



# Examples of foliated harmonic currents and levi-flats on the complex projective plane

Mohamad Alkateeb

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# THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*

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Présentée et soutenue le 11/Décembre/2023 par :

**Mohamad ALKATEEB**

**Exemples de courants harmoniques feuilletés et Levi-plats sur le plan  
projectif complexe**

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# Résumé

Dans cette thèse, nous donnons des exemples de feuilletages sur le plan projectif complexe  $\mathbb{CP}^2$  admettant des courants harmoniques feuilletés positifs dont les supports coïncident avec des levi-plats singuliers qui, à leur tour, peuvent être choisis comme réels-analytiques (mais non-algébriques) ou simplement continus avec nature transversale fractale. De plus, des exemples non triviaux (comme ci-dessus) peuvent déjà être trouvés parmi les feuilletages de degré 2 et 3. De plus, l'espace des courants harmoniques feuilletés positifs pour ces feuilletages est entièrement caractérisé et il contient un unique courant harmonique (non-fermé) dont le support est sur le Levi-plat en question. Enfin, nous donnons également des exemples de feuilletages porteurs de courants fermés feuilletés positifs diffus liés à un théorème dû à Brunella ainsi qu'un critère général pour l'existence de Levi-plats analytiques-réels singuliers pour les feuilletages de Riccati.

## Mots clés

Courants feuilletés, Courants harmoniques, Courants fermés, Equations de Riccati, Groupes Fuchsien, Levi-plats.



# Abstract

In this thesis, we provide examples of foliations on the complex projective plane  $\mathbb{CP}^2$  carrying positive foliated harmonic currents whose supports coincide with singular Levi-flats which, in turn, can be chosen real-analytic (but non-algebraic) or merely continuous with fractal transverse nature. Furthermore, non-trivial examples (as above) can already be found among foliations of degree 2 and 3. In addition, the space of positive foliated harmonic currents for these foliations is fully characterised and it contains a unique harmonic (non-closed) current supported on the Levi-flat in question. Finally, we also provide examples of foliations carrying diffuse positive foliated closed currents related to a theorem due to Brunella as well as a general criterion for the existence of singular real-analytic Levi-flats for Riccati foliations.

## Keywords

Closed currents, Foliated currents, Fuchsian groups, Harmonic currents, Levi-flats, Riccati equations.





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# Contents

<b>Composition du jury</b>	<b>3</b>
<b>Résumé</b>	<b>5</b>
<b>Abstract</b>	<b>7</b>
<b>Acknowledgements</b>	<b>9</b>
<b>1 Prérequis (en français)</b>	<b>13</b>
1.1 Introduction . . . . .	13
1.2 Feuilletages . . . . .	13
1.3 Courants . . . . .	16
1.4 Groupes Fuchsien et Kleinien . . . . .	19
1.5 Présentation des résultats . . . . .	20
<b>2 Building Riccati Foliations and Ahlfors Currents</b>	<b>23</b>
2.1 Introduction . . . . .	23
2.2 Riccati Foliations . . . . .	26
2.3 Examples of Riccati Foliations and Theorem B . . . . .	31
2.3.1 Two Examples of Riccati Foliations Having Degrees 2 or 3 . . . . .	31
2.3.2 High Degree Foliations with Diffuse Positive foliated Closed Currents . . . . .	33
<b>3 Analytic and Continuous Levi-flats and Harmonic Currents</b>	<b>41</b>
3.1 Riccati Equations and Singular Levi-Flats . . . . .	41
3.2 Foliated Harmonic Currents . . . . .	48
<b>4 Geometry of Levi-flats in the Presence of Parabolic Generators</b>	<b>55</b>
4.1 Introduction . . . . .	55
4.2 Local Dynamics of Saddle-Node Singularity . . . . .	55
4.3 Levi-flat in the Presence of Parabolic Generators . . . . .	59
<b>Bibliography</b>	<b>65</b>



# Chapter 1

## Prérequis (en français)

### 1.1 Introduction

Dans ce chapitre, nous allons revoir les notions de base et les théorèmes principaux concernant la théorie des feuilletages, des courants et des Groupes Fuchsien et Kleiniens (c.f. [45] et [2]).

### 1.2 Feuilletages

**Definition 1.2.1.** Soit  $M$  une variété complexe de dimension  $n \geq 2$  et  $k \in \{1, \dots, n-1\}$ . Un feuilletage holomorphe  $\mathcal{F} = (M, \mathcal{L}(\mathcal{F}))$  de dimension  $k$  sur  $M$  est la donnée d'un atlas holomorphe  $\mathcal{L}(\mathcal{F})$  de  $M$  défini par des cartes  $(U_i, \phi_i)_{i \in I}$  dont les changements de carte

$$\phi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \subset \mathbb{C}^n \longrightarrow \phi_i(U_i \cap U_j) \subset \mathbb{C}^n,$$

sont de la forme

$$(z, t) \in \mathbb{C}^k \times \mathbb{C}^{n-k} \longmapsto (\Psi(z, t), \Lambda(t)),$$

où  $\Psi$  et  $\Lambda$  sont des fonctions holomorphes. Les cartes  $(U_i, \phi_i)$  sont appelées des boîtes à flots.

**Definition 1.2.2.** Un feuilletage holomorphe singulier  $\mathcal{F}$  sur  $M$  est la donnée d'un recouvrement ouvert  $(U_i)_{i \in I}$  et de champs de vecteurs holomorphes  $X_i$  non-identiquement nuls sur  $U_i$ , tels que  $\forall i, j \in I$  avec  $U_i \cap U_j \neq \emptyset$ , il existe une fonction holomorphe

$$g_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}^* \text{ telle que } X_i = g_{ij} X_j.$$

On appelle l'ensemble

$$E = \bigcup_{i \in I} \{X_i = 0\}$$

l'espace des singularités de  $\mathcal{F}$ . C'est un sous-ensemble analytique propre de  $M$ . Il est évident que  $\mathcal{F}$  peut être vu comme un feuilletage holomorphe sur  $M \setminus E$ .

**Proposition 1.2.3** (Suspension d'un groupe d'automorphismes). Soit  $S$  une surface de Riemann et  $M$  une variété complexe. On se donne une représentation de groupe fondamental de  $S$ ,

$$\rho : \pi_1(S) \longrightarrow \text{Aut}(M)$$

et

$$\pi : \tilde{S} \longrightarrow S$$

une uniformisation de  $S$ , c'est-à-dire un revêtement universel holomorphe (l'exemple typique est celui d'une surface de Riemann compacte de genre  $g \geq 2$ ,  $\tilde{S} = \mathbb{D}$  et  $M = \mathbb{CP}^1$  avec  $\text{Aut}(M) = \text{PSL}(2, \mathbb{C})$ ). Alors,  $\pi_1(S)$  agit d'une part sur  $\tilde{S}$  par monodromie et sur  $M$  par la représentation  $\rho$ , ainsi il agit comme suit sur le produit

$$g \cdot (z, t) = (g \cdot z, \rho(g)(t)), \quad \forall (z, t) \in \tilde{S} \times M.$$

De plus, son action sur  $\tilde{S}$  est libre et proprement discontinue. On définit alors

$$S_\rho = (\tilde{S} \times M) / \pi_1(S),$$

appelée variété de suspension de  $\rho$ . Le groupe fondamental  $\pi_1(S)$  agissant sur les fibres du revêtement  $\pi$ ,  $S_\rho$  est munie d'une projection

$$p : S_\rho \longrightarrow S$$

définie par

$$p([z, m]) = \pi(z), \text{ pour } (z, m) \in \tilde{S} \times M.$$

Le fait que l'action de  $\pi_1(S)$  soit libre et proprement discontinue implique que  $S_\rho$  est feuilletée par les

$$\{[z, m] \in S_\rho \mid z \in \tilde{S}\}, \text{ pour } m \in M \text{ fixé.}$$

Les feuilles sont donc isomorphes à des revêtements holomorphes de  $S$  en tant que surfaces de Riemann abstraites.

**Exemple 1** (Feuilletages sur  $\mathbb{CP}^n$ ). On considère l'espace projectif complexe  $\mathbb{CP}^n$  de dimension  $n$  doté de l'atlas holomorphe standard  $\mathcal{L}(\mathcal{F})$  donné par les cartes  $(U_j, \phi_j)_{0 \leq j \leq n}$ , où

$$U_j = \{[(z_0, \dots, z_n)]; z_j \neq 0\},$$

et

$$\phi_j([z_0, \dots, z_n]) = \left( \frac{z_0}{z_j}, \dots, \frac{\hat{z}_j}{z_j}, \dots, \frac{z_n}{z_j} \right),$$

pour tout  $j \in \{0, \dots, n\}$ . On considère le champ de vecteurs à coefficients polynomiaux homogènes

$$X(z_0, \dots, z_n) = \sum_{j=0}^n P_j(z_0, \dots, z_n) \frac{\partial}{\partial z_j}$$

sur  $\mathbb{C}^{n+1} \setminus \{0\}$  de degré  $d$ . Rappelons que  $\mathbb{CP}^n$  est le quotient de  $\mathbb{C}^{n+1} \setminus \{0\}$  par l'action diagonale de  $\mathbb{C}^*$ , l'action donnée par les homothéties

$$\begin{aligned} \sigma_\lambda : \mathbb{C}^{n+1} \setminus \{0\} &\rightarrow \mathbb{C}^{n+1} \setminus \{0\} \\ z &\mapsto \sigma_\lambda(z) := \lambda z, \end{aligned}$$

pour  $\lambda \in \mathbb{C}^*$ . De simples calculs donnent

$$D\sigma_\lambda \cdot X = \lambda^{d-1} X,$$

cela veut dire que les directions associées à  $X$  sont invariantes par homothéties. Ainsi  $X$  définit bien un feuilletage singulier de  $\mathbb{CP}^n$ . Une autre manière équivalente de définir un feuilletage holomorphe singulier sur  $\mathbb{CP}^n$  est de considérer un champ de vecteurs polynomial sur  $\mathbb{C}^n$  et de voir qu'il peut être prolongé à un feuilletage holomorphe sur tout  $\mathbb{CP}^n$ . En effet, on peut montrer que tout feuilletage holomorphe dans  $\mathbb{CP}^n$  est induit par un champ de vecteurs polynomial sur  $\mathbb{C}^n$ . On donne dans ce qui suit les détails dans la dimension 2, mais cela se généralise de manière évidente en dimensions supérieures.

On munit  $\mathbb{CP}^2$  de sa structure complexe standard donnée par les cartes

$$\{(U_a, \phi_a), (U_b, \phi_b), (U_c, \phi_c)\}$$

où

$$U_a = \{[(a, b, c)] \in \mathbb{CP}^2; a \neq 0\},$$

$$U_b = \{[(a, b, c)] \in \mathbb{CP}^2; b \neq 0\},$$

$$U_c = \{[(a, b, c)] \in \mathbb{CP}^2; c \neq 0\},$$

et

$$\phi_a([a, b, c]) = \left(\frac{b}{a}, \frac{c}{a}\right),$$

$$\phi_b([a, b, c]) = \left(\frac{a}{b}, \frac{c}{b}\right),$$

$$\phi_c([a, b, c]) = \left(\frac{a}{c}, \frac{b}{c}\right).$$

Soit

$$X(x, y) = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}$$

un champ de vecteurs polynomial en coordonnées affines  $(x, y) = \phi_a([a, b, c])$ . Cela induit naturellement un champ de vecteurs rationnel  $Y$  (resp.  $Z$ ) défini sur les coordonnées affines  $(u, v) = \phi_b([a, b, c])$  (resp.  $(z, w) = \phi_c([a, b, c])$ ). Par exemple,  $Y$  est donné par:

$$\begin{aligned} Y(u, v) &= (\phi_a \circ \phi_b)^*(X(x, y)) \\ &= D(\phi_a \circ \phi_b)^{-1} \cdot X(\phi_a \circ \phi_b(u, v)) \\ &= \begin{pmatrix} -u^2 & 0 \\ -uv & u \end{pmatrix} \cdot \begin{pmatrix} P(1/u, v/u) \\ Q(1/u, v/u) \end{pmatrix} \end{aligned}$$

Comme on l'avait déjà mentionné, ce champ de vecteurs n'est pas holomorphe dans le domaines des coordonnées  $(u, v)$  car il admet des pôles au long de  $\{u = 0\}$ . Cependant, en multipliant  $Y$  par  $u^d$ , le nouveau champ de vecteurs  $u^d Y$  est clairement holomorphe (avec des singularités isolées) dans le domaine de  $(u, v)$ . En rechangeant le système de coordonnées de  $(u, v)$  vers  $(x, y)$  on obtient

$$\begin{aligned} D(\phi_a \circ \phi_b).(u^d Y) &= u^d D(\phi_a \circ \phi_b).Y(u, v) \\ &= u^d D(\phi_a \circ \phi_b).D(\phi_a \circ \phi_b)^{-1}.X(x, y) \\ &= u^d.X(x, y) \end{aligned}$$

En répétant ce procédé pour les autres cartes, on obtient un feuilletage holomorphe de  $\mathbb{CP}^2$ .



Notons que nous avons ainsi construit deux feuilletages holomorphes singuliers (équivalents) sur  $\mathbb{CP}^2$  suivant deux méthodes différentes:

- (1) En se basant sur un champ de vecteurs polynomial homogène sur  $\mathbb{C}^3$ .
- (2) En se basant sur un champ de vecteurs polynomial sur  $\mathbb{C}^2$ .

Plus généralement, dans le cas de dimensions supérieures, on peut construire des feuilletages singuliers holomorphes de  $\mathbb{CP}^n$  à travers des feuilletages singuliers holomorphes de  $\mathbb{C}^n$ . Ceci dit, il est difficile de montrer que les feuilletages de  $\mathbb{C}^n$  donnent naissance à tout les feuilletages de  $\mathbb{CP}^n$ , on a donc le théorème suivant.

**Theorem 1.2.4.** *Soit  $X_0$  un champ de vecteurs à coefficients polynomiaux sur  $\mathbb{C}^n$ . Alors, il existe un unique feuilletage holomorphe singulier  $\mathcal{F} = (\mathbb{CP}^n, \mathcal{L}(\mathcal{F}), E)$  sur  $\mathbb{CP}^n$  tel que  $\text{codim } E \geq 2$  et  $\mathcal{F}$  prolonge le feuilletage sur  $\mathbb{C}^n \simeq U_0$  défini par  $X_0$ . Réciproquement, tout feuilletage  $\mathcal{F} = (\mathbb{CP}^n, \mathcal{L}(\mathcal{F}), E)$  sur  $\mathbb{CP}^n$  vérifiant  $\text{codim } E \geq 2$  est défini sur la carte affine  $U_0$  par un champ de vecteurs polynomial.*

### 1.3 Courants

**Definition 1.3.1.** *Soit  $M$  une variété complexe de dimension  $n$  et  $p \in \{0, \dots, 2n\}$  (resp.  $p, q \in \{0, \dots, n\}$ ). On note  $\mathcal{D}^p(M)$  (resp.  $\mathcal{D}^{p,q}(M)$ ) l'ensemble des  $p$ -formes (resp.  $(p, q)$ -formes) à coefficients  $\mathcal{C}^\infty$  dans toute coordonnée holomorphe sur  $M$  et à support compact, muni de sa topologie usuelle (de semi-normes). Un  $p$ -courant (resp. un  $(p, q)$ -courant) est une application linéaire continue*

$$T : \mathcal{D}^{2n-p}(M) \longrightarrow \mathbb{C}$$

(resp.  $T : \mathcal{D}^{n-p, n-q}(M) \longrightarrow \mathbb{C}$ ). Dans ce cas là,  $T$  est dit courant de degré  $p$  (resp. bidegré  $(p, q)$ ) ou de dimension  $2n - p$  (resp. bidimension  $(n - p, n - q)$ ). On peut définir un  $(p, q)$ -courant  $T$  d'une manière équivalente comme étant un  $(p, q)$ -courant qui satisfait

$$\langle T, \alpha \rangle := T(\alpha) = 0$$

si  $\alpha$  est de bidegré pur  $(p', q') \neq (n - p, n - q)$ .

En analogie avec la théorie des distributions, l'un peut regarder un  $p$ -courant (resp.  $(p, q)$ -courant) comme étant une  $p$ -forme (resp.  $(p, q)$ -forme) avec des coefficients dans l'espace des distributions  $\mathcal{D}'(M)$ .

**Definition 1.3.2.** *Le support d'un  $p$ -courant (resp.  $(p, q)$ -courant)  $T$  est le plus petit fermé  $\text{Supp}(T)$  inclus dans  $M$  tel que  $T|_{\mathcal{D}^{2n-p}(M \setminus \text{Supp}(T))} = 0$  (resp.  $T|_{\mathcal{D}^{n-p, n-q}(M \setminus \text{Supp}(T))} = 0$ ).*

**Definition 1.3.3.** *On dit qu'un courant  $T$  est fermé, si  $dT(\alpha) := T(d\alpha) = 0$ .*

**Definition 1.3.4.** *On dit qu'un courant  $T$  est feuilleté, si  $T(\alpha) = 0$  lorsque  $\alpha$  s'annule sur le fibré tangent du feuilletage  $\mathcal{F}$ .*

**Definition 1.3.5.** *Une  $(p, p)$ -forme  $\alpha$  sur  $M$  est dite positive si  $\alpha$  s'écrit localement comme une combinaison linéaire à coefficients positifs de formes du type*

$$i\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots \wedge i\alpha_p \wedge \bar{\alpha}_p,$$

où les  $(\alpha_j)_{1 \leq j \leq p}$  sont des  $(1, 0)$ -formes.

**Definition 1.3.6.** Un  $(p, p)$ -courant est dit faiblement positif, si pour toute  $(n - p, n - p)$ -forme positive  $\alpha$ ,

$$\langle T, \alpha \rangle \geq 0.$$

**Definition 1.3.7.** On se donne une métrique  $\beta$  hermitienne sur  $M$ . Si  $T$  est un  $(p, p)$ -courant positif,  $T \wedge \beta^{n-p}$  est une mesure positive. On définit ainsi pour un sous-ensemble mesurable  $A$  de  $M$ , la masse de  $T$  sur  $A$  par :

$$\|T\|_A := \int_A T \wedge \beta^{n-p}.$$

**Remark 1.3.8.** Si  $M$  est compacte, alors la masse totale de  $T$  est finie, i.e.,

$$\|T\|_M < +\infty$$

Les courants positifs fermés jouent un rôle très important dans l'interaction entre la dynamique des applications holomorphes ou méromorphes et la géométrie algébrique. Voici donc quelques exemples :

- (1) **Courant associé à une  $p$ -forme.** Soit  $\alpha$  une  $p$ -forme sur  $M$ . On définit donc le  $p$ -courant  $T_\alpha$  par

$$\begin{aligned} T_\alpha : \mathcal{D}^{2n-p}(M) &\rightarrow \mathbb{C} \\ \beta &\mapsto \langle T_\alpha, \beta \rangle := \int_M \alpha \wedge \beta, \end{aligned}$$

$$\text{Supp}(T) = \text{Supp}(\alpha).$$

- (2) **Courant d'intégration.** Soit  $Z$  une sous-variété réelle fermée et orientée de  $M$  de dimension  $k$ . On définit le courant d'intégration sur  $Z$  comme étant le courant de dimension  $k$  noté  $[Z]$  et donné par

$$\begin{aligned} [Z] : \mathcal{D}^k(M) &\rightarrow \mathbb{C} \\ \alpha &\mapsto \langle [Z], \alpha \rangle := \int_Z \alpha, \end{aligned}$$

$\text{Supp}(T) = Z$ . Si  $Z$  est contenu dans une feuille, alors  $[Z]$  est dirigé.

Rappelons maintenant quelques résultats importants de la théorie des courants positifs fermés :

**Proposition 1.3.9.** Soit  $M$  une variété complexe équipée d'une métrique hermitienne  $\omega$ . Alors l'ensemble des  $p$ -courants positifs tels que

$$\int_M T \wedge \omega^{n-p} \leq 1$$

est faiblement compacte.

Citons maintenant trois fameux théorèmes sur les courants positifs fermés dûs à Skoda (Theorem 1.3.10) et à Siu (Theorem 1.3.12 et Theorem 1.3.13) respectivement.

**Theorem 1.3.10** (Théorème de prolongement de Skoda). Soient  $E$  un sous-ensemble analytique d'une variété complexe  $M$  et  $T$  un  $(p, p)$ -courant sur  $X \setminus E$  avec une masse localement finie au voisinage de  $E$ . Alors l'extension triviale  $\tilde{T}$  de  $T$  dans  $M$  est un  $(p, p)$ -courant positif fermé.

Notons que grâce à ce théorème, on peut généraliser la définition du courant d'intégration vers le cas des ensembles analytiques.

Avant de citer le théorème suivant, rappelons la notion de nombre de Lelong qui mesure la singularité d'un courant en un point donné.

**Definition 1.3.11.** Soit  $\Omega \subset \mathbb{C}^n$  un domaine et  $T$  un  $(p, p)$ -courant. Pour  $a \in \Omega$ , et  $r < d(a, \partial\Omega)$ , on pose

$$\sigma_T(a, r) := \int_{\|z-a\| \leq r} T \wedge \beta^{n-p},$$

où  $\beta := i\partial\bar{\partial}\|z\|^2$ . Le nombre de Lelong de  $T$  en  $a$  est défini comme étant:

$$\nu(T, a) := \frac{1}{\pi^{n-p}} \lim_{r \rightarrow 0} \frac{1}{r^{2(n-p)}} \sigma_T(a, r).$$

On peut vérifier que cette notion de nombre de Lelong est bien définie dans le cadre des variétés complexes, car on peut montrer que ces nombres sont invariants par biholomorphismes.

Pour  $c \in \mathbb{R}$ , notons par

$$E_c(T) = \{a \in M ; \nu(T, a) \geq c\},$$

l'ensemble de niveau supérieur de Lelong. En même année, Siu a prouvé le résultat suivant:

**Theorem 1.3.12.** Les ensembles de niveau de Lelong  $E_c(T)$  d'un  $(p, p)$ -courant  $T$  sont des ensembles analytiques de codimension  $\text{codim } E \geq p$ .

Soient  $A$  un sous-ensemble analytique irréductible de  $M$  de codimension  $p$  et  $T$  un  $(p, p)$ -courant sur  $M$ . On définit le nombre de Lelong générique de  $T$  sur  $A$  par

$$\nu(T, A) := \inf\{\nu(T, a); a \in A\}.$$

Maintenant on est en mesure de citer le théorème de décomposition de Siu.

**Theorem 1.3.13** (Théorème de décomposition de Siu). Tout  $(p, p)$ -courant positif  $T$  admet une unique décomposition de la forme

$$T = \sum \lambda_j [A_j] + R$$

où  $\lambda_j$  sont les nombres de Lelong génériques de  $T$  sur les sous-ensembles analytiques irréductibles  $A_j$  de codimension  $p$  et  $R$  un  $(p, p)$ -courant positif tel que  $\text{codim } E_c(R) > p, \forall c > 0$ .

On s'intéresse maintenant aux résultats importants sur les courants dirigés *harmoniques* dans le cadre d'une structure feuilletée singulière  $\mathcal{F} = (M, \mathcal{L}(\mathcal{F}), E)$  sur  $M$ .

Rappelons qu'un courant dirigé  $T$  est dit être *harmonique* si  $dd^c T = 0$ .

**Theorem 1.3.14** (Berndtsson-Sibony). Supposons que  $E$  est localement pluripolaire. Alors, il existe un courant harmonique dirigé positif non-nul sur  $\mathcal{F}$ .

On donne maintenant un résultat intéressant concernant le prolongement des courants harmoniques.

**Proposition 1.3.15** (Dinh-Nguyễn-Sibony). Supposons que  $\mathcal{F}$  n'admet que des singularités isolées et soit  $T$  un courant harmonique dirigé positif sur  $\mathcal{F}$ . Alors  $T$  a une masse finie au voisinage de toute singularité  $a \in E$ . Si  $M$  est compacte,  $T$  peut être prolongé en un  $(n-1, n-1)$ -courant positif  $dd^c$ -fermé sur  $M$ .

## 1.4 Groupes Fuchsien et Kleinien

Rappelons la définition du groupe projectif spécial linéaire qui nous sera utile pour la suite, qu'on note par  $\mathrm{PSL}(2, \mathbb{C})$ . Celui-ci est défini comme le quotient du groupe spécial linéaire par son centre, c'est-à-dire:

$$\mathrm{PSL}(2, \mathbb{C}) = \left\{ \gamma : z \mapsto \gamma(z) = \frac{az + b}{cz + d}; a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\} / \{\pm Id\} \quad (1.1)$$

**Classification des éléments de  $\mathrm{PSL}(2, \mathbb{C})$ .** Un élément  $\gamma \in \mathrm{PSL}(2, \mathbb{C}) \setminus \{\pm Id\}$  est dit:

- Parabolique, si  $|a + d| = 2$ .
- Hyperbolique, si  $|a + d| > 2$ .
- Elliptique, si  $|a + d| < 2$ .

**Définition 1.4.1** (Groupes Kleinien). *Un groupe Kleinien est un sous-groupe discret de type fini du groupe d'isométries de l'espace hyperbolique  $\mathbb{H}^3$  qui préservent l'orientation, ce dernier étant identifiable avec  $\mathrm{PSL}(2, \mathbb{C})$ .*

On se donne un groupe Kleinien  $\Gamma$ . Alors il existe une décomposition naturelle de la frontière

$$\partial\mathbb{H}^3 = S_\infty^2,$$

(qui peut être identifiée avec la sphère usuelle dans le modèle de la boule conforme de  $\mathbb{H}^3$ ) en deux sous-ensembles associée canoniquement à  $\Gamma$ , il s'agit de l'ensemble limite  $\Lambda(\Gamma)$  et l'ensemble de discontinuité  $\Omega(\Gamma)$ . Fixons un point  $x \in \mathbb{H}^3$ , on considère son orbite  $\Gamma x$  sous l'action de  $\Gamma$ , notons son adhérence dans la boule unité fermée par  $\overline{\Gamma x}$ .

**Définition 1.4.2.** *L'ensemble limite est défini par*

$$\Lambda(\Gamma) = \overline{\Gamma x} \setminus \Gamma x = \overline{\Gamma x} \cap S_\infty^2.$$

**Définition 1.4.3.** *Le complémentaire de l'ensemble limite  $\Omega(\Gamma) = S_\infty^2 \setminus \Lambda(\Gamma)$  est donc l'ensemble de discontinuité.*

**Définition 1.4.4.** *Un groupe Kleinien  $\Gamma$  est dit être élémentaire si son ensemble limite  $\Lambda(\Gamma)$  contient au plus deux points.*

Soit  $\Gamma$  un groupe Kleinien, notons par  $C(\Gamma)$  son enveloppe convexe fermé dans  $\mathbb{H}^3 \cup S_\infty^2$ , son intérieur est donc noté par  $\overset{\circ}{C}(\Gamma) = C(\Gamma) \cap \mathbb{H}^3$ . Soit  $M = \mathbb{H}^3/\Gamma$ , notons par  $M_C(\Gamma)$  le quotient  $\overset{\circ}{C}(\Gamma)/\Gamma$ .  $\Gamma$  est dit géométriquement fini si l' $\epsilon$ -voisinage de l'enveloppe convexe  $N_\epsilon(M_C(\Gamma)) \subset M$  est de volume fini.

Rappelons donc ce théorème utile dans la théorie géométrique de mesure dû à Ahlfors.

**Theorem 1.4.5** (Ahlfors). *Si  $\Gamma$  est un groupe Kleinien géométriquement fini, alors  $\Lambda(\Gamma)$  est soit de mesure nulle ou de mesure totale. Si  $\Lambda(\Gamma)$  est de mesure totale, alors l'action de  $\Gamma$  sur  $S_\infty^2$  est ergodique.*

**Définition 1.4.6** (Groupes Quasi-Fuchsien). *Un Sous-groupe discret de  $\mathrm{PSL}(2, \mathbb{C})$  est appelé quasi-fuchsien si son ensemble limite est une Courbe de Jordan non-différentiable.*

**Definition 1.4.7** (Groupes Fuchsien). *Soit  $\Gamma \subset PSL_2(\mathbb{C})$  qui agit de manière invariante sur un disque ouvert  $\Delta \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ , c'est-à-dire que  $\Gamma(\Delta) = \Delta$ . Alors  $\Gamma$  est dit groupe Fuchsien si et seulement si l'une des propriétés équivalentes suivantes est vérifiée:*

- (1)  $\Gamma$  est un sous-groupe discret (pour la topologie usuelle de  $PSL(2, \mathbb{C})$ ).
- (2)  $\Gamma$  agit de façon proprement discontinue en tout point  $z \in \Delta$ .
- (3) Le disque est un sous-ensemble du domaine de discontinuité  $\Omega(\Gamma)$  de  $\Gamma$ .

On distingue deux types de groupes Fuchsien:

- (1) Un groupe Fuchsien est dit du premier type si l'ensemble limite  $\Lambda(\Gamma) = \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ . Ceci est le cas si  $\mathbb{H}/\Gamma$  est de volume fini.
- (2) Un groupe Fuchsien  $\Gamma$  est de second type s'il n'est pas du premier type. Autrement dit, c'est un groupe dont l'ensemble limite  $\Lambda(\Gamma)$  est parfait et nulle part dense dans  $\overline{\mathbb{R}}$  (c'est un ensemble de Cantor).

Voici quelques exemples très répandus de groupes Fuchsien du premier type:

- (1) Tout sous-groupe cyclique de  $PSL(2, \mathbb{R})$  qui est hyperbolique ou parabolique est un groupe Fuchsien.
- (2) Tout sous-groupe cyclique elliptique de  $PSL(2, \mathbb{R})$  est Fuchsien si et seulement s'il est de type fini.
- (3) Tout groupe Fuchsien abélien est cyclique.
- (4) Aucun groupe Fuchsien n'est isomorphe to  $\mathbb{Z} \times \mathbb{Z}$ .
- (5) Soit  $\Gamma$  un groupe Fuchsien non-abélien. Alors le normalisateur de  $\Gamma$  dans  $PSL(2, \mathbb{R})$  est un groupe Fuchsien.
- (6) Tout groupe Fuchsien est Kleinien.

## 1.5 Présentation des résultats

Le résultat suivant correspond au "Theorem A" (c.f. section 2.1)

**Theorem 1.5.1.** *Soit  $\mathcal{F}$  un feuilletage de Riccati sur le plan projectif complexe  $\mathbb{CP}^2$  n'ayant que des droites simples invariantes  $C_1, \dots, C_k$ . Supposons que l'application d'holonomie locale du feuilletage  $\mathcal{F}$  autour de chaque droite invariante  $C_i$  soit un élément elliptique de  $PSL(2, \mathbb{C})$  et que le groupe d'holonomie global  $\Gamma \subset PSL(2, \mathbb{C})$  du feuilletage  $\mathcal{F}$  est un groupe fuchsien (resp. quasifuchsien) de premier type. Alors:*

- (a) *Il existe un ensemble fermé  $\mathcal{L}(\mathcal{F}) \subset \mathbb{CP}^2$  de dimension topologique égale à 3 qui est minimal pour le feuilletage  $\mathcal{F}$ . Autrement dit,  $\mathcal{L}(\mathcal{F})$  est invariant par le feuilletage  $\mathcal{F}$  et chaque feuille du feuilletage  $\mathcal{F}$  en  $\mathcal{L}(\mathcal{F})$  est dense dans  $\mathcal{L}(\mathcal{F})$ .*

- (b) Si  $\Gamma$  est fuchsien, alors  $\mathcal{L}(\mathcal{F})$  est un ensemble non-algébrique analytique-réel avec  $k$  points singuliers, tous de type orbifold. Si  $\Gamma$  est quasifuchsien, alors  $\mathcal{L}(\mathcal{F})$  est une “variété topologique singulière” de dimension de Hausdorff strictement supérieure à 3 et dont l’ensemble singulier est à nouveau constitué de  $k$  points singuliers de type orbifold.
- (c) Il existe un unique courant harmonique feuilleté positif porté par l’ensemble  $\mathcal{L}(\mathcal{F})$ . Ce courant harmonique, en effet, n’est pas fermé.
- (d) L’espace de tous les courants harmoniques feuilletés positifs sur la surface  $\mathbb{CP}^2$  est engendré par  $T$  et par les courants fermés induits par intégration sur chacune des droites invariantes  $C_1, \dots, C_k$ .
- (e) Le courant  $T$  est d’auto-intersection (géométrique) nulle au sens de [21].

Le résultat suivant correspond au “Theorem B” (c.f. section 2.1)

**Theorem 1.5.2.** *Pour tout  $n \in \mathbb{N}$ , il existe un feuilletage  $\mathcal{F}_n$  de degré  $n$  sur le plan projectif complexe  $\mathbb{CP}^2$  satisfaisant les conditions suivantes:*

- *Les points singuliers de  $\mathcal{F}_n$  sont simples sauf l’un parmi eux.*
- *Le feuilletage  $\mathcal{F}_n$  admet une quantité non-dénombrable de courants d’Ahlfors diffus (indépendants). De même, il existe une quantité non-dénombrable d’applications holomorphes (non-constantes) de  $\mathbb{C}$  à valeurs dans  $\mathbb{CP}^2$  qui sont tangentes au feuilletage  $\mathcal{F}_n$  et dont les images sont Zariski denses.*

Ces deux théorèmes sont démontrés dans les deux premiers chapitres (c.f. 2.3.2, 3.1 et 3.2) et correspondent au contenu de l’article [1]. Cependant, le dernier chapitre (qui fait partie d’un travail en cours) est consacré à décrire la géométrie du Levi-plat correspondant dans le cas d’une holonomie Fuchsienne admettant des générateurs paraboliques.



## Chapter 2

# Building Riccati Foliations and Ahlfors Currents

In the present chapter (and the next one), we provide (without modification) the content of the article [1].

### 2.1 Introduction

The purpose of this chapter is to provide examples of (singular holomorphic) foliations on the complex projective plane  $\mathbb{CP}^2$  carrying, dynamically interesting, positive foliated harmonic currents that can be detailed to a significant extent. Currents whose supports are either singular real-analytic Levi-flats or continuous Levi-flats with fractal nature and Hausdorff dimension greater than 3 are included. Throughout this chapter, a current  $T$  is said to be *harmonic* if it verifies  $i\partial\bar{\partial}T = 0$ . In particular, *closed* currents are automatically harmonic. However, in a suitable sense, there are few foliations on the surface  $\mathbb{CP}^2$  admitting non-trivial positive foliated closed currents whereas the existence of positive *foliated harmonic currents* is a rather general phenomenon. For instance, according to Fornæss and Sibony [22], every foliation on the surface  $\mathbb{CP}^2$  having only hyperbolic singularities and leaving no algebraic curve invariant admits unique positive foliated harmonic currents (and no closed ones). In particular, the minimal foliations constructed in [34] admit a unique positive foliated harmonic current and whose supports coincide with all the surface  $\mathbb{CP}^2$ .

In a kind of opposite direction, the simplest examples of foliations on the surface  $\mathbb{CP}^2$  carrying positive foliated *closed* currents are provided by foliations leaving invariant some algebraic curve  $C \subset \mathbb{CP}^2$ : the integration current over the (possibly singular) curve  $C$  is positive foliated and closed. In general, the study of (positive foliated) harmonic/closed currents is a very active area relying on a variety of points of view as confirmed by the large literature including [24], [10], [4], [21], [22], [13], [25], [15], [41], [16], [14] among several others.

However, despite a few remarkable theorems on their existence and uniqueness such as Fornæss-Sibony theorem in [22], very little is known about the actual structure of these harmonic currents. For example, dealing with *non-closed* harmonic currents, basically the only type of information so far available is the mentioned fact that the harmonic currents associated with the foliations of [34] have full support. In particular, this raises the outstanding problem of deciding whether or not these currents are *absolutely continuous* with respect to Lebesgue measure.



Moreover, the complex projective plane  $\mathbb{CP}^2$  is arguably the most interesting surface to investigate the structure of harmonic currents due to their relations with a few outstanding conjectures that are specific for the surface  $\mathbb{CP}^2$ , (c.f. for example [17]). Not surprisingly, constructing suitable foliations on the surface  $\mathbb{CP}^2$  involves special difficulties which, in turn, explains the relative paucity of examples and our focus on this case.

Concerning the study of (non-closed) harmonic currents, the main results of this chapter are summarised by Theorem A which seems to answer a question formulated in [19] and pertains directly some problems mentioned in [17] (with respect to [17] see also our Theorem B). This theorem can also be viewed as a general statement about Riccati foliations on the surface  $\mathbb{CP}^2$ .

**Theorem A.** *Let  $\mathcal{F}$  be a Riccati foliation on the complex projective plane  $\mathbb{CP}^2$  having only simple invariant lines  $C_1, \dots, C_k$ . Assume that the local holonomy map of the foliation  $\mathcal{F}$  around each invariant line  $C_i$  is an elliptic element of  $\mathrm{PSL}(2, \mathbb{C})$  and that the global holonomy group  $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$  of the foliation  $\mathcal{F}$  is a Fuchsian (resp. quasifuchsian) group of first kind. Then, the following holds:*

- (a) *There exists a closed set  $\mathcal{L}(\mathcal{F}) \subset \mathbb{CP}^2$  of topological dimension equal to 3 which is a minimal set for the foliation  $\mathcal{F}$ . In other words,  $\mathcal{L}(\mathcal{F})$  is invariant by the foliation  $\mathcal{F}$  and every leaf of the foliation  $\mathcal{F}$  in  $\mathcal{L}(\mathcal{F})$  is dense in  $\mathcal{L}(\mathcal{F})$ .*
- (b) *If  $\Gamma$  is Fuchsian, then  $\mathcal{L}(\mathcal{F})$  is a real-analytic non-algebraic set with  $k$  singular points, all of them of orbifold-type. If  $\Gamma$  is quasifuchsian, then  $\mathcal{L}(\mathcal{F})$  is a “singular topological manifold” with Hausdorff dimension strictly greater than 3 and whose singular set consists again of  $k$  singular points of orbifold-type.*
- (c) *The set  $\mathcal{L}(\mathcal{F})$  carries a unique positive foliated non-closed harmonic current  $T$  whose support coincides with the whole  $\mathcal{L}(\mathcal{F})$ .*
- (d) *The space of all positive foliated harmonic currents on the surface  $\mathbb{CP}^2$  is generated by  $T$  and by the closed currents induced by integration over each of the invariant lines  $C_1, \dots, C_k$ .*
- (e) *The current  $T$  has zero (geometric) self-intersection in the sense of [21].*

In particular, the current  $T$  in question appears to be the first example of a (non-closed, foliated) positive harmonic current on the surface  $\mathbb{CP}^2$  supported on a set with empty interior or, alternatively, on a set of null Lebesgue measure. In particular, it is interesting to see that this support may be very regular (real-analytic) or merely continuous with transverse fractal nature and Hausdorff dimension strictly greater than 3. We should also mention that foliations as in Theorem A associated with analytic Levi-flats already exist in degree 2 whereas transversely fractal Levi-flats can be found among degree 3 foliations. The reader is referred to Section 2.3 for explicit examples.

When the global holonomy group  $\Gamma$  of the foliation  $\mathcal{F}$  is Fuchsian, the resulting Levi-flat  $\mathcal{L}(\mathcal{F})$  is necessarily non-algebraic thanks to Lebl’s theorem in [32]: if it were algebraic, the leaves of the foliation  $\mathcal{F}$  in  $\mathcal{L}(\mathcal{F})$  would be complex algebraic curves which is clearly not the case since they are dense in  $\mathcal{L}(\mathcal{F})$  (item (a)). It is also interesting to notice that the non-algebraic nature of the leaves of the foliation  $\mathcal{F}$  in  $\mathcal{L}(\mathcal{F})$  is the only obstacle preventing  $\mathcal{L}(\mathcal{F})$  of being algebraic as it follows from [18]. Finally, note that item (b) in Theorem A provides a criterion to ensure the existence of real-analytic Levi-flats. The case where  $\Gamma$  is Fuchsian provides *singular real-analytic* Levi-flats  $\mathcal{L}(\mathcal{F})$  with very simple singularities (orbifold-type): in some sense  $\mathcal{L}(\mathcal{F})$  is “as little singular as possible” which is worth singling out since it might shed some light on the well-known problem about the existence of smooth Levi-flats on the surface  $\mathbb{CP}^2$ .

At this point a comment of *closed foliated currents* is in order. Although they rarely exist, the problem of describing those situations where they are present has attracted significant interest due, among other reasons, to its connection with Kobayashi hyperbolic spaces, (c.f. [12], [38]). With respect to the former, our ideas also enable us to provide examples answering a question by N. Sibony which was motivated by Brunella's result in [8].

First, recall that a (positive foliated) current is said to be *algebraic* if it coincides with the integration current over an algebraic curve invariant by the foliation  $\mathcal{F}$ . Otherwise,  $T$  is said to be *diffuse*. Recall also that a singular point  $p$  of a foliation  $\mathcal{F}$  is called *simple* if the foliation  $\mathcal{F}$  can locally be represented by a holomorphic vector field whose linear part at the singular point  $p$  has at least one eigenvalue different from zero (c.f. Section 2.2). With this terminology, and building on the work of McQuillan [38], Brunella showed that any foliation  $\mathcal{F}$  on the surface  $\mathbb{CP}^2$  having only simple singular points and admitting a diffuse (positive foliated) closed current  $T_A$  of Ahlfors type must have degree at most 4, (c.f. [8]). This has prompted N. Sibony to wonder if some bound on the degree of the foliation  $\mathcal{F}$  may be obtained by dropping (or significantly weakening) the assumption on the corresponding singular points. In this direction, in Section 2.3, the following theorem will be proved:

**Theorem B.** *For every  $n \in \mathbb{N}$ , there exists a degree  $n$  foliation  $\mathcal{F}_n$  on the surface  $\mathbb{CP}^2$  satisfying all of the following conditions:*

- *All but one of the singular points of the foliation  $\mathcal{F}_n$  are simple.*
- *The foliation  $\mathcal{F}_n$  carries uncountably many (independent) diffuse Ahlfors currents. Similarly, there are uncountably many holomorphic maps from  $\mathbb{C}$  to the surface  $\mathbb{CP}^2$  that are tangent to the foliation  $\mathcal{F}_n$  and have Zariski-dense images.*

Note, however, that not all (positive) diffuse foliated closed currents are of Ahlfors type and some examples will also be indicated in Section 2.3.

Let us close this Introduction with an outline of the chapter. Given a discrete group action, the standard *suspension construction* basically yields a foliation defined on a certain fibre bundle. In this sense, suspensions never produce foliations on the surface  $\mathbb{CP}^2$ . Yet, our strategy will consist of building some very special suspensions and then systematically modify the foliation/ambient surface to eventually make the whole structure fit in the surface  $\mathbb{CP}^2$ . Naturally, this will require us to overcome a few difficulties such as the fact that our basic objects (foliated currents, real-analytic sets) *are not* birationally invariant in general.

In Section 2.2, we show how to construct Riccati foliations on  $\mathbb{CP}^1$ -bundles with prescribed holonomy (Proposition 2.2.1), a result going all the way back to Birkhoff [5] and re-discovered in [33]. Besides background and complementary material, Section 2.2 includes a reasonably short proof of Proposition 2.2.1 which is similar to Lins-Neto's argument. However, besides making the chapter self-contained, our proof shows that certain additional choices are always possible and this freedom is sometimes useful in the course of this work.

In Section 2.3, we use Proposition 2.2.1 to provide examples of degree 2 foliations on the surface  $\mathbb{CP}^2$  satisfying the conditions of Theorem A for Fuchsian groups and thus giving rise to real-analytic Levi-flats. Here, it is worth mentioning that the degree 2 examples turn out to be the Halphen vector fields studied in [27]. Similarly, we show how to obtain transversely fractal Levi-flats starting from degree 3. In the second part of this section, we prove Theorem B. Section 2.3 ends with some examples of diffuse positive foliated closed currents that *are not* of Ahlfors type.

Section 3.1 contains the proof of statements (a) and (b) in Theorem A except by a specific lemma

(Lemma 3.1.7) whose proof is deferred to the last section since similar techniques are employed to establish the remaining items in Theorem A as well.

**Terminology.** In the statements of our theorems and throughout the text,  $\mathrm{PSL}(2, \mathbb{C})$  is identified with the automorphism group of  $\mathbb{CP}^1$ . Elements in  $\mathrm{PSL}(2, \mathbb{C})$  are classified as *hyperbolic*, *parabolic*, *elliptic* and the *identity*. In other words, the identity matrix is set aside so that whenever an element of  $\mathrm{PSL}(2, \mathbb{C})$  is said to be elliptic or parabolic it is understood that *this element is different from the identity*.

## 2.2 Riccati Foliations

Let  $R$  be a compact Riemann surface and  $S$  a  $\mathbb{CP}^1$ -bundle over  $R$  whose projection is denoted by  $\mathcal{P} : S \rightarrow R$ . A Riccati foliation is usually defined as a *singular foliation*  $\mathcal{F}$  on the surface  $S$  that is *transverse* to the fibres of  $S$ . This statement can be made accurate as follows. There is a finite set  $\{p_1, \dots, p_k\} \subset R$  such that the following holds:

- The fibres  $C_i = \mathcal{P}^{-1}(p_i)$ , over the points  $p_i$ , are invariant by the foliation  $\mathcal{F}$ ,  $i = 1, \dots, k$ .
- The leaves of the foliation  $\mathcal{F}$ , away from the invariant fibres  $C_i$ , are transverse to the fibres of  $\mathcal{P}$ . In particular, the singular points of the foliation  $\mathcal{F}$  are all contained in the union of the invariant fibres  $C_i$ .

There follows that the restriction of the foliation  $\mathcal{F}$  to the open surface  $S \setminus \{C_1, \dots, C_k\}$ , is regular and, in fact, transverse to the fibres of  $\mathcal{P}$  sitting over points in the (open) Riemann surface  $R \setminus \{p_1, \dots, p_k\}$ . Thus, the standard holonomy representation gives rise to a homomorphism  $\rho : \pi_1(R \setminus \{p_1, \dots, p_k\}) \rightarrow \mathrm{Aut}(\mathbb{CP}^1) \simeq \mathrm{PSL}(2, \mathbb{C})$ . The *global holonomy group*  $\Gamma = \rho(\pi_1(R \setminus \{p_1, \dots, p_k\})) \subset \mathrm{PSL}(2, \mathbb{C})$  encodes all of the transverse dynamics of the foliation  $\mathcal{F}$ .

In this chapter, however, we will focus on *classical Riccati foliations* in which case the base space  $R$  is again the complex projective line  $\mathbb{CP}^1$  so that the surface  $S$  is a  $\mathbb{CP}^1$ -bundle over  $\mathbb{CP}^1$  and, hence, a Hirzebruch surface  $F_n$ ,  $n \geq 0$ . Recall that  $F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$  and that, for  $n \geq 1$ ,  $F_n$  is fully characterised as a  $\mathbb{CP}^1$ -bundle over  $\mathbb{CP}^1$  admitting a rational curve of self-intersection  $-n$  as a section, c.f. [3].

Recall that the surface  $F_n$  possesses a standard atlas consisting of four affine coordinates. More precisely, consider two copies of  $\mathbb{C} \times \mathbb{CP}^1$ . The first copy is endowed with a pair of affine coordinates, namely  $(x, y)$  and  $(\bar{x}, \bar{y})$ , satisfying with  $x = \bar{x}$  and  $y = 1/\bar{y}$ . The second copy is endowed with coordinates  $(u, v)$  and  $(\bar{u}, \bar{v})$  satisfying the analogous relation. The surface  $F_n$  can then be obtained by identifying the point  $(x, y)$  of the first copy with the point  $(u, v) = (1/x, x^n y)$  of the second one. The section of self-intersection  $-n$ , with this identification, is nothing but the rational curve defined by  $\{y = 0\}$  ( $\{v = 0\}$ ) and it will often be referred to as the *null section*.

Consider now a Riccati foliation  $\mathcal{F}$  defined on the first Hirzebruch surface  $F_1$ . The condition of transversality ensures that, in the above mentioned affine coordinates  $(x, y)$ , the foliation  $\mathcal{F}$  is induced by a holomorphic vector field  $X$  having the form

$$X(x, y) = F(x) \frac{\partial}{\partial x} + [c_0(x) + c_1(x)y + c_2(x)y^2] \frac{\partial}{\partial y}, \quad (2.1)$$

where  $F$ ,  $c_0$ ,  $c_1$ , and  $c_2$  are polynomials. In particular, if  $c_0$  is a non-zero constant then the foliation  $\mathcal{F}$  is transverse to the null section of the surface  $F_1$  except maybe at the “point at infinity”. In this regard, it is always possible to choose coordinates where the resulting fibre at infinity is not invariant by the

foliation  $\mathcal{F}$  so that the invariant fibres of the foliation  $\mathcal{F}$  are in natural correspondence with the zeros of the polynomial  $F$ . Moreover, an invariant fibre is said to be *simple* if it corresponds to a simple zero of  $F$ , otherwise it is called a *multiple fibre*.

An alternative definition of *simple fibres* that is slightly more intrinsic since it does not depend on the particular system of coordinates, depends on the notion of eigenvalues of a singular point. Recall that on a complex surface every holomorphic foliation  $\mathcal{F}$  is locally given by the integral curves of a holomorphic vector field  $X$  having only isolated zeros. A vector field satisfying this condition is said to be a *local representative* of the foliation  $\mathcal{F}$ . Now, if  $p$  is a singular point of the foliation  $\mathcal{F}$ , the *eigenvalues* of the foliation  $\mathcal{F}$  at the singular point  $p$  are defined as the eigenvalues of the linear part of a representative vector field  $X$  at the point  $p$ . It follows, since two representative vector fields differ by multiplication by an invertible function, that the eigenvalues of a foliation  $\mathcal{F}$  at the singular point  $p$  are well defined only up to a multiplicative constant, c.f. [30] and [45]. It is immediate to check, with this terminology, that an invariant fibre of a Riccati foliation is simple *if and only if every singular point lying in this fibre has a non-zero eigenvalue in the direction transverse to the invariant fibre itself*. Also, this notion does not depend on the choice of the singular point in the sense that if one singular point has a non-zero eigenvalue in a direction transverse to the invariant fibre then any other singular point lying in the same fibre does too. In this respect, Equation (2.1) shows that an invariant fibre of a Riccati equation contains at least one singular point and at most 2 singular points.

Next, let  $\Gamma$  be a subgroup of  $\mathrm{PSL}(2, \mathbb{C})$  along with a chosen generating set  $\{M_1, \dots, M_{k-1}\}$ . Choose also (pairwise distinct) points  $\{p_1, \dots, p_k\}$  in  $\mathbb{CP}^1$  and set  $B = \mathbb{CP}^1 \setminus \{p_1, \dots, p_k\}$ . The following proposition plays a basic role in our chapters. This proposition goes back to Birkhoff [5] and [6] though an independent treatment was provided by A. Lins-Neto in [33]. For the convenience of the reader, we formulate an accurate statement (Proposition 2.2.1) below and provide a self-contained proof that parallels the one given in [33]. Besides making the chapter more self-contained, the proof given here also singles out a certain amount of flexibility in the construction that will be helpful in the course of the chapter.

**Proposition 2.2.1.** (c.f. [5] and [33]) *With the above notation, there exists a Riccati foliation  $\mathcal{F}$  on the first Hirzebruch surface  $F_1$  satisfying the following conditions:*

- *The foliation  $\mathcal{F}$  leaves invariant exactly  $k$  fibres sitting, respectively, over the points  $\{p_1, \dots, p_k\}$ . These fibres are all simple.*
- *For each  $i \in \{1, \dots, k-1\}$ , the local holonomy map arising from a small simple loop around the point  $p_i$  coincides with the automorphism of  $\mathbb{CP}^1$  identified with the matrix  $M_i$ .*
- *The local holonomy map arising from a small simple loop around the point  $p_k$  coincides with the automorphism of  $\mathbb{CP}^1$  arising from the matrix  $M_k = (M_{k-1} \dots M_1)^{-1}$ .*

*In particular, the global holonomy group of the foliation  $\mathcal{F}$  coincides with  $\Gamma$ .*

*Proof.* For each  $i \in \{1, \dots, k\}$ , let  $\gamma_i \subset \mathbb{CP}^1$  be a small simple loop around the point  $p_i \in \mathbb{CP}^1$ . The fundamental group  $\pi_1(B)$  of  $B = \mathbb{CP}^1 \setminus \{p_1, \dots, p_k\}$  is generated by  $\gamma_1, \dots, \gamma_k$  along with the relation  $\gamma_1 * \dots * \gamma_k = \mathrm{id}$ . We then define a representation  $\rho$  from  $\pi_1(B)$  in  $\mathrm{PSL}(2, \mathbb{C})$  by letting  $\rho(\gamma_i) = M_i$  for  $i = 1, \dots, k$ . The homomorphism  $\rho$  is well defined since  $M_k = (M_{k-1} \dots M_1)^{-1}$  so that  $M_k M_{k-1} \dots M_1 = \mathrm{id}$ . We also have, by construction, that  $\rho(\pi_1(B)) = \Gamma$ .

Next, we use the standard *suspension* construction to obtain a  $\mathbb{CP}^1$ -bundle  $N$  over  $B$  equipped with a foliation  $\mathcal{D}$  which is transverse to its fibres and whose global holonomy group is  $\Gamma$ . In fact, the holonomy

map associated to  $\mathcal{D}$  and arising from a small loop  $\sigma_i \subset B$  encircling the missing point  $p_i$  is precisely the automorphism induced by  $M_i$ .

The manifold  $N$  is clearly open since the basis  $B$  is so. To obtain a compact manifold and an (extended) singular foliation, we will “fill in” each of the missing fibres in  $N$ . Clearly, denoting by  $\mathcal{P} : N \rightarrow B$  the bundle projection, it suffices to show how to “fill in” the fibre  $\mathcal{P}^{-1}(p_1)$  over the point  $p_1$ . For this, we consider a small disc  $D$  around  $p_1$  whose boundary  $\partial D$  is identified with the loop  $\sigma_1$ . By means of a local coordinate  $u$ ,  $D$  can be thought of as a disc around the origin  $0 \in \mathbb{C}$ . First, we will construct a Riccati foliation  $\mathcal{F}_1$  on the product  $D \times \mathbb{CP}^1$  having a single invariant fibre, which sits over  $0 \in D \subset \mathbb{C}$ , and whose holonomy is given by the matrix  $M_1$ . To construct the foliation  $\mathcal{F}_1$ , we then consider coordinates  $(u, v)$  on  $D \times \mathbb{CP}^1$  where  $u$  is as above and  $v$  is an affine coordinate on  $\mathbb{CP}^1$ . Then the foliation  $\mathcal{F}_1$  is given by the integral curves of the following vector fields:

$$\begin{cases} u \frac{\partial}{\partial u} - (\frac{1}{2\pi i}) \frac{\partial}{\partial v}, & \text{if } M_1 \text{ is parabolic,} \\ u \frac{\partial}{\partial u} + av \frac{\partial}{\partial v}, & \text{otherwise.} \end{cases}$$

It is clear that the foliation above admits a holomorphic extension as a Riccati foliation to  $D \times \mathbb{CP}^1$ . Moreover, if  $M_1$  is not parabolic, then the corresponding holonomy map fixes the point  $v = 0$  and a direct inspection shows that the multiplier of this fixed point is  $e^{2\pi i a}$  ( $a \in \mathbb{C}^*$  and  $\Re(a) \in [-1, 1]$ , where  $\Re(a)$  stands for the real part of  $a$ ). In turn, when  $M_1$  is parabolic, the holonomy map is given by  $v \mapsto v - 1$  so that it has a single fixed point corresponding to  $v = \infty$  whose multiplier equals 1. In particular, note that the choice  $a = \pm 1$  leads to trivial holonomy.

To complete the “filling” of the “missing” fibres, it suffices to show that it is possible to glue together the foliations  $\mathcal{D}$  and  $\mathcal{F}_1$  over the punctured disc  $D^*$ . Gluing these foliations together amounts to constructing a holomorphic diffeomorphism  $h$  from  $D^* \times \mathbb{CP}^1$  to  $\mathcal{P}^{-1}(D^*)$  taking the foliation  $\mathcal{F}_1$  to the foliation  $\mathcal{D}$ . Though not indispensable, at this point it is convenient to remind the reader that every holomorphic  $\mathbb{CP}^1$ -bundle is holomorphically locally trivial owing to a theorem due to Fisher and Grauert [20]. To begin, we use the common coordinate  $u$  and fix a base point  $q \in D^*$ . The fibre  $(q, \mathbb{CP}^1)$  in  $D^* \times \mathbb{CP}^1$  is endowed with the coordinate  $v$ . To identify  $(q, \mathbb{CP}^1)$  with the fibre  $\mathcal{P}^{-1}(q)$  we need to choose a affine coordinate  $w$  on  $\mathcal{P}^{-1}(q)$ . The choice of  $w$  is made as follows:

- If  $a = \pm 1$ , i.e., the holonomy map coincides with the identity. Then  $w$  is an arbitrary affine coordinate in  $\mathcal{P}^{-1}(q)$  and the identification is  $v = w$ .
- If the holonomy map is neither the identity nor parabolic. Then we choose  $a$  and  $w$  so that the origin of  $w$  coincides with a fixed point of the holonomy map having multiplier equal to  $e^{2\pi i a}$  and set  $v = w$ .
- If the holonomy map is parabolic. Then the origin of  $w$  is the fixed point of the holonomy map and the identification is  $w = 1/v$ .

The diffeomorphism  $h$  is then defined on  $(q, \mathbb{CP}^1)$  by declaring that  $h$  takes points of  $(q, \mathbb{CP}^1)$  to points in  $\mathcal{P}^{-1}(q)$  in accordance with the identification of the corresponding fibres. It then remains to extend  $h$  to  $D^* \times \mathbb{CP}^1$ . This is done by following the standard lifting path method for the leaves of the foliations  $\mathcal{F}_1$  and  $\mathcal{D}$  (the latter restricted to  $\mathcal{P}^{-1}(D^*)$ ). The path liftings are clearly possible since the restriction of the bundle projection to any (regular) leaf of either  $\mathcal{F}_1$  or  $\mathcal{D}$  is a covering map of the base (the foliations are Riccati). Furthermore, the diffeomorphism  $h$  is globally well defined since its extension over a loop around  $0 \in \mathbb{C}$  coincides with the initial definition on  $(q, \mathbb{CP}^1)$ . In fact,  $h$  on  $(q, \mathbb{CP}^1)$  is determined by

the identification provided by the coordinates  $v$  and  $w$  so that it conjugates the corresponding holonomy maps.

Summarising what precedes, the gluing construction described above allows us to define a Riccati foliation, still denoted by  $\mathcal{D}$ , on a compact surface  $N'$  that happens to be a  $\mathbb{CP}^1$ -bundle over  $\mathbb{CP}^1$ . Moreover, the Riccati foliation  $\mathcal{D}$  possesses exactly  $k$  invariant fibres and satisfies the required holonomy conditions. Finally, being a  $\mathbb{CP}^1$ -bundle over  $\mathbb{CP}^1$ , the surface  $N'$  is a Hirzebruch surface  $F_n$ ,  $n \geq 0$ . It only remains, to complete the proof of the present proposition, to check that we can assume without loss of generality that  $N'$  is actually first Hirzebruch surface  $F_1$ .

Assuming the resulting surface  $F_n$  is different from  $F_1$ , we will show how the surface can be modified to become the surface  $F_1$  while keeping all the conditions on the transformed foliation (still denoted by  $\mathcal{D}$ ). This will be done by constructing a convenient birational map between the surfaces  $F_1$  and  $F_n$ . This birational map is obtained by composing successive (birational) maps from  $F_n$  to  $F_{n-1}$  which are defined as follows. Pick a base point  $q \in F_n$  which is away from the null section (i.e., the section of self-intersection  $-n$ ) and lies in a fibre  $\mathcal{P}^{-1}(q_0)$  transversely intersecting the foliation  $\mathcal{D}$ . The surface  $F_n$  is then blown up at  $q$  so that the new surface  $\tilde{F}_n$  possesses now two rational curves of self-intersection  $-1$ . Namely, the exceptional divisor  $E$  and the transform  $\widetilde{\mathcal{P}^{-1}(q_0)}$  of the fibre  $\mathcal{P}^{-1}(q_0)$ . The blown up surface also possesses a rational curve of self-intersection  $-n$  given by the transform of the null section. Next collapse the  $(-1)$ -curve  $\widetilde{\mathcal{P}^{-1}(q_0)}$  to obtain a new surface. We claim the following:

*Claim.* The surface obtained by collapsing the transform of  $\mathcal{P}^{-1}(q_0)$  is  $F_{n-1}$  and the transform of  $E$  is a fibre of the corresponding fibration. The foliation  $\mathcal{D}$  induces a new foliation  $\mathcal{D}_1$  on  $F_{n-1}$  having an additional invariant fibre, namely the transform of  $E$ . This invariant fibre, however, has trivial holonomy.

*Proof of the Claim.* It is clear that by collapsing of  $\widetilde{\mathcal{P}^{-1}(q_0)}$ , the transform of  $E$  fits inside the previous fibration of  $F_n$  so that the resulting surface is again a  $\mathbb{CP}^1$ -bundle over  $\mathbb{CP}^1$  and, hence, a Hirzebruch surface. However, since the transform of the null section on  $\tilde{F}_n$  intersects  $\widetilde{\mathcal{P}^{-1}(q_0)}$  at a single point and transversely, its image on the resulting Hirzebruch yields a section of self-intersection  $-n+1$ . Thus, the surface obtained at the end of the procedure is, indeed,  $F_{n-1}$ , (c.f. [3]).

As to the foliation induced by  $\mathcal{D}$  on  $F_{n-1}$ , note that  $E \subset \tilde{F}_n$  is invariant by the blown up foliation  $\tilde{\mathcal{D}}$  since it comes from blowing up a regular point of  $\mathcal{D}$ . For the same reason,  $E$  contains exactly one singular point of  $\mathcal{D}$  so that the regular part of  $E$  (leaf of  $\tilde{\mathcal{D}}$ ) is simply connected and, hence, has trivial holonomy. The remainder of the claim follows immediately from these two remarks.  $\square$

By successively applying the previous construction, we obtain a Riccati foliation  $\mathcal{D}'$  on the surface  $F_1$  which satisfies the required conditions in terms of holonomy but which possesses finitely many additional invariant fibres, each of them carrying trivial holonomy. We just need, to finish the proof, to show how these invariant fibres with trivial holonomy can be “eliminated”. This means that the foliation  $\mathcal{D}'$  can be modified, without changing its local holonomy maps, to yield a new foliation for which the previous invariant fibre becomes a regular transverse fibre. The proof of this assertion, however, is rather simple. It is again a gluing procedure. Fix then one such fibre  $\mathcal{P}^{-1}(q)$ . By means of a local coordinate  $u$  on the base, we identify  $\mathcal{P}^{-1}(q)$  to the product  $D \times \mathbb{CP}^1$ . In turn, on  $D \times \mathbb{CP}^1$ , we consider the horizontal foliation  $\mathcal{H}$ . The proof consists, again, of gluing together the foliations  $\mathcal{D}'$  and  $\mathcal{H}$  over  $\mathcal{P}^{-1}(D^*)$  and  $D^* \times \mathbb{CP}^1$ . The construction of the gluing diffeomorphism  $h : D^* \times \mathbb{CP}^1 \rightarrow \mathcal{P}^{-1}(D^*)$  is, however, identical to the previous construction. In fact, the local holonomy maps arising from  $\mathcal{D}'$  and from  $\mathcal{H}$  around the fibres in question are both trivial so that the construction carried out above still applies. This ends the proof of Proposition (2.2.1).  $\square$

**Remark 2.2.2.** Note that the invariant fibres for the Riccati foliation  $\mathcal{F}$  constructed in Proposition 2.2.1 are all simple since the singular points of the foliation  $\mathcal{F}$  always have a non-zero eigenvalue associated with the direction transverse to the invariant fibre.

The chapter being mainly devoted to constructing foliations on the complex projective plane  $\mathbb{CP}^2$ , we need to make accurate what is understood by a *Riccati foliation on the surface*  $\mathbb{CP}^2$ . Since the blow up of the surface  $\mathbb{CP}^2$  at an arbitrary point leads to a surface isomorphic to the surface  $F_1$ , the definition becomes very straightforward.

**Definition 2.2.3.** A foliation on the complex projective plane  $\mathbb{CP}^2$  is said to be a *Riccati foliation* if it is obtained from a Riccati foliation  $\mathcal{F}$  on the first Hirzebruch surface  $F_1$  by collapsing the  $(-1)$ -rational curve in the surface  $F_1$ . In other words, a foliation  $\mathcal{F}_{\mathbb{P}}$  on the surface  $\mathbb{CP}^2$  is a *Riccati foliation* if it possesses a singular point whose blow up leads to a Riccati foliation on the surface  $F_1$ .

In what follows, we will mostly be interested in Riccati foliations on the complex projective plane  $\mathbb{CP}^2$  or on the first Hirzebruch surface  $F_1$ , whose global holonomy groups are certain specific subgroups of  $\mathrm{PSL}(2, \mathbb{C})$  along with a particular choice of generating set. Compared to the statement of Proposition 2.2.1, the choice of points  $p_1, \dots, p_k$  corresponding to the invariant fibres is, however, of little importance. Thus, fixed a group  $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$  along with a generating set  $\{M_1, \dots, M_{k-1}\}$ , the phrase *the Riccati foliation arising from  $\Gamma$  and  $\{M_1, \dots, M_{k-1}\}$*  will be used to refer to the Riccati foliation  $\mathcal{F}$  on the surface  $F_1$  obtained from Proposition 2.2.1 for an arbitrary choice of points  $p_1, \dots, p_k$ . Alternatively, the reader may fix a particular choice throughout the text, for example, setting  $p_1 = 0$ ,  $p_2 = 1$ ,  $p_3 = \infty$ , and then  $p_i = i$  for  $i = 4, \dots, k$ . Similarly, we will refer to the Riccati foliation on the surface  $\mathbb{CP}^2$  (arising from  $\Gamma$  and  $\{M_1, \dots, M_{k-1}\}$ ) meaning the foliation on the surface  $\mathbb{CP}^2$  obtained by blowing down the previous Riccati foliation on the surface  $F_1$ . In terms of notation, Riccati foliations on the surface  $F_1$  will typically be denoted by  $\mathcal{F}$  whereas  $\mathcal{F}_{\mathbb{P}}$  will stand for Riccati foliations on the surface  $\mathbb{CP}^2$ .

Let us close this section with a comment about the *degree of foliations* on the surface  $\mathbb{CP}^2$ . First, note that every homogeneous polynomial vector field  $X$  on  $\mathbb{C}^3$  induces a foliation on the surface  $\mathbb{CP}^2$  unless  $X$  is multiple of the radial vector field

$$R = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}.$$

In fact,  $X$  being homogeneous, the direction associated with the vector  $X(p)$ ,  $p \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}$  does not change over the radial line of  $\mathbb{C}^3$  passing through  $p$ . If, moreover,  $X$  is not a multiple of the radial vector field, then for a generic point  $p$  the vector  $X(p)$  induces a well defined direction on  $T_{q=P(p)}\mathbb{CP}^2$  where

$$P : \mathbb{C}^3 \setminus \{(0, 0, 0)\} \rightarrow \mathbb{CP}^2$$

stands for the canonical projection. In particular, the vector field  $X$  defines a singular foliation on  $\mathbb{CP}^2$ . Conversely, every singular holomorphic foliation on the surface  $\mathbb{CP}^2$  can be obtained out of a homogeneous polynomial vector field on  $\mathbb{C}^3$  by means of the preceding construction, (c.f. for example [30]). If, in addition, we require the zero set of the homogeneous vector field  $X$  to have codimension at least 2, then every pair of homogeneous vector fields  $X$  and  $X'$  inducing the same foliation on the surface  $\mathbb{CP}^2$  must have the same degree. Thus, the following definition makes sense:

**Definition 2.2.4.** The *degree of a foliation  $\mathcal{F}$  on the surface  $\mathbb{CP}^2$*  is the degree of a homogeneous polynomial vector field  $X$  on  $\mathbb{C}^3$  having singular set of codimension at least 2 and inducing the foliation  $\mathcal{F}$  on the surface  $\mathbb{CP}^2$  by means of the canonical projection  $\mathbb{C}^3 \setminus \{(0, 0, 0)\} \rightarrow \mathbb{CP}^2$ .

In closing this section, let us mention that the degree of a foliation  $\mathcal{F}$  on the surface  $\mathbb{CP}^2$  can alternatively be defined as the number of tangencies of  $\mathcal{F}$  with a generic projective line in the surface  $\mathbb{CP}^2$ , c.f. [30] and [45]. Finally, in the special case of Riccati foliations the following lemma is well known and its straightforward proof can safely be left to the reader.

**Lemma 2.2.5.** *Let  $\mathcal{F}$  be a Riccati foliation on the surface  $F_1$  and denote by  $\mathcal{F}_{\mathbb{P}}$  the corresponding Riccati foliation on the surface  $\mathbb{CP}^2$ . Assume that  $\mathcal{F}$  has exactly  $k$  invariant fibres, all of them being simple. Then the degree of  $\mathcal{F}_{\mathbb{P}}$  as a foliation on the surface  $\mathbb{CP}^2$  equals  $k - 1$ .*

## 2.3 Examples of Riccati Foliations and Theorem B

The purpose of this section is to detail two applications of Proposition 2.2.1. First, we will show how examples of foliations of degree 2 on the surface  $\mathbb{CP}^2$  satisfying the conditions of Theorem A with real-analytic Levi-flats can be obtained. Similarly, we will also construct foliations of degree 3 as in Theorem A having transversely fractal Levi-flats. Then we will turn to Theorem B and apply again Proposition 2.2.1 to construct the corresponding foliations.

### 2.3.1 Two Examples of Riccati Foliations Having Degrees 2 or 3

Let us start with *triangular groups* or more precisely, with triangular groups *without cusps*. This is as follows. Let  $m_i \in \mathbb{Z}_+^*$ ,  $i = 1, 2, 3$ , be three positive integers satisfying the condition

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1, \quad (2.2)$$

and set  $m = (m_1, m_2, m_3)$ . It is well known that, up to conjugation, there is exactly one triplet  $(\xi_1, \xi_2, \xi_3)$  of elements in  $\mathrm{PSL}(2, \mathbb{C})$  verifying the relations

$$\xi_1 \xi_2 \xi_3 = \xi_1^{m_1} = \xi_2^{m_2} = \xi_3^{m_3} = \mathrm{id}. \quad (2.3)$$

The subgroup  $\Gamma$  of  $\mathrm{PSL}(2, \mathbb{C})$  generated by  $\xi_1$ ,  $\xi_2$ , and  $\xi_3$  is, therefore, unique up to inner automorphisms of  $\mathrm{PSL}(2, \mathbb{C})$ . The group  $\Gamma$  is called the *triangular group associated with the unordered triplet  $m = (m_1, m_2, m_3)$* . These triangular groups happen to be discrete and conjugate to a subgroup of  $\mathrm{PSL}(2, \mathbb{R})$ . In other words, triangular groups  $\Gamma$  as above are Fuchsian groups. Clearly, the quotient of the hyperbolic disc  $\mathbb{D}$  by  $\Gamma$  is naturally a *spherical orbifold* with three singular points, (c.f. for example [31]).

**Example 2.** Choose  $m = (m_1, m_2, m_3)$  with  $m_i \in \mathbb{Z}_+^*$ ,  $i = 1, 2, 3$ , satisfying condition (2.2) and denote by  $\mathcal{F}^{(m)}$  the Riccati foliation on the surface  $F_1$  obtained by applying Proposition 2.2.1 to the group  $\Gamma$  with generating set  $\{\xi_1, \xi_2, \xi_3\}$ .

The first family of Riccati foliations on the surface  $\mathbb{CP}^2$  to be considered here is, therefore,  $\mathcal{F}_{\mathbb{P}}^{(m)}$ . In other words, for each fixed  $m$ ,  $\mathcal{F}_{\mathbb{P}}^{(m)}$  is the foliation obtained from  $\mathcal{F}^{(m)}$  by collapsing the  $(-1)$ -curve in the surface  $F_1$ .

Note that the entire family of foliations  $\mathcal{F}_{\mathbb{P}}^{(m)}$  is constituted by foliations of degree 2 as it follows from the combination of Remark 2.2.2 and Lemma 2.2.5. Alternatively, we can consider the 3-parameter



family

$$\begin{aligned} X_{(\alpha_1, \alpha_2, \alpha_3)} &= [\alpha_1 z_1^2 + (1 - \alpha_1)(z_1 z_2 + z_1 z_3 - z_2 z_3)] \frac{\partial}{\partial z_1} \\ &\quad + [\alpha_2 z_2^2 + (1 - \alpha_2)(z_1 z_2 - z_1 z_3 + z_2 z_3)] \frac{\partial}{\partial z_2} \\ &\quad + [\alpha_3 z_3^2 + (1 - \alpha_3)(-z_1 z_2 + z_1 z_3 + z_2 z_3)] \frac{\partial}{\partial z_3}, \end{aligned}$$

of quadratic vector fields on  $\mathbb{C}^3$  which corresponds to the family of Halphen vector fields thoroughly studied by A. Guillot in [27]. In particular, the existence of a Levi-flat for the foliation  $\mathcal{F}_{\mathbb{P}}^{(m)}$  is already pointed out in [27]. As a family of homogeneous vector fields of degree 2 on  $\mathbb{C}^3$ , the foliations they induce on the surface  $\mathbb{CP}^2$  are all of degree 2. To obtain the foliation  $\mathcal{F}_{\mathbb{P}}^{(m)}$  it suffices to choose the parameters  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  so that

$$m_i = \frac{\alpha_1 + \alpha_2 + \alpha_3 - 2}{\alpha_i}.$$

**Remark 2.3.1.** A case of Halphen vector fields also discussed in [27] and escaping the setting of Theorem A occurs for  $\alpha_1 = -6$ ,  $\alpha_2 = -4$ , and  $\alpha_3 = 0$ . In this case, the resulting triangular group acquires a cusp and, in fact, is isomorphic to  $\mathrm{PSL}(2, \mathbb{Z})$ . In other words,  $\xi_1$  and  $\xi_2$  are elliptic of orders 2 and 3 but  $\xi_3$  is *parabolic*. This example still possesses a singular Levi-flat whose structure is, however, different from what is described in Theorem A. Also the description of the corresponding positive foliated harmonic currents requires significantly different arguments and will be discussed in a future work.

We can now proceed with an example of Riccati foliation of degree 3 on the surface  $\mathbb{CP}^2$  giving rise to a transversely fractal Levi-flat.

**Example 3.** In this example, we begin by considering a spherical orbifold with 4 singular points (of orbifold type). We then consider the Fuchsian group  $\Gamma'$  arising from this orbifold by means of Riemann uniformisation theorem. In slightly more concrete terms, we can use Poincaré theorem to ensure the existence of these orbifolds. In fact, if  $m_1$ ,  $m_2$ ,  $m_3$  and  $m_4$  are the orders of the orbifold-type singular points, then the condition for the existence of the desired Fuchsian group  $\Gamma'$  becomes (c.f. for example [31])

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4} < 2. \quad (2.4)$$

The construction of one such Fuchsian group is simple enough to be recalled. Consider four radii issued from  $0 \in \mathbb{D} \subset \mathbb{C}$  with adjacent radii defining an angle of  $\pi/2$  (for example  $\mathbb{R}_+$ ,  $i\mathbb{R}_+$ ,  $\mathbb{R}_-$  and  $i\mathbb{R}_-$ ). Now for  $\tau \in (0, 1)$ , choose a point in each radii such that its Euclidean distance to  $0 \in \mathbb{D} \subset \mathbb{C}$  is  $\tau$ . Next define a quadrilateral  $\mathcal{Q}$  by joining points lying in adjacent sides by segments of hyperbolic geodesic. The sides of  $\mathcal{Q}$  are denoted by  $l_1$ ,  $l_2$ ,  $l_3$  and  $l_4$ . Then on top of each side  $l_i$  we construct an isosceles triangle whose angle opposed to the side  $l_i$ , i.e., the angle defined by the two sides of the triangle having the same length, equals  $2\pi/m_i$ . The result of this construction is, therefore, a star-shaped hyperbolic octagon whose sides are labeled  $\xi_1, \xi'_1, \dots, \xi_4, \xi'_4$  (for the orientation, c.f. figure 17 on page 93 of [31]). Furthermore, by considering the asymptotic values of the area of this octagon when  $\tau \rightarrow 0$  and  $\tau \rightarrow 1$ , it becomes clear that  $\tau$  can be chosen so that the area in question equals

$$2\pi \left[ 2 - \sum_{i=1}^4 \frac{1}{m_i} \right]. \quad (2.5)$$

By construction, the sides  $\xi_i$  and  $\xi'_i$  have the same length ( $i = 1, 2, 3, 4$ ). In particular, there are orientation-preserving automorphisms  $H_i$  of  $\mathbb{D}$  such that  $H_i(\xi'_i) = \xi_i$ . Let then  $\Gamma' \subset \text{Aut}(\mathbb{D}) \simeq \text{PSL}(2, \mathbb{R}) \subset \text{PSL}(2, \mathbb{C})$  generated by  $H_1, H_2, H_3$  and  $H_4$ . The conditions of Poincaré's polygon theorem (c.f. [31], page 92) are, hence, matched so that  $\Gamma'$  is discrete, and, hence, Fuchsian, moreover the quotient of  $\mathbb{D}$  by  $\Gamma'$  identifies with the surface (orbifold) obtaining by following the above indicated gluing of the sides of the octagon. It is straightforward checking that this quotient is topologically a sphere with four marked (singular) points corresponding to the vertices of the octagon arising from the isosceles triangles (the vertices with angles  $2\pi/m_i$ ). This completes the construction of a Fuchsian group  $\Gamma'$  with the desired properties.

The Kleinian group  $\Gamma$  desired for this example is not the Fuchsian group  $\Gamma'$  but rather some *quasi-conformal* deformation of  $\Gamma'$ . Recall that a discrete subgroup of  $\text{PSL}(2, \mathbb{C})$  is said to be quasifuchsian if it leaves invariant a Jordan curve  $\mathcal{J} \subset \mathbb{CP}^1$  which is not a real projective circle (in which case the group would be Fuchsian). In particular, the curve  $\mathcal{J}$  will be nowhere differentiable and of Hausdorff dimension strictly greater than 1 owing to [7].

Now, whereas the theory of quasiconformal deformations and Bers Simultaneous Uniformisation theorem are better known in the case of Riemann surfaces, similar statements still hold in the case of Riemann surface orbifolds. The reader is referred to [40], Section 4.4, for the deformation theory of Fuchsian groups containing elliptic elements. In particular, there follows that the Teichmüller space of these spherical orbifolds with four singular points, or equivalently the Teichmüller space of a Fuchsian group as  $\Gamma'$  above, is of complex dimension 1.

Let then  $R$  denote the spherical orbifold arising from the Fuchsian group  $\Gamma'$ . Bers Simultaneous Uniformisation theorem assigns to each point  $p$  in the Teichmüller space of  $R$  a quasifuchsian group  $\Gamma$ , unique up to conjugation, satisfying the following conditions:

- The group  $\Gamma$  leaves invariant a Jordan curve  $\mathcal{J}$ . The connected components of  $\mathbb{CP}^1 \setminus \mathcal{J}$  are then denoted by  $D_1$  and  $D_2$ .
- The quotient  $D_1/\Gamma$  is isomorphic to the orbifold  $R$  whereas the quotient  $D_2/\Gamma$  is isomorphic to the spherical orbifold parametrised by the point  $p$ .

Summarising what precedes, fix four positive integers  $m_1, m_2, m_3$  and  $m_4$  satisfying estimate (2.4), there exists a complex 1 parameter family of quasifuchsian groups  $\Gamma$ , pairwise non-conjugate, and parametrising spherical orbifolds with four singular points whose orders are precisely  $m_1, m_2, m_3$  and  $m_4$ .

Finally, for our example of Riccati foliations with transversely fractal Levi-flats, we consider the family of Riccati foliations  $\mathcal{F}^Q$  on the surface  $F_1$  arising from the above family of Fuchsian groups by means of Proposition 2.2.1. Then the desired examples arise as the corresponding Riccati foliations  $\mathcal{F}_{\mathbb{P}}^Q$  on the surface  $\mathbb{CP}^2$  obtained by collapsing the  $(-1)$ -rational curve in the surface  $F_1$ . In particular, the global holonomy of the foliations  $\mathcal{F}_{\mathbb{P}}^Q$  is given by the mentioned quasifuchsian groups. The reader will also notice that all the foliations  $\mathcal{F}_{\mathbb{P}}^Q$  are of degree 3 as follows from Lemma 2.2.5 since they have exactly four invariant lines, all of them being simple.

### 2.3.2 High Degree Foliations with Diffuse Positive foliated Closed Currents

In this section we will prove Theorem B by building on some of the previous ideas related to Riccati foliations and to Proposition 2.2.1. In the course of the section, we will also accurately formulate several notions involving currents and foliations appearing both in Theorems A and B.

Let us begin with some basic notions about currents on a compact complex surface  $M$ . Let  $\mathcal{D}^k$  denote the space of (smooth) differential forms of degree  $k$  on the surface  $M$ . The space  $\mathcal{D}'_k$  of *currents of dimension  $k$*  (where  $0 \leq k \leq 4$ ) is the  $C^\infty$ -topological dual of  $\mathcal{D}^k$ . The *degree* of a current  $T$  is also defined as the difference between the real dimension of  $M$  and the dimension of  $T$ . In local coordinates, a current  $T$  of dimension  $k$  acts as a  $(4-k)$ -form on the coefficients of a  $k$ -differential form  $\omega$ . In fact, a current  $T$  of dimension  $k$  can be represented as a  $(4-k)$ -differential form with distributional coefficients.

The exterior differential operator  $d$ , as well as the standard operators  $\partial$  and  $\bar{\partial}$ , are defined on currents by letting  $\langle dT, \omega \rangle = (-1)^{k'+1} \langle T, d\omega \rangle$ , where  $T$  is a current of degree  $k'$  and  $\omega$  a  $(3-k')$ -differential form. The action of  $\partial$  and  $\bar{\partial}$  on currents is analogously defined. In particular, the space of currents can be stratified with bidegrees  $p, q$ . A current  $T$  will also be called *closed* if  $dT = 0$  and it will be called *harmonic* if  $i\partial\bar{\partial}T = 0$ .

The complex dimension of  $M$  being 2, we will particularly be interested in  $(1,1)$ -currents. Recall that a  $(1,1)$ -form is said to be *weakly positive* if locally it takes on the form

$$\sum_{i,j=1}^2 \alpha_{i,j} dz_i \wedge d\bar{z}_j,$$

with the matrix  $\{\alpha_{i,j}\}$  being positive semi-definite at every point. In turn, a  $(1,1)$ -current  $T$  is said to be *positive* if the coupling  $T \wedge \omega$  is a positive measure for every weakly positive form  $\omega$ .

Assume now that  $M$  is endowed with a (singular) holomorphic foliation  $\mathcal{F}$ . Then, among all currents on  $M$ , we may look for those providing special insight in the structure of the foliation  $\mathcal{F}$ . This gives rise to the notion of *foliated* or *directed* current.

**Definition 2.3.2.** *Let  $M$  be a complex compact surface equipped with a singular holomorphic foliation  $\mathcal{F}$ . A current  $T$  on the surface  $M$  is said to be foliated, or directed by  $\mathcal{F}$ , if we have  $T(\beta) = 0$  whenever  $\beta$  is a 2-form vanishing on the tangent space of the foliation  $\mathcal{F}$ .*

In foliated coordinates  $(z_1, z_2)$  where the foliation  $\mathcal{F}$  is given by  $\{dz_2 = 0\}$ , every 2-form  $\beta$  vanishing on the tangent space of the foliation  $\mathcal{F}$  must have the form  $\alpha_1 dz_2 + \alpha_2 d\bar{z}_2$  for suitable 1-forms  $\alpha_1, \alpha_2$ . From this, we see that a  $(1,1)$ -foliated current  $T$  must take on the local form

$$T = T(z_1, z_2) dz_2 \wedge d\bar{z}_2,$$

where  $T$  is identified with a distribution. In particular,  $T$  is of type  $(1,1)$ . Moreover, in the context of foliated currents, the notion of being a *positive current* becomes particularly clear, it simply means that  $T(z_1, z_2)$  is identified with a positive measure.

Consider now the positive foliated form  $T = T(z_1, z_2) dz_2 \wedge d\bar{z}_2$ . If, in addition,  $T$  is *closed*, then the distributional derivative of  $T(z_1, z_2)$  with respect to  $z_1$  and  $\bar{z}_1$  must vanish. This means that  $T(z_1, z_2)$  is essentially constant over the plaques of the foliation  $\mathcal{F}$  or, equivalently, that  $T(z_1, z_2)$  depends only on  $z_2$  and, hence, it naturally induces a positive measure on the transverse space (a particular case of Rokhlin disintegration).

Similarly, if the current  $T$  is *harmonic*, then  $T(z_1, z_2)$  yields a harmonic function over the plaques of the foliation  $\mathcal{F}$  which is bounded above and below by strictly positive constants. By Rokhlin disintegration, the current  $T$  still defines a measure on the transverse space which is referred to as the *harmonic measure*.

According to Sullivan [46], in the case of a regular foliation defined on a compact manifold, positive foliated closed currents are in one-to-one correspondence with *transversely invariant measures* as defined by Plante [43] and recalled below.

**Definition 2.3.3.** *Given a regular foliation  $\mathcal{F}$  on a compact complex manifold  $M$ , a transversely invariant measure for the foliation  $\mathcal{F}$  consists of the following data:*

- (1) *A (finite) foliated atlas  $\{(U_i, \varphi_i, \Sigma_i)\}$ , where  $\Sigma_i$  stands for the transverse space.*
- (2) *For every  $i$ , a finite measure  $\mu_i$  defined on the transverse space  $\Sigma_i$ .*
- (3) *Whenever  $U_i \cap U_j \neq \emptyset$ , the “gluing map”  $h_{ij}$  defined by  $\varphi_j \circ \varphi_i^{-1}(z_1, z_2) = (f(z_1, z_2), h_{ij}(z_2))$  satisfies  $h_{ij}^* \mu_j = \mu_i$  whenever both sides are defined.*

In the case of a singular foliation, we simply repeat the above definition on the open manifold  $M \setminus \text{Sing}(\mathcal{F})$  while dropping the condition on finiteness of the foliated atlas  $\{(U_i, \varphi_i, \Sigma_i)\}$ . It is then clear that a (positive foliated) closed current still yields a transversely invariant measure for the foliation  $\mathcal{F}$  in the sense of Definition 2.3.3. Only the converse, i.e., the building of such a current out of a given transversely invariant measure, needs further attention.

We are now ready to begin the proof of Theorem B. Fix  $n \in \mathbb{N}$  and consider  $\mathbb{CP}^1$  with  $(n+1)$  marked points  $b_1, \dots, b_{n+1}$ . Fix also an elliptic element  $\xi \in \text{PSL}(2, \mathbb{C})$  representing an irrational rotation and let  $\Gamma$  be the subgroup generated by  $\xi$ . Let  $\tilde{\mathcal{F}}_n$  be the Riccati foliation on the surface  $F_1$  arising from Proposition 2.2.1 with  $(n+1)$  simple invariant fibres  $C_1, \dots, C_{n+1}$ . More precisely, the local holonomy map of the foliation  $\tilde{\mathcal{F}}_n$  around each of the invariant fibres  $C_1, \dots, C_n$  coincides with  $\xi$  whereas the local holonomy map obtained by winding around  $C_{n+1}$  coincides with  $\xi^{-n}$ . Moreover, each invariant fibre  $C_i$  of the foliation  $\tilde{\mathcal{F}}_n$  contains exactly two singular points of this foliation, namely: one point denoted by  $p_i$  where the quotient of the eigenvalues of the foliation  $\tilde{\mathcal{F}}_n$  is positive real (a “sink”) and a point denoted by  $q_i$  where this quotient is negative real (a “saddle”). Thus, the foliation  $\tilde{\mathcal{F}}_n$  possesses  $2n+2$  singular points and all of them are simple. It follows from the local models for the foliation  $\tilde{\mathcal{F}}_n$  around the invariant fibres  $C_i$  described in the proof of Proposition 2.2.1 that each point  $p_i$  (resp.  $q_i$ ) admits exactly two separatrices. Furthermore, one of these separatrices coincides with the invariant fibre  $C_i$  while the other separatrix, which is also smooth, happens to be transverse to the invariant fibre  $C_i$ . In the sequel, we let  $S_{p_i}$  (resp.  $S_{q_i}$ ) denote the separatrix of  $\tilde{\mathcal{F}}_n$  at the singular point  $p_i$  (resp.  $q_i$ ) that is transverse to the invariant fibre  $C_i$ .

**Lemma 2.3.4.** *We can assume, without loss of generality, that all the separatrices  $S_{p_i}$  (resp.  $S_{q_i}$ ) glue together into a global rational curve of self-intersection 1 (resp.  $-1$ ) invariant by the foliation  $\tilde{\mathcal{F}}_n$ .*

*Proof.* Recall that the global holonomy group  $\Gamma$  of the foliation  $\tilde{\mathcal{F}}_n$  is generated by  $\xi$  (i.e.,  $\Gamma = \langle \xi \rangle$ ) and that, up to conjugation, the action of  $\Gamma$  on  $\mathbb{CP}^1$  can be represented in any fibre of the surface  $F_1$  different from the invariant fibres  $C_1, \dots, C_{n+1}$ .

Let  $C_i$  be a invariant fibre and consider a generic (non-invariant) fibre  $\Sigma$  near to  $C_i$  in the following sense: The local separatrices  $S_{p_i}, S_{q_i}$  associated with the singular points  $p_i, q_i$  intersect  $\Sigma$  transversely at well defined points  $x_{p_i}$  and  $x_{q_i}$ . Clearly, the points  $x_{p_i}, x_{q_i}$  are fixed by the automorphism of  $\Sigma$  obtained as the local holonomy around  $C_i$ . This construction, holds for every  $i = 1, \dots, n+1$ . Now, using parallel transport, we represent all of these points in the same fibre  $\Sigma$ , so they must coincide (up to order) into two well defined set points which, incidentally, are the fixed points of (a conjugate of)  $\xi$ . Indeed, all the local holonomy maps around invariant fibres coincide (strictly speaking except for the fibre  $C_{n+1}$  which coincides with the  $n^{\text{th}}$ -power of the previous ones). In other words, given any two invariant fibres  $C_i$  and  $C_j$ , either we have  $x_{p_i} = x_{p_j}$  and  $x_{q_i} = x_{q_j}$  or we have  $x_{p_i} = x_{q_j}$  and  $x_{q_i} = x_{p_j}$ . However, we can assume without loss of generality that the first possibility holds. Indeed, this happens thanks to the local models

for the foliation  $\tilde{\mathcal{F}}_n$  around the invariant fibres  $C_i$  described in the proof of Proposition 2.2.1: the position of the singular points can be permuted without affecting the holonomy. Thus, we can ensure that all the separatrices  $S_{p_i}$ ,  $i = 1, \dots, n+1$ , glue together so as to define a invariant compact curve  $R_p$ . Similarly, the separatrices  $S_{q_i}$ ,  $i = 1, \dots, n+1$ , yield another invariant compact curve denoted by  $R_q$ . Due to the transverse nature of the Riccati foliation with respect to the fibres of the surface  $F_1$ , it is immediate to conclude that both  $R_p$  and  $R_q$  are sections of the surface  $F_1$ . However, since the eigenvalues of the foliation  $\tilde{\mathcal{F}}_d$  at the singular points  $p_i$  (resp.  $q_i$ ) are positive (resp. negative) real numbers, there follows from Camacho-Sad index theorem [9] that  $R_p$  (resp.  $R_q$ ) has positive (resp. negative) self-intersection. Finally, it is well known that the surface  $F_1$  possesses only two sections which, incidentally, are rational curves of self-intersection 1 and  $-1$ , (c.f. [3]). The lemma follows.  $\square$

Again let  $\Sigma$  be a generic fibre of the surface  $F_1$ . The generator  $\xi$  of the global holonomy group  $\Gamma$  of the foliation  $\tilde{\mathcal{F}}_n$  being an elliptic element of infinite order, there exists a continuum of circles contained in  $\Sigma$  that are invariant under  $\xi$ . Fix one of these invariant circles and denote it by  $S^1$ . Note that the normalised Lebesgue measure on  $S^1$  is invariant under  $\xi$ . Furthermore, the points  $x_p$  and  $x_q$  where the section  $R_p$  and  $R_q$  intersect  $\Sigma$  are the fixed points of  $\xi$  and, therefore, lie away from  $S^1$ .

Let  $\overline{\mathcal{F}(S^1)}$  be the closure of the saturated set  $\mathcal{F}(S^1)$  of the invariant circle  $S^1$  by the foliation  $\tilde{\mathcal{F}}_n$ . We have first to show that  $C_i \cap \overline{\mathcal{F}(S^1)} = \{p_i\}$ . To do so, let us begin with the following claim:  $C_i \cap \overline{\mathcal{F}(S^1)} \subset \{p_i, q_i\}$ . To prove the claim we argue by contradiction. Assume it is false. Then  $\mathcal{F}(S^1)$  accumulates on a regular point of the invariant fibre  $C_i$ . Since the foliation  $\tilde{\mathcal{F}}_n$  has a (linearisable) saddle singular point at  $q_i$ , the fact that  $\mathcal{F}(S^1)$  accumulates on a regular point of the invariant fibre  $C_i$  implies that  $\mathcal{F}(S^1)$  must accumulate at regular points of the separatrix  $S_{q_i}$  as well. However, the separatrix  $S_{q_i}$  intersects the generic fibre  $\Sigma$  at a fixed point of  $\xi$  that, in turn, is accumulated by points in  $\mathcal{F}(S^1)$ . However, this is impossible since the intersection of  $\mathcal{F}(S^1)$  and  $\Sigma$  is the initial invariant circle  $S^1$  which lies away from the fixed points of  $\xi$ . The resulting contradiction then establishes the claim.

Next, note that the above argument actually shows that the separatrix  $S_{q_i}$  cannot be contained in  $\overline{\mathcal{F}(S^1)}$ . Thus,  $\mathcal{F}(S^1)$  cannot accumulate on any of the two separatrices of the foliation  $\tilde{\mathcal{F}}_n$  at the singular point  $q_i$ . Since the singular point  $q_i$  lies in the Siegel domain, this implies that  $\mathcal{F}(S^1)$  cannot accumulate on the singular point  $q_i$  itself. Hence, the intersection  $C_i \cap \overline{\mathcal{F}(S^1)}$  is reduced to  $\{p_i\}$  since it cannot be empty. We have then proved that  $C_i \cap \overline{\mathcal{F}(S^1)} = \{p_i\}$  as required.

For each  $i \in \{1, \dots, n+1\}$ , let  $B_i \subset F_1$  be a small neighbourhood around the singular point  $p_i$  along with coordinates  $(z_1^i, z_2^i)$  where the restriction of the foliation  $\tilde{\mathcal{F}}_n$  to  $B_i$  is linear, i.e., the foliation  $\tilde{\mathcal{F}}_n$  is locally given by a vector field having the form

$$z_1^i \frac{\partial}{\partial z_1^i} + \lambda z_2^i \frac{\partial}{\partial z_2^i},$$

where  $\lambda \in \mathbb{R}^+ \setminus \mathbb{Q}$ . Similarly, for each  $B_i$ ,  $i = 1, \dots, d+1$ , let  $V_i \subset B_i$  be a (smaller) neighbourhood of  $p_i$  properly contained in  $B_i$ . The subset  $\overline{\mathcal{F}(S^1)}$  being contained in the regular part of the foliation  $\tilde{\mathcal{F}}_n$  on the complement of  $\bigcup V_i$ , consider a covering of  $F_1 \setminus \bigcup V_i$  by foliated coordinates  $\{(U_j, \varphi_j, \Sigma_j)\}_{j=1}^N$ . In fact, in what follows, it would be enough to consider the restriction of  $\tilde{\mathcal{F}}_n$  to  $\overline{\mathcal{F}(S^1)}$  as a (singular) lamination by Riemann surfaces, (c.f. for example [23]).

Before continuing, recall that a positive foliated closed current is said to be *algebraic* if it coincides with a constant multiple of the current given by integration over a (possibly singular) algebraic invariant curve. Otherwise, the current is said to be *diffuse*. Similarly, given a (non-constant) holomorphic map

$f : \mathbb{C} \rightarrow \mathbb{CP}^2$ , an Ahlfors current is a positive *closed* current of dimension  $(1, 1)$  which is the limit of a sequence

$$\frac{f_*[D_{r_j}]}{\text{Area}(f(D_{r_j}))}$$

where  $r_j \rightarrow \infty$ . Here  $[D_{r_j}]$  stands for the integration current over the disc  $D_{r_j} \subset \mathbb{C}$  of radius  $r_j$  and the *area*  $\text{Area}(f(D_{r_j}))$  is nothing but the area of  $f(D_{r_j})$  computed with respect to any auxiliary Riemannian metric. Alternatively,  $\text{Area}(f(D_{r_j}))$  can be defined as the integral over  $D_{r_j}$  of the pull-back by  $f$  of the standard Fubini-Study form on the surface  $\mathbb{CP}^2$ .

In what follows, a  $(1, 1)$ -positive foliated closed current  $\tilde{T}$  on the surface  $F_1$  that, in addition, is diffuse and of Ahlfors type will be constructed. Basically the construction will be carried out in two steps due to the presence of singular points. To begin, consider a  $(1, 1)$ -differential form  $\omega$  on the surface  $F_1$  and let it be decomposed as  $\omega = \omega_1 + \omega_2$  where  $\omega_1$  and  $\omega_2$  are  $(1, 1)$ -differential forms supported on  $F_1 \setminus \bigcup V_i$  and  $\bigcup B_i$ , respectively.

First, we are going to define  $\tilde{T}$  for forms like  $\omega_1$ , i.e., forms whose supports are contained in  $F_1 \setminus \bigcup V_i$ . This is very much the general construction in [46] since singular points play no role. Indeed, to define  $\tilde{T}$  in this case we proceed as follows. Since the singular points of the foliation  $\tilde{\mathcal{F}}_n$  lie away from the support of  $\omega_1$  fix a partition of the unity  $(a_j)_{j=1}^N$  strictly subordinate to the cover  $(U_j)_{j=1}^N$  of  $F_1 \setminus \bigcup V_i$ . The product  $a_j \omega_1$  is identified with a continuous function on  $\Sigma_j$ . In fact, we can consider a function  $f_j : \Sigma_j \rightarrow \mathbb{C}$  defined by

$$z_2 \mapsto f_j(z_2) = \int_{P_{z_2}} a_j \omega_1(z_2),$$

in terms of foliated local coordinates  $(z_1, z_2)$  in  $U_j$ , the plaques of the restriction of  $\tilde{\mathcal{F}}_n$  to  $U_j$  are of the form  $P_{z_2} \subset \mathbb{C} \times \{z_2\}$  while the transverse section  $\Sigma_j$  becomes  $\Sigma_j \subset \{z_1 = 0\}$ . Then we integrate and sum these functions obtaining the value

$$\tilde{T}(\omega_1) = \sum_{j=1}^N \left( \int_{\Sigma_j} f_j(z_2) d\mu_j(z_2) \right) = \sum_{j=1}^N \int_{\Sigma_j} \left( \int_{P_{z_2}} a_j \omega_1(z_2) \right) d\mu_j(z_2),$$

where  $\{\mu_j\}$  is the transverse invariant measure induced by the normalised Lebesgue measure on  $S^1$ .

Now we have to define  $\tilde{T}$  for forms like  $\omega_2$ , i.e., those whose supports are contained in  $\bigcup B_i$ . Without loss of generality, we can assume that the support of  $\omega_2$  is contained in  $B_1$ . Consider the restriction  $\tilde{\mathcal{F}}_n|_{B_1}$  of  $\tilde{\mathcal{F}}_n$  to  $B_1$  and let  $D_1$  be a transverse section to this foliation. The invariant circle  $S^1 \subset \Sigma$  induces a circle  $S_1^1 \subset D_1$  which is invariant under the holonomy of  $\tilde{\mathcal{F}}_n|_{B_1}$ . Next, define  $\Phi : \mathcal{S} \times S_1^1 \rightarrow B_1$  by

$$\Phi(t, y) = (e^t, ye^{\lambda t}),$$

where  $\mathcal{S} = \{t \in \mathbb{C} : \text{Re}(t) \leq 0, 0 \leq \text{Im}(t) \leq 2\pi\}$ . The leaves of the foliation  $\tilde{\mathcal{F}}_n|_{B_1}$  are then parametrised by  $\Phi$ . More precisely, for  $y \in S_1^1$  fixed, let  $\Phi_y : \mathcal{S} \rightarrow B_1$  be defined by  $\Phi_y(t) = \Phi(t, y)$ . Then set  $L_y = \{\Phi_y(t) : t \in \mathcal{S}\}$ . Clearly, each  $L_y$  is contained in a leaf of  $\tilde{\mathcal{F}}_n|_{B_1}$  and the union of these pieces of leaves over an orbit of the holonomy group yields all the leaf in question. Hence, to have  $\tilde{T}$  well defined, it suffices to show that the (improper) double integral

$$\int_{S_1^1} \left( \int_{L_y} \omega_2 \right) d\mu_1(y) \tag{2.6}$$

converges. In turn, owing to Dominated convergence and to Tonelli theorem, this double integral converges provided that the improper integral

$$\int_{L_y} \omega_2 = \int_S \Phi_y^* \omega_2$$

is uniformly bounded (independent of  $y \in S_1^1$ ). Let us then show that the latter integral is, indeed, uniformly bounded on  $y$ .

To do so, recall first that  $B_1$  is equipped with coordinates  $(z_1^1, z_2^1)$  where the foliation  $\tilde{\mathcal{F}}_n$  is given by the vector field  $z_1^1 \partial / \partial z_1^1 + \lambda z_2^1 \partial / \partial z_2^1$  with  $\lambda \in \mathbb{R}^+ \setminus \mathbb{Q}$  irrational. In particular,  $L_y$  is parametrised by  $\Phi_y(t) = (e^t, ye^{\lambda t}) = (z_1^1, z_2^1)$ ,  $t \in \mathcal{S}$ . (where  $y \in S_1^1$ ). Hence,  $dz_1^1 = e^t dt$ ,  $dz_2^1 = \lambda ye^{\lambda t} dt$ ,  $d\bar{z}_1^1 = e^{\bar{t}} d\bar{t}$ , and  $d\bar{z}_2^1 = \lambda \bar{y} e^{\lambda \bar{t}} d\bar{t}$ . Letting

$$\omega_2 = \sum_{s,k=1}^2 a_{sk} dz_s^1 \wedge d\bar{z}_k^1,$$

we conclude that

$$\Phi_y^* \omega_2 = \left( e^t e^{\bar{t}} a_{11} + \lambda \bar{y} e^t e^{\lambda \bar{t}} a_{12} + \lambda y e^{\bar{t}} e^{\lambda t} a_{21} + \lambda^2 |y|^2 e^{\lambda t} e^{\lambda \bar{t}} a_{22} \right) dt \wedge d\bar{t} = J(t, y) dt \wedge d\bar{t}.$$

Where, in slightly more explicit terms,  $J(t, y)$  is given by

$$e^{2\operatorname{Re}(t)} a_{11} + \lambda \bar{y} e^{(\lambda+1)\operatorname{Re}(t)+i(-\lambda+1)\operatorname{Im}(t)} a_{12} + \lambda y e^{(\lambda+1)\operatorname{Re}(t)+i(\lambda-1)\operatorname{Im}(t)} a_{21} + \lambda^2 |y|^2 e^{2\lambda\operatorname{Re}(t)} a_{22}.$$

Since  $\lambda$  is positive real, there follows that the integrals over  $\mathcal{S}$  of each of the functions

$$e^{2\operatorname{Re}(t)}, \quad e^{(\lambda+1)\operatorname{Re}(t)+i(-\lambda+1)\operatorname{Im}(t)}, \quad e^{(\lambda+1)\operatorname{Re}(t)+i(\lambda-1)\operatorname{Im}(t)} \quad \text{and} \quad e^{2\lambda\operatorname{Re}(t)}$$

are all absolutely convergent. Furthermore,  $|y|$  is also uniformly bounded since  $y \in S_1^1$  and all the coefficients  $a_{sk}$  are also bounded since they are identified with  $C^\infty$  functions supported in  $B_1$ . Thus, the integral in (2.6) is convergent. Hence the coupling of  $\tilde{T}$  and  $\omega_2$  given by

$$\tilde{T}(\omega_2) = \sum_{j=1}^{d+1} \int_{S_j^1} \left( \int_{L_y} \omega_2 \right) d\mu_j(y)$$

is well defined on forms  $\omega_2$  whose support is contained in the union of the sets  $B_i$ ,  $i = 1, \dots, d+1$ . Now, the formula

$$\tilde{T}(\omega) = \tilde{T}(\omega_1 + \omega_2) = \sum_{s=1}^N \int_{\Sigma_s} \left( \int_{P_{z_2}} a_s \omega_1(z_2) \right) d\mu_s(z_2) + \sum_{k=1}^{d+1} \int_{S_k^1} \left( \int_{L_y} \omega_2 \right) d\mu_k(y)$$

yields a well defined current on  $F_1$  since it clearly does not depend on the decomposition  $\omega = \omega_1 + \omega_2$ . Clearly,  $\tilde{T}$  is positive, foliated, and closed.

*Proof of Theorem B.* In the preceding, we have constructed a  $(1, 1)$ -diffuse positive foliated closed current  $\tilde{T}$  on the surface  $F_1$ . It only remains to check that the current  $\tilde{T}$  actually is of Ahlfors type. For this, notice first any leaf  $L$  of the foliation  $\tilde{\mathcal{F}}_n$  contained in  $\overline{\mathcal{F}(S^1)}$  is dense in  $\overline{\mathcal{F}(S^1)}$  since  $\gamma$  is an irrational rotation. Consider then a leaf  $L_0$  of the foliation  $\tilde{\mathcal{F}}_n$  contained in  $\overline{\mathcal{F}(S^1)}$ . Note that  $L_0$  goes through

the singular points  $p_i$ ,  $i = 1, \dots, n+1$  with infinitely many branches. Still, away from these singular points  $p_1, \dots, p_{n+1}$ , the growth type of  $L_0$  is determined by the growth of the global holonomy group  $\Gamma$  of the foliation  $\tilde{\mathcal{F}}_d$ . The group  $\Gamma$  being cyclic, its growth is linear. However, the combination of the previously considered parametrisation of the leaves of  $\tilde{\mathcal{F}}_n$  in the sets  $B_i$  with the linear growth of the holonomy group  $\Gamma$  guarantees the existence of exhaustion  $(D_{r_j})_{j \in \mathbb{N}}$  of the leaf  $L_0$  such that

$$\lim_{j \rightarrow +\infty} \frac{\text{Length}(\partial D_{r_j})}{\text{Area}(D_{r_j})} = 0,$$

where  $\partial D_{r_j}$  stands for the boundary of  $D_{r_j}$  and where length and area are computed with respect to an auxiliary Hermitian metric fixed on  $F_1$ . In particular, as Riemann surface, the leaf  $L_0$  is a quotient of  $\mathbb{C}$ . Similarly, an Ahlfors current  $T_A$  supported on  $\overline{\mathcal{F}(S^1)}$  can be obtained as an accumulation point for the sequence of normalised integration currents associated with the exhaustion  $(D_{r_j})_{j \in \mathbb{N}}$ . Since all currents in the mentioned sequence are clearly foliated, so it will be its accumulation points. Now, as a foliated closed current,  $T_A$  can be disintegrated on  $\Sigma$  to yield a measure on  $S^1$  that is invariant under  $\xi$ . However, the action of  $\xi$  on  $S^1$  being conjugate to an irrational rotation, it admits a unique invariant probability measure thanks to the well-known Weyl's theorem. Hence, up to a constant multiple,  $T_A$  and  $\tilde{T}$  induce the same measure on  $S^1 \subset \Sigma$ . Therefore, they themselves coincide up to multiplication by a positive constant.

To complete the proof of Theorem B we have to adapt the preceding construction conducted on the surface  $F_1$  to the complex projective plane  $\mathbb{CP}^2$ . Consider the collapsing of the  $(-1)$ -rational curve in the surface  $F_1$  leading to the blow-down projection  $\pi : F_1 \rightarrow \mathbb{CP}^2$  and let  $\mathcal{F}_n$  be the resulting foliation on the surface  $\mathbb{CP}^2$ . By definition,  $\mathcal{F}_n$  is a Riccati foliation on the surface  $\mathbb{CP}^2$  and it has exactly  $(n+1)$  invariant lines, all of them being simple. Therefore,  $\mathcal{F}_n$  is a degree  $n$  foliation on the surface  $\mathbb{CP}^2$ , c.f. Lemma 2.2.5. Naturally the push-forward of the current  $\tilde{T}$  by  $\pi$  induces again a diffuse Ahlfors current  $T$  for the foliation  $\mathcal{F}_n$  on the surface  $\mathbb{CP}^2$ . Finally, in terms of the singular points of the foliation  $\mathcal{F}_n$ , recall from Lemma 2.3.4 that the  $(-1)$ -rational curve in the surface  $F_1$  passes through all “saddle” singular points  $q_i$  and avoid all “sink” singular points  $p_i$ ,  $i = 1, \dots, n+1$ . Thus the following holds:

- The  $(n+1)$  simple (sink) singular points  $p_i$  of the foliation  $\tilde{\mathcal{F}}_n$  yield  $(n+1)$  simple singular points for the foliation  $\mathcal{F}_n$ .
- The  $(n+1)$  simple (saddle) singular points  $q_i$  of the foliation  $\tilde{\mathcal{F}}_n$  sitting in the collapsed curve merge together in a (degenerate of order  $n$ ) singular point for the foliation  $\mathcal{F}_n$ .

Away from these  $n+2$  singular points, the foliation  $\mathcal{F}_n$  is regular as a diffeomorphic image of the regular foliation induced by  $\tilde{\mathcal{F}}_n$  on the corresponding Zariski-open set. Theorem B is proved.  $\square$

Let us close this section with some examples showing that the problem of describing diffuse positive foliated closed currents on the surface  $\mathbb{CP}^2$  goes beyond the description of currents that are of Ahlfors type.

**Example 4.** Prototypical examples of holomorphic foliations admitting diffuse positive foliated closed currents that are not of Ahlfors type are provided by *Hilbert modular foliations*. In the sequel, we restrict ourselves to the case of complex surfaces. Let  $N$  be a square-free positive integer and denote by  $K$  the totally real quadratic field  $\mathbb{Q}(\sqrt{N})$ . The ring of integers in  $K$  will be denoted by  $O_K$  while  $\Delta_K$  will stand for the discriminant of  $K$ . The two natural embedding of  $K$  in  $\mathbb{R}$  induce an embedding of  $\text{PSL}(2, K)$  in  $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ . Through this embedding, the *Hilbert modular group*  $\text{PSL}(2, O_K)$  acts on two



copies  $\mathbb{H} \times \mathbb{H}$  of the upper half plane  $\mathbb{H}$ . The quotient of this action can be compactified by adding finitely many cusps to give rise to a normal singular compact complex space of dimension 2. The singularities of this space, however, are of orbifold-type and arise from elliptic elements in  $\mathrm{PSL}(2, O_K)$ . Once they are resolved in a canonical minimal way, we obtain an algebraic surface  $Y(\Delta_K)$  called the *Hilbert modular surface of  $K$* , c.f. [28]

Next note that the two evident foliations of  $\mathbb{H} \times \mathbb{H}$  by vertical and by horizontal upper planes are both preserved by the action of  $\mathrm{PSL}(2, O_K)$ . Thus, the Hilbert surface  $Y(\Delta_K)$  is endowed with a pair of singular foliations, mutually transverse at a Zariski-open set, which are called *Hilbert modular foliations* and are studied, in particular, in [39]. These foliations will be denoted by  $\mathcal{H}$ . Zariski-dense leaves of  $\mathcal{H}$ , i.e., non algebraic leaves, are of hyperbolic type since they are quotients of the upper half plane  $\mathbb{H}$ . In particular,  $\mathcal{H}$  carries no foliated current of Ahlfors type. Yet, we have:

*Claim.* Every Hilbert modular foliation  $\mathcal{H}$  admits a diffuse positive foliated closed current  $T$ .

*Proof.* Note that the transverse space to  $\mathcal{H}$  on  $(\mathbb{H} \times \mathbb{H})/\mathrm{PSL}(2, O_K)$  can naturally be identified with the quotient of  $\mathbb{H}$  by the action of  $\mathrm{PSL}(2, O_K) \subset \mathrm{PSL}(2, \mathbb{R})$ . From this perspective, the subgroup  $\mathrm{PSL}(2, O_K)$  is actually discrete and the corresponding action admits a fundamental domain having finite hyperbolic volume, c.f. [28]. In turn, the hyperbolic measure is invariant under the holonomy since it locally coincides with Möbius transformation (c.f. the assertion that  $\mathcal{H}$  has a transverse projective structure in Theorem 1 of [39]). In other words, the foliation  $\mathcal{H}$  on the open surface  $(\mathbb{H} \times \mathbb{H})/\mathrm{PSL}(2, O_K)$  possesses a natural transversely invariant measure in the sense of Definition 2.3.3. We can then use this transversely invariant measure to produce a foliated closed current for  $\mathcal{H}$  on the corresponding algebraic (compact) Hilbert surface  $Y(\Delta_K)$  following Sullivan general theory. The fact that Sullivan's construction actually yields a well defined current on  $Y(\Delta_K)$  can be checked by means of a straightforward adaptation of the preceding discussion, keeping in mind the structure of the compactification in [28]. Details are left to the reader.  $\square$

The above constructed current is clearly diffuse since the hyperbolic measure on  $\mathbb{H}$  has no atomic part. Finally, to produce examples defined on  $\mathbb{CP}^2$ , we note that rational Hilbert surfaces  $Y(\Delta_K)$  were classified in [29]. They correspond to the cases where  $\Delta_K$  takes on one of the values 5, 8, 12, 13, 17, 21, 24, 28, 33, 60. Therefore, these surfaces are birationally equivalent to the surface  $\mathbb{CP}^2$ . Once again it is straightforward to follow the birational maps in question to make sure that the initial diffuse positive foliated closed current is pushed-forward to a proper diffuse positive foliated closed current for the corresponding foliation on the surface  $\mathbb{CP}^2$ .

## Chapter 3

# Analytic and Continuous Levi-flats and Harmonic Currents

### 3.1 Riccati Equations and Singular Levi-Flats

In this section we begin a direct approach to the proof of Theorem A. In what follows we will provide a general sufficient criterion for a Riccati foliation on the surface  $\mathbb{CP}^2$  to exhibit (singular) Levi-flats, whether they are real-analytic or only continuous with a fractal nature. The content of the section is summarised by Theorem 3.1.1 which establishes some parts of Theorem A.

Consider a Riccati foliation  $\mathcal{F}$  on the first Hirzebruch surface  $F_1$  along with a *simple invariant fibre*  $C$ . Recall that the foliation  $\mathcal{F}$  has either one or two singular points in the invariant fibre  $C$ , c.f. Equation (2.1). Furthermore, the following can easily be checked:

- If the foliation  $\mathcal{F}$  has two singular points in the invariant fibre  $C$ , then at each singular point, the eigenvalue of the foliation  $\mathcal{F}$  associated with the direction tangent to the invariant fibre  $C$  is necessarily different from zero. The invariant fibre  $C$  being, in addition, simple, each of these singular points will have two eigenvalues different from zero. A singular point possessing two eigenvalues  $\lambda_1$  and  $\lambda_2$  different from zero is said to belong to the *Poincaré domain* if the quotient  $\lambda_1/\lambda_2$  lies in  $\mathbb{C} \setminus \mathbb{R}_-$ . If the quotient  $\lambda_1/\lambda_2$  lies in  $\mathbb{R}_-$ , then the singularity is said to belong to the *Siegel domain*. Finally, singularities with two non-zero eigenvalues  $\lambda_1, \lambda_2$  such that  $\lambda_1/\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$  are called hyperbolic. In particular, hyperbolic singularities belong to the Poincaré domain.
- If the foliation  $\mathcal{F}$  has a unique singular point in the invariant fibre  $C$ , then the eigenvalue of the foliation  $\mathcal{F}$  corresponding to the direction of the invariant fibre  $C$  is equal to zero. However, the invariant fibre  $C$  being simple, the foliation  $\mathcal{F}$  has a non-zero eigenvalue in the direction transverse to the invariant fibre  $C$ . This type of singularity is called a *saddle-node*. In other words, saddle-node singularities are those that have one eigenvalue equal to zero and another different from zero.

**Theorem 3.1.1.** *Let  $\mathcal{F}$  be a Riccati foliation on the surface  $\mathbb{CP}^2$  as in Theorem A. In particular, the local holonomy maps of the foliation  $\mathcal{F}$  around the invariant lines are all elliptic elements. Assume that the global holonomy group  $\Gamma$  of the foliation  $\mathcal{F}$  is a Fuchsian (resp. quasifuchsian) group of first kind. Then the following holds:*

- (a) There exists a closed set  $\mathcal{L}(\mathcal{F}) \subset \mathbb{CP}^2$  of topological dimension equal to 3 which is invariant by the foliation  $\mathcal{F}$ .
- (b) If  $\Gamma$  is Fuchsian, then  $\mathcal{L}(\mathcal{F})$  is a real-analytic set with  $k$  singular points, all of them of orbifold-type. If  $\Gamma$  is quasifuchsian, then  $\mathcal{L}(\mathcal{F})$  is a “singular topological manifold” with Hausdorff dimension strictly greater than 3.

**Lemma 3.1.2.** *Let  $\mathcal{F}$  be a Riccati foliation on the surface  $F_1$  along with a simple invariant fibre  $C$  such that the local holonomy map of  $\mathcal{F}$  around the invariant fibre  $C$  is not parabolic. Then the foliation  $\mathcal{F}$  has two singular points in the invariant fibre  $C$  and each of these singular points is associated with two eigenvalues different from zero.*

*Proof.* Given the review at the beginning of the section about the structure of a Riccati foliation near a simple invariant fibre, the proof amounts to checking that the foliation  $\mathcal{F}$  cannot have a unique singular point  $p$  in the fibre  $C$ . Assume aiming at a contradiction that this was the case. Thus,  $p$  is a saddle-node singular point for the foliation  $\mathcal{F}$  with a non-zero eigenvalue associated to a direction transverse to  $C$ . Owing to Dulac’s normal form for saddle-nodes, the foliation  $\mathcal{F}$  admits a separatrix  $S$  at  $p$  that is smooth, transverse to  $C$ , and tangent to the non-zero eigenvalue of  $\mathcal{F}$  at  $p$  (c.f. for example [30] and [45]). The separatrix  $S$  induces a fixed point for the holonomy map of  $\mathcal{F}$  arising from a loop around  $C$ . Moreover, an elementary computation with Dulac’s normal form ([30], [45]) ensures the following holds:

- The multiplier of the holonomy map at this fixed point equals 1.
- This holonomy map cannot coincide with the identity.

It follows from the preceding that the holonomy map of the foliation  $\mathcal{F}$  arising from winding around  $C$  is a parabolic map. The resulting contradiction proves the lemma.  $\square$

At this point it is convenient to remind the reader of some basic normal form theory for singularities of foliations in dimension 2. The material mentioned below is available from most standard texts such as [30], [45].

First consider a foliation  $\mathcal{F}$  defined on a neighbourhood of  $(0, 0) \in \mathbb{C}^2$  and having at the origin two non-zero eigenvalues  $\lambda_1$  and  $\lambda_2$  satisfying  $\lambda_1/\lambda_2 \in \mathbb{R}_-$ . Then there are local coordinates  $(u, v)$  where the foliation  $\mathcal{F}$  is locally given by a vector field having the form

$$\lambda_1 u[1 + (\text{h.o.t})] \frac{\partial}{\partial u} + \lambda_2 v[1 + (\text{h.o.t})] \frac{\partial}{\partial v}. \quad (3.1)$$

where h.o.t stands for “higher order terms”. In particular, it is immediate to check that the foliation  $\mathcal{F}$  possesses *exactly* two separatrices which, incidentally, are smooth and mutually transverse being given in the above coordinates by the axes  $\{u = 0\}$  and  $\{v = 0\}$ .

Next, let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be two holomorphic foliations as above and sharing the same (non-zero) eigenvalues  $\lambda_1$  and  $\lambda_2$  ( $\lambda_1/\lambda_2 \in \mathbb{R}_-$ ). Let  $(u_i, v_i)$ ,  $i = 1, 2$ , be local coordinates where the foliations  $\mathcal{F}_i$  takes on the form indicated in Equation (3.1). Denote by  $h_i$  the local holonomy map of the foliation  $\mathcal{F}_i$  relative to the axis  $\{v_i = 0\}$ ,  $i = 1, 2$ . A theorem due to Mattei-Moussu [37] then states that the foliations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are analytically equivalent if and only if the holonomy maps  $h_1$  and  $h_2$  are analytically conjugate in  $\text{Diff}(\mathbb{C}, 0)$ .

Let us now move to the case where the two eigenvalues  $\lambda_1, \lambda_2$  of the foliation  $\mathcal{F}$  at the origin verify  $\lambda_1/\lambda_2 \in \mathbb{R}_+$  (always with  $\lambda_1\lambda_2 \neq 0$ ). First of all, if  $\lambda_1$  and  $\lambda_2$  are *non-resonant*, i.e., if neither  $\lambda_1/\lambda_2$  nor  $\lambda_2/\lambda_1$  is a positive integer, then the foliation  $\mathcal{F}$  is *linearisable and diagonalisable*. On the other hand, for the resonant case  $n = \lambda_1/\lambda_2 \in \mathbb{N}$  there are two possible outcomes:

- For  $n \geq 2$ . The foliation  $\mathcal{F}$  is either linearisable (and diagonalisable), i.e., there are local coordinates  $(x, y)$  where the foliation  $\mathcal{F}$  is represented by the vector field

$$x \frac{\partial}{\partial x} + ny \frac{\partial}{\partial y},$$

or it is conjugate to the so-called *Poincaré-Dulac normal form*. In the latter case, in suitable local coordinates the foliation  $\mathcal{F}$  is represented by the vector field

$$x \frac{\partial}{\partial x} + (ny + x^n) \frac{\partial}{\partial y}.$$

In particular, the foliation  $\mathcal{F}$  possesses a unique separatrix which is given in Poincaré-Dulac coordinates by  $\{x = 0\}$  (in particular, this separatrix is necessarily smooth).

- Assume now that  $n = 1$ . In this case, the foliation  $\mathcal{F}$  is always linearisable but there are still two possibilities. Namely, the foliation  $\mathcal{F}$  can be diagonalisable, i.e., conjugate to the foliation represented by the vector field

$$x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},$$

or non-diagonalisable. In the latter case, there are Poincaré-Dulac coordinates where the foliation  $\mathcal{F}$  is represented by the vector field

$$x \frac{\partial}{\partial x} + (y + x) \frac{\partial}{\partial y}.$$

Once again, when admitting the Poincaré-Dulac normal form, the foliation  $\mathcal{F}$  possesses a unique separatrix  $\{x = 0\}$  which happens to be smooth.

**Remark 3.1.3.** To complement the information on foliations admitting the Poincaré-Dulac normal form, we note that the local holonomy map  $h$  arising from the (unique) separatrix is such that its  $n^{\text{th}}$ -iterate  $h^n$  is tangent to the identity while never *equal to the identity* as it can directly be checked.

Armed with the above material on normal forms, we can go back to our approach to Theorem 3.1.1. Recall that remarks at the beginning of the section and Lemma (3.1.2) imply that a Riccati foliation exhibiting a hyperbolic or elliptic holonomy map around a simple fibre  $C$  must have exactly two singularities in  $C$ . Moreover, each of these singularities admit two eigenvalues different from zero.

**Lemma 3.1.4.** *Consider a Riccati foliation  $\mathcal{F}$  on the first Hirzebruch surface  $F_1$ . Let  $C$  be a simple invariant fibre and assume that the local holonomy map of the foliation  $\mathcal{F}$  around the invariant fibre  $C$  is elliptic. Denote by  $p, q \in C$  the two singular points of the foliation  $\mathcal{F}$  in the invariant fibre  $C$  and let  $\lambda_1^p, \lambda_2^p$  and  $\lambda_1^q, \lambda_2^q$  be the corresponding (non-zero) eigenvalues (where  $\lambda_1^p, \lambda_1^q$  are associated with the direction of the invariant fibre  $C$ ). Then these eigenvalues satisfy the relation*

$$\frac{\lambda_2^p}{\lambda_1^p} + \frac{\lambda_2^q}{\lambda_1^q} = 0 \tag{3.2}$$

*and both quotients  $\lambda_2^p/\lambda_1^p$ ,  $\lambda_2^q/\lambda_1^q$  lie in  $\mathbb{R}$ . Furthermore, the foliation  $\mathcal{F}$  is linearisable around both  $p$  and  $q$ . Finally, at both singular points, the foliation  $\mathcal{F}$  possesses a smooth separatrix transverse to the invariant fibre  $C$ .*

*Proof.* Equation (3.2) is a direct application of the Camacho-Sad index formula in [9]. Let us then begin by showing that the singular points  $p$  and  $q$  are not hyperbolic, i.e., that the quotient of the eigenvalues is a real number. For this, assume for a contradiction that the singular point  $p$  is hyperbolic. Owing to the index formula (3.2), there follows that so is  $q$ . It follows from Poincaré theorem that the foliation  $\mathcal{F}$  is linearisable at both  $p$  and  $q$ . In particular, the foliation  $\mathcal{F}$  admits a smooth separatrix  $S_p$  transverse to the invariant fibre  $C$  at  $p$ . Again  $S_p$  corresponds to a fixed point of the holonomy map of the foliation  $\mathcal{F}$  around the invariant fibre  $C$ . However,  $p$  being hyperbolic, there follows from an elementary calculation that this fixed point is hyperbolic, i.e., the absolute value of its multiplier is different from 1. This is impossible since this holonomy map is elliptic. The resulting contradiction then shows that the quotients  $\lambda_2^p/\lambda_1^p$  and  $\lambda_2^q/\lambda_1^q$  lie in  $\mathbb{R}$  and are different from zero.

From Formula (3.2), we can assume without loss of generality that  $\lambda_2^p/\lambda_1^p > 0$  and that  $\lambda_2^q/\lambda_1^q < 0$ . Thus, the singular point  $p$  belongs to the Poincaré domain while  $q$  lies in the Siegel domain. In particular, either  $\mathcal{F}$  is linearisable around  $p$  or it is conjugate to a Poincaré-Dulac normal form.

*Claim.* The foliation  $\mathcal{F}$  cannot admit a Poincaré-Dulac normal form around  $p$ .

*Proof of the Claim.* Assume that the claim is false. Thus, the foliation  $\mathcal{F}$  possesses a unique separatrix at  $p$  which must coincide with the fibre  $C$ . In particular,  $\lambda_2^p/\lambda_1^p = n \in \mathbb{N}^*$ . Meanwhile, Formula (3.2) ensures that at the singular point  $q \in C$  of the foliation  $\mathcal{F}$  the eigenvalues  $\lambda_1^q, \lambda_2^q$  can be chosen as  $\lambda_1^q = 1$  and  $\lambda_2^q = -n$ . In particular, there exists a separatrix  $S_q$  of  $\mathcal{F}$  at  $q$  which is transverse to the invariant fibre  $C$ . Again  $S_q$  can naturally be identified with a fixed point of the holonomy map of the foliation  $\mathcal{F}$  around the fibre  $C$ . Since this holonomy map is elliptic, it must be of finite order equal to  $n$ . Indeed, the linear part of this map at the fixed point represented by  $S_q$  is determined by the eigenvalues  $\lambda_1^q = 1$  and  $\lambda_2^q = -n$  and hence it is a rotation of order  $n$ . Thus the  $n^{\text{th}}$ -iterate of the holonomy map in question has a fixed point with multiplier equal to 1. Since the original holonomy map is an elliptic element, we conclude that its  $n^{\text{th}}$ -iterate must coincide with the identity.

Consider the local structure of the foliation  $\mathcal{F}$  around the Siegel singular point  $q$ , the previously mentioned theorem of Mattei-Moussu ensures that the foliation  $\mathcal{F}$  is linearisable at  $q$  since so is the local holonomy map arising from  $S_q$  (a finite order map). Given that the eigenvalues are 1 and  $-n$ , we see that the local holonomy map of the foliation  $\mathcal{F}$  at  $q$  determined by the fibre  $C$  (the other local separatrix of the foliation  $\mathcal{F}$  at  $q$ ) must coincide with the identity. To derive a contradiction proving the claim, just notice that  $C \setminus \{p, q\}$  is a cylinder so that loops winding once around the singular point  $p$  are homotopic to loops winding once around the singular point  $q$  (up to choosing suitable orientations). Thus, the local holonomy map around  $p$  arising from the separatrix of the foliation  $\mathcal{F}$  at  $p$  induced by  $C$  is the identity. In view of Remark 3.1.3, this contradicts the fact that the foliation  $\mathcal{F}$  is locally conjugate to a Poincaré-Dulac vector field. The claim is proved.  $\square$

To complete the proof of the lemma, note first that the claim implies that the foliation  $\mathcal{F}$  is linearisable around  $p$ . In particular, the separatrix  $S_p$  of the foliation  $\mathcal{F}$  at  $p$  which is transverse to the invariant fibre  $C$  does exist. Moreover, in terms of the separatrix induced by  $C$ , its associated (local) holonomy map is linearisable as well as a germ in  $\text{Diff}(\mathbb{C}, 0)$ . Again using the fact that  $C \setminus \{p, q\}$  is a cylinder, there follows that the (local) holonomy map arising from the separatrix induced by  $C$  at the singular point  $q$  is linearisable as well and, hence, Mattei-Moussu's result ensures that  $\mathcal{F}$  is linearisable at  $q$ . The lemma follows at once.  $\square$

Recall that among discrete subgroups of  $\text{PSL}(2, \mathbb{C})$ , i.e., among Kleinian groups, there are some groups that happen to leave invariant a (real analytic) circle  $S^1$  in  $\mathbb{CP}^1$ . These groups are therefore conjugate to discrete subgroups of  $\text{PSL}(2, \mathbb{R}) \subset \text{PSL}(2, \mathbb{C})$  so that they can be viewed as discrete groups acting on the hyperbolic disc and hence are called *Fuchsian groups*. The limit sets of Fuchsian groups are

naturally contained in the invariant circle  $S^1$ . In fact, up to identifying the initial group to a subgroup of  $\mathrm{PSL}(2, \mathbb{R})$ , the invariant circle becomes identified with the boundary of the unit disc. Finally, recall that a Fuchsian group is said to be of *first kind* if its limit set coincides with all the unit circle.

**Lemma 3.1.5.** *Let  $\mathcal{F}$  be a Riccati foliation on the surface  $F_1$  all of whose invariant fibres  $C_1, \dots, C_k$  are simple. Assume that the following conditions hold:*

- *The local holonomy map of the foliation  $\mathcal{F}$  around each invariant fibre  $C_i$  is an elliptic element of  $\mathrm{PSL}(2, \mathbb{C})$ .*
- *The global holonomy group  $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$  of the foliation  $\mathcal{F}$  is a fuchsian/quasifuchsian group.*

*Then the intersection  $C_i \cap \overline{\mathcal{F}(\Lambda)}$  is reduced to the singular point  $p_i \in C_i$  (that belongs to the Poincaré domain) of the foliation  $\mathcal{F}$ . Moreover, the eigenvalues of the foliation  $\mathcal{F}$  at  $p_i$  belong, in fact, to  $\mathbb{Q}_+$  and the foliation is linearisable around the singular point  $p_i$  (where  $\Lambda$  is the limit set of the global holonomy group  $\Gamma$  of the foliation  $\mathcal{F}$ ).*

*Proof.* Let  $C_i$  be a (simple) invariant fibre. According to Lemma 3.1.4 both singular points  $p_i$  and  $q_i$  have real (non-zero) eigenvalues and the foliation  $\mathcal{F}$  is linearisable around each of them. With the preceding notation, these eigenvalues verify  $\lambda_2^{p_i}/\lambda_1^{p_i} = -\lambda_2^{q_i}/\lambda_1^{q_i} \in \mathbb{R}^*$ . Also, the separatrix of  $\mathcal{F}$  at  $p_i$  (resp.  $q_i$ ) transverse to the invariant fibre  $C_i$  will be denoted by  $S_{p_i}$  (resp.  $S_{q_i}$ ). Consider a generic fibre  $\Sigma \subset F_1$  near  $C_i$  in the sense that on a fibred neighbourhood of the invariant fibre  $C_i$  the separatrices  $S_{p_i}, S_{q_i}$  intersect  $\Sigma$  at unequivocally determined points. Then let these points be denoted by  $x_{p_i}, x_{q_i}$ , respectively. As already mentioned, the points  $x_{p_i}, x_{q_i}$  are fixed by the local holonomy map of the foliation  $\mathcal{F}$  around the invariant fibre  $C_i$ . A direct calculation using linearising coordinates for the foliation  $\mathcal{F}$  around  $p_i, q_i$  shows that the multipliers of the (elliptic) holonomy map in question at the fixed point  $x_{p_i}$  (resp.  $x_{q_i}$ ) are given by  $e^{2\pi i \lambda_2^{p_i}/\lambda_1^{p_i}}$  (resp.  $e^{-2\pi i \lambda_2^{q_i}/\lambda_1^{q_i}}$ ). This elliptic holonomy map must, however, be of finite order since it belongs to a discrete subgroup of  $\mathrm{PSL}(2, \mathbb{C})$ . It follows that  $\lambda_2^{p_i}/\lambda_1^{p_i} \in \mathbb{Q}_+^*$ .

It only remains to prove that  $C_i \cap \overline{\mathcal{F}(\Lambda)} = \{p_i\}$ . The argument is similar to the one employed in Section 2.3.2. In the sequel, we take advantage of the fact that the foliation  $\mathcal{F}$  is linearisable at both  $p_i$  and  $q_i$ . First, we claim that  $C_i \cap \overline{\mathcal{F}(\Lambda)} \subset \{p_i, q_i\}$ . If the claim is false then  $\mathcal{F}(\Lambda)$  accumulates on regular points of the invariant fibre  $C_i$ . Since the foliation  $\mathcal{F}$  has a (linearisable) saddle singular point at  $q_i$ , the preceding implies that  $\mathcal{F}(\Lambda)$  accumulates on a regular points of the separatrix  $S_{q_i}$  as well. Hence, the fixed point  $x_{q_i} \in \Sigma$  belongs to  $\overline{\mathcal{F}(\Lambda)}$ . However,  $\Sigma \cap \overline{\mathcal{F}(\Lambda)}$  is nothing but the limit set of  $\Gamma$  represented in the transverse fibre  $\Sigma$ . This is impossible since, in a Fuchsian group, no fixed point of an elliptic element can lie in the corresponding limit set. Thus, we must have  $C_i \cap \overline{\mathcal{F}(\Lambda)} \subset \{p_i, q_i\}$ . However, since the foliation  $\mathcal{F}$  is linearisable around  $q_i$ , if  $\mathcal{F}(\Lambda)$  accumulates at  $q_i$  then it must accumulate at regular points of  $S_{q_i}$  as well so that the argument above rules this possibility out as well. In conclusion, we have proved that  $C_i \cap \overline{\mathcal{F}(\Lambda)} = \{p_i\}$  and the lemma follows.  $\square$

We can now state Proposition (3.1.6) which yields a criterion for a Riccati foliation to possess a *singular real-analytic* Levi-flat where all leaves of the induced (Levi) foliation, when the global holonomy group of the Riccati foliation  $\mathcal{F}$  is a Fuchsian group of  $\mathrm{PSL}(2, \mathbb{C})$  of first kind, are dense (in the Levi-flat itself).

**Proposition 3.1.6.** *Let  $\mathcal{F}$  be a Riccati foliation on the surface  $F_1$  all of whose invariant fibres  $C_1, \dots, C_k$  are simple. Assume that the following conditions hold:*

- The local holonomy map of the foliation  $\mathcal{F}$  around each invariant fibre  $C_i$  is an elliptic element of  $\mathrm{PSL}(2, \mathbb{C})$ .
- The global holonomy group  $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$  of the foliation  $\mathcal{F}$  is a Fuchsian group.

Then there exists a singular real-analytic variety  $\mathcal{L}(\mathcal{F})$  of (real) dimension 3 invariant by the foliation  $\mathcal{F}$ . Furthermore, the singular points of  $\mathcal{L}(\mathcal{F})$  are all isolated and correspond to the singular points of the foliation  $\mathcal{F}$  lying in the Poincaré domain.

*Proof.* Denote by  $\mathcal{F}(\Lambda)$  the saturated set of the limit set  $\Lambda$  by the foliation  $\mathcal{F}$  and  $\overline{\mathcal{F}(\Lambda)}$  the closure of  $\mathcal{F}(\Lambda)$ . Clearly,  $\mathcal{F}(\Lambda)$  is *locally* a real-analytic 3-dimensional manifold invariant by the foliation  $\mathcal{F}$  and, hence, satisfies the Levi condition for flatness. To prove the proposition we need to describe the accumulation of the set  $\mathcal{F}(\Lambda)$  on each invariant fibre  $C_i$ ,  $i = 1, \dots, k$ . In this direction, Lemma (3.1.5) ensures that the intersection  $C_i \cap \overline{\mathcal{F}(\Lambda)}$  is reduced to the singular point  $p_i$  whose eigenvalues  $\lambda_1^{p_i}$  and  $\lambda_2^{p_i}$  satisfy  $\lambda_2^{p_i}/\lambda_1^{p_i} \in \mathbb{Q}_+$ . The proof of the proposition now follows from Lemma 3.1.7 below stating that  $\overline{\mathcal{F}(\Lambda)}$  still is an analytic variety around  $p_i$ .  $\square$

**Lemma 3.1.7.** *With the notation of Proposition 3.1.6, the set  $\overline{\mathcal{F}(\Lambda)}$  is locally real-analytic around  $p_i$ .*

The proof of Lemma 3.1.7 will be deferred to Section 3.2 since it employs some blowing up procedure that, incidentally, will also be instrumental in constructing the harmonic current  $T$  mentioned in items (c) and (d) of Theorem A.

As a by-product of the previous discussion, we also obtain Proposition (3.1.8) below which is valid for Riccati foliations whose global holonomy groups are quasifuchsian groups and which dispenses with Lemma 3.1.7.

**Proposition 3.1.8.** *Let  $\mathcal{F}$  be a Riccati foliation on the surface  $F_1$  all of whose invariant fibres  $C_1, \dots, C_k$  are simple. Assume that the following conditions hold:*

- The local holonomy map  $M_i$  of the foliation  $\mathcal{F}$  around each invariant fibre  $C_i$  is an elliptic element of  $\mathrm{PSL}(2, \mathbb{C})$ .
- The global holonomy group  $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$  of the foliation  $\mathcal{F}$  is a quasifuchsian group.

Then there exists a closed set  $\mathcal{L}(\mathcal{F})$  invariant by the foliation  $\mathcal{F}$  which is a singular topological manifold of (topological) dimension 3 and Hausdorff dimension strictly greater than 3. Furthermore, the singular points of  $\mathcal{L}(\mathcal{F})$  are all isolated and correspond to the singular points of the foliation  $\mathcal{F}$  lying in the Poincaré domain.

*Proof.* The construction of  $\mathcal{L}(\mathcal{F})$  is exactly as in the previously considered Fuchsian case (c.f. Proposition 3.1.6). The additional difficulty of proving that  $\mathcal{L}(\mathcal{F})$  is real-analytic plays no role in the present case. Meanwhile, the only issue that requires explanation is the claim that the Hausdorff dimension of  $\mathcal{L}(\mathcal{F})$  is strictly greater than 3. This, however, follows from Bowen's theorem [7] asserting that the Hausdorff dimension of a quasicircle is strictly greater than 1. Thus,  $\mathcal{L}(\mathcal{F})$  can be pictured as a lamination by Riemann surfaces which transversely has Hausdorff dimension strictly greater than 1 so that the Hausdorff dimension of the laminated space  $\mathcal{L}(\mathcal{F})$  is strictly greater than 3.  $\square$

Recalling that the direct image of a real analytic set by an analytic map may fail to be real analytic, the last ingredient needed in the proof of Theorem 3.1.1 is Lemma 3.1.9 below.

**Lemma 3.1.9.** *Let  $\mathcal{F}$  and  $\overline{\mathcal{F}(\Lambda)}$  be as in Proposition 3.1.8. Denote by  $\mathcal{C}$  the  $(-1)$ -rational curve contained in the first Hirzebruch surface  $F_1$ . Then we have*

$$\overline{\mathcal{F}(\Lambda)} \cap \mathcal{C} = \phi.$$

*Proof.* For each  $i \in \{1, \dots, k\}$ , let  $b_i$  be the (unique) intersection point between the invariant fibre  $C_i$  and the rational curve  $\mathcal{C}$  and set  $\mathcal{C}_0 = \mathcal{C} \setminus \{b_1, \dots, b_k\}$ . Let  $\Sigma$  be a non-invariant fibre of the surface  $F_1$  (hence transverse to  $\mathcal{F}$ ) and denote by  $q$  the intersection point between  $\Sigma$  and  $\mathcal{C}_0$ . Consider then the holonomy representation  $\rho : \pi_1(\mathcal{C}_0) \rightarrow \mathrm{PSL}(2, \mathbb{C})$  and let  $\Gamma = \rho(\pi_1(\mathcal{C}_0))$  be the global holonomy group of the foliation  $\mathcal{F}$  acting on  $\Sigma$ . Let  $\bar{\mathcal{C}}_0$  be the covering space of  $\mathcal{C}_0$  such that  $\pi_1(\bar{\mathcal{C}}_0) = \ker(\rho)$  and denote by  $\mathcal{P} : \bar{\mathcal{C}}_0 \rightarrow \mathcal{C}_0$  the corresponding covering projection.

Fix a base point  $\bar{p}$  in  $\bar{\mathcal{C}}_0$  and let  $p = \mathcal{P}(\bar{p}) \in \mathcal{C}_0$ . Let  $\gamma : [0, 1] \rightarrow \mathcal{C}_0$  be a path joining  $p$  to  $q$  (i.e.,  $\gamma(0) = p$  and  $\gamma(1) = q$ ), then  $\gamma$  can be lifted to the leaf  $L_p$  of the foliation  $\mathcal{F}$  through the point  $p$  thanks to the fact that the foliation  $\mathcal{F}$  is a Riccati foliation. The lift  $\bar{\gamma} : [0, 1] \rightarrow L_p$  joins  $p$  to a point, denoted by  $x = \mathcal{D}(\bar{p})$ , in the intersection  $L_p \cap \Sigma$ .

The correspondence between the point  $\bar{p} \in \bar{\mathcal{C}}_0$  and the point  $x = \mathcal{D}(\bar{p}) \in \Sigma$  can naturally be extended over paths in  $\bar{\mathcal{C}}_0$ . To do so, consider a path  $\bar{\sigma} : [0, 1] \rightarrow \bar{\mathcal{C}}_0$  such that  $\bar{\sigma}(0) = \bar{p}$ . To define  $\mathcal{D}(\bar{\sigma}(1))$ , denote by  $\sigma : [0, 1] \rightarrow \mathcal{C}_0$  the projection of  $\bar{\sigma}$  onto  $\mathcal{C}_0$ ,  $\mathcal{D}(\bar{\sigma}(1))$  can then be defined to be the terminal point of the lift of the path  $\sigma^{-1} * \gamma$  in  $L_{\sigma(1)}$ . Let  $\bar{\sigma}_1 : [0, 1] \rightarrow \bar{\mathcal{C}}_0$  be a deformation of  $\bar{\sigma}$  with fixed endpoints, then it is easy to check that  $\mathcal{D}(\bar{\sigma}(1)) = \mathcal{D}(\bar{\sigma}_1(1))$ . Thus,  $\mathcal{D}$ , in fact, only depends on the homotopic class of  $\bar{\sigma}$ .

We need then to show that  $\mathcal{D}$  is well defined. Let  $\bar{c}$  be a closed loop in  $\bar{\mathcal{C}}_0$  with a base point at  $\bar{p}$ . Then  $c = \mathcal{P}(\bar{c})$  is a closed loop in  $\mathcal{C}_0$  with a base point at  $p$ . The path  $c^{-1}$  can be lifted to  $L_p$ . Thus, the lift  $c_0 : [0, 1] \rightarrow L_p$  joins  $p$  to a point, denoted by  $p_0$ , in the intersection  $L_p \cap \Sigma$ . Meanwhile,  $p_0 = \rho(c^{-1}) \cdot p = \mathrm{id} \cdot p = p$  since  $c^{-1} \in \pi_1(\bar{\mathcal{C}}_0) = \ker(\rho)$ . It follows that the terminal point of the lift of the path  $c^{-1} * \gamma$  in the leaf  $L_p$  is, again, the point  $x$  itself and, hence, the function  $\mathcal{D}$  is well defined.

Let  $G$  be the group of deck transformations acting on  $\bar{\mathcal{C}}_0$ . As an abstract group  $G$  is given as the quotient

$$\pi_1(\mathcal{C}_0)/\pi_1(\bar{\mathcal{C}}_0) = \pi_1(\mathcal{C}_0)/\ker(\rho).$$

On the other hand, the holonomy representation yields

$$\pi_1(\mathcal{C}_0)/\ker(\rho) \simeq \rho(\pi_1(\mathcal{C}_0)) = \Gamma,$$

where  $\Gamma$  is the global holonomy group of the foliation  $\mathcal{F}$ . Thus, these two groups  $G$  and  $\Gamma$  are isomorphic. In other words,  $G$  and  $\Gamma$ , viewed as transformation groups of  $\bar{\mathcal{C}}_0$  and  $\Sigma$  respectively, can be interpreted as two different actions of a same group, namely  $\pi_1(\mathcal{C}_0)/\ker(\rho)$ . Let  $g$  be an element in  $G$  and  $\alpha$  the corresponding element in  $\Gamma$ . It follows from the construction above that

$$\rho(\alpha) \cdot \mathcal{D}(\bar{r}) = \mathcal{D}(g \cdot \bar{r}),$$

for every  $\bar{r} \in \bar{\mathcal{C}}_0$ . In other words, the function  $\mathcal{D}$  is *equivariant* with respect to the actions of the groups  $G$  and  $\Gamma$ .

Let  $U = \mathcal{D}(\bar{\mathcal{C}}_0) \subset \Sigma$ . Then the set  $U$  is open (since the function  $\mathcal{D}$  is open by construction) and invariant under the action of the group  $\Gamma$ . There follows that  $U/\Gamma \simeq \bar{\mathcal{C}}_0/G$  and, since  $\bar{\mathcal{C}}_0/G$  is nothing but  $\mathcal{C} \setminus \{b_1, \dots, b_k\}$ ,  $U/\Gamma$  is a Hausdorff manifold. Hence, the group  $\Gamma$  acts properly discontinuously on  $U$  which, in turn, implies that  $U$  is entirely contained in the domain of discontinuity of the quasifuchsian group  $\Gamma$ . Thus,  $U \cap \Lambda = \phi$  where  $\Lambda$  is the limit set of the group  $\Gamma$ . The lemma then follows since  $\mathcal{F}(\mathcal{C}_0) \cap \Sigma = U$  and  $\overline{\mathcal{F}(\Lambda)} \cap \Sigma = \Lambda$ .  $\square$



*Proof of Theorem 3.1.1.* The statement follows at once from the combination of Propositions 3.1.6 and 3.1.8 with Lemma 3.1.9. In fact, in the case of Fuchsian groups, the Levi-flat  $\mathcal{L}(\mathcal{F}) \subset \mathbb{CP}^2$  is nothing but the image of the (Levi-flat)  $\mathcal{L}(\mathcal{F}) \subset F_1$  whose existence is ensured by Proposition 3.1.6 through the blow down mapping from the surface  $F_1$  to the surface  $\mathbb{CP}^2$ . To show that the resulting set  $\mathcal{L}(\mathcal{F})$  satisfies the conditions of Theorem 3.1.1 just observe that  $\mathcal{L}(\mathcal{F})$  is contained in a compact part of  $F_1 \setminus \mathcal{C}$  thanks to Lemma 3.1.9, where  $\mathcal{C}$  stands for the  $(-1)$ -rational curve in  $F_1$ . It follows that the mentioned blow down mapping is actually a holomorphic diffeomorphism on a neighbourhood of  $\mathcal{L}(\mathcal{F})$  so that the theorem follows. The same argument applies to the case of quasifuchsien groups as discussed in Proposition 3.1.8. Theorem 3.1.1 is proved.  $\square$

## 3.2 Foliated Harmonic Currents

Having essentially proved (in section 3.1) the first two statements of Theorem A, the present section is devoted to establishing statements (c), (d), and (e) in the theorem in question. In other words, all foliated harmonic currents for Riccati foliations on the surface  $\mathbb{CP}^2$  satisfying the conditions of Theorem A will be described.

To avoid useless repetitions, in the course of this section we shall consider that a Fuchsian group is a special case of a quasifuchsian one. In other words, when referring to quasifuchsian groups in the sequel, the possibility that the group in question is actually Fuchsian *is not* ruled out. Along similar lines, we will denote by  $\mathcal{L}(\mathcal{F}) \subset F_1$  the Levi-flat arising from the Riccati foliation considered in Propositions 3.1.6 and 3.1.8 regardless of whether  $\mathcal{L}(\mathcal{F})$  is real-analytic or transversely fractal. Of course, every quasifuchsien/fuchsian group considered in the sequel is assumed to be non-elementary. The notation used in the proofs of Propositions 3.1.6 and 3.1.8 will be resumed in the sequel.

Consider an invariant fibre  $C_i$  of the foliation  $\mathcal{F}$  along with the singular points  $p_i, q_i \in C_i$ , where  $p_i$  lies in the Poincaré domain and  $q_i$  in the Siegel domain. Denote by  $m_i \geq 2$  the order of the (elliptic) holonomy map of the foliation  $\mathcal{F}$  arising from winding around the invariant fibre  $C_i$ . As previously seen, the foliation  $\mathcal{F}$  is linearisable around the point  $p_i$ . Thus, there are local holomorphic coordinates  $(u_i, v_i)$  around the singular point  $p_i \simeq (0, 0)$  where the foliation  $\mathcal{F}$  is locally conjugate to the foliation associated with the vector field

$$m_i u_i \frac{\partial}{\partial u_i} + n_i v_i \frac{\partial}{\partial v_i},$$

with  $\{u_i = 0\} \subset C_i$ . Moreover,  $m_i$  and  $n_i$  are strictly positive integers which can be assumed to satisfy  $1 \leq n_i < m_i$  without loss of generality. In particular, the separatrix given by  $\{v_i = 0\}$  is distinguished as the unique (local, smooth) separatrix that is transverse to the invariant fibre  $C_i$ . Apart from the separatrix induced by  $C_i$ ,  $\{v_i = 0\}$  is the unique separatrix carrying non-trivial holonomy which is conjugate to the rotation of angle  $2\pi n_i/m_i$ . This separatrix is denoted by  $S_p$ .

In the  $(u_i, v_i)$ -coordinates, the foliation  $\mathcal{F}$  admits the function  $(u_i, v_i) \mapsto u_i^{n_i} v_i^{-m_i}$  as a meromorphic first integral so that the leaves are locally algebraic of the form  $u_i^{n_i} = c v_i^{m_i}$  for a suitable constant  $c \in \mathbb{C}$ . Yet a blow up procedure allows to eliminate the indeterminacy point of the mentioned first integral so as to describe the behaviour of these leaves in a way better adapted to our needs. In fact, it is useful carry out this procedure relying on the following elementary observation: the blow up of the foliation in question yields a foliation leaving invariant the resulting exceptional divisor and possessing exactly 2 singular points. Furthermore, both singular points have integer eigenvalues and one of them lies in the Poincaré domain while the other lies in the Siegel domain. The procedure then amounts to carrying on blowing up the singularity lying in the Poincaré domain until we reach a singular point whose eigenvalues

are 1 and 1. Then one last blow up at this singular point will lead to a foliation that is transverse to the corresponding component of the exceptional divisor. In other words, after finitely many blow ups, all centered at singular points lying in the Poincaré domain, we obtain the picture:

- (1) An exceptional divisor  $\mathcal{E}$  consisting of a string of rational curves  $D_1, \dots, D_l$  with transverse intersections.
- (2) A foliation  $\tilde{\mathcal{F}}$  that possesses only singularities lying in the Siegel domain.
- (3) A component  $D_J$  of  $\mathcal{E}$  which is transverse to  $\tilde{\mathcal{F}}$  (in particular in a neighbourhood of  $D_J$ ).
- (4) The remaining components  $D_1, \dots, D_{J-1}, D_{J+1}, \dots, D_l$  are all invariant by  $\tilde{\mathcal{F}}$ .
- (5)  $D_1$  intersects (the transform of)  $C_i$  while (the transform of)  $S_p$  determines a singular point of  $\tilde{\mathcal{F}}$  lying in  $D_l$ .

We are now ready to prove Lemma 3.1.7 and then proceed to the construction of the current  $T$  supported on  $\mathcal{L}(\mathcal{F})$ .

*Proof of Lemma 3.1.7.* We consider a fibred neighbourhood  $U = \mathbb{D} \times C_i$  of  $C_i$  where  $\mathbb{D} \subset \mathbb{C}$ . Fix a fibre  $\Sigma \subset U$  and let  $\sigma : \Sigma \rightarrow \Sigma$  denote the (elliptic) holonomy map arising from winding around the invariant fibre  $C_i$ . We choose projective coordinates on  $\Sigma$  such that  $\mathcal{L}(\mathcal{F}) \cap \Sigma = \mathbb{R} \cup \{\infty\}$ . To prove the lemma, it suffices to construct a meromorphic first integral  $F$  for the foliation  $\mathcal{F}$  on  $U$  such that  $\mathcal{L}(\mathcal{F}) \cap U$  coincides with  $F^{-1}(\mathbb{R} \cup \{\infty\})$ . Indeed, if  $F$  is such an integral, then  $F = \bar{F}$  yields an analytic equation defining  $\mathcal{L}(\mathcal{F}) \cap U$ .

Denote by  $\xi : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$  an elliptic automorphism of order  $m_i$  leaving  $\mathbb{R} \cup \{\infty\}$  invariant. Consider then a non-constant holomorphic map  $f : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$  which is invariant under  $\xi$ . For example, if we change coordinates such that  $\xi$  becomes a rotation around the origin and where  $\mathbb{R} \cup \{\infty\}$  becomes the unit circle, then we can choose  $f(z) = z^n$ . Moving back to the initial coordinates then yields the desired invariant function.

Denote by  $P_i$  and  $Q_i$  the two fixed points of  $\xi$ . Recall that  $S_p$  (resp.  $S_q$ ) is the separatrix issued from the singular point  $p_i$  (resp.  $q_i$ ) which is transverse to the invariant fibre  $C_i$ . As seen in Section 3.1,  $S_p$  meets  $\Sigma$  at the fixed point  $P_i$ . Similarly  $S_q$  meets  $\Sigma$  at  $Q_i$ .

Next apply to the singular point  $p_i$  the blow up procedure described above. We then denote by  $\tilde{U}$  the transform of  $U$ . We denote also by  $\mathcal{E}_1$  the divisor consisting of the string of rational curves going from the invariant fibre  $C_i$  to  $D_{J-1}$ . Similarly,  $\mathcal{E}_2$  is the string of rational curves  $D_{J+1}, \dots, D_l$ .

To complete the proof of the lemma, we will construct a holomorphic mapping  $\tilde{F} : \tilde{U} \rightarrow \mathbb{CP}^1$  which is constant over the leaves of  $\tilde{\mathcal{F}}$  (the transform of  $\mathcal{F}$ ). Clearly, such a map induces the desired meromorphic first integral  $F$  on  $U$  (Levi extension). In turn, to construct the mapping  $\tilde{F} : \tilde{U} \rightarrow \mathbb{CP}^1$  constant over the leaves of  $\tilde{\mathcal{F}}$ , we first set  $\tilde{F}(z) = f(Q_i)$  if  $z \in \mathcal{E}_1$  and  $\tilde{F}(z) = f(P_i)$  if  $z \in \mathcal{E}_2$ . Now, if  $z \in \tilde{U} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$ , then we consider the leaf  $L_z$  of  $\tilde{\mathcal{F}} \simeq \mathcal{F}$  through  $z$  and set  $\tilde{F}(z) = f(L_z \cap \Sigma)$ . It is immediate that  $\tilde{F}$  is well defined since all the value of  $f$  at every intersection point  $L_z \cap \Sigma$  is the same ( $f$  is invariant under  $\xi$ ). Furthermore, since  $\tilde{\mathcal{F}}$  is regular away from  $\mathcal{E}_1 \cup \mathcal{E}_2$ , it is clear that  $\tilde{F}$  is holomorphic on  $\tilde{U} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$ . Finally, to show that  $\tilde{F}$  is holomorphic on all of  $\tilde{U}$ , it suffices to show that this mapping is continuous at  $\mathcal{E}_1$  and at  $\mathcal{E}_2$ .

The continuity of  $\tilde{F}$  at points in  $\mathcal{E}_1 \cup \mathcal{E}_2$ , however, follows from the structure of (linear) Siegel singular points detailed in Section 3.1. In fact, let  $\{z_i\} \subset \tilde{U} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$  be a sequence of points converging towards a

point in, say,  $\mathcal{E}_1$ . Since all singular points of  $\tilde{\mathcal{F}}$  lie in the Siegel domain, it follows that the corresponding leaves  $L_{z_i}$  accumulate on the separatrix  $S_q$ . Thus, the intersection points  $L_{z_i} \cap \Sigma$  cluster around  $Q_i$  so that  $\tilde{F}(z_i)$  converges to  $f(Q_i)$  hence establishing the continuity of  $\tilde{F}$  at points in  $\mathcal{E}_1$ . The analogous argument shows that  $\tilde{F}$  is also continuous at points in  $\mathcal{E}_2$  so that Riemann extension implies that  $\tilde{F}$  is holomorphic on all of  $\tilde{U}$ . The proof of the lemma is completed.  $\square$

The remainder of the section is devoted to constructing the harmonic current  $T$  supported on  $\mathcal{L}(\mathcal{F})$  in order to complete the proof of Theorem A. Let us then go back to the Riccati foliation  $\mathcal{F}$  on the surface  $F_1$  as in Proposition 3.1.8. The global holonomy group  $\Gamma$  of the foliation  $\mathcal{F}$  is a quasifuchsian group of  $\mathrm{PSL}(2, \mathbb{C})$  (the Fuchsian case viewed as a particular one). Furthermore, since  $\Gamma$  is assumed to be of first kind so that its limit set is all the invariant Jordan curve. The Levi-flat  $\mathcal{L}(\mathcal{F}) \subset F_1$  obtained as the closure of the saturated of this Jordan curve by the foliation  $\mathcal{F}$  is a singular topological manifold of dimension 3 whose singular points coincide with the singular points of the foliation  $\mathcal{F}$  belonging to the Poincaré domain. Clearly, every leaf of the foliation  $\mathcal{F}$  contained in  $\mathcal{L}(\mathcal{F})$  is dense in  $\mathcal{L}(\mathcal{F})$  since  $\Gamma$  is of first kind.

Now for each  $i \in \{1, \dots, k\}$ , we apply the blowing up procedure used in the proof of Lemma 3.1.7 to each of the singular points  $p_i \in C_i$  lying in the Poincaré domain. Denote by  $N$  the resulting surface and by  $\tilde{\mathcal{F}}$  the transform of the foliation  $\mathcal{F}$ . Note that the corresponding projection  $\Pi : N \rightarrow F_1$  is a holomorphic diffeomorphism from the complement of the total exceptional divisor in  $N$  to  $F_1 \setminus \{p_1, \dots, p_k\}$ . Then set  $\mathcal{R}_i = C_i \cup \Pi^{-1}(p_i)$  where the invariant fibre  $C_i$  is identified with its transform. Clearly,  $\mathcal{R}_i$  is a string of rational curves containing a unique curve  $D_{J_i}$  which is transverse to  $\tilde{\mathcal{F}}$  while all the other components are invariant by  $\tilde{\mathcal{F}}$ . In the sequel, let  $\widetilde{\mathcal{L}(\mathcal{F})}$  be the transform of the Levi-flat  $\mathcal{L}(\mathcal{F})$ .

**Lemma 3.2.1.**  *$\widetilde{\mathcal{L}(\mathcal{F})}$  is a regular topological manifold. It intersects the divisor  $\mathcal{R}_i$  on a Jordan curve contained in the curve  $D_{J_i}$ .*

*Proof.* With the notation of Lemma 3.1.7,  $\mathcal{R}_i = C_i \cup \mathcal{E}$  and each of the connected components of  $\mathcal{R}_i \setminus D_{J_i}$  are contained in saturated sets of arbitrarily small neighbourhoods of the local monodromy. Therefore, they remain away from the Jordan curve arising as limit set of  $\Gamma$ . Hence,  $\widetilde{\mathcal{L}(\mathcal{F})} \cap \mathcal{R}_i$  is contained in a compact part of  $D_{J_i}$  minus the intersection points with  $D_{J_{i-1}}$  and  $D_{J_{i+1}}$ . In turn, the foliation  $\tilde{\mathcal{F}}$  is regular at these intersection points and the lemma follows.  $\square$

We can easily compare the leaves of  $\tilde{\mathcal{F}}$  in  $\widetilde{\mathcal{L}(\mathcal{F})}$  with the leaves of the foliation  $\mathcal{F}$  in  $\mathcal{L}(\mathcal{F})$ . In fact, it might be more accurate to talk about the *filled leaves* of the foliation  $\mathcal{F}$  which are defined as follows. First, a *leaf* of the foliation  $\mathcal{F}$  in  $\mathcal{L}(\mathcal{F})$  is nothing but a leaf of the non-singular foliation obtained by restricting the foliation  $\mathcal{F}$  to  $F_1 \setminus \{p_1, \dots, p_k\}$ . If  $L$  is one such leaf, then the corresponding *filled leaf*  $\bar{L}$  is defined by adding the singular point  $p_i$  to every local branch of the leaf  $L$  passing through  $p_i$ , where each branch has the local form  $u_i^{n_i} = cv_i^{m_i}$  for suitable  $c \in \mathbb{C}$  and positive integers  $m_i, n_i$ . Now the comparison between leaves of  $\tilde{\mathcal{F}}$  in  $\widetilde{\mathcal{L}(\mathcal{F})}$  and leaves of the foliation  $\mathcal{F}$  in  $\mathcal{L}(\mathcal{F})$  is made accurate by the following lemma.

**Lemma 3.2.2.** *If  $\tilde{L}$  is a leaf of the foliation  $\tilde{\mathcal{F}}$  contained in  $\widetilde{\mathcal{L}(\mathcal{F})}$ , then the restriction of  $\Pi$  to  $\tilde{L}$  is a diffeomorphism between  $\tilde{L}$  and some filled leaf  $\bar{L}$  of the foliation  $\mathcal{F}$ .*

*Proof.* In view of the preceding, the intersection of a leaf  $\tilde{L}$  with the exceptional divisor of  $\Pi$  is contained in the union of the curves  $D_{J_i}$ . These are compact (rational) curves transverse to  $\tilde{\mathcal{F}}$ . Therefore, the

intersection of  $\tilde{L}$  and the exceptional divisor of  $\Pi$  is uniformly transverse (i.e., transverse with angle bounded from below by a strictly positive constant). The lemma follows immediately.  $\square$

Next, we have:

**Lemma 3.2.3.** *The filled leaves  $\bar{L}$  of the foliation  $\mathcal{F}$  in  $\mathcal{L}(\mathcal{F})$  are hyperbolic Riemann surfaces.*

*Proof.* To show that  $\bar{L}$  is a hyperbolic Riemann surface it suffices to check that its volume grows exponentially. For regular foliations everywhere transverse to a fibration, the growth type of leaves is determined by the growth of the global holonomy group of the foliation which is exponential since it is a quasifuchsian group. A minor difficulty here arises from the fact that  $\bar{L}$  fails to be transverse to the invariant fibre  $C_i$  at the point  $p_i$ .

The indicated issue is, however, settled by Lemma 3.2.2. Around each point  $p_i$ , we can place a small ball  $B_i$  such that the away from  $B_i$  the leaf is transverse to the fibres and hence has its volume growth comparable with the exponential growth of the holonomy group  $\Gamma$ . Furthermore, owing to Lemma 3.2.2, the intersection of each local branch of  $\bar{L}$  with  $B_i$  is a disc of small area and small (comparable) diameter. Combining these pieces of information, it becomes clear that the volume of  $\bar{L}$  grows exponentially so that the statement follows.  $\square$

Before stating the next proposition, it is convenient to recall the notion of self-intersection for harmonic currents introduced in [21]. Consider a positive harmonic current  $T$  on a compact Kähler surface. By using Hodge theory, Fornæss and Sibony managed to define the *self-intersection* of  $T$  by showing that the integral

$$\int T \wedge T$$

is well defined for positive harmonic currents. Moreover, this integral coincides with the usual formulation in the case where  $T$  is smooth. In particular, in the case of the surface  $\mathbb{CP}^2$ , they mentioned the problem of computing the quantity

$$\inf \left\{ \int T \wedge T; \ T \geq 0 \text{ and } i\partial\bar{\partial}T = 0 \right\}.$$

Proving that the infimum in question is strictly positive would have major implications in several well-known conjectures about Riemann surface laminations in the surface  $\mathbb{CP}^2$ . A by-product of their second paper [22], however, is that this infimum equals zero. Yet, it seems that no example of positive harmonic current with zero self-intersection and having, say, support with empty interior was previously identified so that applications to the quoted problems could still be envisaged. In this regard, the contribution of Theorem A is summarised by its item (e).

The following proposition is more naturally stated in the language of *laminations*, (c.f. for example [23]).

**Proposition 3.2.4.** *The Riemann surface lamination defined by the restriction of  $\tilde{\mathcal{F}}$  to  $\widetilde{\mathcal{L}(\mathcal{F})}$  possesses a unique positive foliated harmonic current  $\tilde{T}$ . The current  $\tilde{T}$  is not closed and has zero self-intersection. Finally, the support of the current  $\tilde{T}$  coincides with the whole  $\widetilde{\mathcal{L}(\mathcal{F})}$ .*

*Proof.* As shown by the preceding lemmas, the restriction of  $\tilde{\mathcal{F}}$  to  $\widetilde{\mathcal{L}(\mathcal{F})}$  is a regular lamination by Riemann surfaces all of whose leaves are hyperbolic. We claim that this lamination admits no positive foliated closed current. To check the claim, just note that one such closed current would yield a transverse invariant measure for the lamination in question. This measure would project through  $\Pi$  into a

finite measure invariant by the holonomy group  $\Gamma$  on its limit set, which immediately gives rise to a contradiction since  $\Gamma$  is non-elementary. This contradiction proves the claim.

The existence of a unique positive foliated harmonic current  $\tilde{T}$  on  $\widetilde{\mathcal{L}(\mathcal{F})}$  now follows from the main result in [14]. This same theorem also shows that the self-intersection of  $\tilde{T}$  equals zero. Finally, since the action of  $\Gamma$  on its limit set has all orbits dense, it follows that all the leaves of the lamination induced on  $\widetilde{\mathcal{L}(\mathcal{F})}$  by  $\tilde{\mathcal{F}}$  are dense in  $\widetilde{\mathcal{L}(\mathcal{F})}$ . Hence, the support of  $\tilde{T}$  must coincide with the whole  $\widetilde{\mathcal{L}(\mathcal{F})}$ .  $\square$

Recall that the projection  $\Pi : N \rightarrow F_1$  is holomorphic and globally defined on  $N$ . The current  $\tilde{T}$  can then be pushed forward by  $\Pi$  to yield a positive foliated harmonic current  $T = \Pi_* \tilde{T}$ , for  $\mathcal{F}$  supported on all  $\mathcal{L}(\mathcal{F})$  which, in addition, has null self-intersection. The last ingredient needed in the proof of Theorem A is the following proposition:

**Proposition 3.2.5.** *Every  $(1,1)$ -foliated harmonic current  $T$  supported in  $\mathcal{L}(\mathcal{F})$  can be pulled-back by  $\Pi$  to yield a foliated harmonic current for  $\tilde{\mathcal{F}}$  supported on  $\widetilde{\mathcal{L}(\mathcal{F})}$ .*

*Proof.* Let  $T$  be as in the statement and consider a  $(1,1)$ -differential form  $\omega$  on  $N$ . To define the pull-back  $\tilde{T} = \Pi^* T$  it suffices to define a push-forward  $\Pi_* \omega$  for  $\omega$  so that the coupling  $\langle T, \Pi_* \omega \rangle$  makes sense. In more accurate terms,  $\Pi$  is a diffeomorphism between a Zariski-open subset of  $N$  and  $F_1 \setminus \{p_1, \dots, p_k\}$ . Thus,  $\Pi_* \omega$  is naturally a  $(1,1)$ -differential form defined on  $F_1 \setminus \{p_1, \dots, p_k\}$ . In principle, however, the form  $\Pi_* \omega$  may behave wildly near the points  $p_1, \dots, p_k$ . The proposition will follow from checking that the coupling  $\langle T, \Pi_* \omega \rangle$  is, nonetheless, well defined and yields a continuous functional on the space of  $(1,1)$ -forms on  $N$ .

It is enough to work on a neighbourhood of a point  $p_i$ . As previously seen, there are local coordinates  $(u_i, v_i)$  around  $p_i \simeq (0,0)$  where the foliation  $\mathcal{F}$  is locally given by the vector field

$$m_i u_i \frac{\partial}{\partial u_i} + n_i v_i \frac{\partial}{\partial v_i}$$

where  $m_i$  and  $n_i$  are strictly positive integers and such that  $\{u_i = 0\} \subset C_i$ . We then apply the sequence of blow ups described at the beginning of the section to the point  $p_i$  so as to remove the indetermination of the first integral  $u_i^{n_i} v_i^{-m_i}$ . Resuming the notation used in the proof of Lemma 3.1.7, we recall that the transform of  $\mathcal{L}(\mathcal{F})$  intersects transversely the corresponding exceptional divisor at a unique component  $D_{J_i}$ . In fact,  $D_{J_i}$  is a rational curve of self-intersection  $-1$  since it arises from the last (one-point) blow up performed in our blow up procedure. There are, therefore, affine (blow up) coordinates  $(t_i, s_i)$ ,  $\{t_i = 0\} \subset D_{J_i}$  on a neighbourhood of  $D_{J_i}$  satisfying the following conditions:

- (1) The foliation  $\tilde{\mathcal{F}}$  is locally given by the vector field  $\partial/\partial t_i$ .
- (2) The blow down map  $\Pi : N \rightarrow F_1$  is locally given by  $\Pi(t_i, s_i) = (t_i^{m_i} s_i, t_i^{n_i}) = (u_i, v_i)$ .

In particular, the local inverse of  $\Pi$  is determined in ramified coordinates by  $t_i = \sqrt[n_i]{v_i}$  and  $s_i = u_i / \sqrt[n_i]{v_i^{m_i}}$ .

Now let  $\omega$  be a  $(1,1)$ -differential form on the surface  $N$  whose support intersects  $D_{J_i}$ . Set

$$\omega = a_{11} dt_i \wedge d\bar{t}_i + a_{12} dt_i \wedge d\bar{s}_i + a_{21} d\bar{t}_i \wedge ds_i + a_{22} ds_i \wedge d\bar{s}_i$$

in the local coordinates  $(t_i, s_i)$ . Recall that  $\Pi_* \omega$  is defined away from the exceptional divisor of  $\Pi$ . Over the open set  $F_1 \setminus \{p_1, \dots, p_k\}$ , we have

$$\langle T, \Pi_*(a_{12} dt_i \wedge d\bar{s}_i) \rangle = \langle T, \Pi_*(a_{21} d\bar{t}_i \wedge ds_i) \rangle = \langle T, \Pi_*(a_{22} ds_i \wedge d\bar{s}_i) \rangle = 0,$$

since  $T$  is foliated and the corresponding push-forwards vanish identically over the tangent space of the foliation  $\mathcal{F}$ . Thus, it only remains to show that the coupling

$$\langle T, \Pi_*(a_{11}dt_i \wedge d\bar{t}_i) \rangle$$

is well defined on  $F_1 \setminus \{p_1, \dots, p_k\}$ . To do so, recall first that  $T$  is harmonic so that it is represented in flow-boxes by

$$S = \int_{\Sigma_\alpha} h_\alpha[\Delta_\alpha] d\mu(\alpha),$$

where  $\mu(\alpha)$  is a positive Borel measure on the transverse sections  $\Sigma_\alpha$  and where  $h_\alpha$  stand for strictly positive harmonic functions, uniformly bounded above and below by strictly positive constants. Clearly, the functions  $h_\alpha$  are Borel measurable with respect to  $\alpha$ . Thus, away from  $p_i$ , we have

$$\langle T, \Pi_*(a_{11}dt_i \wedge d\bar{t}_i) \rangle = \int_{\Sigma_\alpha} \left( \int_{L_\alpha} h_\alpha \Pi_*(a_{11}dt_i \wedge d\bar{t}_i) \right) d\mu(\alpha),$$

where  $L_\alpha$  is the corresponding leaf of  $\mathcal{F}$ . Thus, it suffices to show that the integral

$$\int_{L_\alpha} \Pi_*(a_{11}dt_i \wedge d\bar{t}_i)$$

is bounded (uniformly on  $\alpha$ ). To check that this is the case, note first that

$$\Pi_*(a_{11}dt_i \wedge d\bar{t}_i) = a_{11}(v_i^{1/n_i}, u_i v_i^{-m_i/n_i}) dt_i \wedge d\bar{t}_i.$$

In particular, the coefficient  $a_{11}(v_i^{1/n_i}, u_i v_i^{-m_i/n_i})$  is identified with a continuous function on a neighbourhood of  $p_i$  since  $u_i v_i^{-m_i/n_i}$  is actually constant over the leaves of the foliation  $\mathcal{F}$ . Meanwhile, the leaves of the foliation  $\mathcal{F}$  are parameterized by a local coordinate  $z_i \in \mathbb{C}$  satisfying  $z_i^{n_i} = v_i$  and  $z_i^{m_i} = u_i$ . Thus, we actually have  $z_i = t_i$ . In other words, the latter integral becomes

$$\int_{\Delta} a_{11}(v_i^{1/n_i}, u_i v_i^{-m_i/n_i}) dz_i \wedge d\bar{z}_i,$$

where  $\Delta$  is a disc of (uniform) positive radius around  $0 \in \mathbb{C}$ . The proposition then follows since  $a_{11}(v_i^{1/n_i}, u_i v_i^{-m_i/n_i})$  is identified with a bounded continuous function.  $\square$

*Proof of Theorem A.* Statements (a) and (b) of Theorem A were proved in Theorem 3.1.1. In turn assertion (c) follows from the combination of Proposition 3.2.4 and 3.2.5. In fact, Proposition 3.2.5 implies that positive foliated harmonic currents for  $\tilde{\mathcal{F}}$  on  $\widetilde{\mathcal{L}(\mathcal{F})}$  are in 1-1 correspondence with positive foliated harmonic currents for  $\mathcal{F}$  on  $\mathcal{L}(\mathcal{F})$  so that the existence and uniqueness of  $T$  follows from Proposition 3.2.4. This proposition also implies that the self-intersection of  $T$  must be zero.

It remains to prove assertion (d). Besides  $T$ , integration over any of the invariant lines  $C_1, \dots, C_k$  also yields foliated currents that are positive and harmonic (indeed closed). Let  $T_{C_1}, \dots, T_{C_k}$  denote these integration currents. Since  $T$  and  $T_{C_1}, \dots, T_{C_k}$  are clearly independent. It suffices to check that any positive foliated harmonic current  $T'$  is a linear combination of the previous currents. If  $T'$  is as above, up to subtracting a suitable linear combination of  $T$  and of  $T_{C_1}, \dots, T_{C_k}$  we can assume that  $T'$  gives mass neither to the invariant lines  $C_1, \dots, C_k$  nor to the Levi-flat  $\mathcal{L}(\mathcal{F})$ . To complete the proof of

Theorem A, we will show that  $T'$  as above is identically zero. For this assume aiming at a contradiction that  $T'$  is not identically zero. Then its support intersects non-trivially  $F_1 \setminus (\mathcal{L}(\mathcal{F}) \cup C_1 \cup \dots \cup C_k)$ . Since  $T'$  is harmonic (equivalently associated with an harmonic measure), there must exist leaves  $L$  of the foliation  $\mathcal{F}$  contained in the invariant open set  $F_1 \setminus (\mathcal{L}(\mathcal{F}) \cup C_1 \cup \dots \cup C_k)$  that are *recurrent*, i.e., that accumulate on themselves (c.f. [24], [26], [10]). However, this is impossible since, by construction, the transverse dynamics of the foliation  $\mathcal{F}$  on the open set in question is equivalent to the dynamics of the quasifuchsian (or Fuchian) group  $\Gamma$  on its discontinuity set and therefore wandering. This ends the proof of Theorem A.  $\square$

## Chapter 4

# Geometry of Levi-flats in the Presence of Parabolic Generators

### 4.1 Introduction

As mentioned, the material in this last chapter is part of a work in progress. To make the discussion relatively self-contained we have included some revision of specific background material, such as Proposition 2.2.1, mostly stemming from [1] but also already reviewed in the previous chapters. Besides basic material on singular holomorphic foliations already used in the previous sections and well covered, for example, in the references [30] and [45], the only additional prerequisite for the reading is some familiarity with saddle-node singularities in dimension 2, including some relatively simple facts about the corresponding Martinet-Ramis moduli space. Whereas the quoted references also cover this topic, the fundamental reference for the Martinet-Ramis moduli space is their paper [35].

### 4.2 Local Dynamics of Saddle-Node Singularity

Let us begin by recalling some basic material on Riccati foliations on the first Hirzebruch surface  $F_1$ , (c.f. [3]).

Let  $\mathcal{F}$  be a Riccati foliation defined on the surface  $F_1$ . This means that in standard affine coordinates  $(x, y)$  for the surface  $F_1$ , the foliation  $\mathcal{F}$  is induced by a holomorphic vector field  $X$  having the form

$$X(x, y) = F(x) \frac{\partial}{\partial x} + [c_0(x) + c_1(x)y + c_2(x)y^2] \frac{\partial}{\partial y}, \quad (4.1)$$

where  $F$ ,  $c_0$ ,  $c_1$  and  $c_2$  are polynomials.

Note that the foliation  $\mathcal{F}$  leaves invariant those fibres of the surface  $F_1$  where  $F$  vanish (in the above coordinates). An invariant fibre of the foliation  $\mathcal{F}$  contains at least one and at most two singular points of the foliation  $\mathcal{F}$ . In fact, singular points on an invariant fibre  $\{x = x_0\}$  are determined as roots of a degree 2 polynomial in  $y$ . When the foliation  $\mathcal{F}$  has, indeed, two singular points on a given (necessarily invariant) fibre, there follows that each of these singular points has one eigenvalue different from zero in the direction tangent to the fibre. Conversely, if there is a unique singular point, then the eigenvalue of



the foliation  $\mathcal{F}$  associated with the direction tangent to the fibre is equal to zero (the above mentioned polynomial has a double root).

An invariant fibre of the foliation  $\mathcal{F}$  is said to be *simple* if every singular point of the foliation  $\mathcal{F}$  lying in this fibre has a non-zero eigenvalue in the direction transverse to the fibre in question. In local coordinates  $(x, y)$  as above, the invariant fibres are given by the roots of  $F$  and the fibre is simple if and only if it corresponds to a *simple root* of  $F$ . In particular, if a Riccati foliation  $\mathcal{F}$  has a unique singular point  $p$  lying in a simple invariant fibre  $C$ , then this singular point has *exactly* one eigenvalue different from zero. This type of singularity is called a *saddle-node*. In this case, the foliation has a (smooth) separatrix associated with the non-zero eigenvalue which is called the *strong invariant manifold* of the foliation  $\mathcal{F}$  at  $p$ . Meanwhile, the eigenvalue associated with the separatrix induced by the invariant fibre  $C$  is equal to zero and the separatrix in question is referred to as the *weak invariant manifold* of (the foliation  $\mathcal{F}$  at  $p$ ). Our first task is to provide an accurate normal form for the foliation  $\mathcal{F}$  around one such saddle-node singular point.

**Lemma 4.2.1.** *Let  $\mathcal{F}$  be a Riccati foliation on the surface  $F_1$  and consider a simple invariant fibre  $C$  of the foliation  $\mathcal{F}$ . Assume that  $p \in C$  is a saddle-node singular point of the foliation  $\mathcal{F}$ . Then there are local holomorphic coordinates  $(u, v)$  around  $p \simeq (0, 0)$  where the foliation  $\mathcal{F}$  is given by a holomorphic vector field  $X$  having the form*

$$X(u, v) = u \frac{\partial}{\partial u} + v^2 \frac{\partial}{\partial v},$$

with  $\{u = 0\} \subset C$ .

*Proof.* It is well known that every saddle-node singularity admits a normal form known as *Dulac's normal form*, c.f., for example [30] and [45]. In other words, there are local holomorphic coordinates  $(u, v)$  around the singular point  $p \in C$  where the foliation  $\mathcal{F}$  is given by a holomorphic vector field  $X$  having the form

$$X(u, v) = [u(1 + \lambda v^r) + vR(u, v)] \frac{\partial}{\partial u} + v^{r+1} \frac{\partial}{\partial v}.$$

Here  $\lambda \in \mathbb{C}$ ,  $r \geq 1$  is an integer and  $R$  is a holomorphic function whose order at the singular point can be made arbitrarily high. Since the saddle-node in question does have a separatrix associated with the zero eigenvalue, namely the separatrix induced by the invariant fibre  $C$ , the coordinates can be chosen so that  $\{u = 0\}$  coincides with this weak separatrix (i.e., it is contained in the invariant fibre  $C$ ). Hence, the function  $R$  is divisible by  $u$ . Moreover, Equation (2.1) implies that  $r = 1$  (the above mentioned degree 2 polynomial has a double root). Dulac's normal form can hence be refined to

$$X(u, v) = [u(1 + \lambda v) + vR(u, v)] \frac{\partial}{\partial u} + v^2 \frac{\partial}{\partial v}. \quad (4.2)$$

Note that  $\{v = 0\}$  is the strong invariant manifold, i.e., the separatrix associated with the non-zero eigenvalue of the foliation  $\mathcal{F}$  at  $p$ . It is a straightforward verification that the local holonomy map associated with the strong invariant manifold of the saddle-node in (4.2) has the form  $h(v) = v + v^2 + \text{h.o.t.}$ , where “h.o.t.” stands for higher order terms. In other words, it is a germ of holomorphic diffeomorphism fixing  $0 \in \mathbb{C}$ , tangent to the identity but which is necessarily different from the identity. The conjugation class in  $\text{Diff}(\mathbb{C}, 0)$  of the holonomy map associated with the strong invariant manifold locally determines the saddle-node singularity thanks to a theorem due to Martinet-Ramis [35]. However, the foliation  $\mathcal{F}$  being Riccati,  $h$  admits a global extension as an automorphism of  $\mathbb{CP}^1$ . Thus,  $h$  has to be conjugate to the time-one map of the vector field  $v^2 \partial / \partial v$ . The time-one map of the vector field  $v^2 \partial / \partial v$  also

arises as the holonomy of the axis  $\{v = 0\}$  for the saddle-node singularity having the canonical form  $u\partial/\partial u + v^2\partial/\partial v$ , as it is immediate to check. In view of the preceding, there follows that the two saddle-node singularities must be conjugate and this establishes the lemma.  $\square$

Next, we will need a simple lemma (Lemma 4.2.2) about the local dynamics of parabolic maps arising from  $\text{PSL}(2, \mathbb{C})$ . Let  $h : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$  be the germ of a parabolic element in  $\text{PSL}(2, \mathbb{C})$  at its fixed point. Equivalently,  $h$  is the time  $t$  map arising from the 1-dimensional vector field  $z^2\partial/\partial z$  around  $0 \in \mathbb{C}$ ,  $t \in \mathbb{R}^*$ . In particular,  $h$  preserves the real axis  $\mathbb{R} \subset \mathbb{C}$ . Consider then the family of circles  $\mathfrak{C}$  consisting of those circles in  $\mathbb{C}$  with centre at the imaginary axis and passing through  $0 \in \mathbb{C}$ .

**Lemma 4.2.2.** *Every germ of smooth real-analytic curve passing through  $0 \in \mathbb{C}$  and invariant by  $h$  coincides either with the real axis  $\mathbb{R} \subset \mathbb{C}$  or with one of the circles in the family  $\mathfrak{C}$ .*

*Proof.* It suffices to consider the case where  $h$  is the time-one map induced by the vector field  $z^2\partial/\partial z$  on  $\mathbb{C}$ . By direct integration, we obtain

$$h(y) = \frac{y}{1-y}$$

which is valid in some neighbourhood  $U$  of the origin  $0 \in \mathbb{C}$ . Given  $z_0 \in U$ , we define the *forward orbit* of  $z_0$  under  $h$  by

$$\mathfrak{D}_h^+(z_0) := \{h^n(z_0) : n \in \mathbb{N} \text{ and } h^i(z_0) \in U \text{ for } i = 1, \dots, n-1\}.$$

The *backward orbit*  $\mathfrak{D}_h^-(z_0)$  of  $z_0$  under  $h$  is analogously defined.

Next, it is a straightforward computation to check that every circle in the family  $\mathfrak{C}$  is left invariant by  $h$ . Since all these circles are tangent to the real axis  $\mathbb{R}$  at the origin  $0 \in \mathbb{C}$ , every point  $z_0 \in U \setminus \mathbb{R}$  sufficiently close to the origin  $0 \in \mathbb{C}$  is such that at least one between  $\mathfrak{D}_h^+(z_0)$  and  $\mathfrak{D}_h^-(z_0)$  is defined for every  $n$ . Without loss of generality, we can assume that it is  $\mathfrak{D}_h^+(z_0)$ . Then, for every  $n \in \mathbb{N}^*$ , we set  $z_n = h^n(z_0) = a_n + ib_n$ . Since the orbit of  $z_0$  is contained in a circle of the family  $\mathfrak{C}$ , it follows that

$$\lim_{n \rightarrow +\infty} \frac{b_n}{a_n} = 0.$$

Thus, every smooth real-analytic curve  $\mathcal{W}$  passing through the origin  $0 \in \mathbb{C}$  and invariant by  $h$  must be tangent to the real axis  $\mathbb{R}$  at the origin  $0 \in \mathbb{C}$ . In particular, any such curve  $\mathcal{W}$  can locally be parameterised by  $x \mapsto (x, f(x))$  where  $x \in ]-\varepsilon, +\varepsilon[$  for some  $\varepsilon > 0$  and where  $f : ]-\varepsilon, +\varepsilon[ \rightarrow \mathbb{R}$  is a real-analytic function. In view of the preceding, we have  $f(0) = f'(0) = 0$ .

To complete the proof of the lemma, it suffices to check that the graph of  $f$  coincides with a circle in  $\mathfrak{C}$  provided that  $f$  is not identically zero. For this, note that the asymptotic behaviour of the sequence

$$(h^n(y_0) = y_n = a_n + ib_n)_{n \geq 1},$$

can be used to compute the higher order derivatives of  $f$  at the origin  $0 \in \mathbb{C}$ . Indeed, it suffices a sequence of points converging to  $0 \in \mathbb{R}$  for us to determine all the higher derivatives of a function provided that we know a priori that this function is smooth. The argument applies therefore to  $h$  as well as to the function  $g : ]-\varepsilon, +\varepsilon[ \rightarrow \mathbb{R}$  that locally parameterises the circle of  $\mathfrak{C}$  containing the orbit  $\mathfrak{D}_h^+(z_0)$ . There follows that  $f$  and  $g$  have the same Taylor series at  $0 \in \mathbb{R}$  and, since they are real analytic, they must coincide on their domain of definition. The lemma is proved.  $\square$

Recalling that a classical result due to [5] establishes that every finitely generated subgroup  $\Gamma$  of  $\mathrm{PSL}(2, \mathbb{C})$  can be realised as the (global) holonomy group of a Riccati foliation. This result was later rediscovered by Lins-Neto [33] and a slightly different proof is provided in [1]. In this sense, Proposition 2.2.1 is borrowed from [1] and provide a sort of “canonical construction”, having in particular simple fibres, once a finite generating set of the group  $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$  is fixed. However, when  $\Gamma$  as in Proposition 2.2.1 happens to be a Fuchsian group, then Lemma 4.2.1 can be made slightly more accurate as follows.

**Lemma 4.2.3.** *Let  $\mathcal{F}$  be a Riccati foliation as in Proposition 2.2.1. Assume that  $C$  is an invariant fibre for the foliation  $\mathcal{F}$  exhibiting a saddle-node singular point. Assume also that the global holonomy group  $\Gamma$  of the foliation  $\mathcal{F}$  is a Fuchsian group. Then there are local coordinates  $(u, v)$  around the saddle-node singular such that the following holds:*

- (1) *The foliation  $\mathcal{F}$  is locally given by a holomorphic vector field  $X$  having the form*

$$X(u, v) = u \frac{\partial}{\partial u} + v^2 \frac{\partial}{\partial v}.$$

- (2) *In a transverse fibre through a point  $(u_0, v_0)$  where  $u_0 \neq 0$ , the limit set  $\Lambda(\Gamma)$  of  $\Gamma$  coincides with the real axis  $\{v \in \mathbb{R}\}$ .*

*Proof.* We begin with coordinates  $(u_1, v_1)$  given by Lemma 4.2.1 where the foliation  $\mathcal{F}$  is given by a holomorphic vector field  $X_1$  having the form

$$X_1(u_1, v_1) = u_1 \frac{\partial}{\partial u_1} + v_1^2 \frac{\partial}{\partial v_1}.$$

The global holonomy group  $\Gamma$  of the foliation  $\mathcal{F}$  being Fuchsian combined to the fact that the local holonomy map  $M$  of the foliation  $\mathcal{F}$  around the invariant fibre  $C$  is a parabolic element in  $\Gamma$ , implies that the fixed point  $p$  of  $M$  belongs to the limit set  $\Lambda(\Gamma)$ . The limit set  $\Lambda(\Gamma)$  being a smooth real-analytic curve invariant by  $M$  and passing through  $p$ , it must coincide either with the real axis  $\{v_1 \in \mathbb{R}\}$  or with one of the circles in the family  $\mathfrak{C}$  (Lemma 4.2.2). Assume then that the limit set coincides with a circle in the family  $\mathfrak{C}$  and let  $a \in \mathbb{C}$  be chosen such that the Möbius transformation

$$\mathfrak{m} : v \mapsto \frac{v}{1 - av},$$

sends the real axis  $\{v \in \mathbb{R}\}$  in the coordinates  $(u, v)$  to the circle locally containing the limit set  $\Lambda(\Gamma)$  in the coordinates  $(u_1, v_1)$ . Then, the change of coordinates  $\psi$  given by

$$\psi(u, v) = (u, \mathfrak{m}(v)) = (u_1, v_1)$$

is such that

$$X(u, v) = (\psi^* X_1)(u, v) = u \frac{\partial}{\partial u} + v^2 \frac{\partial}{\partial v}.$$

Moreover, in  $(u, v)$  coordinates, it is clear that  $\Lambda(\Gamma)$  locally coincides with the real axis  $\{v \in \mathbb{R}\}$ . The lemma is proved  $\square$

### 4.3 Levi-flat in the Presence of Parabolic Generators

Proposition (4.3.1) below, which is considered to be the main content of the present section, provides some general criteria to ensure the existence of (singular) real-analytic Levi-flats for Riccati foliations on the surface  $F_1$ .

**Proposition 4.3.1.** *Let  $\mathcal{F}$  be a Riccati foliation on the surface  $F_1$  and assume that  $\mathcal{F}$  has only simple invariant fibres, denoted by  $C_1, \dots, C_k$ . Assume that the following conditions hold:*

- (a) *The local holonomy map  $M_i$  of the foliation  $\mathcal{F}$  around each invariant fibre  $C_i$  is either an elliptic or a parabolic element of  $\mathrm{PSL}(2, \mathbb{C})$  (identified with the corresponding automorphism of  $\mathbb{CP}^1$ ).*
- (b) *The global holonomy group  $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$  of the foliation  $\mathcal{F}$  is a Fuchsian group of first kind.*

*Then, there exists a set  $\mathcal{L}(\mathcal{F})$  satisfying all the following conditions:*

- (1)  *$\mathcal{L}(\mathcal{F})$  is invariant by the foliation  $\mathcal{F}$  and away from the invariant fibres is smooth and locally an analytic set of dimension 3.*
- (2) *If  $C$  is an invariant fibre associated with an elliptic (local) holonomy map, then the intersection  $C \cap \mathcal{L}(\mathcal{F})$  is reduced to a unique singular point  $p \in C$  of the foliation  $\mathcal{F}$ . Furthermore,  $\mathcal{L}(\mathcal{F})$  is (locally) real-analytic having an orbifold singular point at  $p$ .*
- (3) *If  $C$  is an invariant fibre associated with a parabolic (local) holonomy map, then the entire fibre  $C$  is contained in  $\mathcal{L}(\mathcal{F})$ . Furthermore, away from the (unique) singular point of  $\mathcal{F}$  lying in the invariant fibre  $C$ ,  $\mathcal{L}(\mathcal{F})$  has the structure of a smooth manifold with boundary (the components of the boundary coinciding with all the invariant fibres associated with parabolic holonomy maps).*
- (4) *Finally, concerning the Levi-foliation on  $\mathcal{L}(\mathcal{F})$ , all its leaves are dense apart from those compact leaves arising from the invariant fibres contained in this Levi-flat closed set.*

**Remark 4.3.2.** There follows then, as an immediate consequence of Theorem 1 (c.f. [11]), that the Levi-flat  $\mathcal{L}(\mathcal{F})$  (in Proposition 4.3.1) cannot be real-analytic at the saddle-node singular point  $O$  as the latter does not admit a meromorphic first integral. The corresponding invariant fibre  $C$  being entirely contained in the Levi-flat  $\mathcal{L}(\mathcal{F})$ , Theorem 1 (in question) cannot be used to clarify the structure of the Levi-flat  $\mathcal{L}(\mathcal{F})$  on a neighbourhood of regular point lying in the invariant fibre  $C$ .

The remainder of this section is devoted to the proof of Proposition 4.3.1. Let then  $\mathcal{F}$  be as in this statement. It follows from the local models for the foliation  $\mathcal{F}$  described in the proof of Proposition 2.2.1 that whenever the local holonomy map  $M_i$  of the foliation  $\mathcal{F}$  around the invariant fibre  $C_i$  is an elliptic element of finite order of  $\mathrm{PSL}(2, \mathbb{C})$ , the corresponding invariant fibre  $C_i$  possesses two non-hyperbolic singular points denoted by  $p_i$  and  $q_i$  and the eigenvalues of the foliation  $\mathcal{F}$  at these singular points  $p_i$  and  $q_i$  are all different from zero. In fact, the quotients of these eigenvalues are also real since the local holonomy around the invariant fibre  $C_i$  is elliptic. Thus, we can assume without loss of generality that the two eigenvalues  $\lambda_1^{p_i}$  and  $\lambda_2^{p_i}$  of the foliation  $\mathcal{F}$  at  $p_i$  are such that  $\lambda_1^{p_i}/\lambda_2^{p_i} > 0$  (i.e., the singular point  $p_i$  belongs to the Poincaré domain) while the eigenvalues  $\lambda_1^{q_i}$  and  $\lambda_2^{q_i}$  of  $\mathcal{F}$  at  $q_i$  verify  $\lambda_1^{q_i}/\lambda_2^{q_i} < 0$  (i.e., the singular point  $q_i$  belongs to the Siegel domain). On the other hand, when the local holonomy map  $M_j$  of the foliation  $\mathcal{F}$  around the invariant fibre  $C_j$  is a parabolic element of  $\mathrm{PSL}(2, \mathbb{C})$ , then the corresponding invariant fibre  $C_j$  possesses a unique singular point  $P_j$  which, incidentally, is a saddle-node singularity whose weak invariant manifold coincides with the invariant fibre  $C_j$  itself.

Consider the saturated  $\mathcal{F}(\Lambda(\Gamma))$  set by  $\mathcal{F}$  of the limit set  $\Lambda(\Gamma)$  associated with the global holonomy group  $\Gamma$ . Since the global holonomy group  $\Gamma$  of the foliation  $\mathcal{F}$  is a Fuchsian group, there follows that  $\Lambda(\Gamma)$  is identified with the (smooth, analytic) circle in  $\mathbb{CP}^1$ . Moreover, the action of  $\Gamma$  on  $\Lambda(\Gamma)$  has all orbits dense by assumption ( $\Gamma$  is of first kind). Let then  $\mathcal{L}(\mathcal{F}) = \overline{\mathcal{F}(\Lambda(\Gamma))}$ , i.e.,  $\mathcal{L}(\mathcal{F})$  is the closure of  $\mathcal{F}(\Lambda(\Gamma))$ . It is clear that  $\mathcal{L}(\mathcal{F})$  is invariant by the foliation  $\mathcal{F}$ .

The first step towards the proof of Proposition 4.3.1 consists of describing the intersection of  $\mathcal{L}(\mathcal{F})$  and a given invariant fibre  $C_i$ . The case where the invariant fibre  $C_i$  is associated with a local elliptic holonomy map is simpler in the sense that it goes back to the previous chapter. More precisely, as previously seen, in this case  $C_i$  contains exactly two singular points  $p_i$  and  $q_i$  of the foliation  $\mathcal{F}$  with  $p_i$  belonging to the Poincaré domain and  $q_i$  to the Siegel domain. Since fixed points of elliptic elements in a Fuchsian group cannot lie in the corresponding limit set, the argument used in Proposition 3.1.6 still applies to show  $C_i \cap \mathcal{L}(\mathcal{F})$  is reduced to the singular point  $p_i$ . Furthermore, around  $p_i$ , the set  $\mathcal{L}(\mathcal{F})$  is an analytic set exhibiting a single singular point which happens to be of orbifold type.

Thus, we only need to work out the structure of the intersection  $C_i \cap \mathcal{L}(\mathcal{F})$  in the case where the invariant fibre  $C_i$  contains a single singular point  $P_j$  of saddle-node type, i.e., the case where the local holonomy associated with the invariant fibre  $C_i$  is a parabolic map. Owing to Lemma 4.2.1, there are local holomorphic coordinates  $(x, y)$  on a neighbourhood  $U$  of  $P_j \simeq (0, 0)$  where the foliation  $\mathcal{F}$  is given by a holomorphic vector field  $X$  having the form

$$X(x, y) = x \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}.$$

Set  $\mathcal{K} = C_i \setminus U$  and consider a  $C^\infty$ -tubular neighbourhood  $\mathcal{T}$  of  $\mathcal{K}$ . The corresponding projection will be denoted by  $\pi : \mathcal{T} \rightarrow \mathcal{K}$ . Similarly, let  $\mathcal{F}|_{\mathcal{T}}$  denote the regular foliation on  $\mathcal{T}$  obtained by restriction of  $\mathcal{F}$ . Up to choosing  $\mathcal{T}$  is small enough, the fibres of  $\pi$  are transverse to the leaves of the foliation  $\mathcal{F}|_{\mathcal{T}}$ . In particular, the restriction of  $\pi$  to a leaf of the foliation  $\mathcal{F}|_{\mathcal{T}}$  is a local diffeomorphism from the leaf in question to  $\mathcal{K}$ . Furthermore, since  $\mathcal{K}$  is contractable (diffeomorphic to a disc),  $\mathcal{T}$  is  $C^\infty$ -diffeomorphic to  $\mathcal{K} \times B(\epsilon)$ , where  $B(\epsilon)$  stands for a small disc in  $\mathbb{C}$ . In other words, the projection  $\pi$  realises  $\mathcal{T}$  as a trivial bundle over  $\mathcal{K}$ . On the other hand,  $\mathcal{K}$  is naturally a leaf of  $\mathcal{F}|_{\mathcal{T}}$  which carries no holonomy since it is simply connected. Hence, Reeb Stability Theorem implies that the restriction of  $\pi$  to a leaf  $L$  of the foliation  $\mathcal{F}|_{\mathcal{T}}$  is actually a (global) diffeomorphism from the leaf in question to  $\mathcal{K}$ . Taking into consideration that  $\mathcal{T}$  can be identified with  $\mathcal{K} \times B(\epsilon)$ , this yields the following:

**Lemma 4.3.3.** *With the above notation, every leaf  $L$  of the foliation  $\mathcal{F}|_{\mathcal{T}}$  can be identified with the graph of a function  $\varphi : \mathcal{K} \rightarrow B(\epsilon)$ , i.e.,  $L = \{(x, \varphi(x)); \varphi : \mathcal{K} \rightarrow B(\epsilon)\}$ .*

Let us now consider the structure of the foliation  $\mathcal{F}$  on the above introduced neighbourhood  $U$  where the coordinates  $(x, y)$  are defined. Up to scaling coordinates, we can assume that  $U$  contains the bidisc of  $\mathbb{C}^2$  of radius 1. Next, denote by  $\Sigma_x$  (resp.  $\Sigma_y$ ) a local transverse section passing through the point  $(1, 0) \in \mathbb{C} \times \mathbb{C}$  (resp.  $(0, 1) \in \mathbb{C} \times \mathbb{C}$ ). In other words, set

$$\Sigma_x = \{(1, y), y \in \mathbb{C}\} \quad \text{and} \quad \Sigma_y = \{(x, 1), x \in \mathbb{C}\}.$$

According to Lemma 4.2.3 we can assume, without loss of generality, that the intersection of the limit set  $\Lambda(\Gamma)$  associated with the Fuchsian group obtained from the global holonomy of the foliation  $\mathcal{F}$  with the local section  $\Sigma_x$  coincides with the real axis. In other words, we have

$$\Lambda(\Gamma) \cap \Sigma_x = \{(1, y), y \in \mathbb{R}\}.$$

Now let  $p_0$  be a point in  $\Lambda(\Gamma) \cap \Sigma_x$  and denote by  $L_{p_0}$  the leaf of the foliation  $\mathcal{F}$  passing through  $p_0 = (1, y_0)$  ( $y_0 \in \mathbb{R}$ ). Clearly, if  $y_0 = 0$  then the leaf  $L_{p_0}$  in question is nothing but the strong invariant manifold (locally given by  $\{y = 0\}$ ). This leaf defines a smooth separatrix for the singular point and it represents the (unique) fixed point of the parabolic holonomy map arising from the invariant fibre  $C_i$ .

Let us now consider the more interesting cases where  $y_0 \neq 0$ . To begin with, let  $\pi_x(x, y) = x$  and  $\pi_y(x, y) = y$  be the canonical projections in the coordinates  $(x, y)$ . Then notice that the fibres of these projections are transverse to the leaves of the foliation  $\mathcal{F}$  that are different from the (invariant) coordinate axes. In the sequel, whenever lifting paths in leaves of the foliation  $\mathcal{F}$  are considered, the corresponding lift will be taken with respect to one of these projections (the context will make clear which one should be used).

To discuss the local geometry of the leaf  $L_{p_0}$  on the neighbourhood  $U$ , the parabolic holonomy map  $M$  whose fixed point is naturally represented by axis  $\{y = 0\}$  has to be taken into account. To make the discussion clearer, it is convenient to proceed as follows. First let  $\mathbb{D} \subset \{y = 0\} \simeq \mathbb{C}$  be the unit disc around  $0 \in \mathbb{C}$  and set  $\mathfrak{R} = \mathbb{D} \setminus \sqrt{-1}\mathbb{R}_+$  so that  $\mathfrak{R}$  is simply connected. Now given  $p_0 = (1, y_0)$ , we consider the maximal lift  $\mathfrak{R}_{p_0}$  of  $\mathfrak{R}$  in  $L_{p_0}$  which is initiated at the point  $p_0 = (1, y_0)$ . Here it should be noted that whereas  $\mathfrak{R}_{p_0}$  may not be globally defined (the path-lifting method may lead to paths leaving the neighbourhood  $U$ ), there is a well-defined maximal lift on which the projection  $\pi_x : \mathfrak{R}_{p_0} \rightarrow \mathfrak{R}$  is one-to-one.

We can now investigate the behaviour of  $\mathfrak{R}_{p_0} \subset L_{p_0}$ . To do so, note that the leaves of the foliation  $\mathcal{F}$  on the neighbourhood  $U$  are determined by the equation

$$x = \text{const} \exp\left(\frac{-1}{y}\right),$$

where  $\text{const}$  is a complex constant. In other words, the leaves are the graphs (over the  $y$ -axis) of the function  $y \mapsto \text{const} \exp(-1/y)$ . Letting  $y = \text{Re}(y) + i\text{Im}(y)$ , we obtain

$$|x| = c_1 \exp\left(-\frac{\text{Re}(y)}{|y|^2}\right), \quad (4.3)$$

for a suitable constant  $c_1$ . Next, consider also the path

$$c : [0, 1] \rightarrow \mathbb{R} \subset \mathbb{C}$$

given by

$$c(t) = (1 - t, 0).$$

Denote by  $\tilde{c}_{p_0}$  the lift of the path  $c$  in  $\mathfrak{R}_{p_0} \subset L_{p_0}$ . Recall that we have  $p_0 = (1, y_0)$  with  $y_0 \in \mathbb{R}$  and  $y_0 \neq 0$  (since  $y_0 = 0$  coincides with the invariant axis  $\{y = 0\}$ ). Then two cases can occur, namely:

- (a) If  $\text{Re}(y_0) > 0$ . The lift  $\tilde{c}_{p_0}$  is clearly well defined for every  $t \in [0, 1]$  and converges towards the origin  $0 \in \mathbb{C}$ . In fact, the entire domain  $\mathfrak{R}_{p_0}$  is globally well-defined and diffeomorphic to  $\mathfrak{R}$ . As a matter of fact, the reader will note that the behaviour of the foliation  $\mathcal{F}$  on the region  $\{(x, y) \in U, \text{Re}(y) > 0\}$  is somehow similar to a “sink” (or “node”) in the sense that all leaves converge to the origin.
- (b) If  $\text{Re}(y_0) < 0$  then Formula (4.3) shows that the absolute value of  $y$  increases as the absolute value of  $x$  decreases. Indeed, the subset of  $U$  consisting of those points of the form  $\{(x, y) \in U, \text{Re}(y) < 0\}$

is known as the “saddle” part of the saddle-node singularity. Thus, the lift  $\tilde{c}$  will not be entirely defined since  $\tilde{c}$  will leave the neighbourhood  $U$  and *enter* the region described in Lemma 4.3.3 where the leaves of the foliation  $\mathcal{F}$  are graphs over  $\mathcal{K} = C \setminus U$ . Finally, note also that the points at which  $L_{p_0}$  leaves the neighbourhood  $U$  (thus preventing to further extend  $\mathfrak{R}_{p_0}$ ) accumulate on regular points of the invariant fibre  $C_i$  as  $y_0 \rightarrow 0^-$ . In particular, there follows that the entire invariant fibre  $C_i$  is contained in  $\mathcal{L}(\mathcal{F}) = \overline{\mathcal{F}(\Lambda(\Gamma))}$ .

The main conclusions of the above discussion, that will be needed for the subsequent sections, can be summarised as follows:

- Claim (1) If  $\operatorname{Re}(y_0) > 0$ , then  $\mathfrak{R}_{p_0}$  converges to the saddle-node singular point  $P_i \simeq (0, 0)$ . Moreover, it is straightforward to check that  $\mathfrak{R}_{p_0}$  is uniformly transverse to the fibres of  $\pi_x$ . Similarly, the area of  $\mathfrak{R}_{p_0}$  is comparable to the area of  $\mathfrak{R}$  (here the areas can simply be defined as subsets of  $\mathbb{C}^2$ ).
- Claim (2) If  $\operatorname{Re}(y_0) < 0$ , then the lift  $\mathfrak{R}_{p_0}$  of  $\mathfrak{R}$  is not globally defined since the corresponding leaf  $L_{p_0}$  leaves the neighbourhood  $U$  and enters the tubular neighbourhood  $\mathcal{T}$  of  $\mathcal{K} = C_i \setminus U$  described in Lemma 4.3.3. After “wrapping around”  $\mathcal{K}$ , the corresponding branch of the leaf  $L_{p_0}$  will also intersect the region of  $U$  in which  $\mathcal{F}$  behaves like a “sink” and, then, converge to the origin as described in Claim (1). Note that, in this case, the maximal lifts  $\mathfrak{R}_{p_0}$  are no longer uniformly transverse to the fibres of  $\pi_x$ . Yet, the area of the corresponding branch of  $L_{p_0}$  (after wrapping around the invariant fibre  $C_i$ ) becomes comparable to the area of the invariant fibre  $C_i$  plus the area of  $\mathfrak{R}$ .

We are now able to establish Proposition 4.3.1.

*Proof of Proposition 4.3.1.* As pointed out in item (a), every invariant fibre associated with a parabolic (local) holonomy map is entirely contained in  $\mathcal{L}(\mathcal{F}) = \overline{\mathcal{F}(\Lambda(\Gamma))}$ . In view of the quoted results of the previous two chapters in the case of invariant fibres associated with elliptic (local) holonomy maps, the only point that still need clarification is the structure of  $\mathcal{L}(\mathcal{F})$  on a neighbourhood of a point  $Q$ , which is regular for the foliation  $\mathcal{F}$ , and lies in an invariant fibre  $C_i$  associated with a parabolic holonomy map. For this, we are going to determine the intersection of  $\mathcal{L}(\mathcal{F})$  with the local transverse section  $\Sigma_y \subset U$ . In fact, we will prove below that

$$\Sigma_y \cap \mathcal{L}(\mathcal{F}) = \{(x, 1), x \geq 0\}$$

which, in turn, establishes our claim about the (local) structure of a smooth manifold with boundary for  $\mathcal{L}(\mathcal{F})$  around regular points in the invariant fibre  $C_i$ .

Note that  $\Sigma_y$  is naturally equipped with the coordinate  $x$  so that the point  $(0, 1) \in C_i \cap \Sigma_y$  can be identified with  $0 \in \Sigma_y$ . Let then  $x_0 \simeq (x_0, 1)$  be a point in  $\Sigma_y \cap \mathcal{L}(\mathcal{F})$  and consider the leaf  $L_{x_0}$  of the restriction of the foliation  $\mathcal{F}$  to  $U$  passing through  $x_0$ . Since  $x_0 \simeq (x_0, 1)$  belongs to  $\mathcal{L}(\mathcal{F})$ , there follows that  $L_{x_0}$  intersects the section  $\Sigma_x$  at points lying in the limit set of the global holonomy group. In other words, at points of the form  $(1, y_1)$  with  $y_1 \in \mathbb{R}$ .

Next, recall that  $L_{x_0}$  is given by the equation

$$x = ex_0 \exp\left(\frac{-1}{y}\right)$$

(where the value of the multiplicative constant is determined by the fact that  $(x_0, 1)$  belongs to  $L_{x_0}$ ). The intersection points  $(1, y_1)$  of  $L_{x_0}$  and  $\Sigma_x$  must hence verify

$$y_1 = \frac{1}{\ln(ex_0)}.$$

where the multivaluedness of the logarithm accounts for the effect of the (local) holonomy map around the invariant fibre  $C_i$ . Setting

$$\ln(ex_0) = 1 + \ln(|x_0|) + i \arg(x_0),$$

we conclude that  $y_1$  is real if and only if it is positive real. Therefore,

$$\Sigma_y \cap \mathcal{L}(\mathcal{F}) = \{(x, 1), x \geq 0\},$$

as claimed. The proof of Proposition 4.3.1 is complete. □





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