\theta)\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega \xi(0)\widetilde{T}_k(\theta_0)\,\mathrm{d}x+\int_{Q_T}A(x,t,\theta)\nabla\theta\cdot\nabla T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t\\=\int_{Q_T} fT_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T} g\nabla T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t+\int_{\partial\Omega\times(0,T)} hT_k(\theta)\xi\,\mathrm{d}\sigma(x)\,\mathrm{d}t.\end{aligned} \tag{2.101}$$

Proof. We recall the definition of S_n:

$$\forall r\in\mathbb{R},\quad S_n'(r)=1-\frac{1}{n}|T_{2n}(r)-T_n(r)|,\qquad S_n(r)=\int_0^r S_n'(s)\,\mathrm{d}s.$$

The proof consists in using the admissible test function $S_n'(\theta)T_k(\theta)\xi$ in the renormalized formulation, to use the regularity of u and the by-part integration formula of Lemma 2.3.4 and then to pass to the limit as n goes to infinity.

Let $k>0$ and $\xi\in C_c^\infty([0,T[)$ (ξ does not depend on the x-variable). By taking S_n' in place of S' in (2.16), the function $S_n(\theta)$ verifies, for any $\varphi\in C_c^\infty([0,T)\times\bar{\Omega})$,

$$\begin{aligned}-\int_{Q_T}\varphi_t S_n(\theta)\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\varphi(0)S_n(\theta_0)\,\mathrm{d}x=-\int_{Q_T}A(x,t,\theta)\nabla\theta\cdot\nabla\theta S_n''(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t\\-\int_{Q_T}S_n'(\theta)A(x,t,\theta)\nabla\theta\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t-\int_{Q_T}u\cdot\nabla\theta S_n'(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t\\+\int_{Q_T}fS_n'(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}g\nabla(S_n'(\theta)\varphi)\,\mathrm{d}x\,\mathrm{d}t+\int_{\partial\Omega\times(0,T)}hS_n'(\theta)\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t.\end{aligned} \tag{2.102}$$

It follows that we are in scope of Lemma 2.3.4 with w in place of $S_n(\theta)$ and appropriate f_1, g_1 and h_1.

For $n > k$ we have $T_k(\theta) = T_k(S_n(\theta))$. Then using Lemma 2.3.4 with $G = T_k$ we obtain

$$\begin{aligned} -\int_{Q_T} \xi_t \widetilde{T}_k(\theta)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \xi(0)\widetilde{T}_k(\theta_0)\,\mathrm{d}x = -\int_{Q_T} A(x,t,\theta)\nabla\theta\cdot\nabla\theta S_n''(\theta)\xi T_k(\theta)\,\mathrm{d}x\,\mathrm{d}t \\ -\int_{Q_T} S_n'(\theta)A(x,t,\theta)\nabla\theta\cdot\nabla(T_k(\theta)\xi)\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u\cdot\nabla\theta S_n'(\theta)T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t \\ +\int_{Q_T} fS_n'(\theta)T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\nabla(S_n'(\theta)T_k(\theta)\xi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega\times(0,T)} hS_n'(\theta)T_k(\theta)\xi\,\mathrm{d}\sigma(x)\,\mathrm{d}t. \end{aligned} \tag{2.103}$$

We now pass to the limit as n goes to infinity. Since ξ does not depend on x, recalling that $\mathrm{div}(u) = 0$ and $u \in L^2(0,T;(H_0^1(\Omega))^2)$ the divergence theorem gives

$$\int_{Q_T} u\cdot\nabla\theta S_n'(\theta)T_k(\theta)\xi\,\mathrm{d}x\,\mathrm{d}t = 0.$$

In view of the definition of S_n, by the decay of the truncated energy we have

$$\lim_{n\to+\infty}\int_{Q_T} A(x,t,\theta)\nabla\theta\cdot\nabla\theta S_n''(\theta)\xi T_k(\theta)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Moreover since θ (and the trace of θ) is finite almost everywhere $S_n'(\theta)$ converges to $S'_n(\theta)$ in L^∞ weak-$*$ as n goes to infinity. It follows that passing to the limit in (2.103) allows one to conclude. □

We now give a regularity result of the renormalized solution of (2.2). In the context of Dirichlet boundary conditions, the following lemma is known as the Boccardo-Gallouët estimates, see [13, 97, 20, 49]. In the context of Neumann boundary conditions Poincaré-Wirtinger inequality allows one to obtain similar results. In addition, it is worth noting that, in the parabolic case, it is necessary to have an estimate on $\|\theta\|_{L^\infty(0,T;L^1(\Omega))}$ and an estimate on $\|T_k(\theta)\|_{L^2(0,T;H^1(\Omega))}$: we specify here the impact of the bound on $\|\theta\|_{L^\infty(0,T;L^1(\Omega))}$ and the one on $\|T_k(\theta)\|_{L^2(0,T;H^1(\Omega))}$.

Lemma 2.5.2. *Let θ be an element of $L^\infty(0,T;L^1(\Omega))$. Assume that, for any $k > 0$, $T_k(\theta) \in L^2(0,T;H^1(\Omega))$ and that*

$$\|T_k(\theta)\|^2_{L^2(0,T;H^1(\Omega))} \le kM + L, \tag{2.104}$$

where M and L are two positive constants independent of k. Then

— *for all $1 \le p < 1 + \frac{2}{N}$, $\theta \in L^p(Q_T)$; more precisely there exists a constant C (depending on T, Ω and p) such that*

$$\|\theta\|_{L^p(Q_T)} \le C\Big(\|\theta\|^{\frac{1}{N+1}}_{L^\infty(0,T;L^1(\Omega))}L^{\frac{N}{2(N+1)}} + \|\theta\|^{\frac{2}{N+2}}_{L^\infty(0,T;L^1(\Omega))}M^{\frac{N}{N+2}}\Big) \tag{2.105}$$

— *for all $1 < r < \overline{p}$ where $\overline{p} = +\infty$ if $N = 2$ and $\overline{p} = \frac{2^*}{2}$ if $N \ge 3$, we have*

$$\theta \in L^1(0,T;L^r(\Omega)).$$

More precisely, for all $1 \le r < \overline{p}$

$$\|\theta\|_{L^1(0,T;L^r(\Omega))} \le C\Big(M^{q_1}\|\theta\|^{q_1(\frac{1}{r}-\frac{2}{p})}_{L^\infty(0,T;L^1(\Omega))} + L^{q_2}\|\theta\|^{q_2(\frac{1}{r}-\frac{2}{p})}_{L^\infty(0,T;L^1(\Omega))}\Big), \tag{2.106}$$

where $2r < p$ if $N = 2$ and $p = \frac{2^*}{2}$ if $N \geq 3$, where $C > 0$ is constant depending on T, Ω, p and r, and where

$$q_1 = \frac{1}{1 + \frac{1}{r} - \frac{2}{p}}, \quad \textit{and} \quad q_2 = \frac{1}{2 + \frac{1}{r} - \frac{2}{p}}.$$

— *for all $1 \leq p < \frac{N+2}{N+1}$, $\nabla\theta \in L^p(Q_T)^N$; more precisely there exists a constant C (depending in T, Ω, and p) such that*

$$\|\nabla\theta\|_{L^p(Q_T)} \leq C\Big(\|\theta\|^{\frac{1}{N+2}}_{L^\infty(0,T;L^1(\Omega))} M^{\frac{N+1}{N+2}} + L^{\frac{1}{2}}\Big). \tag{2.107}$$

Remark 2.5.3. *Decoupling in the estimates the bound on θ in $L^\infty(0,T;L^1(\Omega))$ and the one of $T_k(\theta)$ in $L^2(0,T;H^1(\Omega))$ could be interesting in the case of time-dependent systems arising for example in thermo-mechanic. For example for Kelvin-Voigt models (see [21, 43]) an extra information due to the coupling of the motion equation and the balance energy equation gives a bound on the temperature in $L^\infty(0,T;L^1(\Omega))$. This bound is then crucial to obtain existence results without assuming smallness conditions on the data.*

In the case of a single equation, Lemma 2.5.2 is nothing more than the Boccardo-Gallouët estimates [26] (see also [27, 97]). In this case $\|\theta\|_{L^\infty(0,T;L^1(\Omega))}$ stands for $M + L^{1/2}$ where M is related to $\|f\|_{L^1} + \|\theta_0\|_{L^1}$ and where L is related to $\|g\|^2_{L^2}$. We then recognize the classical estimates

$$\begin{aligned}
\forall 1 \leq p < 1 + \frac{2}{N}, &\quad \|\theta\|_{L^p(Q_T)} \leq C(M + L^{1/2}),\\
\forall 1 \leq r < \overline{p}, &\quad \|\theta\|_{L^1(0,T;L^r(\Omega))} \leq C(M + L^{1/2}),\\
\forall 1 \leq p < \frac{N+2}{N+1}, &\quad \|\nabla\theta\|_{L^p(Q_T)} \leq C(M + L^{1/2}),
\end{aligned}$$

Proof. We use similar arguments to [13, 97, 20, 49]. In the whole proof $C(N,\Omega)$, $C(N,\Omega,T)$, $C(N,\Omega,T,p)$ are positive constants that depend only on their arguments, and which can vary from line to line.

Step 1. To obtain (2.105) we firstly show that for any $k > 0$,

$$\begin{aligned}
&k\Big(\mathrm{meas}\{(x,t) \in Q_T; |\theta(x,t)| \geq k\}\Big)^{\frac{N}{N+2}}\\
&\qquad \leq C(N,\Omega)\Big(|Q_T|^{\frac{N^2}{(N+1)(N+2)}} \|\theta\|^{\frac{1}{N+1}}_{L^\infty(0,T;L^1(\Omega))} L^{\frac{N}{2(N+1)}} + \|\theta\|^{\frac{2}{N+2}}_{L^\infty(0,T;L^1(\Omega))} M^{\frac{N}{N+2}}\Big)
\end{aligned} \tag{2.108}$$

Let us recall that, for any $v \in H^1(\Omega)$, when $N = 2$ the Gagliardo-Nirenberg gives

$$\int_\Omega |v|^{\frac{2(N+1)}{N}} dx \leq C(N,\Omega)\|v\|^{\frac{2}{N}}_{L^1(\Omega)} \|v\|^2_{H^1(\Omega)},$$

while, when $N \geq 3$, Sobolev's embedding theorem and interpolation inequality gives

$$\begin{aligned}
\int_\Omega |v|^{\frac{2(N+1)}{N}} dx &\leq \|v\|^{\frac{2}{N}}_{L^1(\Omega)} \|v\|^2_{L^{\frac{2N}{N-2}}(\Omega)}\\
&\leq C(N,\Omega)\|v\|^{\frac{2}{N}}_{L^1(\Omega)} \|v\|^2_{H^1(\Omega)}.
\end{aligned}$$

In both cases, by applying this inequality to $\theta(\cdot,t)$ and by integrating on $(0,T)$ we obtain

$$\begin{aligned}\int_{Q_T}|T_k(\theta)|^{\frac{2(N+1)}{N}}\,\mathrm{d}x &\le C(N,\Omega)\|\theta\|^{\frac{2}{N}}_{L^\infty(0,T;L^1(\Omega))}\|T_k(\theta)\|^2_{L^2(0,T;H^1(\Omega))}\\ &\le C(N,\Omega)\|\theta\|^{\frac{2}{N}}_{L^\infty(0,T;L^1(\Omega))}(kM+L).\end{aligned}\tag{2.109}$$

If follows that, for any $k>0$,

$$k^{\frac{2(N+1)}{N}}\operatorname{meas}\{(x,t)\in Q_T;|\theta(x,t)|\ge k\}\le C(N,\Omega)\|\theta\|^{\frac{2}{N}}_{L^\infty(0,T;L^1(\Omega))}(kM+L).\tag{2.110}$$

To shorten the notations, let us denote $\operatorname{meas}_{Q_T}\{|\theta|\ge k\}=\operatorname{meas}\{(x,t)\in Q_T;|\theta(x,t)|\ge k\}$: inequality (2.110) implies that for any $k>0$

$$\begin{aligned}k\operatorname{meas}_{Q_T}\{|\theta|\ge k\}^{\frac{N}{N+2}} &\le \frac{k}{k^{\frac{2(N+1)}{N+2}}}C(N,\Omega)\|\theta\|^{\frac{2}{N+2}}_{L^\infty(0,T;L^1(\Omega))}(kM+L)^{\frac{N}{N+2}}\\ &\le C(N,\Omega)\|\theta\|^{\frac{2}{N+2}}_{L^\infty(0,T;L^1(\Omega))}\Big(M+\frac{L}{k}\Big)^{\frac{N}{N+2}}\\ &\le C(N,\Omega)\|\theta\|^{\frac{2}{N+2}}_{L^\infty(0,T;L^1(\Omega))}\Big(M^{\frac{N}{N+2}}+\big(\frac{L}{k}\big)^{\frac{N}{N+2}}\Big).\end{aligned}\tag{2.111}$$

Since we also have, for any $k>0$,

$$k\operatorname{meas}_{Q_T}\{|\theta|\ge k\}^{\frac{N}{N+2}}\le k|Q_T|^{\frac{N}{N+2}}\tag{2.112}$$

by minimizing with respect to k the right-hand side of (2.111) and of (2.112), that is by taking

$$k=\frac{L^{\frac{N}{2(N+1)}}\|\theta\|^{\frac{1}{N+1}}_{L^\infty(0,T;L^1(\Omega))}}{|Q_T|^{\frac{N}{2(N+1)}}}$$

we obtain inequality (2.108).

Now, for any real number $1\le p$, we write

$$\begin{aligned}\int_{Q_T}|\theta|^p\,\mathrm{d}x\mathrm{d}t &= \int_0^\infty \operatorname{meas}\{(x,t)\in Q_T:|\theta|^p>s\}\,\mathrm{d}s\\ &\le \int_0^\beta \operatorname{meas}\{(x,t)\in Q_T:|\theta|^p>s\}\,\mathrm{d}s+\int_\beta^\infty \operatorname{meas}\{(x,t)\in Q_T:|\theta|^p>s\}\,\mathrm{d}s\\ &\le \beta|Q_T|+\int_\beta^\infty \operatorname{meas}\{(x,t)\in Q_T:|\theta|^p>s\}\,\mathrm{d}s.\end{aligned}\tag{2.113}$$

In view of (2.108) we have

$$\operatorname{meas}_Q\{|\theta|^p>s\}\le\frac{\Lambda^{\frac{N+2}{N}}}{s^{\frac{N+2}{Np}}}\tag{2.114}$$

where

$$\Lambda=C(N,\Omega)\Big(|Q_T|^{\frac{N^2}{(N+1)(N+2)}}\|\theta\|^{\frac{1}{N+1}}_{L^\infty(0,T;L^1(\Omega))}L^{\frac{N}{2(N+1)}}+\|\theta\|^{\frac{2}{N+2}}_{L^\infty(0,T;L^1(\Omega))}M^{\frac{N}{N+2}}\Big).$$

From (2.113) and (2.114) it follows that, for any $1 \leq p < \frac{N+2}{N}$ and for any $\beta > 0$,

$$\|\theta\|^p_{L^p(Q_T)} \leq \beta |Q_T| + \frac{1}{\frac{N+2}{Np} - 1} \frac{\Lambda^{\frac{N+2}{N}}}{\beta^{\frac{N+2}{Np}-1}}. \tag{2.115}$$

At last, by choosing $\beta = \Lambda^p$, we deduce that, for any $1 \leq p < \frac{N+2}{N}$

$$\|\theta\|^p_{L^p(Q_T)} \leq C(N, \Omega, T, p)\Lambda^p, \tag{2.116}$$

which is (2.105).
Step 2. We now prove estimate (2.106). It is worth noting that the exponents in the x variable and the ones in the t variable are different. To shorten the notations, let us denote, for a.e. t in $(0,T)$, $\mathrm{meas}_\Omega\{|\theta(\cdot,t)| \geq k\} = \mathrm{meas}\{x \in Q_T; |\theta(x,t)| \geq k\}$.

When $N \geq 3$, Sobolev embedding theorem and (2.104) give that, for any $k > 0$,

$$\begin{aligned} k^2 \int_0^T \Big(\mathrm{meas}_\Omega\{|\theta(\cdot,t) \geq k\}\Big)^{\frac{2}{2^*}} \mathrm{d}t &\leq \int_0^T \Big(\int_\Omega |T_k(\theta)|^{2^*} \mathrm{d}x\Big)^{\frac{2}{2^*}} \mathrm{d}t \\ &\leq C(\Omega) \int_0^T \int_\Omega |\nabla T_k(\theta)|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\leq C(\Omega)(kM + L). \end{aligned} \tag{2.117}$$

When $N = 2$, for any $1 < p < \infty$ we have, for any $k > 0$,

$$k^2 \int_0^T \Big(\mathrm{meas}_\Omega\{|\theta(\cdot,t) \geq k\}\Big)^{\frac{2}{p}} \mathrm{d}t \leq C(\Omega, p)\Big(kM + L\Big). \tag{2.118}$$

Let p such that $1 < p < +\infty$ if $N = 2$ and $p = 2^*/2$ if $N \geq 3$. Let $1 < r < \infty$ (r will be chosen later). Using a changement of variable in the following computation we have

$$\begin{aligned} \int_0^T \Big(\int_\Omega |\theta|^r \,\mathrm{d}x\Big)^{1/r} \mathrm{d}t &= \int_0^T \Big(\int_0^\infty \mathrm{meas}_\Omega\{|\theta(\cdot,t)|^r > s\} \,\mathrm{d}s\Big)^{1/r} \mathrm{d}t \\ &= \int_0^T \Big(\int_0^\infty r s^{r-1} \,\mathrm{meas}_\Omega\{|\theta(\cdot,t)| > s\} \,\mathrm{d}s\Big)^{1/r} \mathrm{d}t, \end{aligned} \tag{2.119}$$

and since $s \mapsto \mathrm{meas}_\Omega\{|\theta(\cdot,t)| > s\}$ is a non increasing function on $[0, +\infty[$ we deduce that

$$\begin{aligned} \int_0^T \Big(\int_\Omega |\theta|^r \,\mathrm{d}x\Big)^{1/r} \mathrm{d}t &\leq \int_0^T \int_0^\infty \Big(\mathrm{meas}_\Omega\{|\theta(\cdot,t)| > s\}\Big)^{1/r} \mathrm{d}s\,\mathrm{d}t \\ &\leq \beta T |\Omega|^{1/r} + \int_0^T \int_\beta^\infty \Big(\mathrm{meas}_\Omega\{|\theta(\cdot,t)| > s\}\Big)^{1/r} \mathrm{d}s\,\mathrm{d}t \\ &\leq \beta T |\Omega|^{1/r} + \int_\beta^\infty \int_0^T \Big(\mathrm{meas}_\Omega\{|\theta(\cdot,t)| > s\}\Big)^{1/r} \mathrm{d}t\,\mathrm{d}s, \end{aligned} \tag{2.120}$$

where $\beta > 0$ will be chosen later. If p is such that $2 < p < +\infty$ if $N = 2$ and $p = 2^*$ if $N \geq 3$, since θ belongs to $L^\infty(0,T; L^1(\Omega))$ from (2.117) and (2.118) we have, for a.e. $t \in (0,T)$

$$\begin{aligned} \Big(\mathrm{meas}_\Omega\{|\theta(\cdot,t)| > s\}\Big)^{\frac{1}{r}} &\leq \Big(\mathrm{meas}_\Omega\{|\theta(\cdot,t)| > s\}\Big)^{\frac{1}{r} - \frac{2}{p}} \Big(\mathrm{meas}_\Omega\{|\theta(\cdot,t)| > s\}\Big)^{\frac{2}{p}} \\ &\leq \Big(\frac{\|\theta(\cdot,t)\|_{L^1(\Omega)}}{s}\Big)^{\frac{1}{r} - \frac{2}{p}} \Big(\mathrm{meas}_\Omega\{|\theta(\cdot,t)| > s\}\Big)^{\frac{2}{p}}. \end{aligned}$$

By integrating on $(0,T)$, in view of (2.118) we obtain

$$\int_0^T \Big(\operatorname{meas}_\Omega\{|\theta(\cdot,t)|>s\}\Big)^{\frac{1}{r}} dt \le C(\Omega,p)\Big(\frac{\|\theta\|_{L^\infty(0,T;L^1(\Omega))}}{s}\Big)^{\frac{1}{r}-\frac{2}{p}}\Big(\frac{M}{s}+\frac{L}{s^2}\Big),$$

which gives together with (2.120)

$$\begin{aligned}\int_0^T \Big(\int_\Omega |\theta|^r\,\mathrm{d}x\Big)^{1/r}\,\mathrm{d}t \le \beta T|\Omega|^{1/r}&\\ + C(\Omega,p)\|\theta\|_{L^\infty(0,T;L^1(\Omega))}^{\frac{1}{r}-\frac{2}{p}} \int_\beta^\infty \frac{1}{s^{\frac{1}{r}-\frac{2}{p}}}\Big(\frac{M}{s}+\frac{L}{s^2}\Big)\,\mathrm{d}s.&\end{aligned} \tag{2.121}$$

By remarking that the integral above is convergent if and only if $\frac{1}{r}-\frac{2}{p}>0$, we obtain that for any $r<+\infty$ if $N=2$ (and in this case p is such that $2r<p$) and for any $r<\frac{2^*}{2}$ if $N\ge 3$ the estimate

$$\begin{aligned}\int_0^T \Big(\int_\Omega |\theta|^r\,\mathrm{d}x\Big)^{1/r}\,\mathrm{d}t \le \beta T|\Omega|^{1/r} + \frac{C(\Omega,p)M}{\frac{1}{r}-\frac{2}{p}}\Big(\frac{\|\theta\|_{L^\infty(0,T;L^1(\Omega))}}{\beta}\Big)^{\frac{1}{r}-\frac{2}{p}}&\\ + \frac{C(\Omega,p)L}{\beta(1+\frac{1}{r}-\frac{2}{p})}\Big(\frac{\|\theta\|_{L^\infty(0,T;L^1(\Omega))}}{\beta}\Big)^{\frac{1}{r}-\frac{2}{p}}.&\end{aligned} \tag{2.122}$$

By choosing

$$\beta = \Big(M\|\theta\|_{L^\infty(0,T;L^1(\Omega))}^{\frac{1}{r}-\frac{2}{p}}\Big)^{q_1} + \Big(L\|\theta\|_{L^\infty(0,T;L^1(\Omega))}^{\frac{1}{r}-\frac{2}{p}}\Big)^{q_2} \tag{2.123}$$

with

$$q_1 = \frac{1}{1+\frac{1}{r}-\frac{2}{p}}, \quad \text{and} \quad q_2 = \frac{1}{2+\frac{1}{r}-\frac{2}{p}} \tag{2.124}$$

we obtain

$$\begin{aligned}\int_0^T \Big(\int_\Omega |\theta|^r\,\mathrm{d}x\Big)^{1/r}\,\mathrm{d}t \le C(N,\Omega,T,p)\times&\\ \Big(M^{q_1}\|\theta\|_{L^\infty(0,T;L^1(\Omega))}^{q_1(\frac{1}{r}-\frac{2}{p})} + L^{q_2}\|\theta\|_{L^\infty(0,T;L^1(\Omega))}^{q_2(\frac{1}{r}-\frac{2}{p})}\Big),&\end{aligned} \tag{2.125}$$

which is (2.106)

Step 3. We now turn to (2.107).

For any $\mu>0$ and $k>0$ we have

$$\begin{aligned}\mu \operatorname{meas}_{Q_T}\{|\nabla\theta|>\mu\}^{\frac{N+1}{N+2}} = \mu \operatorname{meas}_{Q_T}\{|\nabla\theta|>\mu, |\theta|\le k\}^{\frac{N+1}{N+2}}&\\ + \mu \operatorname{meas}_{Q_T}\{|\nabla\theta|>\mu, |\theta|>k\}^{\frac{N+1}{N+2}}.&\end{aligned} \tag{2.126}$$

From (2.104) and (2.108) it follows that

$$\begin{aligned}\mu \operatorname{meas}_{Q_T}\{|\nabla\theta|>\mu\}^{\frac{N+1}{N+2}} &\le \mu\Big(\frac{Mk+L}{\mu^2}\Big)^{\frac{N+1}{N+2}} + \frac{\mu}{k^{\frac{N+1}{N}}}\Lambda\\ &\le \frac{C(N)}{\mu^{\frac{N}{N+2}}}\Big(\big(Mk\big)^{\frac{N+1}{N+2}} + L^{\frac{N+1}{N+2}}\Big) + \frac{\mu}{k^{\frac{N+1}{N}}}\Lambda\end{aligned} \tag{2.127}$$

where
$$\Lambda = C(N,\Omega,T)\Big(\|\theta\|^{\frac{1}{N}}_{L^\infty(0,T;L^1(\Omega))}L^{\frac12} + \|\theta\|^{\frac{2(N+1)}{N(N+2)}}_{L^\infty(0,T;L^1(\Omega))}M^{\frac{N+1}{N+2}}\Big).$$
By writing
$$\frac{C(N)}{\mu^{\frac{N}{N+2}}}\Big(\Big(Mk\Big)^{\frac{N+1}{N+2}} + L^{\frac{N+1}{N+2}}\Big) + \frac{\mu}{k^{\frac{N+1}{N}}}\Lambda = C(N)\Big(\frac{k^{\frac{N+1}{N}}}{\mu}\Big)^{\frac{N}{N+2}}M^{\frac{N+1}{N+2}} + \frac{C(N)}{\mu^{\frac{N}{N+2}}}L^{\frac{N+1}{N+2}} + \frac{\mu}{k^{\frac{N+1}{N}}}\Lambda$$
and by choosing k such that
$$\frac{k^{\frac{N+1}{N}}}{\mu} = \|\theta\|^{\frac{1}{N}}_{L^\infty(0,T;L^1(\Omega))}$$
a few computations allow one to deduce that, $\forall \mu > 0$,
$$\mu\,\mathrm{meas}_{Q_T}\{|\nabla\theta| > \mu\}^{\frac{N+1}{N+2}} \le C(N,\Omega,T)\Big(\|\theta\|^{\frac{1}{N+2}}_{L^\infty(0,T;L^1(\Omega))}M^{\frac{N+1}{N+2}} + \frac{L^{\frac{N+1}{N+2}}}{\mu^{\frac{N}{N+2}}} + L^{\frac12}\Big). \tag{2.128}$$
If follows that
$$\begin{aligned}
\sup_{\mu>0}\mu\,\mathrm{meas}_{Q_T}\{|\nabla\theta| > \mu\}^{\frac{N+1}{N+2}} &\le \sup_{0<\mu\le\mu_0}\mu\,\mathrm{meas}_{Q_T}\{|\nabla\theta| > \mu\}^{\frac{N+1}{N+2}}\\
&\quad + \sup_{\mu_0<\mu}\mu\,\mathrm{meas}_{Q_T}\{|\nabla\theta| > \mu\}^{\frac{N+1}{N+2}}\\
&\le \mu_0|Q_T|^{\frac{N+1}{N+2}}\\
&\quad + \sup_{\mu_0<\mu}C(N,\Omega,T)\Big(\|\theta\|^{\frac{1}{N+2}}_{L^\infty(0,T;L^1(\Omega))}M^{\frac{N+1}{N+2}} + \frac{L^{\frac{N+1}{N+2}}}{\mu^{\frac{N}{N+2}}} + L^{\frac12}\Big)\\
&\le \mu_0|Q_T|^{\frac{N+1}{N+2}}\\
&\quad + C(N,\Omega,T)\Big(\|\theta\|^{\frac{1}{N+2}}_{L^\infty(0,T;L^1(\Omega))}M^{\frac{N+1}{N+2}} + \frac{L^{\frac{N+1}{N+2}}}{\mu_0^{\frac{N}{N+2}}} + L^{\frac12}\Big).
\end{aligned}$$
By taking $\mu_0 = L^{1/2}$ in the previous inequality we deduce that, for any $\mu > 0$,
$$\mu\,\mathrm{meas}_{Q_T}\{|\nabla\theta| > \mu\}^{\frac{N+1}{N+2}} \le C(N,\Omega,T)\Big(\|\theta\|^{\frac{1}{N+2}}_{L^\infty(0,T;L^1(\Omega))}M^{\frac{N+1}{N+2}} + L^{\frac12}\Big). \tag{2.129}$$
We are now in the position to obtain (2.107). For any real number $1 \le p$, we write
$$\begin{aligned}
\int_{Q_T}|\nabla\theta|^p\,\mathrm{d}x\mathrm{d}t &= \int_0^\infty \mathrm{meas}_{Q_T}\{|\nabla\theta|^p > s\}\,\mathrm{d}s\\
&\le \int_0^\beta \mathrm{meas}_{Q_T}\{|\nabla\theta|^p > s\}\,\mathrm{d}s + \int_\beta^\infty \mathrm{meas}_{Q_T}\{|\nabla\theta|^p > s\}\,\mathrm{d}s\\
&\le \beta|Q_T| + \int_\beta^\infty \mathrm{meas}_{Q_T}\{|\nabla\theta|^p > s\}\,\mathrm{d}s.
\end{aligned} \tag{2.130}$$
In view of (2.129) we have
$$\mathrm{meas}_Q\{|\nabla\theta|^p > s\} \le \frac{\Sigma^{\frac{N+2}{N+1}}}{s^{\frac{N+2}{(N+1)p}}} \tag{2.131}$$

where

$$\Sigma = C(N,\Omega)\Big(\|\theta\|_{L^\infty(0,T;L^1(\Omega))}^{\frac{1}{N+2}} M^{\frac{N+1}{N+2}} + L^{\frac{1}{2}}\Big).$$

From (2.130) and (2.131) it follows that, for any $1 \le p < \frac{N+2}{N+1}$ and for any $\beta > 0$,

$$\|\nabla\theta\|_{L^p(Q_T)}^p \le \beta|Q_T| + \frac{1}{\frac{N+2}{(N+1)p} - 1} \frac{\Sigma^{\frac{N+2}{N+1}}}{\beta^{\frac{N+2}{(N+1)p}-1}}. \tag{2.132}$$

At last, by choosing $\beta = \Sigma^p$, we deduce that, for any $1 \le p < \frac{N+2}{N+1}$

$$\|\nabla\theta\|_{L^p(Q_T)}^p \le C(N,\Omega,T,p)\Sigma^p, \tag{2.133}$$

which is (2.107). □

2.6 Uniqueness result

In this section, we prove the uniqueness of the renormalized solution to problem (2.2). In addition to the assumptions (2.3), (2.3), (2.4) and (2.9), we assume that A is locally Lipschitz with respect to s, that is for any compact subset H of $\mathbb{R}$, there exists $M_H > 0$ such that

$$|A(x,t,s) - A(x,t,s')| \le M_H|s-s'|, \text{ for a.e. } (x,t) \text{ in } Q_T, \tag{2.134}$$

and for every s and s' in H. Let us now state the uniqueness result.

Theorem 2.6.1. *Assume that* (2.3)-(2.9) *and* (2.134) *hold. Then Problem* (2.2) *has a unique renormalized solution.*

Before proving Theorem 2.6.1, we recall a technical Lemma that is established in [50], (see Lemma 4.1) and which is crucial to control the behavior of some terms that are present in the proof of the uniqueness. This Lemma is related to the decay of the truncated energy, namely condition (2.15) in Definition 2.2.3. It is worth noting that since the matrix A depends on the solution and since we have the term $\mathrm{div}(g)$ in equation (2.2), we cannot prove an L^1 contraction principle. The main difficulty is to find a test function in the difference between the two solutions which allows one to control the time derivative, the terms coming from the operator and the terms coming from $-\mathrm{div}(g)$.

Lemma 2.6.2. *Under assumptions* (2.5)-(2.6) *and* (2.134)*, let θ^1, θ^2 be two renormalized solutions to* (2.2) *and g_2 a function in $L^2(Q_T)$. We define, for any $0 < k < s$,*

$$\begin{aligned}\Gamma(\theta^1,\theta^2,s,k) = &\int_{\{s-k<|\theta^1|<s+k\}} (A(x,t,\theta^1)\nabla\theta^1\nabla\theta^1 + |g_2|^2)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\{s-k<|\theta^2|<s+k\}} (A(x,t,\theta^2)\nabla\theta^2\nabla\theta^2) + |g_2|^2)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.135}$$

Then, we have that

$$\liminf_{s\to\infty}\limsup_{k\to 0}\frac{1}{k}\Gamma(\theta^1,\theta^2,s,k) = 0. \tag{2.136}$$

Proof of Theorem 2.6.1 We proceed by dividing the proof into 3 steps. We will adapt the strategy developed in [23] and [50] for the nonlinear parabolic problem with Dirichlet boundary conditions, to our problem with Neumann boundary conditions.

Step 1. Let θ^1 and θ^2 be two renormalized solutions of (2.2) for the same data f, g, u, h and θ_0. For any real number $s>0$ and $\sigma>0$, we define the function T_s^σ, which is roughly speaking a regularized truncation function, by

$$T_s^\sigma(0)=0,\qquad (T_s^\sigma)'(r)=\begin{cases}1 & \text{for} \quad |r|<s,\\ \dfrac{1}{\sigma}(s+\sigma-|r|) & \text{for} \quad s\leqslant |r|\leqslant s+\sigma,\\ 0 & \text{for} \quad |r|>s+\sigma.\end{cases} \tag{2.137}$$

Since $T_s^\sigma\in W^{2,\infty}(\mathbb{R})$ and supp $(T_s^\sigma)'\subset[-s-\sigma,s+\sigma]$, we can take $S=T_s^\sigma$ (2.16) for θ^1 and θ^2: for any $\varphi\in C_c^\infty([0,T)\times\bar{\Omega})$, the equation (2.16) in θ_i, for $i\in\{1,2\}$, is

$$\begin{aligned}-\int_{Q_T}\varphi_t T_s^\sigma(\theta^i)\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\varphi(0)T_s^\sigma(\theta_0)\,\mathrm{d}x+\int_{Q_T}A(x,t,\theta^i)\nabla\theta^i\cdot\nabla\theta^i(T_s^\sigma)''(\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t\\ +\int_{Q_T}(T_s^\sigma)'(\theta^i)A(x,t,\theta^i)\nabla\theta^i\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}u\cdot\nabla\theta^i(T_s^\sigma)'(\theta^i)\varphi\,\mathrm{d}x\,\mathrm{d}t\\ =\int_{Q_T}f(T_s^\sigma)'(\theta^i)\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}g\nabla((T_s^\sigma)'(\theta^i)\varphi)\,\mathrm{d}x\,\mathrm{d}t+\int_{\partial\Omega\times(0,T)}h(T_s^\sigma)'(\theta^i)\varphi\,\mathrm{d}\sigma(x)\,\mathrm{d}t\end{aligned}$$

By making the difference between the two equations and we can apply Lemma 2.3.4 with $T_s^\sigma(\theta^1)-T_s^\sigma(\theta^2)$ in place of w, $G(r)=T_k(r)/k$. It follows that for any $k>0$ and for any $\xi\in\mathcal{C}_c^\infty([0,T))$ with $0\le\xi\le1$:

$$\begin{aligned}-\frac{1}{k}\int_0^T\xi_t\widetilde{T}_k(T_s^\sigma(\theta^1)-T_s^\sigma(\theta^2))\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{k}(A_{s,k}^\sigma+\tilde{A}_{s,k}^\sigma+B_{s,k}^\sigma)\\ =\frac{1}{k}(F_{s,k}^\sigma+G_{s,k}^\sigma+H_{s,k}^\sigma).\end{aligned} \tag{2.138}$$

where

$$\begin{aligned}A_{s,k}^\sigma&=\int_{Q_T}\xi[(T_s^\sigma)'(\theta^1)A(x,t,\theta^1)\nabla\theta^1-(T_s^\sigma)'(\theta^2)A(x,t,\theta^2)\nabla\theta^2]\\ &\qquad\times\nabla T_k(T_s^\sigma(\theta^1)-T_s^\sigma(\theta^2))\,\mathrm{d}x\ \mathrm{d}t,\\ \tilde{A}_{s,k}^\sigma&=\int_{Q_T}\xi(T_s^\sigma)''(\theta^1)A(x,t,\theta^1)\nabla\theta^1\nabla\theta^1T_k(T_s^\sigma(\theta^1)-T_s^\sigma(\theta^2))\,\mathrm{d}x\ \mathrm{d}t\\ &\quad-\int_{Q_T}\xi(T_s^\sigma)''(\theta^2)A(x,t,\theta^2)\nabla\theta^2\nabla\theta^2T_k(T_s^\sigma(\theta^1)-T_s^\sigma(\theta^2))\,\mathrm{d}x\ \mathrm{d}t,\\ B_{s,k}^\sigma&=\int_{Q_T}\xi u\cdot[\nabla\theta^1(T_s^\sigma)'(\theta^1)-\nabla\theta^2(T_s^\sigma)'(\theta^2)]T_k(T_s^\sigma(\theta^1)-T_s^\sigma(\theta^2))\,\mathrm{d}x\ \mathrm{d}t,\\ F_{s,k}^\sigma&=\int_{Q_T}\xi f[(T_s^\sigma)'(\theta^1)-(T_s^\sigma)'(\theta^2)]T_k(T_s^\sigma(\theta^1)-T_s^\sigma(\theta^2))\,\mathrm{d}x\ \mathrm{d}t,\\ G_{s,k}^\sigma&=\int_{Q_T}\xi g\nabla\Big[\Big((T_s^\sigma)'(\theta^1)-(T_s^\sigma)'(\theta^2)\Big)T_k(T_s^\sigma(\theta^1)-T_s^\sigma(\theta^2))\Big]\,\mathrm{d}x\,\mathrm{d}t,\\ H_{s,k}^\sigma&=\int_{\partial\Omega\times(0,T)}\xi h[(T_s^\sigma)'(\theta^1)-(T_s^\sigma)'(\theta^2)]T_k(T_s^\sigma(\theta^1)-T_s^\sigma(\theta^2))\ \mathrm{d}\sigma(x)\,\mathrm{d}t.\end{aligned}$$

By writing $B^\sigma_{s,k}$ as

$$B^\sigma_{s,k} = \int_{Q_T} \xi u \cdot \widetilde{T}_k(T^\sigma_s(\theta^1) - T^\sigma_s(\theta^2))\,\mathrm{d}x\,\mathrm{d}t,$$

using the fact that $\mathrm{div}(u) = 0$ in Ω and that ξ does not depend on the x-variable, we have by the Stokes formula that

$$B^\sigma_{s,k} = 0.$$

Step 2. This step is devoted to study each term of (2.138) in order to pass to the limit when $\sigma \to 0$, $k \to 0$ and $s \to \infty$.

First, we observe that for a fixed $s > 0$, we have as σ tends to 0

$$T^\sigma_s(\theta^1) \to T_s(\theta^1) \text{ strongly in } L^2(0,T;H^1(\Omega)) \text{ and a.e in } Q_T, \tag{2.139}$$

$$(T^\sigma_s)'(\theta^1) \to \chi_{\{|\theta^1|\leqslant s\}} \text{ weak-}* \text{ in } L^\infty(Q_T) \text{ and a.e in } \Omega\times(0,t), \tag{2.140}$$

and the analog results for θ^2.

In view of the definition of $\widetilde{T}_k$ we have, for any $r \in \mathbb{R}$,

$$\frac{\widetilde{T}_k}{k}(r) = \frac{1}{k}\int_0^r T_k(s)\,\mathrm{d}s \to |r|, \qquad \text{as } k \to 0$$
$$0 \leq \frac{\widetilde{T}_k}{k}(r) \leq |r|.$$

By (2.139) we then obtain

$$\lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}\int_{Q_T}\xi_t\widetilde{T}_k(T^\sigma_s(\theta^1) - T^\sigma_s(\theta^2))\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T}\xi_t|T_s(\theta^1) - T_s(\theta^2)|\,\mathrm{d}x\,\mathrm{d}t. \tag{2.141}$$

We now turn to $A^\sigma_{s,k}$. We prove that for almost any $t \in (0,T)$

$$\liminf_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}A^\sigma_{s,k} \geqslant 0 \text{ for any } s > 0. \tag{2.142}$$

We will adapt the proof developed in [23] and explain the arguments here for completeness. Since $\mathrm{supp}(T^\sigma_s)' \subset [-s-1, s+1]$ for $0 < \sigma < 1$, we get from (2.139) and (2.140) that

$$\begin{aligned}\lim_{\sigma\to 0}\frac{1}{k}A^\sigma_{s,k} = \frac{1}{k}\int_{Q_T}\xi[\chi_{\{|\theta^1|\leqslant s\}}A(x,t,T_{s+1}(\theta^1))\nabla T_{s+1}(\theta^1) - \chi_{\{|\theta^2|\leqslant s\}}A(x,t,T_{s+1}(\theta^2))\nabla T_{s+1}(\theta^2)]\\ \times \nabla T_k(T_s(\theta^1) - T_s(\theta^2))\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

Using the fact that $\chi_{\{|\theta^1|\leqslant s\}}A(x,t,T_{s+1}(\theta^1)) = \chi_{\{|\theta^1|\leqslant s\}}A(x,t,T_s(\theta^1))$ and $\chi_{\{|\theta^2|\leqslant s\}}A(x,t,T_{s+1}(\theta^2)) = \chi_{\{|\theta^2|\leqslant s\}}A(x,t,T_s(\theta^2))$, we can rewrite it as

$$\begin{aligned}\lim_{\sigma\to 0}\frac{1}{k}A^\sigma_{s,k} &= \frac{1}{k}\int_{Q_T}\xi[A(x,t,T_s(\theta^1))\nabla T_s(\theta^1) - A(x,t,T_s(\theta^1))\nabla T_s(\theta^2)]\\ &\qquad\qquad \times \nabla T_k(T_s(\theta^1) - T_s(\theta^2))\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \frac{1}{k}\int_{Q_T}\xi[A(x,t,T_s(\theta^1))\nabla T_s(\theta^2) - A(x,t,T_s(\theta^2))\nabla T_s(\theta^2)]\\ &\qquad\qquad \times \nabla T_k(T_s(\theta^1) - T_s(\theta^2))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.143}$$

Given assumption (2.3), the first term on the right-hand side of (2.143) is non-negative, so that to prove (2.142), we need to show that the second term tends to zero when k tends to 0. By assumption (2.134) we have

$$
\begin{aligned}
&\frac{1}{k}\left|\int_{Q_T}\xi[A(x,t,T_s(\theta^1))\nabla T_s(\theta^2)-A(x,t,T_s(\theta^2))\nabla T_s(\theta^2)]\times\nabla T_k(T_s(\theta^1)-T_s(\theta^2))\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\leq\frac{1}{k}\int_{Q_T}\xi\chi_{\{0<|T_s(\theta^1)-T_s(\theta^2)|<k\}}M_s|T_s(\theta^1)-T_s(\theta^2)||\nabla T_s(\theta^2)|\times|\nabla(T_s(\theta^1)-T_s(\theta^2))|\,\mathrm{d}x\,\mathrm{d}t\\
&\leq\int_{Q_T}\xi\chi_{\{0<|T_s(\theta^1)-T_s(\theta^2)|<k\}}M_s|\nabla T_s(\theta^2)|\times|\nabla(T_s(\theta^1)-T_s(\theta^2))|\,\mathrm{d}x\,\mathrm{d}t,\\
&\leq\int_{Q_T}\xi\chi_{\{0<|T_s(\theta^1)-T_s(\theta^2)|<k\}}M_s|\nabla T_s(\theta^2)|\times(|\nabla T_s(\theta^1)|+|\nabla T_s(\theta^2)|)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where $M_s>0$ is given by (2.134).

Since θ^1 and θ^2 are renormalized solutions of (2.2), by Definition 2.2.3 we know that $|\nabla T_s(\theta^2)|\times(|\nabla(T_s(\theta^1)+|\nabla T_s(\theta^2)|)\in L^1(Q_T)$. In addition, we have that $\chi_{\{0<|T_s(\theta^1)-T_s(\theta^2)|<k\}}$ goes to 0 as k goes to 0 a.e in Q_T. Thus the Lebesgue dominated Theorem implies that

$$
\int_{Q_T}\xi\chi_{\{0<|T_s(\theta^1)-T_s(\theta^2)|<k|\}}M_H|\nabla T_s(\theta^2)|\times|\nabla(T_s(\theta^1)-T_s(\theta^2))|\,\mathrm{d}x\,\mathrm{d}t\longrightarrow 0\text{ as }k\longrightarrow 0,
$$

which gives (2.142).

As far as $\tilde{A}^{\sigma}_{s,k}$ is concerned, we observe that due to the definition (2.137) of T^{σ}_s, the assumption (2.3) and the fact that $\nabla u=0$ almost everywhere on $\{(x,t);\,u(x,t)=r\}$ for any $r\in\mathbb{R}$, that we have for any σ and any $k>0$,

$$
\begin{aligned}
\frac{1}{k}|\tilde{A}^{\sigma}_{s,k}|\leq\frac{1}{\sigma}\Bigg[&\int_{\{s<|\theta^1|<s+\sigma\}}A(x,t,\theta^1)\nabla\theta^1\nabla\theta^1\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{\{s<|\theta^2|<s+\sigma\}}A(x,t,\theta^2)\nabla\theta^2\nabla\theta^2\,\mathrm{d}x\,\mathrm{d}t\Bigg].
\end{aligned}
\tag{2.144}
$$

For the term $F^{\sigma}_{s,k}$, since $f\in L^1(Q_T)$, we have by (2.139), (2.140) and the Lebesgue dominated Theorem that

$$
\lim_{\sigma\to 0}\frac{1}{k}F^{\sigma}_{s,k}=\frac{1}{k}\int_{Q_T}\xi f(\chi_{\{|\theta^1|\leq s\}}-\chi_{\{|\theta^2|\leq s\}})T_k(T_s(\theta^1)-T_s(\theta^2))\,\mathrm{d}x\,\mathrm{d}\tau.
$$

From the point-wise convergence of $\dfrac{T_k}{k}(T_s(\theta^1)-T_s(\theta^2))$ to $\operatorname{sgn}(T_s(\theta^1)-T_s(\theta^2))$ as $k\to 0$ a.e in Q_T, it follows that

$$
\lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}F^{\sigma}_{s,k}=\int_{Q_T}\xi f(\chi_{\{|\theta^1|\leq s\}}-\chi_{\{|\theta^2|\leq s\}})\operatorname{sgn}(T_s(\theta^1)-T_s(\theta^2))\,\mathrm{d}x\,\mathrm{d}t,
$$

where sgn is the usual sign function (i.e., $\operatorname{sgn}(r)=r/|r|$ if $r\neq 0$ and $\operatorname{sgn}(0)=0$). Moreover, since θ^1 and θ^2 are finite a.e. in Q_T the function $f(\chi_{\{|\theta^1|\leq s\}}-\chi_{\{|\theta^2|\leq s\}})$ converges to 0 a.e in Q_T as s tends to $+\infty$ and it is bounded by $2|f|$ which is a $L^1(Q_T)$ function. The Lebesgue-dominated Theorem implies that

$$
\lim_{s\to+\infty}\lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}F^{\sigma}_{s,k}=0.\tag{2.145}
$$

Similarly, since h belongs to L^1 and since the traces of θ^1 and θ^2 are finite a.e. in $\partial\Omega \times (0,T)$ we have

$$\lim_{s\to+\infty} \lim_{k\to 0} \lim_{\sigma\to 0} \frac{1}{k} H^\sigma_{s,k} = 0. \tag{2.146}$$

We now study $G^\sigma_{s,k}$ which is rewritten as

$$G^\sigma_{s,k} = G^{\sigma,1}_{s,k} + G^{\sigma,2}_{s,k}$$

where

$$G^{\sigma,1}_{s,k} = \int_{Q_T} \xi g[(T^\sigma_s)'(\theta^1) - (T^\sigma_s)'(\theta^2)]\nabla T_k(T^\sigma_s(\theta^1) - T^\sigma_s(\theta^2)) \,\mathrm{d}x\,\mathrm{d}t$$
$$G^{\sigma,2}_{s,k} = \int_{Q_T} \xi g\nabla[(T^\sigma_s)'(\theta^1) - (T^\sigma_s)'(\theta^2)]T_k(T^\sigma_s(\theta^1) - T^\sigma_s(\theta^2)) \,\mathrm{d}x\,\mathrm{d}t.$$

For $G^{\sigma,2}_{s,k}$, we argue as in the case of $\tilde{A}^\sigma_{s,k}$ so that, for any σ and any $k > 0$ we have

$$\frac{1}{k}|G^{\sigma,2}_{s,k}| \leqslant \frac{1}{\sigma}\left[\int_{\{s<|\theta^1|<s+\sigma\}} |g||\nabla\theta^1| \,\mathrm{d}x\,\mathrm{d}t + \int_{\{s<|\theta^2|<s+\sigma\}} |g||\nabla\theta^2| \,\mathrm{d}x\,\mathrm{d}t\right].$$

It follows that by (2.3) and Young's inequality we obtain

$$\begin{aligned}\frac{1}{k}|G^{\sigma,2}_{s,k}| \leqslant \frac{\beta}{\sigma}\Bigg[&\int_{\{s<|\theta^1|<s+\sigma\}} (|g|^2 + A(x,t,\theta^1)\nabla\theta^1\nabla\theta^1) \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\{s<|\theta^1|<s+\sigma\}} (|g|^2 + A(x,t,\theta^2)\nabla\theta^2\nabla\theta^2) \,\mathrm{d}x\,\mathrm{d}t\Bigg],\end{aligned} \tag{2.147}$$

where β is a generic constant which depends only on α.

On the other hand, due to (2.139) and (2.140) we have, for any $s > 0$ and any $k > 0$ that

$$\lim_{\sigma\to 0} \frac{1}{k}|G^{\sigma,1}_{s,k}| = \left|\frac{1}{k}\int_{Q_T} \xi[\chi_{\{|\theta^1|\leqslant s\}}g - \chi_{\{|\theta^2|\leqslant s\}}g]\nabla T_k(T_s(\theta^1) - T_s(\theta^2)) \,\mathrm{d}x\,\mathrm{d}t\right|.$$

It follows that

$$\limsup_{\sigma\to 0} \frac{1}{k}|G^{\sigma,1}_{s,k}| \leqslant \overline{G}_{s,k} + \widehat{G}_{s,k},$$

where

$$\overline{G}_{s,k} = \frac{1}{k}\int_{Q_T} \chi_{\{|\theta^1|\leqslant s\}\cap\{|\theta^2|>s|\}}|g||\nabla T_k(\theta^1 - s\,\mathrm{sgn}(\theta^2))| \,\mathrm{d}x\,\mathrm{d}t,$$
$$\widehat{G}_{s,k} = \frac{1}{k}\int_{Q_T} \chi_{\{|\theta^2|\leqslant s\}\cap\{|\theta^1|>s|\}}|g||\nabla T_k(\theta^2 - s\,\mathrm{sgn}(\theta^1))| \,\mathrm{d}x\,\mathrm{d}t.$$

The arguments used in the estimate of $G^{\sigma,2}_{s,k}$ and the fact that

$$\overline{G}_{s,k} \leqslant \frac{1}{k}\int_{\{s-k<|\theta^1|\leqslant s\}} |g||\nabla\theta^1| \,\mathrm{d}x\,\mathrm{d}t \quad \text{and} \quad \widehat{G}_{s,k} \leqslant \frac{1}{k}\int_{\{s-k<|\theta^2|\leqslant s\}} |g||\nabla\theta^2| \,\mathrm{d}x\,\mathrm{d}t,$$

allow us to prove that

$$\overline{G}_{s,k} \leqslant \frac{\beta}{k}\int_{\{s-k<|\theta^1|\leqslant s\}} (|g|^2 + A(x,t,\theta^1)\nabla\theta^1\nabla\theta^1) \,\mathrm{d}x\,\mathrm{d}\tau, \tag{2.148}$$

and

$$\widehat{G}_{s,k} \leqslant \frac{\beta}{k} \int_{\{s-k<|\theta^2|\leqslant s\}} (|g|^2 + A(x,t,\theta^2)\nabla\theta^2\nabla\theta^2)\,\mathrm{d}x\,\mathrm{d}\tau, \tag{2.149}$$

where β is a constant depending upon α.

Step 3. In this step we prove that $\theta^1 = \theta^2$ almost everywhere in Q_T by passing to the infimum limit as σ goes to 0 and then to the infimum limit as k goes to 0 in (2.138).

From (2.144) and (2.147) we explicitly remark that, for any $k > 0$,

$$\frac{1}{k}(|\tilde{A}^{\sigma}_{s,k}| + |G^{\sigma,2}_{s,k}|) \leqslant \frac{M_1}{\sigma}\Gamma(\theta^1,\theta^2,s,\sigma), \tag{2.150}$$

where M_1 is a constant independent of s, k and σ, and where Γ is defined in Lemma 2.6.2

Moreover from the study of $G^{\sigma,1}_{s,k}$ (2.148) and (2.149) imply that

$$\limsup_{\sigma\to 0} \frac{1}{k}|G^{\sigma,1}_{s,k}| \leq \frac{1}{k}(|\overline{G}_{s,k}| + |\widehat{G}_{s,k}|) \leqslant \frac{M_2}{k}\Gamma(\theta^1,\theta^2,s,k), \tag{2.151}$$

where M_2 is a constant independent of s, k and σ.

Passing to the limit in (2.141) allows one to conclude that

$$\begin{aligned}
-\int_{Q_T} \xi_t |T_s(\theta^1) - T_s(\theta^2)|\,\mathrm{d}x\,\mathrm{d}t \leqslant & -\liminf_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}A^{\sigma}_{s,k} + \liminf_{\sigma\to 0}\frac{M_1}{\sigma}\Gamma(\theta^1,\theta^2,s,\sigma),\\
& + \liminf_{k\to 0}\frac{M_2}{k}\Gamma(\theta^1,\theta^2,s,k) + \lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}F^{\sigma}_{s,k},\\
& + \lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}H^{\sigma}_{s,k}
\end{aligned}$$

for any $s > 0$.

It follows that by using (2.142), (2.145), (2.146) we obtain that

$$\liminf_{s\to+\infty} -\int_{Q_T} \xi_t|T_s(\theta^1) - T_s(\theta^2)|\,\mathrm{d}x\,\mathrm{d}t \leqslant (M_1+M_2)\limsup_{k\to 0}\frac{1}{k}\Gamma(\theta^1,\theta^2,s,k) \tag{2.152}$$

By taking a non increasing function ξ and recalling that θ^1 and θ^2 are finite almost everywhere in Q_T, $T_s(\theta^1)$ and $T_s(\theta^2)$ converge a.e. to θ^1 and θ^2 respectively as $s \to +\infty$. Then Fatou's Lemma implies that

$$-\int_{Q_T} \xi_t|\theta^1 - \theta^2|\,\mathrm{d}x\,\mathrm{d}t \leqslant 2M_1 \liminf_{s\to+\infty}\limsup_{k\to 0}\frac{1}{k}\Gamma(\theta^1,\theta^2,s,k). \tag{2.153}$$

Hence, using Lemma 2.6.2 we obtain that, for any non-increasing $\xi \in C^\infty_c([0,T))$ with $0 \leq \xi \leq 1$,

$$-\int_{Q_T} \xi_t|\theta^1 - \theta^2|\,\mathrm{d}x = 0. \tag{2.154}$$

It follows that $\theta^1 = \theta^2$ a.e. in Q_T which gives us the uniqueness of the renormalized solution of (2.2). $\square$

Perspectives

To our knowledge, the approximation by V.F schemes of the (renormalized) solution to the elliptic equation with L^1 data with Dirichlet or Neumann boundary conditions is limited to the linear case (or with a very specific quasi-linear operator $v \mapsto -\operatorname{div}(\lambda(u)\nabla u)$). These models allow one to impose additional orthogonality restrictions on the meshes. A natural perspective is the approximation of non-linear equations such as $-\Delta_p u = f$ (or nonlinear Leray-Lions operators, $v \mapsto -\operatorname{div}(a(x,v,\nabla v))$ with f belonging to L^1 and the link with renormalized solutions. Even in the variational case, the usual method of the F.V. method (see e.g. [60]) does not apply. In dimension 2 it is possible to use Discrete Duality Finite Volume Schemes (see [5]). More recently Gradient Discretisation Methods (see [55]) have been developed and they can handle the p-Laplace operator or more general nonlinear Leray-Lions operator in the variational. Notice that in the context of weak solutions or solutions in the sense of distributions, the arguments given in [6] allow one to consider the approximation of the solution by CFVE or DDVF method of various nonlinear elliptic problems with L^1 data, based on continuity arguments [4] and Young measures.

As far as the parabolic equation is concerned the very first perspective is the generalization to a class of nonlinear operator with p growth. In fact, for the purposes of the system studied in the second part, we have restricted ourselves to the quasi-linear case. Considering the prototype

$$\partial_t b(u) - \operatorname{div}(a(x,t,u,\nabla u)) = f$$

with non homogeneous Neumann boundary condition, where $v \mapsto -\operatorname{div}(a(x,t,v,\nabla v))$ is Leray-Lions operator with p growth, is possible. Depending on the regularity of b (a maximal monotone graph or an increasing function C^1) the difficulties are different.

In connection with the approximation by V.F. of a nonlinear elliptic equation with L^1 data and Neumann boundary conditions in Chapter 1 the analysis of the convergence of V.F. schemes to

$$\begin{cases} \dfrac{\partial\theta}{\partial t} - \operatorname{div}\left(\lambda(\theta)\nabla\theta\right) + u\cdot\nabla\theta = f - \operatorname{div} g & \text{in } Q_T\ , \\ \theta(t=0) = \theta_0 & \text{in } \Omega\ , \\ \Big(\lambda\theta)\nabla\theta - g\Big)\cdot\vec{n} = h & \text{on } \partial\Omega\times(0,T)\ , \end{cases}$$

is a natural extension to Part 1 of the present work. The case of Dirichlet conditions (and without the term $u\cdot\nabla\theta$ has been performed in [74]: the authors show that the V.F. approximations converge, in some sense, to the renormalized solution. As in the static case, one of the main difficulties is that the renormalized formulation is non linear and contains terms like $S''(\theta)\nabla\theta\cdot\nabla\theta$. In [74] the authors mix discrete version of Aubin-Lions compactness results which are proved in [70, 67] and the techniques of parabolic equation with L^1 data (in particular [24, 90]).

At last numerical experiments with test cases should be integrated. Notice that the elliptic case should include a computation of the median of the solution.

Part II

Solutions for systems that model non-isothermal solidification

Chapter 3

Weak-renormalized solutions for a Navier-Stokes-Boussinesq type system in 2D

3.1 Introduction

In this chapter, we are interested in the existence of a weak-renormalized solution, in the two-dimensional case, for a problem that models non-isothermal solidification with melt convection. The most effective way of representing this type of problem is the phase field model, first studied for modeling the solidification process for pure materials and binary alloys (see for instance [36, 37, 83]), and then developed to include melt convection (see for example [3, 12, 63, 92]).

The phase field models describe the phase transition process; in this work, we suppose that the solid phase is stationary and rigid. The model we consider consists of three equations: the phase-field equation where the phase function is equal to 0 in the liquid state and 1 in the solid state, coupled with the heat equation, and the Navier-Stokes equation with variable viscosity for the non-solid regions. More precisely, we study the following coupled nonlinear system

$$
\begin{aligned}
&\varphi_t + u\cdot\nabla\varphi - \xi\Delta\varphi = -\varphi(1-\varphi)(1-2\varphi+c\theta) && \text{in } Q_T, && (3.1)\\
&\theta_t - \operatorname{div}(\kappa(\varphi,\theta)\nabla\theta) + u\cdot\nabla\theta = \nu(\varphi,\theta)D(u):D(u) + g + l\varphi_t && \text{in } Q_T, && (3.2)\\
&u_t + (u\cdot\nabla)u - \operatorname{div}(\nu(\varphi,\theta)D(u)) + \nabla p = f, \quad \operatorname{div} u = 0 && \text{in } Q_T, && (3.3)\\
&\nabla\varphi\cdot\boldsymbol{n} = 0, \quad \kappa(\varphi,\theta)\nabla\theta\cdot\boldsymbol{n} = 0, \quad u = 0 && \text{on } \Sigma, && (3.4)\\
&\varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0(x), \quad u(x,0) = u_0(x) && \text{in } \Omega, && (3.5)
\end{aligned}
$$

where $Q_T = \Omega\times(0,T)$ is the space-time cylinder with lateral surface $\Sigma = \partial\Omega\times(0,T)$, Ω is a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$, $N=2$, $T>0$, and $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$. Here, φ is the phase-field function which represents the solid/liquid phase, ξ is a positive constant related to the width of the transitions layer, c is a positive constant; θ is the temperature, $l>0$ is associated to the latent heat, κ and ν are two positive functions depending on φ and θ, representing the heat diffusion and the kinematic fluid viscosity respectively; u is the velocity field, p is the associated hydrostatic pressure, f is an external field and $D(u)$ is the deformation tensor, i.e. $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$.

Regarding the mathematical analysis of system (3.1)-(3.5), we note three major difficulties. First, the nonlinearity present in the phase equation and represented by the term $-\varphi(1-\varphi)(1-$

$2\varphi+c\theta)$, second the presence of the nonlinear term $D(u):D(u)$ in the energy equation (3.2) and which belongs to L^1 and finally the convection term $u\cdot\nabla\theta$. We note that most of the literature focuses on the linear system where the term $D(u):D(u)$ is neglected and the viscosity is not variable or on the Boussinesq-type system that does not involve the phase equation. To deal with this weak regularity, it is essential to use an appropriate notion of solution. In our work, we use the notion of the renormalized solution first introduced in [52] by DiPerna-Lions and adapted later to the elliptic [47, 81] and parabolic problem [23, 24]. Authors in [11] and [18] used this notion of solution and established an existence result of a weak renormalized solution for a nonlinear Boussinesq type system. For the nonlinear problems that model non-isothermal solidification, C-Vaz and E-Fernández-Cara have studied in [63] and [102] systems similar to the one we consider in this work. However, in both [63] and [102] the heat was considered to be zero on the boundary and the latent heat was assumed to be very small and therefore neglected.

This chapter aims to extend the work of [63], to the case where there is no heat exchange i.e. $\nabla\theta\cdot\boldsymbol{n}=0$ on $\partial\Omega\times(0,T)$ and to show the existence of a weak renormalized solution without assuming any smallness conditions on the constants in the equations. Our proof is based on a usual regularization technique and a priori estimates. We prove estimates using the coupling between the equations and then use the stability result shown in chapter 2 for the renormalized solutions for parabolic problems with Neumann boundary conditions and compactness argument to pass to the limit in the regularized problem.

This chapter is organized as follows. In Section 3.2 we introduce some functional spaces and give preliminary results on the Navier-stokes equations in the two-dimensional case, on the phase-field equation, and the renormalized solution of the heat equation. Then, in Section 3.3 we set assumptions on the data and give the definition of a weak-renormalized solution of the system (3.1)-(3.5). Section 3.4 is devoted to the statement of our main existence result and its proof. This section is divided into three subsections that constitute the proof: the presentation of the regularized problem, the derivation of estimates, and finally the application of the compactness and the stability arguments that allow us to pass to the limit in the regularized problem.

Finally, throughout this work, C, M and L represent various positive constants depending only on known quantities that will be stated.

3.2 Preliminary results and recalls

Let Ω be a bounded open domain of $\mathbb{R}^N$, $N=2$ or $N=3$ with Lipschitz boundary $\partial\Omega$. We note $Q_T=\Omega\times(0,T)$ the space-time cylinder with lateral surface $\Sigma=\partial\Omega\times(0,T)$ and $\boldsymbol{n}$ the unit outward normal to $\partial\Omega$. For $1\leq p\leq+\infty$, we denote by $L^p(\Omega)$ the usual Lebesgue space and by $W^{1,p}(\Omega)$ the usual Sobolev space. We recall in this section, some necessary results on the Navier-Stokes equations in two dimensions, the phase-field equation and the properties of the renormalized solution for parabolic-type equations with Neumann boundary conditions.

3.2.1 Navier-Stokes equations

We begin this subsection by introducing some classical functional spaces essential for the study of the Navier-stokes equations (see e.g. [101]). Let V be the space defined by

$$V = \{v \in C_0^\infty(\Omega)^N : \operatorname{div} v = 0 \text{ in } \Omega\}.$$

We denote by $\mathcal{H}$ and $\mathcal{V}$ the closures of V in $L^2(\Omega)^N$ and $H_0^1(\Omega)^N$ respectively. Then $\mathcal{H}$ is defined by

$$\mathcal{H} = \{v \in L^2(\Omega)^N : \operatorname{div} v = 0 \text{ in } \Omega,\ v \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$

and $\mathcal{V}$ by

$$\mathcal{V} = \{v \in H_0^1(\Omega)^N : \operatorname{div} v = 0 \text{ in } \Omega\}.$$

For f in $L^2(Q_T)$ and u_0 in $\mathcal{H}$, we consider the following Navier-Stokes equation

$$\begin{cases} u_t + (u\cdot\nabla)u - \operatorname{div}(\nu(x,t)D(u)) + \nabla p = f & \text{in } Q_T\ , \\ \operatorname{div} u = 0 & \text{in } Q_T\ , \\ u = 0 & \text{on } \partial\Omega \times (0,T)\ , \\ u(x,0) = u_0(x) & \text{in } \Omega\ , \end{cases} \tag{3.6}$$

where $\nu \in L^\infty(Q_T)$ such that $0 < \nu_1 \le \nu(x,t) \le \nu_2$ a.e. in Q_T.

Let us state the existence and uniqueness result of a weak solution to (3.6) in the two-dimensional case. This result is well known, it is based on the construction of an approximate solution by the Galerkin method (see e.g. [101] or [79] for the classical Navier-Stokes equations). Since we are in the very classical case of dimension 2, the pression p does not appear in Theorem 3.2.1 and is recovered by De Rham's Lemma (see [101]. Indeed, for a.e. $t \in (0,T)$, $\langle u_t + (u\cdot\nabla)u - \operatorname{div}(\nu(x,t)D(u)) - f, w\rangle = 0$ for any $w \in \mathcal{V}$, gives that there exists a distribution p such that $u_t + (u\cdot\nabla)u - \operatorname{div}(\nu(x,t)D(u)) - f = -\nabla p$.

Theorem 3.2.1. *Assume $N = 2$. There exists a unique weak solution u to (3.6) such that*

$$u \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}), \tag{3.7}$$

$$u_t \in L^2(0,T;\mathcal{V}'), \tag{3.8}$$

$$u(t=0) = u_0 \tag{3.9}$$

and

$$\begin{aligned} \int_0^T \langle u_t, v\rangle_{\mathcal{V}',\mathcal{V}}\, \mathrm{d}t + \int_{Q_T} \nu(x,t)\nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} (u\cdot\nabla)u \cdot v \,\mathrm{d}x\,\mathrm{d}t \\ = \int_{Q_T} f v \,\mathrm{d}x\,\mathrm{d}t \qquad \forall v \in L^2(0,T;\mathcal{V}). \end{aligned} \tag{3.10}$$

Moreover, the function u belongs to $C([0,T];\mathcal{H})$ and satisfies the energy equality

$$\frac{1}{2}\int_\Omega |u(t)|^2 \,\mathrm{d}x + \int_0^t \int_\Omega \nu(x,t)|D(u)|^2 \,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_\Omega |u_0|^2 \,\mathrm{d}x + \int_0^t \int_\Omega f\cdot u \,\mathrm{d}x\,\mathrm{d}s,$$

for any $t \in [0,T]$.

3.2.2 Phase-field equation

We give in this subsection an existence and uniqueness result for the following phase-field equation:

$$\begin{cases} \varphi_t - \xi\Delta\varphi + u\cdot\nabla\varphi = -\varphi(1-\varphi)(1-2\varphi+ch) & \text{in } Q_T\ , \\ \nabla\varphi\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega\times(0,T)\ , \\ \varphi(x,0) = \varphi_0(x) & \text{in } \Omega\ . \end{cases} \tag{3.11}$$

Here, we assume that

$$\xi > 0 \text{ and } c > 0, \tag{3.12}$$
$$\varphi_0 \in L^\infty(\Omega) \text{ such that } 0 \le \varphi_0 \le 1, \tag{3.13}$$
$$u \in L^2(0,T;\mathcal{V})\cap L^\infty(0,T;\mathcal{H}) \text{ such that } \operatorname{div} u = 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega\times(0,T), \tag{3.14}$$
$$h \in L^\infty(Q_T). \tag{3.15}$$

Theorem 3.2.2. *Assume that* (3.12)-(3.15) *hold true. Then, there exists a unique solution* φ *to* (3.11) *such that*

$$\varphi \in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)), \tag{3.16}$$
$$\varphi_t \in L^2(0,T;H^1(\Omega)'), \tag{3.17}$$
$$\varphi(t=0) = \varphi_0 \tag{3.18}$$
$$0 \le \varphi \le 1, \tag{3.19}$$

and

$$\begin{aligned} &\int_0^T \langle\varphi_t,\psi\rangle\,\mathrm{d}t + \xi\int_{Q_T}\nabla\varphi\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} uT_1(\varphi)\nabla\psi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{Q_T} -\varphi(1-\varphi)(1-2\varphi+ch)\cdot\psi\,\mathrm{d}x\,\mathrm{d}t \qquad \forall\psi\in L^2(0,T;H^1(\Omega)). \end{aligned} \tag{3.20}$$

Proof. We will apply the Leray-Schauder fixed point theorem to prove the existence of the solution of the problem (3.11). For this, we consider the mapping $\Gamma : L^2(Q_T) \to L^2(Q_T)$ defined by

$$\Gamma(\varphi) = \widetilde{\varphi}, \qquad \forall\varphi\in L^2(Q_T),$$

where $\widetilde{\varphi}$ is the unique solution of the following problem

$$\begin{cases} \widetilde{\varphi}_t - \xi\Delta\widetilde{\varphi} + \operatorname{div}(uT_1(\varphi)) = F_0(\varphi,h) & \text{in } Q_T\ , \\ \nabla\widetilde{\varphi}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega\times(0,T)\ , \\ \widetilde{\varphi}(t=0) = \varphi_0 & \text{in } \Omega\ , \end{cases} \tag{3.21}$$

with $u \in L^2(0,T;H^1(\Omega))$ such that $\operatorname{div}(u) = 0$ in Ω and $u = 0$ on $\partial\Omega$, $\varphi_0 \in L^\infty(\Omega)$ (which verifies $0 \le \varphi_0 \le 1$) and

$$F_0(r,s) = \begin{cases} 0 & \text{for } \ r \le 0, \\ -r(1-r)(1-2r+cs) & \text{for } \ 0 \le r \le 1, \\ 0 & \text{for } \ r \ge 1, \end{cases} \tag{3.22}$$

where $c > 0$.

First, we note that since $u = 0$ on $\partial\Omega \times (0,T)$, $\operatorname{div}(uT_1(\varphi))$ is well defined and belongs to $L^2(0,T;H^1(\Omega)')$. Hence (3.21) has a unique solution $\widetilde{\varphi}$ such that

$$\widetilde{\varphi} \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \tag{3.23}$$
$$\widetilde{\varphi}_t \in L^2(0,T;H^1(\Omega)'), \tag{3.24}$$
$$\widetilde{\varphi}(t=0) = \varphi_0 \tag{3.25}$$

and

$$\begin{aligned}\int_0^T \langle \widetilde{\varphi}_t, \psi \rangle \,\mathrm{d}t + \xi \int_{Q_T} \nabla\widetilde{\varphi} \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} uT_1(\varphi)\nabla\psi \,\mathrm{d}x\,\mathrm{d}t \\ = \int_{Q_T} -\varphi(1-\varphi)(1-2\varphi+ch)\cdot\psi \,\mathrm{d}x\,\mathrm{d}t \qquad \forall\psi \in L^2(0,T;H^1(\Omega)).\end{aligned} \tag{3.26}$$

Consequently, Γ is well defined.
To establish the compactness of Γ, let φ^n be a bounded sequence in $L^2(Q_T)$ and let $\widetilde{\varphi}^n = \Gamma(\varphi^n)$. By multiplying (3.21) by $\widetilde{\varphi}^n$, we get that, for a.e. $t \in (0,T)$,

$$\begin{aligned}\frac{1}{2}\int_\Omega |\widetilde{\varphi}^n(t)| \,\mathrm{d}x + \xi \int_0^t \int_\Omega |\nabla\widetilde{\varphi}^n|^2 \,\mathrm{d}x\,\mathrm{d}s \le \int_0^t \int_\Omega |uT_1(\varphi^n)\nabla\widetilde{\varphi}^n| \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_\Omega |\varphi_0|^2 \,\mathrm{d}x \\ + \int_0^t \int_\Omega |\varphi^n(1-\varphi^n)(1-2\varphi^n+ch)\cdot\widetilde{\varphi}^n| \,\mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{3.27}$$

Then, Young's and Hölder's inequalities imply that, for a.e. $t \in (0,T)$,

$$\begin{aligned}\int_\Omega |\widetilde{\varphi}^n(t)| \,\mathrm{d}x + (2\xi-\beta)\int_0^t \int_\Omega |\nabla\widetilde{\varphi}^n|^2 \,\mathrm{d}x\,\mathrm{d}s \le \frac{1}{\beta}\int_0^t \int_\Omega |u|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega |h|^2 \,\mathrm{d}x\,\mathrm{d}s \\ + \int_\Omega |\varphi_0|^2 \,\mathrm{d}x + \int_0^t \int_\Omega |\widetilde{\varphi}^n|^2 \,\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{3.28}$$

with β is chosen such that $2\xi - \beta > 0$. In view of the definition of h and $\widetilde{\varphi}_0$ and since the second term of the left-hand side of (3.28) is positive and $u \in L^2(Q_T)^2$, the Gronwall Lemma yields that, there exists a positive C independent of n such that for a.e. $t \in (0,T)$,

$$\int_\Omega |\widetilde{\varphi}^n(t)|^2 \,\mathrm{d}x \le C. \tag{3.29}$$

Hence, $\widetilde{\varphi}^n$ is bounded in $L^\infty(0,T;L^2(\Omega))$. It follows from this and (3.28) that $\widetilde{\varphi}^n$ is also bounded in $L^2(0,T;H^1(\Omega))$. In addition, in view of (3.26), $\widetilde{\varphi}^n_t$ is bounded in $L^2(0,T;H^1(\Omega)')$. Therefore, the Aubin-Simon Lemma (see e.g. [99]-[96]) implies that $\widetilde{\varphi}^n$ is compact in $L^2(Q_T)$.

The mapping Γ is also continuous. In fact, let φ^n and φ in $L^2(Q_T)$ such that φ^n converges to φ strongly in $L^2(Q_T)$ and let $\widetilde{\varphi}^n$ belongs to $L^2(Q_T)$ such that $\Gamma(\varphi^n) = \widetilde{\varphi}^n$. By the same arguments used for the compactness of Γ, we show that $\widetilde{\varphi}^n$ is bounded in $L^2(0,T;H^1(\Omega))$ and $\widetilde{\varphi}^n_t$ is bounded in $L^2(0,T;H^1(\Omega)')$. Thus, there exists a subsequence (still denoted by $\widetilde{\varphi}^n$) and a measurable function $\widetilde{\varphi}$ such that

$$\widetilde{\varphi}_n \rightharpoonup \widetilde{\varphi} \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{3.30}$$
$$\widetilde{\varphi}^n_t \rightharpoonup \widetilde{\varphi}_t \quad \text{weakly in } L^2(0,T;H^1(\Omega)'), \tag{3.31}$$
$$\widetilde{\varphi}^n \to \widetilde{\varphi} \quad \text{strongly in } L^2(Q_T), \tag{3.32}$$
$$\widetilde{\varphi}^n \to \widetilde{\varphi} \quad \text{a.e. in } Q_T. \tag{3.33}$$

Using these convergences, we can pass to the limit in the equation satisfied by $\widetilde{\varphi}^n$ and obtain that $\widetilde{\varphi}$ verifies (3.26). Since the solution of (3.21) is unique, we deduce that the whole sequence $\widetilde{\varphi}^n$ converges to $\widetilde{\varphi}$ in $L^2(Q_T)$ such that $\widetilde{\varphi} = \Gamma(\varphi)$. It follows that Γ is continuous.
Moreover, taking φ in the ball $B_{L^2(Q_T)}$ such that $\widetilde{\varphi} = \Gamma(\varphi)$ we obtain from (3.29), that $\|\widetilde{\varphi}\|_{L^\infty(0,T;L^2(\Omega))} \leq C$. Hence, $\widetilde{\varphi}$ is bounded in $L^2(Q_T)$. It follows that, Γ preserves the ball $\bar{B}_{L^2(Q_T)}(0, TC)$ in $L^2(Q_T)$. Applying the Leray-Schauder fixed point theorem, we can conclude that there exists at least one weak solution to the following problem

$$\begin{cases} \varphi_t - \xi\Delta\varphi + \operatorname{div}(uT_1(\varphi)) = F_0(\varphi, h) & \text{in } Q_T \ , \\ \nabla\varphi \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ \varphi(t=0) = \varphi_0 & \text{in } \Omega. \end{cases} \tag{3.34}$$

Let us show now that, for any solution φ of (3.34), if $0 \leq \varphi_0 \leq 1$, then φ is bounded by 0 and 1 almost everywhere in Q_T. To this purpose, we take $(\varphi - 1)^+$ defined by

$$(\varphi - 1)^+ = \begin{cases} \varphi - 1 & \text{if} \quad \varphi - 1 \geq 0, \\ 0 & \text{if} \quad \varphi - 1 \leq 0, \end{cases}$$

as a test function in problem (3.34). Since $u = 0$ on $\partial\Omega \times (0,T)$ and $\operatorname{div} u = 0$ on Ω, we obtain that for a.e. $t \in (0,T)$

$$\frac{1}{2}\int_\Omega |(\varphi - 1)^+(t)|^2 \,\mathrm{d}x \leq \frac{1}{2}\int_\Omega |(\varphi - 1)^+(0)|^2 \,\mathrm{d}x + \int_0^t \int_\Omega F_0(\varphi, h)(\varphi - 1)^+ \,\mathrm{d}x \,\mathrm{d}s. \tag{3.35}$$

Recalling that $0 \leq \varphi_0 \leq 1$, we get from (3.35) that $\varphi \leq 1$ for a.e. $(x,t) \in Q_T$. Then, we take $-\varphi^-$ defined by

$$\varphi^- = \begin{cases} -\varphi & \text{if} \quad \varphi \leq 0, \\ 0 & \text{if} \quad \varphi \geq 0, \end{cases}$$

as a test function in (3.34). Since $\operatorname{div} u = 0$ and $0 \leq \varphi_0 \leq 1$, we obtain that $\varphi \geq 0$ for a.e. $(t,x) \in Q_T$. It follows that $0 \leq \varphi \leq 1$ a.e. in Q_T and then $T_1(\varphi) = \varphi$ in (3.34).

It remains to prove the uniqueness of solutions of problem (3.11). Let φ_1 and φ_2 be two solutions of problem (3.11). Then φ_1 and φ_2 belong to $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ with $0 \leq \varphi_1, \varphi_2 \leq 1$. By subtracting the results equations of φ_1 and φ_2, multiplying by $\varphi := \varphi_1 - \varphi_2$ and then integrating over Ω, we obtain after using the fact that $\operatorname{div} u = 0$, and $0 \leq \varphi_1, \varphi_2 \leq 1$,

$$\frac{d}{dt}\|\varphi(t)\|^2_{L^2(\Omega)} + 2\xi\|\nabla\varphi\|^2_{L^2(\Omega)} \leq \int_\Omega |F(\varphi_1, h) - F(\varphi_2, h)||\varphi| \,\mathrm{d}x. \tag{3.36}$$

Since $h \in L^\infty(Q_T)$ and F is Lipschitz, we get that

$$\frac{d}{dt}\|\varphi(t)\|^2_{L^2(\Omega)} \leq \|\varphi(t)\|^2_{L^2(\Omega)} \tag{3.37}$$

This inequality with Gronwall's Lema yields that $\varphi \equiv 0$, a.e. in Q_T, thus φ is unique. □

Remark 3.2.3. 1. *The uniqueness result is sufficient for our needs to develop a fixed point argument in the next section. It is clear that assumptions could be weakened. However, due to the nonlinearity, generalizing this result to h in $L^2(Q_T)$ is not a straightforward task.*

2. *If we assume the regularity of the boundary $\partial\Omega$ is C^2, then we can improve the regularity of φ using the L^p regularity theory. It is not in the scope of the present work, but when combined with another choice of right-hand side $F(\varphi, \theta)$, it is an interesting question.*

3.2.3 Renormalized solutions for the heat equation

In this subsection, we recall the existence, the uniqueness and the stability result of the solution of the heat equation. We will mainly use the results of Chapter 2, in the case of homogeneous Neumann boundary conditions.

Let us consider the following heat equation

$$\begin{cases} \theta_t - \operatorname{div}(\kappa(x,t)\nabla\theta) + u\cdot\nabla\theta = f - \operatorname{div}(g) & \text{in } Q_T, \\ \theta(t=0) = \theta_0 & \text{in } \Omega, \\ (\kappa(x,t)\nabla\theta - g)\cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{3.38}$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$. We assume that

$$u \text{ is the solution of the Navier-stokes equation,} \tag{3.39}$$

$$g \in L^2(Q_T)^N \text{ and } \theta_0 \in L^1(\Omega). \tag{3.40}$$

Moreover, the function κ is in $L^\infty(Q_T)$ such that $0 < \kappa_0 \le \kappa \le \kappa_1$, with κ_0 and κ_1 are given real number. Finally, we assume that

$$f \in L^1(Q_T). \tag{3.41}$$

Since the right-hand side of equation (3.38) is in $L^1(Q_T) + L^2(0,T;H^1(\Omega)')$, we will refer to the notion of a renormalized solution to our equation. To this purpose we define, for any real number $k \ge 0$, the truncation function T_k at height $\pm k$ by

$$T_k(r) = \begin{cases} -k & \text{if } \quad r \le -k, \\ r & \text{if } \quad -k \le r \le k, \\ k & \text{if } \quad r \ge k, \end{cases}$$

and then $\widetilde{T}_k$ is defined by

$$\widetilde{T}_k(r) = \int_0^r T_k(s)\,\mathrm{d}s.$$

Following Chapter 2 and [24], we recall the definition of renormalized solution for problem (3.38)

Definition 3.2.4. *Let θ be a measurable function on Q_T. Then, θ is a renormalized solution of* (3.38) *if*

$$\theta \in L^\infty(0,T;L^1(\Omega)), \tag{3.42}$$

$$T_k(\theta) \in L^2(0,T;H^1(\Omega)),\ \textit{for any } k>0, \tag{3.43}$$

$$\lim_{n\to+\infty} \frac{1}{n}\int_{\{(x,t)\in Q_T:\, n<|\theta(x,t)|<2n\}} \kappa(x,t)\nabla\theta\cdot\nabla\theta\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{3.44}$$

and for any $\beta \in W^{2,\infty}(\mathbb{R})$ such that β' has a compact support and any $\eta \in C_c^\infty([0,T)\times\bar\Omega)$, θ satisfies

$$\begin{aligned} &-\int_{Q_T} \eta_t\beta(\theta)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \eta(0)\beta(\theta_0)\,\mathrm{d}x + \int_{Q_T} \kappa(x,t)\nabla\theta\cdot\nabla\theta\beta''(\theta)\eta\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{Q_T} \beta'(\theta)\kappa(x,t)\nabla\theta\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} u\cdot\nabla\theta\beta'(\theta)\eta\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{Q_T} f\beta'(\theta)\eta\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\cdot\nabla(\beta'(\theta)\eta)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.45}$$

The existence and uniqueness of a renormalized solution of (3.38) is a consequence of the results established in Chapter 2.

Theorem 3.2.5. *Under assumptions* (3.39)-(3.41)*, there exists a unique renormalized solution of* (3.38) *in the sense of Definition 3.2.4.*

Moreover, we have the following stability result for the renormalized solution, which will be useful in the next section.

Theorem 3.2.6. *For $\varepsilon > 0$, let f_ε belonging to $L^1(Q_T)$, g_ε belonging to $L^2(Q_T)^N$ and $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ such that $\operatorname{div}(u) = 0$ in Q_T. For any $\varepsilon > 0$, Let us consider θ_ε a renormalized solution of*

$$\begin{cases} \theta_{\varepsilon,t} - \operatorname{div}(\kappa(x,t)\nabla\theta_\varepsilon) + u\cdot\nabla\theta_\varepsilon = f_\varepsilon - \operatorname{div}(g_\varepsilon) & \text{in } Q_T,\\ \theta_\varepsilon(t=0) = \theta_0^\varepsilon & \text{in } \Omega,\\ (\kappa(x,t)\nabla\theta_\varepsilon - g_\varepsilon)\cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{3.46}$$

where $\kappa \in L^\infty(Q_T)$ such that $0 < \kappa_0 \le \kappa \le \kappa_1$ a.e. in Q_T.

Moreover, assume that

$$\theta_0^\varepsilon \to \theta_0 \text{ strongly in } L^1(\Omega), \tag{3.47}$$

$$f_\varepsilon \to f \text{ strongly in } L^1(Q_T), \tag{3.48}$$

and

$$g_\varepsilon \to g \text{ strongly in } L^2(Q_T)^N. \tag{3.49}$$

Under these assumptions, up to a subsequence (still indexed by ε), θ_ε converges to θ a.e. in Q_T, as ε goes to zero where θ is a renormalized solution of

$$\begin{cases} \theta_t - \operatorname{div}(\kappa(x,t)\nabla\theta) + u\cdot\nabla\theta = f - \operatorname{div}(g) & \text{in } Q_T,\\ \theta(t=0) = \theta_0 & \text{in } \Omega\ ,\\ (\kappa(x,t)\nabla\theta - g)\cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{3.50}$$

We complete this subsection with the following regularity result. The proof is developed in chapter 2.

Lemma 3.2.7. *Let θ be an element of $L^\infty(0,T;L^1(\Omega))$. Assume that, for any $k > 0$, $T_k(\theta) \in L^2(0,T;H^1(\Omega))$ and that*

$$\|T_k(\theta)\|^2_{L^2(0,T;H^1(\Omega))} \le kM + L, \tag{3.51}$$

where M and L are two positive constants independant of k. Then,

— *for all $1 \le p < 1 + \frac{2}{N}$, $\theta \in L^p(Q_T)$; more precisely there exists a constant C (depending on M, L, T, Ω and p) such that*

$$\|\theta\|_{L^p(Q_T)} \le C \tag{3.52}$$

— *for all $1 < r < \overline{p}$ where $\overline{p} = +\infty$ if $N = 2$ and $\overline{p} = \frac{2^*}{2}$ if $N \ge 3$, we have*

$$\theta \in L^1(0,T;L^r(\Omega)).$$

More precisely, for all $1 \le r < \overline{p}$

$$\|\theta\|_{L^1(0,T;L^r(\Omega))} \le C, \tag{3.53}$$

where $C > 0$ is constant depending on M, L, T, Ω, p and r.

In addition, for $N = 2$, from the interpolation inequalities, (3.52) *and* (3.53)*, for all $1 \leq s < +\infty$ and for all $1 \leq q < \frac{s}{s-1}$, we have*

$$\theta \in L^s(0,T;L^q(\Omega)).$$

More precisely, there exists a constant C depending on M, L, T, Ω, s and q such that

$$\|\theta\|_{L^s(0,T;L^q(\Omega))} \leq C. \tag{3.54}$$

3.3 Assumptions and definitions

In this section, we present the assumptions and definitions necessary for the study of the system. Recall that Ω is a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, $Q_T = \Omega \times (0,T)$ and $T > 0$. We consider the following system

$$\varphi_t + u\cdot\nabla\varphi - \xi\Delta\varphi = F(\varphi,\theta) \quad \text{in } Q_T, \tag{3.55}$$

$$\theta_t - \operatorname{div}(\kappa(\varphi,\theta)\nabla\theta) + u\cdot\nabla\theta = \nu(\varphi,\theta)D(u) : D(u) + g + l\varphi_t \quad \text{in } Q_T, \tag{3.56}$$

$$u_t + (u\cdot\nabla)u - \operatorname{div}(\nu(\varphi,\theta)D(u)) + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } Q_T, \tag{3.57}$$

$$\nabla\varphi\cdot\boldsymbol{n} = 0, \quad \kappa(\varphi,\theta)\nabla\theta\cdot\boldsymbol{n} = 0, \quad u = 0 \quad \text{on } \Sigma, \tag{3.58}$$

$$\varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0(x), \quad u(x,0) = u_0(x) \quad \text{in } \Omega. \tag{3.59}$$

We assume, throughout this chapter, that the following hypotheses holds true

(A1) $\xi > 0$, $F(\varphi,\theta) = -\varphi(1-\varphi)(1-2\varphi+c\theta)$, $c > 0$;

(A2) $l > 0$, $\kappa, \nu \in C^0(\mathbb{R}^2)$ such that $0 < \kappa_0 \leq \kappa \leq \kappa_1$ and $0 < \nu_0 \leq \nu \leq \nu_1$;

(A3) $g \in L^1(Q_T)^2$ and $f \in L^2(Q_T)^2$;

(A4) $\varphi_0 \in L^\infty(\Omega)$ such that $0 \leq \varphi_0 \leq 1$, $\theta_0 \in L^1(\Omega)$ and $u_0 \in \mathcal{H}$.

As mentioned in the introduction, since the right-hand side of equation (3.56) is in $L^1(Q_T)$, we will refer to the notion of a weak-renormalized solution to our system.

Definition 3.3.1. *We call (φ,θ,u) defined on $\Omega\times(0,T)$ a weak-renormalized solution of* (3.55)-(3.59) *if*

$$u \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}), \quad u_t \in L^2(0,T;\mathcal{V}); \tag{3.60}$$

$$\varphi \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \quad \varphi_t \in L^2(0,T;H^1(\Omega)'); \tag{3.61}$$

$$0 \leq \varphi \leq 1; \tag{3.62}$$

$$\varphi(t=0) = \varphi_0, \quad u(t=0) = u_0; \tag{3.63}$$

$$\theta \in L^\infty(0,T;L^1(\Omega)),\ T_k(\theta) \in L^2(0,T;H^1(\Omega)),\ \forall k > 0; \tag{3.64}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{(x,t)\in Q_T :\, n<|\theta(x,t)|<2n\}} \kappa(\varphi,\theta)|\nabla\theta|^2 \,\mathrm{d}x\,\mathrm{d}t = 0; \tag{3.65}$$

and if the following equations hold true

$$\begin{aligned}\int_0^T \langle u_t, v\rangle_{\mathcal{V}',\mathcal{V}} \,\mathrm{d}t + \int_{Q_T} \nu(\varphi,\theta)\nabla u\cdot\nabla v \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} (u\cdot\nabla)u\cdot v \,\mathrm{d}x\,\mathrm{d}t \\ = \int_{Q_T} fv \,\mathrm{d}x\,\mathrm{d}t \qquad \forall v \in L^2(0,T;\mathcal{V}),\end{aligned} \tag{3.66}$$

$$\begin{aligned}\int_0^T \langle\varphi_t,\psi\rangle \,\mathrm{d}t &+ \int_{Q_T} \xi\nabla\varphi\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}(u\cdot\nabla\varphi)\cdot\psi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{Q_T} -\varphi(1-\varphi)(1-2\varphi+c\theta)\psi \,\mathrm{d}x\,\mathrm{d}t \qquad \forall\psi\in L^2(0,T;H^1(\Omega));\end{aligned} \tag{3.67}$$

and if for every $\beta\in W^{2,\infty}(\mathbb{R})$ such that β' has a compact support and for every $\eta\in C_c^\infty([0,T)\times\Omega)$, θ satisfies

$$\begin{aligned}-\int_{Q_T}\eta_t\beta(\theta)\,\mathrm{d}x\mathrm{d}t &+ \int_{Q_T}\kappa(\varphi,\theta)\nabla\beta(\theta)\cdot\nabla\eta\,\mathrm{d}x\mathrm{d}t + \int_{Q_T}\kappa(\varphi,\theta)\nabla\theta\cdot\nabla\beta'(\theta)\eta\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{Q_T} u\cdot\nabla\theta\beta'(\theta)\eta\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}\beta'(\theta)\nu(\varphi,\theta)D(u):D(u)\eta\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\beta'(\theta)\eta\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T\langle l\varphi_t,\beta'(\theta)\eta\rangle\,\mathrm{d}t + \int_\Omega \eta(0)\beta(\theta_0)\,\mathrm{d}x.\end{aligned} \tag{3.68}$$

Remark 3.3.2. *We observe that all the terms in the weak formulation* (3.67) *of the phase equation make sense. Indeed, in view of Lemma 3.2.7, θ belongs to $L^s(0,T;L^q(\Omega))$ for $1\le s\le +\infty$ and $1\le q< s/(s-1)$. Thus, by the Hölder and the Sobolev embedding theorems, and for all $\psi\in L^2(0,T;L^{q'}(\Omega))$, $\forall\, 2\le q'<2^*$, the term $-\varphi(1-\varphi)(1-2\varphi+cT_{1/\eta}(\theta))\psi$ in* (3.67) *belongs to L^1.*

3.4 Existence result for weak-renormalized solution

Our main result in this chapter is the following existence theorem.

Theorem 3.4.1. *Assume that the assumptions (A1)-(A4) hold true and that $N=2$. Then there exists at least a weak-renormalized solution (φ,θ,u) to* (3.55)-(3.59).

The proof of this result is lengthy, we will present it in the upcoming subsections. Let us provide a brief outline here. To establish the existence of a weak renormalized solution to problem (3.55)-(3.59), we use a regularization technique that has been previously used in articles dealing with this type of system, see for example [63], [102]. The authors in [63] showed the existence of such a solution for a phase field model system similar to ours with Dirichlet boundary conditions for the energy equation instead of Neumann boundary conditions. Furthermore, it is worth noting that in [63] the latent heat l was neglected in the energy equation and a different nonlinear term $F(\varphi,\theta)$ in the phase equation was introduced.
In our case, the aim of the regularization is to deal with the presence of the temperature θ in the phase equation, the convection term $u\cdot\nabla\theta$ and the weak regularity of the non-linear term $D(u):D(u)$ of the heat equation. By introducing a positive parameter ε and an approximation of u, and by using the truncation function, we formulate the regularized problem (3.69)-(3.73). The existence of solutions for this regularized problem is obtained by applying the Leray-Schauder fixed point theorem (see Theorem 3.4.2). We then derive in subsection 3.4.2 a priori estimates independent of the parameter ε. The key here is the coupling between the phase and the heat equation to avoid any smallness condition on l. Consequently, using compactness arguments we prove that a limit of a subsequence exists, and then using the stability of the renormalized solution for the heat equation we show that this limit is the weak renormalized solution to the problem (3.55)-(3.59).

3.4.1 Regularized problem

In this subsection, we introduce the regularized problem related to (3.55)-(3.59) and then we show the existence of solutions for this regularized problem by using the Leray–Schauder fixed point theorem.

We choose here a regularizd problem that can solved in dimensions 2 or 3. Therefore we replace the term $(u\cdot\nabla u)u$ in the Navier-Stokes equation by a regularisation, namely $((R_\varepsilon u)\cdot\nabla)u$. More precisely, let's consider a regularizing sequence in $\mathbb{R}^N$ $\{\rho_\varepsilon\}$. For any $\varepsilon > 0$ and any $w \in \mathcal{V}$, we define $R_\varepsilon w$ the approximation of w by

$$R_\varepsilon w := \rho_\varepsilon * \widetilde{w},$$

where $\widetilde{w}$ is the extension by zero of w in $\mathbb{R}^N$. We note that $R_\varepsilon w \in C^\infty(\mathbb{R}^N)^N$ such that $\operatorname{div}(R_\varepsilon w) = 0$ in Ω and

$$\|R_\varepsilon w\|_{W^{m,q}(\Omega)} \leq C(m,q,\varepsilon)\|w\|_{L^2(\Omega)},$$

for all m and q. Note that if $w_\varepsilon \to w$ in $L^q(\Omega)$ and $\operatorname{div} w = 0$ then $R_\varepsilon w_\varepsilon \to w$ in $L^q(\Omega)$.

Let us now define the regularized system. For $\varepsilon > 0$,

$$\varphi_{\varepsilon,t} + u_\varepsilon\cdot\nabla\varphi_\varepsilon - \xi\Delta\varphi_\varepsilon = -\varphi_\varepsilon(1-\varphi_\varepsilon)(1-2\varphi_\varepsilon + cT_{1/\varepsilon}(\theta_\varepsilon)) \quad \text{in } Q_T, \tag{3.69}$$

$$\theta_{\varepsilon,t} - \operatorname{div}(\kappa(\varphi_\varepsilon,\theta_\varepsilon)\nabla\theta_\varepsilon) + u_\varepsilon\cdot\nabla T_{1/\varepsilon}(\theta_\varepsilon) = h_\varepsilon + g_\varepsilon + l\varphi_{\varepsilon,t} \quad \text{in } Q_T, \tag{3.70}$$

$$u_{\varepsilon,t} + ((R_\varepsilon u_\varepsilon)\cdot\nabla)u_\varepsilon - \operatorname{div}(\nu(\varphi_\varepsilon,\theta_\varepsilon)D(u_\varepsilon)) + \nabla p_\varepsilon = f, \quad \operatorname{div} u_\varepsilon = 0 \quad \text{in } Q_T, \tag{3.71}$$

$$\nabla\varphi_\varepsilon\cdot\boldsymbol{n} = 0, \quad \kappa(\varphi_\varepsilon,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\boldsymbol{n} = 0, \quad u_\varepsilon = 0 \quad \text{on } \Sigma, \tag{3.72}$$

$$\varphi_\varepsilon(x,0) = \varphi_0(x), \quad \theta_\varepsilon(x,0) = \theta_{\varepsilon,0}(x), \quad u_\varepsilon(x,0) = u_0(x) \quad \text{in } \Omega, \tag{3.73}$$

with $h_\varepsilon = T_{1/\varepsilon}(\nu(\varphi_\varepsilon,\theta_\varepsilon)D(u_\varepsilon) : D(u_\varepsilon))$, $g_\varepsilon = T_{1/\varepsilon}(g)$ and $\theta_{\varepsilon,0} = T_{1/\varepsilon}(\theta_0)$.

We prove in this subsection the following existence theorem.

Theorem 3.4.2. *Assume that the assumptions (A1)-(A4) hold true. Then, there exists for each $\varepsilon > 0$, at least one solution $(\varphi_\varepsilon,\theta_\varepsilon,u_\varepsilon)$ to (3.69)-(3.73) such that*

$$\varphi_\varepsilon \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \quad \varphi_{\varepsilon,t} \in L^2(0,T;H^1(\Omega)'), \tag{3.74}$$

$$\theta_\varepsilon \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \quad \theta_{\varepsilon,t} \in L^2(0,T;H^1(\Omega)'), \tag{3.75}$$

$$u_\varepsilon \in L^2(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{H}), \quad u_{\varepsilon,t} \in L^2(0,T;\mathcal{V}'); \tag{3.76}$$

$$\varphi_\varepsilon(t=0) = \varphi_0, \quad \theta_\varepsilon(t=0) = \theta_{\varepsilon,0}, \quad u_\varepsilon(t=0) = u_0 \tag{3.77}$$

and

$$\begin{aligned} &\int_0^T \langle\varphi_{\varepsilon,t},\psi\rangle\,\mathrm{d}t + \int_{Q_T}\xi\nabla\varphi_\varepsilon\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}(u_\varepsilon\cdot\nabla\varphi_\varepsilon)\cdot\psi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{Q_T} -\varphi_\varepsilon(1-\varphi_\varepsilon)(1-2\varphi_\varepsilon + cT_{1/\varepsilon}(\theta_\varepsilon))\psi\,\mathrm{d}x\,\mathrm{d}t \qquad \forall\psi \in L^2(0,T;H^1(\Omega)); \end{aligned} \tag{3.78}$$

and

$$\begin{aligned} &\int_0^T \langle\theta_{\varepsilon,t},\phi\rangle\,\mathrm{d}t + \int_{Q_T}\kappa(\varphi_\varepsilon,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}(u_\varepsilon\cdot\nabla T_{1/\varepsilon}(\theta_\varepsilon))\cdot\phi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{Q_T} h_\varepsilon\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g_\varepsilon\phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle l\varphi_{\varepsilon,t},\phi\rangle\,\mathrm{d}t \qquad \forall\phi \in L^2(0,T;H^1(\Omega)); \end{aligned} \tag{3.79}$$

and

$$\begin{aligned}\int_0^T \langle u_{\varepsilon,t}, v\rangle_{\mathcal{V}',\mathcal{V}}\ \mathrm{d}t + \int_{Q_T} \nu(\varphi_\varepsilon,\theta_\varepsilon)\nabla u_\varepsilon\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}(R_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon\cdot v\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{Q_T} f v\,\mathrm{d}x\,\mathrm{d}t \qquad \forall v\in L^2(0,T;\mathcal{V}).\end{aligned} \tag{3.80}$$

Proof. The proof relies on the Leray-Schauder fixed point theorem.

Let $(\widetilde{\varphi},\widetilde{\theta})\in L^2(Q_T)^2$. Arguments in Section 3.2 of this chapter permits to conclude that if Assumptions (A1)-(A4) hold true, there exists a unique solution $(\varphi_\varepsilon,\theta_\varepsilon,u_\varepsilon)$ to the following system

$$\varphi_{\varepsilon,t} + u_\varepsilon\cdot\nabla\varphi_\varepsilon - \xi\Delta\varphi_\varepsilon = -\varphi_\varepsilon(1-\varphi_\varepsilon)(1-2\varphi_\varepsilon + cT_{1/\varepsilon}(\widetilde{\theta})) \qquad \text{in } Q_T, \tag{3.81}$$

$$\theta_{\varepsilon,t} - \mathrm{div}\,(\kappa(\widetilde{\varphi},\widetilde{\theta})\nabla\theta_\varepsilon) + u_\varepsilon\cdot\nabla T_{1/\varepsilon}(\widetilde{\theta}) = \widetilde{h}_\varepsilon + g_\varepsilon + l\varphi_{\varepsilon,t} \qquad \text{in } Q_T, \tag{3.82}$$

$$u_{\varepsilon,t} + ((R_\varepsilon u_\varepsilon)\cdot\nabla)u_\varepsilon - \mathrm{div}\,(\nu(\widetilde{\varphi},\widetilde{\theta})D(u_\varepsilon)) + \nabla p_\varepsilon = f, \quad \mathrm{div}\ u_\varepsilon = 0 \qquad \text{in } Q_T, \tag{3.83}$$

$$\nabla\varphi_\varepsilon\cdot\boldsymbol{n} = 0, \quad \kappa(\widetilde{\varphi},\widetilde{\theta})\nabla\theta_\varepsilon\cdot\boldsymbol{n} = 0, \quad u_\varepsilon = 0 \qquad \text{on } \Sigma, \tag{3.84}$$

$$\varphi_\varepsilon(x,0) = \varphi_0(x), \quad \theta_\varepsilon(x,0) = \theta_{\varepsilon,0}(x), \quad u_\varepsilon(x,0) = u_0(x) \qquad \text{in } \Omega, \tag{3.85}$$

where

$$\widetilde{h}_\varepsilon = T_{1/\varepsilon}(\nu(\widetilde{\varphi},\widetilde{\theta})D(u_\varepsilon):D(u_\varepsilon))$$

and such that φ_ε, $\theta_\varepsilon\in L^2(0,T;H^1(\Omega))$ and $u_\varepsilon\in L^2(0,T;\mathcal{V})$.

Indeed, from Theorem 3.2.1, there exists a unique $u_\varepsilon\in L^2(0,T;\mathcal{V})\cap L^\infty(0,T;\mathcal{H})$ solution of (3.83) such that

$$\begin{aligned}\int_0^T \langle u_{\varepsilon,t}, v\rangle_{\mathcal{V}',\mathcal{V}}\ \mathrm{d}t + \int_{Q_T} \nu(\widetilde{\varphi},\widetilde{\theta})\nabla u_\varepsilon\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}(R_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon\cdot v\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{Q_T} f v\,\mathrm{d}x\,\mathrm{d}t \qquad \forall v\in L^2(0,T;\mathcal{V}).\end{aligned} \tag{3.86}$$

Concerning the phase equation (3.81), since $T_{1/\varepsilon}(\widetilde{\theta})$ is in $L^\infty(Q_T)$ and $u_\varepsilon\in L^2(Q_T)^N$, Theorem 3.2.2 imply that there exists a unique solution φ_ε in $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ such that $0\le\varphi_\varepsilon\le 1$ and φ_ε verifies

$$\begin{aligned}\int_0^T \langle \varphi_{\varepsilon,t}, \psi\rangle\ \mathrm{d}t + \int_{Q_T} \xi\nabla\varphi_\varepsilon\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}(u_\varepsilon\cdot\nabla\varphi_\varepsilon)\cdot\psi\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{Q_T} -\varphi_\varepsilon(1-\varphi_\varepsilon)(1-2\varphi_\varepsilon + cT_{1/\varepsilon}(\widetilde{\theta}))\psi\,\mathrm{d}x\,\mathrm{d}t \qquad \forall\psi\in L^2(0,T;H^1(\Omega)).\end{aligned} \tag{3.87}$$

At last, since $\widetilde{h}_\varepsilon$, g_ε belong to $L^\infty(Q_T)$, $u_\varepsilon\nabla T_{1/\varepsilon}(\widetilde{\theta}) = \mathrm{div}(u_\varepsilon T_{1/\varepsilon}(\widetilde{(\theta)})$ is in $L^2(0,T;H^1(\Omega)')$ and $\varphi_{\varepsilon,t}$ is in $L^2(0,T;H^1(\Omega)')$, classical arguments (see [96]) allow us to get the existence of unique θ_ε in $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ such that

$$\begin{aligned}\int_0^T \langle \theta_{\varepsilon,t}, \phi\rangle\ \mathrm{d}t + \int_{Q_T} \kappa(\widetilde{\varphi},\widetilde{\theta})\nabla\theta_\varepsilon\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}(u_\varepsilon\cdot\nabla T_{1/\varepsilon}(\widetilde{\theta}))\cdot\phi\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{Q_T} \widetilde{h}_\varepsilon\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g_\varepsilon\phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle l\varphi_{\varepsilon,t}, \phi\rangle\ \mathrm{d}t \qquad \forall\phi\in L^2(0,T;H^1(\Omega)).\end{aligned} \tag{3.88}$$

As a result we can consider the mapping Λ_ε: $L^2(Q_T)^2 \to L^2(Q_T)^2$ defined by

$$\Lambda_\varepsilon(\widetilde{\varphi}, \widetilde{\theta}) = (\varphi_\varepsilon, \theta_\varepsilon), \qquad \forall (\widetilde{\varphi}, \widetilde{\theta}) \in L^2(Q_T)^2,$$

where $(\varphi_\varepsilon, \theta_\varepsilon, u_\varepsilon)$ is the unique solution of (3.81)-(3.85) in the sense of (3.86), (3.87), (3.88). In the following, we will show that Λ_ε is compact, continuous and that there exists $B_1 \times B_2$ balls of $L^2(Q_T)^2$ such that $\Lambda_\varepsilon(B_1 \times B_2) \subset B_1 \times B_2$.

Λ_ε **is compact:** Let $(\widetilde{\varphi}^n, \widetilde{\theta}^n)$ be two bounded sequences in $L^2(Q_T)^2$ and let $(\varphi_\varepsilon^n, \theta_\varepsilon^n) = \Lambda_\varepsilon(\widetilde{\varphi}^n, \widetilde{\theta}^n)$. For a fixed $n \geq 1$, $(\varphi_\varepsilon^n, \theta_\varepsilon^n, u_\varepsilon^n)$ is the unique solution of the problem (3.81)-(3.85) with $(\varphi_\varepsilon^n, \theta_\varepsilon^n, u_\varepsilon^n)$ in place of $(\varphi_\varepsilon, \theta_\varepsilon, u_\varepsilon)$. First, we know that u_ε^n is bounded in $L^2(0,T;\mathcal{V})$. Then, by taking φ_ε^n as a test function in (3.87), we obtain, since $\operatorname{div}(u_\varepsilon^n) = 0$ in Ω, $u_\varepsilon^n = 0$ on $\partial\Omega$ and $0 \leq \varphi_\varepsilon^n \leq 1$, that

$$\|\varphi_{\varepsilon,t}^n\|_{L^2(0,T;H^1(\Omega)')} + \|\varphi_\varepsilon^n\|_{L^\infty(0,T;L^2(\Omega))} + \|\varphi_\varepsilon^n\|_{L^2(0,T;H^1(\Omega))} \leq C(\Omega, \varepsilon). \tag{3.89}$$

Furthermore, by choosing θ_ε^n as a test function in (3.88) and by using the fact that $\varphi_{\varepsilon,t}^n$ is bounded in $L^2(0,T;H^1(\Omega)')$, $\widetilde{\theta}^n$ is bounded in $L^2(Q_T)$, $u_\varepsilon = 0$ on $\partial\Omega$, $\operatorname{div}(u_\varepsilon T_{1/\varepsilon}(\widetilde{\theta^n}))$ is bounded in $L^2(0,T;H^1(\Omega)')$ and assumption (A2) on κ, we get with the help of the Gronwall Lemma that

$$\|\theta_\varepsilon^n\|_{L^2(0,T;H^1(\Omega))} \leq C(\Omega, \varepsilon, \kappa_0) + lC(\Omega, \varepsilon). \tag{3.90}$$

In addition, from (3.88) and (3.89) we obtain that

$$\|\theta_{\varepsilon,t}^n\|_{L^2(0,T;(H^1(\Omega))')} \leq C(\Omega, \varepsilon, \kappa_0). \tag{3.91}$$

As a consequence of (3.89), (3.90), (3.91) and since ε is fixed, $(\varphi_\varepsilon^n, \theta_\varepsilon^n)$ have uniform bounds in the space W, defined by

$$W = \{\phi \in L^2(0,T;H^1(\Omega)); \phi_t \in L^2(0,T;H^1(\Omega)')\}.$$

It follows from Aubin-Simon's Lemma (see e.g. [99]) that $(\varphi_\varepsilon^n, \theta_\varepsilon^n)$ belongs to a compact set of $L^2(Q_T)^2$.

Λ_ε **is continuous:** Let $(\widetilde{\varphi}^n, \widetilde{\theta}^n)$ and $(\widetilde{\varphi}, \widetilde{\theta}) \in L^2(Q_T)^2$ such that $(\widetilde{\varphi}^n, \widetilde{\theta}^n) \to (\widetilde{\varphi}, \widetilde{\theta})$ strongly in $L^2(Q_T)^2$. Let $(\varphi_\varepsilon^n, \theta_\varepsilon^n) = \Lambda_\varepsilon(\widetilde{\varphi}^n, \widetilde{\theta}^n)$.

First, we observe that with the same proceeding as for the estimates (3.89), (3.90) and (3.91) we obtain that φ_ε^n, θ_ε^n are bounded in $L^2(0,T;H^1(\Omega))$ and $\varphi_{\varepsilon,t}^n$, $\theta_{\varepsilon,t}^n$ are bounded in $L^2(0,T;H^1(\Omega)')$. Then, up to a subsequence (still denoted by φ_ε^n, and θ_ε^n) there exists two measurable functions φ_ε and θ_ε such that

$$\varphi_\varepsilon^n \rightharpoonup \varphi_\varepsilon \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \text{ strongly in } L^2(Q_T) \text{ and a.e.,} \tag{3.92}$$

$$\theta_\varepsilon^n \rightharpoonup \theta_\varepsilon \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \text{ strongly in } L^2(Q_T) \text{ and a.e.,} \tag{3.93}$$

$$\varphi_{\varepsilon,t}^n \rightharpoonup \varphi_{\varepsilon,t} \quad \text{weakly in } L^2(0,T;H^1(\Omega)'), \tag{3.94}$$

$$\theta_{\varepsilon,t}^n \rightharpoonup \theta_{\varepsilon,t} \quad \text{weakly in } L^2(0,T;H^1(\Omega)'). \tag{3.95}$$

In order to prove that Λ_ε is continuous, we will show that $(\varphi_\varepsilon, \theta_\varepsilon) = \Lambda_\varepsilon(\widetilde{\varphi}, \widetilde{\theta})$ by passing to the limit in (3.81)-(3.85) associated to $(\varphi_\varepsilon^n, \theta_\varepsilon^n, u_\varepsilon^n)$. For that, we have to show the strong convergence of u_ε^n to u_ε solution of (3.83), in $L^2(0,T;\mathcal{V})$.

Indeed, u_ε^n is the solution to the following Naviers-stokes problem

$$\begin{aligned} u_{\varepsilon,t}^n + ((R_\varepsilon u_\varepsilon^n)\cdot\nabla)u_\varepsilon^n - \operatorname{div}(\nu(\widetilde{\varphi}^n, \widetilde{\theta}^n)D(u_\varepsilon^n)) + \nabla p_\varepsilon^n = f, \quad \operatorname{div} u_\varepsilon^n = 0 \quad \text{in } Q_T, \\ u_\varepsilon^n = 0 \quad \text{in } \Sigma. \end{aligned} \tag{3.96}$$

By taking u_ε^n as a test function in (3.96), using Poincaré, Young and Korn inequalities we find the energy identity

$$\frac{1}{2}\int_\Omega |u_\varepsilon^n(T)|^2\,\mathrm{d}x + \int_{Q_T} \nu(\widetilde{\varphi}^n, \widetilde{\theta}^n)|D(u_\varepsilon^n)|^2\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} f\cdot u_\varepsilon^n\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega |u_0|^2\,\mathrm{d}x. \tag{3.97}$$

and then we get that

$$\|u_\varepsilon^n\|_{L^2(0,T;\mathcal{V})} + \|u_\varepsilon^n\|_{L^\infty(0,T;\mathcal{H})} \le C, \tag{3.98}$$

$$\|u_{\varepsilon,t}^n\|_{L^2(0,T;\mathcal{V}')} \le C(\varepsilon), \tag{3.99}$$

where C and $C(\varepsilon)$ depends on Ω, T, $\|f\|_{L^2(Q_T)}$, $\|u_0\|_{\mathcal{H}}$, ν_1 and ν_2. In view of this using Aubin-Lions Lemma we can extract a subsequence still indexed by u_ε^n such that, when n goes to $+\infty$, we have

$$u_\varepsilon^n \rightharpoonup u_\varepsilon \quad \text{weakly in } L^2(0,T;\mathcal{V}), \text{ strongly in } L^2(0,T);\mathcal{H}) \text{ and a.e.}, \tag{3.100}$$

$$u_{\varepsilon,t}^n \rightharpoonup u_{\varepsilon,t} \quad \text{weakly in } L^2(0,T;\mathcal{V}'), \tag{3.101}$$

where u_ε is in $L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V})$, $u_\varepsilon, t \in L^2(0,T;H^1(\Omega)')$ and then in $C([0,T];H)$. Estimate (3.97) on $u_\varepsilon^n(T)$ in $L^2(\Omega)$, (3.100) and (3.101) also give that

$$u_\varepsilon^n(T) \rightharpoonup u_\varepsilon(T) \text{ weakly in } L^2(\Omega)^N. \tag{3.102}$$

Since $(\widetilde{\varphi}^n, \widetilde{\theta}^n)$ converges strongly to $(\widetilde{\varphi}, \widetilde{\theta})$ in $L^2(Q_T)^2$ as n goes to $+\infty$ and $\nu \in L^\infty(Q_T)$ with $0 < \nu_0 \le \nu \le \nu_1$ we get that

$$\nu(\widetilde{\varphi}^n, \widetilde{\theta}^n) \to \nu(\widetilde{\varphi}, \widetilde{\theta}) \quad \text{strongly in } L^p(Q_T)^2, \text{ for } 1 \le p < +\infty \text{ and in } L^\infty(Q_T)\text{weak-}*. \tag{3.103}$$

By using this convergence with (3.100) and (3.101), we can pass to the limit in (3.96) and obtain that u_ε is the solution of

$$\begin{aligned} u_{\varepsilon,t} + ((R_\varepsilon u_\varepsilon)\cdot\nabla)u_\varepsilon - \operatorname{div}(\nu(\widetilde{\varphi}, \widetilde{\theta})D(u_\varepsilon)) + \nabla p_\varepsilon = f, \quad \operatorname{div} u_\varepsilon = 0 \quad \text{in } Q_T, \\ u_\varepsilon = 0 \quad \text{in } \Sigma. \end{aligned} \tag{3.104}$$

and it satisfies the following energy identity

$$\frac{1}{2}\int_\Omega |u_\varepsilon(T)|^2\,\mathrm{d}x + \int_{Q_T} \nu(\widetilde{\varphi}, \widetilde{\theta})|D(u_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} f\cdot u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega |u_0|^2\,\mathrm{d}x. \tag{3.105}$$

It follows by passing to the limit in (3.97) as n goes to $+\infty$ and by using (3.105) that

$$\begin{aligned} \lim_{n\to+\infty} \Big[\frac{1}{2}\int_\Omega |u_\varepsilon^n(T)|^2\,\mathrm{d}x + \int_{Q_T} \nu(\widetilde{\varphi}^n, \widetilde{\theta}^n)|D(u_\varepsilon^n)|^2\,\mathrm{d}x\,\mathrm{d}t\Big] \\ = \int_\Omega |u_\varepsilon(T)|^2\,\mathrm{d}x + \int_{Q_T} \nu(\widetilde{\varphi}, \widetilde{\theta})|D(u_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.106}$$

As a consequence of this and of the weak convergence of $u_\varepsilon^n(T)$ to $u_\varepsilon(T)$ in $\mathcal{H}$,

$$\lim_{n\to+\infty} \|\nu(\widetilde{\varphi}^n,\widetilde{\theta}^n)^{\frac{1}{2}} D(u_\varepsilon^n)\|^2_{L^2(Q_T)^{2\times 2}} = \|\nu(\widetilde{\varphi},\widetilde{\theta})^{\frac{1}{2}} D(u_\varepsilon)\|^2_{L^2(Q_T)^{2\times 2}}, \tag{3.107}$$

which yields the strong convergence of $(u_\varepsilon^n(T), \nu(\widetilde{\varphi}^n,\widetilde{\theta}^n)^{\frac{1}{2}} D(u_\varepsilon^n))$ in the product space $\mathcal{H} \times L^2(Q_T)^{2\times 2}$. Moreover, due to the a.e. convergence of $\nu(\widetilde{\varphi}^n,\widetilde{\theta}^n)$ and since it is bounded, we deduce from the Lebesgue convergence Theorem that $D(u_\varepsilon^n)$ converges strongly in $L^2(Q_T)^{2\times 2}$, to $D(u_\varepsilon)$ as n goes to $+\infty$. Thus, Korn's inequality implies the strong convergence of ∇u_ε^n, and we can conclude that

$$u_\varepsilon^n \to u_\varepsilon \text{ strongly in } L^2(0,T;\mathcal{V}). \tag{3.108}$$

Now, using (3.108), (3.92), (3.93), (3.94), (3.95) and (3.103) we can pass to the limit and get that $(\varphi_\varepsilon, \theta_\varepsilon, u_\varepsilon)$ satisfy (3.81)-(3.85). Indeed, since $T_{1/\varepsilon}(\widetilde{\theta}^n) \to T_{1/\varepsilon}(\widetilde{\theta})$ in L^∞ weak-$*$ as n goes to ∞ and due to (3.108), the stability of the phase equation implies that φ_ε is solution of (3.81). Similarly, θ_ε is a solution of (3.82). As a conclusion, we obtain that $(\varphi_\varepsilon,\theta_\varepsilon) = \Lambda_\varepsilon(\widetilde{\varphi},\widetilde{\theta})$ and consequently Λ_ε is continuous.

There exists two balls $B_1 \times B_2$ of $L^2(Q_T)^2$ such that $\Gamma^\varepsilon(B_1\times B_2) \subset B_1 \times B_2$: Let $\widetilde{\varphi} \in B_{L^2(Q_T)}(0,R_1)$ and $\widetilde{\theta} \in B_{L^2(Q_T)}(0,R_2)$. We know by reasoning similarly to that of the estimations (3.89) and (3.90) that

$$\|\varphi_\varepsilon\|_{L^2(Q_T)} + \|\varphi_{\varepsilon,t}\|_{L^2(0,T;(H^1(\Omega))')} \le C(\Omega,\varepsilon), \tag{3.109}$$

$$\text{and } \|\theta_\varepsilon\|_{L^2(Q_T)} \le C'(\Omega,\varepsilon,\kappa_0) + lC(\Omega,\varepsilon). \tag{3.110}$$

Hence, by choosing R_1 and R_2 such that

$$C(\Omega,\xi) \le R_1,$$

and

$$C'(\Omega,\xi,\kappa_0) + lC(\Omega,\xi) \le R_2,$$

Λ_ε preserve the balls $B(0,R_1)\times B(0,R_2)$.

We conclude that applying Leray-Schauder's theorem yields the existence of the solution to (3.69)-(3.73). □

3.4.2 A priori estimates

This subsection is devoted to the proof of priori estimates uniform awith respect to ε, for the solution $(\varphi_\varepsilon,\theta_\varepsilon,u_\varepsilon)$ to (3.69)-(3.73). We prove successively that

— u_ε is bounded in $L^2(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{H})$ and $u_{\varepsilon,t}$ is bounded in $L^2(0,t;\mathcal{V}')$

— θ_ε is bounded in $L^\infty(0,T;L^1(\Omega))$, using the coupling of the phase and the heat equations

— φ_ε is bounded in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$

— $\forall k>0$, $\|T_k(\theta_\varepsilon)\|^2_{L^2(0,T;H^1(\Omega))} \le kM + L$, where M and L constants independent of k and ε, using again the coupling of the phase and the heat equation

— $\varphi_{\varepsilon,t}$ is bounded in $L^2(0,T;H^1(\Omega)')$ with the help of the Boccardo-Gallouët estimates; see Lemma 3.2.7

First, we begin by multiplying (3.71) by u_ε and integrating the result over Q_T. Since $\mathrm{div}(R_\varepsilon u_\varepsilon) = 0$, we obtain

$$\frac{1}{2}\int_\Omega |u_\varepsilon(T)|^2\,\mathrm{d}x + \int_{Q_T} \nu(\varphi_\varepsilon,\theta_\varepsilon)|D(u_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} f\cdot u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega |u_0|^2\,\mathrm{d}x. \tag{3.111}$$

Then, assumption (A2) on ν, Poincaré, Young and Korn inequalities permits us to deduce that

$$\|u_\varepsilon\|_{L^2(0,T;\mathcal{V})} + \|u_\varepsilon\|_{L^\infty(0,T;\mathcal{H})} + \|u_{\varepsilon,t}\|_{L^2(0,T;\mathcal{V}')} \le C, \tag{3.112}$$

where C denotes a constant depending on ν_0, $\|f\|_{L^2(Q_T)}$ and independent upon ε. Estimate (3.112) implies that

$$h_\varepsilon = T_{1/\varepsilon}(\nu(\varphi_\varepsilon,\theta_\varepsilon)D(u_\varepsilon) : D(u_\varepsilon)) \text{ is uniformly bounded in } L^1(Q_T)$$

.

For the estimates of φ_ε, we choose φ_ε as a test function in (3.78). By using the Green's formula with $\mathrm{div}(u_\varepsilon) = 0$ on Ω, the fact that $0 \le \varphi_\varepsilon \le 1$ and assumption (A4) on φ_0 we get that for any $0 < t \le T$

$$\int_\Omega |\varphi_\varepsilon(t)|^2\,\mathrm{d}x + \int_0^t\int_\Omega |\nabla\varphi_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s \le C(\xi,Q_T) + C(\xi)\int_0^t\int_\Omega |\theta_\varepsilon|\,\mathrm{d}x\,\mathrm{d}s. \tag{3.113}$$

We now turn to the estimates of θ_ε. First, let us show that θ_ε is bounded in $L^\infty(0,T;L^1(\Omega))$. In order to avoid any smallness condition on l, we perform an estimate on θ_ε using the coupling between the phase and the heat equation. We multiply the phase equation (3.69) by the test function $lT_1(\theta_\varepsilon)$ and the heat equation (3.70) by $T_1(\theta_\varepsilon)$. Summing the two result equations and using the fact that $\mathrm{div}(u_\varepsilon) = 0$ on Ω, we get for a.e. $t \in (0,T)$

$$\begin{aligned}
\xi l\int_0^t\int_\Omega \nabla\varphi_\varepsilon\cdot\nabla T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + l\int_0^t\int_\Omega u_\varepsilon\cdot\nabla\varphi_\varepsilon T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \widetilde{T}_1(\theta_\varepsilon)(t)\,\mathrm{d}x&\\
+\int_0^t\int_\Omega \kappa(\varphi_\varepsilon,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega \widetilde{T}_1(\theta_{\varepsilon,0})\,\mathrm{d}x&\\
+\int_0^t\int_\Omega F(\varphi_\varepsilon,T_{1/\varepsilon}(\theta_\varepsilon))\cdot lT_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega h_\varepsilon T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega g_\varepsilon T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s,&
\end{aligned} \tag{3.114}$$

where $\widetilde{T}_k$ is defined by $\widetilde{T}_k(r) = \int_0^r T_k(s)\,\mathrm{d}s$, for any $r \in \mathbb{R}$.

Since h_ε is bounded in $L^1(Q_T)$, $\widetilde{T}_1(r) \le |r|$, $0 \le \varphi_\varepsilon \le 1$, $g \in L^1(Q_T)$ and $\theta_0 \in L^1(\Omega)$ we obtain for a.e. $t \in (0,T)$

$$\begin{aligned}
\xi l\int_0^t\int_\Omega \nabla\varphi_\varepsilon\cdot\nabla T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + l\int_0^t\int_\Omega u_\varepsilon\cdot\nabla\varphi_\varepsilon T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \widetilde{T}_1(\theta_\varepsilon)(t)\,\mathrm{d}x&\\
+\int_0^t\int_\Omega \kappa(\varphi_\varepsilon,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s \le C(\|g\|_{L^1(Q_T)},\|\theta_0\|_{L^1(\Omega)},l,\Omega,t)&\\
+ lC\int_0^t\int_\Omega |\theta_\varepsilon|\,\mathrm{d}x\,\mathrm{d}s,&
\end{aligned} \tag{3.115}$$

Regarding the first term of the left-hand side of (3.115), we get from (3.113), the Hölder and the Young inequalities that, for a.e. $t \in (0,T)$

$$\begin{aligned}
\xi l\int_0^t\int_\Omega \nabla\varphi_\varepsilon\cdot\nabla T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s &\le \xi l\Big(\frac{1}{2\delta}\int_0^t\int_\Omega |\nabla\varphi_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\delta}{2}\int_0^t\int_\Omega |\nabla T_1(\theta_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}s\Big)\\
&\le C + C\frac{1}{2\delta}\int_0^t\int_\Omega |\theta_\varepsilon|\,\mathrm{d}x\,\mathrm{d}s + \xi l\frac{\delta}{2}\int_0^t\int_\Omega |\nabla T_1(\theta_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned} \tag{3.116}$$

where δ is chosen later, C is only depending on ξ, l, t, Ω and $\|\varphi_0\|_{L^2(Q_T)}$.

Similarly, for the second term of the left-hand side of (3.115), the Hölder and the Young inequalities together with (3.112) and (3.113) yield that

$$\begin{aligned} l\int_0^t\int_\Omega u_\varepsilon\cdot\nabla\varphi_\varepsilon T_1(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s &\le \frac{l}{2}\int_0^t\int_\Omega |u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s+\frac{l}{2}\int_0^t\int_\Omega|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\le C+C\int_0^t\int_\Omega|\theta_\varepsilon|\,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{3.117}$$

for a.e. $t\in(0,T)$, where C is independent of ε, depending only on ξ, l, Ω, t ν_0, $\|u_0\|_{\mathcal{H}}$ and $\|\varphi_0\|_{L^2(\Omega)}$.

Thus, recalling that $\widetilde{T}_1(r)\ge|r|-\frac{1}{2}$ and $0<\kappa_0\le\kappa\le\kappa_1$ a.e. in Q_T, we obtain by combining (3.115), (3.116) and (3.117), for a.e. $t\in(0,T)$,

$$\int_\Omega|\theta_\varepsilon(t)|\,\mathrm{d}x+(\kappa_0-2\xi l\delta)\int_0^t\int_\Omega|\nabla T_1(\theta_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}s\le C\int_0^t\int_\Omega|\theta_\varepsilon|\,\mathrm{d}x\,\mathrm{d}s+C \tag{3.118}$$

where δ is chosen such that $\kappa_0-2\xi l\delta$ is positive with C independent of ε. By Gronwall's Lemma and since the second term of the left-hand-side of (3.118) is positive, we deduce that there exists a positive constant independent of ε such that

$$\int_\Omega|\theta_\varepsilon(t)|\,\mathrm{d}x\le C, \tag{3.119}$$

for a.e. $t\in(0,T)$, which implies that θ_ε is bounded in $L^\infty(0,T;L^1(\Omega))$.

As a consequence of (3.119) and (3.113), we find that

$$\|\varphi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}+\|\varphi_\varepsilon\|_{L^2(0,T;H^1(\Omega))}\le C, \tag{3.120}$$

where C is independent of ε.

We now show, by a similar process to the one used for (3.119), that for all $k>0$, there exist M and L two positive constants independent of k and ε such that

$$\|T_k(\theta_\varepsilon)\|^2_{L^2(0,T;H^1(\Omega))}\le kM+L. \tag{3.121}$$

Indeed, by multiplying the phase equation (3.69) by the test function $lT_k(\theta_\varepsilon)$ and the heat equation (3.70) by $T_k(\theta_\varepsilon)$, then summing the two result equations we obtain, for a.e. $t\in(0,T)$,

$$\begin{aligned} &\xi l\int_0^t\int_\Omega\nabla\varphi_\varepsilon\cdot\nabla T_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s+l\int_0^t\int_\Omega u_\varepsilon\cdot\nabla\varphi_\varepsilon T_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s+\int_\Omega\widetilde{T}_k(\theta_\varepsilon)(t)\,\mathrm{d}x \\ &\qquad+\int_0^t\int_\Omega\kappa(\varphi_\varepsilon,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla T_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s=\int_\Omega\widetilde{T}_k(\theta_{\varepsilon,0})\,\mathrm{d}x \\ &\quad+\int_0^t\int_\Omega F(\varphi_\varepsilon,T_{1/\varepsilon}(\theta_\varepsilon))\cdot lT_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_\Omega h_\varepsilon T_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_\Omega g_\varepsilon T_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.122}$$

It follows, since $0\le\widetilde{T}_k(r)\le k|r|$, $0\le\varphi_\varepsilon\le1$, $g_\varepsilon\in L^1(Q_T)$, $\theta_0\in L^1(\Omega)$ and h_ε is uniformly bounded in $L^1(Q_T)$ that

$$\begin{aligned} &\xi l\int_\Omega^t\nabla\varphi_\varepsilon\cdot\nabla T_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s+l\int_0^t\int_\Omega u_\varepsilon\cdot\nabla\varphi_\varepsilon T_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s+\int_\Omega\widetilde{T}_k(\theta_\varepsilon)(t)\,\mathrm{d}x \\ &\qquad+\int_0^t\int_\Omega\kappa(\varphi_\varepsilon,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla T_k(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s\le kM+L+kC\int_0^t\int_\Omega|\theta_\varepsilon|\,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{3.123}$$

where M, C and L are constants independent of ε and k.

Moreover, due to (3.112), (3.120) and the fact that θ_ε is bounded in $L^\infty(0,T;L^1(\Omega))$, the Hölder and the Young inequalities applied on the first and on the second term of (3.123), imply that, for a.e. $t \in (0,T)$,

$$(\kappa_0 - 2\xi l\delta) \int_0^t \int_\Omega |\nabla T_k(\theta_\varepsilon)|^2 \,\mathrm{d}x\,\mathrm{d}s \leq kM + L, \tag{3.124}$$

where δ is obtained from Young's inequality and chosen such that $\kappa_0 - 2\xi l\delta > 0$ and M, L are constants independent of ε and k. Now, since

$$\int_{Q_T} |T_k(\theta_\varepsilon)|^2 \,\mathrm{d}x\,\mathrm{d}t \leq k \int_{Q_T} |\theta_\varepsilon| \,\mathrm{d}x\,\mathrm{d}t,$$

with θ_ε bounded in $L^\infty(0,T;L^1(\Omega))$, we deduce that for all $k > 0$

$$\int_{Q_T} |T_k(\theta_\varepsilon)|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} |\nabla T_k(\theta_\varepsilon)|^2 \,\mathrm{d}x\,\mathrm{d}t \leq kM + L, \tag{3.125}$$

where M and L are independent of ε and k.

It remains to establish an estimation of $\varphi_{\varepsilon,t}$. The main difficulty comes from the term of the right-hand-side of (3.69) because of the presence of θ_ε. However, since θ_ε is bounded in $L^\infty(0,T;L^1(\Omega))$ independently of ε and due to (3.121), we obtain from Boccardo-Gallouet estimates (see Lemma 3.2.7) and the interpolation inequalities that θ_ε is bounded in $L^2(0,T;L^q(\Omega))$, $\forall 1 \leq q < 2$. Hence, using Hölder's inequality, we obtain that $-\varphi_\varepsilon(1-\varphi_\varepsilon)(1-2\varphi_\varepsilon + cT_{1/\varepsilon}(\theta_\varepsilon))$ is bounded in $L^2(0,T;H^1(\Omega)')$. With the fact that $u \cdot \nabla\varphi = \operatorname{div}(u_\varepsilon\varphi_\varepsilon)$ is bounded in $L^2(0,T;H^1(\Omega)')$ we deduce that

$$\|\varphi_{\varepsilon,t}\|_{L^2(0,T;(H^1(\Omega))')} \leq C, \tag{3.126}$$

with C a positive constant independent of ε.

3.4.3 Passing to the limit

In this section, we extract subsequences and we pass to the limit as ε goes to zero in the regularized system (3.69)-(3.73) to get the existence of a weak renormalized solution in the sense of definition 3.3.1.

First, we notice from estimates (3.112) on u_ε and (3.120)-(3.126) on φ_ε, that a classical Aubin-Simon's type lemma (see e.g. [99]) implies that u_ε is in a compact set in $L^2(0,T;\mathcal{H})$ and φ_ε is in a compact set in $L^2(Q_T)$. Therefore, for a subsequence, still indexed by ε, we have

$$u_\varepsilon \rightharpoonup u \text{ weakly in } L^2(0,T;\mathcal{V}), \tag{3.127}$$
$$u_\varepsilon \to u \text{ strongly in } L^2(Q_T)^2 \text{ and a.e. in } Q_T, \tag{3.128}$$
$$u_{\varepsilon,t} \rightharpoonup u_t \text{ weakly in } L^2(0,T;\mathcal{V}'), \tag{3.129}$$
$$\varphi_\varepsilon \rightharpoonup \varphi \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{3.130}$$
$$\varphi_\varepsilon \to \varphi \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \tag{3.131}$$
$$\varphi_{\varepsilon,t} \rightharpoonup \varphi_t \text{ weakly in } L^2(0,T;H^1(\Omega)'), \tag{3.132}$$

where u belongs to $L^2(0,T;\mathcal{V})$ and φ belongs to $L^2(0,T;H^1(\Omega))$.

Furthermore, due to the estimate (3.121) on $T_k(\theta_\varepsilon)$ and since θ_ε is bounded in $L^\infty(0,T;L^1(\Omega))$ and h_ε is bounded in $L^1(Q_T)$ we are allowed to repeat the same procedure as in Chapter 2 (see Step 3, section 3) to show that there exists a subsequence, still indexed by ε, such that

$$\theta_\varepsilon \to \theta \ \text{ a.e. in } Q_T, \tag{3.133}$$

$$T_k(\theta_\varepsilon) \rightharpoonup T_k(\theta) \text{ weakly in } L^2(0,T;H^1(\Omega)). \tag{3.134}$$

In addition, since $0 \le \varphi_\varepsilon \le 1$ and θ_ε is bounded in $L^2(0,T;L^q(\Omega))$, for $1 \le q < 2$ (see Lemma 3.2.7), we obtain from the point-wise convergence of φ_ε and θ_ε that

$$-\varphi_\varepsilon(1-\varphi_\varepsilon)(1-2\varphi_\varepsilon + cT_{1/\varepsilon}(\theta_\varepsilon)) \rightharpoonup -\varphi(1-\varphi)(1-2\varphi+c\theta)$$

in $L^2(0,T;L^q(\Omega))' = L^2(0,T;L^{q'}(\Omega))$. Recalling that the Sobolev embedding of $H^1(\Omega)$ into $L^p(\Omega)$ for any $p < +\infty$, it follows that

$$-\varphi_\varepsilon(1-\varphi_\varepsilon)(1-2\varphi_\varepsilon + cT_{1/\varepsilon}(\theta_\varepsilon)) \rightharpoonup -\varphi(1-\varphi)(1-2\varphi+c\theta) \tag{3.135}$$

in $L^2(0,T;H^1(\Omega)')$.

Now, from the convergences (3.130), (3.131), (3.132) and (3.135) we can pass to the limit in the phase equation (3.69) and obtain that φ solves (3.55) in the sense of (3.67).

Moreover, since $0 < \nu_0 \le \nu(\varphi_\varepsilon,\theta_\varepsilon) \le \nu_1$, we obtain from (3.133) that $\nu(\varphi_\varepsilon,\theta_\varepsilon)$ converges to $\nu(\varphi,\theta)$ in $L^\infty(Q_T)$ weak-$*$. Hence, combining this with the convergences (3.127), (3.128) and (3.129) and then passing to the limit in (3.71) we get that u is a solution of (3.57) in the sense of (3.66) with $\operatorname{div} u = 0$.

It remains to show that θ is the renormalized solution of (3.56). For this purpose, we will use the stability result of the renormalized solution given in Theorem 3.2.6. We must then prove the following two convergences

$$h_\varepsilon \to \nu(\varphi,\theta)D(u) : D(u) \text{ strongly in } L^1(Q_T), \tag{3.136}$$

$$\varphi_{\varepsilon,t} \to \varphi_t \text{ strongly in } L^2(0,T;H^1(\Omega)'). \tag{3.137}$$

Let us start by showing that $D(u_\varepsilon) \to D(u)$ strongly in $L^2(Q_T)^{2\times 2}$, as ε goes to zero, using arguments similar to those in the proof of Theorem 3.4.2. Since $N = 2$, by the uniqueness of the weak solution of the Navier-stokes equations, we get that u is in $L^2(0,T;\mathcal{V})$ and satisfies

$$\frac{1}{2}\int_\Omega |u(T)|^2 \,\mathrm{d}x + \int_{Q_T} \nu(\varphi,\theta)|D(u)|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} f\cdot u \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega |u_0|^2\,\mathrm{d}x. \tag{3.138}$$

Now, by taking u_ε as a test function in (3.71), by integrating over Q_T, and then by using Hölder and Young inequalities together with the fact that $\operatorname{div} u_\varepsilon = 0$, we obtain

$$\frac{1}{2}\int_\Omega |u_\varepsilon(T)|^2 \,\mathrm{d}x + \int_{Q_T} \nu(\varphi_\varepsilon,\theta_\varepsilon)|D(u_\varepsilon)|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} f\cdot u_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega |u_0|^2\,\mathrm{d}x. \tag{3.139}$$

Therefore, due to (3.127), (3.128), we can pass to the limit as ε goes to zero in (3.139) and get

$$\begin{aligned}\lim_{\varepsilon\to 0}\Big[\frac{1}{2}\int_\Omega |u_\varepsilon(T)|^2 \,\mathrm{d}x + \int_{Q_T} \nu(\varphi_\varepsilon,\theta_\varepsilon)|D(u_\varepsilon)|^2 \,\mathrm{d}x\,\mathrm{d}t\Big] \\ = \int_{Q_T} f\cdot u \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega |u_0|^2\,\mathrm{d}x.\end{aligned} \tag{3.140}$$

Comparing (3.140) to (3.138) yields

$$\begin{aligned}\lim_{\varepsilon\to 0}\Big[\frac{1}{2}\int_\Omega |u_\varepsilon(T)|^2\,\mathrm{d}x + \int_{Q_T}\nu(\varphi_\varepsilon,\theta_\varepsilon)|D(u_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t\Big]&\\ = \int_\Omega |u(T)|^2\,\mathrm{d}x + \int_{Q_T}\nu(\varphi,\theta)|D(u)|^2\,\mathrm{d}x\,\mathrm{d}t.&\end{aligned}\tag{3.141}$$

Since $u_\varepsilon(T)\to u(T)$ weakly in $\mathcal{H}$ and $\nu(\varphi_\varepsilon,\theta_\varepsilon)^{1/2}D(u_\varepsilon)\to\nu(\varphi,\theta)^{1/2}D(u)$ weakly in $L^2(Q_T)^{2\times 2}$, we deduce that

$$D(u_\varepsilon)\to D(u) \text{ strongly in } L^2(Q_T)^{2\times 2},\tag{3.142}$$

which leads to (3.136).

Concerning the strong convergence of $\varphi_{\varepsilon,t}$ to φ_t in $L^2(0,T;(H^1(\Omega))')$, we notice that all the terms in the phase equation (3.69) converge strongly in $L^2(0,T;(H^1(\Omega)')$. Indeed, by multiplying (3.69) by φ_ε, by integrating over Q_T and by passing to the limit as ε goes to zero, that

$$\begin{aligned}\lim_{\varepsilon\to 0}\Big[\frac{1}{2}\int_\Omega |\varphi_\varepsilon(T)|^2\,\mathrm{d}x + \int_{Q_T}\xi|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\Big]&\\ = \int_\Omega |\varphi(T)|^2\,\mathrm{d}x + \int_{Q_T} -\varphi(1-\varphi)(1-2\varphi+c\theta)\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t.&\end{aligned}\tag{3.143}$$

Therefore, since $\varphi_\varepsilon(T)$ converges weakly to $\varphi(T)$ in $L^2(Q_T)$ and $\nabla\varphi_\varepsilon$ converges weakly to $\nabla\varphi$ in $L^2(Q_T)$, we obtain the strong convergence of $\nabla\varphi_\varepsilon$ to $\nabla\varphi$ in $L^2(Q_T)$. Moreover, due to (3.128) together with (3.131) and the fact that $0\le\varphi_\varepsilon\le 1$, the term $u_\varepsilon\nabla\varphi_\varepsilon$ converges strongly in $L^2(0,T;(H^1(\Omega))')$. Hence, we get the strong convergence of $\varphi_{\varepsilon,t}$ to φ_t in $L^2(0,T;(H^1(\Omega))')$.

Finally, due to (3.136) and (3.137) together with the strong convergence of g_ε to g in $L^1(Q_T)$, Theorem 3.2.6 allows one to deduce that θ_ε converges to θ a.e. in Q_T, where θ is the renormalized solution of (3.56).

The proof of Theorem 3.4.1 is now complete. $\square$

Chapter 4

Solutions for a Navier-Stokes-Boussinesq type system in 3D

4.1 Introduction

In the present work, we are interested in the mathematical analysis in the three-dimensional case of a phase-field model system for non-isothermal solidification with melt convection. Such a model was first studied in the context of the solidification process for pure materials and binary alloys (see for instance [36, 37, 83]), and then developed to include melt convection (see for example [3, 12, 63, 92]). In this chapter, we study the existence of solutions to the following nonlinear system

$$
\begin{aligned}
&\varphi_t + u\cdot\nabla\varphi - \xi\Delta\varphi = -\varphi(1-\varphi)(1-2\varphi+c\theta) && \text{in } Q_T, && (4.1)\\
&\theta_t - \operatorname{div}(\kappa(\varphi,\theta)\nabla\theta) + \operatorname{div}(u\theta) = \nu(\varphi,\theta)D(u):D(u) + g + l\varphi_t && \text{in } Q_T, && (4.2)\\
&u_t + (u\cdot\nabla)u - \operatorname{div}(\nu(\varphi,\theta)D(u)) + \nabla p = f, \quad \operatorname{div} u = 0 && \text{in } Q_T, && (4.3)\\
&\nabla\varphi\cdot\boldsymbol{n} = 0, \quad \kappa(\varphi,\theta)\nabla\theta\cdot\boldsymbol{n} = 0, \quad u\cdot\boldsymbol{n} = 0, \quad \alpha u_\tau + (\nu(\theta,\varphi)D(u)\boldsymbol{n})_\tau = 0 && \text{on } \Sigma, && (4.4)\\
&\varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0(x), \quad u(x,0) = u_0(x) && \text{in } \Omega, && (4.5)
\end{aligned}
$$

where Ω is $C^{1,1}$ open bounded domain in $\mathbb{R}^3$, $T > 0$ and $Q_T = \Omega \times (0,T)$ with lateral surface $\Sigma = \partial\Omega \times (0,T)$ and $\boldsymbol{n}$ the outer normal normal to the boundary $\partial\Omega$. The system described here consists of three partial differential equations with advection terms in each equation and variable viscosity ν and heat conductivity κ. Equation (4.1) is the phase field equation, the unknown is the function φ which represents the solid/liquid phase, ξ is a positive constant related to the width of the transitions layer and c is a positive constant. Equation (4.2) is the energy equation, the unknown is the temperature θ, $l > 0$ is the latent heat, and κ and ν are two positive functions depending on φ and θ, representing the heat conductivity and the fluid viscosity. Finally, equation (4.3) is the Navier-Stokes equation required for the non-solid region. The unknowns are the velocity u and the pressure p, f is an external field, and $D(u)$ denotes the symmetric part of the gradient of the vector field u i.e. $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$. Here, The fluid is considered incompressible, which explains the equation $\operatorname{div} u = 0$ in Q_T. Concerning the boundary conditions, we consider the case where there is no heat exchange i.e. $\nabla\theta\cdot\boldsymbol{n} = 0$ on Σ and we assume that the boundary is impermeable i.e. $u\cdot\boldsymbol{n} = 0$ on Σ. Moreover, since the viscosity and the heat conductivity is dependent on the temperature and the phase, we consider the slip boundary conditions i.e. $\alpha u_\tau + (\nu(\varphi,\theta)D(u)\boldsymbol{n})_\tau = 0$ on Σ where $\alpha \in [0,\infty)$ and $z_\tau := z-(z\cdot\boldsymbol{n})\boldsymbol{n}$. This choice allows the pressure to be introduced globally as an integrable function (see [66] or [104] for details).

In the three-dimensional case, the analysis of a phase field model for solidification has been studied under the effect of a magnetic field in [58] and for a change Phase of an Alloy with convection in [31]. To our knowledge, the existence of solutions for a phase field model system similar to ours with variable viscosity, variable heat conductivity and a nonlinear term belonging to L^1 in the heat equation has only been studied in the two-dimensional case within the framework of the renormalized solutions; see for instance Chapter 3 in the present document or in [63] in the case of Dirichlet boundary conditions on θ and where the latent heat l is considered very small. However, the N-S-E equations (4.2) and (4.3) without the phase equation have been subject to many studies in the three-dimensional case. Lions in [80] provided two approaches to obtain the existence of the solution of the N-S-E system, in which the viscosity and heat conductivity are considered positive constants. In the case of temperature-dependant viscosity and heat conductivity, and this is the case we are interested in, authors in [32], [34], and [33] established long-time and large data existence result for a quite general class of Newtonian incompressible fluids. NSKTE models, which couples Navier-Stokes equation with turbulent kinetic energy equation, have a similar structure to (4.2)–(4.3) (without the phase equation) and then present the same difficulties (see e.g.[78, 40, 91]).

In the three-dimensional case, the non-uniqueness of the solution to the Navier-Stokes equation is the main obstacle and we cannot expect to solve (4.1)–(4.3) as in the previous chapter. Even if we are able to construct an approximated problem as in Chapter 3 the convergence of $D(u_\varepsilon) : D(u_\varepsilon)$ is an open question. As in [80, 32] we perform a transformation of equations (4.2) and (4.3) to derive a formally equivalent system. It consists in multiplying (4.3) by u and adding the result equation to (4.2), and we obtain

$$\begin{aligned}(\theta + \frac{|u|^2}{2})_{,t} - \operatorname{div}(\kappa(\varphi,\theta)\nabla\theta) + \operatorname{div}(u(\frac{|u|^2}{2} + \theta + p))\\ = \operatorname{div}(\nu(\varphi,\theta)D(u)u) + g + l\varphi_t + f\cdot u.\end{aligned} \tag{4.6}$$

As noticed in [80] equation (4.6) is the "total energy" equation (see also [34] and we add the following entropy inequality

$$\theta_t - \operatorname{div}(\kappa(\varphi,\theta)\nabla\theta) + \operatorname{div}(\theta u) \geq \nu(\varphi,\theta)D(u) : D(u) + g + l\varphi_t. \tag{4.7}$$

Therefore, our system is reformulated as follows.

$$\varphi_t + \operatorname{div}(u\varphi) - \xi\Delta\varphi = F(\varphi,\theta) \qquad \text{in } Q_T, \tag{4.8}$$

$$\begin{aligned}(\theta + \frac{|u|^2}{2})_{,t} - \operatorname{div}(\kappa(\varphi,\theta)\nabla\theta) + \operatorname{div}(u(\frac{|u|^2}{2} + \theta + p))\\ = \operatorname{div}(\nu(\varphi,\theta)D(u)u) + g + l\varphi_t + f\cdot u \qquad \text{in } Q_T,\end{aligned} \tag{4.9}$$

$$\theta_t - \operatorname{div}(\kappa(\varphi,\theta)\nabla\theta) + \operatorname{div}(\theta u) \geq \nu(\varphi,\theta)D(u) : D(u) + g + l\varphi_t \qquad \text{in } Q_T, \tag{4.10}$$

$$u_t + (u\cdot\nabla)u - \operatorname{div}(\nu(\varphi,\theta)D(u)) + \nabla p = f, \quad \operatorname{div} u = 0 \qquad \text{in } Q_T, \tag{4.11}$$

$$\nabla\varphi\cdot\boldsymbol{n} = 0, \quad \kappa(\varphi,\theta)\nabla\theta\cdot\boldsymbol{n} = 0, \quad u\cdot\boldsymbol{n} = 0, \quad \alpha u_\tau + (\nu(\theta,\varphi)D(u)\boldsymbol{n})_\tau = 0 \qquad \text{on } \Sigma, \tag{4.12}$$

$$\varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0(x), \quad u(x,0) = u_0(x) \qquad \text{in } \Omega, \tag{4.13}$$

Our purpose in this work is to get an existence result of solution for this problem. By combining the techniques developed in chapter 3 in the two-dimensional case and the techniques developed in [32] in the case of a Navier-Stokes-Fourier system for compressible fluids, we show the existence of weak suitable solutions to the system (4.1)-(4.5).

The chapter is organized as follows. In Section 4.2 we recall some known results and definitions. In particular, the functional spaces to be used, the Helmholtz decomposition, and some regularity results. In Section 4.3 we state the assumptions and define the solution of the the system. The main result of this chapter is also stated in Section 4.3. Section 4.4 is devoted to the proof of the existence result. It is divided into four subsections. In the first Subsection 4.4.1, we introduce the (ε, η)-approximate system, then in Subsection 4.4.2 we derive uniform estimates on the solution of this approximate system, and at last, in the Subsection 4.4.3 and Subsection 4.4.4, we pass to the limit as $\varepsilon \to 0$ and then as $\eta \to 0$ in the approximate system to obtain the desired existence result. Finally, for completeness, the existence of the solution to the approximate system is established in Section 4.5.

4.2 Preliminary results and recalls

This section is devoted to the recall of some notions and results necessary for the study of the system.

Let Ω be a bounded open domain of $\mathbb{R}^N$ such that $\Omega \in C^{1,1}$. We denote by $D(\Omega)$ the space of C^∞ functions which have compact support and $D'(\Omega)$ the space of distributions on Ω. For $1 \leq p \leq +\infty$, $L^p(\Omega)$ designates the usual Lebesgue spaces, $W^{1,p}(\Omega)$ the usual Sobolev spaces and $W^{-1,p'}(\Omega)$ its dual. We use the symbol $\langle a, b\rangle$ for $a \in X$ and $b \in X^*$ to design $\langle a, b\rangle_{X,X^*}$. We denote by $\operatorname{tr} u$, the trace of a Sobolev function u on the boundary and by T_k, for any real number $k \geq 0$, the truncation function at height $\pm k$, that is

$$T_k(r) = \begin{cases} -k & \text{if} \quad r \leq -k, \\ r & \text{if} \quad -k \leq r \leq k, \\ k & \text{if} \quad r \geq k. \end{cases}$$

Then we define the function $\widetilde{T}_k$ by

$$\widetilde{T}_k(r) = \int_0^r T_k(s)\,\mathrm{d}s.$$

In this chapter, we will be using some interpolation inequalities for the estimates. These inequalities are the result of Hölder's and embeddings inequalities, we will give them below.

$$\text{For } 2 \leq q \leq 6, \;\; \|v\|_{L^q(\Omega)} \leq \|v\|_{L^2(\Omega)}^{\frac{6-q}{2q}} \|v\|_{L^6(\Omega)}^{\frac{3(q-2)}{2q}} \leq C\|v\|_{L^2(\Omega)}^{\frac{6-q}{2q}} \|v\|_{W^{1,2}(\Omega)}^{\frac{3(q-2)}{2q}}, \tag{4.14}$$

$$\text{For } 1 \leq q \leq 6, \;\; \|v\|_{L^q(\Omega)} \leq \|v\|_{L^1(\Omega)}^{\frac{6-q}{5q}} \|v\|_{L^6(\Omega)}^{\frac{6(q-1)}{5q}} \leq C\|v\|_{L^1(\Omega)}^{\frac{6-q}{5q}} \|v\|_{W^{1,2}(\Omega)}^{\frac{6(q-1)}{5q}}. \tag{4.15}$$

In addition, we provide the following regularity results known as the Boccardo-Gallouët estimates in the case of Dirichlet boundary conditions. The proof is given in Chapter 3.

Lemma 4.2.1. *Let v be an element of $L^\infty(0,T;L^1(\Omega))$. Assume that, for any $k > 0$, $T_k(v) \in L^2(0,T;H^1(\Omega))$ and that*

$$\|T_k(v)\|^2_{L^2(0,T;H^1(\Omega))} \leq kM + L, \tag{4.16}$$

where M and L are two positive constants independent of k. Then,

— *for all* $1 \leq p < 1 + \frac{2}{N}$, $v \in L^p(Q_T)$*; more precisely there exists a constant* C *(depending on* M, L, T, Ω *and* p*) such that*

$$\|v\|_{L^p(Q_T)} \leq C \tag{4.17}$$

— *for all* $1 < r < \overline{p}$ *where* $\overline{p} = \frac{2^*}{2}$ *if* $N \geq 3$*, we have*

$$v \in L^1(0,T; L^r(\Omega));$$

More precisely, for all $1 \leq r < \overline{p}$

$$\|v\|_{L^1(0,T;L^r(\Omega))} \leq C \tag{4.18}$$

where $C > 0$ *is constant depending on* M, L, T, Ω, p *and* r.

— *for all* $1 \leq p < \frac{N+2}{N+1}$, $\nabla v \in L^p(Q_T)^N$*; more precisely there exists a constant* C *(depending in* M, L, T, Ω, *and* p*) such that*

$$\|\nabla v\|_{L^p(Q_T)} \leq C. \tag{4.19}$$

In addition, for $N = 3$ *using these estimates and the interpolation inequalities, we have* $u \in L^r(0,T; L^q(\Omega))$*, for all* $1 \leq r < +\infty$ *and* $1 \leq q < \frac{5r}{5r-2}$*; more precisely, there exists a positive constant* C *(depending on* M, L, T, Ω, r, q*) such that*

$$\|v\|_{L^r(0,T;L^q(\Omega))} \leq C. \tag{4.20}$$

Now, we note that for the analysis of the Navier-stokes equations, we need to introduce other functional spaces. Hence, for $p \geq 1$, we define the space $W^{1,p}_{\boldsymbol{n}}$ by

$$W^{1,p}_{\boldsymbol{n}} = \{v \in W^{1,p}(\Omega)^N : \operatorname{tr} v \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$

and $W^{1,p}_{\boldsymbol{n},\mathrm{div}}$ by

$$W^{1,p}_{\boldsymbol{n},\mathrm{div}} = \{v \in W^{1,p}_{\boldsymbol{n}}(\Omega)^N : \operatorname{div} v = 0 \text{ in } \Omega\},$$

with $W^{-1,p'}_{\boldsymbol{n}}$, $W^{-1,p'}_{\boldsymbol{n},\mathrm{div}}$ their dual spaces respectively. Finally, we denote by $L^p_{\boldsymbol{n}}$ the closure of $W^{1,p}_{\boldsymbol{n},\mathrm{div}}$ in $L^p(\Omega)^N$ and for $r > 1$, we write $u \in C([0,T], L^r_{weak}(\Omega)^3)$ to name $u \in L^\infty(0,T; L^r(\Omega)^3)$ sucht that $t \mapsto (u(t), \phi) \in C([0,T])$ for all $\phi \in L^{r'}(\Omega)^3$.

We define now the so-called Helmholtz decomposition. We will use this decomposition in subsection 4.4.1 for the regularized problem and in subsection 4.4.2 for the pressure estimates. Let u be in $W^{1,p}_{\boldsymbol{n}}$, for $1 \leq p \leq +\infty$. Then,

$$u = u_{\mathrm{div}} + \nabla g^u, \tag{4.21}$$

where g^u is a solution to the auxiliary problem

$$\begin{cases} \Delta g^u = \operatorname{div} u & \text{in } \Omega\ , \\ \nabla g^u \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega\ , \\ \displaystyle\int_\Omega g^u \,\mathrm{d}x = 0. \end{cases} \tag{4.22}$$

Consequently, we deduce from this definition that $\operatorname{div} u_{\text{div}} = 0$ a.e. in Ω and if Ω has $C^{1,1}$ regularity, the L^p-theory (see [72, 71])for the problem (4.22) yields that

$$\|g^u\|_{W^{2,q}(\Omega)} \le C_{reg}(\Omega, q)\| \operatorname{div} u\|_{L^q(\Omega)}, \tag{4.23}$$
$$\|u_{\text{div}}\|_{W^{1,q}(\Omega)} \le (C_{reg}(\Omega, q) + 1)\|u\|_{W^{1,q}(\Omega)}, \tag{4.24}$$
$$\|g^u\|_{W^{1,s}(\Omega)} \le C(\Omega, s)\|u\|_{L^s(\Omega)}, \tag{4.25}$$
$$\|u_{\text{div}}\|_{L^s(\Omega)} \le (C(\Omega, s) + 1)\|u\|_{L^s(\Omega)}. \tag{4.26}$$

Finally, we end this section by recalling Korn's inequality in Lemma 4.2.2 and compactness result for some functions on the boundary in Lemma 4.2.3 and Corollary 4.2.4. Their proofs can be found in [34].

Lemma 4.2.2. *[[34]] Let $\Omega \in C^{0,1}$ and $1 \le p \le +\infty$. There exists a positive constant C depending only on Ω and p such that for all $u \in W^{1,p}(\Omega)^N$ with $\operatorname{tr} u \in L^2(\partial\Omega)^N$ there holds*

$$C\|u\|_{W^{1,p}(\Omega)} \le \|D(u)\|_{L^p(\Omega)} + \|\operatorname{tr} u\|_{L^2(\partial\Omega)}. \tag{4.27}$$

Lemma 4.2.3. *[[34]] Let $\Omega \in C^{0,1}$ and $N = 2, 3$. Let $q_1 \ge 1$, r and $q_2 \in]1, +\infty[$ and let S be defined by*

$$S = \{u; u \in L^\infty(0, T; L^2(\Omega)^N) \cap L^r(0, T; W^{1,r}_{\boldsymbol{n}}), u_t \in L^q_1(0, T; W^{-1,q_2}_{\boldsymbol{n},\text{div}})\}.$$

If $\{u^i\}_{i=1}^\infty$ is bounded in S and $r \in]\frac{2(N+1)}{N+2}, 2]$, then $\{\operatorname{tr} u^i\}_{i=1}^\infty$ is precompact in $L^s(0, T; L^s(\partial\Omega)^N)$ for all $s \in [1, r(N + 2)/N - 2/N[$.

Corollary 4.2.4. *[[34]] Assume that $\{u^i\}_{i=1}^\infty$ is bounded in S with $N = 3$ and $r = 2$. Then, $\{\operatorname{tr} u^i\}_{i=1}^\infty$ is precompact in $L^s(0, T; L^s(\partial\Omega)^3)$ for all $s \in 1 \le s < 8/3$, particularly in $L^2(0, T; L^2(\partial\Omega)^3)$.*

4.3 Assumptions, definitions and main result

In this section, we present the assumptions and definitions required for the study of the system. We then state our main existence result.

Let $\Omega \in C^{1,1}$ be an open bounded set of $\mathbb{R}^3$ and $T > 0$. Let $Q_T = \Omega \times (0, T)$ and $\partial\Omega$ the boundary of Ω with $\boldsymbol{n}$ the outer normal to the boundary. We recall that our purpose is to show the existence of (φ, θ, u, p) solution of

$$\varphi_t + u\cdot\nabla\varphi - \xi\Delta\varphi = F(\varphi, \theta) \quad \text{in } Q_T, \tag{4.28}$$
$$\theta_t - \operatorname{div}(\kappa(\varphi, \theta)\nabla\theta) + \operatorname{div}(u\theta) = \nu(\varphi, \theta)D(u) : D(u) + g + l\varphi_t \quad \text{in } Q_T, \tag{4.29}$$
$$u_t + (u\cdot\nabla)u - \operatorname{div}(\nu(\varphi, \theta)D(u)) + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } Q_T, \tag{4.30}$$
$$\nabla\varphi \cdot \boldsymbol{n} = 0, \quad \kappa(\varphi, \theta)\nabla\theta \cdot \boldsymbol{n} = 0, \quad u \cdot \boldsymbol{n} = 0, \quad \alpha u_\tau + (\nu(\theta, \varphi)D(u)\boldsymbol{n})_\tau = 0 \quad \text{on } \Sigma, \tag{4.31}$$
$$\varphi(x, 0) = \varphi_0(x), \quad \theta(x, 0) = \theta_0(x), \quad u(x, 0) = u_0(x) \quad \text{in } \Omega, \tag{4.32}$$

where $\alpha \in [0, \infty)$ and $z_\tau := z - (z \cdot \boldsymbol{n})\boldsymbol{n}$.

As we explained in the introduction, since the terms g and $\nu(\varphi, \theta)D(u) : D(u)$ on the right-hand side of (4.29) belong to L^1 only and u is not sufficiently smooth to be an admissible test function, we reformulate the system (4.28)-(4.32) in the following way. We search for (φ,θ,u,p) solution of

$$\varphi_t + \mathrm{div}\,(u\varphi) - \xi\Delta\varphi = F(\varphi,\theta) \qquad \text{in } Q_T, \quad (4.33)$$

$$(\theta + \frac{|u|^2}{2})_{,t} - \mathrm{div}\,(\kappa(\varphi,\theta)\nabla\theta) + \mathrm{div}\,(u(\frac{|u|^2}{2} + \theta + p))$$
$$= \mathrm{div}(\nu(\varphi,\theta)D(u)u) + g + l\varphi_t + f\cdot u \qquad \text{in } Q_T, \quad (4.34)$$

$$\theta_t - \mathrm{div}\,(\kappa(\varphi,\theta)\nabla\theta) + \mathrm{div}(\theta u) \geq \nu(\varphi,\theta)D(u):D(u) + g + l\varphi_t \qquad \text{in } Q_T, \quad (4.35)$$

$$u_t + (u\cdot\nabla)u - \mathrm{div}\,(\nu(\varphi,\theta)D(u)) + \nabla p = f, \quad \mathrm{div}\ u = 0 \qquad \text{in } Q_T, \quad (4.36)$$

$$\nabla\varphi\cdot\boldsymbol{n} = 0, \quad \kappa(\varphi,\theta)\nabla\theta\cdot\boldsymbol{n} = 0, \quad u\cdot\boldsymbol{n} = 0, \quad \alpha u_\tau + (\nu(\theta,\varphi)D(u)\boldsymbol{n})_\tau = 0 \qquad \text{on } \Sigma, \quad (4.37)$$

$$\varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0(x), \quad u(x,0) = u_0(x) \qquad \text{in } \Omega, \quad (4.38)$$

where we assume that

(1) $\xi > 0$, $F(\varphi,\theta) = -\varphi(1-\varphi)(1-2\varphi+c\theta)$, $c > 0$;

(2) $l > 0$, $\kappa,\nu \in C^0(\mathbb{R}^2)$ such that $0 < \kappa_0 \leq \kappa \leq \kappa_1$ and $0 < \nu_0 \leq \nu \leq \nu_1$;

(3) $g \in L^1(Q_T)$ and $f \in L^2(Q_T)^3$;

(4) $\varphi_0 \in L^\infty(\Omega)$ such that $0 \leq \varphi_0 \leq 1$, $\theta_0 \in L^1(\Omega)$ and $u_0 \in L^2_{\boldsymbol{n}}$.

To introduce the notions of weak and suitable weak solution to (4.33)-(4.38), we must first define the energetic functional $E(t,\cdot)$ by

$$E(t,\phi) := \int_\Omega (\theta(t,x) + \frac{|u(t,x)|^2}{2})\phi(x)\,\mathrm{d}x \quad \forall\phi \in C(\bar{\Omega}),$$

and the initial energetic functional $E_0(\phi)$ by

$$E_0(\phi) := \int_\Omega (\theta_0 + \frac{|u_0|^2}{2})\phi\,\mathrm{d}x \quad \forall\phi \in C(\bar{\Omega}).$$

Now we give the definition of the solution of (4.33)-(4.38).

Definition 4.3.1. *Let $\Omega \in C^{1,1}$. Assume that* (1)-(4) *hold true. We call* (φ,θ,u,p) *a suitable weak solution of* (4.33)-(4.38) *if*

$$u \in C(0,T;L^2_{weak}(\Omega)^3) \cap L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}), \quad u_t \in L^{\frac{5}{3}}(0,T;W_{\boldsymbol{n}}^{-1,\frac{5}{3}}); \qquad (4.39)$$

$$p \in L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}}(\Omega)); \qquad (4.40)$$

$$\varphi \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)), \quad \varphi_t \in L^2(0,T;W^{-1,2}(\Omega)); \qquad (4.41)$$

$$0 \leq \varphi \leq 1; \qquad (4.42)$$

$$\theta \in L^\infty(0,T;L^1(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega)),\ \forall\, 1 \leq p < 5/4; \qquad (4.43)$$

$$\theta \in L^1(0,T;L^r(\Omega)),\ \forall 1 \leq r < \frac{2^*}{2}; \qquad (4.44)$$

$$E(t,\phi) \in C([0,T]), \quad \lim_{t\to 0+} E(t,\phi) = E_0(\phi) \quad \forall\phi \in C(\bar{\Omega}); \qquad (4.45)$$

$$\lim_{t\to 0+} \|u(t) - u_0\|_{L^2(\Omega)} = 0,\ \varphi(t=0) = \varphi_0; \qquad (4.46)$$

and if

$$\int_0^T \langle u_t, v\rangle \,\mathrm{d}t + \int_{Q_T} \nu(\varphi,\theta) D(u)\cdot D(v)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\partial\Omega} \alpha u v \,\mathrm{d}\omega\,\mathrm{d}t$$
$$+ \int_{Q_T} (u\cdot\nabla)u\cdot v\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} p \operatorname{div} v\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} f v\,\mathrm{d}x\,\mathrm{d}t \qquad \forall v\in L^\infty(0,T;W^{1,\infty}_{\boldsymbol{n}}); \tag{4.47}$$

$$\begin{aligned}&\int_0^T \langle \varphi_t, \psi\rangle \,\mathrm{d}t + \int_{Q_T} \xi\nabla\varphi\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u\varphi\nabla\psi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad= \int_{Q_T} -\varphi(1-\varphi)(1-2\varphi+c\theta)\psi\,\mathrm{d}x\,\mathrm{d}t \qquad \forall\psi\in L^2(0,T;W^{1,2}(\Omega));\end{aligned} \tag{4.48}$$

$$\begin{aligned}&-\int_{Q_T}\Big(\frac{|u|^2}{2}+\theta\Big)\phi_t\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u\Big(\frac{|u|^2}{2}+\theta+p\Big)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}\kappa(\varphi,\theta)\nabla\theta\cdot\nabla\phi\,\mathrm{d}t\,\mathrm{d}x\\ &\qquad + \int_0^T\int_{\partial\Omega}\alpha|u|^2\phi\,\mathrm{d}x\,\mathrm{d}\omega - \int_{Q_T}\nu(\varphi,\theta)D(u)u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_{Q_T} g\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}\langle l\varphi_t,\phi\rangle\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} f u\phi\,\mathrm{d}x\,\mathrm{d}t + E_0(\phi) \qquad \forall\phi\in C_c^\infty([0,T)\times\bar\Omega).\end{aligned} \tag{4.49}$$

Moreover,

$$\begin{aligned}&-\int_{Q_T}\theta\phi_t\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u\theta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}\kappa(\varphi,\theta)\nabla\theta\cdot\nabla\phi\,\mathrm{d}t\,\mathrm{d}x - \int_{Q_T}\nu(\varphi,\theta)|D(u)|^2\phi\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad \ge \int_{Q_T} g\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}\langle l\varphi_t,\phi\rangle\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \theta_0\phi_0\,\mathrm{d}x\,\mathrm{d}t \qquad \forall\phi\in C_c^\infty([0,T)\times\bar\Omega), \phi\ge 0.\end{aligned} \tag{4.50}$$

Let us give some remarks concerning Definition 4.3.1.

Remark 4.3.2. *To get* (4.47), (4.48), (4.49) *and* (4.50) *we multiply the equations* (4.33), (4.34), (4.35) *and* (4.36) *by the corresponding test functions, integrate over* Ω *and* $(0,T)$ *and then use the integration by parts Lemma. We remark the presence of the boundary integral in* (4.47). *Indeed, recalling that* $z_\tau := z - (z\cdot\boldsymbol{n})\boldsymbol{n}$ *we have, formally,* $(\nu(\varphi,\theta)D(u)\boldsymbol{n})_\tau := \nu(\varphi,\theta)D(u)\boldsymbol{n} - (\nu(\varphi,\theta)D(u)\boldsymbol{n}\cdot\boldsymbol{n})\boldsymbol{n}$ *on* Σ*. Since* $v\cdot\boldsymbol{n}=0$ *on* $\partial\Omega$ *and* $\alpha u_\tau + (\nu(\theta,\varphi)D(u)\boldsymbol{n})_\tau = 0$ *on* Σ*, we obtain formally by using the Green's formula that*

$$\begin{aligned}&-\int_0^T\int_\Omega \operatorname{div}(\nu(\varphi,\theta)D(u))v\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_\Omega \nu(\varphi,\theta)D(u)\cdot D(v)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\partial\Omega}(\nu(\varphi,\theta)D(u)\boldsymbol{n})\cdot v\,\mathrm{d}\omega\,\mathrm{d}t\\ &= \int_0^T\int_\Omega \nu(\varphi,\theta)D(u)\cdot D(v)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\partial\Omega}(\nu(\varphi,\theta)D(u)\boldsymbol{n})_\tau\cdot v\,\mathrm{d}\omega\,\mathrm{d}t\\ &\quad - \int_0^T\int_{\partial\Omega}((\nu(\varphi,\theta)D(u)\boldsymbol{n})\cdot\boldsymbol{n})\boldsymbol{n}\cdot v\,\mathrm{d}\omega\,\mathrm{d}t\\ &= \int_0^T\int_\Omega \nu(\varphi,\theta)D(u)\cdot D(v)\,\mathrm{d}x\,\mathrm{d}t + \alpha\int_0^T\int_{\partial\Omega} u\cdot v\,\mathrm{d}\omega\,\mathrm{d}t.\end{aligned}$$

Similarly, we obtain the boundary integral in (4.49).

Remark 4.3.3. *We note that all the terms in* (4.47)*,* (4.48)*,* (4.49) *and* (4.50) *are well-defined. Indeed,*

1. *For the boundary integrals, Lemma 4.2.3 with* (4.39) *yields that*

$$\operatorname{tr} u \in L^2(0,T;L^2(\partial\Omega)^3).$$

Hence, all the boundary integrals are finite.

2. *Concerning the nonlinear term* $-\varphi(1-\varphi)(1-2\varphi+c\theta)$ *in the phase formulation* (4.48)*, we recall that* $0 \le \varphi \le 1$ *and since* $\theta \in L^\infty(0,T;L^1(\Omega)) \cap L^1(0,T;L^r(\Omega))$*, for all* $1 \le r < \frac{2^*}{2}$*, we obtain from the interpolation inequalities that* $\theta \in L^2(0,T;L^{\frac{6}{5}}(\Omega))$*. Then, by the Hölder's inequality we get that*

$$\theta\psi \in L^1(Q_T) \text{ for all } \psi \in L^2(0,T;W^{1,2}(\Omega)).$$

3. *For the second term in* (4.49)*, since* $u \in L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}})$ *we obtain due to the interpolation inequalities that* $u \in L^{\frac{10}{3}}(Q_T)^3$*. Hence, from Hölder's inequality and since* $p \in L^{\frac{5}{3}}(Q_T)$*, we get that*

$$u(\frac{|u|^2}{2} + p) \in L^{\frac{10}{9}}(Q_T)^3.$$

Furthermore, using the fact that $u \in L^{\frac{10}{3}}(Q_T)^3$ *and* $\theta \in L^q(Q_T)$*, for all* $1 \le q < 5/3$ *we obtain from Hölder's inequality that*

$$u\theta \in L^p(Q_T)^3 \text{ for all } 1 \le p < 10/9.$$

It follows that the second integral of (4.49) *is well defined.*

As a consequence of (4.39)-(4.43) *the other terms in* (4.47)*,* (4.48)*,* (4.49) *and* (4.50) *are also well defined.*

Now, we state our main existence result.

Theorem 4.3.4. *Let* $\Omega \in C^{1,1}$ *be an open bounded domain in* $\mathbb{R}^3$*. Assume that* (1)-(4) *hold true. Then, there exists a suitable weak solution* (φ,θ,u,p) *to the system* (4.33)-(4.34) *in the sense of Definition 4.3.1*

4.4 Proof of the existence result

The proof of Theorem 4.3.4 will be developed in the following subsections. It is based on an approximation technique. We first construct an approximate system with the help of two positive parameters ε and η. Then, we derive priori estimates on the solution of this system. Finally, we pass to the limit in the approximate system first when ε goes to zero, then as η goes to zero. We will combine the techniques developed in [32] for a Navier-Stokes-Fourier system with temperature-dependent material coefficients and the ideas developed in the two-dimensional case. Thus, due to the presence of $l\varphi_t$ in the heat equation and to avoid any smallness conditions on l and to get the estimates on the temperature we use here the coupling between the phase and the heat equations.

Subsection 4.4.1 is devoted to the (ε, η)-approximate system (4.53)-(4.58). The idea is to replace the velocity u in the convective and the advection terms of the heat and the phase equations by a suitable free divergence function u_η for $\eta > 0$ and replace the temperature θ in the nonlinear term of the phase equation by $T_{1/\eta}(\theta)$ for $\eta > 0$. Moreover, we choose to construct the pressure from the beginning, thus perturbing for all $\varepsilon > 0$ the incompressible constraint, $\operatorname{div} u = 0$, by the Neumann problem $-\varepsilon\Delta p = -\operatorname{div} u$, allows to provide the pressure since as we can see the pressure is a solution of this problem. Concerning the existence of a solution to the (ε, η)-system, for all $\varepsilon > 0$, $\eta > 0$, it is shown using the Galerkin method. We develop its proof in Section 4.5 of this chapter.

In Subsection 4.4.2, we derive estimates on the solution of the approximate system. These estimates are uniform with respect to ε and η. For the estimates of the pressure we use the Helmholtz decomposition and for the estimates of the temperature and the phase we use the coupling between the heat and the phase equation. In this subsection, we will notice the utility of the Navier's boundary condition and the choice of Ω in $C^{1,1}$.

Finally, in Subsection 4.4.3, we use the estimates that are independent with respect to ε and some compactness arguments to pass to the limit as ε goes to 0. Then, in Subsection 4.4.4 we let η go to $0+$ and obtain that (φ, θ, u, p) is a solution in the sense of Definition 4.3.1.

4.4.1 Regularized problem

We define in this subsection the (ε, η)-regularized problem. For simplicity of notation, we write (φ, θ, u, p) instead of $(\varphi^{\varepsilon,\eta}, \theta^{\varepsilon,\eta}, u^{\varepsilon,\eta}, p^{\varepsilon,\eta})$.

Let ε and η be fixed positive small numbers. We first introduce the so-called quasi-compressible approximation problem. For $\varepsilon > 0$, we will perturb the incompressibility constraint, $\operatorname{div} u = 0$, in the following way

$$\begin{cases} \varepsilon\Delta p = \operatorname{div} u & \text{in } Q_T\ , \\ \nabla p \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega \times (0,T)\ , \\ \displaystyle\int_\Omega p \, \mathrm{d}x = 0. \end{cases} \tag{4.51}$$

Then, as in [32] we modify the convective term by a suitable $\eta-$ approximation. We use here this suitable $\eta-$ approximation in the advection terms of the phase and heat equations. Let $u \in W^{1,q}_{\boldsymbol{n}}$ and $\eta > 0$ be fixed. We define u_η the η-approximation of u by

$$u_\eta := ((\rho_\eta u) * \omega_\eta)_{\mathrm{div}},$$

where the symbol $(\cdot)_{\mathrm{div}}$ comes from the Helmholtz decomposition (4.22), the symbol $u * \omega_\eta$ is the standard regularization of an integrable function u with kernel ω_η having the support in a ball of radii η and ρ_η is the function defined by

$$\rho_\eta := \begin{cases} 0 & \text{if dist } (x, \partial\Omega) \le 2\eta, \\ 1 & \text{elsewhere.} \end{cases} \tag{4.52}$$

We observe that this definition of u_η yields that

$$\int_\Omega ((u_\eta)\cdot\nabla)u \cdot u \, \mathrm{d}x = \int_\Omega u_\eta \cdot \nabla \frac{|u|^2}{2} \, \mathrm{d}x = -\frac{1}{2}\int_\Omega \operatorname{div} u_\eta |u|^2 \, \mathrm{d}x = 0.$$

Moreover, if $u^n \to u$ in $L^q(\Omega)$ as n goes to $+\infty$ and $\operatorname{div} u = 0$ then $u^n_\eta \to u$ in $L^q(\Omega)$ as n goes to $+\infty$ and η goes to 0_+.

Using (4.51) instead of $\operatorname{div} u = 0$, $T_{1/\eta}(\theta)$ instead of θ in the nonlinear term of the phase equation and u_η instead of u in the convective terms of the phase, the temperature, and the Navier-Stokes equations, we define the (ε, η)-regularized problem. For $\eta > 0$, $\varepsilon > 0$

$$\varphi_t + u_\eta \cdot \nabla\varphi - \xi\Delta\varphi = -\varphi(1-\varphi)(1-2\varphi + cT_{1/\eta}(\theta)) \quad \text{in } Q_T, \tag{4.53}$$

$$\theta_t - \operatorname{div}(\kappa(\varphi,\theta)\nabla\theta) + \operatorname{div}(u_\eta\theta) = T_{1/\varepsilon}(\nu(\varphi,\theta)D(u) : D(u) + g_\varepsilon + l\varphi_t \quad \text{in } Q_T, \tag{4.54}$$

$$u_t + ((u_\eta)\cdot\nabla)u - \operatorname{div}(\nu(\varphi,\theta)D(u)) + \nabla p = f \quad \text{in } Q_T, \tag{4.55}$$

$$-\varepsilon\Delta p = \operatorname{div} u \quad \text{in } Q_T \tag{4.56}$$

$$\nabla\varphi\cdot\boldsymbol{n} = 0, \quad \kappa(\varphi,\theta)\nabla\theta\cdot\boldsymbol{n} = 0, \quad u\cdot\boldsymbol{n} = 0, \ \alpha u_\tau + (\nu(\varphi,\theta)D(u)\boldsymbol{n})_\tau = 0 \quad \text{on } \Sigma, \tag{4.57}$$

$$\varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0^\varepsilon(x), \quad u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{4.58}$$

with $g_\varepsilon = T_{1/\varepsilon}(g)$ and $\theta_0 = T_{1/\varepsilon}(\theta_0)$. We explicitly remark that all the terms in (4.28)-(4.32) which have "low" regularities, are regularized in (4.53)-(4.58). It follows that we can expect to solve (4.53)-(4.58) in the variational framework.

Theorem 4.4.1. *Let $\Omega \in C^{1,1}$. Assume that assumptions (1)-(4) hold. Then, there exists for each $\varepsilon > 0$, $\eta > 0$, a solution (φ, θ, u, p) to (4.53)-(4.58), such that*

$$u \in C(0,T; L^2(\Omega)^3) \cap L^2(0,T; W^{1,2}_{\boldsymbol{n}}), \quad u_t \in L^{\frac{5}{3}}(0,T; W^{-1,2}_{\boldsymbol{n}}); \tag{4.59}$$

$$p \in L^2(0,T; W^{1,2}(\Omega)); \tag{4.60}$$

$$\varphi \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)), \quad \varphi_t \in L^2(0,T; W^{-1,2}(\Omega)); \tag{4.61}$$

$$0 \le \varphi \le 1; \tag{4.62}$$

$$\theta \in L^\infty(0,T; L^1(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)); \tag{4.63}$$

$$u(t=0) = u_0,\ \varphi(t=0) = \varphi_0,\ \theta(t=0) = \theta_0^\varepsilon; \tag{4.64}$$

and

$$\begin{aligned} &\int_0^T \langle u_t, v\rangle \,\mathrm{d}t + \int_{Q_T} \nu(\varphi,\theta)D(u)\cdot D(v)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega} \alpha uv\,\mathrm{d}\omega\,\mathrm{d}t \\ &\quad + \int_{Q_T} (u_\eta\cdot\nabla)u\cdot v\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} p\operatorname{div} v\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} fv\,\mathrm{d}x\,\mathrm{d}t \qquad \forall v \in L^2(0,T; W^{1,2}_{\boldsymbol{n}}); \end{aligned} \tag{4.65}$$

$$-\varepsilon\int_\Omega \nabla p(t)\cdot\nabla h \ \mathrm{d}x = \int_\Omega h\operatorname{div}(u(t)) \qquad \forall h \in W^{1,2}(\Omega) \text{ and a.e. } t \in (0,T); \tag{4.66}$$

$$\begin{aligned} &\int_0^T \langle \varphi_t, \psi\rangle \,\mathrm{d}t + \int_{Q_T} \xi\nabla\varphi\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u_\eta\varphi\nabla\psi\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{Q_T} -\varphi(1-\varphi)(1-2\varphi + cT_{1/\eta}(\theta))\psi\,\mathrm{d}x\,\mathrm{d}t \qquad \forall\psi \in L^2(0,T; W^{1,2}(\Omega)); \end{aligned} \tag{4.67}$$

$$\begin{aligned} &\int_0^T \langle \theta_t, \phi\rangle \,\mathrm{d}t - \int_{Q_T} u_\eta\theta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \kappa(\varphi,\theta)\nabla\theta\cdot\nabla\phi\,\mathrm{d}t\,\mathrm{d}x \\ &\qquad - \int_{Q_T} T_{1/\varepsilon}(\nu(\varphi,\theta)|D(u)|^2)\phi\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{Q_T} g_\varepsilon\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \langle l\varphi_t, \phi\rangle \,\mathrm{d}x\,\mathrm{d}t \qquad \forall\phi \in L^2(0,T; W^{1,2}(\Omega)). \end{aligned} \tag{4.68}$$

This theorem will be proven in Section 4.5 at the end of this Chapter.

4.4.2 A Priori estimates

In this subsection we establish estimates for $(\varphi, \theta, u, p) := (\varphi_\varepsilon^\eta, \theta_\varepsilon^\eta, u_\varepsilon^\eta, p_\varepsilon^\eta)$ that are uniform with respect to η and ε and also estimates uniform only with respect to ε. We note $Q_t = (0,t)\times\Omega$ and C a constant independent of ε and η and $C(\eta)$ a constant depending on η, which can vary from line to line.

Estimates on u and p

We follow the arguments developed in [32].
First, we start by multiplying (4.65) by u and (4.66) by p then we add the resulting equations. Since on the boundary, $p \cdot \boldsymbol{n} = 0$ and $\alpha u_\tau + \nu(\varphi,\theta)(D(u)\boldsymbol{n})_\tau = 0$, we obtain for all $t \in (0,T)$

$$\begin{aligned} &\frac{1}{2}\int_\Omega |u(t)|^2 \,\mathrm{d}x + \int_{Q_t} \nu(\varphi,\theta)|D(u)|^2 \,\mathrm{d}x\,\mathrm{d}s + \varepsilon\|\nabla p\|^2_{L^2(Q_T)} \\ &\qquad + \alpha\int_0^t \|u\|^2_{L^2(\partial\Omega)} \,\mathrm{d}s = \int_{Q_T} f\cdot u \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_\Omega |u_0|^2 \,\mathrm{d}x. \end{aligned} \tag{4.69}$$

From assumption (2), we know that $0 < \nu_0 \le \nu(\varphi,\theta) \le \nu_1$. Hence (4.69) together with the Korn's inequality (4.27) imply that, for all $t \in (0,T)$,

$$\begin{aligned} &\frac{1}{2}\int_\Omega |u(t)|^2 \,\mathrm{d}x + \int_{Q_t} \nu(\varphi,\theta)|D(u)|^2 \,\mathrm{d}x\,\mathrm{d}s + \varepsilon\|\nabla p\|^2_{L^2(Q_T)} \\ &\qquad + \alpha\int_0^t \|u\|^2_{W^{1,2}_n(\Omega)} \,\mathrm{d}s \le C\left(\int_{Q_T} f\cdot u \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_\Omega |u_0|^2 \,\mathrm{d}x\right). \end{aligned} \tag{4.70}$$

where $C > 0$, independent of ε, η. It follows, by using Young's inequality to (4.70) that

$$\sup_{t\in(0,T)} \|u(t)\|^2_{L^2(\Omega)} + \int_{Q_T} \nu(\varphi,\theta)|D(u)|^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon\|\nabla p\|^2_{L^2(Q_T)} + \int_0^T \|u\|^2_{W^{1,2}_n(\Omega)} \,\mathrm{d}t \le C. \tag{4.71}$$

Consequently, (4.71) imply that u is bounded in $L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;W^{1,2}_{\boldsymbol{n}})$. Due to this and from the interpolation inequalities, we deduce that

$$\int_0^T \|u\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\Omega)} \le C. \tag{4.72}$$

We now want to establish uniform estimates on the pressure p with respect to ε and then uniform estimates with respect to ε and η. For this purpose, as in [32] and [34] we consider $h^{\varepsilon,\eta} = h$ such that h solves the following Neumann problem, for some $\beta \in (1,2]$ (specified later),

$$\begin{cases} \Delta h = |p|^{\beta-2}p - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega |p|^{\beta-2}p \,\mathrm{d}x & \text{in } \Omega\ , \\ \nabla h \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega\ , \\ \displaystyle\int_\Omega h \,\mathrm{d}x = 0. \end{cases} \tag{4.73}$$

From the L^p-theory of the Neumann problem for the Laplace equation (see for example [72, 71]) we have

$$\|\nabla h\|^{\beta'}_{W^{1,\beta'}(\Omega)} \leq C\|p\|^{\beta}_{L^{\beta}(\Omega)}, \tag{4.74}$$

where $C > 0$ is independent of ε and η.
Choosing ∇h as a test function in (4.65), and comparing with (4.73), we obtain

$$\begin{aligned}\int_0^T \|p\|^{\beta}_{L^{\beta}(\Omega)}\,\mathrm{d}t &= \int_0^T < u_t, \nabla h > \,\mathrm{d}t - \int_{Q_T}(u_\eta\otimes u)\cdot\nabla^2 h\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}\nu(\varphi,\theta)D(u).D(\nabla h)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\alpha\int_0^T\int_{\partial\Omega} u\cdot\nabla h\,\mathrm{d}\omega\,\mathrm{d}t - \int_{Q_T} f\cdot\nabla h\,\mathrm{d}x\,\mathrm{d}t\\ &= I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned}$$

We observe first, that I_1 is negative.
Indeed, by taking a sequence of smooth function u^n such that $u^n_t \to u_t$ strongly in $L^2(0,T;W^{-1,2}_{\boldsymbol{n}})$ and $u^n \to u$ strongly in $L^2(0,T;W^{1,2}_{\boldsymbol{n}})$. Then, by taking a sequence p^n such that p^n is the solution of $\varepsilon\Delta p^n = \operatorname{div} u^n$, with $\nabla p^n\cdot\boldsymbol{n} = 0$ on the boundary and $\int_\Omega p^n\,\mathrm{d}x = 0$. We define h^n as a solution of (4.73) where p is replaced by p^n.
Now, by applying the Helmholtz decomposition (4.22) to u^n and since $\operatorname{div} u^n_{\mathrm{div}} = 0$ with $\nabla h^n\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, we obtain

$$I^n_1 = \int_0^T < u^n_{\mathrm{div},t} + \nabla g^{u^n}_t, \nabla h^n > \,\mathrm{d}t = \int_0^T < \nabla g^{u^n}_t, \nabla h^n > \,\mathrm{d}t = -\int_0^T\int_\Omega g^{u^n}_t\cdot\Delta h^n\,\mathrm{d}x\,\mathrm{d}t. \tag{4.75}$$

Comparing (4.56) with (4.22), the uniqueness of the solution of the Laplace equation and the fact that $p^n(0) = -\frac{1}{\varepsilon}\operatorname{div} u^n_0 = 0$ imply that

$$\begin{aligned}I^n_1 &= -\varepsilon\int_0^T < p^n_t, |p^n|^{\beta-2}p - \frac{1}{\Omega}\int_\Omega |p^n|^{\beta-2}p\,\mathrm{d}x > \,\mathrm{d}t\\ &= -\frac{\varepsilon}{\beta}\|p^n(T)\|^{\beta}_{L^{\beta}(\Omega)} + \frac{\varepsilon}{\beta}\|p^n(0)\|^{\beta}_{L^{\beta}(\Omega)}\\ &= -\frac{\varepsilon}{\beta}\|p^n(T)\|^{\beta}_{L^{\beta}(\Omega)} \leq 0.\end{aligned} \tag{4.76}$$

Therefore, by the density of the smooth functions, we have u^n_t converges strongly to u_t in $L^2(0,T;W^{-1,2}_{\boldsymbol{n}})$, p^n converges strongly to p in $L^2(Q_T)$ and then ∇h^n converges to ∇h in $L^2(Q_T)^N$. It follows that $I_1 \leq 0$.

For I_2, I_3, I_4 and I_5, since $\beta \leq 2$ and $0 < \nu_0 \leq \nu(\varphi,\theta) \leq \nu_1$, the Hölder's and Young's inequalities provide the following estimates

$$I_2 \leq \int_{Q_T}|u_\eta\otimes u|^\beta\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{8}\int_0^T \|p\|^{\beta}_{L^{\beta}(\Omega)}\,\mathrm{d}x \tag{4.77}$$

$$I_3 \leq C\int_{Q_T}|D(u)|^\beta\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{8}\int_0^T \|p\|^{\beta}_{L^{\beta}(\Omega)}\,\mathrm{d}t, \tag{4.78}$$

$$I_4 \leq \alpha\int_0^T \|\operatorname{tr} u\|_{L^2(\Omega)}\|\operatorname{tr}\nabla h\|_{L^2(\Omega)}\,\mathrm{d}t \tag{4.79}$$

$$\leq \int_0^T C\|\operatorname{tr} u\|^2_{L^2(\Omega)} + \|\nabla h\|^2_{W^{1,2}(\Omega)} \tag{4.80}$$

$$\leq C\int_0^T (1 + \|tr(u)\|^2_{L^2(\partial\Omega)})\,\mathrm{d}t + \frac{1}{8}\int_0^T \|p\|^{\beta}_{L^{\beta}(\Omega)}\,\mathrm{d}t, \tag{4.81}$$

$$I_5 \le C \int_{Q_T} |f|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{8}\int_0^T \|p\|^\beta_{L^\beta(\Omega)}\,\mathrm{d}t + C. \tag{4.82}$$

Choosing $\beta = 2$, (4.77) imply that

$$\begin{aligned} I_2 &\le \left(\int_{Q_T} |u_\eta|^5 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{5}} \left(\int_{Q_T} |u|^{\frac{10}{3}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{3}{5}} + \frac{1}{8}\int_0^T \|p\|^\beta_{L^\beta(\Omega)}\,\mathrm{d}x \\ &\le C(\eta)\left(\int_{Q_T} |u|^{\frac{10}{3}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{3}{5}} + \frac{1}{8}\int_0^T \|p\|^\beta_{L^\beta(\Omega)}\,\mathrm{d}x, \end{aligned} \tag{4.83}$$

Thus, from (4.71), (4.72) and from the previous estimates (4.76), (4.83), (4.78), (4.81), (4.82) we get

$$\int_0^T \|p\|^2_{L^2(\Omega)}\,\mathrm{d}t \le C(\eta). \tag{4.84}$$

It then follows from (4.84), (4.71) and (4.72) that

$$\|u_t\|_{L^2(0,T;W_n^{-1,2}(\Omega))} \le C(\eta). \tag{4.85}$$

Now to obtain the estimates on p independent of both ε and η, we proceed similarly as before and take $\beta = \frac{5}{3}$. Indeed, using (4.26), (4.72) and Young's inequality for I_2, we get

$$\begin{aligned} I_2 &\le \int_{Q_T} |u_\eta \otimes u|^\beta \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{8}\int_0^T \|p\|^\beta_{L^\beta(\Omega)}\,\mathrm{d}x \\ &\le \left(\int_{Q_T} |u_\eta|^{\frac{10}{3}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_{Q_T} |u|^{\frac{10}{3}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} + \frac{1}{8}\int_0^T \|p\|^\beta_{L^\beta(\Omega)}\,\mathrm{d}x \\ &\le (C(\Omega, \frac{10}{3}) + 1)\int_{Q_T} |u|^{\frac{10}{3}} \,\mathrm{d}x\,\mathrm{d}t) + \int_{Q_T} |u|^{\frac{10}{3}} \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{8}\int_0^T \|p\|^\beta_{L^\beta(\Omega)}\,\mathrm{d}x, \\ &\le C + \frac{1}{8}\int_0^T \|p\|^\beta_{L^\beta(\Omega)}\,\mathrm{d}x. \end{aligned}$$

We then obtain the following estimate

$$\int_0^T \|p\|^{\frac{5}{3}}_{L^{\frac{5}{3}}(\Omega)}\,\mathrm{d}t \le C. \tag{4.86}$$

From this estimate, (4.71) and (4.72) we have

$$\|u_t\|_{L^{\frac{5}{3}}(0,T;W_n^{-1,\frac{5}{3}}(\Omega))} \le C. \tag{4.87}$$

Estimates on φ and θ

We now turn to the estimates of φ and θ. We first multiply the phase equation (4.67) by φ as a test function. Due to the assumption (4) on φ_0 and since $0 \le \varphi \le 1$ and $\operatorname{div} u_\eta = 0$ on Ω and $|T_{1/\eta}(\theta)| \le |\theta|$, we obtain

$$\int_\Omega |\varphi(T)|^2 \,\mathrm{d}x + \int_{Q_T} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \le C + C\int_{Q_T} |\theta| \,\mathrm{d}x\,\mathrm{d}t. \tag{4.88}$$

Then, we choose $T_1(\theta)$ as test function in (4.68) and $lT_1(\theta)$ as a test function in (4.67). Summing the result equations and using the fact that $\mathrm{div}(u_\eta) = 0$ on Ω, we get for all $t \in (0, T)$

$$\begin{aligned} \xi l \int_{Q_t} \nabla\varphi \cdot \nabla T_1(\theta)\,\mathrm{d}x\,\mathrm{d}s + l \int_{Q_t} u_\eta \cdot \nabla\varphi T_1(\theta)\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \widetilde{T}_1(\theta)(t)\,\mathrm{d}x + \int_{Q_t} \kappa(\varphi,\theta)\nabla\theta \cdot \nabla T_1(\theta)\,\mathrm{d}x\,\mathrm{d}s \\ = \int_\Omega \widetilde{T}_1(\theta_0)\,\mathrm{d}x + \int_{Q_t} -\varphi(1-\varphi)(1-2\varphi + cT_{1/\eta}(\theta)) \cdot lT_1(\theta)\,\mathrm{d}x\,\mathrm{d}s \\ + \int_{Q_t} \nu(\varphi,\theta) D(u) : D(u) T_1(\theta)\,\mathrm{d}x\,\mathrm{d}s + \int_{Q_t} g_\eta T_1(\theta)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{4.89}$$

We observe that due to (4.71) and (2), $T_{1/\varepsilon}(\nu(\varphi,\theta)D(u) : D(u))$ is bounded uniformly in $L^1(Q_T)$. Moreover, $\widetilde{T}_1(\theta_0) \le |\theta_0|$ with $\theta_0 \in L^1(\Omega)$, $g \in L^1(Q_T)$ and $0 \le \varphi \le 1$. Thus,

$$\begin{aligned} \xi l \int_{Q_t} \nabla\varphi \cdot \nabla T_1(\theta)\,\mathrm{d}x\,\mathrm{d}s + l \int_{Q_t} u_\eta \cdot \nabla\varphi T_1(\theta)\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \widetilde{T}_1(\theta)(t)\,\mathrm{d}x \\ + \int_{Q_t} \kappa(\varphi,\theta)\nabla\theta \cdot \nabla T_1(\theta)\,\mathrm{d}x\,\mathrm{d}s \le C + lC \int_{Q_t} |\theta|\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{4.90}$$

Applying Hölder's and Young's inequalities on the first and the second term of the left-hand side of (4.90) and using (4.71) and (4.88) yield that

$$\begin{aligned} \xi l \int_{Q_t} \nabla\varphi \cdot \nabla T_1(\theta)\,\mathrm{d}x\,\mathrm{d}s &\le \xi l \Big(\frac{1}{2\delta} \int_{Q_t} |\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\delta}{2} \int_{Q_t} |\nabla T_1(\theta)|^2\,\mathrm{d}x\,\mathrm{d}s\Big) \\ &\le C + C\frac{\xi l}{2\delta} \int_{Q_t} |\theta|\,\mathrm{d}x\,\mathrm{d}s + \xi l \frac{\delta}{2} \int_{Q_t} |\nabla T_1(\theta)|^2\,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{4.91}$$

where δ is chosen later and

$$\begin{aligned} l \int_{Q_t} u_\eta \cdot \nabla\varphi T_1(\theta)\,\mathrm{d}x\,\mathrm{d}s &\le \frac{l}{2} \int_{Q_t} |u_\eta|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{l}{2} \int_{Q_T} |\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\le C + C \int_{Q_t} |\theta|\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{4.92}$$

Combining (4.90), (4.91) and (4.92) together with the fact that $\widetilde{T}_1(r) \ge |r| - \frac{1}{2}$ and $0 < \kappa_0 \le \kappa \le \kappa_1$, we obtain, for all $t \in (0, T)$

$$\int_\Omega |\theta(t)|\,\mathrm{d}x + (\kappa_0 - 2\xi l\delta) \int_{Q_t} |\nabla T_1(\theta)|^2\,\mathrm{d}x\,\mathrm{d}s \le C \int_{Q_t} |\theta|\,\mathrm{d}x\,\mathrm{d}s + C \tag{4.93}$$

where δ is chosen such that $\kappa_0 - 2\xi l\delta > 0$. Since the second term of the left-hand-side of (4.93) is positive, Gronwall's inequality permits us to conclude that there exists a constant C independent of ε and η such that for a.e. $t \in (0, T)$

$$\int_\Omega |\theta(t)|\,\mathrm{d}x \le C, \tag{4.94}$$

which implies that θ is bounded uniformly in $L^\infty(0, T; L^1(\Omega))$.
It follows from this and (4.88) that

$$\|\varphi\|_{L^\infty(0,T;L^2(\Omega))} + \|\varphi\|_{L^2(0,T;W^{1,2}(\Omega))} \le C. \tag{4.95}$$

We will next prove that

$$\|T_k(\theta)\|^2_{L^2(0,T;W^{1,2}(\Omega))} \leq kM + L, \, \forall k > 0; \tag{4.96}$$

where $M > 0$ and $L > 0$ two constants independent of ε and η. To this end, we argue as in the proof of (4.94) by taking $T_k(\theta)$ as a test function in (4.68) and $lT_k(\theta)$ as a test function in (4.67). Therefore, by summing the result equations we get, for all $t \in (0,T)$

$$\begin{aligned}\xi l \int_{Q_t} \nabla\varphi \cdot \nabla T_k(\theta) \,\mathrm{d}x\,\mathrm{d}s + l \int_{Q_t} u_\eta \cdot \nabla\varphi T_k(\theta) \,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \widetilde{T}_k(\theta)(t) \,\mathrm{d}x + \int_{Q_t} \kappa(\varphi,\theta)\nabla\theta \cdot \nabla T_k(\theta) \,\mathrm{d}x\,\mathrm{d}s \\ = \int_{Q_t} -\varphi(1-\varphi)(1-2\varphi + cT_{1/\eta}(\theta)) \cdot lT_k(\theta) \,\mathrm{d}x\,\mathrm{d}s \\ + \int_\Omega \widetilde{T}_k(\theta_0) \,\mathrm{d}x + \int_{Q_t} \nu(\varphi,\theta) D(u) : D(u) T_k(\theta) \,\mathrm{d}x\,\mathrm{d}s + \int_{Q_t} g_\varepsilon T_k(\theta) \,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{4.97}$$

Recalling that $T_{1/\varepsilon}(\nu(\varphi,\theta)D(u) : D(u))$ is bounded uniformly in $L^1(Q_T)$, $g \in L^1(Q_T)$, $\theta_0 \in L^1(\Omega)$, $0 \leq \varphi \leq 1$ and $0 \leq \widetilde{T}_k(r) \leq k|r|$, we obtain from (4.97) that, for all $t \in (0,T)$,

$$\begin{aligned}\xi l \int_{Q_t} \nabla\varphi \cdot \nabla T_k(\theta) \,\mathrm{d}x\,\mathrm{d}s + l \int_{Q_t} u_\eta \cdot \nabla\varphi T_k(\theta) \,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \widetilde{T}_k(\theta)(t) \,\mathrm{d}x \\ + \int_{Q_t} \kappa(\varphi,\theta)\nabla\theta \cdot \nabla T_k(\theta) \,\mathrm{d}x\,\mathrm{d}s \leq kM + L + kC \int_{Q_t} |\theta| \,\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{4.98}$$

where $M > 0$ and $L > 0$ two constants independent of ε and η.
Applying Young's and Hölder's inequalities on the first and the second term of the left-hand side of (4.98) and since θ is bounded in $L^\infty(0,T;L^1(\Omega))$, we get,

$$(\kappa_0 - 2\xi l\delta) \int_{Q_t} |\nabla T_k(\theta)|^2 \,\mathrm{d}x\,\mathrm{d}s \leq kM + L, \tag{4.99}$$

where δ is chosen such that $\kappa_0 - 2\xi l\delta > 0$. Again taking into account that θ is bounded in $L^\infty(0,T;L^1(\Omega))$, we deduce that, for all $k > 0$,

$$\int_{Q_T} |T_k(\theta)|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} |\nabla T_k(\theta)|^2 \,\mathrm{d}x\,\mathrm{d}t \leq kM + L, \tag{4.100}$$

where $M > 0$ and $L > 0$ independent of ε and η.
In view of this result and since θ is bounded in $L^\infty(0,T;L^1(\Omega))$, Lemma 4.2.1 implies that

$$\theta \text{ is bounded in } L^q(0,T;L^q(\Omega)), \, \forall 1 \leq q < 5/3, \tag{4.101}$$

$$\nabla\theta \text{ is bounded in } L^s(0,T;L^s(\Omega)), \, \forall 1 \leq s < 5/4. \tag{4.102}$$

Hence,

$$\theta \text{ is bounded in } L^p(0,T;W^{1,p}(\Omega)), \, \forall 1 \leq p < 5/4. \tag{4.103}$$

We finish this subsection by the estimates on φ_t and θ_t.

Recall that θ is bounded in $L^\infty(0,T;L^1(\Omega))$ and due to Lemma 4.2.1 we have that θ is bounded in $L^1(0,T;L^r(\Omega))$ for all $1 \leq r < \bar{r}$ where $\bar{r} = 2^*/2$. Thus, from the usual interpolation results, we get that

$$\theta \text{ is bounded in } L^m(0,T;L^n(\Omega)) \; \forall 1 < m < +\infty, \, \forall 1 < n < \bar{n}(m), \tag{4.104}$$

where $\bar{n} = 3m/(3m-2)$. In particular, we get that θ is bounded in $L^2(0,T;L^{\frac{6}{5}}(\Omega))$. Thus, by the Hölder and the Sobolev embedding inequalities, we deduce that the term $-\varphi(1-\varphi)(1-2\varphi+cT_{1/\eta}(\theta))$ is bounded in $L^2(0,T;W^{-1,2}(\Omega))$. In addition, due to (4.71) and since $0\le\varphi\le 1$, we get that $u_\eta\cdot\nabla\varphi$ is bounded in $L^2(0,T;W^{-1,2}(\Omega))$. Therefore,

$$\|\varphi_t\|_{L^2(0,T;W^{-1,2}(\Omega))} \le C. \tag{4.105}$$

As far as θ_t is concerned, we note that $L^1(\Omega)\hookrightarrow W^{-1,\alpha'}(\Omega)$, $\forall\alpha > N$, hence the term $T_{1/\varepsilon}(\nu(\varphi,\theta)D(u):D(u))$ and g_ε are bounded in $L^1(0,T;W^{-1,\alpha'}(\Omega))$ for all $\alpha > N$. On the other hand, from (4.103) we have that $\nabla\cdot(\kappa(\varphi,\theta)\nabla\theta)$ is bounded in $L^p(0,T;W^{-1,p'}(\Omega))$, $\forall p > 5$ and since the term $u_\eta\cdot\nabla T_{1/\eta}(\theta)$ is bounded in $L^1(0,T;W^{-1,q'}(\Omega))$ for all $q > 10$. It follows that

$$\|\theta_t\|_{L^1(0,T;W^{-1,\alpha'}(\Omega))} \le C, \tag{4.106}$$

where α is sufficiently large.

4.4.3 Passing to the limit as ε goes to 0

In this section, we want to pass to the limit as ε tends to zero in the regularized system (4.53)-(4.58). We write $(\varphi^\varepsilon,\theta^\varepsilon,u^\varepsilon,p^\varepsilon)$ rather than $(\varphi^{\varepsilon,\eta},\theta^{\varepsilon,\eta},u^{\varepsilon,\eta},p^{\varepsilon,\eta})$.

First, we observe from estimates (4.71), (4.72) (4.85), (4.84), (4.95), (4.105), (4.103), (4.101) and (4.106) and with the help of Aubin-Lions Lemma (see for instance [99]), that for a subsequence, still indexed by ε, we have

$$u^\varepsilon \rightharpoonup u \text{ weakly in } L^2(0,T;W^{1,2}_{\boldsymbol{n}}(\Omega)), \tag{4.107}$$
$$u^\varepsilon \to u \text{ strongly in } L^q(0,T;L^q(\Omega)^3) \text{ for } 1\le q<10/3 \text{ and a.e. in } Q_T, \tag{4.108}$$
$$u^\varepsilon_t \rightharpoonup u_t \text{ weakly in } L^2(0,T;W^{-1,2}_{\boldsymbol{n}}(\Omega)), \tag{4.109}$$
$$p^\varepsilon \rightharpoonup p \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{4.110}$$
$$\varphi^\varepsilon \rightharpoonup \varphi \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{4.111}$$
$$\varphi^\varepsilon \to \varphi \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \tag{4.112}$$
$$\varphi^\varepsilon_t \rightharpoonup \varphi_t \text{ weakly in } L^2(0,T;W^{-1,2}(\Omega)), \tag{4.113}$$
$$\theta^\varepsilon \rightharpoonup \theta \text{ weakly in } L^p(0,T;W^{1,p}(\Omega)) \text{ for } 1\le p<5/4, \tag{4.114}$$
$$\theta^\varepsilon \to \theta \text{ strongly in } L^s(Q_T) \text{ for } 1\le s<5/3 \text{ and a.e. in } Q_T. \tag{4.115}$$

Furthermore, from Corollary 4.2.4 we obtain

$$\operatorname{tr} u^\varepsilon \to \operatorname{tr} u \text{ strongly in } L^2(0,T;L^2(\partial\Omega)^3). \tag{4.116}$$

Consequently, using (4.107),(4.108), (4.109), (4.110), (4.132) and assumption ((2)) on ν, we can pass to the limit in (4.65) as ε goes to zero and get that, for all $v\in L^2(0,T;W^{1,2}_n)$

$$\begin{aligned}&\int_0^T < u_t, v > \,\mathrm{d}t + \int_{Q_T}\nu(\varphi,\theta)\nabla u\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}(u_\eta\cdot\nabla)u\cdot v\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega}\alpha uv\,\mathrm{d}w\,\mathrm{d}t\\ &= \int_{Q_T} p\operatorname{div} v\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} fv\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.117}$$

where (u,θ,φ,p) in this equation is $(u^\eta,\theta^eta,p^\eta)$.

Recall that $p^\varepsilon\cdot n=0$ on the boundary, from (4.71) we know that for $\psi\in L^2(0,T;W^{1,2}(\Omega))$ we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\Big|\int_0^T(\operatorname{div}u^\varepsilon,\psi)\mathrm{d}t\Big| &= \lim_{\varepsilon\to 0}\Big|\varepsilon\int_0^T\int_\Omega\nabla\psi\nabla p^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\Big|\\ &= \lim_{\varepsilon\to 0}\sqrt{\varepsilon}\left(\int_{Q_T}|\nabla\psi|^2\,\mathrm{d}x\mathrm{d}t\right)^{\frac12}\left(\int_{Q_T}\varepsilon|\nabla p^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12}.\end{aligned}$$

Thus,due to (4.71), we can pass to the limit as ε goes to 0 and we deduce that

$$\operatorname{div}u=0\ \text{ a.e. in } Q_T.$$

Now for passing to the limit in the phase equation, we note from (4.115), (4.112) and since $|T_{1/\eta}(\theta^\varepsilon)|\le 1/\eta$ that the Lebesgue convergence theorem yields that

$$\lim_{\varepsilon\to 0}\int_{Q_T}-\varphi^\varepsilon(1-\varphi^\varepsilon)(1-2\varphi^\varepsilon+cT_{1/\eta}(\theta^\varepsilon))\,\mathrm{d}x\,\mathrm{d}t=\int_{Q_T}-\varphi(1-\varphi)(1-2\varphi+cT_{1/\eta}(\theta))\,\mathrm{d}x\,\mathrm{d}t. \tag{4.118}$$

Hence, it follows from this, (4.111), (4.112) and (4.113) that we can pass to the limit as ε goes to zero in (4.67) and obtain that for all ψ in $L^2(0,T;W^{1,2}(\Omega))$, φ verifies

$$\begin{aligned}&\int_0^T<\varphi_t,\psi>\ \mathrm{d}t+\int_{Q_T}\xi\nabla\varphi\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}(u_\eta\cdot\nabla\varphi)\cdot\psi\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^T-\varphi(1-\varphi)(1-2\varphi+cT_{1/\eta}(\theta))\psi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.119}$$

where (u,θ,φ,p) in this equation is $(u^\eta,\theta^\eta,\varphi^\eta,p^\eta)$.

Finally, to pass to the limit in (4.54) we need to show that

$$D(u^\varepsilon)\to D(u)\ \text{strongly in } L^2(0,T;L^2(\Omega)^{3\times 3})\ \text{as } \varepsilon\to 0. \tag{4.120}$$

In view of assumption (2) on ν and from the convergences (4.107), (4.112) and (4.114), we know that $\nu(\varphi^\varepsilon,\theta^\varepsilon)^{1/2}D(u^\varepsilon)$ converges weakly in $L^2(0,T;L^2(\Omega)^{3\times3})$ to $\nu(\varphi,\theta)^{1/2}D(u)$ as $\varepsilon\to 0$. Hence,

$$\int_{Q_T}\nu(\varphi,\theta)|D(u)|^2\,\mathrm{d}x\,\mathrm{d}t\le\liminf_{\varepsilon\to 0}\int_{Q_T}\nu(\varphi^\varepsilon,\theta^\varepsilon)|D(u^\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.121}$$

Moreover, using equation (4.65) and then passing to the limit as ε goes to 0 we deduce that

$$\begin{aligned}\int_{Q_T}\nu(\varphi^\varepsilon,\theta^\varepsilon)|D(u^\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t &\le\int_{Q_T}\nu(\varphi^\varepsilon,\theta^\varepsilon)|D(u^\varepsilon)|^2+\varepsilon|\nabla p^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le-\frac{\|u^\varepsilon(T)\|^2_{L^2(\Omega)}}{2}+\int_{Q_T}f\cdot u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^T\alpha\|u^\varepsilon\|^2_{L^2(\partial\Omega)}\,\mathrm{d}t+\frac{\|u_0\|^2_{L^2(\Omega)}}{2}\\ &\le-\frac{\|u(T)\|^2_{L^2(\Omega)}}{2}+\int_{Q_T}f\cdot u\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\alpha\|u\|^2_{L^2(\partial\Omega)}\,\mathrm{d}t\\ &\quad+\frac{\|u_0\|^2_{L^2(\Omega)}}{2}\\ &=\int_{Q_T}\nu(\varphi,\theta)|D(u)|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Comparing this and (4.121) we obtain (4.120). Now, by using (4.120), (4.114), (4.115), assumption (2) on κ and ν and by taking ε to 0 we deduce that for all $\phi \in D(0,T;W^{1,q}(\Omega))$, for q sufficiently large

$$\begin{aligned} &-\int_{Q_T} \theta\phi_t \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} \theta u_\eta \nabla\phi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \kappa(\varphi,\theta)\nabla\theta\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_\Omega \theta_0\phi(0)\,\mathrm{d}x + \int_{Q_T} \nu(\varphi,\theta)|D(u)|^2\phi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\phi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \langle l\varphi_t,\phi\rangle \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.122}$$

4.4.4 Passing to the limit as η goes to 0

We want now to pass to the limit as η goes to 0+ and show the existence of a solution (φ,θ,u,p) in the sense of definition 4.3.1.

First, from the estimates (4.71), (4.72), (4.86), (4.87) on u, u_t and p and the estimates (4.101), (4.103), (4.95), (4.105), (4.106) on θ, θ_t, φ, φ_t (all derived in subsection 4.4.2 and all uniform with respect to η) and Aubin-Lions Lemma, we obtain that for a subsequence, still indexed by η,

$$u^\eta \rightharpoonup u \text{ weakly in } L^2(0,T;W^{1,2}_{\boldsymbol{n}}), \tag{4.123}$$

$$u^\eta \to u \text{ strongly in } L^q(0,T;L^q(\Omega)^3) \text{ for } 1\le q<10/3 \text{ and a.e. in } Q_T, \tag{4.124}$$

$$u^\eta_t \rightharpoonup u_t \text{ weakly in } L^2(0,T;W^{-1,2}_{\boldsymbol{n}}), \tag{4.125}$$

$$p^\eta \rightharpoonup p \text{ weakly in } L^{\frac53}(0,T;L^{\frac53}\Omega)), \tag{4.126}$$

$$\varphi^\eta \rightharpoonup \varphi \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{4.127}$$

$$\varphi^\eta \to \varphi \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \tag{4.128}$$

$$\varphi^\eta_t \rightharpoonup \varphi_t \text{ weakly in } L^2(0,T;(W^{-1,2}(\Omega)), \tag{4.129}$$

$$\theta^\eta \rightharpoonup \theta \text{ weakly in } L^p(0,T;(W^{1,p}(\Omega)) \text{ for } 1\le p<5/4, \tag{4.130}$$

$$\theta^\eta \to \theta \text{ strongly in } L^s(0,T;L^s(\Omega)) \text{ for } 1\le s<5/3. \tag{4.131}$$

Moreover using Corollary 4.2.4 we have

$$\operatorname{tr} u^\eta \to \operatorname{tr} u \text{ strongly in } L^2(0,T;L^2(\partial\Omega)^3). \tag{4.132}$$

We know from the previous subsection that div(u^η)= 0 and u^η verifies

$$\begin{aligned} &\int_0^T < u^\eta_t, v > \,\mathrm{d}t + \int_{Q_T} \nu(\varphi,\theta)\nabla u^\eta\cdot\nabla v \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} (u^\eta_\eta\cdot\nabla)u^\eta\cdot v \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega} \alpha u^\eta v \,\mathrm{d}w\,\mathrm{d}t \\ &= \int_{Q_T} p \operatorname{div} v \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} f v \,\mathrm{d}x\,\mathrm{d}t, \qquad \forall v\in L^2(0,T;W^{1,2}_{\boldsymbol{n}}). \end{aligned} \tag{4.133}$$

Then, using the convergences (4.123), (4.124), (4.125) and (4.126), we can pass to the limit in (4.133) as η goes to 0+ and obtain

$$\begin{aligned} &\int_0^T \langle u_t, v\rangle \,\mathrm{d}t + \int_{Q_T} \nu(\varphi,\theta)D(u)\cdot D(v) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega} \alpha uv \,\mathrm{d}\omega\,\mathrm{d}t \\ &\quad + \int_{Q_T} (u\cdot\nabla)u\cdot v \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} p\operatorname{div} v \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} fv \,\mathrm{d}x\,\mathrm{d}t \qquad \forall v\in L^\infty(0,T;W^{1,\infty}_{\boldsymbol{n}}). \end{aligned} \tag{4.134}$$

Now, we know that for all ψ in $L^2(0,T;W^{1,2}(\Omega))$, φ^η verifies

$$\begin{aligned}&\int_0^T < \varphi_t^\eta, \psi > \,\mathrm{d}t + \int_{Q_T} \xi \nabla\varphi^\eta \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} (u_\eta^\eta \cdot \nabla\varphi^\eta) \cdot \psi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T < -\varphi^\eta(1-\varphi^\eta)(1-2\varphi^\eta + cT_{1/\eta}(\theta^\eta)), \psi > \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.135}$$

Using the convergences (4.127), (4.128), (4.128) we can pass to the limit as η goes to 0+ in all the terms of the right-hand side of (4.135). Concerning the nonlinear term, we have that θ^η is bounded in $L^2(0,T;L^q(\Omega))$, for $1 \le q < 5/4$ (see Lemma 4.2.1), it follows from the point-wise convergence of φ^η and θ^η that

$$-\varphi^\eta(1-\varphi^\eta)(1-2\varphi^\eta + cT_{1/\eta}(\theta^\eta)) \rightharpoonup -\varphi(1-\varphi)(1-2\varphi + c\theta)$$

in $L^2(0,T;L^{q'}(\Omega))$. Recalling the Sobolev embedding of $W^{1,2}(\Omega)$ into $L^p(\Omega)$ for any $2 \le p \le 2^*$, thus

$$-\varphi^\eta(1-\varphi^\eta)(1-2\varphi^\eta + cT_{1/\eta}(\theta^\eta)) \rightharpoonup -\varphi(1-\varphi)(1-2\varphi + c\theta)$$

in $L^2(0,T;W^{-1,2}(\Omega))$. It follows that φ verifies

$$\begin{aligned}&\int_0^T \langle \varphi_t, \psi \rangle \,\mathrm{d}t + \int_{Q_T} \xi \nabla\varphi \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u\varphi\nabla\psi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{Q_T} -\varphi(1-\varphi)(1-2\varphi + c\theta)\psi \,\mathrm{d}x\,\mathrm{d}t \qquad \forall\psi \in L^2(0,T;W^{1,2}(\Omega)).\end{aligned} \tag{4.136}$$

It remains to prove the equation (4.49) and the inequality (4.50). For this purpose, we take $\phi := u^\eta\phi$, for all $\phi \in C_c^\infty([0,T) \times \bar{\Omega})$ as test function in (4.133) and ϕ a test function in the following equation satisfied by θ^η,

$$\begin{aligned}&-\int_{Q_T} \theta^\eta \phi_t \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} \theta^\eta u_\eta^\eta \nabla\phi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \kappa(\varphi^\eta, \theta^\eta)\nabla\theta^\eta \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_\Omega \theta_0\phi(0)\,\mathrm{d}x + \int_{Q_T} \nu(\varphi^\eta,\theta^\eta)|D(u^\eta)|^2\phi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\phi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \langle l\varphi_t^\eta, \phi\rangle \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.137}$$

Adding the two result equations we obtain

$$\begin{aligned}&-\int_{Q_T} (\frac{|u^\eta|^2}{2} + \theta^\eta)\phi_t \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} (u_\eta(\frac{|u^\eta|^2}{2} + \theta^\eta) + u^\eta p^\eta)\nabla\phi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \kappa(\varphi^\eta,\theta^\eta)\nabla\theta^\eta \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t \\ &+ \alpha\int_0^T \int_{\partial\Omega} |u^\eta|^2\phi \,\mathrm{d}\omega\,\mathrm{d}t + \int_{Q_T} \nu(\varphi^\eta,\theta^\eta)D(u^\eta)u^\eta\phi \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} fu^\eta\phi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{Q_T} g\phi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \langle l\varphi_t^\eta, \phi\rangle \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \theta_0\phi(0)\,\mathrm{d}x + \int_\Omega \frac{|u_0|^2}{2}\phi(0)\,\mathrm{d}x.\end{aligned} \tag{4.138}$$

Now passing to the limit as η goes to 0 in (4.138) and using the convergences (4.123), (4.124), (4.126), (4.130) and (4.131) we obtain (4.49).

Finally, for the inequality (4.50) choosing $\Psi \in D(0,T;C^\infty(\Omega))$ such that $\Psi \ge 0$ a.e. in Q_T, as a test function in (4.137)., we obtain

$$\begin{aligned}&-\int_{Q_T} \theta^\eta \Psi_t \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u_\eta^\eta \theta^\eta \cdot \nabla\Psi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \kappa(\varphi^\eta,\theta^\eta)\nabla\theta^\eta \cdot \nabla\Psi \,\mathrm{d}t\,\mathrm{d}x \\ &\ge \int_{Q_T} \nu(\varphi^\eta,\theta^\eta)|D(u^\eta)|^2\Psi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g\Psi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \langle l\varphi_t^\eta, \Psi\rangle \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \theta_0\phi_0 \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.139}$$

We know due to (4.123) and the assumption on ν that

$$\nu(\varphi^\eta,\theta^\eta)^{1/2}D(u^\eta) \rightharpoonup \nu(\varphi,\theta)^{1/2}D(u) \text{ weakly in } L^2(0,T;L^2(\Omega)^{3\times 3}).$$

Therefore, from the weak-lower-semicontinuity of the L^2 norm and since $\Psi \geq 0$ a.e. in Q_T we have that

$$\int_0^T\int_\Omega \nu(\varphi,\theta)|D(u)|^2\Psi \,\mathrm{d}x\,\mathrm{d}t \leq \liminf_{\eta\to 0+}\int_0^T\int_\Omega \nu(\varphi^\eta,\theta^\eta)|D(u^\eta)|^2\Psi\,\mathrm{d}x\,\mathrm{d}t.$$

It follows, using this and the above convergences that we can pass to the limit in (4.139) and obtain (4.50).

The proof of Theorem (4.3.4) is now complete. □

4.5 Proof of Theorem 4.4.1

We give here the proof of the existence of a solution to the approximate system (4.53)-(4.58) via Galerkin approximations. We will adapt the proof derived in [32] and explain the modifications.

Construction of the Galerkin approximations

As in [32], we start by defining a linear map $\mathcal{F} : W^{1,2}_{\boldsymbol{n}} \to W^{2,2}(\Omega)$, such that for any $\boldsymbol{u} \in W^{1,2}_{\boldsymbol{n}}$, $\mathcal{F}(\boldsymbol{u}) = p \in W^{2,2}(\Omega)$, where p is the solution of the following problem

$$\begin{cases} \varepsilon\Delta p = \operatorname{div}\boldsymbol{u} & \text{in } Q_T \ , \\ \nabla p\cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega\times(0,T) \ , \\ \displaystyle\int_\Omega p\,\mathrm{d}x = 0. & \end{cases} \tag{4.140}$$

Since $\Omega \in \mathcal{C}^{1,1}$, we have from the regularity theory for the Neumann problem that p exists and the mapping $\mathcal{F} : W^{1,2}_{\boldsymbol{n}} \to W^{2,2}(\Omega)$ is continuous.

Now, let $\{\mathbf{w}_j\}_{j=1}^\infty$ be an orthogonal basis ($\int_\Omega \mathbf{w}_i\cdot\mathbf{w}_j\,\mathrm{d}x = \delta_{ij}$) of $W^{1,2}_{\boldsymbol{n}}$ that is orthonormal in $L^2(\Omega)^3$, and let $\{w_j\}_{j=1}^\infty$ be a basis of $W^{1,2}(\Omega)$ which is again orthonormal in the space $L^2(\Omega)$. For every $j, \mathbf{w}_j$ belongs to $W^{1,6}_{\boldsymbol{n}}$. This basis can be formed by the eigenvectors of the problem $-\Delta\boldsymbol{u}+\boldsymbol{v} = \lambda\boldsymbol{u}$ in Ω, and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ and $((\nabla\boldsymbol{u})\boldsymbol{n})_\tau + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = 0$ on $\partial\Omega$ (see [87] for details)

Following [79, 96] (see also [34]), we construct the Galerkin approximations $\left\{\boldsymbol{u}^{N,M},\theta^{N,M},\varphi^{N,M},p^{N,M}\right\}_{N,M}^\infty$ such that

$$\begin{aligned} \boldsymbol{u}^{N,M} &:= \sum_{i=1}^N c_i^{N,M}(t)\mathbf{w}_i, \\ \theta^{N,M} &:= \sum_{i=1}^M d_i^{N,M}(t)w_i, \\ \varphi^{N,M} &:= \sum_{i=1}^M e_i^{N,M}(t)w_i, \\ p^{N,M} &:= \mathcal{F}\left(\boldsymbol{v}^{N,M}\right) \end{aligned}$$

where $\boldsymbol{c}^{N,M} := \left(c_1^{N,M}, \dots, c_N^{N,M}\right), \mathbf{d}^{N,M} := \left(d_1^{N,M}, \dots, d_M^{N,M}\right)$ and $\boldsymbol{e}^{N,M} := \left(e_1^{N,M}, \dots, e_N^{N,M}\right)$ solve the system of ordinary differential equations

$$\begin{aligned} &\int_\Omega \frac{\mathrm{d}}{\mathrm{dt}} \boldsymbol{u}^{N,M} \cdot \mathbf{w}_j \, \mathrm{d}x - \int_\Omega \boldsymbol{u}_\eta^{N,M} \otimes \boldsymbol{u}^{N,M} \cdot \nabla \mathbf{w}_j \, \mathrm{d}x + \int_\Omega \nu(\varphi^{N,M}, \theta^{N,M}) \mathbf{D}(\boldsymbol{u}^{N,M}) \cdot \nabla \mathbf{w}_j \, \mathrm{d}x \\ &\quad + \alpha \int_{\partial\Omega} \boldsymbol{u}^{N,M} \cdot \mathbf{w}_j \, \mathrm{d}\omega - \int_\Omega \mathcal{F}(\boldsymbol{u}^{N,M}) \cdot \operatorname{div} \mathbf{w}_j \, \mathrm{d}x = \int_\Omega f \mathbf{w}_j \, \mathrm{d}x, \quad j = 1, 2, \dots, N, \end{aligned} \tag{4.141}$$

$$\begin{aligned} &\int_\Omega \frac{\mathrm{d}}{\mathrm{dt}} \theta^{N,M} \cdot w_k \, \mathrm{d}x - \int_\Omega \boldsymbol{u}_\eta^{N,M} \theta^{N,M} \cdot \nabla w_k \mathrm{d}x + \int_\Omega \kappa(\varphi^{N,M}, \theta^{N,M}) \nabla \theta^{N,M} \cdot \nabla w_k \, \mathrm{d}x \\ &\quad = \int_\Omega T_{1/\varepsilon}(\nu(\varphi^{N,M}, \theta^{N,M}) |\mathbf{D}(\boldsymbol{u}^{N,M})|^2) \cdot w_k \, \mathrm{d}x + \int_\Omega T_{1/\varepsilon}(g) \cdot w_k \, \mathrm{d}x \\ &\quad\quad + l \left\langle \frac{\mathrm{dt}}{\mathrm{dt}} \varphi^{N,M}, w_k \right\rangle, \quad k = 1, 2, \dots, M, \end{aligned} \tag{4.142}$$

$$\begin{aligned} &\int_\Omega \frac{\mathrm{d}}{\mathrm{dt}} \varphi^{N,M} \cdot w_k \, \mathrm{d}x - \int_\Omega \boldsymbol{u}_\eta^{N,M} \varphi^{N,M} \cdot \nabla w_k \, \mathrm{d}x + \int_\Omega \xi \nabla \varphi^{N,M} \cdot \nabla w_k \, \mathrm{d}x \\ &\quad = \int_\Omega F(\varphi^{N,M}, T_{1/\eta}(\theta^{N,M})) \cdot w_k \, \mathrm{d}x, \quad k = 1, 2, \dots, M. \end{aligned} \tag{4.143}$$

For the initial conditions, we assume that

$$\boldsymbol{u}^{N,M}(\cdot, 0) = \boldsymbol{u}_0^{N,M}, \quad \theta^{N,M}(\cdot, 0) = \theta_0^{N,M}, \quad \varphi^{N,M}(\cdot, 0) = \varphi_0^{N,M},$$

where where $\boldsymbol{u}_0^{N,M} := \sum_{j=1}^N c_0^N \mathbf{w}_j$ are the projections of $\boldsymbol{u}_0$ onto linear hulls of $\{\mathbf{w}_j\}_{j=1}^N$, for $\theta_0^{N,M}$, as in [32], we take $\theta_0^{N,M} := (\omega_{1/N} * \theta_0)$ and $0 \le \varphi_0^{N,M} \le 1$. It is worth noting that the function $F(r, s)$ is defined as

$$F(r, s) = \begin{cases} -r(1-r)(1-2r+cs) & \text{if } 0 \le r \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

which allows one to derive estimates and the property $0 \le \varphi \le 1$.

By applying the projection onto the linear hull of $\{w_j\}_{j=1}^M$, we obtain that $\theta_0^{N,M}$ has the form $\theta_0^{N,M} := \sum_{j=1}^M d_0^M w_j$ and $\varphi_0^{N,M} := \sum_{j=1}^M e_0^M w_j$. Note that

$$\begin{aligned} &\theta_0^{M,N} \to_{M\to\infty} \theta_0^N \quad \text{strongly in } L^2(\Omega), \\ &\boldsymbol{u}_0^N \to_{N\to\infty} u_0 \quad \text{strongly in } L^2(\Omega)^3, \\ &\theta_0^N \to_{N\to\infty} \theta_0 \quad \text{strongly in } L^1(\Omega). \end{aligned} \tag{4.144}$$

The short time existence of solution to (4.141), (4.142) and (4.143) is shown via Classical Caratheodory Theory (see [32], [103]).

A priori estimates

In this subsection, we derive a priori estimates. We note that C is a positive constant independent of N and M which may change from line to line.

Uniform estimates on $\boldsymbol{u}^{N,M}$

As in [32], we first multiply the j-th equation in (4.141) by $c_j^{N,M}$, then we sum over $j = 1,\dots,M$ and integrate over $(0,T)$. Due to the definition of $\mathcal{F}$ and since the viscosity verifies (2) we obtain

$$\begin{aligned}
&\frac{1}{2}\sup_{t\in(0,T)}\|\boldsymbol{u}^{N,M}(t)\|^2_{L^2(\Omega)} + \int_0^T \nu_0\|D(\boldsymbol{u}^{N,M})\|^2_{L^2(\Omega)}\,\mathrm{d}t + \int_{Q_T}\boldsymbol{u}_\eta^{N,M}\otimes\boldsymbol{u}^{N,M}\cdot\nabla\boldsymbol{u}^{N,M}\,\mathrm{d}x\,\mathrm{d}t\\
&+\alpha\int_0^T\|\boldsymbol{u}^{N,M}\|^2_{L^2(\partial\Omega)}\,\mathrm{d}t + \int_0^T\varepsilon\|\nabla p^{N,M}\|^2_{L^2(\Omega)}\mathrm{d}t\\
&\le\int_0^T\|f\|_{L^2(\Omega)}\|\boldsymbol{u}^{N,M}\|_{L^2(\Omega)}\,\mathrm{d}t + \frac{1}{2}\|\boldsymbol{u}_0^{N,M}\|^2_{L^2(\Omega)}.
\end{aligned}\tag{4.145}$$

Now since $\operatorname{div}\boldsymbol{u}_\eta^{N,M} = 0$ on Ω and $\boldsymbol{u}^{N,M}\cdot\boldsymbol{n} = 0$ on the boundary, we have

$$\begin{aligned}
\int_{Q_T}\boldsymbol{u}_\eta^{N,M}\otimes\boldsymbol{u}^{N,M}\cdot\nabla\boldsymbol{u}^{N,M}\,\mathrm{d}x\,\mathrm{d}t &= \int_{Q_T}\boldsymbol{u}_\eta^{N,M}\cdot\nabla\frac{|\boldsymbol{u}^{N,M}|}{2}\,\mathrm{d}x\,\mathrm{d}t\\
&= -\int_{Q_T}\operatorname{div}\boldsymbol{u}_\eta^{N,M}\cdot\frac{|\boldsymbol{u}^{N,M}|^2}{2}\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned}\tag{4.146}$$

It follows from the equation above that

$$\begin{aligned}
&\frac{1}{2}\sup_{t\in(0,T)}\left\|\boldsymbol{u}^{N,M}(t)\right\|^2_{L^2(\Omega)} + \int_0^T\|D(\boldsymbol{u}^{N,M})\|^2_{L^2(\Omega)}\,\mathrm{d}t + \alpha\int_0^T\|\boldsymbol{u}^{N,M}\|^2_{L^2(\partial\Omega)}\,\mathrm{d}t\\
&\quad\le C\int_0^T\|f\|_{L^2(\Omega)}\|\boldsymbol{u}^{N,M}\|_{L^2(\Omega)}\,\mathrm{d}t + \|\boldsymbol{u}_0^{N,M}\|^2_{L^2(\Omega)}.
\end{aligned}\tag{4.147}$$

Applying Korn and Young's inequalities on (4.147) we get

$$\sup_{t\in(0,T)}\|\boldsymbol{v}^{N,M}(t)\|_2^2 + \int_0^T\|\boldsymbol{u}^{N,M}\|^2_{W^{1,2}_{\boldsymbol{n}}}\,\mathrm{d}t + \varepsilon\int_0^T\|\nabla p^{N,M}\|^2_{L^2(\Omega)}\,\mathrm{d}t \le C.\tag{4.148}$$

Moreover, using the interpolation inequality we deduce that

$$\int_0^T\|\boldsymbol{u}^{N,M}\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\Omega)}\,\mathrm{d}t\le C.\tag{4.149}$$

Finally, by multiplying (4.141) by $\frac{\mathrm{d}}{\mathrm{d}t}c_j^{N,M}$, summing over $j = 1,\dots,N$ and integrating over $(0,T)$ we get using (4.148) that

$$\int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{c}^{N,M}\right)^2\,\mathrm{d}t\le C(N).\tag{4.150}$$

Uniform estimates on $\theta^{N,M}$ and $\varphi^{N,M}$

We start with the estimates on $\varphi^{N,M}$. By multiplying the k-th equation (4.143) by $e_k^{N,M}$, summing over $k = 1,\dots,M$, then integrating over $t\in(0,T)$, and using the fact that $\operatorname{div}u_\eta^{N,M} = 0$ in Q_T we obtain that

$$\begin{aligned}
&\frac{1}{2}\|\varphi^{N,M}(t)\|^2_{L^2(\Omega)} + \xi\int_0^t\|\nabla\varphi^{N,M}\|^2_{L^2(\Omega)}\,\mathrm{d}s\\
&\quad\le\int_{Q_t}|F(\varphi^{N,M},T_{1/\eta}(\theta^{N,M}))\varphi^{N,M}|\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\|\varphi_0^{N,M}\|_{L^2(\Omega)}\\
&\le C\Big(1+\int_{Q_t}|\varphi^{N,M}|^2\,\mathrm{d}x\,\mathrm{d}s\Big).
\end{aligned}\tag{4.151}$$

Thus, by applying Gronwall's Lemma to (4.151) we get that

$$\|\varphi^{N,M}\|_{L^\infty(0,T;L^2(\Omega))} + \|\varphi^{N,M}\|_{L^2(0,T;W^{1,2}(\Omega))} \leq C. \tag{4.152}$$

Moreover, using the fact that $\left\langle \varphi_t^{N,M}, \psi(t) \right\rangle = \left\langle \varphi_t^{N,M}, \boldsymbol{P}^M \psi(t) \right\rangle$, where $\boldsymbol{P}^M$ is the projection of $W^{1,2}$ onto the linear hull of $\{w_i\}_{i=1}^M$, we obtain due to the above estimates that

$$\|\varphi_t^{N,M}\|_{L^2(0,T;W^{-1,2}(\Omega))} \leq C. \tag{4.153}$$

As far as θ^N is concerned, we multiply the k-th equation (4.142) by $d_k^{N,M}$, then we sum over $k = 1, \ldots, M$ and integrate over $t \in (0,T)$. Since $\operatorname{div} \boldsymbol{u}_\eta^{N,M} = 0$ in Q_T, we get

$$\begin{aligned}\frac{1}{2}\|\theta^{N,M}(t)\|^2_{L^2(\Omega)} + \kappa_0 \int_0^t \|\nabla\theta^{N,M}\|^2_{L^2(\Omega)} \,\mathrm{d}s \leq &\int_0^t \int_\Omega |T_{1/\varepsilon}(\nu(\varphi^{N,M}, \theta^{N,M})|\mathbf{D}(u^{N,M})|^2)||\theta^{N,M}| \,\mathrm{d}x \,\mathrm{d}s \\ &+ \frac{1}{2}\|\theta_0^{N,M}\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega |T_{1/\varepsilon}(g)\theta^{N,M}| \,\mathrm{d}x \,\mathrm{d}s \\ &+ \int_0^t | \left\langle l\varphi_t^{N,M}, \boldsymbol{P}^M \theta^{N,M} \right\rangle | \,\mathrm{d}s.\end{aligned} \tag{4.154}$$

For a fixed $\varepsilon > 0$, we know that $T_{1/\varepsilon}(\nu(\varphi^{N,M}, \theta^{N,M})|\mathbf{D}(u^{N,M})|^2) \leq 1/\varepsilon$ and since g is in $L^1(Q_T)$ and $\varphi_t^{N,M}$ is bounded in $L^2(0,T;W^{-1,2}(\Omega))$ we get by applying Hölder's and Young's inequalities on (4.154), that

$$\|\theta^{N,M}(t)\|^2_{L^2(\Omega)} + C_1 \int_0^t \|\nabla\theta^{N,M}\|^2_{L^2(\Omega)} \,\mathrm{d}s \leq C(\|\theta_0^N\|^2_{L^2(\Omega)}, \varepsilon, g, Q_t, \eta, l)(1 + \int_{Q_t} |\theta^{N,M}|^2 \,\mathrm{d}x \,\mathrm{d}s). \tag{4.155}$$

It follows from the Gronwall's lemma that

$$\|\theta^{N,M}\|_{L^\infty(0,T;L^2(\Omega))} + \|\theta^{N,M}\|_{L^2(0,T;W^{1,2}(\Omega))} \leq C. \tag{4.156}$$

In addition, using the fact that $\left\langle \theta_t^{N,M}, \psi(t) \right\rangle = \left\langle \theta_t^{N,M}, \boldsymbol{P}^M \psi(t) \right\rangle$, where $\boldsymbol{P}^M$ is the projection of $W^{1,2}$ onto the linear hull of $\{w_i\}_{i=1}^M$, we obtain due to the above estimates that

$$\|\theta_t^{N,M}\|_{L^2(0,T;W^{-1,2}(\Omega))} \leq C. \tag{4.157}$$

Limit as $M \to \infty$

Due to the estimates (4.148)-(4.157) developed in the previous subsection, there exists a subsequence $\{\boldsymbol{c}^{N,M}, \theta^{N,M}, \varphi^{N,M}\}_{M=1}^\infty$ (still denoted as before) such that for $M \to \infty$ we have

$$\boldsymbol{c}_{,t}^{N,M} \rightharpoonup \boldsymbol{c}_{,t}^N \text{ weakly in } L^2(0,T), \tag{4.158}$$

$$\boldsymbol{c}^{N,M} \rightharpoonup \boldsymbol{c}^N \text{ weakly-}* \text{ in } L^\infty(0,T), \tag{4.159}$$

$$\theta^{N,M} \rightharpoonup \theta^N \text{ weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{4.160}$$

$$\theta^{N,M} \rightharpoonup \theta^N \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{4.161}$$

$$\theta_t^{N,M} \rightharpoonup \theta_t^N \text{ weakly in } L^2(0,T;W^{-1,2}(\Omega)) \tag{4.162}$$

$$\varphi^{N,M} \rightharpoonup \varphi^N \text{ weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{4.163}$$

$$\varphi^{N,M} \rightharpoonup \varphi^N \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{4.164}$$

$$\varphi_t^{N,M} \rightharpoonup \varphi_t^N \text{ weakly in } L^2(0,T;W^{-1,2}(\Omega)). \tag{4.165}$$

In addition, due to the embeddings $W^{1,2}(0,T) \hookrightarrow \mathcal{C}^{\frac{1}{2}}(0,T) \hookrightarrow\hookrightarrow \mathcal{C}(0,T)$ and from the estimates (4.148) and (4.150) we have that

$$\boldsymbol{c}^{N,M} \to \boldsymbol{c}^N \quad \text{strongly in } \mathcal{C}(0,T). \tag{4.166}$$

It follows from (4.166) and the choice of the basis that

$$\boldsymbol{u}^{N,M} \to \boldsymbol{u}^N \quad \text{strongly in } L^4(0,T;W^{1,4}_{\boldsymbol{n}}), \tag{4.167}$$

. Since $\mathcal{F}$ is continuous, (4.167) imply that

$$p^{N,M} \to p^N \quad \text{strongly in } L^2(0,T;W^{1,2}(\Omega)). \tag{4.168}$$

Now using these convergences, we can pass to the limit as M goes to ∞, in (4.141) and obtain that $(\boldsymbol{u}^N, p^N)$ satisfy for almost all $t \in (0,T)$

$$\begin{aligned}\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}^N \cdot \mathbf{w}_j \,\mathrm{d}x - \int_\Omega \boldsymbol{u}^N_\eta \otimes \boldsymbol{u}^N \cdot \nabla \mathbf{w}_j \,\mathrm{d}x + \int_\Omega \nu(\varphi^N, \theta^N)\mathbf{D}(\boldsymbol{u}^N)\cdot \nabla \mathbf{w}_j \,\mathrm{d}x \\ + \alpha \int_{\partial\Omega} \boldsymbol{u}^N \cdot \mathbf{w}_j \,\mathrm{d}\omega - \int_\Omega p^N \cdot \operatorname{div} \mathbf{w}_j \,\mathrm{d}x = \int_\Omega f \mathbf{w}_j \,\mathrm{d}x, \quad j = 1,2,\dots,N,\end{aligned} \tag{4.169}$$

and

$$-\varepsilon(\nabla p^N(t), \nabla \Psi) = (\Psi, \operatorname{div} \boldsymbol{u}^N(t)) \qquad \forall \Psi \in W^{1,2}(\Omega). \tag{4.170}$$

Furthermore, the convergences we have here also permit us to pass to the limit as M goes to ∞ in (4.142) and (4.143) and to obtain that for a.a $t \in (0,T)$ θ^N satisfies

$$\begin{aligned}&\left\langle \theta^N_t, \Phi \right\rangle \mathrm{d}x - \int_\Omega \boldsymbol{u}^N_\eta \theta^N \nabla \Phi \mathrm{d}x + \int_\Omega \kappa(\varphi^N, \theta^N)\nabla\theta^N \cdot \nabla \Phi \,\mathrm{d}x \\ &= \int_\Omega T_{1/\varepsilon}(\nu(\varphi^N,\theta^N)|\mathbf{D}(\boldsymbol{u}^N)|^2)\Phi \,\mathrm{d}x + \int_\Omega T_{1/\varepsilon}(g)\Phi \,\mathrm{d}x + l \left\langle \varphi^N_t, \Phi \right\rangle, \qquad \forall \Phi \in W^{1,2}(\Omega),\end{aligned} \tag{4.171}$$

and for a.a. $t \in (0,T)$, φ^N satisfies

$$\begin{aligned}\left\langle \varphi^N_t, \psi \right\rangle - \int_\Omega \boldsymbol{u}^N_\eta \varphi^N \nabla \psi \,\mathrm{d}x + \int_\Omega \xi \nabla \varphi^N \cdot \nabla \psi \,\mathrm{d}x \\ = \int_\Omega F(\varphi^N, T_{1/\eta}(\theta^N))\psi \,\mathrm{d}x, \qquad \forall \psi \in W^{1,2}(\Omega).\end{aligned} \tag{4.172}$$

Finally for the initial conditions, due to (4.166) and since $\boldsymbol{c}^{N,M}(0) = \boldsymbol{c}^N_0$ for all M, it is clear that

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}^N_0 \tag{4.173}$$

Moreover, we have that θ^N and φ^N in $C(0,T;L^2(\Omega))$. Thus, we can define

$$\theta^N(0,\cdot) = \theta^N_0 \quad \text{and } \varphi^N(0,\cdot) = \varphi^N_0. \tag{4.174}$$

Estimates independent of N

In this subsection, we can establish on $(\varphi^N, \theta^N, \boldsymbol{u}^N, p^N)$ the same estimates as in subsection 4.5. Thus, by using the same arguments, we obtain that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^N(t)\|_2^2 + \int_0^T \|\boldsymbol{u}^N\|^2_{W^{1,2}_{\boldsymbol{n}}} \,\mathrm{d}t + \varepsilon \int_0^T \|\nabla p^N\|^2_{L^2(\Omega)} \,\mathrm{d}t \le C. \tag{4.175}$$

It follows from (4.175) that

$$\|\boldsymbol{u}_t^N\|_{L^2(0,T;W^{1,2}_{\boldsymbol{n}}(\Omega))} \leq C(\varepsilon,\eta). \tag{4.176}$$

Furthermore, concerning θ^N and φ^N we have

$$\|\theta^N\|_{L^\infty(0,T;L^2(\Omega))} + \|\theta^N\|_{L^2(0,T;W^{1,2}(\Omega))} + \|\theta_t^N\|_{L^2(0,T;W^{-1,2}(\Omega))} \leq C, \tag{4.177}$$

and

$$\|\varphi^N\|_{L^\infty(0,T;L^2(\Omega))} + \|\varphi^N\|_{L^2(0,T;W^{1,2}(\Omega))} + \|\varphi_t^N\|_{L^2(0,T;W^{-1,2}(\Omega))} \leq C. \tag{4.178}$$

Limit as $N \to \infty$

Using the estimates developed in the previous subsection with the help of a generalized version of the Aubin-Lions lemma (see [99]) we obtain the existence of $(\varphi, \boldsymbol{u}, \theta, p)$ such that

$$\boldsymbol{u}_t^N \rightharpoonup \boldsymbol{u}_t \text{ weakly in } L^2(0,T;W^{-1,2}_{\boldsymbol{n}}), \tag{4.179}$$

$$\boldsymbol{u}^N \to \boldsymbol{u} \text{ weakly in } L^2(0,T;W^{1,2}_{\boldsymbol{n}}), \tag{4.180}$$

$$\boldsymbol{u}^N \to \boldsymbol{u} \text{ strongly in } L^r(0,T;L^r(\Omega)^3)\ \forall r \in [1,10/3), \tag{4.181}$$

$$p^N \rightharpoonup p \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{4.182}$$

$$\theta^N \rightharpoonup \theta \text{ weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{4.183}$$

$$\theta^N \rightharpoonup \theta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{4.184}$$

$$\theta_t^N \rightharpoonup \theta_t \text{ weakly in } L^2(0,T;W^{-1,2}(\Omega)), \tag{4.185}$$

$$\varphi^N \rightharpoonup \varphi \text{ weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{4.186}$$

$$\varphi^N \rightharpoonup \varphi \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{4.187}$$

$$\varphi_t^N \rightharpoonup \varphi_t \text{ weakly in } L^2(0,T;W^{-1,2}(\Omega)). \tag{4.188}$$

Consequently, from Lemma 4.2.3 we have that

$$\operatorname{tr} \boldsymbol{u}^N \to \operatorname{tr} \boldsymbol{u} \quad \text{strongly in } L^2(0,T;L^2(\partial\Omega)^3). \tag{4.189}$$

Moreover, due to the assumption on ν with the a.e. convergences of θ^N, φ^N and the weak convergence of u^N in $L^2(0,T;W^{1,2}_{\boldsymbol{n}})$, we get that

$$\nu(\varphi^N,\theta^N)^{\frac{1}{2}}\mathbf{D}(u^N) \rightharpoonup \nu(\varphi,\theta)^{\frac{1}{2}}\mathbf{D}(u) \text{ weakly in } L^2(0,T;L^2(\Omega)^{3\times 3}). \tag{4.190}$$

Now, by using these convergences we can pass to the limit in (4.169) and get

$$\begin{aligned}
&\int_0^T \langle \boldsymbol{u}_t, v\rangle \,\mathrm{d}t + \int_{Q_T} \nu(\varphi,\theta)\mathbf{D}(\mathbf{u})\cdot\mathbf{D}(v)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega} \alpha\boldsymbol{u}v\,\mathrm{d}\omega\,\mathrm{d}t \\
&\quad + \int_{Q_T} (\boldsymbol{u}_\eta\cdot\nabla)\boldsymbol{u}\cdot v\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} p\operatorname{div} v\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} fv\,\mathrm{d}x\,\mathrm{d}t \qquad \forall v \in L^2(0,T;W^{1,2}_{\boldsymbol{n}});
\end{aligned} \tag{4.191}$$

and also obtain

$$-\int_\Omega \varepsilon\nabla p(t)\cdot\nabla\Psi\,\mathrm{d}x = \int_\Omega \Psi\operatorname{div}(\boldsymbol{u}(t))\,\mathrm{d}x, \qquad \forall\Psi \in W^{1,2}(\Omega) \text{ and a.e. } t \in (0,T). \tag{4.192}$$

For the initial condition $\boldsymbol{u_0}$, we know that $\boldsymbol{u} \in C(0,T;L^2_{weak}(\Omega)^3)$ consequently, we have that $\boldsymbol{u}(0) = \boldsymbol{u}_0$. In addition, taking $v = \boldsymbol{u}$ in (4.191) we obtain for all $t \in [0,T]$

$$\begin{aligned} \frac{1}{2}\|\boldsymbol{u}(t)\|_2^2 + \int_0^t \int_\Omega \nu(\varphi,\theta)|\mathbf{D}(\boldsymbol{u})|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \varepsilon\|\nabla p\|^2_{L^2(\Omega)}\,\mathrm{d}s + \alpha\int_0^t \|\boldsymbol{u}\|^2_{L^2(\partial\Omega)}\,\mathrm{d}w \\ = \int_0^t \int_\Omega f\boldsymbol{u}\mathrm{d}s + \frac{1}{2}\|\boldsymbol{u}_0\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.193}$$

Furthermore, observe that $T_{1/\eta}(\theta^N)$ converges weakly-$*$ in $L^\infty(Q_T)$ to $T_{1/\eta}(\theta)$. Thus, the above convergences are sufficient to pass to the limit in (4.172) as N goes to ∞ and get that

$$\begin{aligned} &\int_0^T \langle \varphi_t, \psi\rangle \,\mathrm{d}t + \int_{Q_T} \xi\nabla\varphi\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} u_\eta\varphi\nabla\psi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{Q_T} F(\varphi, T_{1/\eta}(\theta))\psi\,\mathrm{d}x\,\mathrm{d}t \qquad \forall\psi\in L^2(0,T;W^{1,2}(\Omega)). \end{aligned} \tag{4.194}$$

Finally, to pass to the limit in (4.171), we need to show that

$$\nu(\varphi^N,\theta^N)^{\frac{1}{2}}\mathbf{D}(u^N) \to \nu(\varphi,\theta)^{\frac{1}{2}}\mathbf{D}(u) \text{ strongly in } L^2(0,T;L^2(\Omega)^{3\times 3}). \tag{4.195}$$

Passing to the limit in all the other terms in (4.171) is standard. To prove (4.195) we proceed as follow, using the lower semicontinuity of L^2 norms for the quantities ∇p, $\nu(\varphi,\theta)^{1/2}\boldsymbol{D}(\boldsymbol{u})$, we obtain by comparing to the equation (4.193), that for all $t\in(0,T)$

$$\begin{aligned} &\int_0^t\int_\Omega u(\theta)|\mathbf{D}(\boldsymbol{u})|^2\,\mathrm{d}x\,\mathrm{d}s \le \liminf_{N\to\infty}\int_0^t\int_\Omega \nu(\varphi^N,\theta^N)|\mathbf{D}(\boldsymbol{u}^N)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &= \liminf_{N\to\infty}\Big(\int_0^t -\varepsilon\|\nabla p^N\|^2_{L^2(\Omega)} - \alpha\|\boldsymbol{u}^N\|^2_{L^2(\partial\Omega)} + \int_0^t\int_\Omega f\boldsymbol{u}^N\,\mathrm{d}x\,\mathrm{d}s\frac{1}{2}\|\boldsymbol{u}_0^N\|^2_{L^2(\Omega)} - \frac{1}{2}\|\boldsymbol{u}^N(t)\|^2_{L^2(\Omega)}\Big) \\ &\quad \le \int_0^t -\varepsilon\|\nabla p\|^2_{L^2(\Omega)} - \alpha\|\boldsymbol{u}\|^2_{L^2(\partial\Omega)} + \int_0^t\int_\Omega f\boldsymbol{u}\mathrm{d}s\mathrm{d}t + \frac{1}{2}\|\boldsymbol{u}_0\|^2_{L^2(\Omega)} - \frac{1}{2}\|\boldsymbol{u}(t)\|^2_{L^2(\Omega)} \\ &= \int_0^t\int_\Omega \nu(\varphi,\theta)|\mathbf{D}(\boldsymbol{u})|^2\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{4.196}$$

Hence, we can pass to the limit in (4.171) and get

$$\begin{aligned} &\int_0^T\langle\theta_t,\phi\rangle\,\mathrm{d}t - \int_{Q_T} u_\eta\theta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}\kappa(\varphi,\theta)\nabla\theta\cdot\nabla\phi\,\mathrm{d}t\,\mathrm{d}x \\ &\qquad - \int_{Q_T} T_{1/\varepsilon}(\nu(\varphi,\theta)|D(u)|^2)\phi\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{Q_T} g_\varepsilon\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}\langle l\varphi_t,\phi\rangle\,\mathrm{d}x\,\mathrm{d}t \qquad \forall\phi\in L^2(0,T;W^{1,2}(\Omega)). \end{aligned} \tag{4.197}$$

The proof of Theorem 4.4.1 is now complete. □

Perspectives

The very first continuation of the present work concerns the numerical analysis of the system and starting with the two-dimensional case is a reasonable goal. Indeed in dimension 2 since it seems possible to have V.F schemes for parabolic equations with L^1 data and Neumann boundary conditions, writing a scheme for the system and proving the convergence is an interesting question. For the Navier-Stokes we can use nonconforming finite elements as in [70] and for the phase field equation a V.F. scheme. It is clear that if we assume smallness conditions on the data (in particular l should be small enough) we can avoid the use of the coupling in deriving the estimates of the temperature in $L^\infty(0,T;L^1(\Omega))$. However, without any smallness condition, one of the main difficulties is to recover some properties of the (continuous) solution on the discrete scheme and in particular the estimates that allow one to have compactness results.

As mentioned in Chapter 3, the right-hand side for the phase field equation is $F(\varphi,\theta) = -\varphi(1-\varphi)(1-2\varphi+c\theta)$, as in [31, 58]. In the literature a very common choice is $F_1(\varphi,\theta) = -\varphi(1-\varphi)(1-2\varphi)+c\theta$ (see e.g. [36, 63]). From a mathematical point of view the case $F(\varphi,\theta)$ has some advantages: it allows to conserve the property $0 \leq \varphi \leq 1$ which is satisfactory from a physical point of view and allows one to avoid smallness condition on the term $l\varphi_t$ in the conservation energy equation. With the right-hand side $F_1(\varphi,\theta)$ due to the strong coupling and the nonlinearity it seems that we are losing the structure condition, and then having estimates on the phase and the temperature is more difficult. Using the L^p-maximal regularity, at the cost of assuming that Ω is more regular, and in dimension $N=2$, is a possible direction and, in parallel, an in-depth study of modeling.

In Chapter 3 and 4 we have assumed that the density ρ is constant. Introducing a variable density should be accessible in dimension 2 and is a challenge in dimension 3. It introduces a new equation

$$\rho_t + \operatorname{div}(\rho u) = 0$$

and, without the phase equation, it modifies the Navier-Stokes equation and conservation energy equation into

$$\begin{gathered}\rho\theta_t - \operatorname{div}\left(\kappa(\rho,\theta)\nabla\theta\right) + \rho u{\cdot}\nabla\theta = \nu(\rho,\theta)D(u) : D(u) + g,\\ \rho\Big[u_t + (u\cdot\nabla)u\Big] - \operatorname{div}\left(\nu(\rho,\theta)D(u)\right) + \nabla p = f.\end{gathered}$$

The very first step is to consider the case of dimension 2. Here we have to develop a notion of a renormalized solution, or at most a solution in a weak sense, for the energy equation which has an L^1 forcing term and the unusual term $\rho\partial_t\theta$. Concerning the Navier-Stokes equation with variable density we refer [101, 80, 62]. In the case of Dimension 3 and with Navier-Slip boundary condition the main obstacle is the control of the pressure p. Indeed in Chapter 4 we use strongly the Helmholtz decomposition and and the L^p regularity theory. With the presence of the variable density, it seems that the method developed in [34, 32] which is used in Chapter 4 to control the pressure p in $L^{5/3}$ and then u_t and θ_t in some appropriate spaces does not work. At last it is possible to consider the full system with the phase field equation

$$\varphi_t + u \cdot \nabla \varphi - \xi \Delta \varphi = -\varphi(1-\varphi)(1-2\varphi+c\theta)$$

and the latent heat term $l\varphi_t$ in the balance energy equation.

Fluid thermomechanics is a large source of strongly coupled partial differential equations systems. If the dissipation energy is not neglected in the balance energy equation, we then have to handle equations with L^1 right-hand for which all the techniques can help.

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