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Polarizations in phase space: Boundary conditions and gravitational charges

Affidavit

I, undersigned, Gloria Odak, hereby declare that the work presented in this manuscript is my own work, carried out under the scientific supervision of Simone Speziale, in accordance with the principles of honesty, integrity and responsibility inherent to the research mission. The research work and the writing of this manuscript have been carried out in compliance with both the french national charter for Research Integrity and the Aix-Marseille University charter on the fight against plagiarism.

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Liste de publications et participation aux conférences

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- **G. Odak and S. Speziale**, *Brown-York charges with mixed boundary conditions*, JHEP **11** (2021) 224, 2109.02883.
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- **G. Odak, A. Rignon-Bret and S. Speziale**, *General gravitational charges on null hypersurfaces*, (2023), 2309.03854.

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- Quantum Gravity Around the Corner, Waterloo, Canada
- Loops conference, Lyon, France
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- Gravitational Waves - A new window to the Universe

Résumé

L'édifice de la relativité générale (RG) repose sur un principe fondamental : l'invariance par difféomorphisme [1]. Ce principe stipule que les lois de la physique doivent rester invariantes sous des transformations de coordonnées lisses. Il témoigne de l'universalité de l'interaction gravitationnelle et constitue un principe fondamental de la théorie. Mais que signifie l'invariance par difféomorphisme et pourquoi occupe-t-elle une place si importante en RG ?

Fondamentalement, l'invariance par difféomorphisme reflète la nature de jauge de la relativité générale. Elle résume l'idée selon laquelle le système de coordonnées précis que nous utilisons pour décrire l'espace-temps n'a pas de signification physique fondamentale. En d'autres termes, tous les choix de coordonnées sont équivalents dans leur capacité à décrire les phénomènes physiques. Cette perspective met en évidence une rupture magnifique et radicale avec la mécanique classique : en RG, l'espace-temps n'est pas une simple toile de fond passive, mais un acteur actif. Il peut se déformer, se tordre et se courber en réponse à la présence de matière ou d'énergie.

En RG, ce que perçoit un observateur peut être radicalement différent du point de vue d'un autre, en fonction de leur mouvement relatif et de leur position dans le champ gravitationnel. Étant donné que les différents observateurs sont représentés mathématiquement comme des choix de coordonnées différents, et que la covariance générale est fondamentale, cela soulève la question fondamentale de savoir ce qui constitue véritablement une "observable" en RG. Les réponses ne sont ni directes ni intuitives, et exigent une combinaison subtile de mathématiques et de connaissances physiques.

L'une des manifestations les plus captivantes de la RG est la propagation des ondes gravitationnelles, c'est-à-dire des ondulations dans le tissu de l'espace-temps causées par des événements astrophysiques tels que la fusion de trous noirs ou d'étoiles à neutrons. Ces phénomènes, prédits par Einstein il y a plus d'un siècle [2], ont échappé à l'observation directe jusqu'à récemment [3]. Einstein lui-même s'est battu avec cette idée, doutant parfois de l'existence physique de ces ondes. La question était de savoir si les ondes gravitationnelles étaient simplement un produit de la théorie linéaire qui pouvait être éliminé par un changement de coordonnées, ou si elles pouvaient potentiellement transférer de l'énergie sur de grandes distances dans le cadre de la théorie non linéaire. Ce débat,

alimenté par les difficultés inhérentes à la formulation de la question dans un contexte significatif, s'est étendu sur quatre décennies [4].

La difficulté consistait à isoler les aspects propagatifs des dix composantes du champ métrique, en particulier en l'absence d'une prescription canonique que la relativité générale ne pouvait pas fournir. L'idée clé est apparue dans les années 1960 lorsque Bondi, Metzner, van der Burg [5], et Sachs [6, 7] ont établi un cadre robuste. Ils ont conçu un système de coordonnées qui tire parti du fait que le rayonnement gravitationnel se déplace le long des cônes de lumière. Un résultat notable est la formule de perte de masse de Bondi, qui fournit une preuve théorique de l'existence des ondes gravitationnelles, même dans le cadre d'une théorie non linéaire. Ces résultats ont été d'une grande importance, car ils ont non seulement clarifié la nature des ondes gravitationnelles, mais aussi modifié notre compréhension des symétries dans les régions asymptotiques de l'espace-temps. Dans les régions éloignées des sources gravitationnelles, où l'espace-temps retrouve sa forme plate, on pouvait s'attendre à ce que seules les isométries exactes de l'espace de Minkowski (le groupe de Poincaré) soient présentes. Mais ils ont révélé un groupe de symétrie élargi. En effet, ils ont découvert que le groupe de symétrie asymptotique, désormais connu sous le nom de groupe BMS, allait au-delà du groupe de Poincaré : les quatre translations de Poincaré ont été étendues à un sous-groupe normal infini de supertranslations, qui sont des translations avec une dépendance angulaire, révélant des symétries de l'espace-temps bien plus complexe que ce qui avait été anticipé.

Les observations expérimentales d'ondes gravitationnelles, cependant, n'ont été faites que récemment [3], témoignant du pouvoir prédictif de la RG et des avancées technologiques d'observatoires tels que LIGO et Virgo. Leur détection a été saluée comme un triomphe, non seulement de la prouesse expérimentale, mais aussi comme un témoignage du pouvoir prédictif de la RG. Une prédiction qui, autrefois, restait à l'horizon de la testabilité, a maintenant été mise en lumière grâce aux détections révolutionnaires d'observatoires tels que LIGO et Virgo. Pourtant, ces ondes portent en elles des empreintes subtiles, des indices des mécanismes qui les ont produites et qui peuvent être étudiés en utilisant les symétries sous-jacentes et leurs charges.

Lorsque les bras de LIGO vibrent en réponse à une onde gravitationnelle, il ne s'agit pas seulement d'un événement isolé sur Terre. Géométriquement, ces bras sont positionnés à l'infini futur de type lumière d'un espace-temps idéalisé et asymptotiquement plat. Le cataclysme à l'origine de ces ondulations - peut-être la fusion de deux trous noirs - se déroule dans les profondeurs de cet espace-temps. Pourtant, ce sont ces événements lointains qui envoient des échos gravitationnels, culminant dans les minuscules oscillations que les instruments de LIGO, d'une sensibilité exceptionnelle, peuvent détecter. La modélisation d'un tel système n'est pas triviale. L'imposition de frontières et de conditions aux limites associées est cruciale pour cette description. Celles-ci décrivent le comportement des champs gravitationnels (et de tout autre champ) lorsqu'ils s'approchent de ces limites et

interagissent avec elles. Mais lorsqu'on introduit des frontières dans la partie, il faut tenir compte d'importantes subtilités liées aux symétries de jauge [8, 9, 10].

En l'absence de frontières, les symétries de jauge servent principalement à signifier les redondances dans notre description des systèmes physiques. Elles encodent la notion que de nombreuses configurations différentes des champs fondamentaux - que ce soit le tenseur métrique de la relativité générale ou les champs de jauge de la physique des particules - peuvent représenter le même état physique [11]. Cela revient à dire que des coordonnées différentes peuvent décrire le même point dans l'espace. Dans un tel contexte, les symétries de jauge ne sont pas directement observables. Elles sont un sous-produit de nos descriptions mathématiques plutôt que des caractéristiques inhérentes au monde physique. Cependant, l'introduction de frontières modifie considérablement cette image. Lorsque notre espace-temps, ou tout autre système décrit par une théorie de jauge, possède des frontières, certaines symétries de jauge acquièrent une signification physique. Pourquoi ? Parce qu'en présence de ces frontières, elles ont une charge non-nulle. Cela signifie que ces transformations, au lieu d'être de simples acteurs d'arrière-plan, deviennent physiquement pertinentes. Elles peuvent faire la distinction entre des états physiques distincts, en étiquetant de véritables degrés de liberté.

On pourrait alors se demander : pourquoi cela est-il important ? Pour le comprendre, il faut se rappeler que dans la compréhension conventionnelle des théories de jauge, les symétries de jauge sont de simples artefacts mathématiques. Elles ne correspondent pas à des transformations physiques réelles. Elles apparaissent comme des redondances dans notre description, ce qui implique que plusieurs configurations de champs mathématiquement distinctes peuvent représenter le même état physique. Habituellement, ces redondances sont éliminées. En identifiant les transformations de jauge qui ont des implications physiques, nous comprenons et différencions ce qui peut être mesuré, de ce qui peut être observé et de ce qui reste un simple choix de coordonnées et peut être éliminé en toute sécurité.

Cette transition de la jauge à la physique peut être décrite comme l'émergence de "modes de bord" ou de degrés de liberté associés aux frontières, qui peuvent porter des informations physiques et même médier les interactions entre l'intérieur et la frontière du système [12, 13, 14, 15, 16]. Ces symétries physiques induites par les limites jouent un rôle crucial dans plusieurs domaines de la physique théorique. Par exemple, dans le contexte de la physique de la matière condensée, elles peuvent conduire à des états exotiques de la matière tels que les isolants topologiques, où les états de la frontière ou du bord transportent du courant alors que l'intérieur reste isolant. De même, dans le contexte gravitationnel, ces symétries permettent de mieux comprendre la structure de l'espace-temps, ce qui peut avoir des implications importantes en théorie quantique.

Nous constatons que la définition du comportement d'un système sur ses bords est aussi cruciale que la compréhension de son comportement à l'intérieur. Le comportement

du système sur ses bords est régi par des conditions aux limites. Bien que souvent subtiles dans leurs implications, elles jouent un rôle indispensable dans la formulation des théories physiques, en donnant une image plus claire de la dynamique globale. Elles garantissent que les problèmes sont bien posés, en guidant les systèmes dans leurs interactions avec les contraintes environnantes et en apportant des solutions significatives et cohérentes. Classiquement, les conditions aux limites ont été classées en différents types : Dirichlet, où les valeurs des variables de champ sont fixées aux bords ; Neumann, où les flux des champs sont spécifiés ; et Robin, un hybride des deux premiers [17]. Chaque type offre des perspectives uniques sur la façon dont un système se comporte et interagit avec son environnement, chacune étant adaptée à des scénarios spécifiques et à des motivations physiques. Ces conditions servent de lignes directrices pour la résolution des équations différentielles, garantissant que les solutions sont à la fois uniques et physiquement significatives. D'où l'intérêt d'étudier comment les conditions aux limites affectent la compréhension de la dynamique gravitationnelle.

Historiquement, dans le contexte de la physique gravitationnelle, les conditions aux limites de Dirichlet ont été le choix conventionnel. Cependant, ce choix n'est pas incriticable. Pour apprécier les nuances, il est utile d'opposer la relativité générale aux théories présentant une structure fixe de l'espace-temps. Dans ce dernier scénario, où le champ métrique est immuable et sert de simple toile de fond, les conditions de Dirichlet et de Neumann sont toutes deux appropriées du point de vue du problème de la valeur limite initiale (IBVP) [18, 19, 20]. Elles garantissent la bonne pose du problème de la valeur initiale et sont relativement simples dans leurs implications. Cependant, l'absence de champs d'arrière-plan fixes complique cette situation. Anderson et An [21, 22] soutiennent que, contrairement aux théories ayant une structure spatio-temporelle fixe, en relativité générale, l'utilisation directe de données aux limites de Dirichlet ou de Neumann donne lieu à des problèmes mal posés pour la gravité dans le vide. Il faut chercher des conditions aux limites qui respectent la dynamique et les contraintes inhérentes à la relativité générale, et une proposition notable dans ce contexte provient de York dans les années 1980 [23]. Il a défini un ensemble de conditions aux limites qui combinent des éléments des types Dirichlet et Neumann. Au lieu de restreindre complètement la métrique induite à la frontière, les conditions de York fixent la métrique conforme et son moment conjugué, qui est proportionnel à la courbure extrinsèque. Cette approche offre non seulement une solution géométriquement élégante, mais garantit également une dynamique cohérente avec les principes de la relativité générale.

En laissant de côté l'IBVP, l'étude des conditions aux limites nous conduit naturellement à la question des Lagrangiens de bord. Ces termes spécifiques aux limites de l'action apparaissent de manière évidente lorsque l'on modifie les conditions aux limites. Dans le cas de la relativité générale, pour tous les types de conditions aux limites étudiés, ils sont proportionnels à la divergence de la normale de bord. Dans les cas non nuls, cela se

traduit par la trace de la courbure extrinsèque, et pour les conditions de Dirichlet, cela se manifeste par le terme bien connu de Gibbons-Hawking-York [24, 25].

L'étude des systèmes classiques se divise traditionnellement en deux cadres principaux : l'approche lagrangienne et l'approche hamiltonienne. Alors que l'approche lagrangienne se concentre sur le principe d'action et les équations du mouvement, la méthode hamiltonienne met l'accent sur l'espace des phases, les variables canoniques et leur évolution. Le concept de l'espace de phase covariant (covariant phase space, CPS) [26, 27, 28, 29, 30, 31, 32, 33, 34], un cadre élégant qui combine les points forts des deux méthodes, jette un pont entre ces deux méthodologies. Le cadre CPS s'écarte des vues traditionnelles dans sa conceptualisation de l'espace des phases. Au lieu de le comprendre comme un espace de conditions initiales (typique de la mécanique hamiltonienne), il est conçu comme l'espace des solutions aux équations du mouvement [28]. Cette réinterprétation, subtile mais profonde, présente une série d'avantages. Notamment, elle élimine la nécessité d'identifier une direction temporelle ou une foliation spécifique. Au contraire, elle préserve naturellement la nature covariante du système, en adhérant aux principes de la relativité et en garantissant que les descriptions restent inchangées sous l'effet des transformations de coordonnées.

Un tel changement conceptuel affecte également la façon dont nous abordons les conditions aux limites. Dans l'espace des phases covariant, les conditions aux limites ne sont pas simplement des contraintes imposées au système, mais peuvent être considérées comme un choix de polarisation dans le collecteur symplectique représentant l'espace des phases. Cette perspective permet une compréhension plus intrinsèque des conditions aux limites, en les associant aux propriétés géométriques de l'espace des champs.

Le théorème de Noether est un pilier essentiel de la physique théorique qui trouve une nouvelle voix dans le contexte de l'espace des phases covariant. Posés à l'origine par Emmy Noether au début du 20e siècle, ses théorèmes établissent un lien profond entre les symétries d'un système et ses quantités conservées. Lorsqu'ils sont reformulés dans le langage de l'espace de phase covariant, les théorèmes de Noether acquièrent une certaine élégance. Ils élucident la relation complexe entre les symétries des solutions dans l'espace des phases et les quantités conservées qui leur sont associées. La nature de jauge des difféomorphismes est encodée dans le fait qu'ils sont des modes zéro de la forme pré-symplectique – les charges qui les génèrent sont des termes de bord purs qui, en l'absence de frontières, s'évanouissent lorsque les équations du mouvement sont satisfaites. Lorsque l'on considère des régions avec des bords, les transformations de jauge peuvent être divisées en deux catégories : celles qui correspondent toujours à des dégénérescences de la forme symplectique et celles qui acquièrent des charges non nulles supportées à la frontière. Les premières sont parfois appelées "transformations de jauge larges", rappelant le fait qu'elles sont de jauge à l'intérieur et que la présence d'une structure de fond sous la

forme d'une frontière est ce qui rompt la covariance et les rend physiques et, en principe, observables.

En substance, l'exploration du CPS sert de testament à l'adaptabilité et à la résilience de la mécanique classique, prouvant sa pertinence même dans des contextes modernes. En entremêlant les principes fondamentaux et les interprétations contemporaines, elle offre un terrain fertile pour de nouvelles perspectives, enrichissant davantage notre compréhension des systèmes physiques. Cependant, malgré la clarté qu'apporte la formulation covariante, elle n'est pas exempte de nuances. Plusieurs ambiguïtés apparaissent lorsque l'on tente de définir les quantités conservées dans ce cadre. La résolution de ces ambiguïtés nécessite certaines prescriptions afin de garantir que les quantités dérivées ont une interprétation physique claire. Une procédure de définition des charges dans ce cadre a été présentée dans le travail fondateur de Wald et Zoupas [33]. Ces charges satisfont une modification de l'équation de Hamilton dans laquelle la forme symplectique évaluée sur une variation de difféomorphisme donne la variation de la charge, plus un terme supplémentaire représentant le flux. Elles correspondent aux charges ADM et BMS, associées aux bords asymptotiques de type temps et lumière respectivement, et ce procédé peut être étendu aux bords à distance finie. Cette séparation entre la charge et le flux n'est pas unique et une prescription est nécessaire pour la fixer. Dans le cadre WZ, cela se fait en utilisant certains choix physiquement justifiés qui nécessitent des hypothèses de covariance qui ne sont pas toujours satisfaites [35].

Un cadre plus général a été développé récemment par de nombreux auteurs [34, 36, 37, 38, 39, 40, 14] exploitant diverses ambiguïtés [41] survenant dans la définition du potentiel symplectique du lagrangien pour donner une définition des charges de surface basée sur le choix de la polarisation de l'espace des phases. La charge est définie sur un ensemble de conditions aux limites conservatrices (par exemple, [40, 42]) ou sur un sous-espace de configurations de champs stationnaires (par exemple, [36]), et cette définition est ensuite étendue au reste de l'espace des phases, définissant de manière unique la division charge-flux pour une polarisation donnée. Comme la prescription utilise la structure de bord pour sélectionner les ambiguïtés dans la définition de la charge de Noether, nous l'appelons *charge de Noether améliorée*.

Une autre solution au problème de l'intégrabilité de la charge peut être obtenue en étendant l'espace des phases par l'introduction d'un ensemble de champs dynamiques supplémentaires qui proviennent de l'encastrement des coins dans l'espace-temps et sont connus sous le nom de modes de bord [14, 43, 44]. Dans cette approche, même si le système gravitationnel est ouvert et peut avoir des degrés de liberté qui quittent et entrent dans le système, les charges de symétrie sont toujours hamiltoniennes. Ceci est possible parce que les degrés de liberté dissipatifs dans l'évolution du coin sont encodés dans les variations non triviales des encastrements. Indépendamment de la perspective choisie, les

deux approches doivent être deux manières différentes de comprendre la même physique sous-jacente.

La prescription WZ reproduit non seulement les célèbres charges d'Arnowitt-Deser-Misner (ADM) pour l'infini spatial, mais étend également son application aux charges de Geroch et de Dray-Streubel associées à l'infini futur nul. Des travaux récents ont encore élargi cette approche, en considérant des hypersurfaces nulles à des distances finies et des horizons localement stationnaires [36, 45]. D'autre part, en utilisant la prescription de charge de Noether améliorée, [40, 38] ont montré comment inclure des difféomorphismes arbitraires dépendant du champ et des anomalies – des quantités dépendantes du fond dont la transformation de l'espace de champ sous les difféomorphismes diffère de la dérivée de Lie. L'inclusion de difféomorphismes dépendant des champs et d'anomalies dans l'espace des phases covariant est cruciale. Ils apparaissent dans divers contextes, tels que les extensions des vecteurs de symétrie de bord. Ils jouent également un rôle important dans l'approche canonique. Les anomalies, qui apparaissent souvent lorsque des structures d'arrière-plan sont impliquées, sont cruciales pour les observables de le bord. Le fait de les reconnaître et de les traiter systématiquement peut considérablement accroître l'utilité du CPS. La prescription WZ peut-elle être appliquée dans ce contexte plus général ? Il est utile de répondre à cette question pour de futures recherches, mais aussi pour mieux comprendre la relation précise entre la littérature récente et [33]. En particulier, les difféomorphismes dépendent du champ et les anomalies apparaissent dans l'étude du groupe BMS. Cela soulève une question – comment se fait-il que Wald et Zoupas aient pu dériver les charges BMS sans inclure l'une ou l'autre de ces deux caractéristiques dans leur description ? Nous avons découvert que, bien que WZ n'ait pas spécifiquement pris en compte ces éléments, leur approche reposait intrinsèquement sur certaines hypothèses de covariance. En traduisant ces hypothèses, nous révélons que la méthode WZ peut traiter certains difféomorphismes dépendant des champs et anomalies, mais seulement pour des conditions spécifiques. Nous appelons les anomalies acceptables "anomalies douces".

En poursuivant notre étude, nous juxtaposons la prescription WZ aux charges de Noether améliorées et montrons que les charges WZ peuvent en effet être reformulées comme des charges de Noether améliorées, à condition que des conditions spécifiques soient remplies. Cette réconciliation accentue l'adaptabilité et la robustesse inhérentes au cadre WZ dans les interprétations modernes. En termes pratiques, nous avons testé nos conclusions à l'aide de quatre scénarios. Le test le plus complexe a été réalisé à l'infini futur de type lumière, où nous avons pu montrer de BMS standard et mieux comprendre comment WZ obtenait des résultats sans traiter directement les anomalies. Bien que la dérivation des charges BMS en tant que charges de Noether améliorées ait été discutée précédemment dans [46] et [38], ces articles s'appuyaient sur des connaissances préalables des charges BMS. Notre approche est plus fondamentale et s'appuie directement sur les conditions WZ.

En raison de sa dépendance explicite à l'égard des lagrangiens de bord, la prescription améliorée de la charge de Noether rend l'étude de l'effet du changement des conditions de frontière simple. Un fait important en RG est que l'énergie n'est pas strictement conservée en raison de la dissipation causée par le rayonnement gravitationnel. Cependant, si l'on se concentre uniquement sur les espaces de temps non radiatifs, l'idée d'une énergie conservée dans l'espace des phases devient réalisable. Une notion d'énergie conservée est donnée par la charge ADM à l'infini spatial, que les ondes gravitationnelles n'atteignent jamais, ainsi que sa contrepartie quasi-locale, la charge de Brown-York (BY). Toutefois, cette conception de l'énergie n'est pas universelle. Elle dépend fortement des conditions aux limites spécifiques utilisées pour éliminer la dissipation. En effet, les calculs des charges ADM et BY nécessitent l'imposition de conditions aux limites de Dirichlet, que ce soit aux limites asymptotiques ou finies. Au chapitre 8, nous étudions ce qu'il advient des valeurs d'énergie lorsque nous passons des conditions de Dirichlet aux conditions de Neumann, en passant par les conditions mixtes de York. Ce processus, qui implique différents degrés de fixation de la métrique induite et de sa quantité de mouvement (la courbure extrinsèque), tend à réduire les énergies. Par exemple, dans le cas de l'espace-temps de Kerr, les valeurs de l'énergie sont respectivement de M, $2M/3$ et $M/2$.

Pour parvenir à ces conclusions, deux méthodes principales ont été utilisées. La première méthode est l'espace des phases covariant, particulièrement puissant car il peut décrire à la fois des scénarios radiatifs et non radiatifs. Ici, l'objectif est de comprendre comment la restriction du flux symplectique à la frontière sélectionne un potentiel symplectique particulier, conduisant à des charges intégrables. C'est l'idée de base de la prescription de charge de Noether améliorée, qui fournit une approche universelle pour les charges, valable pour diverses conditions aux limites.

Notre deuxième outil est une analyse canonique directe fondée sur la décomposition ADM. Cette méthode est particulièrement adaptée au contexte non radiatif. Notamment, lorsque nous étendons l'analyse pour inclure différentes conditions aux limites, nous observons que cette méthode canonique produit des charges qui s'alignent parfaitement avec celles déduites de la prescription de charge de Noether améliorée. Cet alignement renforce la validité de la formule de charge de Noether améliorée.

Pour aller plus loin, nous introduisons une intersection non orthogonale entre la frontière temporelle et les hypersurfaces spatiales. Lorsque ces coins non orthogonaux sont introduits, le principe variationnel nécessite un terme 2D supplémentaire qui, dans le cas de conditions limites de Dirichlet, est donné par le lagrangien de Hayward. Étant donné que notre objectif principal dans la construction de charges de surface covariantes tourne autour d'un principe variationnel bien défini, il est essentiel de vérifier si ce lagrangien de coin influence également la formule de charge. Pour répondre à cette question, les calculs utilisant la prescription de charge de Noether améliorée ont été répétés, cette fois en tenant compte des coins non orthogonaux, et les résultats ont été juxtaposés

aux résultats canoniques. Le calcul canonique a mis en évidence le fait qu'un coin non orthogonal introduit des changements à la fois physiques et mathématiques. D'un point de vue physique, on peut considérer deux catégories distinctes d'observateurs : ceux qui sont au repos par rapport à la foliation de l'espace-temps et ceux qui sont au repos le long de la frontière temporelle. A' l'intersection entre la frontière de type temps et la foliation (les coins), ces deux observateurs sont liées par une transformation de Lorentz boosté locale, ce qui rend les différentes notions d'énergie directement comparables. Mathématiquement, cela se manifeste par des "termes d'inclinaison" supplémentaires dans les charges, qui dépendent explicitement de l'angle hyperbolique local entre les normales. Il est important de noter que nos résultats ont confirmé la cohésion entre les résultats covariants et canoniques, même dans le contexte de coins non orthogonaux, attestant ainsi de la robustesse de la prescription de charge de Noether améliorée.

En nous concentrant sur les frontières de genre lumière, nous ne sommes plus limités aux conditions limites conservatives et nous pouvons également explorer l'aspect dynamique. Les développements récents de Chandrasekaran, Flanagan et Prabhu (CFP) ont caractérisé le groupe de symétrie de la relativité générale sur les hypersurfaces génériques de genre lumière comme une extension du groupe BMS pour tenir compte des difféomorphismes arbitraires et des transformations de Weyl sur n'importe quelle section transversale de type espace 2d [36].[1] En utilisant la procédure Wald-Zoupas (WZ), ils ont dérivé des charges qui maintiennent la conservation sur des hypersurfaces sans cisaillement et sans expansion, les horizons non expansifs (NEH) étant un excellent exemple. Une exploration détaillée ultérieure des NEH a été présentée dans [45], mettant en lumière la façon dont la composante globale des transformations de Weyl équivaut à la charge de surface. Les mêmes transformations de Weyl jouent également un rôle prépondérant dans l'étude des éventuels cheveux doux des trous noirs par Hawking, Perry et Strominger [47].

Un défaut des charges CFP est que certaines ne sont pas conservées dans un espace-temps plat, ce qui signifie qu'une partie de leur flux ne provient pas du rayonnement ou d'autres processus physiques. Cela limite leur applicabilité - par exemple, un effondrement sphérique verrait un cône de lumière plat se plier dans un horizon des événements, et ce processus sera mal décrit par une charge qui évolue déjà avant tout influx de matière. Cela soulève la question de savoir s'il existe une prescription différente pour les charges qui soit exempte de cette limitation. En effet, les charges de Noether ainsi que les charges de Wald-Zoupas ne sont pas garanties comme étant universelles, et peuvent dépendre d'un choix de polarisation fait lors de l'écriture du potentiel symplectique [32, 34, 14, 42]. La construction CFP est basée sur un choix spécifique de conditions aux limites de Dirichlet, et sur un ensemble spécifique de restrictions des variations, correspondant à

[1] Le même groupe qui peut être obtenu à l'infini de genre lumière en relaxant les conditions de chute d'une manière compatible avec la renormalisation du potentiel symplectique [46].

une certaine structure universelle construite selon les lignes directrices de ce qui a été fait avec succès à l'infini futur nul (voir par exemple [48] pour une revue). Nous montrons dans le chapitre 9 qu'il existe un choix différent de polarisation conduisant à des charges conservées dans un espace-temps plat ainsi que des hypersurfaces sans cisaillement et sans expansion. Cette polarisation, que nous appelons *polarisation conforme*, a déjà été examinée dans [49], et est similaire à la polarisation de York discutée dans le cas non nul. Les charges résultantes ne sont pas anormales, préservées sur les cônes lumineux plats, et peuvent être utilisées pour étudier les processus dynamiques de formation des trous noirs, comme anticipé dans [50]. Nous montrons que cette polarisation conduit à des conditions aux limites conformes sur une hypersurface nulle qui fournissent une résolution alternative aux ambiguïtés de l'action gravitationnelle avec des limites de genre lumière [51] basées sur des conditions aux limites de Dirichlet.

Une des restrictions sur les variations considérées dans l'article CFP concerne l'inaffinité de la normale à l'hypersurface nulle. Cette restriction joue un rôle important, et la relâcher réintroduit les ambiguïtés de [51] et empêche une implémentation complète de la procédure WZ. Néanmoins, les auteurs ont remarqué que cette restriction supprime des directions de l'espace des phases le long desquelles la double forme symplectique n'est pas dégénérée. Par conséquent, elle peut correspondre à une variation physiquement pertinente, et la question a été laissée ouverte de savoir si l'on pouvait construire des charges WZ dans cet espace de phase plus large. Nous soutenons que cette question a une réponse positive : Il existe une généralisation précise de la procédure WZ qui permet de construire des charges dans l'espace des phases élargi qui inclut des variations de l'inaffinité. La généralisation par rapport à la procédure originale de Wald-Zoupas concerne la notion de stationnarité, tandis que l'exigence de covariance reste inchangée. À notre avis, le fait de permettre différentes notions de stationnarité constitue une flexibilité avantageuse du formalisme, et c'est la covariance par rapport à toute structure universelle d'arrière-plan qui est le principe clé de la prescription de Wald-Zoupas. Cette procédure sélectionne à nouveau la polarisation conforme, et les nouvelles charges satisfont aux lois d'équilibre des flux sans anomalie. Cette réponse positive découle des meilleures propriétés de covariance de la polarisation conforme, que nous rendons explicites par une analyse détaillée des anomalies des différentes polarisations sous différentes restrictions sur les variations.

La thèse est structurée comme suit :

Chapitre 2 : Géométrie Dans ce chapitre, nous donnons un aperçu des hypersurfaces intégrées dans l'espace-temps. Ceci a un double objectif : premièrement, établir fermement la notation utilisée tout au long de la thèse ; et deuxièmement, rassembler les diverses propriétés des hypersurfaces qui sont dispersées dans différents ouvrages, offrant ainsi un point de référence complet. Compte tenu de leurs attributs géométriques uniques, les hypersurfaces de genre lumière font l'objet d'une attention particulière.

Chapitre 3 : Symétries Ce chapitre offre une vue d'ensemble concise des théorèmes de Noether sous leur forme conventionnelle. Pour mettre en évidence les implications des théorèmes, nous passons en revue une série d'exemples simples.

Chapitre 4 : Espace de phase covariant Nous donnons ici un aperçu du formalisme du CPS dans le langage moderne. Nous revisitons la charge de Noether dans ce nouveau cadre, et discutons des problèmes liés à la définition des charges hamiltoniennes. Le chapitre introduit également le concept d'anomalies.

Chapitre 5 : Conditions aux limites et termes limites Ce chapitre présente un examen complet des conditions aux limites. Nous commençons par les aborder du point de vue des équations aux dérivées partielles. Cette approche est ensuite combinée avec des idées tirées du principe d'action. Nous concluons ce chapitre par une discussion sur les conditions aux limites admissibles pour la gravité d'Einstein-Hilbert.

Chaptire 6 : Groupes de symétrie Dans ce chapitre, nous étudions les anomalies du potentiel symplectique dans différentes polarisations. Nous considérons diverses restrictions de variations et dérivons les groupes de symétrie correspondants.

Chapitre 7 : Prescriptions de charge Dans ce chapitre, nous dressons la liste des diverses ambiguïtés associées à la définition de la charge. Nous présentons les prescriptions de charge de WZ et de Noether améliorées, et entreprenons une comparaison entre les deux approches.

Chapitre 8 : Charges de polarisations différentes avec des frontières non nulles S'orientant vers un territoire plus spécialisé, ce chapitre explore la généralisation des charges ADM et BY dans le contexte de différentes conditions aux limites. En outre, nous étudions les coins non orthogonaux et les analysons dans les cadres covariant et canonique.

Chapitre 9 : Charges avec des polarisations différentes avec des frontières de genre lumière. En nous appuyant sur le chapitre précédent, nous nous concentrons sur les différentes charges des charges dans le cas de frontières de genre lumière. Nous dérivons des charges avec des propriétés intéressantes en utilisant la polarisation conforme.

Chapitre 10 : Conclusions. Ce chapitre de conclusion fournit un résumé cohérent des chapitres précédents et propose des pistes de recherche pour l'avenir.

Abstract

In this thesis we study the relationship between symmetries, boundary conditions, and conservation or flux-balance laws in General Relativity with the covariant phase space formalism. Non-trivial symmetries occur in arbitrary spacetimes if they admit a boundary, and the nature of the symmetries and of the charges one can construct depends on the chosen boundary conditions. These charges offer a refined solution to the issue of quasi-local observables in general relativity, although one must resolve potential ambiguities in their definitions. To address this, we conduct a critical comparison between the Improved Noether Charge and Wald-Zoupas prescriptions, specifically clarifying the role of anomalies introduced by the boundary. This is particularly relevant for understanding BMS symmetries. Our work introduces an independent definition of Wald-Zoupas charges at future null infinity, allowing for a more straightforward computation of BMS charges. We also considered charges at finite distances, supported on both time-like and null boundaries. For time-like ones, we examined the dependency of the expression for the energy on boundary conditions, and proposed new Brown-York-type charges for Neumann and York's boundary conditions. A comparison with canonical treatments confirmed a perfect agreement. For null boundaries, it is possible to consider leaky boundary conditions in a non-ambiguous way. We study the most general phase space permitting arbitrary metric variations, identify a one-parameter family of covariant symplectic potentials, and explain how restricting some of the variations is necessary for the symplectic potential to satisfy physical requirements of stationarity. This allows us to not only recover previous charge expressions, but introduce a new set of charges that extends the stationarity property to flat light-cones, with promising implications for dynamical entropies. Overall, we are very happy with the work we did, which was very non-trivial, required a lot of effort both in studying the literature and doing computations, and produced intriguing new results which I hope will be of interest to researchers in the field.

Résumé

Dans cette thèse, j'ai étudié la relation entre les symétries, les conditions aux bords et les lois de conservation ou de bilan de flux en relativité générale avec le formalisme de l'espace des phases covariant. Des symétries non triviales se produisent dans des espaces-temps arbitraires s'ils admettent une bord, et la nature des symétries et des charges que l'on peut construire dépend des conditions aux bords choisies. Les charges offrent une solution élégante au problème des observables quasi-locales en relativité générale, mais pour cela, il faut traiter les ambiguïtés potentielles dans leur définition. Pour clarifier cela, nous avons fait une comparaison critique entre les prescriptions de la Improved Noether Charge et des charges de Wald-Zoupas, en clarifiant en particulier le rôle et les anomalies que la bord peut introduire. Ceci est particulièrement pertinent pour comprendre les symétries BMS, et notre travail fournit une définition indépendante des charges de Wald-Zoupas à l'infini nul futur qui permet de calculer les charges BMS de manière plus directe. Nous avons également considéré des charges à des distances finies, dans les deux contextes différents des bords temporelles et nulles. Pour les premières, nous avons examiné comment l'expression de l'énergie dépend des conditions aux bords et nous avons proposé de nouvelles charges de type Brown-York pour les conditions de Neumann et de York. Nous avons comparé nos résultats à un traitement canonique et trouvé un accord parfait. Pour les bords nulles, il est possible de considérer des conditions aux bords "fuyantes" de manière non ambiguë. Nous avons étudié l'espace des phases le plus général qui permet des variations métriques arbitraires, identifié une famille à un paramètre de potentiels symplectiques covariants, et expliqué comment la restriction de certaines des variations est nécessaire pour que le potentiel symplectique satisfasse aux exigences physiques de stationnarité. Nous retrouvons les expressions précédentes pour les charges, mais introduisons également un nouvel ensemble qui étend la propriété de stationnarité aux cônes de lumière plats et présente des applications prometteuses pour les entropies dynamiques. Dans l'ensemble, nous sommes très satisfaits du travail que nous avons accompli, qui loin d'être banal, a nécessité beaucoup d'effort à la fois pour étudier la littérature et pour effectuer des calculs, et a produit des résultats nouveaux et intrigants que j'espère seront d'intérêt pour les chercheurs dans le domaine.

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Introduction

1

The edifice of general relativity (GR) stands atop a cornerstone principle – diffeomorphism invariance [1]. This principle dictates that the laws of physics should remain invariant under smooth coordinate transformations. It stands as a testament to the universality of the gravitational interaction and serves as a foundational tenet of the theory. But what does diffeomorphism invariance signify, and why does it hold such a paramount place in GR?

At its heart, diffeomorphism invariance reflects the gauge nature of general relativity [52]. It encapsulates the idea that the precise coordinate system we use to describe spacetime is of no fundamental physical meaning. In other words, all coordinate choices are equivalent in their ability to describe physical phenomena. This perspective highlights a beautiful and radical departure from classical mechanics: in GR, spacetime is not just a passive backdrop but an active player. It can warp, twist, and curve in response to the presence of matter or energy. This coordinate freedom generates a conceptual challenge when one tries to define what can be truly observed in the theory. In simpler terms, if different coordinate systems give rise to divergent representations of the same physical process, what quantities or variables can be said to have invariant meaning across all frames of reference?

In GR, what one observer perceives can be starkly different from another's perspective, depending on their relative motion and position in the gravitational field. Since different observers are mathematically represented as different coordinate choices, and we are holding general covariance sacred, this brings forth the fundamental question of what truly constitutes an *observable* in GR. The answers are neither straightforward nor intuitive, demanding a symphony of mathematics and physical insight.

One of the most captivating manifestations of GR is the propagation of gravitational waves – ripples in the fabric of spacetime caused by astrophysical events like merging black holes or neutron stars. These phenomena, predicted by Einstein over a century ago [2], eluded direct observation until recently [3]. Einstein himself wrestled with the idea, at times doubting the physical existence of such waves. The question was whether gravitational waves were merely a product of the linear theory that could be eliminated by a change of coordinates, or was there potential for them to transfer energy over vast distances in the nonlinear theory? This debate, fueled by the inherent challenges of framing the question in a meaningful context, spanned four decades [4].

The complication was in isolating the propagative aspects from the ten components of the metric field, especially in the absence of a canonical prescription that General Relativity couldn't provide. The breakthrough came in the 1960s when Bondi, Metzner, van der Burg [5], and Sachs [6, 7] established a robust framework. They devised a coordinate system that leveraged the fact that gravitational radiation moves along light cones. A notable result was the Bondi mass loss formula, providing a theoretical proof of

the existence of gravitational waves even within a non-linear theory. Their findings were of great importance, not only clarifying the nature of gravitational waves but also reshaping our understanding of the symmetry structure at the asymptotic regions of spacetime. One could initially expect, in the regions far from gravitational sources where spacetime reverts back to its flat form, only the exact isometries of Minkowski space – the Poincaré group. But they revealed an enriched symmetry structure. Namely, they found that the asymptotic symmetry group, now known as the BMS group, went beyond the Poincaré group – the four translations of Poincaré were expanded into an infinite-dimensional normal subgroup of supertranslations, which are angular-dependent translations, revealing a far more complex structure of spacetime symmetries than previously anticipated.

Experimental observations of gravitational waves, however, only came to be recently [3], standing as a testament to the predictive power of GR and the technological advancements of observatories like LIGO and Virgo. Their detection was hailed as a triumph, not just of experimental prowess, but as a testament to the predictive power of GR. A prediction that once lingered at the horizons of testability, has now been brought to light with the groundbreaking detections by observatories like LIGO and Virgo. Yet, these waves bring with them subtle imprints, clues to the mechanisms that produced them that can be studied using the underlying symmetries and their charges

When the arms of LIGO vibrate in response to an incoming gravitational wave, it's more than just an isolated event on Earth. Geometrically, these arms are positioned at the future null infinity of an idealized, asymptotically flat spacetime. The drama that causes these ripples - perhaps the cataclysmic merging of two black holes - unfolds deep in the bulk of this spacetime. Yet, it's these distant events that send out gravitational echoes, culminating in the minute oscillations that LIGO's exquisitely sensitive instruments can detect. The modeling of such a system isn't trivial. Crucial to this description is the imposition of boundaries and associated boundary conditions. These describe how the gravitational (and any other) fields behave as they approach and interact with these boundaries. But when boundaries are introduced into the play, one needs to account for important subtleties related to gauge symmetries [8, 9, 10].

In the absence of boundaries, gauge symmetries serve primarily to signify redundancies in our description of physical systems. They encode the notion that many different configurations of the fundamental fields – be it the metric tensor of general relativity or the gauge fields of particle physics – can represent the same physical state [11]. This is akin to how different coordinates can describe the same point in space. In such a context, gauge symmetries are not directly observable. They are a byproduct of our mathematical descriptions rather than inherent features of the physical world. However, the introduction of boundaries significantly changes this picture. When our spacetime, or any other system described by a gauge theory, has boundaries, certain gauge symmetries acquire a physical significance. Why? In a system without boundaries, Wilson loops are commonly used

as gauge-invariant observables. A Wilson loop is a trace of the holonomy of the gauge field around a closed loop. When a boundary is introduced into the system, things get more complicated. Now, the Wilson loops can be 'cut' by the boundary, destroying the gauge invariance. A cut Wilson loop becomes a Wilson line whose end points transform in some representation of the gauge grop. This introduces the necessity of keeping track of the ends of the Wilson loops at the boundary, giving rise to what are called *edge modes* – fictitious fields that exist at the boundary and are necessary for maintaining gauge invariance. They can carry physical information and even mediate interactions between the bulk and the boundary of the system [12, 13, 14, 15, 16]. Mathematically speaking, in the presence of boundaries, these gauge transformations aquire a non-vanishing charge. This means that, rather than being mere background players, they become physically relevant. They can distinguish between distinct physical states, labelling genuine degrees of freedom. These boundary-induced physical symmetries play a crucial role in several domains of theoretical physics. For instance, in the context of condensed matter physics, they can lead to exotic states of matter like topological insulators, where the boundary or edge states carry current while the bulk remains insulating. Similarly, in the gravitational context, these symmetries offer a deeper understanding of the structure of spacetime potentially having important implications for the quantum theory.

Now one might ask: Why does this matter? To comprehend this, one needs to remember that in the conventional understanding of gauge theories, gauge symmetries are purely mathematical artefacts. They don't correspond to real physical transformations. They emerge as redundancies in our description, implying that many mathematically distinct field configurations might represent the same physical state. The act of 'gauging away' these redundancies has long been an accepted routine. By identifying which gauge transformations have physical implications, we anchor our understanding of what can be measured, what can be observed, and what remains a mere coordinate choice and can safely be gauged away.

Building on this foundational insight into the role of gauge transformations, let us turn our attention to the literal 'edges' of the problem—the boundary conditions. To be able to sift through mathematical redundancies to find the physically relevant information, understanding boundary conditions turns out to be crucial. While often subtle in their implications, they play an indispensable role in formulating physical theories, painting a clearer picture of the overall dynamics. They ensure the well-posedness of problems, guiding systems in their interactions with surrounding constraints and rendering meaningful, consistent solutions. Classically, boundary conditions have been classified into distinct types: Dirichlet, where the values of the field variables are fixed at the boundary; Neumann, where the fluxes of the fields are specified; and Robin, a hybrid of the former two [17]. Each type offers unique insights into how a system behaves and interacts with its surroundings, each tailored to specific scenarios and physical motivations.

These conditions serve as guidelines for solving differential equations, ensuring solutions are both unique and physically meaningful. This motivates the interest in studying how boundary conditions affect the understanding of gravitational dynamics.

Historically, in the context of gravitational physics, Dirichlet boundary conditions have been the conventional choice. However, it's not without its critiques. To appreciate the nuances here, it's beneficial to contrast general relativity with theories featuring a fixed spacetime structure. In the latter scenario, where the metric field is immutable and serves as a backdrop, both Dirichlet and Neumann conditions are suitable from the point of view of the initial-boundary value problem (IBVP) [18, 19, 20]. They ensure well-posedness for the IBVP and are relatively straightforward in their implications. However, the absence of fixed background fields complicates this situation. Anderson and An [21, 22] argue that, contrary to theories with a fixed space-time structure, in general relativity, the direct use of either Dirichlet or Neumann boundary data results in ill-posed problems for vacuum gravity. One should look for boundary conditions that respect the dynamics and constraints inherent in general relativity, and one notable proposition in this context originates from York in the 1980s [23]. He defined a set of boundary conditions that combine elements of both Dirichlet and Neumann types. Instead of completely restricing the induced metric at the boundary, York's conditions fix the conformal metric and its momentum conjugate, which is proportional to the extrinsic curvature. This approach not only offers a geometrically elegant solution but also ensures well-behaved dynamics consistent with the principles of general relativity. Keeping the IBVP aside, the study of boundary conditions naturally leads us to the question of boundary Lagrangians. These specific boundary terms in the action emerge conspicuously when one alters the boundary conditions. In the case of general relativity, for all types of boundary conditions studied, they are proportional to the divergence of the normal to the boundary. In non-null cases, this translates to the trace of the extrinsic curvature, and for Dirichlet conditions, it manifests as the well-recognized Gibbons-Hawking-York term [24, 25].

The study of classical systems traditionally bifurcates into two primary frameworks: the Lagrangian and Hamiltonian approaches. While the Lagrangian approach focuses on the action principle and the equations of motion, the Hamiltonian method emphasizes the phase space, canonical variables, and their evolution. Bridging these two methodologies is the concept of the *covariant phase space* (CPS) [26, 27, 28, 29, 30, 31, 32, 33, 34], an elegant framework that combines the strengths of both. The CPS framework deviates from traditional views in its conceptualization of phase space. Instead of understanding it as a space of initial conditions (typical in Hamiltonian mechanics), it's conceived as the space of solutions to the equations of motion [28]. This reinterpretation, subtle yet profound, leads to a series of advantages. Notably, it eradicates the need to single out a specific time direction or foliation. Instead, it naturally preserves the covariant nature of the system, adhering to the principles of relativity and ensuring that the descriptions remain

unaltered under coordinate transformations. Such a conceptual shift also affects the way we approach boundary conditions. Within the covariant phase space, boundary conditions are not merely constraints imposed on the system but can be thought of as a choice of polarization in the symplectic manifold representing the phase space. This perspective allows for a more intrinsic understanding of the boundary conditions, intertwining them with the geometric properties of the space of fields.

As the presence of boundaries breaks covariance, in this setting diffeomorphisms might generate non-vanishing charges. This inspires us to revisit Noether's theorem, giving it a renewed voice in the context of the covariant phase space. Originally posited by Emmy Noether in the early 20th century, her theorems establish a profound connection between symmetries of a system and its conserved quantities. When reformulated in the language of covariant phase space, Noether's theorems acquire a certain elegance. They highlight the intricate relationship between the symmetries of the solutions in the phase space and the conserved quantities associated with them. The gauge nature of diffeomorphisms is encoded in the fact that they are zero-modes of the pre-symplectic form – the charges generating them are pure boundary terms which, in the absence of boundaries, vanish on-shell. This is a well known feature of gauge symmetries.

In essence, the exploration of the CPS serves as a testament to the adaptability and resilience of classical mechanics, proving its relevance even in modern contexts. By intertwining foundational principles with contemporary interpretations, it provides a fertile ground for novel insights, further enriching our understanding of physical systems. However, despite the clarity the covariant formulation provides, it's not without its nuances. Various ambiguities arise when attempting to define conserved quantities in this framework. Addressing these ambiguities requires certain prescriptions to ensure that the derived quantities have a clear physical interpretation. A procedure for defining charges in this framework was put out in the seminal work by Wald and Zoupas [33]. These charges satisfy a modification of Hamilton's equation in which the symplectic form evaluated on a diffeomorphism variation yields the variation of the charge, plus an additional term representing the flux. This split between the charge and flux is not unique and a prescription is needed to fix it. This is usually referred to as *the problem of charge integrability*. In the WZ framework this is done using certain physically justified choices which require assumptions of covariance which are not always satisfied [35].

A more general framework has been developed recently by many authors [34, 36, 37, 38, 39, 40, 14] exploiting various ambiguities [41] arising in the definition of the symplectic potential from the Lagrangian to give a definition of surface charges based on the choice of phase space polarization.[1] The charge is defined on a subspace where the flux vanishes

[1] Here we use the term polarization in a loose sense to refer to the choice of the symplectic potential in the 'diagonal' pdq form. We explain our viewpoint on the relationship between polarization and boundary conditions in Section 5.

– either on shell of conservative boundary conditions (e.g. [40, 42]) or on a subspace of stationary field configurations (e.g. [36]), and this definition is then extended to the rest of the phase space, uniquely defining the charge-flux split for a given polarization. As the prescription is using the boundary structure to select the ambiguities in the definition of the Noether charge, we refer to it as the *(boundary-)improved Noether charge*.

An alternative resolution of the issue of charge integrability can be achieved by extending the phase space by introducing a set of additional dynamical fields which come from the embedding of corners in spacetime and are known as edge modes [14, 43, 44]. In this approach, even though the gravitational system is open and can have degrees of freedom leaving and entering the system, the symmetry charges are always hamiltonian. This is possible because the dissipative degrees of freedom in the evolution of the corner are encoded in the non-trivial variations of the embeddings. Independently of the perspective on the topic, the two approaches must be two different ways of understanding the same underlying physics.

The WZ prescription not only reproduces the famous Arnowitt-Deser-Misner (ADM) charges for spatial infinity but also extends its application to the Geroch and Dray-Streubel charges associated with future null infinity. Recent works have further expanded on this, considering null hypersurfaces at finite distances and non-expanding horizons [36, 45]. On the other hand, using the improved Noether charge prescription, [40, 38] have shown how to include arbitrary field-dependent diffeomorphisms and anomalies – background-depedent quantities whose field-space transformation under diffeomorphisms differs from the Lie derivative. Including field-dependent diffeomorphisms and anomalies in the covariant phase space is crucial. They appear in various contexts, such as boundary symmetry vectors extensions and in certain symmetry enhancements. They also play a significant role in the canonical approach. Anomalies, which often show up when background structures are involved, are important for boundary observables. Recognizing and addressing them systematically can significantly expand the utility of the CPS. Can the WZ prescription be applied in this more general context? Answering this question is useful for future research, but also to better understand the precise relation between the recent literature and [33]. In particular, both field-dependent diffeomorphisms and anomalies appear in the study of the BMS group. This prompts a question – how come Wald and Zoupas were able to derive the BMS charges without including either of these two features in their description? We found that while WZ didn't specifically consider these elements, their approach inherently made certain covariance assumptions. By translating these assumptions, we reveal that the WZ method can handle some field-dependent diffeomorphisms and anomalies, but only under specific conditions. We term the acceptable anomalies as *soft anomalies*.

Furthering our investigation, we juxtapose the WZ prescription against the improved Noether charges and show that the WZ charges can indeed be reformulated as improved Noether charges, provided specific conditions are met. This reconciliation accentuates the

inherent adaptability and robustness of the WZ framework within modern interpretations. In practical terms, we tested our findings using four scenarios. The most complex test was at future null infinity, where we could derive standard BMS charges and better understand how WZ got results without directly handling anomalies. While the derivation of BMS charges as improved Noether charges was previously discussed in [46] and [38], those papers relied on prior knowledge of BMS charges. Our approach is more foundational, relying directly on the WZ conditions.

Due to its explicit dependence on boundary Lagrangians, the improved Noether charge prescription makes studying the effect of changing boundary conditions straightforward. An important fact in GR is that energy is not strictly conserved due to the dissipation caused by gravitational radiation. However, when focusing solely on non-radiative spacetimes, an idea of conserved energy in the phase space becomes feasible. One notion of conserved energy is given by the ADM charge at spatial infinity, where gravitational waves never reach, as well as its quasi-local counterpart, the Brown-York (BY) charge. Still, this understanding of energy is not universal. It heavily relies on the specific boundary conditions used to eliminate dissipation. Namely, both the ADM and BY charge computations require imposition of Dirichlet boundary conditions, be it at asymptotic or finite boundaries. In Chapter 8 we study what happens to energy values as we transition from Dirichlet, through York's mixed conditions, to Neumann's. This process, which involves varying degrees of fixation of the induced metric and its momentum (the extrinsic curvature), tends to result in reduced energies. Illustratively, in the case of the Kerr spacetime, the energy values are found to be M, $2M/3$, and $M/2$, respectively.

To arrive at these conclusions, two primary methodologies were used. The first method is the covariant phase space, especially powerful as it can describe both radiative and non-radiative scenarios. Here, the focus is on understanding how restricting symplectic flux at the boundary selects a distinct symplectic potential, leading to integrable charges. This is the basic idea of the improved Noether charge prescription which provides a universal approach for charges that holds true across various boundary conditions. Our second tool is a direct canonical analysis founded on the ADM decomposition. This method is particularly apt for the non-radiative context. Notably, when we extend the analysis to include different boundary conditions, we observe that this canonical method produces charges that align seamlessly with those deduced from the improved Noether charge prescription. This alignment fortifies the validity of the improved Noether charge formula.

Going a step further, we introduce a non-orthogonal intersection between the time-like boundary and the space-like hypersurfaces. When these non-orthogonal corners are introduced, the variational principle requires an additional 2D term, which, in the case of Dirichlet boundary conditions, is given by the Hayward Lagrangian. Given

that our primary aim in constructing covariant surface charges revolves around a well-defined variational principle, it is vital to ascertain whether this corner Lagrangian also influences the charge formula. To answer this, calculations using the improved Noether charge prescription were repeated, this time factoring in non-orthogonal corners, and the outcomes were juxtaposed with canonical results. The canonical calculation highlighted that a non-orthogonal corner introduces both physical and mathematical shifts. From a physical perspective, one could consider two distinct observer categories – those at rest with respect to the space-like foliation of spacetime, and those at rest along the time-like boundary. At the corner, these are related by a boost transformation, making the different notions of energy directly comparable. Mathematically this manifests itself as the additional 'tilting terms' in the charges depending explicitly on the boost angle between the normals. Importantly, our findings confirmed the cohesion between the covariant and canonical results, even in the context of non-orthogonal corners, thereby attesting to the robustness of the improved Noether charge prescription.

Moving our focus to null boundaries, we are no longer limited to conservative boundary conditions and can probe into leaky boundaries as well. Recent developments by Chandrasekaran, Flanagan, and Prabhu (CFP) characterized the symmetry group of general relativity on generic null hypersurfaces as an extension of the BMS group to accommodate arbitrary diffeomorphisms and Weyl transformations on any 2d space-like cross-section [36].[2] Using the Wald-Zoupas (WZ) procedure, they derived charges which maintain conservation across shear-free and expansion-free hypersurfaces, with non-expanding horizons (NEHs) being a prime example. A subsequent detailed exploration into NEHs was presented in [45], shedding light on how the global sector of the Weyl transformations equates to the area charge. The same Weyl transformations play also a prominent role in the investigation of possible black hole soft hairs by Hawking, Perry and Strominger [47].

A shortcoming of the CFP charges is that some of them are not conserved in flat spacetime, meaning a portion of their flux doesn't stem from radiation or other physical processes. This limits their applicability – e.g. a spherical collapse would see a flat light-cone bend into an event horizon, and this process will be poorly described by a charge that is already evolving prior to any matter infalling. This raises the question whether a different prescription for the charges exists that is free of this limitation. Indeed, the Noether charges as well as the Wald-Zoupas charges are not guaranteed to be universal, and may depend on a choice of polarization made in writing down the symplectic potential [32, 34, 14, 42]. The CFP construction is based on a specific choice of Dirichlet boundary conditions, and on a specific set of restrictions of the variations, corresponding to a certain universal structure constructed along the guidelines of what has successfully been done

[2] The same group that can be obtained at null infinity relaxing the fall-off conditions in a way compatible with renormalization of the symplectic potential [46].

at future null infinity (see e.g. [48] for a review). We show in Chapter 9 that there is a different choice of polarization leading to charges that are conserved in flat spacetime together with shear-free and expansion-free hypersurfaces. This polarization, we refer to as the *conformal polarization*, was previously considered in [49], and is similar to the York polarization discussed in the non-null case. The resulting charges are not anomalous, preserved on flat light-cones, and can be used to study dynamical processes of black hole formation, as anticipated in [50]. We show that this polarization leads to conformal boundary conditions on a null hypersurface that provide an alternative resolution to the ambiguities of the gravitational action with null boundaries [51] based on Dirichlet boundary conditions.

One of the restrictions on the variations considered in the CFP paper concerns the inaffinity of the normal to the null hypersurface. This plays an important role, and relaxing it reintroduces the ambiguities of [51] and prevents a complete implementation of the WZ procedure. Nonetheless, the authors remarked that this restriction removes directions of the phase space along which the symplectic two-form is not degenerate. Therefore it may correspond to a physically relevant variation, and the question was left open whether one could construct WZ charges in this larger phase space. We argue that this question has a positive answer: There is a precise generalization of the WZ procedure that allows the construction of charges in the larger phase space that includes variations of the inaffinity. The generalization with respects to the original WZ procedure concerns the notion of stationarity, whereas the requirement of covariance is untouched. In our opinion allowing different notions of stationarity is an advantageous flexibility of the formalism, and it is covariance with respect to any background universal structure that is the key tenet of the WZ prescription. This procedure selects again the conformal polarization, and the new charges satisfy anomaly-free flux-balance laws. This positive answer follows from the better covariance properties of the conformal polarization,which we make explicit through a detailed analysis of the anomalies of the different polarizations under different restrictions on the variations.

The thesis is structured as follows:

Chapter 2: Geometry In this chapter, we give an overview of hypersurfaces embedded into spacetime. This serves dual purposes: firstly, to firmly establish the notation used throughout the thesis; and secondly, to collect various properties of hypersurfaces that are dispersed across different literature, offering a comprehensive reference point. Given their unique geometrical attributes, null hypersurfaces are given special focus.

Chapter 3: Symmetries This chapter offers a concise overview of Noether's theorems in their conventional form. To highlight the theorems' implications, we walk through a range of simple examples.

Chapter 4: Covariant Phase Space Here, we give an overview of the CPS formalism in the modern language. We revisit Noether's charge within this refreshed

framework, and discuss problems with the definition of Hamiltonian charges. The chapter also introduces the concept of anomalies.

Chapter 5: Boundary Conditions and Boundary Terms A comprehensive examination of boundary conditions is done in this chapter. We start by approaching them from the point of view of partial differential equations. This is then combined with insights drawn from the action principle. We conclude this chapter with a discussion on permissible boundary conditions for Einstein-Hilbert gravity.

Chapter 6: Leaky Boundary Conditions and Boundary Symmetry Groups In this chapter, we perform a detailed study of the covariance properties of the gravitational symplectic potential on a null boundary, and of the different polarizations that can be used. We emphasise the importance of keeping track of anomalies in the covariant phase space, with the exact nature of the anomalies contingent on the selected leaky boundary conditions. We study the symmetry groups that arise with different phase space prescriptions, and determine the fields that have anomalous transformations.

Chapter 7: Charge Prescriptions In this chapter, we list various ambiguities associated with charge definition. We present both the WZ and improved Noether charge prescriptions, and undertake a comparison between the two approaches.

Chapter 8: Charges with Different Polarizations with Timelike Boundaries Heading into more specialized territory, this chapter explores the generalization of ADM and BY charges in the context of different boundary conditions. Additionally, we study non-orthogonal corners and analyze them within both covariant and canonical frameworks.

Chapter 9: Charges with Different Polarizations with Null Boundaries. Building on the previous chapter, we focus on the nuances of charges in the case of null boundaries. We derive charges with nice properties using the conformal polarization.

Chapter 10: Conclusions. This concluding chapter provides a coherent summary of preceding chapters and lays out potential directions for future research.

Notation

The spacetime manifold is denoted M, the Lorenzian metric $g_{\mu\nu}$ obeys the mostly positive convention $(-,+,+,+)$. When being general, its dimension is denoted by n but in practice we take it to be $n=4$. A generic non-null hypersurface with unit normal n is denoted Σ with signature $s=n^2=\pm 1$. When the signature is specified, symbol Σ is kept for a spacelike hypersurface $s=n^2=-1$, and $\mathcal{T}$ is a timelike hypersurface with normal $\bar{n}$, $s=\bar{n}^2=1$. A null hypersurface is denoted $\mathcal{N}$, its normal l, and the rigging vector n. The corner is denoted S, it can be treated as a hypersurface in Σ, $\mathcal{T}$, or $\mathcal{N}$, in which cases we use pairs of normals (n,u), $(\bar{n},\bar{u})$, and (l,n), respectively.

Greek letters are spacetime indices $\mu,\nu,...\in\{0,1,2,3\}$. If a $3+1$ foliation is present, 0 is the coordinate along the direction of the foliation, and lowercase Latin letters are coordinates on the leaves $a,b,...\in\{1,2,3\}$. Uppercase Latin letters are used to denote compact angular coordinates along the corner $A,B,...\in\{2,3\}=\{\theta,\phi\}$. In all cases, $(,)$ denotes symmetrization, $\langle,\rangle$ trace-free symmetrization, and $[,]$ antisymmetrization.

The volume form on spacetime is given by $\epsilon := \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\, dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma$, with $\epsilon_{0123} := \sqrt{-g}$. An arrow under a p-form denotes the pullback to the boundary $\mathcal{B}$, and $\overset{\mathcal{B}}{=}$ is an equality holding at the boundary only. Equality holding on-shell of the equations of motion is denoted $\hat{=}$, and the equality holding on-shell of boundary conditions is denoted by $\overset{\text{b.c.}}{=}$. We use units $c=1=G$ and define the gravitational Lagrangian without the 16π prefactor.

Geometry

2

In this chapter we briefly review the geometry of hypersurfaces and corners. This is useful to fix the notation, but it will also allows us to offer a review of various properties that are often overlooked. We will in particular review the distinction between intrinsic and extrinsic geometries, using spacetime covariant notation and offering the translation to hypersurface indices on the one hand and to Newman-Penrose (NP) notation for the case of null hypersurfaces. We will recall the notion of class-III and class-I invariance to talk about quantities which are independent respectively of the choice of normal and of rigging vector, and finally some useful expressions that arise when working with the special coordinate system provided by affine coordinates.

This chapter doesn't contain any new research; all the information included is derived from existing publications. The section on non-null hypersurfaces primarily relies on Chapter 3 of [53]. We are using notation and conventions established in the appendix of [42], which are also reiterated in this text. As for the section concerning null hypersurfaces, various sources have studied its content and will be acknowledged throughout the text. Although the way this information is organized and presented is unique and was previously published in [54], the actual material is well-established but often fragmented across different publications, each with its own set of conventions and notations.

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A *hypersurface* Σ is a submanifold of codimension-1 embedded in some ambient manifold M. In all cases of interest here the ambient manifold will be a four-dimensional Lorenztian spacetime, and hypersurfaces will be three-dimensional. Some examples we will consider are boundaries of some region of interest, Cauchy slices, lightcones, etc. These hypersurfaces can be spacelike (Σ), timelike ($\mathcal{T}$), or null ($\mathcal{N}$). Due to their starkly different geometry we will dedicate a separate section to null hypersurfaces.

A *corner* S is a codimension-2 spacelike submanifold embedded in an ambient spacetime M. It can be thought of as a 'boundary of the boundary',[1] and it can be defined as the intersection of initial/final Cauchy slice and the lateral boundary. We will always assume the corner to be closed, we will treat it as a topological 2-sphere embedded in the four-dimensional spacetime.

[1] In this definition the corner is a hypersurface (codimension-1) embedded in the boundary (of dimension d-1).

2.1 Spacelike and timelike hypersurfaces

A hypersurface Σ can be defined by a Cartesian equation

$$\Phi(x^\mu) = 0, \tag{2.1.1}$$

as a level-set of a scalar field Φ, or by a set of parametric equations

$$x^\mu = x^\mu(y^a), \tag{2.1.2}$$

where x^μ are spacetime coordinates ($\mu = 0, 1, ..., d-1$), and y^a are coordinates intrinsic to the hypersurface ($a = 1, ..., d-1$). From (2.1.1) one naturally defines the normal one-form $\partial_\mu \Phi$, while from (2.1.2) one naturally defines $d-1$ tangent vectors $e^\mu_a := \frac{\partial x^\mu}{\partial y^a}$. We introduce the unit normal to the hypersurface

$$n_\mu = s\frac{\partial_\mu \Phi}{\sqrt{|g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi|}}, \quad s = n_\mu n^\mu = \begin{cases} +1 \text{ timelike} \\ -1 \text{ spacelike} \end{cases} \tag{2.1.3}$$

Notice that $e^\mu_a n_\mu = 0$. This simply means that Φ is constant along the hypersurface, which we know by definition.

2.1.1 Intrinsic geometry

Taking the spacetime line element and confining the displacement to the hypersurface, we define the *induced metric* q of Σ as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}\frac{\partial x^\mu}{\partial y^a}dy^a\frac{\partial x^\nu}{\partial y^b}dy^b \tag{2.1.4}$$

$$= g_{\mu\nu}e^\mu_a e^\nu_b dy^a dy^b =: q_{ab}dy^a dy^b, \tag{2.1.5}$$

$$q_{\mu\nu} := q_{ab}e^a_\mu e^b_\nu = g_{\mu\nu} - sn_\mu n_\nu. \tag{2.1.6}$$

We will refer to $q_{\mu\nu}$ as the induced metric as well, slightly abusing language but significantly simplifying notation by never using q_{ab} or latin indices whatsoever.[2] This way indices are always raised and lowered by the spacetime metric $g_{\mu\nu}$. By raising one index in (2.1.6), we define the projector

$$q^\mu_\nu = \delta^\mu_\nu - sn^\mu n_\nu, \quad q^\mu_\rho q^\rho_\nu = q^\mu_\nu, \quad q^\mu_\nu n^\nu = 0, \tag{2.1.7}$$

[2] On the other hand, one should keep in mind that the actual induced metric is the pull-back of the projector $q_{\mu\nu}$, so later when we study variations we will keep in mind that variations which keep the induced metric fixed satisfy $\underleftarrow{\delta q_{\mu\nu}} = 0$, but not necessarily $\delta q_{\mu\nu} = 0$.

mapping spacetime vectors to vectors intrinsic to the hypersurface

$$q(v) = v_{||}, \text{for } TM \ni v = v_{||} + v_{\perp} n, v_{||} \in T\Sigma, v_{\perp} \in \mathbb{R}.^{3} \tag{2.1.8}$$

Using the projector, we define the *induced covariant derivative*

$$D_{\mu}\left(q_{\nu\rho} v^{\rho}\right) = q_{\mu}^{\sigma} \nabla_{\sigma}\left(q_{\nu\rho} v^{\rho}\right), \tag{2.1.9}$$

for any spacetime vector v. Generalization to higher rank tensors is straightforward. We use this to define the induced curvature tensor

$$\left[D_{\mu}, D_{\nu}\right] v^{\rho} = {}^{(3)}R^{\rho}_{\sigma\mu\nu} v^{\sigma}. \tag{2.1.10}$$

We denote the spacetime volume 4-form by

$$\epsilon := \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma}\, dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma}, \qquad \epsilon_{0123} := \sqrt{-g}. \tag{2.1.11}$$

The induced volume 3-form on Σ is

$$\epsilon_{\Sigma} \equiv d\Sigma := i_n \epsilon = s n^{\mu} d\Sigma_{\mu}, \qquad \epsilon = s n \wedge \epsilon_{\Sigma}, \qquad i_n \epsilon_{\Sigma} \tag{2.1.12}$$

where $d\Sigma_{\mu} := s n_{\mu} d\Sigma = \sqrt{-sq}\; d^3 y$ is the oriented volume element in the conventions of [51]. Accordingly, the pull-back of a 3-form on Σ reads

$$\underleftarrow{\theta} = \frac{s}{3!} \theta_{\mu\nu\rho} \epsilon^{\mu\nu\rho\sigma} n_{\sigma} d\Sigma = -\frac{s}{3!} \theta_{\mu\nu\rho} \epsilon_{\Sigma}^{\mu\nu\rho} d\Sigma = \frac{1}{3!} \theta_{\mu\nu\rho} \epsilon^{\mu\nu\rho\sigma} d\Sigma_{\sigma} =: \theta^{\mu} d\Sigma_{\mu}. \tag{2.1.13}$$

To give some explicit intuition about these conventions, for Σ and $\mathcal{T}$ defined in Minkowski space respectively by $t = const.$ and $r = const.$, we would have

$$\epsilon_{\Sigma} = dr \wedge d\theta \wedge d\phi, \qquad \epsilon_{\mathcal{T}} = -dt \wedge d\theta \wedge d\phi,$$

with pull-backs on their boundaries

$$\underleftarrow{i_{\partial_r} \epsilon_{\Sigma}} = d\theta \wedge d\phi = \epsilon_S, \qquad \underleftarrow{i_{\pm\partial_t} \epsilon_{\mathcal{T}}} = \mp d\theta \wedge d\phi = \mp \epsilon_S.$$

2.1.2 Extrinsic geometry

The *extrinsic curvature* of Σ is defined as

$$K_{\mu\nu} := q_{\mu}^{\rho} q_{\nu}^{\sigma} \nabla_{\rho} n_{\sigma} \tag{2.1.14}$$

[3] slight abuse of notation here: since we are using the covariant notation, the tangent space $T\Sigma$ is seen as embedded in TM.

This quantity captures how the hypersurface is embedded in the ambient manifold, and depends explicit on the 4d metric via the normal components $n_\mu \Gamma^\mu_{\rho\sigma}$. From torsion-freeness of the metric it follows that the extrinsic curvature is symmetric $K_{\mu\nu} = K_{\nu\mu}$, and it can be expressed as the Lie derivative of the induced metric along the normal

$$K_{\mu\nu} = \frac{1}{2} q^\rho{}_\mu q^\sigma_\nu \mathcal{L}_n g_{\rho\sigma} = \frac{1}{2} q^\rho_\mu q^\sigma_\nu \mathcal{L}_n q_{\rho\sigma} = \frac{1}{2} \mathcal{L}_n q_{\mu\nu} \tag{2.1.15}$$

where we used $n^\mu \mathcal{L}_n q_{\mu\nu} = -q_{\mu\nu} \mathcal{L}_n n^\nu = 0$. The trace can be take with either g or q as it is a hypersurface-intrinsic tensor, and it is the divergence of the normal

$$K = g^{\mu\nu} K_{\mu\nu} = q^{\mu\nu} \nabla_\mu n_\nu = \nabla_\mu n^\mu \tag{2.1.16}$$

or in terms of the Lie derivative

$$K = \frac{1}{2} q^{\mu\nu} \mathcal{L}_n g_{\mu\nu} = \frac{1}{2} g^{\mu\nu} \mathcal{L}_n q_{\mu\nu} = \mathcal{L}_n \ln \sqrt{|q|}. \tag{2.1.17}$$

We define the *acceleration* of the normal as

$$a^\mu := n^\nu \nabla_\nu n^\mu. \tag{2.1.18}$$

Just like the extrinsic curvature, it captures geometry extrinsic to the hypersurface but it is a hypersurface-intrinsic vector field, $n \cdot a = 0$.

2.1.3 Corner geometry

We define the corner S as the level set of both Φ and another scalar field $\bar{\Phi}$ defined (at least) along Σ,

$$\Phi(x^\mu) = 0 = \bar{\Phi}(x^\mu). \tag{2.1.19}$$

We can think of this as an intersection of two hypersurfaces, Σ defined as a level set of Φ, and $\bar{\Sigma}$ defined as a level set of $\bar{\Phi}$. We can treat the corner as a hypersurface in Σ and define the pair normals (n, u) where the unit normal $u \in T\Sigma$[4] is given by

$$u_\mu = -s \frac{q^\nu_\mu \partial_\nu \bar{\Phi}}{\sqrt{|q^{\mu\nu} \partial_\mu \bar{\Phi} \partial_\nu \bar{\Phi}|}}, \quad u \cdot n = 0,\ u^2 = -s. \tag{2.1.20}$$

[4] Here it is important to notice that a codimension-2 surface comes with two normals that are not unique. However, fixing one of them (e.g. by saying that S as a hypersurface in Σ, we are taking n as one of the normals to S), and demanding they are orthonormal, the other one is fixed up to orientation.

We can also treat it as a hypersurface in $\bar{\Sigma}$, and define the pair $(\bar{n}, \bar{u})$, where $\bar{n}$ is defined by a barred version of (2.1.3), and $\bar{u} \in T\bar{\Sigma}$ is given by

$$\bar{u}_\mu = -\bar{s}\frac{\bar{q}^\nu_\mu \partial_\nu \Phi}{\sqrt{|\bar{q}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi|}}, \quad u\cdot\bar{n} = 0,\ u^2 = -\bar{s}. \tag{2.1.21}$$

We will distinguish two possibilities: (i) *orthogonal corners* occur when the two hypersurfaces intersect orthogonally, i.e. $n\cdot\bar{n} \overset{\text{s}}{=} 0$, and (ii) *non-orthogonal corners* whose angle of intersection is $\beta := \arcsin(n\cdot\bar{n})$. In that case, boost relations between normals are summarized below.

$$n\cdot\bar{n} = \sinh\beta, \qquad u\cdot\bar{n} = \cosh\beta, \qquad \bar{u}\cdot n = -\cosh\beta, \qquad \lambda := (\cosh\beta)^{-1} \tag{2.1.22}$$

$$\begin{pmatrix} n \\ u \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}\begin{pmatrix} \bar{u} \\ \bar{n} \end{pmatrix}, \qquad \begin{pmatrix} \bar{u} \\ \bar{n} \end{pmatrix} = \begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix}\begin{pmatrix} n \\ u \end{pmatrix} \tag{2.1.23}$$

$$u^\mu = \lambda q^\mu_\nu \bar{n}^\nu = \lambda(\bar{n}^\mu + \sinh\beta n^\mu), \qquad \bar{u}^\mu = \lambda \bar{q}^\mu_\nu n^\nu = \lambda(n^\mu - \sinh\beta\bar{n}^\mu) \tag{2.1.24}$$

Intrinsic geometry of the corner

We can also define the 2-dimensional induced metric as

$$\gamma_{\mu\nu} = q_{\mu\nu} - u_\mu u_\nu = g_{\mu\nu} + n_\mu n_\nu - u_\mu u_\nu = g_{\mu\nu} - \bar{n}_\mu\bar{n}_\nu + \bar{u}_\mu\bar{u}_\nu. \tag{2.1.25}$$

The induced volume 2-form is

$$\epsilon_S \equiv dS := -s i_u d\Sigma = -n^\rho u^\sigma dS_{\rho\sigma}, \qquad \epsilon_\Sigma = u\wedge\epsilon_S, \qquad \epsilon = sn\wedge u\wedge\epsilon_S, \tag{2.1.26}$$

where $dS_{\rho\sigma} = 2n_{[\rho}u_{\sigma]}dS$ is the oriented surface element with both outgoing normals. Accordingly, the pull-back of a 2-form α on $\partial\Sigma$ is

$$\underleftarrow{\alpha} = -\frac{s}{2}\alpha_{\mu\nu}\epsilon^{\mu\nu\rho\sigma}n_\rho u_\sigma dS = -\frac{s}{2}(\star\alpha)^{\mu\nu}dS_{\mu\nu}. \tag{2.1.27}$$

Foliations: For Section 8.4, we take the boundaries Σ and $\mathcal{T}$ to be part of foliations defined by the level sets of two scalar fields $\varphi^0(x)$ and $\varphi^1(x)$ respectively. Without loss of generality we can adapt our coordinates such that $\varphi^0 = t$ and $\varphi^1 = r$ in spherical coordinates, so that the corners defined by the intersections of the two foliations are spheres parametrized by θ and ϕ. In these adapted coordinates, the presence of a non-orthogonal corner is directly parametrized by one of the metric components,

$$n = -\frac{1}{\sqrt{-g^{tt}}}dt, \qquad \bar{n} = \frac{1}{\sqrt{g^{rr}}}dr, \qquad \sinh\beta = -\frac{g^{tr}}{\sqrt{-g^{tt}}\sqrt{g^{rr}}}. \tag{2.1.28}$$

This identification can be used to provide a bulk extension of the function β.

We further have

$$\sqrt{-g} = N\sqrt{q}, \qquad \sqrt{-\bar{q}} = N\lambda\sqrt{\gamma}, \tag{2.1.29}$$

in terms of the ADM variables $N = -1/\sqrt{-g^{tt}}$, and

$$q_{ab} = g_{ab} = \begin{pmatrix} q_{rr} & q_{rA} \\ & \gamma_{AB} \end{pmatrix}, \qquad \bar{q}_{\bar{a}\bar{b}} = \begin{pmatrix} -N^2 + q_{ab}N^aN^b & q_{Ab}N^b \\ & \gamma_{AB} \end{pmatrix}. \tag{2.1.30}$$

In this set-up, the orthogonal corner case corresponds to a partial gauge-fixing in which one component of the shift vector vanishes, $N^r = 0$. It is also possible to consider a more general set-up, in which the timelike boundary is not a level set of one of the coordinates. In this case one can describe both orthogonal and non-orthogonal cases without gauge fixing.

Extrinsic geometry of the corner

We can repeat the definition of the extrinsic curvature for the corner by treating it as a hypersurface in Σ and $\bar{\Sigma}$. Below we summarize relations between the different extrinsic curvatures:

$$k_{\mu\nu} := \gamma^\rho_\mu D_\rho u_\nu = \gamma^\rho_\mu q^\sigma_\nu D_\rho u_\sigma, \qquad \bar{k}_{\mu\nu} := \gamma^\rho_\mu \bar{D}_\rho \bar{u}_\nu \tag{2.1.31}$$

$$k = g^{\mu\nu}k_{\mu\nu} = \gamma^{\mu\nu}\nabla_\mu u_\nu = \nabla_\mu u^\mu + n^\mu n^\nu \nabla_\mu u_\nu = \lambda(\bar{K} + \sinh\beta K + \nabla_{\bar{u}}\beta - \bar{n}\cdot a) \tag{2.1.32}$$

$$\bar{k} = g^{\mu\nu}\bar{k}_{\mu\nu} = \gamma^{\mu\nu}\nabla_\mu \bar{u}_\nu = \nabla_\mu \bar{u}^\mu - \bar{n}^\mu \bar{n}^\nu \nabla_\mu \bar{u}_\nu = \lambda(K - \sinh\beta \bar{K} + \nabla_u \beta + n\cdot \bar{a}) \tag{2.1.33}$$

Defined this way, k and $\bar{k}$ are quantities that are extrinsic to S but intrinsic to Σ and $\bar{\Sigma}$, respectively. The corner can equally be treated as a spacelike cut of a null hypersuface $\mathcal{N}$. This case will be analysed in the next section.

2.2 Null hypersurfaces

We consider a null hypersurface $\mathcal{N}$ defined by a cartesian equation $\Phi = 0$, and denote its normal

$$l_\mu :\overset{\mathcal{N}}{=} -f\partial_\mu \Phi, \qquad l^2 \overset{\mathcal{N}}{=} 0, \tag{2.2.1}$$

with $f > 0$ as to have the vector future-pointing. The corresponding vector l^μ is null and hypersurface orthogonal, hence it is also tangent to $\mathcal{N}$ and geodetic,

$$l^\mu \nabla_\mu l^\nu \overset{\mathcal{N}}{=} k l^\nu. \tag{2.2.2}$$

The hypersurface is thus naturally fibrated by null geodesics, and $k = 0$ if they are affinely parametrized. The chosen tangent vector is referred to as generator of $\mathcal{N}$. Taking the language from fibre bundles, the direction along the fibre is referred to as vertical, and

the others as horizontal. We will assume that $\mathcal{N}$ has topology $I \times S$, where $S = S^2$ and I is some interval in $\mathbb{R}$. If $I = \mathbb{R}$ in affine coordinates the hypersurface is called complete, and in this case all null geodesics extend indefinitely in both directions. It is called semi-complete if it extends indefinitely in one direction only, and has a boundary in the other direction caused for instance by the formation of caustics or crossings. We will often use adapted coordinates (Φ, x^a), where x^a, $a = 1, 2, 3$ are coordinates on the leaves of the Φ foliation. The condition that $\mathcal{N}$ is null then induces a partial gauge-fixing of the metric given by

$$g^{\Phi\Phi} \overset{\mathcal{N}}{=} 0. \tag{2.2.3}$$

For spacelike and timelike hypersurfaces, there is a canonical choice of normal with unit norm. This makes the normal independent of the embedding of the hypersurface, i.e. invariant under a change of parametrization that preserve the location of the boundary: $\Phi \mapsto \Phi' = \Phi F(x)$, with F smooth at $\mathcal{N}$. No such preferred choice exists in the null case. The function f is thus arbitrary, and one has to check on a case by case basis whether a given quantity is independent of the embedding or not. On the other hand, the geometry of the hypersurface is sensitive only to the equivalence class $[l_\mu = A l_\mu]$ of normals identified up to an arbitrary rescaling. This rescaling can be obtained in two independent ways. First, by changing the choice of f, and second, by changing the embedding, effectively multiplying f by F.

A good example of a non-geometric quantity is the inaffinity k appearing in (2.2.2) since it depends on the parametrization via f and Φ. In fact, an explicit calculation using (2.2.1) gives

$$k = \pounds_l \ln f - \frac{f}{2} \partial_\Phi g^{\Phi\Phi}, \tag{2.2.4}$$

written in adapted coordinates. If follows that k depends on the perpendicular derivative of the metric, therefore it contains information about the extrinsic geometry. We can rewrite (2.2.4) in a more covariant form if we parametrize an arbitrary extension of the normal as $l = -f d\Phi + \Phi\, v$, then

$$k = \pounds_l \ln f - \frac{1}{2} \pounds_n l^2 + \frac{1}{f} l \cdot v, \tag{2.2.5}$$

where n is any null vector such that $n \cdot l = -1$. This expression is slightly misleading because it may give the impression that k depends on v, but this dependence cancels out between the second and third term to give back (2.2.4).[5] From this general expression one can also read off the special values when the extension is null everywhere ($l^2 = 0$, $v \neq 0$), when it is hypersurface-orthogonal everywhere ($v = 0$), or both. A typical example of the first special case is Kerr's principal null direction, hypersurface orthogonal only at

[5] Independence of k from the extension of l can be also checked showing that $l' = l + \Phi v$ gives the same k as l, and it means that different k imply different f at fixed spacetime metric.

the horizon. For the second special case, l is normal to a foliation that has a single null leaf, and for the third, l is normal to a null foliation.

The null vector n introduced in (2.2.5) is known as *the rigging vector,* and it is a convenient tool to work on null hypersurfaces. It allows one the use of covariant expressions at all times and allow us to avoid hypersurface indices, thus making the relation to spacetime objects transparent. It also allows one to use the Newman-Penrose (NP) formalism and the numerous results that have been derived in that language. To that end, we complete the pair (l, n) to a doubly-null NP tetrad $(l, n, m, \bar{m})$ on $\mathcal{N}$.

The downside of the rigging vector approach is its reliance on an arbitrary choice of auxiliary vector. But it is quite easy to check which quantities are independent of this choice. The arbitrariness is a 2-parameter family given by

$$n \to n + \bar{a}m + a\bar{m} + |a|^2 l, \qquad m \to m + al, \qquad a \in \mathbb{C}. \tag{2.2.6}$$

Quantities which are invariant under (2.2.6) are independent of the choice of auxiliary rigging vector. For instance, it is easy to check that (2.2.5) is invariant. The map (2.2.6) is an internal Lorentz transformations of the NP tetrad that corresponds to the two translations of the ISO(2) little group stabilizing l. We will refer to it as a class-I transformation (of the NP tetrad), following [55]. In this classification, class-II transformations are the two null translations of the ISO(2) little group stabilizing n. They change l and disalign it from the normal to the hypersurface, and will not be considered in the rest of the paper. The remaining two internal transformations are the class-III spin-boost transformations

$$(l, n, m, \bar{m}) \to (Al, A^{-1}n, e^{i\varphi}m, e^{-i\varphi}\bar{m}). \tag{2.2.7}$$

The boost transformation acts as a rescaling of the normal by an arbitrary real function A. Therefore, quantities invariant under this boost are independent of the choices of f and of the embedding used when writing (2.2.1). This is not the case for the inaffinity, which transforms as $k \to A(k + \pounds_l A)$.

An important result of [36] is that the equivalence class

$$[l_\mu, k] = [Al_\mu, A(k + \pounds_l A)] \tag{2.2.8}$$

can be taken to be the universal background structure in the covariant phase space of metrics with a null hypersurface.[6] That is, any two metrics with a null hypersurface

[6] This is referred to as universal boundary structure. We prefer the adjective background to emphasize that these are quantities that will not be varied in the phase space. The paper [36] also introduces a notion of 'universal intrinsic structure', based on purely intrinsic quantities, and which we do not consider here.

admit a coordinate system in which they have the same (2.2.8). Elements in the universal structure (2.2.8) must thus be class-III invariant.

These internal Lorentz transformations are practical tools to discern the quantities that depend solely on the geometry of $\mathcal{N}$ from those that depend on additional structures or choices. To reiterate, class-I invariance means independence from the choice of rigging vector,[7] and class-III invariance guarantees independence from rescaling the normal, namely from the choice of f and from reparametrizations $\Phi \mapsto \Phi' = \Phi F(x)$. One theme of this paper will be that lack of class I and class III invariance translates to anomalies in the covariant phase space.

If need be to select a specific rigging vector, there are two natural ways to do so that are common in the literature. The first is to require it to be parallel-transported along l on $\mathcal{N}$, see e.g. [57]. This choice is unique, and fixes the class-I transformation so that the NP spin-coefficient π vanishes.[8] The second way is to require it to be adapted to a given $2+1$ foliation of $\mathcal{N}$. This choice is again unique once the foliation is given. In this case the class-I transformation is fixed setting to zero two of the three components of the pull-back of n, thus making it hypersurface orthogonal within $\mathcal{N}$. It follows that the 2d planes spanned by $(m, \bar{m})$ integrate to the leaves of the foliation.

A volume form on $\mathcal{N}$ can be defined from the spacetime volume form ϵ via

$$\epsilon = -l \wedge \epsilon_{\mathcal{N}}. \tag{2.2.9}$$

The conventional minus sign here follows from assuming an outgoing normal l , and would be plus if it was instead chosen as incoming. The volume form $\epsilon_{\mathcal{N}}$ is class-I invariant but not class-III invariant because it depends on f. This formula defines an equivalence class of volume forms, related by adding any 3-form containing l. A convenient representative of this equivalence class can be chosen using the rigging vector as

$$\epsilon_{\mathcal{N}} := \underleftarrow{i_n \epsilon} = \frac{\sqrt{-g}}{f} d^3x, \tag{2.2.10}$$

where the second equality uses adapted coordinates (Φ, x^a), and the arrow under the form means pull-back on $\mathcal{N}$. We will make this choice from now on. Written in this way, class-I invariance may not appear as obvious, but it follows from the pull-back and the fact that m^μ is tangent to $\mathcal{N}$.

[7] The extended structure of a null hypersurface plus a specific choice of normal is called Carollian structure in some literature, and the further extension including a specific choice of rigging vector a 'ruled' or rigged Carollian structure, see e.g. [56].

[8] Requiring $(m, \bar{m})$ to be also parallel-transported will further fix the spin part of the class-III so that the NP coefficient ϵ is real. This is the same letter used below for the volume form, but being the first a scalar and the second a form no confusion should hopefully arise.

On $\mathcal{N}$, we also define the spacelike area form of the corner S as

$$\epsilon_S := i_l \epsilon_{\mathcal{N}} = i\underleftarrow{m \wedge \bar{m}}, \qquad i_l \epsilon_S = 0, \qquad \epsilon_{\mathcal{N}} = -n \wedge \epsilon_S. \tag{2.2.11}$$

It is class-I invariant and defined independently of any choice of foliation of $\mathcal{N}$. It satisfies

$$d\epsilon_S = \theta \epsilon_{\mathcal{N}}, \tag{2.2.12}$$

where θ is the expansion of l, as defined below. Notice that ϵ_S so defined contains also vertical components, even if n is adapted to a foliation and $(m, \bar{m})$ are integrable. Only choosing affine coordinates will make the area 2-form purely horizontal. From this equation one derives the following useful identity,

$$(\pounds_l + \theta) X \, \epsilon_{\mathcal{N}} = d(X \epsilon_S). \tag{2.2.13}$$

Using the rigging vector, a local projector on 2d spacelike planes can be introduced

$$\gamma_{\mu\nu} := g_{\mu\nu} + 2 l_{(\mu} n_{\nu)} = 2 m_{(\mu} \bar{m}_{\nu)}. \tag{2.2.14}$$

Its pull-back to the null surface $\underleftarrow{\gamma}_{\mu\nu}$, or γ_{ab} in hypersurface indices, coincides with the pull-back of the spacetime metric.[9] This is the (degenerate) induced metric, whose null direction is given by l^μ itself. The class-III invariant pair $(\underleftarrow{\gamma}_{\mu\nu}, l^\mu \epsilon_{\mathcal{N}})$ contains six independent quantities, which are the analogue of the induced geometry in the non-degenerate case.

For the extrinsic geometry, we look at the pull-back of the gradient of the normal vector. This quantity gives the extrinsic curvature in the case of a spacelike or timelike hypersurface. In the null case, $W_\mu{}^\nu := \nabla_{\underleftarrow{\mu}} l^\nu$ defines a purely hypersurface-intrinsic objet, satisfying $n^\mu W_\mu{}^\nu = 0 = W_\mu{}^\nu l_\nu$, and $l^\mu W_\mu{}^\nu = k l^\nu$. It is related to the Weingarten map, which is the reason for the notation W. The actual map is given using hypersurface indices as in [36, 39], but that definition is equivalent to ours in terms of covariant 4d indices. To see the geometric content of this map, it is convenient to use the rigging vector and decompose it as follows,

$$\begin{aligned} W_\mu{}^\nu := \nabla_{\underleftarrow{\mu}} l^\nu &\overset{\mathcal{N}}{=} \omega_{\underleftarrow{\mu}} l^\nu + \gamma^\nu_\rho B_{\underleftarrow{\mu}}{}^\rho \\ &= \big((\bar{\alpha} + \beta) \bar{m}_{\underleftarrow{\mu}} - \epsilon n_{\underleftarrow{\mu}}\big) l^\nu - (\sigma \bar{m}_{\underleftarrow{\mu}} + \rho m_{\underleftarrow{\mu}}) \bar{m}^\nu + \text{cc.}^{10} \end{aligned} \tag{2.2.15}$$

[9] The pull-back to S will give the induced metric of the corner like in the non-null case.

The second line makes reference to the NP formalism (with mostly-plus signature, we use the conventions of [58]), and the various tensors there appearing are:

$$B_{\mu\nu} := \gamma^\rho_\mu \gamma^\sigma_\nu \nabla_\rho l_\sigma = \frac{1}{2}\gamma^\rho_\mu \gamma^\sigma_\nu \pounds_l \gamma_{\rho\sigma} \overset{\mathcal{N}}{=} \sigma_{\mu\nu} + \frac{1}{2}\gamma_{\mu\nu}\theta, \tag{2.2.16a}$$

$$\sigma_{\mu\nu} := \gamma^\rho_{\langle\mu}\gamma^\sigma_{\nu\rangle}\nabla_\rho l_\sigma = -\bar{m}_\mu \bar{m}_\nu \sigma + cc, \qquad \theta := 2m^{(\mu}\bar{m}^{\nu)}\nabla_\mu l_\nu = -2\rho, \tag{2.2.16b}$$

$$\omega_\mu := -\eta_\mu - k n_\mu, \qquad \eta_\mu := \gamma^\rho_\mu n^\sigma \nabla_\rho l_\sigma = -(\alpha + \bar{\beta})m_\mu + cc, \qquad l^\mu \omega_\mu = k = 2\mathrm{Re}(\epsilon). \tag{2.2.16c}$$

Here B is the deformation tensor, whose antisymmetric part vanishes because l is hypersurface orthogonal at $\mathcal{N}$, σ is the shear and θ the expansion; ω is the rotational 1-form of isolated and non-expanding horizons [58, 59], satisfies $\omega \cdot l = k$; η is the connection 1-form on the normal timelike planes spanned by (l, n), it provides the connection term of the covariant derivative $\eth$ used in NP calculus [60, 61], and it is sometimes called Hajicek 1-form [62], or twist, since it is related to the non-integrability of the normal planes via

$$\gamma_{\mu\nu}[n, l]^\nu = \eta_\mu - \gamma^\nu_\mu (l^\rho \nabla_\rho n_\nu - \partial_\nu \ln f). \tag{2.2.17}$$

The Weingarten map depends on a specific choice of normal and not on the equivalence class. It is nonetheless useful to describe the geometry of the null hypersurface. From (2.2.16a), we see that the shear and expansion are entirely determined by the induced metric and a choice of l, so they are part of the intrinsic geometry. The dependence on the scaling of l can be eliminated if we look at the densitized expressions $\sigma\epsilon_{\mathcal{N}}$ and $\theta\epsilon_{\mathcal{N}}$ which are class-III invariant.

Perpendicular derivatives of the metric enter the inaffinity k and the twist η_μ. These quantities could be taken as the analogue of the extrinsic geometry, but they are ambiguous since they depend on the choice of l representative and not on the equivalence class. This dependence can be partially removed if we consider the following shifts,

$$\bar{k} := k - l^\mu \partial_\mu \ln f = -\frac{f}{2}\partial_\Phi g^{\Phi\Phi}, \tag{2.2.18}$$

$$\bar{\eta}_\mu := \eta_\mu + \gamma^\nu_\mu \partial_\nu \ln f = \gamma_{\mu\nu}([n, l]^\nu + l^\rho \nabla_\rho n^\nu) = m_\mu(\bar{m}_\nu [n, l]^\nu + \pi) + \mathrm{cc}, \tag{2.2.19}$$

where π here is one of the NP coefficients. $\bar{\eta}_\mu$ and $\bar{k}\epsilon_{\mathcal{N}}$ are invariant under changes of f, but not under changes of embedding $\Phi \to \Phi F(x)$. Therefore they are still not class-III invariant, but at least satisfy the weaker requirement of being independent of the choice of normal representative at fixed embedding.[11] If we keep the embedding fixed, $\bar{k}\epsilon_{\mathcal{N}}$ is fully unambiguous. However $\bar{\eta}_\mu$ is not, because it inherits a dependence on the rigging vector from η_μ, hence it is still not a genuine measure of the extrinsic geometry of $\mathcal{N}$. In

[11] This is consistent with the statement in [63] that a quantity like $\bar{k}\epsilon_{\mathcal{N}}$ here is class-III boost invariant, because that paper works with a fixed 2+2 foliation.

quantity	Rigging-vector independence	Rescaling independence	Boost weight	Spin weight
$\sigma_{\mu\nu}$	✗	✗	1	0
σ	✓	✗	1	2
θ	✓	✗	1	0
$\epsilon_{\mathcal{N}}$	✓	✗	-1	0
k	✓	✗	1+inhom	0
η_μ	✗	✗	0+inhom	0
$\alpha + \bar{\beta}$	✗	✗	0+inhom	-1
$\gamma_{\mu\nu}[l,n]^\nu$	✗	✓	0	0

Table 2.1.: *Behaviour under class-I and class-III transformations. Quantities that are not invariant under (2.2.7) can be characterized in terms of their boost and spin weights, respectively a and b, defined by $X \to A^a e^{ib\varphi} X$ (up to possible inhomogeneous terms) under (2.2.7). The boost weight can also be interpreted as a conformal weight, for instance in the case of future null infinity where the normal is the gradient of the conformal rescaling of the metric.*

fact, even though the Weingarten map is independent of the choice of rigging vector, the decomposition we used on the right-hand side of (2.2.15) introduces a dependence on it: only θ, k and (the scalar contraction) σ are class-I invariant, whereas $\sigma_{\mu\nu}$, η_μ and ω_μ are not. For convenience, the transformation properties of all quantities are summarized in Table 2.1, with the details reported in Appendix B.1.

The only case in which (the pull-back of) $\bar{\eta}_\mu$ is class-I invariant is on a non-expanding horizon with $k = 0$. And in fact it characterizes the shape of a non-expanding horizon via the Noether charge construction [45]. To use it as a measure of the extrinsic geometry of a general $\mathcal{N}$, one has to fix the class-I gauge freedom. If we do so taking n parallel transported by l the NP spin coefficient π vanishes and can identify the twist $\bar{\eta}_\mu$ (or equivalently η_μ with a gradient normal representative) with the non-integrability of the timelike planes, thanks to (2.2.19). Sometimes it is more convenient to fix n to be adapted to a foliation of $\mathcal{N}$ instead. We will show in Chapter 9 that $\bar{\eta}_\mu$ determines the evolving Noether charges associated with the leaves of that foliation.

We conclude with two more remarks about the Weingarten map. First, its trace is given by

$$W := W_{\underset{\leftarrow}{\mu}}{}^{\mu} = \nabla_\mu l^\mu + \frac{1}{2}\partial_n l^2 = \theta + k, \tag{2.2.20}$$

and provides the boundary term for the variational principle with Dirichlet boundary conditions on a null hypersurface [64, 51, 65, 66], the equivalent of the Gibbons-Hawking-York term. The discrepancy between the trace of the Weingarten map and the divergence of the normal may look unfamiliar, but it would occur also in the timelike case if the normal τ is not of unit-norm off the hypersurface: $K = \nabla_\mu \tau^\mu + \frac{1}{2}\partial_\tau \tau^2$, where $K_{\mu\nu} := q^\rho_\mu \nabla_\rho \tau_\nu$.

Second, an alternative covariant construction of the Weingarten map can be given in terms of the 'half-projector' $\Pi_\mu{}^\nu := \gamma_\mu^\nu - n_\mu l^\nu$, defining $\tilde{W}_\mu{}^\nu := \Pi_\mu{}^\rho \nabla_\rho l^\nu$. This tensor is rigging-vector dependent, but not its pull-back on the hypersurface. This pull-back is the definition used in [39], and coincides with (2.2.15). The trace also coincides with (2.2.20), namely $\tilde{W}_\mu{}^\mu = W$.

2.2.1 Foliations

The volume form $\epsilon_\mathcal{N}$ is not class-III invariant, and depends on the full spacetime metric determinant $\sqrt{-g}$. On non-degenerate hypersurfaces choosing a unit-norm normal makes the volume form depend only on the determinant of the induced metric. The unit-norm option does not exist in the null case, but one can achieve a similar result introducing a $2+1$ foliation of $\mathcal{N}$. The foliation can be arbitrary, provided that its leaves are spacelike. We take it to be defined by the level sets of some scalar function λ, and denote $x^a = (\lambda, x^A)$ the coordinates adapted to it.

Note that if we take this choice together with the foliation defined by Φ, we obtain spacetime a coordinate system (Φ, λ, x^A) adapted to a 2+2 foliation of spacetime (see e.g. [67]). Our choice of letters for these coordinates is meant to preserve generality of the formalism with respect to common applications. For example, to make the link the Schwarzschild metric in retarded Bondi coordinates we would take $(\lambda, \Phi) = (u, r - 2M)$ and $\mathcal{N}$ is the white hole horizon, or using advanced time instead $(\lambda, \Phi) = (v, 2M - r)$ and $\mathcal{N}$ is the black hole horizon. We can also keep assuming l^μ future pointing namely $g^{\Phi\lambda} < 0$ without loss of generality. In the first case this leads to $g^{ur} < 0$, in the second case to $g^{vr} > 0$. Or if $\mathcal{N}$ is a null cone in Minkowski in a doubly-null foliation, we can identify $\Phi = u := t - r$ and $\lambda = v := t + r$. Since λ is a (null) time, we will refer to ∂_λ as a time derivative, and use a dot to indicate it.

In these coordinates,

$$\sqrt{-g} = -\frac{1}{g^{\Phi\lambda}}\sqrt{\gamma}, \tag{2.2.21}$$

where γ is the determinant of the spacelike metric γ_{AB} on the 2d leaves. Hence,

$$\epsilon_\mathcal{N} = \frac{\sqrt{\gamma}}{l^\lambda} d\lambda d^2x. \tag{2.2.22}$$

We see that it is a completely intrinsic quantity, but it is still not class-III invariant and contains more information than the 2d area form γ: it depends also on the extent of l via l^λ. If we now choose $f = -1/g^{\Phi\lambda}$, we obtain $l^\lambda = 1$ and

$$\epsilon_\mathcal{N} = \sqrt{\gamma} d\lambda d^2x. \tag{2.2.23}$$

Notice that λ does not need to be a parameter along the null geodesics. In general after making these choices,

$$l^a = (1, -b^A), \qquad \underleftarrow{g_{ab}} = \begin{pmatrix} \gamma_{AB} b^A b^B & \gamma_{AB} b^B \\ & \gamma_{AB} \end{pmatrix}, \tag{2.2.24}$$

and the vector b^A acts as a shift vector for the 2+1 foliation defined by λ. If we partially fix the coordinate gauge requiring that x^A are conserved along the generators, then we are setting the shift vector to zero, and $l^a = (1, 0, 0)$. In terms of the spacetime metric, this partial gauge-fixing reads $g^{\Phi A} \overset{\mathcal{N}}{=} 0$. We refer to it as partial Bondi gauge, as in [68, 69]. The foliation-dependent choice $f = -1/g^{\Phi\lambda}$ was referred to as 'canonical' normalization in [66], for its analogy with the ADM spacelike case, since $1/g^{\Phi\lambda}$ plays the role of lapse in the $3+1$ decomposition with null slices [70].

The simplification (2.2.23) gives to the volume form a similar structure to the one of non-degenerate hypersurfaces (albeit in terms of a codimension-2 determinant), and it is often used in the literature, e.g. [53]. It is valid only in the foliation chosen, but in the partial Bondi gauge it remains valid for any new foliation obtained by a super-translation $\lambda' = \lambda + T(x^A)$.

A common choice of $2+1$ foliation is the one induced by the intersections of $\mathcal{N}$ with a spacelike foliation. In this case the cross-sections of $\mathcal{N}$ provide the boundary $\partial\Sigma$ of each 3d spacelike leaf Σ. Let us denote by τ the unit-norm normal to the spacelike foliation, and parametrize the scalar product as follows,

$$l \cdot \tau \overset{\mathcal{N}}{=} -\frac{1}{\sqrt{2}} e^{-\hat{\beta}}. \tag{2.2.25}$$

The overall minus sign is due to the fact that both vectors are future pointing. The quantity $\hat{\beta}$ has no geometric meaning per se, since it is not class-III invariant. It can be used to measure the change of geometric tilt between $\mathcal{N}$ and Σ only if l is kept fixed. The unit-norm normal to the cross-section within $T\Sigma$ is

$$\hat{r}^\mu \overset{\mathcal{N}}{=} \pm\sqrt{2} e^{\hat{\beta}} q^\mu_\nu l^\nu, \qquad q_{\mu\nu} := g_{\mu\nu} + \tau_\mu \tau_\nu, \tag{2.2.26}$$

where the sign is plus if $\mathcal{N}$ is the outgoing null hypersurface from the boundary of Σ, and minus if it is the incoming one. It can be used to define a rigging vector adapted to $\partial\Sigma$, which is given by

$$n = \frac{1}{\sqrt{2}} (e^{\hat{\beta}} \tau \mp e^{-\hat{\beta}} \hat{r}). \tag{2.2.27}$$

Now (l, n) and $(\tau, \hat{r})$ provide two possible bases for the timelike plane normal to $\partial\Sigma$. This change of basis is used to determine the corner terms required in the action by the variational principle.

2.2.2 Affine coordinates

The fact that a null hypersurface is ruled by null geodesics endows it with a preferred class of foliations, in which λ is a parameter along the geodesics, $l^\mu \partial_\mu \lambda = 1$. To use this parameter as one of the coordinates, we fix an initial cross-section of $\mathcal{N}$, say at $\lambda = 0$, define angular coordinates x^A there and then Lie-drag them along $\mathcal{N}$. This defines a coordinate system with vanishing shift vector,

$$l^a = (1, 0, 0), \qquad \underleftarrow{g_{ab}} = \begin{pmatrix} 0 & 0 \\ & \gamma_{AB} \end{pmatrix}. \tag{2.2.28}$$

These coordinates satisfy the partial Bondi gauge. We have $l_\mu = (g_{\lambda\Phi}, 0, 0, 0)$ and $g_{\Phi\lambda} = 1/g^{\Phi\lambda}$, therefore this choice of tangent vector corresponds to the 'canonical normalization' for f. This is an example of a situation in which f is metric-dependent. We can complete this partial gauge fixing on $\mathcal{N}$ with a fourth condition, for instance redefining Φ so that $g_{\lambda\Phi} \overset{\mathcal{N}}{=} -1$. The metric now satisfies

$$g_{\lambda\lambda} = O(\Phi), \qquad g_{\Phi\lambda} = -1 + O(\Phi), \qquad g_{\lambda A} = O(\Phi), \tag{2.2.29}$$

and it is fully gauge-fixed on $\mathcal{N}$.

The coordinate system can be further specialized if we require the parameter to be affine, namely

$$l_o := \frac{d}{d\lambda}, \qquad l_o^\mu \nabla_\mu l_o^\nu \overset{\mathcal{N}}{=} 0. \tag{2.2.30}$$

This condition fixes the first-order extension of the metric component $g_{\lambda\lambda}$ so that $\partial_\Phi g_{\lambda\lambda} \overset{\mathcal{N}}{=} 2\partial_\lambda g_{\Phi\lambda}$.[12] Since one can always choose the adapted coordinate Φ such that $g_{\Phi\lambda} = -1 + O(\Phi)$, in that gauge we have $g_{\lambda\lambda} = g^{\Phi\Phi} = O(\Phi^2)$.[13] At this point,

$$g_{\lambda\lambda} = O(\Phi^2), \qquad g_{\Phi\lambda} = -1 + O(\Phi), \qquad g_{\lambda A} = O(\Phi), \tag{2.2.31}$$

and the rest of the metric is arbitrary. The condition of affinity can always be imposed via gauge-fixing, but we see that it is not a characteristic of the hypersurface coordinates alone, since it involves the first-order extension of the metric.

In the affine coordinate system, any normal vector in the equivalence class satisfies

$$l^\mu \overset{\mathcal{N}}{=} f l_o^\mu \tag{2.2.32}$$

[12] This follows from $\Gamma^\mu_{\lambda\lambda} \overset{\mathcal{N}}{=} 0$, which by invertibility of the metric is equivalent to $2\partial_\lambda g_{\mu\lambda} - \partial_\mu g_{\lambda\lambda} \overset{\mathcal{N}}{=} 0$. This is identically satisfied by (2.2.28) for $\mu = (\lambda, A)$, and thus reduces to the single equation given in the text.

[13] Notice that this would be a 'generalized' diffeomorphism, not invertible at the hypersurface, similar to how going from static Schwarzschild coordinates to Eddington-Finkelstein is singular at the horizon.

and

$$k = l^\mu \partial_\mu \ln f. \tag{2.2.33}$$

Hence it is affine iff f is chosen constant in λ, namely $\pounds_l f = 0$. Furthermore since $\partial_\Phi g^{\Phi\Phi} \overset{\mathcal{N}}{=} 0$, any extension of l with $v \cdot l \overset{\mathcal{N}}{=} 0$ satisfies $\partial_n l^2 \overset{\mathcal{N}}{=} 0$, namely it is null at first-order off the hypersurface. We also notice that in affine coordinates $\bar{k} = 0$.

This coordinate system can be extended to a neighbourhood of $\mathcal{N}$ as follows (see e.g. [45]). We shoot geodesics off $\mathcal{N}$, and Lie drag x^a along them. Namely, we have

$$n_o^\mu = \frac{\partial}{\partial \Phi}, \qquad n_o^\mu \nabla_\mu n_o^\nu = k_{n_o} n_o^\nu, \qquad \pounds_{n_o} x^a = 0. \tag{2.2.34}$$

We can then completely fix the bulk coordinate gauge freedom if we require that (i) n_o is null everywhere, (ii) Φ is affine (hence $k_{n_o} = 0$), and (iii) it is the gradient of the foliation of constant λ on $\mathcal{N}$, namely $n_o \overset{\mathcal{N}}{=} -d\lambda$. The last condition in particular means that n_o gives a choice of rigging vector for l_o adapted to the λ foliation. In terms of metric components, (i) fixes $g_{\Phi\Phi} = 0$, then (ii) requires $\Gamma^\mu_{\Phi\Phi} = 0$, which in turns implies $\partial_\Phi g_{\Phi\mu} = 0$. Finally, (iii) fixes $g_{\Phi\mu} \overset{\mathcal{N}}{=} (-1, 0, 0, 0)$. The resulting coordinates (λ, Φ, x^A) are defined in a caustic-free open neighbourhood of $\mathcal{N}$, in which the metric reads

$$g_{\mu\nu} = \begin{pmatrix} \Phi^2 F & -1 & \Phi P_A \\ & 0 & 0 \\ & & \gamma_{AB} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 \\ & -\Phi^2(F - P^2) & \Phi P^A \\ & & \gamma^{AB} \end{pmatrix}, \tag{2.2.35}$$

where F, P_A and γ_{AB} are arbitrary metric coefficients. We stress that what makes λ an affine parameter on $\mathcal{N}$ is not so much $g_{\lambda\Phi} = -1$ but $g_{\lambda\lambda} = O(\Phi^2)$. This coordinate system can always be reached, and if one restricts the residual diffeomorphisms to preserve it, the whole extension of ξ is fixed. On the other hand, if one relaxes it and requires only the minimal conditions (2.2.31), only the first order extension $\hat{\xi}^\Phi$ is fixed, whereas $\hat{\xi}^\lambda$ and $\hat{\xi}^A$ remain arbitrary. In any case, charges and fluxes dont depend on these two extensions that can be kept arbitrary.

We can now choose an extension of l such that

$$l^\mu = f l_o^\mu \tag{2.2.36}$$

everywhere in the chart, and not only at $\mathcal{N}$. This is achieved taking $v = \Phi F d\lambda + P_A dx^A$. It satisfies $l \cdot v \overset{\mathcal{N}}{=} 0$ hence it is null at first order around $\mathcal{N}$. This extension is not hypersurface-orthogonal nor null nor geodesic, except at $\mathcal{N}$.

Summarizing, affine coordinates on $\mathcal{N}$ depend on extrinsic properties of the metric, and give us (2.2.31) and (2.2.33). The normal in these coordinates is given by (2.2.32), and it is in general not null at first order, but this can be easily achieved choosing for instance the extension (2.2.36). Another convenient extension is $dl \overset{\mathcal{N}}{=} 0$, which implies instead $v = -df + \Phi v'$, namely $l = -d(f\Phi) + \Phi^2 v'$ is a gradient on $\mathcal{N}$, and $v \cdot l \overset{\mathcal{N}}{=} -\partial_l f$. Then choosing $\Phi' = f\Phi$ or more generally f time independent is also enough to have $v \cdot l \overset{\mathcal{N}}{=} 0$ hence $\partial_n l^2 \overset{\mathcal{N}}{=} 0$. We see that taking $dl \overset{\mathcal{N}}{=} 0 = \pounds_l f$ is an alternative way to have $\partial_n l^2 \overset{\mathcal{N}}{=} 0$ without imposing (2.2.36).

Symmetries

3

In this chapter, the conventional form of Noether's theorems is explored, highlighting their implications in various examples. The chapter begins by introducing Noether's theorems and their significance in the study of symmetries. Through a range of simple examples, the theorems' implications are demonstrated, showcasing their relevance in understanding the conservation laws associated with symmetries.

While the chapter draws extensively from pre-existing literature, some of the calculations are original. However, these calculations are straightforward enough that we assume they must already exist somewhere – it was simply more efficient to perform them anew rather than search for them. For a more exhaustive review complete with further examples, the reader is directed to [71].

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3.1 Active vs passive transformations

Symmetries provide insights into the fundamental properties and behaviors of physical systems. When discussing symmetries, it is important to distinguish between two fundamental ways transformations can affect a system: active and passive transformations. The distinction between the two is important for our understanding of how symmetries relate to the dynamics of a system and the viewpoint of an observer. An *active transformation* involves physically changing the system itself, while keeping the observer or coordinate system fixed. In other words, the transformation is directly applied to the system's elements. Active transformations represent the real evolution of a physical system under a symmetry operation. Let us consider an example to illustrate this concept.

Consider a scalar field $\phi(x)$. An active transformation is mathematically represented as:

$$\phi'(x') = \phi(x). \tag{3.1.1}$$

The transformed field value ϕ' at the transformed point x' is that same as the non-transformed field value ϕ at the non-transformed point x – we have kept the coordinates fixed and transformed the field configuration.

On the other hand, *passive transformations* are concerned with changing the observer's perspective or coordinate system while keeping the physical system unchanged. In this case, the transformation affects the way we describe the system rather than the system itself. This is a transformation of the coordinates only, it neither moves points nor affects the value of the field. An enlightening example showing the relationship between active and passive transformations is given in Section 2.2.4 of [72]. For a discussion on importance of invariance under *active* diffeomorphisms for GR, see also Sections 2.2.5–2.3.2.

When dealing with diffeomorphisms, active and passive transformation perspectives are mutually interchangeable. However, the passive approach may not be universally applicable, as not every transformation can be represented by a coordinate change. This limitation will be demonstrated through the example of the conformal particle Lagrangian in Section 3.3. Another significant difference between active and passive transformations should be noted: while passive transformations affect coordinate changes and thereby influence all quantities defined on spacetime, active transformations may only apply to dynamical fields. In physical theories where background structures are present, it is crucial to ensure these structures remain constant while only the dynamical fields transform. This necessity becomes particularly relevant in the context of gravitational phase space, which we will study in depth in Chapters 4 and onward. Understanding the suptleties between active and passive transformations enhances our understanding of symmetry-related conservation laws. With this idea, in subsequent sections, employing the active transformation perspective we will explain the mathematical framework of symmetries, and show how Noether's theorem enriches our grasp of the dynamics.

3.2 Noether's theorem

Noether's theorem [73] has far-reaching implications in physics, providing deep insights into the fundamental conservation laws that govern various physical phenomena. It plays a crucial role in understanding the underlying structure of physical theories and has been successfully applied in various areas of physics, from classical mechanics to quantum field theory and beyond. By revealing the intimate connection between symmetries and conservation laws, Noether's theorem has become a fundamental tool for deriving and understanding the conservation principles inherent in physical theories, offering a systematic way to link symmetries to measurable quantities that remain constant throughout a system's evolution.

To formulate the theorem, it is important to distinguish two types of variations. The first is a concept of an action symmetry. To illustrate this concept, let us consider a simple example of a free particle described by the action:

$$S[q(t)] = \int dt \frac{\dot{q}^2}{2} \tag{3.2.1}$$

This action is clearly invariant under translations, as $S[q(t) + q_0] = S[q(t)]$ for any constant q_0. This is one example of an action symmetry. In a more general setting, a *symmetry variation* is defined by a function $\delta_\epsilon q$, in which a transformation of the trajectory from $q(t)$ to $q(t) + \delta_\epsilon q(t)$ is encoded, keeping the action invariant up to a boundary term

$$\delta S[q, \delta_\epsilon q] = S[q + \delta_\epsilon q] - S[q] = \int dY. \tag{3.2.2}$$

This equation is to be understood as an equation for $\delta_\epsilon q(t)$. These functions can be used to simplify the EL equations, and if, for a system with n degrees of freedom, we can find $2n$ functions $\delta_\epsilon q(t)$ satisfying this equation for a given action, then we have effectively solved the equations of motion for the system.[1] An important aspect of action symmetries is that they hold without using the EL equation, i.e. (3.2.2) is true for any $q(t)$. This statement might seem trivial now but it will be important later.

On the other hand, *on-shell variations* represent a different restriction of a general action variation. Here, the fields $q(t)$ are constrained to satisfy the Euler-Lagrange equations, while the variations δq are arbitrary. The variation of the action can be expressed as:

$$\begin{aligned}
\delta S[q, \delta q] &= \int dt \left(\frac{\partial L}{\partial q} \delta q + \frac{\delta L}{\delta \dot{q}} \delta \dot{q} \right) && (3.2.3)\\
&= \int dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q + \int dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) && (3.2.4)\\
&\hat{=} \int d \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) =: \delta S[\bar{q}, \delta q], && (3.2.5)
\end{aligned}$$

This equation holds for any variation δq, but in the last line the fields $q(t)$ are restricted so that the bulk term vanishes. We emphasise this by introducing notation $\bar{q}$ to denote the solutions of the Euler-Lagrange equations. This (redundant) notation will be used only in this chapter, in the remainder we will use the usual $\hat{=}$ for equalities holding *on-shell* of the equations of motion.

[1] The transformed equations after applying these symmetries will often have fewer variables and derivatives, making the system more manageable to solve. In some cases, these transformed equations can be fully solved algebraically, leading to explicit solutions in terms of constants of motion. In this case the system is said to be completely integrable.

Both symmetry variations (3.2.2) and on-shell variations (3.2.5) are special cases of the variation of the action which, in its general form, is given by (3.2.3). The combination of these two types of variations leads to Noether's theorem. We have seen that both give rise to boundary terms when acting on the action, but for different reasons. Noether's symmetry $\delta S[q, \delta_\epsilon q]$ arises due to the equations satisfied by δq, while on-shell variation $\delta S[\bar{q}, \delta q]$ takes its form thanks to the equations satisfied by q. When both q and δq are restricted, Noether's theorem comes into play, revealing a quantity Q that remains constant along the trajectory. Namely, setting $q = \bar{q}$ in (3.2.2) and $\delta q = \delta_\epsilon q$ in (3.2.3) the two equations are the same and subtracting them we obtain the conservation law

$$\frac{d}{dt} Q \hat{=} 0, \quad Q := \frac{\partial L}{\partial \dot{q}} \delta_\epsilon q - Y \tag{3.2.6}$$

This is Noether's first theorem: given a symmetry, the charge Q defined above is conserved *on-shell*.

3.3 The conformal particle

Consider a particle in an inverse quadratic potential

$$S[q] = \int dt \left(\frac{\dot{q}^2}{2} - \frac{1}{q^2} \right) \tag{3.3.1}$$

The equation of motion is $\ddot{q} = \frac{2}{q^3}$. An interesting property of this action is that is has enough symmetry so that we can solve its equation of motion without ever integrating it. One of the symmetries of the action is the time-translation symmetry, which gives rise to the conservation of energy as a Noether charge. To demonstrate this, we consider an infinitesimal time translation $t \to t' = t + \epsilon$, where ϵ is a small parameter. Since $q'(t') = q(t)$,

$$\delta_\epsilon q(t) = q(t + \epsilon) - q(t) = -\epsilon \dot{q}, \tag{3.3.2}$$

and the action changes by a boundary term as follows:

$$\delta_\epsilon \dot{q}(t) = \dot{q}(t + \epsilon) - \dot{q}(t) = -\epsilon \ddot{q} \tag{3.3.3}$$

$$\delta S[q, \delta_\epsilon q] = -\int dt \left(\dot{q} \delta_\epsilon \dot{q} + \frac{2}{q^3} \delta_\epsilon q \right) = -\int dt \left(\epsilon \dot{q} \ddot{q} + 2\epsilon q^{-3} \dot{q} \right) = -\epsilon \int dt \frac{d}{dt} \left(\frac{1}{2} \dot{q}^2 - \frac{1}{q^2} \right) \tag{3.3.4}$$

$$\delta S[\bar{q}, \delta q] = \int d\left(\dot{q} \delta q \right) \tag{3.3.5}$$

$$Q = -\frac{1}{q^2} - \dot{q}^2 + \frac{1}{2} \dot{q}^2 =: -E. \tag{3.3.6}$$

Notice that the symmetry is represented by the function $\delta_\epsilon q(t) = \epsilon \dot{q}(t)$, which depends on time t at just one instance. This is where our active vs passive discussion becomes important. Essentially, in the active approach, we are converting time translation into a change in the function $q(t)$. We will adopt this perspective where symmetries are seen as alterations in the fields rather than changes in the coordinates. The change $\delta_\epsilon q(t) = q'(t) - q(t)$ is simply the difference between two functions evaluated at the same time t. This implies that $\delta_\epsilon\left(\frac{d}{dt}q(t)\right) = \frac{d}{dt}\delta_\epsilon q(t)$ and ensures that the variations of time derivatives are treated consistently.

The conservation equation $\dot{E} \hat{=} 0$ provides an algebraic relationship between q and $\dot{q}$. Furthermore, the action of the conformal particle is invariant under dilatations of the form:

$$t \to t' = \lambda t, \quad q \to q'(t') = \sqrt{\lambda} q(t). \tag{3.3.7}$$

The infinitesimal version reads

$$q'\left((1+\epsilon)t\right) = \left(1 + \frac{\epsilon}{2}\right) q(t) \Rightarrow \delta_\epsilon q(t) = \frac{\epsilon}{2} q - \epsilon t \dot{q}, \tag{3.3.8}$$

with $\lambda = 1 + \epsilon$, $\epsilon \ll 1$. The velocity transforms as $\delta_\epsilon \dot{q}(t) = \frac{d}{dt}\delta_\epsilon q(t) = -\epsilon \dot{q} - \epsilon t \ddot{q} + \frac{\epsilon}{2}\dot{q}$, and the variation of the action is

$$\delta S[q, \delta_\epsilon q] = \int dt \left(\dot{q}\delta_\epsilon \dot{q} + \frac{2}{q^3}\delta_\epsilon q\right) \tag{3.3.9}$$

$$= \epsilon \int dt \left(-t\ddot{q}\dot{q} - \frac{1}{2}\dot{q}^2 + -2t\frac{\dot{q}}{q^3} + \frac{1}{q^2}\right) \tag{3.3.10}$$

$$= \epsilon \int dt \left[\frac{d}{dt}\left(\frac{t}{q^2} - \frac{t\dot{q}^2}{2}\right)\right], \tag{3.3.11}$$

and the conserved charge is

$$Q = \frac{t}{q^2} - \frac{t\dot{q}^2}{2} - \dot{q}(-t\dot{q} + \frac{1}{2}q) = t\left(\frac{\dot{q}^2}{2} + \frac{1}{q^2}\right) - \frac{q\dot{q}}{2} =: D. \tag{3.3.12}$$

Now we have another algebraic equation for q and $\dot{q}$. From the two equations we can obtain q as a function of time in terms of integration constants E and D without ever integrating the equation of motion.[2] Indeed, the conservation laws obtained through Noether's theorem not only provide profound insights into the principles governing a system but also offer practical advantages in terms of simplifying the analysis of that system. Notice that, since the system is one-dimensional, at least one of the conserved charges must be an explicit function of time – otherwise the system would be static,

[2] To spell this out, $D = tE - \frac{q\dot{q}}{2}$ and $E = \frac{\dot{q}^2}{2} + \frac{1}{q^2}$ lead to $q(t) = \pm\sqrt{\frac{1+2(tE-D)^2}{E}}$. One can explicitly confirm that $\frac{d^2}{dt^2}\sqrt{\frac{1+2(tE-D)^2}{E}} = \frac{2}{q^3}$, i.e. $q(t)$ satisfies the EL equation.

essentially lacking dynamics. The presence of at least one time-dependent conserved charge ensures that the system has some evolving, non-trivial behavior.

Since it is not at all obvious, it's worthwhile to check that this charge is indeed conserved

$$\frac{d}{dt}D = \left(\frac{\dot{q}^2}{2} + \frac{1}{q^2}\right) + t\left(\dot{q}\ddot{q} - \frac{2}{q^3}\dot{q}\right) - \frac{q\ddot{q}}{2} - \frac{\dot{q}^2}{2} \tag{3.3.13}$$

$$\hat{=} \left(\frac{\dot{q}^2}{2} + \frac{1}{q^2}\right) + t\left(\dot{q}\ddot{q} - \ddot{q}\dot{q}\right) - \frac{2}{q^2} - \frac{\dot{q}^2}{2} = 0. \tag{3.3.14}$$

As a matter of fact, the absence of dimensional constants in the conformal particle action, given by (3.3.1), implies the existence of an even larger symmetry known as projective or Möbius invariance. Under this transformation, both time and the trajectory undergo a Möbius transformation, defined by

$$t \to t' = \frac{at+b}{ct+d}, \quad q(t) \to q'(t') = (ct+d)^{-1}q(t),\text{[3]} \tag{3.3.15}$$

where $ad - bc = 1$, and a, b, c, d are constants. To demonstrate the invariance of the action under this transformation, we can perform an expansion around $a = 1 + \epsilon_a$, $b = \epsilon_b$, $c = \epsilon_c$, and $d = 1 - \epsilon_a$ as follows

$$t' = t + (\epsilon_b + t(2\epsilon_a - \epsilon_c t)) \tag{3.3.16}$$

$$q'(t) = q'(t') - \dot{q}(t)\left(\epsilon_b + 2\epsilon_a t - \epsilon_c t^2\right) \tag{3.3.17}$$

$$= (1 + \epsilon_a - \epsilon_c t)q(t) - \dot{q}(t)\left(\epsilon_b + 2\epsilon_a t - \epsilon_c t^2\right) \tag{3.3.18}$$

$$\delta_{abc}q(t) = (\epsilon_a - \epsilon_c t)q(t) - (\epsilon_b + (2\epsilon_a - \epsilon_c t)t)\,\dot{q}(t) \tag{3.3.19}$$

$$\delta_{abc}\dot{q}(t) = -\epsilon_c q(t) - (\epsilon_a - \epsilon_c t)\dot{q}(t) - (\epsilon_b + (2\epsilon_a - \epsilon_c t)t)\,\ddot{q}(t) \tag{3.3.20}$$

$$\tag{3.3.21}$$

The action transforms as

$$\delta S[q, \delta_{abc}q] = \int dt\left(\dot{q}\delta_{abc}\dot{q} + \frac{2}{q^3}\delta_{abc}q\right) \tag{3.3.22}$$

$$= \int dt\Big[-\epsilon_c q\dot{q} - (\epsilon_a - \epsilon_c t)\dot{q}^2 - (\epsilon_b + (2\epsilon_a - \epsilon_c t)t)\,\dot{q}\ddot{q} \tag{3.3.23}$$

$$+ 2(\epsilon_a - \epsilon_c t)\frac{1}{q^2} - 2\left(\epsilon_b + (2\epsilon_a - \epsilon_c t)t\right)\frac{\dot{q}}{q^3}\Big] \tag{3.3.24}$$

$$\int d\left[-\frac{1}{2}\epsilon_c q^2 + (\epsilon_b + (2\epsilon_a - \epsilon_c t)t)\left(-\frac{\dot{q}^2}{2} + \frac{1}{q^2}\right)\right]. \tag{3.3.25}$$

[3] Notice that this transformation is not a diffeomorphism and cannot be implemented from the passive point of view. If we tried

Now we can calculate the conserved charge

$$
\begin{aligned}
Q_{abc} &= -\frac{1}{2}\epsilon_c q^2 + (\epsilon_b + (2\epsilon_a - \epsilon_c t)t)\left(-\frac{\dot{q}^2}{2} + \frac{1}{q^2}\right) - I_{abc}\theta && (3.3.26)\\
&= -\frac{1}{2}\epsilon_c q^2 + (\epsilon_b + (2\epsilon_a - \epsilon_c t)t)\left(-\frac{\dot{q}^2}{2} + \frac{1}{q^2}\right) && (3.3.27)\\
&\quad - (\epsilon_a - \epsilon_c t)q\dot{q} + (\epsilon_b + (2\epsilon_a - \epsilon_c t)t)\dot{q}^2 && (3.3.28)\\
&= -\frac{1}{2}\epsilon_c q^2 + (\epsilon_b + (2\epsilon_a - \epsilon_c t)t)\, E - (\epsilon_a - \epsilon_c t)q\dot{q}. && (3.3.29)
\end{aligned}
$$

It is conserved for any choice of $\epsilon_a, \epsilon_b, \epsilon_c$

$$
\begin{aligned}
\dot{Q}_{abc} &= -\epsilon_c q\dot{q} + 2(\epsilon_a - \epsilon_c t)E + \epsilon_c q\dot{q} - (\epsilon_a - \epsilon_c t)(\dot{q}^2 + q\ddot{q}) && (3.3.30)\\
&\hat{=} (\epsilon_a - \epsilon_c t)\left(2E - \dot{q}^2 - \frac{2}{q^2}\right) = 0. && (3.3.31)
\end{aligned}
$$

Three interesting choices[4] are obtained by fixing the following parameters

- $\epsilon_a = 0 = \epsilon_c, \epsilon_b = \epsilon \quad \rightarrow \quad Q_\epsilon/\epsilon = E$
- $\epsilon_b = 0 = \epsilon_c, \epsilon_a = 2\epsilon \quad \rightarrow \quad Q_\epsilon/\epsilon = D$
- $\epsilon_a = 0 = \epsilon_b, \epsilon_b = \epsilon \quad \rightarrow \quad Q_\epsilon/\epsilon = -\frac{1}{2}q^2 - t^2 E + tq\dot{q} =: K$

We will discuss their geometric interpretation in Section 3.4 when we introduce their algebra.

3.4 Noether's theorem from a Hamiltonian perspective

The distinction between the Lagrangian and Hamiltonian approaches to mechanics often brings to mind the dichotomy between geometric and algebraic perspectives. The Lagrangian formalism, with its emphasis on action integrals, is inherently geometric. It provides a holistic view of the system's dynamics and is often more intuitive when considering physical 'stories' like the path a particle takes under the influence of a force. On the other hand, the Hamiltonian formulation is more algebraic, dealing with phase space, Poisson brackets, and symplectic structures. This approach offers a high degree of mathematical elegance and tractability, especially when it comes to transformations and symmetries. However, this elegance sometimes comes at the cost of obscuring the physical intuition that may be more transparent in the Lagrangian picture.

[4] Of course, given that this theory has only 2 integration constants, the three charges are not independent, they are related by

$$2KE + 2D^2 = -1. \qquad (3.3.32)$$

With that said, let us delve into how Noether's theorem extends to the Hamiltonian framework. One could argue that this is the natural setting for the theorem, as the algebraic richness of the Hamiltonian formalism provides a robust platform for understanding symmetries and conservation laws. To demonstrate this, consider a general Hamiltonian action given by:

$$S[p,q] = \int dt(p\dot{q} - H(p,q)) \tag{3.4.1}$$

where p and q are the generalized coordinates and momenta, respectively, and $H(p,q)$ is the Hamiltonian function.

The equations of motion for this Hamiltonian action are

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \tag{3.4.2}$$

which can be written using the Poisson bracket notation as

$$\dot{q} = \{q, H\}, \quad \dot{p} = \{p, H\}. \tag{3.4.3}$$

The time derivative of any phase space function can be expressed in terms of the bracket as

$$\frac{df(p,q,t)}{dt} = \{f, H\} + \frac{\partial f}{\partial t}. \tag{3.4.4}$$

Now, suppose we have a conserved charge $Q(p,q,t)$, meaning that

$$\frac{dQ(p,q,t)}{dt} = 0 \Rightarrow \{Q, H\} = -\frac{\partial Q}{\partial t}. \tag{3.4.5}$$

We want to show that this conserved charge generates a symmetry of the action. Let us define the infinitesimal transformation $\delta_\epsilon q$ and $\delta_\epsilon p$ as follows

$$\delta_\epsilon q = \{q, \epsilon Q\} = \epsilon \frac{\partial Q}{\partial p}, \quad \delta_\epsilon p = \{p, \epsilon Q\} = -\epsilon \frac{\partial Q}{\partial q} \tag{3.4.6}$$

where ϵ is an infinitesimal parameter.

We can now calculate the variation of the action (3.4.1) under this transformation

$$\delta_\epsilon S = \int dt \left[p\delta_\epsilon \dot{q} + \dot{q}\delta_\epsilon p - \frac{\partial H}{\partial q}\delta_\epsilon q - \frac{\partial H}{\partial p}\delta_\epsilon p \right] \tag{3.4.7}$$

$$= \int \epsilon dt \left[p\frac{d}{dt}\delta_\epsilon q - \frac{\partial Q}{\partial q}\dot{q} - \frac{\partial H}{\partial q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial Q}{\partial q} \right] \tag{3.4.8}$$

$$= \int \epsilon dt \left[\frac{d}{dt}\left(p\delta_\epsilon q\right) - \frac{\partial Q}{\partial p}\dot{p} - \frac{\partial Q}{\partial q}\dot{q} + \{Q, H\} \right] \tag{3.4.9}$$

$$= \int \epsilon dt \left[\frac{d}{dt}\left(p\delta_\epsilon q - Q\right) + \frac{\partial Q}{\partial t} + \{Q, H\} \right] \tag{3.4.10}$$

$$= \int \epsilon d\left(p\delta_\epsilon q - Q\right). \tag{3.4.11}$$

Thanks to (3.4.5), the variation is a total derivative and hence a symmetry. Furthermore, using (3.2.6) we can identify the Noether charge as $p\delta_\epsilon q - p\delta_\epsilon q + Q = Q$.

Going back to the algebraic properties, suppose now that we have two conserved charges Q_1 and Q_2. Then it follows from Jacobi identity that the commutator $\{Q_1, Q_2\}$ is also conserved:

$$\frac{d}{dt}\{Q_1, Q_2\} = \big\{\{Q_1, Q_2\}, H\big\} + \frac{\partial}{\partial t}\{Q_1, Q_2\} \tag{3.4.12}$$

$$= -\big\{\{H, Q_1]\}, Q_2\big\} - \big\{\{Q_2, H\}, Q_1\big\} + \frac{\partial}{\partial t}\{Q_1, Q_2\} \tag{3.4.13}$$

$$= -\big\{\frac{\partial}{\partial t}Q_1, Q_2\big\} + \big\{\frac{\partial}{\partial t}Q_2, Q_1\big\} + \frac{\partial}{\partial t}\{Q_1, Q_2\} = 0. \tag{3.4.14}$$

Thus $\{Q_1, Q_2\}$ is also a conserved charge, and as a consequence generates another symmetry. Now, $\{Q_1, Q_2\}$ may be zero, may be a new charge Q_3, or it may be proportional to Q_1 or Q_2. In any case, the conclusion is that a complete set of conserved charges $Q_a = Q_1, Q_2, Q_3, ...$ satisfies a Lie algebra

$$\{Q_a, Q_b\} = f_{ab}^{\ c}\, Q_c. \tag{3.4.15}$$

for some structure constants $f_{ab}^{\ c}$. This algebra is especially interesting for the quantum theory. We now go back to the conformal particle, as an explicit example.

The conformal particle in Hamiltonian form

Applying these concepts to the conformal particle, we reexamine the system from a Hamiltonian perspective. The conformal particle action in Hamiltonian form is given by:

$$S = \int \left(p\dot{q} - \left(\frac{p^2}{2} + \frac{1}{q^2} \right) \right) dt. \tag{3.4.16}$$

We identified the conserved charges as

$$E = \frac{p^2}{2} + \frac{1}{q^2} = H, \tag{3.4.17}$$

$$D = tE - \frac{1}{2}pq, \text{and} \tag{3.4.18}$$

$$K = t^2E - 2tD - \frac{1}{2}q^2. \tag{3.4.19}$$

Under the Poisson bracket, these charges satisfy the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra:

$$\{D, H\} = H, \tag{3.4.20}$$

$$\{D, K\} = -K, \tag{3.4.21}$$

$$\{H, K\} = 2D. \tag{3.4.22}$$

This Lie algebraic structure is crucial for understanding the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence [74], where these conserved charges are related to symmetries of the dual gravitational system. To give these charges a geometric interpretation, let us consider the projective action P of $\mathfrak{sl}(2,\mathbb{R})$ on the Minkowski plane. An arbitrary $A \in \mathfrak{sl}(2,\mathbb{R})$ acts on a Minkowski vector $X = (t, x)$ as

$$P(A)X = P\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\begin{bmatrix} t \\ x \end{bmatrix} = \frac{1}{ct+d}\begin{bmatrix} at+b \\ x \end{bmatrix}. \tag{3.4.23}$$

We have seen in Section 3.3 that our charges correspond to the following parameter choices

- H: $a = 1 = d, c = 0, b \in \mathbb{R}$
- W: $b = 0 = c, d = \frac{1}{a}, a \in \mathbb{R}$
- D: $a = 1 = d, b = 0, c \in \mathbb{R}$

We have already interpreted $H = E$ as the energy, and indeed the transformation

$$P\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} t+b \\ x \end{bmatrix}. \tag{3.4.24}$$

is a time translation by b. The action of the dilatation charge D is

$$P\left(\begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix}\right)\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} a^2t \\ ax \end{bmatrix}. \tag{3.4.25}$$

the dilatation of the Minkowski plane, no surprises there. The third charge generates a more complicated transformation. Its action is given by

$$P\left(\begin{bmatrix}1 & 0\\ c & 1\end{bmatrix}\right)\begin{bmatrix}t\\ x\end{bmatrix} = \frac{1}{ct+1}\begin{bmatrix}t\\ x\end{bmatrix}. \tag{3.4.26}$$

and is usually referred to as a *special conformal transformation*. This is a composition of an inversion $x^\mu \to x^\mu/x^2 =: y^\mu$, a translation $y^\mu \to y^\mu - c^\mu =: z^\mu$, and another inversion $z^\mu \to z^\mu/z^2 =: x'^\mu$. A special conformal transformation of a coordinate grid in a plane is shown in Figure 3.1.

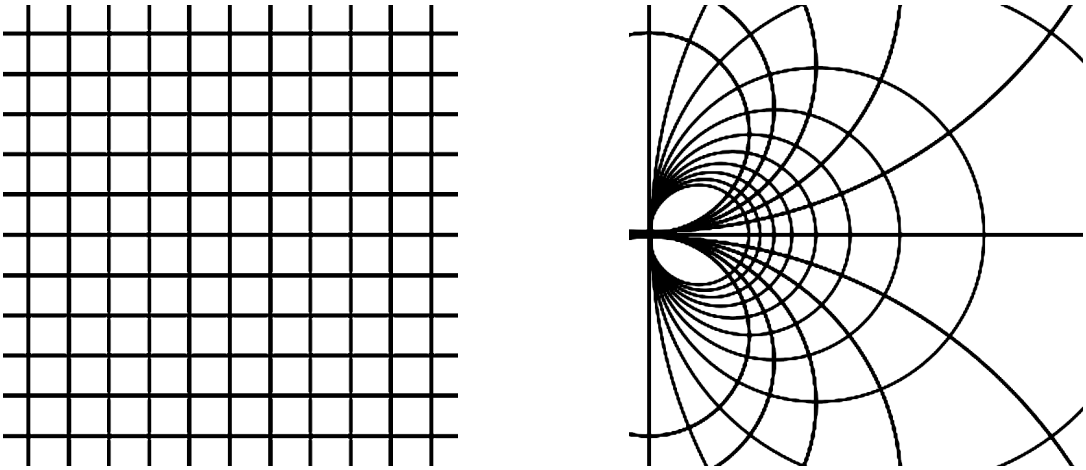

Fig. 3.1.: Left panel: *A coordinate grid prior to a special conformal transformation.*
Right panel: *The same grid after a special conformal transformation.*

3.5 Noether's theorem in field theory

In field theory, Noether's theorem extends naturally from its particle mechanics counterpart. Let us consider a free scalar field with the action given by

$$S[\phi(x)] = \frac{1}{2}\int d^4x\, \partial_\mu\phi\partial^\mu\phi. \tag{3.5.1}$$

Just like the free particle, this action in invariant under constant translations $\phi(x) \to \phi(x) + \phi_0$, indeed, $S[\phi(x) + \phi_0] = S[\phi(x)]$. The symmetry now acts on the field $\phi(x)$.

As in particle mechanics, the set of symmetries is defined by all infinitesimal functions $\delta_\xi\phi(x)$ such that, for arbitrary field configurations $\phi(x)$, the variation of the action is expressed as

$$S[\phi(x), \delta_\xi\phi(x)] = \int \sqrt{|g|}d^nx\partial_\mu Y^\mu_\xi \tag{3.5.2}$$

where $\sqrt{|g|}d^n$ is the volume form[5] and Y^μ_ξ is a vector field on spacetime. The derivation of Noether's theorem in field theory follows the same path as in particle mechanics. Field theories are described by actions of the form $S[\phi(x)] = \int \sqrt{|g|}d^nx\mathcal{L}(\phi, \partial_\mu\phi) =: \int L$, and the corresponding Euler-Lagrange equations are given by

$$E(\phi(x)) \equiv \partial_\mu\left(\frac{\partial L}{\partial\phi_{,\mu}}\right) - \frac{\partial L}{\partial\phi} = 0\ . \tag{3.5.3}$$

Here we have absorbed the volume form in the defition of the Lagrangian which we will be treating as a top-form from now on: $L = \mathcal{L}\sqrt{|g|}d^nx$. The on-shell variation is computed as

$$\begin{aligned}\delta S[\phi, \delta\phi] &= \int\left(\frac{\partial L}{\partial\phi}\delta\phi + \frac{\partial L}{\partial\phi_{,\mu}}\delta\phi_{,\mu}\right)\\ &= \int E\delta\phi + \int\partial_\mu\left(\frac{\partial L}{\partial\phi_{,\mu}}\delta\phi\right)\\ &\hat{=} \int d\theta\ = \delta S[\bar{\phi}, \delta\phi],\end{aligned} \tag{3.5.4}$$

where in the last line we have used that the field $\bar{\phi}$ satisfies its Euler-Lagrange equations.

Again the same logic as in the finite-dimensional case: (3.5.2) is valid for any ϕ, in particular for $\bar{\phi}$, and (3.5.4) is valid for any $\delta\phi$, in particular for $\delta_\xi\phi$. Thus, inserting $\bar{\phi}$ into (3.5.2) and $\delta_\xi\phi$ into (3.5.4) the left hand sides are equal. Subtracting both equations we obtain the conserved current equation

$$\partial_\mu j^\mu_\xi \hat{=} 0, \quad j^\mu_\xi := I_\xi\theta^\mu - Y^\mu_\xi, \quad \theta^\mu = \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\delta\phi. \tag{3.5.5}$$

[5] In Chapter 2 we denoted the volume form by ϵ. We use the alternative notation $\sqrt{|g|}d^nx$ in present chapter only, not to risk confusion with the infinitesimal parameter ϵ.

This is Noether's first theorem in field theory.

3.6 Noether's theorems in gauge theories

To understand the application of Noether's theorem to gauge theories, let us start by considering the Maxwell action with a source term

$$S = -\int d^4x \left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu\right), \quad F_{\mu\nu} := 2\partial_{[\mu}A_{\nu]}. \tag{3.6.1}$$

Its variation reads

$$\delta S[A_\mu, \delta A_\mu] = -\int \left[d^4x F^{\mu\nu}\partial_\mu \delta A_\nu + J^\mu \delta A_\mu\right] \tag{3.6.2}$$

$$= \int d^4x \left[E^\mu \delta A_\nu - \partial_\mu(F^{\mu\nu}\delta A_\nu)\right] \tag{3.6.3}$$

The field equations are given by $E^\mu := \partial_\nu F^{\mu\nu} - J^\mu \hat{=} 0$. The action is invariant under gauge transformations of form

$$\delta_\varphi A_\mu = \partial_\mu \varphi. \tag{3.6.4}$$

Choosing a φ with compact support, combining (3.6.4) with (3.6.3) yields the *off-shell* identity

$$0 = \int d^4x E^\mu \partial_\mu \varphi = -\int d^4x \partial_\mu E^\mu \varphi. \tag{3.6.5}$$

This is valid for any φ thus we have found an *off-shell* realtion between the field equations:

$$\partial_\mu E^\mu = 0. \tag{3.6.6}$$

This is a general result that is contained in Noether's second theorem – existence of gauge symmetries implies mutual dependence of the EL equations. The general proof follows the procedure above: An arbitrary variation of a general Lagrangian L is $\delta L = E\delta\phi + d\theta$. Take a gauge transformation by a local function $\varphi = \varphi(x)$. Specialize to symmetry variation, $\delta_\varphi L = E\delta_\varphi\phi + I_\varphi d\theta$. Take compactly supported φ, $\delta_\varphi L = E\delta_\varphi\phi$. Since this is a symmetry,

$$E\delta_\varphi\phi = 0. \tag{3.6.7}$$

The next question one may ask is what happens if φ is supported at the boundary? A common understanding is that Noether charges associated with gauge symmetries are trivial, given that gauge transformations don't affect the physics of the theory. However,

Noether's first theorem is valid for any symmetry, not just global ones. The conserved current[6] associated to (3.6.4) is given by

$$j^\mu_\varphi = -F^{\mu\nu}\partial_\nu\varphi. \tag{3.6.8}$$

Haven't we just found the Noether current of a gauge symmetry that has no reason to vanish in general? Yes, we have. It seems a bit strange as we have actually found infinitely many conserved currents for infinitely many arbitrary functions φ, but these are all real conserved currents. As a matter of fact, the vanishing of Noether charges for local symmetries holds only in the absence of boundaries. When boundaries are present, non-trivial 'gauge' transformations at the boundaries can lead to non-trivial Noether charges associated with 'gauge' symmetries.

When local transformations are non-trivial on the boundary, the Noether charges become surface integrals over the boundary of space, rather than bulk integrals. This is a consequence of both first and second Noether theorem [73]. In the literature this is often regarded to as the second theorem since it is included in her general expression for the invariance of the action under a continuous group, but it is in fact a combination of the two. These surface charges are important in understanding boundary and infrared effects in gauge theories and have significant implications in the study of observables and symmetries in general relativity. Recognizing the significance of surface charges and non-trivial gauge transformations at the boundary is essential for a comprehensive understanding of gauge theories and gravity in the presence of boundaries.

Let us go back to the Maxwell example. Using the EL equations, the current

$$j^\mu_\varphi = -\partial_\nu(F^{\mu\nu}\varphi) + \partial_\nu F^{\mu\nu}\varphi \mathrel{\hat{=}} -\partial_\nu(F^{\mu\nu}\varphi), \tag{3.6.9}$$

is indeed a boundary term. While the current itself is not trivial when $\varphi \neq 0$ at the boundary, its conservation law is – it is a consequence of symmetry $\partial_\mu\partial_\nu(F^{\mu\nu}\varphi) = 0$.

Assuming trivial spacetime cohomology, for any conserved current $\partial_\mu j^\mu \mathrel{\hat{=}} 0$, one can find an antisymmetric 2-form $q^{\mu\nu}$ such that $j^\mu \mathrel{\hat{=}} \partial_\nu q^{\mu\nu}$. The non-trivial result of Noether for gauge theories is that this 2-form q is locally constructed out of the fields ϕ and the gauge field φ.[7] To prove this, let us first make a few observations:

(i) j^μ_φ itself is locally constructed out of fields ϕ and φ,

(ii) $j^\mu_\varphi = 0$ whenever $\varphi = 0$ and $\partial_\mu\varphi = 0$,

(iii) $j^\mu_\varphi + j^\mu_{\varphi'} = j^\mu_{\varphi+\varphi'}$.

[6] See [75] for a more in-depth review focusing on electromagnetic Noether charges (and more).

[7] For non-triviality of the statement it is important for q to be *locally* constructed out of the fields. In general when $\partial_\mu j^\mu = 0$ and $\partial_\mu q^{\mu\nu} = j^\nu$, one can always find q as the integral $q^{\mu\nu} = \int_{x_0}^{x_1} dx^{[\mu} j^{\nu]}$ along an arbitrary path connecting x_0 and x_1. Deforming the path will change q by an exact form.

All of them follow from the construction of j^μ by performing an infinitesimal gauge transformation on the fields (it must be linear in φ) and integrating by parts.

Now consider a volume V whose boundary consists in two parts $\partial V = \Sigma_1 \cup \Sigma_2$ such that $\partial\Sigma_1 = S = \partial\Sigma_2$.

$$0 \hat{=} \int_V \epsilon_V \nabla_\mu j^\mu = \int_{\Sigma_1} \epsilon_\Sigma \nabla_\mu j^\mu + \int_{\Sigma_2} \epsilon_\Sigma \nabla_\mu j^\mu \tag{3.6.10}$$

Choosing a φ such that $\varphi(\Sigma_1) = 0$ and $\partial_\mu\varphi(\Sigma_2) = 0$,

$$\int_{\Sigma_1} \epsilon_\Sigma \nabla_\mu j^\mu \hat{=} -\int_{\Sigma_2} \epsilon_\Sigma \nabla_\mu j^\mu \hat{=} \int_S q_\varphi \hat{=} 0, \quad \text{if } \varphi = 0 = \partial_\mu\varphi \text{ on } S \tag{3.6.11}$$

Since (iii) implies $q^{\mu\nu}_\varphi + q^{\mu\nu}_{\varphi'} = q^{\mu\nu}_{\varphi+\varphi'}$ we have shown that

$$\int_S q_\varphi \hat{=} \int_S q_{\varphi'}, \quad \text{if } \varphi \overset{S}{=} \varphi' \text{ and } \partial_\mu\varphi \overset{S}{=} \partial_\mu\varphi'. \tag{3.6.12}$$

Looking at the middle equality of (3.6.11), because q depends locally on φ, we can replace φ on the RHS by φ' such that $\varphi' \overset{S}{=} \varphi$ and $\partial_\mu\varphi' \overset{S}{=} \partial_\mu\varphi$, and $\varphi' = 0$ outside a small neighbourhood of S. Now the RHS only depends on field variables arbitrarily close to S because it vanishes whenever φ' and its derivative vanish, and therefore the LHS only depends on field variables arbitrarily close to S, i.e. q is locally constructed out of the fields.

Let us illustrate the results of this section on the example of the Einstein-Hilbert Lagrangian

$$L^{\text{EH}}[g_{\mu\nu}] = R\epsilon, \quad \epsilon = \sqrt{-g}d^4x, \tag{3.6.13}$$

which is invariant under general diffeomorphisms $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$, $\delta_\xi L^{\text{EH}} = di_\xi L =: \epsilon\nabla_\mu Y^\mu_\xi$.

$$\delta L^{\text{EH}} = \left[\left(R^{\mu\nu} + \frac{1}{2}Rg^{\mu\nu}\right)\delta g_{\mu\nu} + g_{\mu\nu}\delta R^{\mu\nu}\right]\epsilon \tag{3.6.14}$$

$$= \left(G^{\mu\nu}\delta g_{\mu\nu} + \nabla_\mu\theta^\mu\right)\epsilon, \quad \theta^\mu := g_{\mu\nu}\delta R^{\mu\nu} = 2g^{\mu[\rho}g^{\nu]\sigma}\nabla_\nu\delta g_{\rho\sigma}, \tag{3.6.15}$$

where we used the Palatini identity

$$\delta R_{\mu\nu} = 2\nabla_{[\rho}\delta\Gamma^\rho_{\nu]\mu}. \tag{3.6.16}$$

The Noether current is

$$j^\mu_\xi = I_\xi \theta^\mu - Y^\mu_\xi = -R\xi^\mu + 4g^{\mu[\rho}g^{\nu]\sigma}\nabla_\nu\nabla_{(\rho}\xi_{\sigma)} \tag{3.6.17}$$

$$= -R\xi^\mu + \Box\xi^\mu + R^\mu_\nu\xi^\nu - \nabla^\mu\nabla_\nu\xi^\nu \tag{3.6.18}$$

$$= 2G^{\mu\nu}\xi_\nu - 2\nabla_\nu\nabla^{[\mu}\xi^{\nu]} \tag{3.6.19}$$

$$\hat{=} 2\nabla_\nu\nabla^{[\nu}\xi^{\mu]} =: \nabla_\nu q^{\nu\mu}, \tag{3.6.20}$$

a total divergence of the Komar 2-form $q := \frac{1}{2}q^{\mu\nu}\epsilon_{\mu\nu\rho\sigma}dx^\rho \wedge dx^\sigma$. If ξ is a Killing vector of a particular solution $\mathcal{L}_\xi g_{\mu\nu} = 0$, it will satisfy

$$\nabla \cdot \xi = 0, \tag{3.6.21}$$

$$\Box\xi_\mu = -R_{\nu\mu}\xi^\nu, \tag{3.6.22}$$

$$\nabla_\mu\nabla_\nu\xi_\rho = R^\sigma_{\mu\nu\rho}\xi_\sigma. \tag{3.6.23}$$

In that case,

$$j^\mu_\xi = R\xi^\mu - \Box\xi^\mu - R^\mu_\nu\xi^\nu + \nabla^\mu\nabla_\nu\xi^\nu \hat{=} 0, \tag{3.6.24}$$

This observation implies that the surface over which the charge is integrated can be reshaped or deformed without altering the overall value of the integral. Coupled with the fact that q depends locally on ξ and $g_{\mu\nu}$, this is a very powerful property. Take for example the Kerr metric. It is asymptotically flat and one can use the isometries of the Minkowski metric at infinity to define conserved charges which will give the Komar mass and angular momentum for $\xi = \partial_t$ or ∂_ϕ. (In)famously, some of these Komar charges will be off by relative factors of 2. Such discrepancies can be resolved by employing more intricate constructions of surface charges using the covariant phase space formalism. Wald's seminal paper [31] accomplishes this and provides a Noetherian proof of the first law of black hole thermodynamics. However, there are various ambiguities in defining these surface charges and which remain a subject of debate, with different authors using distinct approaches in various contexts. Further details on this matter will be discussed in the next chapter.

Covariant Phase Space

4

This chapter serves as an in-depth review and pedagogical account of the covariant phase space formalism, which has been extensively covered in the literature, recently nicely reviewed in [76]. We start our discourse by revisiting calculus on spacetime, focusing particularly on the de Rham complex. Following that, we extend these ideas to calculus on field space, thereby providing a fresh mathematical context for understanding variations and other constructs introduced earlier in Chapter 3. We continue to discuss diffeomorphism invariance and introduce the anomaly operator. Armed with this mathematical framework, we proceed to redefine Noether charges within the realm of covariant phase space, making connections to familiar concepts and explaining their utility in a more generalized setting. The chapter further goes into Hamiltonian charges, underscoring the intricacies and potential pitfalls that may show up in their formulation.

While the chapter does not introduce any original results, it aims to serve as a comprehensive guide, offering a unified perspective on well-established theories. The presentation tries to bridge the gap between abstract mathematical structures and their practical application in theoretical physics, providing a more nuanced understanding of symmetries, conservation laws, and their interplay in the covariant phase space formalism.

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The phase space is commonly understood as the space of initial conditions for a dynamical system, characterized by coordinates q and momenta p. This space, endowed with Poisson brackets, is typically constructed as a cotangent bundle over some configuration space—the space of q's (see e.g. [77]). While powerful, this standard formulation is inherently non-covariant, necessitating the explicit breaking of covariance to establish such a space. This approach presents challenges when dealing with covariant theories.

To overcome these limitations, the concept of *covariant phase space* was introduced [27, 28, 32, 33, 34] as the space of all solutions to the field equations of the theory under consideration. This new formulation obviates the need to single out a specific time direction, offering a fully covariant description. In this setting, the phase space becomes a space of entire trajectories rather than just initial conditions. To rigorously develop this idea, we will explore the essential principles of symplectic geometry. This chapter aims

to equip the reader with the mathematical tools needed to understand this modern and covariant concept of phase space.

4.1 Spacetime calculus

For completeness, we start by giving an overview of the usual calculus on spacetime. We define a one-form as a linear map from the tangent bundle TM to the space of smooth funcitons $C^\infty(M)$. The space of all one-forms is dual to the space of vectors and we denote it T^*M. A p-form is a linear, antisymmetric map from p copies of the tangent bundle T^pM to the space of smooth functions $C^\infty(M)$. The space of all p-forms is denoted $\Omega^p(M,\mathbb{R}) = \bigwedge^p T^*M$, where $\wedge$ denotes the antisymmetric (exterior or wedge) product. Due to antisymmetry, the maximal degree of a form on an n-dimensional manifold is n, and such forms are called *top-forms*. We define 0-forms to be functions $\bigwedge^0 T^*M = C^\infty(M)$. The space of all forms is called the *de Rham complex* and is defined as

$$\Omega(M,\mathbb{R}) := \bigoplus_{p=0}^{n} \bigwedge^p T^*M. \tag{4.1.1}$$

On this space we define three derivatives:

(i) *the interior product* is defined as contraction of a differential form with a vector field. For $\xi \in TM$, it is defined as

$$i_\xi : \Omega^p(M,\mathbb{R}) \to \Omega^{p-1}(M,\mathbb{R}) \tag{4.1.2}$$

$$i_\xi\omega(\zeta_1,...,\zeta_{p-1}) = \omega(\xi,\zeta_1,...,\zeta_{p-1}) \quad \forall\ \zeta_1,...,\zeta_{p-1} \in TM. \tag{4.1.3}$$

We see from the definition that it decreases the degree of a form by 1. It is a unique anti-derivation of degree -1 such that for one-forms α,

$$i_\xi\alpha = \xi\cdot\alpha = \xi^\mu\alpha_\mu, \tag{4.1.4}$$

where in the last equality we introduced local coordinates $(x_1,...,x_n)$, in which $\xi = \xi^\mu\partial_\mu$ and $\alpha = \alpha_\mu dx^\mu$. For a p-form $\omega = \omega_{\mu_1...\mu_p}dx^{\mu_1}\wedge...\wedge dx^{\mu_n}$ the components of (4.1.3) are

$$(i_\xi\omega)_{\mu_1...\mu_{p-1}} = \sum_{q=1}^{p}(-1)^{q-1}\xi^\mu\omega_{\nu_1...\nu_p}\delta^{\nu_1}_{\mu_1}\cdots\delta^{\nu_q}_{\mu}\cdots\delta^{\nu_p}_{\mu_{p-1}}. \tag{4.1.5}$$

The interior product satisfies the graded Leibniz rule – for a p-form α and a q-form β,

$$i_\xi(\alpha\wedge\beta) = i_\xi\alpha\wedge\beta + (-1)^p\alpha\wedge i_\xi\beta. \tag{4.1.6}$$

It is antisymmetric, $i_\xi i_\zeta = -i_\zeta i_\xi$, which implies $i_\xi i_\xi = 0$.

(ii) *the exterior derivative* is an anti-derivation of degree 1

$$d : \Omega^p(M, \mathbb{R}) \to \Omega^{p+1}(M, \mathbb{R}). \tag{4.1.7}$$

For 0-forms, $f \in C^\infty(M)$, df is the differential of f. It is nilpotent $d^2 = 0$ and satisfies the graded Leibniz rule

$$d(\alpha \wedge \beta) = i_\xi \alpha \wedge \beta + (-1)^p \alpha \wedge d\beta. \tag{4.1.8}$$

In local coordinates, the exterior derivative of a p-form $\omega = \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$ is given by

$$d\omega = \partial_\mu \omega_{\mu_1 \dots \mu_p} dx^\mu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} \tag{4.1.9}$$

(iii) *the Lie derivative* is a derivation of degree 0 – it does not change the degree of forms. It is defined by *Cartan's magic formula* as the anti-commutator of exterior derivative and interior product

$$\mathcal{L}_\xi = i_\xi d + d i_\xi. \tag{4.1.10}$$

We can use this formula to derive the full algebra satisfied by the three derivatives

$$[\mathcal{L}_\xi, i_\xi] = 0 = [\mathcal{L}_\xi, d], \quad [\mathcal{L}_\xi, i_\zeta] = i_{[\xi,\zeta]} = [i_\xi, \mathcal{L}_\zeta] \tag{4.1.11}$$

Even though no reference to the metric was made in the definition, the Lie derivative of any tensor along a vector field can be expressed through the covariant derivatives of that tensor and vector field.

$$\begin{aligned} &\mathcal{L}_\xi T^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} = \xi^\rho \left(\nabla_\rho T^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \right) - \\ &\left(\nabla_\rho \xi^{\mu_1} \right) T^{\rho \dots \mu_r}_{\nu_1 \dots \nu_s} - \dots - \left(\nabla_\rho \xi^{\mu_r} \right) T^{\mu_1 \dots \mu_{r-1} \rho}_{\nu_1 \dots \nu_s} + \\ &\left(\nabla_{\nu_1} \xi^\rho \right) T^{\mu_1 \dots \mu_r}_{\rho_1 \dots \nu_s} + \dots + \left(\nabla_{\nu_s} \xi^\rho \right) T^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_{s-1} \rho} \end{aligned} \tag{4.1.12}$$

The expression for the Lie derivative remains unaffected if we replace the covariant derivative ∇_μ with any torsion-free connection $\tilde{\nabla}_\mu$, or locally, with the coordinate-dependent derivative ∂_μ. This property illustrates that the Lie derivative is independent of the metric. However, using the covariant derivative is often more convenient since it possesses the useful property of commuting with raising and lowering indices.

In the context of general relativity, the Lie derivative finds crucial applications, especially in the study of spacetime symmetries where tensors or other geometric

objects are preserved. A particularly common type of symmetry encountered in the analysis of spacetimes is the Killing symmetry, or *isometry*, which implies that the metric tensor remains unchanged under Lie dragging. The Lie derivative formula gives the condition that a vector field must satisfy to generate an isometry

$$\mathcal{L}_\xi g_{\mu\nu} = 2\nabla_{[\mu}\xi_{\nu]} = 0 \tag{4.1.13}$$

We call the *volume form* the unique (up to multiplication by a function) top form of the space time. In local coordinates, it can be expressed in terms of the determinant of the metric as

$$\epsilon = \sqrt{-g}dx^1 \wedge \cdots \wedge dx^n = \sqrt{-g}d^n x = \star 1, \tag{4.1.14}$$

where $\star$ is the Hodge dual mapping p-forms into $(n-p)$-forms defined as

$$\star\omega := \frac{\sqrt{-g}}{(n-p)!}\omega_{\mu_1,\ldots,\mu_p}\epsilon^{\mu_1,\ldots,\mu_p}{}_{\nu_1,\ldots,\nu_{n-p}}\ \mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_{n-p}} \tag{4.1.15}$$

The volume form is naturally integrated over spacetime to give its volume. In general, a differential p-form can be integrated over a p-dimensional (oriented) manifold. For example, a one-form is integrated over a curve, a two-form over a surface and so on.

We say a differential form α is *closed* if its exterior derivative vanishes $d\alpha = 0$, and it is *exact* if it is the exterior derivative of another differential form $\alpha = d\beta$. In other words, a form is exact if it is in the image of d, and it is closed if it is in the kernel of d. Because $d^2 = 0$, everty exact form is necessarily closed.

Stokes' theorem states that the integral of a differential form ω over the boundary ∂M of some orientable manifold M is equal to the integral of its exterior derivative $d\omega$ over the whole of M. Mathematically, this can be expressed as:

$$\int_{\partial M} \omega = \int_M d\omega\,. \tag{4.1.16}$$

4.2 Field space calculus

The field space Γ is the space of all field configurations and it is assumed to be a differentiable manifold. On this space we can define a structure analogous to the de Rham complex from the previous section.

A one-form is a linear map from the tangent bundle $T\Gamma$ to the space of smooth functionals $F = C^\infty(\Gamma)$. The space of all one-forms is dual to the space of vectors and we denote it $T^*\Gamma$. A p-form is a linear, antisymmetric map from p copies of the tangent bundle $T^p\Gamma$ to the space of smooth functions $C^\infty(\Gamma)$. The space of all p-forms is denoted

$\Omega^p(\Gamma, F) = \curlywedge^p T^*M$, where $\curlywedge$ denotes the wedge product.[1] 0-forms to be functionals $\curlywedge^0 T^*\Gamma = F$. The space of all forms is called the *the variational complex* and is defined as

$$\Omega(\Gamma, F) := \bigoplus_{p=0}^{\dim\Gamma} \curlywedge^p T^*\Gamma \tag{4.2.1}$$

The three derivatives in this space are defined as follows:

(i) *the interior product* = field contraction

$$I_X : \Omega^p(\Gamma, F) \to \Omega^{p-1}(\Gamma, F) \tag{4.2.2}$$

is antisymmetric $I_X I_Y = -I_Y I_X \quad \Rightarrow \quad I_X I_X = 0$.

(ii) *the exterior derivative* = field variation

$$\delta : \Omega^p(\Gamma, F) \to \Omega^{p+1}(\Gamma, F). \tag{4.2.3}$$

is nilpotent $\delta^2 = 0$

(iii) *the field-space Lie derivative*

$$\delta_X = I_X \delta + \delta I_X. \tag{4.2.4}$$

Again, the full algebra satisfied by the three derivatives

$$[\delta_X, I_X] = 0 = [\delta_X, \delta], \quad [\delta_X, I_Y] = I_{[X,Y]} = [I_X, \delta_Y]. \tag{4.2.5}$$

The variational complex can be put together with the de Rham complex to form the *variational bicomplex* (M, Γ) consisting of (p, q)-forms which are p-forms in spacetime and q-forms in field space. The exterior derivative on the bicomplex is $\delta + d$. Now there are two distinct choices made in the literature:

- the exterior derivative is nilpotent $(\delta + d)^2 = 0$, which is equivalent to anti-commutativity of the two exterior derivatives $\delta d + d\delta = 0$
- the two derivatives commute $\delta d = d\delta$.

Since we will never use the combined exterior derivative and since there is no physical reason for anti-commutativity of spacetime derivatives and variations, we will take the second approach. The two will yield the same results, so we can see this as making the choice with less chances of making sign mistakes.

[1] To keep the notation clean, when there is no risk of confusion and no need to emphasise it, we will drop the symbod for field space wedge product and leave it implicit.

We want to apply this formalism to a physical theory described by the action

$$S = \int L, \tag{4.2.6}$$

where L is the Lagrangian $(d, 0)$-form. We have already calculated variations of the action, for example in (3.2.3), and we have seen that we can always decompose it into a bulk term that is proportional to the equations of motion and a boundary term

$$\delta L[\phi] = E_i \delta\phi^i + d\theta \approx d\theta. \tag{4.2.7}$$

Now we have the mathematical foundation to analyse the boundary term. It is a (d-1,1)-form called the *(pre-)symplectic potential current* which we integrate to get the *(pre-)symplectic potential*. We further define a (d-1,2)-form ω called the *(pre-)symplectic 2-form current* and its integral over a codimension-1 hypersurface Ω, a $(0, 2)$-form called the *(pre-)symplectic 2-form*

$$\Omega := \int_\Sigma \omega := \int_\Sigma \delta\theta, \tag{4.2.8}$$

which is the main ingredient of the formalism as it encodes the dynamics of the physical system via the Poisson bracket. When being pedantic, we use lowercase letters for the currents, namely the integrands, and capital letters for the integrated quantities. However, we will loosely speak of both as the symplectic potential and 2-form, for ease of language.

The symplectic form has three important properties. First, it is closed in field space $\delta\omega = 0$. This follows directly from the definition (4.2.8) as a variation of θ. Second, it is non-degenerate. This is why we used the prefix *pre-* before – to remind us that this form is possibly degenerate, and the actual phase space is defined by quotienting this pre-symplectic space by the action of the group of zero modes of the pre-symplectic form.[2] Finally, the symplectic form is closed in spacetime on-shell of the EL equations

$$d\omega = d\delta\theta = \delta d\theta \mathrel{\hat{=}} \delta\delta L = 0. \tag{4.2.9}$$

This will be important when we study the conservation of charges.

[2] Since two equivalence class representatives differ by a zero mode, it follows that Ω is a function of class and, since we eliminated the degeneracy, it is indeed symplectic on the quotient space.

4.3 Symmetries and charges

In Section 4.1 we used the spacetime Lie derivative to define isometries as the directions preserving the metric. There is no metric on field space but one can define its isometries as directions $X \in T\Gamma$ that preserve the symplectic form

$$\delta_X \omega = 0. \tag{4.3.1}$$

Such vector fields X are called *symplectomorphisms* and they are the symmetries of the symplectic form in the same sense in which Killing vectors are symmetries of the metric.[3] Using the definition of ω and the nilpotency of the variation, this implies (assuming trivial cohomology on $T^*\Gamma$)

$$0 = \delta_X \omega = \delta I_X \omega = 0 \quad \Rightarrow \quad I_X \omega := -\delta h_X, \tag{4.3.2}$$

for some (2,1)-form h_X. Integrating on a codimension-1 surface,

$$I_X \Omega := -\delta H_X, \quad H_X = \int_\Sigma h_X. \tag{4.3.3}$$

A vector field $X \in T\Gamma$ satisfying this equation is called the *Hamiltonian vector field* (HVF), and the H_X is the *canonical charge*. For trivial cohomology, all symplectomorphisms are Hamiltonian and vice versa. The symplectic form carries a representation of the charge algebra via the Poisson bracket

$$\{H_X, H_Y\} = \delta_X H_Y. \tag{4.3.4}$$

up to a possible central extension in which case the representation is projective, we show this in Appendix A.1.

In radiative problems or in the presence of dissipation, the situation is made more complicated by the fact that we are interested in transformations that correspond to field space vector fields that are not Hamiltonian, or equivalently the "infinitesimal Hamiltonian" generating these transformations is "not integrable". The interest in such non-Hamiltonian vector fields arises in situations when the symplectic 2-form is not conserved from one Cauchy slice to another. This is a situation characteristic for field theory and cannot happen in finite-dimensional cases, where instead of (4.2.9) one has

[3] There is a slight abuse of language here: Symplectomorphisms are usually defined as isometries of Ω rather than ω. If we do not consider dynamical embedding fields, the variation and integration will commute and one can do manipulations like $\delta\Omega = \delta \int \omega = \int \delta\omega = 0$. This is a non-trivial assumption whose violations lead to the appearance of edge modes [78, 79] and are an interesting direction of research that has received a lot of attention recently but we will not consider them here. We will always assume that the symmetries of ω are also symmetries of Ω by treating embeddings as a part of the background.

$\dot{\omega} = 0$. The symplectic form being closed only guarantees that the difference between the initial and final slice will be equal to the flux through the lateral boundary. In the case of conservative boundary conditions, this flux vanishes, and (4.2.9) indeed boils down to $\int_{\Sigma_0} \omega = \int_{\Sigma_1} \omega$, and the generator is Hamiltonian. If the system is open, and there are physical degrees of freedom leaking through the boundary, and in order to talk about charges and their non-conservation, a prescription needs to be chosen to restore integrability and identify the would-be-hamiltonian generator. We will come back to this in Sec. 4.3.3.

Another problem that can occur is the fact that *finiteness* of charges is only guaranteed when the boundary is at a finite distance. Asymptotic boundaries typically lead to divergent charges because the action itself could diverge. This problem is generally solved by the method of *symplectic renormalization* [80, 46, 81] which boils down to adding boundary counterterms to the action to render θ and ω finite. This is not always possible, and sometimes the only option is to renormalize θ directly. We will perform symplectic renormalization explicitly in Chapter 8 when we study charges at spacelike infinity.

4.3.1 Anomalies

We will be using the CPS formalism to study covariant theories whose symmetries are spacetime diffeomorphisms. We first recall that the variation of a dynamical field under an infinitesimal diffeomorphism is given by the Lie derivative

$$\delta_\xi \phi = \mathcal{L}_\xi \phi. \tag{4.3.5}$$

To make contact with our newly defined formalism, we notice that the spacetime diffeomorphism encoded in a Lie derivative $\mathcal{L}_\xi$ is a field space vector field, and the above expression is equivalent to

$$\delta_{\mathcal{L}_\xi} \phi = I_{\mathcal{L}_\xi} \delta\phi. \tag{4.3.6}$$

Just like in spacetime, on field space we can choose local coordinates $\left\{\phi^i\right\}$ so that $\left\{\delta\phi^i\right\}$ is a basis of $T^*\Gamma$ and $\left\{\frac{\delta}{\delta\phi^i}\right\}$ is a basis of $T\Gamma$. In this basis, the field space vector field generating spacetime diffeos is given by

$$\mathcal{L}_\xi = \int d^n x \mathcal{L}_\xi \phi^i(x) \frac{\delta}{\delta\phi^i(x)}. \tag{4.3.7}$$

However, since confusion might arise between this Lie derivative acting on dynamical fields only and the passive transformation that also transforms the background fields, this notation is not used and instead two common ways to denote this vector field that show up in the literature are $\hat{\xi}$ and X_ξ. Not to clutter the notation we choose to remove the hat or the X entirely and denote $I_\xi := I_{\hat{\xi}}$ and $\delta_\xi := \delta_{\hat{\xi}}$. We risk no confusion as the spacetime

vector $\xi \in TM$ and the field space vector $\xi \in T\Gamma$ can be distinguished by the different notation for derivations on M and on Γ. We will restore the hat when necessary.

We mentioned that δ_ξ acts only on dynamical fields ϕ, what about the background fields χ? Well, they are constant it field space so $\delta\chi = 0$ which implies $\delta_\xi\chi = 0$. Nevertheless, they don't have to be constant in spacetime and $\mathcal{L}_\xi\chi \neq 0$ in general. This tells us that (4.3.5) will be true only for quantities not depending on the background fields. We usually call these quantities *covariant* and using the covariant phase space formalism, we can measure non-covariance by calculating the difference $\delta_\xi - \mathcal{L}_\xi$. In the presence of dynamical diffeomorphisms $\delta\xi \neq 0$, and for field space forms, a more useful measure of covariance is the *anomaly operator*, defined as

$$\Delta_\xi := \delta_\xi - \mathcal{L}_\xi - I_{\delta\xi}. \tag{4.3.8}$$

The third term in (4.3.8) is relevant when acting on functionals of the fields that are forms in field space, as for example on the symplectic potential.[4] We will use the term *anomalous* to refer to non-covariant quantities.

4.3.2 Noether charges

We can now be more precise about mathematical objects introduced in Chapter 3. First notice that the Noether symmetry variation is nothing other than a field space Lie derivative δ_ξ. On-shell variation is always equal to the (exterior derivative of the) symplectic potential θ and the Noether charge was given by $j_\xi = I_\xi\theta - Y_\xi$. For a covariant Lagrangian, $\delta_\xi L = \mathcal{L}_\xi L = di_\xi L$, and

$$j_\xi = I_\xi\theta - i_\xi L. \tag{4.3.10}$$

We are interested in the general situations that include field-dependent diffeomorphisms, $\delta\xi \neq 0$, and anomalies. This lack of covariance can occur in the presence of background structures which are described by spacetime fields but constant under variations in field space, and it will be necessary to understand the charges on null hypersurfaces. The only restriction we make is that the anomaly of the Lagrangian should at most be a boundary term, this is to say that there exist choices of Lagrangians such that the background structure that may break covariance only enters through exact forms.[5] Ac-

[4] For a 1-form $F(\phi,\chi)\delta\phi$ we have

$$\delta_\xi(F\delta\phi) = \partial_\phi F\delta_\xi\phi\delta\phi + F\delta\delta_\xi\phi = \partial_\phi F\pounds_\xi\phi\delta\phi + F\delta\pounds_\xi\phi = \pounds_\xi(F\delta\phi) - \partial_\chi F\pounds_\xi\chi\delta\phi + F\pounds_{\delta\xi}\phi, \tag{4.3.9}$$

where we used $[\delta,\delta_\xi] = 0$ in the first equality, and $[\delta,\pounds_\xi] = \pounds_{\delta\xi}$ in the last, and the definition $I_{\delta\xi}\delta\phi = \pounds_{\delta\xi}\phi$.

[5] This includes the treatment of anomalous bulk Lagrangians like ADM, since it differs from the covariant Einstein-Hilbert Lagrangian by a boundary term.

cordingly, $L = L^{\text{cov}} + d\ell$ with $\Delta_\xi L^{\text{cov}} = 0$ and $\Delta_\xi L = da_\xi$, and we define the Lagrangian and symplectic anomalies a_ξ and A_ξ via

$$\Delta_\xi \ell = a_\xi, \qquad \Delta_\xi \theta = \delta a_\xi - a_{\delta\xi} + dA_\xi. \tag{4.3.11}$$

If such anomalies are present, they show up in the formula for the Noether charges as $\delta_\xi L = \mathcal{L}_\xi L + da_\xi$, and so

$$j_\xi := I_\xi \theta - i_\xi L - a_\xi \hat{=} dq_\xi, \tag{4.3.12}$$

To understand the meaning of a_ξ in this formula, consider the pull-back on a given hypersurface. If the hypersurface is a boundary used to define the covariant phase space, then the relevant symmetry vectors ξ are those tangent to it, since they are the only ones preserving the boundary and thus the phase space. Then the pull-back of $i_\xi L = 0$ vanishes, and the variation of the Noether charge along the boundary has two contributions: one is the symplectic flux of the symmetry, and this is the contribution due to physical degrees of freedom crossing the hypersurface. The other is the anomaly. This term induces a charge variation caused by the background structure, thus introducing a non-dynamical contribution to the flux. For instance, this term is non-zero if one uses a normal that depends on the foliation to which the boundary belongs.[6].

4.3.3 Integrability of the 'Hamiltonian' charge

The presence of radiation guarantees that every diffeomorphism moving the corner necessarily defines a non-HVF, it does not preserve the symplectic form, and as a consequence the charges it generates are not conserved. The definitions of both the charge and the flux are ambiguous and a prescription needs to be chosen to disentengle them. To emphasise that the right-hand side of (4.3.2) is not guaranteed to be integrable, we introduce the thermodynamical notation [82]

$$\not{d} h_\xi := -I_\xi \omega. \tag{4.3.13}$$

[6] A different situation occurs if the background structure breaks diffeomorphism invariance entirely, for instance if we have matter fields but the (curved) metric is treated as a fixed background, the anomaly term a_ξ is nothing but the energy-momentum tensor of matter, and one recovers the non-general-covariant notion of bulk Noether charge. This observation allows one to reverse the standard viewpoint that sees Noether charges as global, becoming surface charges in the special case of local gauge symmetries; and consider instead that all Noether charges are surface charges, becoming global only in the presence of anomalies introduced by background structures [50]

We expand this to pinpoint the obstructions to integrability:

$$\begin{aligned}
\not{d}h_\xi = -I_\xi \delta\theta &= -\delta_\xi\theta + \delta I_\xi \theta && (4.3.14)\\
&= -\mathcal{L}_\xi\theta - (\delta a_\xi - a_{\delta\xi} + dA_\xi) - I_{\delta\xi}\theta + \delta(j_\xi + i_\xi L + a_\xi) && (4.3.15)\\
&= -di_\xi\theta - i_\xi d\theta - (-a_{\delta\xi} + dA_\xi) - I_{\delta\xi}\theta + \delta j_\xi + i_\xi \delta L + i_{\delta\xi} L && (4.3.16)\\
&\hat{=} -di_\xi\theta - (-a_{\delta\xi} + dA_\xi) + \delta dq_\xi - dq_{\delta\xi} - a_{\delta\xi} && (4.3.17)\\
&= d\left(\delta q_\xi - i_\xi\theta - q_{\delta\xi} - A_\xi\right) && (4.3.18)
\end{aligned}$$

where in the second line we used (4.3.11) and (4.3.12).

In (4.3.18), the term $i_\xi\theta + q_{\delta\xi} + A_\xi$ is not necessarily a total variation and presents a potential obstruction to integrability of the Hamiltonian charge. As discussed above, the non-integrability happens because the property (4.2.9) is not enough to guarantee that the symplectic form is conserved between two Cauchy slices. When there is flux leaking through the lateral boundary, (4.3.13) will not define a Hamiltonian generator, and one needs a prescription for the charges. One way to understand this is that these diffeomorphisms move the corner, and by doing so, one is sensitive to degrees of freedom that could be entering or escaping the causal domain of the initial Cauchy slice. One is thus dealing with an open system, and the construction of canonical generators is more subtle. The obvious choice of taking the Noether charge via (4.3.12) for the Einstein-Hilbert action leads to the Komar formulae. These have various useful properties, but also shortcomings that have been known for a long time (such as wrong factors of 2 in the energy at both spatial and future null infinity, generically non-invariant flux-balance laws, and so forth, see e.g. [83, 84]), hence the motivation for a different prescription.

We are going to split (4.3.18) between the integrable part, which we will identify as the charge, and the non-integrable part, which is the flux[7]:

$$\not{d}h_\xi = \delta h_\xi + f_\xi. \tag{4.3.19}$$

This split is not unique and there are multiple proposals as to how to fix it. A choice that lies at the heart of the improved Noether charge prescription [34, 85, 40, 38] is taking h_ξ to be a Noether charge for a specific choice of the boundary Lagrangian, which can be justified by the action principle. A similar result can be achieved via the Wald-Zoupas (WZ) procedure [33], choosing the split so that the conservation of the charge is guaranteed under certain physically motivated assumptions. We will review both of these approaches in Chapter 7, where we will also show under which conditions the resulting WZ charges can be identified as the Noether charges for an appropriate choice of the boundary

[7] In [40] the first term in (4.3.18) is identified as the symplectic flux $\mathcal{F}^\theta_\xi := \int_S i_\xi\theta$ associated with ξ and full obstruction to integrabitlity is called the Noetherian flux $\mathcal{F}_\xi := \int_S i_\xi\theta + q_{\delta\xi} + A_\xi$. They can be related as $\mathcal{F}_\xi = \mathcal{F}^\theta_\xi - \mathcal{F}^\theta_{\delta\xi}$

Lagrangian. We see that the Lagrangian and symplectic anomalies enter respectively the Noether charge q_ξ and the (infinitesimal) Hamiltonian generator dh_ξ. Notice that the Hamiltonian generator depends only on θ, whereas the Noether charge depends on θ but also explicitly on the boundary Lagrangian via its anomaly. This hints to a deep connection of the improved Noether charge with the boundary conditions which will be discussed in detail in the subsequent chapters.

Various other interesting proposals on how to achieve integrability can be found in the literature. For example one can keep track of how the boundary is embedded into the bulk by extending the phase space introducing additional dynamical fields supported at the boundary whose dynamics is defined so that the outgoing flux is absorbed into the definition of the charge [78, 79]. Another approach uses a new Leibnizian bracket with respect to which the charges are integrable [86]. At least in non-dynamical cases, there is also the possibility of obtaining integrability by finding an appropriate field-dependence of ξ so that the $q_{\delta\xi}$ term cancels the obstruction, a procedure known as 'slicing', see e.g. [87]. We will not consider these alternative constructions here.

Boundary conditions and boundary terms

5

The behavior of dynamic systems is mathematically described by partial differential equations (PDEs) which elegantly encapsulate the dynamic evolution of various physical quantities across space and time. An important question in the theory of PDEs is the imposition of boundary conditions, which govern the system's behavior at its edges, and are essential for obtaining well-posed problems and meaningful solutions. Categorized into various types like Dirichlet, Neumann, and Robin, they offer a glimpse into how a system interacts with its surroundings.

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In the study of differential equations, a boundary-value problem (BVP) is a differential equation that comes with specific conditions known as boundary conditions. Solving a BVP means finding a solution that satisfies both the differential equation and the given boundary conditions. BVPs have practical significance in various branches of physics because most physical situations require them for describing the interaction with the environment. For instance, problems involving the wave equation, which deal with phenomena like normal modes, are often formulated as boundary-value problems.

Depending on the physical situation on hand, different types of boundary conditions can be imposed: a boundary condition that stipulates the function's actual value is referred to as a *Dirichlet boundary condition*, often termed a first-type boundary condition. For instance, if one end of an iron rod is maintained at absolute zero temperature, this boundary condition imparts knowledge of the function's value at that particular point in space. Conversely, a boundary condition that sets the value of the function's normal derivative is known as a *Neumann boundary condition*, designated as a second-type boundary condition. To illustrate, imagine a situation where an iron rod is connected to a heater at one end, resulting in continuous energy addition. Although the precise temperature may not be known, the rate of energy input is defined by this boundary condition. One can also define a *mixed boundary value problem*, where the solution is required to satisfy a Dirichlet or a Neumann boundary condition in a mutually exclusive way on disjoint parts of the boundary. This is similar to a *Robin boundary condition* which requires a linear combination, possibly with pointwise variable coefficients of the Dirichlet

and the Neumann boundary value conditions to be satisfied on the whole boundary of a given domain.

For these problems to be useful in real-world applications, they need to be well-posed. This means that, given certain input values, a unique solution should exist, and this solution should change continuously with the input. A considerable amount of theoretical effort in the field of partial differential equations goes into proving that boundary-value problems, arising in various scientific contexts, satisfy this well-posedness criterion.

However, for the purpose of this discussion, we will adopt a simpler, more intuitive definition of well-posedness. Specifically, when we discuss *allowed* boundary conditions, we are referring to those that are compatible with the constraints set forth by the action principle. This concept will be elaborated upon in the subsequent section, highlighting the fundamental role of the action principle in determining suitable boundary conditions. We will not go into further constraints that are to be imposed in order to have a well-defined boundary-value problem.

5.1 Variational principle

The action principle is a foundational concept for describing motion in both classical mechanics and field theory. In this section we will look beyond its well-known role in generating equations of motion, into its capacity to derive viable boundary conditions. This exploration unfolds against the backdrop of boundary-value problems but keeps their well-posedness aside. Essentially, we are examining the minimum requirements for boundary conditions to uphold a coherent action principle, and our considerations can be seen as zeroth level constraints on what can be fixed on a boundary of a physical theory.

Through the principle of least action, systems evolve along paths that extremize the action. This minimization or maximization process, yields the equations of motion that govern the system's behavior in the bulk. When we manipulate the action variation to get rid of variations of the field derivatives and derive the Euler-Lagrange equations, we integrate by parts and boundary terms emerge. For the action to be extremal, these terms must vanish, leading to constraints in form of boundary conditions. The vanishing of boundary terms becomes a key requirement for extremal action. This demand for boundary terms to disappear directly translates to constraints on the boundary conditions. A particular set of boundary conditions ensures that these boundary terms vanish, and in turn, the action is extremal. By performing Legendre transformations, i.e. integrating by parts in field space, we transition between various boundary conditions, modifying the boundary Lagrangian in accordance with the chosen boundary conditions but preserving the bulk Lagrangian and along with it the equations of motion.

To be more concrete, take as a starting point a Lagrangian L^{D} that leads to a stationary action under Dirichlet boundary conditions $\delta\phi^a \overset{\mathcal{B}}{=} 0$, we can express the variation as follows

$$\begin{aligned}
\delta L^{\mathrm{D}} &= E_a \delta\phi^a + d\theta && (5.1.1)\\
&= E_a \delta\phi^a + d(\tilde{\pi}_a \delta\phi^a d^{d-1}x) && (5.1.2)\\
&\overset{\text{b.c.}}{=} 0, && (5.1.3)
\end{aligned}$$

where E_a are the Euler-Lagrange equations, θ is the symplectic potential, and $\tilde{\pi}_a$ is the momentum conjugate to ϕ^a. In this context, the tilde notation indicates a boundary density, where $\tilde{a} = \sqrt{|q|}a$ if the boundary volume form is given by $\epsilon_{\mathcal{B}} = \sqrt{|q|}d^{d-1}y$. To transition to Neumann boundary conditions while ensuring a well-posed action principle, we introduce a new Lagrangian

$$L^{\mathrm{N}} = L^{\mathrm{D}} + d\ell, \tag{5.1.4}$$

such that

$$\delta L^{\mathrm{N}} = \delta L^{\mathrm{D}} + d\delta\ell \overset{\text{b.c.}}{=} 0, \tag{5.1.5}$$

with Neumann boundary conditions defined as $\delta\tilde{\pi}_a \overset{\mathcal{B}}{=} 0$.[1] The simplest choice of ℓ is given by

$$\delta L^{\mathrm{N}} \approx d(\tilde{\pi}_a \delta\phi^a d^{d-1}x) + d\delta\ell \sim \#^a \delta\tilde{\pi}_a \quad \Rightarrow \quad \ell \overset{\mathcal{B}}{=} -\pi_a \phi^a \epsilon_{\mathcal{B}}. \tag{5.1.6}$$

Even more concretely, let us consider the scalar field Lagrangian, $L = \frac{1}{2}d^4x\partial_\mu\phi\partial^\mu\phi$. Its variation reads

$$\delta L = d^4x\partial^\mu\phi\partial_\mu\delta\phi = d^4x\left[\partial_\mu(\partial^\mu\phi\delta\phi) - \Box\phi\delta\phi\right]. \tag{5.1.7}$$

The second term encodes the field equation $\Box\phi = 0$, and the first term tells us to impose Dirichlet boundary condition $\delta\phi|_{\mathcal{B}} = 0$ so we can denote $L^{\mathrm{D}} := L$. To use Neumann boundary conditions, we follow (5.1.6) to get $L^{\mathrm{N}} = \frac{1}{2}d^4x\partial_\mu\phi\partial^\mu\phi - d(\underleftarrow{d^3x}\phi\partial_n\phi)$.

Just as the boundary conditions in a scalar field theory led us to different forms of the Lagrangian, they serve as a precursor to the different types of polarizations in the associated phase space. This is important for both the classical dynamics and the eventual quantum description of the system. Therefore, understanding polarizations in phase space is as integral to the study of dynamical systems as understanding the boundary conditions. In the next section we discuss the connection between the two for a generic theory.

[1] Notice that we are defining Neumann boundary conditions as fixing the momentum, and not the velocity (or the normal derivative of the field). This is the input we are getting from the action.

5.2 Boundary Conditions and Polarization of Phase Space

In classical mechanics, the Hamiltonian framework describes the dynamics of a system using a symplectic manifold, typically represented by the cotangent bundle of a configuration space, T^*M. However, the notion of phase space extends to more general symplectic manifolds. This abstraction transcends the traditional split between configuration variables and their conjugate momenta. In this sense, the conventional classification of boundary conditions into Dirichlet, Neumann, and Robin may be seen as overly restrictive, as it relies on a fixed separation between configuration space and momentum space, which we might not want to have and one might want to look for different ways to categorize them. In the context of a covariant phase space, one might argue that this categorization breaks covariance as the definition of the momentum requires a choice of time or a Hamiltonian.

A more consistent understanding of boundary conditions in phase space is gained through the concept of *polarization*. A phase space polarization is a foliation by Lagrangian submanifolds, where each leaf is, by definition, isotropic, i.e. spanned by Poisson-commuting coordinates. We will refer to the coordinates along each leaf as the configuration variables q^a.[2] Once these are chosen, one can define the conjugate momenta p_a locally thanks to the Darboux theorem, so that together $\{q^a, p_b\}$ form Darboux coordinates and $\theta = p_a \delta q^a$. This way the choice of a polarization of the phase space is, via a choice of adapted coordinates that diagonalize it, encoded in the choice of the symplectic potential. Connecting to the discussion in the previous section, we note that by fixing the coordinates of a Lagrangian submanifold, one can define boundary conditions without reference to a predetermined separation between positions and momenta. In other words, various types of boundary conditions can be understood as fixing the 'configuration' variable for different definitions of this variable, namely instead of thinking about different types of boundary conditions one can think about different flavours of Dirichlet. This offers an alternative approach to classifying boundary conditions, with equivalent results but from a different point of view.

In the ensuing discussion, we will use the term polarization in reference to a selection of symplectic potential. This should however be understood as an abuse of language, because even though a choice of polarization induces a natural choice of symplectic potential, we don't expect any bijection. To understand why, let us start from the canonical symplectic

[2] To justify the name, let us go back to the special case of the cotangent bundle T^*M, where exists a natural polarization called the *vertical polarization* [88]. In local coordinates it is spanned by the vectors ∂_p tangent to the fibers of T^*M and the leaves can be identified with M. Since the definition of a Lagrangian submanifold is coordinate independent, namely we can define it as the maximal submanifold on which the pullback of the symplectic form vanishes, we are going to use the term *configuration variable* to refer to *any* choice of coordinates on the leaves.

potential $\theta = p_a \delta q^a$. This is in diagonal form, the momenta p_a Poisson commute. If we now integrate by parts in field space one or more pairs, we obtain an alternative valid polarization. This may be associated to new meaningful boundary conditions, even though only after analysis of the field equations one knows whether the new boundary conditions are generic or restrict the space of solutions. Now consider instead a generic symplectic potential for the same polarization, of the form $\theta = P_a(q,p)\delta q^a$. The P_a are not necessarily conjugate variables, nor they necessarily commute among each other. Even though such a symplectic potential appears to be a bad choice, in practice this is something that can easily happen. For instance, it happens in the case of Lagrange's spinning top if we use the generators of rotations as momenta, see e.g. [77], as it is done in geometric actions [89, 90]. More closely related to our thesis, it is what happens when one gives the pull-back of the symplectic potential of general relativity a geometric interpretation in terms of the first and second fundamental forms of the hypersurface. If the symplectic potential is in this more general form, integrating by part in field space needs more care.

Once a choice of the symplectic potential corresponding to meaningful boundary conditions is made, its structure can be used to determine the boundary terms needed for the associated variational principle. However open questions remain. First, the definition of the symplectic potential from the variation of the Lagrangian exhibits a degree of ambiguity which allows a certain level of freedom in choosing its precise form. Integration by parts in field space uses one of these ambiguities – it corresponds to adding a boundary term to the Lagrangian which does not affect the field equations or the symplectic form, and we have seen in the previous section that it corresponds to the change of boundary conditions. However, we can also add an arbitrary spacetime-exact 3-form to a θ satisfying (4.2.7) without spoiling its relationship with the Lagrangian. This means that the equivalence of the boundary Lagrangians and choices of polarization is defined only up to corner terms. Second, it is unclear if the relationship between boundary conditions and boundary Lagrangians is bijective. We defined (5.1.6) as the simplest choice of boundary Lagrangian for a given set of boundary conditions, and we don't know if it is unique. We will see in Chapter 8 an example of a boundary Lagrangian appearing to lead two distinct choices of boundary conditions where additional input had to be used to disregard one of them. The question remains whether there exist a general principle, or the relationship between boundary conditions and polarizations requires a case-by-case analysis. The experience gained with this thesis suggests the second option.

5.3 Conservative boundary conditions for GR

In this section we review the different boundary conditions for metric gravity that will be consider in subsequent chapters. We start from the Einstein-Hilbert (EH) Lagrangian,

$$L^{\text{EH}} = R\epsilon, \tag{5.3.1}$$

where $\epsilon := \sqrt{-g}d^4x$ is the volume 4-form. The variation gives

$$\delta L^{\text{EH}} = G_{\mu\nu}\delta g^{\mu\nu}\epsilon + d\theta^{\text{EH}}, \tag{5.3.2}$$

where

$$\theta^{\text{EH}} = \frac{1}{3!}\theta^\mu \epsilon_{\mu\nu\rho\sigma}\, dx^\nu \wedge dx^\rho \wedge dx^\sigma, \qquad \theta^\mu = 2g^{\rho[\sigma}\delta\Gamma^{\mu]}_{\rho\sigma} = 2g^{\mu[\rho}g^{\nu]\sigma}\nabla_\nu \delta g_{\rho\sigma}. \tag{5.3.3}$$

To study possible polarizations, the symplectic potential needs to be pulled-back to the boundary and put in a diagonal '$p\delta q$' form. Due to their starkly different geometric properties and interpretations, we will study boundary conditions on timelike and null lateral boundaries separately.

5.3.1 Timelike boundaries

Stating with the simpler of the two, consider a hypersurface Σ, with unit normal n_μ, and boundary $\partial\Sigma = S$, with unit normal u_μ within $T^*\Sigma$ that $u\cdot n = 0$. The corresponding volume forms are ϵ_Σ and ϵ_S, related among each other and to ϵ as in Chapter 2. In this section we consider both cases of space-like or time-like Σ at once, and accordingly we define $s := n^2 = \mp 1 = -u^2$. We will later introduce different notation for the different boundaries for the sake of clarity. We denote $q_{\mu\nu} := g_{\mu\nu} - s n_\mu n_\nu$ the projector, whose pull-back on Σ gives the induced metric, with determinant q; and $K_{\mu\nu} := q^\rho_\mu \nabla_\rho n_\nu$ the extrinsic curvature, with $K := g^{\mu\nu}K_{\mu\nu} = \nabla_\mu n^\mu$. Taking the pull-back of the boundary variation on Σ one has (see e.g. [91, 51, 66])

$$\underleftarrow{\theta}^{\text{EH}} = s\left(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K\right)\epsilon_\Sigma + d\vartheta^{\text{EH}} = s\, q_{\mu\nu}\delta\tilde{\Pi}^{\mu\nu}d^3x + d\vartheta^{\text{EH}}, \tag{5.3.4}$$

$$\underleftarrow{\vartheta}^{\text{EH}} := -u_\mu \delta n^\mu \epsilon_S = u^\mu n^\nu \delta g_{\mu\nu}\epsilon_S. \tag{5.3.5}$$

In the second equality of (5.3.4) we introduced the gravitational momentum

$$\tilde{\Pi}^{\mu\nu} := \sqrt{q}(K^{\mu\nu} - q^{\mu\nu}K), \qquad \tilde{\Pi} := g_{\mu\nu}\tilde{\Pi}^{\mu\nu} = -2\sqrt{q}K, \tag{5.3.6}$$

familiar from the ADM analysis, here written as a spacetime tensor.

The geometric decomposition (5.3.4) of the boundary variation allows us to study the variational principle in a finite region of spacetime like the one in Fig. 5.1. The

variational principle is well-defined only if the total boundary contribution vanishes, so that the action is correctly extremized on-shell.[3] Let us see when this happens. Gluing (5.3.4) along the boundary, one obtains the corner variation [92][4]

$$\vartheta^{\text{EH}}|_{\partial\Sigma} + \vartheta^{\text{EH}}|_{\partial\mathcal{T}^{\pm}} = \mp 2\delta\beta\epsilon_S, \tag{5.3.7}$$

where $\sinh\beta$ is the scalar product between the time-like and space-like outgoing normals, see Fig. 5.1 and Chapter 2 for further relations. Let us first consider the case of orthogonal corners, $\beta = 0$. The corner variation vanishes, and we are left with the 3d bulk terms, which vanish if $\delta\tilde{\Pi}^{\mu\nu} \overset{\partial M}{=} 0$, as can be seen from the second equality of (5.3.4). The Einstein-Hilbert action thus has a well-defined variational principle if we fix the momentum on the boundary, i.e. if we use Neumann boundary conditions. Note that this is only true in four dimensions, which is the only case in which the second equality of (5.3.4) holds. The general Neumann boundary Lagrangian can be found in [93] and will be dimension-dependent.

Switching focus to Dirichlet boundary conditions, namely fixing the induced metric so that $\delta q_{\mu\nu} = 0$, there is a left-over term in the first equality of (5.3.4). To cancel it, we need to add the Gibbons-Hawking-York boundary Lagrangian [24, 25]

$$\ell^{\text{GHY}} = 2sK\epsilon_\Sigma. \tag{5.3.8}$$

The Lagrangian with a well-defined Dirichlet variational principle is thus

$$L^{\text{GHY}} = L^{\text{EH}} + d\ell^{\text{GHY}}. \tag{5.3.9}$$

These are the two most commonly considered options.[5] However as pointed out in [95, 96], a better choice of boundary conditions is York's set of mixed boundary conditions [97, 23], which fix the conformal metric $\hat{q}_{\mu\nu} := q^{-1/3}q_{\mu\nu}$ and K, because, unlike Dirichlet and Neumann, they lead to a well posed initial boundary value problem (See also [98, 99, 100, 22]). With this choice, one needs to add a boundary Lagrangian given by [23]

$$\ell^{\text{Y}} = \frac{2}{3}sK\epsilon_\Sigma. \tag{5.3.10}$$

[3] The time-like and space-like parts of the boundary variation have different conceptual status. To define the phase space, it is enough to ensure that the time-like variation vanishes. Vanishing of the space-like variation will then correspond to picking a specific solution in the phase space.

[4] See also [51] for a more recent derivation, and [65, 66] to see how the derivation is simplified if one uses tetrad instead of metric variables.

[5] Alternatively, one may wish to work with a field space in which variations are left arbitrary, and deduce the appropriate variational principle for each pair of bulk-boundary Lagrangian interpreting the boundary variation as an equation of motion. For example, L^{GHY} induces the homogeneous Neumann boundary condition $\tilde{\Pi}^{\mu\nu} = 0$, whereas the EH Lagrangian would lead to the inadmissible degenerate condition $q_{\mu\nu} = 0$. This approach, emphasized and studied in [94], will not be pursued here.

This can be easily shown observing that

$$s\left(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K\right)\sqrt{q} = -s\tilde{P}^{\mu\nu}\delta\hat{q}_{\mu\nu} - s\tilde{P}_K\delta K - s\frac{2}{3}\delta(K\sqrt{q}), \tag{5.3.11}$$

where

$$\tilde{P}^{\mu\nu} := q^{1/3}(\tilde{\Pi}^{\mu\nu} - \frac{1}{3}q^{\mu\nu}\tilde{\Pi}), \qquad \tilde{P}_K = \frac{4}{3}\sqrt{q} \tag{5.3.12}$$

are the five plus one momenta conjugated respectively to the conformal metric and trace of the extrinsic curvature.[6]

York's 'mostly-Dirichlet' mixed boundary conditions include the conformally flat initial data often used in numerical relativity [102]. A peculiarity of this choice is that the momenta don't commute, since next to the canonical pair $(\tilde{P}_K, K)$, the remaining five pairs satisfy (omitting $\delta^{(3)}(x, x')$ to shorten the expressions)

$$\{\hat{q}_{\mu\nu}, \hat{q}_{\rho\sigma}\} = 0, \qquad \{\hat{q}_{\mu\nu}, \tilde{P}^{\rho\sigma}\} = \delta^{\rho\sigma}_{(\mu\nu)} - \frac{1}{3}q_{\mu\nu}q^{\rho\sigma}, \qquad \{\tilde{P}^{\mu\nu}, \tilde{P}^{\rho\sigma}\} = \frac{1}{3}\left(\hat{q}^{\rho\sigma}\tilde{P}^{\mu\nu} - \hat{q}^{\mu\nu}\tilde{P}^{\rho\sigma}\right). \tag{5.3.13}$$

The non-commutativity also implies that the flipped option of taking 'mostly-Neumann' mixed boundary conditions with fixed traceless momentum and the induced metric determinant is not admissible since the momenta do not form a Lagrangian submanifold and as such do not define a polarization of the phase space.[7]

The three options considered above can be simultaneously treated parametrizing the boundary Lagrangian

$$\ell^b := bsK\epsilon_\Sigma, \tag{5.3.14}$$

with parameter $b = 2, 2/3, 0$ respectively for Dirichlet, mixed and Neumann boundary conditions.

[6] The general n-dimensional version of this formula is

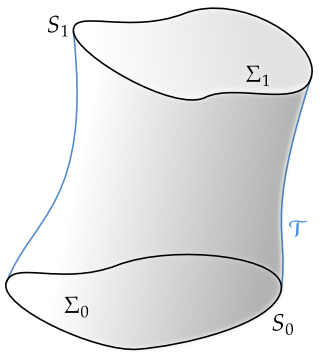
$$\left(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K\right)\sqrt{q} = -q^{\frac{1}{n-1}}\left(\tilde{\Pi}^{\mu\nu} - \frac{1}{n-1}q^{\mu\nu}\tilde{\Pi}\right)\delta\hat{q}_{\mu\nu} - 2\frac{n-2}{n-1}\sqrt{q}\delta K - \frac{2}{n-1}\delta(K\sqrt{q}).$$

For $n = 3$, the required boundary term is thus $K\sqrt{q}$, namely one-half the GHY term. This is the numerical coefficient deduced in [101] using asymptotic fall-off conditions for AdS and flat spacetimes. The nature of such asymptotic conditions was left as an open question there. The equation above suggests that they were of York's type. However, since the Neumann boundary term in n dimensions [93] is given by $(4-n)K$, for $n = 3$ it also gives one half of the GHY term. We think that in this case York's boundary conditions can be disregarded as ill-defined since, according to the uniformization theorem, the 2d conformal metric does not contain enough information about the boundary. Hence, the one-half GHY term should be considered of Neumann type.

[7] Which is good, because

$$\tilde{\Pi}^{\mu\nu}\delta q_{\mu\nu} = -\hat{q}_{\mu\nu}\delta\tilde{P}^{\mu\nu} + \frac{2}{3}\tilde{\Pi}\,\delta\ln\sqrt{q}$$

shows that the EH Lagrangian would lead to a well-defined variational principle in this case as well, breaking the expected injective relation between choice of boundary Lagrangian and choice of boundary conditions.



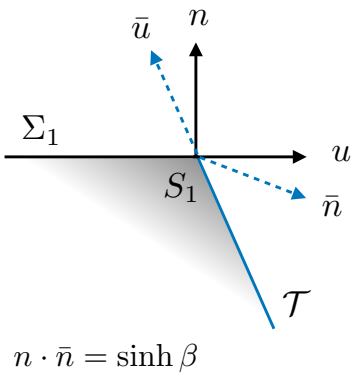


Fig. 5.1.: Left panel: *A finite region of spacetime and the notation used for the boundaries.* Right panel: *The four normals generically present at the corner, here the one in the future of $\mathcal{T}$. Unbarred quantities refer to the space-like hypersurface, and barred quantities to the time-like one.*

	metric	s	normal	extrinsic curvature	boundary normal
M	g				
Σ	q	-1	n	K	u
$\mathcal{T}$	$\bar{q}$	1	$\bar{n}$	$\bar{K}$	$\bar{u}$
S	γ		(n, u) $(\bar{n}, \bar{u})$	k	

Table 5.1.: *Notations for the different manifolds:* spacetime M, spacelike hypersurface Σ, timelike hypersurface $\mathcal{T}$, and corner S.

In the following, it will be convenient to use different notations for the space-like and time-like boundaries. In doing so, we keep the notation (n, u) for the space-like quantities, and introduce bars to distinguish the time-like boundaries, $(\bar{n}, \bar{u})$. These notations are the ones summarized in Fig. 5.1 and Table 5.1. Accordingly, the action principle with orthogonal corners reads

$$S = \int_M L^{\text{EH}} + d\ell^b = \int_M R\epsilon - b \int_{\Sigma_0}^{\Sigma_1} K\epsilon_\Sigma + b \int_{\mathcal{T}} \bar{K}\epsilon_{\mathcal{T}}. \tag{5.3.15}$$

Non-orthogonal corners

If one allows for non-orthogonal corners, additional 2d boundary terms are potentially needed in the action principle, to compensate for variations like (5.3.7). The precise form of these variations depends on the type of corner considered, see [51] for a comprehensive analysis. We restrict attention here to the corner between a space-like and a time-like boundary, as in Fig. 5.1.

To cancel the corner variation (5.3.7) we have two options: either we fix β so that $\delta\beta = 0$, or we fix the induced metric so that $\delta\epsilon_S = 0$. In the first case, no boundary term is needed. In the second case, we need to add the Hayward boundary term

$$\ell^{\mathrm{H}} = 2\beta\epsilon_S. \tag{5.3.16}$$

The second option is consistent with Dirichlet boundary conditions, since fixing the induced metric q also fixes ϵ_S. This is indeed how Dirichlet boundary conditions with non-orthogonal corners are usually treated, see e.g. [92]. Fixing β instead has the flavour of a Neumann-type condition, since it is easy to see that β captures a metric component that is not part of the induced metric. But in fact, it is also not part of the momentum, but a combination of lapse and shift instead, see (2.1.28). Hence, it is an *additional* condition to be provided. This additional condition is a priori not needed for the well-posedness of the initial value boundary problem. In fact, the corner contribution can always be thought of as part of the space-like hypersurface, and then its variation corresponds simply to a change in the state, and not of the boundary conditions [34]. Furthermore, it is a change of state associated to a different choice of lapse and shift, and which can thus be considered irrelevant to characterize different physical solutions.

On the other hand, if the finite boundary is considered as a part of the characterization of the observer, then a solution with different β would be on the same status as, say, a Kerr solution with different values of the asymptotic lapse and shift, namely corresponding to boosted or rotated black holes. Hence, within the context of thinking of the gauge degrees of freedom broken by the boundary as physical, it is of interest to consider β as part of the phase space. Indeed, this choice will be justified by the canonical analysis of the boundary terms done in Section 8.4.

A similar logic can be applied to the case of York's mixed boundary conditions. Since they leave the determinant of the induced metric free, it seems reasonable to us to take $\delta\beta = 0$ also in this case, even though it is not required by the well-posedness of the initial value problem [95, 22]. Again, this will be justified by the canonical analysis reported in Section 8.4. These choices are summarized in Table **??**.

As before, we can treat all cases with a generic corner Lagrangian

$$\ell^c = c\beta\epsilon_S, \tag{5.3.17}$$

with c needing to be $2, 0$ and 0 respectively for Dirichlet, mixed and Neumann boundary conditions. The action principle with non-orthogonal corners thus reads

$$S = \int R\epsilon - b\int_{\Sigma_0}^{\Sigma_1} K\epsilon_\Sigma + b\int_{\mathcal{T}} \bar{K}\epsilon_{\mathcal{T}} + c\int_{S_0}^{S_1} \beta\,\epsilon_S. \tag{5.3.18}$$

boundary conditions	*quantity fixed on boundary*	*value of b*	*quantity fixed at corner*	*value of c*
Dirichlet	$q_{\mu\nu}$	2	γ	2
York	$(\hat{q}_{\mu\nu}, K)$	2/3	β	0
Neumann	$\tilde{\Pi}^{\mu\nu}$	0	β	0

Table 5.2.: *Different boundary conditions and their boundary and corner Lagrangians.*
For Dirichlet boundary conditions, (5.3.15) and (5.3.18) are referred to as trace-K actions in the literature. Accordingly, we will refer to them as b-generalized trace-K actions.

5.3.2 Null boundaries

The most general expression for the pull-back of (5.3.3) on a null hypersurface reads [66][8]

$$\underleftarrow{\theta}^{\text{EH}} = \left[(\sigma^{\mu\nu} + \frac{\theta}{2}\gamma^{\mu\nu})\delta\gamma_{\mu\nu} - 2\omega_\mu \delta l^\mu + 2\delta\left(\theta + k\right)\right]\epsilon_{\mathcal{N}} + d\vartheta^{\text{EH}}. \tag{5.3.19}$$

This expression holds for arbitrary variations on the null-hypersurface: the only restriction made is to preserve the null nature of the hypersurface, namely

$$l_\mu \delta l^\mu \overset{\mathcal{N}}{=} 0. \tag{5.3.20}$$

In particular, it is valid for a field-dependent f, so that

$$\delta l_\mu \overset{\mathcal{N}}{=} \delta \ln f \, l_\mu \tag{5.3.21}$$

doesn't need to vanish. If f is field-independent, $\delta l_\mu \overset{\mathcal{N}}{=} 0$ or equivalently $n^\mu \delta l_\mu \overset{\mathcal{N}}{=} 0$.

The expression (5.3.19) does not depend on rescalings of l nor on the choice of auxiliary vector n. Independence from rescalings follows from its invariance under the class-III spin-boost transformations (2.2.7), and implies independence from changes of embeddings $\Phi \to \Phi'(\Phi)$. Independence from n follows from the invariance under class-I Lorentz transformations (2.2.6). See App. B.1 for proofs. The invariance is only a property of the full expression. The individual quantities are not, as summarized in Table 2.1. This is relevant for various considerations that we do below.

[8] With ω here defined with opposite sign, as to match [58, 59]. We took the time to accurately translate notations to prove that it is indeed equivalent to the one computed in [64], including the corner term, thus answering the question left open in [66]. This expression generalizes the one of [51] which assumes $\delta l^\mu = 0$, and the one of [63] which assumes $\delta l_\mu = -n^\rho l^\sigma \delta g_{\rho\sigma} l_\mu$, the latter implying that $n_\mu \delta l^\mu = 0$. It also generalizes the one of [85] – contrarily to what there stated –, which assumes $\delta l_\mu = 0$. We will come back to these restrictions and their motivations below. We hope that no confusion arises because the same letter θ appears as both the symplectic potential integrand and the expansion of l. The risk should be reduced by the fact that the letter used for the symplectic potential always comes with labels such as ^{EH} or $'$.

The variation of the inaffinity can be written in terms of δl_μ and δl^μ:

$$\begin{aligned}\delta k &= k n^\mu \delta l_\mu - 2 n^\mu \nabla_{(\mu} l_{\nu)} \delta l^\nu - 2 n^\mu l^\nu \nabla_{(\mu} \delta l_{\nu)} + \frac{1}{2} n^\mu \nabla_\mu \delta l^2 \\ &= (k n^\mu + \frac{1}{2} n^\nu \nabla_\nu l^\mu - \frac{1}{2} l^\mu n^\nu \nabla_\nu - n^\mu l^\nu \nabla_\nu) \delta l_\mu - (n^\nu \nabla_\mu l_\nu + \frac{1}{2} n^\nu \nabla_\nu l_\mu - \frac{1}{2} l_\mu n^\nu \nabla_\nu) \delta l^\mu .\end{aligned} \tag{5.3.22}$$

Notice that presence of normal derivatives on the variations, which imply that δk varies even if we fix both l_μ and l^μ on the hypersurface:

$$\delta l^\mu = \delta l_\mu \overset{\mathcal{N}}{=} 0 \qquad \Rightarrow \qquad \delta k = -\frac{1}{2} n^\mu \nabla_\mu (l_\nu l_\rho \delta g^{\nu\rho}). \tag{5.3.23}$$

This vanishes if we restrict the variations to preserve affine coordinates. Therefore δk is an independent variation because it captures the possibility of varying the metric between the form (2.2.29) and (2.2.31), which are both consistent with fixing l_μ and l^μ on the hypersurface.

Finally, the corner term is [63, 66]

$$\vartheta^{\text{EH}} := n^\mu \delta l_\mu \epsilon_S - i_{\delta l} \epsilon_{\mathcal{N}} = (n^\mu \delta l_\mu + n_\mu \delta l^\mu)\, \epsilon_S - n \wedge i_{\delta l} \epsilon_S . \tag{5.3.24}$$

The last term vanishes if we pull-back on a space-like cross section with n adapted to it.

Phase space polarizations

We would like to manipulate the RHS of (5.3.19) so to put it in the form

$$\underset{\leftarrow}{\theta} = \theta' - \delta \ell + d\vartheta , \tag{5.3.25}$$

where $\theta' = p\delta q$ for a given choice of polarization of the (kinematical) phase space on the boundary.[9] This form will be useful to discuss two different but related contexts: the variational principle, and the definition of charges using covariant phase space methods.

If we attempt to interpret (5.3.19) as a symplectic potential in $p\delta q$ form, we see that the q's appearing are not independent, since $\gamma_{\mu\nu}$ also determines θ through (2.2.16a). To put it in diagonal form, we need two steps. The first is to observe that the variation of the volume form has two components,

$$\delta \epsilon_{\mathcal{N}} = \left(\frac{1}{2} \gamma^{\mu\nu} \delta \gamma_{\mu\nu} + n_\mu \delta l^\mu \right) \epsilon_{\mathcal{N}} , \tag{5.3.26}$$

[9] This notion of polarization should be used only at the level of the kinematical phase space. In fact, on a null-hypersurface the momenta do not depend on velocities, hence there are second class constraints and the actual symplectic structure is given by a Dirac bracket and not the initial, kinematical Poisson bracket, see e.g. [103, 70].

as can be established starting from the identity

$$\frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}\gamma^{\mu\nu}\delta\gamma_{\mu\nu} + n_\mu\delta l^\mu - n^\mu\delta l_\mu. \tag{5.3.27}$$

Using (5.3.26), the pull-back (5.3.19) can be rewritten in the form

$$\underleftarrow{\theta}^{\mathrm{EH}} = \left[\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu\delta l^\mu + 2\delta(\theta + k)\right]\epsilon_{\mathcal{N}} + \theta\delta\epsilon_{\mathcal{N}} + d\vartheta^{\mathrm{EH}}, \tag{5.3.28}$$

where

$$\pi_\mu := -2\left(\omega_\mu + \frac{\theta}{2}n_\mu\right) = 2\left(\eta_\mu + \left(k - \frac{\theta}{2}\right)n_\mu\right). \tag{5.3.29}$$

The second step is to integrate by parts in field space the third term. This leads to

$$\underleftarrow{\theta}^{\mathrm{EH}} = \theta^{\mathrm{D}} - \delta\ell^{\mathrm{D}} + d\vartheta^{\mathrm{EH}}, \tag{5.3.30}$$

where

$$\theta^{\mathrm{D}} := \sigma^{\mu\nu}\delta\gamma_{\mu\nu}\epsilon_{\mathcal{N}} + \pi_\mu\delta l^\mu\epsilon_{\mathcal{N}} - (\theta + 2k)\delta\epsilon_{\mathcal{N}}, \tag{5.3.31}$$

and

$$\ell^{\mathrm{D}} := -2W\epsilon_{\mathcal{N}} = -2(\theta + k)\epsilon_{\mathcal{N}} = -2k\epsilon_{\mathcal{N}} - 2d\epsilon_S. \tag{5.3.32}$$

The last equality follows from (2.2.12), and can be used to simplify the boundary term reabsorbing its dependence on the expansion in the corner term, and work with

$$\ell^{\mathrm{D}\prime} = -2k\epsilon_{\mathcal{N}}, \qquad \vartheta^{\mathrm{EH}\prime} = \vartheta^{\mathrm{EH}} + 2\delta\epsilon_S. \tag{5.3.33}$$

We can now identify $\theta' = \theta^{\mathrm{D}}$, which is diagonal form $p\delta q$ with $q = (\underleftarrow{\gamma}_{\mu\nu}, l^\mu)$. The q terms only involve the intrinsic geometry, therefore the symplectic potential is in the form of a Dirichlet polarization, whence the D label. The diagonalization obtained involves the sum of three pairs of configuration variables and momenta that can be characterized as spin 2, 1 and 0, as discussed for instance in [63]. Denoting $\varepsilon := \sqrt{-g}/f$, we can write the conjugate momenta as densities,

$$\tilde{\pi}^{\mu\nu} := \varepsilon\sigma^{\mu\nu}, \qquad \tilde{\pi}_\mu := 2\varepsilon\left(\eta_\mu + \left(k - \frac{\theta}{2}\right)n_\mu\right), \qquad \tilde{\pi} := -\varepsilon\left(\theta + 2k\right). \tag{5.3.34}$$

The first term in (5.3.31) can be equally written as

$$\sigma^{\mu\nu}\delta\gamma_{\mu\nu}\,\epsilon_{\mathcal{N}} = -\sigma_{\mu\nu}\delta(\gamma^{\mu\nu}\,\epsilon_{\mathcal{N}}) = -\gamma_{\mu\nu}\delta\sigma^{\mu\nu}\epsilon_{\mathcal{N}}, \tag{5.3.35}$$

where $\gamma^{\mu\nu}\,\epsilon_{\mathcal{N}}$ is manifestly conformal invariant. This is the spin-2 pair.

The spin-1 pair has three components, two 'transverse' ones whose momentum is the twist, and a 'longitudinal' one proportional to $n_\mu\delta l^\mu$. The longitudinal variation is

there to compensate the dependence of $\gamma^{\mu\nu}\delta\gamma_{\mu\nu}$ on the choice of rigging vector. This dependence makes it a non-exact 1-form in field space, see (5.3.26), and can be removed if we restrict the variations to satisfy

$$n_\mu \delta l^\mu \overset{\mathcal{N}}{=} 0 \qquad \Rightarrow \qquad \frac{1}{2}\gamma^{\mu\nu}\delta\gamma_{\mu\nu}\epsilon_{\mathcal{N}} \overset{\mathcal{N}}{=} \delta\epsilon_{\mathcal{N}}. \tag{5.3.36}$$

This restriction can be achieved in two ways. The first is to choose the metric-dependence of f such that $\delta l_\mu \overset{\mathcal{N}}{=} -n^\rho l^\sigma \delta g_{\rho\sigma} l_\mu$ [63], from which (5.3.36) follows. The second is to choose first a foliation, and then the 'canonical' normalization for f [66].

The split into spin pairs is appealing, but it is not canonical: the spin-2 pair is class-III invariant but not class-I invariant, and the spin-1 and spin-0 pairs mix up under class I and class III transformations, as can be easily seen using the formulas given in App. B.1. In other words, the spin pairs are not singled out by the geometry of the hypersurface, but require additional non-geometric choices of normal representative and of rigging.[10]

There is however a more serious problem with θ^{D}: it is not class-III invariant even as a whole. This can be checked explicitly, but it can be more easily read off the fact that the standard symplectic potential is class-III invariant, whereas the boundary Lagrangian and corner potential are not:

$$\delta\ell^{\mathrm{D}} \to \delta\ell^{\mathrm{D}} - 2\delta(\pounds_l \ln A\, \epsilon_{\mathcal{N}}), \qquad \vartheta^{\mathrm{EH}} \to \vartheta^{\mathrm{EH}} - 2\delta \ln A\, \epsilon_S. \tag{5.3.37}$$

Adding up and using (2.2.13), we find

$$\theta^{\mathrm{D}} \to \theta^{\mathrm{D}} - 2(\pounds_{\delta l} \ln A - \theta\delta \ln A)\epsilon_{\mathcal{N}} - 2\pounds_l \ln A\, \delta\epsilon_{\mathcal{N}}. \tag{5.3.38}$$

We will see below how this issue affects both the variational principle and the covariant phase space.

Notice that insofar as the rescaling is field-independent, $\delta A = 0$ and the non-invariance of θ^{D} stems from the boundary Lagrangian alone. Let us first restrict attention to this case. One way to obtain class-III invariance is to restrict the variations so that δl^μ and $\delta\epsilon_{\mathcal{N}}$ vanish. The first is possible without loss of generality by a restriction on the coordinates, whereas the second is a strong restriction on the dynamics, and we are interested in avoiding it.

One way to do so is to impose $\delta k = 0$, as was done in the CFP paper [36]. If we do so, then

$$\underset{\leftarrow}{\theta}^{\mathrm{EH}}|_{\delta k=0} = \left[\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu + 2\delta\theta\right]\epsilon_{\mathcal{N}} + \theta\delta\epsilon_{\mathcal{N}} + d\vartheta^{\mathrm{EH}} = \theta^{\mathrm{CFP}} - \delta\ell^{\mathrm{CFP}} + d\vartheta^{\mathrm{EH}}, \tag{5.3.39}$$

[10] They are thus singled out by a rigged Carollian structure.

where

$$\theta^{\text{CFP}} := \left(\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu\right)\epsilon_{\mathcal{N}} - \theta\delta\epsilon_{\mathcal{N}}, \qquad \ell^{\text{CFP}} := -2\theta\epsilon_{\mathcal{N}}. \tag{5.3.40}$$

This choice makes the boundary Lagrangian and the symplectic potential class-III invariant.

If we impose instead $\delta l_\mu = 0$, we can replace k with $\bar{k}$ in (5.3.28) and obtain

$$\underset{\leftarrow}{\theta}^{\text{EH}}|_{\delta l_\mu = 0} = \theta^{\bar{\text{D}}} - \delta\ell^{\bar{\text{D}}} + d\vartheta^{\text{EH}}, \tag{5.3.41}$$

where

$$\theta^{\bar{\text{D}}} := \sigma^{\mu\nu}\delta\gamma_{\mu\nu}\epsilon_{\mathcal{N}} + (\pi_\mu + 2\partial_\mu \ln f)\delta l^\mu \epsilon_{\mathcal{N}} - (\theta + 2\bar{k})\delta\epsilon_{\mathcal{N}}, \qquad \ell^{\bar{\text{D}}} := -2(\theta + \bar{k})\epsilon_{\mathcal{N}}. \tag{5.3.42}$$

This boundary Lagrangian is independent of f, but not of reparametrizations of Φ. This is only a partial resolution of the ambiguities, and class-III invariance is not achieved.

Remarkably, there is a solution that does not require any restriction on the variations, but instead performing an integration by parts in field space of the spin-0 term:

$$-(\theta + 2k)\delta\epsilon_{\mathcal{N}} = -\delta[(\theta + 2k)\epsilon_{\mathcal{N}}] + \delta(\theta + 2k)\epsilon_{\mathcal{N}}. \tag{5.3.43}$$

This leads to

$$\underset{\leftarrow}{\theta}^{\text{EH}} = \theta^{\text{Conf}} - \delta\ell^{\text{Conf}} + d\vartheta^{\text{EH}}, \tag{5.3.44}$$

where now

$$\theta^{\text{Conf}} := [\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu + \delta(\theta + 2k)]\epsilon_{\mathcal{N}}, \tag{5.3.45}$$

and

$$\ell^{\text{Conf}} := -\theta\epsilon_{\mathcal{N}} = -d\epsilon_S. \tag{5.3.46}$$

The new boundary Lagrangian and symplectic potential are class-III invariant. This shows the type of valuable insights that can be obtained using the freedom of changing potential via (7.1.1).

The decomposition (5.3.45) was considered also in [49], while looking for thermodynamical interpretations of the symplectic potential. It corresponds to a change of polarization in the phase space that identifies as configuration variables the conformal class of the 2d metric – equivalently the shear, recall (5.3.35) and discuss around there –, the tangent vector, and the spin-0 momentum $\theta + 2k$ instead of the volume form. We will show that it leads to a Noether flux-balance law with no anomaly term. This is related to that fact that this choice makes the boundary Lagrangian unambiguous even without any restriction on the variations. Conformal boundary conditions appear thus to be better behaved, something argued for also in the time-like case [95].

There are other integrations by parts in field space that could be considered. One could change the spin-0 sector with different numerical factors. For instance the investigations of black hole entropy by Chandrasekaran and Speranza (CS) in [85] motivates the choice

$$\ell^{\mathrm{CS}} = -(k + 2\theta)\epsilon_{\mathcal{N}} = \ell^{\mathrm{D}} + k\epsilon_{\mathcal{N}}. \tag{5.3.47}$$

The motivation from black hole entropy will be briefly explained below, but notice that this choice is not class-III invariant. On the other hand, a change of polarization in the spin-1 pair seems of little use, since we cannot treat η_μ as an independent configuration variable from the spin-2 pair. This is due to the fact that the constraint equations on $\mathcal{N}$ relates η_μ to the radiative data which are contained in the spin-2 pair. For completeness, we report in App. B.2 an exploration of alternative polarizations and their boundary conditions.

Our goal is to will study how changing the boundary Lagrangian as in the examples above affects the variational principle and the construction of gravitational charges. With the exception of (5.3.42), the boundary Lagrangians that we consider have the same functional dependence, and differ only by numerical factors. This is similar to what we had in the case of a time-like boundary [42], and allows us to treat all cases at once writing the boundary Lagrangian in parametric form as

$$\ell^{(b,c)} = -(bk + c\theta)\epsilon_{\mathcal{N}}. \tag{5.3.48}$$

A boundary Lagrangian of this family is class-III invariant for $b = 0$ and any value of c. The specific examples described above correspond to:

	D	CFP	Conf	CS
b	2	0	0	1
c	2	2	1	2

(5.3.49)

The option (5.3.42) could be included adding a third parameter, but we have seen that it is only a partial solution and will thus be considered less in the following. The symplectic potential corresponding to this family is

$$\theta^{(b,c)} = \left[\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu + (2-b)\delta k + (2-c)\delta\theta\right]\epsilon_{\mathcal{N}} - (bk + (c-1)\theta)\delta\epsilon_{\mathcal{N}}, \tag{5.3.50}$$

and we repeat for convenience of comparison the four particular cases discussed earlier:

$$\theta^{\mathrm{D}} := (\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu)\epsilon_{\mathcal{N}} - (\theta + 2k)\delta\epsilon_{\mathcal{N}}, \tag{5.3.51a}$$

$$\theta^{\mathrm{CFPk}} := \big(\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu + 2\delta k\big)\epsilon_{\mathcal{N}} - \theta\delta\epsilon_{\mathcal{N}}, \tag{5.3.51b}$$

$$\theta^{\mathrm{Conf}} := [\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu + \delta(\theta + 2k)]\epsilon_{\mathcal{N}}, \tag{5.3.51c}$$

$$\theta^{\mathrm{CS}} := \big(\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu + \delta k\big)\epsilon_{\mathcal{N}} - (k + \theta)\delta\epsilon_{\mathcal{N}}. \tag{5.3.51d}$$

Here CFPk stands for the extension of the CFP case to $\delta k \neq 0$. Although both θ^{CFPk} and θ^{Conf} are III-invariant, only the latter is in diagonal form for the general case with $\delta k \neq 0$, since in the former the volume form appears both as q and as p. More in general, any potential with $b = 0$ is III-invariant, but only the one with $c = 1$ is in diagonal form. For $\delta k = 0$, both θ^{CFP} and θ^{Conf} are diagonal and class III-invariant for $\delta A = 0$.

So far we have assumed that $\delta A = 0$. This condition is satisfied if we restrict the class of allowed normals to satisfy $\delta l^\mu \overset{\mathcal{N}}{=} 0$ or $\delta l_\mu \overset{\mathcal{N}}{=} 0$. If one relaxes both conditions and allows for $\delta A \neq 0$, then a class-III invariant boundary Lagrangian is no longer sufficient to have a class-III invariant symplectic potential, because of the contribution from ϑ^{EH}. A possibility would then be to absorb $d\vartheta^{\mathrm{EH}}$ in the definition of θ'. This is possible of course, however it would spoil the idea that θ' should be in diagonal $p\delta q$ form. Furthermore, we will see that there are other reasons to impose $\delta l^\mu \overset{\mathcal{N}}{=} 0$ or $\delta l_\mu \overset{\mathcal{N}}{=} 0$, as well as to keep ϑ^{EH} out of θ'. For this reason we keep ϑ^{EH} in the corner term.

Conservative boundary conditions and the variational principle

In the study of the variational principle, one wants to find suitable boundary conditions that make the variation of the action vanish everywhere on-shell, including at the boundary. In this context, (7.1.1) is useful to identify the required boundary and corner terms to be added to the action for the allowed boundary conditions. Suppose that we find a decomposition like (7.1.1) with a certain $\theta' = p\delta q$, and such that adding ϑ to the contribution coming from the part of the boundary complementary to $\mathcal{N}$ we get a total variation, call it δc. We can then conclude that the boundary conditions identified by $\delta q = 0$ provide a well-defined variational principle, once the action is supplemented with the boundary term ℓ as well as the corner term c.

In this section we show how the different polarizations of the null symplectic potential give a variational principle with different boundary conditions. We first review two known but non-trivial facts about Dirichlet boundary conditions, namely that one has to fix one more condition than the intrinsic geometry [64], and that the resulting boundary terms are ambiguous [51]. We then show that the alternative conformal boundary conditions improve this problem.

Dirichlet boundary conditions and their ambiguity

Dirichlet boundary conditions hold fixed the intrinsic geometry. In the case of a null hypersurface, we could take this to mean

$$\gamma^{\mu\rho}\gamma^{\nu\sigma}\delta\gamma_{\rho\sigma} \overset{\mathcal{N}}{=} 0, \qquad \delta l^{\mu} \overset{\mathcal{N}}{=} 0. \tag{5.3.52}$$

The first condition is class-III invariant, but the second only if $\delta A = 0$. Nonetheless they imply $\delta\epsilon_{\mathcal{N}} \overset{\mathcal{N}}{=} 0$ thanks to (5.3.26), therefore they fix entirely the intrinsic geometry. On-shell of these conditions (5.3.19) gives

$$\underset{\leftarrow}{\theta}^{\mathrm{EH}} = -\delta\ell^{\mathrm{D}} + \partial_n l^2\, n^{\mu}\delta l_{\mu}\epsilon_{\mathcal{N}} + d\vartheta^{\mathrm{EH}}, \tag{5.3.53}$$

with $\vartheta^{\mathrm{EH}} = n^{\mu}\delta l_{\mu}\epsilon_S$. The first term on the RHS is a total variation, and can be eliminated if the boundary Lagrangian (5.3.32) is added to the initial action. This is the equivalent of the Gibbons-Hawking-York term, and can even be written in exactly the same form as the divergence of the normal using (2.2.20). Notice also that (5.3.52) also imply $\delta\theta \overset{\mathcal{N}}{=} 0$, hence the only relevant term in (5.3.32) is the inaffinity k. It is then equivalent to work with this boundary Lagrangian or the alternative choice (5.3.33).

The second term requires

$$\partial_n l^2\, n^{\mu}\delta l_{\mu} \overset{\mathcal{N}}{=} 0. \tag{5.3.54}$$

This is an extra condition on top of (5.3.52). But it can be satisfied playing with the freedom of choosing the extension of l, without adding any further boundary conditions. Let us do so for now.

The third term on the RHS is not a total variation, but it can be shown that once it is added to the contribution coming from the rest of the boundary, one obtains a total variation [104, 92, 51, 65, 66]. For instance if the null boundary is joined to a space-like boundary Σ,

$$\vartheta^{\mathrm{EH}} + \vartheta^{\mathrm{EH}}_{\Sigma} = -2\delta\hat{\beta}\epsilon_S, \tag{5.3.55}$$

where $\hat{\beta}$ is defined by (2.2.25). This is a total variation under Dirichlet boundary conditions, since the first of (5.3.52) implies $\delta\epsilon_S \overset{\mathcal{N}}{=} 0$. It can thus be compensated by the corner Lagrangian

$$l^{\mathrm{H}} := 2\hat{\beta}\epsilon_S. \tag{5.3.56}$$

Here H stands for Hayward. See [92, 51] for other examples of joints and their corner terms.

We see that the Dirichlet variational principle is not well-defined for the Einstein-Hilbert action with a null boundary, and one needs to supplement the action with the boundary terms ℓ^{D} and ℓ^{H}, in analogy with what happens with other types of boundaries. This is the result that one typically finds in the literature [64, 51, 65]. The problem

that was raised in [51] is that these boundary and corner terms are ambiguous and non-geometric: they involve quantities like k and $\hat{\beta}$ which depend on the choice of normal representative and on changes of embedding. In our language, they are not class-III invariant. A solution proposed in [51] was to add the non-local boundary term $(\theta \ln \theta)\epsilon_{\mathcal{N}}$. An alternative solution that doesn't involve the non-local counterterm would be to work with class-III invariant quantities only.

As we have seen at the end of the previous section, this can be achieved in various ways. One is to add the condition $\delta k \overset{\mathcal{N}}{=} 0$. The boundary Lagrangian for this variational principle is the class-III invariant choice ℓ^{CFP} and it is unambiguous. In fact, it is spacetime exact, hence it can be reabsorbed in the corner term, leading to a variational principle without any boundary term along the null hypersurface. The problem raised in [51] is however not completely solved, because there is still the need for corner terms like (5.3.56), which maintain their ambiguity.

Next, let us consider (5.3.54) again. If the extension of the normal is kept arbitrary, in particular not null at first order, the boundary conditions (5.3.52) are *not* sufficient for a well-posed variational principle, and must be supplemented by the additional restriction $\delta l_\mu \overset{\mathcal{N}}{=} 0$. This additional restriction seems quite natural, because as we recalled above, the induced metric and tangent vector characterize the intrinsic geometry only at fixed f. The conditions $\delta l^\mu = \delta l_\mu \overset{\mathcal{N}}{=} 0$ are in fact equivalent to $l_\mu \delta g^{\mu\nu} \overset{\mathcal{N}}{=} 0$ and as such, manifestly class III-invariant. A boundary Lagrangian for the Dirichlet variational principle supplemented by $\delta l_\mu \overset{\mathcal{N}}{=} 0$ is given in (5.3.42), but as remarked there, this removes only the ambiguity under change of f at fixed embedding, and not under changes of embedding. This would then only be a partial resolution to the problem of ambiguities. The reason why we have class-III invariant boundary conditions but fail to have a fully class-III invariant boundary Lagrangian is that $\ell^{\bar{D}}$ transforms under class-III with a inhomogeneous term proportional to $\pounds_l \ln A$, and its variation is zero under the above conditions.

As for the corner terms, the additional condition means that (5.3.24) vanishes. This does not remove the Hayward corner term (5.3.56) from the variational principle, since it is still needed to cancel a contribution to the variation coming from the space-like boundary. But it removes it in the case of a corner between two null boundaries with the same boundary conditions. In the latter case there is no corner ambiguity.

It follows that if we take both additional restrictions,

$$\gamma^{\mu\rho}\gamma^{\nu\sigma}\delta\gamma_{\rho\sigma} = \delta l^\mu = \delta l_\mu = \delta k \overset{\mathcal{N}}{=} 0, \tag{5.3.57}$$

this strengthened Dirichlet variational principle is well-defined with no contribution from the null boundary, and all potential ambiguities reduced to a choice of normal at the corner between a null and a non-null boundary.

The importance of choosing the right boundary term goes beyond the variational principle. In [51], it was discussed its relevance to in the context of the 'action=complexity' proposal in AdS/CFT holography. Below, we will see how it affects the charges constructed with covariant phase space methods. In this application, only the ambiguity of the boundary Lagrangian matters, and not the one of the corner terms used in the variational principle.

Conformal boundary conditions

Consider now the alternative polarization (5.3.45). This vanishes for the conformal boundary conditions

$$\delta\sigma^{\mu\nu} \overset{\mathcal{N}}{=} 0, \qquad \delta l^{\mu} \overset{\mathcal{N}}{=} 0, \qquad \delta(\theta + 2k) \overset{\mathcal{N}}{=} 0. \tag{5.3.58}$$

The first two are equivalent to Dirichlet except for $\delta\epsilon_{\mathcal{N}} \overset{\mathcal{N}}{=} 0$, which is replaced by the third restriction above. Since θ is intrinsic and k depends on perpendicular derivatives of the metric, the last condition is of Robin type. Including (5.3.54) through a first-order null extension, θ^{Conf} vanishes on-shell of (5.3.58), and (5.3.44) gives

$$\underset{\leftarrow}{\theta}^{\text{EH}} = -\delta\ell^{\text{Conf}} + d\vartheta^{\text{EH}}. \tag{5.3.59}$$

With respect to the general family (5.3.50), θ^{Conf} corresponds to $b = 0$ and $c = 1$, and it is easy to see that it is the only possibility that would allow conservative boundary conditions with $\delta\epsilon_{\mathcal{N}} \neq 0$.

The interesting remark is that this boundary Lagrangian is class-III invariant, hence geometric and not ambiguous. Changing polarization resolves the problem of ambiguity of the boundary Lagrangian without adding counter-terms, or any of the additional restrictions $\delta l_{\mu} = 0$ or $\delta k = 0$ considered above. If we do include the $\delta k = 0$ restriction, then the conformal and Dirichlet polarization boil down to the same boundary conditions, and their boundary Lagrangians (5.3.46) and (5.3.40) are indeed the same up to corner terms.

Next, we look at the corner terms. ϑ^{EH} is the same as before, therefore we still have (5.3.55) when looking at the joint between a null and a space-like boundary. This is no longer a total variation, because (5.3.58) does not imply $\delta\epsilon_S \overset{\mathcal{N}}{=} 0$. Therefore, to have a well-defined variational principle we need to add the corner boundary condition

$$\delta\epsilon_S \overset{S}{=} 0. \tag{5.3.60}$$

The need for an additional condition on top of (5.3.58) seems reasonable because the expansion is the derivative of the 2d metric, hence one is missing an initial datum when providing boundary data in terms of the expansion. Upon doing so, the required action

corner terms in the variational principle are the same as in the Dirichlet case, e.g. (5.3.56). Therefore even though these boundary conditions eliminate the ambiguity of the null boundary Lagrangian, they do not eliminate the ambiguity of the corner terms, at least in so far as they are completed with (5.3.60). If we further add $\delta l_\mu = 0$ then as before we remove the need of corner terms between null boundaries.

The boundary Lagrangian ℓ^{Conf} is spacetime exact, hence it could also be absorbed into a modified ϑ. In this case the conformal boundary conditions require no boundary Lagrangian at all. This however does not change the ambiguity of the corner terms, since one is adding a non-ambiguous term to the existing ambiguous one.

Having found a polarization with a class-III invariant boundary Lagrangian will be very useful for the construction of charges in the covariant phase space, which is what we turn to next. In that context, having corner terms in the action principle which are ambiguous is not important, because these don't enter neither the expression for the Noether current nor that for the Hamiltonian generator. However from the point of view of the variational principle one may be interested in going further, and see whether it exists a completely unambiguous variational principle including the corner terms. A possibility would be to replace (5.3.60) with its conjugated corner variable, namely set $\delta\hat{\beta} \overset{S}{=} 0$. No corner term in the action would then be needed. To that end, one should first study whether $\delta(\theta + 2k) \overset{S}{=} 0$ has any bearing on $\hat{\beta}$. We leave further investigations of this idea for future work.

Leaky boundary conditions and boundary symmetry groups

6

In this chapter, we perform a detailed study of the covariance properties of the gravitational symplectic potential on a null boundary, and of the different polarizations that can be used. Using Sachs' framework for constraint-free data on a null hypersurface, we draw a distinction between physical and gauge degrees of freedom. This separation aids in determining which conservative boundary conditions can be relaxed to permit the flux of physical degrees of freedom through the boundary. These leaky boundary conditions are instrumental in the construction of a covariant phase space that captures the dynamics of gravitational systems when there is radiation through the lateral boundary. The symplectic potential's nature is influenced by both the intrinsic and extrinsic geometry of the hypersurface and non-geometric parameters like the the extension of the normal. We discuss how different splits can introduce dependencies on the scaling of the normal and the choice of the rigging vector. These dependencies manifest in the form of field anomalies in the covariant phase space, with the exact nature of the anomalies contingent on the selected leaky boundary conditions. We study the symmetry groups that arise with different phase space prescriptions, and determine the fields that have anomalous transformations. The results presented in this chapter have been published in [54]. A largely overlaping analysis has simultaneously been published in [105].

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Sachs' identification of constraint-free data on a null hypersurface [106] can be used to cut a distinction between physical and gauge degrees of freedom, and in turn understand which part of the conservative boundary conditions can be relaxed in order to allow flux of physical degrees of freedom through the boundary. Such leaky boundary conditions are useful in order to construct a covariant phase space that describes the evolution of dynamical gravitational systems using the flux defined by the symplectic potential. As we have seen, the symplectic potential depends on the intrinsic and extrinsic geometry of the hypersurface, as well as on non-geometric quantities such as the choice of extension of the normal. Furthermore any split like (7.1.1) can introduce a dependence of the individual terms on the scaling of the normal and on the choice of rigging vector. This dependence shows up in the covariant phase space in the form of anomalous transformations of the fields, or anomalies for short. The exact nature of the anomalies depends on the leaky boundary conditions chosen. Different types have been explored in the literature, leading

to different residual gauge transformations and thus different boundary symmetry groups. In this section we characterize the different symmetry groups and compute their anomalies. In the following section we will see how the anomalies enter the gravitational flux and the Noether charges, and study how the flux-balance laws are reorganized when changing polarization of the symplectic potential.

Let us first talk about the variations that enter the symplectic potential. The shear and expansion are determined by the induced metric, and the twist is also expected to be determined by the induced metric on-shell of the Einstein's equations. Therefore, the symplectic potential contains substantially only four independent variations: $\delta\gamma_{\mu\nu}$, δl^μ, δl_μ and δk. The radiative data are contained in the first variation, so that one should definitely be left free in leaky boundary conditions allowing for gravitational flux. The question is then what to do with the remaining three.

On the hypersurface, we can always choose coordinates such that the tangent vector l^μ has the simple form (2.2.28). Therefore $\delta l^\mu \overset{\mathcal{N}}{=} 0$ can be interpreted as a restriction of the phase space to variations preserving this choice of coordinates. Furthermore, l_μ is now metric dependent, unless we fix the Φ coordinate to have $g_{\Phi\lambda} = -1$, or equivalently we take the 'canonical' normalization. With the first option, $\delta l_\mu \overset{\mathcal{N}}{=} 0$ is also a restriction of the phase space preserving a certain choice of coordinates. Both coordinate choices are always achievable and don't restrict the physics of the system: they can be taken as part of the universal structure. Therefore it seems reasonable to impose both restrictions, and this is indeed the conclusion reached through the careful analysis done in [36].

The situation with δk is a bit more subtle, because one may expect that having eliminated the variations coming from f, this is a genuine variation of the extrinsic geometry that contains physics. But as we see from (5.3.23), this variation captures the perpendicular derivative of $l_\mu l_\nu \delta g^{\mu\nu}$, whose vanishing means that the coordinates are affine. Hence restricting δk to vanish or not means that the symmetry group of the covariant phase space preserves or not affine coordinates, namely metrics in the form (2.2.31) as opposed to (2.2.29). And this seems merely a gauge statement.

This question was also left partially open in [36]. In the main body of the paper δk is fixed to vanish, on the account that any two metrics with a null hypersurface $\mathcal{N}$ can be made to have the same pair (l^μ, k) via a diffeomorphism on $\mathcal{N}$, suggesting that k should be taken as part of the universal background structure. Vanishing δk was also used in order to complete the Wald-Zoupas procedure and identify a non-ambiguous notion of charges. On the other hand, it was pointed out that the symplectic 2-form is not degenerate along any of the boundary diffeomorphisms. If one takes zero-modes of the symplectic 2-form with boundary terms included to be a definition of gauge, then every diffeomorphism of the boundary should be considered as a physical transformation. This motivates the investigation of an enlarged phase space in which k is allowed to vary, and

possibly even l^μ. As we will see, enlarging the phase space affects the symmetry groups and the associated transformations in field space, in particular their anomalies.

6.1 Anomalies and class-III invariance

The variations studied in Section 5.3.2 keep the boundary fixed and null: $\delta\Phi = 0$ and $l_\mu \delta l^\mu \overset{\mathcal{N}}{=} 0$, or $\delta g^{\Phi\Phi} \overset{\mathcal{N}}{=} 0$ in adapted coordinates. This means that Φ and $g^{\Phi\Phi}|_{\mathcal{N}}$ are fixed background structures, whereas the rest of the metric can be varied freely. The split between dynamical and background structures introduces a delicate aspect of anomalies in the construction of the phase space, as we reviewed in Sec. 4.3.1. The anomaly operator defined there coincides with the non-covariance for field-independent diffeomorphisms and for field-space scalar functionals. From this formula we see that anomaly-freeness means

$$\partial_\chi F \pounds_\xi \chi = 0. \tag{6.1.1}$$

Namely, F should either not depend on the background fields χ, or if it does, the symmetry group should be made only of isometries of χ. The symmetry group is typically required to preserve some universal background structure. If this is described by the background fields χ, then the symmetry group coincides with those diffeomorphisms that are isometries of χ, and there are no anomalies. But if the background structure is described equivalence classes of background fields, the situation is different, because isometries of the background structure don't need to be isometries of individual representative fields, which are the quantities entering (6.1.1). We also see that the notion of covariance given by matching Lie derivatives can be stated equivalently as

$$\partial_\chi F \pounds_\xi \chi = F \pounds_{\delta\xi} \phi. \tag{6.1.2}$$

It means that representatives of the equivalence class for which the symmetry group is not an isometry can still be allowed, provided it carries a specific field-dependence. Even though it seems natural to talk about covariance when the two Lie derivatives match, it is the notion of anomaly-freeness that carries the most direct interpretation in terms of independence from background structures. We will come back to this difference in Section 9.6 at the end.

When we apply this technology to a null boundary, the background fields are Φ and $g^{\Phi\Phi}|_{\mathcal{N}}$. Their anomalies are

$$\Delta_\xi \Phi = -\pounds_\xi \Phi = -\xi^\Phi, \qquad \Delta_\xi g^{\Phi\Phi} = -\pounds_\xi g^{\Phi\Phi} \overset{\mathcal{N}}{=} \xi^\Phi \partial_\Phi g^{\Phi\Phi} + 2 g^{\Phi a} \partial_a \xi^\Phi, \tag{6.1.3}$$

and vanish if we restrict attention to diffeomorphisms that satisfy $\xi^{\Phi} \overset{\mathcal{N}}{=} 0$. These are the tangent diffeomorphisms, and don't move the boundary. They can be parametrized as

$$\xi = \xi^a_{\mathcal{N}} \partial_a + \Phi \bar{\xi}^\mu \partial_\mu \in \mathrm{Diff}(\mathcal{N}), \qquad \xi \cdot l \overset{\mathcal{N}}{=} 0, \tag{6.1.4}$$

with $\bar{\xi}$ smooth on $\mathcal{N}$. We will refer to $\bar{\xi}^\mu$ as the *extension* of the symmetry vector outside of the boundary and into the bulk, and to the specific component $\bar{\xi}^{\Phi} \overset{\mathcal{N}}{=} -(f\Phi)^{-1}\xi \cdot l$ as the perpendicular extension. Since $\pounds_\xi g^{\Phi\Phi} \overset{\mathcal{N}}{=} 0$ for $\xi \in T\mathcal{N}$, we can write covariantly

$$\hat{\xi} := \int \pounds_\xi g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}}, \tag{6.1.5}$$

without the need to treat separately $g^{\Phi\Phi}$ and the dynamical components of the metric.

However, anomalies are present even for tangent diffeomorphisms if we have to deal with normal derivatives of the background fields. This is precisely the case at hand, since the pull-back of the symplectic potential depends on the normal 1-form. For a tangent diffeomorphism, we have

$$\Delta_\xi l_\mu \overset{\mathcal{N}}{=} -w_\xi l_\mu, \qquad \Delta_\xi l^\mu \overset{\mathcal{N}}{=} -w_\xi l^\mu, \qquad w_\xi := (\pounds_\xi - \delta_\xi) \ln f + \bar{\xi}^{\Phi}. \tag{6.1.6}$$

We see that the anomaly w_ξ depends on both the choice of normal representative, through the non-covariance of f, and on the diffeomorphism considered, through the perpendicular extension of the symmetry vector field, namely its Φ component. As far as both quantities are arbitrary, one can choose them so that anomalies are vanishing, for instance taking $f = 1$ and $\bar{\xi}^{\Phi} = 0$. However, while the choice $f = 1$ is always acceptable (but may not be the best choice to study a specific problem), $\bar{\xi}^{\Phi} = 0$ is not, because in most cases of interest this extension is fixed to a non-vanishing value determined by the parameters $\xi^a_{\mathcal{N}}$. These include the case of isometries, asymptotic symmetries at future null infinity, and it would exclude from the symmetries the possibility of a Killing vector, whose perpendicular extension is fixed and non-vanishing. More in general, asymptotic symmetries at future null infinity as well as on a physical null hypersurface and a non-expanding horizon all require to fix the perpendicular extension of the symmetry vectors non a non-vanishing value determined by the parameters ξ^a. We will review below why.

To give further intuition about the meaning of w_ξ, consider the case of a non-null boundary. We still have (6.1.3), hence anomalies only appear for quantities like the normal, once we restrict attention to diffeomorphisms that are tangent to the boundary. For an arbitrary normal, (6.1.6) is also still valid. But if we choose a unit-norm normal, then w_ξ vanishes identically. If we recall that a unit-norm normal has the property of being independent of the embedding of the boundary, we see that anomalies arise not so much from the presence of a boundary, but rather from a foliation-dependence in its description. In other words, the equivalent of class-III invariance in the time-like case is

achieved through invariance under Φ reparametrization only, because f is fixed. Coming back to the case of a null boundary, there is no foliation-independent description, and no canonical normalization for the normal, hence anomalies become relevant.

Let us point out that (6.1.6) is the main anomaly that we have to worry about, but not the only one. A second source of anomalies is the rigging vector, which is also a non-dynamical background quantity. Its anomalies are less important in the end, but will appear in some intermediate calculations and it is useful to track them as well. For an arbitrary choice of rigging,

$$\Delta_\xi n_\mu = w_\xi n_\mu + Z_\mu, \qquad Z \cdot l = Z \cdot n = 0, \tag{6.1.7}$$

where the proportionality to w_ξ of the first term follows from $l \cdot n = -1$, and the vector Z parametrizes the rigging anomaly. Its explicit form depends on the specific choice of n, and we can leave it unspecified in the following. For instance, the projector $\gamma_{\mu\nu}$ is manifestly class-III invariant but not class-I invariant. It has an anomaly determined by (6.1.7) as

$$\Delta_\xi \gamma_{\mu\nu} = 2 l_{(\mu} Z_{\nu)}. \tag{6.1.8}$$

The anomalies (6.1.6) and (6.1.7) correspond to the non-invariance under infinitesimal class-I and class-III transformations with parameters $A = e^{-w_\xi} \simeq 1 - w_\xi$ and $a = m \cdot Z$. Further anomalies appear for quantities with a non-vanishing spin weight, since these depend on the background structure m associated with the choice of n. However these will not be relevant for us, we will always compute anomalies of quantities that can be expressed in terms of l and n alone. For these, it is easy to prove that class-I and class-III invariance implies anomaly-free, see Appendix B.1.3.

A subtle point to highlight is that anomaly-freeness requires class-III invariance in the general sense of a field-dependent rescaling. For instance, $n^\mu \delta l_\mu$ is manifestly class-III invariant if the rescaling is field-independent, but not otherwise: $n^\mu \delta l_\mu \to n^\mu \delta l_\mu - \delta \ln A$. It is in fact anomalous,

$$\Delta_\xi (n^\mu \delta l_\mu) = -\Delta_\xi \delta \ln f = \delta w_\xi - w_{\delta\xi}. \tag{6.1.9}$$

We remark for later use that this specific anomaly vanishes if the variations are restricted by $\delta l_\mu \overset{\mathcal{N}}{=} 0$, since ξ is tangent to $\mathcal{N}$. But it vanishes also if $\delta l_\mu \neq 0$ provided that $\delta l^\mu \overset{\mathcal{N}}{=} 0$. This may not look obvious, but it follows from (6.1.6), and can be deduced also looking at (5.3.27).

Similarly, a quantity that is only partially class-III invariant like $\bar{k}\epsilon_\mathcal{N}$ (we recall that it is independent of f but not of invariant under reparametrizations of Φ) is also anomalous,

$$\Delta_\xi (\bar{k}\epsilon_\mathcal{N}) = -\pounds_l \bar{\xi}^\Phi \epsilon_\mathcal{N}. \tag{6.1.10}$$

6.2 Anomalies of the boundary Lagrangians

As a first application of this formalism, we compute the anomaly of the boundary Langrangians (5.3.48). Using (6.1.6) we find

$$\Delta_\xi \epsilon_{\mathcal{N}} = w_\xi \epsilon_{\mathcal{N}}, \qquad \Delta_\xi \theta = -w_\xi \theta, \qquad \Delta_\xi \epsilon_S = 0, \qquad \Delta_\xi k = -(\pounds_l + k) w_\xi. \tag{6.2.1}$$

From the last one, we also deduce that

$$\pounds_l w_\xi = (\pounds_\xi - w_\xi) k - \delta_\xi k. \tag{6.2.2}$$

Adding up these contributions we have

$$a_\xi^{(\mathrm{b,c})} := \Delta_\xi \ell^{(\mathrm{b,c})} = b \pounds_l w_\xi \, \epsilon_{\mathcal{N}} = b \, dw_\xi \wedge \epsilon_S, \tag{6.2.3}$$

where in the last equality we used (2.2.11) and the fact that w_ξ is only defined on $\mathcal{N}$. As expected, any member with $b \neq 0$ is not class-III invariant and it is anomalous. The family of covariant boundary Lagrangians is (5.3.48) with $b = 0$ and c arbitrary. This includes in particular the Conf and CFP choices. In the latter case, notice that the statement about covariance is valid also if $\delta k \neq 0$. The anomalous Lagrangians include the Dirichlet choice with $b = 2$. Such anomalies would only vanish in the special case $\pounds_l w_\xi = 0$. Looking at (6.2.2), we see that this would occur for instance if the phase space is restricted to satisfy $k = \delta k = 0$. Finally for the excluded member (5.3.42), we have

$$a_\xi^{\bar{\mathrm{D}}} := \Delta_\xi \ell^{\bar{\mathrm{D}}} = 2 \pounds_l \bar{\xi}^{\Phi} \, \epsilon_{\mathcal{N}}, \tag{6.2.4}$$

which captures explicitly its dependence on reparametrizations of Φ.

This result shows that Dirichlet boundary conditions require an anomalous boundary Lagrangian, whereas conformal boundary conditions admit a covariant one. The situation is the same if we move the expansion term to the corner and work with the primed boundary Lagrangians. In this case the conformal boundary Lagrangian vanishes and its covariance is obvious. Recalling the earlier discussion on ambiguities, we see that the anomaly here keeps track of the dependence of the Dirichlet boundary Lagrangian on non-geometric structures, hence of its ambiguity. This comes from its dependence on the inaffinity and failure of being class-III invariant, and the problem is resolved switching to the conformal polarization instead.

Lagrangian anomalies appear in the study of central extensions of charge algebras. As shown in [85, 40], the Lagrangian anomaly can be used to compute the cocyle that appears on the right-hand side of the Barnich-Troessaert bracket [107]. This approach was applied in [85] to investigate whether one can obtain a central charge that would be relevant to understand black hole entropy from super-rotations as proposed in [108]. It

was found that one can indeed reproduce the functional dependence of the entropy on the horizon's area, but with a wrong numerical factor. The right numerical factor would require as boundary Lagrangian $k\epsilon_{\mathcal{N}}$ instead of $2k\epsilon_{\mathcal{N}}$.[1]

6.3 Anomalies of the symplectic potentials

Next, we look at the anomalies of the symplectic potential. The standard symplectic potential θ^{EH} is manifestly anomaly-free, since it depends only on the metric and its derivatives. So does its pull-back, since the anomaly operator commutes with taking the pull-back for tangent diffeomorphisms. Decomposing it as in (5.3.19) introduces the background structure given by the reference NP tetrad used, which captures the choice of normal representative and of rigging vector. This step does not introduce anomalies, since as we proved (5.3.19) is both class I and III invariant.

Anomalies can instead appear in the preferred choice of θ'. From (7.1.1) we have

$$0 = \Delta_\xi \theta' - \Delta_\xi \delta\ell + d\Delta_\xi \vartheta. \tag{6.3.1}$$

Using this formula we can easily compute the anomaly for the family (5.3.48). The corner term is (5.3.24) in all cases, and its anomaly is given by

$$\Delta_\xi \vartheta^{\text{EH}} = 2\Delta_\xi(n^\mu \delta l_\mu)\epsilon_S. \tag{6.3.2}$$

Using this and (6.2.3),

$$\Delta_\xi \theta^{(b,c)} = b\, dw_\xi \wedge \delta\epsilon_S + (b-2)d\Delta_\xi(n^\mu \delta l_\mu) \wedge \epsilon_S - 2\theta\Delta_\xi(n^\mu \delta l_\mu)\epsilon_{\mathcal{N}}. \tag{6.3.3}$$

The relation of this formula to the lack of class-III invariance is straightforwardly obtained with the replacement $\ln A = -w_\xi$.

For conservative boundary conditions with $\delta\epsilon_S = 0$, the anomaly vanishes for $b = 2$ (the case of Dirichlet boundary conditions) if $\theta = 0$, and for any b and any θ if $\Delta_\xi(n^\mu \delta l_\mu)$ vanishes, which we recall follows from either $\delta l_\mu \overset{\mathcal{N}}{=} 0$ or $\delta l^\mu \overset{\mathcal{N}}{=} 0$.

What if we want to impose leaky boundary conditions instead, with $\delta\epsilon_S \neq 0$? We need either b or dw_ξ to vanish. The family with $b = 0$ is covariant under a minimal set of restrictions:

$$\delta l^\mu \overset{\mathcal{N}}{=} 0 \quad or \quad \delta l_\mu \overset{\mathcal{N}}{=} 0 \qquad \Rightarrow \qquad \Delta_\xi \theta^{(0,c)} = 0. \tag{6.3.4}$$

[1] This alternative boundary Lagrangian corresponds to the boundary condition $\delta k = (\theta + k)\delta \ln \varepsilon$. They do not impose $\delta l^\mu = 0$ however, instead the super-rotations considered in [108, 85] stem from asymptotic symmetries of an auxiliary AdS_3 space that appears under a special coordinate transformation of the near-horizon geometry.

These are the most generic conditions that guarantee that the symplectic potential is anomaly-free for all ξ's, namely independent of the choice of normal representative. If $b \neq 0$, we can use

$$dw_\xi \wedge \delta\epsilon_S = \pounds_{\delta l} w_\xi \epsilon_{\mathcal{N}} + \pounds_l w_\xi \delta\epsilon_{\mathcal{N}}. \tag{6.3.5}$$

This vanishes if

$$\delta l^\mu = \delta k = k \overset{\mathcal{N}}{=} 0 \qquad \Rightarrow \qquad \Delta_\xi \theta^{(\mathrm{b,c})} = 0, \tag{6.3.6}$$

however with these conditions the terms in b drop out completely. We conclude that the only relevant case is $b = 0$.

We see that the conformal polarization is the only choice of *diagonal* symplectic potential that is anomaly-free upon imposing only the minimal condition that $\delta f = 0$. All other choices considered are either not in diagonal form, or require additional restrictions on the phase space. This makes the conformal polarization best suitable to study more general leaky boundary conditions without introducing anomalies, which is important for the integrability of Hamiltonian charges.

The anomaly of the symplectic potential can also be derived summing up the anomalies of each spin pair, which we report here for completeness. The shear tensor $\sigma^{\mu\nu}$ is neither class-I nor class-III invariant, and it carries both anomalies:

$$\Delta_\xi \sigma^{\mu\nu} = -w_\xi \sigma^{\mu\nu} + 2 l^{(\mu} \sigma^{\nu)\rho} Z_\rho \tag{6.3.7}$$

(whereas the NP scalar being rigging-independent only carries the first anomaly, $\Delta_\xi \sigma = -w_\xi \sigma$). Using this and (6.1.8) in the spin-2 pair of the symplectic potential, we find

$$\begin{aligned} \Delta_\xi(\sigma^{\mu\nu}\delta\gamma_{\mu\nu}\epsilon_{\mathcal{N}}) &= \Delta_\xi(\sigma^{\mu\nu}\epsilon_{\mathcal{N}})\delta\gamma_{\mu\nu} + \sigma^{\mu\nu}\epsilon_{\mathcal{N}}\Delta_\xi\delta\gamma_{\mu\nu} = 2 l^\mu \sigma^{\nu\rho} Z_\rho \delta\gamma_{\mu\nu}\epsilon_{\mathcal{N}} \\ &= \delta l^\mu (\theta Z_\mu - 2 Z^\rho \nabla_\nu l_\rho)\epsilon_{\mathcal{N}}, \end{aligned} \tag{6.3.8}$$

where we used $\Delta_{\delta\xi} g_{\mu\nu} = 0$ and the orthogonality properties of Z. Next, we have

$$\Delta_\xi \pi_\mu = 2(\partial_\mu + l_\mu \partial_n) w_\xi + 2 Z^\rho \nabla_\mu l_\rho - \theta Z_\mu + 2 l_\mu (Z^\rho n^\nu + n^\rho Z^\nu)\nabla_\nu l_\rho, \tag{6.3.9}$$

so the spin-1 pair has anomaly

$$\begin{aligned} \Delta_\xi(\pi_\mu \delta l^\mu \epsilon_{\mathcal{N}}) &= \Delta_\xi \pi_\mu \delta l^\mu \epsilon_{\mathcal{N}} - \pi \cdot l\,(\delta w_\xi - w_{\delta\xi})\epsilon_{\mathcal{N}} \\ &= (2\partial_\mu w_\xi + 2 Z^\sigma \nabla_\mu l_\sigma - \theta Z_\mu)\delta l^\mu \epsilon_{\mathcal{N}} + (2k - \theta)\Delta_\xi(n^\mu \delta l_\mu)\epsilon_{\mathcal{N}}. \end{aligned} \tag{6.3.10}$$

where we used $\pi \cdot l = \theta - 2k$. The last term in the RHS depends on δl_μ but vanishes for $\delta l^\mu \overset{\mathcal{N}}{=} 0$ as discussed below (6.1.9). Finally for the spin-0 part let us consider the cases of Dirichlet and conformal polarizations as examples.

$$-\Delta_\xi[(\theta + 2k)\delta\epsilon_{\mathcal{N}}] = -(\theta + 2k)\Delta_\xi(n^\mu \delta l_\mu)\epsilon_{\mathcal{N}} + 2\pounds_l w_\xi \delta\epsilon_{\mathcal{N}}. \tag{6.3.11}$$

Adding up the three contributions we recover (6.3.3) for $b = 2$,

$$\Delta_\xi \theta^{\mathrm{D}} = 2dw_\xi \wedge \delta\epsilon_S - 2\theta\Delta_\xi(n^\mu \delta l_\mu)\epsilon_{\mathcal{N}}. \tag{6.3.12}$$

Switching to conformal polarization,

$$\Delta_\xi[\delta(\theta + 2k)\epsilon_{\mathcal{N}}] = -(\theta + 2k)\Delta_\xi(n^\mu \delta l_\mu)\epsilon_{\mathcal{N}} - 2(\pounds_{\delta l} w_\xi + \pounds_l \Delta_\xi(n^\mu \delta l_\mu))\epsilon_{\mathcal{N}}. \tag{6.3.13}$$

In the last term we can use

$$\pounds_l \Delta_\xi(n^\mu \delta l_\mu))\epsilon_{\mathcal{N}} = d\Delta_\xi(n^\mu \delta l_\mu)) \wedge \epsilon_S. \tag{6.3.14}$$

Adding up we recover (6.3.3) for $b = 0$,

$$\Delta_\xi \theta^{\mathrm{Conf}} = -2d[\Delta_\xi(n^\mu \delta l_\mu) \wedge \epsilon_S]. \tag{6.3.15}$$

This derivation allows one to appreciate that the potential anomaly coming from the term δk present in θ^{Conf} cancels out with a contribution coming from the spin-1 term. The subtle point is that even though the anomaly of k does not vanish for $\delta l^\mu = 0$, the anomaly of $\delta k\, \epsilon_{\mathcal{N}}$ does.

6.4 Boundary symmetry groups

In this section we review the different boundary symmetry groups that have been considered in the literature, with emphasis on the different background structures kept fixed. We show how they can be derived in a simple way using δ_ξ and w_ξ. We highlight how each additional restriction on the variations affects the symmetry group, the extension of the symmetry vector fields, and the anomalies.

The minimal requirement that we consider is that the boundary should be a null surface, which restricts the variations to satisfy $l_\mu \delta l^\mu \overset{\mathcal{N}}{=} 0$. The residual diffeomorphisms that preserve the boundary and the condition that it is null must satisfy $\pounds_\xi \Phi \overset{\mathcal{N}}{=} 0$ and

$$l_\mu \delta_\xi l^\mu = l_\mu(\pounds_\xi l^\mu + \Delta_\xi l^\mu) \overset{\mathcal{N}}{=} l_\mu l_\nu \pounds_\xi g^{\mu\nu} = f^2 \pounds_\xi g^{\Phi\Phi} \overset{\mathcal{N}}{=} 0. \tag{6.4.1}$$

These equations are solved by $\xi^\Phi \overset{\mathcal{N}}{=} 0$. This equation is valid for an arbitrary diffeomorphism tangent to the boundary, therefore the symmetry group with this minimal

background structure is the full $\mathrm{Diff}(\mathcal{N})$. For later convenience, we parametrize an arbitrary tangent vector as

$$\xi = \tau(\lambda, x^A)\partial_\lambda + Y^A(\lambda, x^B)\partial_A + \dots \tag{6.4.2}$$

in affine coordinates.[2] The ellipsis here denotes the extension of the vector field off the hypersurface, which is for the moment arbitrary. The anomalies for this symmetry group are determined via (6.1.6) by the choice of f and extension component $\bar{\xi}^\Phi$. As explained earlier, one can use the freedom in choosing these two quantities to eliminate partially or completely the anomalies.

Next, we add the restriction that the tangent vector be fixed, $\delta l^\mu \overset{\mathcal{N}}{=} 0$, as done for instance in [51]. The residual diffeomorphisms must satisfy

$$\delta_\xi l^\mu \overset{\mathcal{N}}{=} 0. \tag{6.4.3}$$

The analysis of these three conditions can be split in two cases, corresponding to the vertical and the horizontal components:

$$n_\mu \delta_\xi l^\mu = n_\mu l_\nu \pounds_\xi g^{\mu\nu} - \delta_\xi \ln f \overset{\mathcal{N}}{=} 0 \qquad \Rightarrow \qquad \text{restricts the extension} \tag{6.4.4}$$

$$m_\mu \delta_\xi l^\mu = m_\mu \pounds_\xi l^\mu \overset{\mathcal{N}}{=} 0 \qquad \Rightarrow \qquad \text{restricts the allowed diffeos} \tag{6.4.5}$$

To understand the first condition as a restriction on the extension, observe that in adapted coordinates it contains the term $\partial_\Phi \xi^\Phi$. Hence this can be seen as an equation for the component $\bar{\xi}^\Phi$ of the extension, and leaves the symmetry group $\mathrm{Diff}(\mathcal{N})$ unchanged. To understand the second restriction, it is easiest to take a coordinate system (λ, Φ, x^A) with λ affine parameter, so that (2.2.31) holds. Then

$$\delta_\xi l^\mu = (\delta_\xi \ln f - \bar{\xi}^\Phi) l^\mu - f\partial_\lambda \xi^\mu \overset{\mathcal{N}}{=} 0 \quad \Rightarrow \quad \begin{cases} \mu = \Phi & \xi \in T\mathcal{N} \\ \mu = \lambda & \bar{\xi}^\Phi = \delta_\xi \ln f - \partial_\lambda \tau \\ \mu = A & \partial_\lambda Y^A = 0 \end{cases} \tag{6.4.6}$$

The λ component shows that in these coordinates the extension is restricted via a function of the time derivative of $\tau := \xi^\lambda$ and of $\delta_\xi f$.[3] Comparing with (6.1.6), we see that the anomalies for these residual diffeos read

$$w_\xi = \pounds_\xi \ln f - \dot{\tau}. \tag{6.4.7}$$

[2] Here τ is a free function and not the NP coefficient. NP notation will not be used in the rest of the thesis.

[3] It should be clear that the equation can not be solved taking f as a function of an arbitrary extension, because that would make the scaling of the normal dependent on the diffeomorphism considered.

They are thus determined by the parameters of the symmetry vector field plus the choice of f. The cross-section components A show that the tangential diffeomorphisms must be time-independent, $\dot{Y}^A = 0$. Namely they become $\mathrm{Diff}(S^2)$ super-rotations as in [109, 81]. On shell of (6.4.6), the symmetry vector fields (6.4.2) reduce to

$$\xi = \tau(\lambda, x^A)\partial_\lambda + Y^A(x^B)\partial_A - \Phi(\dot{\tau} - \delta_\xi \ln f)\partial_\Phi + \dots, \tag{6.4.8}$$

where the dots include the part of the extension left arbitrary.

These vector fields however do *not* close under the spacetime Lie bracket, see Appendix B.3 for a proof. It happens only if we require $\delta_\xi f = 0$, or if $\Delta_\xi f = 0$. The first option is a priori more general since it can be achieved without any further restriction on ξ simply requiring $\delta l_\mu \overset{\mathcal{N}}{=} 0$. Having done so, the vector fields are given by

$$\xi = \tau\partial_\lambda + Y^A\partial_A - \Phi\dot{\tau}\partial_\Phi + \dots, \qquad \tau(\lambda, x^A), \quad Y^A = Y^A(x^B). \tag{6.4.9}$$

The extension $\bar{\xi}^\Phi$ no longer depends on f, and is entirely determined by the parameters of the symmetry vector fields at $\mathcal{N}$. These vector fields close under the Lie bracket, and span the subgroup

$$G^{\mathrm{aST}} := \mathrm{Diff}(S) \ltimes \mathbb{R}^{\mathcal{N}} \subset \mathrm{Diff}(\mathcal{N}). \tag{6.4.10}$$

Here aST stands for 'arbitrary time-dependent super-translations'. The semi-direct product structure follows from (B.3.2), and it means in particular that the identification of the parameters with Y^A and τ is not canonical, but relies on the choice of affine coordinates we made. This is a situation familiar from the BMS and CFP groups.

There is also another way to understand the importance of adding the condition $\delta l_\mu \overset{\mathcal{N}}{=} 0$. Without it, the four conditions (6.4.1) and (6.4.6) imply only three restrictions of the metric variations, because varying f does not affect the metric. Therefore the residual diffeomorphisms (6.4.8) do not correspond to a complete gauge fixing like (2.2.29). Imposing $\delta l_\mu \overset{\mathcal{N}}{=} 0$ turns the four conditions into four conditions on the metric variations, given by:

$$\delta l^\mu = \delta l_\mu \overset{\mathcal{N}}{=} 0 \quad \Rightarrow \quad l^\mu \delta g_{\mu\nu} \overset{\mathcal{N}}{=} 0. \tag{6.4.11}$$

Notice that the last equation is class-III invariant, hence the symmetry group satisfying $l^\mu \delta_\xi g_{\mu\nu} \overset{\mathcal{N}}{=} 0$ depends only on the equivalence class of normals and not on a choice of representative. Diffeomorphisms preserving this condition satisfy $l^\mu \pounds_\xi g_{\mu\nu} = 0$. In affine coordinates, and restricting ξ to be tangential, we get

$$\xi^\Phi \overset{\mathcal{N}}{=} 0, \qquad \partial_\Phi \xi^\Phi + \partial_\lambda \xi^\lambda \overset{\mathcal{N}}{=} 0, \qquad \partial_\lambda \xi^A \overset{\mathcal{N}}{=} 0, \tag{6.4.12}$$

that coincide with the restriction of (6.4.6) to $\delta_\xi f = 0$.[4]

Let us now see the effect of adding $\delta k = 0$ on top of the previous conditions. From (5.3.23) we have

$$\delta_\xi k = -\frac{1}{2} n^\mu \nabla_\mu (l_\nu l_\rho \pounds_\xi g^{\nu\rho}) \overset{\mathcal{N}}{=} 0. \tag{6.4.13}$$

This equation involves the first derivatives off the hypersurface of the metric. What it does is to further restrict the diffeomorphisms to preserve the condition of affine coordinates, namely the metric in the form (2.2.31), as opposed to the more general form (2.2.29). It is easy to see that in affine coordinates the equation simplifies to

$$\ddot{\tau} = 0. \tag{6.4.14}$$

This means that we can write $\tau = T(x^A) + \lambda W(x^A)$ in terms of a supertranslation with parameter T and a Weyl transformation with parameter W. We have thus recovered the symmetry group of [36],[5]

$$G^{\text{CFP}} := (\text{Diff}(S) \ltimes \mathbb{R}^S_W) \ltimes \mathbb{R}^S_T, \tag{6.4.15}$$

with vector fields

$$\xi = T\partial_\lambda + Y^A \partial_A + W(\lambda\partial_\lambda - \Phi\partial_\Phi) + \dots. \tag{6.4.16}$$

As for the anomaly, this is still given by (6.4.7), now

$$w_\xi = T\partial_\lambda \ln f + W(\lambda\partial_\lambda \ln f - 1) + Y^A \partial_A \ln f. \tag{6.4.17}$$

To be precise, this is the symmetry group if $\mathcal{N}$ is complete. If it is semi-complete instead, super-translations must be dropped because they do to preserve the boundary of $\mathcal{N}$ [36]. This happens for instance if $\mathcal{N}$ is the boundary of a causal diamond [110] or a light-cone.

If the covariant phase space is restricted to describe only NEHs, then the background structure can be strengthen even more to allow for a constant rescaling only of the normal, and then the group is restricted to constant W's [45],

$$G^{\text{NEH}} := (\text{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^S_T) \times R_W. \tag{6.4.18}$$

Let us comment on the method we used to derive the symmetry groups G^{aST} and G^{CFP}. This was based on identifying the diffeomorphisms that preserve the variations required to vanish, as opposed to the more common approach in the literature that

[4] Notice that it is necessary to restrict upfront to tangential diffeomorphisms, otherwise one obtains the larger set of solutions with $\xi^\Phi \overset{\mathcal{N}}{=} f(x^A)$. We also point out that the weaker set $l^\mu \delta g_{\mu \underset{\leftarrow}{\nu}} = 0$ misses $\partial_\lambda \xi^\mu + \partial_\Phi \xi^\Phi = 0$, namely (6.4.4).

[5] The same group appears also at future null infinity under generalized fall-off conditions compatible with finiteness of the symplectic potential [46], and was in that context called BMSW group.

consists on identifying the isometries of the background structure. But it is easy to prove the equivalence of our method in this context. From (6.1.6), we see that $\delta_\xi l^\mu \overset{\mathcal{N}}{=} 0$ is equivalent to $\pounds_\xi l^\mu \overset{\mathcal{N}}{=} w_\xi l^\mu$, namely the diffeomorphisms that preserve the equivalence class $[l^\mu = A l^\mu]$. Same story for $\delta_\xi l_\mu \overset{\mathcal{N}}{=} 0$. For $\delta_\xi k \overset{\mathcal{N}}{=} 0$, we see from (6.2.1) that is equivalent to $\pounds_\xi k \overset{\mathcal{N}}{=} w_\xi(k + \pounds_l \ln w_\xi)$, namely the diffeomorphisms that preserve the equivalence class (2.2.8). The three equations for the Lie derivatives of l^μ, l_μ and k are indeed the conditions used in [36].

We also remark the importance of using affine coordinates in order to solve for the vector fields that satisfy the phase space restrictions. The conditions (6.4.3) and (6.4.22) are in fact complicated in arbitrary coordinates, and boil down to simple statements about time-independence in affine coordinates. More importantly, the ξ's solving these equations in arbitrary coordinates depend explicitly on the metric, and can be characterized in a metric-independent way only in affine coordinates. What makes affine coordinates special is that they corresponding to a gauge fixing whose preservation coincides with preserving the background structure. Otherwise one cannot describe the symmetry vector fields in a universal way, and must work with field-dependent diffeomorphisms.[6]

How about the Robin-type condition (5.3.58)? In this case, (6.4.13) is replaced by

$$\delta_\xi k = -\frac{1}{2}\pounds_l(\nabla \cdot \xi), \tag{6.4.19}$$

which in affine coordinates becomes

$$\ddot{\tau} = \frac{1}{2}\partial_\lambda(\tau\theta + D_A Y^A). \tag{6.4.20}$$

The novelty with respect to the previous cases is the metric-dependence of the equation in affine coordinates. For instance the RHS vanishes for a NEH, but not in general. We will study an explicit non-zero solution below. We don't know the general solution to this equation, but it is clear that it will be a metric-dependent function $\tau = \tau(\lambda, T, W, Y; g)$, where $T(x^A)$ and $W(x^A)$ are integration constants. The space of parameters is the same as (6.4.15), however without the explicit solution we cannot even check whether they close under the Lie bracket and form a subgroup of diffeomorphisms. Working with these

[6] This observation should be contrasted with the analysis in the CFP paper, where the vector fields where characterized in a metric-independent way independently of the choice of coordinates. We believe that the reason for this is that their characterization is done directly in terms of intrinsic quantities on the hypersurface only, and therefore in a metric-independent way (as shown by (6.4.11), it is only by looking at spacetime restrictions that the symmetry group can be seen as preserving a metric gauge-fixing). We also remark that when they construct a spacetime diffeomorphism representative of the symmetry, they *define* it to match the intrinsic diffeomorphism when restricted to $\mathcal{N}$, see their (5.6). We can do the same here: once the metric-dependent vector fields are found in arbitrary coordinates, we can do an intrinsic diffeomorphism on $\mathcal{N}$ that maps them to the metric-independent one in affine coordinates.

Robin boundary conditions one seems therefore to be forced to deal with field-dependent diffeomorphisms, which complicates things significantly.

We have shown what happens when one implements one by one the vanishing of $n_\mu \delta l^\mu$, δl^μ, δl_μ and δk, and how it determines the symmetry groups $\mathrm{Diff}(\mathcal{N})$, G^{aST} and G^{CFP} and their anomalies. We proceeded in this specific order, because it is the one that appears the most useful to us, but with the same method one can consider any mixture of partial implementations. Let us briefly comment on a few of these partial alternatives.

The condition (6.4.14) can be derived also without imposing the condition $\delta l_\mu \overset{\mathcal{N}}{=} 0$. To see that, we start from the general formula (5.3.22). Restricting to $\delta l^\mu \overset{\mathcal{N}}{=} 0$, we have

$$\delta k = \frac{1}{2} l^\mu l^\nu n^\rho \nabla_\rho \delta g_{\mu\nu} - n^\mu l^\nu l^\rho \nabla_\rho \delta g_{\mu\nu}. \tag{6.4.21}$$

The residual diffeomorphisms preserving this condition must satisfy

$$\delta_\xi k = l^\mu l^\nu n^\rho (R^\sigma{}_{\mu\nu\rho} \xi_\sigma - \nabla_\mu \nabla_\nu \xi_\rho) \overset{\mathcal{N}}{=} 0. \tag{6.4.22}$$

The term in bracket can be recognized as a property of a Killing vector, but the allowed ξ's are here more general since only a specific scalar contraction of that term is being imposed to vanish, and on the hypersurface only. The ξ's solving this equation are complicated functions of the metric in general, but in affine coordinates it gives back (6.4.14). As explained earlier though, in the extended phase space with $\delta l_\mu \neq 0$ the vector fields don't close under the spacetime Lie bracket, and do not correspond to the residual gauge fixings preserving (2.2.31).

Suppose now that we require $\delta l_\mu \overset{\mathcal{N}}{=} 0$ and/or $\delta k = 0$ without fixing l^μ. Imposing $\delta_\xi l_\mu \overset{\mathcal{N}}{=} 0$ can only be solved if $\delta f = 0$, which imposese no restriction on the symmetry vector fields. Without $n_\mu \delta_\xi l^\mu = 0$ the extension $\bar{\xi}^\Phi$ is left arbitrary, and so is the anomaly which is given by the general formula (6.1.6). Therefore the charges depend not only on the parameters of the symmetry on $\mathcal{N}$, but also on $\bar{\xi}^\Phi$. The relevant symmetry vector fields are thus $\xi^a \partial_a + \Phi \bar{\xi}^\Phi \partial_\Phi$. They close under the Lie bracket and span the subgroup $\mathrm{Diff}(\mathcal{N}) \otimes \mathbb{R}$ parametrized by four free functions on $\mathcal{N}$. The physical relevance of a symmetry group not intrinsic to the boundary is unclear to us. It means in particular that the charges can have arbitrary values even though the restriction of the vector field to the boundary vanishes. Therefore this enlarged phase space appears not to provide a good handle for the study of dynamical geometric properties of a null hypersurface.[7]

Preserving k on top of $\delta_\xi l_\mu \overset{\mathcal{N}}{=} 0$ leads to

$$\delta_\xi k \overset{\mathcal{N}}{=} 2 l_\rho \nabla^{(\mu} \xi^{\rho)} n^\nu \nabla_\mu l_\nu - l^\mu l^\nu n^\rho \nabla_\rho \nabla_\mu \xi_\nu. \tag{6.4.23}$$

[7] A different viewpoint is taken in [105], where it is argued that this additional free parameter should be taken seriously as a characterization of the near-null hypersurface geometry.

This equation is not class-III invariant, therefore the diffeos solving this equations depend on the choice of normal, and cannot be characterized in purely geometric terms. For instance for a vertical diffeo $\xi = \tau\partial_\lambda$, we get $k(\pounds_l + k)\tau/f = 0$, which is solved by any τ for $k = 0$, and by $\dot{\tau} = (\dot{f} - k)\tau/f$ for $k \neq 0$. The situation is similar if we preserve k without preserving l_μ, just the above equation become more complicated, and remains not class-III invariant. We conclude that imposing $\delta k \overset{\mathcal{N}}{=} 0$ without $\delta l^\mu \overset{\mathcal{N}}{=} 0$ leads to symmetry groups which depend on structures unrelated to the geometry of $\mathcal{N}$. Related boundary conditions and symmetry groups appear also in [111, 112, 113, 87, 114].

Charge prescriptions

7

In this chapter, we explore two widely-recognized prescriptions for defining charges: the Improved Noether Charge Prescription [34, 40, 36, 115, 94] and the Wald-Zoupas Prescription [33]. Both of these methods provide frameworks for solving the integrability problem by choosing a charge-flux split, an issue we discussed in Section 4.3.3. After outlining the basics of each prescription, we put forth a new comparison between them. Specifically, we demonstrate that the Wald-Zoupas approach is applicable to gravitational charges even when anomalies and field-dependent diffeomorphisms are present, but only if there is a particular relationship between these elements. We use BMS symmetries as a case study to highlight the kinds of anomalies that are permissible within this framework. These allowable anomalies are revealed to correspond to 'soft terms' in the charges. We also investigate whether the WZ can be aligned with the improved Noether Charge. We find that this alignment is possible under certain conditions defined by a specific differential equation. Our findings clarify why the WZ prescription remains effective even in the face of the anomalous behavior associated with BMS transformations. This should serve as a useful reference for those seeking to navigate the complex literature concerning surface charges. The results of this chapter have been published in [35].

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In order to understand the prescriptions, it is important to recall that the covariant phase space constructed in Chapter 4 is not unique, because of the existing freedom in choosing the symplectic potential. This freedom is two-fold [41]: first, we can add an arbitraty spacetime-exact 3-form $d\alpha$ to a θ satisfying (4.2.7) without spoiling its relationship with the Lagrangian; and second, we can add a boundary term to the Lagrangian, without affecting the field equations or the symplectic structure. These two cohomological ambiguities (one in spacetime and one in field space) are summarized by

$$L \to L + dY, \qquad \theta \to \theta + d\alpha + \delta Y, \qquad \omega \to \omega + \delta d\alpha. \tag{7.0.1}$$

In particular, the Noether charge defined via (4.3.12) depends on the choice of representative, and transforms as

$$q_\xi \to q_\xi + i_\xi Y + I_\xi \alpha. \tag{7.0.2}$$

For a given L, we refer to the choice of θ obtained simply removing d as the 'bare' choice. This choice follows if the symplectic potential is defined using Anderson's homotopy

operator [83, 116, 117, 10, 14], which is the approach taken in [40]. Another mathematical way to eliminate the freedom is to require the Noether current (4.3.12) to be weakly vanishing [10]. These choices are convenient for bookkeeping and can always be made, but they are *not needed* to obtain the results used and derived here. In the rest of the chapter, we will consider arbitrary θ's, without any a priori mathematical prescription.

7.1 Improved Noether charge from phase space polarization

The main idea that Wald and Zoupas had is that the ambiguities in the definition of the charges should be resolved by deciding under which physical requirements the charges are to be conserved. Mathematically, this can be controlled trading the initial symplectic potential θ (be it the bare one or any other chosen one) for a symplectic potential such that its pull-back on the lateral boundary $\mathcal{B}$ vanishes in the subset of the phase space corresponding to a desirable physical requirement, such as a choice of conservative boundary conditions, or a choice of stationarity conditions. In practice, one takes the pull-back to the lateral boundary and decomposes it as follows,

$$\underset{\leftarrow}{\theta} = \theta' - \delta\ell + d\vartheta, \tag{7.1.1}$$

where θ' is required to be in the form $p\delta q$ for some choice of polarization of the phase space. The new θ' corresponds to $L' := L + d\ell$, namely a theory with the same field equations, and is equivalent to θ under the freedom (7.0.1). The idea of changing from the initial θ to a physically motivated θ' dates back to [32] and [33], was generalized in [118] and [34] to include the corner potential ϑ, and takes a central role in various follow-up works [14, 85, 94, 40, 38].

The terms ℓ and ϑ appearing above are produced by the manipulations needed to put (the pull-back of) θ in the chosen θ' form. The explicit form of ϑ depends also on the representatives chosen for θ and θ'. Since we require $\theta' = p\delta q$, ℓ is manifestly the boundary term to be added to the Lagrangian to have a well-defined variational principle with those boundary conditions. However (7.1.1) does not identify a unique ℓ, since the condition is still satisfied under the replacement

$$(\ell, \vartheta) \to (\ell + dc, \vartheta + \delta c). \tag{7.1.2}$$

Therefore, for a given representative θ, one can compute a unique ϑ only once a choice for θ' *and* ℓ is made.[1] For instance, a non-vanishing ϑ occurs for Dirichlet boundary

[1] Unique ϑ up to addition of exact 2-forms, but these will be irrelevant in the following since we will only look at compact corners. Accordingly, we will ignore all $2d$-exact forms in the rest of the paper. Notice also that fixing both θ' and ℓ can be equivalently seen as fixing θ' and ϑ. This is the viewpoint taken

conditions if L is the Einstein-Hilbert Lagrangian and ℓ is the Gibbons-Hakwing-York term, as established as early as [119]. As observed there, the resulting ϑ shifts the symplectic 2-form,

$$\omega' = \delta\theta' = \omega - d\delta\vartheta. \tag{7.1.3}$$

We want to characterize the physical situations in which the new symplectic potential vanishes on the lateral boundary $\mathcal{B}$, namely

$$\theta' \overset{\mathcal{B}}{=} 0. \tag{7.1.4}$$

Since it is in the form $\theta' = p\delta q$, we can distinguish two cases, depending on whether it is δq or p to vanish, and which we name following [33].

Case I: We impose *conservative* boundary conditions $\delta q \overset{\mathcal{B}}{=} 0$. The new symplectic 2-form ω' also vanishes on the lateral boundary,

$$\underleftarrow{\omega} \overset{\mathcal{B}}{=} 0, \tag{7.1.5}$$

and therefore is preserved between the initial and final space-like hypersurfaces. This makes the system conservative, hence the name. In other words, the system is in case I within each cotangent space at fixed q, but not for trajectories that vary both p and q.

Clearly, different choices of conservative boundary conditions are possible, corresponding to different choices of polarizations, and this turns out to affect the charges. This will be the topic of the next two chapters.

Case II: There exist solutions for which $p \overset{\mathcal{B}}{=} 0$. They provide a notion of *stationary* backgrounds, whose precise nature depends on the form of p, namely on the polarization chosen. We can distinguish two situations, one in which all p's vanish, and one in which only some vanish, and (7.1.4) is achieved by the vanishing of the complementary δq's. Either way, the symplectic 2-form ω' is not conserved,

$$\underleftarrow{\omega} \overset{\mathcal{B}}{\neq} 0, \tag{7.1.6}$$

because there are no restrictions on the variations δp and at least some of the δq's.

Notice that one typically imposes some boundary conditions also in case II, weaker than the conservative ones, and needed to preserve a certain boundary structure of physical relevance, for instance in order to characterize graviational radiation. The boundary

in [38], where the chosen quantities are referred to respectively as boundary and corner (symplectic) fluxes. The freely choosable ϑ's have to be related by (7.1.2), just like the freely choosable θ's have to be related by (7.1.1).

structure shared by a certain class of metrics is referred to as their *universal structure*. The conservative boundary conditions of case I are of the same type that are used in the variational principle, while we refer to the generic, weaker set of boundary conditions of case II as radiative or *leaky*.

It should be stressed that we are making these characterizations with the goal of resolving the ambiguities in the charges, and not of restricting the phase space. Once the corresponding θ' is chosen, we compute the associated charges, and then we use them in the *full* phase space. This means that charges defined using conservative boundary conditions will not be conserved in the full phase space, and charges defining using a specific stationarity condition will not be conserved when evaluated around any other solution not respecting it. Clearly, charges constructed using the different perspectives of Case I and II but corresponding to the *same* polarization are equal and have equal properties.

The formulas for charges associated with the new symplectic potential θ' take exactly the same form as before, namely [40]

$$j'_\xi := I_\xi \theta' - i_\xi L' - a'_\xi \hat{=} dq'_\xi, \tag{7.1.7}$$

$$\not{d} h_\xi := -I_\xi \omega' \hat{=} d\left(\delta q'_\xi - i_\xi \theta' - q'_{\delta\xi} - A'_\xi\right), \tag{7.1.8}$$

where $L' = L + d\ell$ and $\omega' = \delta\theta'$. In other words, the formalism allows one to treat all choices on equal footing. Again, the infinitesimal Hamiltonian generator depends only on θ', whereas the Noether charge depends on θ' but also explicitly on the boundary Lagrangian ℓ via its anomaly: $(\theta', \ell) \mapsto q'_\xi$. In particular, q'_ξ depends on any corner term that may be present in the choice of ℓ, which is not visible from L' and θ'. The relation between the Noether charges associated to (θ', ℓ) and the initial ones is[2]

$$q'_\xi = q_\xi + i_\xi \ell - I_\xi \vartheta. \tag{7.1.9}$$

Keeping the primed notation is useful if we want to compare boundary-improved charges to specific bare charges. For instance, take L to be the Einstein-Hilbert Lagrangian, and θ its bare symplectic potential. Then q_ξ is the original Noether charge [83], given by the Komar formulas and their limitations. If we add the boundary Lagrangian ℓ given by the Gibbons-Hawking-York term and choose the Dirichlet polarization for θ', the improved Noether charges give the Brown-York formulas [32, 34], as we will show in

[2] The reader may notice a notational hiccup at this point. Logically, it would make more sense to denote the boundary Lagrangian ℓ', so that one can use ℓ to refer to whatever choice of corner was present in the initial q_ξ. Accordingly, one should add primes on both ℓ and ϑ on the right-hand side of (7.1.1), and following formulas. We choose not to do so and instead follow the notation of [40]. This allows us to keep the notation lighter, and also refer to that paper for all proofs. Notice also that the practical use we will make of the unprimed notation will be to specialize to the bulk covariant Lagrangian with no boundary term, so no confusion will arise as to what ℓ refers to.

Chapter 8. In [42] we referred to this prescription as Freidel-Geiller-Pranzetti formula, since we used the notation of [14], but given the number of authors contributing to these developments, it seems fair to simply talk about *improved Noether charges*. The improvement with respect to the original, 'bare' Noether charges, is two-fold. First, the Brown-York formulas give the correct ADM charges at spatial infinity [120], unlike the Komar formulas. Second, (7.1.9) can be made invariant under the cohomological ambiguities (7.0.1) [94, 38]. Indeed, if we require that the choice of polarization θ' is kept fixed under (7.0.1), we have $(\ell, \vartheta) \to (\ell - Y, \vartheta + \alpha)$, therefore even if q_ξ changes as in (7.0.2), q'_ξ is invariant. In other words, it is the prescription of working with a unique θ' that eliminates these ambiguities.

On the other hand, fixing θ' alone is not sufficient to obtain a unique charge, because as anticipated above, q'_ξ depends also on the boundary Lagrangian chosen. This can be seen explicitly observing that (7.1.9) is affected by the corner ambiguity (7.1.2), which leads to [38]

$$q'_\xi \to q'_\xi - \Delta_\xi c. \tag{7.1.10}$$

Therefore even if the cohomology ambiguities (Y, α) are fixed by the choice of θ', there is still an ambiguity in the charge if anomalies are present. This ambiguity is removed if one does not prescribe only θ' but also a specific choice of ℓ, thus fixing c.[3] At least in the case of the WZ prescription, what we will find is that the corner shift is not related to corner terms in the action principle, but rather in removing anomalies from the boundary Lagrangian.

Let us now discuss how the restriction (7.1.4) affects integrability, and the ensuing relation between the improved Noether charge and the Hamiltonian generator. Consider first the familiar case of no anomalies and $\delta\xi = 0$. When (7.1.4) holds, the improved Noether charge q'_ξ is conserved (this follows because the ξ's allowed in the covariant phase space are tangent to the boundary, and thus the second term in (7.1.7) vanishes taking the pull-back). The Hamiltonian generator (7.1.8) is integrable, and the Hamiltonian coincides with q'_ξ, up to constant terms in field space. Such terms can be fixed for instance looking at a reference solution [33], requiring the Hamiltonian charges to vanish there. Having established this, one can take the prescription of using the improved Noether charges in the full phase space. This prescription gives charges that by definition have the useful property of being conserved and Hamiltonian generators in the conservative or stationary subspace. Notice that this prescription is equivalent to defining the charges starting

[3] This may be taken as a suggestion that what matters to get unique charges is prescribing a specific action principle including boundary and corner terms, as pointed out in [94, 38]. However more work is needed in our opinion before this suggestion is borne out, because counter-examples exist, both ways. Going one way, one can think of the example of adding an anomalous corner term (hence changing the charge) but which is globally defined (hence not entering as corner terms in the action principle). Going the other way, the example of time-like boundaries with non-orthogonal corners reviewed below in Section 8.3 shows that there is no corner shift needed to get the BY charges corresponding to the WZ prescription even though there are corner terms in the action principle.

from the Hamiltonian generator and subtracting the flux, since $-I_\xi\omega' + di_\xi\theta' = d\delta q'_\xi$. The important point to stress is that q'_ξ is not associated to an arbitrary choice of symplectic potential, but to the physically preferred θ'. This removes any ambiguity in the procedure.

In the general case with $\delta\xi \neq 0$ and anomalies, we can compare the Hamiltonian generator and the improved Noether charge as follows. When (7.1.4) holds, the Hamiltonian generators are integrable iff there exists a functional X such that

$$dY = \delta X, \tag{7.1.11}$$

where

$$Y = -q'_{\delta\xi} - A'_\xi. \tag{7.1.12}$$

This requirement means that

$$X = ds_\xi + C_\xi, \qquad \delta s_\xi = Y, \tag{7.1.13}$$

and C_ξ is a constant in field-space. If this condition is satisfied, we can again prescribe the charges on the full phase space subtracting the symplectic flux, via

$$\delta dh_\xi := -I_\xi\omega' + di_\xi\theta' \hat{=} d\left(\delta q'_\xi - q'_{\delta\xi} - A'_\xi\right) = \delta d(q'_\xi + s_\xi). \tag{7.1.14}$$

The first equality follows from (7.1.8), and the second from (7.1.13). This formula provides a definition for the Hamiltonian charge associated to the physically selected θ', and works only if the anomalies satisfy the descent-type equation (7.1.11). This is not yet the WZ prescription but a generalization thereof, since as we will review in the next section, the WZ prescription makes additional requirements than just a specific $p\delta q$ form of θ'.

From the definition (7.1.14) it follows that

$$h_\xi = q'_\xi + s_\xi, \tag{7.1.15}$$

up to field-space constants as before (the constant C_ξ above drops out on the other hand). Because of the extra term s_ξ, the prescription (7.1.14) associated with the chosen θ' does *not* coincide in general with the improved Noether charge q'_ξ associated with a given (θ', ℓ). However, there are two interesting remarks to make at this point. First, the formula (7.1.8) is invariant under the corner Lagrangian shift (7.1.2), unlike the improved Noether charge which changes according to (7.1.10). Therefore, we can change the boundary Lagrangian by a corner term without affecting the Hamiltonian generator, and use this freedom to find a corner-improved Noether charge that matches the Hamiltonian charge. In other words, one can ask whether there is a choice of ℓ compatible with (7.1.1) such

that its Noether charges match the Hamiltonian prescription. Comparing (7.1.10) with (7.1.15), we see that the matching is possible if there exists a corner term c whose anomaly reproduces the integrable anomalies appearing in (7.1.13), namely

$$\Delta_\xi c = -s_\xi. \tag{7.1.16}$$

If such c exists, the corner-improved Noether charge associated with $\theta^{\mathrm{c}} = \theta'$ and $\ell^{\mathrm{c}} = \ell + dc$ matches the Hamiltonian charge,[4]

$$q^{\mathrm{c}}_\xi = q'_\xi + s_\xi \equiv h_\xi. \tag{7.1.17}$$

To be precise, the last equivalence is only up to the field-space constants mentioned above, since these can be freely added to h_ξ in order to satisfy special vanishing requirements, but not to q^{c}_ξ which is defined uniquely. The condition (7.1.16) is a partial differential equation that relates the corner improvement to the allowed anomalies of the symmetry vectors ξ. We will see below in the case of future null infinity an example of this equation and of its solution. In general, we don't know whether it is always possible to solve it. Whenever it is, the generalized WZ charge (7.1.14) can always be derived as an improved Noether charge. We will show in the next section that the WZ additional requirements allows us to get a more explicit form for s_ξ, and we will make more comments about solving it then.

The second remark is that the flux of this corner-improved Noether charge is still anomalous, since it is given by

$$\underleftarrow{dq^{\mathrm{c}}_\xi} = I_\xi \theta^{\mathrm{c}} - a^{\mathrm{c}}_\xi = I_\xi \theta' - a'_\xi + ds_\xi. \tag{7.1.18}$$

This provides also the flux of the Hamiltonian charge (up to the usual field-space constants), since the δ-variation of the above expression must match (7.1.14). Therefore, the charges are *not* automatically conserved when (7.1.4) holds. Clearly, additional physical requirements could be useful to achieve conservation when θ' vanishes on the lateral boundary. We will see next that the WZ prescription provides precisely such missing requirements, by forcing $a^{\mathrm{c}}_\xi = 0$. As a result, $I_\xi \theta'$ gives the flux responsible for the variation of the charges, and one obtains charges that are conserved under the desired circumstances for which θ' vanishes.[5]

[4] The notation ^{c} stands for corner-improved, and should not be confused with the notation for covariant used in [38]. We don't use any specific notation for covariant quantities, although typically we will associate them with the initial, unprimed quantities.

[5] Even if the variation of the charges is given in the end by $I_\xi \theta'$, it is still preferable to characterize the physical requirements such as stationarity in terms of θ', as requirements on $I_\xi \theta'$ may be ambiguous. We will see in Chapter 9 an example of such an ambiguity.

7.2 Wald-Zoupas prescription

We now review the WZ prescription from [33] and highlight the additional inputs that are brought in with respect to the previous general discussion. The prescription is based on removing the radiative part from the symplectic flux, identified making use of a background structure that can be attributed to the lateral boundary. To do so, one selects a symplectic potential $\bar{\theta}$ based on three criteria:

0. It must be a potential for the pull-back of the symplectic 2-form on the boundary, namely
$$\underset{\leftarrow}{\omega} = \delta\bar{\theta}. \tag{7.2.1}$$

1. It must be a local and covariant functional of the dynamical fields and background structure. This is sometimes assumed to imply vanishing anomalies and field-independent diffeomorphisms, but we will see shortly that it is more general than that – and this is crucial for understanding the future null infinity results.

2. It should vanish for conservative boundary conditions, case I presented earlier, or for arbitrary perturbations around stationary solutions, case II. The latter means that it must be of the form $F(g)\delta g$ where $F(g_{stationary}) = 0$. In reference to the earlier discussion, if we think of $\bar{\theta}$ as a certain $p\delta q$ polarization, then WZ stationarity is of the type $p = 0$.

Ideally, these criteria should be enough to single out a unique choice for $\bar{\theta}$, and this is indeed the case in the examples that we will review below. In case II, the preferred $\bar{\theta}$ satisfying all criteria is identified as the radiative symplectic flux, namely, a quantity whose vanishing means that all metrics sharing the background structure agree that the solution is stationary. As a consequence of the requirements made, one typically obtains

$$\bar{\theta} = \underset{\leftarrow}{\theta} + \delta b, \tag{7.2.2}$$

for some non-vanishing b defined on $\mathcal{N}$. We can interpret this formula as a special case of (7.1.1), where $\theta' = \bar{\theta}$ has to satisfy the WZ requirements above, b is the pull-back of a boundary Lagrangian up to the corner ambiguity (7.1.2), and ϑ vanishes or is at most a total variation so that it can be reabsorbed in b. An arbitrary $d\vartheta$ cannot be present because it would violate (7.2.1) hence condition 0.

The WZ prescription for the integrable charges is then to subtract the radiative flux on $\mathcal{N}$,

$$\oint \underset{\leftarrow}{dq}_\xi^{\mathrm{WZ}} := -I_\xi\omega + di_\xi\bar{\theta} \hat{=} d\left(\delta\bar{q}_\xi - \bar{q}_{\delta\xi} - \bar{A}_\xi\right). \tag{7.2.3}$$

This is the same formula that we discussed in the previous section. The novelty is the additional restriction given by conditions 0 and 1. To study the most general situation

under which the WZ prescription works (namely if we can replace $\not\delta$ with δ), let us look closely at the covariance requirement. This property, as spelled out in footnote 9 of [33], translates in our notation to[6]

$$\Delta_\xi \bar\theta + I_{\delta\xi}\bar\theta = 0 \qquad \Leftrightarrow \qquad (\delta_\xi - \pounds_\xi)\bar\theta = 0. \tag{7.2.4}$$

In other words, the anomaly of $\bar\theta$ and the allowed field-dependent diffeomorphisms are constrained, so that $\bar\theta$ is covariant: its field-space derivative coincides with the spacetime Lie derivative.[7] This condition is indeed sufficient to guarantee integrability, since we can rewrite (7.2.3) using

$$-I_\xi\omega + di_\xi\bar\theta = \delta I_\xi\bar\theta - \Delta_\xi\bar\theta - I_{\delta\xi}\bar\theta \qquad \Rightarrow \qquad \not\delta \underleftarrow{dq_\xi^{\mathrm{WZ}}} = \underleftarrow{\delta dq_\xi^{\mathrm{WZ}}} = \delta I_\xi\bar\theta, \tag{7.2.5}$$

namely

$$\underleftarrow{dq_\xi^{\mathrm{WZ}}} = I_\xi\bar\theta \tag{7.2.6}$$

up to (spacetime exact) field-space constants, that can be used as described earlier if one needs to set the charges to zero for a specific reference solution.

On the other hand, the WZ covariance requirement does *not* imply that all anomalies vanish. However, it implies some restrictions. Indeed, we know that :

$$\Delta_\xi\bar\theta = \Delta_\xi \underleftarrow{\theta} + \delta\Delta_\xi b - \Delta_{\delta\xi} b = \delta\bar a_\xi - \bar a_{\delta\xi} + d\bar A_\xi \tag{7.2.7a}$$

$$\underleftarrow{I_{\delta\xi}\bar\theta} = d\bar q_{\delta\xi} + a_{\delta\xi}. \tag{7.2.7b}$$

[6] Referring to the background fields as χ and the dynamical fields as ϕ, the requirement spelled out in that footnote is $\bar\theta(\chi, \varphi_*\phi, \varphi_*\delta\phi) = \varphi_*\bar\theta(\chi, \phi, \delta\phi)$, and assumes the transformation law $\delta\phi \mapsto \varphi_*\delta\phi$. This transformation law is fine if the diffeomorphism is field-independent, but if it is field-dependent, one has to use $\delta\phi \mapsto \delta(\varphi_*\phi)$ in order for the total system background+perturbation to be physically equivalent after the diffeomorphism. Accordingly, the WZ requirement should be modified to $\bar\theta(\chi, \varphi_*\phi, \delta(\varphi_*\phi)) = \varphi_*\bar\theta(\chi, \phi, \delta\phi)$. At the linearized level, this gives

$$\bar\theta(\chi, \pounds_\xi\phi, \delta\phi) + \bar\theta(\chi, \phi, \delta\pounds_\xi\phi) = \pounds_\xi\bar\theta(\chi, \phi, \delta\phi) = \bar\theta(\pounds_\xi\chi, \phi, \delta\phi) + \bar\theta(\chi, \pounds_\xi\phi, \delta\phi) + \bar\theta(\chi, \phi, \pounds_\xi\delta\phi),$$

from which (7.2.4) follows. In a previous version of the paper on arXiv, we considered a stronger condition in which $\Delta_\xi\bar\theta$ and $I_{\delta\xi}\bar\theta$ vanish individually. This is not necessary, and the correct version leads us to simpler equations when applied to the BMS analysis.

[7] However, it can be convenient to define the non-convariance by the vanishing of the anomaly rather than the equality between the spacetime lie derivative and the phase space derivative. Indeed, even the Eintein-Hilbert lagrangian is not covariant with the latter definition while it is anomaly free. Howeevr, if we interpret the covariant condition of Wald and Zoupas by setting $\Delta_\xi\bar\theta = 0$ only, then $-I_\xi\omega + di_\xi\bar\theta = \delta I_\xi\bar\theta - I_{\delta\xi}\bar\theta$ is still a boundary term but not necessarily an exact form in field space anymore and so we cannot write $-I_\xi\omega + di_\xi\bar\theta = \delta dq_\xi^{WZ}$ anymore. In other words, we are not even guaranteed to be able to write $-I_\xi\omega = \delta dh_\xi$ in the non radiative case, this is why we considered the equality between the phase space and spacetime Lie derivatives as a better notion of covariance for the WZ procedure including field dependent diffeomorphisms.

In the first we used $\underset{\leftarrow}{\Delta_\xi \theta} = \Delta_\xi \underset{\leftarrow}{\theta}$ which is valid for tangent ξ. Hence, (7.2.4) gives :

$$\Delta_\xi \bar{\theta} + I_{\delta\xi}\bar{\theta} = \delta\bar{a}_\xi + d(\bar{A}_\xi + \bar{q}_{\delta\xi}) = 0 \tag{7.2.8}$$

or :

$$\delta\bar{a}_\xi = -d(\bar{q}_{\delta\xi} + \bar{A}_\xi). \tag{7.2.9}$$

In the above formula one can freely replace $\bar{A}_\xi$ with A_ξ, since anyways there is no corner difference between $\bar{\theta}$ and θ. This relation is a special case of (7.1.11), in which X is determined by the Lagrangian anomaly $\bar{a}_\xi$. As a consequence,

$$\bar{a}_\xi = ds_\xi + C_\xi, \qquad \delta s_\xi = -\bar{q}_{\delta\xi} - A_\xi. \tag{7.2.10}$$

We see that the WZ covariance requirements (7.2.4) are compatible with the presence of field-dependent diffeomorphisms and anomalies, provided $\delta\xi$, a_ξ and $\Delta_\xi b$ are related by (7.2.9). We refer to such WZ-compatible anomalies as mild or *soft* anomalies. We will see below an example that justifies this name. This is the most general situation allowed by the WZ requirements, and as seen above it is enough to guarantee integrability of their prescription for the charges. If we compare with the generalized WZ prescription (7.1.14), we see that (7.2.9) is a special case of the integrability condition (7.1.11). The restriction comes from having added conditions 0 and 1.

We stress that we have done nothing new concerning the charges: we have merely re-derived the same formula of WZ, namely (7.2.6), under the same conditions as they did. Our only contribution is to point out that such conditions, and therefore the derivation, do admit anomalies, provided they are soft in the above sense.

Now that we have clarified that the WZ prescriptions also works in the presence of the soft anomalies, we can ask if it is possible to interpret the resulting charges as improved Noether charges for some specific choice of boundary Lagrangian. The reason why this is not obvious is that since $\bar{a}_\xi \neq 0$, we have

$$\underset{\leftarrow}{I_\xi \bar{\theta}} = d\bar{q}_\xi + \bar{a}_\xi, \tag{7.2.11}$$

where

$$\bar{q}_\xi := q_\xi + i_\xi b, \qquad \bar{a}_\xi := a_\xi + \Delta_\xi b. \tag{7.2.12}$$

This can be proved from (7.2.2), or read off directly from (7.1.7) using the fact that we can interpret b as a boundary Lagrangian.[8] The compatibility of this equation with (7.2.6) follows from (7.2.9), and if $\bar{a}_\xi \neq 0$ there is a mismatch.

Accordingly, we can distinguish three situations, depending on what anomalies are present:

[8] That fact that b is only defined on the boundary, namely that $db \equiv 0$, does not affect the derivation.

(a) The preferred symplectic flux $\bar{\theta}$ is associated to a total Lagrangian $L + db$ without anomalies. Then $\bar{a}_\xi = 0$, and we have

$$q_\xi^{\mathrm{WZ}} = \bar{q}_\xi = q_\xi + i_\xi b. \tag{7.2.13}$$

In this case, the WZ charge coincides with an improved Noether charge with boundary Lagrangian $\ell = b$ and vanishing ϑ. The flux formula (7.2.6) is consistent with (7.2.11) since the anomaly vanishes.

Remark: covariance of both θ and $\bar{\theta}$ is not enough to guarantee $\bar{a}_\xi = 0$, see (4.3.11).

(b) There are soft anomalies, and $C_\xi = 0$. Then

$$\bar{a}_\xi = ds_\xi, \qquad \delta s_\xi = -\bar{q}_{\delta\xi} - A_\xi. \tag{7.2.14}$$

Then

$$q_\xi^{\mathrm{WZ}} = q_\xi + i_\xi b + s_\xi. \tag{7.2.15}$$

It differs from the improved Noether charge $\bar{q}_\xi$ that would be immediately associated with (7.2.2), namely with boundary Lagrangian $\ell = b$ and $\vartheta = 0$. Notice that the additional term s_ξ is precisely the shift in the charge required so that the anomaly is removed from its flux, and this is how (7.2.11) is mapped to (7.2.6). When anomalies are present, the WZ prescription eliminates them from the flux, and puts them in the definition of the charge.

Next, we can ask if there exists a choice of boundary Lagrangian whose improved Noether charges reproduce the same shift. This is possible if $\bar{q}_\xi$ and q_ξ^{WZ} are related by (7.1.9), namely if we can find a corner term that satisfies (7.1.16), the same general equation of the previous section applies here. If a solution to this equation exists, then the WZ charge is the improved Noether charge q^{c} with boundary Lagrangian $\ell^{\mathrm{c}} = b + dc$. This fixes the corner ambiguity in the charges. Notice that $\Delta_\xi \ell^{\mathrm{c}} = -a_\xi$ and that ℓ^{c} is not unique, since any further shift by an anomaly-free corner term will also work, and produce the same charges.

Remark: If we start from a covariant Lagrangian, $a_\xi = 0$ and the required shift can be identified computing the anomaly of b,

$$\bar{a}_\xi = \Delta_\xi b = ds_\xi. \tag{7.2.16}$$

As a consequence, the corner-improved boundary Lagrangian is covariant,

$$\Delta_\xi \ell^{\mathrm{c}} = \Delta_\xi b + d\Delta_\xi c = 0. \tag{7.2.17}$$

If furthermore the starting θ is also covariant, then $A_\xi = 0$ and

$$\delta s_\xi = -\bar{q}_{\delta\xi}. \tag{7.2.18}$$

This means that $\bar{q}_{\delta\xi}$ is integrable, and exposes an interesting interplay that occurs between soft anomalies and field-dependent diffeomorphisms. This interplay will be crucial below to understand why one can do calculations for the BMS group passing over anomalies.

From this analysis we deduce that when it is possible to reproduce the WZ charges as improved Noether charges, the latter can be identified a priori as those associated with a covariant choice of bulk *and* boundary Lagrangians. All allowed anomalies can be restricted to corner terms. Again ℓ^c is not unique, since adding anomaly-free corner terms will give the same charges. Therefore, it is enough to pick any representative in the class of anomaly-free Lagrangians.

(c) A general soft anomaly is present, including C_ξ. The flux of the improved Noether charge contains an extra term with respect to the WZ flux. The two equations are still compatible because (7.2.6) is valid up to field-space constant terms. If C_ξ is spacetime-exact it can be reabsorbed in the Hamiltonian charge, and the matching with an improved Noether charge can be obtained following the same analysis as case (b). If C_ξ is not spacetime-exact, the matching is not possible. We could not find any examples in which this situation occurs, but we have no arguments to rule it out either. Lacking both, we refrain from drawing any conclusions about this case.

Notice that (7.2.18) can also be proven directly from Noether's theorem, as follows. For an arbitrary $\delta\xi$ and assuming no anomalies in the initial Lagrangian and symplectic potential, we have

$$\begin{aligned}\underleftarrow{dq_{\delta\xi}} &= \underleftarrow{I_{\delta\xi}\theta} - \underleftarrow{i_{\delta\xi}L} = I_{\delta\xi}\underleftarrow{\theta} - \underleftarrow{i_{\delta\xi}L} = (\delta_\xi - \mathcal{L}_\xi - \Delta_\xi)\underleftarrow{\theta} - \underleftarrow{i_{\delta\xi}L} \\ &= (\delta_\xi - \mathcal{L}_\xi)(\bar{\theta} - \delta b) - \underleftarrow{i_{\delta\xi}L} = -(\delta_\xi - \pounds_\xi - \Delta_\xi)\delta b - \underleftarrow{i_{\delta\xi}L} = -\delta\Delta_\xi b - \pounds_{\delta\xi} b - \underleftarrow{i_{\delta\xi}L}.\end{aligned} \tag{7.2.19}$$

If $\delta\xi$ is now restricted to a symmetry vector, it is tangent and thus the last term vanishes. Using this and $\pounds_{\delta\xi} b = d i_{\delta\xi} b$, we recover (7.2.18). This alternative derivation highlights that a non-vanishing $q_{\delta\xi}$ means that one is working with a symplectic potential that is not covariant, in spite of not being anomalous, because $I_{\delta\xi}\theta \neq 0$. In this case, the covariance requirement of the WZ flux means trading

$$\Delta_\xi\theta = 0, \quad I_{\delta\xi}\theta \neq 0 \qquad \Rightarrow \qquad \Delta_\xi\bar{\theta} = -I_{\delta\xi}\bar{\theta} \neq 0. \tag{7.2.20}$$

Summarizing, the WZ covariance requirement is enough to guarantee that the anomaly and field-dependent diffeomorphism contributions to (4.3.13) are integrable. The WZ prescription (7.2.3) can be interpreted as an improved Noether charge constructed so to have the anomalous term $\bar{a}_\xi$ shifted from the flux to the definition of the charge. This shift can be identified a priori if it possible to find a covariant boundary Lagrangian. In other words, it is premature to conclude from (7.2.2) that b is the boundary Lagrangian. If $\Delta_\xi b \neq 0$, one should rather look for a covariant $\ell^c = b + dc$.

This leads to the following independent definition of the WZ charges. First, evaluate (7.2.2), choosing a covariant $\bar{\theta}$, namely such that

$$(\delta_\xi - \mathcal{L}_\xi)\bar{\theta} = 0, \tag{7.2.21a}$$

plus the conservative or stationarity requirement chosen as described in Cases I and II. Second, identify a corner term c such that

$$\ell^c := b + dc, \qquad \Delta_\xi \ell^c = 0. \tag{7.2.21b}$$

Finally, compute the improved Noether charge associated with $(\bar{\theta}, \ell^c)$, or in other words with the split

$$\underset{\leftarrow}{\theta} = \bar{\theta} - \delta\ell^c + d\delta c. \tag{7.2.22}$$

7.3 Extending the WZ prescription to non-field-exact corners

This prescription can be immediately generalizing condition 0 to (7.1.1), as pointed out in [34]. All that will change is that the identification of ℓ^c will start from (7.1.1) instead of from b. We repeat the procedure in this case, for the sake of clarity and ease of reference.

First, evaluate (7.1.1), choosing θ' such that

$$(\delta_\xi - \mathcal{L}_\xi)\theta' = 0, \tag{7.3.1a}$$

plus the conservative or stationarity requirement chosen as described in Cases I and II. Second, identify a corner term c such that

$$\ell^c := \ell + dc, \qquad \Delta_\xi \ell^c = 0. \tag{7.3.1b}$$

Finally, compute the improved Noether charge associated with (θ', ℓ^c), or in other words with the split

$$\underset{\leftarrow}{\theta} = \theta' - \delta\ell^c + d(\vartheta + \delta c). \tag{7.3.2}$$

This means computing

$$q^{\mathrm{c}}_\xi = q'_\xi - \Delta_\xi c = q_\xi + i_\xi \ell^{\mathrm{c}} - I_\xi \vartheta^{\mathrm{c}}, \qquad \vartheta^{\mathrm{c}} := \vartheta + \delta c, \tag{7.3.3}$$

where q_ξ is the Noether charge associated with the covariant bulk Lagrangian, and q'_ξ is the Noether charge associated with the pair (θ', ℓ). The resulting flux is anomaly-free, $dq^{\mathrm{c}}_\xi = I_\xi \theta' - i_\xi L'$, as opposed to the anomalous flux of q'_ξ, given by (7.1.7). Of course, if we choose directly an anomaly-free ℓ, then $q'_\xi \equiv q^{\mathrm{c}}_\xi$.

7.4 BMS charges at future null infinity

At future null infinity, using Bondi coordinates,[9] one has

$$\underset{\leftarrow}{\theta} = -\Big(2\delta M - \frac{1}{2}\delta(D_A D_B C^{AB}) + \frac{1}{2} N_{AB}\delta C^{AB} - \frac{1}{8}\delta(N_{AB}C^{AB})\Big)\epsilon_{\mathcal{I}}. \tag{7.4.1}$$

See Appendix C.1 for definitions and some details. At a first sight, one may identify the physical flux with the non-integrable third term, namely the Bondi news $N_{AB} := \partial_u C_{AB}$ contracted with the variation of the shear C_{AB}. The stationarity requirement is then satisfied by all spacetimes with vanishing news, for arbitrary variations. The issue though is that this term does *not* satisfy the covariance requirement, because N_{AB} is not covariant. The resolution of this issue was found by Geroch [123] with the introduction of a background tensor ρ_{AB} carrying his name, and whose transformation property is $\Delta_\xi \rho_{\langle AB\rangle} = \Delta_\xi N_{AB}$, so that

$$\hat{N}_{AB} := N_{AB} - \rho_{\langle AB\rangle} \tag{7.4.2}$$

is covariant. The Wald-Zoupas criteria thus single out as preferred potential [33][10]

$$\bar{\theta} = -\frac{1}{2}\hat{N}_{AB}\delta C^{AB}\,\epsilon_{\mathcal{I}}. \tag{7.4.3}$$

The remainder is a total variation and identifies

$$b = \Big(2M - \frac{1}{2}\bar{D}_A\bar{D}_B C^{AB} - \frac{1}{8}N_{AB}C^{AB} + \frac{1}{2}\rho_{AB}C^{AB}\Big)\epsilon_{\mathcal{I}}. \tag{7.4.4}$$

Therefore all three conditions for the WZ prescription are met.

[9] For descriptions with geometric quantities only and avoiding reference to Bondi coordinates, see e.g. [121, 48, 122].

[10] In [33], the covariant news $\hat{N}$ are denoted N, referred to as Bondi news, and one keeps in mind that the expression $\partial_u C_{AB}$ is only valid in the special set of Bondi frames. This is indeed a better nomenclature in our opinion. We maintain however the N and $\hat{N}$ notation here to match more easily with the contemporary literature, where Geroch's analysis seems to have been forgotten at some point. Notice also that $\hat{N}$ can be defined in geometric terms as the Lie derivative of the shear [122], which makes its covariance manifest. Geroch's construction on the other hand used the Schouten tensor of the (pull-back of the) unphysical Riemann tensor, hence a non-covariant quantity, and which coincides with $\partial_u C_{AB}$ in Bondi coordinates.

A key property of $\rho_{\langle AB\rangle}$ is to vanish identically when the background metric is the round 2-sphere. This makes the choice $N\delta C$ numerically correct in such Bondi frames, however one should keep in mind that the correct potential secretly depends on the Geroch tensor in order to secure covariance. This is relevant for us, because calculation of the anomaly involves derivatives in field space, and the anomaly of b would be different if we forgot the term necessary to the covariance of $\bar{\theta}$.

From (7.4.4) and the anomaly-freeness of the Einstein-Hilbert Lagrangian we compute

$$\bar{a}_\xi = \Delta_\xi b = ds_\xi, \qquad where \qquad s_\xi := \frac{1}{4} C^{AB} \bar{D}_A \bar{D}_B \tau, \tag{7.4.5}$$

and $\tau := \xi^u = T + \frac{u}{2} D_A Y^A$. Details of this calculation are in Appendix C. We see that we are in case (b): there is a shift, caused by the fact that the 'naive' boundary Lagrangian b has an anomaly. The shift term s_ξ is independent of the (global) conformal Killing vector Y^A if we restrict to a round sphere, but it should not be neglected even in these frames because it still contributes to the flux of supertranslation charges. For stationary spacetimes, this term is irrelevant on a round sphere, but not in arbitrary frames.

The shift can furthermore be obtained from the corner ambiguity. In fact, using the anomalous transformations reported in Appendix C.1, it is easy to find a local functional c solving (7.1.16):

$$c := \frac{1}{16} C_{AB} C^{AB} \epsilon_S, \qquad \Delta_\xi c = -s_\xi. \tag{7.4.6}$$

Therefore the BMS charges obtained from the WZ prescription can also be obtained as improved Noether charges, choosing an anomaly-free boundary Lagrangian such as

$$\ell^c = b + dc = \Big(2M - \frac{1}{2} \bar{D}_A \bar{D}_B C^{AB} + \frac{1}{2} \rho_{AB} C^{AB}\Big) \epsilon_{\mathcal{I}}, \qquad \Delta_\xi \ell^c = 0. \tag{7.4.7}$$

This choice of boundary Lagrangian is of course not unique: any further shift by an anomaly-free corner term would give the same charges. In this case, it means that we can add an arbitrary contribution proportional to

$$(N_{AB} C^{AB} - 2\bar{D}_A \bar{D}_B C^{AB}) \epsilon_{\mathcal{I}}, \tag{7.4.8}$$

which is both a corner term and anomaly-free. In other words, one can equivalently use any element in the family

$$\ell^c_x = \Big(2M - \frac{1+x}{2} \bar{D}_A \bar{D}_B C^{AB} + \frac{x}{4} N_{AB} C^{AB} + \frac{1}{2} \rho_{AB} C^{AB}\Big) \epsilon_{\mathcal{I}}, \qquad x \in \mathbb{R}. \tag{7.4.9}$$

Summarizing, the WZ charges for the BMS group can be obtained as improved Noether charges following the prescription (7.2.21), namely starting from the covariant Einstein-Hilbert Lagrangian and bare symplectic potential, and choosing $(\bar{\theta}, \ell^c)$ given by (7.4.3) and (7.4.9) respectively. The resulting corner term in (7.2.22) is (7.4.6).

The fact that the the BMS charges can be obtained as improved Noether charges is consistent with what observed in [46, 38], where relevant boundary Lagrangians were identified a posteriori. The novelty of our derivation is the identification of the boundary Lagrangians and charges from first principles, thanks to the attention paid to anomalies.[11]

As a final remark, the WZ-compatible anomaly (7.4.5) has the structure of the soft term in the flux-balance laws for the BMS charges. This example provides a physical example of the meaning of the anomaly contribution to the variation of the improved Noether charge (7.1.7): an improved Noether charge with boundary Lagrangian (7.4.4) as opposed to (7.4.7), or with an arbitrary corner improvement not selected by the covariance requirement (7.2.17), would differ from the standard BMS charges by soft terms. One consequence would be that they would measure different memory effects, another that the boost part of the charges would fail to be conserved on stationary spacetimes in arbitrary frames. The relation between the soft terms and anomalies is further explained by the detailed calculations reported in the Appendix C.1, which highlight how the bridge between the two lies in the first-order extension of the symmetry vector fields. We also report there the calculation of the charges (App. C.1), as well as the explanation of how (7.2.18) allows one to do the calculation à la Wald-Zoupas without the need to take explicitly into account the anomalies (App. C.2).

[11] When comparing our quantitative results with the literature, some attention is however needed. The corner term (7.4.5) differs from the one used in [46] by a factor of 2 (mind the different units used, $16\pi G = 1$ here, $8\pi G = 1$ there). This follows from the fact that [46] uses the tetrad Lagrangian, whose bare symplectic potential differs from the Einstein-Hilbert one by a corner term [124, 66]. We explain this comparison in App. C.3, and our results here perfectly agree with those of [46]. We find on the contrary a disagreement with the conclusions of [38], which find (1) no anomalous shift, namely they claim $q_\xi^{\rm wz} = \bar{q}_\xi + i_\xi b$, and (2) no restriction of the boundary Lagrangian to be anomaly-free, namely they consider four different options of which only their (6.17) is in our family, with $x = 0$. The remaining (6.9), (6.18) and (6.18) are anomalous. Disagreement (1) is in our opinion due to a computational mistake, we believe that their equation (6.15) lacks a factor 1/2 in the third term, the one that reads $\mathcal{UD}f$. Their numerical factor would indeed make the anomalous shift unnecessary in order to obtain the BMS charges, but it is in contradiction with our calculation reported in App. C.1, as well as with the calculations of [107, 81] which use the $q_{\delta\xi}$ term, and which we report in App. C.2. Since the presence of a non-zero anomaly is crucial to our paper, we made multiple checks of our calculations and the presence of this contribution. But of course we welcome further feedback on this point, should the mistake be on our end. Disagreement (2) is on the other hand not an issue, provided (1) is fixed. Every time their boundary Lagrangian is anomalous, they redefine the charge by hand to remove what the anomalous contribution to the flux, via the quantity they denote $\tilde{h}_\xi$. What we have shown here is that $\tilde{h}_\xi$ is in general not an improved Noether charge in the sense of (7.1.9), and that there is no need to do this redefinition by hand, because it is possible to identify the charge uniquely working with a covariant pair of bulk and boundary Lagrangians.

Charges on timelike boundaries

8

In this chapter we compute the surface charges of gravity for a family of conservative boundary conditions, that include Dirichlet, Neumann, and York's mixed boundary conditions defined by holding fixed the conformal induced metric and the trace of the extrinsic curvature. We show that for all boundary conditions considered, canonical methods give the same answer as covariant phase space methods improved by a boundary Lagrangian, a prescription introduced in Chapter 7 and thus supported by our results. The procedure also suggests a new integrable charge for the Einstein-Hilbert Lagrangian, different from the Komar charge for non-Killing and non-tangential diffeomorphisms. We study how the energy depends on the choice of boundary conditions, showing that both the quasi-local and the asymptotic expressions are affected. Finally, we generalize the analysis to non-orthogonal corners, confirm the matching between covariant and canonical results without any change in the prescription, and discuss the subtleties associated with this case. This work was published in [42].

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It is a fundamental property of general relativity that energy is not conserved, but dissipated by gravitational radiation. A notion of conserved energy in phase space can be introduced if one restricts attention to non-radiative spacetimes. An example of conserved energy is the ADM charge [8] at spatial infinity, or its quasi-local version the Brown-York (BY) charge [120] (see [84] for a review on quasi-local charges). This notion of energy is however not universal, and depends on the way the system is made conservative, namely on the specific choice of boundary conditions used to eliminate dissipation. The ADM and BY formulas for instance, are based on Dirichlet boundary conditions. In this Chapter we study how the value of the energy changes as we move from Dirichlet to York's mixed boundary conditions, to Neumann's. This corresponds to fixing less components of the induced metric, and more components of its momentum, namely the extrinsic curvature, as reviewed in Chapter 5. We will see that the neat effect of this process is to produce smaller values of the energy. For instance applying the formula to the Kerr spacetime, we find respectively M, $2M/3$ and $M/2$.

To obtain these results, we use two different methods. The first is the covariant phase space. In the non-radiative/conservative case, the improved Noether charge prescription can be used to obtain Hamiltonians studying how imposing vanishing symplectic flux at the boundary selects a specific symplectic potential leading to integrable charges. We have seen that the charge is defined in an unambiguous way, and depends on both the bulk Lagrangian and the boundary term required by the variational principle associated with the chosen boundary conditions, as anticipated in [118]. The improved Nother charge prescription, given by equation (7.1.9), is thus a perfect tool to investigate the question raised above.

The second method is the straightforward canonical analysis based on the ADM decomposition, which has no problem in dealing with the non-radiative context. The calculation we present is a simple extension of the analysis done in [125, 91] for Dirichlet boundary conditions. A nice feature of this extension is to see explicitly how the boundary term changes the kinetic part of the ADM Lagrangian to recast it in the form consistent with the symplectic potential associated with the chosen boundary conditions. We find that the canonical method reproduces exactly the same charges obtained with the improved Noether charge prescription in the covariant method, for all cases considered. A consequence of our results is thus to offer support to the prescription of [14].

These results are based on the simplest set-up with an orthogonal corner between the time-like boundary and the space-like hypersurfaces, a situation where 3d boundary Lagrangians are sufficient to make the variational principle well-defined. We also investigate a more general context with non-orthogonal corners. In the presence of non-orthogonal corners, the variational principle requires an additional 2d term [104, 92] (see also [51, 65, 66] for recent work). Since the rationale for constructing the covariant surface charges is to use an action with a well-posed variational principle, one may wonder if the corner Lagrangian contributes to the formula for the charges as well.

To address this question, we repeat the calculations using the improved Noether charge prescription with different boundary conditions, this time allowing for non-orthogonal corners, and compare the results with the ones obtained with canonical methods. Using canonical methods [125, 91], it is known that the presence of a non-orthogonal corner has both a physical and a mathematical consequence. The physical consequence is that one can consider two different classes of observers, those at rest with respect to the space-like foliation of spacetime, and those at rest along the time-like boundary. At the corner, these are related by a boost transformation, making the different notions of energy directly comparable. The mathematical consequence is that the charges pick up 'tilting terms', namely they depend explicitly on the boost between the normals. This dependence allows us a further, non-trivial test passed by the improved Noether charge: The covariant and canonical results match also for non-orthogonal corners, without any amendment to the formula. This matching is however subtle, because in order to obtain it, one needs

to take into account that the boundary term contains a kinetic term that has to be independently put in Hamiltonian form via a Legendre transform [91, 34]. The role of the corner Lagrangian is only to adjust the boundary kinetic term to match the chosen boundary conditions.

The notation for the different boundaries and their geometric objects are summarized in Fig. 5.1 and Table 5.1.

8.1 Charges from covariant phase space

The first method we are going to use to compute the charges associated with the different boundary conditions is the covariant phase space, in particular the improved Noether charge prescription (7.1.9). Recall that the idea is that Hamiltonians can be constructed if we restrict the variations to preserve some given boundary conditions along $\mathcal{T}$. Consider the boundary Lagrangian ℓ required by a well-defined variational principle with given boundary conditions at $\mathcal{T}$. By construction, restricting the variations to those preserving the boundary conditions, the boundary variation of L must be equal and opposite to the variation of ℓ, up at most to a corner term: $\theta^{\mathrm{L}} \overset{\mathcal{T}}{=} -\delta\ell + d\vartheta$. As a consequence, it is possible to redefine the symplectic potential and 2-form as

$$\theta := \theta^{\mathrm{L}} + \delta\ell - d\vartheta, \qquad \omega = \delta\theta = \delta\theta^{\mathrm{L}} - d\delta\vartheta. \tag{8.1.1}$$

In this way, one automatically has vanishing symplectic flux across the boundary, when the boundary conditions are imposed:

$$\underset{\leftarrow}{\theta} \overset{\text{b.c.}}{=} 0 \quad \Leftrightarrow \quad \theta^{\mathrm{L}} \overset{\text{b.c.}}{=} -\delta\ell + d\vartheta \qquad \Rightarrow \qquad \omega \overset{\text{b.c.}}{=} 0. \tag{8.1.2}$$

This condition guarantees that the system is closed, and the defining equation (7.1.9) integrable to yield a Hamiltonian generator.[1] The new quantities in (8.1.1) depend on the pair (L, ℓ) of bulk-boundary Lagrangians, dependence which we don't make explicit in order to keep the notation light. The resulting Hamiltonian charge is[2] (7.1.9) up to a constant of integration in field space that we will come back to below in Section 8.1.3.

[1] The last equation in (8.1.2) is Wald's sufficient condition for integrability. It can also be derived requiring the condition that an Hamiltonian vector field (in field space) preserves the symplectic form, $\delta_\xi\omega = \delta(I_\xi\omega) = 0$, which implies $I_\xi\omega = \delta h_\xi$, and if there are no anomalies, we also have $\delta_\xi\omega = \mathcal{L}_\xi\omega \approx d(i_\xi\omega)$.

[2] This can be seen evaluating

$$-I_\xi\omega = -I_\xi\delta\theta = -I_\xi\delta\theta^{\mathrm{L}} + dI_\xi\delta\vartheta \approx d(\delta q_\xi - i_\xi\theta^{\mathrm{L}} - q_{\delta\xi} + i_\xi d\vartheta - \delta I_\xi\vartheta + I_{\delta\xi}\vartheta),$$

where we used

$$I_\xi\delta\vartheta = \delta_\xi\vartheta - \delta I_\xi\vartheta = \mathcal{L}_\xi\vartheta + I_{\delta\xi}\vartheta - \delta I_\xi\vartheta = d(i_\xi\vartheta) + i_\xi d\vartheta - \delta I_\xi\vartheta + I_{\delta\xi}\vartheta.$$

For field-independent diffeomorphisms and on-shell of (8.1.2) we obtain (7.1.9). Further details can be found in e.g. [34, 126]. Here we assumed that no anomalies are present, but the resulting formula

Let us briefly comment on the equivalence with the WZ prescription in this case: a good candidate for the preferred symplectic potential $\bar{\theta}$ is the non-integrable term in (7.2.2). It satisfies condition 0 with $b = \ell$, and condition 2 with conservative boundary conditions $q^{\mu\rho}q^{\nu\sigma}\delta q_{\rho\sigma} \overset{\mathcal{T}}{=} 0$, or with a notion of stationarity given by $\Pi = 0$. To discuss its covariance, we evaluate

$$(\delta_\xi - \pounds_\xi)\,(\Pi_{\mu\nu}\delta q^{\mu\nu}) = \Delta_\xi \Pi_{\mu\nu}\delta q^{\mu\nu} + \Pi_{\mu\nu}\delta\Delta_\xi q^{\mu\nu} + \Pi_{\mu\nu}\pounds_{\delta\xi} q^{\mu\nu}. \tag{8.1.3}$$

The residual diffeomorphisms that preserve the phase space must preserve the boundary, hence be tangent to it. As a consequence $\delta_\xi q_{\mu\nu} = \pounds_\xi q_{\mu\nu}$, and $\Delta_\xi n_\mu = 0$ provided we work with a unit-norm normal (see Appendix D). Therefore

$$\Delta_\xi q_{\mu\nu} = \Delta_\xi \epsilon_{\mathcal{T}} = \Delta_\xi K_{\mu\nu} = 0. \tag{8.1.4}$$

We conclude that this $\bar{\theta}$ is covariant for field-independent diffeomorphisms. The boundary symmetry group is $\mathrm{Diff}(\mathcal{T})$, and the charges will be conserved for arbitrary variations around solutions with $\Pi = 0$, and around arbitrary solutions but only for variations restricted to preserve the boundary conditions. In the latter case the only allowed symmetries of the conservative subset of the phase space are the Killing vectors of the boundary metric.

Following the prescription used in [32] and adopted in the WZ paper, we take conservative boundary conditions, as in Case I. Now that we have chosen $\bar{\theta}$ and identified b with the Gibbons-Hawking-York (GHY) term, the next step is to look at the anomalies. We have $a_\xi = 0$ from the initial choice of the Einstein-Hilbert Lagrangian, and $\Delta_\xi b^{\mathrm{WZ}} = 0$[3] from (8.1.4). Therefore all anomalies vanish and we are in case (a). According to (7.2.13), the WZ charge is given by the improved Noether charge $q_\xi + i_\xi b^{\mathrm{WZ}}$. One can easily check that this is indeed the WZ charge computed in [32, 33], and which gives the Brown-York formulas at finite distance.

8.1.1 Corner symplectic potential

The only aspect of the prescription (7.1.9) that requires some care is the determination of the corner symplectic potential. It can be in principle computed using Anderson's homotopy operator as argued in [14], but in practise it is simpler to derive it taking the variation of the boundary Lagrangian, and arranging it in such a way that the boundary field equations are consistent with the boundary conditions one is imposing. Let us see

(7.1.9) is valid also in the anomalous case. This was shown in [126], and one has to use the fact that boundary conditions are consistent, namely $\Delta_\xi \theta|_{\text{b.c.}} \overset{\mathcal{T}}{=} 0$.

[3] In this Chapter we will use b^{WZ} when referring to the WZ prescription to avoid notational clash with the parameter b in the action.

explicitly this strategy at play with ℓ^b. To compute its variation, we use the standard result

$$\delta K = -\frac{1}{2}K^{\mu\nu}\delta g_{\mu\nu} + g^{\rho[\sigma}n^{\mu]}\nabla_\mu \delta g_{\rho\sigma} + \frac{1}{2}q^\rho_\mu \nabla_\rho \left(q^\mu_\nu \delta n^\nu\right). \tag{8.1.5}$$

The second term is proportional the symplectic potential θ^{EH}, see (5.3.3), and the third term can be written in terms of the induced covariant derivative on the hypersurface D_μ.[4] Applying this formula to (5.3.14), we find

$$\begin{aligned}\delta\ell^b &= \frac{b}{2}s\left(\left(Kq^{\mu\nu} - K^{\mu\nu}\right)\delta g_{\mu\nu} + 2g^{\alpha[\beta}n^{\lambda]}\nabla_\lambda \delta g_{\alpha\beta} + D_\mu\left(q^\mu_\nu \delta n^\nu\right)\right)\epsilon_\Sigma \\ &= \frac{b}{2}\left(-\theta^{\text{EH}} + s(K_{\mu\nu} - q_{\mu\nu}K)\delta q^{\mu\nu}\epsilon_\Sigma + d\vartheta^{\text{EH}}\right).\end{aligned} \tag{8.1.6}$$

To determine the corner symplectic potential of ℓ^b, the bulk term must be consistent with the boundary conditions we want to impose, as to reproduce (the first of) (8.1.2). Rearranging the terms in (8.1.6), we find

$$\begin{aligned}\theta^{\text{EH}} + \delta\ell^b &= (1 - \frac{b}{2})\theta + \frac{b}{2}s(K_{\mu\nu} - q_{\mu\nu}K)\delta q^{\mu\nu}\epsilon_\Sigma + \frac{b}{2}d\vartheta^{\text{EH}} \\ &= s\left((K_{\mu\nu} - \frac{b}{2}q_{\mu\nu}K)\delta q^{\mu\nu} + (b-2)\delta K\right)\epsilon_\Sigma + d\vartheta^{\text{EH}}.\end{aligned} \tag{8.1.7}$$

We can explicitly check that the term in bracket in the second equality vanishes accordingly to the boundary conditions chosen: $\delta q_{\mu\nu} \overset{\mathcal{T}}{=} 0$ for $b = 2$, $\delta\tilde{\Pi}^{\mu\nu} \overset{\mathcal{T}}{=} 0$ for $b = 0$ (see (5.3.4)), and $\delta\hat{q}_{\mu\nu} \overset{\mathcal{T}}{=} 0 \overset{\mathcal{T}}{=} \delta K$ for $b = 2/3$ (see (5.3.11)). Therefore, we conclude that the corner symplectic potential of ℓ^b is precisely ϑ^{EH}, irrespectively of these values of b.

The fact that this construction yields a consistent non-vanishing ϑ for $b = 0$ is quite remarkable. It leads to the suggestion of taking a non-vanishing corner symplectic potential also for Neumann boundary conditions, even if the boundary Lagrangian is zero in this case. One may discard this possibility, but as we will see below, keeping it allows one to introduce an integrable charge for the Einstein-Hilbert action valid also for non-tangential diffeos, and which reduces to the Komar expression in the case of isometries.

8.1.2 Charges for Dirichlet, York and Neumann polarizations

We now apply the prescription (7.1.9) to the Lagrangian $L^b := L^{\text{EH}} + d\ell^b$. This requires evaluating the three terms in (7.1.9) and their pull-backs on the corner S

[4] Here we used

$$\int_\Sigma D_\mu(q^\mu_\nu \delta n^\nu)d\Sigma = \int_{\partial\Sigma} q^\mu_\nu \delta n^\nu dS_\mu = -s\int_{\partial\Sigma} u_\mu \delta n^\mu dS \qquad dS_\mu = -su_\mu dS.$$

intersection of Σ and $\mathcal{T}$. We consider the corner in the future of $\mathcal{T}$, so that the outgoing time-like normal is future-pointing. Our conventions for the volume forms, orientations and pull-backs are reported in Chapter 2. The symplectic potential of L^{EH} is given in (5.3.4), and its Noether charge is the Komar 2-form [83]

$$q_\xi^{\text{EH}} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho\xi^\sigma dx^\rho \wedge dx^\sigma \overset{S}{=} 2n_\mu u_\nu \nabla^{[\mu}\xi^{\nu]}\epsilon_S. \tag{8.1.8}$$

Next, the pull-back of ℓ^b on the time-like boundary $\mathcal{T}$ gives

$$i_\xi \ell^b \overset{\mathcal{T}}{=} b\bar{K} i_\xi \epsilon_{\mathcal{T}} = \frac{b}{2}\bar{K}u^\mu\xi^\nu\epsilon_{\mu\nu\rho\sigma}dx^\rho \wedge dx^\sigma \overset{S}{=} b\bar{K}\xi\cdot n\,\epsilon_S. \tag{8.1.9}$$

The last ingredient is the corner symplectic potential just evaluated. Its pull-back on the future boundary of $\mathcal{T}$ gives

$$\vartheta = \vartheta^{\text{EH}} \overset{\partial\mathcal{T}^+}{=} -u_\mu\delta n^\mu(-\epsilon_S) = n_\mu u_\nu \delta g^{\mu\nu}\epsilon_S, \qquad I_\xi\vartheta = -2n_\mu u_\nu \nabla^{(\mu}\xi^{\nu)}\epsilon_S. \tag{8.1.10}$$

Adding up the three terms, the Hamiltonian charge is found to be

$$\begin{aligned} H_\xi^b &= \int_S q_\xi^{\text{EH}} + i_\xi\ell^b - I_\xi\vartheta^b = \int_S 2n_\mu u_\nu(\nabla^{[\mu}\xi^{\nu]} + \frac{b}{2}\bar{K}\xi^\mu u^\nu + \nabla^{(\mu}\xi^{\nu)})\epsilon_S \\ &= 2\int_S n_\mu u_\nu(\nabla^\mu\xi^\nu + \frac{b}{2}\bar{K}\xi^\mu u^\nu)\epsilon_S \\ &= -2\int_S n^\mu\xi^\nu(\bar{K}_{\mu\nu} - \frac{b}{2}\bar{q}_{\mu\nu}\bar{K})\epsilon_S, \end{aligned} \tag{8.1.11}$$

where in the last step we used that for orthogonal corners we can take $u \equiv \bar{n}$, therefore $\xi\cdot u = 0$, $n^\mu = \bar{q}^{\mu\nu}n_\nu$ and $n^\mu\nabla_\mu u_\nu = n^\mu\bar{K}_{\mu\nu}$.

From this general formula we can read the three special cases we have been discussing so far. For Dirichlet boundary conditions, $b = 2$, we have

$$H_\xi^{\text{BY}} = -2\int_S n^\mu\xi^\nu(\bar{K}_{\mu\nu} - \bar{q}_{\mu\nu}\bar{K})\epsilon_S = -2\int_S n^\mu\xi^\nu\bar{\Pi}_{\mu\nu}\epsilon_S. \tag{8.1.12}$$

This is the result of [34] (see also [33, 14]): the Hamiltonian generating the boundary symmetries in the covariant phase space is the Brown-York surface charge. For York's mixed boundary conditions, we find

$$H_\xi^{\text{Y}} = -2\int_S n^\mu\xi^\nu(\bar{K}_{\mu\nu} - \frac{1}{3}\bar{q}_{\mu\nu}\bar{K})\epsilon_S = -2\int_S n^\mu\xi^\nu\bar{\Pi}_{\langle\mu\nu\rangle}\epsilon_S, \tag{8.1.13}$$

namely the surface charge is the traceless part of the ADM momentum on the time-like boundary. This result appeared recently in [22]. Finally for Neumann boundary conditions, the trace part of the extrinsic curvature drops out and we are left with

$$H_\xi^{\mathrm{N}} = -2\int_S n^\mu \xi^\nu \bar{K}_{\mu\nu}\,\epsilon_S = -2\int_S n^\mu \xi^\nu (\bar{\Pi}_{\mu\nu} - \frac{1}{2}\bar{q}_{\mu\nu}\bar{\Pi})\epsilon_S. \tag{8.1.14}$$

The last expression can be taken as definition of integrable charge for the Einstein-Hilbert action valid also for diffeomorphisms non-tangential to the corner S, for which the usual prescription fails, as remarked below (4.3.12). It is quite a non-trivial step, since with the usual prescription one obtains integrable charges only for tangential diffeomorphisms, and follows from taking seriously the improved Noether charge prescription and the construction (8.1.7) of the corner symplectic potential. The new Einstein-Hilbert charge reduces to the Komar expression for (arbitrary) tangential diffeomorphisms. This can be seen starting from the second line of (8.1.11) with $b = 0$, and using $n{\cdot}u = n{\cdot}\xi = u{\cdot}\xi = 0$ to prove that the symmetrization in n and u vanishes. We also notice that for non-tangential diffeomorphisms, (8.1.14) reduces to the Komar expression in the case of isometries. This can be immediately seen again from the second line of (8.1.11) with $b = 0$ and using the Killing equation. Hence, (8.1.14) does provide an extension of the Komar formula to non-isometric diffeomorphisms endowed with an interpretation of Hamiltonian generator for Neumann boundary conditions.

The charges (8.1.11) can be split into energy and angular momentum, introducing a decomposition of the diffeomorphism as

$$\xi^\mu = N n^\mu + N^\mu, \qquad N \cdot n = 0. \tag{8.1.15}$$

Notice that since we are already restricting the diffeos to satisfy (8.1.24), N^μ only has components tangent to the corner. Then,

$$n^\mu N^\nu \bar{K}_{\nu\mu} = n^\mu N^\nu \bar{q}^\rho_\nu \nabla_\rho u_\mu = -u^\mu N^\nu \bar{q}^\rho_\nu \nabla_\rho n_\mu = -u^\mu N^\nu \gamma^\rho_\nu \nabla_\rho n_\mu = -u^\mu N^\nu K_{\nu\mu}. \tag{8.1.16}$$

As for the piece proportional to lapse, it can be expressed in terms of the 2d extrinsic curvature $k = \bar{K} + n^\mu n^\nu \nabla_\nu u_\nu$ (see (2.1.32) in the Appendix, here adapted to orthogonal corners). Adding up, and using $\xi \cdot n = -N$, (8.1.11) gives

$$H_\xi^b = -2\int_S \left(N\Big(k + \frac{b-2}{2}\bar{K}\Big) - N^\mu u^\nu K_{\mu\nu} \right) \epsilon_S. \tag{8.1.17}$$

In Section 8.4, we will reproduce this expression using canonical methods.

The term proportional to N is the energy, whereas the term proportional to N^μ is the corner diffeomorphism charge and contains the angular momentum. We see that changing the boundary conditions leaves the angular momentum invariant but changes the notion

of energy of the system. The fact that integrable charges obtained through the imposition of boundary conditions depend on the latter was expected [32], and we are seeing here the results of a quantitative analysis. This dependence is after all understandable: in the open case there is no general notion of energy, so it makes sense that when we close it, the notion of energy depends on *how* we close the system. See also [127] for earlier discussions on the relation between energy and boundary conditions in general relativity.

The quasi-local charges (8.1.11) are defined on the surface corner S of a finite region of spacetime. To study what happens for asymptotic charges, we need to first consider the required subtraction terms. We will do so in the next subsection, and in the next section we will use the Kerr spacetime to explore the explicit quasi-local and asymptotic values of the charges and see how they are affected by the choice of boundary conditions.

8.1.3 Subtraction terms and symplectic renormalization

The quasi-local expressions (8.1.11) or equivalently (8.1.17) are fine as quasi-local charges at finite distance, but they diverge when the corner is pushed to spatial infinity. This is a familiar result from the Brown-York analysis, and the standard procedure is to remove the divergence with a subtraction term depending on a background solution, typically Minkowski. From the covariant phase space perspective, this is a natural procedure that amounts to the simple fact that when integrating (7.1.9) one can take into account a non-vanishing constant of integration in field space [33]. This reference or background solution can be taken to be Minkowski, and produces the subtraction term of the Brown-York analysis leading to finite expressions as pointed out in [128].[5] We now show that this procedure can be generalized to $b \neq 2$, and that it can also be understood in the framework of symplectic renormalization, which plays an important role for subtracting analoguous divergences at null infinity [81, 80, 46]. Namely instead of removing the divergence via a background solution, we can renormalize the charge using the prescription (7.1.9) and subtracting the contribution that would come from a boundary Lagrangian ℓ_{div} that captures the divergences of L, namely

$$H_\xi^{\text{R}} = H_\xi - \int_S i_\xi \ell_{\text{div}} - I_\xi \vartheta_{\text{div}}. \tag{8.1.18}$$

In the present context, we restrict attention to asymptotically flat metrics at spatial infinity, and the leading divergence of H_ξ^b comes from the Minkowskian behaviour of the charge. Therefore,

$$\ell^b_{\text{div}} = b\bar{K}_\eta\, \epsilon_\Sigma, \qquad i_\xi \ell^b_{\text{div}} \overset{S}{=} b\bar{K}_\eta\, \xi \cdot n\, \epsilon_S, \qquad \vartheta^b_{\text{div}} = 0. \tag{8.1.19}$$

[5] There the finiteness was attributed to the use of tetrads in the first order formalism, but we believe the result applies to any formulation, and follows from the fact that by plugging in the fall-off condition on the variables in the variational formula (4.3.12), the leading order Minkowski contribution is eliminated directly being a fixed background.

Then the renormalized charge is

$$
\begin{aligned}
H_\xi^{\mathrm{R}b} &= H_\xi^b - \int_S i_\xi \ell^b_{\mathrm{div}} \\
&= -2\int_S n^\mu \xi^\nu \Big(\bar{K}_{\mu\nu} - \frac{b}{2}\bar{q}_{\mu\nu}(\bar{K} - \bar{K}_\eta)\Big)\epsilon_S \\
&= -2\int_S \left(N\Big(k - \frac{b}{2}k_\eta + \frac{b-2}{2}\bar{K}\Big) + N^\mu n^\nu \bar{K}_{\mu\nu}\right)\epsilon_S,
\end{aligned}
\tag{8.1.20}
$$

where in the last step we used (8.1.15) and the fact that in Minkowski,

$$
\bar{K}_\eta = k_\eta = \frac{2}{r}. \tag{8.1.21}
$$

Restoring the $16\pi G$ factors, we define the $b-$generalized energy and angular momentum as follows,

$$
E = H_n^{\mathrm{R}b} = -\frac{1}{8\pi G}\int (k - \frac{b}{2}k_0 + \frac{b-2}{2}\bar{K})\epsilon_S \tag{8.1.22}
$$

$$
J = H_{\partial_\phi}^b = -\frac{1}{8\pi G}\int n^\mu \bar{K}_{\mu\phi}\epsilon_S, \tag{8.1.23}
$$

which correspond respectively to the generator of unit-lapse hypersurface-orthogonal diffeomorphisms, and rotations around a fiducial vertical axis fixed say by asymptotic flatness. We see that the boundary conditions do not affect the angular momentum, as expected since this charge does not see the symplectic flux, and furthermore is independent of renormalization since it coincides with its quasi-local value. The energy on the other hand depends explicitly on both the boundary conditions and the renormalization.

8.1.4 Residual diffeomorphisms

Let us give a few more details about the allowed diffeomorphisms. They must preserve the boundary as well as the boundary conditions whose imposition makes the charges integrable. This means that (i) they cannot move the boundary $\mathcal{T}$, namely

$$
\xi \in T\mathcal{T} \quad \Leftrightarrow \quad \xi \cdot \bar{n} = 0, \tag{8.1.24}
$$

which equals $\xi \cdot u = 0$ in the case of orthogonal corners, and (ii), $\delta_\xi F(g_{\mu\nu}) = 0$, where F are the boundary conditions chosen.

The set of diffeomorphisms that respect Dirichlet boundary conditions satisfy

$$
\bar{q}^\rho_\mu \bar{q}^\sigma_\nu \delta_\xi g_{\rho\sigma}|_{\mathcal{T}} = 0 \quad \Leftrightarrow \quad \bar{D}_{(\mu}\xi_{\nu)} = 0. \tag{8.1.25}
$$

These are boundary Killing vectors. These are not required to be isometries of the whole spacetime, so this condition does not restrict the bulk metric.

In the case of York's mixed boundary conditions, fixing the conformal metric is preserved by conformal Killing vectors (CKV) of the boundary,

$$(\bar{q}^{\rho}_{\mu}\bar{q}^{\sigma}_{\nu} - \frac{1}{3}\bar{q}_{\mu\nu}\bar{q}^{\rho\sigma})\delta_{\xi}g_{\rho\sigma} \overset{\mathcal{T}}{=} 0 \quad \Leftrightarrow \quad \bar{D}_{(\mu}\xi_{\nu)} - \frac{1}{3}\bar{q}_{\mu\nu}\bar{D}\cdot\xi \overset{\mathcal{T}}{=} 0. \tag{8.1.26}$$

While the condition on the trace of the extrinsic curvature is given by $\mathcal{L}_{\xi}K = 0$.

Here we should point out how much using unit-norm normal to describe the boundary simplifies the calculation of residual diffeomorphisms. Had we used a different normal, anomalies would inevitably come into play. As a matter of fact, we did perform these calculations in [42], before understanding the nice results of Appendix D. To do this, we started from the variation (8.1.5). Specializing to a diffeomorphism, it can be written in the following form,

$$\delta_{\xi}K = -2(K^{\mu\nu} - \frac{s}{2}Kn^{\mu}n^{\nu})\nabla_{\mu}\xi_{\nu} - n^{\mu}(g_{\mu\nu}\Box + R_{\mu\nu})\xi^{\nu} + sn^{\mu}n^{\nu}n^{\rho}\nabla_{\mu}\nabla_{\nu}\xi_{\rho}, \tag{8.1.27}$$

which makes it conveniently manifest that it would vanish exactly for a Killing vector. Using the orthogonality and conformal boundary Killing properties of ξ, it reduces to

$$\delta_{\xi}K = -\frac{1}{3}KD\cdot\xi - R_{\mu\nu}\xi^{\nu}n^{\mu} + D_{\mu}(K^{\mu}_{\sigma}\xi^{\sigma}). \tag{8.1.28}$$

Imposing this to be zero, recalling the anomaly-freeness for the case of unit-norm normal, using the CKV condition (and restoring the bars and $s = 1$ appropriate to the time-like boundary we are interested in), we obtain a slightly nicer expression

$$\xi_{\sigma}D_{\mu}K^{<\mu\sigma>} = R_{\mu\nu}\xi^{\nu}n^{\mu}. \tag{8.1.29}$$

This scalar equation between components of ξ completes the restriction given by (8.1.26).

In [42] we left the question of residual diffeomorphisms in the case of Neumann boundary conditions open as the calculations were getting too messy. Taking the anomaly-freeness of the unit-norm normal into account, we can now spell them out as

$$\mathcal{L}_{\xi}\tilde{\Pi}^{\mu\nu} = 0 \quad \Rightarrow \quad \mathcal{L}_{\xi}\Pi^{\mu\nu} = D\cdot\xi\Pi^{\mu\nu}. \tag{8.1.30}$$

8.2 Kerr example

In this section we restore the $16\pi G$ factors. To get some further intuition about the meaning of these different charges, we consider their explicit values in the case of the Kerr

solution. Integrating the Komar form (8.1.8) on a 2-sphere at constant (t, r) for the two Killing vectors ∂_t and ∂_ϕ one gets

$$Q^{\mathrm{EH}}_{\partial_t} = \frac{M}{2G}, \qquad Q^{\mathrm{EH}}_{\partial_\phi} = -\frac{Ma}{G}. \tag{8.2.1}$$

This result is independent of r since the Noether current j^{EH}_ξ vanishes in vacuum for a Killing vector. This is the standard Noether charge for the Einstein-Hilbert Lagrangian. We now compute the Hamiltonian charge (7.1.9) associated with different boundary conditions. First of all, we observe that $I_\xi \vartheta$ is proportional to the Killing equation and thus vanishes for both ∂_t and ∂_ϕ. Since ∂_ϕ is tangential to the corner, $i_{\partial_\phi}\ell^b = 0$, and the Hamiltonian charge coincides with the Komar expression, $H_{\partial_\phi} = -Ma/G$. However, this is not the case for the charge generated by ∂_t. Using $n_t := n \cdot \partial_t = -(-g^{tt})^{-1/2}$, we evaluate

$$\int_{S^2} i_{\partial_t}\ell^b = \frac{b}{16\pi G}\int_{S^2} \bar{K} n_t \epsilon_S = -\frac{b}{4G}\left(r - M + \frac{\Delta}{a}\operatorname{arctanh}\frac{a}{r}\right), \tag{8.2.2}$$

where $\Delta := r^2 - 2Mr + a^2$. Adding this up to the Komar expression according to (7.1.9), we find

$$\begin{aligned} H_{\partial_t} &= \frac{b+2}{4G}M - \frac{b}{4G}\left(r + \frac{\Delta}{a}\operatorname{arctanh}\frac{a}{r}\right) \\ &= -\frac{b}{2G}r + \frac{3b+2}{4G}M - \frac{b}{6G}\frac{a^2}{r} + O(r^{-2}). \end{aligned} \tag{8.2.3}$$

This expression diverges linearly, as discussed above. We also notice that for $b = 0$ the expression coincides with the value of the Komar charge alone (8.2.1), as to be expected from the equivalence in the case of isometries of the Neumann charge discussed below (8.1.14). Adding the subtraction term, we arrive at the renormalized charge

$$H^{\mathrm{R}}_{\partial_t} = H_{\partial_t} - \frac{b}{2G} r n_t = \frac{b+2}{4G}M - \frac{b}{4Gr}\left(M^2 + \frac{2a^2}{3}\right) + O(r^{-2}), \tag{8.2.4}$$

where we used

$$n_t = -1 + \frac{M}{r} + \frac{M^2}{2r^2} + O(r^{-3}). \tag{8.2.5}$$

We see that the asymptotic value of the renormalized energy still depends on the choice of boundary conditions. For $b = 2$ we recover the usual energy of the Kerr spacetime, namely M. For mixed boundary conditions the asymptotic energy reduces to $2/3M$, and for Neumann boundary conditions to $M/2$.

We also remark that for all values of b, the quasi-local charge for the time-diffeomorphisms is r-dependent, a result familiar from the Brown-York papers. In this derivation based on (7.1.9), the r-dependence is introduced by the contribution of the boundary Lagrangian,

and captures the fact that the full quasi-local charge does not descend from an on-shell vanishing current as the Komar term alone.

In this Kerr example we found it natural to evaluate the energy using the (not hypersurface-orthogonal) Killing vector ∂_t, but a more general choice for the energy is to take the hypersurface orthogonal time-like vector Nn, which can always be introduced. For Kerr, these two choices asymptotically align and require the same subtraction term. The resulting value of the energy is also very similar: the difference turns out to appear only at order $O(r^{-3})$. To eliminate all reference to the choice of diffeomorphism, we can also use the definition (8.1.22) of generalized BY energy, which correspond to the generator of unit-lapse hypersurface-orthogonal diffeomorphisms. The resulting expression is slightly more involved than (8.2.4),[6] and with the help of Mathematica we find

$$
\begin{aligned}
E = &-\frac{r}{4G}\sqrt{\frac{\Delta}{a^2+r^2}}\left(b-2-2b\sqrt{\frac{a^2+r^2}{\Delta}}+\frac{br(r-M)}{\Delta}\frac{\sqrt{a^2+r^2}}{a}\operatorname{arctanh}\frac{a}{\sqrt{a^2+r^2}}\right. \\
&\left.+\frac{\sqrt{2}\,(r\Delta-M(r^2-a^2))}{a\Delta\sqrt{M\,(r\Delta+2M(2r^2+a^2))}}\operatorname{arctanh}\sqrt{\frac{2Ma^2}{a^2(2M+r)+r^3}}\right) \\
\overset{b=2}{=}\ &-\frac{r}{4G}\sqrt{\frac{\Delta}{a^2+r^2}}\left(\frac{2r(r-M)}{\Delta}\frac{\sqrt{a^2+r^2}}{a}\operatorname{arctanh}\frac{a}{\sqrt{a^2+r^2}}\right. \\
&\left.+\frac{\sqrt{2}\,(r\Delta-M(r^2-a^2))}{a\Delta\sqrt{M\,(r\Delta+2M(2r^2+a^2))}}\operatorname{arctanh}\sqrt{\frac{2Ma^2}{a^2(2M+r)+r^3}}-4\sqrt{\frac{a^2+r^2}{\Delta}}\right) \\
\overset{a=0}{=}\ &\frac{r}{G}\left(1-\sqrt{1-\frac{2M}{r}}\right)=\frac{M}{G}-\frac{M^2}{2Gr}+O(r^{-2}).
\end{aligned}
\tag{8.2.6}
$$

The expression with $b=2$ and $a=0$ can be recognized as the familiar BY result for Schwarzschild. Expanding the general expression (8.2.6) at spatial infinity, we find

$$
E=\frac{b+2}{4G}M-\frac{1}{6Gr}\left(3M^2+ba^2\right)+O(r^{-2}) \tag{8.2.7}
$$

The different choice of diffeomorphism is reflected by the different subleading terms, but the asymptotic value is the same as (8.2.4), in particular the dependence on the boundary conditions is the one already discussed.

[6] This can be understood because the choice of a Killing vector sets to zero the $I_\xi\vartheta$ contribution to the charge, whereas (8.1.22) sees this contribution as well.

8.3 Covariant surface charges with non-orthogonal corners

In this section we look at covariant phase space charge in the case of non-orthogonal corners, $\beta \neq 0$. We can distinguish two classes of observers, those at rest with respect to the space-like foliation Σ, and those at rest along the time-like boundary. We may refer to them as respectively unbarred and barred observers, as in [91]. At the corner, these are related by the boost transformation with rapidity β, see Fig. 5.1 and (2.1.23). Canonical methods, which fail to take into account the presence of leakage at the time-like boundary, can be used to compute charges for either class of observes [91]. However this may not be the case for covariant phase space methods. As reviewed earlier, the condition for the integrability of the covariant phase space charges requires a vanishing symplectic flux through the time-like boundary. Imposing boundary conditions at a non-orthogonal time-like boundary means that symplectic flux can a priori leak though the time-like evolution of an unbarred observer at the corner (for instance through late time null trajectories or time-like trajectories), see Fig. 8.1. Therefore, we will only construct covariant phase space charges for the barred observers.

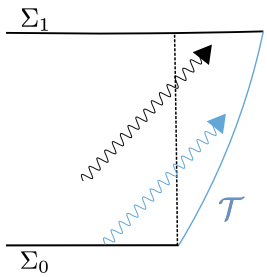

Fig. 8.1.: *If the time-like hypersurface at which the boundary conditions are imposed is tilted with respect to the time-like evolution of the corner of Σ, symplectic flux can a priori leak through it, as in the top (and darker) arrow. In that case, the procedure to get integrable charges applies only to the 'barred observers', namely those at rest with respect to a foliation of $\mathcal{T}$.*

Apart from this conceptual difference, there is also a technical mathematical question raised by non-orthogonal corners. The formula (7.1.9) can be applied straightforwardly, however which boundary Lagrangian and corner symplectic potential should we use? As shown by (5.3.18), the presence of non-orthogonal corners requires additional boundary terms in the action principle. How should these be taken into account in the prescription for the charges?

To address this question, we compute the charges using (7.1.9) with the same 3d boundary Lagrangian and corner symplectic potential as in the orthogonal case, simply

ignoring any possible additional terms. We will then compare the result with the one obtained with canonical methods. The discrepancy will provide the answer to our question. Notice that even using the same 3d boundary Lagrangian, a non-vanishing β shows up in the normals that appear in the formula. In particular, we can pull-back at the corner using the barred basis $(\bar{u}, \bar{n})$ instead of the unbarred basis (n, u). The three ingredients for the charge are then

$$q_\xi^{\text{EH}} \overset{S}{=} 2n_\mu u_\nu \nabla^{[\mu}\xi^{\nu]}\epsilon_S = -2\bar{n}_\mu \bar{u}_\nu \nabla^{[\mu}\xi^{\nu]}\epsilon_S, \tag{8.3.1}$$

$$i_\xi \ell^b = b\bar{K} i_\xi \epsilon_{\mathcal{T}} = \frac{b}{2}\bar{K}\bar{n}^\mu \xi^\nu \epsilon_{\mu\nu\rho\sigma} dx^\rho \wedge dx^\sigma \overset{S}{=} b\bar{K}\xi \cdot \bar{u}\, \epsilon_S, \tag{8.3.2}$$

$$I_\xi \vartheta^b \overset{S}{=} -2n_\mu u_\nu \nabla^{(\mu}\xi^{\nu)}\epsilon_S = -2\bar{n}_\mu \bar{u}_\nu \nabla^{(\mu}\xi^{\nu)}\epsilon_S. \tag{8.3.3}$$

Adding up using the barred expressions, we find

$$\begin{aligned} H_\xi^b &= \int_S q_\xi^{\text{EH}} + i_\xi \ell^b - I_\xi \vartheta^b = \int_S -2\bar{n}_\mu \bar{u}_\nu (\nabla^{[\mu}\xi^{\nu]} - \frac{b}{2}\bar{K}\bar{n}^\mu \xi^\nu - \nabla^{(\mu}\xi^{\nu)})\epsilon_S \\ &= -2\int_S \xi^\mu \bar{u}^\nu (\bar{K}_{\mu\nu} - \frac{b}{2}\bar{q}_{\mu\nu}\bar{K})\epsilon_S. \end{aligned} \tag{8.3.4}$$

It is very similar to the orthogonal result, with the same dependence on the extrinsic curvature, but this time taken along $\bar{u} = \cosh\beta n - \sinh\beta u$: The normal to Σ in the orthogonal case is replaced by its projection along the non-orthogonal boundary $\mathcal{T}$.

To manipulate further this expression, we use the decomposition (8.1.15) of the diffeomorphism, in terms of unbarred lapse and shift to facilitate the comparison with the canonical result below. The restriction (8.1.24) to tangential diffeomorphisms implies a relation between the unbarred shift and lapse:

$$\xi \cdot \bar{n} = 0 \quad \Rightarrow \quad N^\mu \bar{n}_\mu = -N\sinh\beta. \tag{8.3.5}$$

The other scalar products give

$$\xi \cdot \bar{u} = -N\lambda, \qquad \xi \cdot n = -N, \qquad \xi \cdot u = -N\lambda\sinh\beta. \tag{8.3.6}$$

It is also convenient to decompose the shift vector as

$$N^\mu = N \cdot u\, u^\mu + N_{\text{S}}^\mu, \qquad N \cdot u = \xi \cdot u = -N\lambda\sinh\beta, \tag{8.3.7}$$

and N_{S}^μ are the components tangent to the corner. From the first of (8.3.6) it follows that the second integrand in (8.3.4) gives

$$-\frac{b}{2}\xi \cdot \bar{u}\bar{K} = \frac{b}{2}N\lambda\bar{K}. \tag{8.3.8}$$

For the first integrand, we use

$$\bar{u}^\nu \nabla_\mu \bar{n}_\nu = n^\nu \nabla_\mu u_\nu + \partial_\mu \beta, \tag{8.3.9}$$

which follows from the boost transformations (2.1.23), to rewrite in terms of (8.1.15)

$$\begin{aligned} \xi^\mu \bar{u}^\nu \bar{K}_{\mu\nu} &= \xi^\mu \bar{q}^\rho_\mu \bar{u}^\nu \nabla_\rho \bar{n}_\nu = \xi^\mu (n^\nu \nabla_\mu u_\nu + \partial_\mu \beta) \\ &= N(k - \nabla_\mu u^\mu) - N^\mu u^\nu K_{\mu\nu} + \dot{\beta}, \end{aligned} \tag{8.3.10}$$

where we used $\xi \cdot \bar{n} = 0$ in the second equality, and (2.1.32) as well as $\xi = \partial_t$ in the second line. From $u^\mu = \lambda (q\bar{n})^\mu$, see (2.1.24), we have that

$$\nabla_\mu u^\mu = \lambda \bar{K} + \lambda \nabla_{\bar{u}} \beta + \lambda \sinh \beta K. \tag{8.3.11}$$

Replacing this expression in (8.3.10), and using

$$N \lambda \nabla_{\bar{u}} = \nabla_t - N^\mu_{\mathrm{S}} \nabla_\mu \tag{8.3.12}$$

which follows from (8.3.7), we arrive at

$$\begin{aligned} \xi^\mu \bar{u}^\nu \bar{K}_{\mu\nu} &= N\Big(k - \lambda \bar{K} - \lambda \sinh \beta (K - u^\mu u^\nu K_{\mu\nu})\Big) - N^\mu_{\mathrm{S}} (u^\nu K_{\mu\nu} - \partial_\mu \beta) \\ &= N\Big(k - \lambda \bar{K} - \lambda \sinh \beta \, \jmath_{\vdash}\Big) + N^\mu_{\mathrm{S}} (\jmath_{\mathrm{S}\mu} + \partial_\mu \beta), \end{aligned} \tag{8.3.13}$$

where

$$\jmath_{\vdash} := -u^\mu u^\nu \Pi_{\mu\nu} = K - u^\mu u^\nu K_{\mu\nu} = \gamma^{\mu\nu} K_{\mu\nu}, \tag{8.3.14}$$

$$\jmath^\mu_{\mathrm{S}} := -\gamma^{\mu\rho} u^\nu \Pi_{\rho\nu} = -\gamma^{\mu\rho} u^\nu K_{\rho\nu}, \tag{8.3.15}$$

are respectively the radial (or normal) and tangential momentum [91]. Adding up with (8.3.8), we find

$$H^b_\xi = -2 \int_S \left(N\Big(k + \frac{b-2}{2} \lambda \bar{K} - \lambda \sinh \beta \, \jmath_{\vdash}\Big) + N^\mu_{\mathrm{S}} (\jmath_{\mathrm{S}\mu} + \partial_\mu \beta) \right) \epsilon_S. \tag{8.3.16}$$

This formula provides the energy-momentum decomposition of the $b-$generalized Brown-York quasi-local charge with boundaries at non-orthogonal corners, in terms of (8.1.15) and using (8.3.5). We remark that it depends on the boundary Lagrangian via b, but not on c and thus not on the corner Lagrangian. For $b = 2$, it reproduces correctly (4.6) of [91].[7] Notice that the contribution of the radial momentum becomes

[7] To see the equivalence, we first observe that the notational translation from our paper to theirs is $(\beta, \lambda, \tanh \beta, u^\mu, K_{\mu\nu}, k) \mapsto (-\theta, 1/\gamma, -v, n^\mu, -K_{\mu\nu}, -k)$, and then recall their definitions $\bar{N} = N/\gamma, \bar{\varepsilon} = \gamma k - \gamma v \jmath_{\vdash}, \bar{\jmath}^\mu_{\mathrm{S}} = \jmath^\mu_{\mathrm{S}} - \gamma^{\mu\nu} \partial_\nu \theta$.

mixed with the energy because the radial shift component $N \cdot u$ is proportional to lapse, as a consequence of (8.3.5). To keep track of the momentum components separately from the energy, we can also rewrite (8.3.16)as

$$H^b_\xi = -2 \int_S \left(N \left(k + \frac{b-2}{2} \lambda \bar{K} \right) + N \cdot u \, \jmath_\vdash + N^\mu_{\mathrm{S}} (\jmath_{\mathrm{S}\mu} + \partial_\mu \beta) \right) \epsilon_S. \tag{8.3.17}$$

A few brief comment about the WZ charges are in order. Even with non-orthogonal corners they are still improved Noether charges with boundary Lagrangian b^{WZ}, namely case (a). In fact, the presence of a second normal in ϑ will make some of the boundary diffeomorphisms anomalous, but (8.1.4) still holds, and the GHY Lagrangian remains covariant even with non-orthogonal corners. Therefore no corner shift is needed to obtain the WZ charges. The BY formulas follow indeed from (7.2.13) with b given by the trace-K Lagrangian also with non-orthogonal corners [34, 42].[8] Let us also comment about the importance of the contribution of [34]. If we relax the corner-orthogonality condition, we have $\vartheta \neq 0$ hence condition 0 is violated. This brings us outside of the hypothesis used in [32, 33]. However conditions 1 and 2 are still valid. The crucial insight of [34] was to show that the modification (7.1.3) of the symplectic two-form caused by ϑ is not only acceptable, but indeed leads to the correct Brown-York formulas in the case of non-orthogonal corners. This calculation is an example of the generalized WZ prescription (7.3.3), and the insight on the relevance of the redefined symplectic form ω' played an important role in the general developments reviewed in Section 7.1.

8.4 Surface charges from canonical methods

In this section we review the canonical construction of [120, 125, 91], and show that it extends to the mixed and Neumann boundary conditions. We consider directly the general case of non-orthogonal corners. In particular, we will show that (i) the boundary terms recast the kinetic terms in the form appropriate to the chosen boundary conditions, and (ii) one reproduces the same expressions obtained with covariant phase space methods for orthogonal corners.

We start from the b-generalized trace-K action (5.3.15), and replace

$$R\epsilon = L^{\mathrm{ADM}} + 2\nabla_\mu (n^\mu K - a^\mu)\epsilon. \tag{8.4.1}$$

[8] This should be compared with the $3+1$ canonical calculation done in the next section, where the 2d Hayward corner term is needed in order to obtain the BY formulas with non-orthogonal corners. Its role is to secure the right Legendre transform on the boundary. There appears to be no relation between the Hayward term in the action and the corner shift (7.1.10) in the covariant improved Noether charge, which we use only to remove non-covariance from the boundary Lagrangian and satisfy the WZ conditions, and is not needed here.

From Stokes theorem,

$$\int_M \nabla_\mu(n^\mu K - a^\mu)\epsilon = \int_{\Sigma_0}^{\Sigma_1} K\epsilon_\Sigma + \int_{\mathcal{T}} (\sinh\beta\, K - \bar{n}\cdot a)\epsilon_{\mathcal{T}}. \tag{8.4.2}$$

Therefore, (5.3.15) can be rewritten as follows,

$$S = \int L^{\text{ADM}} + (2-b)\int_{\Sigma_0}^{\Sigma_1} K\epsilon_\Sigma + \int_{\mathcal{T}} (b\bar{K} + 2\sinh\beta\, K - 2\bar{n}\cdot a)\epsilon_{\mathcal{T}} + c\int_{S_0}^{S_1} \beta\epsilon_S. \tag{8.4.3}$$

The first two integrals above give the bulk terms on the space-like slices. The ADM Lagrangian density is

$$\tilde{L}^{\text{ADM}} := N\sqrt{q}(\mathcal{R} + K_{\mu\nu}^2 - K^2) = \tilde{\Pi}^{\mu\nu}\dot{q}_{\mu\nu} - N\tilde{\mathcal{H}} - N^a\tilde{\mathcal{H}}_\mu - 2\partial_\mu(\tilde{\Pi}^{\mu\nu}N_\nu), \tag{8.4.4}$$

where

$$\tilde{\mathcal{H}} := \frac{1}{\sqrt{q}}(\tilde{\Pi}_{\mu\nu}^2 - \frac{1}{2}\tilde{\Pi}^2) - \sqrt{q}\mathcal{R}, \qquad \tilde{\mathcal{H}}_\mu := -2\tilde{q}_{\mu\nu}D_\rho\Pi^{\nu\rho} \tag{8.4.5}$$

are the Hamiltonian and spatial diffeo constraints, and the boundary term is the one giving rise to the ADM momentum

$$P(\vec{N}) := 2\int_S \Pi^{\mu\nu}u_\mu N_\nu\, \epsilon_S. \tag{8.4.6}$$

The second integral can be rewritten as the spacetime integral of $(2-b)\partial_t(K\sqrt{q}) = \frac{1}{2}(b-2)\partial_t\tilde{\Pi}$. This combines with the kinetic term of the ADM Lagrangian, giving

$$\tilde{\Pi}^{\mu\nu}\dot{q}_{\mu\nu} + \frac{b-2}{2}\dot{\tilde{\Pi}} = \begin{cases} \tilde{\Pi}^{\mu\nu}\dot{q}_{\mu\nu} & b=2 \\ -q_{\mu\nu}\dot{\tilde{\Pi}}^{\mu\nu} & b=0 \\ \tilde{P}^{\mu\nu}\dot{q}_{\mu\nu} + \frac{4}{3}\sqrt{q}\dot{K} & b=2/3 \end{cases} \tag{8.4.7}$$

The resulting polarization is in agreement with the discussion in Section 5.3.1. In other words, different choices of b correspond to different boundary conditions, and the role of the second integral in (8.4.3) is to adjust the kinetic term to the chosen coordinate-momentum pair. As we will see shortly, the Hayward term plays the same role for the boundary kinetic term.

The last two integrals in (8.4.3) will reveal the value of the sourface term corresponding to the energy. To make this explicit, we use first Stokes theorem to rewrite

$$\int_S^{S'} \beta\,\epsilon_S = \int_{\mathcal{T}} \bar{D}_\mu(\beta\bar{u}^\mu)\,\epsilon_{\mathcal{T}} = \int dt \int_S d^2x\, \partial_t(\beta\sqrt{\gamma}), \tag{8.4.8}$$

where in the last step we decomposed $\mathcal{T}$ into its t foliation with space-like leaves S. Then, using the relation (2.1.32) between extrinsic curvatures, the last two integrals in (8.4.3) give

$$\int_{\mathcal{T}} \ell_{\mathcal{T}} := \int_{\mathcal{T}} \left(2\cosh\beta\, k + (b-2)\bar{K} + (c-2)\nabla_{\bar{u}}\beta + c\beta\bar{D}_\mu\bar{u}^\mu\right)\epsilon_{\mathcal{T}}. \tag{8.4.9}$$

However, this surface term is not yet the contribution to the energy, because it is not in Hamiltonian form: both $\nabla_{\bar{u}}$ and the Hayward term contain time derivatives, and a Legendre transform is needed to read the correct Hamiltonian. This procedure was explained in [91] for $c = 2$, and in [34] for $c = 0$.[9] To make time derivatives explicit, we use (8.3.12) and the second equality in (8.4.8). Then (8.4.9) gives

$$\int_{\mathcal{T}} \ell_{\mathcal{T}} = \int dt \int_S d^2x \left[(c-2)\sqrt{\gamma}\dot{\beta} + c\beta\dot{\sqrt{\gamma}} + \sqrt{\gamma}\Big(N(2k + (b-2)\lambda\bar{K}) + 2N_{\mathrm{S}}^\mu\partial_\mu\beta\Big)\right], \tag{8.4.10}$$

where we used the relation (2.1.29) between determinants to replace $\epsilon_{\mathcal{T}}$ with $N\lambda\sqrt{\gamma}$. The first two terms give a polarization of the phase space according to the chosen boundary conditions:

$$(c-2)\sqrt{\gamma}\dot{\beta} + c\beta\dot{\sqrt{\gamma}} = \begin{cases} -2\sqrt{\gamma}\dot{\beta} & c = 0 \\ 2\beta\dot{\sqrt{\gamma}} & c = 2 \end{cases} \tag{8.4.11}$$

For $c = 2$, $\ell_{\mathcal{T}}$ contains the boundary kinetic term $2\beta\partial_t\sqrt{\gamma}$, which is in Dirichlet form pdq. The boundary momentum is then $p_\gamma = 2\beta$, and the Legendre transform gives

$$\ell_{\mathcal{T}} = p_\gamma\dot{\sqrt{\gamma}} - \mathcal{H}_{\mathcal{T}}, \tag{8.4.12}$$

$$\mathcal{H}_{\mathcal{T}} = -2\sqrt{\gamma}\Big[N\Big(k + \frac{b-2}{2}\lambda\bar{K}\Big) + N_{\mathrm{S}}^\mu\partial_\mu\beta\Big]. \tag{8.4.13}$$

For $c = 0$, the boundary kinetic term is instead $-2\sqrt{\gamma}\partial_t\beta$, which is of Neumann/York form $-qdp$, according to Table **??**. The momentum is now $p_\beta = -2\sqrt{\gamma}$, and the Legendre transform gives

$$\ell_{\mathcal{T}} = p_\beta\dot{\beta} - \mathcal{H}_{\mathcal{T}}, \tag{8.4.14}$$

with precisely the same Hamiltonian (8.4.13) again. This is the correct surface contribution to the Hamiltonian, and it is independent of c, as in the covariant phase space result. The

[9] This point was overlooked in the first preprint version of [42], and led us to the erroneous conclusion that there was a discrepancy between the covariant and canonical charges for non-orthogonal corners.

only effect of the Hayward term is to adjust the boundary kinetic term from Dirichlet to Neumann/mixed form. The total action is thus

$$S = \int dt \left[\int_\Sigma d^3x \left(\tilde{\Pi}^{\mu\nu} \dot{q}_{\mu\nu} + \frac{b-2}{2} \dot{\tilde{\Pi}} - N\tilde{\mathcal{H}} - N^a \tilde{\mathcal{H}}_\mu \right) \right. \\ \left. + \int_S d^2x \Big((c-2)\sqrt{\gamma}\dot{\beta} + c\beta\dot{\sqrt{\gamma}} - \sqrt{\gamma}\Big(E(N,\beta) + P(\vec{N},\beta) \Big) \Big) \right], \tag{8.4.15}$$

where

$$E(N,\beta) = -2 \int_S N \left(k + \frac{b-2}{2} \lambda \bar{K} \right) \epsilon_S, \tag{8.4.16}$$

$$P(\vec{N},\beta) = 2 \int_S (\Pi^{\mu\nu} u_\mu N_\nu - N_{\mathrm{s}}^\mu \partial_\mu \beta)\, \epsilon_S = -2 \int_S \Big(N \cdot u\, \jmath_\vdash + N_{\mathrm{s}}^\mu (\jmath_{\mathrm{s}\mu} + \partial_\mu \beta) \Big) \epsilon_S. \tag{8.4.17}$$

The charges coincide perfectly with (8.3.17), for all values of b and β. The canonical and covariant results thus match for all boundary conditions considered.

As a final remark, let us say that a discrepancy for non-orthogonal corners could have been expected, since a corner Lagrangian is needed for the variational principle and it was not included in the application of (7.1.9). The reason why it does not happen is that the corner term can always be considered as part of the space-like boundary instead of the time-like boundary, as thus it enters the specification of the state, and not of the phase space [34]. We confirm this, and what we have seen is that by keeping track of the space-like boundary terms, one can read the form of the boundary kinetic terms associated with the chosen boundary conditions.

Charges on null boundaries

9

Having discussed the different symmetry groups associated with a larger or smaller background structure, in this Chapter we review how the covariant phase space allows one to associate Noether charges and Hamiltonian generators to these symmetries. In particular, we will discuss how the choice of polarization affects the definition of charges and their fluxes, and highlight the role played by anomalies.

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In this chapter we address the problem of non-integrability of the charge using the Wald-Zoupas procedure as described in Chapter 7, which aims at prescribing a (possibly unique) set of charges requiring them to coincide with the canonical generators when a physically identified flux vanishes. The preferred flux is selected from the equivalence class (7.1.1) based on covariance and physical criteria, for instance such that the charges are constant under conservative boundary conditions, or for perturbations around special solutions corresponding to stationary spacetimes. We then showed how this procedure can be extended to include corner contribution, anomalies and field-dependent diffeomorphisms. We also showed under which conditions the resulting WZ charges can be identified as Noether charges for a specific choice of boundary Lagrangian (see also [34, 85, 40, 38]).

In Chapter 7, we defined covariance as the equality of field space and spacetime Lie derivatives. As the writing of this thesis was coming to an end, we have realised that anomaly-freeness is, in fact, a better notion of covariance and we use it in this chapter. For the moment, we consider only field-independent diffeomorphisms, because this is sufficient to understand the symmetry groups described in Chapter 6. In this case anomaly-freeness coincides with the equality of the two Lie derivatives, and we can follow the procedure outlined in Chap. 7. The case with field-dependent diffeomorphism is discussed at the end in Sec 9.6, where we explain why we changed our mind regarding the appropriate definition of covariance.

Starting from $\theta = \theta^{\mathrm{EH}}$ and a given hypersurface, we select a preferred symplectic potential θ' in the equivalence class (7.1.1) satisfying the three criteria listed in Sec 7.2. If the preferred θ' and its Lagrangian $L' = L + d\ell$ are covariant, then the formulas (4.3.12)

and (4.3.18) are still valid with primes everywhere. The importance of the condition 2 is then clear: when θ' vanishes, the Noether charges coincide with the canonical generator for field-independent diffeomorphisms. Furthermore, they are automatically conserved in the subset of the phase space satisfying the conditions of case I or II. The new Noether charges are related to those of θ by (7.1.9)

However, there is a caveat. In spite of the covariance requirement of condition 1, the selection process may introduce anomalies. This happens if the preferred θ' is associated to a new Lagrangian $L' = L + d\ell$ whose boundary term is anomalous: $a'_\xi := \Delta_\xi \ell \neq 0$. In this case the new charges do not satisfy Noether's theorem in its original form (4.3.10), because that relies on the covariance of L, specifically on the fact that $\delta_\xi L = \pounds_\xi L = d i_\xi L$. If the boundary Lagrangian is anomalous we have instead $\delta_\xi L' = d(i_\xi L' + a_\xi)$ and the formula becomes

$$j'_\xi := I_\xi \theta' - i_\xi L' - a'_\xi \hat{=} dq'_\xi, \qquad dj'_\xi \hat{=} 0. \tag{9.0.1}$$

Condition 2 is no longer sufficient to guarantee the conservation of the Noether charges q'_ξ. This potential problem is avoided thanks to condition 1. In fact, the pull-back of the Hamiltonian generators on the lateral boundary gives

$$-\underleftarrow{I_\xi \omega} = \delta I_\xi \theta' - (\delta_\xi - \pounds_\xi)\theta' - d i_\xi \theta' \hat{=} \delta(dq'_\xi + a'_\xi) - (\delta_\xi - \pounds_\xi)\theta' - d i_\xi \theta'. \tag{9.0.2}$$

Here we used condition 0 and $d\theta' \equiv 0$ (that follows since θ' is only defined after pull-back) in the first equality, and (9.0.1) in the second. If condition 1 holds the generator is integrable once the preferred flux is subtracted:

$$-I_\xi \omega + d i_\xi \theta' = \delta I_\xi \theta' \hat{=} \delta(dq'_\xi + a'_\xi). \tag{9.0.3}$$

Furthermore, condition 1 also implies that the Lagrangian anomaly must be spacetime-exact, specifically that $a'_\xi = ds_\xi$ where $\delta s_\xi = -A'_\xi$ and A'_ξ is the symplectic anomaly of the preferred θ' [40]. This makes it possible to define the WZ charges

$$q^{\mathrm{WZ}}_\xi := q'_\xi + s_\xi. \tag{9.0.4}$$

They satisfy the flux-balance laws

$$dq^{\mathrm{WZ}}_\xi \hat{=} I_\xi \theta', \qquad d\delta q^{\mathrm{WZ}}_\xi \hat{=} -\underleftarrow{I_\xi \omega} + d i_\xi \theta'. \tag{9.0.5}$$

It follows that they are conserved *and* provide Hamiltonian generators when θ' vanishes, be it for conservative boundary conditions, or leaky boundary conditions around stationary configurations.

This is the Wald-Zoupas prescription. We stress that its keystone is condition 1. Without condition 1, we would in fact be stuck with (9.0.1), without the possibility to use

(9.0.2) to justify and be guaranteed that the anomaly can be reabsorbed in the definition of the charge. With a'_ξ (or even just part of it) still on the LHS of (9.0.1), stationarity of θ' would fail to give conserved charges, hence condition 2 would entirely lose its physical relevance.

We also stress that even if the selected θ' is covariant and the Lagrangian anomaly a'_ξ drops out of the flux-balance laws in the end, anomalous transformations can still be present, since

$$I_\xi \theta' = p\pounds_\xi q + p\Delta_\xi q. \tag{9.0.6}$$

This anomaly contribution is physically correct, because it is the right quantity to have so that background structures don't contribute to the flux: remember in fact that $\delta_\xi l^\mu \overset{\mathcal{N}}{=} 0$ but $\pounds_\xi l^\mu \neq 0$, for instance.

The question we posed in Chapter 7 is whether the WZ charges (9.0.4) can always be interpreted as Noether charges (7.1.9) for some boundary Lagrangian. The answer was yes iff $s_\xi = -\Delta_\xi c$ for some local 2-form c constructed out of the fields and the background structure, and in this case the correct boundary Lagrangian is the anomaly-free choice $\ell + dc$. So we have two approaches to constructing charges: the Noether charge, based on selecting specific θ' and ℓ. And the WZ prescription, based on selecting only a preferred θ'. The convergence of the two approaches is obtained when the WZ charges can be derived as improved Noether charges with the choice of ℓ determined by a condition of covariance.

An important point to appreciate here is that the charges are anomalous even if the symplectic potential is not. This is simply because the charges depend on the symmetry vector fields ξ which are generally anomalous. What one should require then is that the charge anomaly is sourced only by the ξ's, namely that

$$\Delta_\chi q_\xi = \frac{\partial g}{\partial \xi}\Delta_\chi \xi = -\frac{\partial g}{\partial \xi}\pounds_\chi \xi = -q_{[\chi,\xi]}. \tag{9.0.7}$$

This is precisely what is guaranteed for the WZ charges thanks to the covariance requirements of symplectic potential and boundary Lagrangian.

Note that is also possible to drop condition 0, and consider a generalized WZ prescription based on 1 and 2 alone [34, 35]. This generalization will not be needed here, but it is necessary in order to obtain Brown-York charges at finite time-like boundaries with non-orthogonal corners as shown explicitly in Chapter 8, and for the generalized angular momentum of super-rotations at future null infinity [81]. The WZ charges are still given by (9.0.4), but where (7.1.9) has a non-trivial ϑ term, and ω' replaces ω in (9.0.5).

Summarizing, our viewpoint as put forward in Chapter 7 and also in [35] is that the crux of the WZ procedure is really condition 1 (or its alternative version as anomaly-freeness). Condition 0 can be dropped, and condition 2 should be interpreted as a

framework rather than a unique set-up, meaning that different notions of conservative boundary conditions or stationarity conditions can be considered, in order to describe different physical problems.

9.1 Wald-Zoupas conditions on null hypersurfaces

We now study which of the family of symplectic potentials $\theta^{(b,c)}$ in (5.3.50) satisfies the WZ conditions. We will recover the results of [36], show how they change for different polarizations, and how they can be extended to the relaxed phase space with $\delta k \neq 0$.

$$\theta^{(b,c)} = \left[\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu + (2-b)\delta k + (2-c)\delta\theta\right]\epsilon_{\mathcal{N}} - (bk + (c-1)\theta)\delta\epsilon_{\mathcal{N}}, \tag{9.1.1}$$

Condition 0. Looking at (5.3.24), we see that it requires

$$\delta l^\mu = \delta l_\mu \overset{\mathcal{N}}{=} 0. \tag{9.1.2}$$

These can be satisfied without any restriction on the dynamics, and correspond to the gauge fixing (6.4.11).

Condition 1. From Section 6.3 we have have identified the family $b = 0$ and c arbitrary as covariant for either one of the two conditions in (9.1.2).

Condition 2, case I. We can distinguish two options for conservative boundary conditions. If we impose $\delta\epsilon_{\mathcal{N}} = 0$, then (5.3.50) vanishes for $\delta l^\mu = \delta\sigma^{\mu\nu} = 0$, which in turns imply $\delta\theta = 0$, and $(2-b)\delta k = 0$. This gives us Dirichlet boundary conditions for $b = 2$, and strengthened Dirichlet conditions including $\delta k = 0$ for $b \neq 2$. If we don't impose $\delta\epsilon_{\mathcal{N}} = 0$, then necessarily $b = 0$ and $c = 1$, and we find the conformal boundary conditions (5.3.58) of the York polarization.

Condition 2, case II. If we want stationarity to correspond to a shear and expansion-free surface (equivalent to a NEH in vacuum), as in [36], then we need

$$\left[\pi_\mu \delta l^\mu + (2-b)\delta k + (2-c)\delta\theta\right]\epsilon_{\mathcal{N}} - bk\delta\epsilon_{\mathcal{N}} = 0. \tag{9.1.3}$$

The most general solution of this equation is

$$b = 0, \qquad \delta l^\mu \overset{\mathcal{N}}{=} 0, \qquad \delta k \overset{\mathcal{N}}{=} \frac{c-2}{2}\delta\theta. \tag{9.1.4}$$

For $c = 2$, we recover the result of [36]: the symplectic potential

$$\theta^{\text{CFP}} = \sigma^{\mu\nu}\delta\gamma_{\mu\nu}\epsilon_{\mathcal{N}} - \theta\delta\epsilon_{\mathcal{N}} \tag{9.1.5}$$

meets all the WZ criteria (and was argued in [36] to be unique under these conditions). For arbitrary c, we find a 1-parameter family of covariant WZ potentials that satisfy the same stationarity condition,

$$\theta^{c} = \sigma^{\mu\nu}\delta\gamma_{\mu\nu}\epsilon_{\mathcal{N}} - (c-1)\theta\delta\epsilon_{\mathcal{N}}. \tag{9.1.6}$$

This includes the conformal polarization for $c = 1$. These potentials are associated with a phase space in which inaffinity is allowed to vary, but in a way fully constrained by the expansion via (9.1.4). These properties holds also if we further relax $\delta l_\mu \overset{\mathcal{N}}{=} 0$. This defines a new phase space that could be interesting to further explore. The difficulty with this generalization of the CFP result is that the symmetry vector fields appear to be field-dependent and not universal, as we saw in Section 6.4.

9.2 Stationarity on flat light-cones

The notion of stationarity as shear and expansion-free used above is solidly based on physical grounds: shear and expansion-free hypersurfaces capture the idea that no radiation is going through the surface, and include standard stationary examples such as non-expanding horizons and Killing horizons. However, this notion is not exhaustive of stationarity understood as lack of radiation, as there are plenty of null hypersurfaces which possess shear and expansion even in the absence of gravitational waves. Consider for instance a light-cone in flat Minkowski space: its expansion grows, hence the CFP flux (9.1.5) is non-zero, even though there is no actual dynamics taking place. This is an objectable feature, which disconnects charge conservation from absence of radiation. It motivates the question whether one can find a different potential leading to a vanishing flux on both non-expanding horizons and flat light-cones. This is not possible within the framework above, because the CFP symplectic potential is unique under the requests of covariance and stationarity on NEH.What we propose is to relax the notion of stationarity, from $\theta' = 0$ to:

Case III: $I_\xi\theta' = 0$ for *every* symmetry vector field on the stationary solutions.

This condition is weaker than $\theta' = 0$, therefore the immediate consequence is a lost of uniqueness. Why this appears bad at first sight, our point is that it can be compensated by the larger set of solutions that can be included. For the family of anomaly-free potentials,

$$I_\xi\theta^{c} = \left[\sigma^{\mu\nu}\pounds_\xi\gamma_{\mu\nu} + \pi_\mu\delta_\xi l^\mu + 2\delta_\xi k + (2-c)\delta_\xi\theta\right]\epsilon_{\mathcal{N}} - (c-1)\theta\delta_\xi\epsilon_{\mathcal{N}}. \tag{9.2.1}$$

For this to vanish on a NEH, we need (9.1.4) with δ replaced by δ_ξ. But if we further require that the symmetry group preserves the NEH, then we must have $\delta_\xi\theta = 0$. We thus recover the conditions $\delta_\xi l^\mu = \delta_\xi k = 0$ and the CFP group. Having done so, the flux

vanishes on a NEH for any c, hence the weaker stationarity condition leaves an ambiguity. But this ambiguity is eliminated because we can now extend the set of stationary solutions. Consider a flat light-cone. The CFP group is reduced to Weyl transformations and super-rotations only, because the surface is only semi-complete. The shear vanishes, and

$$I_\xi \theta^c = (2-c)\delta_\xi \theta \epsilon_{\mathcal{N}} - (c-1)\theta \delta_\xi \epsilon_{\mathcal{N}} = (2-c)\pounds_\xi \theta \epsilon_{\mathcal{N}} - (c-1)\theta \pounds_\xi \epsilon_{\mathcal{N}} - \theta w_\xi \epsilon_{\mathcal{N}}. \quad (9.2.2)$$

To evaluate this flux, we can exploit the fact that the potential is class-III invariant to make a convenient choice of normal. We take affine coordinate λ and

$$l = \lambda \partial_\lambda, \qquad \Rightarrow \qquad \theta = 2, \qquad \epsilon_{\mathcal{N}} = \frac{\sqrt{-g}}{f} d^3x = \lambda d\lambda \wedge \epsilon_S. \quad (9.2.3)$$

Then the first term in (9.2.2) vanishes, and so does the anomaly, see (6.4.17). Only super-translations have non-vanishing anomaly, but these are not part of the symmetry vector field. Finally for the second term we have $\pounds_\xi \epsilon_{\mathcal{N}} = \xi^\lambda d^3x = W\epsilon_{\mathcal{N}}$. This term doesn't vanish, but can be eliminated if we set $c = 1$.

We conclude that the flux (9.2.1) vanishes for NEH *and also* for a flat light cone if $c = 1$. Stationarity in the weaker sense of case III allows one to solve the problem of a non-vanishing flux on a flat light-cone. This process selects again a unique potential, and this is the conformal one instead of the CFP one.[1]

9.3 CFP phase space with varying inaffinity

We have seen that the covariance condition is satisfied also in the larger phase space with δk left arbitrary. For this space the symmetry vector fields span the group G^{aST} given in (6.4.10), with $\delta l_\mu = 0$ required to have closure under the Lie bracket, while $n_\mu \delta_\xi l^\mu = 0$ is required to make the anomaly 'canonical', namely depend on the symmetry parameters and f but not on the extension, see (6.4.7).

On the other hand the stationarity condition is violated for both the original WZ definition and the weaker definition of case III. In other words, the vector fields in G^{aST} which are not in G^{CFP} have a non-vanishing flux on a non-expanding horizon, given by $I_\xi \theta = 2(\pounds_\xi k - \pounds_l w_\xi - k w_\xi)\epsilon_{\mathcal{N}}$. In affine coordinates this reduces to $f\ddot{\tau}$ and vanishes only for the CFP vectors.

[1] One may wonder whether this idea can be taken one step further to identify a symplectic potential with vanishing flux on shear-full hypersurfaces in flat spacetime. We don't know if this could be done, but it seems difficulty since even a non-radiative shear can introduce a dynamical evolution of the null hypersurface. The case of a light-cone is special in this sense because even if the area changes, the expansion has a constant representative.

9.4 Charges

In this section we use the formula (7.1.9) to write the charges for arbitrary variations, arbitrary ξ and any choice of (b, c) in the boundary Lagrangian. We will give some general observations and then comment on the special features that occur when the additional restrictions are added, and specific values of b and c chosen, in parallel with the discussion on the fluxes of the previous section. In particular we will explain how to reduce them to the Wald-Zoupas prescription.

In the following we take n adapted to the cross sections S of the null boundary on which we are evaluating the charges. This fixes the class-I ambiguity, and it is a choice useful to simplify various expressions. Namely,

$$n = \frac{1}{fg^{\lambda\Phi}} d\lambda \tag{9.4.1}$$

where λ is an arbitrary parameter labelling the cross sections of a space-like foliation of λ. We can then write the pull-backs as follows. For the Komar charge,

$$q_\xi = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\nabla^\mu\xi^\nu dx^\rho \wedge dx^\sigma \overset{S}{=} 2n_\mu l_\nu \nabla^{[\mu}\xi^{\nu]}\epsilon_S. \tag{9.4.2}$$

For the boundary Lagrangian,

$$i_\xi \ell^{(\mathrm{b,c})} = -(bk + c\theta) i_\xi \epsilon_{\mathcal{N}} \overset{S}{=} (bk + c\theta)\xi \cdot n\, \epsilon_S. \tag{9.4.3}$$

For the corner symplectic potential,

$$I_\xi \vartheta^{\mathrm{EH}} \overset{S}{=} (n_\mu \delta_\xi l^\mu + n^\mu \delta_\xi l_\mu)\epsilon_S = (n_\mu \pounds_\xi l^\mu + n^\mu \pounds_\xi l_\mu + 2w_\xi)\epsilon_S, \tag{9.4.4}$$

where we used (6.1.6). The Lie derivatives satisfy the following identity,

$$n^\mu \pounds_\xi l_\mu + n_\mu \pounds_\xi l^\mu = 2n_\mu l_\nu \nabla^{[\mu}\xi^{\nu]} + 2n_\mu \xi^\nu \nabla_\nu l^\mu. \tag{9.4.5}$$

The first term on the RHS coincides with the pull-back of the Komar 2-form. The second is a contraction of the Weingarten map, thanks to the restriction of ξ to be tangent.

Adding up according to (7.1.9), we get

$$\begin{aligned} q^{\mathrm{c}}_\xi &= -2[n^\mu \xi^\nu (W_{\nu\mu} - \frac{1}{2}(bk + c\theta) g_{\mu\nu}) + w_\xi]\epsilon_S \\ &= -[2n^\mu \xi^\nu (W_{\nu\mu} - W g_{\mu\nu}) + \xi \cdot n((2-b)k + (2-c)\theta) + 2w_\xi]\epsilon_S \\ &= -[2\xi^\mu(\eta_\mu - \theta n_\mu) + \xi \cdot n((2-b)k + (2-c)\theta) + 2w_\xi]\epsilon_S. \end{aligned} \tag{9.4.6}$$

These are the Noether charges for the full group $\mathrm{Diff}(\mathcal{N})$, and any polarization in the family (5.3.48). No restrictions except for $l_\mu \delta l^\mu = 0$. The anomaly is also responsible for the shift between these charges and a Brown-York-like expression based on the Weingarten map alone, as was found in [39] for the Dirichlet polarization. One of our initial motivation was to study whether this shift could be removed changing polarization. As we can see from the general expression (9.4.6), this is not the case for the polarizations considered. They only affect the numerical coefficients that would give rise to the trace term W, and not the anomaly contribution. But the anomaly term is on the other hand very important: it leads to the area being the charge associated with a constant Weyl rescaling, arguably the most famous gravitational charge for horizons.

The c-term in the boundary Lagrangian is a total derivative and could have been moved to ϑ. This move leaves the charges invariant, because corner shifts in the boundary Lagrangian only matter if the shift is anomalous [38, 35], and $\theta\epsilon_{\mathcal{N}}$ is not. Using ℓ^{D} versus $\ell^{\mathrm{D}\prime}$ in the case of Dirichlet polarization, or ℓ^{Conf} versus nothing in the case of conformal polarization, is irrelevant.

We can now make the earlier discussion on the need of a physical prescription for the charges concrete. First of all, they are not class-III invariant, and depend explicitly on the choice of normal representative taken. Secondly, they depend on the extension $\bar{\xi}^{\Phi}$ of the symmetry vector fields through the anomaly w_ξ. Therefore if this extension is a free parameter, the charges can be given an arbitrary value even if the intrinsic parameters on the hypersurface are kept fixed. Further problems appear if we look at their flux, which is given by

$$\underleftarrow{dq}_\xi^{(\mathrm{b,c})} \hat{=} I_\xi \theta^{(\mathrm{b,c})} - a_\xi^{(\mathrm{b,c})} \tag{9.4.7}$$

$$\begin{aligned} &= [\sigma^{\mu\nu} \pounds_\xi \gamma_{\mu\nu} + \pi_\mu \pounds_\xi l^\mu + (2-b)\pounds_\xi k - 2\pounds_l w_\xi \\ &+ (2-c)\pounds_\xi \theta - 2\theta w_\xi]\epsilon_{\mathcal{N}} - [bk + (c-1)\theta]\pounds_\xi \epsilon_{\mathcal{N}}. \end{aligned} \tag{9.4.8}$$

The problem with this flux is the same one that plagues the Komar charges: it can be non-zero flux even on a NEH or in Minkowski space for generic diffeomorphisms tangent to a generic null hypersurface! This is thus a good example of the earlier discussion, that a generic version of Noether theorem may be unpractical, and one needs some additional input to reorganize it in a more useful way. We know that the dynamical content of the flux-balance laws is the constraint equations, namely for a null hypersurface the Raychaudhuri and Damour equations, as discussed for instance in [49]. These can be derived from (9.4.7) for for $\xi = l$ and for $\xi = Y$ respectively. The point is that for arbitrary b and c the terms in $\dot{\theta}$ and $\dot{\bar{\eta}}_\mu$ will appear scattered on both LHS and RHS, and that without phase space restrictions there will be gauge-dependent terms in both charge and flux that cancel out in the final equation.

We now impose $n_\mu \delta l^\mu = 0$, which allows us to rewrite the anomaly as in (6.4.7). The term $\pounds_l \ln f$ obtained in this way can be expanded out and reabsorbed into the shifts of k and η_μ to $\bar{k}$ and $\bar{\eta}_\mu$, giving

$$q_\xi^{(\mathrm{b,c})} = -2[\xi \cdot \bar{\eta} - \xi \cdot n(\frac{c}{2}\theta + \frac{b}{2}k - \bar{k}) - \dot{\tau}]\epsilon_S. \tag{9.4.9}$$

It is interesting to discuss the behaviour of this quantity under class-III transformations. The terms in $\bar{k}$ and $\bar{\eta}_\mu$ are partially invariant, namely they change only if the rescaling is induced by a reparametrization of Φ. But the same reparametrization changes also the anomaly, see (6.1.6)! The two changes perfectly compensate, and these terms are fully class-III invariant. The only non-class-III invariant part is the term in k, and this was to be expected since for $b \neq 0$ we already know that the flux cannot be covariant for $k \neq 0$. If we restrict to $b = 0$ we thus have an expression for the charges

$$q_\xi^{\mathrm{c}} = -2[\xi \cdot \bar{\eta} - \xi \cdot n(\frac{c}{2}\theta - \bar{k}) - \dot{\tau}]\epsilon_S \tag{9.4.10}$$

which is perfectly class-III invariant. This discussion shows the importance of the role played by the anomaly piece in the charge expression.

Furthermore for $b = 0$ the boundary Lagrangian is anomaly-free, hence these charges are Wald-Zoupas, provided we satisfy the covariance and stationarity condition. The covariance can be satisfied taking $\delta l_\mu = 0$, which does not restrict the symmetry group. Hence so far the group is still the full $\mathrm{Diff}(\mathcal{N})$. However as explained earlier, the stationarity condition requires restrictions on the variations. In particular we need $\delta l^\mu = \delta k = 0$, whether we use the original or the weaker condition of stationarity. This step alone is responsible for reducing the symmetry group to G^{CFP}. We also remark that the restriction (6.4.14) implies that $\pounds_l \bar{\xi}^\Phi = 0$, therefore the partially class-III invariant quantities $\bar{k}$ and $\bar{\eta}_\mu$ become fully class-III invariant, and accordingly, their anomaly vanishes. If we use affine coordinates, $\bar{k} = 0$ and with the parametrization (6.4.16) of the symmetry vector fields we can write

$$q_\xi^{\mathrm{c}} = -2[Y \cdot \bar{\eta} + cT\theta_\lambda + (\frac{c}{2}\lambda\theta_\lambda - 1)W]\epsilon_S, \tag{9.4.11}$$

where θ_λ denotes the expansion of the affine generator. In particular: the super-rotation charge aspect is the shifted twist $\bar{\eta}$, the super-translation charge aspect is the expansion, and the Weyl charge aspect is the area minus the expansion, and reduces to the area on a NEH.

For $c = 2$, we recover the CFP charges of [36]. On a NEH they reduce to

$$q_\xi^{\mathrm{c}} \overset{\mathrm{NEH}}{=} -2[Y \cdot \bar{\eta} - W]\epsilon_S, \tag{9.4.12}$$

matching the result of [45], and are conserved and independent of the polarization parameter c. They never vanish for any finite area, and one has to invoke a limiting procedure to argue that they vanish in Minkowski. On a flat light-cone we have $\bar{\eta}_\mu = 0$, and they reduce to

$$q^c_{W,Y} \overset{l.c.}{=} -2[(c-1)W]\epsilon_S. \tag{9.4.13}$$

They are not conserved unless $c = 1$, in which case they vanish. The conformal polarization does not only has better stationarity property, but it also makes it more natural to assess that the charges vanish in flat spacetime.

For the larger groups G^{aST} and $\text{Diff}(\mathcal{N})$ the charges are well defined but do not satisfy the stationarity condition neither in the original sense of case II, nor in the weaker sense of case III.

9.5 Second-order perturbations around flat light-cones

In the previous section, we identified covariant charges and fluxes which are conserved on a flat light-cone, and vanish exactly on each cross sections. This occurs for the special choice of polarization $b = 0$ and $c = 2$, and for symmetry vector fields belonging to the CFP group. We now study their evolution when the light-cone is perturbed by gravitational radiation. Since the charges and fluxes are covariant, we can use any normal representative, and we pick the choice (9.2.3) with a constant expansion in flat light-cones and vanishing anomaly. The flux-balance law is

$$dq_\xi = (\sigma^{\mu\nu}\pounds_\xi\gamma_{\mu\nu} + \pounds_\xi\theta)\epsilon_{\mathcal{N}}, \tag{9.5.1}$$

where we remember that only super-rotations Y and Weyl super-translations W are allowed as symmetries. For a pure Weyl transformation,

$$dq_W = W(2\sigma^2_{\mu\nu} + \pounds_l\theta)\epsilon_{\mathcal{N}}. \tag{9.5.2}$$

We now solve for $\pounds_l\theta$ using the Raychaudhuri equation, in a perturbative expansion around a NEH. We assume that the shear is infinitesimal, and write $\theta = 2 + \theta_1 + O(\sigma^4)$. Linearizing the Raychaudhuri equation we find

$$\pounds_l\theta_1 = -\theta_1 - \sigma^2_{\mu\nu}, \tag{9.5.3}$$

whose solution is

$$\theta_1(\lambda, x^A) = -\frac{1}{\lambda}\int_{\lambda_0}^{\lambda}\sigma^2_{\mu\nu}(\lambda', x^A)d\lambda'. \tag{9.5.4}$$

Here λ_0 can be taken as the value of affine parameter after which the perturbation enters the light-cone, with the tip located at $\lambda_0 = 0$. Plugging this result in (9.5.2) and integrating over a region $\Delta\mathcal{N}$ of the null hypersurface, we get

$$\Delta q_W = \int_{\Delta\mathcal{N}} W \left(\lambda^3 \sigma_\lambda^2 + \int_{\lambda_0}^{\lambda} \lambda'^2 \sigma_{\lambda'}^2 d\lambda' \right) d\lambda \wedge \epsilon_S + O(\sigma^4), \tag{9.5.5}$$

where $\sigma_\lambda := \sigma/\lambda$ is the shear of the affinely parametrized normal. The flux is made of two pieces. The first one, proportional to the shear squared, represents the energy of weak gravitational waves entering the light cone locally. It is a tidal heating term. The second piece is related to the gravitational waves which have entered the light-cone since λ_0. Unlike the first term, this terms is not local, and depend on the history of the gravitational waves which entered the outgoing light from $\lambda = 0$. Hence, even in the absence of local shear, we expect a variation of the charge if some weak gravitational waves have previously entered the outgoing light cone. This is because if it is the case, spacetime is not flat anymore in the local surrounding, as there is some energy localized inside the outgoing light cone, the energy of the gravitational waves which have previously entered. Hence, this second term is a memory effect. Furthermore, the flux of the future pointing diffeomorphisms is positive, and so the charge increases, underlying the fact the gravitational waves carry positive energy.

Next, we take a pure super-rotation, so $\xi \in TS$. In affine coordinates, $\xi^\mu \partial_\mu = Y^A \partial_A$ and the charge density associated to this tangent vector is just given by $q_Y = -2\bar{\eta}_\mu \xi^\mu \epsilon_S = Y^A P_A \epsilon_S$, with variation

$$dq_Y = d(Y^A P_A \epsilon_S) = -2\pounds_l(\xi^\mu \bar{\eta}_\mu). \tag{9.5.6}$$

We can now compute the flux of the charge using our flux balance law. We make use of the linearized Raychaudhuri equation (9.5.3) to express the linearized expansion θ_1 in terms of the shear (9.5.4). For a tangent diffeomorphism $\xi^\mu \partial_\mu = Y^A \partial_A$, we have for small perturbations around the flat light cone

$$\begin{aligned} I_\xi \theta^Y &= \sigma^{\mu\nu} \pounds_\xi \gamma_{\mu\nu} \epsilon_{\mathcal{N}} + \pounds_\xi \theta \epsilon_{\mathcal{N}} \\ &= 2D_\mu(\sigma^{\mu\nu} v_\nu)\epsilon_{\mathcal{N}} - 2\xi^\mu D_\nu \sigma^\nu_\mu \epsilon_{\mathcal{N}} - \left(\frac{1}{\lambda} \int_0^\lambda \sigma^{\mu\nu} \xi^\alpha D_\alpha \sigma_{\mu\nu} d\lambda' \right) \epsilon_{\mathcal{N}} + O(\theta_1^2) \\ &= -2Y^A D_B \sigma^B_A \epsilon_{\mathcal{N}} + O(\sigma_{\mu\nu}^2) \end{aligned} \tag{9.5.7}$$

where we disregarded the first term in the equation (9.5.7) because it was a total divergence which does not give any contribution upon integration on the compact cross sections. Therefore, at leading order, we find that the charge variation is given by the angular derivative of the shear along the cross sections. We notice that the charge

variation is proportional to the shear at leading order, not the square of the shear. The coefficient P_A appearing in the charge is the coefficient of the first order expansion of g_{uA} in affine coordinate, and so it has the interpretation of an angular momentum for small perturbations around the flat background. Therefore the charge associated to the tangent diffeomorphsims is modified by the the angular momentum of the weak gravitational waves crossing the outgoing light cone. The equation relating the charge variation (9.5.6) to the flux is a linearization of the more general Damour equation [129].

9.6 Wald-Zoupas prescription with field-dependent diffeomorphisms

We have seen that if the symmetry vector fields include field-dependent diffeomorphisms, the notions of covariance defined by matching Lie derivatives and by anomaly-freeness can give different answers. The question is then which of the two should be used as condition 1 of the WZ prescription. In Chapter 7 we used the matching of Lie derivatives. Following discussions with Chandrasekaran and Flanagan on the topics presented in [54, 105] we were motivated to reconsider this choice. In this section we compare the two options, explain the logic that motivated our choice in [35], and give a new argument that tilts the balance in favour of the anomaly-free option which is the one favoured by Chandrasekaran and Flanagan. We also briefly explain why this difference was in the end not important to understand the application of the formalism to the BMS group at future null infinity studied in Chapter 7.

To answer this question, we look at (9.0.2), which is still valid if $\delta\xi \neq 0$. This formula suggests to take the matching-Lie-derivative options. In fact if we require $(\delta_\xi - \pounds_\xi)\theta' = 0$ (9.0.3) is still valid, so we can proceed as before subtracting the preferred flux to obtain an integrable generator. Furthermore, condition 1 also implies that the Lagrangian anomaly must be spacetime-exact, specifically that $a'_\xi = ds_\xi$ where now $\delta s_\xi = -q'_{\delta\xi} - A'_\xi$ [35]. The WZ charges are then defined as in (9.0.4) with this new s_ξ, and still satisfy the flux-balance laws (9.0.5). This notion of covariance appears thus naturally when talking about integrability of the charges.

Let us consider now requiring instead $\Delta_\xi\theta' = 0$, which as we discussed in Section 9.6 is a simpler notion of background-independence. Furthermore, it is this property that is satisfied by the standard symplectic potential, $\Delta_\xi\theta^{\text{EH}} = 0$, while $I_{\delta\xi}\theta^{\text{EH}} \neq 0$ in general. Imposing the anomaly-free condition, the term $I_{\delta\xi}\theta'$ appears in the RHS of (9.0.2), and the previous procedure no longer works if this is not zero. We can then attempt a charge definition subtracting this new term as well, so that (9.0.3) is replaced by

$$-\underleftarrow{I_\xi\omega} + I_{\delta\xi}\theta' + di_\xi\theta' = \delta I_\xi\theta' \hat{=} \delta(dq'_\xi + a'_\xi). \tag{9.6.1}$$

However we are no longer guaranteed that a'_ξ is spacetime exact. Therefore this condition alone is not sufficient to define the charges, and must be supplemented by the additional condition that

$$I_{\delta\xi}\theta' = dX \tag{9.6.2}$$

be spacetime exact. This additional property suffices to obtain WZ charges when the symmetry vectors are field dependent. The charges are still given by (9.0.4), this time with $\delta s_\xi = -q'_{\delta\xi} - A'_\xi - X$, and are as before conserved and Hamiltonian generators when θ' vanishes. Notice that (9.6.2) is guaranteed if the final boundary Lagrangian is covariant. So one can rephrase the two independent prescriptions $\Delta_\xi\theta' = 0$ and (9.6.2) also as $a'_\xi = A'_\xi = 0$. We conclude that even if the notion of anomaly-freeness may appear more natural, it is less economical, in that it is not sufficient per se to guarantee the existence of the WZ charges, and one has to require also (9.6.2).

We now show that the stronger covariance requirement just stated is actually necessary if one goes beyond the flux-balance properties (9.0.5), and requires also that the charges gives a faithful representation of the symmetry group under Poisson brackets. This can be already seen requiring as before that the only anomaly of the charges comes from the vector field ξ. If they are field-dependent, (9.0.7) is replaced by

$$\Delta_\chi q_\xi = \frac{\partial g}{\partial \xi}\Delta_\chi \xi = \frac{\partial g}{\partial \xi}(\delta_\chi - \pounds_\chi)\xi = q_{\delta_\xi \chi} - q_{[\![\chi,\xi]\!]} = q_{\Delta_\chi \xi}. \tag{9.6.3}$$

Now applying the anomaly operator to the flux formula (9.0.1) we get [40]

$$\Delta_\chi q_\xi \mathrel{\hat{=}} -q_{[\![\chi,\xi]\!]} + q_{\delta_\xi \chi} + I_\xi A_\chi + i_\xi a_\chi. \tag{9.6.4}$$

If the charge is of the WZ type, then $A_\xi = a_\xi = 0$, and we recover (9.0.7). Conversely if we choose a covariant potential, $\Delta_\xi\theta = dA_\xi = (\delta_\xi - \pounds_\xi - I_{\delta\xi})\theta = (\delta_\xi - \pounds_\xi)\theta - dq_{\delta\xi}$. From this we have $I_\xi A_\chi = -q_{\delta_\chi \xi}$ and $\Delta_\chi q_\xi \mathrel{\hat{=}} -q_{[\![\chi,\xi]\!]}$, failing to reproduce (9.0.7). Therefore even if the covariance criterium seems more economical, and in some cases easier to define, it is only the anomaly-free criterium that guarantees the correct behaviour of the charge algebra in the case of field-dependent diffeomorphisms.

Conclusions 10

Overview of the reviewed content

While the primary focus of the thesis was on original contributions, the earlier chapters served to lay down the foundation by reviewing and gathering scattered knowledge into a coherent narrative. Chapter 2 offers a self-contained review of the geometry of hypersurfaces embedded in spacetime. While the results reviewed here are well-known, having them systematically compiled serves a twofold purpose. First, it provides a unified reference point for readers, allowing for seamless continuity in understanding the material that follows. And second, it simplifies the often arduous task of navigating through various treatments of these concepts in existing literature—especially when it comes to null hypersurfaces. Different authors often introduce their own notational and definitional conventions, which can make comparative reading difficult. Our review aims to serve as a cohesive primer, thus simplifying this task.

The third chapter is dedicated to a review of Noether's theorem in its standard formulation. We go beyond a cursory treatment by emphasising some of the subtler aspects, particularly concerning active versus passive transformations. Examples are provided to offer a deeper understanding of the subject. The chapter also explores Noether's theorem in the Hamiltonian framework, and reviews Noether's theorem for gauge theories that will be of particular relevance for generally covariant theories which will later become the center of our attention.

In the fourth chapter, we explore the covariant phase space formalism—a framework that unifies the Lagrangian and Hamiltonian descriptions in a manner that emphasizes their complementary features. This forms the basis upon which our subsequent original research is constructed. Additionally, we introduce the notion of anomalies, which serve as a measure of non-covariance. The concept of anomalies is not only fascinating in its own right but also plays a critical role in the topics that will follow.

In summary, the initial chapters of this dissertation serve as a comprehensive foundation, both consolidating existing knowledge and introducing important frameworks and concepts. This groundwork equips the reader with the essential tools and knowledge to appreciate the original contributions made in the following chapters. By assembling an integrated review and background, we grease the wheels for a more focused and effective discussion on gravitational charges, which are the main topic of the rest of the text. In the chapters that follow, we build on this strong foundation to present our original contributions, which will be detailed in the rest of this conclusion.

Boundary conditions

The fifth chapter serves as a pivot, transitioning from the review of foundational principles to our original contributions. In this chapter, we study allowed boundary conditions from the point of view of the action principle, specifically in the context of General Relativity. While the section on non-null boundaries discusses well-known results, the section on null boundaries goes beyond mere restatement. It decomposes the null symplectic potential in the most general form, emphasizing the importance of geometrically motivated quantities for defining polarization by highlighting the importance of Class-III-invariance. This approach allows for a careful analysis of constraints put forth by different authors, clearing the path for meaningful comparisons.

The central focus of this chapter is the issue of allowed polarizations. We identify multiple options, each previously discussed in different contexts. In the timelike case, the York option, which we favor, has recently gained interest from both mathematical [21] and physical perspectives [130]. We scrutinize the commonly used null Dirichlet polarization for null boundaries for its shortcomings, as outlined in [51]. Our examination of the non-Class-III-invariant nature of the total action in this polarization leads us to propose a covariant alternative: the conformal polarization. This eliminates the need for the counterterm suggested in [51], resolving the problematic dependence on parametrization of the null generator of the boundary. This finding has implications for the 'action=complexity' proposal, where the non-uniqueness of the null counterterm has sparked debate [131, 132]. Our work offers a way to bypass this controversy, eliminating the need for a counterterm and its associated issues. However, we note that open questions remain—specifically concerning the corner contribution to the action. The Hayward Lagrangian exhibits parametrization dependence in this context. In the timelike York polarization, which bears geometric similarity to the null conformal one, we argue mathematically and physically that adding the Hayward Lagrangian may be unnecessary. We suggest fixing $\delta\beta = 0$ at the corner as part of the York boundary conditions. This position is supported by our assertion that β is not solely part of either the induced metric or the extrinsic geometry, but rather a combination of lapse and shift. Thus, different β configurations can be viewed as equivalent to differently rotated or boosted observers. For null boundaries, we refrain from making these claims as the nature of β is less clear, and it remains to be seen whether the restriction $\delta(2k + \theta) = 0$ affects variations of β in some way. The interplay between intrinsic and extrinsic geometry on null hypersurfaces, and the corner term difference between fixing the 2d induced metric versus fixing the shear, motivates us to tread cautiously and leave this issue for future research.

Using Sachs' constraint-free data on a null hypersurface, we are able to disentangle physical degrees of freedom from gauge which helps in determining which conservative boundary conditions can be relaxed to allow for gravitational flux through the boundary. In this case one can still impose boundary conditions and here their role is not eliminating

the flux as in the conservative case, but rather preserving certain boundary structures. In Chapter 6 we have highlighted the role of various different symmetry groups that arise as variations are restricted and the universal structure is strengthened: $\mathrm{Diff}(\mathcal{N})\otimes\mathbb{R}$, $\mathrm{Diff}(\mathcal{N})$, G^{aST}, G^{CFP} and G^{NEH}, plus the case of field-dependent diffeomorphisms (6.4.19).

We have used a spacetime description for all quantities, and have found it convenient to describe everything using a NP tetrad. This introduces non-dynamical background quantities that transform non-covariantly in the phase space, but we have shown that independence from the background structure can be easily kept under control. Quantities independent of the choice of NP tetrad are covariant, and can be identified from their invariance under a joint class-I and class-III transformation of the tetrad. Furthermore the extra structures are relevant to the Carroll literature, hence our formalism can be immediately used in that context.

Understanding covariance as independence from the choice of NP representative has immediate application as clarification of the ambiguities of null boundary terms that arise if Dirichlet boundary conditions are imposed in a weak way. We have further discussed why reducing the ambiguity requires working with strengthened Dirichlet boundary conditions, whose meaning is to preserve a choice of affine coordinates on the boundary. Alternatively, the ambiguity can be reduced allowing the inaffinity to change but in a way fixed by the rate of change of the boundary area, namely by the expansion. This choice provides a definition of conformal boundary conditions on null hypersurfaces.

Charges and fluxes

Continuing on Chapter 4's discussion on integrability of hamiltonian charges, in Chapter 7 we present two ways to obtain integrable charges by choosing the charge-flux split – the Wald-Zoupas and the improved Nother charge prescriptions. We analyse the physical requirements of the WZ prescription, namely the need for covariance in relation to background structures and the conceptual utility of 'stationarity' in radiative scenarios, as spelled by conditions 1 and 2 in Section 7.2. We point out that, the WZ approach also imposes limitations, specifically concerning what we termed as 'condition 0,' which restricts variations in corner symplectic potential ϑ. This limitation has been critiqued and extended in the literature [34, 14] by introducing the improved Noether charge, useful notably when aiming to recover the Brown-York charges with non-orthogonal corners, as detailed in Chapter 8, or when exploring the generalizations of the BMS group [133].

We show two significant results. First, we highlight that the WZ charges do not map directly onto the improved Noether charges in a general sense, unless specific criteria are met, namely if (7.1.16) admits solutions. We leave the question of existence of general solutions to this equation as an open question, answering which would provide a valuable insight into the universality or limitations of the WZ framework. Second important result that we found is that the WZ prescription is capable of accommodating anomalies

and field-dependent diffeomorphisms under particular covariance conditions[1], enriching its applicability and explaining why it works in situations one would not expect it to work prior to this analysis. We referred to anomalies allowed in the WZ approach as soft because of their physical meaning at future null infinity. We have shown that these anomalies can be related to the allowed field-dependent diffeomorphisms, via (7.2.9) which has an important consequence – there are situations in which the anomalies can be completely ignored. Explicitly, this explains the validity of the WZ's computation of the BMS charges, the lack of understanding of which was in part the motivation for this whole analysis. Nonetheless, we believe it does not impact the relevance of taking anomalies into account. We hope that our new derivation of the BMS charges in the main text and in Appendix C shows that using the formalism with anomalies enriches our understanding of the mathematics as well as the physics.

One example of what can be learned is how anomalies capture the difference between future null infinity and a null hypersurface at a finite distance, such as an event horizon. It is well known that the BMS symmetries are different from the symmetries of a null hypersurface in spacetime. For example, in BMS, dilations are not independent while on a finite null hypersurface they are independent and their associated charge is given by the area. Matching the two symmetries is possible relaxing the fall-off conditions so that the BMS group is enhanced to the group G^{CFP} or BMSW, as it was called in [46]. The key elements of this generalization have been worked out in earlier in [107], but with different restrictions that do not allow the full Diff(S) symmetry. They were studied further in [134, 80]. See also [112, 135, 87, 136] for related work on charges at horizons.

However, it was argued [44] that the two copies of the BMSW group, one at finite boundary, and the other at $\mathcal{I}$, are actually representations of different sectors of the universal corner algebra, defined from purely geometric considerations. We know that in both cases the anomaly comes from the boundary normal. While at a finite distance the background structure only provides the location of the boundary, at future null infinity it also provides the compactification factor. As a consequence, the metric on the cross-section is anomaly-free at finite distance, but not on $\mathcal{I}$. This introduces a second source of anomalous transformations, given by the inhomogeneous terms of the metric functionals on $\mathcal{I}$. It would be interesting to study the limit of a finite null boundary to $\mathcal{I}$, and compare how the two versions of G^{CFP} fit into the big picture painted in [44].

In Chapter 8, we looked into the role of conservative boundary conditions in Hamiltonians generated via covariant phase space methods. Our study extends the existing Dirichlet-based analyses [32, 34, 14] to both York and Neumann conditions. We demonstrated that these charges strongly depend on selected boundary conditions. More

[1] Remember that in this context, covariance was defined as $\delta_\xi = \mathcal{L}_\xi$, which boils down to $\Delta_\xi = -I_{\delta\xi}$. It would be interesting to compare WZ charges obtained using this condition with those coming from the employing the slicing method [87] as they seem similar in spirit with anomalies being compensated for by a carefully chosen field dependence of the generators.

boundary conditions	*quantity held fixed*	*value of b*	*quasi-local energy*	*Kerr (renormalized)*
Dirichlet	$q_{\mu\nu}$	2	k	M
York	$(\hat{q}_{\mu\nu}, K)$	$2/3$	$k - 2\bar{K}/3$	$2M/3$
Neumann	$\tilde{\Pi}^{\mu\nu}$	0	$k - \bar{K}$	$M/2$

Table 10.1.: *Different values of the energy computed as the generator of time-diffeomorphisms at conservative boundaries.*

explicitly, in the example of Kerr spacetime, we found that choosing boundary conditions with fewer components of the induced metric fixed, while fixing more components of the momentum, results in a lower energy value for time-like boundaries with positive extrinsic curvature. One may try to interpret this by saying that holding the extrinsic geometry fixed at the boundary instead of the intrinsic, results in energy is being stored on the boundary itself and removed from the system. These findings are summarized in Table 10.1.

It's important to emphasize that this dependence on boundary conditions is a unique feature of field theory, and is especially interesting for general relativity in particular. In finite-dimensional systems, energy does not depend on the choice of boundary conditions.[2] However, already in the case of the scalar field, whose variational principle we considered in (5.1.7), there is a discrepancy between Dirichlet and Neumann energies which manifests itself as a boundary term.[3] This observation led us to investigate analogous boundary condition dependencies in gauge theories. In these theories, as discussed in Chapter 3, the bulk charge serves purely as a constraint, rendering boundary terms as the only interesting component. Since the boundary terms are so sensitive to the choice of polarization, we checked how the charges respond to the use of alternative polarizations for Einstein-Hilbert gravity.

We performed calculations using both covariant and canonical methods, extending our analysis to include non-orthogonal corners. In this context, we identified two distinct classes of observers: those at rest with respect to the space-like foliation of spacetime,

[2] We can see this on the example of the Dirichlet and Neumann Lagrangians for a point particle, $L^{\text{D}} = \dot{x}^2/2$ and $L^{\text{N}} = -x\ddot{x}/2 = L^{\text{D}} + d\ell$, with $\ell = -x\dot{x}/2$. To compute the energy in the latter case, one can use the method of Ostrogradsky (see e.g. [137] for a modern description) and define two momenta

$$p_1 := \frac{\partial L^{\text{N}}}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L^{\text{N}}}{\partial \ddot{x}} = \frac{1}{2}\dot{x}, \qquad p_1 := \frac{\partial L^{\text{N}}}{\partial \ddot{x}} = -\frac{1}{2}x.$$

The energy is then given by

$$E^{\text{N}} := p_1\dot{x} + p_2\ddot{x} - L^{\text{N}} = \frac{1}{2}\dot{x}^2 \equiv E^{\text{D}},$$

matching the standard expression obtained with L^{D}.

[3] This discrepancy is given by

$$E^D = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2, \quad E^N = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 - \partial_a\left(\phi\partial^a\phi\right) = E^D - \partial_a\left(\phi\partial^a\phi\right). \tag{10.0.1}$$

and those at rest along the time-like boundary. At the corner, these two types are related by a boost transformation, which allows for a direct comparison of different notions of energy. The charges pick up 'tilting terms', indicating an explicit dependence on the boost between the normals. This phenomenon serves as another test passed by the improved Noether charge prescription. The covariant and canonical results align perfectly for both orthogonal and non-orthogonal corners, without requiring any formula adjustments. However, to achieve this match, in the canonical approach it was crucial to recognize that the boundary term contains an independent kinetic term, which must be transformed into Hamiltonian form via a Legendre transform. This example highlights the elegance of the covariant phase space formalism where the formulas automatically worked and gave the correct results. While the canonical calculations were technically involved, they served not only to convince us in the power of the CPS, but also to provide additional rationale for incorporating Neumann-type corner contributions in York's boundary conditions, as discussed in Chapter 2.

In thermodynamics, various forms of energy – such as internal energy, enthalpy, and free energy – are tied to distinct boundary conditions. For instance, setting the pressure is associated with defining enthalpy, while constraining the temperature is linked to free energy. These varying boundary conditions not only correlate with specific physical scenarios but also correspond to unique mathematical phase spaces. In this sense, the interpretation of energy might extend beyond the field's energy to include a component related to the boundary-condition-controlling device. So far all we have are analogies and we think that further discussions are necessary to clarify interpretation of these gravitational surface charges and their sensitivity to boundary conditions.

In any case, our results in Chapter 8 highlight a strong dependence of charges on boundary representation in general relativity, something already observed when changing variables and formulations (see e.g. discussions in [138, 124, 14, 139, 140, 141, 142]), and here found when changing boundary conditions. We anticipate that the impact of such a dependence at the classical level would be even more pronounced in quantum theory.

In Chapter 9 we have presented a general analysis of the charges and fluxes on an arbitrary null hypersurface, with arbitrary variations of the metric allowed. We have studied polarizations of the symplectic potential, their transformation and conservation properties, and explained their relation to the Wald-Zoupas prescription. The key idea we hold on to is that of covariance. Specifically, we prioritize quantities that do not rely on any background structures other than those that are intrinsic to the definition of the phase space itself. With this in mind, we identify a one-parameter family of covariant symplectic potentials.

Imposing stationarity as in the original Wald-Zoupas prescription, one recovers the unique symplectic potential of Chandrasekaran, Flanagan and Prabhu. The associated charges are all conserved on non-expanding horizons, but not on flat spacetime. We point

out the use of a weaker notion of stationarity that allows for selecting a unique set of charges that are conserved, and vanish on a flat light-cone, as opposed to charges obtained following the strong stationarity condition.

Furthermore, the flux of future-pointing diffeomorphisms at leading-order around an outgoing flat light-cone is positive and reproduces the tidal heating term plus a memory-like term. This general analysis allows applicability to a wide range of physical scenarios. It provides a clean slate from which one can accurately analyze gravitational phenomena, including charges, fluxes, dynamical notions of entropy, and are useful to clarify the interplay between different boundary conditions, charge prescriptions, and symmetry groups that can be associated with a null boundary.

Appendix: Charge algebra

A

A.1 Poisson algebra

The derivation in this Section follows [76]. The Poisson bracket between two Hamiltonians H_X and H_Y is given by:

$$\{H_X, H_Y\} = \delta_Y H_X \ , \tag{A.1.1}$$

where X, Y are symplectomorphisms in $T\Gamma$. It can be shown that this Poisson bracket is skew-symmetric due to the properties of symplectomorphisms. The explicit calculations are:

$$\begin{aligned} \{H_X, H_Y\} &= I_Y \delta H_X \\ &= -I_Y I_X \Omega \\ &= I_X I_Y \Omega \\ &= -I_X \delta H_Y \\ &= -\{H_Y, H_X\} \ . \end{aligned} \tag{A.1.2}$$

Moreover, the Poisson bracket is entirely determined by the symplectic 2-form Ω, as:

$$\{H_X, H_Y\} = I_X I_Y \Omega \ . \tag{A.1.3}$$

Now, we also examine spacetime vector fields ξ, ζ in TM. Their associated Poisson bracket is:

$$\{H_\xi, H_\zeta\} = \delta_\zeta H_\xi \ . \tag{A.1.4}$$

And the Lie bracket between these spacetime vector fields is:

$$[\xi, \zeta] = \mathcal{L}_\xi \zeta \ . \tag{A.1.5}$$

We explore how the Lie bracket of spacetime vector fields relates to the Poisson bracket of charges. For this, we use the field-space version of the second equation of (4.1.11)

$$I_{[\xi,\zeta]} = [\delta_\xi, I_\zeta] \ . \tag{A.1.6}$$

Upon applying this to Ω, it simplifies to:

$$I_{[\xi,\zeta]} \Omega = \delta I_\xi I_\zeta \Omega \ , \tag{A.1.7}$$

which introduces a new Poisson bracket:

$$I_{[\xi,\zeta]}\Omega = \delta\{H_\xi, H_\zeta\} \ . \tag{A.1.8}$$

Note that the identity

$$I_\xi \Omega = -\delta H_\xi \tag{A.1.9}$$

links the Noether charge H_ξ with the field space vector field ξ, via the symplectic 2-form Ω. If we consider two Noether charges like H_ξ and $H_\xi + \kappa$, they both relate to the same vector field ξ as long as $\delta\kappa = 0$. This relationship is cohomological, meaning it is invariant under such additions. We can also conclude that the Poisson bracket $\{H_\xi, H_\zeta\}$ and $\{H_\xi, H_\zeta\} + \kappa_{\xi,\zeta}$ correspond to the same Lie bracket vector field $[\xi, \zeta]$ as long as $\delta\kappa_{\xi,\zeta} = 0$. Now, by combining this with the identity (A.1.8) we find that

$$\{H_\xi, H_\zeta\} = -H_{[\xi,\zeta]} + \kappa_{\xi,\zeta} \tag{A.1.10}$$

where $\delta\kappa_{\xi,\zeta} = 0$.

This tells us that the Poisson bracket of charges can be thought of as a projective representation of the Lie bracket of symmetries. In other words, the charge algebra mimics the symmetry algebra but may include extra terms, known as central extensions. These central extensions are constants in the field space and commute with the rest of the algebra's generators.

Appendix: Phase space of null boundaries

B

B.1 Internal Lorentz transformations

The behaviour of all geometric quantities under a class-I transformation (2.2.6) can be easily computed, or read from [55] using the NP formalism. To that end, recall that

$$\sigma_{\mu\nu} = -\sigma\bar{m}_\mu\bar{m}_\nu + cc, \qquad \eta_\mu = -(\alpha+\bar{\beta})m_\mu + cc. \qquad \theta = -2\mathrm{Re}(\rho), \qquad k = 2\mathrm{Re}(\epsilon). \tag{B.1.1}$$

The fact that l is hypersurface-orthogonal hence geodesic fixes the two NP coefficients $\kappa = 0$ and $\rho = -\theta/2$. Apart from this, the formulas are general.

B.1.1 Class-I

Under (2.2.6), we have:

$$\gamma_{\mu\nu} \to \gamma_{\mu\nu} + 2\left(al_{(\mu}\bar{m}_{\nu)} + \bar{a}l_{(\mu}m_{\nu)} + |a|^2 l_\mu l_\nu\right), \qquad \epsilon_{\mathcal{N}} \to \epsilon_{\mathcal{N}}, \tag{B.1.2}$$

$$\sigma_{\mu\nu} \to \sigma_{\mu\nu} - 2(\sigma\bar{a}\bar{m}_{(\mu}l_{\nu)} + cc) - 2\mathrm{Re}(\sigma\bar{a}^2)l_\mu l_\nu \qquad \sigma \to \sigma, \qquad \theta \to \theta, \qquad k \to k, \tag{B.1.3}$$

$$\eta_\mu \to \eta_\mu - [a\bar{\sigma} + \bar{a}\,(k+\rho)]m_\mu + [\bar{a}\eta\cdot m - a(a\bar{\sigma} + \bar{a}\,(k+\rho))]l_\mu + cc. \tag{B.1.4}$$

We see that η_μ is invariant on a non-expanding horizon iff $k = 0$.

We now check invariance of the pulled-back standard symplectic potential (5.3.19). We first notice that the corner term ϑ^{EH} is invariant thanks to $m^\mu\delta l_\mu = 0$ and the pull-back. Of the bulk term, the third and fourth are invariant. Plugging the above transformations in the first and second term, and using $l_\mu\delta l^\mu = 0$, we obtain

$$(\sigma^{\mu\nu} + \frac{\theta}{2}\gamma^{\mu\nu})\delta\gamma_{\mu\nu} \to \mathrm{idem} + 2[(a\bar{\sigma} + \bar{a}\rho)m_\mu\delta l^\mu + cc], \tag{B.1.5}$$

$$2(\eta_\mu + kn_\mu)\delta l^\mu \to \mathrm{idem} - 2[(a\bar{\sigma} + \bar{a}\rho)m_\mu\delta l^\mu + cc], \tag{B.1.6}$$

from which the invariance of $\underset{\leftarrow}{\theta}^{\mathrm{EH}}$ follows immediately. The result holds also if $\delta a \neq 0$.

B.1.2 Class-III

Under (2.2.7), we have

$$\gamma_{\mu\nu} \to \gamma_{\mu\nu}, \qquad \epsilon_{\mathcal{N}} \to A^{-1}\epsilon_{\mathcal{N}}, \qquad \sigma^{\mu\nu} \to A\,\sigma^{\mu\nu}, \qquad \theta \to A\,\theta, \qquad k \to A(k + \pounds_l \ln A), \tag{B.1.7}$$

$$\eta_\mu \to \eta_\mu - \gamma^\nu_\mu\nabla_\nu \ln A, \qquad \omega_\mu \to \omega_\mu + \partial_\mu \ln A + l_\mu \pounds_n \ln A. \tag{B.1.8}$$

We now check invariance of the pulled-back standard symplectic potential (5.3.19). Using these transformations, we first derive

$$\vartheta^{\mathrm{EH}} \to \mathrm{idem} - 2\delta \ln A\, \epsilon_S. \tag{B.1.9}$$

It is class-III invariant for a field-independent rescaling, but not for a field-dependent one. The first of the bulk terms is invariant. The others give

$$-2\omega_\mu \delta l^\mu \epsilon_{\mathcal{N}} \to \mathrm{idem} - 2(k\delta \ln A + \pounds_{\delta l} \ln A)\epsilon_{\mathcal{N}}, \tag{B.1.10}$$

$$2\delta(\theta + k)\epsilon_{\mathcal{N}} \to \mathrm{idem} + 2\Big((\theta + k)\delta \ln A + \frac{1}{A}\delta(\pounds_l A)\Big)\epsilon_{\mathcal{N}}. \tag{B.1.11}$$

Adding up, we obtain

$$\underleftarrow{\theta}^{\mathrm{EH}} \to \mathrm{idem} + 2(\pounds_l + \theta)\delta \ln A\, \epsilon_{\mathcal{N}} - 2d(\delta \ln A\, \epsilon_S). \tag{B.1.12}$$

This is manifestly invariant for a field-independent rescaling. For field-dependent ones invariance follows from the cancellation between the bulk and corner terms thanks to the identity (2.2.13).

B.1.3 Anomalies and NP representatives

The background structure we use to describe a null hypersurface is a choice of NP tetrad. In this Appendix we prove that quantities that are independent of the choice of NP representative, namely invariant under both class I and III transformations, are also anomaly-free. Consider a generic functional F of the dynamical fields $\phi = g_{\mu\nu}$ and the background fields (Φ, l_μ, n_μ). Anomaly-freeness with respect to Φ is achieved restricting the diffeomorphisms to be tangent, so we assume to have done that in the following. The variation of F under a change of tetrad is

$$\delta_{(a,\alpha)} F = \frac{\partial F}{\partial l}\delta_{(a,\alpha)} l + \frac{\partial F}{\partial n}\delta_{(a,\alpha)} n, \tag{B.1.13}$$

where

$$\delta_{(a,\alpha)} l = \alpha l, \qquad \delta_{(a,\alpha)} n = -\alpha n + \bar{a} m + a\bar{m}, \qquad a \ll 1, \qquad \alpha \ll 1, \tag{B.1.14}$$

is the infinitesimal version of (2.2.6) and (2.2.7). This coincides with the anomalies (6.1.6) and (6.1.7) for $\alpha = -w_\xi$ and $a = m \cdot Z$. Taking this special values,

$$\delta_{(a,\alpha)} F = \frac{\partial F}{\partial l}\Delta_\xi l + \frac{\partial F}{\partial n}\Delta_\xi n = \Delta_\xi F. \tag{B.1.15}$$

Therefore the vanishing of the LHS implies that F is anomaly-free.

B.2 Alternative polarizations

Different choices of θ' can be obtained integrating by parts in field space writing $p\delta q = \delta(pq) - q\delta p$ for one or more canonical pairs, or by integrating by parts on the hypersurface and thus moving terms in and out of ϑ. Not all such manipulations are useful when looking for admissible boundary conditions, because they may lead to a symplectic potential which is not in diagonal form, or whose δq are not independent. In the main text we restricted attention to changes of polarization in the spin-0 sector only. In this Appendix we present two more changes that affect the boundary Lagrangian. We will assume that (5.3.54) holds, and don't consider integrations by part in spacetime, namely the corner term ϑ^{EH} is always the same. We can then start from (5.3.31).

Changing polarization in the spin-1 sector can be done using

$$\pi_\mu \delta l^\mu \, \epsilon_{\mathcal{N}} = \delta((\theta - 2k)\epsilon_{\mathcal{N}}) - l^\mu \delta(\pi_\mu \epsilon_{\mathcal{N}}). \tag{B.2.1}$$

This manipulation changes the boundary Lagrangian. It remains in the family (5.3.48), but with different numerical coefficients. As mentioned in the main text, the momentum π_μ is determined in terms of the shear of the null hypersurface by the Einstein's equations. Therefore it cannot be specified independently from the shear, hence boundary conditions based on this polarization would be consistent only if the boundary equations of motions are satisfied.

For the second change, we observe that the spin-2 and spin-0 sectors can be written in terms of a single tensor, so that

$$\underleftarrow{\theta}^{\text{D}} = \left[\Pi^{\mu\nu}\delta\gamma_{\mu\nu} + 2(\eta_\mu - \theta n_\mu)\delta l^\mu\right]\epsilon_{\mathcal{N}}, \tag{B.2.2}$$

where

$$\Pi^{\mu\nu} = B^{\mu\nu} - (\theta + k)\gamma^{\mu\nu} = \sigma^{\mu\nu} - \frac{1}{2}\left(\theta + 2k\right)\gamma^{\mu\nu}. \tag{B.2.3}$$

Notice the change in the spin-1 momentum, due to (5.3.26). Written in this form, one can consider a change in polarization of the first term, which is given by

$$\Pi^{\mu\nu}\delta\gamma_{\mu\nu}\epsilon_{\mathcal{N}} = -\gamma_{\mu\nu}\delta\tilde{\Pi}^{\mu\nu}d^3x - \delta\big(\left(\theta + 2k\right)\epsilon_{\mathcal{N}}\big). \tag{B.2.4}$$

One can also combine this with a change in the spin-1 sector via

$$(\eta_\mu - \theta n_\mu)\delta l^\mu \epsilon_{\mathcal{N}} = \delta(\theta\epsilon_{\mathcal{N}}) - l^\mu\delta\big((\eta_\mu - \theta n_\mu)\epsilon_{\mathcal{N}}\big), \tag{B.2.5}$$

and get another element of the same family. If we now restrict the variations to $\delta l^\mu = 0$ we have

$$\underleftarrow{\theta}^{\text{EH}} = -\gamma_{\mu\nu}\delta\tilde{\Pi}^{\mu\nu}d^3x - \delta(\theta\epsilon_{\mathcal{N}}) + d\vartheta^{\text{EH}}. \tag{B.2.6}$$

This is reminiscent of the Neumann form of the symplectic potential in the non-null case, however Π misses the η part of the extrinsic geometry. One could try to resolve this rewriting in terms of $W_{\mu\nu}$, which gives

$$\underleftarrow{\theta}^{\text{EH}} = [\gamma^{\mu\nu}\delta W_{\mu\nu} + \delta W + (2\eta_\mu + kn_\mu)\delta l^\mu + (\partial_n l^2 + k)n^\mu \delta l_\mu]\epsilon_{\mathcal{N}} + d\vartheta^{\text{EH}}. \tag{B.2.7}$$

So even if $W_{\mu\nu}$ contains the eta term missing in Π, it drops out because $\eta_\mu l_\nu \delta\gamma^{\mu\nu} = 0$.

B.3 Closure of Lie brackets

In this Appendix we study the conditions under which symmetry vector fields are closed under the spacetime Lie bracket. We consider first the vector fields (6.4.8). First of all we check that they close as intrinsic vectors on $\mathcal{N}$. Namely we define

$$\hat{\xi}^\mu := \tau(\lambda, x^B)\partial_\lambda + Y^A(x^B)\partial_A. \tag{B.3.1}$$

Then we have

$$[\hat{\xi}_1, \hat{\xi}_2] = \hat{\xi}_{12}, \qquad \tau_{12} := \tau_1\dot{\tau}_2 + Y_1[\tau_2] - (1 \leftrightarrow 2), \qquad Y_{12} = [Y_1, Y_2]_S. \tag{B.3.2}$$

The algebra closes and has a semi-direct product structure with $\mathrm{Diff}(S)$ acting on the space $\mathbb{R}^{\mathcal{N}}$ of functions of $\mathcal{N}$ as scalar densities of weight $-\dot{\tau}$.

Next, recall that the condition $n_\mu \delta_\xi l^\mu \overset{\mathcal{N}}{=} 0$ partially constraints the extension ξ of the intrinsic vectors $\bar{\xi}$ off of $\mathcal{N}$, specifically the component $\xi^\Phi = \Phi\bar{\xi}^\Phi = \Phi(\delta_\xi \ln f - \dot{\tau})$. This contraint makes closure of the ξ's under the spacetime Lie bracket not automatic. The non-trivial component to check is

$$\begin{aligned}[\xi_1, \xi_2]^\Phi &= \Phi((\pounds_{\xi_1}\delta_{\xi_2} - \pounds_{\xi_2}\delta_{\xi_1})\ln f - \dot{\tau}_{[\xi_1,\xi_2]}) + O(\Phi^2) \\ &= \Phi(\pounds_{[\xi_1,\xi_2]}\ln f + (\pounds_{\xi_1}\Delta_{\xi_2} - \pounds_{\xi_2}\Delta_{\xi_1})\ln f - \dot{\tau}_{[\xi_1,\xi_2]}) + O(\Phi^2)\end{aligned} \tag{B.3.3}$$

Closure of the algebra requires $(\pounds_\xi\Delta_\chi - \pounds_\chi\Delta_\xi)\ln f = \Delta_{[\xi,\chi]}\ln f$. It works if $\ln f$ has no anomaly. Likewise we have

$$\begin{aligned}(\pounds_\xi\delta_\chi - \pounds_\chi\delta_\xi)\ln f &= \pounds_\xi\left(\frac{\partial \ln f}{\partial g_{\mu\nu}}\pounds_\chi g_{\mu\nu}\right) - \pounds_\chi\left(\frac{\partial \ln f}{\partial g_{\mu\nu}}\pounds_\xi g_{\mu\nu}\right) \\ &= \frac{\partial \ln f}{\partial g_{\mu\nu}}(\pounds_\xi\pounds_\chi - \pounds_\chi\pounds_\xi)g_{\mu\nu} + \pounds_\xi\frac{\partial \ln f}{\partial g_{\mu\nu}}\pounds_\chi g_{\mu\nu} - \pounds_\chi\frac{\partial \ln f}{\partial g_{\mu\nu}}\pounds_\xi g_{\mu\nu} \\ &= \delta_{[\xi,\chi]}\ln f + \pounds_\xi\frac{\partial \ln f}{\partial g_{\mu\nu}}\pounds_\chi g_{\mu\nu} - \pounds_\chi\frac{\partial \ln f}{\partial g_{\mu\nu}}\pounds_\xi g_{\mu\nu}\end{aligned} \tag{B.3.4}$$

and the last term does not have any reason to vanish as well. therefore the algebra do not close.

Adding the condition $\delta l_\mu \overset{\mathcal{N}}{=} 0$ eliminates the extra term, and the algebra closes. Notice that it closes for τ an arbitrary function on $\mathcal{N}$, namely for the group G^{aST} associated with the relaxed CFP phase space with δk arbitrary. This result may appear in tension with the result of [43], where it was proved that the largest *corner* subalgebra that closes at first order under the Lie bracket includes at most a linear dependence in time, as opposed to the arbitrary time dependence of $\tau(\lambda, x^A)$ here. The difference is that the corner subalgebra includes two independent sets of super-translations, corresponding to the two null times that tick off the corner. Restricting the diffeomorphisms to be tangent to $\mathcal{N}$ eliminates from the algebra those elements that prevent an arbitrary time dependence.

Finally, we consider the case with an explicit field dependence as in (6.4.20). Taking the divergence of the vector Y_3^A,

$$\begin{aligned} D_A Y_3^A &= Y_1^B D_A D_B Y_2^A - Y_2^B D_A D_B Y_1^A \\ &= (Y_1^B \partial_B \dot{\tau}_2 - Y_2^B \partial_B \dot{\tau}_1) - (Y_1^B \partial_B \tau_2 - Y_2^B \partial_B \tau_1)\theta_l - (\tau_2 Y_1^B \partial_B \theta_l - \tau_1 Y_2^B \partial_B \theta_l) - w'(x^A) \end{aligned} \tag{B.3.5}$$

where $w'(x^A) = -Y_1^B \partial_B w_2 - Y_2^B \partial_B w_1$. Now, we have to compare (B.3.5) to $\dot{\tau}_3 - \tau_3 \theta_l$ and so we write

$$\begin{aligned} \dot{\tau}_3 - \tau_3 \theta_l - D_A Y_3^A &= \tau_1 \ddot{\tau}_2 - \tau_2 \ddot{\tau}_1 - (\tau_1 \dot{\tau}_2 - \tau_2 \dot{\tau}_1)\theta_l + (\tau_2 Y_1^B D_B \theta_l - \tau_1 Y_2^B D_B \theta_l) + Y_1^B \partial_B w_2 + Y_2^B \partial_B w_1 \\ &= \tau_1 w_2(x^A) - \tau_2 w_1(x^A) + (\tau_2 Y_1^B \partial_B \theta_l - \tau_1 Y_2^B \partial_B \theta_l) - Y_1^B \partial_B w_2 - Y_2^B \partial_B w_1 \end{aligned} \tag{B.3.6}$$

We observe that in general this is not a constant in the null parameter λ (except if both τ_1 and τ_2 vanish) so it means that the Lie algebra does not close.

Appendix: Wald - Zoupas and anomalies

C

C.1 BMS anomalies

We review here some basic formulas of the BMS transformations, and show how to compute the associated anomalies and the shift between the WZ and improved Noether charges. We follow [46] for the notation. While the general logic remains the same described in the main text, performing the calculations explicitly requires paying attention to two special features. The limit to $\mathcal{I}$ and the difference between a symmetry vector field on $\mathcal{I}$ and its bulk extension, and the fact that we choose to work with a specific coordinate system. Working in Bondi coordinates (u, r, θ, ϕ) and with conformal factor $\Omega := 1/r$, the asymptotic Killing vectors are

$$\xi := \tau\partial_u + Y^A\partial_A + \Omega(\dot{\tau}\partial_\Omega - \partial^A\tau\partial_A) + O(\Omega^2). \tag{C.1.1}$$

For the BMS group, $\tau = T + \frac{u}{2}D_AY^A$, where $T(\theta, \phi)$ is the supertranslation parameter, and $Y^A(\theta\,\phi)$ a conformal Killing vector on the two-sphere. For the BMSW enlargement [46], which encompasses both extended [143] and generalized [109, 81] BMS groups, $\tau = T(\theta, \phi) + uW(\theta, \phi)$, and Y^A is an arbitrary vector, which we take to be globally defined. From

$$\Delta_\xi\tilde{g}_{\mu\nu} = -g_{\mu\nu}\pounds_\xi\Omega^2 = -\frac{2}{\Omega}\xi^\Omega\tilde{g}_{\mu\nu} = -2\dot{\tau}\tilde{g}_{\mu\nu}, \tag{C.1.2}$$

we see that

$$w_\xi = -\dot{\tau}. \tag{C.1.3}$$

The covariant phase space at future null infinity is parametrized by the functionals $\bar{q}_{AB}, C_{AB}$, respectively the leading and first sub-leading orders of the 2d metric, and the mass and angular momentum aspects $M, \bar{P}_A$. The parametrization is chosen so that $\bar{P}_A$ coincide with the definition of Dray and Streubel in these coordinates. All quantities

depend on (u, θ, ϕ), except for the background metric $\bar{q}_{AB}$ which is constant in u.[1] The phase space transformations generated by the asymptotic BMSW symmetries are [46]

$$\delta_\xi \bar{q}_{AB} = (\pounds_Y - 2\dot{\tau})\bar{q}_{AB}, \tag{C.1.4a}$$

$$\delta_\xi C_{AB} = (\tau\partial_u + \pounds_Y - \dot{\tau})C_{AB} - 2\bar{D}_{\langle A}\partial_{B\rangle}\tau, \tag{C.1.4b}$$

$$\delta_\xi N_{AB} = (\tau\partial_u + \pounds_Y)N_{AB} - 2\bar{D}_{\langle A}\partial_{B\rangle}\dot{\tau}, \tag{C.1.4c}$$

$$\delta_\xi M = (\tau\partial_u + \pounds_Y + 3\dot{\tau})M + \left(\frac{1}{2}\bar{D}_A N^{AB} + \partial^B \bar{F}\right)\partial_B\tau + \frac{1}{4}\partial_u(C^{AB}\bar{D}_A\partial_B\tau), \tag{C.1.4d}$$

$$\begin{aligned}\delta_\xi \bar{P}_A = {} & (\tau\partial_u + \pounds_Y + 2\dot{\tau})\bar{P}_A + 3M\partial_A\tau - \frac{1}{8}N_{BC}C^{BC}\partial_A\tau + \frac{1}{2}\left(C_A^C N_{BC}\right)\partial^B\tau \\ & + \frac{3}{4}\left(\bar{D}_A\bar{D}_C C_B{}^C - \bar{D}_B\bar{D}_C C_A C\right)\partial^B\tau + \frac{1}{4}\partial_A\left(C^{BC}\bar{D}_B\bar{D}_C\tau\right) \\ & + \frac{1}{2}\bar{D}_{\langle A}\bar{D}_{B\rangle}\tau\bar{D}_C C^{BC} + C_{AB}\left(\bar{F}\partial^B\tau + \frac{1}{4}\partial^B\Delta\tau\right).\end{aligned} \tag{C.1.4e}$$

Here $\pounds_Y$ is a slight abuse of notation and should be understood as the Lie derivative for tensors on the two-sphere. The functionals transform in general not as scalars but rather as densities in the u variable, because of the $\dot{\tau}$ shifts, and as tensors on the sphere in the A indices, plus inhomogeneous terms. Because of this algebraic structure, $\mathcal{I}$ is endowed with the structure of a fiber bundle $S^2 \times \mathbb{R}$ in which the fibers are the conformal weights. The density shifts and the inhomogeneous terms are responsible for the anomalies.

To see that explicitly, we need first to explain how the covariant Lie derivative is mapped to a gauge-fixed description associated with the Bondi coordinates used above. Consider a 3-form on $\mathcal{I}$. This is a spacetime covariant quantity, which in Bondi coordinates will read like a scalar on the 2-sphere times the volume form, e.g. $v_A w^A \epsilon_{\mathcal{I}}$. For an example, see the symplectic potential at $\mathcal{I}$ given by (7.4.1). Using the asymptotic symmetry vectors (C.1.1), we have

$$\pounds_\xi(v_A w^A \epsilon_{\mathcal{I}}) = \pounds_\xi(v_A w^A)\epsilon_{\mathcal{I}} + v_A w^A \pounds_\xi \epsilon_{\mathcal{I}}. \tag{C.1.5}$$

Now we can write

$$\pounds_\xi(v_A w^A) = \xi^\mu\partial_\mu(v_A w^A) = \tau\partial_u(v_A w^A) + Y^B\partial_B(v_A w^A) = \tau\partial_u(v_A w^A) + \pounds_Y(v_A w^A) = L_\xi(v_A w^A),$$

where we introduced the Bondi-frame Lie derivative

$$L_\xi := \tau\partial_u + \pounds_Y, \tag{C.1.6}$$

or Bondi Lie derivative for short. Its action is that of a Lie derivative on the conformal bundle of u-dependent 2-sphere tensors.

[1] With Penrose's definition of asymptotic flatness (see e.g. [33, 48]), one can always choose a conformal factor satisfying the Bondi condition $\tilde{\nabla}_\mu n^\mu = 0$, and then $\pounds_n \bar{q}_{AB} = 0$. This is the case with the choice of Ω taken here, from which the asymptotic Einstein's equations impose $\partial_u \bar{q}_{AB} = 0$.

The anomalies of the phase space functionals are thus given by $\Delta_\xi = \delta_\xi - L_\xi$ (the last term from the definition (4.3.8) drops out because we are acting on field-space scalars), namely

$$\Delta_\xi\, \bar{q}_{AB} = -2\dot{\tau}\bar{q}_{AB}, \tag{C.1.7a}$$

$$\Delta_\xi\, C_{AB} = -\dot{\tau}C_{AB} - 2\bar{D}_{\langle A}\partial_{B\rangle}\tau, \tag{C.1.7b}$$

$$\Delta_\xi\, N_{AB} = -2\bar{D}_{\langle A}\partial_{B\rangle}\dot{\tau}, \tag{C.1.7c}$$

$$\Delta_\xi\, M = 3\dot{\tau}M + \left(\frac{1}{2}\bar{D}_A N^{AB} + \partial^B\bar{F}\right)\partial_B\tau + \frac{1}{4}\partial_u(C^{AB}\bar{D}_A\partial_B\tau), \tag{C.1.7d}$$

and similarly for $\bar{P}_A$, which won't be needed in the following. These formulas are identical for BMSW and BMS, with the only simplification for BMS being that $\dot{\tau} = \bar{D}_A Y^A/2$. Here $\bar{F} = \bar{R}/4$. If we choose a Bondi frame, namely a round sphere, then $\bar{F} = 1/2$ and one term in $\Delta_\xi M$ drops out. From now on, we restrict attention to the BMS case, but we will allow for general frames.

The anomaly of the background metric is familiar from the BMS literature: the 2d metric is a background structure, hence $\delta\bar{q}_{AB} = 0$, while an asymptotic symmetry changes it by a conformal transformation given by $2\dot{\tau} = \bar{D}_A Y^A$. Hence the RHS of (C.1.7a). In the generalized BMS and in BMSW the phase space is enlarged to include the Bondi frame as a variable, hence $\delta\bar{q}_{AB} \neq 0$, but the resulting anomaly is again just a conformal transformation, albeit with an arbitrary factor instead of just the Lorentz boost $\bar{D}_A Y^A$. From this expression we can also derive $\Delta_\xi\sqrt{\bar{q}} = \sqrt{\bar{q}}\bar{q}^{AB}\Delta_\xi\bar{q}_{AB}/2 = -2\dot{\tau}\sqrt{\bar{q}}$.

The anomaly of the news (C.1.7c) is also familiar from the seminal work of Geroch [123], which introduced the tensor carrying his name, and whose traceless part is

$$\rho_{\langle AB\rangle} := -2\bar{D}_{\langle A}\varphi\bar{D}_{B\rangle}\varphi - 2\bar{D}_{\langle A}\bar{D}_{B\rangle}\varphi, \tag{C.1.8}$$

where 2φ is the conformal factor relating the metric $\bar{q}_{AB}$ to a round 2-sphere metric. From this expression and the condition $\partial_u\bar{q}_{AB} = 0$ we deduce that $\pounds_\xi\rho_{\langle AB\rangle} = -2\bar{D}_{\langle A}\bar{D}_{B\rangle}\dot{\tau}$. Geroch also proves that this tensor is universal. Hence $\delta\rho_{\langle AB\rangle}$=0, and

$$\Delta_\xi\, \rho_{AB} = -2\bar{D}_{\langle A}\partial_{B\rangle}\dot{\tau} \equiv \Delta_\xi N_{AB}. \tag{C.1.9}$$

It follows that $\hat{N}_{AB} := N_{AB} - \rho_{AB}$ is covariant, i.e. its anomaly vanishes. These formulas trivialize if we take a round 2-sphere, because then $\rho_{<AB>}$ vanishes, and so does the anomaly of N_{AB} thanks to special properties of the (global) conformal Killing vectors. On the other hand, we will see that the anomaly of b does not vanish even in a round sphere. Notice that we have defined the Geroch tensor with an overall opposite sign as the literature, just for convenience of writing the covariant news as a difference. Concerning its behaviour under Lie derivative, this uses crucially the fact that ρ_{AB} has a trace part

that does not vanish on a round 2-sphere, otherwise this Lie derivative would vanish as well. Finally, we also note without proof that $\bar{D}^A \rho_{AB} = -\partial_B \bar{R}$.

The anomaly of the volume form $\epsilon_{\mathcal{I}}$ is given by (D.0.11) with (C.1.3), namely[2]

$$\Delta_\xi \epsilon_{\mathcal{I}} = -3\dot{\tau}\epsilon_{\mathcal{I}}. \tag{C.1.10}$$

Putting together these results, we find

$$\begin{aligned}(\delta_\xi - \pounds_\xi)\left(\hat{N}_{AB}\delta C^{AB}\epsilon_{\mathcal{I}}\right) &= (\delta_\xi - L_\xi)\hat{N}_{AB}\delta C^{AB}\epsilon_{\mathcal{I}} + \hat{N}_{AB}(\delta_\xi - L_\xi)\delta C^{AB}\epsilon_{\mathcal{I}} + \hat{N}_{AB}\delta C^{AB}(\delta_\xi - L_\xi)\epsilon_{\mathcal{I}} \\ &= \Delta_\xi \hat{N}_{AB}\delta C^{AB}\epsilon_{\mathcal{I}} + \hat{N}_{AB}\delta\Delta_\xi C^{AB}\epsilon_{\mathcal{I}} + \hat{N}_{AB}L_{\delta\xi}C^{AB} + \hat{N}_{AB}\delta C^{AB}\Delta_\xi\epsilon_{\mathcal{I}} \\ &= \hat{N}_{AB}\delta\Delta_\xi C^{AB}\epsilon_{\mathcal{I}} - 3\dot{\tau}\hat{N}_{AB}\delta C^{AB}\epsilon_{\mathcal{I}} = 0. \end{aligned} \tag{C.1.11}$$

In the third equality we used $L_{\delta\xi}C^{AB} = 0$, since $\delta\xi = O(\Omega^2)$ as follows from (C.1.1). In the last we used $\Delta_\xi C^{AB} = 3\dot{\tau}C^{AB} - 2\bar{D}^{\langle A}\partial^{B\rangle}\tau$ which follows from (C.1.7). As a consequence, the non-integrable term that can be read naively from (7.4.1) is not covariant, whereas $\bar{\theta}$ defined in (7.4.3) is.

Next, we compute the anomaly of (7.4.4), here copied for convenience:

$$b = \left(2M + D_A\bar{U}^A - \frac{1}{8}N_{AB}C^{AB} + \frac{1}{2}\rho_{AB}C^{AB}\right)\epsilon_{\mathcal{I}}, \tag{C.1.12}$$

with $\bar{U}^A = -\frac{1}{2}\bar{D}_B C^{AB}$. For this term, we have[3]

$$\begin{aligned}\Delta_\xi(\bar{D}_A\bar{U}^A\epsilon_{\mathcal{I}}) &= \bar{D}_A(\Delta_\xi\bar{U}^A)\epsilon_{\mathcal{I}} + \bar{D}_A\bar{U}^A\Delta_\xi\epsilon_{\mathcal{I}} + [\Delta_\xi, \bar{D}_A]\bar{U}^A\epsilon_{\mathcal{I}} \\ &= \bar{D}_A(\Delta_\xi\bar{U}^A)\epsilon_{\mathcal{I}} - 3\dot{\tau}\bar{D}_A\bar{U}^A\epsilon_{\mathcal{I}} + (\dot{\bar{U}}^A\partial_A\tau - 2\bar{U}^A\partial_A\dot{\tau})\epsilon_{\mathcal{I}} \\ &= \bar{D}_A\big((\Delta_\xi - 3\dot{\tau})\bar{U}^A\big)\epsilon_{\mathcal{I}} + \frac{1}{2}\partial_u(C^{AB}\bar{D}_A\bar{D}_B\tau). \end{aligned} \tag{C.1.13}$$

[2] It is also possible to derive this writing $\epsilon_{\mathcal{I}} = du \wedge \epsilon_S$. The 1-form du is an anomalous quantity on the scale bundle, with anomaly given by $\Delta_\xi du = -\pounds_\xi du = -\dot{\tau}du$, and $\Delta_\xi\epsilon_S = -2\dot{\tau}\epsilon_S$. Care is needed when writing $\epsilon_S = \sqrt{\bar{q}}d^2\theta$ and using the anomaly for $\sqrt{\bar{q}}$ previously derived. This is because $\sqrt{\bar{q}}$ is a density, therefore we should remember that $d^2\theta$ is an invariant. This is a familiar result for Lie derivatives of volume forms: if we write $\epsilon = \sqrt{-g}d^4x$, where $d^4x := \frac{1}{n!}\epsilon_{\mu\nu\rho\sigma}dx^\mu \wedge d^\nu \wedge dx^\rho \wedge dx^\sigma$ is a density, we have $\pounds_\xi\sqrt{-g} = \sqrt{-g}\nabla_\mu\xi^\mu$ and $\pounds_\xi d^4x = 0$.

[3] The second equality below follows from the fact that for BMS

$$[\Delta_\xi, \bar{D}_A] = -[\pounds_\xi, \bar{D}_A],$$

which can be computed from

$$[\tau\partial_u, \bar{D}_A]f^B = -\partial_u f^B\bar{D}_A\tau, \qquad [\mathcal{L}_Y, \bar{D}_A]f^A = f^A\bar{D}_A\bar{D}_C Y^C,$$

and we also observe that $[\partial_u, \mathcal{L}_Y] = 0$.

The total derivatives on the sphere can be dropped. As a consequence, we don't need to know the explicit form of the anomaly of $\bar{U}^A$. For the interested reader, it can be found in [46]. For the other terms in b, we have

$$\Delta_\xi\Big((2M - \frac{1}{8}N_{AB}C^{AB} + \frac{1}{2}\rho_{AB}C^{AB})\epsilon_{\mathcal{I}}\Big) = \left(-\frac{1}{4}\partial_u(C^{AB}\bar{D}_A\bar{D}_B\tau) + \frac{1}{2}\partial^A\bar{R}\partial_A\tau\right)\epsilon_{\mathcal{I}} + \frac{1}{2}\rho_{AB}\Delta_\xi(C^{AB}\epsilon_{\mathcal{I}}).$$

Using the property relating the divergence of Geroch's tensor to the gradient of the 2d curvature, the last two terms here cancel out after integration by parts. Collecting the results, we conclude that

$$\Delta_\xi b = ds_\xi, \qquad s_\xi = \frac{1}{4}C^{AB}\bar{D}_A\bar{D}_B\tau\,\epsilon_S. \tag{C.1.14}$$

This proves (7.4.5) used in the main text.[4] As for (7.4.6), this follows immediately computing the anomaly of $\bar{\beta} := -\frac{1}{32}C_{AB}C^{AB}$, which gives

$$c := -2\bar{\beta}\epsilon_S, \qquad \Delta_\xi c = -\frac{1}{4}C^{AB}\bar{D}_A\bar{D}_B\tau\,\epsilon_S. \tag{C.1.15}$$

Let's check that the shift (7.4.5) indeed reproduces the known expressions of the WZ charges at $\mathcal{I}$. The expansion of the Komar two-form gives (to lighten the notation, we drop in the following the sphere indices A)

$$q_\xi = [\tau(2M - D\bar{U} + \frac{1}{8}CN - \frac{1}{4}DDC) + 2Y(-r\bar{U} + \bar{P} + \partial\bar{\beta}) + 2W(-r^2 + 2\bar{\beta})]\epsilon_S. \tag{C.1.16}$$

The divergent terms vanish for BMS, and can be renormalized away for BMSW [144], so we will drop them in the following.[5] Then we have

$$i_\xi\epsilon_{\mathcal{I}} = -\xi\cdot l\,\epsilon_S, \qquad -\xi\cdot l = \tau, \qquad i_\xi b = \tau\Big(2M + \bar{D}\bar{U} - \frac{1}{8}NC + \frac{1}{2}\rho C\Big)\epsilon_S, \tag{C.1.17}$$

hence

$$q_\xi + i_\xi b = \left(\tau(4M - \frac{1}{4}\bar{D}\bar{D}C + \frac{1}{2}\rho C) + 2Y\bar{P}\right)\epsilon_S. \tag{C.1.18}$$

This shows that without the right corner shift, the improved Noether charge with b as boundary Lagrangian doesn't give the standard BMS charges. The difference is the soft term $\bar{D}\bar{D}C$. If we used these charges, we would not not measure the standard memory

[4] Notice that the contribution of the density weights to the anomaly drops out, and it is only the inhomogeneous terms of (C.1.7) that matter in the end. Hence the calculations are consistent with those of [46], where the densities were not included in the definition of the anomaly. This alternative option is perfectly fine, and amounts to treating the functional in the covariant phase space as densities as opposed to as tensors on the sphere.

[5] Taking the on-shell value of $\bar{U}$, the second and fourth terms add up to $-\frac{1}{2}\bar{D}\bar{U}$. Our $\bar{U}$ coincides with the $\mathcal{U}$ used in [38], and our $1/2$ instead of their 1 in the third term of their (6.15) is the mismatch we referred to in the main text.

effects, and their flux is determined not only by the physical symplectic potential $\bar{\theta}$, but by the anomalous contribution as well. This leads to two types of problems. First, if one uses an arbitrary frame in which the ρC term does not vanish, the boost charges would not be conserved in stationary spacetimes. The problem would not occur for supertranslations because $\dot{T} = 0$. Second, if one uses a round sphere, the extra term $\bar{D}\bar{D}C$ is time-independent so it does not affect the flux, however one would still have the problem of non-zero Minkowski charges for supertranslations.

Finally, adding up (C.1.14), which after a trivial integration by parts on the 2-sphere can be rewritten as

$$s_\xi = -\frac{\tau}{2}\bar{D}\bar{U}\epsilon_S, \tag{C.1.19}$$

we obtain the desired result[6]

$$q_\xi^{\text{WZ}} = q_\xi + i_\xi b + s_\xi = \left(4\tau\hat{M} + 2Y\bar{P}\right)\epsilon_S, \qquad \hat{M} = M + \frac{1}{8}\rho_{AB}C^{AB}. \tag{C.1.20}$$

These charges vanish exactly on the Minkowski solution for a round sphere with arbitrary supertranslations, therefore there is no need of any shift by field-space integration constants. But the most important property is that their flux vanish in an arbitrary frame when the covariant news vanish, unlike for (C.1.18).

The calculation proves that the WZ charges can be obtained without ever talking about Hamiltonian generators, but just as an improved Noether charge with the prescriptions (7.2.21). The anomaly-free boundary Lagrangian can be read from (C.1.15) to be (7.4.7), which we report here for convenience,

$$\ell^c = \left(2M + \bar{D}\bar{U} + \frac{1}{2}\rho C\right)\epsilon_{\mathcal{I}}. \tag{C.1.21}$$

We also notice that

$$\left(\bar{D}\bar{U} + \frac{1}{4}CN\right)\epsilon_{\mathcal{I}} = d\left(\frac{1}{8}C^2\epsilon_S + i_{\bar{U}}\epsilon_{\mathcal{I}}\right), \qquad \bar{U}^\mu := (0, 0, \bar{U}^A). \tag{C.1.22}$$

This corner term is also anomaly-free once we integrate on the 2-sphere to get rid of the total derivatives that appear when using (C.1.13). We conclude that the WZ charges can be obtained starting from the family of boundary Lagrangians (7.4.9), that all differ from (C.1.21) by a term proportional to (C.1.22).

C.2 Charges' archeology

In this Appendix we comment on the importance of the interplay relation (7.2.18). This allows one to understand how Wald and Zoupas were able to get away without ever

[6] Recall we are using the notation from [46] and units $16\pi G = 1$. The relation to the angular momentum aspect used in [107, 81] is $N_A = \bar{P}_A + \partial_A\bar{\beta}$.

talking about anomalies, and will also be the opportunity for us to add some comments about [107, 145, 122] that we think may be useful to the reader. If we start from the Einstein-Hilbert Lagrangian there are no anomalies, and

$$-I_\xi\omega \hat{=} d(\delta q_\xi - q_{\delta\xi} - i_\xi\theta). \tag{C.2.1}$$

Then, the WZ prescription (7.2.3) gives

$$\begin{aligned} -I_\xi\omega + di_\xi\bar{\theta} &\hat{=} d(\delta q_\xi - q_{\delta\xi} - i_\xi\theta + i_\xi\bar{\theta}) = d(\delta q_\xi - q_{\delta\xi} + i_\xi\delta b) \\ &= d(\delta(q_\xi + i_\xi b) - q_{\delta\xi} - i_{\delta\xi}b). \end{aligned} \tag{C.2.2}$$

If we take the bare Eistein-Hilbert θ, this is covariant and q_ξ is Komar; the covariance requirement for $\bar{\theta}$ guarantees not only (7.2.9) but also (7.2.18). Therefore, $\delta s_\xi = -\bar{q}_{\delta\xi} = -q_{\delta\xi} - i_{\delta\xi}b$. Using this equality in (C.2.2) we recover the calculation of the charges done at the end of the previous Section, namely adding s_ξ as computed from the anomaly of b. But we can also forget about the anomalous origin of s_ξ, and compute directly $q_{\delta\xi}$ and $i_{\delta\xi}b$ in (C.2.2). On first thought, one may imagine that these vanish, since there is no field dependence in ξ at zeroth or first order, see (C.1.1). However, it had been observed as early as [146] that the limit of the Komar 2-form to future null infinity depends on the second-order extension as well, and in fact it even depends on the *third* order insofar as the radial component is concerned. This can be trivially checked using for instance Bondi coordinates and $\Omega = 1/r$. The Komar formula then contains $\partial_r\xi^r$, which when integrated against the r^2 area 2-form fishes a contribution $O(r^{-1})$ in ξ^r, which is $O(\Omega^3)$. But then, the second and higher-order terms are generically field-dependent. Using the Tamburino-Winicour extension, equivalent to preserving the bulk Bondi coordinates used in the previous Section, we have

$$\delta\xi = \left(\frac{\Omega^2}{2}\delta(C^{AB}\partial_B\tau) + O(\Omega^3)\right)\partial_A + \left(\frac{\Omega^3}{2}\delta(\bar{D}_A C^{AB}\partial_B\tau + \frac{1}{2}C^{AB}\bar{D}_A\partial_B\tau) + O(\Omega^4)\right)\partial_r. \tag{C.2.3}$$

This vector gives a vanishing contribution when hooked with b, but not when plugged in the Komar form. There, it replaces a divergent term that was a total divergence on the sphere (hence integrating to zero) if ξ was used, with a finite term that is no longer a total 2d-divergence, but rather gives on the cross-sections

$$-q_{\delta\xi} = \frac{1}{4}\delta(\tau\bar{D}\bar{D}C)\epsilon_S. \tag{C.2.4}$$

This is precisely the same contribution of s_ξ, as expected from the general equivalence (7.2.18). As a consequence, one can do the calculation using the first line of (C.2.2), and obtain the correct result without ever talking about anomalies, and instead properly taking

into account the $q_{\delta\xi}$ term (C.2.4). This is the way the calculation is done for instance in [81], even though the contribution of the term (C.2.4) is not explicitly reported.[7] Notice also that the neat result of this term is to make $I_\xi\omega$ independent of the field-dependent extension, because (C.2.4) cancels the $O(\Omega^3)$ term that appears when computing δq_ξ, As for the second-order terms in q_ξ, they drop out when taking the pull-back on a fixed u cross-section of $\mathcal{I}$. The final result depends only on the zeroth and first orders of ξ, which are field-independent.

This term is also taken into account in the formula used in [107], following [10, 147], and this is for us the only reference in the literature where all aspects of the calculation of the BMS charges are properly and explicitly discussed.[8]

Coming back to the WZ paper, there are actually two difficulties with the way the BMS calculations are presented. The first is that since they assume $\delta\xi = 0$, they write $-I_\xi\omega = \delta q_\xi - i_\xi\theta$. This is not too bad, because it can be easily corrected: the effective consequence of the $q_{\delta\xi}$ term in (C.2.2) is that one should take the variation of q_ξ treating ξ as a c-number even if it is field-dependent. With this caveat in mind, the calculations are correct. Otherwise, (94) of [33] is missing an additional finite term coming from the $O(\Omega^2)$ terms of ξ. Notice that WZ discuss the independence of (C.2.1) from the arbitrary part of the extension of the asymptotic symmetry vector, below their equation (22). This independence is taken there as a definition of equivalent representatives, but it can be proved explicitly as done in [122], Lemma 5.2. The proof is given there only for field-independent higher-order extensions, but can be trivially generalized to our case if δq_ξ is replaced by $\delta q_\xi - q_{\delta\xi}$. Or alternatively, with the caveat that ξ is always a c-number for δ. This way of understanding the action of δ and the RHS of $I_\xi\omega$ for field-dependent diffeomorphisms was made more explicit shortly after in [148]. We suppose that this is the approach taken also in [122], even though it is nowhere explicitly stated, otherwise some of their calculations are missing intermediate terms that cancel out in the end result. We remark that having extended the proof of independence from higher-order extensions to the field-dependent case, one can also compute the RHS of (C.2.1) ignoring such terms, instead of computing them and see that they cancel out. This means in particular ignoring the $q_{\delta\xi}$ term altogether. This provides another way of interpreting the results of of [33, 122] as correct. With these caveats in mind, [122] is a very clear and explicit paper, and has the further advantage of presenting the calculations in two different gauges as well as in covariant language.

The second difficulty of the WZ paper concerns the boost charges. Inspection of (C.1.1) shows that these get a contribution from the vertical part, and therefore are not generated purely by a vector tangential to the cross-section. In other words, restricting

[7] We thank Adrien Fiorucci for sharing his calculations.

[8] Mind however that [107] does not start from $-I_\xi\omega$ but adds to it a term proportional to the Killing equation, see e.g. (9.10) in [66]. This additional term has vanishing limit to $\mathcal{I}$.

ξ to be tangential is a stronger condition than setting the super-translation parameter to zero. Nonetheless, Wald and Zoupas tried to recover all Lorentz charges, rotations as well as boosts, from a purely tangential vector. The interest in doing so is possibly that for a field-independent and purely tangential vector, the Hamiltonian generator is integrable since the pull-back of $i_\xi\theta$ vanishes, and one does not need any prescription. The result is the Komar formula, which they knew gives the Dray-Streubel charges for angular momentum but not for boosts, unless the extension is chosen to satisfy the Geroch-Winicour condition. So what Wald and Zoupas set up to do is to prove that the variation of the Komar formula is unchanged if the Geroch-Winicour condition is imposed, because then they can claim that the Dray-Streubel charges are recovered when they further impose the condition that all charges vanish in Minkowski spacetime. This is arguably a more tortuous path than straightforwardly including the vertical part of the vector in the boost contribution, which is the reason the calculation works no matter what extension is taken.

The same result of [107] then appeared again in [145]. Both papers use the Tamburino-Winicour extension described above. However [145] claims that the result matches Dray-Streubel because the Geroch-Winicour condition $\nabla_\mu\xi^\mu = 0$ can be relaxed to $\nabla_\mu\xi^\mu = O(\Omega^2)$, which is satisfied by the Tamburino-Winicour extension that they use. This argument is wrong in our opinion, because the Tamburino-Winicour extension precisely requires the linkage term in order to reproduce the right boost charges [149]. The reason why [145] gets the right charges is for us not that the linkage is not needed because of the chosen extension, but because of the correct inclusion of the vertical term, just as in [107].

As a final comment, notice that (C.2.4) shows that $I_{\delta\xi}\theta \neq 0$ for the Einstein-Hilbert bare potential, by consistency with the Noether theorem (7.1.7) with ξ replaced by $\delta\xi$. To verify this explicitly some care is needed, because $\delta\xi$ is *not* a symmetry vector.[9] In particular, $\delta_{\delta\xi}$ does not exist on the asymptotic phase space. Instead, we can use the general formula (7.2.19) and take the limit to infinity. Since $\delta\xi = O(\Omega^2)$, the last two terms vanish and we find

$$\lim_{r\to\infty} \underleftarrow{I_{\delta\xi}\theta} = -\delta\Delta_\xi b = -d\delta s_\xi. \tag{C.2.5}$$

This result together with $q_{\delta\xi} = -\delta s_\xi$ proves the consistency of (C.2.4) with Noether's theorem.

The reason why field-dependent extensions of the diffeomorphisms matter only at $\mathcal{I}$ and not in computing the charges at finite distance, is because the Komar formula

[9] It vanishes on $\mathcal{I}$, and the Tambourino-Winicour extension of the trivial vector on $\mathcal{I}$ vanishes everywhere, unlike $\delta\xi$. The latter is more akin to the difference between two different bulk representatives of the same asymptotic Killing vector.

depends on first derivatives of ξ, which are field-independent at finite distance, but involve higher orders at $\mathcal{I}$, which see the field-dependence.

C.3 Tetrad variables

There are three useful remarks to make if one uses tetrad variables. First, the bare symplectic potential differs from the Einstein-Hilbert one by an exact 3-form [124, 66]. Second, if one fixes the same physical θ and boundary Lagrangian, the improved Noether charge is the same [37]. Furthermore, the DPS exact 3-form is anomaly-free, therefore one can use the same covariant boundary Lagrangian as in the metric case to evaluate the Wald-Zoupas prescription for the BMS charges.

The bare symplectic potential given by the Einstein-Hilbert Lagrangian differs from the tetrad one by an exact 3-form [124, 66],

$$\theta = \theta^e + d\alpha^{\text{DPS}}, \qquad \alpha^{\text{DPS}} = \star(e_I \wedge \delta e^I). \tag{C.3.1}$$

As a consequence, the bare Noether charges computed without adding any boundary Lagrangian are also different, and we have

$$q_\xi = q^e_\xi + I_\xi \alpha^{\text{DPS}}, \tag{C.3.2}$$

where q_ξ is Komar, and $q^e_\xi = \frac{1}{2}\epsilon_{IJKL} e^I \wedge e^J \, i_\xi \omega^{KL}$. The improved Noether charges can be made to coincide if one chooses the boundary Lagrangian ℓ and θ' to match the metric choices, as pointed out in [37]:

$$q^{e\prime}_\xi = q^e_\xi + i_\xi \ell - I_\xi \vartheta^e = q_\xi + i_\xi \ell - I_\xi \vartheta = q'_\xi. \tag{C.3.3}$$

This is a perfect example of the value of working with the improved Noether charge, ambiguities such as picking a representative of the equivalence class become irrelevant once attention is switched to the physically preferred symplectic potential.[10]

Both θ^e and α^{DPS} are anomaly-free. Furthermore, α^{DPS} becomes field-space-exact at $\mathcal{I}$,

$$\underleftarrow{d\alpha}^{\text{DPS}} = \delta(DU + \frac{1}{8}CN). \tag{C.3.4}$$

Therefore condition 0 of the WZ prescription is satisfied, with

$$\underleftarrow{\theta^e} = -\Big(\delta(2M + 2D\bar{U}) + \frac{1}{2}N_{AB}\delta C^{AB}\Big)\epsilon_{\mathcal{I}}. \tag{C.3.5}$$

[10] When the authors of [14] write the table of different corner symmetry algebras associated with the ADM, EH, EC and ECH Lagrangians, they are looking at the bare Noether charges q_ξ associated with the bare symplectic potential and no boundary Lagrangian, as selected by the homotopy prescription. Should they switch to the improved Noether charges q'_ξ selected in each case by the same θ' and the same ℓ, they would of course obtain the same algebra in each case.

Taking the same $\bar{\theta}$ as before, we have

$$b^e = \Big(2M + 2D\bar{U} + \frac{1}{2}\rho C\Big)\epsilon_{\mathcal{I}}. \tag{C.3.6}$$

The anomaly of this boundary Lagrangian can be computed as shown before and gives twice the metric one, $\Delta_\xi b^e = ds^e_\xi = 2ds_\xi$ with s_ξ given by (C.1.19). On the one hand, this is the right result to get the correct WZ charge, since using the results of [46],

$$q^e_\xi + i_\xi b^e = \big(\tau(4M + D\bar{U}) + 2Y\bar{P}\big)\,\epsilon_S, \qquad q^e_\xi + i_\xi b^e + 2s_\xi = \big(4\tau M + 2Y\bar{P}\big)\,\epsilon_S. \tag{C.3.7}$$

On the other hand, this means that the corner shift needed to get this result from an improved Noether charge is also twice the metric one,

$$c^e = 2c = -4\bar{\beta}\epsilon_S = \frac{1}{8}C^2\epsilon_S, \tag{C.3.8}$$

$$\ell^e = b^e + dc^e = \Big(2M + 2D\bar{U} + \frac{1}{4}CN + \frac{1}{2}\rho C\Big)\epsilon_{\mathcal{I}}, \qquad \Delta_\xi \ell^e = 0. \tag{C.3.9}$$

Notice that this anomaly-free boundary Lagrangian differs from the metric one (7.4.7) by the anomaly-free corner term (C.1.22). It thus belong to the same anomaly-free class, and can indeed be recognized as (7.4.9) with $x = 1$. That it belongs to the same family of anomaly-free boundary Lagrangians was to be expected, since $\Delta_\xi \alpha^{\text{DPS}} = 0$.

In the same anomaly-free class of tetrad boundary Lagrangians we find, taking $x = -1$,

$$\ell^{\text{BMSW}} = \Big(2M - \frac{1}{4}CN + \frac{1}{2}\rho C\Big)\epsilon_{\mathcal{I}}, \tag{C.3.10}$$

which is the one used in [46]. Those results are thus perfectly compatible with the ones here presented, and the novelty is that we now know how to identify this boundary Lagrangian a priori, without having to deduce it from already knowing the WZ charges.

As a final remark, notice that there is no incompatibility between the fact that (C.1.22) and (C.3.4) are both anomaly-free in spite of having different relative factors, because $[\Delta_\xi, \delta] = -\Delta_{\delta\xi} \neq 0$. This calculation cannot however be done explicitly without providing a definition for $\delta_{\delta\xi}$, which in turns requires an extension of b^e.

Appendix: Background fields and anomalies

D

In this Appendix we review how to compute the anomaly associated with a background structure, and prove the absence of anomalies in the case of time-like boundaries parametrized by a unit-normal, the result used in Chapter 8.

Anomalies arise when the covariant phase space contains background structures. Let us denote by ϕ the dynamical fields, and by χ the background fields. For the dynamical fields we define $\delta_\xi \phi := \pounds_\xi \phi$, whereas the background fields satisfy $\delta_\xi \chi = 0$, whence the anomaly $\Delta_\xi \chi = -\pounds_\xi \chi$. To understand the third term in (4.3.8), consider a functional of the fields that is a one-form in field space, namely $F(\phi, \chi)\delta\phi$. In this case we have

$$\delta_\xi(F\delta\phi) = \partial_\phi F \delta_\xi \phi \delta\phi + F\delta\delta_\xi\phi = \partial_\phi F \pounds_\xi \phi \delta\phi + F\delta\pounds_\xi\phi = \pounds_\xi(F\delta\phi) - \partial_\chi F \pounds_\xi \chi \delta\phi + F\pounds_{\delta\xi}\phi, \tag{D.0.1}$$

where we used $[\delta, \delta_\xi] = 0$ in the first equality, and $[\delta, \pounds_\xi] = \pounds_{\delta\xi}$ in the last. Hence,

$$\Delta_\xi(F\delta\phi) = -\partial_\chi F \pounds_\xi \chi \delta\phi = (\delta_\xi - \pounds_\xi - I_{\delta\xi})F\delta\phi. \tag{D.0.2}$$

The first example of background structure we consider in the following is a spacetime boundary $\mathcal{B}$. We define it by its Cartesian equation as $\chi(x^\mu) = 0$, and associated with it a normal 1-form $n_\mu := -f\partial_\mu\chi$. The field χ is a fixed background structure, with $\delta\chi = 0$. Since $\pounds_\xi\chi = \xi^\mu\partial_\mu\chi$, every diffeomorphism that does not preserve the boundary is anomalous. When constructing the covariant phase space associated to this boundary, the only relevant diffeomorphisms are those that preserve the boundary, namely

$$\xi^\mu n_\mu \overset{\mathcal{B}}{=} 0 \qquad \Rightarrow \qquad \xi^\mu = \bar{\xi}^\mu + \chi\hat{\xi}^\mu, \tag{D.0.3}$$

where $\bar{\xi}^\mu n_\mu = 0$. The boundary is shared by all metrics in the phase space. The diffeomorphisms that preserve the boundary are also called residual diffeomorphisms, or symmetry vector fields, hinting at the physical relevance that boundary diffeomorphisms can acquire. In different situations, one may add additional background structure on top of the presence of the boundary, still shared by all metrics in the phase space and usually referred to as *universal* structure. Any additional requirement in the universal structure can restrict the symmetry group.

From (D.0.3) it follows that $\pounds_\xi\chi \overset{\mathcal{B}}{=} 0$, and therefore $\Delta_\xi\chi = 0$: the boundary is covariant with respect to the diffeomorphisms that preserve it. However, anomalies can

still appear when we look at derivatives of χ, for instance through the normal 1-form. In fact, a simple calculation shows that

$$\Delta_\xi n_\mu = w_\xi n_\mu, \qquad w_\xi := \Delta_\xi \ln f - \hat{\xi}^\mu \partial_\mu \chi. \tag{D.0.4}$$

If we take a gradient as normal, say $f = 1$, then the anomaly comes entirely from $\hat{\xi}^\mu \partial_\mu \chi$, namely from how much the extension of ξ off $\mathcal{B}$ does not preserve the neighbouring leaves of the χ foliation. However, as long as the foliation is not null, the anomaly associated with a non-trivial extension $\hat{\xi}$ can be eliminated choosing f so that the normal is unit-norm: in this case in fact,

$$n_\mu = s \frac{\partial_\mu \chi}{\sqrt{s g^{\rho\sigma} \partial_\rho \chi \partial_\sigma \chi}}, \qquad n^2 = s := \pm 1, \tag{D.0.5}$$

and

$$\begin{aligned} \Delta_\xi n_\mu &= -s \frac{1}{\sqrt{s g^{\rho\sigma} \partial_\rho \chi \partial_\sigma \chi}} \left(\pounds_\xi \partial_\mu \chi - s g^{\nu\lambda} \frac{\partial_\nu \chi \pounds_\xi \partial_\lambda \chi}{s g^{\rho\sigma} \partial_\rho \chi \partial_\sigma \chi} \partial_\mu \chi \right) \\ &= -s \frac{1}{\sqrt{s g^{\rho\sigma} \partial_\rho \chi \partial_\sigma \chi}} \left(\delta^\nu_\mu - s n^\nu n_\mu \right) \pounds_\xi \partial_\nu \chi = -q^\nu_\mu \pounds_\xi n_\nu = 0 \end{aligned} \tag{D.0.6}$$

because of the condition that ξ preserves the boundary. Recalling that unit-norm means foliation independence of the normal, we see that what this anomaly is capturing is not so much the presence of the boundary, but rather any foliation-dependence in its description, namely non-invariance under $\chi \mapsto \chi'(\chi, x^\mu)$.

In the case of a null hypersurface, there is no choice of f that would make the normal foliation-independent, hence anomalies (D.0.4) are generically present. Furthermore, in order to distinguish physical solutions on the covariant phase space, one typically reduces the allowed variations to preserve a certain universal structure [33, 36]. This reduces the symmetry group and can lead to a fixed, non-vanishing first-order extension, hence anomalies. An interesting difference arises between a null hypersurface at a finite distance and future null infinity. In both cases, we have a background field describing the presence of the boundary. But at future null infinity, the same structure is used as conformal factor Ω in the compactification. As a consequence, reparametrizing χ at finite distance changes the normal 1-form n_μ, but reparametrizing Ω changes both the normal and the unphysical metric which induces the metric on the cross-sections, leading to two sources of anomalies. To see this difference in formulas, consider the volume elements. At finite distance we have

$$\epsilon_{\mathcal{N}} = \underleftarrow{i_l \epsilon} = -l \wedge \epsilon_S, \tag{D.0.7}$$

where $l \cdot n = -1$ is the auxiliary vector, hence (D.0.4) implies [85]

$$\Delta_\xi \epsilon_{\mathcal{N}} = -w_\xi \epsilon_{\mathcal{N}}, \qquad \Delta_\xi \epsilon_S = 0. \tag{D.0.8}$$

But the volume element of future null infinity is determined from the unphysical metric $\tilde{g}_{\mu\nu} := \Omega^2 g_{\mu\nu}$, which is anomalous:

$$\Delta_\xi \Omega = 0, \qquad \Delta_\xi n_\mu = w_\xi n_\mu, \qquad \Delta_\xi \tilde{g}_{\mu\nu} = 2 w_\xi \tilde{g}_{\mu\nu}, \qquad \Delta_\xi \tilde{\epsilon} = 4 w_\xi \tilde{\epsilon}. \tag{D.0.9}$$

Therefore taking

$$\epsilon_{\mathcal{I}} = \underleftarrow{i_l \tilde{\epsilon}} = -l \wedge \epsilon_S, \tag{D.0.10}$$

we have

$$\Delta_\xi \epsilon_{\mathcal{I}} = 3 w_\xi \epsilon_{\mathcal{N}}, \qquad \Delta_\xi \epsilon_S = 2 w_\xi \epsilon_S. \tag{D.0.11}$$

The first difference is that the anomalous dimension of the induce volume element changes from -1 to $+3$, and the second difference is that tensors on the cross-sections are now anomalous as well, unlike in the finite dimensional case. This comes as explained above from the fact that the background structure has the double role of determining the boundary and providing the unphysical metric.

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