Analysis, control and simulation of non-Newtonian fluids

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THESIS DISSERTATION

Analysis, control, and simulation of non-Newtonian fluid flows

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La passion et les rêves sont comme le temps: rien ne peut les arrêter; et il en sera ainsi tant qu'il y aura des gens prêts à donner un sens au mot "Liberté".

Gol D. Roger, *One Piece, Chapitre 100* (Eiichiro Oda).

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Albert Einstein, *Comment je vois le monde.*

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Foreword

This dissertation has been written with the aim of providing as self-contained as possible a presentation of the mathematical analysis of a class of non-Newtonian flows, in order to present the research work carried out by the present author and his collaborators as part of a doctoral thesis, on the one hand, and with a pedagogical aspiration, on the other, so that a student a priori not specialized in the field can read and interpret the various results without needing, at first, to go looking for information in an accompanying document. First, let us draw up an outline of the dissertation, which the reader will be able to refer to in order to simplify reading.

The dissertation begins with an introduction in French (hereafter entitled "Introduction (French)"), and the same introduction is then translated into English so that the non-French-speaking reader can benefit from the entire presentation without loss of information (hereafter entitled "Summary of the dissertation").

In Chapter 1, we first present in Section 1.1 the theory of nonlinear monotone operators, as well as that of variational inequalities, and the links between the two. This section is intended to provide an introduction to the tools and notions used in the analysis of the non-Newtonian flows we consider in this dissertation, but its reading can also be passed by the informed reader, its role once again being purely pedagogical. In a second step, we will present in Section 1.2 a state-of-the-art of existing results using these same theories. We have chosen to present a variety of results, both old and new, in order to show how the various conditions and hypotheses operate, and how research has evolved. Secondly, we will show a solution existence result (Theorem 1.2.18) and a finite stopping time result (Theorem 1.3.3), both obtained in collaboration with Laurent Chupin and Nicolae Cîndea (see [44]). To illustrate our finite stopping time result, we will present an analogous result in the case of the p-Laplacian parabolic equation.

In Chapter 2, we will present control theory for quasilinear equations. Section 2.1 will be devoted to the presentation of the Hilbert Uniqueness Method, which we will be using later, and via which we show an approximate control regularity result (Theorem 2.1.3) obtained in collaboration with Nicolae Cîndea (see [49]). Then, in Section 2.2, we will review the state of the art of various controllability results for quasilinear parabolic equations, and demonstrate the exact and approximate controllability results obtained with Nicolae Cîndea (Theorem 2.2.6, Theorem 2.2.11, Corollary 2.2.12, Corollary 2.2.13, and Corollary 2.2.14) also obtained in [49]. For pedagogical purposes, we will briefly introduce the mixed finite element methods and Hsieh-Clough-Tocher finite elements, which we use in the simulations of our results, which come from [49] as well.

Chapter 3 is devoted to the analysis of a sedimentation problem. In Section 3.1, we briefly present the theory of non-homogeneous Besov spaces on the torus, which will be invaluable in the analysis of the model, in particular to show the existence of weak solutions in the critical case. In Section 3.2, we recall some results concerning transport equations. These two sections can therefore be skipped by the experienced reader. In Section 3.3, we will establish a new result, obtained in collaboration with Dimitri A. Cobb concerning the existence of solutions for the sedimentation model in a power-law fluid whose viscosity depends on a possibly degenerate density coefficient (Theorem 3.0.2), to be prepublished in [54].

The final chapter of this dissertation concerns the study of an applied problem taken from medicine, the aim of which is to determine the diffusion coefficients of the porous membrane of a dialyzer fiber. To this end, we present simulations to determine the coefficients, based on a model previously studied by Julien Aniort, Laurent Chupin and Nicolae Cîndea in [7]. This study was carried out in collaboration with Julien Aniort and his team at the Gabriel Montpied University Hospital in Clermont-Ferrand.

Functional spaces and certain related prerequisites are defined and presented in the appendix. By way of example, tools that are commonplace in fluid mechanics but are nevertheless frequently used throughout the dissertation are presented, such as the spaces constructed from zero divergence distributions $\mathcal{D}_\sigma(\Omega)$ which are defined in Section A.2, the Aubin-Lions theorem is presented in Section A.1.1, or De Rham's theorem in Section A.1.2.

A final appendix presents the models studied in this dissertation, and this can be passed on by the discerning reader. More precisely, it is is an introduction to rational continuum mechanics, with a dual objective. Firstly, it highlights results from mathematical physics, which offers a framework that we hope will be easy to grasp without any prior physical knowledge (other than classical mechanics), and in which it is possible to find properties analogous to the laws of classical mechanics. In some cases, in particular for the action-reaction principles described by Newton's third law in classical mechanics, these properties are demonstrable in the axiomatics of continuum mechanics, whereas they are postulated in classical mechanics. The second objective is to show how the models we study mathematically are obtained, and how to interpret them. A student of mathematics will sometimes be confronted, when reading certain articles, with terms that are not necessarily part of his or her lexical field: simple fluid, shear, ... We therefore thought it would be useful to offer a few clarifications concerning the various macroscopic models studied. Reading this chapter is entirely optional, as it is presented solely for educational purposes. The results presented in this chapter are essentially drawn from [167, 166, 48].

Notations

Depending on the context, different notations will be used in this dissertation. Unless expressly stated otherwise, dimension of space will be designated by the letter N.

The symbol $\lesssim$ (respectively $\gtrsim$) will be used to express the fact that the inequality is true to within one multiplicative (positive) constant, when the parameters on which the constant depends have no influence on the computations. Where necessary, we will denote such a constant by the generic letter C.

For a fixed $p \in [1, +\infty]$, we will use either the notation p' or q to designate the element $r \in [1, +\infty]$ satisfying

$$\frac{1}{p} + \frac{1}{r} = 1.$$

When we use the letter q for any other purpose, it will always be specified in the statement in question.

In Chapter 3, we will refer to functional spaces without specifying spatio-temporal dependencies, in order to lighten the notation. More precisely, we will denote by $X(Y)$ the space $X(\mathbb{R}_+, Y(\mathbb{T}^N))$, and by $X_T(Y)$ the space $X((0,T), Y(\mathbb{T}^N))$, where X and Y describe functional spaces and $\mathbb{T}^N$ is the N-dimensional torus. For example, for $s, r \geq 1$ the space $L^s(L^r)$ denotes the space $L^s(\mathbb{R}_+, L^r(\mathbb{T}^N))$.

In this dissertation, the figures are sometimes inspired by different works. Specifically, Figure C.6 is inspired by [48, Fig. 1.4-1.], while Figures C.9, C.10, C.11, II.7, II.5, 1.1, and 1.2 in the present

dissertation are reproduced from the respective models [48, Fig. 1.8-3., Fig. 2.3-2., Fig. 2.3-3.], [59, Fig. 8.1., Fig. 4.3.], [175, Figure 32.5], and [156, Figure 11.]. We point out that Figures II.7,II.5 are the same as Figures I.3, I.1.

List of research work and prepublications

- Laurent Chupin, Nicolae Cîndea, Geoffrey Lacour. "Variational inequality solutions and finite stopping time for a class of shear-thinning flows." arXiv preprint arXiv:2112.02871 (2022). *Submitted.*
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- Julien Aniort, Nicolae Cîndea, Geoffrey Lacour. "Numerical calibration of a chemical transfert in hemodialysis." (Tentative title) *Work in process of finalization* (2023).

Car c'est tout à fait de quelqu'un qui aime à savoir, ce sentiment, s'étonner: il n'y a pas d'autre point de départ de la quête du savoir que celui-là.

Socrate, *Théétète* (Platon).

Introduction (french)

Ce manuscrit est consacré à l'analyse mathématique de certains fluides non-newtoniens et de certaines propriétés qui leurs sont propres. Avant toute chose, précisons qu'un fluide est "un milieu matériel parfaitement déformable", c'est-à-dire, pour simplifier, qu'il s'agit d'un corps déformable. De très nombreux matériaux sont des fluides, et nous y sommes confrontés quotidiennement. On pensera facilement aux gaz comme l'air, aux liquides, l'eau par exemple, ou même encore aux plasmas. En revanche, ce qui est moins connu, c'est que de nombreux fluides adoptent des comportements qui dépendent des contraintes qu'ils subissent. Prenons l'exemple de la mayonnaise: lorsque celle-ci subit des contraintes, par exemple lorsqu'elle est fouettée, elle se comporte comme un liquide au sens usuel du terme. Cependant, lorsqu'on la laisse au repos sans la soumettre à des forces extérieures non naturelles, elle reste figée, comme un solide. En fait, si l'on veut être plus précis, il faut dépasser un certain niveau de forces appliquées, un certain seuil de contraintes, pour que la mayonnaise viennent à se comporter de manière semblable à un liquide. Les fluides qui ont ce type de comportement sont appelés des fluides à seuil, dont on distinguera par exemple les fluides de *Bingham*. Ainsi, si nous voulons préciser notre définition d'un fluide, nous pouvons rajouter qu'il s'agit d'un corps qui peut subir des déformations non rigides. En d'autres termes, ce n'est pas un solide[a]. Pour insister encore un peu sur l'exemple de la mayonnaise, celle-ci s'écoule d'autant plus facilement que les contraintes qui lui sont imposées sont importantes. On parle alors de comportement *rhéofluidifiant*. On trouve aussi des fluides, comme une solution peu diluée de fécule de maïs par exemple, qui ont le comportement inverse: plus les contraintes imposées sont conséquentes, plus le fluide tend à se solidifier. On parle alors de comportement *rhéoépaississant*. L'étude des propriétés liées à l'écoulement des fluides s'appelle la *rhéologie*[b], et peut être abordée de multiples façons. Une première approche est la compréhension des interactions microscopiques donnant lieu à des écoulements particuliers. Bien que cette approche ne soit pas celle que nous aborderons dans ce manuscrit, elle permet de donner un cadre assez précis permettant de distinguer différents types d'écoulements. Citons quelques exemples.

Les suspensions: Il s'agit d'un cadre général de fluides. Intuitivement, lorsque de toutes petites particules, disons solides, se trouvent dans un fluide, on parle de celles-ci comme étant en suspension dans le fluide, et par suite on dit que le mélange est une suspension. De très nombreux exemples de suspensions se retrouvent dans la vie courante, comme par exemple le béton, le ciment, les peintures, mais c'est également le cas des potages (qui sont des suspensions de légumes dans l'eau), ou encore des fluides biologiques comme le sang (qui est, grossièrement, une suspension de globules et de plaquettes dans la plasma sanguin). Lorsque la force de gravité agissant sur une suspension est plus importante que les autres forces auxquelles celle-ci est soumise, on parle de *sédimentation*. Nous étudierons un tel phénomène au Chapitre 3. On observe alors sur le long terme une *séparation de phases*, ce qui se traduit par une variation de la densité du fluide selon la zone considérée.

Les colloïdes: Lorsque les particules en suspension sont très petites (disons de diamètre inférieur

[a] Si l'on veut être précis, un fluide est un matériau égalitaire qui n'est pas un solide, voir [167, Section IV.17.].

[b] Des racines grecques *rheos* qui signifie "courant" ou "écoulement" et de *logos* qui signifie "recherche" ou "théorie".

à 1 μm), certaines interactions qui étaient jusqu'alors négligeables prennent de l'ampleur, il s'agit d'interactions entre les particules en suspension dans le liquide, que l'on qualifie d'*interactions colloïdales*. Il s'agit essentiellement des forces de Van der Waals, qui tendent à faire s'agréger les différentes particules en suspension. Ainsi, les colloïdes donnent souvent lieu à des mélanges non homogènes, car les particules tendent à s'agglutiner. D'un point de vue pratique, on stabilise en général les interactions colloïdales afin d'éviter un tel phénomène et pour rendre le colloïde utilisable. De nombreux fluides sont de nature colloïdale, citons par exemple une nouvelle fois les peintures, les ciments, mais aussi des dentifrices, des crèmes cosmétiques, des argiles, etc. Aussi, on retrouve un certains nombre de colloïdes qui ont ce comportement de fluide de Bingham, c'est-à-dire qu'il est nécessaire d'y appliquer des contraintes suffisantes pour que le fluide entre en mouvement, s'écoule.

Les polymères: Le lecteur pourra s'interroger quant à la présence des polymères dans cette série d'exemple, tant ceux-ci sont utilisés sous leur forme solide dans la vie quotidienne. Cependant, dans le processus de fabrication de ces matériaux polymériques, on place des polymères en suspension dans un liquide. Un polymère, ou chaîne polymérique, est donc une molécule formée d'un grand nombre de particules (plusieurs milliers voire plusieurs millions) identiques associées par leur atome de carbone. Une des propriétés très intéressante est que cette chaîne admet donc énormément de configurations possibles, car elle peut s'étendre ou se recroqueviller. Cela induit que le fluide obtenu en mettant des polymères en suspension (et par suite le matériau qui en résulte, en général) a d'excellentes propriétés d'élasticité. On parle alors de fluide *viscoélastique*, et le mélange obtenu se comporte, pour parler simplement, comme un solide élastique lorsqu'on impose une contrainte, puis très rapidement s'écoule comme un liquide. Ainsi, il est commun que les solutions polymériques soient rhéofluidifiantes.

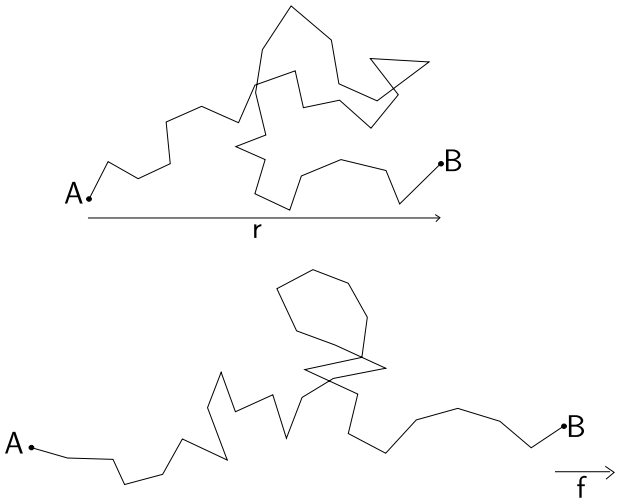


Figure I.1: Représentation d'une chaîne polymérique. La longueur de la chaîne r est la distance euclidienne entre ses extrémités. Elle peut s'étendre lorsqu'elle soumise à une contrainte f.

Les émulsions: Dans les exemples précédents, nous nous sommes essentiellement focalisés sur des suspensions de particules solides dans un liquide. En fait, il arrive qu'un fluide soit en fait un "mélange" de plusieurs fluides distincts, ayant tous des propriétés différentes. Lorsque l'on est dans une telle configuration, on parle en général de *fluide complexe*. Dans certains cas, on souhaite placer des petites particules d'un fluide en suspension (c'est-à-dire qu'elles soient très dispersées) dans un autre fluide, par exemple pour aider à la pénétration dans un matériau poreux. Lorsqu'une telle suspension a lieu, on parle alors d'une *mise en émulsion*. Ce terme d'usage commun se retrouve à juste titre dans la vie quotidienne, en particulier dans la sphère culinaire. Cela

nous permet de revenir à notre exemple initial qu'est la mayonnaise, qui est la mise en émulsion dans une solution oléagineuse (parfois aussi aqueuse, au choix du préparateur) de jaunes d'oeufs et de moutarde (et peut être quelques assaisonnements complémentaires). Comme nous l'avons évoqué, les émulsions peuvent alors avoir des comportements rhéofluidifiants, et aussi être des fluides à seuil. On retrouvera de nouveau des exemples d'émulsions dans de multiples domaines, en allant de l'agroalimentaire (vinaigrettes, beurre, etc.) au génie civil (fluides pétroliers) en passant par la cosmétique (crèmes par exemple).

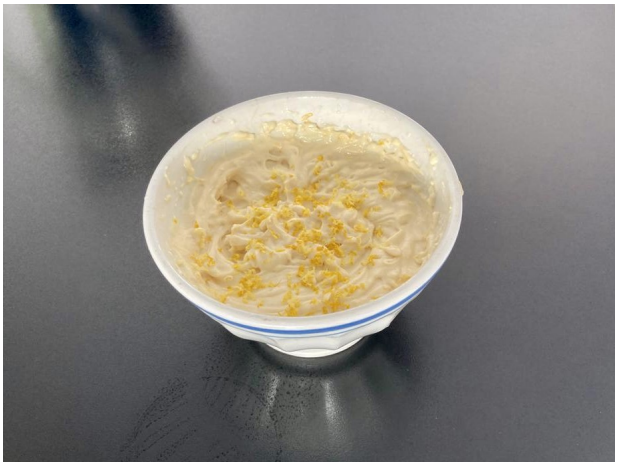


Figure I.2: La mayonnaise est une émulsion, qui se comporte comme un fluide de Bingham: sans être soumise à des contraintes suffisamment importantes, elle reste figée.

Il existe bien d'autres exemples de fluides, nous renvoyons le lecteur à [59] (dont la plupart des exemples précédents sont issus) ainsi qu'à [19, 105] pour des explications précises et détaillées.

Cependant, comme nous le mentionnions précédemment, un fluide est un corps qui n'est pas solide, au sens où il peut subir des déformations non rigides. Une question se pose alors: qu'entend-t-on lorsque l'on parle de corps? Une première approche serait d'utiliser la définition de la mécanique classique, en disant qu'un corps est un ensemble de points pondérés obéissants aux lois de la mécanique classique. Bien que cette approche ne soit pas fausse, elle présente de nombreuses difficultés. D'une part, si un point représente une particule de fluide, il est difficile de comprendre comment les interactions à de petites échelles, disons mésoscopiques, vont être considérées. Ainsi, doivent elles être omises, ou sinon prises en considération, mais comment? Il est alors préférable de définir un cadre simplifié qui rende compte du comportement macroscopique de l'écoulement, sans avoir à rentrer dans des considérations annexes comme les interactions de Van der Waals par exemple. Cela conduit à donner une définition différente d'un corps, comme étant un élément d'un ensemble, généralement un sous-ensemble ouvert de $\mathbb{R}^N$, $N \geq 2$.

I.1 Analyse des écoulements non-newtoniens

Nous commençons par présenter un modèle générique d'écoulement pour un fluide visqueux incompressible, puis nous étudierons l'existence de solutions pour celui-ci.

I.1.1 Position du problème et résultats antérieurs

Dans l'Appendice C, nous présenterons un cadre rigoureux permettant de conduire à cette dernière approche, que nous qualifierons d'approche macroscopique. Ce cadre est celui de la *mécanique*

des milieux continus, qui est une alternative simplifiée pour l'étude de certains fluides ou solides, particulièrement adapté à des matériaux déformables. Nous présenterons donc des résultats de physique-mathématique établis dans le cadre des recherches associées au sixième problème de Hilbert (formalisation mathématique de la physique), dont les résultats pourront être retrouvés dans [167]. Les objectifs de ce chapitre sont multiples:

- Premièrement, il sera mis en avant quels sont les objets considérés dans le cadre de la mécanique des milieux continus et par conséquent, dans le cadre de la mécanique des fluides telle que considérée dans la suite du manuscrit.
- Cela permettra de mettre en évidence comment les déformations induites par l'écoulement sont considérées dans notre cadre, et donc comment l'on approche les observations mentionnées précédemment.
- Enfin, en mettant en évidence que des alternatives aux lois de la mécanique classique sont disponibles dans le cadre de la mécanique des milieux continus, nous pourront établir le modèle générique des équations décrivant l'écoulement d'un fluide visqueux incompressible.

Ces dernières équations rendent compte d'une moyennisation locale des propriétés du fluide, et sont les suivantes:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \text{dans } \mathbb{R}_+ \times \Omega \\ \rho\left(\partial_t u + (u \cdot \nabla)u\right) - \operatorname{div}\left(\tau(\rho, D(u))\right) + \nabla \pi(\rho) = \rho f & \text{dans } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{dans } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{sur } \mathbb{R}_+ \times \partial\Omega \\ \rho_{|t=0} = \rho_0 & \text{dans } \Omega \\ u_{|t=0} = u_0 & \text{dans } \Omega \end{cases} \tag{I.1.1}$$

où les équations sont posées dans $\mathbb{R}_+ \times \Omega$, c'est-à-dire que la variable temporelle évolue dans $\mathbb{R}_+$ et la variable spatiale dans l'ouvert $\Omega \subset \mathbb{R}^N$. La première équation est appelée *équation de continuité*, et l'inconnue ρ est la densité du fluide. La seconde est l'équation de l'écoulement, dans laquelle u est le champ de vitesses du fluide, et π la pression. La condition $\operatorname{div}(u) = 0$ traduit quant à elle l'incompressibilité du fluide, c'est-à-dire que pour tout petit volume de fluide considéré, celui-ci reste constant en fonction du temps. Enfin, le tenseur τ est le tenseur des contraintes internes associées à l'écoulement. Lorsque ce dernier s'exprime de façon linéaire relativement à $D(u)$, où $D(u) = \frac{\nabla u + \nabla u^T}{2}$ est le tenseur des taux de déformations du fluide, on dit que le fluide est *newtonien*, d'après les hypothèses énoncées par Isaac Newton[c]. Dans le cas contraire, l'écoulement est dit *non-newtonien*. Dans le cas d'un fluide incompressible, les écoulements newtoniens sont décrits par exemple par les équations d'Euler (cas d'un fluide dit parfait) ou de Navier-Stokes.

Néanmoins, les écoulements newtoniens ne recouvrent pas les cas précédemment décrits, comme par exemple la rhéofluidification ou le rhéoépaississement. Autrement dit, de tels comportements sont non-newtoniens et par conséquent cela se traduit par le fait que le tenseur des contraintes s'exprime de façon nonlinéaire relativement à $D(u)$. Ainsi, (I.1.1) donne un cadre précis pour l'étude macroscopique d'un écoulement non-newtonien, mais, bien que cela ne soit pas l'approche que nous présenterons dans ce manuscrit, mentionnons qu'il existe des approches alternatives à cette dualité entre le macroscopique et le mésoscopique: on parle alors de modèle micro-macro. De tels modèles ont été décrits et analysés avec succès, par exemple pour les suspensions polymériques diluées. On pourra référer à [127, Chapitre 4] ainsi qu'à [118] par exemple pour la modélisation

[c] Voir par exemple [31, Section 1.4.2.] pour davantage de détails.

et l'analyse de modèles d'haltères.

Nous focaliserons dans un premier temps notre étude sur les équations de la forme (I.1.1) pour une densité constante, c'est-à-dire que $\rho = \rho_0 > 0$. Dans ce cas, le système générique se réécrit, avec un bon choix d'unités:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \operatorname{div} \tau(D(u)) + \nabla \pi = f & \text{dans } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{dans } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{sur } \mathbb{R}_+ \times \partial\Omega \\ u_{|t=0} = u_0 & \text{dans } \Omega. \end{cases} \tag{I.1.2}$$

Il reste alors une question en suspens: bien qu'on sache que le tenseur des contraintes τ décrit une loi non linéaire, on ne sait pas comment elle peut s'exprimer. Autrement dit, on ne sait pas quels sont les modèles que nous pouvons considérer. En fait, comme on pourrait s'y attendre, ces modèles sont essentiellement établis sur la base d'expérimentations physiques. Pour cela, on utilise un appareil permettant d'estimer les différentes propriétés du fluide en fonction des contraintes qui lui sont appliquées, appelé un *rhéomètre*. Les mesures relevées sont ensuite approchées par des courbes décrivant des fonctions, relatives à la déformation du fluide, et l'on obtient alors la description de τ.

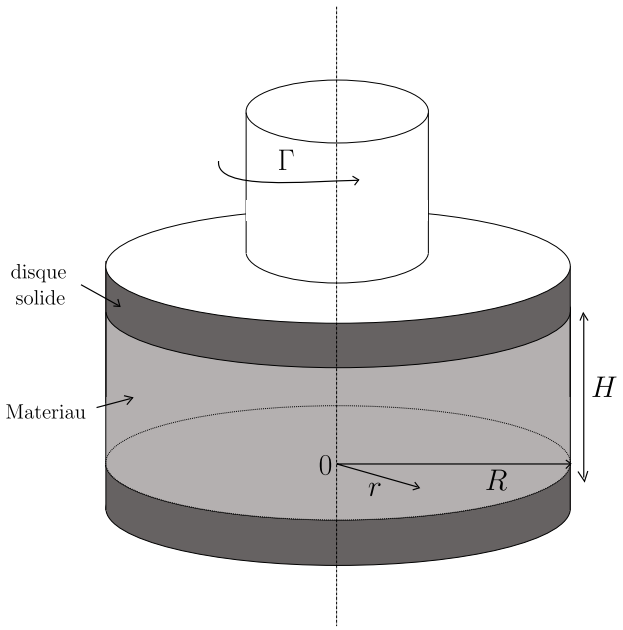


Figure I.3: Un rhéomètre à disques parallèles: une plaque est fixe puis le matériau est placé entre celle-ci et une plaque supérieure, dont la mobilité est assurée par la rotation relative à un axe, provoquant un cisaillement, que l'on mesure en fonction des contraintes appliquées.

Citons quelques exemples de lois pour τ, où $\kappa \in \{0, 1\}$ selon les considérations sur la viscosité du fluide, ce qui revient à dire que l'on ajoute ou pas de la diffusion dans l'équation.

- **Carreau**:

$$\tau(M) = \kappa M + (1 + |M|^2)^{\frac{p-2}{2}} M, \; p > 1,$$

- **Carreau-Yasuda**:

$$\tau(M) = \kappa M + (1 + |M|^{\alpha})^{\frac{p-2}{\alpha}} M,\ p > 1,\ \alpha > 0,$$

- **Ostwald-De Waele (loi puissance)**:

$$\tau(M) = \kappa M + |M|^{p-2} M,\ p > 1,$$

- **Cross**:

$$\tau(M) = \kappa M + \frac{M}{1 + |M|^{2-p}},\ p > 1,$$

- **Bingham**:

$$\tau(M) = \kappa M + \tilde{\tau}(M), \quad \tilde{\tau}(M) = \begin{cases} \frac{M}{|M|} & \text{si } |M| > 0 \\ [0,1] & \text{sinon,} \end{cases}$$

- **Herschel-Bulkley**:

$$\tau(M) = \kappa M + \tilde{\tau}(M), \quad \tilde{\tau}(M) = \begin{cases} \frac{M}{|M|} + |M|^{p-2} M,\ p > 1 & \text{si } |M| > 0 \\ [0,1] & \text{sinon.} \end{cases}$$

En observant les courbes associées aux tenseurs τ précités, on se rend compte que lorsque le paramètre p vérifie $1 < p < 2$, le fluide est rhéofluidifiant, et rhéoépaississant si $p > 2$. Dans le cas des fluides de Carreau, de Carreau-Yasuda, d'Ostwald-De Waele, et de Cross, lorsque $p = 2$, τ est linéaire en M et alors le fluide est newtonien. Une autre observation intéressante concerne les lois de Carreau-Yasuda et de Cross dans le cas rhéofluidifiant: lorsque le fluide est peu déformé ou énormément, i.e. lorsque $|M|$ est très petit ou très grand, on s'attend alors à ce que celui-ci adopte un comportement relativement semblable à un fluide newtonien. Ce sera en effet le cas, ce qui peut s'observer par exemple pour le sang, ce qui sera utilisé notamment au Chapitre 4. Cette remarque demeure valable pour un fluide de Carreau, qui est un fluide de Carreau-Yasuda pour $\alpha = 2$. Le cas de la loi de Bingham est, formellement, une loi d'Ostwald-De Waele pour le cas limite $p = 1$. Il convient cependant de remarquer dans ce cas que la partie non diffusive du tenseur des contraintes $\tilde{\tau}$ est discontinue en $M = 0$, puisque $\tilde{\tau}(M) = \frac{M}{|M|}$. Dans l'étude mathématique d'une telle équation, la notion appropriée est celle de fonction multivaluée. Quoiqu'il en soit, lorsque $|\tilde{\tau}(M)|$ prend une valeur différente de 1, cela impose à M de prendre la valeur $M = 0$. Cela revient à dire que le fluide a un comportement *viscoplastique*, autrement dit il se comporte comme un solide lorsqu'il ne subit pas assez de contraintes et s'écoule dès lors qu'un certain seuil de contraintes est dépassé: on parle alors de fluide à seuil. Avant de poursuivre, rappelons les définitions suivantes, où $\alpha \in \mathbb{N}^k$ est un multi-indice.

Définition I.1.1. *(Types de nonlinéarité) On dit qu'une équation aux dérivées partielles est:*

(i) Semilinéaire si, pour des fonctions $(a_\alpha)_{|\alpha|=k}$, elle s'écrit sous la forme:

$$\sum_{|\alpha|=k} a_\alpha(x) \nabla^\alpha u + a_0\left(\nabla^{k-1} u, \dots, \nabla u, u, x\right) = 0$$

(ii) Quasilinéaire si, pour des fonctions $(a_\alpha)_{|\alpha|=k}$, elle s'écrit sous la forme:

$$\sum_{|\alpha|=k} a_\alpha(\nabla^{k-1} u, \dots, \nabla u, u, x) \nabla^\alpha u + a_0\left(\nabla^{k-1} u, \dots, \nabla u, u, x\right) = 0$$

Remarquons alors que les équations décrites pour les tenseurs que nous considérons sont alors des équations *quasilinéaires*[d] suivant leur variable spatiale, ce qui revient à dire que celle-ci s'exprime comme une fonction linéaire des termes dont les dérivées sont du plus grand ordre.

Ainsi, l'analyse mathématique des fluides non-newtoniens se trouve à l'interface de l'analyse non linéaire et de la mécanique des fluides. Pour cette raison, nous décrirons dans le Chapitre 1 les outils nécessaires à l'analyse des équations quasilinéaires. Une notion incontournable dans ce cadre est celle d'*opérateur monotone*, c'est-à-dire qu'il s'agit d'un opérateur $A : X \to X^*$, où X est un espace de Banach réflexif et X^* est son dual topologique, qui vérifie:

$$\forall (u, v) \in X^2,\ \langle A(u) - A(v), u - v \rangle_{X^* \times X} \geq 0$$

.

Par exemple, le laplacien sur un domaine borné Ω défini par $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ est un opérateur monotone car il vérifie

$$\int_\Omega (-\Delta u + \Delta v)(u - v)\, dx = \int_\Omega |\nabla (u - v)|^2\, dx \geq 0.$$

La terminologie d'opérateur monotone peut être pensée en se rappelant qu'une fonction d'une variable réelle $f : I \to \mathbb{R}$ est croissante sur $I \subset \mathbb{R}$ si et seulement si:

$$\forall (t, s) \in I^2,\ (f(t) - f(s))\,(t - s) \geq 0,$$

l'inégalité étant inversée dans le cas d'un fonction décroissante. Ceci étant dit, on pensera également (et surtout) au fait que f est convexe sur I si et seulement si elle vérifie l'inégalité de convexité:

$$\forall (t, s) \in I^2,\ \left(f'(t) - f'(s)\right)(t - s) \geq 0.$$

On retrouvera alors une relation très étroite entre la notion d'opérateur monotone et la notion de convexité, il s'agit des opérateurs dits à *potentiel* convexes, et leur caractérisation est liée au le théorème de Kachurovskii, qui étend les caractérisations usuelles des fonctions convexes aux fonctionnelles. Dans le cas du laplacien, cela revient à dire que

$$(-\Delta u, v)_{L^2} = \left(J'(u), v\right)_{L^2},$$

où

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2\, dx,$$

est une fonctionnelle convexe. De manière générale, la plupart des lois introduites précédemment dérivent d'un potentiel convexe, à l'exception des fluides à seuil (Bingham ou Herschel-Bulkley). Remarquons que dans ce cas, le caractère rhéoépaississant ou rhéofluidifiant des fluides dont l'écoulement est associé aux lois citées pour τ (Carreau, Cross, Ostwald-De Waele) est déterminé par un paramètre $p > 1$. En fait, de ce paramètre dépend la régularité a priori de la solution (au moins si $\kappa = 0$). Pour simplifier, nous considérerons dans cette introduction que:

[d] Notons que la nomenclature des équations semilinéaires et quasilinéaires ne vient pas des notions d'application des mêmes noms. On pourra par exemple penser à une application décrivant une paraboloïde qui est une application quasilinéaire mais n'est pas décrite par une fonction linéaire en sa dernière variable.

$$\tau(M) \approx \kappa M + (\mu + |M|^2)^{\frac{p-2}{2}} M, \quad \kappa \in \{0,1\}, \mu \geq 0,$$

c'est-à-dire que τ se comporte comme le tenseur sus-mentionné dans les estimations, et garderons ces considérations à l'esprit par la suite.

En utilisant à bon escient la notion d'opérateur monotone, il est possible de montrer l'existence de solutions pour les équations (I.1.2). Citons par exemple le résultat suivant (voir [141]):

Théorème I.1.2 (Málek-Nečas-Ružička). *Considérons un domaine $\Omega \subset \mathbb{R}^3$, ayant une frontière C^3, $p \in [2,3)$, $u_0 \in W^{1,p}_{0,\sigma}(\Omega)$, $f \in L^2((0,T), L^2(\Omega))$, un temps arbitrairement fixé $T \in \mathbb{R}^*_+$, tels que le tenseur des contraintes τ vérifie de bonnes hypothèses de monotonie. Alors il existe une solution à* (I.1.2) *au sens faible. De plus, si $p \geq \frac{9}{4}$, la solution u est unique et vérifie:*

$$u \in C((0,T), L^2_\sigma(\Omega)) \cap L^\infty((0,T), W^{1,p}_{0,\sigma}(\Omega)) \cap L^{\frac{2}{p-1}}((0,T), W^{2,\frac{6}{p+1}}(\Omega)) \text{ et } \partial_t u \in L^2((0,T), L^2(\Omega)).$$

Remarque I.1.3. *Nous avons ici omis de spécifier les hypothèses précises du théorème, en particulier concernant le tenseur des contraintes τ, ce que nous ferons également par la suite concernant les résultats liés aux Chapitres 1 et 2. Ce choix est effectué afin de ne pas alourdir la présentation des résultats, puisque selon le résultat, les hypothèses concernant le système considéré ou le tenseur des contraintes sont multiples et très variables selon les auteurs. Cependant, les hypothèses et résultats concernés seront cités avec précision dans les Chapitres 1 et 2. Dans le cas présent, nous renvoyons le lecteur aux hypothèses (S1)–(S4) dans le Chapitre 1.*

Le théorème I.1.2 (qui correspond au Theorem 1.2.4 dans le manuscrit) nous indique donc qu'il existe des solutions dans le cas d'un écoulement rhéoépaississant, et que de plus il est possible de montrer que la solution est unique dès lors que le caractère de rhéoépaississement du fluide est suffisamment conséquent.

Le cas des fluides rhéofluidifiants est plus délicat, car le terme non linéaire peut parfois être singulier, notamment avec une loi d'Ostwald-De Waele ou de Bingham. Il est néanmoins toujours possible, en faisant appel à des techniques d'analyse fine, de montrer l'existence de solutions pour (I.1.2). On a par exemple le résultat suivant (voir [72] pour le résultat, les hypothèses sont (SW1),(SW2'),(SW3'), (SW4), et sont détaillées dans le Chapitre 1).

Théorème I.1.4 (Diening-Ružička-Wolf). *Soit Ω un sous-ensemble ouvert borné (régulier[e]) de $\mathbb{R}^N$, où $N \geq 2$, et soit $0 < T < +\infty$ fixé arbitrairement. De plus, en considérant certaines hypothèses de monotonie sur τ (que nous omettons ici[f]) et*

$$\frac{2N}{N+2} < p < +\infty,$$

alors, si $u_0 \in L^2_\sigma(\Omega)$ et $f \in L^q((0,T)\times\Omega)$, il existe une solution faible $u \in L^p\left((0,T), W^{1,p}_{0,\sigma}(\Omega)\right) \cap C_w([0,T], L^2_\sigma(\Omega))$ de (I.1.2).

Dans le cas bidimensionnel, le Théorème I.1.4 couvre l'ensemble des lois que nous avions alors mentionnées, exceptée celle pour l'écoulement de Bingham. Cela vient simplement du fait que l'opérateur non linéaire associé à un tel écoulement n'est pas monotone. Deux approches sont alors possibles : premièrement, l'on peut affaiblir la notion de solutions associées au problème, sinon on peut généraliser la notion d'opérateur monotone. La première approche conduit à la notion d'*inégalité variationnelle*. Formellement, on obtient une telle inégalité en multipliant la solution

[e] Nous reviendrons sur l'hypothèse de régularité au Chapitre 1

[f] Voir Remarque I.1.3

par $\varphi - u$, où φ est une fonction test (c'est-à-dire qu'elle appartient à un espace de fonctions dont la régularité est adaptée au cadre de l'équation). Dans ce cas, on obtient, toujours formellement, une expression du terme non linéaire comme dérivée d'un potentiel, ou comme limite de telles dérivées. L'idée est alors d'utiliser les propriétés de convexité d'un potentiel convexe, donc d'un opérateur monotone d'après le théorème de Kachurovskii, pour établir une inégalité impliquant le champ de vitesses (i.e. la solution). Dans cette inégalité, on ne fait alors plus apparaître la dérivée d'un potentiel, mais une différence l'impliquant.

Illustrons à titre d'exemple avec un calcul simple. Pour approcher le 1-laplacien, qui est formellement l'opérateur $u \mapsto |\nabla u|^{-1}\nabla u$, nous pouvons par exemple utiliser l'opérateur:

$$A_\varepsilon(u) = \frac{1}{1+\varepsilon}\int_\Omega |\nabla u|^{1+\varepsilon}\, dx,$$

qui est un potentiel convexe, et donc jouit de propriétés de convexité d'après le théorème de Kachurovskii. Formellement, l'opérateur terme A'_ε converge vers le 1-laplacien, noté Δ_1, lorsque $\varepsilon \to 0$. En particulier, cela signifie que l'on s'attend à ce que ce dernier bénéficie des mêmes inégalités de convexité que l'opérateur A'_ε. Plus précisément, toujours formellement:

$$\underbrace{\langle A'_\varepsilon(u) - A'_\varepsilon(v), u - v\rangle}_{\geq 0} \underset{\varepsilon\to 0}{\longrightarrow} \underbrace{\langle \Delta_1(u) - \Delta_1(v), u - v\rangle}_{\geq 0}.$$

Ce raisonnement heuristique montre que pour de tels problèmes pour lesquels la convexité est dégénérée, on cherche à faire apparaître une inégalité, le théorème de Kachurovskii nous permettant d'écrire:

$$\begin{aligned}\langle A'_\varepsilon(u) - A'_\varepsilon(v), u - v\rangle \geq 0 \Leftrightarrow A_\varepsilon(u) - A_\varepsilon(v) \geq \langle A'_\varepsilon(u), u - v\rangle \\ \underset{\varepsilon\to 0}{\longrightarrow} A_0(u) - A_0(v) \geq \langle \Delta_1(u), u - v\rangle \quad \text{(I.1.3)}\end{aligned}$$

Les calculs étant formels, le principe est alors de donner un sens à la dernière limite, donc à Δ_1 dans notre exemple. Les inégalités de ce type sont nommées *inégalités variationnelles*, et permettent de définir une notion de solution faible aux problèmes singuliers.

En appliquant ce même principe de passage à la limite au système de Bingham pour $\kappa = 1$, qui est parabolique, le résultat suivant peut être obtenu (voir [50] ainsi que [77] pour la preuve, et (D1)–(D4) pour le détail des hypothèses données dans le Chapitre 1).

Théorème I.1.5 (Cioranescu-Duvaut-J.L.Lions). *Soit $k \in \mathbb{N}$, $k \geq 1$, et Ω un domaine de $\mathbb{R}^N$ à frontière lisse, $N \in \{2, 3\}$. On considère $(p_i)_{1\leq i\leq k}$, $(q_i)_{1\leq i\leq k}$, tels que*

$$p_i \geq \frac{3N}{N+2} \quad \textit{and} \quad q_i \geq \frac{2p_i}{p_i - 1}$$

et $f \in \bigcap_{i=1}^{k} L^{\frac{q_i}{q_i-1}}((0,T), W_\sigma^{-1,\frac{p_i}{p_i-1}}(\Omega))$. Alors, sous certaines hypothèses, l'on a:

$$u \in \left(\bigcap_{i=1}^{k} L^{q_i}((0,T), H^s_{0,\sigma}(\Omega))\right) \cap L^\infty((0,T), L^2_\sigma(\Omega))$$

tel que $\partial_t u \in \bigcap_{i=1}^{k} L^{\frac{q_i}{q_i-1}}((0,T), H^{-s}_{\sigma}(\Omega))$ *satisfasse pour tout* φ *dans un ensemble de fonctions admissibles:*

$$\int_0^T \left(\langle \partial_t \varphi, \varphi - u \rangle + \langle -\Delta u, \varphi - u \rangle + \int_\Omega (u \otimes u) : D(\varphi - u)\, dx + A(\varphi) - A(u) \right) dt \geq \int_0^T \langle f, \varphi - u \rangle\, dt, \quad \text{(I.1.4)}$$

le résultat étant vrai pour le modèle de Bingham.

Illustrons notre propos en reprenant notre exemple précédent:A rend compte du terme non linéaire, qui a beaucoup d'égard se comporte comme le 1-laplacien, mais il existe cependant quelques différences. En effet, en prenant la limite $\varepsilon \to 0$ dans (I.1.3) avec un opérateur d'approximation tel que présenté précédemment, on n'obtient pas nécessairement exactement le 1-laplacien mais une variante de ce dernier. On retrouve une approche similaire du Théorème I.1.5.

Expliquons le sens de l'inégalité variationnelle (I.1.4), pour cela commençons par remarquer que l'inégalité variationnelle

$$\int_0^T \left(\langle \partial_t u, \varphi - u \rangle + \langle -\Delta u, \varphi - u \rangle + \int_\Omega (u \otimes u) : D(\varphi - u)\, dx + A(\varphi) - A(u) \right) dt \geq \int_0^T \langle f, \varphi - u \rangle\, dt \quad \text{(I.1.5)}$$

permet de recouvrir une solution classique, au moins formellement, si u est régulière, en testant contre $\varphi = (1+\lambda)u$ puis $\varphi = (1-\lambda)u$, pour $\lambda > 0$, puis divisant par λ et prenant la limite $\lambda \to 0$. Les auteurs du ThéorèmeI.1.5 ont montré l'existence d'une inégalité de la forme (I.1.5) dans le cadre bidimensionnel, y compris pour Bingham (voir [77]), mais pas dans le cas tridimensionnel. L'inégalité (I.1.4) présentée dans le Théorème I.1.5 n'est pas identique, puisqu'il apparaît un terme de dérivée temporelle sur φ et non sur u. Cela définit une notion de solution très faible pour le problème.

Notons également qu'un des avantages des inégalités variationnelles réside dans le fait que celles-ci se prêtent relativement bien aux simulations numériques (voir par exemple [158, 100]).

Une approche alternative aux inégalités variationnelles est donc de donner un sens plus général aux opérateurs monotones, permettant en particulier la prise en considération des multivaluations[g]. Cette généralisation est donnée par la notion d'*opérateur maximal monotone*, que nous verrons en détail dans le Chapitre 1. Dans ce cadre, il est alors possible d'étendre les solutions de monotonie précitées (voir [35], les hypothèses sont données dans le Chapitre 1 par (B1)–(B4)).

Théorème I.1.6 (Bulíček-Gwiazda-Málek-Świerczewska)**.** *On considère un domaine* Ω *de* $\mathbb{R}^N$, $N \in \{1,2\}$, *suffisamment régulier, et on suppose que le tenseur* τ *vérifie certaines hypothèses de monotonie maximale (voir Theorem 1.2.12) pour* $p > \frac{2N}{N+2}$. *Soit une N-fonction* φ *vérifiant la* Δ_2*-condition, la* ∇_2*-condition (voir Définition A.5.5), et telle qu'il existe* $C_1, C_2, C_3, C_4 > 0$ *telles que*

$$C_1 t^p - C_2 \leq \varphi(t) \leq C_3 t^r + C_4 \qquad r \in [p, +\infty)$$

Alors, il existe une solution faible de (I.1.2) *pour la condition de bord* $u \cdot n = 0$ *au lieu de la condition de Dirichlet homogène.*

[g]Rappelons que dans le cas d'un écoulement de Bingham, nous avions dit que le cadre approprié pour traiter le tenseur $\tilde{\tau}(M) = \frac{M}{|M|}$ est celui des fonctions multivaluées, c'est à cet endroit qu'elles font leur apparition

Le Théorème I.1.6 (qui correspond au Theorem 1.2.12 dans le manuscrit) est une alternative au Théorème I.1.4 pour les opérateurs maximaux monotones (mais avec la condition de bord $u \cdot n = 0$). Notons que les solutions obtenues appartiennent à des espaces dont la régularité dépend d'une N-fonction, appelés *espaces d'Orlicz*. Nous renvoyons à la Section A.5 en appendice pour plus de détails concernant ces espaces. Notons tout de même que, bien que les opérateurs maximaux monotones soient considérés dans le Théorème I.1.6, le cas des fluides à seuil (c'est-à-dire Bingham ou Herschel-Bulkley) ne semble pas pris en compte, nous renvoyons aux remarques suivant le Theorem 1.2.12 dans le Chapitre 1 pour davantage de détails.

I.1.2 Résultats de la thèse

Plusieurs questions se posent alors:

- Est-il possible de construire des solutions dans le cas tridimensionnel, qui prennent en considération les fluides à seuil?
- Si de telles solutions existent, sont-elles en relation avec des solutions numériquement utilisées?
- Quel est l'impact du terme non linéaire sur la régularité des solutions?
- Ces solutions rendent-elles compte de certaines propriétés expérimentales propres aux écoulement rhéofluidifiants?

Comme nous l'avons mentionné précédemment, les solutions sous forme d'inégalités variationnelles sont souvent utilisées pour les simulations numériques d'écoulements à seuil, soit en prenant la limite sur le paramètre $p > 1$ dans Ostwald-De Waele, soit en ajoutant une petite constante que l'on fait tendre vers 0 dans le simulations. Il convient alors de s'intéresser à l'existence théorique de telles solutions, idéalement sous des hypothèses ne faisant pas intervenir explicitement des objets d'analyse fonctionnelle complexes à définir. Le point crucial réside dans le fait de donner un sens à la limite dans l'approximation du terme non linéaire propre au caractère non-newtonien du fluide, ce qui peut se faire de plusieurs façons, par exemple en faisant apparaître un *gradient de Clarke* (voir Chapitre 1), ce qui donne la notion d'*inégalité hémivariationnelle*, ou encore en utilisant une convergence de type *Kuratowski* (voir Section A.4 en appendice).

Le cas singulier $p = 1$ (qui correspond aux fluides de Bingham et d'Herschel-Bulkley) est particulièrement difficile à traiter, tout spécialement dans le cas d'un fluide où l'on ne considère pas de terme de diffusion, correspondant à $\kappa = 0$. Dans ce cas, il faut alors considérer l'opérateur dans l'espace des fonctions à déformations bornées $BD(\Omega)$ (nous renvoyons au Chapitre 1 ainsi qu'à la Section A.3 en appendice pour davantage de précisions, et à [6] pour une discussion détaillée). Une approche intermédiaire mais qui permet néanmoins d'appréhender certaines des difficultés du problème est de considérer un terme de diffusion (avec un laplacien), ce qui revient à considérer $\kappa = 1$. On a alors un tenseur de la forme

$$-\operatorname{div}(\tau) := -\Delta - \Delta_1$$

et la conservation de l'énergie permet d'assurer que l'opérateur est bien défini sur $H^1_0(\Omega) \cap BD(\Omega)$, et ainsi l'opérateur $-\Delta - \Delta_1$ est défini sur un sous-domaine du laplacien. Plus généralement, et toujours dans cet esprit, nous considérons des tenseurs de contraintes visqueuses de la forme:

$$\tau(M) = M + F(|M|)M,$$

ce qui revient à considérer le système suivant:

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u - \operatorname{div}\left(F(|D(u)|)D(u)\right) + \nabla\pi = f & \text{dans } \mathbb{R}_+\times\Omega \\ \operatorname{div}(u) = 0 & \text{dans } \mathbb{R}_+\times\Omega \\ u = 0 & \text{sur } \mathbb{R}_+\times\partial\Omega \\ u_{|t=0} = u_0 & \text{dans } \Omega. \end{cases} \tag{I.1.6}$$

On fait les hypothèses suivantes sur le tenseur par des hypothèses sur F, qui est une fonction d'une variable réelle.

(C1) $F : (0,+\infty) \to (0,+\infty)$;

(C2) $F \in W^{1,\infty}_{loc}\left((0,+\infty)\right)$;

(C3) $t \mapsto tF(t)$ est croissante sur $(0,+\infty)$;

(C4) il existe $p \in [1,2]$, $t_0 > 0$ et $K > 0$ tels que pour tout $t \geq t_0$, $F(t) \leq Kt^{p-2}$.

On obtient alors le résultat suivant, issu de [44].

Théorème I.1.7. *On considère que F vérifie les hypothèses (C1)-(C4) et que $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$, est un domaine borné à frontière Lipschitz, $T > 0$ et soit une donnée initiale $u_0 \in H^1_{0,\sigma}(\Omega)$ ainsi qu'un terme de force $f \in L^2((0,T), H^{-1}_\sigma(\Omega))$. Alors, il existe une solution faible u de* (I.1.6) *satisfaisant la régularité suivante*

$$u \in C_w\left((0,T), L^2_\sigma(\Omega)\right) \cap L^2\left((0,T), H^1_{0,\sigma}(\Omega)\right) \quad \text{et} \quad \partial_t u \in L^{\frac{4}{N}}((0,T), H^{-1}_\sigma(\Omega)).$$

Dans le Théorème I.1.7, la notion de solution faible est donnée par la définition suivante.

Définition I.1.8. *On dit que la fonction $u \in L^2\left((0,T), H^1_{0,\sigma}(\Omega)\right) \cap C_w((0,T), L^2_\sigma(\Omega))$ telle que $\partial_t u \in L^{\frac{4}{N}}\left((0,T), H^{-1}_\sigma(\Omega)\right)$ est une solution faible de* (I.1.6) *si elle vérifie $u_{|t=0} = u_0 \in H^1_{0,\sigma}(\Omega)$, eet pour tout $\varphi \in C^\infty((0,T)\times\Omega)$:*

$$\begin{aligned} &\int_0^T \langle \partial_t u(t), \varphi(t)\rangle \, dt + \frac{1}{2}\left(\|u_0\|^2_{L^2(\Omega)} - \|u(T)\|^2_{L^2(\Omega)}\right) + \int_0^T \int_\Omega D(u(t)) : D(\varphi(t) - u(t)) \, dx \\ &- \int_0^T \int_\Omega (u(t)\cdot\nabla u(t))\cdot\varphi(t) \, dx \, dt + \int_0^T \int_\Omega G\left(|D(\varphi(t))|\right) - G\left(|D(u(t))|\right) \, dx \, dt \\ &\geq \int_0^T \langle f(t), \varphi(t) - u(t)\rangle \, dt. \end{aligned} \tag{I.1.7}$$

Dans l'inégalité variationnelle (I.1.7), la fonction G est définie de sorte à donner un sens à la limite de la procédure d'approximation, tout comme dans notre raisonnement heuristique impliquant (I.1.3), sur le terme non linéaire, et joue un rôle analogue à celui de la fonction A dans le Théorème I.1.5. Intuitivement parlant, la fonction G joue le rôle de primitive de la fonction F sans en être toutefois une par rapport aux raisons évoquées précédemment. Notons alors que le Théorème I.1.7 prend en compte les écoulements à seuil, et que cela s'obtient comme limite des modèles utilisés pour les simulations numériques. De plus, remarquons que la régularité des solutions est optimale au sens où l'on recouvre la régularité des solutions pour les équations de Navier-Stokes incompressible, ce qui était suggéré par le caractère bien posé de l'opérateur non linéaire. Notons que F n'est pas a priori bien définie en $s = 0$, il convient alors de donner un sens à sa limite dans ce cadre, comme nous l'avions précédemment mentionné.

Notons qu'il existe encore au moins une approche alternative à celles mentionnées, celle des solutions dissipatives. Le principe associé à ces dernières est de faire intervenir un tenseur d'énergie dépendant de la quantité d'énergie que le système a dissipé jusqu'au temps présent, ce qui permet de définir des solutions très faibles. Le cas de telles solutions a été établi avec succès dans de nombreux cas, comme par exemple pour l'équation d'Euler incompressible. Nous renvoyons le lecteur intéressé à [138, Section 4.4.] pour plus de détails dans ce dernier cas. Dans le cas de l'équation (I.1.2), il est possible de montrer l'existence de solutions dissipatives, y compris dans le cadre tridimensionnel pour les fluides à seuil (Théorème d'Abbatiello-Feireisl), nous renvoyons le lecteur à [1] pour plus de détails dans ce cas.

Dans une dernière partie de cette section, nous nous arrêtons sur une des propriétés majeures des écoulements rhéofluidifiants qui que leur flot devient statique au bout d'un intervalle de temps fini, on parle d'arrêt en temps fini. Autrement dit, si le fluide n'est soumis à aucune force extérieure, il s'immobilise. Cette propriété se retrouve naturellement dans des équations d'évolution présentant des termes non linéaires semblables à celui de la loi d'Ostwald-De Waele, par exemple lorsqu'on considère les solutions de l'équation:

$$\begin{cases} \partial_t u - \Delta_p u = 0 & \text{dans } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{sur } \mathbb{R}_+ \times \partial\Omega \\ u_{|t=0} = u_0 & \text{dans } \Omega \end{cases} \tag{I.1.8}$$

où l'opérateur Δ_p, défini ici pour $p > 1$ par

$$\Delta_p : \left| \begin{array}{l} W_0^{1,p}(\Omega) \longrightarrow W^{-1,q}(\Omega) \\ \\ u \longmapsto \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \end{array} \right. \tag{I.1.9}$$

est appelé le p-laplacien, s'arrêtent en temps fini lorsque $1 < p < 2$. Mieux que cela, il est possible d'estimer le temps d'arrêt en fonction de la norme de la donnée initiale, c'est l'objet du Théorème d'arrêt en temps fini de Di-Benedetto, qui correspond au Theorem 1.3.1 dans le manuscrit. On peut alors se demander si il est possible de montrer un tel résultat dans le cas des solutions établies dans le Théorème I.1.7. La réponse est positive et donnée par le résultat suivant, également issu de [44].

Théorème I.1.9. *Considérons que les hypothèses du Théorème I.1.7 sont vérifiées, que $T > 0$ est choisi suffisamment grand, et soit $p \in [1, 2)$. De plus, nous considérons deux constantes strictement positives λ et $T_1 < T$ telles que*

$$F(t) \geq \lambda t^{p-2} \text{ pour tout } t \in (0, +\infty) \qquad \text{et} \qquad f = 0 \quad \text{presque partout dans } (T_1, T). \tag{I.1.10}$$

Alors, il existe un temps d'arrêt fini $T_0 \in (0, T)$ pour u, c'est-à-dire que $u(t) \equiv 0$ pour presque tout $t \geq T_0$.

Le Théorème I.1.9 (qui correspond au Theorem 1.3.3 dans le manuscrit), exprime le fait que lorsque l'on arrête de soumettre le fluide à des forces extérieures, celui-ci fini par se stabiliser dans un état stationnaire en un temps fini. Pour revenir à notre exemple initial, lorsque l'on arrête de fouetter la mayonnaise, alors celle-ci se fige.

I.2 Contrôlabilité des équations paraboliques quasilinéaires

Comme souvent, une solution trouvée appelle une question nouvelle. En l'occurrence, sachant que cet arrêt en temps fini existe, est-il possible d'appliquer une force, localement (en espace), de

sorte que l'on puisse faire en sorte que le temps d'arrêt soit aussi petit qu'on le souhaite? Ou, en d'autres termes, est-il possible de faire en sorte qu'à partir d'un temps prescrit, le fluide soit forcément à l'arrêt, en appliquant une force localisée? C'est l'objet du Chapitre 2, et ce type de question relève de la *théorie du contrôle*.

Notons qu'il y a différentes notions de contrôlabilité, citons-en trois.

- La contrôlabilité approchée,
- La contrôlabilité exacte,
- La contrôlabilité optimale.

Prenons un exemple illustrant ces trois notions. Imaginons que vous souhaitez vous rendre de Paris à New-York. Faire du contrôle approché, c'est montrer qu'il existe un moyen de se rendre de Paris à un endroit qui est arbitrairement proche de New-York, et lorsqu'il est possible de montrer que l'on peut se rendre exactement à New-York, alors on a fait du contrôle exact. Maintenant, lorsque l'on a étudié un problème de contrôlabilité exacte ou approchée, on peut par exemple s'être donné un temps pour faire le trajet, disons quinze heures. Peut être que notre solution nous dit que se rendre de Paris à New-York est possible en ce laps de temps si l'on prend un avion de chasse. Cependant, il y a fort à parier que l'empreinte carbone de ce dernier soit conséquente, et a fortiori que la consommation de carburant le soit aussi. S'intéresser à la contrôlabilité optimale, c'est donc se dire que sachant que nous allons de Paris à New-York, nous souhaitons nous y rendre en minimisant un coût, qui peut par exemple être la consommation de carburant. Cela peut être sujet à des contraintes, comme par exemple le temps, où le chemin suivi. Dans ce cadre, ce qui est important est donc de minimiser une fonctionnelle de coût. Pour notre exemple, si l'on se donne une contrainte de quinze heures, il semble que prendre un avion de ligne soit une alternative plus intéressante, du point de vue du contrôle optimal, que celle de l'avion de chasse.

I.2.1 Position du problème et résultats précédents

La question que nous avons évoquée est donc une question relative à la contrôlabilité exacte ou approchée. Pour simplifier, nous considérerons la contrôlabilité d'équations paraboliques quasilinéaires (où les solutions sont à valeurs scalaires), dont un exemple est donné par (I.1.8).

Prenons le temps de faire un commentaire sur ces équations. D'un point de vue historique, l'analyse des équations paraboliques quasilinéaires et des propriétés de leurs solutions a pris son essor dans les années 1960, avec les travaux pionniers [125, 132, 9]. Ces équations ont fait l'objet d'un grand nombre de publications au cours des dernières décennies. Comme dans le cas des équations paraboliques linéaires, la question de la bornitude de la solution ou de son gradient se pose naturellement. De tels résultats sont maintenant bien connus et ont été établis pour de larges classes de systèmes quasi-linéaires, nous pouvons nous référer à [144, 78, 9, 37, 36, 73, 26, 151] dans le cas d'un domaine borné avec une frontière régulière. Cependant, cela reste une source d'activité de recherche importante, en particulier pour l'étude des systèmes singuliers ou dégénérés. Parmi les exemples, on peut citer les résultats récents de régularité au second ordre pour le p-Laplacien parabolique (voir par exemple [45, 87]) et son alternative dans le cadre du gradient symétrisé avec la méthode d'A-approximation (voir [25]). Dans le cas d'une équation à valeurs scalaires non dégénérée et non singulière, il est bien connu que l'on obtient des solutions lisses (nous renvoyons par exemple à [43, Theorem 3.4.1., Sections 3.1.4. and 0.10.]). Notons que ce fait est utilisé, par exemple, pour approcher la solution de l'équation du p-Laplacien parabolique (I.1.8) (voir par exemple [87, Section 4] ou [130]). Le cas $p = 1$ est plus difficile, mais il est encore possible de montrer dans certains cas non singuliers des propriétés d'existence et de régularité intéressantes

(voir par exemple [145, 170, 82]) ; dans ce dernier cas, nous soulignons que le caractère lisse des solutions de viscosité a été prouvé dans [122], ce qui peut être lié à des raisonnements tels que ceux présentés dans [119]. Nous devons également souligner que l'effet régularisant des équations paraboliques quasi-linéaires est connu dans de nombreux cas (voir par exemple [106, Chapitre IV] ou [145]). La littérature sur le sujet des équations et systèmes paraboliques quasi-linéaires est extrêmement vaste, c'est pourquoi nous renvoyons principalement le lecteur intéressé aux monographies [132, 125, 131, 169, 153, 176, 43, 123, 4] pour l'étude des propriétés de ces équations et systèmes.

La contrôlabilité des équations quasilinéaires est un sujet de recherches très actif, on se référera par exemple à [39, 41, 40], dans le cadre du contrôle optimal, ou [140, 139, 88] dans le cadre de la contrôlabilité exacte. Dans ces derniers articles, les résultats de la contrôlabilité exacte s'appliquent aux systèmes où le terme non linéaire dépend de la solution du système, mais pas de son gradient. Les résultats qui en sont issus sont présentés en détail dans le Chapitre 2.

Afin d'illustrer nos propos, nous considérons désormais l'alternative suivante à (I.1.8), où $\omega \subsetneq \Omega$ est un sous-domaine et φ un terme de forces extérieures:

$$\begin{cases} \partial_t u - \Delta_p u = \chi_\omega \varphi & \text{dans } (0,T)\times\Omega \\ u = 0 & \text{sur } (0,T)\times\partial\Omega \\ u_{|t=0} = u_0 & \text{dans } \Omega \end{cases} \tag{I.2.1}$$

Toujours dans le cas des équations quasilinéaires, lorsque le terme non linéaire dépend du gradient de la solution, seul un résultat de contrôlabilité locale avait jusqu'alors été démontré dans l'article récent [91], plus précisément le résultat suivant est vérifié (voir (F1)–(F3) dans le Chapitre 2 pour les hypothèses exactes).

Théorème I.2.1 (Fernández–Cara-Límaco-Thamsten-Menezes)**.** *Soit Ω un domaine de $\mathbb{R}^N$ à frontière lisse, $\omega \subsetneq \Omega$ un ouvert, $y_0 \in H_0^1(\Omega)\cap H^5(\Omega)$ et vérifiant la condition de compatibilité:*

$$(\Delta y_0, \Delta^2 y_0) \in H_0^1(\Omega)^2$$

Alors, il existe $\eta > 0$ tel que si

$$\|y_0\|_{H^5(\Omega)} \leq \eta$$

on ait l'existence d'un contrôle $\varphi \in L^2((0,T)\times\omega)$ pour des équations du type (I.2.1) *impliquant $y(T) = 0$.*

Le Théorème I.2.1 montre donc un résultat de contrôlabilité locale exacte, le terme local décrivant qu'il y a une condition de petitesse pour la norme de la donnée initiale. Ce théorème s'applique à une classe d'équations comprenant le p-laplacien parabolique, et nous renvoyons à l'énoncé précis de celui-ci (Theorem 2.2.5) dans le manuscrit pour les hypothèses précises concernant les équations prises en considération.

I.2.2 Résultats de la thèse

Nous nous intéresserons aux équations de la forme suivante:

$$\begin{cases} \partial_t y - \operatorname{div}(F(|\nabla y|)\nabla y) = \chi_\omega \varphi & \text{dans } Q_T \\ y = 0 & \text{sur } \Sigma_T \\ y_{|t=0} = y^0 & \text{dans } \Omega, \end{cases} \tag{I.2.2}$$

où $\Omega \subset \mathbb{R}^N$ est un domaine ouvert et borné dont la frontière est lisse $\partial\Omega$, $Q_T = (0,T)\times\Omega$, $\Sigma_T = (0,T)\times\partial\Omega$, la donnée initiale y^0 appartient à $L^2(\Omega)$, $\omega \subsetneq \Omega$ est également un domaine régulier, et il existe $p > 1$ tel que la fonction $F : \mathbb{R}_+ \to \mathbb{R}_+^*$ vérifie les hypothèses suivantes:

(A1) $F \in W^{1,\infty}(\mathbb{R}_+) \cap C^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+) \cap L^{\frac{p}{p-1}}(\mathbb{R}_+)$;

(A2) Le potentiel défini pour tout $t \in \mathbb{R}_+$ par $\Phi(t) = \int_0^t sF(s)\, ds$ est convexe et satisfait $\Phi \in W^{1,\frac{p}{p-1}}(\mathbb{R}_+)$;

(A3) Il existe $C_1, C_2, \mu, \nu > 0$ et $k_1, k_2 \geq 0$ tels que pour tout $t \in \mathbb{R}_+$,nous ayons

$$k_1 + C_1(\mu + t^2)^{\frac{p-2}{2}} \leq F(t) \leq k_2 + C_2(\nu + t^2)^{\frac{p-2}{2}}.$$

Le contrôle φ agit dans un ouvert borné et non vide $\omega \subset \Omega$. Plus précisément, on note $\chi_\omega \in C^\infty(\overline{\Omega})$ une fonction régulière telle que

$$\chi_\omega(x) = \begin{cases} 1 & \text{for } x \in \omega_\delta \\ 0 & \text{for } x \in \overline{\Omega \setminus \omega}, \end{cases}$$

où $\omega_\delta = \{x \in \omega \text{ tels que } \operatorname{dist}(x, \partial\omega) > \delta\}$ pour une constante $\delta > 0$ donnée suffisamment petite.

L'objectif est le suivant: on considère le système linéaire associé à (I.2.2), donné par:

$$\begin{cases} \partial_t u - \operatorname{div}(a(t,x)\nabla u) = \chi_\omega \varphi & \text{dans } (0,T) \times \Omega \\ u = 0 & \text{sur } \partial\Omega \\ u_{|t=0} = u^0 & \text{dans } \Omega \end{cases} \tag{I.2.3}$$

et l'on montre que ce dernier est approximativement contrôlable, avec un contrôle lisse. Pour ce faire, il convient en premier lieu d'adapter une méthode connue sous le nom de *Méthode d'Unicité de Hilbert* (abrégé par l'acronyme HUM par la suite). Cette méthode repose sur un argument de dualité de type Fenchel-Rockafellar et est présentée en détail dans le Chapitre 2. Nous référons également à [30] pour une présentation générale et très détaillée de la méthode dans le cadre des équations paraboliques. Ainsi, on montre le résultat suivant (voir [49, Theorem 1.1.]).

Théorème I.2.2. *Soit Ω un sous ensemble ouvert borné de $\mathbb{R}^N$ à frontière Lipschitz, $u^0 \in L^2(\Omega)$, $a \in C^\infty(\overline{Q_T})$ vérifiant*

$$0 < \rho_\star \leq a(t,x) \qquad ((t,x) \in \overline{Q_T}),$$

et $T > 0$. Alors, pour tout $\varepsilon > 0$ il existe un contrôle approché $\varphi \in C^\infty(\overline{Q_T})$ au sens où la solution correspondante u de (I.2.3) *vérifie*

$$\|u(T)\|_{L^2(\Omega)} < \varepsilon.$$

De plus, le contrôle φ dépend de manière Lipschitz continue du coefficient de diffusion a pour la norme $\|\cdot\|_{L^2(Q_T)}$.

Le lecteur pourra à raison s'interroger quant à l'intérêt de la dépendance continue de la solution φ relativement au paramètre de diffusion. La raison est la suivante: à partir de la solution contrôlée de l'équation linéaire, on souhaite montrer l'existence d'un contrôle approché pour l'équation quasilinéaire, en appliquant un théorème de point fixe de type Schauder, mais pour une topologie faible. Il est alors nécessaire d'avoir une dépendance continue sur le point fixe dans un espace bien choisi pour effectuer cette méthode de point fixe. Appliquant cette stratégie, on peut alors montrer le résultat de contrôlabilité approchée suivant (voir [49, Theorem 1.2.]).

Théorème I.2.3. *Soit F vérifiant les hypothèses (A1)–(A3) et y^0 appartenant à $L^2(\Omega)$ choisis tels qu'il existe une unique solution à* (I.2.2). *Alors, il existe un contrôle distribué φ, dont la régularité est donnée par le Théorème I.2.2, tel que* (I.2.2) *soit approximativement contrôlable à zéro en $T > 0$, i.e., pour tout $y^0 \in L^2(\Omega)$ et tout $\varepsilon > 0$ il existe un contrôle $\varphi \in C^\infty(\overline{Q_T})$ tel que la solution y de* (I.2.2) *satisfasse*

$$\|y(T)\|_{L^2(\Omega)} \leq \varepsilon.$$

On remarquera que le cas des équations singulières ou dégénérées comme le p-laplacien pour $p \neq 2$ ne sont a priori pas pris en considération dans le précédent théorème. Cependant, il est possible de prendre une famille de solutions arbitrairement proches du cas du p-laplacien parabolique qui vérifient les hypothèses du Théorème I.2.3. On obtient alors immédiatement le corollaire suivant (voir [49, Corollary 1.1.]).

Corollaire I.2.4. *Soit $u^0 \in L^2(\Omega)$ et $\frac{3}{2} < p < 3$. Alors* (I.2.1) *est approximativement contrôlable à zéro pour tout temps $T > 0$, i.e., pour tout $\varepsilon > 0$ il existe un contrôle $\varphi \in C^\infty(\overline{Q_T})$ tel que la solution u of* (I.2.1) *vérifie*

$$\|u(T)\|_{L^2(\Omega)} \leq \varepsilon.$$

En fait, dans le cas où l'on se concentre sur les équations ayant une solution qui s'arrête en temps fini, il est possible de déduire du résultat précédent un résultat de contrôlabilité exacte globale. Plus précisément, on a (voir [49, Theorem 1.3.]):

Théorème I.2.5. *Considérons que F vérifie les hypothèses (A1)–(A3), et que y est la solution de* (I.2.2) *associée à la donnée initiale $y^0 \in L^2(\Omega)$, $y^0 \neq 0$. De plus, on considère que y s'arrête en temps fini, c'est-à-dire que, si $\varphi = 0$, alors il existe $T_s \in (0, T)$ tel que:*

$$\|y(T_s)\|_{L^2(\Omega)} = 0. \tag{I.2.4}$$

Dans ce cas, on peut choisir un terme de force φ tel que y soit exactement contrôlable à zéro pour tout temps $T^\star \in (0, T)$.

Comme nous l'avons mentionné, le p-laplacien parabolique s'arrête en temps fini, il s'ensuit donc immédiatement le résultat suivant (voir [49, Corollary 1.2.]).

Corollaire I.2.6. *Soit $u^0 \in L^\infty(\Omega)$ et $\frac{3}{2} < p < 2$. Alors, en choisissant la solution positive de* (I.2.1)*, le problème* (I.2.1) *est contrôlable à zéro pour tout temps $T > 0$,* i.e.*, il existe un contrôle $\varphi \in C^\infty(\overline{Q_T})$ telle que la solution u de* (I.2.1) *vérifie*

$$u(T) = 0.$$

I.3 Solutions faibles d'un système de Stokes-transport non newtonien

Jusqu'à présent, nous avons considéré des fluides en mouvement qui finissent par s'arrêter complètement. Dans les Chapitres 1 et 2, nous nous sommes concentrés sur des modèles où la densité du fluide était constante (en considérant les modèles du chapitre 2 comme une façon de rendre compte de modèles proches de la mécanique des fluides), mais cela n'est pas adapté pour étudier des problèmes de sédimentation ou les fluides comportant des particules en suspension par exemple.

I.3.1 Position du problème et résultats précédents

Dans ce cas, nous mènerons notre étude en nous concentrant sur le cas d'un fluide *quasi-stationnaire*, c'est-à-dire que le fluide n'évolue plus suffisamment pour considérer le terme de dérivée temporelle dans le modèle. Dans le Chapitre 3, nous étudierons un problème de sédimentation dans un fluide en loi puissance, en considérant l'équation suivante:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{dans } \mathbb{R}_+ \times \Omega \\ -\operatorname{div}\left(\nu(\rho)|D(u)|^{p-2}D(u)\right) + \nabla \pi = \rho g & \text{dans } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{dans } \mathbb{R}_+ \times \Omega \\ \rho_{|t=0} = \rho_0 & \text{dans } \Omega. \end{cases} \tag{I.3.1}$$

Le système (I.3.1) n'est plus un problème parabolique mais un système couplé hyperbolique-elliptique, ce qui appelle des méthodes de résolution très différentes. Comme nous le verrons, l'équation (I.3.1) est une équation dite scalaire active, c'est-à-dire que l'inconnue est une fonction scalaire ρ qui est solution d'une équation de transport donnée par

$$\partial_t \rho + u \cdot \nabla \rho = 0,$$

où u est un champ de vitesses exprimé comme une fonction de ρ (dans notre cas, par le biais de la résolution d'un problème elliptique non linéaire). Un premier cas important d'équation scalaire active, qui a été la source de nombreuses recherches récentes, est donné par le cas de l'équation de Stokes-transport

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{dans } \mathbb{R}_+ \times \Omega \\ -\Delta u + \nabla \pi = \rho g & \text{dans } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{dans } \mathbb{R}_+ \times \Omega \\ \rho_{|t=0} = \rho_0 & \text{dans } \Omega, \end{cases} \tag{I.3.2}$$

qui correspond à (I.3.1) pour $p = 2$, et lorsque ν est constante suivant la densité ρ. Notons que l'absence de terme convectif est cohérente du point de vue physique, puisque pour un fluide quasi-statique, on considère un transport par convection négligeable relativement au transport par diffusion, ce qui revient à dire que l'on considère un écoulement à très petit *nombre de Reynolds*. Nous renvoyons à [105, Chapter 9] pour davantage de détails. Notons que ce système s'obtient rigoureusement comme limite d'un grand nombre de particules sphériques en sédimentation (en dynamique non inertielle) plongées dans un fluide de Stokes (voir par exemple [113, 114] ainsi que [115] pour l'analyse de l'impact de la viscosité d'Einstein). Précisons également de tels raisonnements sont valables dans d'autres cadres, comme par exemple en remplaçant le modèle de Stokes par un modèle de type Vlasov-Stokes (voir [112]). L'analyse du modèle (I.3.2) a été effectuée dans le cadre d'un domaine borné général, il convient d'ajouter une condition garantissant l'unicité des solutions, comme une condition de bord. Par exemple, avec la condition de Dirichlet homogène, le système générique (I.3.2) devient

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{dans } \mathbb{R}_+ \times \Omega \\ -\Delta u + \nabla \pi = \rho g & \text{dans } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{dans } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{sur } \mathbb{R}_+ \times \partial\Omega \\ \rho_{|t=0} = \rho_0 & \text{dans } \Omega \end{cases} \tag{I.3.3}$$

Il peut être montré (voir [61]), que pour un champ de vitesse $u \in L^1(W^{1,\infty})$, la régularité spatiale induite par la donnée initiale peut être préservée pour des estimations sur l'équation de transport, et que l'on a une perte de régularité pour un champ de vitesses de régularité moindre. Pour espérer obtenir l'unicité d'un couple de solutions dans le cadre d'une équation scalaire active, il faut donc choisir une donnée initiale qui permette a priori de conduire à un couple de solutions dont le champ de vitesses a une régularité $W^{1,\infty}$ en espace. Dans le cas où la régularité spatiale du

champ de vitesses est supérieure, nous parlons de régime sous-critique, de régime sur-critique si la régularité est moindre, et de régime critique si la donnée initiale conduit exactement a un champ de vitesses de régularité spatiale Lipschitz (ou, à la rigueur, Log-Lipschitz). Dans le cas d'un régime sous-critique, il a été montré [128, Theorem 1.1.] le résultat ci-après.

Théorème I.3.1 (Leblond). *Soit Ω un domaine de frontière C^2 de $\mathbb{R}^N$, $N \in \{2,3\}$, et $\rho_0 \in L^\infty(\Omega)$. Alors il existe une solution $(\rho,u) \in L^\infty(\mathbb{R}_+, L^\infty(\Omega)) \times L^\infty(\mathbb{R}_+, W^{1,\infty}(\Omega))$ de* (I.3.3) *au sens des distributions.*

On peut alors s'interroger quant à la véracité d'un tel résultat sur un domaine non borné. Il est alors parfois possible de montrer l'existence de solutions, par exemple pour le système:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{dans } \mathbb{R}_+ \times ((0,1) \times \mathbb{R}) \\ -\Delta u + \nabla \pi = \rho g & \text{dans } \mathbb{R}_+ \times ((0,1) \times \mathbb{R}) \\ \operatorname{div}(u) = 0 & \text{dans } \mathbb{R}_+ \times ((0,1) \times \mathbb{R}) \\ u = 0 & \text{sur } \mathbb{R}_+ \times (\{0,1\} \times \mathbb{R}) \\ \int_0^1 u_1(x,y)\, dy = 0 & \text{dans } \mathbb{R}_+ \times \mathbb{R} \\ \rho_{|t=0} = \rho_0 & \text{dans } ((0,1) \times \mathbb{R})\,. \end{cases} \tag{I.3.4}$$

Théorème I.3.2 (Leblond). *Soit $\rho_0 \in L^\infty(\Omega)$ et $\Omega = (0,1) \times \mathbb{R}$. Alors il existe une solution $(\rho,u) \in L^\infty(\mathbb{R}_+, L^\infty(\Omega)) \times L^\infty(\mathbb{R}_+, W^{1,\infty}(\Omega))$ de* (I.3.4) *au sens des distributions.*

Dans le cas tridimensionnel de l'espace entier, c'est-à-dire pour $\Omega = \mathbb{R}^3$, il est nécessaire d'ajouter une condition de décroissance à l'infini en espace pour le champ de vitesses, soit:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{dans } \mathbb{R}_+ \times \mathbb{R}^3 \\ -\Delta u + \nabla \pi = \rho g & \text{dans } \mathbb{R}_+ \times \mathbb{R}^3 \\ \operatorname{div}(u) = 0 & \text{dans } \mathbb{R}_+ \times \mathbb{R}^3 \\ \lim\limits_{|x| \to +\infty} |u| = 0 & \text{dans } \mathbb{R}^3 \\ \rho_{|t=0} = \rho_0 & \text{dans } \mathbb{R}^3 \end{cases} \tag{I.3.5}$$

Ainsi, il été établi le résultat suivant [142, Theorem 2.1.].

Théorème I.3.3 (Mecherbet-Sueur). *Soit $q \geq 3$ et ρ_0 une mesure de probabilité sur $L^q(\mathbb{R}^3)$. En notant E_q le sous-espace de $L^q(\mathbb{R}^3)$ composé de telles mesures, il existe une unique solution au sens des distributions (u,π,ρ) à* (I.3.5) *telle que:*

- *Si $q > 3$, $(u,\pi,\rho) \in C\left(\mathbb{R}_+, W^{2,q}(\mathbb{R}^3) \times W^{1,q}(\mathbb{R}^3) \times E_q\right)$*
- *Si $q = 3$, $(u,\pi,\rho) \in C\left(\mathbb{R}_+, \dot{W}^{2,3}(\mathbb{R}^3) \times \dot{W}^{1,3}(\mathbb{R}^3) \cap \bigcap\limits_{r \in (3,+\infty)} W^{1,r}(\mathbb{R}^3) \times W^{1,3}(\mathbb{R}^3) \times E_q\right)$.*

Le résultat de ce théorème étant optimal au sens où le cas $q = 3$ correspond au régime critique (et $q > 3$ au régime sous-critique). Nous renvoyons à [142] pour davantage de précisions. Une autre question importante est de savoir si les solutions obtenues sont asymptotiquement stables relativement à une perturbation sur la densité initiale ρ_0. Un premier résultat en ce sens a été obtenu dans un cadre semblable à (I.3.3) en remplaçant le laplacien par l'identité, c'est-à-dire en regardant une équation d'écoulement en milieu poreux. On obtient alors le système de type Darcy-transport suivant.

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{dans } \mathbb{R}_+ \times \Omega \\ u + \nabla \pi = \rho g & \text{dans } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{dans } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{sur } \mathbb{R}_+ \times \partial\Omega \\ \rho_{|t=0} = \rho_0 & \text{dans } \Omega \end{cases} \tag{I.3.6}$$

Notons que ce système peut se réécrire, pour $\Omega := \mathbb{R}^2$:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{dans } \mathbb{R}_+ \times \mathbb{R}^2 \\ u = R_1 R^\perp \rho & \text{dans } \mathbb{R}_+ \times \mathbb{R}^2 \\ \operatorname{div}(u) = 0 & \text{dans } \mathbb{R}_+ \times \mathbb{R}^2 \\ \rho_{|t=0} = \rho_0 & \text{dans } \mathbb{R}^2 \end{cases} \tag{I.3.7}$$

où $R^\perp = (R_2, -R_1)$, et R_1 etR_2 correspondent aux transformations de Riesz respectivement suivant la première et la seconde variable spatiales. Ainsi, on peut visualiser l'équation (I.3.7) comme une équation quasi-géostrophique de surface (usuellement désignée par l'acronyme SQG), pour laquelle le champ de vitesses est donné par $u = -\nabla^\perp(-\Delta)^{-\frac{1}{2}}\rho$. Toutes deux font partie de la classe des équations quasi-géostrophiques de surface généralisées, pour lesquelles nous définissons

$$u = \nabla^\perp(-\Delta)^{-\frac{\beta}{2}} m(D)\rho,$$

où $\beta \in \mathbb{R}$ et $m(D)$ est un multiplicateur de Fourier positif, défini de sorte que $m(D) = \log(\mathrm{Id} - \Delta)^\mu$ $\mu \in \mathbb{R}$. Nous renvoyons à [53, Section 1.1.] pour davantage de détails et d'explications sur les modèles associés. On a alors le résultat suivant (voir [80, Theorem 1.3.]).

Théorème I.3.4 (Elgindi). *Soit $\eta(y) := y$. Il existe $\varepsilon_0 > 0$ tel que si l'on considère la solution de* (I.3.7) *associée à la donnée initiale $\rho_0 = \tilde{\rho}_0 + \eta$ où $\|\tilde{\rho}_0\|_{W^{7,1}(\mathbb{R}^2)} + \|\tilde{\rho}_0\|_{H^s(\mathbb{R}^2)} \leq \varepsilon \leq \varepsilon_0$, $s \geq 20$, alors la solution ρ vérifie pour tout $t > 0$:*

(1) $\|\rho(t) - \eta\|_{H^3(\mathbb{R}^2)} \leq \varepsilon t^{-\frac{1}{4}}$

(2) $\|u_1(t)\|_{H^3(\mathbb{R}^2)} \leq \varepsilon t^{-\frac{3}{4}}$

(3) $\|u_2(t)\|_{H^3(\mathbb{R}^2)} \leq \varepsilon t^{-\frac{5}{4}}$

où $u := (u_1, u_2) = R_1 R^\perp \rho$.

Notons qu'un résultat similaire peut être établi pour le tore bidimensionnel (voir [80, Theorem 1.4.]). Il est en fait possible de montrer, pour $\Omega = \mathbb{T} \times (0,1)$, que les solutions de (I.3.3) sont aussi asymptotiquement stables relativement à une perturbation sur la donnée initiale ρ_0 (voir [60, Theorem 1.1.]).

Théorème I.3.5 (Dalibard-Guillod-Leblond). *Il existe $\varepsilon_0 > 0$ petit tel que pour toute donnée initiale $\rho_0 \in H^6(\mathbb{T} \times (0,1))$ vérifiant $\|\rho_0 - \theta\|_{H^6(\mathbb{T}\times(0,1))} \leq \varepsilon_0$ et $(\rho_0 - \theta) \in H^2_0(\mathbb{T} \times (0,1))$, la solution ρ de* (I.3.3) *satisfasse:*

$$\|\rho - \rho_\infty\|_{L^2(\mathbb{T}\times(0,1))} \leq \varepsilon_0 (1+t)^{-1} \qquad et \qquad \|\rho - \rho_\infty\|_{H^4(\mathbb{T}\times(0,1))} \leq \varepsilon_0$$

où ρ_∞ est donné par le réarrangement vertical décroissant de ρ_0, c'est-à-dire:

$$\rho_\infty(z) := \int_0^\infty \mathbb{1}_{0 \leq z \leq |\{\rho_0 \geq \lambda\}|}\, d\lambda,$$

et où la fonction $\theta : \mathbb{T} \times (0,1) \to (0,1)$ *qui dépend seulement de sa seconde variable, est appelée profil de densité stratifié et est donnée par* $\theta(z) = 1 - z$.

On peut alors s'interroger quant à l'existence de résultats d'existence de solutions pour une classe de problèmes incluant des problèmes du type (I.3.5) et (I.3.7), et, par suite, l'existence de résultats de stabilité sur ceux-ci. On considère l'équation de Stokes-transport-fractionnaire suivante:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{dans } \mathbb{R}_+ \times \mathbb{R}^N \\ (-\Delta)^{\frac{\alpha}{2}} u + \nabla \pi = \rho g & \text{dans } \mathbb{R}_+ \times \mathbb{R}^N \\ \operatorname{div}(u) = 0 & \text{dans } \mathbb{R}_+ \times \mathbb{R}^N \\ \lim\limits_{|x| \to +\infty} |u| = 0 & \text{dans } \mathbb{R}^N \\ \rho_{|t=0} = \rho_0 & \text{dans } \mathbb{R}^N. \end{cases} \tag{I.3.8}$$

Notons que (I.3.8) défini bien une équation scalaire active, car en appliquant le projecteur de Leray $\mathbb{P} = \mathrm{Id} + \nabla(-\Delta)^{-1} \operatorname{div}$, à savoir la projection orthogonale L^2 sur l'espace des champs de vecteurs à divergence nulle à la seconde équation de (I.3.8), on obtient directement

$$u = (-\Delta)^{-\frac{\alpha}{2}} \mathbb{P}(\rho g).$$

Il est alors possible de montrer l'existence et l'unicité de solutions globales pour (I.3.8).

Théorème I.3.6 (Cobb). *Soit* $\alpha \in (0, N)$ *et un exposant* q *tel que*

$$\frac{2}{1 + \frac{\alpha}{N}} < q < \frac{N}{\alpha}. \tag{I.3.9}$$

Alors pour toute donnée initiale $\rho_0 \in L^q(\mathbb{R}^N)$*, il existe une solution faible globale* $\rho \in L^\infty(\mathbb{R}_+, L^q(\mathbb{R}^N))$ *de* (I.3.8) *associée à la donnée initiale* ρ_0*. Si de plus:*

- $\alpha \in [0, N)$ *et que les indices* $s > 0$ *et* $q \in [1, +\infty]$ *sont tels que*

  $$q < \frac{N}{\alpha}, \qquad \text{et} \qquad s \geq 1 - \alpha.$$

  Alors si $\rho_0 \in B^s_{\infty,1}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$*, il existe un temps* $T > 0$ *tel que la solution de* (I.3.8) *soit unique et vérifie* $\rho \in C([0,T), B^s_{\infty,1}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N))$.

- $N \geq 2$ *et* $\alpha \in (1, N)$*. Alors pour* $q \in \left[1, \frac{N}{\alpha}\right)$ *et* $r = \frac{N}{\alpha - 1}$*, et pour toute donnée initiale* ρ_0 *dans l'espace* $L^q(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$*,* (I.3.8) *admet une unique solution globale* $\rho \in L^\infty(\mathbb{R}_+, L^q(\mathbb{R}^N) \cap L^r(\mathbb{R}^N))$.

- $N \geq 2$ *et* $\alpha \in [1, N]$*. Alors pour* $q = \frac{N}{\alpha}$ *et* $r = \frac{N}{\alpha - 1}$*, et pour toute donnée initiale* $\rho_0 \in \dot{B}^0_{p,1}(\mathbb{R}^N) \cap B^0_{r,1}(\mathbb{R}^N)$*,* (I.3.8) *admet une unique solution globale* $\rho \in C(\mathbb{R}_+; \dot{B}^0_{q,1}(\mathbb{R}^N) \cap B^0_{r,1}(\mathbb{R}^N))$.

Nous renvoyons le lecteur à [53, Theorem 1.1., Theorem 1.3., Theorem 1.6.] pour une preuve de ce résultat. Le Théorème I.3.6 assure donc l'existence de solutions (locales et globales) ainsi que l'unicité, pour une importante famille d'équations scalaires actives correspondant à la sédimentation dans des fluides newtoniens, en couvrant à la fois les régimes critique et sous-critique.

I.3.2 Résultats de la thèse

Il est alors pertinent de s'interroger quant à la possibilité de démontrer un tel résultat dans le cadre d'un fluide non-newtonien, par exemple un fluide suivant une loi d'Ostwald-De Waele. C'est précisément l'objet du problème posé en (I.3.1).

Plus précisément, nous considérons le problème suivant sur le tore N-dimensionnel.

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{dans } \mathbb{R}_+ \times \mathbb{T}^N \\ -\operatorname{div}\left(\nu(\rho)|D(u)|^{p-2}D(u)\right) + \nabla \pi = \rho g & \text{dans } \mathbb{R}_+ \times \mathbb{T}^N \\ \operatorname{div}(u) = 0 & \text{dans } \mathbb{R}_+ \times \mathbb{T}^N \\ \fint_{\mathbb{T}^N} u \, \mathrm{d}x = 0 & \text{dans } \mathbb{R}_+ \\ \rho_{|t=0} = \rho_0 & \text{dans } \mathbb{T}^N \end{cases} \tag{I.3.10}$$

où ν est une fonction positive de la densité ρ qui peut potentiellement dégénérer. Commentons brièvement la physique de ce système, en notant un tenseur τ dépendant de $D(u)$. On note $\overline{\rho}$ la densité du mélange, de sorte que celle-ci s'écrive:

$$\overline{\rho} = \tilde{\rho} + \rho,$$

où $\tilde{\rho}$ est la densité du fluide porteur, qui est supposée constante, c'est-à-dire que le fluide porteur est homogène, et ρ est la densité des particules en suspensions à laquelle on soustrait la densité du fluide porteur. On observe alors que la densité ρ des particules en suspensions, à laquelle on soustrait la densité du fluide porteur, peut être négative, selon qu'elle est plus ou moins importante relativement à la densité du fluide porteur. Plus précisément, lorsque celle-ci est négative, les particules remontent vers la surface (phénomène de crémage), sinon elles descendent vers le fond suivant le vecteur de gravité (phénomène de sédimentation). Nous regrouperont, comme il est usuel de le faire, ces deux terminologies en parlant de sédimentation. Les équations que nous avons initialement sont donc données par:

$$\begin{cases} \partial_t \overline{\rho} + u \cdot \nabla \overline{\rho} = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^N \\ -\operatorname{div}\left(\overline{\nu}(\overline{\rho})|D(u)|^{p-2}D(u)\right) + \nabla \overline{\pi} = \overline{\rho} g & \text{in } \mathbb{R}_+ \times \mathbb{T}^N \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^N \\ \fint_{\mathbb{T}^N} u \, \mathrm{d}x = 0 & \text{in } \mathbb{R}_+ \\ \overline{\rho}_{|t=0} = \overline{\rho_0} & \text{in } \mathbb{T}^N \end{cases} \tag{I.3.11}$$

En fait, il est possible de voir que la première ligne se réécrit directement comme fonction de ρ, puisque $\tilde{\rho}$ est constante. Dans la deuxième équation, on peut faire l'observation suivante: $\overline{\rho} g = \tilde{\rho} g + \rho g = \nabla\left(\tilde{\rho} g \cdot x\right) + \rho g$, et ainsi on peut réécrire $\pi = \overline{\pi} + \tilde{\rho} g \cdot x$, ce qui permet de définir un nouveau terme de pression. En posant $\rho_{|t=0} = \overline{\rho_0} - \tilde{\rho} := \rho_0$, et en redéfinissant $\overline{\nu}(\overline{\rho}) = \nu(\rho)$, il s'ensuit le système (I.3.10). Notons que, d'un point de vue mathématique, les systèmes (I.3.11) et (I.3.10) sont identiques. Cependant, la justification précédente permet de comprendre pourquoi nous avons a priori une densité qui peut être négative, en fait cela vient du fait qu'il y a des termes microscopiques présents dans (I.3.11) qui vont disparaître et conduire, via le procédé précédemment décrit, à (I.3.10). Nous référons à [115] pour davantage de détails.

Revenons à l'analyse mathématique du problème. Il n'est pas évident que le problème défini une équation scalaire active, à cause notamment du théorème nonlinéaire. En fait, en multipliant formellement l'équation de Stokes non-newtonienne par $D(u)$, nous n'obtenons pas de borne uniforme sur u en $W^{1,p}$ comme nous l'aurions si imposions à $\nu(\rho)$ de rester strictement positive. Nous pouvons montrer par des estimations a priori une majoration plus faible, mais insuffisante pour

raisonner directement par compacité. Nous travaillons alors a priori en régime sur-critique. Pour montrer que (I.3.10) est bien une équation scalaire active, il convient d'observer que l'opérateur:

$$A_\rho(u) = \frac{1}{p}\int_{\mathbb{T}^N} \nu(\rho)|D(u)|^p \, dx$$

est un opérateur à potentiel strictement convexe et coercif sur un espace de Banach réflexif, en l'occurence sur un sous-espace de $L^p(\nu(\rho)dx)$, qui est l'espace L^p associé à la mesure à densité $\nu(\rho)dx$. Il est alors nécessaire d'effectuer un découplage dans le système (I.3.10) en étudiant séparément les propriétés de l'équation de Stokes non-newtonienne et de l'équation de transport (ou équation de continuité). Il faut alors trouver un schéma adapté à ce découplage pour effectuer une méthode d'approximation de type Friedrich, l'objectif étant de montrer l'existence de solutions faibles, définies comme suit.

Definition I.3.7. *On considère $N \geq 2$ la dimension et un exposant $q \in [1, +\infty]$. Soit $\rho_0 \in L^q$ une donnée initiale. La fonction $\rho \in L^\infty(L^q)$ est une solution faible du problème de Stokes-transport non-newtonien* (I.3.10) *associée à la donnée initiale ρ_0 si les conditions suivantes sont satisfaites:*

(i) Il existe un champ de vitesses $u : \mathbb{R}_+ \times \mathbb{T}^N \longrightarrow \mathbb{R}^N$ tel que $\nu(\rho)|D(u)|^{p-1} \in L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{T}^N)$ et qui, pour presque tout $t \in \mathbb{R}_+$, est solution de l'équation de Stokes: en d'autres termes $\operatorname{div}(u) = 0$ dans $\mathcal{D}(\mathbb{R}_+ \times \mathbb{T}^N)$ et quelle que soit $\phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{T}^N, \mathbb{R}^N)$ à divergence nulle, nous avons

$$\iint \nu(\rho)|D(u)|^{p-2}D(u) : D(\phi)\,\mathrm{d}x\,\mathrm{d}t = \iint \rho g \cdot \phi \,\mathrm{d}x\,\mathrm{d}t \,;$$

(ii) Le champ de vitesses est $u \in L^\infty(L^{q'})$ et ρ est une solution faible de l'équation de transport pour la donnée initiale ρ_0, c'est-à-dire que pour tout $\phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{T}^N, \mathbb{R})$, nous avons

$$\iint \Big(\rho\partial_t\phi + \rho u \cdot \nabla\phi\Big)\,\mathrm{d}x\,\mathrm{d}t + \int \rho_0\phi(0)\,\mathrm{d}x = 0.$$

Montrer l'existence de solutions, même approchées, n'est pas immédiat: on ne peut pas appliquer tel quel un théorème de Cauchy-Lipschitz. La régularité a priori étant insuffisante, il est nécessaire de coupler un argument par une méthode de Minty dans l'équation de Stokes non-newtonienne, et de faire usage de la théorie de Di Perna-Lions pour estimer le terme de densité. Ce faisant, on peut montrer l'existence de solutions approchées globales via l'usage du théorème de Cauchy-Peano, et montrer la convergence de ces solutions. On obtient ainsi le résultat suivant (voir [54] et Theorem 3.0.2), qui est issu d'un travail réalisé en collaboration entre Dimitri A. Cobb et le présent auteur lors d'un séjour de recherche à l'Université de Bonn (Allemagne).

Théorème I.3.8 (Theorem 3.0.2)**.** *Soit $N \geq 2$ la dimension. On considère $p \in (1, +\infty)$ et une fonction $\nu \in C^\infty(\mathbb{R} \setminus \{0\}) \cap C^{0,\overline{\gamma}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ telle que $\nu(|r|) \gtrsim |r|^\gamma$ pour tout $r \in [-1, 1]$, $\overline{gamma} = \min(\gamma, 1)$, $\gamma \geq 0$.*

On considère également des exposants $q \in (1, 2)$ et $\sigma \in [1, +\infty]$ tels que l'une des conditions suivantes soit vérifiée:

(i) Cas sous-critique: ou bien nous avons l'inégalité stricte

$$\frac{1}{p}\left(1 + \frac{\gamma}{\sigma}\right) + \frac{1}{q} - \frac{1}{N} < 1 \,; \tag{I.3.12}$$

(ii) Cas critique: ou bien nous avons l'égalité

$$\frac{1}{p}\left(1 + \frac{\gamma}{\sigma}\right) + \frac{1}{q} - \frac{1}{N} = 1 \,, \tag{I.3.13}$$

et de plus la condition $q \geq \frac{2N}{N+2}$ est vraie. En particulier, cela est toujours vrai si $N = 2$.

Alors, pour toute donnée initiale $\rho_0 \in L^q(\mathbb{T}^N)$ *telle que* $1/\rho_0 \in L^\sigma(\mathbb{T}^N)$*, il existe une solution faible* $\rho \in L^\infty\big(\mathbb{R}_+, L^q(\mathbb{T}^N)\big)$*. De plus, si* $\rho_0 \in L^q(\mathbb{T}^N) \cap L^r(\mathbb{T}^N)$ *pour un* $r \in [1, +\infty]$*, alors*

$$\|\rho\|_{L^\infty(L^r)} = \|\rho_0\|_{L^r}.$$

Nous attirons l'attention du lecteur sur le fait que les terminologies "cas critique" et "cas sous-critique" dans l'énoncé du Théorème I.3.8 ne sont pas en lien avec le régime sous-critique et critique pour l'unicité des solutions de l'équation scalaire, mais sont en rapport avec la régularité a priori critique pour montrer l'existence de solutions faibles.

Une question est de savoir si les résultats des Théorèmes I.3.8 et I.2.5 peuvent conduire à un résultat de contrôlabilité exacte interne des solutions de (I.3.10). Nous ne discutons pas ici du bien fondé physique de cette assertion, puisque a priori dès lors que l'on applique des forces externes suffisamment importantes au fluide, celui-ci serait conduit à sortir de son état quasi-statique, et par la suite la cohérence du modèle serait discutable. On pourrait pallier ce problème en définissant des contrôles admissibles qui permettraient de garder une cohérence physique pour ce dernier. Remarquons que dans le cas de (I.3.5), il a été établi l'existence de contrôles internes pour l'équation de continuité et de Stokes (voir [142, Theorem 2.5.]).

À titre d'exemple, remarquons que le modèle (I.3.10) peut rendre compte, par exemple, d'un phénomène de sédimentation des globules rouges dans le plasma sanguin.

I.4 Détermination d'un coefficient de diffusion en hémodialyse

Nous conclurons ce manuscrit par l'étude d'un problème appliqué au domaine médical concernant l'hémodialyse, réalisé en collaboration avec le Centre Hospitalier Universitaire Gabriel Montpied de Clermont-Ferrand. Comme nous l'évoquions précédemment, le sang est un fluide non-newtonien rhéofluidifiant, et celui-ci se modélise assez bien par des modèles de Carreau ou de Cross. En d'autres termes, une des particularités du sang est qu'il se comporte, lorsqu'il est soumis à d'importantes déformations, comme un fluide newtonien.

Plus précisément, le sang est une suspension d'érythrocytes (globules rouges), de leucocytes (globules blancs, comprenant les granulocytes éosinophiles, basophiles, et neutrophiles, ainsi que les lymphocytes et les monocytes), et de de thrombocytes (plaquettes), dans un liquide aqueux appelé le plasma, qui est également constitué de divers éléments comme des protéines (albumine, immunoglobulines, fibrinogènes, transferrine, α_1-antitrypsine, macroglobulines, lipoprotéines, céruloplasmine, etc.) ainsi que d'espèces chimiques variées et autres substances organiques ou inorganiques (sels minéraux comme le calcium, le sodium, le potassium, ou encore le magnésium, par exemple). Ainsi, le sang est une suspension aqueuse de particules très hétérogènes, tant en taille qu'en représentations volumiques.

Principalement, les érythrocytes représentent en moyenne environ 40% du volume sanguin, et jouissent de propriétés d'élasticité importantes. La concentration volumique en érythrocytes dans le sang est appelée l'hématocrite (souvent abrégée en Ht), et se calcule par centrifugation du sang dans un tube, en estimant le volume obtenu formé par l'agrégat d'érythrocytes. Notons qu'en général l'agrégat obtenu pour la mesure de l'hématocrite est constitué en moyenne de 96% d'érythrocytes, le reste étant du plasma retenu dans l'agrégat. Ainsi, la mesure de l'hématocrite doit être comprise avec une certaine marge d'erreur due à la mesure. Notons également que l'hématocrite joue un rôle prépondérant dans le caractère rhéofluidifiant du sang, il est même établi qu'en-dessous d'un certain seuil d'hématocrite, le sang n'est plus rhéofluidifiant (voir [33]) ce qui peut être interprété dans le sens où en général, le plasma est une suspension tellement diluée qu'on peut le considérer comme un fluide newtonien. Cependant, certaines pathologies induisent, sous une augmentation significative de certaines protéines comme des fibrinogènes ou

des immunoglobulines, une modification du plasma telle que sa viscosité ne corresponde plus à un modèle newtonien (voir par exemple [74]).

Ainsi, la viscosité du sang est très complexe et dépend de plusieurs paramètres. Un fait intéressant est que lorsque le sang circule dans un capillaire, c'est-à-dire formellement un long tube de petit diamètre avec comme frontière une membrane partiellement perméable, alors plus le diamètre de celui-ci est petit, plus la viscosité diminue: il s'agit de l'effet Fåhræus-Lindqvist (voir [85, 101, 152]). Cependant, cet effet s'inverse en dessous d'un certain diamètre (inférieur à $6\,\mu$m, et significativement en dessous de $3\,\mu$m).

Dans [7], ce phénomène est mis en évidence en comparant les simulations numériques obtenues en considérant des modèles d'écoulements non-newtoniens et newtoniens, dès lors que le sang s'écoule dans les petites fibres d'un dialyseur: les dimensions associées à une telle fibre sont par exemple donnés par un rayon $R = 2.3 \times 10^{-4}$ m et une longueur $L = 2.3 \times 10^{-1}$ m, le rayon du cylindre interne dans lequel circule le sang étant alors $R_1 = 10^{-4}$ m et l'épaisseur de la membrane d'environs de $0,4 \times 10^{-4}$ m.

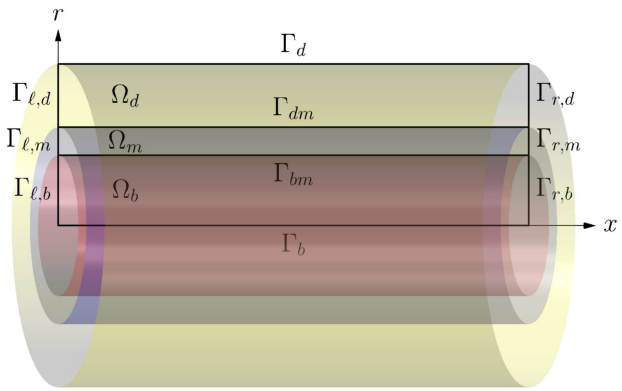


Figure I.4: Représentation tridimensionnelle d'une fibre de dialyseur. Ω_d, Ω_m, et Ω_b représentent respectivement les régions associées au dialysat, à la membrane poreuse, et au sang.

L'objectif est alors le suivant: à partir de prélèvements réalisés sur des patients, peut-on déterminer les valeurs des coefficients de diffusion intramembranaires, dits coefficients de *Beavers-Joseph*? Pour ce faire, on considère que la membrane est assujettie à une loi de Darcy, et l'on prend en considération les réactions chimiques ayant lieu dans le dialyseur, modélisées par le système d'évolution suivant, où les valeurs $(c_i)_{1\leq i\leq 5}$ correspondent aux concentrations des cinq espèces chimiques en présence et prises en considération.

$$\partial_t c_i + S_i\left(U_x\partial_x c_i + U_r\partial_r c_i\right) - \frac{1}{r}\partial_r\left(rD_i\partial_r c_i\right) - \partial_x\left(D_i\partial_x c_i\right) = F_i(\boldsymbol{c}) \text{ dans } (0,T)\times\Omega \tag{I.4.1a}$$

$$c_i = \overline{c_{i,\text{ input}}} \text{ sur } (0,T)\times(\Gamma_{\ell,b}\cup\Gamma_{r,d}) \tag{I.4.1b}$$

$$\partial_{\boldsymbol{n}} c_i = 0 \text{ sur } (0,T)\times\Gamma_n \tag{I.4.1c}$$

$$c_i(0) = c_{i,0} \text{ dans } \Omega. \tag{I.4.1d}$$

Dans (I.4.1), les coefficients $(D_i)_{1\leq i\leq 5}$ représentent les coefficients de diffusion intramembranaires pour les espèces $(c_i)_{1\leq i\leq 5}$, $(S_i)_{1\leq i\leq 3}$ est un coefficient utilisé pour éliminer les espèces ne traversant pas la membrane (Albumine et Calcium Albumine), soit

$$S_i(x) = \begin{cases} 1, \text{ si } x\in\Omega_b \\ 0, \text{ si } x\in\Omega\setminus\Omega_b \end{cases}$$

et $S_i = 1$ si $i\in\{1,\ 4,\ 5\}$.

L'approche que nous aborderons pour déterminer les coefficients de diffusion repose sur une méthode de descente de gradient couplée avec une méthode de Newton, et sera présentée en détails dans le Chapitre 4.

Remarquons que dans le modèle précédent, nous considérons pour simplifier une membrane poreuse et une diffusion intramembranaire suivant une loi de Darcy. Potentiellement, on peut dans certains cas observer une agrégation de thrombocytes sur le bord de la membrane du dialyseur: ce phénomène est appelé l'hémostase primaire. Cela pourrait venir d'un détachement du collagène présent dans les cellules endothéliales (c'est-à-dire des cellules composant l'endothélium, la couche interne des vaisseaux sanguins en contact direct avec le sang) qui se fixent sur la membrane ou, le cas échéant, circulent dans le sang. Les thrombocytes se fixent alors aux molécules de collagène, ce qui induit une variabilité dans la porosité de la membrane du dialyseur. Une éventualité pour pallier de tels problèmes serait de considérer une loi de Darcy généralisée davantage adaptée aux échanges de fluides par la membrane poreuse, comme ce qui a été établi dans [157]. Pour une discussion approfondie concernant la biomécanique du sang, nous renvoyons le lecteur à [97, 148].

Notons pour conclure cette introduction que la connaissance des coefficients est intéressante car elle permet d'ouvrir une question de contrôle optimal: le fait d'effectuer une dialyse a pour but de rétablir un équilibre souhaitable entre différentes espèces chimiques intrasanguines chez le patient. Connaissant les coefficients de diffusion et l'état cible, quelle est alors la bonne concentration des espèces chimiques à mettre dans le dialysat pour atteindre la cible souhaitée? Cela dépend des données propres à chaque patient, l'objectif étant d'atteindre de manière optimale les concentrations souhaitées à la sortie du dialyseur.

[La science] monte très haut dans l'infiniment grand, elle descend très bas dans l'infiniment petit. Entre ces deux extrêmes, qui se répondent l'un à l'autre, elle explore ce que nous appelons le monde réel – c'est-à-dire le nôtre.

Jean d'Ormesson, *La Science* (Guide des égarés).

Summary of the dissertation

This dissertation is devoted to the mathematical analysis of certain non-Newtonian fluids and their specific properties. First of all, a fluid is "a perfectly deformable material medium", i.e., to put it simply, a deformable body. Many materials are fluids, and we are confronted with them on a daily basis. We can easily think of gases such as air, liquids such as water, or even plasmas. What is less well known, however, is that many fluids adopt behaviors that depend on the stresses they are subjected to. Take mayonnaise, for example: when subjected to stress, such as when whipped, it behaves like a liquid in the usual sense. However, when left at rest without being subjected to unnatural external forces, it remains fixed, like a solid. In fact, to be more precise, you need to exceed a certain level of applied forces, a certain threshold of stress, for mayonnaise to start behaving like a liquid. Fluids with this type of behavior are called threshold fluids, with the example of Bingham fluids. So, if we want to refine our definition of a fluid, we can add that it is a body that can undergo non-rigid deformations. In other words, it is not a solid[h]. To emphasize the example of mayonnaise, it flows all the more easily the greater the constraints imposed on it. This is called shear-thinning behavior. There are also fluids, such as a slightly diluted solution of cornstarch, which behave in the opposite way: the greater the constraints imposed, the more the fluid tends to solidify. This is known as shear-thickening behavior. The study of fluid flow properties is called *rheology*[i], and can be approached in a number of ways. A first approach is to understand the microscopic interactions that give rise to particular flows. Although this is not the approach we will be taking in this dissertation, it does provide a fairly precise framework for distinguishing different types of flow. Here are a few examples.

Suspensions: This is a general framework for fluids. Intuitively, when very small particles, say solids, are in a fluid, we speak of them as being suspended in the fluid, and consequently we say that the mixture is a suspension. Many examples of suspensions can be found in everyday life, such as concrete, cement and paint, but also in soups (which are suspensions of vegetables in water), or biological fluids such as blood (which is, roughly speaking, a suspension of blood cells and platelets in blood plasma). When the force of gravity acting on a suspension is greater than the other forces to which it is subjected, we speak of sedimentation. We will be looking at this phenomenon in Chapter 3. In the long term, phase separation is observed, which translates into a variation in fluid density depending on the zone under consideration.

Colloids: When the particles in suspension are very small (say, less than 1 μm in diameter), certain previously negligible interactions become more pronounced. These are called colloidal interactions. These are essentially Van der Waals forces, which tend to cause the various suspended particles to aggregate. As a result, colloids often give rise to inhomogeneous mixtures, as particles tend to clump together. From a practical point of view, colloidal interactions are usually stabilized to avoid this phenomenon and to make the colloid usable. Many fluids are colloidal in nature, again including paints and cements, but also toothpastes, cosmetic creams, clays and so on. A number of colloids also exhibit Bingham fluid behavior, i.e. it is necessary to apply sufficient constraints for the fluid to move and flow.

[h] If we want to be precise, a fluid is an egalitarian material that is not a solid, see [Section IV.17.]Truesdell-1991.

[i] from the Greek roots *rheos* meaning "current" or "flow" and *logos* meaning "research" or "theory"

Polymers: Readers may wonder why polymers are included in this series of examples, given their widespread use in solid form in everyday life. However, in the manufacturing process for these polymeric materials, polymers are suspended in a liquid. A polymer, or polymer chain, is therefore a molecule made up of a large number of identical particles (several thousand or even several million) associated by their carbon atom. One of the most interesting properties of a polymer chain is that it can take on an enormous number of possible configurations, as it can expand or contract. This means that the fluid obtained by suspending polymers (and consequently the resulting material, in general) has excellent elasticity properties. The resulting mixture behaves, to put it simply, like an elastic solid when subjected to stress, and then very quickly flows like a liquid. It is therefore common for polymer solutions to be shear-thinning .

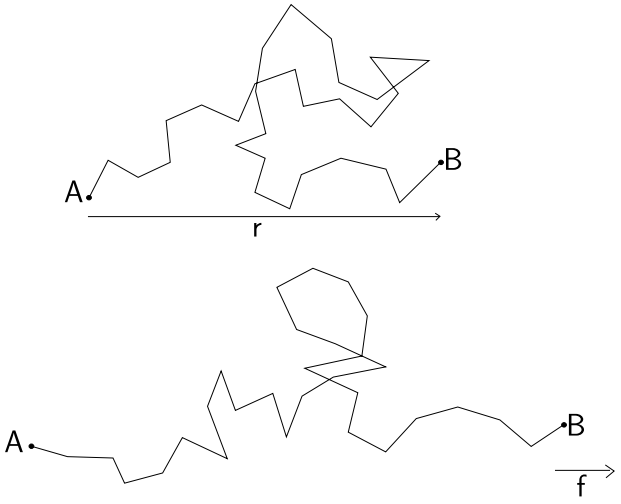


Figure II.5: Representation of a polymeric chain. The length of the chain r is the Euclidean distance between its ends. It can expand when subjected to a constraint f.

Emulsions: In the previous examples, we focused mainly on suspensions of solid particles in a liquid. In fact, sometimes a fluid is actually a "mixture" of several distinct fluids, all with different properties. In such a configuration, we generally speak of a *complex fluid*. In some cases, it is desirable to place small particles of one fluid in suspension (i.e. highly dispersed) in another fluid, for example to aid penetration of a porous material. When such a suspension takes place, it is referred to as en *emulsion setting*. This commonly-used term is rightly found in everyday life, particularly in the culinary sphere. This brings us back to our initial example of mayonnaise, which is the emulsification of egg yolks and mustard (and perhaps a few additional seasonings) in an oilseed solution (sometimes also aqueous, as the maker chooses). As mentioned above, emulsions can exhibit shear-thinning behavior, and can also be threshold fluids. Once again, examples of emulsions can be found in a wide range of fields, from food processing (salad dressings, butter, etc.) to civil engineering (petroleum fluids) and cosmetics (creams, for example).

There are many other examples of fluids, and we refer the reader to [59] (from which most of the above examples are taken) as well as [19, 105] for precise and detailed explanations.

However, as we mentioned earlier, a fluid is a body that is not solid, in the sense that it can undergo non-rigid deformations. The question then arises: what do we mean when we speak of a body? A first approach would be to use the definition of classical mechanics, saying that a body is a set of weighted points obeying the laws of classical mechanics. While this approach is not wrong, it does present a number of difficulties. On the one hand, if a point represents a fluid particle, it is difficult to understand how interactions at small, say mesoscopic, scales are

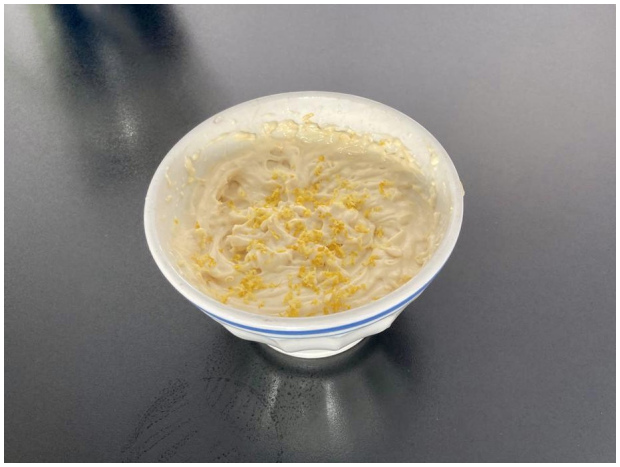


Figure II.6: Mayonnaise is an emulsion, which behaves like a Bingham fluid: without being subjected to sufficiently high stresses, it remains set.

to be considered. So, should they be omitted, or otherwise taken into consideration, but how? In such cases, it is preferable to define a simplified framework that accounts for the macroscopic behavior of the flow, without having to enter into ancillary considerations such as Van der Waals interactions, for example. This leads to a different definition of a body, as an element of a set, generally an open subset of $\mathbb{R}^N$.

II.1 Analysis of non-Newtonian flows

We begin by presenting a generic flow model for an incompressible viscous fluid, then study the existence of solutions for it.

II.1.1 Problem position and previous results

In Appendix C, we will present a rigorous framework leading to the latter approach, which we will refer to as the macroscopic approach. This framework is that of continuum-mechanics, which is a simplified alternative for the study of certain fluids or solids, particularly well-suited to deformable materials. We will therefore present physics-mathematics results established within the framework of research associated with Hilbert's sixth problem (mathematical formalization of physics), the results of which can be found in [167]. The aims of this chapter are manifold:

- Firstly, it will be highlighted which objects are considered within the framework of continuum mechanics and, consequently, within the framework of fluid mechanics as considered in the remainder of the dissertation.
- This will highlight how flow-induced deformations are considered in our framework, and thus how we approach the observations mentioned above.
- Finally, by demonstrating that alternatives to the laws of classical mechanics are available within the framework of continuum mechanics, we will be able to establish the generic model of the equations describing the flow of an incompressible viscous fluid.

The latter equations reflect a local averaging of fluid properties, and are as follows.

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ \rho\left(\partial_t u + (u \cdot \nabla)u\right) - \operatorname{div}\left(\tau(\rho, D(u))\right) + \nabla \pi(\rho) = \rho f & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \\ \rho_{|t=0} = \rho_0 & \text{in } \Omega \\ u_{|t=0} = u_0 & \text{in } \Omega \end{cases} \tag{II.1.1}$$

where the equations are set in $\mathbb{R}_+ \times \Omega$, i.e. the time variable evolves in $\mathbb{R}_+$ and the space variable in the open $\Omega \subset \mathbb{R}^N$. The first equation is called the continuity equation, and the unknown ρ is the fluid density. The second is the flow equation, where u is the fluid velocity field, and π is the pressure. The condition $\operatorname{div}(u) = 0$ reflects the incompressibility of the fluid, i.e. for any small volume of fluid considered, it remains constant as a function of time. Finally, the tensor τ is the tensor of internal stresses associated with the flow. When the latter is expressed linearly with respect to $D(u)$, where $D(u) = \frac{\nabla u + \nabla u^T}{2}$ is the tensor of the fluid's deformation rates, the fluid is said to be *Newtonian*, according to the assumptions made by Isaac Newton[j]. Otherwise, the flow is said to be non-Newtonian. In the case of an incompressible fluid, Newtonian flows are described, for example, by the Euler equations (for a perfect fluid) or the Navier-Stokes equations.

Nevertheless, Newtonian flows do not cover the cases described above, such as shear-thinning or shear-thickening behaviors. In other words, such behaviors are non-Newtonian and consequently result in the stress tensor being expressed non-linearly relative to $D(u)$. Thus, (II.1.1) gives a precise framework for the macroscopic study of a non-Newtonian flow, but, although this is not the approach we will present in this dissertation, let us mention that there are alternative approaches to this duality between the macroscopic and the mesoscopic: we then speak of a micro-macro model. Such models have been successfully described and analyzed, for example for dilute polymeric suspensions. Reference can be made to [127, Chapter 4] and [118] for the modeling and analysis of dumbbell models, for example.

Initially, we will focus on equations of the form (II.1.1) for constant density, i.e. $\rho = \rho_0 > 0$. In this case, the generic system is rewritten, with a good choice of units:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \operatorname{div} \tau(D(u)) + \nabla \pi = f & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \\ u_{|t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{II.1.2}$$

This leaves us with an open question: although we know that the stress tensor τ describes a nonlinear law, we do not know how it can be expressed. In other words, we do not know which models we can consider. In fact, as you would expect, these models are essentially based on physical experiments. To do this, we use a device called a *rheometer* to estimate the various properties of the fluid as a function of the stresses applied to it. The measurements taken are then approximated by curves describing functions, relative to the deformation of the fluid, and the τ description is obtained.

[j] See for example [31, Section 1.4.2.] for more details.

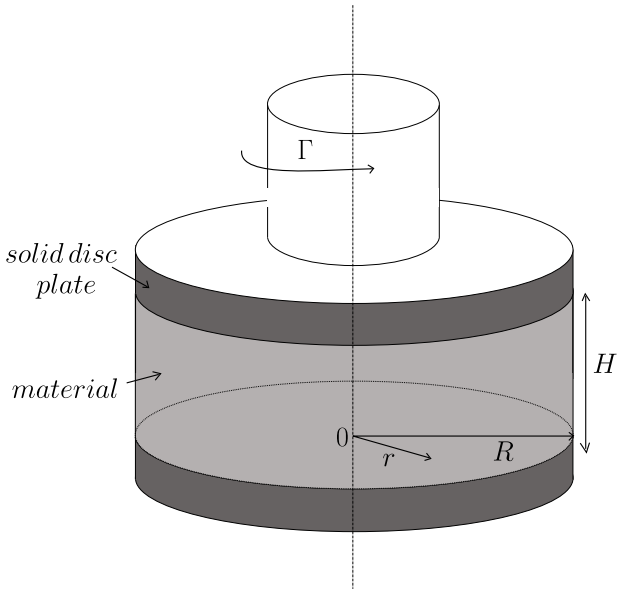


Figure II.7: A parallel-disk rheometer: a fixed plate is placed between the material and an upper plate, whose mobility is ensured by rotation relative to an axis, causing shear, which is measured as a function of the stresses applied.

Let us take a look at some examples of laws for τ, where $\kappa \in \{0, 1\}$ according to considerations on fluid viscosity, which comes down to whether or not we add diffusion to the equation.

- **Carreau**:
$$\tau(M) = \kappa M + (1 + |M|^2)^{\frac{p-2}{2}} M,\ p > 1,$$
- **Carreau-Yasuda**:
$$\tau(M) = \kappa M + (1 + |M|^\alpha)^{\frac{p-2}{\alpha}} M,\ p > 1,\ \alpha > 0,$$
- **Ostwald-De Waele (power law)**:
$$\tau(M) = \kappa M + |M|^{p-2} M,\ p > 1,$$
- **Cross**:
$$\tau(M) = \kappa M + \frac{M}{1 + |M|^{2-p}},\ p > 1,$$
- **Bingham**:
$$\tau(M) = \kappa M + \tilde{\tau}(M), \quad \tilde{\tau}(M) = \begin{cases} \frac{M}{|M|} & \text{si } |M| > 0 \\ [0, 1] & \text{otherwwise,} \end{cases}$$
- **Herschel-Bulkley**:
$$\tau(M) = \kappa M + \tilde{\tau}(M), \quad \tilde{\tau}(M) = \begin{cases} \frac{M}{|M|} + |M|^{p-2} M,\ p > 1 & \text{si } |M| > 0 \\ [0, 1] & \text{otherwise.} \end{cases}$$

By observing the curves associated with the above-mentioned τ tensors, we realize that when the parameter p verifies $1 < p < 2$, the fluid is shear-thinning, and shear-thickening if $p > 2$. In the

case of Carreau, Carreau-Yasuda, Ostwald-De Waele, or Cross fluids, when $p = 2$, τ is linear in M and the fluid is Newtonian. Another interesting observation concerns the Carreau-Yasuda and Cross laws in the shear-thinning case: when the fluid is little deformed or enormously deformed, i.e. when $|M|$ is very small or very large, then we expect it to adopt a behavior relatively similar to a Newtonian fluid. This will indeed be the case, as can be observed for blood, for example, which will be used in Chapter 4. This remark remains valid for a Carreau fluid, which is a Carreau-Yasuda fluid for $\alpha = 2$. The case of Bingham's law is, formally, an Ostwald-De Waele law for the limiting case $p = 1$. Note, however, that the non-diffusive part of the stress tensor $\tilde{\tau}$ is discontinuous at $M = 0$, since $\tilde{\tau}(M) = \frac{M}{|M|}$. In the mathematical study of such an equation, the appropriate notion is that of a multivalued function. In any case, when $|\tilde{\tau}(M)|$ takes a value other than 1, this forces M to take the value $M = 0$. In other words, it behaves like a solid when subjected to insufficient stress, and flows when a certain stress threshold is exceeded: this is known as a threshold fluid. Before continuing, let us recall the following definitions, where $\alpha \in \mathbb{N}^k$ is a multi-index.

Definition II.1.1. *(Nonlinearity types) A partial differential equation is said to be:*

(i) Semilinear if, for functions $(a_\alpha)_{|\alpha|=k}$, it can be written as:

$$\sum_{|\alpha|=k} a_\alpha(x)\nabla^\alpha u + a_0\left(\nabla^{k-1}u, \dots, \nabla u, u, x\right) = 0$$

(ii) Quasilinear if, for functions $(a_\alpha)_{|\alpha|=k}$, it can be written as:

$$\sum_{|\alpha|=k} a_\alpha(\nabla^{k-1}u, \dots, \nabla u, u, x)\nabla^\alpha u + a_0\left(\nabla^{k-1}u, \dots, \nabla u, u, x\right) = 0$$

Note that the equations described for the tensors we are considering are then quasilinear equations[k] according to their spatial variable, which means that the latter is expressed as a linear function of the terms whose derivatives are of the highest order.

Thus, the mathematical analysis of non-Newtonian fluids lies at the interface of nonlinear analysis and fluid mechanics. For this reason, in Chapter 1 we describe the tools needed to analyze quasilinear equations. An essential notion in this context is that of a monotone operator, i.e. an operator $A : X \to X^*$, where X is a reflexive Banach space and X^* is its topological dual, which verifies:

$$\forall (u, v) \in X^2,\ \langle A(u) - A(v), u - v\rangle_{X^* \times X} \geq 0.$$

For example, the Laplacian on a bounded domain Ω defined by $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is a monotone operator because it verifies

$$\int_\Omega (-\Delta u + \Delta v)(u - v)\, dx = \int_\Omega |\nabla(u - v)|^2\, dx \geq 0.$$

The terminology of monotone operator can be thought of by remembering that a function of one real variable $f : I \to \mathbb{R}$ is increasing on $I \subset \mathbb{R}$ if and only if:

[k] Note that the nomenclature of semilinear and quasilinear equations is not derived from the notions of applications with the same names. For example, an application describing a paraboloid is a quasilinear application, but is not described by a linear function in its last variable.

$$\forall (t,s) \in I^2,\ (f(t)-f(s))\,(t-s) \geq 0,$$

the inequality being reversed in the case of a decreasing function. Having said that, we should also (and above all) remember that f is convex on I if and only if it satisfies the convexity inequality:

$$\forall (t,s) \in I^2,\ \left(f'(t)-f'(s)\right)(t-s) \geq 0.$$

We then find a very close relationship between the notion of monotone operator and the notion of convexity: these are the convex *potential* operators, and their characterization is linked to Kachurovskii's theorem, which extends the usual characterizations of convex functions to functionals. In the case of the Laplacian, this means that

$$(-\Delta u, v)_{L^2} = \left(J'(u), v\right)_{L^2},$$

where

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx,$$

is a convex functional. In general, most of the laws introduced above derive from a convex potential, with the exception of threshold fluids (Bingham or Herschel-Bulkley). Note that in this case, the shear-thickening or shear-thinning behaviour of fluids whose flow is associated with the laws cited for τ (Carreau, Cross, Ostwald-De Waele) is determined by a parameter $p > 1$. In fact, the a priori regularity of the solution depends on this parameter (at least if $\kappa = 0$). For simplicity's sake, we will assume in this introduction that:

$$\tau(M) \approx \kappa M + (\mu + |M|^2)^{\frac{p-2}{2}} M, \quad \kappa \in \{0,1\}, \mu \geq 0,$$

i.e. τ behaves like the above-mentioned tensor in the estimates, and we will keep these considerations in mind as we move forward.

By making good use of the notion of monotone operator, it is possible to show the existence of solutions for equations (II.1.2). Take, for example, the following result (see [141]):

Theorem II.1.2 (Málek-Nečas-Ružička)**.** *Consider a domain $\Omega \subset \mathbb{R}^3$, with a C^3 boundary, $p \in [2,3)$, $u_0 \in W^{1,p}_{0,\sigma}(\Omega)$, $f \in L^2((0,T), L^2(\Omega))$, a time $T \in \mathbb{R}^*_+$ arbitrarily fixed, such that the stress tensor τ verifies good monotonicity assumptions. Then there is a solution to* (II.1.2) *in the weak sense. Moreover, if $p \geq \frac{9}{4}$, the solution u is unique and verifies:*

$$u \in C((0,T), L^2_\sigma(\Omega)) \cap L^\infty((0,T), W^{1,p}_{0,\sigma}(\Omega)) \cap L^{\frac{2}{p-1}}((0,T), W^{2,\frac{6}{p+1}}(\Omega)) \text{ et } \partial_t u \in L^2((0,T), L^2(\Omega)).$$

Remark II.1.3. *We have omitted here to specify the precise assumptions of the theorem, in particular concerning the τ stress tensor, which we will also do later concerning the results linked to Chapters 1 and 2. This choice has been made so as not to make the presentation of the results too cumbersome, since depending on the result, the assumptions concerning the system under consideration or the stress tensor are multiple and vary considerably from one author to another. However, the assumptions and results concerned will be cited with precision in Chapters 1 and 2. In this case, we refer the reader to the following hypotheses (S1)–(S4) into the Chapter 1.*

Theorem II.1.2 (which corresponds to Theorem 1.2.4 in the dissertation) thus tells us that there are solutions in the case of a shear-thickening flow, and that moreover it is possible to show

that the solution is unique as soon as the shear-thickening character of the fluid is sufficiently consistent.

The case of shear-thinning fluids is more delicate, as the nonlinear term can sometimes be singular, especially with an Ostwald-De Waele or Bingham law. However, it is always possible to show the existence of solutions for (II.1.2), using fine analysis techniques. For example, we have the following result (see [72] for the result, the assumptions being given by (SW1),(SW2'),(SW3'), (SW4), and can be found in Chapter 1).

Theorem II.1.4 (Diening-Ružička-Wolf). *Let Ω be an open bounded subset (regular[l]) of $\mathbb{R}^N$, where $N \geq 2$, and and arbitrarily fixed $0 < T < +\infty$. In addition, under certain assumptions of monotonicity on τ (which we omit here[m]), and considering*

$$\frac{2N}{N+2} < p < +\infty,$$

if $u_0 \in L^2_\sigma(\Omega)$ and $f \in L^q((0,T)\times\Omega)$, there is a weak solution $u \in L^p\left((0,T), W^{1,p}_{0,\sigma}(\Omega)\right) \cap C_w([0,T], L^2_\sigma(\Omega))$ of (II.1.2).

In the two-dimensional case, Theorem II.1.4 covers all the laws we mentioned at the time, except the one for Bingham flow. This is simply because the nonlinear operator associated with such a flow is not monotone. Two approaches are then possible: first, we can weaken the notion of solutions associated with the problem, otherwise we can generalize the notion of monotone operator. The first approach leads to the notion of a variational inequality. Formally, such an inequality is obtained by multiplying the solution by $\varphi - u$, where φ is a test function (i.e. it belongs to a space of functions whose regularity is adapted to the framework of the equation). In this case, we obtain, still formally, an expression for the nonlinear term as the derivative of a potential, or as the limit of such derivatives. The idea is then to use the convexity properties of a convex potential, i.e. a monotone operator according to Kachurovskii's theorem, to establish an inequality involving the velocity field (i.e. the solution). In this inequality, we no longer show the derivative of a potential, but a difference involving it.

Let us illustrate with a simple calculation. To approximate the 1-laplacian, which is formally the operator $u \mapsto |\nabla u|^{-1}\nabla u$, we can use the operator:

$$A_\varepsilon(u) = \frac{1}{1+\varepsilon}\int_\Omega |\nabla u|^{1+\varepsilon}\, dx,$$

which is a convex potential, and therefore enjoys convexity properties according to Kachurovskii's theorem. Formally, the term operator A'_ε converges to the 1-Laplacian, denoted Δ_1, when $\varepsilon \to 0$. In particular, this means that the latter is expected to benefit from the same convexity inequalities as the operator A'_ε. More precisely, still formally:

$$\underbrace{\langle A'_\varepsilon(u) - A'_\varepsilon(v), u - v\rangle}_{\geq 0} \underset{\varepsilon\to 0}{\longrightarrow} \underbrace{\langle \Delta_1(u) - \Delta_1(v), u - v\rangle}_{\geq 0}$$

This heuristic reasoning shows that for such problems, for which convexity is degenerate, we seek to bring out an inequality, Kachurovskii's theorem allowing us to write:

$$\langle A'_\varepsilon(u) - A'_\varepsilon(v), u - v\rangle \geq 0 \Leftrightarrow A_\varepsilon(u) - A_\varepsilon(v) \geq \langle A'_\varepsilon(u), u - v\rangle \underset{\varepsilon\to 0}{\longrightarrow} A_0(u) - A_0(v) \geq \langle \Delta_1(u), u - v\rangle \quad \text{(II.1.3)}$$

[l] We will come back to the assumption of regularity in Chapter 1

[m] See Remark II.1.3

Since the calculations are formal, the principle is to give meaning to the last limit, Δ_1 in our example. Inequalities of this type are called *variational inequalities*, and allow us to define a notion of weak solution to singular problems.

By applying the same principle of boundary crossing to the Bingham system for $\kappa = 1$, which is parabolic, the following result can be obtained (see [50] and [77] for a proof, the detailed assumptions being given by (D1)–(D4) in the Chapter 1)).

Theorem II.1.5 (Cioranescu-Duvaut-J.L.Lions). *Let $k \in \mathbb{N}$, $k \geq 1$, and Ω be a domain of $\mathbb{R}^N$ with smooth boundary, $N \in \{2, 3\}$. We consider $(p_i)_{1\leq i\leq k}$, $(q_i)_{1\leq i\leq k}$, such as*

$$p_i \geq \frac{3N}{N+2} \quad \text{and} \quad q_i \geq \frac{2p_i}{p_i - 1}$$

and $f \in \bigcap_{i=1}^{k} L^{\frac{q_i}{q_i-1}}((0,T), W_\sigma^{-1,\frac{p_i}{p_i-1}}(\Omega))$. Then, under certain assumptions, we have:

$$u \in \left(\bigcap_{i=1}^{k} L^{q_i}((0,T), H^s_{0,\sigma}(\Omega)) \right) \cap L^\infty((0,T), L^2_\sigma(\Omega))$$

such that $\partial_t u \in \bigcap_{i=1}^{k} L^{\frac{q_i}{q_i-1}}((0,T), H_\sigma^{-s}(\Omega))$ satisfies for all φ in a set of admissible functions:

$$\int_0^T \left(\langle \partial_t \varphi, \varphi - u \rangle + \langle -\Delta u, \varphi - u \rangle + \int_\Omega (u \otimes u) : D(\varphi - u)\, dx + A(\varphi) - A(u) \right) dt \geq \int_0^T \langle f, \varphi - u \rangle\, dt, \quad \text{(II.1.4)}$$

the result being true for the Bingham model.

Let us illustrate our point by returning to our previous example: A accounts for the nonlinear term, which in many respects behaves like the 1-laplacian, but there are a few differences. Indeed, taking the limit $\varepsilon \to 0$ in (II.1.3) with an approximation operator as presented above, we do not necessarily obtain exactly the 1-laplacian but a variant of it. A similar approach can be found in Theorem II.1.5.

To explain the meaning of the variational inequality (II.1.4), let us start by noting that the variational inequality

$$\int_0^T \left(\langle \partial_t u, \varphi - u \rangle + \langle -\Delta u, \varphi - u \rangle + \int_\Omega (u \otimes u) : D(\varphi - u)\, dx + A(\varphi) - A(u) \right) dt \geq \int_0^T \langle f, \varphi - u \rangle\, dt \quad \text{(II.1.5)}$$

allows us to recover a classical solution, at least formally, if u is regular, by testing against $\varphi = (1 + \lambda)u$ then $\varphi = (1 - \lambda)u$, for $\lambda > 0$, then dividing by λ and taking the limit $\lambda \to 0$. The authors of Theorem II.1.5 have shown the existence of an inequality of the form (II.1.5) in the two-dimensional frame, including for Bingham (see [77]), but not in the three-dimensional case. The inequality (II.1.4) presented in Theorem II.1.5 is not identical, since a time derivative term appears on φ and not on u. This defines a very weak notion of solution for the problem.

It should also be noted that one of the advantages of variational inequalities is that they lend themselves relatively well to numerical simulations (see for example [158, 100]).

An alternative approach to variational inequalities is therefore to give a more general meaning to monotone operators, allowing in particular the consideration of multivaluations[n]. This generalization is given by the notion of *maximal monotone operator*, which we will see in detail in Chapter 1. Within this framework, it is then possible to extend the aforementioned monotonicity solutions (see [35], the assumptions being given in Chapter 1 by (B1)–(B4)).

Theorem II.1.6 (Bulíček-Gwiazda-Málek-Świerczewska)**.** *Consider a domain Ω of $\mathbb{R}^N$, $N \in \{1,2\}$, sufficiently regular, and we assume that the tensor τ verifies some maximal monotonicity assumptions (see Theorem 1.2.12) for $p > \frac{2N}{N+2}$. Let be an N-function φ checking the Δ_2-condition, the ∇_2-condition (see Definition A.5.5), and such that $C_1, C_2, C_3, C_4 > 0$ such as*

$$C_1 t^p - C_2 \leq \varphi(t) \leq C_3 t^r + C_4 \qquad r \in [p, +\infty)\,.$$

Then there is a weak solution of (I.1.2) *for the boundary condition $u \cdot n = 0$ instead of the homogeneous Dirichlet condition.*

The Theorem II.1.6 (which corresponds to the Theorem 1.2.12 in the dissertation) is an alternative to Theorem II.1.4 for maximal monotone operators (but with the boundary condition $u \cdot n = 0$). Note that the solutions obtained belong to spaces whose regularity depends on a N-function, called *Orlicz-spaces*. We refer you to Section A.5 in the appendix for more details on these spaces. Note, however, that although maximal monotone operators are considered in Theorem II.1.6, the case of threshold fluids (i.e. Bingham or Herschel-Bulkley) does not seem to be taken into account, we refer to the remarks following Theorem 1.2.12 in Chapter 1 for further details.

II.1.2 Thesis results

This raises several questions:

- Is it possible to construct solutions in the three-dimensional case that take threshold fluids into account?
- If such solutions exist, are they related to numerical solutions?
- What impact does the nonlinear term have on the regularity of solutions?
- Do these solutions account for certain experimental properties specific to shear-thinning flows?

As mentioned above, solutions in the form of variational inequalities are often used for numerical simulations of threshold flows, either by taking the limit on parameter $p > 1$ in Ostwald-De Waele, or by adding a small constant which is made to tend towards 0 in the simulations. The theoretical existence of such solutions then needs to be investigated, ideally under assumptions that do not explicitly involve complex-to-define objects of functional analysis. The crucial point lies in making sense of the limit in the approximation of the nonlinear term specific to the non-Newtonian nature of the fluid, which can be done in a number of ways, for example by making a *Clarke gradient* appear (see Chapter 1), giving the notion of *hemivariational inequality*, or using *Kuratowski* convergence (see Section A.4 in the appendix).

[n] Remind that in the case of a Bingham flow, we had said that the appropriate framework for treating the tensor $\tilde{\tau}(M) = \frac{M}{|M|}$ is that of multivalued functions, which is where they make their appearance

The singular case $p = 1$ (corresponding to Bingham and Herschel-Bulkley fluids) is particularly difficult to deal with, especially in the case of a fluid where we do not consider any diffusive term, corresponding to $\kappa = 0$. In this case, we have to consider the operator in the space of functions with bounded deformations $BD(\Omega)$ (we refer to Chapter 1 and to Section A.3 in the appendix for further details, and to [6] for a detailed discussion). An intermediate approach, which nevertheless allows us to grasp some of the difficulties of the problem, is to consider a diffusion term (with a Laplacian), which amounts to considering $\kappa = 1$, giving us a tensor of the form

$$-\operatorname{div}(\tau) := -\Delta - \Delta_1$$

and the conservation of energy ensures that the operator is well defined on $H_0^1(\Omega) \cap BD(\Omega)$, and thus the operator $-\Delta - \Delta_1$ is defined on a subdomain of the Laplacian. More generally, and still in this spirit, we consider viscous stress tensors of the form:

$$\tau(M) = M + F(|M|)M$$

This leads to the following system:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u - \operatorname{div}\left(F(|D(u)|)D(u)\right) + \nabla\pi = f & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \\ u_{|t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{II.1.6}$$

Assumptions about the tensor are made by assumptions about F, which is a function of a real variable.

(C1) $F : (0, +\infty) \to (0, +\infty)$;

(C2) $F \in W^{1,\infty}_{loc}((0, +\infty))$;

(C3) $t \mapsto tF(t)$ is non-decreasing on $(0, +\infty)$;

(C4) there is $p \in [1, 2]$, $t_0 > 0$ and $K > 0$ such that for any $t \geq t_0$, $F(t) \leq Kt^{p-2}$.

The following result is obtained from [44].

Theorem II.1.7. *We consider that F verifies hypotheses (C1)-(C4) and that $\Omega \subset \mathbb{R}^N$, $N \in \{2, 3\}$, is a bounded domain with Lipschitz boundary, $T > 0$ and an initial data $u_0 \in H^1_{0,\sigma}(\Omega)$ as well as a force term $f \in L^2((0, T), H^{-1}_\sigma(\Omega))$. So there is a weak solution u of* (II.1.6) *satisfying the following regularity*

$$u \in C_w\left((0, T), L^2_\sigma(\Omega)\right) \cap L^2\left((0, T), H^1_{0,\sigma}(\Omega)\right) \quad \text{and} \quad \partial_t u \in L^{\frac{4}{N}}((0, T), H^{-1}_\sigma(\Omega)).$$

In Theorem II.1.7, the notion of a weak solution is given by the following definition.

Definition II.1.8. *We say that the function $u \in L^2\left((0, T), H^1_{0,\sigma}(\Omega)\right) \cap C_w((0, T), L^2_\sigma(\Omega))$ such that $\partial_t u \in L^{\frac{4}{N}}\left((0, T), H^{-1}_\sigma(\Omega)\right)$ is a weak solution of* (II.1.6) *if it checks $u_{|t=0} = u_0 \in H^1_{0,\sigma}(\Omega)$, and for all $\varphi \in C^\infty((0, T) \times \Omega)$:*

$$\int_0^T \langle \partial_t u(t), \varphi(t) \rangle \ dt + \frac{1}{2}\left(\|u_0\|^2_{L^2(\Omega)} - \|u(T)\|^2_{L^2(\Omega)} \right) + \int_0^T \int_\Omega D(u(t)) : D(\varphi(t) - u(t)) \ dx$$
$$- \int_0^T \int_\Omega (u(t) \cdot \nabla u(t)) \cdot \varphi(t) \ dx \ dt + \int_0^T \int_\Omega G\left(|D(\varphi(t))|\right) - G\left(|D(u(t))|\right) \ dx \ dt$$
$$\geq \int_0^T \langle f(t), \varphi(t) - u(t) \rangle \ dt. \tag{II.1.7}$$

In the variational inequality (II.1.7), the function G is defined so as to give meaning to the limit of the approximation procedure, just as in our heuristic reasoning involving (II.1.3), on the nonlinear term, and plays a role analogous to that of the function A in the Theorem II.1.5. Intuitively speaking, the function G plays the role of a primitive of the function F, without however being one, for the reasons given above. Note then that Theorem II.1.7 takes threshold flows into account, and that this is obtained as a limit of the models used for numerical simulations. Furthermore, we note that the regularity of solutions is optimal in the sense that we recover the regularity of solutions for incompressible Navier-Stokes equations, which was suggested by the well-posedness of the nonlinear operator. Note that F is not a priori well-defined in $s = 0$, so we need to make sense of its limit in this setting, as previously mentioned.

It should be noted that there is at least one alternative approach to those mentioned above, that of dissipative solutions. The principle associated with the latter is to involve an energy tensor that depends on the amount of energy the system has dissipated up to the present time, enabling very weak solutions to be defined. The case of such solutions has been successfully established in many cases, such as for the incompressible Euler equation. We refer the interested reader to [138, Section 4.4.] for more details in the latter case. In the case of the equation (II.1.2), it is possible to show the existence of dissipative solutions, including in the three-dimensional framework for threshold fluids (Abbatiello-Feireisl Theorem), we refer the reader to [1] for more details in this case.

In the last part of this section, we focus on one of the major properties of shear-thinning flows: their flow becomes static after a finite time interval, known as finite-time standstill. In other words, if the fluid is not subjected to any external force, it comes to a standstill. This property is naturally found in evolutionary equations with nonlinear terms similar to the Ostwald-De Waele law, for example when considering solutions to the equation:

$$\begin{cases} \partial_t u - \Delta_p u = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \\ u_{|t=0} = u_0 & \text{in } \Omega \end{cases} \tag{II.1.8}$$

where the operator Δ_p, defined here for $p > 1$ by

$$\Delta_p : \left| \begin{array}{l} W_0^{1,p}(\Omega) \longrightarrow W^{-1,q}(\Omega) \\ \\ u \longmapsto \operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right) \end{array} \right. \tag{II.1.9}$$

is called the p-laplacien, stops in finite time when $1 < p < 2$. Better than that, it is possible to estimate the stopping time as a function of the norm of the initial data, this is the object of the Di Benedetto Theorem, which corresponds to the Theorem 1.3.1 in the dissertation. We may then ask whether it is possible to show such a result in the case of the solutions established in Theorem II.1.7. The answer is positive and given by the following result, also from [44].

Theorem II.1.9. *Consider that the assumptions of Theorem I.1.7 are verified, that $T > 0$ is chosen to be sufficiently large, and $p \in [1, 2)$. In addition, we consider two strictly positive constants λ et $T_1 < T$ such as*

$$F(t) \geq \lambda t^{p-2} \text{ for all } t \in (0, +\infty) \qquad \text{and} \qquad f = 0 \quad \text{almost everywhere in } (T_1, T). \tag{II.1.10}$$

Then there is a finite stopping time $T_0 \in (0, T)$ for u, i.e. $u(t) \equiv 0$ for almost every $t \geq T_0$.

The Theorem II.1.9 (which corresponds to Theorem 1.3.3 in the dissertation), expresses the fact that when we stop subjecting the fluid to external forces, it eventually stabilizes in a stationary state in a finite time. To return to our initial example, when we stop whipping mayonnaise, it freezes.

II.2 Controllability of quasilinear parabolic equations

As is often the case, a solution that has been found leads to a new question. In this case, knowing that this finite-time stop exists, is it possible to apply a force, locally (in space), so that the stop time can be made as small as desired? Or, to put it another way, is it possible to ensure that, from a prescribed time onwards, the fluid is necessarily at a standstill, by applying a localized force? This is the subject of Chapter 2, and this type of question falls within the scope of *control theory*.

Note that there are different notions of controllability:

- Approximate controllability,
- Exact controllability,
- Optimal controllability.

Let us take an example to illustrate these three concepts. Let us say you want to get from Paris to New York. Approximate control means showing that there is a way to get from Paris to a place that is arbitrarily close to New York, and when it is possible to show that you can get exactly to New York, then you have done exact control. Now, when we have studied a problem of exact or approximate controllability, we may, for example, have given ourselves a time to make the journey, say fifteen hours. Perhaps our solution tells us that getting from Paris to New York is possible in that time if we take a fighter jet. However, it is a safe bet that the latter's carbon footprint is substantial, let alone its fuel consumption. To look at optimal controllability, we need to know that we are going from Paris to New York, and we want to get there while minimizing a cost, which could be fuel consumption. This may be subject to constraints, such as time, or the route taken. In this context, what is important is to minimize a cost functional. For our example, if we give ourselves a constraint of fifteen hours, it seems that taking an airliner is a more interesting alternative, from the point of view of optimal control, than taking a fighter jet.

II.2.1 Problem position and previous results

The question we have raised is therefore one of exact or approximate controllability. For simplicity's sake, we will consider the controllability of quasilinear parabolic equations (where solutions are scalar-valued), an example of which is given by (II.1.8).

Let us take a moment to comment on these equations. From a historical point of view, the analysis of quasilinear parabolic equations and the properties of their solutions took off in the

1960s, with the pioneering work of [125, 132, 9]. These equations have been the subject of a large number of publications in recent decades. As in the case of linear parabolic equations, the question of the bornitude of the solution or its gradient naturally arises. Such results are now well known and have been established for large classes of quasi-linear systems, we can refer to [144, 78, 9, 37, 36, 73, 26, 151] in the case of a bounded domain with a regular boundary. However, this remains an important source of research activity, particularly for the study of singular or degenerate systems. Examples include recent second-order regularity results for the p-parabolic Laplacian (see for example [45, 87]) and its alternative in the symmetrical gradient framework with the A-approximation method (see [25]). In the case of a nondegenerate, nonsingular scalar-valued equation, it is well known that smooth solutions are obtained (we refer, for example, to [43, Theorem 3.4.1., Sections 3.1.4. and 0.10.]). Note that this fact is used, for example, to approximate the solution of the p-Laplacian parabolic equation (II.1.8) (see e.g. [87, Section 4] or [130]). The $p = 1$ case is more difficult, but it is still possible to show interesting existence and regularity properties in certain non-singular cases (see for example [145, 170, 82]) ; in the latter case, we would point out that the smoothness of viscosity solutions has been proven in [122], which can be linked to reasoning such as that presented in [119]. We should also point out that the smoothing effect of quasilinear parabolic equations is known in many cases (see for example [106, Chapitre IV] or [145]). The literature on the subject of quasi-linear parabolic equations and systems is extremely vast, so we mainly refer the interested reader to the monographs [132, 125, 131, 169, 153, 176, 43, 123, 4] to study the properties of these equations and systems.

The controllability of quasilinear equations is a subject of very active research, see for example [39, 41, 40], for optimal control, or [140, 139, 88] in the context of exact controllability. In the latter papers, the exact controllability results are applied to systems where the nonlinear term depends on the solution of the system, but not on its gradient. The resulting results are presented in detail in Chapter 2.

To illustrate, we now consider the following alternative to (II.1.8), where $\omega \subsetneq \Omega$ is a subdomain and φ is an external force term:

$$\begin{cases} \partial_t u - \Delta_p u = \chi_\omega \varphi & \text{in } (0,T) \times \Omega \\ u = 0 & \text{on } (0,T) \times \partial\Omega \\ u_{|t=0} = u_0 & \text{in } \Omega \end{cases} \tag{II.2.1}$$

Still in the case of quasilinear equations, when the nonlinear term depends on the gradient of the solution, only one local controllability result had so far been demonstrated in the recent paper [91], more precisely, the following result is verified (see (F1)–(F3) in Chapter 2 for exact assumptions).

Theorem II.2.1 (Fernández–Cara-Límaco-Thamsten-Menezes)**.** *Let Ω be a domain of $\mathbb{R}^N$ with smooth boundary, $\omega \subsetneq \Omega$ an open set, $y_0 \in H_0^1(\Omega) \cap H^5(\Omega)$ and verifying the compatibility condition:*

$$(\Delta y_0, \Delta^2 y_0) \in H_0^1(\Omega)^2$$

Then there exists $\eta > 0$ such that if

$$\|y_0\|_{H^5(\Omega)} \leq \eta$$

we have the existence of a control $\varphi \in L^2((0,T) \times \omega)$ for equations of the type (II.2.1) *involving $y(T) = 0$.*

The Theorem II.2.1 shows an exact local controllability result, the local term describing that there is a smallness condition for the norm of the initial data. This theorem applies to a class of equations including the p-parabolic Laplacian, and we refer to its precise statement (Theorem 2.2.5) in the dissertation for precise assumptions concerning the equations taken into consideration.

II.2.2 Thesis results

We are interested in equations of the following form.

$$\begin{cases} \partial_t y - \operatorname{div}\left(F(|\nabla y|)\nabla y\right) = \chi_\omega \varphi & \text{in } Q_T \\ y = 0 & \text{on } \Sigma_T \\ y_{|t=0} = y^0 & \text{in } \Omega, \end{cases} \tag{II.2.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain with a smooth boundary $\partial\Omega$, $Q_T = (0,T)\times\Omega$, $\Sigma_T = (0,T)\times\partial\Omega$, the initial data y^0 belongs to $L^2(\Omega)$, $\omega \subsetneq \Omega$ is also a regular domain, and there exists $p > 1$ such that the function $F : \mathbb{R}_+ \to \mathbb{R}_+^*$ verifies the following assumptions:

(A1) $F \in W^{1,\infty}(\mathbb{R}_+) \cap C^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+) \cap L^{\frac{p}{p-1}}(\mathbb{R}_+)$;

(A2) The potential defined for any $t \in \mathbb{R}_+$ par $\Phi(t) = \int_0^t sF(s)\,ds$ is convex and satisfies $\Phi \in W^{1,\frac{p}{p-1}}(\mathbb{R}_+)$;

(A3) There exists $C_1, C_2, \mu, \nu > 0$ and $k_1, k_2 \geq 0$ such that for any $t \in \mathbb{R}_+$, we have

$$k_1 + C_1(\mu + t^2)^{\frac{p-2}{2}} \leq F(t) \leq k_2 + C_2(\nu + t^2)^{\frac{p-2}{2}}.$$

The control φ acts in a bounded and non empty open subset $\omega \subset \Omega$. More precisely, we denote $\chi_\omega \in C^\infty(\overline{\Omega})$ a regular function such that

$$\chi_\omega(x) = \begin{cases} 1 & \text{for } x \in \omega_\delta \\ 0 & \text{for } x \in \overline{\Omega \setminus \omega}, \end{cases}$$

where $\omega_\delta = \{x \in \omega \text{ such that } \operatorname{dist}(x, \partial\omega) > \delta\}$ for a constant $\delta > 0$ sufficiently small.

The objective is as follows: consider the linear system associated with (II.2.2), given by:

$$\begin{cases} \partial_t u - \operatorname{div}(a(t,x)\nabla u) = \chi_\omega \varphi & \text{in } (0,T)\times\Omega \\ u = 0 & \text{on } \partial\Omega \\ u_{|t=0} = u^0 & \text{in } \Omega \end{cases} \tag{II.2.3}$$

and we show that the latter is approximately controllable, with smooth control. To do this, we first adapt a method known as *Hilbert Uniqueness Method* (abbreviated by the acronym HUM hereafter). This method is based on a Fenchel-Rockafellar duality argument and is presented in detail in Chapter 2. We also refer to [30] for a general and very detailed presentation of the method in the context of parabolic equations. Thus, we show the following result (see [49, Theorem 1.1.]).

Theorem II.2.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary, $u^0 \in L^2(\Omega)$, $a \in C^\infty(\overline{Q_T})$ checking*

$$0 < \rho_\star \leq a(t,x) \qquad ((t,x) \in \overline{Q_T}),$$

and $T > 0$. So, for all $\varepsilon > 0$ there exists an approximate control $\varphi \in C^\infty(\overline{Q_T})$ in the sense that the corresponding solution u of (II.2.3) *verifies*

$$\|u(T)\|_{L^2(\Omega)} < \varepsilon.$$

In addition, the control φ depends in a continuous Lipschitz manner on the diffusion coefficient a for the norm $\|\cdot\|_{L^2(Q_T)}$.

The reader may rightly wonder about the interest of the continuous dependence of the solution φ on the diffusion parameter. The reason is as follows: starting from the controlled solution of the linear equation, we wish to show the existence of an approximate control for the quasilinear equation, by applying a Schauder-type fixed-point theorem, but for a weak topology. It is then necessary to have a continuous dependence on the fixed point in a well-chosen space to perform this fixed-point method. Applying this strategy, we can then show the following approximate controllability result (see [49, Theorem 1.2.]).

Theorem II.2.3. *Let F satisfy the hypotheses (A1)-(A3) and y^0 belonging to $L^2(\Omega)$ chosen such that there exists a unique solution to* (II.2.2). *Then there exists a distributed control φ, whose regularity is given by Theorem II.2.2, such that* (II.2.2) *is approximately null controllable in $T > 0$, i.e., for all $y^0 \in L^2(\Omega)$ and all $\varepsilon > 0$ there exists a control $\varphi \in C^\infty(\overline{Q_T})$ such that the solution y of* (II.2.2) *satisfies*

$$\|y(T)\|_{L^2(\Omega)} \leq \varepsilon.$$

Note that the case of singular or degenerate equations such as the p-Laplacian for $p \neq 2$ are a priori not taken into consideration in the previous theorem. However, it is possible to take a family of solutions arbitrarily close to the case of the parabolic p-Laplacian which verify the hypotheses of Theorem II.2.3. We then immediately obtain the following corollary (see [49, Corollary 1.1.]).

Corollary II.2.4. *Let $u^0 \in L^2(\Omega)$ and $\frac{3}{2} < p < 3$. Then* (II.2.1) *is approximately null controllable for any time $T > 0$, i.e., for any $\varepsilon > 0$ there exists a control $\varphi \in C^\infty(\overline{Q_T})$ such that the solution u of* (II.2.1) *verifies*

$$\|u(T)\|_{L^2(\Omega)} \leq \varepsilon.$$

In fact, in the case where we focus on equations with a solution that stops in finite time, it is possible to deduce a global exact controllability result from the previous result. More precisely, we have (see [49, Theorem 1.3.]):

Theorem II.2.5. *Consider that F verifies the assumptions (A1)-(A3), and that y is the solution of* (II.2.2) *associated with the initial data $y^0 \in L^2(\Omega)$, $y^0 \neq 0$. Furthermore, y is assumed to stop in finite time, i.e. if $\varphi = 0$, then there exist $T_s \in (0, T)$ such that:*

$$\|y(T_s)\|_{L^2(\Omega)} = 0. \tag{II.2.4}$$

In this case, we can choose a force term φ such that y is exactly null controllable for any time $T^\star \in (0, T)$.

As we mentioned, the parabolic p-Laplacian stops in finite time, so the following result immediately follows (voir [49, Corollary 1.2.]).

Corollary II.2.6. *Let $u^0 \in L^\infty(\Omega)$ and $\frac{3}{2} < p < 2$. Then, by choosing the positive solution of* (II.2.1), *the problem* (II.2.1) *is null controllable for any time $T > 0$,* i.e., *there exists a control $\varphi \in C^\infty(\overline{Q_T})$ such that the solution u of* (II.2.1) *verifies*

$$u(T) = 0.$$

II.3 Weak solutions of a non-Newtonian Stokes-transport system

So far, we have considered moving fluids that eventually come to a complete stop. In Chapters 1 and 2, we concentrated on models where the fluid density was constant (considering the models in Chapter 2 as a way of accounting for models close to fluid mechanics), but this is not suitable for studying sedimentation problems or fluids with suspended particles, for example.

II.3.1 Problem position and previous results

In this case, we will focus our study on the case of a *quasi-stationary fluid*, i.e. the fluid no longer evolves sufficiently to consider the time derivative term in the model. In Chapter 3, we will study a sedimentation problem in a power-law fluid, considering the following equation:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ -\operatorname{div}\left(\nu(\rho)|D(u)|^{p-2}D(u)\right) + \nabla \pi = \rho g & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ \rho_{|t=0} = \rho_0 & \text{in } \Omega. \end{cases} \tag{II.3.1}$$

The (II.3.1) system is no longer a parabolic problem but a coupled hyperbolic-elliptic system, which calls for very different solution methods. As we shall see, the (II.3.1) equation is an active scalar equation, i.e. the unknown is a scalar function ρ which is a solution of a transport equation given by

$$\partial_t \rho + u \cdot \nabla \rho = 0,$$

where u is a velocity field expressed as a function of ρ (in our case, by solving a nonlinear elliptic problem). A first important case of an active scalar equation, which has been the source of much recent research, is given by the case of the Stokes-transport equation

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ -\Delta u + \nabla \pi = \rho g & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ \rho_{|t=0} = \rho_0 & \text{in } \Omega, \end{cases} \tag{II.3.2}$$

which corresponds to (II.3.1) for $p = 2$, and when ν is constant according to the density ρ. Note that the absence of a convective term is consistent from a physical point of view, since for a quasi-stationary fluid, we consider transport by convection to be negligible in relation to transport by diffusion, which means that we are considering a flow with very small Reynolds number. We refer to [105, Chapter 9] for further details. Note that this system is rigorously obtained as the limit of a large number of sedimenting spherical particles (in non-inertial dynamics) immersed in a Stokes fluid (see for example [113, 114] as well as [115] for analysis of the impact of Einstein viscosity). We should also point out that such reasoning is valid in other settings, such as when replacing the Stokes model with a Vlasov-Stokes-type model (see [112]). As the analysis of the (II.3.2) model has been carried out in the framework of a general bounded domain, it is appropriate to add a condition guaranteeing the uniqueness of solutions, such as an edge condition. For example, with the homogeneous Dirichlet condition, the generic system (II.3.2) becomes

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ -\Delta u + \nabla \pi = \rho g & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \\ \rho_{|t=0} = \rho_0 & \text{in } \Omega. \end{cases} \tag{II.3.3}$$

It can be shown (see [61]), that for a velocity field $u \in L^1(W^{1,\infty})$, the spatial regularity induced by the initial data can be preserved for estimates on the transport equation, and that there is a loss of regularity for a velocity field of lesser regularity. In order to obtain the uniqueness of a pair of solutions within the framework of an active scalar equation, it is therefore necessary to

choose an initial datum that allows a priori to lead to a pair of solutions whose velocity field has $W^{1,\infty}$ regularity in space. If the spatial regularity of the velocity field is higher, we speak of a sub-critical regime, a super-critical regime if the regularity is lower, and a critical regime if the initial data leads exactly to a velocity field with Lipschitz (or, in a pinch, Log-Lipschitz) spatial regularity. In the case of a sub-critical regime, the following result has been shown [128, Theorem 1.1.].

Theorem II.3.1 (Leblond). *Let Ω be a domain with C^2 boundary of $\mathbb{R}^N$, $N \in \{2,3\}$, and $\rho_0 \in L^\infty(\Omega)$. So there is a solution $(\rho, u) \in L^\infty(\mathbb{R}_+, L^\infty(\Omega)) \times L^\infty(\mathbb{R}_+, W^{1,\infty}(\Omega))$ of* (II.3.3) *in the sense of distributions.*

This raises the question of the veracity of such a result on an unbounded domain. It is sometimes possible to show the existence of solutions, for example for the system:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{in } \mathbb{R}_+ \times ((0,1) \times \mathbb{R}) \\ -\Delta u + \nabla \pi = \rho g & \text{in } \mathbb{R}_+ \times ((0,1) \times \mathbb{R}) \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times ((0,1) \times \mathbb{R}) \\ u = 0 & \text{on } \mathbb{R}_+ \times (\{0,1\} \times \mathbb{R}) \\ \int_0^1 u_1(x,y)\, dy = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R} \\ \rho_{|t=0} = \rho_0 & \text{in } ((0,1) \times \mathbb{R}). \end{cases} \tag{II.3.4}$$

Theorem II.3.2 (Leblond). *Let $\rho_0 \in L^\infty(\Omega)$ and $\Omega = (0,1) \times \mathbb{R}$. So there is a solution $(\rho, u) \in L^\infty(\mathbb{R}_+, L^\infty(\Omega)) \times L^\infty(\mathbb{R}_+, W^{1,\infty}(\Omega))$ of* (II.3.4) *in the sense of distributions.*

In the three-dimensional case of whole space, i.e. for $\Omega = \mathbb{R}^3$, it is necessary to add a condition of decay to infinity in space for the velocity field, i.e.:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3 \\ -\Delta u + \nabla \pi = \rho g & \text{in } \mathbb{R}_+ \times \mathbb{R}^3 \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3 \\ \lim\limits_{|x| \to +\infty} |u| = 0 & \text{in } \mathbb{R}^3 \\ \rho_{|t=0} = \rho_0 & \text{in } \mathbb{R}^3. \end{cases} \tag{II.3.5}$$

The following results were obtained [142, Theorem 2.1.].

Theorem II.3.3 (Mecherbet-Sueur). *Let $q \geq 3$ and ρ_0 be a probability measure on $L^q(\mathbb{R}^3)$. Denoting E_q the subspace of $L^q(\mathbb{R}^3)$ composed of such measures, there exists a unique solution in the sense of the distributions (u, π, ρ) to* (II.3.5) *such that:*

- *If $q > 3$, $(u, \pi, \rho) \in C\left(\mathbb{R}_+, W^{2,q}(\mathbb{R}^3) \times W^{1,q}(\mathbb{R}^3) \times E_q\right)$*
- *If $q = 3$, $(u, \pi, \rho) \in C\left(\mathbb{R}_+, \dot{W}^{2,3}(\mathbb{R}^3) \times \dot{W}^{1,3}(\mathbb{R}^3) \cap \bigcap\limits_{r \in (3,+\infty)} W^{1,r}(\mathbb{R}^3) \times W^{1,3}(\mathbb{R}^3) \times E_q\right)$.*

The result of this theorem is optimal in the sense that the case $q = 3$ corresponds to the critical regime (and $q > 3$ to the sub-critical regime). We refer to [142] for more details. Another important question is whether the solutions obtained are asymptotically stable relative to a perturbation on the initial density ρ_0. A first result in this direction was obtained in a framework similar to (II.3.3) by replacing the Laplacian by the identity, i.e. by looking at a flow equation in a porous medium. We then obtain the following Darcy-transport-type system:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ u + \nabla \pi = \rho g & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \\ \rho_{|t=0} = \rho_0 & \text{in } \Omega. \end{cases} \tag{II.3.6}$$

Note that this system can be rewritten for $\Omega := \mathbb{R}^2$:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^2 \\ u = R_1 R^\perp \rho & \text{in } \mathbb{R}_+ \times \mathbb{R}^2 \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^2 \\ \rho_{|t=0} = \rho_0 & \text{in } \mathbb{R}^2, \end{cases} \tag{II.3.7}$$

where $R^\perp = (R_2, -R_1)$, and R_1 andR_2 correspond to the Riesz transforms along the first and second spatial variables respectively. Thus, we can visualize the equation (II.3.7) as a surface quasi-geostrophic equation (usually referred to by the acronym SQG), for which the velocity field is given by $u = -\nabla^\perp(-\Delta)^{-\frac{1}{2}}\rho$. Both belong to the class of generalized surface quasi-geostrophic equations, for which we define

$$u = \nabla^\perp(-\Delta)^{-\frac{\beta}{2}} m(D)\rho,$$

where $\beta \in \mathbb{R}$ and $m(D)$ is a positive Fourier multiplier, defined so that $m(D) = \log(\mathrm{Id} - \Delta)^\mu$ $\mu \in \mathbb{R}$. We refer to [53, Section 1.1.] for further details and explanations of the associated models. We then have the following result (see [80, Theorem 1.3.]).

Theorem II.3.4 (Elgindi). *Let $\eta(y) := y$. There exists $\varepsilon_0 > 0$ such that if we consider the solution of* (II.3.7) *associated with the initial data $\rho_0 = \tilde{\rho}_0 + \eta$ where $\|\tilde{\rho}_0\|_{W^{7,1}(\mathbb{R}^2)} + \|\tilde{\rho}_0\|_{H^s(\mathbb{R}^2)} \leq \varepsilon \leq \varepsilon_0$, $s \geq 20$, then the solution ρ verifies for all $t > 0$:*

(1) $\|\rho(t) - \eta\|_{H^3(\mathbb{R}^2)} \leq \varepsilon t^{-\frac{1}{4}}$

(2) $\|u_1(t)\|_{H^3(\mathbb{R}^2)} \leq \varepsilon t^{-\frac{3}{4}}$

(3) $\|u_2(t)\|_{H^3(\mathbb{R}^2)} \leq \varepsilon t^{-\frac{5}{4}}$

où $u := (u_1, u_2) = R_1 R^\perp \rho$.

Note that a similar result can be established for the two-dimensional torus (see [80, Theorem 1.4.]). It is in fact possible to show, for $\Omega = \mathbb{T} \times (0,1)$, that the solutions of (II.3.3) are also asymptotically stable relative to a perturbation on the initial ρ_0 (see [60, Theorem 1.1.]).

Theorem II.3.5 (Dalibard-Guillod-Leblond). *There exists a small $\varepsilon_0 > 0$ such that for any initial data $\rho_0 \in H^6(\mathbb{T} \times (0,1))$ verifying $\|\rho_0 - \theta\|_{H^6(\mathbb{T}\times(0,1))} \leq \varepsilon_0$ and $(\rho_0 - \theta) \in H_0^2(\mathbb{T} \times (0,1))$, the solution ρ of* (II.3.3) *satisfies:*

$$\|\rho - \rho_\infty\|_{L^2(\mathbb{T}\times(0,1))} \leq \varepsilon_0 (1+t)^{-1} \qquad \text{and} \qquad \|\rho - \rho_\infty\|_{H^4(\mathbb{T}\times(0,1))} \leq \varepsilon_0$$

where ρ_∞ is given by the decreasing vertical rearrangement of ρ_0, i.e.:

$$\rho_\infty(z) := \int_0^\infty \mathbb{1}_{0 \leq z \leq |\{\rho_0 \geq \lambda\}|}\, d\lambda,$$

and where the function $\theta : \mathbb{T} \times (0,1) \to (0,1)$ which depends only on its second variable, is called stratified density profile and is given by $\theta(z) = 1 - z$.

This raises the question of the existence of solution results for a class of problems including problems of the type (II.3.5) and (II.3.7), and, consequently, the existence of stability results on these. We observe that the fractional Stokes-transport equation:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^N \\ (-\Delta)^{\frac{\alpha}{2}} u + \nabla \pi = \rho g & \text{in } \mathbb{R}_+ \times \mathbb{R}^N \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^N \\ \lim\limits_{|x| \to +\infty} |u| = 0 & \text{in } \mathbb{R}^N \\ \rho_{|t=0} = \rho_0 & \text{in } \mathbb{R}^N \end{cases} \tag{II.3.8}$$

Note that (II.3.8) does define an active scalar equation, since by applying the Leray projector $\mathbb{P} = \mathrm{Id} + \nabla(-\Delta)^{-1} \operatorname{div}$, namely the orthogonal projection L^2 onto the space of zero divergence vector fields to the second equation of (II.3.8), we obtain directly

$$u = (-\Delta)^{-\frac{\alpha}{2}} \mathbb{P}(\rho g).$$

It is then possible to show the existence and uniqueness of global solutions for (II.3.8).

Theorem II.3.6 (Cobb). *Let $\alpha \in (0, N)$ and an exponent q such that*

$$\frac{2}{1 + \frac{\alpha}{N}} < q < \frac{N}{\alpha}. \tag{II.3.9}$$

Then for any initial data $\rho_0 \in L^q(\mathbb{R}^N)$, there exists a global weak solution $\rho \in L^\infty(\mathbb{R}_+, L^q(\mathbb{R}^N))$ of (II.3.8) *associated with the initial data ρ_0. If, in addition:*

- *$\alpha \in [0, N)$ and the indices $s > 0$ and $q \in [1, +\infty]$ are such that*

  $$q < \frac{N}{\alpha}, \qquad \text{and} \qquad s \geq 1 - \alpha.$$

  Then if $\rho_0 \in B^s_{\infty,1}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, there exists a time $T > 0$ such that the solution of (II.3.8) *is unique and verifies $\rho \in C([0,T), B^s_{\infty,1}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N))$.*

- *$N \geq 2$ and $\alpha \in (1, N)$. So for $q \in \left[1, \frac{N}{\alpha}\right)$ and $r = \frac{N}{\alpha - 1}$, and for any initial data ρ_0 in the space $L^q(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$,* (II.3.8) *has a unique global solution $\rho \in L^\infty(\mathbb{R}_+, L^q(\mathbb{R}^N) \cap L^r(\mathbb{R}^N))$.*

- *$N \geq 2$ and $\alpha \in [1, N]$. So for $q = \frac{N}{\alpha}$ and $r = \frac{N}{\alpha - 1}$, and for any initial data $\rho_0 \in \dot{B}^0_{p,1}(\mathbb{R}^N) \cap B^0_{r,1}(\mathbb{R}^N)$,* (II.3.8) *has a unique global solution $\rho \in C(\mathbb{R}_+; \dot{B}^0_{q,1}(\mathbb{R}^N) \cap B^0_{r,1}(\mathbb{R}^N))$.*

We refer the reader to [53, Theorem 1.1., Theorem 1.3., Theorem 1.6.] for a proof of this result. Theorem I.3.6 thus ensures the existence of solutions (local and global) as well as uniqueness, for an important family of active scalar equations corresponding to sedimentation in Newtonian fluids, covering both critical and subcritical regimes.

II.3.2 Thesis results

It is then pertinent to ask whether such a result can be demonstrated in the context of a non-Newtonian fluid, such as one following an Ostwald-De Waele law. This is precisely the problem posed in (II.3.1).

More precisely, we consider the following problem on the N-dimensional torus.

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^N \\ -\operatorname{div}\left(\nu(\rho)|D(u)|^{p-2}D(u)\right) + \nabla \pi = \rho g & \text{in } \mathbb{R}_+ \times \mathbb{T}^N \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^N \\ \fint_{\mathbb{T}^N} u \, \mathrm{d}x = 0 & \text{in } \mathbb{R}_+ \\ \rho_{|t=0} = \rho_0 & \text{in } \mathbb{T}^N \end{cases} \tag{II.3.10}$$

where ν is a positive function of the density ρ that can potentially degenerate. Let us briefly comment on the physics of this system, noting a tensor τ depending on $D(u)$. Let us note $\overline{\rho}$ the density of the mixture, so that it can be written:

$$\overline{\rho} = \tilde{\rho} + \rho,$$

where $\tilde{\rho}$ is the density of the carrier fluid, which is assumed to be constant, i.e. the carrier fluid is homogeneous, and ρ is the density of the suspended particles from which the density of the carrier fluid is subtracted. It can then be seen that the density ρ of the suspended particles, from which the density of the carrier fluid is subtracted, can be negative, depending on whether it is greater or lesser relative to the density of the carrier fluid. More precisely, when the latter is negative, the particles rise to the surface (creaming phenomenon), otherwise they sink to the bottom following the gravity vector (sedimentation). As is customary, we will combine these two terminologies and refer to them as sedimentation. Our initial equations are therefore given by:

$$\begin{cases} \partial_t \overline{\rho} + u \cdot \nabla \overline{\rho} = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^N \\ -\operatorname{div}\left(\overline{\nu}(\overline{\rho})|D(u)|^{p-2}D(u)\right) + \nabla \overline{\pi} = \overline{\rho} g & \text{in } \mathbb{R}_+ \times \mathbb{T}^N \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^N \\ \fint_{\mathbb{T}^N} u \, \mathrm{d}x = 0 & \text{in } \mathbb{R}_+ \\ \overline{\rho}_{|t=0} = \overline{\rho_0} & \text{in } \mathbb{T}^N \end{cases} \tag{II.3.11}$$

In fact, we can see that the first line can be rewritten directly as a function of ρ, since $\tilde{\rho}$ is constant. In the second equation, we can make the following observation: $\overline{\rho} g = \tilde{\rho} g + \rho g = \nabla\left(\tilde{\rho} g \cdot x\right) + \rho g$, and so we can rewrite $\pi = \overline{\pi} + \tilde{\rho} g \cdot x$, which allows us to define a new pressure term. By positing $\rho_{|t=0} = \overline{\rho_0} - \tilde{\rho} := \rho_0$, and redefining $\overline{\nu}(\overline{\rho}) = \nu(\rho)$, we obtain the system (II.3.10). Note that, from a mathematical point of view, the systems (II.3.11) and (II.3.2) are identical. However, the preceding justification allows us to understand why we a priori have a density that can be negative, in fact it comes from the fact that there are microscopic terms present in (II.3.11) that will disappear and lead, via the process previously described, to (I.3.10). We refer to [115] for more details.

Let us go back to the mathematical analysis of the problem. It is not obvious that the problem defines an active scalar equation, in particular because of the nonlinear term. In fact, by formally multiplying the non-Newtonian Stokes equation by $D(u)$, we do not obtain a uniform bound on u in $W^{1,p}$ as we would if we imposed $\nu(\rho)$ to remain strictly positive. We can use a priori estimates to show a smaller majorization, but it is not enough to reason directly by compactness. We then

work a priori in the over-critical regime. To show that (II.3.10) is indeed an active scalar equation, we need to observe that the operator:

$$A_\rho(u) = \frac{1}{p}\int_{\mathbb{T}^N} \nu(\rho)|D(u)|^p \, dx$$

is a strictly convex and coercive potential operator on a reflexive Banach space, in this case on a subspace of $L^p(\nu(\rho)dx)$, which is the L^p space associated with the $\nu(\rho)dx$ density measure. It is then necessary to decouple the (II.3.10) system by studying the properties of the non-Newtonian Stokes equation and the transport equation (or continuity equation) separately. We then need to find a scheme adapted to this decoupling to perform a Friedrich-type approximation method, the aim being to show the existence of weak solutions, defined as follows.

Definition II.3.7. *Let $d \geq 2$ be the dimension and an exponent $q \in [1, +\infty]$. Let $\rho_0 \in L^q$ be initial data. The function $\rho \in L^\infty(L^q)$ is a weak solution of the non-Newtonian Stokes-transport problem* (II.3.10) *associated with the initial data ρ_0 if the following conditions are satisfied:*

(i) There exists a velocity field $u : \mathbb{R}_+ \times \mathbb{T}^N \longrightarrow \mathbb{R}^N$ such that $\nu(\rho)|D(u)|^{p-1} \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{T}^N)$ and which, for almost any $t \in \mathbb{R}_+$, is a solution of the Stokes equation: in other words, $\mathrm{div}(u) = 0$ *in $\mathcal{D}(\mathbb{R}_+ \times \mathbb{T}^N)$ and whatever $\phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{T}^d, \mathbb{R}^N)$ with null divergence, we have*

$$\iint \nu(\rho)|D(u)|^{p-2}D(u) : D(\phi)\,\mathrm{d}x\,\mathrm{d}t = \iint \rho g \cdot \phi \,\mathrm{d}x\,\mathrm{d}t \,;$$

(ii) The velocity field is $u \in L^\infty(L^{q'})$ and ρ is a weak solution of the transport equation for the initial data ρ_0, i.e. for all $\phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{T}^N, \mathbb{R})$, we have

$$\iint \Big(\rho\partial_t\phi + \rho u \cdot \nabla\phi\Big)\,\mathrm{d}x\,\mathrm{d}t + \int \rho_0\phi(0)\,\mathrm{d}x = 0.$$

Showing the existence of solutions, even approximate ones, is not immediate: we cannot apply a Cauchy-Lipschitz theorem as it stands. As a priori regularity is insufficient, it is necessary to couple an argument by a Minty method into the non-Newtonian Stokes equation, and make use of Di Perna-Lions theory to estimate the density term. In doing so, we can show the existence of global approximate solutions via the use of the Cauchy-Peano theorem, and demonstrate the convergence of these solutions. This leads to the following result (see [54] and Theorem 3.0.2), which stems from collaborative work between Dimitri A. Cobb and the present author during a research stay at the University of Bonn (Germany).

Theorem II.3.8 (Theorem 3.0.2)**.** *Let $d \geq 2$ be the dimension. Consider $p \in (1, +\infty)$ and a function $\nu \in C^\infty(\mathbb{R} \setminus \{0\}) \cap C^{0,\overline{\gamma}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\nu(|r|) \gtrsim |r|^\gamma$ for all $r \in [-1, 1]$, $\overline{\gamma} = \min(\gamma, 1)$, $\gamma \geq 0$. We also consider exponents $q \in (1, 2)$ and $\sigma \in [1, +\infty]$ such that one of the following conditions is satisfied:*

(i) Subcritical case: either we have the strict inequality

$$\frac{1}{p}\left(1 + \frac{\gamma}{\sigma}\right) + \frac{1}{q} - \frac{1}{N} < 1 \,; \tag{II.3.12}$$

(ii) Critical case: either we have equality

$$\frac{1}{p}\left(1 + \frac{\gamma}{\sigma}\right) + \frac{1}{q} - \frac{1}{N} = 1 \,, \tag{II.3.13}$$

and the condition $q \geq \frac{2N}{N+2}$ is true. In particular, this is always true if $N = 2$.

Then, for any initial data $\rho_0 \in L^q(\mathbb{T}^N)$ *such that* $1/\rho_0 \in L^\sigma(\mathbb{T}^N)$*, there exists a weak solution* $\rho \in L^\infty\big(\mathbb{R}_+; L^q(\mathbb{T}^N)\big)$*. Furthermore, if* $\rho_0 \in L^q(\mathbb{T}^N) \cap L^r(\mathbb{T}^N)$ *for some* $r \in [1, +\infty]$*, then*

$$\|\rho\|_{L^\infty(L^r)} = \|\rho_0\|_{L^r}.$$

We draw the reader's attention to the fact that the terminologies "critical case" and "subcritical case" in the statement of Theorem II.3.8 are not related to the subcritical and critical regime for the uniqueness of solutions of the scalar equation, but are related to the a priori critical regularity for showing the existence of weak solutions.

One question is whether the results of Theorems II.3.8 and II.2.5 can lead to a result of exact internal controllability of (II.3.10) solutions. We won't discuss the physical validity of this assertion here, since a priori, as soon as sufficiently large external forces are applied to the fluid, the latter would be led out of its quasi-stationary state, and subsequently the consistency of the model would be questionable. We could alleviate this problem by defining admissible controls that would enable us to maintain physical consistency. Note that in the case of (II.3.5), the existence of internal controls for the continuity and Stokes equations has been established (see [142, Theorem 2.5.]).

As an example, let us note that the model (II.3.10) can account, for example, for the sedimentation of red blood cells in plasma.

II.4 Determination of a diffusion coefficient in hemodialysis

We conclude this dissertation with the study of a problem applied to the medical field of hemodialysis, carried out in collaboration with the Gabriel Montpied University Hospital in Clermont-Ferrand. As mentioned above, blood is a shear-thinning non-Newtonian fluid, and can be modeled fairly well by Carreau or Cross models. In other words, one of the peculiarities of blood is that, when subjected to large deformations, it behaves like a Newtonian fluid.

More specifically, blood is a suspension of erythrocytes (red blood cells), leukocytes (white blood cells, including eosinophil granulocytes, basophils and neutrophils, as well as lymphocytes and monocytes), and thrombocytes (platelets), in an aqueous fluid called plasma, which is also made up of proteins (albumin, immunoglobulins, fibrinogen, transferrin, α_1-antitrypsin, macroglobulins, lipoproteins, ceruloplasmin, etc.) as well as various chemical species and other organic or inorganic substances (mineral salts such as calcium, sodium, potassium and magnesium, for example). Blood is thus an aqueous suspension of highly heterogeneous particles, both in size and in volume.

On average, erythrocytes account for around 40% of blood volume, and boast significant elasticity properties. The volume concentration of erythrocytes in blood is called the hematocrit (often abbreviated to Ht), and is calculated by centrifuging blood in a tube, estimating the resulting volume formed by the erythrocyte aggregate. It should be noted that, in general, the aggregate obtained for hematocrit measurement is made up of an average of 96% of erythrocytes, the remainder being plasma retained in the aggregate. Thus, hematocrit measurement must be understood with a certain margin of error due to the measurement process. It should also be noted that hematocrit plays a major role in the shear-thinning properties of blood, and it has even been established that below a certain hematocrit threshold, blood is no longer shear-thinning (see [33]), which can be interpreted in the sense that, in general, plasma is such a dilute suspension that it can be considered as a Newtonian fluid. However, certain pathologies induce, under a significant increase in certain proteins such as fibrinogens or immunoglobulins, a modification of plasma such that its viscosity no longer corresponds to a Newtonian model (see for example [74]).

Thus, the viscosity of blood is highly complex and depends on a number of parameters. An interesting fact is that when blood flows through a capillary, i.e. formally a long tube with a small

diameter and a partially permeable membrane as boundary, then the smaller is the diameter, the lower is the viscosity: this is the Fåhræus-Lindqvist effect (see [85, 101, 152]). However, this effect is reversed below a certain diameter (below $6\,\mu$m, and significantly below $3\,\mu$m).

In [7], this phenomenon is highlighted by comparing numerical simulations obtained by considering non-Newtonian and Newtonian flow models, when blood flows through the small fibers of a dialyzer: the dimensions associated with such a fiber are given, for example, by a radius $R = 2.3 \times 10^{-4}$ m and a length $L = 2.3 \times 10^{-1}$ m, the radius of the inner cylinder in which the blood flows being then $R_1 = 10^{-4}$ m and the membrane thickness around 0.4×10^{-4} m.

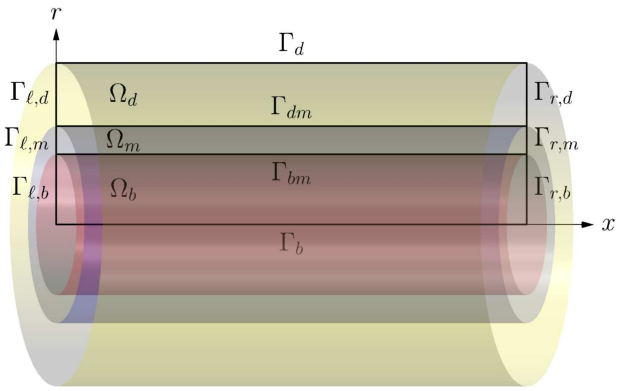


Figure II.8: Three-dimensional representation of a dialyzer fiber. Ω_d, Ω_m, and Ω_b represent the regions associated with dialysate, porous membrane and blood respectively.

The aim is to determine the values of intra-membrane diffusion coefficients, known as *Beavers-Joseph* coefficients, from patient samples. To do this, we consider that the membrane is subject to a Darcy law, and we take into account the chemical reactions taking place in the dialyzer, modeled by the following evolution system, where the values $(c_i)_{1\leq i\leq 5}$ correspond to the concentrations of the five chemical species present and taken into consideration.

$$\partial_t c_i + S_i\left(U_x\partial_x c_i + U_r\partial_r c_i\right) - \frac{1}{r}\partial_r\left(rD_i\partial_r c_i\right) - \partial_x\left(D_i\partial_x c_i\right) = F_i(\boldsymbol{c}) \text{ in } (0,T)\times\Omega \tag{II.4.1a}$$

$$c_i = \overline{c_{i,\text{ input}}} \text{ on } (0,T)\times(\Gamma_{\ell,b}\cup\Gamma_{r,d}) \tag{II.4.1b}$$

$$\partial_{\boldsymbol{n}} c_i = 0 \text{ on } (0,T)\times\Gamma_n \tag{II.4.1c}$$

$$c_i(0) = c_{i,0} \text{ in } \Omega. \tag{II.4.1d}$$

In (II.4.1), the coefficients $(D_i)_{1\leq i\leq 5}$ represent the intramembrane diffusion coefficients for the species $(c_i)_{1\leq i\leq 5}$, $(S_i)_{1\leq i\leq 3}$ is a coefficient used to eliminate species that do not cross the membrane (Albumin and Calcium Albumin), i.e.

$$S_i(x) = \begin{cases} 1, & \text{if } x\in\Omega_b \\ 0, & \text{if } x\in\Omega\setminus\Omega_b \end{cases}$$

and $S_i = 1$ if $i\in\{1,\ 4,\ 5\}$.

Our approach to determining diffusion coefficients is based on a gradient descent method coupled with a Newton method, and will be presented in detail in Chapter 4.

Note that in the previous model, for simplicity's sake, we consider a porous membrane and intramembrane diffusion according to Darcy's law. In some cases, thrombocytes may aggregate at the edge of the dialyzer membrane, a phenomenon known as primary hemostasis. This may be due to a detachment of the collagen present in the endothelial cells (i.e. cells making up

the endothelium, the inner layer of blood vessels in direct contact with the blood) which attach themselves to the membrane or, as the case may be, circulate in the blood. Thrombocytes then attach themselves to the collagen molecules, resulting in variability in the porosity of the dialyzer membrane. One way of overcoming such problems would be to consider a generalized Darcy's law more suited to fluid exchange via the porous membrane, as established in [157]. For an in-depth discussion of blood biomechanics, we refer the reader to [97, 148].

To conclude this introduction, it should be noted that knowledge of the coefficients is interesting because it opens up a question of optimal control: the aim of performing dialysis is to restore a desirable equilibrium between different intrasanguineous chemical species in the patient. Knowing the diffusion coefficients and the target state, what is the right concentration of chemical species to put in the dialysate to reach the desired target? This depends on the data specific to each patient, the aim being to optimally achieve the desired concentrations at the outlet of the dialyzer.

Écoulements et transformations renouvellent le monde constamment, comme le cours ininterrompu du temps maintient toujours nouvelle la durée infinie. Dans ce fleuve, de quel objet parmi ceux qui passent en courant, pourrait-on faire cas, puisqu'il n'est pas possible de s'arrêter sur chacun?

Marc-Aurèle, *Pensées pour moi-même*, Livre VI.

1

Mathematical analysis of non-Newtonian fluid flows

This chapter is dedicated to the mathematical analysis of non-Newtonian flows of the shear-thinning or shear-thickening type. First, we introduce the tools needed to understand and master the methods used in this context. We begin by studying the theory of nonlinear monotone operators and some of their properties, before turning to variational inequalities. To illustrate the importance of the latter, we will highlight some of their applications, in control theory for example. Secondly, we present some existence results for non-Newtonian flows, first using monotonicity methods and then variational inequalities. We will conclude this part of existence with Theorem 1.2.18 proved by the present author and his advisors in [44]. Finally, we will establish the existence of a finite stopping time for shear-thinning flows of the Ostwald-De Waele and Bingham type, which is the subject of Theorem 1.3.3 proved by the same authors.

1.1 Analysis tools

In this section, we introduce the basic tools and notions that are usually used in the mathematical analysis of non-Newtonian fluids. First, we briefly introduce the theory of nonlinear monotone operators, and then we discuss the notion of variational inequality and the link between these two notions.

1.1.1 Nonlinear monotone operators

Let us generically consider a reflexive Banach space X and denote its topological dual X^*. In a first time, we will consider operators $A : X \to X^*$ that are not a priori linear. We recall that, by definition, A is bounded if it maps any bounded subset of X into a bounded subset of X^*. Let us introduce some classical definitions generalizing those of the linear case (for an introduction to the linear theory, we mainly refer to [174]). The aim of this section is to show some results leading to the existence of solutions to the equation:

$$A(u) = f, \tag{1.1.1}$$

where u belongs to a reflexive separable Banach space X, f belongs to its topological dual X^*, and $A : X \to X^*$. Thus, showing the existence of solutions for any fixed f is the same as showing

that the operator A is onto.

Definition 1.1.1 (Monotonicity modes). *Let X being a reflexive separable Banach space, and denote X^* its dual. We say that the operator $A : X \to X^*$ is:*

- **Monotone** *if :* $\forall (u,v) \in X^2,\ \langle A(u) - A(v), u - v\rangle \geq 0$
- **Strictly monotone** *if A is monotone and satisfies:* $\forall (u,v) \in X^2,\ u \neq v \Rightarrow \langle A(u) - A(v), u - v\rangle > 0$
- **Pseudomonotone** *if it is bounded and for every sequence $(u_n)_{n\in\mathbb{N}} \subset X$ satisfying:*

  (A1) $u_n \underset{n\to+\infty}{\rightharpoonup} u$;

  (A2) $\overline{\lim\limits_{n\to+\infty}} \langle A(u_n), u_n - u\rangle \leq 0$,

  then it satisfies: $\forall v \in X,\ \langle A(u), u - v\rangle \leq \underline{\lim\limits_{n\to+\infty}} \langle A(u_n), u_n - v\rangle$
- **d-monotone** *with respect to a seminorm $|\cdot|$ if for an increasing function $d : [0,+\infty) \to \mathbb{R}$, it satifies:*

$$\forall (u,v) \in X^2,\ \langle A(u) - A(v), u - v\rangle \geq \left(d\left(|u|\right) - d\left(|v|\right)\right)\left(|u| - |v|\right)$$

- **Uniformly monotone** *if there exists an increasing function $\varphi : [0,+\infty) \to [0,+\infty)$ such that:*

$$\forall (u,v) \in X^2,\ \langle A(u) - A(v), u - v\rangle \geq \varphi(\|u - v\|)\|u - v\|$$

- **Strongly monotone** *if it is uniformly monotone relatively to $\varphi : t \mapsto Ct$, $C > 0$.*

Let us now explain two examples to illustrate some notions of monotonicity given in the previous definition. Note that when we consider a differential operator associated to some boundary conditions, these are in general encoded in the space on which the operator is defined.

Example 1.1.2. *Let H be a separable Hilbert space and $P : H \to H$ be the projection onto its closed unit ball $\overline{B}_H$. We recall that for $u \in H$, $P(u)$ is the unique element of $\overline{B}_H$ satisfying:*

$$\langle P(u) - u, P(u) - v\rangle \leq 0 \qquad \forall v \in \overline{B}_H$$

Also, P is a monotone operator since for all $v \in H$:

$$\langle P(u) - P(v), u - v\rangle = \langle P(u) - P(v), u - P(u)\rangle + \|P(u) - P(v)\|_H^2 + \langle P(u) - P(v), P(v) - v\rangle \geq 0$$

Example 1.1.3 (Monotonicity of the p-Laplacian). *Let us consider $\Omega \subset \mathbb{R}^N$ a bounded domain and the p-Laplacian operator Δ_p in the homogeneous Dirichlet setting given for $p > 1$ by:*

$$\Delta_p : \left| \begin{array}{l} W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega) \\ u \mapsto \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right). \end{array} \right. \tag{1.1.2}$$

Then, Δ_p is d-monotone for $d : [0,+\infty) \to [0,+\infty)$, $t \mapsto t^{p-1}$. Moreover, it is strictly monotone for $p > 1$, and even uniformly monotone for $p \geq 2$. See [156, Example 2.83 p.75] and [175, Example 25.5 p.502].

Note that a number of properties presented in this section remain true for operators that do not necessarily verify one of the modes of monotonicity presented above, but are of type M. Since we do not need such a degree of generality for our presentation, we refer the interested reader to [161, Chapter II]. Let us now introduce the different modes of continuity usually used together with the different modes of monotonicity presented above.

Definition 1.1.4 (Continuity modes)**.** *Let X be a reflexive separable Banach space, X^* its dual and $A : X \to X^*$. Then, we say that A is:*

- **Hemicontinuous** *if for all $(u, v, w) \in X^3$, the map $t \mapsto \langle A(u + tv), w\rangle$ is continuous, that is A is directionally weakly continuous*
- **Radially continuous** *if for all $(u, v) \in X^2$, the map $t \mapsto \langle A(u+tv), v\rangle$ is continuous, that is if the previous definition holds only for $w = v$ instead of every w*
- **Demicontinous** *if for all $v \in X$, the map $u \mapsto \langle A(u), v\rangle$ is continuous*
- **Weakly continuous** *if for all $v \in X$, the map $u \mapsto \langle A(u), v\rangle$ is weakly continuous*
- **Strongly continuous** *if for $(u_n)_{n\in\mathbb{N}} \subset X$ such that $u_n \underset{n\to+\infty}{\rightharpoonup} u \in X$, then $A(u_n) \underset{n\to+\infty}{\longrightarrow} A(u)$*

We point out that using the previous definitions, an operator $A : X \to X^*$ is demicontinuous if it is continuous from X endowed with its strong topology to X^* endowed with its weak topology, and weakly continuous if it is continuous from X to X^* both endowed with their weak topologies. Let us now give some usual properties using the previous definitions. The first property is known as *Minty's trick*.

Lemma 1.1.5 (Minty's trick)**.** *Let X be a reflexive separable Banach space, and $A : X \to X^*$ be radially continuous. Assume moreover that there exists $u \in X$ and $f \in X^*$ such that it satisfies:*

$$\langle f - A(v), u - v\rangle \geq 0 \qquad \forall v \in X. \tag{1.1.3}$$

Then $A(u) = f$ in X^*.

Proof. Consider $v = u + rw$ for $w \in X$ and $r > 0$. Then (1.1.3) leads to:

$$\langle f - A(u + rw), -rw\rangle \geq 0 \qquad \forall w \in X. \tag{1.1.4}$$

Then we cand divide by $r > 0$ into (1.1.4) and pass to the limit over r, using the radial continuity of A, to get:

$$\langle f - A(u), w\rangle \leq 0 \qquad \forall v \in X. \tag{1.1.5}$$

Since the result holds for w as for $-w$, and for every $w \in X$, the result follows. □

The following properties hold.

Proposition 1.1.6. *Let X be a reflexive separable Banach space and $A : X \to X^*$. Then, we have:*

(i) If A is pseudomonotone, then it is demicontinuous

(ii) If A is monotone, it is locally bounded in the following sense:

$$\forall u \in X,\ \exists \varepsilon > 0,\ \exists C \geq 0,\ \forall v \in X,\ \|u - v\|_X \leq \varepsilon \Rightarrow \|A(v)\|_{X^*} \leq C \tag{1.1.6}$$

(iii) If A is monotone and radially continuous, then it is demicontinuous

(iv) If A is monotone and hemicontinuous, then it is pseudomonotone

(v) If A is strongly continuous, then it is pseudomonotone.

Proof. (i) Let $(u_n)_{n\in\mathbb{N}}$ be a sequence converging to $u \in X$. By definition, the sequence $(A(u_n))_{n\in\mathbb{N}}$ is bounded in X^*. Then, as X is assumed separable, we get up to extract that $A(u_n) \rightharpoonup w$ for some $w \in X^*$. It follows that

$$\langle A(u_n), u_n - u\rangle \underset{n\to+\infty}{\longrightarrow} \langle w, 0\rangle = 0,$$

and thus from the definition of a pseudomonotone operator, it leads to, for every $v \in X$:

$$\langle A(u), u - v\rangle \leq \underline{\lim}_{n\to+\infty} \langle A(u_n), u_n - v\rangle = \langle w, u - v\rangle.$$

Setting successively $v = 0$ and $v = 2u$ leads to $A(u) = w$ in X^*. So w is uniquely determined and therefore the whole sequence converges, which proves the result.

(ii) Suppose that the result does not hold, thus assume that the converse is true. Up to translate and without loss of generality we can consider $u = 0$. Hence there exists $(u_n)_{n\in\mathbb{N}}$ such that $u_n \underset{n\to+\infty}{\longrightarrow} 0$ and satisfying $\|A(u_n)\|_{X^*} \underset{n\to+\infty}{\longrightarrow} +\infty$. Let us now consider $(\beta_n)_{n\in\mathbb{N}}$ defined by

$$\beta_n := 1 + \|A(u_n)\|_{X^*}\|u_n\|_X.$$

Thanks to the monotonicity of A, we can write:

$$\begin{aligned}\langle \beta_n^{-1} A(u_n), v\rangle &\leq \beta_n^{-1}\left(\langle A(u_n), u_n\rangle + \langle A(v), v - u_n\rangle\right)\\ &\leq 1 + \|A(v)\|_{X^*}(\|v\|_X + \|u_n\|_X)\\ &\underset{n\to+\infty}{\longrightarrow} 1 + \|A(v)\|_{X^*}\|v\|_X.\end{aligned}$$

But we can now prove the same inequality with $-v$ instead of v, which leads to thefact that $\overline{\lim}_{n\to+\infty} |\langle \beta_n^{-1} A(u_n), v\rangle| < +\infty$, this last being true for every $v \in X$. Applying Banach-Steinhaus theorem, we then get that there exists a constant $M > 0$ such that $\|A(u_n)\|_{X^*} \leq M$, so that:

$$\|A(u_n)\|_{X^*} \leq M\beta_n = M\left(1 + \|A(u_n)\|_{X^*}\|u_n\|_X\right),$$

which in turn implies

$$\|A(u_n)\|_{X^*} \leq \frac{M}{1 - M\|u_n\|_X} \underset{n\to+\infty}{\longrightarrow} M,$$

but this is a contradiction, thus the result is true.

(iii) We choose some $u \in X$ such that there is a sequence $(u_n)_{n\in\mathbb{N}}$ in X which converges toward u. By the previous point (ii), the sequence $(A(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in X^* then up to, an extraction we have that $A(u_n) \rightharpoonup w \in X^*$. Now from the monotonicity of A, we get for every $v \in X$ that:

$$0 \leq \langle \lim_{n\to+\infty} \langle A(u_n) - A(v), u_n - v\rangle = \langle w - A(v), u - v\rangle.$$

Now from the radial continuity of A, one can apply Lemma 1.1.5 which leads to $A(u) = w$. Since w is uniquely determined, the whole sequence $(A(u_n))_{n\in\mathbb{N}}$ converges weakly for every subsequence initially considered.

(iv) Let us consider a sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \underset{n\to+\infty}{\rightharpoonup} u$ for some $u \in X$, and such that:

$$\overline{\lim_{n\to+\infty}} \langle A(u_n), u_n - u\rangle \leq 0. \tag{1.1.7}$$

From the monotonicity of A, we get that:

$$\langle A(u_n), u_n - u\rangle \geq \langle A(u), u_n - u\rangle. \tag{1.1.8}$$

The right-hand side of the above inequality goes to zero as $n \to \infty$, also combining (1.1.7) and (1.1.8) leads to:

$$\langle A(u_n), u_n - u\rangle \underset{n\to\infty}{\longrightarrow} 0.$$

Setting now $w = u + \lambda(v - u)$ for some $\lambda > 0$, we have since A is monotone that:

$$\langle A(u_n) - A(w), u_n - w\rangle \geq 0,$$

which leads to:

$$\lambda\langle A(u_n), u - v\rangle \geq \langle -A(u_n), u_n - u\rangle + \langle A(w), u_n - u\rangle + \lambda\langle A(w), u - v\rangle.$$

Letting $n \to \infty$ in this inequality and then dividing by $\lambda > 0$ leads, for every $v \in X$, to:

$$\underline{\lim_{n\to+\infty}} \langle A(u_n), u_n - v\rangle \geq \langle A(w), u - v\rangle.$$

Now since A is hemicontinuous, we can take the limit $\lambda \to 0$ in this last and get:

$$\underline{\lim_{n\to+\infty}} \langle A(u_n), u_n - v\rangle \geq \langle A(u), u - v\rangle,$$

namely A is a pseudomonotone operator.

(v) Assume that A is strongly continuous, then considering a sequence in X denoted $(u_n)_{n\in\mathbb{N}}$ such that $u_n \underset{n\to+\infty}{\rightharpoonup} u$, then $A(u_n) \underset{n\to+\infty}{\longrightarrow} A(u)$. Thus, it follows for every $v \in X$ that:

$$\langle A(u), u - v\rangle = \lim_{n\to+\infty} \langle A(u_n), u_n - v\rangle,$$

and so A is pseudomonotone.

□

Note that the different modes of continuity can be expected to relate to each other. Although it is not easy to establish these, the following theorem shows us that it is sometimes possible, with the help of the monotonicity of the operator, to link some of them.

Theorem 1.1.7 (Kato)**.** *Let X be a reflexive separable Banach space, denote X^* its topological dual, and consider a monotone operator $A : X \to X^*$. Then, A is hemicontinuous over $D(A)$ if and only if it is demicontinuous over $D(A)$*

Proof. We refer to [120] or [51, Chapter V Theorem 1.6] for a proof. □

Let us now recall the following definition.

Definition 1.1.8 (Coercivity)**.** *Let X be a reflexive separable Banach space, and consider $A : X \to X^*$. Then, A is* **coercive** *if there exists $\psi : [0, +\infty) \to [0, +\infty)$ satisfying $\psi(t) \underset{t\to+\infty}{\longrightarrow} +\infty$ such that:*

$$\langle A(u), u\rangle \geq \psi(\|u\|_X)\|u\|_X \qquad \forall u \in X \tag{1.1.9}$$

If A only satisfies $\|A(u)\|_{X^*} \underset{\|u\|_X\to+\infty}{\longrightarrow} +\infty$, then we say that A is **weakly coercive**.

We point out that (1.1.9) is equivalent to:

$$\frac{\langle A(u), u\rangle}{\|u\|_X} \underset{\|u\|_X\to+\infty}{\longrightarrow} +\infty. \tag{1.1.10}$$

As is usual in calculus of variations, the notion of coercivity plays an important role in establishing the existence of solutions to (1.1.1). We have the following important theorem:

Theorem 1.1.9 (Brézis)**.** *Let X be a reflexive separable Banach space and $A : X \to X^*$ a coercive pseudomonotone operator, then A is onto.*

Sketch of the proof. The proof is rather long so we just give a sketch of it here. We refer the interested reader to [156, Theorem 2.6.] for details. The proof is based on a nonlinear Galerkin method.

First step: We proceed to a Galerkin approximation, which is possible thanks to the separability of the space. Then, using the properties of the operator, we show the existence of Galerkin solutions using reasoning by the absurd (we assume that "desirable" solutions do not exist), based on the application of Brouwer's fixed-point theorem.

Second step: Using a priori estimates, we show that we can pass to the limit with respect to the Galerkin parameter and thus obtain the desired result. □

Theorem 1.1.9 means that there exists always a solution to (1.1.1) under the wished assumptions. Let us now exhibit some other properties linked to coercivity.

Proposition 1.1.10. *Let X being a separable reflexive Banach space and $A : X \to X^*$ a monotone, radially continuous, bounded and coercive operator. Then:*

(i) A is onto

(ii) If A is also strictly monotone, then A^{-1} exists and is a bounded demicontinuous strictly monotone operator. Moreover, if A is d-monotone and if X is is uniformly convex, then A^{-1} is continuous

(iii) If, in addition, A is uniformly (respectively strongly) monotone, then A^{-1} is uniformly (respectively Lipschitz) continuous.

Proof. (i) First, from Proposition 1.1.6 (iii), A is a pseudomonotone operator. By Theorem 1.1.9, since A is assumed to be coercive, then it is onto. Moreover, we can prove that the set of solutions to $A(u) = f$ is a convex subset of X: from Proposition 1.1.6 (i), it is also a demicontinuous operator. Thus, the set of solutions to $A(u) = f$ is closed in X. Now, setting $u = \frac{\psi+\psi'}{2}$, where η and η' are both solutions to the previous equation, one can see that:

$$\langle f - A(v), u - v\rangle = \frac{1}{2}\langle A(\eta) - A(v), \eta - v\rangle + \frac{1}{2}\langle A(\eta') - A(v), \eta' - v\rangle \geq 0,$$

since $A(\eta) = A(\eta') = f$, and using the monotonicity of A. The Minty's trick given by Lemma 1.1.5 leads to the fact that $A(u) = f$ and the set of solutions is then convex.

(ii) We have for all couple of solutions u and v to $A(u) = f$ that

$$\langle A(u) - A(v), u - v\rangle = \langle f - f, u - v\rangle = 0,$$

but from the strict monotonicity of A, this should lead to a contradiction as soon as $u \neq v$. Then, the solution is unique, so the inverse A^{-1} exists. Let us consider $(u, v) \in X^2$, $u \neq v$, and $(f, g) \in (X^*)^2$ such that $A(u) = f$ and $A(v) = g$ respectively. Then the strict monotonicity of A^{-1} is rather direct, writing:

$$\langle f - g, A^{-1}(f) - A^{-1}(g)\rangle = \langle A(u) - A(v), u - v\rangle > 0.$$

Using the coercivity of A, there exists by definition $\psi : \mathbb{R}_+ \to \mathbb{R}$ such that $\psi(t) \underset{t\to+\infty}{\longrightarrow} +\infty$ and satisfying for all $u \in X$:

$$\langle A(u), u\rangle \geq \|u\|_X \psi\left(\|u\|_X\right).$$

But this implies, for u solution to $A(u) = f$, that:

$$\psi\left(\|u\|_X\right)\|u\|_X \leq \langle A(u), u\rangle = \langle f, u\rangle \leq \|f\|_{X^*}\|u\|_X,$$

and thus it follows:

$$\psi\left(\|A^{-1}(f)\|_X\right) \leq \|f\|_{X^*},$$

which means that A^{-1} maps bounded sets of X^* into bounded sets of X, and so this is a bounded operator.

Let us now consider a sequence $(f_n)_{n\in\mathbb{N}}$ into X^* such that $f_n \underset{n\to+\infty}{\rightharpoonup} f$ for $f \in X^*$. From the boundedness of A^{-1}, we have that $(A^{-1}(u_n))_{n\in\mathbb{N}}$ is bounded, and thus up to extract we have that $A^{-1}(f_n) = u_n \underset{n\to+\infty}{\rightharpoonup} u$ in X. Then, A^{-1} is demicontinuous if we can prove that $A(u) = f$, but arguing by monotonicity, we have for every $v \in X$:

$$0 \leq \langle A(u_n) - A(v), u_n - v\rangle = \langle f_n - A(v), u_n - v\rangle.$$

Letting $n \to +\infty$ in the previous inequality leads to:

$$0 \geq \langle f - A(v), u - v\rangle,$$

and Minty's trick Lemma 1.1.5 leads to $A(u) = f$. If we assume now that A is d-monotone, we can write for some $(u_n)_{n\in\mathbb{N}}$ which weakly converges toward u in X:

$$\begin{aligned}\left(d\left(\|u_n\|_X\right) - d\left(\|u\|_X\right)\right) \leq \langle A(u_n) - A(u), u_n - u\rangle = \langle f_n - A(u), u_n - u\rangle \\ \underset{n\to+\infty}{\longrightarrow} \langle f - A(u), u - u\rangle = 0,\end{aligned}$$

which leads, since $d : \mathbb{R} \to \mathbb{R}$ is increasing, to $\|u_n\|_X \underset{n\to+\infty}{\longrightarrow} \|u\|_X$. Now, since $u_n \underset{n\to\infty}{\rightharpoonup} u$ and both belongs to a closed convex set by (i), we can apply Proposition A.7.5 which leads to $u_n \underset{n\to\infty}{\longrightarrow} u$ in X. Also, A^{-1} is continuous.

(iii) By definition of uniform monotonicity, there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that we can write, for every $(u, v) \in X^2$, $u \neq v$, satisfying $A(u) = f$ and $A(v) = g$:

$$\begin{aligned}\varphi\left(\|u - v\|_X\right)\|u - v\|_X &\leq \langle A(u) - A(v), u - v\rangle = \langle f - g, u - v\rangle \\ &\leq \|f - g\|_{X^*}\|u - v\|_X,\end{aligned}$$

thus A^{-1} is uniformly continuous, from the assumptions over φ. If A is strictly monotone, then $\varphi = \mathrm{Id}$, and thus the previous inequality leads to:

$$\|A^{-1}(f) - A^{-1}(g)\|_{X^*} = \|u - v\|_X \leq \|f - g\|_{X^*},$$

and so the result is proved. □

Finally, another important result of existence of solutions to (1.1.1) is the following:

Theorem 1.1.11. *(Browder-Minty) Let X be a reflexive separable Banach space and $A : X \to X^*$ a coercive, monotone, and radially continuous operator. Then A is onto.*

Proof. We apply a Galerkin method, by considering subspaces $X_k \subset X$ such that $\cup_{k\in\mathbb{N}} X_k$ is dense in X, we is valid from the separability of the spaces considered. Then one can define a projector $\mathbb{P}_k : X^* \to X_k^*$ such that for every $v^k \in X_k$:

$$\langle \mathbb{P}_k(f), v^k\rangle = \langle f, \mathbb{P}_k(v^k)\rangle = \langle f, v^k\rangle.$$

Then for a Galerkin solution $u_k \in X_k$, one can write for $v^k \in X_k$:

$$\langle A(u_k), v^k\rangle = \langle f, v^k\rangle,$$

from the coercivity of A and setting $v = u_k$ in the previous inequality leads to the boundedness of $(A(u_k))_{k\in\mathbb{N}}$ and the one of $\|u_k\|_X$, thus one can extract a subsequence such that $A(u_k) = \mathbb{P}k(f) \underset{k\to+\infty}{\rightharpoonup} f$.
Hence, the monotonicity of A leads to:

$$\langle A(v^k) - A(u_k), v^k - u_k\rangle = \langle A(u_k), v^k - u_k\rangle - \langle f, v^k - u_k\rangle \geq 0,$$

and considering a sequence v^k such that $v^k \underset{n\to+\infty}{\longrightarrow} v$ in X (arguing by density), we get since A is demicontinuous from Proposition 1.1.6 (ii) that A is strongly continuous, and so one can pass to the limit over $k \to +\infty$ to get for every $v \in X$:

$$\langle A(v) - f, v - u\rangle \geq 0.$$

Setting now $v = u + \lambda v'$ for an arbitrary chosen $v' \in X$, and use Minty's trick Lemma 1.1.5 leads to the result. □

In view of the previous examples, one would be tempted to believe that the classical theory of monotone operators (in the sense of the different modes of monotonicity considered) lends itself more easily to the framework of operators depending on a single variable, or even of elliptic operators. In fact, this is not the case: as the following propositions show, the time derivative operator is a monotone operator, and moreover the different modes of monotonicity enjoy an additive stability.

Proposition 1.1.12 (Monotonicity of time derivative). *Let us consider a reflexive Banach space X, its topological dual X^*, and a separable Hilbert space H such that:*

$$X \subseteq H \subseteq X^*$$

Moreover, let us denote the time derivative operator $\mathcal{T}$ defined as:

$$\mathcal{T} : \left| \begin{array}{l} D(\mathcal{T}) \longrightarrow X^* \\ \\ u \longmapsto \partial_t u \end{array} \right. \tag{1.1.11}$$

where we denoted $D(\mathcal{T}) := \left\{ u \in W^{1,p}([0,T], X) / u_{|t=0} = u_0 \in H \right\}$. Then, $\mathcal{T}$ is a monotone operator.

Proof. The proof is rather direct. First, $\mathcal{T}$ is a linear operator. Its monotonicity follows from the estimate:

$$\int_0^T \langle \mathcal{T}(u) - \mathcal{T}(v), u - v \rangle \, dt = \frac{1}{2} \|(u-v)(T)\|_H^2 \geq 0.$$

□

Proposition 1.1.13 (Sum of monotone operators). *Assume that X is a reflexive separable Banach space, X^* is its dual, and consider $A, B : X \to X^*$. Then:*

(i) If both A and B are monotone, then $A + B$ is monotone.

(ii) If both A and B are pseudomonotone, then $A + B$ is pseudomonotone.

Proof. (i) This is true by definition.
(ii) See [175, Proposition 27.6 (e)]. □

Multivalued theory

The theory of nonlinear monotone operators that we have just introduced, which we could call classical, is very efficient to show the existence of solutions of nonlinear partial differential equations. However, it happens in mechanics, especially in plasticity theory, to encounter multivalued operators. In this case, the theory presented so far fails, and it is then necessary to extend the notion of monotonicity. This is the subject of the end of this section. If Y is a Banach space, we denote 2^Y its power set

Definition 1.1.14 (Multivalued mapping). *Let us consider a reflexive separable Banach space X and X^* its topological dual. Assume that $A : X \to 2^{X^*}$ is a multivalued mapping, that is it assigns to every $u \in X$ a subset $A(u)$ of X^*. Then we define the:*

- **Effective domain** *of A as being the set:*

$$D(A) := \{ u \in X / \ A(u) \neq \emptyset \}$$

- **Range** *of A as being the set:*

$$R(A) := \bigcup_{u \in X} A(u)$$

- **Graph** *of A as being the set:*

$$G(A) := \{(u, v) \in X \times X^* / \ u \in D(A), v \in A(u)\}$$

- **Inverse** *of A as being the mapping given by:*

$$A^{-1} : \left| \begin{array}{l} X^* \to 2^X \\ \\ v \mapsto \{u \in X / \ v \in A(u)\} \end{array} \right.$$

Moreover, if $B : X \to 2^{X^*}$, then for every couple of scalars (α, β), we define the **linear combination** *of A and B as being the multivalued mapping given by:*

$$(\alpha A + \beta B)(u) = \left\{ \begin{array}{cl} \alpha A(u) + \beta B(u) & \text{if } u \in D(A) \cap D(B) \\ \emptyset & \text{otherwise} \end{array} \right.$$

One can see from the previous definitions that we get $D(\alpha A + \beta B) = D(A) \cap D(B)$. Furthermore, let us point out that every single-valued map $A : X \to X^*$ can be identified with a multivalued map, setting:

$$\overline{A} : \left| \begin{array}{l} X \to 2^{X^*} \\ \\ u \mapsto \left\{ \begin{array}{cl} \{A(u)\} & \text{if } u \in D(A) \\ \emptyset & \text{otherwise} \end{array} \right. \end{array} \right.$$

We then have that $R\left(\overline{A}\right) = R(A)$ and $D\left(\overline{A}\right) = D(A)$ for the usual definitions of the domain and the range of a single-valued mapping. Also, we point out that we always have by definition that $D\left(A^{-1}\right) = R(A)$.

Definition 1.1.15 (Proper mapping, extension)**.** *Let us consider a reflexive separable Banach space X and denote its topological dual X^*. Also, we consider $A, B : X \to 2^{X^*}$. We say that the operator A is:*

(i) **Proper** *if $D(A) \neq \emptyset$, that is A is not identically equal to ∞*

(ii) An **extension** *of B if $G(B) \subseteq G(A)$*

We are now able to give the main definition of this section.

Definition 1.1.16 (Maximal monotonicity)**.** *Consider a reflexive separable Banach space X, X^* its dual, and $A : X \to 2^{X^*}$. Then, we say that A is a:*

(i) **Monotone** *operator if*

$$\langle u^* - v^*, u - v \rangle \geq 0 \qquad \forall \left((u, u^*), (v, v^*)\right) \in G(A)^2$$

(ii) **Maximal monotone** *operator if it is monotone and it satifies*

$$(u, u^*) \in X \times X^*, \ \forall (v, v^*) \in G(A), \ \langle u^* - v^*, u - v \rangle \geq 0 \Rightarrow (u, u^*) \in G(A)$$

In terms of set theory, the monotonicity (respectively maximal monotonicity) of A is equivalent to the monotonicity (respectively maximal monotonicity) of its graph $G(A)$. We mainly refer to [175, Section 32.2.] for the details.

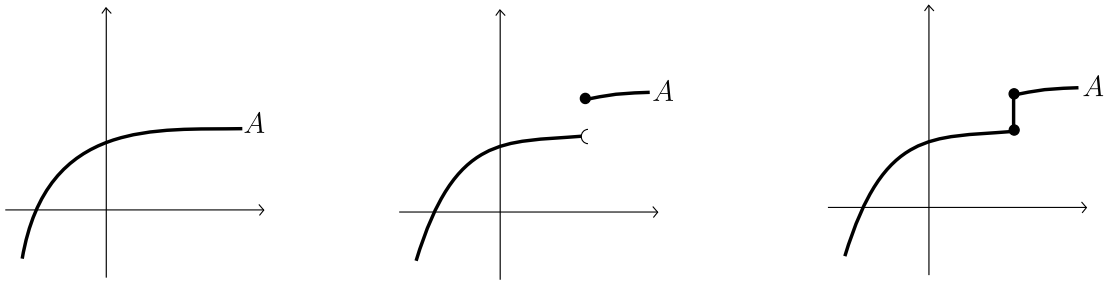


Figure 1.1: From the left to the right, A is respectively monotone and maximal monotone, monotone but not maximal monotone, and maximal monotone.

Let us give some properties concerning multivalued monotone and maximal monotone operators.

Proposition 1.1.17. *Let us consider X a real reflexive separable Banach space, X^* its topological dual, and $A : X \to 2^{X^*}$. Assume also that $B : X \to X^*$ is a monotone operator. Then, the following statements hold.*

(i) A is maximal monotone if and only if its inverse is maximal monotone

(ii) If A is maximal monotone, then for each $u \in X$, the set $A(u)$ is a weakly-$*$ closed convex subset of X^*

(iii) If A is monotone, it admits a maximal monotone extension

(iv) If B is hemicontinuous, then it is maximal monotone

(v) If X is a (real) Hilbert space, then B is maximal monotone if and only if $R(B + I) = X$

Proof. (i) See [175, Proposition 32.5.].
(ii) See [175, Proposition 32.6.].
(iii) It is a matter of applying Zorn's theorem.
(iv) See [175, Proposition 32.7.].
(v) See [175, Proposition 32.8.]. □

Let us emphasize that the temporal derivation operator introduced in Proposition 1.1.12 is not a problem and also fits the multivalued framework.

Proposition 1.1.18 (Maximal monotonicity of time derivative)**.** *The operator $\mathcal{T}$ introduced in Proposition 1.1.12 is maximal monotone.*

Proof. [175, Proposition 32.10.] □

Subdifferentials

To close this section, let us exhibit a prime example of an operator which, under certain conditions, is maximal monotone: the subdifferential of a functional. To express ourselves formally, the notion of subdifferential aims at generalizing the notion of derivative of a convex functional. Let us recall the following result.

Theorem 1.1.19 (Kachurovskii)**.** *Let X be a reflexive separable Banach space, X^* its topological dual. Moreover, let us consider a functional $J : X \to \overline{\mathbb{R}}$ such that $D(J)$ is a convex subset of X and such that it admits a Gâteaux-derivative J' for every $u \in D(J)$. The following statements are equivalent:*

(a) J *is convex.*

(b) $J(v) - J(u) \geq \langle J'(u), v - u \rangle \qquad \forall (u, v) \in D(J)^2.$

(c) $\langle J'(u) - J'(v), u - v \rangle \geq 0 \qquad \forall (u, v) \in D(J)^2.$

Proof. Assume (a). Then one can write for every $t \in (0, 1]$ and every $(u, v) \in D(J)^2$:

$$J(v + t(u - v)) = J(tu + (1 - t)v) \leq tJ(u) + (1 - t)J(v),$$

which in turn leads to, after substracting $J(v)$ from each side and then dividing by $t > 0$:

$$\frac{J(v + t(u - v)) - J(v)}{t} \leq J(u) - J(v).$$

Letting $t \to 0$ into the previous inequality leads to:

$$\langle J'(v), u - v \rangle \leq J(u) - J(v),$$

which is nothing else than (b). Let us now consider that (b) holds, then one can write for every $(u, v) \in D(J)^2$:

$$\begin{aligned} \langle J'(v), u - v \rangle &\leq J(u) - J(v) \\ \langle J'(u), v - u \rangle &\leq J(v) - J(u). \end{aligned}$$

Summing the two inequalities above leads to (c), after reordering the terms. Finally, if we assume that (c) holds, we set the function $\varphi(t) = J(tu + (1 - t)v)$, for $t \in [0, 1]$. Then, we get:

$$\varphi'(t) = \langle J'(v + t(u - v)), u - v \rangle,$$

and moreover for every $s \in [0, t)$:

$$\big(\varphi'(t) - \varphi'(s)\big)\,(t - s) \geq 0.$$

Hence, φ' is a non-decreasing function and it follows from the mean-value theorem that:

$$\frac{\varphi(t) - \varphi(0)}{t} \leq \frac{\varphi(t) - \varphi(1)}{t - 1} = \frac{\varphi(1) - \varphi(t)}{1 - t},$$

this last inequality being equivalent to:

$$\varphi(t) \leq t\varphi(1) + (1 - t)\varphi(0),$$

which in turn can be rewritten:

$$J(tu + (1 - t)v) \leq tJ(u) + (1 - t)J(v),$$

and so J is convex, namely (c) implies (a), and the result is proved. □

Based on the relation in Theorem 1.1.19-(b), it is legitimate to try to find, in the case of functionals which present, for example, a lack of regularity, some elements $u^* \in X^*$ which verify the following relation :

$$J(v) - J(u) \geq \langle u^*, v - u \rangle \qquad \forall v \in X. \tag{1.1.12}$$

This motivates the following definition.

Definition 1.1.20 (Subgradient, subdifferential). *Assume that X is a real reflexive separable Banach space and denote X^* its topological dual. We consider a functional $J : X \to \overline{\mathbb{R}}$.*

(i) An element $u^* \in X^*$ is called a **subgradient** *of J at $u \in X$ if $|J(u)| \neq +\infty$ and if it satisfies* (1.1.12).

(ii) The **subdifferential** *of J at $u \in X$ is the set of all its subgradients at u, and is denoted $\partial J(u)$.*

From a geometric point of view, in $\mathbb{R}^2$ one can visualize the notion of subgradient at $u \in \mathbb{R}$ by noticing that it coincides with the supporting hyperplanes of the epigraph of J at u.

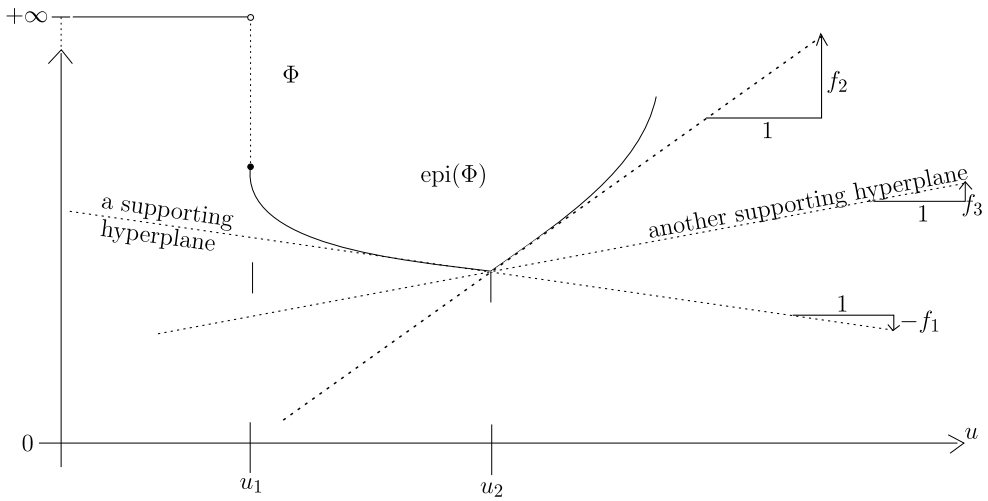


Figure 1.2: Subdifferential of a convex lower semi-continuous function; $\partial\Phi(u_1) = \emptyset$, $f_3 \in [f_1, f_2] = \partial\Phi(u_2)$.

The following proposition shows that the generalization we have made does make sense.

Proposition 1.1.21. *Let us consider a reflexive separable Banach space X, X^* its topological dual, and a functional $J : X \to \mathbb{R}$. The following statements hold.*

(i) Assume that J is convex and admits a Gâteaux-derivative at $u \in X$. Then, we have: $\partial J(u) = \{J'(u)\}$.

(ii) If the subdifferential $\partial J : X \to X^*$ is single-valued and hemicontinuous, then J is Gâteaux-differentiable over X and for all $u \in X$, we have $\partial J(u) = \{J'(u)\}$.

Proof. Let us prove (i), that is we assume that J has a Gâteaux-derivative at $u \in X$. Then, since J is convex, one can apply Kachurovskii Theorem 1.1.19 and write for every $v \in X$:

$$J(u+v) - J(u) \geq \langle J'(u), v\rangle,$$

namely $J'(u) \in \partial J(u)$. Conversely, let us consider $u^* \in \partial J(u)$, then we can write by definition, for every $v \in X$ and $t > 0$:

$$J(u+tv) - J(u) \geq \langle u^*, tv\rangle.$$

Dividing the above inequality by $t > 0$, and then taking the limit as $t \to 0$ leads to $\langle J'(u), v\rangle \geq \langle u^*, v\rangle$ for all $v \in X$, that is setting $-v$ instead of v leads to the equality, for all $v \in X$:

$$\langle J'(u), v\rangle = \langle u^*, v\rangle.$$

Thus, $J'(u) = u^*$ and the result follows.

Let us now prove (ii). Since J is assumed to be single-valued, we can write for every $(u, v) \in X^2$ and $t > 0$:

$$\begin{aligned} J(u+tv) - J(u) &\geq \langle \partial J(u), v\rangle, \\ J(u) - J(u+tv) &\geq -\langle \partial J(u+tv), tv\rangle. \end{aligned}$$

Now, using both inequalities above, the hemicontinuity of ∂J leads to, for all $v \in X$:

$$\langle \partial J(u), v\rangle \leq \underline{\lim}_{t\to 0} \frac{J(u+tv) - J(u)}{t} \leq \overline{\lim}_{t\to 0} \frac{J(u+tv) - J(u)}{t} \leq \lim_{t\to 0} \langle \partial J(u+tv), v\rangle = \langle \partial J(u), v\rangle.$$

Hence, we get that $J'(u)$ does exist and satisfies $J'(u) = \partial J(u)$, which is the desired result. □

As we said before, the subdifferential of a functional can provide a good example of a maximal monotone operator, this is the object of the following theorem.

Theorem 1.1.22 (Brønsted-Rockafellar)**.** *Let X be a real reflexive separable Banach space and denote X^* its topological dual. We consider a lower semicontinuous convex functional $J : X \to (-\infty, +\infty]$ such that $D(J) \neq \emptyset$. Then, its subdifferential $\partial J : X \to 2^{X^*}$ is a maximal monotone operator.*

The proof of Theorem 1.1.22 is rather hard and is derived from the results of [34] to which we refer the interested reader.

Example 1.1.23 (Duality map)**.** *Let us give an example of a commonly used maximal monotone operator. Consider a real reflexive separable Banach space X and the functional:*

$$\varphi : \left| \begin{array}{l} X \to \mathbb{R} \\ \\ u \mapsto \frac{1}{2}\|u\|_X^2. \end{array} \right.$$

Then, the **duality map** *of φ is defined as being:*

$$J : \left| \begin{array}{l} X \to 2^{X^*} \\ \\ u \mapsto \partial\varphi(u) \end{array} \right.$$

If X is only supposed to be a real Banach space, then the duality map is a maximal monotone operator (see e.g. [175, Proposition 32.21]). Furthermore, under our assumptions and assuming moreover that X^* is strictly convex, then the duality map J is a bounded single-valued demicontinuous coercive maximal monotone operator (see e.g. [175, Proposition 32.22 (a)]).

Let us emphasize that the sum of two monotone maximal operators is not necessarily a monotone maximal operator anymore. The following theorem provides a very convenient condition to ensure additive stability.

Theorem 1.1.24 (Rockafellar). *Let us consider a reflexive separable Banach space X, X^* its dual and $A, B : X \to 2^{X^*}$ two maximal monotone operators. Then if $D(A) \cap D(B) \neq \emptyset$, $A + B$ is maximal monotone.*

Proof. We refer to [15, Theorem 2.6.] or [174, Theorem 47.F]. □

As we have seen, a single-valued operator can always be identified with a multivalued operator. A natural question is then to know if, from a monotone single-valued operator, it is possible to construct a maximal monotone operator. This result holds on real Hilbert spaces and, together with Theorem 1.1.24, allows to use the full potential of the theory of maximal monotone operators. It is the subject of the following proposition.

Proposition 1.1.25. *Let X be a real reflexive separable Banach space, X^* its topological dual, and H a real Hilbert space such that:*

$$X \subseteq H \subseteq X^*$$

Moreover, let us consider a bounded hemicontinuous monotone operator $A : X \to 2^{X^*}$, a convex lower semicontinuous functional $\varphi : X \to (-\infty, +\infty]$, such that $D(\varphi) \neq \emptyset$, and assume that there exists $u_0 \in X$ such that $\partial\varphi(u_0) \neq 0$ and:

$$\frac{\langle A(u), u - u_0\rangle_{X^* \times X}}{\|u\|_X} \underset{\|u\|_X \to +\infty}{\longrightarrow} +\infty.$$

Then, the mapping :

$$B : \left| \begin{array}{l} H \to 2^H \\ \\ u \mapsto \left\{ \begin{array}{cc} \{A(u) + \partial\varphi(u)\} \cap H & \text{for } u \in X \\ \emptyset & \text{for } u \in H \backslash X \end{array} \right. \end{array} \right. \tag{1.1.13}$$

is maximal monotone.

Proof. We refer the interested reader to [174, Chapter 55]. □

We end this section with a major result in the framework of multivalued theory, related to the existence of solutions for equations defined in this framework, which is the main objective of this last. First, let us point out that if X is, as usual, a real reflexive separable Banach space, and X^* its topological dual, then considering $A : X \to 2^{X^*}$ a maximal monotone operator, then the equation $A(u) = f$ does not a priori make sense on set-valued parts. For this reason, we generally denote $f \in A(u)$ instead. This being the case, we will sometimes abuse the notation, when it is not confusing, to use the writing in equation form.

Theorem 1.1.26 (Browder). *Let us consider a real reflexive separable Banach space X with its topological dual denoted X^*, $\Omega \subseteq X$ a nonempty closed convex subset of X, $A : \Omega \to 2^{X^*}$ a maximal monotone operator, and $B : \Omega \to X^*$ a bounded and demicontinuous pseudomonotone operator. Then the problem:*

$$f \in A(u) + B(u), \qquad f \in X^*$$

has a solution.

The proof of Theorem 1.1.26 is beyond the scope of our study and we refer the interested reader to [175, Theorem 32.A].

1.1.2 Variational inequalities and applications

We saw with Theorem 1.1.19 that in the framework of convex analysis, a characterization of the convexity of a functional $J : X \to \mathbb{R}$ was the inequality:

$$J(v) - J(u) \geq \langle J'(u), v - u \rangle \qquad \forall (u, v) \in D(J)^2. \tag{1.1.14}$$

This same theorem shows that in this framework, it is still equivalent to the monotonicity of the Gâteaux-derivative J' as an operator. In the previous section, we have seen how the different modes of monotonicity can lead to the existence of solutions for general problems. An alternative approach is given by that of variational inequalities, which generalize in some way the principle of inequalities of the same type as that of (1.1.14). The objective of this approach is to be able to apply the principles of convex analysis to functionals which are not convex a priori, but also to be able to circumvent the problems linked to a lack of regularity or to a multivaluation. In this section, we first briefly introduce the principle of variational inequality and some properties motivating its use. Then, in order to further motivate the close link between variational inequalities and monotone operators, we emphasize the connection between potential operators and the calculus of variations. Finally, we will cite some results that demonstrate the success of using variational inequalities in optimal control theory.

Definition and basic properties

One of the first theorems motivating the development of the theory is the following.

Theorem 1.1.27 (J.L.Lions-Stampacchia)**.** *Consider a real Hilbert space H, H^* its dual, and Ω a nonempty closed convex subset of H. Also, we assume that $a : H \times H \to \mathbb{R}$ is bilinear. Then, the following statements hold.*

(i) If a is coercive, then for each $w \in H^*$ there exists a unique $u \in \Omega$ such that:

$$a(u, v - u) \geq \langle w, v - u \rangle \qquad \forall v \in \Omega \tag{1.1.15}$$

Moreover, the map:

$$\psi : \left| \begin{array}{l} H^* \to \Omega \\ \\ w \mapsto u \end{array} \right.$$

is continuous.

(ii) If a is not coercive, but satisfies the non negativity condition $a(v, v) \geq 0$ for all $v \in \Omega$, then the set of solutions is a closed convex subset of H (but can be empty).

Proof. See [137] and [161, Theorem 2.3., Section I.2.]. □

There is no absolute consensus about the definition of a variational inequality: it is more an idea to come back to a result comparable to the Theorem 1.1.27. In this sense, such an inequality is generally defined as an inequality similar to (1.1.14) or (1.1.15), involving an unknown, a test function, and some auxiliary terms. However, in order to make the presentation as clear as possible, let us give a very general definition of what we can consider as a variational inequality, making it clear that this definition will not be of any use other than for the structure of the presentation.

Definition 1.1.28 (Variational type inequality). *Let us assume that X and Y are two real reflexive separable Banach spaces, denote X^* the topological dual of X, Ω a subset of X, and consider $A : \Omega \to 2^{X^*}$, $B : \Omega \to Y$. Moreover, consider Z another real reflexive separable Banach space, U a subset of Z and $F : U \times Y \to \mathbb{R}$. Then, a variational inequality with unknown $u \in \Omega$ is an inequality of the form:*

$$\langle A(u), v - u\rangle + F\left(y, B(u)\right) \geq 0 \qquad (y, v) \in U \times \Omega \tag{1.1.16}$$

Usually, one could think of $Y = \mathbb{R} \times \mathbb{R}^N$, $U = [0, T] \times \tilde{\Omega}$, A as a differential operator and B as an operator leading to some norms or seminorms of u in the functional spaces involved. We point out that when the subset Ω into Definition 1.1.28 depends of the unknown u, the inequality is generally called **quasivariational**, but we will not use this designation throughout our presentation.

As previously mentioned, there are close links between operators with a monotonicity mode and variational inequalities, which are mainly a very efficient practical tool for the analysis associated with this type of operators. To further highlight this connection, we are interested in the study of evolutionary equations. As a first step, let us mention that if H is a real separable Hilbert space, $A : H \to 2^H$ is a maximal monotone operator, and $f \in W^{1,2}([0, T], H)$, then the problem:

$$\begin{cases} f \in \partial_t u + A(u) - cu \\ u_{|t=0} = u_0 \end{cases} \tag{1.1.17}$$

where $t \in [0, T]$ and $c \in \mathbb{R}$, has a solution. The proof of this result comes from the so called nonlinear Yosida approximation, which in turn is related to the notion of resolvent and accretive operator. These notions not being necessary for our presentation, we refer the interested reader to [15, Chapter 3] or [174, Sections 55.1 and 55.2].

Finally, let us give a classical result concerning quadratic-type evolution variational inequalities.

Proposition 1.1.29. *Let us consider a real reflexive separable Banach space X, X^* being its topological dual, and H a real separable Hilbert space such that:*

$$X \subseteq H \subseteq X^*$$

Moreover, let us consider a bounded bilinear form $a : X \times X \to \mathbb{R}$, a convex lower semicontinuous functional $\varphi : X \to (-\infty, +\infty]$ with $D(\varphi) \neq \emptyset$, $u_0 \in X$ and $f \in W^{1,2}([0, T], H)$ satifying:

(i) $\exists(\alpha, \beta) > 0,\ \forall v \in X,\ a(v, v) \geq \alpha\|v\|_X^2 - \beta\|v\|_H^2$

(ii) $\varphi(u_0) < +\infty$

(iii) $\forall w \in H,\ \forall v \in X,\ a(u_0, v - u_0) + \varphi(v) - \varphi(u_0) \geq (w, v - u_0)_H$

where we denoted $(\cdot, \cdot)_H$ the inner product over H. Then, the variational inequality:

$$\forall v \in X,\ (\partial_t u, v - u)_H + a(u, v - u) - \langle f, v - u\rangle_{X^* \times X} + \varphi(v) \geq \varphi(u) \tag{1.1.18}$$

admits a solution $u \in W^{1,2}((0, T], H)$ satisfying $u_{|t=0} = u_0$.

Proof. Use theorems above and [173, Proposition 55.5]. □

Monotonicity and potential

In the words of the late Jacques-Louis Lions "The main application that [Guido] Stampacchia had in mind at the beginning of this theory [of variational inequalities] was to potential theory" (see [136]). It is therefore natural, in view of the history of the theory of variational inequalities as well as Theorem 1.1.19, to introduce the notion of potential operator and the formulation in variational inequality which is linked to it.

More precisely, if X is a real reflexive separable Banach space, we would like to solve an equation of type

$$A(u) = f$$

where $A : X \to X^*$ is an operator following some mode of monotonicity. Thus, we are interested in Gâteaux-differentiable functionals $J : X \to \mathbb{R}$ verifying the equation

$$J'(u) = A(u) - f$$

i.e. we are looking to investigate the validity of the relation

$$J'(u) = 0,$$

which is equivalent to the issue to find the critical points of J. This problem, under the assumption of sufficient regularity, is equivalent to minimizing the functional J. The theory of critical points, consisting in the study of such problems, is one of the fundamental aspects of the variational analysis of partial differential equations. We refer the interested reader to [84, Chapter 8], [121] or [164] for an introduction to the different methods applicable in this context. This section is devoted to the study of the link between variational inequalities and the existence of a potential operator. As before, we first present the single-valued framework and then return to the multivalued theory. This motivates the following notion.

Definition 1.1.30 (Potential operator)**.** *Let us consider a real reflexive separable Banach space X, X^* its topological dual and $A : X \to X^*$. Then, A is said to be a* **potential operator** *if there exists a Gâteaux-differentiable functional $J : X \to \mathbb{R}$, called the potential, such that $A = J'$.*

Example 1.1.31 (p-Laplacian)**.** *The p-Laplacian operator, given by* (1.1.2)*, is a potential operator for $1 < p$, whose potential being the functional:*

$$J_{\Delta_p} : \left| \begin{array}{l} W_0^{1,p}(\Omega) \to \mathbb{R} \\ \\ u \mapsto \frac{1}{p} \displaystyle\int_\Omega |\nabla u|^p \, dx \end{array} \right. \tag{1.1.19}$$

Example 1.1.32 (Symmetrized p-Laplacian)**.** *The symmetrized p-Laplacian, defined as:*

$$S\Delta_p : \left| \begin{array}{l} W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega) \\ u \mapsto \operatorname{div}\left(|D(u)|^{p-2} D(u)\right), \end{array} \right. \tag{1.1.20}$$

is well-defined in view of Section A.3 and is a potential operator associated to the potential:

$$J_{S\Delta_p} : \left| \begin{array}{l} W_0^{1,p}(\Omega) \to \mathbb{R} \\ \\ u \mapsto \frac{1}{p} \displaystyle\int_\Omega |D(u)|^p \, dx. \end{array} \right. \tag{1.1.21}$$

First, let us establish a link between the convexity of the potential and its associated operator.

Proposition 1.1.33. *Let us assume that X is a real reflexive separable Banach space, X^* is its dual, and $A : X \to X^*$ is a potential operator, whose potential being the functional $J : X \to \mathbb{R}$. Then the following statements are equivalent.*

(i) The potential J is convex (respectively strictly convex).

(ii) A is monotone (respectively strictly monotone).

Proof. Actually, it is simply a reformulation of a part of Kachurovskii theorem 1.1.19, since assuming that J is convex, it leads to the fact that for every $(u, v) \in X^2$:

$$\langle J'(u) - J'(v), u - v\rangle,$$

namely J' is monotone. Equivalently, if the previous inequality is satisfied, then J is convex, once again from Theorem 1.1.19. □

Let us now express a first link between the potential minimization problem, the associated Euler-Lagrange equation (characterizing its critical points), and the variational inequality it generates.

Proposition 1.1.34. *Let X be a real reflexive separable Banach space, denote X^* its topological dual, Ω be a subset of X, and consider a potential operator $A : \Omega \to X^*$ with potential $J : \Omega \to \mathbb{R}$. Moreover, assume that u is a solution to the problem:*

$$J(u) = \min_{v\in\Omega} J(v). \tag{1.1.22}$$

Then, the following statements hold:

(i) If Ω is open, the u solves: $A(u) = 0$.

(ii) If Ω is convex, then u is a solution to the variational inequality:

$$\langle A(u), v - u\rangle \geq 0 \qquad \forall v \in \Omega.$$

Proof. Let us first prove (ii), namely we assume that Ω is convex. Then, if $\overline{u}$ is a minimum of J, we get that it is attained while we set $t = 0$ in $J(\overline{u} + t(v - \overline{u}))$. Differentiating this last at $t = 0$, we get that:

$$\langle J'(\overline{u} + t(v - \overline{u})), v - \overline{u}\rangle_{|t=0} = \langle J'(\overline{u}), v - \overline{u}\rangle = \langle A(\overline{u}), v - \overline{u}\rangle \geq 0,$$

and the assertion is proved. Now let us turn to prove (i). It is a matter of observing that one can take an arbitrary v in a neighborhood of $\overline{u}$, thus take $-v$ instead of v leads to the result. □

This result can be clarified when we strengthen the assumptions. Specifically, we have:

Proposition 1.1.35. *Let us consider a real reflexive separable Banach space X, X^* being its topological dual, $\Omega \subseteq X$, $A : \Omega \to X^*$ a potential operator, whose potential being the functional $J : \Omega \to \mathbb{R}$. Furthermore, assume that J is twice Gâteaux-differentiable, and there exists $\psi : [0, +\infty) \to [0, +\infty)$ verifying $\psi(t) \underset{t\to+\infty}{\longrightarrow} +\infty$ such that :*

$$J''(u)(h, h) \geq \psi(\|h\|)\|h\| \qquad \forall (u, h) \in \Omega^2.$$

Then, if $\Omega = X$:

(i) There exists at least a minimum of J over X.

(ii) $u \in X$ is a minimum of J over X if and only if it is a critical point of J.

(iii) If ψ is increasing, A is a uniformly monotone operator.

(iv) If furthermore $\psi > 0$ on $[0, +\infty)$, then A is strictly monotone over X and the minimum is unique.

Moreover, if Ω is a nonempty closed convex subset of X, then u is a minimum of the potential J if and only if it satifies the variational inequality

$$\langle A(u), v - u \rangle \geq 0 \qquad \forall v \in \Omega.$$

Proof. We refer to [175, Proposition 25.22 and Corollary 25.23] for a proof of this result. □

In variational analysis, it is common to try to show that the functional to be minimized is sequentially weakly lower semicontinuous. For a potential operator, it is sometimes possible to show, depending on the assumptions, that this is the case for the potential associated to it. This is the subject of the following result.

Proposition 1.1.36. *Let us consider a real reflexive separable Banach space X, its topologial dual X^*, and a bounded pseudomonotone potential operator $A : X \to X^*$ with potential $J : X \to \mathbb{R}$. If one of the following assumption is fulfilled:*

(a) A is monotone,

(b) A is locally bounded and pseudomonotone,

(c) A is bounded demicontinuous and satisfies $u_n \to u$ in X for every $(u_n)_{n\in\mathbb{N}}$ such that:

$$u_n \underset{n\to+\infty}{\rightharpoonup} u \qquad \text{and} \qquad \overline{\lim_{n\to+\infty}} \langle A(u_n) - A(u), u_n - u \rangle \leq 0,$$

(d) J is convex and lower semicontinuous,

(e) J is twice Gâteaux-differentiable and satisfies $J''(u)(h, h) \geq 0$ for every $(u, h) \in X^2$,

then the potential J is sequentially weakly lower semicontinuous.

Proof. First, let us observe that to show points (a), (b), and (c), it suffices to show the following property: whatever the sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \underset{n\to+\infty}{\rightharpoonup} u$ where $u \in X$, then we have:

$$\overline{\lim_{n\to+\infty}} \langle A(u_n), u_n - u \rangle \geq 0.$$

and A is locally bounded.

Assuming that A is monotone, we can write for $u_n \underset{n\to+\infty}{\rightharpoonup} u$:

$$\langle A(u_n), u_n - u \rangle \geq \langle A(u), u_n - u \rangle,$$

and the result follows by taking the upper limit in the previous inequality, and observing that A is locally bounded according to the Proposition 1.1.6 (ii).

If A is pseudomonotone, the result is obtained by noting that for such a sequence u_n satisfying the criterion for defining pseudomonotonicity, we have for all $v \in X$:

$$\overline{\lim_{n\to+\infty}} \langle A(u_n), u_n - u\rangle \leq 0,$$
$$\langle A(u), u - v\rangle \leq \underline{\lim_{n\to+\infty}} \langle A(u_n), u_n - v\rangle.$$

By setting $u = v$ in the second inequality, we then obtain:

$$\langle A(u_n), u_n - u\rangle \underset{n\to+\infty}{\longrightarrow} 0,$$

which also establishes the condition, since we assume that A is locally bounded.

The case of (c) implies that A is pseudomonotone, since we get for $(u_n)_{n\in\mathbb{N}}$ such that the assumptions of (c) are fulfilled:

$$\overline{\lim_{n\to+\infty}} \langle A(u_n) - A(u), u_n - u\rangle = \overline{\lim_{n\to+\infty}} \langle A(u_n), u_n - u\rangle \leq 0,$$

which implies $u_n \underset{n\to+\infty}{\longrightarrow} u$ in X, so by demicontiuity $A(u_n) \underset{n\to+\infty}{\rightharpoonup} A(u)$, which in turn leads to, for every $v \in X$:

$$\langle A(u_n), u_n - v\rangle \underset{n\to+\infty}{\longrightarrow} \langle A(u), u - v\rangle,$$

so that A is bounded pseudomonotone and we got the wished property.

Thus, let us prove that the announced property implies the result. We proceed by contradiction, namely we assume that J is not weakly sequentially lower continuous at $u \in X$. Then there exists a constant $\gamma > 0$ and a sequence $(u_n)_{n\in\mathbb{N}}$ in X such that $u_n \underset{n\to+\infty}{\rightharpoonup} u$ in X, satisfying:

$$\begin{aligned} J(u_n) - J(u) &< -\gamma, \\ \lim_{n\to+\infty} J(u_n) - J(u) &\leq -\gamma. \end{aligned} \tag{1.1.23}$$

Applying the mean value theorem to the mpping $t \mapsto J(u + tv)$ where $t \in [0, 1]$, for every fixed $v \in X$, we get that there exists $\theta \in (0, 1)$ satisfying:

$$J(u + v) - J(u) = \langle A(u + \theta v), v\rangle.$$

Combining this fact with (1.1.23) leads for $\varepsilon > 0$ to:

$$J(u_n) - J(u + \varepsilon(u_n - u)) + J(u + \varepsilon(u_n - u)) - J(u) < -\gamma,$$

and the existence of a sequence $(\theta_n(\varepsilon))_{n\in\mathbb{N}}$ in $(0, 1)$, such that:

$$J(u + \varepsilon(u_n - u)) - J(u) = \varepsilon\langle A(u + \varepsilon\theta_n(\varepsilon)(u_n - u)), u_n - u\rangle.$$

Then, we highlight that $(u_n - u)_{n\in\mathbb{N}}$ is bounded since $u_n \underset{n\to+\infty}{\rightharpoonup} u$, and the local boundedness of A at $u \in X$ leads, for ε small enough, to:

$$|J(u + \varepsilon(u_n - u)) - J(u)| \lesssim \varepsilon,$$

this inequality being true for all $n \in \mathbb{N}$. Following the same line of arguments, we can get for $\varepsilon > 0$ small enough:

$$|J(u_n) - J(u + \varepsilon(u_n - u))| \lesssim \varepsilon,$$

and thus using (1.1.23) to this last leads to the existence of a sequence $(\beta_n(\varepsilon))_{n\in\mathbb{N}}$ in $(0,1)$ such that:

$$\begin{aligned}&\langle A(u_n + \beta_n(\varepsilon)(1-\varepsilon)(u_n - u)), (1-\varepsilon)(u_n - u)\rangle \\ &\qquad = \left(\frac{1-\varepsilon}{1+\beta_n(\varepsilon)(1-\varepsilon)}\right)\langle A(u_n + \beta_n(\varepsilon)(1-\varepsilon)(u_n - u)), u_n - u + \beta_n(\varepsilon)(1-\varepsilon)(u_n - u)\rangle.\end{aligned}$$

Now, we have by assumption that

$$u_n + \beta_n(\varepsilon)(1-\varepsilon)(u_n - u) \underset{n\to+\infty}{\rightharpoonup} u,$$

and so we get that:

$$\begin{aligned}&\langle A(u_n + \beta_n(\varepsilon)(1-\varepsilon)(u_n - u)), u_n - u + \beta_n(\varepsilon)(1-\varepsilon)(u_n - u)\rangle \\ &\qquad = \langle A(u_n + \beta_n(\varepsilon)(1-\varepsilon)(u_n - u)), u_n - u + \beta_n(\varepsilon)(1-\varepsilon)(u_n - u)\rangle < 0,\end{aligned}$$

which is a contradiction with the assumptions. Hence J is weakly sequentially lower semicontinuous.

To prove (d), we also argue by contradiction. Let us assume that J is not weakly sequentially lower semi-continuous, hence there exists $(u_n)_{n\in\mathbb{N}}$ in X such that $u_n \underset{n\to+\infty}{\rightharpoonup} u$ and so that there are $\mu > 0$ and $n_0 \in \mathbb{N}$ satisfying for every $n \geq n_0$:

$$J(u) > \mu \geq J(u_n),$$

which implies:

$$J(u) > \mu \geq \underline{\lim}_{n\to+\infty} J(u_n),$$

and so this is in contradiction with the fact that the set $\{u \in X \ / \ J(u) \leq \mu\}$ is weakly relatively closed. Hence, J is weakly lower continuous.

The last point, (d), follows from the fact that it implies that J is a convex potential, and so from Proposition 1.1.33 A is a monotone operator, then (a) leads to the result. □

Let us exhibit an interesting property, highlighting the interest of weak coercivity in the variational analysis of potential operators. The following two theorems establish this relationship, the first with the monotonicity and the second with the pseudomonotonicity of the operator.

Theorem 1.1.37. *Let us consider a real reflexive separable Banach space X, X^* its topological dual, and a monotone potential operator $A : X \to X^*$ with weakly coercive potential $J : X \to \mathbb{R}$. Then, $u \in X$ is a minimum of J over X if and only if it is a critical point of J, and there exists at least a minimum of J. Furthermore, if A is strictly monotone, then the minimum is unique.*

Proof. We refer to [175, Theorem 25.F.]. □

Theorem 1.1.38. *Let us consider a real reflexive separable Banach space X, X^* its topological dual, and a bounded pseudomonotone potential operator $A : X \to X^*$ with C^1 weakly coercive potential $J : X \to \mathbb{R}$. Then, $u \in X$ is a minimum of J over X if and only if it is a critical point of J, and there exists at least a minimum of J.*

Proof. We refer to [175, Theorem 25.G.]. □

Let us return to the motivation explained at the beginning of this section, the Theorem 1.1.19, by establishing an alternative version of this result in the framework of monotone potential operators.

Theorem 1.1.39. *Assume that X is a real reflexive separable Banach space, denote X^* its topological dual, and consider $A : X \to X^*$. The following statements are equivalent:*

(a) A is a monotone potential operator.

(b) A is monotone hemicontinuous and satisfies:

$$\int_0^1 \langle A(tu), u\rangle \, dt - \int_0^1 \langle A(tv), v\rangle \, dt = \int_0^1 \langle A(v + t(u - v)), u - v\rangle \, dt \qquad \forall (u, v) \in X^2.$$

(c) A is a potential operator whose potential $J : X \to \mathbb{R}$ is a weakly lower semicontinuous convex functional.

Proof. We refer to [174, Proposition 42.11]. □

Before establishing the theory in a multivalued framework, let us state without proof the following result.

Proposition 1.1.40. *Assume that X is a real reflexive separable Banach space, denote X^* its topological dual, and consider a monotone potential operator $A : X \to X^*$. Then A is demicontinuous.*

Let us now very briefly introduce the multivalued framework for potential operators. First of all, let us note that the multivalued theory for such operators can be approached by several aspects, with different generalizations of the notion of differentiation of a functional. A lot of work has been done in this sense, and although it could possibly be fruitful in the framework of the mathematical analysis of non-Newtonian fluids, it is impossible for us to mention all of them here, which would force us to deviate from the research that has been done so far. However, we refer the interested reader to [12], [10], or [63] for some alternative approaches.

Definition 1.1.41 (Superpotential operator)**.** *Let X be a reflexive separable Banach space, X^* its topological dual, and $A : X \to 2^{X^*}$. We say that A is a* **superpotential** *operator if there exists a functional $J : X \to \overline{\mathbb{R}}$ such that $A = \partial J$.*

Example 1.1.42 (1-Laplacian)**.** *The case of the 1-Laplacian, being the extremal case of the p-Laplacian when $p = 1$, is not a potential operator in sense of Definition 1.1.30. However, it can be shown that it is a potential operator in sense of Definition 1.1.41, associated to the functional:*

$$J_{\Delta_1} : \left| \begin{array}{l} L^2(\Omega) \to \overline{\mathbb{R}} \\ \\ u \mapsto \left\{ \begin{array}{ll} V(u, \Omega) + \displaystyle\int_{\partial\Omega} |\gamma_0(u)| \, d\sigma & \text{if } \ u \in BV(\Omega) \cap L^2(\Omega) \\ +\infty & \text{otherwise ,} \end{array} \right. \end{array} \right. \tag{1.1.24}$$

where

$$V(u,\Omega) := \sup\left\{ s \in \mathbb{R}/s = \int_\Omega u \operatorname{div}(\varphi)\, dx,\ \varphi \in C_0^1(\Omega,\mathbb{R}^N), \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\}$$

is the total variation of u *over* $\Omega \subset \mathbb{R}^N$, $BV(\Omega) := \left\{ u \in L^1(\Omega)/V(u,\Omega) < +\infty \right\}$ *is the space of functions with bounded variations over* Ω, γ_0 *is the trace on the boundary and* σ *the Hausdorff measure. Then,* ∂J_{Δ_1} *is not everywhere well defined over its domain* $D(J_{\Delta_1})$*, but is a potential maximal monotone operator. Let us point out that the* 1*-Laplacian is not generally well-defined over* $W^{1,1}(\Omega)$*, even if there is some approches in that case. We refer the interested reader to [17, 83, 66, 65, 67] for an overview of this operator. Moreover, we point out that the operator:*

$$J_{\Delta_1} : \left| \begin{array}{l} H_0^1(\Omega) \cap H^2(\Omega) \to H^{-1}(\Omega) + H^{-2}(\Omega) \\ \\ u \mapsto \Delta u + \Delta_1 u \end{array} \right. \tag{1.1.25}$$

is well defined (see [16, Section 4.3.]).

Example 1.1.43 (Symmetrized 1-Laplacian)**.** *The same conclusions holds for the symmetrized* 1*-Laplacian. However, since from Proposition A.3.9 we get that* $BV(\Omega) \neq BD(\Omega)$*, we must replace in the previous example* $BV(\Omega)$ *by* $BD(\Omega)$*.*

In general, we use the term potential operator to refer to both potential and superpotential operators. Once again, a natural question is whether, like Theorem 1.1.19, there is a link between the convexity of the potential and the monotonicity of the operator. The following result is available for this purpose.

Theorem 1.1.44. *Let* X *be a real reflexive separable Banach space, with topological dual given by* X^**, and let* $A : X \to 2^{X^*}$ *be a potential operator (in sense of Definition 1.1.41), such that its potential* $J : X \to \overline{\mathbb{R}}$ *is a proper convex functional. Then the following statements hold:*

(i) A *is a closed-valued, convex-valued, and monotone operator.*

(ii) If J *is lower semicontinuous, then* A *is maximal monotone.*

(iii) If J *is coercive,* A *is onto.*

Proof. See [156, Theorem 5.3.]. We point out that (iii) can be proved using the duality map introduced earlier as an example. □

As previously mentioned, one of the interests of the theory presented so far is to be able to establish results in an a priori non-convex framework. As an example, let us mention the following result.

Theorem 1.1.45. *Consider* X *a real reflexive separable Banach space, its topological dual* X^**,* $A : X \to 2^{X^*}$ *and* $B : X \to X^*$ *two potential operators, respectively associated to* $J_A : X \to \overline{\mathbb{R}}$ *and* $J_B : X \to \mathbb{R}$*. If* J_A *is a proper convex lower semicontinuous functional and* J_B *a Gâteaux-differentiable and weakly lower semicontinuous functional, then for every* $f \in X^*$*, there exists a solution* $u \in X$ *to the variational inequality:*

$$J_A(v) + \langle B(u), u - v \rangle \geq J_A(u) + \langle f, v - u \rangle \qquad \forall v \in X,$$

that is $A + B$ *is onto.*

Proof. See [156, Theorem 5.4.]. □

We have tried to make the close relationship between variational inequalities and operators presenting some mode of monotonicity as clear as possible for the reader. However, a major advantage of variational inequalities is that they allow us to considerably relax the notion of solution, by making it possible to give meaning to solutions as soon as a certain type of limit exists. In this way, a number of solutions in the form of variational inequalities can be derived from a potential, but not in the sense of a classical subdifferential as previously presented. A classic example is given by the following approach.

Definition 1.1.46 (Clarke gradient, hemivariational inequality)**.** *Let X be a real reflexive separable Banach space, X^* its topological dual, and consider a functional $J : X \to \overline{\mathbb{R}}$. Then, we define the* **Clarke gradient** *as being the set:*

$$\partial_C J(u) := \left\{ f \in X^*,\ \forall v \in X,\ \overline{\lim_{\substack{w \to u \\ \varepsilon \to 0 \\ \varepsilon > 0}}} \left(\frac{J(w + \varepsilon v) - J(w)}{\varepsilon} \right) \geq \langle f, v \rangle \right\}.$$

A multivalued inclusion involving a Clarke gradient is called an **hemivariational inequality**.

As with quasivariational inequalities, hemivariational inequalities will be referred to as variational inequalities if the case arises.

Applications to optimal control theory

Variational inequalities lend themselves particularly well to the study of controllability problems. This is due in particular to the possibility of establishing such solutions under very relaxed assumptions. In particular, as far as optimal control is concerned, it is worth noting that assumptions on the cost function may sometimes not require differentiability at all (see e.g. [94, 95]). To illustrate this point, let us put forward the following example of optimal control for a parabolic equation.

Consider the problem, for $\Omega \subset \mathbb{R}^N$, $\psi \in H^2(\Omega)$ and $f \in L^2(\Omega)$:

$$\begin{cases} B(u) + f \in \partial_t u + A(u) + M(u - \psi) & \text{on } (0,T) \times \Omega \\ u_{|t=0} = u_0 & \text{on } \Omega, \end{cases} \tag{1.1.26}$$

where the following assumptions are satisfied:

(PC1) There is an Hilbert space V with topological dual V^* such that the following inclusions hold:

$$V \subset\subset L^2(\Omega) \subset V^*.$$

(PC2) $A : D(A) \subset V \to V^*$ is a symmetric linear continuous operator such that there exists $(\alpha, \beta) \in (0, +\infty) \times \mathbb{R}$ satisfying:

$$\langle A(u), u \rangle \geq \alpha \|u\|_V^2 - \beta \|u\|_{L^2(\Omega)}^2 \qquad \forall u \in V.$$

(PC3) M is a potential operator, whose potential being given by $J : \mathbb{R} \to \overline{\mathbb{R}}$. Moreover, defining the sequence $(M_\varepsilon)_{\varepsilon > 0}$ as being:

$$M_\varepsilon : \left| \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ \\ t \mapsto \frac{t - (\mathrm{Id} + \varepsilon M)^{-1}(t)}{\varepsilon} \end{array} \right.$$

we assume that for every monotone increasing function $\Lambda \in C^1(\mathbb{R})$ such that $\Lambda(0) = 0$, there exists a positive constant C which does not depend of ε such that the following inequalities hold:

$$\langle A(u), M_\varepsilon(u-\psi)\rangle \geq -C\left(1 + \|M_\varepsilon(u-\psi)\|_{L^2(\Omega)}\right)(1 + \|u\|_{L^2(\Omega)}),$$
$$\langle A(u), \Lambda \circ u\rangle \geq -C\left(1 + \|\Lambda \circ u\|_{L^2(\Omega)}\right)(1 + \|u\|_{L^2(\Omega)}),$$

for all $\varepsilon > 0$ and $u \in D_2(A)$, where $D_2(A)$ is the domain of A identified as an operator over $L^2(\Omega)$.

(PC4) B is a linear continuous operator from V to $L^2(\Omega)$.

Next, we give ourselves a cost functional defining the optimal control problem.

$$G : \left| \begin{array}{l} D(G) \subset D(A) \times L^2((0,T),V) \to \overline{\mathbb{R}} \\ \\ (u,v) \longmapsto \int_\Omega g(t,u(t)) + h(v(t))\, dt + \phi(u(T)). \end{array} \right. \tag{1.1.27}$$

One way of visualizing this functional is to imagine that we wish to reach the proximity of a certain state of the system given by (1.1.26) at time T, given by the function ϕ, but that the means of doing so has a systemic cost given by the function g, and we wish to make the control v, whose action is given by the cost h, act in order to minimize the cost of reaching at time T a state close to that predicted by the system.

We assume that the following assumptions are satisfied for the functional (1.1.27).

(PF1) $h : V \to \overline{\mathbb{R}}$ is a convex lower semicontinuous functional, and there exists $(\gamma, \mu) \in (0, +\infty) \times \mathbb{R}$ such that:

$$h(v) \geq \gamma \|v\|_V^2 - \mu \|v\|_{L^2(\Omega)}^2 \qquad \forall v \in V,$$

(PF2) $g : [0,T] \times L^2(\Omega) \to [0, +\infty)$ is a measurable function over its first variable, and for every $r > 0$, there exists $C_r > 0$ which does not depend of t such that $g(t,0) \in L^\infty([0,T])$ and

$$|g(t,y) - g(t,z)| + |\phi(y) - \phi(z)| \leq C_r \|y - z\|_{L^2(\Omega)} \qquad \forall (y,z) \in L^2(\Omega)^2,\ \|y\|_{L^2(\Omega)} + \|z\|_{L^2(\Omega)} \leq r.$$

Then, one can see that, setting:

$$\rho : \left| \begin{array}{l} L^2(\Omega) \to \overline{\mathbb{R}} \\ \\ u \longmapsto \int_\Omega J(u - \psi)\, dx, \end{array} \right.$$

we have the equivalent formulation of (1.1.26) given by the Cauchy problem

$$\left\{ \begin{array}{ll} B(u) + f \in \partial_t u + A(u) + \partial\rho(u) & \text{on } (0,T) \\ u_{|t=0} = u_0 & \text{on } D(\rho) \cap V = \left\{u \in V / J(u-\psi) \in L^1(\Omega)\right\}, \end{array} \right. \tag{1.1.28}$$

where $\partial\rho$ is maximal monotone from Theorem 1.1.22. One could think of an homogeneous Dirichlet problem given $V = H_0^1(\Omega)$, $V^* = H^{-1}(\Omega)$, and $A = -\Delta$, for example. Finally, the optimal control problem is given by:

$$\text{Find} \min_{v \in L^2((0,T),V)} G(u,v) \text{ where } u \text{ solves (1.1.28)}, u_0 \in D(\rho) \cap V.$$

Then, the following result holds.

Theorem 1.1.47. *For every $u_0 \in \overline{D(\rho) \cap V}$, the optimal control problem admits at least a solution.*

We refer to [14, Chapter 5] for a proof of this result. Let us point out that this result is not valid for highly singular problems, but there are other results along these lines, see e.g. [18]. Also, there exists several approaches to these kind of problems, sometimes without potential. We should e.g. refer to [11, 117], which respectively deal with local and optimal controllability.

1.2 Applications to the study of non-Newtonian flows

In this section, we explain how the theory of monotone operators or variational inequalities can be used to construct solutions to problems arising from non-Newtonian fluid mechanics. First, we present results specific to shear-thickening fluids, for which it is generally possible not only to show the existence and uniqueness of solutions, but also, if the fluid is sufficiently shear-thickening, to construct strong solutions. In a second step, we will present some results for the existence of solutions to equations modeling shear-thinning fluids, first presenting results obtained by monotonicity, then in the form of variational inequality. The results presented in this section are intended to give the reader a broad overview of the methods used and results obtained in our study, but as the literature is very extensive, we do not claim to be exhaustive.

1.2.1 Shear-thickening flows

Initially, we are interested in a shear-thickening fluid model, using the p-Laplacian operator directly in the modeling, which remains consistent given that we will assume a homogeneous Dirichlet condition, i.e. a no-slip boundary condition. More precisely, let us consider the following system.

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \nabla \pi = f & \text{in } (0,T) \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } (0,T) \times \Omega \\ u = 0 & \text{on } (0,T) \times \partial\Omega \\ u_{|t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{1.2.1}$$

We define a weak solution of the previous system as follows.

Definition 1.2.1. *Let us assume that $u_0 \in L^2_\sigma(\Omega)$ and $f \in L^q((0,T), H^{-1}_\sigma(\Omega))$, where $q \geq 1$ is a real number satisfying:*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

$p \geq \frac{N+2}{2}$ and $\Omega \subset \mathbb{R}^N$ is an open bounded domain. Then we say that $u \in L^p((0,T), H^1_{0,\sigma}(\Omega)) \cap L^\infty((0,T), L^2_\sigma(\Omega))$ is a weak solution of (1.2.1) if it satisfies for all $\varphi \in C^\infty_0((0,T), C^\infty_{0,\sigma}(\Omega))$:

$$\int_0^T \left(\langle \partial_t u, \varphi \rangle + \int_\Omega \nabla u : \nabla \varphi \, dx + \int_\Omega u \otimes u : \nabla \varphi \, dx + \int_\Omega |\nabla u|^{p-2} \nabla u : \nabla \varphi \, dx \right) dt = \int_0^T \langle f, \varphi \rangle \, dt \tag{1.2.2}$$

Then the following result holds.

Theorem 1.2.2 (J.L.Lions). *Let us assume that $u_0 \in L^2_\sigma(\Omega)$ and $f \in L^q((0,T), H^{-1}_\sigma(\Omega))$, where $q \geq 1$ is a real number satisfying:*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and $\Omega \subset \mathbb{R}^N$ is an open bounded domain. Moreover, we assume that p satisfies:

$$p \geq \frac{N+2}{2}. \tag{1.2.3}$$

Then, there exists a unique weak solution to (1.2.1) *in the sense of Definition 1.2.1.*

Proof. We refer to [132, Théorème 5.2.] for a complete proof. Let us just point out that the uniqueness follows from the monotonicity of the operator and suitable a priori estimates. □

Note that similar conclusions are still valid for more general systems. In the following we focused in the three-dimlensional case. Consider, for example, the three-dimensional one.

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \operatorname{div}(\mathbb{S}(D(u))) + \nabla \pi = f & \text{in } (0,T) \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } (0,T) \times \Omega \\ u = 0 & \text{on } (0,T) \times \partial\Omega \\ u_{|t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{1.2.4}$$

Where the stress tensor $\mathbb{S} \in \mathbb{R}^{3\times 3}_{\text{sym}}$ associated to the flow is assumed to derive from a potential $J : \mathbb{R}^{3\times 3}_{\text{sym}} \to [0, +\infty)$, such that $J = \Phi(|\cdot|^2)$ for the usual euclidiean matrix norm, and such that there exists $p \in [2, +\infty)$ satisfying for every $(A, B) \in \left(\mathbb{R}^{3\times 3}_{\text{sym}}\right)^2$ and every $(i, j, k, l) \in [1, 3] \cap \mathbb{N}$:

(S1) $\Phi(0) = \partial_i \partial_j J(0) = 0$.

(S2) $\mathbb{S}(A)_{i,j} = \partial_i \partial_j J(A)$.

(S3) $\exists C_1 > 0,\ \partial_i \partial_j \partial_k \partial_l J(A) B_{i,j} B_{k,l} \geq C_1 (1 + |A|)^{p-2} |B|^2$.

(S4) $\exists C_2 > 0,\ |\partial_i \partial_j \partial_k \partial_m J(A)| \leq C_2 (1 + |A|)^{p-2}$.

We define a weak solution of (1.2.4) as follow.

Definition 1.2.3. *Let us consider a bounded domain $\Omega \subset \mathbb{R}^3$, with C^3 boundary, a fixed time $T \in \mathbb{R}^*_+$, $p \in [2, 3)$, $u_0 \in W^{1,p}_{0,\sigma}(\Omega)$, $f \in L^2((0,T), L^2(\Omega))$, and that $\mathbb{S}$ satisfies assumptions* (S1) - (S4). Then, we define a weak solution to (1.2.4) as being a function $u \in L^\infty((0,T), L^2_\sigma(\Omega) \cap L^p((0,T), W^{1,p}_{0,\sigma}(\Omega))$ satisfying for all $\varphi \in C^\infty_0((-\infty, T), C^\infty_{0,\sigma}(\Omega))$:

$$\int_0^T \left(-\langle u, \partial_t \varphi \rangle + \int_\Omega \mathbb{S}((D(u)) : D(\varphi)\, dx + \int_\Omega u \otimes u : \nabla \varphi\, dx \right) dt = \int_0^T \langle f, \varphi \rangle\, dt + \int_\Omega u_0 \cdot \varphi(0)\, dx.$$

Then, the following result holds.

Theorem 1.2.4 (Málek-Nečas-Ružička). *Let us consider a bounded domain $\Omega \subset \mathbb{R}^3$, with C^3 boundary, a fixed time $T \in \mathbb{R}^*_+$, $p \in [2, 3)$, $u_0 \in W^{1,p}_{0,\sigma}(\Omega)$, $f \in L^2((0,T), L^2(\Omega))$, and that $\mathbb{S}$ satisfies assumptions* (S1) - (S4). Then, there exists a weak solution to (1.2.4) in sense of Definition 1.2.3. Moreover, if $p \geq \frac{9}{4}$, the solution u is unique and satisfies:

$$u \in C((0,T), L^2_\sigma(\Omega)) \cap L^\infty((0,T), W^{1,p}_{0,\sigma}(\Omega)) \cap L^{\frac{2}{p-1}}((0,T), W^{2,\frac{6}{p+1}}(\Omega)) \text{ and } \partial_t u \in L^2((0,T), L^2(\Omega))$$

We will admit the proof of this result, for which we refer to [141]. In fact, under other assumptions, which are not necessarily more restrictive, a similar result can be shown for $p > 2$, see [24, Theorem 3.4.].

Remark 1.2.5. *In the case of the* 3*-laplacian operator with the symmetrized gradient, we obtain the Smagorinsky model for Large Eddy Simulation of turbulence (see e.g. [22]).*

1.2.2 Shear-thinning flows

We can now present a set of results concerning shear-thinning fluids. Because of the possible singularity present in the deviatoric part of the stress tensor, the mathematical analysis of these can be harder. Initially, we will present results whose main idea is the use of the monotonicity of the operator. In particular, this has the advantage of being able to handle equations with no Laplacian. Next, we will present solutions in the form of variational inequalities, derived from our work (see [44]).

Some monotonicity results

Let us first consider the following system.

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \operatorname{div}\left(\mathbb{S}_W(t,x,D(u))\right) + \nabla \pi = -\operatorname{div}(f) & \text{in } (0,T)\times\Omega \\ \operatorname{div}(u) = 0 & \text{in } (0,T)\times\Omega \\ u = 0 & \text{on } (0,T)\times\partial\Omega \\ u_{|t=0} = u_0 & \text{in } \Omega \end{cases} \tag{1.2.5}$$

We assume that there exists for $1 \leq p < +\infty$ such that the following assumptions over $\mathbb{S}_W$ are verified, with $q = \frac{p}{p-1}$ if $p > 1$ or $q = +\infty$ if $p = 1$:

(SW1) The tensor $\mathbb{S}_W : (0,T)\times\Omega\times\mathbb{R}^{N\times N}_{\text{sym}} \to \mathbb{R}^{N\times N}_{\text{sym}}$ is a Carathéodory function.

(SW2) There exists $C_0 > 0$ and a non negative function $g \in L^q((0,T)\times\Omega)$ such that:

$$\forall A \in \mathbb{R}^{N\times N}_{\text{sym}},\ a.e.\ (t,x) \in (0,T)\times\Omega,\ |\mathbb{S}_W(t,x,A)| \leq C_0|A|^{p-1} + g.$$

(SW3) There exists $C_1 > 0$ and a non negative function $h \in L^1((0,T)\times\Omega)$ such that:

$$\forall A \in \mathbb{R}^{N\times N}_{\text{sym}},\ a.e.\ (t,x) \in (0,T)\times\Omega,\ C_1|A|^{p-1} - h \leq \mathbb{S}_W(t,x,A) : A.$$

(SW4) The operator $A \longmapsto \mathbb{S}_W(t,x,A)$ is strictly monotone for *a.e.* $(t,x) \in (0,T)\times\Omega$.

Also, let us define the notion we will consider later as a weak solution of (1.2.5).

Definition 1.2.6. *Let us assume that* $\frac{2N}{N+2} \leq p < +\infty$ *and that (SW1)-(SW2) holds. Then, for* $u_0 \in L^2_\sigma(\Omega)$*,* $f \in L^1((0,T)\times\Omega)$*, we say that* $u \in L^p((0,T), W^{1,p}_{0,\sigma}(\Omega)) \cap L^\infty((0,T), L^2_\sigma(\Omega))$ *is a weak solution tp* (1.2.5) *if for every* $\varphi \in C^\infty((0,T), C^\infty_{0,\sigma}(\Omega))$ *such that* $\operatorname{supp}(\varphi) \subset\subset [0,T)$*, we have:*

$$\int_0^T \left(-\int_\Omega u \cdot \partial_t \varphi \, dx + \int_\Omega (\mathbb{S}_W(t,x,D(u)) - u\otimes u) : D(\varphi)\, dx \right) dt = \int_0^T \int_\Omega f : \nabla\varphi \, dx\, dt + \int_\Omega u_0 \cdot \varphi(0)\, dx.$$

The following result holds.

Theorem 1.2.7 (Wolf). *Let us assume that $\Omega \subseteq \mathbb{R}^N$, $1 + \frac{N}{N+2} < p < +\infty$, a fixed time $T \in \mathbb{R}_+^*$, $u_0 \in L^2_\sigma(\Omega)$, $f \in L^q(\Omega)$ and that $\mathbb{S}_W$ satisfies assumptions (SW1)-(SW4). Then, there exists a weak solution $u \in L^p((0,T), W^{1,p}_{0,\sigma}(\Omega)) \cap C_w((0,T), L^2_\sigma(\Omega))$ to (1.2.5) in sense of Definition 1.2.6.*

This result has several advantages: firstly, it ensures the existence of solutions for tensors that do not necessarily take into account a Laplacian term. Furthermore, there is no a priori limitation on the Ω domain, which may or may not be bounded or regular. However, if the author of the theorem insists on this point, we should note that this lack of restriction must be understood in a prescribed sense: while there is no need to specify that the set must be measurable, let us mention, for example, that the author nevertheless seems to employs Korn inequality (see Theorem A.3.3) in an underlying way (through embbeddings, for example). However, it is actually known that Korn inequality is not valid on cuspidal domains, for example (see [2]).

Example 1.2.8. *The following tensors can be considered by Theorem 1.2.7, for $1 + \frac{N}{N+2} < p < +\infty$:*

- $\mathbb{S}_W(t,x,A) := C|A|^{p-2}A$ *(Perfect power-law)*
- $\mathbb{S}_W(t,x,A) := C(1+|A|^2)^{\frac{p-2}{2}}A$ *(Perfect Carreau law)*
- $\mathbb{S}_W(t,x,A) := C|A|^{p-2}A + CA$ *(Power-law)*
- $\mathbb{S}_W(t,x,A) := C(1+|A|^2)^{\frac{p-2}{2}}A + CA$ *(Carreau law)*

Sketch of the proof of Theorem 1.2.7. Firstly, it is established that there is an integrable pressure term even in the case where the tensor $\mathbb{S}_W$ is non-diffusive, in the sense that it cannot be written as the sum of a Laplacian and another term (vulgarly, we can sometimes say that we do not consider "kinematic viscosity", to make the link with the Navier-Stokes equations). The proof then breaks down into several steps.

First step: We show the existence of local solutions for an approximate decoupled system, in which the convective term is expressed according to another function and is penalized by a smooth cut-off function. In particular, a general version of the Theorem B.0.1 is applied. We show that we can find local solutions to the system by applying a Schauder fixed-point theorem, which is done via the use of a harmonic decomposition and the Minty's trick Lemma 1.1.5. This leads to the existence of weak solutions on bounded domains.

Second step: We show the existence of global solutions for bounded domains, which is made possible by estimates.

Third step: We show that it is possible to extend the result to unbounded domains approximated by a sequence of bounded domains, and by showing that the estimates for this sequence of domains are uniform.

We refer to [171] for the details. □

In fact, there is still room for improvement. First of all, it is possible to slightly relax assumptions (SW2)-(SW3) assuming that there exists $\delta \geq 0$ as follows.

(SW2') There exists $C_0 > 0$ and a non negative function $g \in L^q((0,T)\times\Omega)$ such that:

$$\forall A \in \mathbb{R}^{N\times N}_{\text{sym}},\ a.e.\ (t,x) \in (0,T)\times\Omega,\ |\mathbb{S}_W(t,x,A)| \leq C_0(\delta + |A|)^{p-2}|A| + g.$$

(SW3') There exists $C_1 > 0$ and a non negative function $h \in L^1((0,T)\times\Omega)$ such that:

$$\forall A \in \mathbb{R}^{N\times N}_{\text{sym}},\ a.e.\ (t,x) \in (0,T)\times\Omega,\ C_1(\delta + |A|)^{p-2}|A| - h \leq \mathbb{S}_W(t,x,A) : A.$$

We then have the following result, which is an improvement on Theorem 1.2.7.

Theorem 1.2.9 (Diening-Ružička-Wolf). *Let us consider Ω an open bounded subset of $\mathbb{R}^N$, where $N \geq 2$, and a fixed time $T \in \mathbb{R}^*_+$. Moreover, we assume that $\mathbb{S}$ satisfies assumptions (SW1),(SW2'),(SW3'), and (SW4) for*

$$\frac{2N}{N+2} < p < +\infty,$$

and some $\delta \geq 0$. Then, if $u_0 \in L^2_\sigma(\Omega)$ and $f \in L^q((0,T)\times\Omega)$, there exists a weak solution $u \in L^p\left((0,T), W^{1,p}_{0,\sigma}(\Omega)\right) \cap C_w([0,T], L^2_\sigma(\Omega))$ of (1.2.5) in the sense of Definition 1.2.6.

Proof. The proof is based on the Lipschitz truncation method, where the domain is studied using a Whitney decomposition. We refer to [72] for more details. □

Remark 1.2.10. *We point out that Theorem 1.2.9 is optimal in the sense that for $p < \frac{2N}{N+2}$, the Sobolev embbedding*

$$L^p((0,T), W^{1,p}_{0,\sigma}(\Omega)) \cap L^\infty((0,T), L^2_\sigma(\Omega)) \hookrightarrow L^{\frac{(N+2)p}{N}}((0,T)\times\Omega)$$

does not hold. Even if the convective term in the weak formulation can still be well defined, there is a lack of monotonicity over $W^{1,p}_0(\Omega)$ (in space variable) in this case, and it is not obvious that the considered methods can still be used when dealing with $W^{1,p}_{0,\sigma}(\Omega) \cap L^2(\Omega)$ (see [72, Remark 1.4.6.]).

A result similar to Theorem 1.2.9 can be obtained under slightly different assumptions for Navier boundary conditions. More precisely, consider the following system:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \operatorname{div}\left(\mathbb{S}_B(t,x,D(u))\right) + \nabla\pi = f & \text{in } (0,T)\times\Omega \\ \operatorname{div}(u) = 0 & \text{in } (0,T)\times\Omega \\ u\cdot n = 0 & \text{on } (0,T)\times\partial\Omega \\ \mathbb{S}_B - (\mathbb{S}_B\cdot n)n = -a\left(u - (u\cdot n)n\right) = 0 & \text{on } (0,T)\times\partial\Omega \\ u_{|t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{1.2.6}$$

where $a \geq 0$ and n is a unit outer normal vector to $\partial\Omega$. The Navier boundary conditions state that the tangential component of the velocity is proportional to the tangential stresses applied at the boundary (we refer to [98] for details concerning these last). The following assumptions are made.

(B1) For *a.e.* $(t,x) \in (0,T)\times\Omega$, $0 \in \mathbb{S}_B(t,x,0)$.

(B2) The map $A \mapsto \mathbb{S}_B(t,x,A)$ is monotone on $\mathbb{R}^{N\times N}_{\text{sym}}$.

(B3) The map $A \mapsto \mathbb{S}_B(t,x,A)$ is maximal monotone on $\mathbb{R}^{N\times N}_{\text{sym}}$.

(B4) There exists a non negative function $g \in L^1((0,T)\times\Omega)$, a constant $C_0 > 0$, and an N-function φ (see Definition A.5.1) such that:

$$\forall A \in \mathbb{R}^{N\times N}_{\text{sym}},\ \mathbb{S}_B(t,x,A) : A \geq -g + C_0\left(\varphi(|A|) + \varphi^*\left(|\mathbb{S}_B(t,x,A)|\right)\right) \quad a.e.\ (t,x) \in (0,T)\times\Omega.$$

At this point, we feel it is important to make an initial comment on the hypotheses presented here. These are equivalent to those proposed by the authors (see [35]), but they are presented in a different form. Firstly, the authors wish to highlight an implicit relationship between the deviatoric part of the stress tensor and the strain rate tensor. Thus, the authors do not establish the hypotheses on $\mathbb{S}_B$ but on a graph $\mathcal{A}(t,x)$. The fact that a couple (D,S) belongs to the graph $\mathcal{A}(t,x)$ is then equivalent to a relationship of the form $G(D,S)=0$. This choice of writing allows us to take a different look at the constitutive relations. However, the authors' assumption (A3) of maximal monotonicity is equivalent to the existence of a maximal monotone operator describing the evolution of the graph (see [175, Section 32.2.]), and joining this to their explicit assumption (A5), we obtain that it is equivalent to the explicit relation we have chosen to evoke in our assumptions, in order to maintain a certain consistency with the rest of the results presented here.

Definition 1.2.11. *Let us consider a domain Ω of $\mathbb{R}^N$ with $C^{1,1}$ boundary, a fixed time $T\in\mathbb{R}_+^*$ and $p>1$. Moreover, we assume that $u_0\in L^2_\sigma(\Omega)$ satisfies $u_0\cdot n=0$ on $\partial\Omega$, and $f\in L^{\frac{p}{p-1}}\left((0,T)\times W_\sigma^{-1,\frac{p}{p-1}}(\Omega)\right)$ satisfies the same assumption on the boundary. We say that u is a weak solution of (1.2.6) if the following statements hold for every test function φ.*

- *There exists $\pi\in L^1((0,T)\times\Omega)$ such that equation*

  $$\langle\partial_t u,\varphi\rangle+\langle\mathbb{S}_B(t,x,D(u)),D(\varphi)\rangle-\langle u\otimes u,D(\varphi)\rangle+a\int_{\partial\Omega}\gamma_0(u)\cdot\gamma_0(\varphi)\,d\sigma=\langle f,\varphi\rangle+\int_\Omega\pi\mathrm{div}(\varphi)\,dx$$

  is satisfied for a.e. $t\in(0,T)$.
- *$u\in C_w((0,T),L^2_\sigma(\Omega))\cap L^p((0,T),W^{1,p}_\sigma(\Omega))$ such that $u\cdot n=0$ in the sense of both spaces.*
- *There exists an N-function such that the map $A\mapsto\mathbb{S}_B(t,x,A)$ belongs to $L^{\varphi^*}((0,T)\times\Omega)$ and $D(u)$ belongs to $L^\varphi((0,T)\times\Omega)$.*
- $\|u(t)-u_0\|_{L^2(\Omega)}\underset{\substack{t\to0\\t>0}}{\longrightarrow}0$.

Here are used Orlicz spaces (see Section A.5). We then have the following result.

Theorem 1.2.12 (Bulíček-Gwiazda-Málek-Świerczewska)**.** *Let us consider that the assumptions (B1)-(B4) as the assumptions of Definition 1.2.11 are fulfilled for some $p>\frac{2N}{N+2}$ as for an N-function φ satisfying the Δ_2-condition, the ∇_2-condition (see Definition A.5.5), and such that there exist $C_1,C_2,C_3,C_4>0$ verifying*

$$C_1t^p-C_2\leq\varphi(t)\leq C_3t^r+C_4\qquad\text{for some }\ r\in[p,+\infty)\,.$$

Then, there exists a weak solution to (1.2.6) in the sense of Definition 1.2.11.

Let us take a moment to understand the ins and outs of this result. First of all, it is worth noting that a number of power-law models are taken into account here, which the authors of the article take the time to point out (see [35, Lemma 1.1.]). That said, threshold flow models, such as (perfect) Bingham or Herschel-Bulkley for example, are not taken into account in this result. Indeed, we note that the identity function is not an N-function, and therefore cannot define an Orlicz space. Note also that the proof of the result uses the reflexivity of Orlicz spaces, which is true by Theorem A.5.10, Proposition A.5.8 and Theorem A.5.6, since the N-function under consideration satisfies the Δ_2- and ∇_2-conditions. Note that the restriction $p>\frac{2N}{N+2}$ must be understood, particularly in view of Remark 1.2.10, as the limitation of the (perfect) power-law

models considered.

A considerable advantage of this result is that the use of Orlicz spaces and the Navier boundary condition allows the authors to obtain an integrable pressure. As a result, using the Lipschitz truncation method, the authors deduce that the weak formulation is verified almost everywhere. While the authors are content to announce the triviality of this assertion, we note that this is due to the fact that it is possible to integrate in the approximation on any subdomain, and to apply a Lebesgue differentiation theorem.

Variational inequality solutions

As we have established, there is a close link between the notion of monotone operator and that of variational inequality. It is therefore also natural to look for solutions in this form for shear-thinning flow problems, especially as such solutions lend themselves relatively well to numerical analysis (see for example [100]) or control theory, as mentioned above. Consider the following system:

$$\begin{cases} \partial_t u - A(u) + (u\cdot\nabla)u - \operatorname{div}\left(\mathbb{S}_C(D(u))\right) + \nabla\pi = f & \text{in } (0,T)\times\Omega \\ \operatorname{div}(u) = 0 & \text{in } (0,T)\times\Omega \\ u = 0 & \text{on } (0,T)\times\partial\Omega \\ u_{|t=0} = 0 & \text{in } \Omega, \end{cases} \tag{1.2.7}$$

where $\mathbb{S}_C(D(u)) = \partial\Phi(D(u))$ under the following assumptions:

(D1) There exists a family of Gâteaux-differentiable proper lowersemicontinuous convex functionals $(\Phi_i)_{1\le i\le k}$ and a proper lower semicontinuous convex functional ψ over $L^2(\Omega)$ such that the functional:

$$\Phi : \left| \begin{array}{l} L^2(\Omega) \to (-\infty, +\infty] \\ \\ u \mapsto \displaystyle\sum_{i=1}^{k} \Phi_i + \psi \end{array} \right.$$

satisfies $\Phi(0) = 0$.

(D2) The functionals $(\Phi_i)_{1\le i\le k}$ satisfy, for some $p > 1$, $r \ge p$:

$$\left(\int_\Omega |\Phi_i'(u)|^{\frac{p}{p-1}} |D(u)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \le C\|D(u)\|_{L^p(\Omega)}^{\frac{pr}{2}}.$$

(D3) There exists a family of functionals $(\psi_i)_{i\in\mathbb{N}}$ which admit Gâteaux-derivatives and such that for every $\varphi \in L^2((0,T)\times\Omega)$, and a sequence $(\varphi_j)_{j\in\mathbb{N}}$ which is weakly convergent toward φ in $L^2((0,T)\times\Omega)$ we have:

$$\int_0^T \psi_j(\varphi) - \psi(\varphi)\, dt \underset{j\to+\infty}{\longrightarrow} 0 \quad \text{and} \quad \psi_j'(0) = 0$$
$$\int_0^T \psi_j(\varphi_j) \le C \Rightarrow \lim_{j\to+\infty} \int_0^T \psi_j(\varphi_j)\, dt \ge \lim_{j\to+\infty} \int_0^T \psi(\varphi_j)\, dt.$$

(D4) The map

$$A : \left| \begin{array}{l} W^{1,p}_{0,\sigma}(\Omega) \to W^{-1,q}_{\sigma}(\Omega) \\ \\ u \mapsto \displaystyle\sum_{i=1}^{k} A_i(u). \end{array} \right.$$

where $(A_i)_{1\leq i\leq k}$ is a family of coercive (for $\psi = \mathrm{Id}$ relatively to Definition 1.1.8) hemicontinuous monotone operators defined such that

$$\langle A_i(u), \varphi\rangle = \int_\Omega \Phi_i'(u) D_{i,j}(\varphi)\, dx.$$

From Proposition 1.1.13 it is clear that the operator A defined above is monotone and hemicontinuous, and we can think of it as a generalization of the Laplacian. Now let us give the following result for the existence of a solution of (1.2.7).

Theorem 1.2.13 (Cioranescu-Duvaut-J.L.Lions)**.** *Let us consider for $k \in \mathbb{N}$, $k \geq 1$ some $(p_i)_{1\leq i\leq k}$, $(q_i)_{1\leq i\leq k}$ and a fixed time $T \in \mathbb{R}_+^*$, being such that*

$$p_i \geq \frac{3N}{N+2} \quad \textit{and} \quad q_i \geq \frac{2p_i}{p_i - 1}$$

and $f \in \displaystyle\bigcap_{i=1}^{k} L^{\frac{q_i}{q_i-1}}((0,T), W_\sigma^{-1,\frac{p_i}{p_i-1}}(\Omega))$, where Ω is a domain of $\mathbb{R}^N$ with smooth boundary. Then, assuming that assumptions (D1)-(D4) are fulfilled, there exists:

$$u \in \left(\bigcap_{i=1}^{k} L^{q_i}((0,T), H^s_{0,\sigma}(\Omega)) \right) \cap L^\infty((0,T), L^2_\sigma(\Omega))$$

such that $\partial_t u \in \displaystyle\bigcap_{i=1}^{k} L^{\frac{q_i}{q_i-1}}((0,T), H_\sigma^{-s}(\Omega))$ satisfying for every φ in an admissible set of test functions:

$$\int_0^T \left(\langle \partial_t \varphi, \varphi - u\rangle + \langle A(u), \varphi - u\rangle + \int_\Omega (u\otimes u) : D(\varphi - u)\, dx + \psi(\varphi) - \psi(u) \right) dt \geq \int_0^T \langle f, \varphi - u\rangle\, dt, \quad (1.2.8)$$

the result being true for the Bingham model.

The previous result allows us to obtain the existence of solutions in the form of variational inequality, where the emphasis is on convexity by systematically testing against the term $u - \varphi$. However, as pointed out in [77], it is not possible, with this weak formulation, to recover the equation even formally (that is admitting smoothness). Secondly, the singular term is approximated by penalizing the power into the functional, which is not generally the preferred approach from a numerical point of view. We refer the interested reader to [77, 50] for a proof of this result.

This brings us to the following question: is it possible to construct solutions in the form of variational inequalities for fluids with shear-thinning behavior, using the approximation commonly employed for numerical simulations? Of course, ideally we would need solutions that at least formally allow us to find solutions to the equation. From Remark 1.2.10, it is clear that we need to

introduce a Laplacian so as not to lose too much regularity, otherwise definitional problems could arise. That said, this is not a problem in the sense that it does not contravene the physical reality of the models under consideration (on the contrary). Also, a point that may seem important in view of the previous theorems is to show the existence of solutions under simple assumptions, so that the result can be understood by a wide community including mathematicians as well as physicists and engineers.

Let us consider the following system

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi - \operatorname{div}\left(F\left(|D(u)|\right) D(u)\right) = f & \text{in } (0,T) \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } (0,T) \times \Omega \\ u = 0 & \text{on } [0,T) \times \partial\Omega \\ u = u_0 & \text{on } \{0\} \times \Omega \end{cases} \tag{1.2.9}$$

Where Ω is an open subset of $\mathbb{R}^N$, $N \in \{2,3\}$ having a Lipschitz boundary, and F satisfies the following assumptions.

(C1) $F : [0,+\infty) \to [0,+\infty)$;

(C2) $F \in W^{1,\infty}_{loc}\left((0,+\infty)\right)$;

(C3) $t \mapsto tF(t)$ is non-decreasing on $(0,+\infty)$;

(C4) there exist $p \in [1,2]$, $t_0 > 0$ and $K > 0$ such that for every $t \geq t_0$, $F(t) \leq Kt^{p-2}$.

Remark 1.2.14. *Assumption (C3) is equivalent to the fact that for all $\varepsilon \geq 0$, the function $t \mapsto tF\left(\sqrt{\varepsilon + t^2}\right)$ is non-decreasing. Indeed, we can write:*

$$\forall t \in (0,+\infty),\ tF\left(\sqrt{\varepsilon + t^2}\right) = \left(\frac{t}{\sqrt{\varepsilon + t^2}}\right)\sqrt{\varepsilon + t^2}F\left(\sqrt{\varepsilon + t^2}\right).$$

Hence, $t \mapsto tF\left(\sqrt{\varepsilon + t^2}\right)$ is the product of two non-negative and non-decreasing functions, so it is a non-decreasing function. The opposite implication being obvious by setting $\varepsilon = 0$.

It is then necessary to be able to give a very weak meaning to the solutions, giving variational inequalities their full meaning. Before giving the notion of definition that we will be considering, let us introduce the regularization functional, derived from a potential, that we will be using to give meaning to our solutions.

We fix for the moment $0 \leq \varepsilon \leq \delta$ and we define a function $G_\epsilon : (0,+\infty) \to (0,+\infty)$ and a functional $j_\varepsilon : H^1_{0,\sigma}(\Omega) \to \mathbb{R}$ by

$$G_\varepsilon(t) = \int_0^t sF(\sqrt{\varepsilon + s^2})\, ds \qquad \text{for every } t \in (0,+\infty) \tag{1.2.10}$$

and

$$j_\varepsilon(v) = \int_\Omega G_\varepsilon(|D(v)|)\, dx, \quad (v \in H^1_{0,\sigma}(\Omega)), \tag{1.2.11}$$

respectively. We also denote $j = j_0$ and $G = G_0$. One can check that G_ε is a convex functional for ε small enough. Indeed,

$$G'_\varepsilon(t) = tF(\sqrt{\varepsilon + t^2}), \qquad \text{for every } t \in (0,+\infty),$$

and applying the hypothesis (C3) the convexity of G follows immediately.

Lemma 1.2.15. *For every $\varepsilon > 0$ the functional j_ε defined by (1.2.11) is convex and verifies*

$$\langle j'_\varepsilon(v), w\rangle_{-1,1} = \int_\Omega F\left(\sqrt{\varepsilon + |D(v)|^2}\right)(D(v) : D(w))\ dx \qquad (v, w \in H^1_{0,\sigma}(\Omega)). \tag{1.2.12}$$

Proof. The convexity of j_ε is immediately obtained from the hypothesis (C3) and (1.2.11). For every $t \in \mathbb{R}$ we have

$$\begin{aligned}\frac{d}{dt}\left(G_\varepsilon\left(|D(v+tw)|\right)\right) &= G'_\varepsilon\left(|D(v+tw)|\right)\frac{d}{dt}\left(|D(v+tw)|\right)\\ &= F\left(\sqrt{\varepsilon + |D(v+tw)|^2}\right)|D(v+tw)|\left(\frac{D(v+tw) : D(w)}{|D(v+tw)|}\right)\\ &= F\left(\sqrt{\varepsilon + |D(v+tw)|^2}\right)D(v+tw) : D(w).\end{aligned}$$

Hence

$$\begin{aligned}\langle j'_\varepsilon(v+tw), w\rangle = \frac{d}{dt}j_\varepsilon(v+tw) &= \int_\Omega \frac{d}{dt}\left(G_\varepsilon\left(|D(v+tw)|\right)\right)\ dx\\ &= \int_\Omega F\left(\sqrt{\varepsilon + |D(v+tw)|^2}\right)D(v+tw) : D(w)\ dx.\end{aligned}$$

Letting t going to 0 we obtain (1.2.12). □

Remark 1.2.16. *We point out that j' is well defined. Firstly, by our assumptions (C2) and (C3), we can deduce that for all $\beta \in \left(0, \frac{1}{2}\right)$, there exists δ_0 such that:*

$$F(t) \le t^{-(1+\beta)} \quad \text{for every } t \in (0, \delta_0).$$

Indeed, assume that this last inequality does not hold, then for every $\delta_0 > 0$, there exists $t_0 \in (0, \delta_0)$ such that:

$$F(t_0) > t_0^{-(1+\beta)}.$$

We can consider without loss of generality that $\delta_0 < \min\left(1, F(1)^{-\frac{1}{\beta}}\right)$, which implies, using our assumption (C3):

$$\delta_0^{-\beta} < t_0^{-\beta} < t_0 F(t_0) \le F(1).$$

This contradiction shows the result. We recall Korn's L^2 equality for divergence free vector fields:

$$\int_\Omega |D(\varphi)|^2\ dx = \frac{1}{2}\|\varphi\|^2_{H^1_0}, \quad (\varphi \in H^1_{0,\sigma}(\Omega)).$$

Using these last results and applying Cauchy Schwarz's and Hölder's inequalities, we get:

$$
\begin{aligned}
&|\langle j'(u),\varphi\rangle_{-1,1}| = \left|\int_\Omega F(|D(u)|)D(u) : D(\varphi)\ dx\right| \\
&\leq \frac{1}{\sqrt{2}}\left(\int_\Omega F(|D(u)|)^2|D(u)|^2\ dx\right)^{\frac{1}{2}} \|\varphi\|_{H_0^1} \\
&= \frac{1}{\sqrt{2}}\left(\int_{\{|D(u)|\leq\delta_0\}} F(|D(u)|)^2|D(u)|^2\ dx + \int_{\{|D(u)|>\delta_0\}} F(|D(u)|)^2|D(u)|^2\ dx\right)^{\frac{1}{2}} \|\varphi\|_{H_0^1} \\
&\leq \frac{1}{\sqrt{2}}\left(\int_{\{|D(u)|\leq\delta_0\}} |D(u)|^{-2\beta}\ dx + \int_{\{|D(u)|>\delta_0\}} F(|D(u)|)^2|D(u)|^2\ dx\right)^{\frac{1}{2}} \|\varphi\|_{H_0^1} \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{1-2\beta}\int_{\{|D(u)|\leq\delta_0\}}\int_0^{|D(u)|} s^{1-2\beta}\ ds\ dx + \int_{\{|D(u)|>\delta_0\}} F(|D(u)|)^2|D(u)|^2\ dx\right)^{\frac{1}{2}} \|\varphi\|_{H_0^1}.
\end{aligned}
$$

And so j' is well-defined.

We are now able to give the following definition.

Definition 1.2.17 (Weak solution of (2.2.8))**.** *We say that a function $u \in L^2\left((0,T), H^1_{0,\sigma}(\Omega)\right) \cap C_w((0,T), L^2_\sigma(\Omega))$ such that $\partial_t u \in L^{\frac{4}{N}}\left((0,T), H_\sigma^{-1}(\Omega)\right)$ is a weak solution of (2.2.8) if and only if u verifies $u_{|t=0} = u_0 \in H^1_{0,\sigma}(\Omega)$, and for all $\varphi \in C^\infty((0,T)\times\Omega)$:*

$$
\begin{aligned}
&\int_0^T \langle \partial_t u(t), \varphi(t)\rangle\ dt + \frac{1}{2}\left(\|u_0\|^2_{L^2(\Omega)} - \|u(T)\|^2_{L^2(\Omega)}\right) + \int_0^T\int_\Omega D(u(t)) : D(\varphi(t)-u(t))\ dx \\
&- \int_0^T\int_\Omega (u(t)\cdot\nabla u(t))\cdot\varphi(t)\ dx\ dt + \int_0^T\int_\Omega G\left(|D(\varphi(t))|\right) - G\left(|D(u(t))|\right)\ dx\ dt \\
&\geq \int_0^T \langle f(t), \varphi(t)-u(t)\rangle\ dt. \qquad (1.2.13)
\end{aligned}
$$

Let us start by quickly checking that we can formally find equation (2.2.8) from this definition. First, we point out that since u belongs to $C_w((0,T), L^2_\sigma(\Omega))$, Definition 1.2.17 makes sense. Then, if we consider that the Lebesgue measure of the set

$$\{(t,x)\in(0,T)\times\Omega \ \mid\ |D(u)(t,x)|\leq\delta\}$$

is equal to zero for a small $\delta>0$, we have, from an argument similar to the one in Lemma 1.2.15 that:

$$\int_0^T \langle j'(u),\varphi\rangle\, dt = \int_0^T\int_\Omega F\left(|D(u)|\right)\left(D(u) : D(\varphi)\right)\ dx\, dt.$$

Now, if we replace φ by $u+s\varphi$, with $s>0$, in the variational inequality (1.2.13), we obtain after dividing by s:

$$
\begin{aligned}
&\int_0^T\int_\Omega D(u) : D(\varphi)\ dx\, dt + \int_0^T\int_\Omega \frac{G\left(|D(u+s\varphi)|\right) - G\left(|D(u)|\right)}{s}\ dx\, dt \\
&\geq \int_0^T\int_\Omega \langle f - \partial_t u, \varphi\rangle\, dt - \int_0^T\int_\Omega (u\cdot\nabla u)\cdot\varphi\ dx\, dt.
\end{aligned}
$$

Since j admits a Fréchet-derivative, it also admits a Gâteaux-derivative and both are the same. Hence, taking the limit as $s \to 0$:

$$\begin{aligned}&\int_0^T \int_\Omega D(u) : D(\varphi)\, dx\, dt + \int_0^T \int_\Omega F\left(|D(u)|\right)\left(D(u) : D(\varphi)\right)\, dx\, dt \\ &\geq \int_0^T \int_\Omega \langle f - \partial_t u, \varphi\rangle\, dt - \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \varphi\, dx\, dt.\end{aligned}$$

Repeating once again the previous reasoning but writing $u - s\varphi$ instead of $u + s\varphi$, we get the following equality:

$$\begin{aligned}&\int_0^T \int_\Omega D(u) : D(\varphi)\, dx\, dt + \int_0^T \int_\Omega F\left(|D(u)|\right)\left(D(u) : D(\varphi)\right)\, dx\, dt \\ &= \int_0^T \int_\Omega \langle f - \partial_t u, \varphi\rangle\, dt - \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \varphi\, dx\, dt.\end{aligned}$$

Therefore, assuming that u is regular enough, we obtain

$$\begin{aligned}&-\frac{1}{2}\int_0^T \int_\Omega \Delta u \cdot \varphi\, dx\, dt - \int_0^T \int_\Omega \operatorname{div}\left(F\left(|D(u)|\right) D(u)\right) \varphi\, dx\, dt \\ &= \int_0^T \int_\Omega (f - \partial_t u - u \cdot \nabla u) \cdot \varphi\, dx\, dt.\end{aligned}$$

Furthermore De Rham's theorem for a domain with Lipschitz boundary states that there exists a pressure term π such that $f = \nabla \pi$ into some well chosen Sobolev space (see [72, section 2] for details). Considering such a function and also the two previous observations, we can write:

$$\int_0^T \int_\Omega \left(\partial_t u + (u.\nabla)u - \frac{1}{2}\Delta u + \nabla p - \operatorname{div}\left(F\left(|D(u)|\right) D(u)\right) - f\right) \varphi\, dx\, dt = 0, \quad \varphi \in C^\infty((0,T)\times\Omega),$$

which is almost everywhere equivalent to the equation (2.2.8) up to the multiplicative dynamic viscosity constant $\frac{1}{2}$. We have omitted this constant in Definition 1.2.17 for convenience, and note that it is enough to add the constant 2 in front of the term $\int_0^T \int_\Omega D(u) : D(u - \varphi)\, dx\, dt$ in order to find exactly (2.2.8).

We have now shown that these very weak solutions formally allow us to find a solution to the (2.2.8) system. We now need to show that such solutions exist. This is the subject of the following theorem.

Theorem 1.2.18. *Assume that the function F satisfies the hypotheses (C1)-(C4) and that $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$, is a bounded domain with a Lipschitz boundary, $T > 0$ and consider an initial datum $u_0 \in H^1_{0,\sigma}(\Omega)$ and a force term $f \in L^2((0,T), H^{-1}_\sigma(\Omega))$. Then, there exists a weak solution u of (2.2.8) having the following regularity*

$$u \in C_w\left((0,T), L^2_\sigma(\Omega)\right) \cap L^2\left((0,T), H^1_{0,\sigma}(\Omega)\right) \quad \text{and} \quad \partial_t u \in L^{\frac{4}{N}}((0,T), H^{-1}_\sigma(\Omega)).$$

Proof of Theorem 1.2.18. The proof of this result is based on a nonlinear Galerkin method and is divided into four steps. First, we briefly establish the Galerkin solutions for the regularized system, with the regularization usually used in numerical methods. Next, we carry out energy estimates to derive, in a third step, weak convergence properties. Finally, we demonstrate the theorem by making use of properties specific to variational inequalities.

First step: Galerkin scheme

We apply here the usual Galerkin method using the Stokes operator in homogeneous Dirichlet setting, and we use its eigenfunctions $(w_i)_{i\in\mathbb{N}}$ as an orthogonal basis of $H^1_{0,\sigma}(\Omega)$ and orthonormal basis of $L^2_\sigma(\Omega)$ (see [84] for details about this property, and [155, Section 2.3] for details concerning the Stokes operator).

For every positive integer m, we denote by P_m the projection of $L^2_\sigma(\Omega)$ onto $\mathrm{Span}\left((w_i)_{1\leq i\leq m}\right)$. We would like to formally define our Galerkin system as follows.

$$\begin{cases} \partial_t u_m + P_m(u_m\cdot\nabla u_m) + \nabla P_m(\pi) & -\Delta u_m - P_m\left(\mathrm{div}\left(F\left(|D(u_m)|\right)D(u_m)\right)\right) = P_m f \\ \mathrm{div}(u_m) = 0 & \text{on } (0,T)\times\Omega \\ u_m = 0 & \text{on } [0,T)\times\partial\Omega \\ u_m = P_m(u_0) & \text{on } \{0\}\times\Omega. \end{cases} \tag{1.2.14}$$

In order to avoid the issue posed by the nonlinear term in domains for which the fluid is not deformed we consider the following regularized Galerkin system:

$$\begin{cases} \partial_t u_{m,\varepsilon} - P_m\Big(\mathrm{div}\Big(F\Big(\sqrt{\varepsilon+|D(u_{m,\varepsilon})|^2}\Big)D(u_{m,\varepsilon})\Big)\Big) & \\ \qquad + P_m(u_{m,\varepsilon}\cdot\nabla u_{m,\varepsilon}) + \nabla P_m(\pi) - \Delta u_{m,\varepsilon} = P_m f & \text{in } (0,T)\times\Omega \\ \mathrm{div}(u_{m,\varepsilon}) = 0 & \text{in } (0,T)\times\Omega \\ u_{m,\varepsilon} = 0 & \text{on } [0,T)\times\partial\Omega \\ u_{m,\varepsilon|t=0} = P_m(u_0) & \text{in } \Omega, \end{cases} \tag{1.2.15}$$

with $0<\varepsilon<1$. Applying a Galerkin method, we can see that, writing $u_{m,\varepsilon}(t) = \sum_{i=1}^m d_m^i(t)w_i$, we obtain the ordinary differential system for all $1\leq i\leq m$:

$$\begin{aligned} {d_m^i}'(t) &= \langle f, w_i\rangle - \int_\Omega \frac{1}{2}\|w_i\|^2_{H^1_0} d_m^i(t)\,dx - \int_\Omega D(u_0):D(w_i)\,dx \\ &- \int_\Omega \frac{1}{2}\|w_i\|^2_{H^1_0} F\left(\sqrt{\varepsilon+\sum_{j=1}^m \frac{1}{2}\|w_j\|^2_{H^1_0}(d_m^j(t))^2 + 2(D(w_j):D(u_0))d_m^j(t) + \frac{1}{2}\|u_0\|^2_{H^1_0}}\right) d_m^i(t)\,dx \\ &- \int_\Omega F\left(\sqrt{\varepsilon+\sum_{j=1}^m \frac{1}{2}\|w_j\|^2_{H^1_0}(d_m^j(t))^2 + 2(D(w_j):D(u_0))d_m^j(t) + \frac{1}{2}\|u_0\|^2_{H^1_0}}\right)(D(u_0):D(w_i))\,dx \\ &\qquad - \sum_{j=1}^m \int_\Omega w_j\cdot\nabla w_i d_m^i(t) d_m^j(t)\,dx, \end{aligned} \tag{1.2.16}$$

completed with initial condition $d_m^i(0) = (u_0,w_i)_{H^1_0}$. This system is described by a locally Lipschitz continuous function with respect to d_m. Indeed, applying the hypothesis (C2), the function $\psi:\mathbb{R}^m\to\mathbb{R}$ defined by

$$\psi(x) = F\left(\sqrt{\varepsilon^2+\sum_{j=1}^m \frac{1}{2}\|w_j\|^2_{H^1_0}x_j^2 + 2(D(w_j):D(u_0))x_j + \frac{1}{2}\|u_0\|^2_{H^1_0}}\right) \qquad \forall x\in\mathbb{R}^m$$

is locally Lipschitz. The Picard-Lindelöf theorem shows the existence of a solution for system (1.2.15).

Second step : Energy estimates

We recall that the solution $u_{m,\varepsilon}$ of (1.2.15) belongs to Span $((w_i)_{1\leq i\leq m})$, for $(w_i)_{i\in\mathbb{N}}$ the basis of $H^1_{0,\sigma}(\Omega)$ which are the eigenfunctions of the Stokes operator in the homogeneous Dirichlet setting.

In order to clarify our presentation, we specify that we consider the following notion of solution.

Definition 1.2.19 (Solution of (1.2.15))**.** *We say that $u_{m,\varepsilon} \in L^2((0,T),H^1_{0,\sigma}(\Omega))$, $\partial_t u_{m,\varepsilon} \in L^2((0,T),H^{-1}(\Omega))$ is a weak solution of (1.2.15) if for every $\varphi \in C^\infty((0,T)\times\Omega)$ and for a.e. $t\in(0,T)$ it satisfies*

$$\langle \partial_t u_{m,\varepsilon},\varphi\rangle + \int_\Omega D(u_{m,\varepsilon}) : D(\varphi)\,dx + \langle j'_\varepsilon(u_{m,\varepsilon}),\varphi\rangle - \int_\Omega (u_{m,\varepsilon}\cdot\nabla u_{m,\varepsilon})\cdot\varphi\,dx = \langle f,\varphi\rangle. \quad (1.2.17)$$

We also say that (1.2.17) is the formulation in space of the solution of (1.2.15) when the time is fixed. We point out that this definition makes sense since we are studying smooth finite dimensional Galerkin solutions. Then, in order to obtain weak limits into the Galerkin formulation, we establish some estimates.

Proposition 1.2.20. *Assume that $u_{m,\varepsilon}$ is a solution of (1.2.15) in the sense of Definition 1.2.19. Then, up to some multiplicative positive constants depending on p, Ω, N, T, $\|u_0\|_{L^2(\Omega)}$ and $\|f\|_{L^2((0,T),H^{-1}(\Omega))}$, the following estimates hold:*

1. $\|u_{m,\varepsilon}\|^2_{L^\infty((0,T),L^2_\sigma)} + \frac{1}{2}\|u_{m,\varepsilon}\|^2_{L^2\left((0,T),H^1_{0,\sigma}\right)} \lesssim \|f\|^2_{L^2((0,T),H^{-1})} + \|u_0\|^2_{L^2}$;
2. $\|j'_\varepsilon(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})} \lesssim \left(1+\|f\|_{L^2((0,T),H^{-1})} + \|u_0\|_{L^2}\right)^{p-1}$;
3. $\|\partial_t u_{m,\varepsilon}\|_{L^{\frac{4}{N}}((0,T),H^{-1})} \lesssim \|f\|^2_{L^2((0,T),H^{-1})} + \|u_0\|^2_{L^2} + \left(\|f\|^2_{L^2((0,T),H^{-1})} + \|u_0\|^2_{L^2}\right)^2$
$$+ \left(1+\|f\|_{L^2((0,T),H^{-1})} + \|u_0\|_{L^2}\right)^{p-1}.$$

Before the proof of Proposition 1.2.20, we state the following result.

Lemma 1.2.21. *Let X be a Banach space, and $\gamma \geq \frac{1}{2}$. Then, the following inequality holds:*

$$\forall (u,v)\in X^2,\ \|u+v\|^\gamma_X \leq 2^{\left(\gamma-\frac{1}{2}\right)}\left(\|u\|^\gamma_X + \|v\|^\gamma_X\right).$$

Proof. Using the convexity of $t\mapsto t^{2(2-p)}$ and triangle's inequality of the norm, we get:

$$\|u+v\|^{2\gamma}_X = 2^{2\gamma}\left\|\frac{u+v}{2}\right\|^{2\gamma}_X \leq 2^{2\gamma-1}\left(\|u\|^{2\gamma}_X + \|v\|^{2\gamma}_X\right).$$

Applying now the well-known inequality: $\forall (a,b)\in[0,+\infty)^2,\ \sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$, we get the result. □

Proof of Proposition 1.2.20.

1. Setting $\varphi = u_{m,\varepsilon}$ in the weak formulation, we get:

$$\frac{1}{2}\frac{d}{dt}\|u_{m,\varepsilon}\|^2_{L^2} + \int_\Omega |D(u_{m,\varepsilon})|^2\,dx + \underbrace{\langle j'_\varepsilon(u_{m,\varepsilon}),u_{m,\varepsilon}\rangle}_{\geq 0} - \underbrace{\int_\Omega (u_{m,\varepsilon}\cdot\nabla u_{m,\varepsilon})\cdot u_{m,\varepsilon}\,dx}_{=0} = \langle f,u_{m,\varepsilon}\rangle.$$

Using the well-known Korn's L^2 equality for divergence free vectors fields, we get

$$\frac{d}{dt}\|u_{m,\varepsilon}(t)\|_{L^2}^2 + \|u_{m,\varepsilon}(t)\|_{H_0^1}^2 \leq 2\left\langle f(t), u_{m,\varepsilon}(t)\right\rangle.$$

Moreover, we have:

$$2\left\langle f(t), u_{m,\varepsilon}(t)\right\rangle \leq 2\|f(t)\|_{H^{-1}}^2 + \frac{1}{2}\|u_{m,\varepsilon}(t)\|_{H_0^1}^2.$$

Then, using the above inequality and integrating on $(0,t)$ we get

$$\|u_{m,\varepsilon}(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \|u_{m,\varepsilon}\|_{H_0^1}^2\, dt \leq 2\int_0^t \|f\|_{H^{-1}}^2\, dt + \|u_0\|_{L^2}^2. \tag{1.2.18}$$

Indeed, we recall that $(P_m(u_0), w_i)_{L^2} = (u_0, P_m w_i)_{L^2} = (u_0, w_i)_{L^2}$, and the conclusion follows. From now on, we will omit to detail this last part which is usual.

2. We have, using Cauchy-Schwarz's inequality and Korn's equality in the divergence free L^2 setting:

$$\begin{aligned}\left\langle j_\varepsilon'(u_{m,\varepsilon}), \varphi\right\rangle &= \int_\Omega F\left(\sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2}\right) D(u_{m,\varepsilon}) : D(\varphi)\, dx \\ &\leq \frac{1}{\sqrt{2}}\left(\int_\Omega F\left(\sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\, dx\right)^{\frac{1}{2}} \|\varphi\|_{H_0^1}. \end{aligned}\tag{1.2.19}$$

From hypothesis (C4), setting $A = \Omega \cap \{|D(u_{m,\varepsilon})| \leq t_0\}$ and B its complement in Ω, we obtain

$$\begin{aligned}\int_\Omega F\left(\sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\, dx &= \int_A F\left(\sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\, dx \\ &\quad + \int_B F\left(\sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\, dx.\end{aligned}$$

Let's estimate these two integrals independently. By assumption (C3), we have that the application $t \mapsto t^2 F\left(\sqrt{\varepsilon + t^2}\right)^2$ is non-decreasing, and we obtain directly:

$$\begin{aligned}\int_A F\left(\sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\, dx &\leq F\left(\sqrt{\varepsilon + {t_0}^2}\right)^2 {t_0}^2 |A| \\ &\leq F\left(\sqrt{\varepsilon + {t_0}^2}\right)^2 {t_0}^2 |\Omega| \\ &\leq F\left(\sqrt{1 + {t_0}^2}\right)^2 \sqrt{1 + {t_0}^2} |\Omega| \\ &\lesssim 1.\end{aligned}$$

Then we have, using again (C4):

$$\begin{aligned}\int_B F\left(\sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\, dx &\lesssim \int_B \frac{|D(u_{m,\varepsilon})|^2}{\left(\varepsilon + |D(u_{m,\varepsilon})|^2\right)^{2-p}}\, dx \\ &\lesssim \int_B |D(u_{m,\varepsilon})|^{2(p-1)}\, dx \\ &\lesssim K\int_B |\nabla u_{m,\varepsilon}|^{2(p-1)}\, dx \\ &\lesssim \|u_{m,\varepsilon}\|_{H_0^1}^{2(p-1)},\end{aligned}$$

where we used Jensen's inequality in the concave setting with $t \mapsto t^{p-1}$ in the last line. So, we obtain:

$$\left(\int_\Omega F\left(\sqrt{\varepsilon+|D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\,dx\right)^{\frac{1}{2}} \lesssim \left(1+\|u_{m,\varepsilon}\|_{H_0^1}^{2(p-1)}\right)^{\frac{1}{2}}. \tag{1.2.20}$$

Thus, combining the inequality (1.2.19)-(1.2.20) and using Lemma 1.2.21 with $\gamma = \frac{2}{N}$, we get:

$$\|j_\varepsilon'(u_{m,\varepsilon})\|_{H^{-1}}^{\frac{4}{N}} \lesssim 1+\|u_{m,\varepsilon}\|_{H_0^1}^{\frac{4(p-1)}{N}}.$$

Therefore, integrating in time over $(0,T)$:

$$\|j_\varepsilon'(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})}^{\frac{4}{N}} \lesssim 1+\|u_{m,\varepsilon}\|_{L^{\frac{4(p-1)}{N}}((0,T),H_0^1)}^{\frac{4(p-1)}{N}}.$$

Then, since $0 < \frac{4(p-1)}{N} \leq 2$, we get, using the embbedding $L^2 \hookrightarrow L^{\frac{4(p-1)}{N}}$ and Lemma 1.2.21 with $X := H_0^1$, $q = \frac{4(p-1)}{N}$ and $p = 2$ on $\|u_{m,\varepsilon}\|_{L^{\frac{4(p-1)}{N}}((0,T),H_0^1)}$:

$$\|j_\varepsilon'(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})}^{\frac{4}{N}} \lesssim 1+\|u_{m,\varepsilon}\|_{L^2((0,T),H_0^1)}^{\frac{4(p-1)}{N}}.$$

Using the first point of the proposition for $t = T$, and since $\frac{4(p-1)}{N} \geq 0$, we get:

$$\|j_\varepsilon'(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})}^{\frac{4}{N}} \lesssim 1+(\|f\|_{L^2((0,T),H^{-1})}+\|u_0\|_{L^2})^{\frac{4(p-1)}{N}}.$$

Then, using the exponent $\frac{N}{4}$ on both sides and applying once again Lemma 1.2.21 with $\gamma = \frac{N}{4}$ on the right-hand side in the ineuality above leads us to:

$$\|j_\varepsilon'(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})} \lesssim 1+(\|f\|_{L^2((0,T),H^{-1})}+\|u_0\|_{L^2})^{p-1}.$$

This is the wished result.

3. From the weak formulation (1.2.17) we get

$$\langle \partial_t u_{m,\varepsilon}, \varphi\rangle = -\int_\Omega D(u_{m,\varepsilon}) : D(\varphi)\,dx - \langle j_\varepsilon'(u_{m,\varepsilon}), \varphi\rangle + \int_\Omega (u_{m,\varepsilon}\cdot\nabla u_{m,\varepsilon})\cdot\varphi\,dx + \langle f, \varphi\rangle. \tag{1.2.21}$$

Let us point out that

$$\int_\Omega D(u_{m,\varepsilon}) : D(\varphi)\,dx = \frac{1}{2}\int_\Omega \nabla u_{m,\varepsilon}\cdot\nabla\varphi\,dx \leq \frac{1}{2}\|u_{m,\varepsilon}\|_{H_0^1}\|\varphi\|_{H_0^1}. \tag{1.2.22}$$

Also, setting $p = 4$, $k = 0$, $l = 1$, $r = q = 2$, and $s = 2$ into Theorem A.6.1, we get that up to a multiplicative positive constant which only depends on N and Ω, we have:

$$\|u\|_{L^4} \lesssim \|\nabla u\|_{L^2}^{\frac{N}{4}}\|u\|_{L^2}^{\frac{4-N}{4}}$$

and thus:

$$\|u\|_{L^4}^2 \lesssim \|\nabla u\|_{L^2}^{\frac{N}{2}}\|u\|_{L^2}^{\frac{4-N}{2}}. \tag{1.2.23}$$

Then, we have:

$$\begin{aligned}\left|\int_\Omega (u_{m,\varepsilon}.\nabla u_{m,\varepsilon}).\varphi\, dx\right| &\leq \frac{1}{2}\int_\Omega |u_{m,\varepsilon}|^2.|\nabla\varphi|\, dx \\ &\leq \|u_{m,\varepsilon}\|_{L^4}^2\|\nabla\varphi\|_{L^2} \\ &\lesssim \|u_{m,\varepsilon}\|_{L^2}^{\frac{4-N}{2}}\|u_{m,\varepsilon}\|_{H_0^1}^{\frac{N}{2}}\|\varphi\|_{H_0^1}. \end{aligned} \tag{1.2.24}$$

So, putting (1.2.22)–(1.2.24) and the second estimate of the Proposition 1.2.20 in (1.2.21), we obtain

$$\begin{aligned}\langle \partial_t u_{m,\varepsilon}, \varphi\rangle \lesssim \frac{1}{2}\|u_{m,\varepsilon}\|_{H_0^1}\|\varphi\|_{H_0^1} + \|j_\varepsilon'(u_{m,\varepsilon})\|_{H^{-1}}\|\varphi\|_{H_0^1} + \|u_{m,\varepsilon}\|_{L^2}^{\frac{4-N}{2}}\|u_{m,\varepsilon}\|_{H_0^1}^{\frac{N}{2}}\|\varphi\|_{H_0^1} \\ + \|f\|_{H^{-1}}\|\varphi\|_{H_0^1},\end{aligned}$$

and, therefore,

$$\|\partial_t u_{m,\varepsilon}(t)\|_{H^{-1}} \lesssim \frac{1}{2}\|u_{m,\varepsilon}\|_{H_0^1} + \|j_\varepsilon'(u_{m,\varepsilon})\|_{H^{-1}} + \|u_{m,\varepsilon}\|_{L^2}^{\frac{4-N}{2}}\|u_{m,\varepsilon}\|_{H_0^1}^{\frac{N}{2}} + \|f\|_{H^{-1}}.$$

Now, using the following convexity inequality

$$\forall k\in\mathbb{N},\ \forall (x_i)_{1\leq i\leq k}\in(0,+\infty)^k,\ \left(\sum_{i=1}^k x_i\right)^{\frac{4}{N}} \lesssim \sum_{i=1}^k x_i^{\frac{4}{N}}$$

we get:

$$\|\partial_t u_{m,\varepsilon}(t)\|_{H^{-1}}^{\frac{4}{N}} \lesssim \|u_{m,\varepsilon}\|_{H_0^1}^{\frac{4}{N}} + \|j_\varepsilon'(u_{m,\varepsilon})\|_{H^{-1}}^{\frac{4}{N}} + \|u_{m,\varepsilon}\|_{L^2}^{\frac{8-2N}{N}}\|u_{m,\varepsilon}\|_{H_0^1}^2 + \|f\|_{H^{-1}}^{\frac{4}{N}}.$$

Since $N\in\{2,3\}$, we have $\frac{4}{N}\leq 2$. Hence, integrating in time over $(0,T)$ and using the embedding $L^2(\Omega)\hookrightarrow L^{\frac{4}{N}}(\Omega)$:

$$\begin{aligned}\|\partial_t u_{m,\varepsilon}\|_{L^{\frac{4}{N}}((0,T),H^{-1})}^{\frac{4}{N}} &\lesssim \|u_{m,\varepsilon}\|_{L^2((0,T),H_0^1)}^{\frac{4}{N}} + \|j_\varepsilon'(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})}^{\frac{4}{N}} \\ &\quad + \|u_{m,\varepsilon}\|_{L^\infty((0,T),L^2)}^{\frac{8-2N}{N}}\|u_{m,\varepsilon}\|_{L^2((0,T),H_0^1)}^{\frac{4}{N}} + \|f\|_{L^2((0,T),H^{-1})}^{\frac{4}{N}}.\end{aligned}$$

Using the previously given convexity inequality and the first and second points of the proposition we obtain the desired result.

□

Third step: weak convergence

We are now interested in the weak convergence with respect to the energy estimates established in Proposition 1.2.20. Here, we prove such convergences by passing to the limit with respect to the parameter ε in a first time, then by passing to the limit with respect to the Galerkin parameter m.

Before proving Theorem 1.2.18, we establish several useful lemmas.

Lemma 1.2.22. *Consider that $\varphi \in L^2((0,T),H_0^1(\Omega))$, then there exists a constant $C(\varepsilon,\varphi)>0$ which goes to zero as ε does, such that the following inequality holds:*

$$j_\varepsilon(\varphi)+C(\varepsilon,\varphi)\geq j(\varphi), \tag{1.2.25}$$

where j_ε and j are defined by (1.2.11).

Proof. Recalling that the assumption (C3) states that $t\mapsto tF(t)$ is increasing, we get:

$$\begin{aligned}
j(\varphi) &:= \int_\Omega \int_0^{|D(\varphi)|} sF(s)\,ds\,dx \\
&\leq \int_\Omega \int_0^{\sqrt{\varepsilon}} sF(s)\,ds\,dx + \int_\Omega \int_{\sqrt{\varepsilon}}^{\sqrt{\varepsilon}+|D(\varphi)|} sF(s)\,ds\,dx \\
&\leq \varepsilon\sqrt{\varepsilon}F(\varepsilon)|\Omega| + \int_\Omega \int_0^{\sqrt{2|D(\varphi)|\sqrt{\varepsilon}+|D(\varphi)|^2}} sF(\sqrt{\varepsilon+s^2})\,ds\,dx \\
&\leq \underbrace{\varepsilon\sqrt{\varepsilon}F(\varepsilon)|\Omega| + \int_\Omega \int_{|D(\varphi)|}^{2^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}|D(\varphi)|^{\frac{1}{2}}+|D(\varphi)|} sF(\sqrt{\varepsilon+s^2})\,ds\,dx}_{:=C(\varepsilon,\varphi)} + j_\varepsilon(\varphi),
\end{aligned}$$

which is the wished result. □

Lemma 1.2.23. *Consider Ω an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, and a sequence $(w_n)_{n\in\mathbb{N}}$ such that $\|w_n\|_{L^2((0,T),H^1_{0,\sigma}(\Omega)}\lesssim 1$. Then, for almost all $(t,x)\in(0,T)\times\Omega$, the following inequality holds:*

$$\underline{\lim}_{n\to+\infty}\ |D(w_n)(t,x)|\geq |D(w)(t,x)|.$$

Proof. Firstly, let us recall that Eberlein-Šmulyan theorem leads up to an extraction to $w_n \rightharpoonup w$ in $L^2((0,T),H_0^1(\Omega))$ then, for all Lebesgue points $t_0\in(0,T)$ and $x_0\in\Omega$, for all $\delta>$ and $R>0$ small enough, we have $w_n\rightharpoonup w$ in $L^2((t_0-\delta,t_0+\delta),H^1(B(x_0,R))$. Indeed, we have for all test function φ :

$$\int_0^T\int_\Omega \nabla w_n\cdot\nabla\varphi\,dt\,dx \underset{n\to+\infty}{\longrightarrow} \int_0^T\int_\Omega \nabla w\cdot\nabla\varphi\,dt\,dx.$$

Hence, we can take φ, which belongs to $C_0^\infty((t_0-\delta,t_0+\delta)\times B(x_0,R))$ (up to arguing by density thereafter), satisfying:

$$\nabla\varphi=\begin{cases}\nabla\psi & \text{on } (t_0-\delta,t_0+\delta)\times B(x_0,R)\\ 0 & \text{on } (0,T)\times\Omega\backslash(t_0-\delta,t_0+\delta)\times B(x_0,R)\end{cases}$$

and so this leads to:

$$\int_{t_0-\delta}^{t_0+\delta}\int_{B(x_0,R)} \nabla w_n\cdot\nabla\psi\,dt\,dx \underset{n\to+\infty}{\longrightarrow} \int_{t_0-\delta}^{t_0+\delta}\int_{B(x_0,R)} \nabla w\cdot\nabla\psi\,dt\,dx.$$

That is $w_n\rightharpoonup w$ in $L^2((t_0-\delta,t_0+\delta),H^1(B(x_0,R)))$. Also, from Korn's L^2 equality and Lebesgue's differentiation theorem over (δ,R) after dividing by $2\delta|B(x_0,R)|$, one gets that for every Lebesgue point $(t_0,x_0)\in(0,T)\times\Omega$:

$$|D(w_n(t_0,x_0))|^2 \lesssim 1.$$

Following the same line of arguments, we find that:

$$\underline{\lim}_{n\to+\infty} \int_{t_0-\delta}^{t_0+\delta}\int_{B(x_0,R)} |D(w_n)|^2 \, dx\, dt \geq \int_{t_0-\delta}^{t_0+\delta}\int_{B(x_0,R)} |D(w)|^2 \, dx\, dt.$$

Dividing each side by $2\delta|B(x_0,R)|$, we get:

$$\underline{\lim}_{n\to+\infty} \fint_{t_0-\delta}^{t_0+\delta}\fint_{B(x_0,R)} |D(w_n)|^2 \, dx\, dt \geq \fint_{t_0-\delta}^{t_0+\delta}\fint_{B(x_0,R)} |D(w)|^2 \, dx\, dt$$

then letting $(\delta, R) \to (0,0)$ leads to the result, after applying a dominated convergence theorem. □

The following lemma gives the convergence of $u_{m,\varepsilon}$ when ε goes to zero.

Lemma 1.2.24. *With the hypotheses of Proposition 1.2.20 there exists* $v_m \in L^2\left((0,T), H^1_{0,\sigma}(\Omega)\right) \cap L^\infty\left((0,T), L^2_\sigma(\Omega)\right)$ *with* $\partial_t v_m \in L^{\frac{4}{N}}\left((0,T), H^{-1}_\sigma(\Omega)\right)$ *such that, up to subsequences:*

1. $\partial_t u_{m,\varepsilon} \rightharpoonup v_m'$ *in* $L^{\frac{4}{N}}\left((0,T), H^{-1}_\sigma(\Omega)\right)$;
2. $u_{m,\varepsilon} \rightharpoonup v_m$ *in* $L^2\left((0,T), H^1_{0,\sigma}(\Omega)\right)$;
3. $u_{m,\varepsilon} \to v_m$ *in* $L^2((0,T), L^2_\sigma(\Omega))$;
4. $u_{m,\varepsilon} \overset{*}{\rightharpoonup} v_m$ *in* $L^\infty\left((0,T), L^2_\sigma(\Omega)\right)$.

Moreover, v_m *satisfies, for all* $\psi \in C^\infty((0,T)\times\Omega)$*:*

$$\frac{1}{2}\left(\|v_m(T)\|^2_{L^2} - \frac{1}{2}\|u_0\|^2_{L^2}\right) - \int_0^T \langle \partial_t v_m, \psi\rangle \, dt + \int_0^T\int_\Omega D(v_m) : D(v_m - \psi)\, dx\, dt + \int_0^T j(v_m) - j(\psi)\, dt - \int_0^T\int_\Omega (v_m\cdot\nabla v_m)\cdot\psi \, dx\, dt \leq \int_0^T \langle f, v_m - \psi\rangle \, dt. \tag{1.2.26}$$

Proof. The first and second points follow from the reflexivity of $L^{\frac{4}{N}}\left((0,T), H^{-1}_\sigma(\Omega)\right)$ and $L^2((0,T), H^1_{0,\sigma}(\Omega))$ respectively, the third one from Aubin-Lions' Lemma, and the last one by Banach-Alaoglu-Bourbaki's theorem.

Then, since $u_{m,\varepsilon}$ is a solution of (1.2.15), it satisfies (1.2.17). Testing against $\varphi = u_{m,\varepsilon} - \psi$ in (1.2.17) for a test function ψ, we have:

$$\langle \partial_t u_{m,\varepsilon}, u_{m,\varepsilon} - \psi\rangle + \int_\Omega D(u_{m,\varepsilon}) : D(u_{m,\varepsilon} - \psi)\, dx + \langle j_\varepsilon'(u_{m,\varepsilon}), u_{m,\varepsilon} - \psi\rangle - \int_\Omega (u_{m,\varepsilon}\cdot\nabla u_{m,\varepsilon})\cdot\psi \, dx = \langle f, u_{m,\varepsilon} - \psi\rangle. \tag{1.2.27}$$

Applying Lemma 1.2.15 leads to the well-known convexity inequality:

$$j_\varepsilon(u_{m,\varepsilon}) - j_\varepsilon(\psi) \leq \langle j_\varepsilon'(u_{m,\varepsilon}), u_{m,\varepsilon} - \psi\rangle. \tag{1.2.28}$$

Using now Lemma 1.2.22 for $u_{m,\varepsilon}$ in (1.2.28), we get:

$$j(u_{m,\varepsilon}) - C(\varepsilon, u_{m,\varepsilon}) - j_\varepsilon(\psi) \leq \langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon} - \psi \rangle$$

and then, by (C3) applied to $u_{m,\varepsilon}$ for the convergence toward v_m, we get:

$$j(u_{m,\varepsilon}) - C(\varepsilon, u_{m,\varepsilon}) - j_\varepsilon(\psi) \leq \langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon} - \psi \rangle.$$

Then, we can write (see [84] part 5.9. for details):

$$\forall \varphi \in H^1_{0,\sigma}(\Omega),\ \int_\Omega u_{m,\varepsilon}(T)\varphi\, dx = \langle u_{m,\varepsilon}(T), \varphi \rangle = \int_0^T \langle \partial_t u_{m,\varepsilon}(t), \varphi \rangle\, dt + \langle u_0, \varphi \rangle\,. \tag{1.2.29}$$

Now, we also have, using Proposition 1.2.20:

$$\begin{aligned}\int_0^T \langle \partial_t u_{m,\varepsilon}(t), \varphi \rangle\, dt + \langle u_0, \varphi \rangle &\lesssim \|\partial_t u_{m,\varepsilon}\|_{L^{\frac{4}{N}}((0,T),H^{-1})} \left(\int_0^T \|\varphi\|_{H^1_0}^{\frac{4}{4-N}}\, dt \right)^{\frac{4-N}{4}} + \|u_0\|_{L^2}\|\varphi\|_{H^1_0} \\ &\lesssim \left(T^{\frac{4-N}{4}} + \|u_0\|_{L^2} \right) \|\varphi\|_{H^1_0}.\end{aligned}$$

In the above inequality we considered φ as a function in $L^\infty((0,T), H^1_0(\Omega))$, so it belongs to $L^{\frac{4}{4-N}}((0,T), H^1_0(\Omega))$ and its left-hand side defines a linear form over $L^{\frac{4}{N}}((0,T), H^{-1}(\Omega))$.
Also, the weak convergence leads to:

$$\int_0^T \langle \partial_t u_{m,\varepsilon}(t), \varphi \rangle\, dt \underset{\varepsilon\to 0}{\longrightarrow} \int_0^T \langle \partial_t v_m(t), \varphi \rangle\, dt. \tag{1.2.30}$$

Finally, (1.2.29) and (1.2.30) imply, up to apply a dominated convergence theorem, to:

$$u_{m,\varepsilon}(T) \underset{\varepsilon\to 0}{\rightharpoonup} v_m(T) \quad \text{in} \quad L^2(\Omega). \tag{1.2.31}$$

Then, (1.2.31) implies:

$$\underline{\lim}_{\varepsilon\to 0} \frac{1}{2} \left(\|u_{m,\varepsilon}(T)\|^2_{L^2} - \|P_m(u_0)\|^2_{L^2} \right) \geq \frac{1}{2} \left(\|v_m(T)\|^2_{L^2} - \|P_m(u_0)\|^2_{L^2} \right) \tag{1.2.32}$$

Also, from usual estimates (see [155, Chapter 4]), since $u_{m,\varepsilon} \underset{\varepsilon\to 0}{\rightharpoonup} v_m$ in $L^2((0,T), H^1_{0,\sigma}(\Omega))$, we have up to extract:

$$\int_0^T \int_\Omega |D(u_{m,\varepsilon})|^2\, dx\, dt \underset{\varepsilon\to 0}{\longrightarrow} \int_0^T \int_\Omega |D(v_m)|^2\, dx\, dt \tag{1.2.33}$$

and

$$\int_0^T \int_\Omega (u_{m,\varepsilon} \cdot \nabla u_{m,\varepsilon}) \cdot \psi\, dx\, dt \underset{\varepsilon\to 0}{\longrightarrow} \int_0^T \int_\Omega (v_m \cdot \nabla v_m) \cdot \psi\, dx\, dt. \tag{1.2.34}$$

Integrating in time (1.2.27), and passing to the limit over ε, combining with (1.2.33), (1.2.34), (1.2.32) and Lemma 1.2.23 leads to (1.2.26). □

Arguing in the same way, we obtain the following result.

Lemma 1.2.25. *Under the assumptions of Proposition 1.2.20, there exists* $u \in L^2\left((0,T),H^1_{0,\sigma}(\Omega)\right)\cap L^\infty\left((0,T),L^2_\sigma(\Omega)\right)$ *with* $\partial_t u \in L^{\frac{4}{N}}\left((0,T),H^{-1}_\sigma(\Omega)\right)$ *such that the function* v_m *given by Lemma 1.2.24 verifies.*

1. $\partial_t v_m \rightharpoonup \partial_t u$ *in* $L^{\frac{4}{N}}\left((0,T),H^{-1}_\sigma(\Omega)\right)$;
2. $v_m \to u$ *in* $L^2\left((0,T),L^2_\sigma(\Omega)\right)$;
3. $v_m \rightharpoonup u$ *in* $L^2((0,T),H^1_{0,\sigma}(\Omega))$;
4. $v_m \overset{*}{\rightharpoonup} u$ *in* $L^\infty\left((0,T),L^2_\sigma(\Omega)\right)$.

Moreover, we point out that $u \in C_w((0,T),L^2_\sigma(\Omega))$ from the above estimates (see [31, Proposition V.1.7. p.363] for details).

Fourth step: proof of the Theorem

We point out that the coefficients of v_m given by Lemma 1.2.24 satisfy an ODE as (1.2.16) with $\varepsilon = 0$, then v_m is still smooth in space and time. Moreover, we can take up again the method previously used, that is we can write :

$$\forall \varphi \in H^1_{0,\sigma}(\Omega),\ \int_\Omega v_m(T)\varphi\,dx = \langle v_m(T),\varphi\rangle = \int_0^T \langle \partial_t v_m(t),\varphi\rangle\,dt + \langle P_m(u_0),\varphi\rangle\,. \tag{1.2.35}$$

Using Proposition 1.2.20 then leads to:

$$\begin{aligned}\int_0^T \langle \partial_t v_m(t),\varphi\rangle\,dt + \langle u_0,\varphi\rangle &\lesssim \|\partial_t v_m\|_{L^{\frac{4}{N}}((0,T),H^{-1})}\left(\int_0^T \|\varphi\|_{H^1_0}^{\frac{4}{4-N}}\,dt\right)^{\frac{4-N}{4}} + \|u_0\|_{L^2}\|\varphi\|_{H^1_0}\\ &\lesssim \left(T^{\frac{4-N}{4}} + \|u_0\|_{L^2}\right)\|\varphi\|_{H^1_0}.\end{aligned} \tag{1.2.36}$$

Then, the weak convergence leads to:

$$\int_0^T \langle \partial_t v_m(t),\varphi\rangle\,dt \underset{\varepsilon\to 0}{\longrightarrow} \int_0^T \langle \partial_t v_m(t),\varphi\rangle\,dt. \tag{1.2.37}$$

Finally, (1.2.35) and (1.2.37) imply:

$$v_m(T) \underset{\varepsilon\to 0}{\rightharpoonup} u(T) \quad \text{in} \quad L^2(\Omega). \tag{1.2.38}$$

Then, (1.2.31) implies:

$$\varliminf_{m\to+\infty} \frac{1}{2}\left(\|v_m(T)\|^2_{L^2} - \|P_m(u_0)\|^2_{L^2}\right) \geq \frac{1}{2}\left(\|u(T)\|^2_{L^2} - \|u_0\|^2_{L^2}\right) \tag{1.2.39}$$

Using once again usual estimates for Navier-Stokes equation, since $v_m \underset{m\to+\infty}{\longrightarrow} u$ in $L^2((0,T),H^1_{0,\sigma}(\Omega))$, we have:

$$\int_0^T\int_\Omega |D(v_m)|^2\,dx\,dt \underset{m\to+\infty}{\longrightarrow} \int_0^T\int_\Omega |D(u)|^2\,dx\,dt \tag{1.2.40}$$

and

$$\int_0^T \int_\Omega (v_m \cdot \nabla v_m) \cdot \psi \, dx \, dt \underset{m \to +\infty}{\longrightarrow} \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \psi \, dx \, dt. \tag{1.2.41}$$

Applying lemma 1.2.23 with our assumption (C3) and passing to the limit over m, we get:

$$\underline{\lim}_{m \to +\infty} \int_0^T j(v_m) \, dt \geq j(u). \tag{1.2.42}$$

Passing to the limit over m in (1.2.36), combining with (1.2.40), (1.2.41), (1.2.42) and (1.2.39) leads to:

$$\frac{1}{2}\left(\|u(T)\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)}\right) - \int_0^T \langle \partial_t u, \psi \rangle \, dt + \int_0^T \int_\Omega D(u) : D(u - \psi) \, dx \, dt + \int_0^T j(u) - j(\psi) \, dt - \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \psi \, dx \, dt \leq \int_0^T \langle f, u - \psi \rangle \, dt \tag{1.2.43}$$

which is the desired result, that is u is a weak solution of (2.2.8). □

The previous theorem, proved in [44], therefore shows the existence of solutions for a class of shear-thinning non-Newtonian flows, taken as the limit by the regularization usually studied for numerical simulations (see, for example, [158]). We can cite, for example, some possible choices of function F satisfying assumptions (C1)-(C4) and leading to non-Newtonian behavior.

1. Firstly, in order to describe power-law fluids (also known as Ostwald-De Waele or Norton-Hoff flows), we can consider functions $(F_p)_{1<p<2}$ given by:

$$F_p : \left| \begin{array}{l} (0, +\infty) \to (0, +\infty) \\ \\ t \longmapsto t^{p-2}. \end{array} \right.$$

2. Considering functions $(F_{\mu,p})_{\mu>0, p \in [1,2)}$ of the form

$$F_{\mu,p} : \left| \begin{array}{l} (0, +\infty) \to (0, +\infty) \\ \\ t \longmapsto (\mu + t^2)^{\frac{p-2}{2}} \end{array} \right.$$

   leads to Carreau flows.

3. Cross fluids are obtained by choosing function $(F_{\gamma,\alpha})_{\gamma>0, p \in [1,2)}$ given by:

$$F_{\gamma,p} : \left| \begin{array}{l} (0, +\infty) \to (0, +\infty) \\ \\ t \longmapsto \gamma + t^{p-2}. \end{array} \right.$$

4. Another possible choice is to take functions $(F_{p,\beta,\gamma})$ given

$$F_{p,\beta,\gamma}: \left| \begin{array}{l} (0,+\infty) \to (0,+\infty) \\ \\ t \longmapsto \left\{ \begin{array}{ll} t^{p-2}\log(1+t)^{-\beta} & \text{If } t \in (0,\gamma] \\ \\ \log(1+\gamma)^{-\beta} t^{p-2} & \text{If } t \in (\gamma,+\infty) \end{array} \right. \end{array} \right.$$

for $1 < p < 2$ and some $\beta, \gamma > 0$ with γ small enough.

Remark 1.2.26. *The reader may wonder whether similar results exist when density is not constant. In this context, for the two-dimensional Bingham case, the existence of solutions in the form of a variational inequality of the type of Theorem 1.2.18 was established in [28]. In addition, it is possible to show the existence of similar solutions with some measure regularity for the power and Bingham laws in a three-dimensional setting, see [90]. However, this last result is outside the scope of our study, as the spaces considered are part of a microlocal analysis, and we refer to [5] for an introduction to the analysis of such spaces.*

1.3 Finite stopping time property

An interesting property of shear-thinning fluids is that their flow stops in finite time. This means that there is a time after which the fluid no longer flows, provided it is not subjected to any external force inducing displacement. This phenomenon has been studied for classes of parabolic equations. Take, for example, the case of the parabolic p-Laplacian equation for $1 < p < 2$.

$$\left\{ \begin{array}{ll} \partial_t u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \\ u_{|t=0} = u_0 & \text{in } \Omega \end{array} \right. \tag{1.3.1}$$

More precisely, the following result holds.

Theorem 1.3.1 (Di Benedetto (stopping time))**.** *Let Ω be a bounded domain of $\mathbb{R}^N$ and $u \in C(\mathbb{R}_+, L^2(\Omega)) \cap L^p(\mathbb{R}_+, W_0^{1,p}(\Omega))$ be the unique non negative weak solution of* (1.3.1) *with initial datum $u_0 \in L^\infty(\Omega)$ assumed to be non negative. Then, there exists a finite time T^* depending only upon N, p, and u_0 such that $u(\cdot, t) \equiv 0$ for every $t \geq T^*$.*

Proof. We refer to [71, Chapter VII, Proposition 2.1.]. □

This raises the question of whether such a finite stopping time result can be established for solutions in the form of a variational inequality as in Theorem 1.2.18. First, let us give a sense to the existence of a finite stopping time.

Definition 1.3.2 (Finite stopping time)**.** *Let u be a weak solution in the sense of Definition 1.2.17 of the system* (2.2.8)*. We say that $T_0 \in (0,T)$ is a finite stopping time for u if:*

$$\|u(T_0)\|_{L^2(\Omega)} = 0.$$

In order to prove the existence of a finite stopping time for the solution u provided by Theorem 1.2.18, we do not make any assumption on the initial velocity field, but we assume that after a certain time the fluid is no longer subjected to any external force. More exactly we make some more assumption on F as stated by the following theorem.

Theorem 1.3.3 (Existence of a finite stopping time). *Assume that the hypotheses of Theorem 1.2.18 are verified, that $T > 0$ is choosen large enough, and let $p \in [1,2)$. Moreover, we assume that there exists two positive constants κ and $T_1 < T$ such that*

$$F(t) \geq \kappa t^{p-2} \text{ for every } t \in (0,+\infty) \qquad \text{and} \qquad f = 0 \quad \text{almost everywhere on } (T_1, T). \tag{1.3.2}$$

Then, there exists a finite stopping time $T_0 \in (0,T)$ for u in the sense of Definition 1.3.2.

We point out that the proof being well-known in the two-dimensional case (see [70]) and can be in that last case a direct application of the Korn's inequality and Sobolev's embbedding theorem. For this reason, we only give a proof in the three-dimensional setting.

Proof of Theorem 1.3.3. Let $u_{m,\epsilon}$ be the solution of (1.2.15). Choosing $\varphi = u_{m,\varepsilon}$ in (1.2.17) we get:

$$\langle u'_{m,\varepsilon}, u_{m,\varepsilon}\rangle + \int_\Omega |D(u_{m,\varepsilon})|^2\, dx + \langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon}\rangle - \underbrace{\int_\Omega (u_{m,\varepsilon}\nabla u_{m,\varepsilon})\, u_{m,\varepsilon}\, dx}_{=0} = \langle f, u_{m,\varepsilon}\rangle. \tag{1.3.3}$$

Combining (1.2.11) and (1.3.2), we obtain

$$\langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon}\rangle \geq \int_\Omega |D(u_{m,\varepsilon})|^2 \left(\varepsilon + |D(u_{m,\varepsilon})|^2\right)^{\frac{p-2}{2}}\, dx \tag{1.3.4}$$

Now, observing that:

$$|D(u_{m,\varepsilon})|^2 = \varepsilon + |D(u_{m,\varepsilon})|^2 - \varepsilon \qquad \text{and} \qquad \left(\varepsilon + |D(u_{m,\varepsilon})|^2\right)^{\frac{p-2}{2}} \leq \varepsilon^{\frac{p-2}{2}},$$

and since:

$$|D(u_{m,\varepsilon})|^p \leq (\varepsilon + |D(u_{m,\varepsilon})|^2)^{\frac{p}{2}},$$

we get from (1.3.3) and (1.3.4), we get:

$$\frac{1}{2}\frac{d}{dt}\left(\|u_{m,\varepsilon}(t)\|^2_{L^2(\Omega)}\right) + \int_\Omega |D(u_{m,\varepsilon})|^2\, dx + \|D(u_{m,\varepsilon})\|^p_{L^p(\Omega)} \lesssim \langle f, u_{m,\varepsilon}\rangle + |\Omega|\varepsilon^{\frac{p}{2}}. \tag{1.3.5}$$

Then, using successively the embbedding $L^p(\Omega) \hookrightarrow L^1(\Omega)$, assumption (1.3.2) and the Theorem A.6.3, we get from (1.3.5), for $t \in (T_1, t)$:

$$\frac{1}{2}\frac{d}{dt}\left(\|u_{m,\varepsilon}(t)\|^2_{L^2(\Omega)}\right) + \int_\Omega |D(u_{m,\varepsilon})|^2\, dx + \|u_{m,\varepsilon}\|^p_{L^{\frac{3}{2}}(\Omega)} \lesssim |\Omega|\varepsilon^{\frac{p}{2}}. \tag{1.3.6}$$

Now, from the embbedding $\left\{u \in H^1_0(\Omega)/\|D(u)\|_{L^2(\Omega)} < +\infty\right\} \hookrightarrow L^6(\Omega)$ which can be obtained using Korn's L^2 equality and Sobolev embbedding $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$ we get from (1.3.6):

$$\frac{1}{2}\frac{d}{dt}\left(\|u_{m,\varepsilon}(t)\|^2_{L^2(\Omega)}\right) + \left(\|u_{m,\varepsilon}\|^2_{L^6(\Omega)} + \|u_{m,\varepsilon}\|^p_{L^{\frac{3}{2}}(\Omega)}\right) \lesssim |\Omega|\varepsilon^{\frac{p}{2}}. \tag{1.3.7}$$

Now, we can apply Lemma A.6.2 with $r = \frac{3p}{p+4}$ to get:

$$\|u_{m,\varepsilon}\|_{L^2(\Omega)}^{1-2\left(\frac{2-p}{p+4}\right)} \leq \left(\frac{4}{p+4}\right)\|u_{m,\varepsilon}\|_{L^{\frac{3}{2}}(\Omega)}^{p} + \left(\frac{p}{p+4}\right)\|u_{m,\varepsilon}\|_{L^6(\Omega)}^2. \tag{1.3.8}$$

Then, (1.3.8) combined with (1.3.7) leads to, for $t \in (T_1, T)$:

$$\frac{1}{2}\frac{d}{dt}\left(\|u_{m,\varepsilon}(t)\|_{L^2(\Omega)}^2\right) + \|u_{m,\varepsilon}\|_{L^2(\Omega)}^{1-2\left(\frac{2-p}{p+4}\right)} \lesssim |\Omega|\varepsilon^{\frac{p}{2}}. \tag{1.3.9}$$

Assume that for all $t \in (T_1, T)$, we get that $\|u_{m,\varepsilon}\|_{L^2(\Omega)} \geq \frac{1}{\mu}|\Omega|\varepsilon^{\frac{p}{2}}$, for some $\mu > 0$ small enough. Then dividing by $\|u_{m,\varepsilon}(t)\|_{L^2(\Omega)}$ the both sides of (1.3.9), we obtain for almost all $t \in (T_1, T)$:

$$\frac{d}{dt}\left(\|u_{m,\varepsilon}(t)\|_{L^2(\Omega)}\right) + \|u_{m,\varepsilon}(t)\|_{L^2(\Omega)}^{-2\left(\frac{2-p}{p+4}\right)} \lesssim |\Omega|\varepsilon^{\frac{p}{2}},$$

which is equivalent to:

$$\frac{d}{dt}\left(\|u_{m,\varepsilon}(t)\|_{L^2(\Omega)}^{\frac{8-p}{p+4}}\right) \lesssim \mu - 1.$$

Up to take $\mu < 1$, integrating over (T_1, t), we have for almost all $t \in (T_1, T)$:

$$\|u_{m,\varepsilon}(t)\|_{L^2(\Omega)}^{\frac{8-p}{p+4}} \lesssim \|u_{m,\varepsilon}(T_1)\|_{L^2(\Omega)}^{\frac{8-p}{p+4}} + (1-\mu)(T_1 - t),$$

which leads to $\|u_{m,\varepsilon}(t)\|_{L^2(\Omega)}^{\frac{8-p}{p+4}} < 0$ for t large enough, up to take T large enough. This is a contradiction, so, there exists a time $T_0 \in (0, T)$ such that $\|u_{m,\varepsilon}(t)\| \leq \frac{1}{\mu}|\Omega|\varepsilon^{\frac{p}{2}}$, and so letting $\varepsilon \to 0$ leads to $\|v_m(t)\|_{L^2(\Omega)} = 0$ for all $t \in [T_0, T)$ since it leads to $\|v_m\|_{L^2((T_0,T),L^2(\Omega))} = 0$ from the weak convergence in Lemma 1.2.24. The same line of arguments shows the existence of a finite stopping time for u in the sense of Definition 1.3.2. This concludes the proof. □

It should be pointed out that finite stopping time for parabolic systems can be shown for a wide class of systems, under varying assumptions. We refer the interested author to [8, Sections 2.2.3 and 3.7] for an in-depth discussion of the subject.

D'après ce que j'ai entendu dire de certaines tentatives dans un sens novateur, je suis convaincu qu'on peut surmonter ces limitations de la science actuelle, qu'on peut donc développer une science qui soit directement et constamment subordonnée à nos besoins et nos désirs, dans laquelle il n'y aurait plus de séparation arbitraire entre l'activité scientifique et l'ensemble de nos modes de connaissance, où il n'y aurait plus de séparation arbitraire entre la science et notre vie.

Alexandre Grothendieck, *Allons-nous continuer la recherche scientifique?.*

2

Control theory for quasilinear parabolic partial differential equations

This chapter is devoted to the study of the null controllability of quasilinear parabolic equations with p-growth. First, we present in Section 2.1 the Hilbert Uniqueness Method introduced in [135], adapted to a particular framework of diffusive linear parabolic equations. This method makes it possible to show the existence of regular controls for such linear equations, which is the object of the well-known Theorem 2.1.2. We will show that it is in fact possible to adapt the method in order to obtain smooth approximate controls depending Lipschitz continuously on the diffusion coefficient, which is the main novelty of this section and is the subject of Theorem 2.1.3. Secondly, in Section 2.2, we present controllability results on quasilinear parabolic systems. To this end, we will first present in subsection 2.2.1 known results in the case where the nonlinearity depends on the state, i.e. on the solution of the equation. Then we present in Subsection 2.2.2 known results for control of such equations in the case of a dependence of the gradient of the state in the nonlinear term, first from an optimal control point of view and then from an exact or approximate control point of view. Apart from Theorem 2.2.3 and Theorem 2.2.5, the results presented in the Subsection 2.2.2 are new and are the main subject of our work [49].

2.1 Hilbert Uniqueness Method

Here, we present the Hilbert Uniqueness Method, which we shall hereafter abbreviate by the acronym HUM. Historically, this method was introduced in the 1980s, and it is often accepted that the first explicit presentation of this method can be found in [135]. Since then, this method has been the source of many developments, one of the major issues being to show observability inequalities, an example of which we present in Proposition 2.1.1 (we refer, for example, to [168, 11] for a discussion on this subject). The proof of such inequalities relies essentially on the existence of Carleman-type inequalities (see e.g. [89, 96, 129]). Note that HUM can be employed in many settings, much more general than the one presented here. For a detailed presentation in the parabolic case, we refer mainly to [30, 99] (see also [89]) while we can refer the interested reader to [133, 134] for a presentation in a hyperbolic setting.

2.1.1 Approximate controllability and regularity

We consider the following generic system for a domain Ω with Lipschitz boundary.

$$\begin{cases} \partial_t u - \operatorname{div}(a(t,x)\nabla u) = \chi_\omega \varphi & \text{in } (0,T)\times\Omega \\ u = 0 & \text{on } \partial\Omega \\ u_{|t=0} = u^0 & \text{in } \Omega \end{cases} \tag{2.1.1}$$

where $a \in C^\infty(\overline{Q_T})$ satisfies:

$$0 < \rho_\star \leq a(t,x) \qquad ((t,x) \in \overline{Q_T}), \tag{2.1.2}$$

where $\rho_\star > 0$ is a constant. We will say that (2.1.1) is approximately null controllable at time $T > 0$ if there exists a force term φ, whose regularity we will discuss later, such that we have

$$\|u(T)\|_{L^2(\Omega)} \leq \varepsilon \tag{2.1.3}$$

It is natural that the force term φ depends on ε and, a fortiori, so does the solution u. We have omitted these dependencies here for the sake of simplicity. The distributed controllability of equation (2.1.1) is a well studied subject (see for example [96] or the more recent review paper [89]). The existence of an optimal distributed control, in the sense of minimal L^2-norm, can be obtained by applying the Hilbert Uniqueness Method (HUM) introduced in [135]. The main idea of the method, that we present here through an example, is to consider the dual final boundary value problem of (2.1.1) given by:

$$\begin{cases} \partial_t \varphi + \operatorname{div}\left(a(t,x)\nabla\varphi\right) = 0 & \text{in } Q_T \\ \varphi = 0 & \text{on } \Sigma_T \\ \varphi(T) = \varphi^0 & \text{in } \Omega \end{cases} \tag{2.1.4}$$

for some $\varphi^0 \in L^2(\Omega)$ and then to deduce the controllability of (2.1.1). In this chapter, we will denote by $\|\cdot\|$ the usual norm over $L^2(\Omega)$, associated to the inner product that we will denote by $(\cdot,\cdot)$. Also, for every $\delta \geq 0$ we consider the functional:

$$J^a_\delta(\varphi^0) = \frac{1}{2}\iint_{q_T} \chi_\omega |\mathcal{S}^a(M_\delta(\varphi^0))|^2 \, dx\, dt + \frac{\varepsilon}{2}\|\varphi^0\|^2 + (\mathcal{S}^a(M_\delta(\varphi^0))(0), y^0), \tag{2.1.5}$$

where $\mathcal{S}^a(\varphi^0)$ is the solution of (2.1.4) with a diffusion coefficient a satisfying (2.1.2).

Let $\varepsilon > 0$. For every $\delta > 0$ we denote $M_\delta(\varphi^0) \in C^\infty(\Omega)$ a mollification of some $\varphi^0 \in L^2(\Omega)$ (see [32]) such that $\|M_\delta(\varphi^0) - \varphi^0\| \to 0$ when $\delta \to 0$. Following this notation we set $M_0(\varphi^0) := \varphi^0$. We then recall the following result.

Proposition 2.1.1 (Observability inequality [89, Theorem 1.5.]). *There exists a constant $C_0 > 0$, depending of Ω, T, ω, and $\|a\|_{L^\infty(Q_T)}$, such that the following inequality holds:*

$$\|\mathcal{S}^a(M_\delta(\varphi^0))(0)\|^2 \leq C_0 \iint_{q_T} \chi_\omega |\mathcal{S}^a(M_\delta(\varphi^0))|^2 \, dx\, dt \qquad (\varphi^0 \in L^2(\Omega)). \tag{2.1.6}$$

We then have the following result.

Theorem 2.1.2. *Let us denote φ^0 the minimizer of J^a_0 in $L^2(Q_T)$. Then, $\mathcal{S}^a(\varphi^0)$ is an approximate control of the solution u of (2.1.1) in the sense of (2.1.3).*

Proof. Let us first consider $\psi^0 \in L^2(\Omega)$, then testing into the weak formulation of (2.1.1), one gets:

$$\int_0^T \left(\partial_t u, \mathcal{S}^a\left(\psi^0\right)\right)\, dt + \iint_{Q_T} a\nabla u \nabla \mathcal{S}^a(\psi^0)\, dx\, dt = \iint_{q_T} \chi_\omega \mathcal{S}^a(\overline{\varphi^0})\, dx\, dt, \tag{2.1.7}$$

Where $\overline{\varphi^0}$ is the minimizer of J_0^a, that is of (2.1.5) with $\delta = 0$. Now, we point out that we have (at least formally, it can be rigorously proved by regularity of the solutions):

$$\int_0^T \left(\partial_t u, \mathcal{S}^a\left(\psi^0\right)\right)\, dt = \left(u(T), \psi^0\right) - \left(u^0, \mathcal{S}^a\left(\psi^0\right)(0)\right) - \int_0^T \left(u, \partial_t \mathcal{S}^a\left(\psi^0\right)\right)\, dt. \tag{2.1.8}$$

However, testing against u into the weak formulation of (2.1.4) leads to the formula:

$$-\int_0^T \left(u, \partial_t \mathcal{S}^a\left(\psi^0\right)\right)\, dt = -\iint_{Q_T} a\nabla u \nabla \mathcal{S}^a(\psi^0)\, dx\, dt. \tag{2.1.9}$$

Combining (2.1.8) and (2.1.9), it comes from (2.1.7):

$$\left(u(T), \psi^0\right) - \left(u^0, \mathcal{S}^a\left(\psi^0\right)(0)\right) = \iint_{q_T} \chi_\omega \mathcal{S}^a\left(\overline{\varphi^0}\right) \mathcal{S}^a(\psi^0)\, dx\, dt. \tag{2.1.10}$$

Then, since $\overline{\varphi^0}$ is the minimizer of J_0^a, it satisfies its associted Euler-Lagrange formulation, so testing into this last against ψ^0 leads to:

$$\iint_{q_T} \chi_\omega \mathcal{S}^a\left(\overline{\varphi^0}\right) \mathcal{S}^a(\psi^0)\, dx\, dt + \varepsilon\left(\overline{\varphi^0}, \psi^0\right) + \left(\mathcal{S}^a(\psi^0)(0), u^0\right) = 0. \tag{2.1.11}$$

Adding now (2.1.10) and (2.1.11) leads to:

$$\left(u(T) + \varepsilon\overline{\varphi^0}, \psi^0\right) = 0. \tag{2.1.12}$$

The last equality being true for every $\psi^0 \in L^2(\Omega)$, it then follows that

$$u(T) = -\varepsilon\overline{\varphi^0}, \tag{2.1.13}$$

and thus

$$\|u(T)\| \leq \varepsilon \|\overline{\varphi^0}\|. \tag{2.1.14}$$

It remains to prove that $\|\overline{\varphi^0}\|$ remains bounded uniformly relatively to ε. But, from (2.1.11) with $\psi^0 = \overline{\varphi^0}$ combined with (2.1.6), one gets:

$$\left\|\chi_\omega \mathcal{S}^a\left(\overline{\varphi^0}\right)\right\|^2_{L^2(Q_T)} \lesssim \|u^0\|^2. \tag{2.1.15}$$

Now, testing against $\mathcal{S}^a\left(\overline{\varphi^0}\right)$ into the weak formulation of (2.1.4) associated to the final datum $\overline{\varphi^0}$ leads to, using once again (2.1.6)

$$\|\overline{\varphi^0}\|^2 \lesssim \left\|\chi_\omega \mathcal{S}^a\left(\overline{\varphi^0}\right)\right\|^2_{L^2(Q_T)}. \tag{2.1.16}$$

Combining (2.1.14), (2.1.15), and (2.1.16) leads to the result. □

Clearly, the previous result leads, in the case of a linear system, to the existence of exact control of the solution when $\varepsilon \to 0$. Also, there is several ways to define a suitable functional in this sense. However, we are not going to deal with this case here. The previous theorem has many generalizations. We refer you mainly to [30] for a detailed presentation in a general framework, clarifying the duality link we have implied. More precisely, the preceding result is closely related to the Fenchel-Rockafellar duality, which applies to variational solutions induced by functionals (for which we refer to [79, Part two] for a detailed discussion).

We now aim to show here that minimizing J^a_δ over $L^2(\Omega)$ for $\delta > 0$ we obtain a control φ_δ which is now in $C^\infty(\overline{Q_T})$ which is close to the one obtained by minimizing J^a_0, so, for $\delta > 0$ small enough and for this regular control, the solution u of (2.1.1) still verifies (2.1.3). More precisely, we have the following result.

Theorem 2.1.3 ([49, Theorem 1.1.]). *Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, $u^0 \in L^2(\Omega)$, $a \in C^\infty(\overline{Q_T})$ satisfying (2.1.2) and $T > 0$. Then, for every $\varepsilon > 0$ there exists an approximate control $\varphi \in C^\infty(\overline{Q_T})$ in the sense that the corresponding solution u of (2.1.1) verifies*

$$\|u(T)\| < \varepsilon.$$

Moreover, the control φ depends Lipschitz continuously to the diffusion coefficient a for the norm $\|\cdot\|_{L^2(Q_T)}$.

For every $\delta > 0$ we denote $\overline{\varphi^0_\delta}$ the minimum of J^a_δ. Therefore, for every $\psi^0 \in L^2(\Omega)$ we have

$$\iint_{q_T} \chi_\omega \mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta}))\mathcal{S}^a(M_\delta(\psi^0))\, dx\, dt + \varepsilon(\overline{\varphi^0_\delta}, \psi^0) + (\mathcal{S}^a(M_\delta(\psi^0))(0), y^0) = 0 \tag{2.1.17}$$

$$\iint_{q_T} \chi_\omega \mathcal{S}^a(\overline{\varphi^0})\mathcal{S}^a(\psi^0)\, dx\, dt + \varepsilon(\overline{\varphi^0}, \psi^0) + (\mathcal{S}^a(\psi^0)(0), y^0) = 0. \tag{2.1.18}$$

Lemma 2.1.4. *With the above notation, there exists a constant $C_0 > 0$ such that*

$$\|\overline{\varphi^0_\delta}\| \leq C_0 \|y^0\| \tag{2.1.19}$$

and for every $f \in C([0,T], L^2(\Omega))$ the following convergences occur

$$(f, \mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta})) - \mathcal{S}^a(\overline{\varphi^0_\delta}))_{L^2(Q_T)} \to 0,$$

when $\delta \to 0$

Proof. Remark that $J^a_\delta(\overline{\varphi^0_\delta}) \leq J^a_\delta(0) = 0$. This implies that $(\overline{\varphi^0_\delta})_{\delta>0}$ verifies (2.1.19). Then we can extract a subsequence, still denoted $(\overline{\varphi^0_\delta})_{\delta>0}$, weakly converging to φ^0 in $L^2(\Omega)$. Let us observe that $\mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta}) - \varphi^0_\delta) = \mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta}) - M_\delta(\varphi^0)) + \mathcal{S}^a(M_\delta(\varphi^0) - \overline{\varphi^0_\delta})$.

Now, in the weak formulation of $\mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta}) - M_\delta(\varphi^0))$, one can write for the final datum term:

$$(\mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta}) - M_\delta(\varphi^0))(T), f(T))_{L^2(\Omega)} = (\overline{\varphi^0_\delta}, M_\delta(f(T)))_{L^2(\Omega)} - (M_\delta(\varphi^0), f(T))_{L^2(\Omega)} \underset{\delta\to 0}{\longrightarrow} 0. \tag{2.1.20}$$

Hence, $\lim_{\delta\to 0} \mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta}) - M_\delta(\varphi^0))$ is the weak solution of (2.1.4) associated to the null final datum, namely arguing by uniqueness $\mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta}) - M_\delta(\varphi^0)) \underset{\delta\to 0}{\rightharpoonup} \mathcal{S}^a(0) = 0$. Using a similar argument, we get that $\mathcal{S}^a(M_\delta(\varphi^0) - M_\delta(\overline{\varphi^0_\delta})) \underset{\delta\to 0}{\rightharpoonup} 0$ and the result follows. □

More exactly, we aim to prove the following result.

Proposition 2.1.5. *With the above notation, we have*

$$\iint_{q_T} \chi_\omega \left| \mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta})) - \mathcal{S}^a(\overline{\varphi^0}) \right|^2 \, dx \, dt + \varepsilon \|\overline{\varphi^0_\delta} - \overline{\varphi^0}\|^2 \to 0$$

when $\delta \to 0$.

Proof. We choose $\psi^0 = \overline{\varphi^0_\delta} - \overline{\varphi^0}$ in (2.1.17)-(2.1.18) and we subtract these relations:

$$\iint_{q_T} \chi_\omega \mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta}))\mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta} - \overline{\varphi^0})) \, dx \, dt - \iint_{q_T} \chi_\omega \mathcal{S}^a(\overline{\varphi^0})\mathcal{S}^a(\overline{\varphi^0_\delta} - \overline{\varphi^0}) \, dx \, dt$$
$$+\varepsilon \|\overline{\varphi^0_\delta} - \overline{\varphi^0}\|^2 + (\mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta} - \overline{\varphi^0}))(0) - \mathcal{S}^a(\overline{\varphi^0_\delta} - \overline{\varphi^0})(0), y^0) = 0.$$

Using the linearity of the equation (2.1.4) (hence of $\mathcal{S}^a$) and of M_δ, the above equality writes as follows:

$$\iint_{q_T} \chi_\omega |\mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta}))|^2 \, dx \, dt + \iint_{q_T} \chi_\omega |\mathcal{S}^a(\overline{\varphi^0})|^2 \, dx \, dt - \iint_{q_T} \chi_\omega \mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta}))\mathcal{S}^a(M_\delta(\overline{\varphi^0})) \, dx \, dt$$
$$- \iint_{q_T} \chi_\omega \mathcal{S}^a(\overline{\varphi^0_\delta})\mathcal{S}^a(\overline{\varphi^0}) \, dx \, dt + \varepsilon \|\overline{\varphi^0_\delta} - \overline{\varphi^0}\|^2 + (\mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta} - \overline{\varphi^0}) - (\overline{\varphi^0_\delta} - \overline{\varphi^0}))(0), y^0) = 0.$$

Finally, we get

$$\iint_{q_T} \chi_\omega |\mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta})) - \mathcal{S}^a(\overline{\varphi^0})|^2 \, dx \, dt - \iint_{q_T} \chi_\omega \mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta}))\mathcal{S}^a(M_\delta(\overline{\varphi^0}) - \overline{\varphi^0}) \, dx \, dt$$
$$+ \iint_{q_T} \chi_\omega \mathcal{S}^a(\overline{\varphi^0})\mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta}) - \overline{\varphi^0_\delta}) \, dx \, dt + \varepsilon \|\overline{\varphi^0_\delta} - \overline{\varphi^0}\|^2 + (\mathcal{S}^a(M_\delta(\overline{\varphi^0_\delta} - \overline{\varphi^0}) - (\overline{\varphi^0_\delta} - \overline{\varphi^0}))(0), y^0) = 0.$$

The result follows applying Lemma 2.1.4. □

We now give the proof of Theorem 2.1.3.

Proof of Theorem 2.1.3. The existence of approximate controls $\varphi \in C^\infty(Q_T)$ is obtained by minimizing the functional J^a_δ applying the standard HUM method, the regularity being derived from the usual regularity in the linear parabolic case (see, for example, [32, Theorem 10.1]). We therefore focus on proving the Lipschitz L^2 continuity with respect to the diffusion coefficient a.

Let us consider two diffusion coefficients a and b in $C^\infty(\overline{Q_T})$ verifying (2.1.2). Then, we denote $\varphi_a = \mathcal{S}^a(M_\delta(\overline{\varphi^0_{a,\delta}}))$ and $\varphi_b = \mathcal{S}^b(M_\delta(\overline{\varphi^0_{b,\delta}}))$ with $\overline{\varphi^0_{a,\delta}}$ being the minimum of J^a_δ and $\overline{\varphi^0_{b,\delta}}$ being the minimum of J^b_δ. Writing $w := \varphi_a - \varphi_b$, we get that w satisfies the following equation:

$$\begin{cases} \partial_t w + \operatorname{div}(a(t,x)\nabla w) = -\operatorname{div}((a-b)\nabla\varphi_b) & \text{in } Q_T \\ w = 0 & \text{on } \Sigma_T \\ w(T) = M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}) & \text{in } \Omega. \end{cases} \tag{2.1.21}$$

An energy estimate over (2.1.21) leads:

$$\frac{1}{2}\|w(T)\|^2 + (\rho_\star - s)\|w\|^2_{L^2((0,T),H^1_0(\Omega))} \leq \frac{1}{4s}\|\varphi_b\|^2_{W^{1,\infty}(\Omega)}\|a-b\|^2_{L^2(Q_T)} + \frac{1}{2}\|M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}})\|^2. \tag{2.1.22}$$

According to the Young's inequality for convolution (see [32, Theorem 4.15.]):

$$\|M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}})\| \leq \|\rho_\delta\|_{L^1(\Omega)}\|\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}\| \leq \|\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}\| \tag{2.1.23}$$

where $(\rho_\delta)_{\delta>0}$ is the mollifier used to define M_δ. Then, we get from Euler-Lagrange formula:

$$\iint_{q_T} \chi_\omega \varphi_a \mathcal{S}^a(M_\delta(\psi^0))\, dx\, dt + \varepsilon(\overline{\varphi^0_{a,\delta}}, \psi^0) + (\mathcal{S}^a(M_\delta(\psi^0))(0), y^0) = 0 \tag{2.1.24}$$

$$\iint_{q_T} \chi_\omega \varphi_b \mathcal{S}^b(M_\delta(\psi^0))\, dx\, dt + \varepsilon(\overline{\varphi^0_{b,\delta}}, \psi^0) + (\mathcal{S}^b(M_\delta(\psi^0))(0), y^0) = 0. \tag{2.1.25}$$

Subtracting the relations (2.1.24)-(2.1.25), we obtain:

$$\begin{aligned}\iint_{q_T} \chi_\omega(\varphi_a \mathcal{S}^a(M_\delta(\psi^0)) - \varphi_b \mathcal{S}^b(M_\delta(\psi^0)))\, dx\, dt + \varepsilon(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}, \psi^0)& \\ +(\mathcal{S}^a(M_\delta(\psi^0))(0) - \mathcal{S}^b(M_\delta(\psi^0))(0), y^0) = 0.&\end{aligned} \tag{2.1.26}$$

Now, one can write:

$$\begin{aligned}\iint_{q_T} \chi_\omega(\varphi_a \mathcal{S}^a(M_\delta(\psi^0)) - \varphi_b \mathcal{S}^b(M_\delta(\psi^0)))\, dx\, dt = \iint_{q_T} \chi_\omega |\varphi_a - \varphi_b|^2\, dx\, dt& \\ + \iint_{q_T} \chi_\omega(\varphi_a - \varphi_b)(\mathcal{S}^a(M_\delta(\psi^0)) - \varphi_a + \varphi_b)\, dx\, dt + \iint_{q_T} \chi_\omega \varphi_b(\mathcal{S}^a(M_\delta(\psi^0)) - \mathcal{S}^b(M_\delta(\psi^0)))\, dx\, dt.&\end{aligned} \tag{2.1.27}$$

Setting $\psi^0 := \overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}$ into (2.1.26) and using (2.1.27) leads to:

$$\begin{aligned}\iint_{q_T} \chi_\omega |\varphi_a - \varphi_b|^2\, dx\, dt + \varepsilon\|\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}\|^2 + (\mathcal{S}^a(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}))(0) - \mathcal{S}^b(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}))(0), y^0) =& \\ - \iint_{q_T} \chi_\omega(\varphi_a - \varphi_b)(\mathcal{S}^a(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}})) - \varphi_a + \varphi_b)\, dx\, dt& \\ - \iint_{q_T} \chi_\omega \varphi_b(\mathcal{S}^a(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}})) - \mathcal{S}^b(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}})))\, dx\, dt.&\end{aligned} \tag{2.1.28}$$

Here, writing $\beta := \mathcal{S}^a(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}})) - \mathcal{S}^b(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}))$, we have that it solves:

$$\begin{cases} \partial_t \beta + \operatorname{div}(a\nabla\beta) = \operatorname{div}\left((a-b)\nabla\mathcal{S}^b(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}})\right) & \text{in } Q_T \\ \beta = 0 & \text{on } \Sigma_T \\ \beta(T) = 0 & \text{in } \Omega. \end{cases} \tag{2.1.29}$$

Testing against β into the weak formulation of (2.1.29) leads to, after applying Hölder's and Young's inequality for $0 < s < \rho_\star$:

$$\frac{1}{2}\|\beta(0)\|^2 + (\rho_\star - s)\iint_{Q_T} |\nabla\beta|^2\, dx\, dt \leq \frac{1}{4s}\|\mathcal{S}^b(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}))\|^2_{W^{1,\infty}(Q_T)}\|a-b\|^2_{L^2(Q_T)}. \tag{2.1.30}$$

From Poincaré's inequality (2.1.30) leads to:

$$\|\beta\|_{L^2(Q_T)} \leq \left(\frac{\lambda_1(\Omega)^{-1}}{4s(\rho_\star - s)}\right)^{\frac{1}{2}} \|\mathcal{S}^b(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}})\|_{W^{1,\infty}(Q_T)} \|a-b\|_{L^2(Q_T)}, \tag{2.1.31}$$

where $\lambda_1(\Omega)^{-1}$ is the sharp Poincaré constant, and $\rho_\star$ comes from (2.1.2). Using (2.1.31) into (2.1.28), then Young's inequality and an energy estimate, we then get for $0 < s < 1$:

$$\begin{aligned}(1-s)\iint_{q_T} \chi_\omega |\varphi_a - \varphi_b|^2 \, dx\, dt + \varepsilon \|\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}\|^2 \leq \frac{1}{4s}\|\mathcal{S}^a(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}})) - \varphi_a + \varphi_b\|^2_{L^2(q_T)} \\ + \|\varphi_b\|_{L^2(q_T)} \|\mathcal{S}^a(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}})) - \mathcal{S}^b(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}))\|_{L^2(q_T)} \\ + \|y^0\| \left(\frac{\lambda_1(\Omega)^{-1}}{4s(\rho_\star - s)}\right)^{\frac{1}{2}} \|\mathcal{S}^b(M_\delta(\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}})\|_{W^{1,\infty}(Q_T)} \|a-b\|_{L^2(Q_T)}.\end{aligned} \tag{2.1.32}$$

Now, setting $\mathsf{W} = \mathcal{S}^a(M_\delta(\psi^0)) - \varphi_a + \varphi_b$ and $\mathsf{w} = \mathcal{S}^a(M_\delta(\psi^0)) - \mathcal{S}^b(M_\delta(\psi^0))$, we get that these respectively solve:

$$\begin{cases} \partial_t \mathsf{W} + \operatorname{div}(a\nabla \mathsf{W}) = \operatorname{div}((a-b)\nabla\varphi_b) & \text{in } Q_T \\ \mathsf{W} = 0 & \text{on } \Sigma_T \\ \mathsf{W}(T) = 0 & \text{in } \Omega \end{cases} \tag{2.1.33}$$

$$\begin{cases} \partial_t \mathsf{w} + \operatorname{div}(a\nabla \mathsf{w}) = -\operatorname{div}((a-b)\nabla\mathcal{S}^b(\psi^0)) & \text{in } Q_T \\ \mathsf{w} = 0 & \text{on } \Sigma_T \\ \mathsf{w}(T) = 0 & \text{in } \Omega \end{cases} \tag{2.1.34}$$

and following exactly the same argument as for (2.1.31) leads to, for some $0 < s < \rho_\star$:

$$\begin{cases} \|\mathsf{W}\|_{L^2(q_T)} \leq \left(\frac{\lambda_1(\Omega)^{-1}}{4\rho_\star(1-s)s}\right)^{\frac{1}{2}} \|\varphi_b\|^2_{W^{1,\infty}(\Omega)} \|a-b\|^2_{L^2(Q_T)} \\ \\ \|\mathsf{w}\|_{L^2(q_T)} \leq \left(\frac{\lambda_1(\Omega)^{-1}}{4\rho_\star(1-s)s}\right)^{\frac{1}{2}} \|\mathcal{S}^b(\psi^0)\|^2_{W^{1,\infty}(\Omega)} \|a-b\|^2_{L^2(Q_T)}. \end{cases} \tag{2.1.35}$$

Combining (2.1.32) to (2.1.35), finally leads to the existence of a positive constant $C > 0$ such that:

$$\begin{aligned}\varepsilon\|\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}\|^2 &\leq (1-s)\iint_{q_T} \chi_\omega |\varphi_a - \varphi_b|^2 \, dx\, dt + \varepsilon\|\overline{\varphi^0_{a,\delta}} - \overline{\varphi^0_{b,\delta}}\|^2 \\ &\lesssim \|a-b\|^2_{L^2(Q_T)} + \|a-b\|_{L^2(Q_T)}.\end{aligned} \tag{2.1.36}$$

By dividing each member of (2.1.36) by $\varepsilon > 0$ and combining this with (2.1.23) and (2.1.22), we deduce the result. □

At this point, it is worth noting that the Lipschitz constant in L^2 of the control is strongly dependent on the parameter $\varepsilon > 0$. Indeed, the Lipschitz constant thus obtained explodes as ε tends towards 0. Consequently, there is no reason to conclude that the exact control, of minimal L^2 norm, depends continuously in the L^2 sense on the diffusion coefficient.

2.2 Controllability of quasilinear parabolic equations

Let us now turn to the study of the controllability of quasilinear equations. As we said, we will first highlight some existing results on the controllability of quasilinear equations whose nonlinearity depends on the state, and then focus on the case of nonlinear terms depending on the gradient of the state. To highlight recent advances in this last field, we begin by presenting an optimal controllability result given by the Theorem 2.2.3, and then we present a local controllability result, namely the Theorem 2.2.5. Recall that local controllability is to be understood in the sense that, if the initial data is small enough according to a certain norm, then it is possible to control the equation. This result is presented independently from the other results that follow it, from [49]. The principle of these last results is to apply the Theorem 2.1.3 in order to apply a Schauder-type fixed-point theorem, but for continuity in a weak topology.

2.2.1 Equations with nonlinear term involving the state

In this section, we focus on controllability results for a class of quasilinear equations where nonlinearity is applied to the solution of the equation (i.e. not to its gradient). We consider the following internal control problem.

$$\begin{cases} \partial_t y - \operatorname{div}(a(y)\cdot\nabla y) = \mathbb{1}_\omega \varphi & \text{in } Q_T \\ y = 0 & \text{on } \Sigma_T \\ y_{|t=0} = y^0 & \text{in } \Omega \end{cases} \tag{2.2.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with C^3 boundary, $\omega \subsetneq \Omega$ is a nonempty open set, $Q_T = (0,T)\times\Omega$, and $\Sigma_T = (0,T)\times\partial\Omega$. As usual, $\mathbb{1}_\omega$ should refer to the characteristic function of ω. Here, $a : \mathbb{R} \to \mathbb{R}^{N\times N}_{\text{sym}}$ is of class C^2 and satisfies the coercivity assumption:

$$\sum_{i,j=1}^{N} a_{i,j}(s)\xi_i\xi_j \gtrsim |\xi|^2 \quad (s,\xi)\in\mathbb{R}\times\mathbb{R}^N \tag{2.2.2}$$

Moreover, we set:

$$A = \sum_{i,j=1}^{N}\left(1 + \sup_{|s|\le 1}|a_{i,j}(s)|^2 + \sup_{|s|\le 1}|a'_{i,j}(s)|^2\right) \quad \text{and} \quad \nu = \max_{(i,j)\in[1,N]^2\cap\mathbb{N}^2}|a'_{i,j}(s)|^2 \tag{2.2.3}$$

Then, the following result holds.

Theorem 2.2.1 (Liu-Zhang). *Let us consider Ω a bounded domain of $\mathbb{R}^N$ with C^3 boundary, $\omega \subsetneq \Omega$ a nonempty open set, and a constant $\gamma > 0$. Moreover, we assume that $a : \mathbb{R} \to \mathbb{R}^{N\times N}_{\text{sym}}$ satisfies the coercivity assumption given by (2.2.2). Then, for any initial value $y_0 \in C^{2+\frac{1}{2}}(\overline{\Omega})$ satisfying*

$$\|y_0\|_{C^{2+\frac{1}{2}}(\overline{\Omega})} \le \gamma$$

and the first order compatibility condition, one can find a control

$$\varphi \in C^{\frac{1}{2},\frac{1}{4}}(\overline{Q_T})$$

with $\operatorname{supp}(\varphi) \subseteq [0,T]\times\omega$ *such that the solution y of (2.2.1) satisfies $y(T) = 0$ in Ω.*

Moreover, $\|\varphi\|_{C^{\frac{1}{2},\frac{1}{4}}(\overline{Q_T})} \leq C_1 e^{e^{C_2 A}} \|y_0\|_{L^2(\Omega)}$ *where* C_1 *depends only on* C_0, ν, N, Ω *and* T, *and* C_2 *depends only on* C_0, N, Ω *and* T, *the constants* A *and* ν *being given in* (2.2.3).

Proof. The proof is based on an observability inequality. More precisely, considering the dual system of the linearized (2.2.1), the authors obtain a Carleman inequality involving only the C^1 norm of the diffusion coefficient. The authors deduce an observability inequality that allows local controllability to be induced using Schauder theory applied to parabolic equations. We refer to [140] for the proof of this result. □

If, instead of the previous assumptions, we consider the alternative version of (2.2.1)

$$\begin{cases} \partial_t y - \operatorname{div}(a(y)\nabla y) = \chi_\omega \varphi & \text{in } Q_T \\ y = 0 & \text{on } \Sigma_T \\ y_{|t=0} = y^0 & \text{in } \Omega \end{cases} \tag{2.2.4}$$

where χ is a smooth cutoff function over ω, $N \leq 3$, and the diffusion coefficient verifies:

(L1) $a \in C^3(\mathbb{R})$.

(L2) $\exists(\rho_0, \rho_1) > 0$, $\forall x \in \mathbb{R}^N$, $\rho_0 \leq a(x) \leq \rho_1$.

(L3) $\forall(\beta, x) \in \mathbb{N} \times \mathbb{R}^N$, $\beta \leq 3 \Rightarrow |a^{(\beta)}(x)| \lesssim 1$.

Then, the following result holds.

Theorem 2.2.2 (Fernández–Cara-Límaco-Marin–Gayte)**.** *Let* Ω *be a domain of* $\mathbb{R}^N$, *with smooth boundary,* $N \in \{2, 3\}$, $\omega \subsetneq \Omega$ *an open set,* $y_0 \in H_0^1(\Omega) \cap H^3(\Omega)$ *and we assume that assumptions (L1)-(L3) hold. Then, there exists* $\eta > 0$ *such that if*

$$\|y_0\|_{H^3(\Omega)} \leq \eta$$

we have the existence of a control $\varphi \in L^2((0,T) \times \Omega)$ *leading to* $y(T) = 0$.

Proof. See [88] □

The proof of this result is based on a Ljusternik-type inversion method. We should point out that the authors state that this result extends to cases where the coefficient a in (2.2.4) is not homogeneous, in the sense that it may also depend on spatio-temporal variables. We refer the reader to the article [88] for further details.

2.2.2 Equations with nonlinear term involving the gradient of the state

We devote this section to the study of the controllability of quasilinear parabolic equations whose nonlinear term depends on the gradient of the state. First, we present an optimal controllability result. Then, in line with our main focus, we examine the exact and approximate controllability of such equations. It should be stressed, however, that for the sake of unity of presentation, we concentrate our presentation on the distributed controllability.

Optimal controllability

Consider the optimal control problem consisting of minimizing the cost function

$$J(f) := \int_{Q_T} L(t,x,y(t,x),f(t,x))\,dx\,dt \tag{2.2.5}$$

over a nonempty convex set $K \subset L^r(Q_T)$, for $1 \leq r < +\infty$, where we denoted $L : Q_T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and y (which depends of f) solves the following equation.

$$\begin{cases} \partial_t y - \operatorname{div}(a(t,x,\nabla y)) + b(t,x,y) = f & \text{in } Q_T \\ y = 0 & \text{on } \Sigma_T \\ y_{|t=0} = y_0 & \text{in } \Omega \end{cases} \tag{2.2.6}$$

Let us state suitable assumptions over the different parameters involved, where we assume that there exist $p > \max\left(\frac{2N}{N+2}, 1\right)$ a positive increasing function ψ and $k > 0$ such that these are satisfied:

(CA1) $a : Q_T \times \mathbb{R}^N \to \mathbb{R}^N$ is such that for every $1 \leq i \leq N$, $(t,x) \mapsto a_i(t,x,u)$ is a measurable function over Q_T for every $u \in \mathbb{R}^N$, and for almost every $(t,x) \in Q_T$, the mapping $u \mapsto a_i(t,x,u)$ belongs to $C^1(\mathbb{R}^N)$.

(CA2) $b : Q_T \times \mathbb{R} \to \mathbb{R}$ is such that $(t,x) \mapsto b(t,x,v)$ is a measurable function over Q_T and $v \mapsto b(t,x,v)$ belongs to $C^1(\mathbb{R})$ for almost every $(t,x) \in Q_T$

(CA3) $\forall (\xi, v) \in \mathbb{R}^N \times \mathbb{R},\ \nabla a(t,x,v) : (\xi \otimes \xi) \gtrsim (k + |v|)^{p-2}|\xi|^2$.

(CA4) $\forall (t,x,v) \in Q_T \times \mathbb{R},\ |\nabla a(t,x,v)| \lesssim (k+|v|)^{p-2}$.

(CA5) $\forall (t,x,v) \in Q_T \times \mathbb{R},\ 0 \leq \partial_v a(t,x,v) \leq \psi(|v|)$.

(CA6) $\forall (t,x) \in Q_T,\ \forall i \in [1,N] \cap \mathbb{N},\ a_i(t,x,0) = b(t,x,0) = 0$.

(CA7) The map $(v,u) \mapsto L(t,x,u,v)$ belongs to $C^1(\mathbb{R}^2)$ and $L(t,x,0,0) \in L^1(Q_T)$ for all $(t,x) \in Q_T$.

(CA8) $\forall M > 0,\ \exists (h_0^M, h_1^M, C_M) \in L^{\left(\frac{r}{r-1}\right)}(Q_T) \times L^2(Q_T) \times (0,+\infty)$,

$$\begin{aligned} |\partial_v L(t,x,u,v)| &\leq h_0^M(t,x) + C_M|v|^{r-1} \qquad (t,x,u,v) \in Q_T \times [-M,M] \times \mathbb{R} \\ |\partial_u L(t,x,u,v)| &\leq h_1^M(t,x) + C_M|v|^{\frac{r}{2}} \qquad (t,x,u,v) \in Q_T \times [-M,M] \times \mathbb{R}. \end{aligned}$$

(CA9) The map $v \mapsto L(t,x,u,v)$ is convex for every $(t,x,u) \in Q_T \times \mathbb{R}$.

(CA10) There exists $f_0 \in K$ such that the set $\{f \in K /\ J(f) \leq J(f_0)\}$ is bounded in $L^r(Q_T)$.

Then, the following result holds.

Theorem 2.2.3 (Casas-Fernández-Yong). *Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary and $0 < T < +\infty$. We assume that there exists $p > \max\left(\frac{2N}{N+2}, 1\right)$ a positive increasing function ψ and $k > 0$ such that assumptions (CA1)-(CA10) are fulfilled. Then, there exists a solution to the optimal control problem* (2.2.5)-(2.2.6).

Proof. We refer to [41, Theorem 3.1] for a proof of this result. Let us point out that, slightly modifying the assumptions, the result still holds for $p = +\infty$. □

When we talk about the existence of a solution to the optimal control problem, we need to understand the existence of a solution with usual regularity of the equation (2.2.6), as well as the existence of a minimizer f of the functional J defined by (2.2.5) given in the space $L^r(Q_T)$. The following example gives an idea of the scope of Theorem 2.2.3.

Example 2.2.4. *We point out that Theorem 2.2.3 does not take into account the case of parabolic p-Laplacian, since assumptions (CA1) or (CA3) could not be satisfied for example. Nevertheless, one can set $a(t,x,\nabla y) = (k+|\nabla y|)^{p-2}\nabla y$, and the previous result leads to the optimal controllability of such an equation.*

Note, however, that there are alternative results to the aforementioned Theorem 2.2.3, whose study focuses on the p-Laplacian. Reference can be made, for example, to [42], for results relating to this framework. In general, however, these are not results whose controllability is internal. For the case of an optimal internal control of the p-Laplacian with $p \geq 2$, see [150], for example.

Exact and approximate controllability

Let us start by presenting a local controllability result, whose proof is an adaptation of that of Theorem 2.2.2. Consider the following quasilinear equation.

$$\begin{cases} \partial_t y - \operatorname{div}\left(a(\nabla y)\nabla y\right) = \mathbb{1}_\omega \varphi & \text{in } Q_T \\ y = 0 & \text{on } \Sigma_T \\ y_{|t=0} = y_0 & \text{in } \Omega \end{cases} \tag{2.2.7}$$

We assume the following assumptions for $a : \mathbb{R}^N \to \mathbb{R}$.

(F1) $a \in C^4(\mathbb{R}^N)$.

(F2) $\exists \rho_0 > 0,\ \forall x \in \mathbb{R}^N,\ \rho_0 \leq a(x)$.

(F3) $\exists r \in [1,+\infty),\ \forall (\beta,x) \in \mathbb{N}^N \times \mathbb{R}^N,\ |\beta| \leq 4 \Rightarrow |\nabla^\beta a(x)| \lesssim 1 + |x|^{(r-|\beta|)_+}$.

Then, the following result holds.

Theorem 2.2.5 (Fernández–Cara-Límaco-Thamsten-Menezes)**.** *Let Ω be a domain of $\mathbb{R}^N$ with smooth boundary, $\omega \subsetneq \Omega$ an open set, $y_0 \in H_0^1(\Omega) \cap H^5(\Omega)$ satisfying the compatibility condition:*

$$(\Delta y_0, \Delta^2 y_0) \in H_0^1(\Omega)^2$$

and we assume that assumptions (F1)-(F3) hold. Then, there exists $\eta > 0$ such that if

$$\|y_0\|_{H^5(\Omega)} \leq \eta$$

we have the existence of a control $\varphi \in L^2((0,T)\times\Omega)$ leading to $y(T) = 0$.

Proof. See [91] □

As in Theorem 2.2.2, the proof of this result relies on a local inversion theorem applied to the linearized system, as well as a regularity argument.

We now turn to the main result of this section. We consider the following system:

$$\begin{cases} \partial_t y - \operatorname{div}\left(F(|\nabla y|)\nabla y\right) = \chi_\omega \varphi & \text{in } Q_T \\ y = 0 & \text{on } \Sigma_T \\ y(0) = y^0 & \text{in } \Omega, \end{cases} , \tag{2.2.8}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with a smooth boundary $\partial\Omega$, $Q_T = (0,T) \times \Omega$, $\Sigma_T = (0,T) \times \partial\Omega$, the initial data y^0 belongs to $L^2(\Omega)$, and there exists $p > 1$ such that the function $F : \mathbb{R}_+ \to \mathbb{R}_+^*$ verifies the following assumptions:

(A1) $F \in W^{1,\infty}(\mathbb{R}_+) \cap C^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+) \cap L^{\frac{p}{p-1}}(\mathbb{R}_+)$;

(A2) The potential defined for every $t \in \mathbb{R}_+$ by $\Phi(t) = \int_0^t sF(s)\ ds$ is convex and satisfies $\Phi \in W^{1,\frac{p}{p-1}}(\mathbb{R}_+)$;

(A3) There exists $C_1, C_2, \mu, \nu > 0$ and $k_1, k_2 \geq 0$ such that for every $t \in \mathbb{R}_+$, we have that

$$k_1 + C_1(\mu + t^2)^{\frac{p-2}{2}} \leq F(t) \leq k_2 + C_2(\nu + t^2)^{\frac{p-2}{2}}.$$

The control φ acts in the open and non empty set $\omega \subset \Omega$. More precisely, we denote by $\chi_\omega \in C^\infty(\overline{\Omega})$ a regular function such that

$$\chi_\omega(x) = \begin{cases} 1 & \text{for } x \in \omega_\delta \\ 0 & \text{for } x \in \overline{\Omega \setminus \omega}, \end{cases}$$

where $\omega_\delta = \{x \in \omega \text{ such that } \operatorname{dist}(x, \partial\omega) > \delta\}$ for a given $\delta > 0$ small enough. We also denote $q_T = (0,T) \times \omega$.

Let us underline that assumption (A2) means that $\tilde{\Phi}(u) = \Phi(|u|)$ is a convex potential and thus the operator $A : X \to X^*$ defined for every $\phi \in X$ by

$$A\phi = -\operatorname{div}\left(F(|\nabla\phi|)\nabla\phi\right) \tag{2.2.9}$$

is monotone from Proposition 1.1.33, where X is a reflexive Banach space compactly and densely embedded in $L^2(\Omega)$ and X^* is its dual with respect to the pivot space $L^2(\Omega)$. Having this in mind, we point out that the existence of a unique weak solution to equation (2.2.8) is a direct consequence of Theorem B.0.1 applied to the nonlinear monotone operator A given by (2.2.9).

We aim to prove the following result.

Theorem 2.2.6 ([49, Theorem 1.2.]). *Assume that F satisfies assumption (A1)-(A3) and y^0 belongs to $L^2(\Omega)$ are chosen such that there exists a unique solution of (2.2.8). Then, there exists a distributed control φ, whose regularity is given by Theorem 2.1.3, such that (2.2.8) is approximately null controllable in any time $T > 0$, i.e., for every $y^0 \in L^2(\Omega)$ and every $\varepsilon > 0$ there exists a control $\varphi \in C^\infty(\overline{Q_T})$ such that the solution y of (2.2.8) satisfies*

$$\|y(T)\| \leq \varepsilon.$$

In order to extend the controllability properties of the linear equation (2.1.1) to the quasilinear equation (2.2.8), we aim to apply a fixed point theorem. In this purpose, we first consider the result given by Theorem 2.1.3, which allows to obtain regular approximate controls for the equation (2.1.1) which are Lipschitz continuous with respect to the diffusion coefficient a. The existence of approximate regular controls for the linear equation (2.1.1) makes possible to define

an application associating to the diffusion coefficient a the quantity $F_\delta(|\nabla u_{a,\delta}|)$ from a bounded closed convex set with values in itself, $u_{a,\delta}$ being the controlled solution of the regularized version of (2.1.1) and F_δ being a regularization of the function F still verifying hypotheses (A1)–(A3). The objective is then to show the continuity of such applications on some weakly sequentially compact sets, in order to apply a suitable fixed point theorem. Remark that, since we aim to prove only a null approximate controllability result, it is not necessary to take the limit with respect to the regularization parameter δ. Here and henceforth we denote by "$*$" the convolution product.

We define a regularisation process $R_\delta : L^1(Q_T) \to C^\infty(\overline{Q_T})$. More exactly, for every $g \in L^1(Q_T)$ we define $R_\delta(g)$ by

$$R_\delta(g) = \nu_\delta * (\chi_\delta g) + \delta, \tag{2.2.10}$$

where $(\nu_\delta)_\delta$ is a mollifier, $\chi_\delta : \mathbb{R}^{N+1} \to \mathbb{R}$ is a smooth cutoff function with $\text{supp}(\chi_\delta) = Q_T$. We can see that $R_\delta(g) \underset{\delta \to 0}{\longrightarrow} g$ in $L^1(Q_T)$ for every $g \in L^1(Q_T)$.

From now on, we should denote $L^2_+(Q_T)$ the subspace of non negative functions in $L^2(Q_T)$. Then, we consider the following bounded convex closed set

$$K_\delta = \left\{ f \in L^2_+(Q_T) \text{ satisfying } \|f\|^2_{L^2(Q_T)} \leq 2(C_L + 1) \left(\frac{1}{\delta} + \frac{C_0}{2\delta^2} \right) \|y^0\|^2 + \delta |\Omega| T \right\}, \tag{2.2.11}$$

where C_L is the Lipschitz constant of F from assumption (A1) and C_0 is the observability constant given in (2.1.6). Moreover, let us observe that R_δ is continuous over $L^2_+(Q_T)$. Taking $a \in K_\delta$ and $h \in L^2(Q_T)$ such that $a + h \in K_\delta$, we can write:

$$R_\delta(a + h) - R_\delta(a) = \nu_\delta * (\chi_\delta h) \tag{2.2.12}$$

and the continuity of R_δ follows from Hölder's inequality.

Let us now define the function $G_\delta : K_\delta \to L^2(Q_T)$ by

$$G_\delta(a) = F(|\nabla v_{a,\delta}|), \tag{2.2.13}$$

where $v_{a,\delta}$ is the weak solution to:

$$\begin{cases} \partial_t v_{a,\delta} - \text{div}\,(R_\delta(a)(t,x)\nabla v_{a,\delta}) = \chi_\omega \varphi_a & \text{in } Q_T \\ v_{a,\delta} = 0 & \text{on } \Sigma_T \\ v_{a,\delta}(0) = M_\delta(y^0) & \text{in } \Omega, \end{cases} \tag{2.2.14}$$

with $\varphi_a \in C^\infty(\overline{Q_T})$ being the approximate control provided by Theorem 2.1.3 applied to (2.2.14) which is nothing else than an alternative version of (2.1.1) with a regularised operator $L_\delta : u \mapsto -\text{div}\,(R_\delta(a)\nabla u)$.

Let us recall the following fixed point theorem (see [126] for a proof).

Theorem 2.2.7 ([126, Theorem 2.1])**.** *Let K be a closed convex subset of a Banach space Y. Let us consider $G : K \to K$ such that for all sequence $(a_n)_n \subset K$ which converges weakly toward a, then $(G(a_n))_n$ admits a subsequence which converges strongly toward $G(a)$. Hence, if G is continuous and $G(K)$ is weakly compact, G admits a fixed point.*

Before proving the Theorem 2.2.6, let us prove the following lemma.

Lemma 2.2.8. *For every $\delta > 0$ the application $G_\delta : K_\delta \to L^2(Q_T)$ defined by (2.2.13) is continuous and verifies $G(K_\delta) \subset K_\delta$.*

Proof. First, let us show the fact that $G_\delta(K_\delta) \subset K_\delta$. Since K_δ contains $0_{L^2(Q_T)}$, from (A1) one can write:

$$\begin{aligned}\|G_\delta(a) - G_\delta(0)\|^2_{L^2(Q_T)} &= \|F(|\nabla v_{a,\delta}|) - F(|\nabla v_{0,\delta}|)\|^2_{L^2(Q_T)} \\ &\leq C_L \|v_{a,\delta} - v_{0,\delta}\|^2_{L^2((0,T),H^1_0(\Omega))}. \end{aligned} \tag{2.2.15}$$

Now, we point out that for every $a \in K_\delta$, $\|v_{a,\delta}\|_{L^2((0,T),H^1_0(\Omega))}$ is bounded by $\|M_\delta(y^0)\|\sqrt{\frac{1}{\delta} + \frac{C_0}{2\delta^2}}$, this bound following from the energy estimate:

$$\frac{1}{2}\|v_{a,\delta}\|^2_{L^2(Q_T)} + \frac{\delta}{2}\|v_{a,\delta}\|^2_{L^2((0,T),H^1_0(\Omega))} \leq \frac{1}{2}\|M_\delta(y^0)\|^2 + \frac{1}{4\delta}\|\chi_\omega \varphi_a\|^2_{L^2(q_T)}\,, \tag{2.2.16}$$

since testing against φ^0 into the Euler-Lagrange equation associated to (2.1.5), then applying the observability inequality as given in Proposition 2.1.1 and Young's inequality leads to the inequality:

$$\|\chi_\omega \varphi_a\|^2_{L^2(q_T)} \leq C_0 \|M_\delta(y^0)\|^2\,, \tag{2.2.17}$$

and combining (2.2.16) together with (2.2.17) implies

$$\|v_{a,\delta}\|^2_{L^2((0,T),H^1_0(\Omega))} \leq \frac{1}{\delta}\|M_\delta(y^0)\|^2 + \frac{1}{2\delta^2}\|\chi_\omega \varphi_a\|^2_{L^2(q_T)} \leq \left(\frac{1}{\delta} + \frac{C_0}{2\delta^2}\right)\|M_\delta(y^0)\|^2. \tag{2.2.18}$$

Now, from (2.2.15), we can write:

$$\|G_\delta(a)\|^2_{L^2(Q_T)} \leq C_L\left(\|v_{a,\delta}\|^2_{L^2((0,T),H^1_0(\Omega))} + \|v_{0,\delta}\|^2_{L^2((0,T),H^1_0(\Omega))}\right) + \|G_\delta(0)\|^2_{L^2(Q_T)}. \tag{2.2.19}$$

On the other hand, we have:

$$\|G_\delta(0)\|^2_{L^2(Q_T)} \leq \|v_{0,\delta}\|^2_{L^2((0,T),H^1_0(\Omega)} + \delta|\Omega|T. \tag{2.2.20}$$

Hence, combining (2.2.20), (2.2.19) and (2.2.18), we get that $G_\delta(K_\delta) \subset K_\delta$.

Let us now show that G_δ is continuous. Since F is globally Lipschitz from assumption (A1), one have:

$$\|G_\delta(a+h) - G_\delta(a)\|^2_{L^2(Q_T)} = \|F(|\nabla v_{a+h,\delta}|) - F(|\nabla v_{a,\delta}|)\|^2_{L^2(Q_T)} \leq C_L\|\nabla v_{a+h,\delta} - \nabla v_{a,\delta}\|^2_{L^2(Q_T)}. \tag{2.2.21}$$

Then, we have that $w_{a,h} := v_{a+h,\delta} - v_{a,\delta}$ solves:

$$\begin{cases} \partial_t w_{a,h} - \operatorname{div}\left(R_\delta(a+h)\nabla w_{a,h}\right) = \chi_\omega\left(\varphi_{a+h} - \varphi_a\right) - \operatorname{div}\left(\left(R_\delta(a+h) - R_\delta(a)\right)\nabla v_{a+h}\right) & \text{in } Q_T \\ w_{a,h} = 0 & \text{on } \Sigma_T \\ w_{a,h}(0) = 0 & \text{in } \Omega. \end{cases} \tag{2.2.22}$$

Hence, an energy estimate over (2.2.22) leads to, using parametrized Young's inequality and Poincaré's inequality:

$$\frac{1}{2}\|w_{a,h}\|^2_{L^\infty((0,T),L^2(\Omega))}+\frac{\delta}{2}\|w_{a,h}\|^2_{L^2((0,T),H^1_0(\Omega))}\leq\frac{1}{\delta}\|\varphi_{a+h}-\varphi_a\|^2_{L^2(Q_T)}$$
$$+\frac{1}{\delta}\|\nabla v_{a+h}\|^2_{L^\infty(Q_T)}\|R_\delta(a+h)-R_\delta(a)\|^2_{L^2(Q_T)}.$$

Using Theorem 2.1.3 combined to (2.2.21), and since, for a k large enough with respect to N, $H^k_0(\Omega)$ is continuous embedded in $W^{1,\infty}_0(\Omega)$, we get that:

$$\|G_\delta(a+h)-G_\delta(a)\|^2_{L^2(Q_T)}\leq\frac{C(\delta)}{\delta^2}\left(\|h\|^2_{L^2(Q_T)}+\|M_\delta(y^0)\|^2_k\|R_\delta(a+h)-R_\delta(a)\|^2_{L^2(Q_T)}\right), \tag{2.2.23}$$

which proves the lemma. □

Remark 2.2.9. *We can avoid the global regularity assumptions over F as in (A1) by considering that*

$$F\in W^{1,\infty}_{loc}(\mathbb{R}_+)\cap L^2_{loc}(\mathbb{R}_+)\cap L^{\frac{p}{p-1}}_{loc}(\mathbb{R}_+).$$

It is then necessary to introduce an additional regularization process. First, for every $\delta>0$ we define a regularisation process $r_\delta: L^1_{loc}(\mathbb{R}_+)\cap W^{1,\infty}_{loc}(\mathbb{R}_+)\to C^\infty(\mathbb{R}_+)\cap W^{1,\infty}(\mathbb{R}_+)$ by:

$$r_\delta(F):=\zeta_\delta*(\sigma_\delta F)+\delta\,, \tag{2.2.24}$$

for every $F\in L^1(\mathbb{R}_+)\cap W^{1,\infty}(\mathbb{R}_+)$, where $(\zeta_\delta)_\delta$ is a mollifier and $\sigma_\delta:\mathbb{R}\to\mathbb{R}$ is the smooth cutoff function satisfying:

$$\sigma_\delta=\begin{cases}1 & \text{in } \left[\delta,\frac{1}{\delta}\right]\\ 0 & \text{in } (-\infty,0]\cup\left[\frac{1}{\delta}+\delta,+\infty\right).\end{cases} \tag{2.2.25}$$

Then, we see that such a regularization process also holds over $L^1_{\text{loc}}(\mathbb{R}_+)\cap W^{1,\infty}(\mathbb{R}_+\backslash\{0\})$, in the sense that for every $F\in L^1_{\text{loc}}(\mathbb{R}_+)\cap W^{1,\infty}(\mathbb{R}_+\backslash\{0\})$, such an $r_\delta(F)$ leads to a globally Lipschitz function, i.e., *there exists $C_{L,\delta}>0$ such that*

$$|r_\delta(F)(t)-r_\delta(F)(s)|\leq C_{L,\delta}|t-s|, \tag{2.2.26}$$

for every $t,\ s>0$.

Remark 2.2.10. *We point out that is essential here to consider solutions to the regularized equation* (2.2.14)*, since the space of functions which are essentially positively lower and upper bounded do not give rise to regular enough solutions of* (2.1.1) *(namely, at least Hölder continuous) since we can construct discontinuous solutions with respect to the space variable of* (2.1.1) *for some diffusion coefficient in this space, given by Serrin's example (see [160]).*

We are now able to prove Theorem 2.2.6.

Proof of Theorem 2.2.6. Let $\delta>0$ to choose later. In order to apply Theorem 2.2.7, we first show that if $(a_n)_n$ is a sequence which converges weakly toward a, then $G_\delta(a_n)$ converges weakly, up to a subsequence, toward $G_\delta(a)$. First, let us observe that the weak convergence of $(a_n)_n$ implies that $R_\delta(a_n)\underset{n\to+\infty}{\longrightarrow}R_\delta(a)$ strongly in K_δ, by definition of R_δ. Also, arguing by continuity, one can see that the associated controls provided by Theorem 2.1.3 in (2.2.14) leads to

$$\|\chi_\omega\varphi_{R_\delta(a_n)}-\chi_\omega\varphi_{R_\delta(a)}\|_{L^2(Q_T)}\underset{n\to+\infty}{\longrightarrow}0.$$

From this, an energy estimate leads to, setting $w_n := v_{a,\delta} - v_{a_n,\delta}$, where $v_{a,\delta}$ and $v_{a_n,\delta}$ are respectively solutions to (2.2.14) associated to the diffusion coefficient $R_\delta(a)$ and $R_\delta(a_n)$ and to the controls $\varphi_{R_\delta(a)}$ and $\varphi_{R_\delta(a_n)}$, respectively:

$$\frac{1}{2}\|w_n\|^2_{L^\infty((0,T),L^2(\Omega))} + \frac{\delta}{2}\|w_n\|^2_{L^2((0,T),H^1_0(\Omega))} \leq \frac{1}{\delta}\|\chi_\omega(\varphi_{R_\delta(a)} - \varphi_{R_\delta(a_n)})\|^2_{L^2(q_T)} + \frac{C\|y^0\|_k}{\delta}\|R_\delta(a_n) - R_\delta(a)\|^2_{L^2(Q_T)}. \quad (2.2.27)$$

Then, we can write, from (2.2.21):

$$\|G_\delta(a_n) - G_\delta(a)\|^2_{L^2(Q_T)} = \|F(|\nabla v_{a_n,\delta}|) - F(|\nabla v_{a,\delta}|)\|^2_{L^2(Q_T)} \leq C_L\|w_n\|^2_{L^2(0,T,H^1_0(\Omega))}. \quad (2.2.28)$$

Then, (2.2.27) combined with (2.2.28) leads to the fact that $\|G_\delta(a_n) - G_\delta(a)\|_{L^2(Q_T)} \underset{n\to+\infty}{\longrightarrow} 0$. Applying now Lemma 2.2.8, we get from Theorem 2.2.7 that G_δ admits a unique fixed point in K_δ. Namely, we get that in the solution to the equation

$$\begin{cases} \partial_t \mathsf{v}_\delta - \operatorname{div}\left(R_\delta\left(F(|\nabla \mathsf{v}_\delta|)\right)\nabla \mathsf{v}_\delta\right) = \chi_\omega\varphi & \text{in } Q_T \\ \mathsf{v}_\delta = 0 & \text{on } \Sigma_T \\ \mathsf{v}_\delta(0) = M_\delta(y^0) & \text{in } \Omega. \end{cases} \quad (2.2.29)$$

φ can be chosen as an approximate control of (2.2.29), from Theorem 2.1.3. For the sake of simplicity, we denote $R_\delta(F(|\cdot|))$ as $F_\delta(|\cdot|)$. Next, we denote $w := y - \mathsf{v}_\delta$ with y the solution to (2.2.8). Writing:

$$F(|\nabla y|)\nabla y - F_\delta(|\nabla \mathsf{v}_\delta|)\nabla \mathsf{v}_\delta = F(|\nabla y|)\nabla y - F(|\nabla \mathsf{v}_\delta|)\nabla \mathsf{v}_\delta + (F(|\nabla \mathsf{v}_\delta|) - F_\delta(|\nabla \mathsf{v}_\delta|))\nabla \mathsf{v}_\delta, \quad (2.2.30)$$

an energy estimate leads to:

$$\frac{1}{2}\|w(T)\|^2 + \iint_{Q_T}(F(|\nabla y|)\nabla y - F(|\nabla \mathsf{v}_\delta|)\nabla \mathsf{v}_\delta)\cdot(\nabla y - \nabla \mathsf{v}_\delta)\,dxdt + \iint_{Q_T}(F(|\nabla \mathsf{v}_\delta|)\nabla \mathsf{v}_\delta - F_\delta(|\nabla \mathsf{v}_\delta|)\nabla \mathsf{v}_\delta)\cdot(\nabla y - \nabla \mathsf{v}_\delta)\,dx\,dt = \frac{1}{2}\|M_\delta(y^0) - y^0\|^2. \quad (2.2.31)$$

Then, (2.2.31) leads, using the monotonicity of the operator (assumption (A3) see [132, Chapitre 2, section 1.3.] and [175, section 25.3]) to:

$$\begin{aligned} \frac{1}{2}\|w(T)\|^2 &\leq \frac{1}{2}\|w(T)\|^2 + \iint_{Q_T}(F(|\nabla y|)\nabla y - F(|\nabla \mathsf{v}_\delta|)\nabla \mathsf{v}_\delta)\cdot(\nabla y - \nabla \mathsf{v}_\delta)\,dxdt \\ &\leq \left|\iint_{Q_T}(F(|\nabla \mathsf{v}_\delta|)\nabla \mathsf{v}_\delta - F_\delta(|\nabla \mathsf{v}_\delta|)\nabla \mathsf{v}_\delta)\cdot(\nabla y - \nabla \mathsf{v}_\delta)\,dx\,dt\right| + \frac{1}{2}\|M_\delta(y^0) - y^0\|^2 \\ &= \left|\iint_{Q_T}(\nabla\Phi(|\nabla \mathsf{v}_\delta|) - \nabla\Phi_\delta(|\nabla \mathsf{v}_\delta|))\cdot\nabla(y - \mathsf{v}_\delta)\,dx\,dt\right| + \frac{1}{2}\|M_\delta(y^0) - y^0\|^2. \end{aligned} \quad (2.2.32)$$

It remains to prove that the first term in the right hand side goes to zero as δ does. First, let us remark that we can write:

$$\begin{aligned}\nabla\Phi_\delta(|x|) &:= F_\delta(|x|)x\\ &= R_\delta(F)(|x|)x\\ &= (\nu_\delta * (\chi_\delta F)(|x|) + \delta)\, x\\ &= \nu_\delta * (\chi_\delta F)(|x|)x + \delta x\\ &= \nu_\delta * (\chi_\delta F(|x|)x) + \delta x\\ &= \nabla(\Phi)^\delta(|x|) + \delta x.\end{aligned} \tag{2.2.33}$$

Here, we denoted by $(\Phi)^\delta$ a regularization (by mollification) of Φ. Combining (2.2.32) and (2.2.33), we easily get:

$$\begin{aligned}\frac{1}{2}\|w(T)\|^2 \leq \left|\iint_{Q_T} (\nabla\Phi(|\nabla \mathsf{v}_\delta|) - \nabla(\Phi)^\delta(|\nabla \mathsf{v}_\delta|)) \cdot \nabla(y - \mathsf{v}_\delta)\, dx\, dt\right| + \left|\iint_{Q_T} \delta\nabla \mathsf{v}_\delta \cdot (\nabla y - \nabla \mathsf{v}_\delta)\, dx\, dt\right|\\ + \frac{1}{2}\|M_\delta(y^0) - y^0\|^2.\end{aligned} \tag{2.2.34}$$

Evaluating the second integral term in (2.2.34), we obtain:

$$\begin{aligned}\left|\iint_{Q_T} \delta\nabla \mathsf{v}_\delta \cdot (\nabla y - \nabla \mathsf{v}_\delta)\, dx\, dt\right| \leq \delta\|y - \mathsf{v}_\delta\|^2_{L^2((0,T),H^1_0(\Omega)}\\ + \delta\|y\|_{L^{\frac{p}{p-1}}\left((0,T),W_0^{1,\frac{p}{p-1}}(\Omega)\right)} \|y - \mathsf{v}_\delta\|_{L^p\left((0,T),W_0^{1,p}(\Omega)\right)}.\end{aligned} \tag{2.2.35}$$

We get, since the solutions are regular enough, that the term present in (2.2.35) goes to zero as δ does. Now, from assumption (A2), then using (2.2.10) and (2.2.24), we get the following estimate for the first integral term in (2.2.34):

$$\begin{aligned}\left|\iint_{Q_T} (\nabla\Phi(|\nabla \mathsf{v}_\delta|) - \nabla(\Phi)^\delta(|\nabla \mathsf{v}_\delta|)) \cdot \nabla(y - \mathsf{v}_\delta)\, dx\, dt\right| \leq \left\|\nabla\Phi(|\nabla \mathsf{v}_\delta|) - \nabla(\Phi)^\delta(|\nabla \mathsf{v}_\delta|)\right\|_{L^{\frac{p}{p-1}}(Q_T)}\\ \|y - \mathsf{v}_\delta\|_{L^p(0,T,W_0^{1,p}(\Omega))}.\end{aligned} \tag{2.2.36}$$

Let us now formally denote $\nabla \mathsf{v} = \underline{\lim}_{\delta\to 0} \nabla \mathsf{v}_\delta$ (which leads to a term that can be estimated even if this limit was infinite, thanks to assumptions (A1)–(A3), even if it can be proven that it is finite under suitable assumptions, see e.g. [153, Theorem 5.2.1.] or [176, Theorem 2.3.1. and Theorem 2.4.1.]). Then, we can write:

$$\begin{aligned}\nabla\Phi(|\nabla \mathsf{v}_\delta|) - \nabla(\Phi)^\delta(|\nabla \mathsf{v}_\delta|) = \quad &\nabla\Phi(|\nabla \mathsf{v}_\delta|) - \nabla\Phi(|\nabla \mathsf{v}|)\\ &+\nabla\Phi(|\nabla \mathsf{v}|) - (\nabla\Phi)^\delta(|\nabla \mathsf{v}|)\\ &+(\nabla\Phi)^\delta(|\nabla \mathsf{v}|) - \nabla(\Phi)^\delta(|\nabla \mathsf{v}|)\\ &+\nabla(\Phi)^\delta(|\nabla \mathsf{v}|) - \nabla(\Phi)^\delta(|\nabla \mathsf{v}_\delta|).\end{aligned} \tag{2.2.37}$$

Integrating the left-hand side in (2.2.37), we get by definition that the first and the fourth term in the obtained right-hand side goes to zero as δ does thanks to assumptions (A1)–(A3), the second term does too using assumption (A2) and since it is a classical mollification. It remains to deal with the third term, but it still goes to zero as δ does applying Friedrich's Lemma (see e.g. [43, Section 1.5.4.], [116, Lemma 17.1.5] or [31, Section 2.2.]).

Moreover, an energy estimate leads to, testing against v_δ into the weak formulation:

$$\frac{1}{2}\|\mathsf{v}_\delta\|^2_{L^\infty((0,T),L^2(\Omega))} + \iint_{Q_T} F_\delta(|\nabla \mathsf{v}_\delta|)|\nabla \mathsf{v}_\delta|^2\, dx\, dt = \iint_{q_T} \chi_\omega\varphi\nabla \mathsf{v}_\delta\, dx\, dt + \frac{1}{2}\|M_\delta(y^0)\|^2_{L^2(\Omega)}. \tag{2.2.38}$$

We get from assumptions (A1)–(A2), using Young's inequality for $0 < s < 1$ and the fact that $\|M_\delta(y^0)\|_{L^2(\Omega)} \leq \|y^0\|_{L^2(\Omega)}$:

$$\begin{aligned}\tfrac{1}{2}\|\mathsf{v}_\delta\|^2_{L^\infty((0,T),L^2(\Omega))} + 2\delta\|\mathsf{v}_\delta\|^2_{L^2((0,T),H^1_0(\Omega))} + & \quad \iint_{Q_T} F(|\nabla \mathsf{v}_\delta|)|\nabla \mathsf{v}_\delta|^2 \, dx\, dt \\ & \leq \left(\frac{p-1}{p^{\frac{p}{p-1}} s^{\frac{1}{p-1}}}\right) \|\chi_\omega \varphi\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(q_T)} + s\|\mathsf{v}_\delta\|^p_{L^p((0,T),W^{1,p}_0(\Omega)} \\ & + \|\nabla\Phi(|\nabla \mathsf{v}_\delta|)\cdot\nabla \mathsf{v}_\delta - (\nabla\Phi)^\delta(|\nabla \mathsf{v}_\delta|)\cdot\nabla \mathsf{v}_\delta\|_{L^1(Q_T)} \\ & + \tfrac{1}{2}\|y^0\|^2_{L^2(\Omega)}.\end{aligned} \tag{2.2.39}$$

We point out that from assumptions (A2)–(A3), the product $\nabla\Phi(|\nabla \mathsf{v}_\delta|)\cdot\nabla \mathsf{v}_\delta$ involved in the above inequality is non negative. Let us focus ourselves to the case $1 < p < 2$, the case $p \geq 2$ being rather direct. From assumption (A4), since we get:

$$\begin{aligned}\iint_{Q_T} F(|\nabla \mathsf{v}_\delta|)|\nabla \mathsf{v}_\delta|^2 \, dx\, dt & \geq \iint_{Q_T} (\mu + |\nabla \mathsf{v}_\delta|^2)^{\frac{p-2}{2}} |\nabla \mathsf{v}_\delta|^2 \, dx\, dt \\ & = \iint_{Q_T} (\mu + |\nabla \mathsf{v}_\delta|^2)^{\frac{p}{2}} \, dx\, dt - \iint_{Q_T} \mu(\mu + |\nabla \mathsf{v}_\delta|^2)^{\frac{p-2}{2}} \, dx\, dt \ , \\ & \geq \|\mathsf{v}_\delta\|^p_{L^p((0,T),W^{1,p}_0(\Omega))} - |\Omega| T \mu^{\frac{p}{2}}.\end{aligned} \tag{2.2.40}$$

then one can write from (2.2.39) and (2.2.40) for $0 < s < \frac{1}{2}$:

$$\begin{aligned}(1-s)\|\mathsf{v}_\delta\|^p_{L^p((0,T),W^{1,p}_0(\Omega))} \leq & \quad \left(\frac{p-1}{p^{\frac{p}{p-1}} s^{\frac{1}{p-1}}}\right) \|\chi_\omega \varphi\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(q_T)} \\ & + \|\nabla\Phi(|\nabla \mathsf{v}_\delta|)\cdot\nabla \mathsf{v}_\delta - (\nabla\Phi)^\delta(|\nabla \mathsf{v}_\delta|)\cdot\nabla \mathsf{v}_\delta\|_{L^1(Q_T)} \\ & + \tfrac{1}{2}\|y^0\|^2_{L^2(\Omega)} + |\Omega| T \mu^{\frac{p}{2}}.\end{aligned} \tag{2.2.41}$$

Using Young's inequality, we then get from (2.2.41):

$$\begin{aligned}(1-2s)\|\mathsf{v}_\delta\|^p_{L^p((0,T),W^{1,p}_0(\Omega))} \leq & \quad \left(\frac{p-1}{p^{\frac{p}{p-1}} s^{\frac{1}{p-1}}}\right) \|\chi_\omega \varphi\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(q_T)} \\ & + \left(\frac{p-1}{p^{\frac{p}{p-1}} s^{\frac{1}{p-1}}}\right) \|\nabla\Phi(|\nabla \mathsf{v}_\delta|) - (\nabla\Phi)^\delta(|\nabla \mathsf{v}_\delta|)\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(Q_T)} \\ & + \tfrac{1}{2}\|y^0\|^2_{L^2(\Omega)} + |\Omega| T \mu^{\frac{p}{2}}.\end{aligned} \tag{2.2.42}$$

And so the uniform bound over δ of $\|\mathsf{v}_\delta\|_{L^p((0,T),W^{1,p}_0(\Omega))}$ follows since every term in the right-hand side of (2.2.42) is uniformly bounded over δ (this last being chosen small enough). Combining this fact with (2.2.36), we get that the left-hand side of (2.2.36) goes to zero as δ does. Thus, up to take δ small enough, from (2.2.34)–(2.2.36) combined to (2.2.42), we obtain the wished approximate controllability. □

In fact, it is possible to derive an exact controllability result from the Theorem refthm:global-approximate-control-quasi as soon as we have the existence of a finite stopping time for the solutions of (2.2.8). More precisely, we have that:

Theorem 2.2.11 ([49, Theorem 1.3.]). *Assume that F satisfies assumptions (A1)–(A3), and that y is the solution of (2.2.8) associated to an initial data $y^0 \in L^2(\Omega)$. Moreover, let us consider that y stops in finite time, which is that, if $\varphi = 0$ then, there exists $T_s \in (0,T)$, $\gamma > 0$ and $\mu > 0$ such that:*

$$\|y(T_s)\| = 0 \quad \text{and} \quad T_s \leq \mu \|y^0\|^\gamma. \tag{2.2.43}$$

Then, one can choose the force term φ such that y is exactly null controllable in any time $T^\star \in (0,T)$.

Proof of Theorem 2.2.11. Let $T^\star \in (0,T)$ and $\varepsilon = \left(\frac{T^\star}{2\mu}\right)^{\frac{1}{\gamma}}$. Applying Theorem 2.2.6 for an approximate control in time $\frac{T^\star}{2}$ there exists a control $\varphi_c \in L^2((0,\frac{T^*}{2})\times\Omega)$ such that the solution y of (2.2.8) with the control given by

$$\varphi(t) = \begin{cases} \varphi_c(t) & \text{for } t \in \left(0, \frac{T^\star}{2}\right) \\ 0 & \text{for } t \geq \frac{T^\star}{2} \end{cases}$$

verifies

$$\left\| y\left(\frac{T^\star}{2}\right)\right\| \leq \varepsilon.$$

Combining the above inequality to the estimate (2.2.43) we obtain that $y(T^\star) = 0$, which is the desired result. □

The results given by Theorem 2.2.6 and Theorem 2.2.11 can in fact be demonstrated for a larger class of equations, typically for the parabolic p-Laplacian (as given by (1.3.1) in the homogeneous setting) which is given by

$$\begin{cases} \partial_t \mathsf{v} - \operatorname{div}\left(|\nabla \mathsf{v}|^{p-2}\nabla \mathsf{v}\right) = \chi_\omega \varphi & \text{in } Q_T \\ \mathsf{v} = 0 & \text{on } \Sigma_T \\ \mathsf{v}(0) = y^0 & \text{in } \Omega, \end{cases} \tag{2.2.44}$$

with $\frac{3}{2} < p < 3$. We then have the following results.

Corollary 2.2.12 ([49, Corollary 1.1.]). *Let $y^0 \in L^2(\Omega)$ and $\frac{3}{2} < p < 3$. Then (2.2.44) is approximate null controllable in any time $T > 0$, i.e., for every $\varepsilon > 0$ there exists a control $\varphi \in C^\infty(\overline{Q_T})$ such that the solution v of (2.2.44) verifies*

$$\|\mathsf{v}(T)\| \leq \varepsilon.$$

Corollary 2.2.13 ([49, Corollary 1.2.]). *Let $y^0 \in L^\infty(\Omega)$ and $\frac{3}{2} < p < 2$. Then, by always choosing a non-negative solution to (2.2.44), the problem (2.2.44) is null controllable in any time $T > 0$,* i.e., *there exists a control $\varphi \in C^\infty(\overline{Q_T})$ such that the solution v of (2.2.44) verifies*

$$\mathsf{v}(T) = 0.$$

Proofs of Corollary 2.2.12 and Corollary 2.2.13. The proof is in fact quite immediate. It is enough to note that the solutions of (2.2.44) can be approximated by the solutions of

$$\begin{cases} \partial_t \mathsf{y} - \mu\Delta \mathsf{y} - \operatorname{div}\left(\left(\mu + |\nabla \mathsf{y}|^2\right)^{\frac{p-2}{2}} \nabla \mathsf{y}\right) = \chi_\omega \varphi & \text{in } Q_T \\ \mathsf{y} = 0 & \text{on } \Sigma_T \\ \mathsf{y}(0) = y^0 & \text{in } \Omega. \end{cases} \tag{2.2.45}$$

(see e.g. [130]) which is approximately controllable according to Theorem 2.2.6. It is possible to see that, for example, by observing that the approximation operator in $\mu > 0$ has the M-property (see [176, Lemma 3.2.2.], [132, Chapitre 2 Remarque 2.1.], and [175, Proposition 31.5.]) and converges in the sense of $L^p((0,T), W_0^{1,p}(\Omega))$ to the p-Laplacian operator. Then, Corollary 2.2.13 is an immediate consequence of Theorem 2.2.11 applied to [71, Proposition 2.1.] and [132, Exemple 1.5.2.], setting $X := W_0^{1,p}(\Omega) \cap L^2(\Omega)$. □

It is also possible to apply these results in a similar way to obtain the exact controllability of other quasilinear parabolic equations stopping in finite time and whose structure is close to that of the p-Laplacian, as for example for the following equation.

$$\begin{cases} \partial_t \mathsf{u} - \Delta \mathsf{u} - \operatorname{div}\left(|\nabla \mathsf{u}|^{p-2}\nabla \mathsf{u}\right) = \chi_\omega \varphi & \text{in } Q_T \\ \mathsf{u} = 0 & \text{on } \Sigma_T \\ \mathsf{u}(0) = y^0 & \text{in } \Omega \end{cases} \tag{2.2.46}$$

with $\frac{3}{2} < p < 2$. We point out that the operator $A : \mathsf{u} \mapsto -\Delta \mathsf{u} - \operatorname{div}\left(|\nabla \mathsf{u}|^{p-2}\nabla \mathsf{u}\right)$ is well-defined and monotone over $X := H^2(\Omega) \cap H_0^1(\Omega)$ (see e.g. [16, Section 4.3.]).

Then, we have the following result.

Corollary 2.2.14. *Let $y^0 \in L^2(\Omega)$, and $\frac{3}{2} < p < 2$. Then (2.2.46) is null controllable in any time $T > 0$, i.e., there exists a control $\varphi \in C^\infty(\overline{Q_T})$ such that the solution u of (2.2.46) verifies*

$$\mathsf{u}(T) = 0.$$

The proof of Corollary 2.2.14 can be obtained in similar way to that of Corollary 2.2.13, applying an stopping time result as [8, Theorem 2.1.], setting $X := H_0^1(\Omega) \cap H^2(\Omega)$.

2.3 Mixed finite element method

When numerically studying a constrained minimization problem, it is sometimes possible to reduce it to what is known as a mixed formulation. More precisely, we then consider two coupled weak formulations, defined using two bilinear forms, and whose sought-after minima coincide with the minima of the initial problem. This makes it possible to successfully apply a finite element method to nonlinear problems.

As mentioned in the summary, we first set out some reminders in Subsection 2.3.1 as well as at the beginning of Subsection 2.3.2.

2.3.1 Mixed Lagrangian formulation

Let be two Hilbert spaces H_1 and H_2. Consider the problem of finding $(u, \lambda) \in H_1 \times H_2$ satisfying the following equation:

$$\begin{cases} a(u,\varphi) + b(\varphi,\lambda) = \langle f, \varphi \rangle & \forall \varphi \in H_1 \\ b(u,\phi) = \langle g, \phi \rangle & \forall \phi \in H_2 \end{cases} \tag{2.3.1}$$

where $a : H_1 \times H_1 \to \mathbb{R}$ is a continuous and symetric bilinear form, and $b : H_2 \times H_1 \to \mathbb{R}$ is a continuous bilinear form.

We consider the following two associated linear forms A and B given by Riesz representation theorem:

$$A : \left| \begin{array}{l} H_1 \to H_1' \\ \\ u \mapsto A(u) \end{array} \right. \tag{2.3.2}$$

where

$$a(u,\varphi) = \langle A(u), \varphi \rangle, \quad \forall (u,\varphi) \in H_1^2,$$

$$B : \left| \begin{array}{l} H_1 \to H_2' \\ \\ \lambda \mapsto B(\lambda) \end{array} \right. \tag{2.3.3}$$

where

$$b(\phi,\lambda) = \langle B(\lambda), \phi \rangle, \quad \forall (\lambda,\phi) \in H_1 \times H_2.$$

Here and thereafter, we should denote $A^* : H_1 \to H_1'$ and $B^* : H_2 \to H_1'$ their respective adjoint linear forms for the duality bracket. Thus, problem (2.3.1) can be reformulated as follows.

$$\left\{ \begin{array}{ll} A(u) + B^*\lambda = f & \text{in } H_1' \\ B(u) = g & \text{in } H_2' \end{array} \right. \tag{2.3.4}$$

We point out that since a is symmetric, then the problem (2.3.1) reduces to the constrained minimization problem

$$\min_{B(\varphi)=g} \frac{1}{2} a(\varphi,\varphi) - \langle f, \varphi \rangle, \tag{2.3.5}$$

and so λ stands as the Lagrange multiplier of (2.3.1) and its associated saddle-point problem is given by the condition

$$\inf_{\varphi \in H_1} \sup_{\phi \in H_2} \left(\frac{1}{2} a(\varphi,\varphi) + b(\phi,\lambda) - \langle f, \varphi \rangle - \langle g, \phi \rangle \right). \tag{2.3.6}$$

The reader should not worry if the considerations of this approach are not yet very clear. These will become clearer as we return to our case study. Let us denote by a_K the restriction of a to the subspace $\mathrm{Ker}(B) \times \mathrm{Ker}(B)$, and by A_K the function defined, as for A, by:

$$A_K : \left| \begin{array}{l} \mathrm{Ker}(B) \to \mathrm{Ker}(B)' \\ \\ u \mapsto A_K(u) \end{array} \right. \tag{2.3.7}$$

where

$$a_K(u,\varphi) = \langle A_K(u), \varphi \rangle, \quad \forall (u,\varphi) \in \mathrm{Ker}(B)^2.$$

Let us now set out one of the main results of this section.

Theorem 2.3.1. *Let H_1 and H_2 be two real Hilbert spaces. Moreover, we consider two continuous bilinear forms $a : H_1 \times H_1 \to \mathbb{R}$ and $b : H_2 \times H_1 \to \mathbb{R}$, a being symmetric. We define the continuous linear functions $A : H_1 \to H_1'$, $B : H_2 \to H_1'$ and $A_K : \mathrm{Ker}(B) \to \mathrm{Ker}(B)'$ respectively by* (2.3.2), (2.3.3), *and* (2.3.7). *Then, the problem* (2.3.1) *has a unique solution for every $(f,g) \in H_1' \times H_2'$ if and only if both of the following conditions hold:*

(SP1) A_K is an isomorphim from $\mathrm{Ker}(B)$ *over* $\mathrm{Ker}(B)'$.

(SP2) B is onto, i.e. $R(B) = H_2'$.

Proof. See [27, Theorem 4.2.2.]. □

We point out that a similar result holds replacing assumption (SP1) by a coercivity condition over a_K. However, we then lose the uniqueness (see [27, Theorem 4.2.1.]).

2.3.2 The Ladyzhenskaya-Babuška-Brezzi conditions

The purpose of this section is to show that it is possible to find conditions equivalent to (SP1) and (SP2) of the Theorem 2.3.1, but which will often be preferable in practice.

From the Corollary A.8.2, we can say that condition (SP2) is equivalent to the existence of a constant $\eta > 0$ such that for $v \neq 0$:

$$\eta \leq \frac{\|B^*(v)\|_{H_2'}}{\|v\|_{H_1}}. \tag{2.3.8}$$

Now, if we want to optimize the constant η, we write from (2.3.8):

$$\eta \leq \inf_{\substack{v \in H_1 \\ v \neq 0}} \frac{\|B^*(v)\|_{H_2'}}{\|v\|_{H_1}} = \inf_{\substack{v \in H_1 \\ v \neq 0}} \sup_{\substack{u \in H_2 \\ u \neq 0}} \frac{|b(u,v)|}{\|v\|_{H_1}\|u\|_{H_2}}. \tag{2.3.9}$$

Then (2.3.9) is equivalent to (SP2). Arguing in the same way, we can prove that assumption (SP1) is equivalent, form Corollary A.8.2 and the symmetry of a, to the existence of $\gamma > 0$ such that the following inequality holds:

$$\gamma \leq \inf_{\substack{v \in H_1 \\ v \neq 0}} \sup_{\substack{w \in H_1 \\ w \neq 0}} \frac{|a(v,w)|}{\|v\|_{H_1}\|w\|_{H_1}}. \tag{2.3.10}$$

Remark 2.3.2. *We point out that in the case when the bilinear map a is not symmetric,* (2.3.10) *should be replace by the fact that there exists $\gamma > 0$ such that both of the following inequalities hold.*

$$\begin{cases} \gamma \leq \inf\limits_{\substack{v \in H_1 \\ v \neq 0}} \sup\limits_{\substack{w \in H_1 \\ w \neq 0}} \frac{|a(v,w)|}{\|v\|_{H_1}\|w\|_{H_1}}, \\ \gamma \leq \inf\limits_{\substack{w \in H_1 \\ w \neq 0}} \sup\limits_{\substack{v \in H_1 \\ v \neq 0}} \frac{|a(v,w)|}{\|v\|_{H_1}\|w\|_{H_1}}. \end{cases} \tag{2.3.11}$$

Since Corollary A.8.2 should be applied in the aim to establish the isomorphic equivalence.

The equivalences (2.3.10) and (2.3.9) of (SP1) and (SP2) respectively are called the **Ladyzhenskaya-Babuška-Brezzi conditions**.

Thus, we get the following alternative version of Theorem 2.3.1.

Theorem 2.3.3. *Let H_1 and H_2 be two real Hilbert spaces. Moreover, we consider two continuous bilinear forms $a : H_1 \times H_1 \to \mathbb{R}$ and $b : H_2 \times H_1 \to \mathbb{R}$, a being symmetric. We define the continuous linear functions $A : H_1 \to H_1'$, $B : H_2 \to H_1'$ and $A_K : \mathrm{Ker}(B) \to \mathrm{Ker}(B)'$ respectively by* (2.3.2),

(2.3.3), *and* (2.3.7). *Then, the problem* (2.3.1) *has a unique solution for every* $(f,g) \in H_1' \times H_2'$ *if and only if there exists two pisitive constants* γ *and* η *such that* (2.3.10) *and* (2.3.9) *hold.*

Example 2.3.4. *Let us consider the elliptic Stokes problem with non-slip boundary conditions for a domain* Ω *of* $\mathbb{R}^2$ *with smooth boundary given by finding* $(u,p) \in H_0^1(\Omega) \times L^2(\Omega)$ *such that:*

$$\begin{cases} \int_\Omega D(u) : D(\varphi)\, dx - \int_\Omega p \operatorname{div}(\varphi)\, dx = \int_\Omega v \cdot f\, dx & v \in H_0^1(\Omega) \\ \int_\Omega \phi \operatorname{div}(u)\, dx = 0 & \phi \in L^2(\Omega). \end{cases} \tag{2.3.12}$$

System (2.3.12) *is of the same kind as* (2.3.1) *setting* $H_1 = H_0^1(\Omega)$ *and* $H_2 = L^2(\Omega)$. *Moreover, one can prove that* (2.3.10) *and* (2.3.9) *holds, the last being harder to prove. We refer for example to [31, Chapter IV, Section 3.1.] for a proof. Then, Theorem 2.3.3 implies the existence of a unique solution to* (2.3.12).

2.4 Simulations using Hsieh-Clough-Tocher finite elements

Following the presentation done in [46, 23], we now recall how to define the structure of reduced Hsieh-Clough-Tocher (subsequently abbreviated to HCTr) finite elements and how we make use of them to illustrate our results with numerical simulations. As a reminder, a finite element of $\mathbb{R}^N$ is a triple (K,P,Σ) such that the following assertions are verified.

(FE1) K is a subset of $\mathbb{R}^N$ with a nonempty interior and Lipschitz boundary

(FE2) P is a finite dimensional vector space whose elements are real-valued functions defined over K

(FE3) Σ is a set of $\dim(P)$ linear forms over P, which is P-unisolvent, namely:

$$\forall \left((\alpha_i)_{1\le i\le \dim(P)}, (\phi_i)_{1\le i\le \dim(P)}\right) \in \mathbb{R}^{\dim(P)} \times \Sigma^{\dim(P)},\ \exists p \in P,\ \phi_i(p) = \alpha_i \tag{2.4.1}$$

and sucht that such a p is unique.

Clearly, we have that $|\Sigma| = \dim(P)$. In addition, the linear forms $(\phi_i)_{1\le i\le \dim(P)}$ that describe the set Σ are called the degrees of freedom of the finite element.

A HCTr is a finite element (K,P,Σ) built as follows:

(HCTr1) K is a triangle with vertices a_1, a_2, and a_3.

(HCTr2) For every $a \in \overset{\circ}{K}$, we denote by K_i the triangle with vertices a, $a_{\overline{i+1}}$, $a_{\overline{i+2}}$, where $\overline{i}$ stands as the representation of i in $\mathbb{Z}/3\mathbb{Z}$. Moreover, we denote by O_i the opposite side of a_i into K. Then, we set:

$$P := \left\{p \in C^1(K) /\ \forall i \in [1,3] \cap \mathbb{N},\ p_{|K_i} \in \mathbb{P}_3(K_i),\ \partial_n p_{|O_i} \in \mathbb{P}_1(O_i)\right\}. \tag{2.4.2}$$

(HCTr3) The set Σ is given by:

$$\Sigma := \left\{\phi \in L(P,\mathbb{R}) /\ \exists (i,j) \in ([1,3] \cap \mathbb{N})^2,\ \phi = \phi_{i,j}\right\},$$

where $(\phi_{i,j})_{1\le i,j\le 3}$ is given for every $i \in [1,3] \cap \mathbb{N}$ by

$$\begin{cases} \phi_{i,1} : p \mapsto p(a_i) \\ \phi_{i,2} : p \mapsto p'(a_i)(a_{\overline{i+1}} - a_i) \\ \phi_{i,3} : p \mapsto p'(a_i)(a_{\overline{i+2}} - a_i). \end{cases}$$

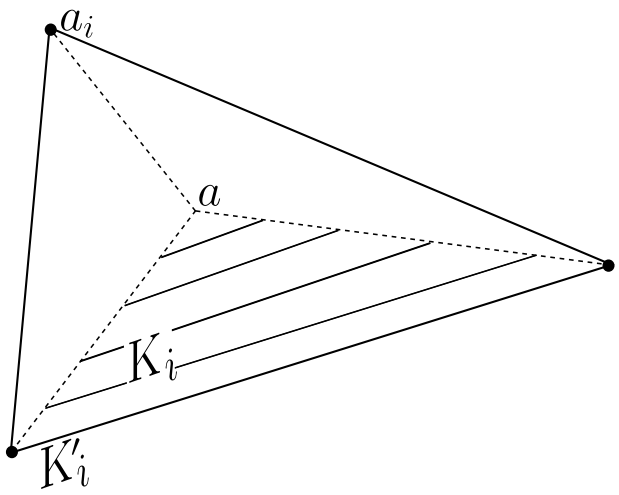


Figure 2.1: A representation of a reduced Hsieh-Clough-Tocher triangle

Then, the following result holds.

Theorem 2.4.1 (Ciarlet)**.** *The triple (K, P, Σ) defined by (HCTr1)-(HCTr3) is a finite element, namely Σ is P-unisolvent.*

Proof. See [46, Theorem 1]. □

We now wish to illustrate the results of Theorem 2.2.6 with numerical simulations. To this end, we follow a mixed finite element method as presented in section 2.3. More precisely, we propose a numerical strategy for the computation of an approximate null control for quasilinear equations (2.2.8). In a first step we approach an approximate control φ for the linear equation (2.1.1) by solving a mixed formulation in order to approach the solution of the optimality condition (2.1.18). Such a method is established in [143] in order to approach an approximated control by solving a mixed formulation of the following type: find $(\varphi, \lambda) \in \Phi \times L^2(Q_T)$ solution to

$$\begin{cases} \mathsf{a}(\varphi, \overline{\varphi}) + \mathsf{b}(\overline{\varphi}, \lambda) = l(\overline{\varphi}) & (\overline{\varphi} \in \Phi) \\ \mathsf{b}(\varphi, \overline{\lambda}) = 0 & (\overline{\lambda} \in L^2(Q_T)), \end{cases} \tag{2.4.3}$$

where

$$\mathsf{a} : \Phi \times \Phi \to \mathbb{R}, \quad \mathsf{a}(\varphi, \overline{\varphi}) = \iint_{q_T} \chi_\omega \varphi \overline{\varphi}\, dx\, dt + \varepsilon(\varphi(T), \overline{\varphi}(T)) \tag{2.4.4}$$

$$\mathsf{b} : \Phi \times L^2(Q_T) \to \mathbb{R}, \quad \mathsf{b}(\varphi, \lambda) = -\iint_{Q_T} (\partial_t \varphi + \operatorname{div}(a \nabla \varphi)) \lambda\, dx\, dt \tag{2.4.5}$$

$$l : \Phi \to \mathbb{R}, \quad l(\varphi) = -(\varphi(0), y^0). \tag{2.4.6}$$

The space Φ appearing in the above relations is the completion with respect to the norm

$$|||\varphi|||^2 = \iint_{q_T} \chi_\omega |\varphi|^2\, dx\, dt + \varepsilon \|\varphi(T)\|^2 + \eta \|\partial_t \varphi + \operatorname{div}(a \nabla \varphi)\|^2_{L^2(Q_T)}$$

of the following space:

$$W = \left\{ \varphi \in C^2(\overline{Q_T}),\ \varphi(T) \in C^\infty(\Omega),\ \varphi = 0 \text{ on } \Sigma_T \right\}.$$

Using Theorem 2.3.3, it can be shown that the mixed formulation (2.4.3) is well-posed, φ being the solution of (2.1.4) corresponding to the final data obtained as the minimum of the functional J_0^a given by (2.1.5) (we refer to [143] for details).

In order to numerically compute an approximate control for the quasilinear equation (2.2.8) we employ the mixed formulation of the control problem combined to a fixed point strategy. This approach is illustrated by several examples in the one-dimensional setting in the space variable. For the remaining part of this section we consider $\Omega = (0,1)$, $\omega = (0.1, 0.5)$ and $T = 0.5$.

From a practical point of view, the proposed strategy needs to efficiently compute the solutions of mixed formulations of the form (2.4.3). In order to numerically approach the solutions of such mixed formulations, we consider structured triangulations $\mathcal{T}_h$ of the domain Q_T with $h > 0$ being the diameter of triangles forming $\mathcal{T}_h$. Then we define the finite dimensional sub-spaces $\Phi_h \subset \Phi$ and $\Lambda_h \subset L^2(Q_T)$ as follows:

$$\Phi_h = \left\{ \phi_h \in C^1(\overline{Q_T}) \ : \ \phi_h|_K \in \mathbb{P}(K) \ \forall K \in \mathcal{T}_h, \ \phi_h = 0 \text{ on } \Sigma_T \right\}, \tag{2.4.7}$$

where $\mathbb{P}(T)$ denotes the reduced Hsieh-Clough-Tocher (HCTr for short) C^1 finite element space, and

$$\Lambda_h = \left\{ \lambda_h \in C(\overline{Q_T}) \ : \ \lambda_h|_K \in \mathbb{P}_1(K) \ \forall K \in \mathcal{T}_h \right\}, \tag{2.4.8}$$

with $\mathbb{P}_1(T)$ being the space of affine functions with respect to both x and t. We then approach the mixed formulation (2.4.3) by its following discrete version: find $(\varphi_h, \lambda_h) \in \Phi_h \times \Lambda_h$ solution to

$$\begin{cases} \mathsf{a}(\varphi_h, \overline{\varphi}_h) + \mathsf{b}(\overline{\varphi}_h, \lambda_h) = l(\overline{\varphi}_h) & (\overline{\varphi}_h \in \Phi_h) \\ \mathsf{b}(\varphi_h, \overline{\lambda}_h) = 0 & (\overline{\lambda}_h \in \Lambda_h). \end{cases} \tag{2.4.9}$$

Remark that for every $h > 0$ the mixed-formulation (2.4.9) is well posed. Nevertheless, in order to have a convergence of the solutions (φ_h, λ_h) to the solution (φ, λ) a discrete inf-sup should be verified for the discrete mixed-formulation (2.4.9) with a inf-sup constant uniform with respect to h. Proving such a uniform inf-sup condition is generally a difficult question. An alternative avoiding the necessity of this condition is to stabilize the mixed formulation (2.4.9) by an appropriate term.

We denote by N_x the number of right triangles in the triangulation $\mathcal{T}_h$ having one side on the boundary $\Omega \times \{0\}$ and by N_y the number of right triangles having one side on the boundary $\{0\} \times (0,T)$. We take N_y such that the vertical side h_y of every triangle in $\mathcal{T}_h$ is much smaller than h_x where h_x is the length of the horizontal side of the triangle. Then $h_x = 1/N_x$ and $N_y = N_x \gamma^{-1} T$ with $\gamma \in (0,1]$ being such that N_y is an integer. Two such triangulations are represented in Figure 2.2.

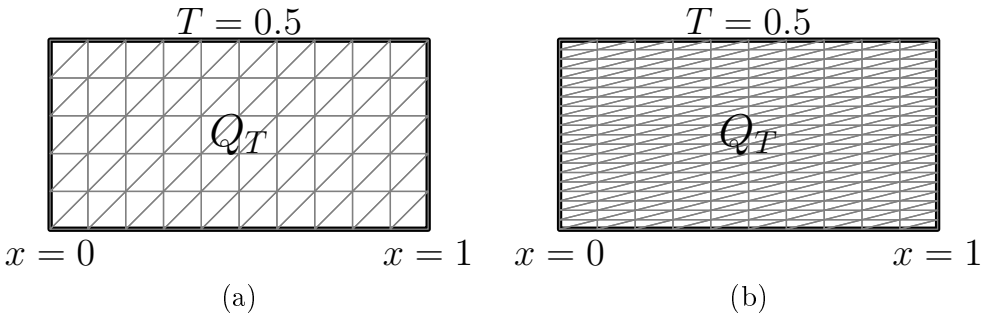


Figure 2.2: Two structured triangulations of Q_T with $N_x = 10$. (a) $\gamma = 1$. (b) $\gamma = 0.25$.

Since the controls of minimal L^2 norm for the heat equation oscillate in time near the control time T, for all the simulations discussed in this work we consider meshes that are more thin in time than in space. More exactly, we take $N_y = 320$ and $N_x \in \{20,\ 40,\ 80,\ 160\}$.

2.4.1 Approximation of controls for linear parabolic equations

Let us consider a non-homogeneous diffusion coefficient given by

$$a(t,x) = \frac{1}{10}\left(1 + x^2 + t\right). \tag{2.4.10}$$

In order to compute an approximate control for the equation (2.1.1) we numerically aproach the minima of the functional J_0^a by solving the mixed formulation (2.4.9).

In what follows, we consider two examples of regular initial data to control.

Example 1: linear equation with $y^0(x) = \sin(\pi x)$

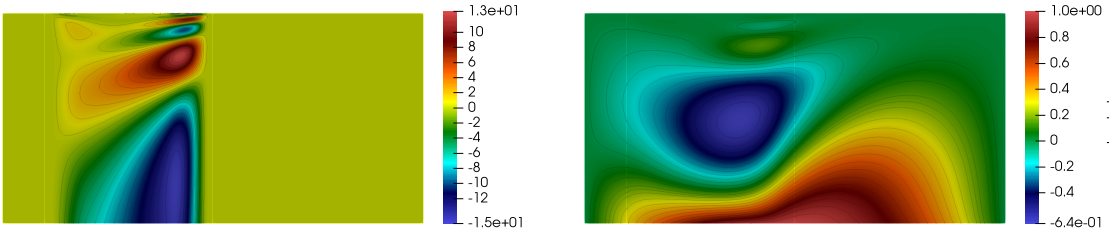
As a first example we consider the approximate control of the linear equation (2.1.1) with initial data given by

$$y^0(x) = \sin(\pi x). \tag{2.4.11}$$

In Table 2.1 we gather the L^2 norm of the approximate control $\chi_\omega \varphi$ obtained for different meshes and three different values of ε. We observe that the norm of the control converges with respect to the size of the mesh for each value of $\varepsilon \in \{10^{-2i}$ with $1 \leq i \leq 6\}$. We observe that norm of the control are larger for smaller valuer of ε and they seem to converge with respect to N_x and ε. The control $\chi_\omega \varphi$ and its associated controlled solution λ computed for $N_x = 160$ and $\varepsilon = 10^{-12}$ are displayed in Figure 2.3.

ε	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}
$N_x = 20$	0.943	1.946	2.495	2.690	2.698	2.698
$N_x = 40$	0.930	1.895	2.422	2.659	2.678	2.678
$N_x = 80$	0.935	1.905	2.437	2.690	2.717	2.718
$N_x = 160$	0.936	1.908	2.442	2.699	2.730	2.730

Table 2.1: Example 1: $L^2(q_T)$ norm of the control of the linear equation (2.1.1) with a diffusion coefficient a given by (2.4.10) and initial data (2.4.11) as a function of ε and N_x.



(a) (b)

Figure 2.3: Example 1: (a) Control $\chi_\omega \varphi$ for the linear equation (2.1.1) with a given by (2.4.10) and initial data (2.4.11) computed for $N_x = 160$ and $\varepsilon = 10^{-12}$. (b) The corresponding controlled solution λ.

Example 2: linear equation with $y^0(x) = \chi_{(0.6,0.9)}(x)$

As a second example we consider a localized but still regular initial data to control:

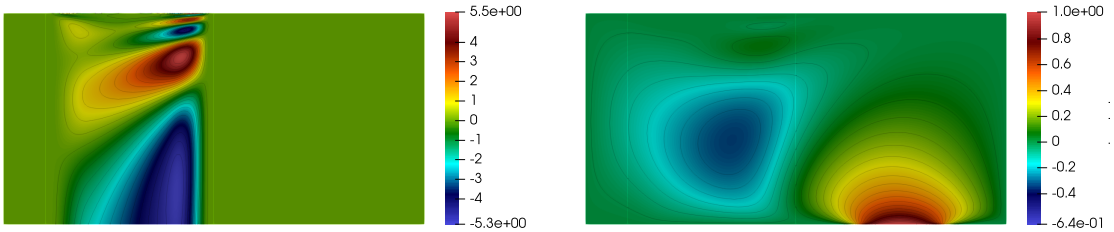
$$y^0(x) = \chi_{(0.6,0.9)}(x) = \begin{cases} 1, & \text{if } x \in [0.6+\delta, 0.9-\delta] \\ 0, & \text{if } x \in (0,1) \setminus (0.6, 0.9) \\ e^{\alpha\left(\frac{1}{\delta^2} - \frac{1}{(x-0.6)(0.6+2\delta-x)}\right)} & \text{if } x \in (0.6, 0.6+\delta) \\ e^{\alpha\left(\frac{1}{\delta^2} - \frac{1}{(x-0.9+2\delta)(0.9-x)}\right)} & \text{if } x \in (0.9-\delta, 0.9), \end{cases} \tag{2.4.12}$$

with $\delta = 0.1$ and $\alpha = 0.02$.

We obtain results similar to the ones in the Example 1 described in Section 2.4.1. The L^2 norm of the obtained control are listed in Table 2.2. We also depict the control and corresponding controlled solution computed on the mesh with $N_x = 160$ and $\varepsilon = 10^{-6}$ in Figure 2.4.

ε	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}
$N_x = 20$	0.250	0.615	0.854	0.947	0.950	0.950
$N_x = 40$	0.242	0.591	0.820	0.931	0.941	0.941
$N_x = 80$	0.244	0.595	0.826	0.946	0.959	0.959
$N_x = 160$	0.244	0.596	0.827	0.950	0.964	0.965

Table 2.2: Example 2: $L^2(q_T)$ norm of the control of the linear equation (2.1.1) with a diffusion coefficient a given by (2.4.10) and initial data (2.4.12) as a function of ε and N_x.



(a) (b)

Figure 2.4: Example 2: (a) Control $\chi_\omega \varphi$ for the linear equation (2.1.1) with a given by (2.4.10) and initial data (2.4.12) computed for $N_x = 160$ and $\varepsilon = 10^{-12}$. (b) The corresponding controlled solution λ.

2.4.2 Approximation of controls for quasilinear equations

We now focus on the controllability of quasilinear equations. First we consider the following nonlinearity:

$$F(X) = \frac{1}{10}\left(1 + (1+X^2)^{-\frac{1}{2}}\right). \tag{2.4.13}$$

Remark that this nonlinear function F satisfies assumptions (A1)–(A3).

In order to numerically approach the control and the corresponding controlled solution we employ a fixed-point algorithm combined to the strategy proposed in Section 2.4.1 for the approximation of controls for linear parabolic equations. More exactly, the following algorithm is employed for the computation of an approximate null control for the quasilinear equation (2.2.8):

In what follows we consider the same initial data as in Section 2.4.1 for the control of the quasilinear equation (2.2.8) corresponding to this choice of F. We consider different levels of meshes

Algorithm 1 Fixed point algorithm for the approximation of the control and the controlled solution for the quasilinear problem

Require: F, y^0, T, ε, $i_{\max}$ ▷ $i_{\max}$ is the maximal number of iterations
  $a \leftarrow 1$
  $i \leftarrow 0$
  err $\leftarrow +\infty$
  Compute the control and controlled solution (φ_0, λ_0) for the linear problem.
  while $i \leq i_{\max}$ and err $>$ tol **do** ▷ The tolerance tol is taken equal to h^2
    $a \leftarrow F(|\nabla \lambda_i|)$
    $i \leftarrow i + 1$
    Compute the control and controlled solution (φ_i, λ_i) for the linear problem.
    err $\leftarrow \|\chi_\omega(\varphi_i - \varphi_{i-1})\|_{L^2(q_T)}$
  end while
  if err $\leq$ tol **then**
    The algorithm converged.
    The control and solution of the quasiliinear problem are $(\varphi, \lambda) \leftarrow (\varphi_i, \lambda_i)$.
  end if

and several values of the penalization parameter ε. For each mesh of the domain Q_T and every value of ε we compute the L^2 norm of the control provided by Algorithm 1 and we report the number of iterations needed for its convergence.

Example 3: quasilinear equation with $y^0(x) = \sin(\pi x)$

In this section we consider again the control of initial data (2.4.11) in the case of the quasilinear equation (2.2.8) with F given by (2.4.13). The first question we would want to investigate is related to the convergence of Algorithm 1. In this purpose we list in Table 2.3 the number of iterations needed for the convergence of the fixed point algorithm for four levels of meshes and for four different values of the penalization parameter ε. We observe that, for every $\varepsilon \in \{10^{-2i} \text{ with } 1 \leq i \leq 6\}$ the number of iterations needed for the convergence slightly increases with N_x. This is probably due to the fact that tolerance parameter in the algorithm is smaller for larger values of N_x. The second observation is that the fixed point algorithm does not converge for small values of ε and fine enough meshes.

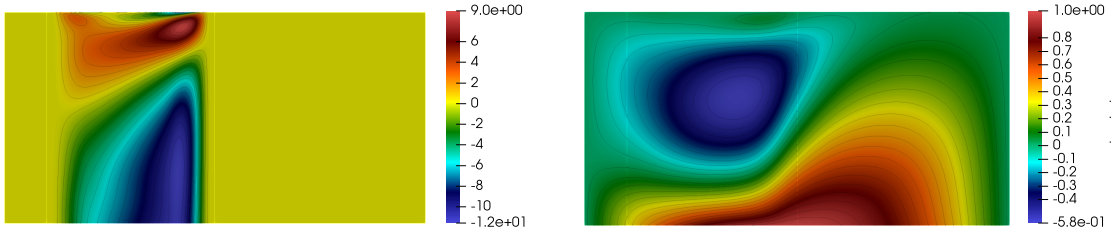
ε	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}
$N_x = 20$	4	5	6	6	6	7
$N_x = 40$	4	6	7	7	8	8
$N_x = 80$	5	7	8	9	9	-
$N_x = 160$	5	8	9	-	-	-

Table 2.3: Example 3: The number of iterations needed for the convergence of Algorithm 1 as a function of ε and N_x for the control of quasilinear equation (2.2.8) whith F given by (2.4.13) and initial data (2.4.11).

In Table 2.4 we gather the norm of the approximate control computed for different of values of N_x and ε. As for the control of the linear equation we observe a convergence of the norm of the control with respect to h. The control obtained for $N_x = 160$ and $\varepsilon = 10^{-6}$ and its associated controlled solution are illustrated in Figure 2.5.

ε	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}
$N_x = 20$	0.485	1.423	2.366	3.011	3.356	3.389
$N_x = 40$	0.486	1.390	2.301	2.931	3.317	3.402
$N_x = 80$	0.488	1.395	2.316	2.956	3.359	-
$N_x = 160$	0.489	1.396	2.319	-	-	-

Table 2.4: Example 3: $L^2(q_T)$ norm of the control of the quasilinear equation (2.2.8) with F given by (2.4.13) and initial data (2.4.11) as a function of ε and N_x.



(a) (b)

Figure 2.5: Example 3: (a) Control $\chi_\omega \varphi$ of the quasilinear equation (2.2.8) with F given by (2.4.13), initial data given by (2.4.11) and for $N_x = 160$, $\varepsilon = 10^{-6}$. (b) The corresponding controlled solution λ.

Example 4: quasilinear equation with $y^0(x) = \chi_{(0.6,0.9)}(x)$

This last example consider the numerical approximation of the approximate null control for equation (2.2.8) with F given by (2.4.13) and initial data (2.4.12). For this choice of initial data we conduct the same experiments as for Example 3. We obtain similar results with the difference that Algorithm 1 has a better convergence for this initial data. As reported in Table 2.5 the fixed point algorithm converge for $\varepsilon = 10^{-12}$ and for all the values of the discretization parameter N_x. Nevertheless, the number of iterations augment for $N_x = 160$ and the convergence will probably deteriorate for smaller values of ε.

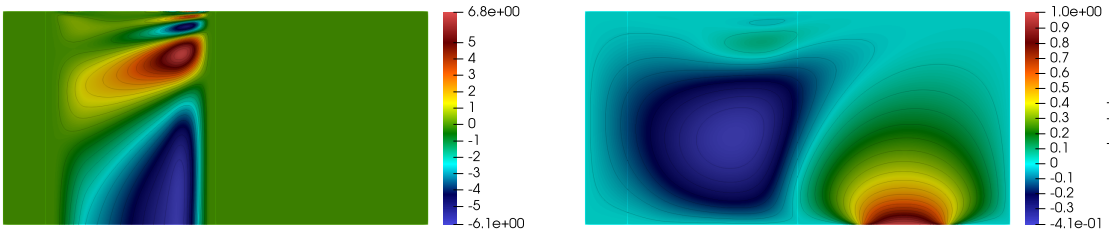
ε	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}
$N_x = 20$	4	5	6	6	6	6
$N_x = 40$	4	5	6	7	7	7
$N_x = 80$	5	6	7	8	8	8
$N_x = 160$	6	7	8	9	9	-

Table 2.5: Example 4: the number of iterations needed for the convergence of Algorithm 1 as a function of ε and N_x for the control of quasilinear equation (2.2.8) whith F given by (2.4.13) and initial data (2.4.12).

The values of the L^2 norm of the computed controls, reported in Table 2.6, indicate that controls converge with respect to N_x for fixed values of ε. This convergence seems faster for larger value of the penalization parameter ε. The control and the corresponding controlled solution associated to the initial data (2.4.12) are displayed in Figure 2.6.

ε	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}
$N_x = 20$	0.127	0.419	0.748	1.013	1.166	1.181
$N_x = 40$	0.124	0.405	0.720	0.976	1.147	1.185
$N_x = 80$	0.124	0.406	0.723	0.983	1.161	1.213
$N_x = 160$	0.124	0.406	0.723	0.984	1.163	-

Table 2.6: Example 4: $L^2(q_T)$ norm of the control of the quasilinear equation (2.2.8) with F given by (2.4.13) and initial data (2.4.12) as a function of ε and N_x.



(a) (b)

Figure 2.6: Example 4. (a) Control $\chi_\omega \varphi$ of the quasilinear equation (2.2.8) with F given by (2.4.13), initial data given by (2.4.12) and for $N_x = 160$, $\varepsilon = 10^{-10}$. (b) The corresponding controlled solution λ.

Si donc c'est le continu, l'étant est multiple. En effet, le continu est divisible indéfiniment. Mais il y a une difficulté à propos de la partie et de la totalité, qui n'est sans doute pas relative au propos présent mais qui vaut en elle-même : est-ce que la partie et la totalité sont une ou multiples, et comment sont-elles une ou multiples, et si elles sont multiples, comment sont elles multiples? Et à propos des parties qui ne sont pas continues la difficulté est aussi que, si chacune de ces parties est une avec la totalité en tant qu'elle est indivisible, elles le seront aussi les unes par rapport aux autres. Mais si c'est l'un comme indivisible, il n'aura en rien ni quantité ni qualité, et par conséquent l'étant ne sera ni infini comme le dit Mélissos, ni non plus fini comme le dit Parménide. Car c'est la limite qui est indivisible, non le fini.

Aristote, *Physique.*

3

Sedimentation in a quasi-static non-Newtonian fluid

This chapter is devoted to the study of the existence of global weak solutions for the sedimentation problem in an Ostwald-De Waele fluid, modeled by the following system.

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^N \\ -\operatorname{div}\left(\nu(\rho)|D(u)|^{p-2}D(u)\right) + \nabla \pi = \rho g & \text{in } \mathbb{R}_+ \times \mathbb{T}^N \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^N \\ \fint_{\mathbb{T}^N} u \, \mathrm{d}x = 0 & \text{in } \mathbb{R}_+ \\ \rho_{|t=0} = \rho_0 & \text{in } \mathbb{T}^N \end{cases} \tag{3.0.1}$$

In Section 3.1, we will first present the prerequisites concerning Besov spaces on the N-dimensional torus, which will be useful for analyzing the model in critical regularity. Then, in Section 3.2, we will recall some results on transport equations, which we will also use. First, let us define the notion of a weak solution associated with (3.0.1). Throughout this chapter, in the interests of simplicity of notation, we regularly omit to specify integration domains where this is unambiguous. In this case, we will use a double integral to denote integration over $\mathbb{R}_+ \times \mathbb{T}^N$ and a single integral to denote integration over $\mathbb{T}^N$. Similarly, where there is no ambiguity, we regularly omit to specify the underlying variables of integration.

Definition 3.0.1. *We consider a dimension $N \geq 2$ and an exponent $q \in [1, +\infty]$. Let $\rho_0 \in L^q$ be an initial datum. We say that a function $\rho \in L^\infty(L^q)$ is a weak solution of the Stokes-Transport problem* (3.0.1) *associated to the initial datum ρ_0 if the following conditions are satisfied:*

(i) There exists a velocity field $u : \mathbb{R}_+ \times \mathbb{T}^N \longrightarrow \mathbb{R}^N$ such that $\nu(\rho)|Du|^{p-1} \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{T}^N)$ and which is, for almost every time $t \in \mathbb{R}_+$, a weak solution of the Stokes equation: in other words $\operatorname{div}(u) = 0$ in $\mathcal{D}(\mathbb{R}_+ \times \mathbb{T}^N)$ and, for any divergence-free $\phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{T}^N, \mathbb{R}^N)$, we have

$$\iint \nu(\rho)|Du|^{p-2}Du : D\phi \, \mathrm{d}x \, \mathrm{d}t = \iint \rho g \cdot \phi \, \mathrm{d}x \, \mathrm{d}t \, ;$$

(ii) The velocity field is $u \in L^\infty(L^{q'})$ satisfies $\fint u \, \mathrm{d}x = 0$ and ρ is a weak solution of the transport equation with initial datum ρ_0, that is, for all $\phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{T}^N, \mathbb{R})$, we have

$$\iint \left(\rho \partial_t \phi + \rho u \cdot \nabla \phi\right) \mathrm{d}x \, \mathrm{d}t + \int \rho_0 \phi(0) \, \mathrm{d}x = 0.$$

We aim to prove the following result, which is the main theorem of this Chapter.

Theorem 3.0.2 ([54]). *We work in dimension $N \geq 2$. Consider $p \in (1, +\infty)$ and a function $\nu \in C^\infty(\mathbb{R} \setminus \{0\}) \cap C^{0,\overline{\gamma}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\nu(|r|) \gtrsim |r|^\gamma$ for all $r \in [-1, 1]$, $\overline{\gamma} = \min(\gamma, 1)$, $\gamma \geq 0$.*

Consider exponents $q \in (1, 2)$ and $\sigma \in [1, +\infty]$ such that one of the following conditions is satisfied:

(i) Sub-critical case: either we have the strict inequality

$$\frac{1}{p}\left(1 + \frac{\gamma}{\sigma}\right) + \frac{1}{q} - \frac{1}{N} < 1 \, ; \tag{3.0.2}$$

(ii) Critical case: either we have equality

$$\frac{1}{p}\left(1 + \frac{\gamma}{\sigma}\right) + \frac{1}{q} - \frac{1}{N} = 1 \, , \tag{3.0.3}$$

and in addition the condition $q \geq \frac{2N}{N+2}$ holds. In particular, this is always true when $N = 2$.

Then, for any initial datum $\rho_0 \in L^q(\mathbb{T}^N)$ such that $1/\rho_0 \in L^\sigma(\mathbb{T}^N)$, there exists a weak solution $\rho \in L^\infty\big(\mathbb{R}_+; L^q(\mathbb{T}^N)\big)$. In addition, if $\rho_0 \in L^q(\mathbb{T}^N) \cap L^r(\mathbb{T}^N)$ for some $r \in [1, +\infty]$, then

$$\|\rho\|_{L^\infty(L^r)} = \|\rho_0\|_{L^r}.$$

Section 3.3 will be devoted to the proof of this last. We point out that (3.0.1) being an active scalar equation, we can express the velocity field u as a function of ρ.

Remark 3.0.3. *The present theorem is the result of joint work carried out with Dimitri A. Cobb by the present author, and was the subject of a research stay at the University of Bonn (Germany). This work, together with other results, will be the subject of a pre-publication.*

3.1 Introduction to Besov spaces on the torus

In this section, we introduce the notion of Besov space, focusing on the cases which are relevant in view of a definition over the N-dimensional torus $\mathbb{T}^N$. First, and in order not to lose generality, we establish the results on $\mathbb{R}^N$, via the non-homogeneous Littlewood-Paley decomposition and the associated Besov spaces. Finally, we will see that this approach immediately substitutes on the torus.

First of all, let us recall the notion of Fourier multiplier which will be used throughout this chapter.

Definition 3.1.1 (Fourier multiplier). *Let $m : \mathbb{R}^N \to \mathbb{C}$ be a bounded function. Then, the Fourier multiplier associated to m is the operator denoted $m(D)$ and defined, when it makes sense, by:*

$$\forall u \in \mathcal{S}'(\mathbb{R}^N), \ \forall \xi \in \mathbb{R}^N, \ \widehat{m(D)u}(\xi) = m(\xi)\hat{u}(\xi).$$

We also say that m is the symbol associated to $m(D)$.

Let us present the notion of Littlewood-Paley decomposition some of its well-known properties. This decomposition is based on a dyadic partition of unity in the frequency variable, in the sense that if we choose some compactly supported and radially symmetric smooth function $\chi \in \mathcal{D}$ such that $r \mapsto \chi(re)$ is decreasing for all $e \in \mathbb{R}^N$, such that

$$\chi(x) = 1 \text{ for } |x| \leq 1 \qquad \text{and} \qquad \chi(x) = 0 \text{ for } |x| \geq 2,$$

then, by setting $\varphi(\xi) = \chi(\xi) - \chi(2\xi)$ and $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, we obtain the following partition of unity in the frequency variable (see [13, Proposition 2.10.])

$$1 = \chi(\xi) + \sum_{j\geq 0} \varphi_j(\xi). \tag{3.1.1}$$

This last holding for all $\xi \in \mathbb{R}^d$. This partition of unity gives rise to a set of operators: we define the Littlewood-Paley blocks by

$$\begin{array}{ll} \Delta_j = 0 & \text{if } j \leq -2, \\ \Delta_{-1} = \chi(D), & \\ \Delta_j = \varphi_j(D) & \text{for } j \geq 0. \end{array} \tag{3.1.2}$$

We also define the homogeneous low frequency cut-off operator $S_j = \chi(2^{j-1}D)$. Let us also make the following observation: the operators Δ_j and S_j are respectively scaled versions of $\varphi(D)$ and Δ_{-1}, and thus are uniformly bounded in the $L^q \longrightarrow L^q$ topology for all $q \in [1, +\infty]$.

Hence, we can write, at least formally for instance

$$\mathrm{Id} = \sum_{j\geq -1} \Delta_j.$$

The above relation being known as the non homogeneous Littlewood-Paley decomposition, and holds over the space of tempered distributions $\mathcal{S}'(\mathbb{R}^N)$ (see [13, Proposition 2.12., Proposition 2.13.]).

One of the main properties linked with Littlewood-Paley decomposition is given by the fact that it is possible to estimate the derivatives of a distribution in terms of its frequencies. More precisely, given a distribution u, the Fourier transform of $\Delta_j u$ has its support included in an annulus of size more or less 2^j, thus the derivative of u will act as a multiplication by 2^j. This is the object of the following result.

Proposition 3.1.2 (Bernstein inequalities)**.** *Let $0 < r < R$. There exists a constant $C > 0$ such that for any nonnegative integer k, any couple (p, q) in $[1, +\infty]^2$, with $p \leq q$, and any function $u \in L^p$, we have, for all $\lambda > 0$,*

$$\begin{aligned} \mathrm{supp}\,(\widehat{u}) \subset B(0, \lambda R) \quad &\Longrightarrow \quad \|\nabla^k u\|_{L^q} \leq C^{k+1}\, \lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)}\, \|u\|_{L^p}\,; \\ \mathrm{supp}\,(\widehat{u}) \subset \{\xi \in \mathbb{R}^N \,|\, r\lambda \leq |\xi| \leq R\lambda\} \quad &\Longrightarrow \quad C^{-k-1}\, \lambda^k \|u\|_{L^p} \leq \|\nabla^k u\|_{L^p} \leq C^{k+1}\, \lambda^k \|u\|_{L^p}\,. \end{aligned}$$

Proof. See [13, Lemma 2.1.] □

We are now able to define the Besov spaces.

Definition 3.1.3 (Non-homogeneous Besov space)**.** *Let $s \in \mathbb{R}$ and $1 \leq p, r \leq +\infty$. The non-homogeneous Besov space $B^s_{p,r} = B^s_{p,r}(\mathbb{R}^N)$ is defined as the set of tempered distributions $f \in \mathcal{S}'$ for which*

$$\|f\|_{B^s_{p,r}} := \left\| \left(2^{js}\, \|\Delta_j f\|_{L^p}\right)_{j\geq -1} \right\|_{\ell^r} < +\infty\,.$$

We then have the following property.

Proposition 3.1.4. *Let us consider (s, p, r) as in Definition 3.1.3. Then, $B^s_{p,r}$ endowed with the norm $\|\cdot\|_{B^s_{p,r}}$ is a Banach space.*

It can be proved, even if we will not use this notion, (see [165, Section 2.3.3., Section 2.5.7.]) that for $0 < p < 1$, the space $B^s_{p,r}$ can still be well-defined but is no more a Banach space (nevertheless, it is a quasi-Banach space).

As with the Sobolev spaces $W^{s,p}$, the exponent $s \in \mathbb{R}$ acts as a regularity index and p as an integrability exponent. In fact, a consequence of the Bernstein inequalities are the embeddings

$$B^k_{p,1} \subset W^{k,p} \subset B^k_{p,\infty}, \tag{3.1.3}$$

which holds for all $k \in \mathbb{N}$ and all $p \in [1, +\infty]$. Moreover, another consequence of Proposition 3.1.2 is that it leads to some embeddings between Besov spaces. More precisely, the following result holds.

Proposition 3.1.5. *Consider $s_1 \in \mathbb{R}$ and $q_1, q_2, r_1, r_2 \in [1, +\infty]$ such that $q_1 \leq q_2$ and $r_1 \leq r_2$, then the inclusion $B^{s_1}_{q_1,r_1} \subset B^{s_2}_{q_2,r_2}$ holds with*

$$s_2 := s_1 - d\left(\frac{1}{q_1} - \frac{1}{q_2}\right).$$

Proof. See [13, Proposition 2.71]. □

Moreover, the following interpolation inequality holds.

Proposition 3.1.6. *Consider real numbers $s_1 \leq s_2$ and $p, r \in [1, +\infty]$. Then, for all $\theta \in [0, 1]$ and $s := \theta s_1 + (1-\theta)s_2$, we have*

$$\forall f \in B^{s_2}_{p,r}, \qquad \|f\|_{B^s_{p,r}} \leq \|f\|^{\theta}_{B^{s_1}_{p,r}} \|f\|^{1-\theta}_{B^{s_2}_{p,r}}.$$

Proof. For $f \in B^{s_2}_{p,r}$, we write

$$2^{jsr}\|\Delta_j f\|^r_{L^p} = \underbrace{2^{jsr\theta}\|\Delta_j f\|^{r\theta}_{L^p}}_{\in \ell^{\frac{r}{\theta}}} \underbrace{2^{jsr(1-\theta)}\|\Delta_j f\|^{r(1-\theta)}_{L^p}}_{\in \ell^{\frac{r}{1-\theta}}}.$$

Then it leads to the result using the Hölder inequality on the sequences with exponents $\frac{1}{r} = \frac{\theta}{r} + \frac{1-\theta}{r}$. □

Let us also point out that there exists some embeddings that are more precise than those given in (3.1.3). In particular, we will devote a special attention to the role of the third index r of the Besov space $B^s_{p,r}$.

Proposition 3.1.7. *Consider $q \in [1, 2]$ and $r \in [2, +\infty[$. Then we have the following continuous embeddings:*

$$L^q \subset B^0_{q,2} \qquad \text{and} \qquad B^0_{r,2} \subset L^r.$$

and also

$$B^0_{q,q} \subset L^q \qquad \text{and} \qquad L^r \subset B^0_{r,r}.$$

Proof. See [13, Theorems 2.40 and 2.41]. □

Proposition 3.1.8. *Consider a Fourier multiplier $m(D)$ whose symbol is a smooth function away from the origin $m \in C^\infty(\mathbb{R}^N \setminus \{0\})$ such that there is an order $M \in \mathbb{R}$ with the following property: for all $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq d + 2$, there exists a constant $C_\alpha > 0$ with*

$$\forall \xi \neq 0, \qquad |\partial^\alpha m(\xi)| \leq C_\alpha |\xi|^{M-|\alpha|}.$$

Then, for any Lebesgue exponent $q \in [1, +\infty]$ and any $f \in \mathcal{S}'$, we have

$$\forall j \geq 0, \qquad \|m(D)\Delta_j f\|_{L^q} \leq C(d) 2^{Mj} \|\Delta_j f\|_{L^q}.$$

Proof. See [13, Lemma 2.2]. □

Proposition 3.1.8 allows us to show continuity properties on Fourier multipliers, except that the proposition only deals with dyadic Δ_j blocks with $j \geq 0$. To understand low-frequency behavior, other techniques are required, such as the use of Bernstein inequalities. We can also turn to the theory of Calderón-Zygmund, whose essential result is quoted below. The theorem will be used elsewhere.

Theorem 3.1.9 (Calderón-Zygmund)**.** *Consider a Fourier multiplier $m(D)$ whose symbol is a homogeneous function of degree zero $m \in C^\infty(\mathbb{R}^d \backslash \{0\})$. Then for any $1 < q < +\infty$, the operator $m(D) : \mathcal{S} \longrightarrow \mathcal{S}'$ has a unique bounded extension*

$$m(D) : L^q \longrightarrow L^q.$$

Proof. This is a consequence of [102, Theorem 4.2.2.]. □

Let us now introduce the paraproduct operator, which relies on the observation that the product of two elements belonging into $\mathcal{S}'(\mathbb{R}^N)$, denoted thereafter u and v, can be written

$$u\,v \;=\; \mathcal{T}_u(v) \,+\, \mathcal{T}_v(u) \,+\, \mathcal{R}(u,v)\,, \tag{3.1.4}$$

where we set

$$\mathcal{T}_u(v) \,:=\, \sum_j S_{j-1}u\Delta_j v, \qquad \text{and} \qquad \mathcal{R}(u,v) \,:=\, \sum_j \sum_{|j'-j|\leq 1} \Delta_j u\, \Delta_{j'} v\,.$$

Where $\mathcal{T}$ and $\mathcal{R}$ are respectively called the paraproduct and the remainder operators. The equation (3.1.4) is called the Bony decomposition, and follows from [29]. In the end, we can think of (3.1.4) as a kind of Leibniz formula for temperate distributions. The following result holds.

Proposition 3.1.10. *For any $(s,p,r) \in \mathbb{R} \times [1,+\infty]^2$ and $t > 0$, the paraproduct operator $\mathcal{T}$ maps continuously $L^\infty \times B^s_{p,r}$ in $B^s_{p,r}$ and $B^{-t}_{\infty,\infty} \times B^s_{p,r}$ in $B^{s-t}_{p,r}$. Moreover, the following estimates hold:*

$$\|\mathcal{T}_u(v)\|_{B^s_{p,r}} \,\leq\, C\,\|u\|_{L^\infty}\,\|\nabla v\|_{B^{s-1}_{p,r}}$$

as well as

$$\|\mathcal{T}_u(v)\|_{B^{s-t}_{p,r}} \,\leq\, \frac{C}{t}\|u\|_{B^{-t}_{\infty,\infty}}\,\|\nabla v\|_{B^{s-1}_{p,r}} \qquad \textit{and} \qquad \|\mathcal{T}_u(v)\|_{B^{s-t}_{p,r}} \,\leq\, \frac{C}{t}\|u\|_{B^{-t}_{p,\infty}}\,\|\nabla v\|_{B^{s-1}_{\infty,r}}.$$

For any (s_1,p_1,r_1) and (s_2,p_2,r_2) in $\mathbb{R} \times [1,+\infty]^2$ such that $s_1 + s_2 > 0$, $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ and $\frac{1}{r} := \frac{1}{r_1} + \frac{1}{r_2} \leq 1$, the remainder operator $\mathcal{R}$ maps continuously $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$ into $B^{s_1+s_2}_{p,r}$. Also, we have the estimates over the remainder:

$$\|\mathcal{R}(u,v)\|_{B^{s_1+s_2}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}$$

In the case $s_1 + s_2 = 0$, provided $r = 1$, operator $\mathcal{R}$ is continuous from $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$ with values in $B^0_{p,\infty}$.

Proof. See [13, Theorem 2.85.] □

Note that so far we have only mentioned results valid on $\mathbb{R}^N$. However, we are (also) interested in Besov spaces on the torus. In fact, the construction of Besov spaces on the torus is in every respect similar to that on $\mathbb{R}^N$. Let us start by noting that on the torus, we have $\mathcal{S}'(\mathbb{T}^N) = \mathcal{D}'(\mathbb{T}^N)$. It then suffices to consider the decomposition of an element u of $\mathcal{D}'(\mathbb{T}^N)$ into a Fourier series, i.e.:

$$u(x) = \sum_{k\in\mathbb{Z}^N} \hat{u_k} e^{ik\cdot x} \quad \text{where} \quad \hat{u_k} = \fint_{\mathbb{T}^N} e^{-ik\cdot y} u(y)\, dy,$$

where we point out that $\mathbb{Z}^N$ is the dual lattice of $\mathbb{T}^N$. Then we can define the periodic dyadic blocks and the low frequency cut-off operator as follows

$$\Delta_j^{\mathrm{per}} u = \sum_{k\in\mathbb{Z}^N} \varphi(2^{-j}k)\hat{u_k} e^{ik\cdot x} = \fint_{\mathbb{T}^N} h_j(y) u(x-y)\, dy, \tag{3.1.5}$$

$$S_j^{\mathrm{per}} u = \hat{u_0} + \sum_{l\leq j-1} \Delta_l^{\mathrm{per}} u = \sum_{k\in\mathbb{Z}^N} \chi(2^{-j}k)\hat{u_k} e^{ik\cdot x}, \tag{3.1.6}$$

where we wrote $h_j(y) = \sum_{k\in\mathbb{Z}^N} \varphi(2^{-j}k) e^{ik\cdot y}$. Also, one can associate by duality to every u in $\mathcal{D}'(\mathbb{T}^N)$ a unique periodic distribution u^{per} defined as

$$u^{\mathrm{per}}(x) = \sum_{\gamma\in(2\pi\mathbb{Z})^N} u(x+\gamma) \qquad \text{for } x \in [-\pi,\pi]^N.$$

Over the last, both periodic and nonhomogeneous Littlewood-Paley decomposition holds. Moreover, periodic blocks and nonhomogeneous blocks coincides in the following sense.

Proposition 3.1.11. *For every u belonging to $\mathcal{S}'(\mathbb{R}^N)$, we have:*

$$\forall j \in \mathbb{Z},\ \big(\varphi(2^{-j}D)u\big)^{\mathrm{per}} = \Delta_j^{\mathrm{per}}\, u^{\mathrm{per}}.$$

Proof. The proof relies to an application of Poisson formula. We refer to [62, Proposition 1.2.2.] for details. □

With this in mind, we can then see that all the results hitherto presented on $\mathbb{R}^N$ remain valid on $\mathbb{T}^N$. It should be pointed out, however, that this is not the only way to establish a periodic theory from that on $\mathbb{R}^N$. In fact, it is possible to establish a link between certain periodic distributions and distributions that damp out at high frequencies, which amounts to considering weighted spaces. We refer the interested reader to [165, Sections 9.1.3. and 9.1.4.] for more details on the latter approach, which we won't be using in our study.

The reader may wonder why we do not use here the notion homogeneous Besov spaces. The reason is that one of the major interests of such spaces is the very fine study of low frequencies, i.e. around zero. However, there is no special interest here in proceeding in this way: since frequency space is a lattice, analysis of low frequencies is reduced to $k = 0$. In particular, functions f having a null mean value

$$\hat{f}(0) = \int_{\mathbb{T}^N} f\, \mathrm{d}x = 0$$

have no low frequency component.

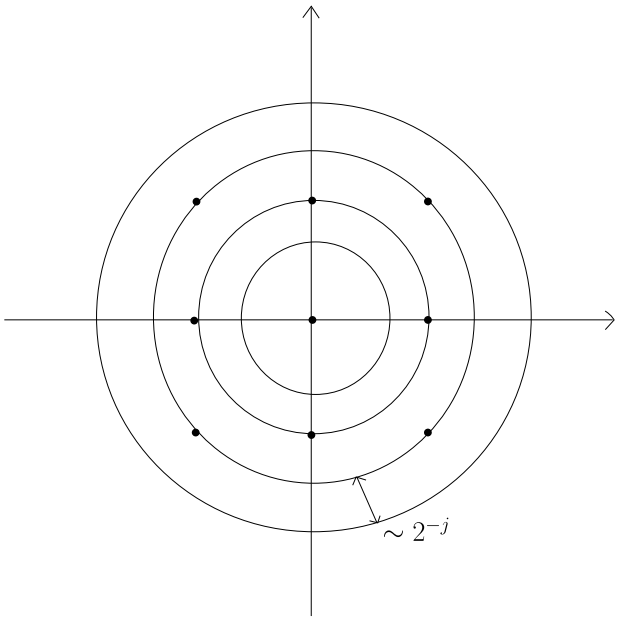


Figure 3.1: Homogeneous decomposition for the torus

We refer the interested reader to [13, 55, 159] for further discussions about Besov spaces theory, particularly in the whole space case.

3.2 Some results for transport equations

In this section, we set out some results concerning transport equations, mainly taken from [75]. We are concerned with the linear transport equation

$$\begin{cases} \partial_t f + \operatorname{div}(fu) = 0 \\ f(0) = f_0 \end{cases} \tag{3.2.1}$$

where u is a divergence-free vector field $\operatorname{div}(u) = 0$, and will be assumed to have regularity $u \in L^1_T(W^{1,\beta})$ with respect to space, for some $\beta \geq 1$. The first result we mention is a well-posedness theorem for (3.2.1).

Theorem 3.2.1 (Di Perna-P.L.Lions)**.** *Let $q \geq \beta'$ and consider an initial datum $f_0 \in L^q$. Then there exists a unique weak solution $f \in L^\infty(L^q)$ of the transport equation.*

The proof of this Theorem relies on a regularization procedure: let $(\psi_\epsilon)_{\epsilon>0}$ be a mollification sequence on $\mathbb{T}^N$. Then, by taking the convolution of the transport equation, we obtain the system

$$\partial_t f_\epsilon + \operatorname{div}(f_\epsilon u) = \big[u \cdot \nabla, \psi_\epsilon * \big] f,$$

where $f_\epsilon = \psi_\epsilon * f$ and the commutator is to be understood in the weak sense, namely we have

$$\big[u \cdot \nabla, \psi_\epsilon * \big] f = \operatorname{div}(f_\epsilon u) - \psi_\epsilon * \operatorname{div}(fu).$$

Then it is a matter of proving that the commutator goes to 0 as the regularization parameter does. We refer to [75, Proposition II.1. and Theorem II.2.] for details. The following lemma shows that the commutator in fact converges to zero as the mollification parameter does.

Lemma 3.2.2. *Assume that $u \in L^1(W^{1,\beta})$ and fix a function $g \in L^\infty(L^q)$ for some $q \geq \beta'$. We then have*

$$\big[u \cdot \nabla, \psi_\epsilon * \big] g \longrightarrow 0 \qquad \text{in } L^1(L^\alpha),$$

where α is given by $\frac{1}{\alpha} = \frac{1}{\beta} + \frac{1}{q}$.

Proof. See [75, Lemma II.1.] □

Finally, we cite one last result, which has to do with propagation of regularity: the solution f can be as regular as the velocity field and the initial datum allows.

Theorem 3.2.3. *Consider $s > 1 + \frac{N}{2}$ and assume that $u \in L^1(H^s)$ and $f_0 \in H^s$. Then the unique solution f of the transport equation (3.2.1) given by theorem 3.2.1 has regularity $f \in L^\infty_{\text{loc}}(H^s)$ and we have, for every $T > 0$,*

$$\|f\|_{L^\infty_T} \leq \|f_0\|_{H^s} \exp\left(C \int_0^T \|\nabla u\|_{H^{s-1}} \, \mathrm{d}t \right).$$

Proof. See e.g. [13, Theorem 3.19.]. □

3.3 Existence of weak solutions

The aim of this section is to show the existence of global weak solutions of (3.0.1), that is we aim to prove Theorem 3.0.2. The whole proof will be divided in several steps. First, we establish some a priori estimates in Subsection 3.3.1, then we will show that the steady problem is well-posed in Subsection 3.3.2, in order to prove that it defines in a suitable sense an active scalar equation, we establish some continuity property in Subsection 3.3.3. Then in Subsection 3.3.4, we prove that there exists some approximate solutions, and we show the strong convergence of densities in Subsection 3.3.5. Finally, we conclude the proof of the Theorem in Subsection 3.3.6.

In this section, we will exceptionally use the letter d to denote the dimension of the ambient space, the letter N being dedicated to another use for the sake of clarity. Also, for any $p \geq 1$, we will note in this section p' the real number satisfying:

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

which was hitherto noted by the letter q, which will be used here to another effect as well.

3.3.1 A priori estimates

Let us start by establishing a few a priori estimates that will be useful later on.

Proposition 3.3.1. *Consider a smooth initial datum ρ_0 associated to a smooth solution ρ and smooth velocity and pressure fields u and π. Then, under the assumptions of Theorem 3.0.2, we have the following inequalities:*

$$\|\rho\|_{L^\infty(L^q)} \leq \|\rho_0\|_{L^q} \tag{3.3.1}$$

and

$$\|D(u)\|_{L^\beta} \lesssim \big\|1/\rho_0\big\|_{L^\sigma}^{\frac{\gamma}{p-1}} \|\rho_0\|_{L^q}^{\frac{1}{p-1}}, \tag{3.3.2}$$

where $1 < \beta < +\infty$ is defined by the relation $\frac{1}{\beta} = \frac{1}{p}\left(1 + \frac{\gamma}{\sigma}\right)$. In addition, we also have the inequality

$$\|u\|_{L^{q'}} \lesssim \|D(u)\|_{L^\beta}. \tag{3.3.3}$$

Remark 3.3.2. *Notice that, when the viscosity coefficient $\nu(\rho)$ is degenerate, that is when $\gamma > 0$ and $\sigma < +\infty$, then we may experience a loss of regularity when compared to the usual p-Laplacian estimates: the velocity field is not $W^{1,p}$ but only $W^{1,\beta}$.*

Remark 3.3.3. *Inequality* (3.3.2) *is consistent with the equation's "scaling " properties. Since the viscosity coefficient behaves, at worse, as $\nu_*|\rho|^\gamma$ in a neighborhood of $\{\rho = 0\}$, we see that the Stokes equation*

$$-\operatorname{div}\big(\nu(\rho)|D(u)|^{p-2}D(u)\big) + \nabla\pi = \rho g$$

behaves roughly as (in physicist notation)

$$\big[density\big]^\gamma \times \big[velocity\big]^{p-1} = \big[density\big],$$

so that the velocity is "homogeneous" to a density at the power $\frac{1-\gamma}{p-1}$. This is absolutely consistent with the righthand-side of (3.3.2)*, which also scales as a density to the power $\frac{1-\gamma}{p-1}$.*

Proof (of Proposition 3.3.1). First of all, thanks to the fact that the velocity field is assumed to be divergence-free, the flow map associated to u preserves the Lebesgue measure, and therefore all Lebesgue norms. We deduce that

$$\|\rho(t)\|_{L^q} = \|\rho_0\|_{L^q} \qquad \text{for all } t \in \mathbb{R}_+. \tag{3.3.4}$$

For the same reasons, we also have

$$\big\|1/\rho(t)\big\|_{L^\sigma} = \big\|1/\rho_0\|_{L^\sigma}. \tag{3.3.5}$$

Now, let us focus on the finding estimates for the velocity field. For this, we perform an energy estimate in the elliptic equation: by taking the scalar product of the Stokes equation by u, we obtain

$$\int \nu(\rho)|D(u)|^{p-2}D(u) : \nabla u = \int \rho g \cdot u.$$

Because the symmetric and skew-symmetric spaces form orthogonal subspaces for the $d\times d$ matrix scalar product, we deduce that

$$\int \nu(\rho)|D(u)|^p = \int \rho g \cdot u. \tag{3.3.6}$$

There will be two steps in our estimates. First of all, we will show how the norm $\|D(u)\|_{L^\beta}$ can be controlled by the lefthand term in this equality. Once we have done that, we will focus on the righthand side term and examine how to close the estimates.

We then start by finding an upper bound for $\|D(u)\|_{L^\beta}$. The idea is to introduce the viscosity multiplier $\nu(\rho)$ in order to obtain force the apparition of the lefthand side term of (3.3.6). We have:

$$D(u) = \nu(\rho)^{1/p}D(u)\frac{1}{\nu(\rho)^{1/p}}.$$

We remark that the L^p norm of the first factor $\nu(\rho)^{1/p}D(u)$ is exactly (up to the $1/p$-th power) the lefthand side of (3.3.6). On the other hand, because of our assumption on the viscosity multiplier $\nu(r)$, we may bound the last factor in this product by a power of $1/\rho$, namely

$$\frac{1}{\nu(\rho)^{1/p}} \leq \frac{1/\nu_*^{1/p}}{|\rho|^{\gamma/p}}.$$

so that the $L^{\sigma p/\gamma}$ power of $\nu(\rho)^{-1/p}$ will exactly be the L^σ norm of $1/\rho$ (again, up to the $1/\sigma$-th power). Applying Hölder's inequality, we immediately obtain

$$\begin{aligned}\|D(u)\|_{L^\beta} &\leq \big\|\nu(\rho)D(u)\big\|_{L^p}\big\|1/|\rho|^{\gamma/p}\big\|_{L^{\sigma p/\gamma}} \\ &\lesssim \left(\int \nu(\rho)|D(u)|^p\right)^{1/p}\big\|1/\rho\big\|_{L^\sigma}^{\gamma/p},\end{aligned} \tag{3.3.7}$$

where the exponent β is defined by the relation

$$\frac{1}{\beta} = \frac{1}{p} + \frac{\gamma}{\sigma p} = \frac{1}{p}\left(1 + \frac{\gamma}{\sigma}\right).$$

Note that this inequality is *homogeneous* in the sense of Remark 3.3.3.

The second step is now to bound the righthand side in the energy balance equation (3.3.6). Because ρ is bounded in L^q, we will try to show that u is bounded in $L^{q'}$. With that in mind, we express u as a function of the strain rate tensor: keeping in mind that $\operatorname{div}(u) = 0$, we may write

$$u = -2(-\Delta)^{-1}\operatorname{div}(D(u)). \tag{3.3.8}$$

In other words, u is the image of $D(u)$ by a Fourier multiplier of order -1. This makes possible expressing the regularity of u in terms of that $D(u) \in L^\beta$. Two cases must be distinguished, according to whether the sub-critical or the critical assumptions of Theorem 3.0.2 hold.

First case: critical case. First of all, note that the condition $q \geq \frac{2d}{d+2}$ is equivalent to the inequality $\frac{1}{q} - \frac{1}{d} \leq \frac{1}{2}$. In particular, we have $\beta \leq 2$ thanks to the relation $\frac{1}{\beta} + \frac{1}{q} - \frac{1}{d} = 1$. Therefore, applying the Besov embeddings of Proposition 3.1.7, we obtain that $L^\beta \subset B^0_{\beta,2}$. Propositions 3.1.5 and 3.1.7 therefore provide the chain of embeddings

$$B^1_{\beta,2} \subset B^0_{q',2} \subset L^{q'},$$

since the exponents satisfy $\frac{1}{q'} = \frac{1}{\beta} - \frac{1}{d}$. This leads us to the bound $\|u\|_{L^{q'}} \lesssim \|D(u)\|_{L^\beta}$, and so, by plugging this in the energy balance equation (3.3.6), we have

$$\int \nu(\rho)|D(u)|^p = \int \rho g \cdot u \leq \|\rho\|_{L^q}\|u\|_{L^{q'}} \lesssim \|\rho\|_{L^q}\|D(u)\|_{L^\beta}.$$

Second case: sub-critical case. In that case, the Proposition 3.1.5 provide the inclusion

$$B^1_{\beta,\infty} \subset B^s_{q',\infty} \subset L^{q'}$$

with $s = d\left(1 - \frac{1}{\beta} - \frac{1}{q} + \frac{1}{d}\right) > 0$. In particular, thanks to Proposition 3.1.8, we obtain the same inequality as above:

$$\int \nu(\rho)|D(u)|^p = \int \rho g \cdot u \leq \|\rho\|_{L^q}\|u\|_{L^{q'}} \lesssim \|\rho\|_{L^q}\|D(u)\|_{L^\beta}.$$

In both cases, we may use these estimates in inequality (3.3.7) in order to finish the proof of the proposition: first we write

$$\begin{aligned}\|D(u)\|_{L^\beta} &\lesssim \left(\int \nu(\rho)|D(u)|^p\right)^{1/p}\big\|1/\rho\big\|_{L^\sigma}^{\gamma/p} \\ &\lesssim \|D(u)\|_{L^\beta}^{1/p}\|\rho\|_{L^q}^{1/p}\big\|1/\rho\big\|_{L^\sigma}^{\gamma/p^2},\end{aligned}$$

and then we apply Young's inequality $ab \lesssim \epsilon a^p + \frac{1}{\epsilon} b^{p'}$ in order to absorb the $\|D(u)\|_{L^\beta}$ factor in the lefthand side, and finally get,

$$\begin{aligned} \|D(u)\|_{L^\beta} &\lesssim \|\rho\|_{L^q}^{p'/p} \big\|1/\rho\big\|_{L^\sigma}^{\gamma p'/p^2} \\ &\lesssim \|\rho\|_{L^q}^{\frac{1}{p-1}} \big\|1/\rho\big\|_{L^\sigma}^{\frac{\gamma}{p-1}}. \end{aligned} \tag{3.3.9}$$

The upper bound can be evaluated at initial time $t = 0$ thanks to (3.3.4) and (3.3.5). □

3.3.2 Well-posedness for the Stokes system

Having this in mind, in order to show that we do define an active scalar equation via system (3.0.1), let us begin by proving that the non-Newtonian Stokes problem

$$\begin{cases} -\operatorname{div}\big(\nu(\rho)|D(u)|^{p-2}D(u)\big) + \nabla\pi = \rho g \\ \operatorname{div}(u) = 0. \end{cases} \tag{3.3.10}$$

has a unique solution satisfying suitable estimates. Namely, the following Proposition holds.

Proposition 3.3.4. *Consider $1 < p, q < +\infty$. For any $\rho \in L^q$ there is a unique solution $u \in W^{1,\beta}$ such that $⨍ u\,\mathrm{d}x = 0$ and satisfying*

$$\int \nu(\rho)|D(u)|^p \lesssim \|\rho\|_{L^q}^{\frac{1}{p-1}} \big\|1/\rho\big\|_{L^\sigma}^{\frac{\gamma}{p-1}}. \tag{3.3.11}$$

We introduce the following notation: we call $\Psi(\rho)$ the unique solution associated to ρ, thus defining a nonlinear map $\Psi : L^q \longrightarrow W^{1,\beta}$.

Proof. We introduce the real reflexive Banach space

$$Y_\rho := \left\{ v \in W^{1,1} \,/\, ⨍ v\,\mathrm{d}x = 0,\ \operatorname{div}(v) = 0,\ \int \nu(\rho)|D(v)|^p\,\mathrm{d}x < +\infty \right\}, \tag{3.3.12}$$

endowed with the norm

$$\|v\|_{Y_\rho} := \left(\int \nu(\rho)|D(u)|^p\,\mathrm{d}x \right)^{\frac{1}{p}}, \tag{3.3.13}$$

which is the norm involved on Y_ρ by the one defined over $L^p(\mathbb{T}^d, \nu(\rho)\,\mathrm{d}x)$, since $\nu(\rho)\,\mathrm{d}x$ is a measure from the positivity almost everywhere and the boundedness of $\nu(\rho)$. We point out that the divergence free and mean value integrals assumption arising in the definition of Y_ρ make sense since from our a priori estimates we have $Y_\rho \subset W^{1,\beta} \subset \mathcal{D}'$. Then, let us point out that the functional A_ρ defined over Y_ρ by

$$A_\rho(u) := \frac{1}{p} \int \nu(\rho)|D(u)|^p\,\mathrm{d}x - \int \rho g \cdot u\,\mathrm{d}x, \tag{3.3.14}$$

which can be rewritten as

$$A_\rho(u) := \frac{1}{p}\|u\|_{Y_\rho}^p - \int \rho g \cdot u\,\mathrm{d}x, \tag{3.3.15}$$

so that A_ρ is a strictly convex functional, since for every $(u, v) \in Y_\rho$, $u \neq v$ and every $\theta \in (0, 1)$:

$$\begin{aligned} A_\rho(\theta u + (1-\theta)v) &= \frac{1}{p}\|\theta u + (1-\theta)v\|_{Y_\rho}^p - \int \rho g \cdot (\theta u + (1-\theta)v)\ \mathrm{d}x \\ &< \theta\left(\frac{1}{p}\|u\|_{Y_\rho}^p - \int \rho g \cdot u\,\mathrm{d}x\right) + (1-\theta)\left(\frac{1}{p}\|v\|_{Y_\rho}^p - \int \rho g \cdot v\,\mathrm{d}x\right) \\ &= \theta A_\rho(u) + (1-\theta)A_\rho(v). \end{aligned}$$

The second inequality above being true since the function $t \mapsto t^p$ defined over $\mathbb{R}_+$ is strictly convex. The Euler-Lagrange formulation associated to the minimization of A_ρ leads, computing the derivative in the direction of a test function, to the fact that each of its minimizers are a weak solution of (3.3.10), and vice versa (thanks to the convexity of A_ρ). It remains to prove the existence and uniqueness of such a minimizer, but we underline that A_ρ is coercive over Y_ρ, since we get from (3.3.15), using Hölder's inequality

$$A_\rho(u) \geq \frac{1}{p}\|u\|_{Y_\rho}^p - \|\rho\|_{L^q}\|u\|_{L^{q'}}. \tag{3.3.16}$$

Then, thanks to (3.3.3), we get from (3.3.16)

$$A_\rho(u) \gtrsim \|u\|_{Y_\rho}^p - \|\rho\|_{L^q}\|D(u)\|_{L^\beta}. \tag{3.3.17}$$

Using once again our a priori estimates it follows from (3.3.17) that, combining (3.3.5) and (3.3.7) :

$$A_\rho(u) \gtrsim \|u\|_{Y_\rho}^p - C(\rho_0)\left(\int \nu(\rho)|D(u)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{3.3.18}$$

Finally, Young's inequality applied in the right-hand side of (3.3.18) leads to the wished result, namely

$$A_\rho(u) \gtrsim (1-\varepsilon)\,\|u\|_{Y_\rho}^p - C(\varepsilon, p', \rho_0). \tag{3.3.19}$$

The previous computations mean that A_ρ is a p-coercive functional over Y_ρ. Hence, it follows from [84, Section 8.2.2., Theorem 2, Theorem 3] that it admits a unique minimum in Y_ρ, once again denoted u, this last being a weak solution of (3.3.10), which is hence unique. Testing now into the weak formulation against u, then using both estimates (3.3.3) and (3.3.9) leads to (3.3.11).

□

Remark 3.3.5. *Observe that Proposition 3.3.4 can also be proved much more quickly, by noting that A_ρ defines a strictly convex potential over Y_ρ, and is therefore a strictly monotone operator according to Proposition 1.1.33. The result is then an immediate consequence of Theorem 1.1.37. However, the proof we proposed here is elementary.*

3.3.3 Continuity of the inverse nonlinear Stokes map

In the above section, we have shown that the nonlinear Stokes-type equation can be uniquely solved: with the notations of Theorem 3.0.2, for any $\rho \in L^q$, there exists a unique u solution of the Stokes equation (3.3.10) which lies in the space $W^{1,\beta}$, where $\frac{1}{\beta} = \frac{1}{p}\left(1 + \frac{\gamma}{\sigma}\right)$. In other words, we have defined a nonlinear map

$$\Psi : \left| \begin{array}{l} L^q \longrightarrow W^{1,\beta} \\ \\ \rho \longmapsto u. \end{array} \right. \tag{3.3.20}$$

The purpose of this section is to prove a form of continuity property for the map Ψ. Of course, it is too much to expect that Ψ is norm continuous $L^q \longrightarrow W^{1,\beta}$. However, by relaxing the topologies, we may still obtain a continuity result. The following Proposition is based on the Minty's trick (see Lemma 1.1.5).

Proposition 3.3.6. *Let us assume that p, q, γ, σ are defined as in Theorem 3.0.2 and β as in Proposition 3.3.1. We consider a sequence $(\rho_n)_{n\in\mathbb{N}}$ such that there exists $\rho_n \to \rho$ in L^q. Then, one have for Ψ given by* (3.3.20)

$$\Psi(\rho_n) \rightharpoonup \Psi(\rho) \qquad \text{in } W^{1,\beta}. \tag{3.3.21}$$

In other words, the nonlinear map Ψ is $L^q \longrightarrow W^{1,\beta}_w$ continuous, where $W^{1,\beta}_w$ is the Sobolev space $W^{1,\beta}$ equipped with its weak topology.

Proof. First, let us introduce for $\phi \in \mathcal{D}_{\rm div} := \{\varphi \in \mathcal{D} \,/\, \mathrm{div}(\varphi) = 0\}$ the functional

$$X_n(\phi) := \int \nu(\rho_n)\Big(|D(u_n)|^{p-2}D(u_n) - |D(\phi)|^{p-2}D(\phi)\Big) : (D(u_n) - D(\phi))\,\mathrm{d}x \tag{3.3.22}$$

where $u_n = \Psi(\rho_n)$ is the unique solution of (3.3.10) associated to ρ_n, as given by Proposition 3.3.4. Thanks to inequality (3.3.11) we are sure that the functional X_n is well-defined on the space $\mathcal{D}_{\rm div}$. In addition, the functionals X_n are nonnegative due to the following lemma.

Lemma 3.3.7. *Consider $n \in \mathbb{N}$ and $1 < p < +\infty$. Then for every regular and divergence-free function $\phi \in \mathcal{D}_{\rm div}$, we have a monotonicity inequality:*

$$X_n(\phi) \geq 0. \tag{3.3.23}$$

Proof. We begin with the observation that the functional

$$A_n(\phi) := \frac{1}{p}\int \nu(\rho_n)|D(\phi)|^p\,\mathrm{d}x \tag{3.3.24}$$

is convex because $p > 1$. Furthermore, A_n acts as a potential with respect to X_n. This means that $X_n(\phi)$ is formally given by the relation

$$X_n(\phi) = \big\langle A_n'(u_n) - A_n'(\phi), u_n - \phi\big\rangle, \tag{3.3.25}$$

where $A_n'(f)$ is the differential of A_n evaluated at a point f. Still formally, the convexity of A_n shows that the above is non-negative $X_n(\phi)$.

The argument as it is is not entirely complete: this is because while $\phi \in C^\infty$ is very regular, it may not be the case of $u_n = \Psi(\rho_n)$ which is defined as the unique solution of a nonlinear elliptic problem. However, we may argue by density: the functional X_n is continuous on the space

$$Y_n := \Big\{f \in W^{1,1}, \quad Df \in L^p\big(\nu(\rho_n)\,\mathrm{d}x\big)\Big\},$$

and the viscosity multiplier $\nu(r)$ is bounded, so C^∞ is a dense subspace of $L^p\big(\nu(\rho_n)\,\mathrm{d}x\big)$. It is therefore enough to show that the brackets in (3.3.25) are non-negative when assuming that u_n is smooth. □

Using the weak formulation of (3.3.10) into (3.3.23), we get for all $\phi \in \mathcal{D}_{\rm div}$:

$$X_n(\phi) = \int \rho_n(u_n - \phi)g\,\mathrm{d}x - \int \nu(\rho_n)|D(\phi)|^{p-2}D(\phi) : (D(u_n) - D(\phi))\,\mathrm{d}x \geq 0 \tag{3.3.26}$$

We can now pass to the limit on n in this expression. First, we have that $\rho_n \to \rho$ a.e. since $\rho_n \longrightarrow \rho$ in L^q, and $D(u_n) \rightharpoonup D(u)$ in L^β, in view of (3.3.7) and (3.3.11). Also we have that $\nu(\rho_n) \longrightarrow \nu(\rho)$ in $L^{\beta'}$. Thus we get:

$$\int \nu(\rho_n)|D(\phi)|^{p-2}D(\phi) : (D(u_n) - D(\phi))\,\mathrm{d}x \underset{n\to+\infty}{\longrightarrow} \int \nu(\rho)|D(\phi)|^{p-2}D(\phi) : (D(u) - D(\phi))\,\mathrm{d}x \tag{3.3.27}$$

Moreover, we recall that from (3.3.7) and (3.3.11) we have the boundedness $(u_n) \subset W^{1,\beta} \subset L^{q'}$ which leads to weak convergence $u_n \rightharpoonup u$ in $L^{q'}$, up to an extraction. Also, let us consider a family of functions $(\eta_k)_{k>0}$ chosen as:

$$\begin{cases} \eta_k \in C_b^1(\mathbb{R}) \\ 0 < \eta_k' \leq 1 \text{ in } \mathbb{R} \\ \eta_k(r) = r \text{ in } [-k, k] \end{cases} \tag{3.3.28}$$

the bounds given into the previous definition being uniform over k. Then, one can write:

$$\rho_n u_n - \rho u = (\rho_n - \eta_k(\rho_n))\, u_n + \eta_k(\rho_n)u_n - \eta_k(\rho)u + (\eta_k(\rho) - \rho)\, u \tag{3.3.29}$$

Then, let us point out that we have by dominated convergence Theorem:

$$\|\eta_k(\rho) - \rho\|_{L^q}^q \lesssim \int_{\{|\rho|>k\}} |\rho|^q \,\mathrm{d}x \underset{k\to+\infty}{\longrightarrow} 0 \tag{3.3.30}$$

and that $\eta_k(\rho_n) \underset{n\to+\infty}{\longrightarrow} \eta_k(\rho)$ in L^d, so that $\eta_k(\rho_n) - \rho_n \to \eta_k(\rho) - \rho$ in L^q. Then, it follows that:

$$\eta_k(\rho_n)u_n \underset{n\to+\infty}{\rightharpoonup} \eta_k(\rho)u \quad \text{in } L^\beta. \tag{3.3.31}$$

It follows that such a convergence also holds in the topology of $\mathcal{D}'$. We therefore deduce from , (3.3.29), (3.3.30), (3.3.31) and (3.3.3):

$$\begin{aligned} |\langle \rho_n u_n - \rho u, \psi\rangle| &\leq \|\rho_n - \eta_k(\rho_n)\|_{L^q}\|u_n\|_{L^{q'}}\|\psi\|_{L^\infty} + \|\eta_k(\rho_n)u_n - \eta_k(\rho)u\|_{L^\beta}\|\psi\|_{L^{\beta'}} \\ &\quad + \|\eta_k(\rho) - \rho\|_{L^q}\|u\|_{L^{q'}}\|\psi\|_{L^\infty} \\ &\lesssim \|\rho_n - \eta_k(\rho_n)\|_{L^q}\|D(u_n)\|_{L^\beta}\|\psi\|_{L^\infty} + \|\eta_k(\rho_n)u_n - \eta_k(\rho)u\|_{L^\beta}\|\psi\|_{L^{\beta'}} \\ &\quad + \|\eta_k(\rho) - \rho\|_{L^q}\|D(u)\|_{L^\beta}\|\psi\|_{L^\infty} \\ &\lesssim C(\rho_0)\Big(\|\rho_n - \eta_k(\rho_n)\|_{L^q}\|\psi\|_{L^\infty} + \|\eta_k(\rho_n)u_n - \eta_k(\rho)u\|_{L^\beta}\|\psi\|_{L^{\beta'}} \\ &\quad + \|\eta_k(\rho) - \rho\|_{L^q}\|\psi\|_{L^\infty}\Big) \end{aligned} \tag{3.3.32}$$

where in the previous equation the bracket stands as the duality bracket over $\mathcal{D}' \times \mathcal{D}$. Taking the upper limit over $n \to +\infty$ into (3.3.32) leads, thanks to (3.3.30), to

$$\rho_n u_n \underset{n\to+\infty}{\rightharpoonup} \rho u \quad \text{in } \mathcal{D}' \tag{3.3.33}$$

From (3.3.33), (3.3.27) and (3.3.26), we get for all $\phi \in \mathcal{D}_{\mathrm{div}}$:

$$X_n(\phi) \underset{n\to+\infty}{\longrightarrow} X(\phi) = \int \rho(u-\phi)g\,\mathrm{d}x - \int \nu(\rho)|D(\phi)|^{p-2}D(\phi) : (D(u) - D(\phi))\,\mathrm{d}x \geq 0 \tag{3.3.34}$$

In the previous expression, the limit must be understood in a weak sense, in view of the convergence in measures. Moreover, one can check that:

$$\|\nu(\rho_n)|D(u_n)|^{p-1}\|_{L^{p'}}^{p'} \leq \|\nu\|_{L^\infty}^{p'-1} \int \nu(\rho_n)|D(u_n)|^p \,\mathrm{d}x \leq \|\nu\|_{L^\infty}^{p'-1} \|\rho_0^{-1}\|_{L^\sigma}^{\sigma\left(1-\frac{p'}{r}\right)} \|\rho_0\|_{L^q}^{p'} \tag{3.3.35}$$

Then up to an extraction we get from (3.3.35) that there exists $\chi \in L^{p'}$ such that:

$$\nu(\rho_n)|D(u_n)|^{p-2}D(u_n) \rightharpoonup \chi \quad \text{in} \quad L^{p'} \tag{3.3.36}$$

The main objective is now to show that $\chi = \nu(\rho)|D(u)|^{p-2}D(u)$, and that we have the convergence desired in the equation. Then, testing into (3.3.10) we get that for all test function $\psi \in \mathcal{D}_{\mathrm{div}}$:

$$\int \nu(\rho_n)|D(u_n)|^{p-2}D(u_n) : D(\psi) \,\mathrm{d}x = \int \rho_n g\psi \,\mathrm{d}x \tag{3.3.37}$$

Thus, from (3.3.33) and (3.3.36) we get in $\mathcal{D}'_{\mathrm{div}}$:

$$\int \chi : D(\psi) \,\mathrm{d}x = \int \rho g\psi \,\mathrm{d}x \tag{3.3.38}$$

Combining (3.3.38) together with (3.3.34) we get:

$$X(\phi) = \int \Big(\chi : D(u-\phi) - \nu(\rho)|D(\phi)|^{p-2}D(\phi) : (D(u) - D(\phi))\Big) \,\mathrm{d}x \geq 0 \tag{3.3.39}$$

The resulting functional X is the derivative of a convex potential, tested against $u-\phi$. To demonstrate our result, we would now like to test into (3.3.39) against some well-chosen ϕ functions. So, choosing $\phi = u + \lambda\psi$, for $\psi \in \mathcal{D}_{\mathrm{div}}$ and $\lambda > 0$ in (3.3.39), then divide by λ leads to:

$$-\int \Big(\chi : D(\psi) + \nu(\rho)|D(u+\lambda\psi)|^{p-2}D(u+\lambda\psi) : D(\psi)\Big) \,\mathrm{d}x \geq 0 \tag{3.3.40}$$

Passing to the limit as $\lambda \to 0$ into (3.3.40) leads to:

$$\int \nu(\rho)|D(u)|^{p-2}D(u) : D(\psi) \,\mathrm{d}x \geq \int \chi : D(\psi) \,\mathrm{d}x \tag{3.3.41}$$

Following the same line of arguments with $\phi = u - \lambda\psi$ leads to the converse inequality, and then to the equality:

$$\int \nu(\rho)|D(u)|^{p-2}D(u) : D(\psi) \,\mathrm{d}x = \int \chi : D(\psi) \,\mathrm{d}x \tag{3.3.42}$$

Now, thanks to (3.3.37), we obtain by means of (3.3.42) that u satisfies the weak formulation of (3.3.10), namely

$$\int \big(\nu(\rho)|D(u)|^{p-2}D(u) : D(\psi) - \rho g\psi\big) \,\mathrm{d}x = 0$$

□

Remark 3.3.8. *We point out that the structure of the proof of Proposition 3.3.6 is quite robust. In particular, if we had added to the Stokes equation an elliptic term, say $-\nu_0\Delta^k$ for some constant $\nu_0 > 0$, so as to have instead*

$$\begin{cases} -\nu_0\Delta^k u - \operatorname{div}\big(\nu(\rho)|D(u)|^{p-2}D(u)\big) + \nabla\pi = \rho g \\ \operatorname{div}(u) = 0, \end{cases}$$

then the result would still apply: one could use the methods of Proposition 3.3.4 to obtain, for every $\rho \in L^q$, the existence and uniqueness of a solution $u \in L^p\big(\nu(\rho)\,\mathrm{d}x\big) \cap H^k$, noted $\Psi_0(\rho)$, and a straightforward adaptation of the proof immediately above would yield continuity $\Psi_0 : L^q \longrightarrow W^{1,\beta}_w$ regardless of the value of ν_0.

3.3.4 Approximate solutions

In this part, we will construct a family of functions regular (ρ_n) that solve an approximate system, namely

$$\begin{cases} \partial_t\rho_n + \operatorname{div}(\rho_n u_n) = 0 \\ u_n = S_n v_n \\ -\operatorname{div}\big(\nu(\rho_n)|D(v_n)|^{p-2}D(v_n)\big) + \nabla\pi_n = \rho_n g \\ \operatorname{div}(v_n) = 0. \end{cases} \tag{3.3.43}$$

In the above, the operator S_n is the Littlewood-Paley approximation operator defined in (3.1.6). We equip system (3.3.43) with the following initial values: for every n,

$$\rho_n(0) = S_n\rho_0. \tag{3.3.44}$$

It follows that the approximate initial data are uniformly bounded in L^q with respect to q, that is $\|S_n\rho_n\|_{L^q} \lesssim \|\rho_0\|_{L^q}$. Our reasoning is the following: formally, taking the limit $n \to +\infty$ lets us recover the non-Newtonian Stokes-Transport system (3.0.1). In addition, the velocity fields u_n are smooth functions with respect to the space variable (they are in fact trigonometric polynomials), and so the solutions ρ_n are also smooth functions. In particular, the *a priori* estimates of Proposition 3.3.1 hold for the approximate solutions.

However, the approximate system (3.3.43) is a set of highly nonlinear PDEs, so that even the existence of solutions to (3.3.43) is not obvious! In the next proposition, we make sure that (3.3.43) does indeed have global smooth solutions.

Proposition 3.3.9. *Consider any $\ell \geq 0$. Then the initial value problem* (3.3.43)-(3.3.44) *has a solution $\rho_n \in L^\infty(L^2) \cap W^{1,\infty}_{\mathrm{loc}}(H^\ell)$ such that, for almost every time $t \geq 0$, the function v_n is a solution of the Stokes system* (3.3.10) *with datum ρ_n. In particular, inequality* (3.3.10) *holds: for almost all times,*

$$\int \nu(\rho_n)|D(v_n)|^p \lesssim \|\rho_n\|_{L^q}^{\frac{1}{p-1}} \big\|1/\rho_n\big\|_{L^\sigma}^{\frac{\gamma}{p-1}},$$

and so inequality (3.3.7) *provides $v_n \in L^\infty(W^{1,\beta})$.*

Proof. In order to show existence of solutions to the approximate system (3.3.43), we introduce another approximate system, for which solutions will be shown to exists through the Cauchy-Peano theorem. Let us consider, for any fixed n and for all $N \geq 1$, the system

$$\begin{cases} \partial_t\rho_N + E_N \operatorname{div}(\rho_N u_N) = 0 \\ u_N = S_n v_N \\ -\frac{1}{N}\Delta^k v_N - \operatorname{div}\big(\nu(\rho_N)|D(v_N)|^{p-2}D(v_N)\big) + \nabla\pi_N = \rho_N g \\ \operatorname{div}(v_N) = 0, \end{cases} \tag{3.3.45}$$

where E_N is a Fourier truncation operator defined by

$$\forall f \in \mathcal{S}, \qquad \widehat{E_N f}(\xi) = \mathbb{1}_{|\xi| \leq N} \widehat{f}(\xi).$$

and $k \geq 1$ is an exponent which will be fixed later on. In fact, E_N is the L^2-orthogonal projection on the space of functions whose Fourier transform is supported in the ball $B(0, N)$. We equip system (3.3.45) with the initial datum

$$\rho_N(0) = E_N S_n \rho_0. \tag{3.3.46}$$

STEP 1: defining an ODE system. First of all, we show that for any $N \geq 1$, system (3.3.45) possesses a global solution ρ_N. We will do this by means of the Cauchy-Peano theorem applied in a subspace of L^2. We define a map $F_N : L^2 \longrightarrow L^2$ by the following steps, which will be justified immediately below:

1. For any $\rho_N \in L^2$, we define v_N to be the unique solution of a Stokes system with improved viscosity:
$$\begin{cases} \frac{-1}{N}\Delta^k v_N - \operatorname{div}\left(\nu(\rho_N)|D(v_N)|^{p-2}D(v_N)\right) + \nabla \pi_N = \rho_N g \\ \operatorname{div}(v_N) = 0\,; \end{cases} \tag{3.3.47}$$
2. We then define the trigonometric polynomial u_N by the relation $u_N = S_n v_N$;
3. Finally, we set $F_N(\rho_N) := \operatorname{div}(\rho_N u_N)$, which is a trigonometric polynomial, and hence in L^2.

In the three steps above, only the first one requires additional explanations, as Proposition 3.3.4 does not give *per se* solutions to this modified Stokes equation. However, the proof of Proposition 3.3.4 can be adapted with straightforward modifications, simply by replacing the functional (3.3.14) with

$$A_\rho(u) = \frac{1}{2N}\int |\nabla^k u|^2 + \frac{1}{p}\int \nu(\rho)|D(u)|^p - \int \rho g \cdot u$$

and by working in the space $Y_\rho \cap H^k$ instead of (3.3.12). Because we have assumed that $q < 2$, the same computations as in the proof of Proposition 3.3.4 show that the functional A_ρ is strictly convex and coercive. We deduce, in the same way, that for every $\rho_N \in L^2$, system (3.3.47) possesses a unique weak solution $v_N \in H^k \cap Y_\rho$ that satisfies $\int v_N = 0$ and

$$\|v_N\|_{H^k} \lesssim \sqrt{N}\|\rho_N\|_{L^2}.$$

Remark 3.3.10. *Introducing a penalized term as an improved viscosity by adding $-\frac{1}{N}\Delta^k$ in the Stokes problem is a necessary step: it allows v_N to be estimated in some space by an upper bound that does not involve $\|1/\rho_N\|_{L^\sigma}$, thus enabling the Cauchy-Peano theorem to work in the energy space L^2.*

STEP 2: applying the Cauchy-Peano theorem. If we wish to apply the Cauchy-Peano theorem to the ODE $\partial_t \rho_N = F_N(\rho_N)$, we must check that $F_N : L^2 \longrightarrow L^2$ is continuous. Consider then two functions $\rho_N, \rho'_N \in L^2$ to which we associate v_N and v'_N solutions of the Stokes problem (3.3.47), and $(u_N, u'_N) = S_n(v_N, v'_N)$. Then, we have

$$\begin{aligned} \left\|F_N(\rho_N) - F_N(\rho'_N)\right\|_{H^M} &\lesssim C(N)\left\|E_N \operatorname{div}\left((\rho_N - \rho'_N)u_N\right)\right\|_{H^M} \\ &\quad + C(N)\left\|E_N \operatorname{div}\left((u_N - u'_N)\rho'_N\right)\right\|_{H^M} \\ &\lesssim C(N)\|\rho_N - \rho'_N\|_{L^2}\|u_N\|_{L^\infty} + C(N)\|u_N - u'_N\|_{L^\infty}\|\rho'_N\|_{L^2}. \end{aligned} \tag{3.3.48}$$

We turn our attention to the difference $u_N - u'_N$. We subtract the elliptic equation satisfied by v_N by the one satisfied by v'_N, multiply by the function $v_N - v'_N$ and integrate by parts so as to obtain

$$\frac{1}{N}\int \left|\nabla^k v_N - \nabla^k v'_N\right|^2 + \int \Big(\nu(\rho_N)|D(v_N)|^{p-2}D(v_N) - \nu(\rho'_N)|D(v'_N)|^{p-2}D(v'_N)\Big) : D(v_N - v'_N) \\ = \int (\rho_N - \rho'_N) g \cdot (v_N - v'_N). \tag{3.3.49}$$

Let us decompose the large integral above into two parts, in order to benefit from the monotinicity inequalities fulfilled by the Stokes operator: we have

$$\int \Big(\nu(\rho_N)|D(v_N)|^{p-2}D(v_N) - \nu(\rho'_N)|D(v'_N)|^{p-2}D(v'_N)\Big) : D(v_N - v'_N) \\ = \int \nu(\rho_N)\Big(|D(v_N)|^{p-2}D(v_N) - |D(v'_N)|^{p-2}D(v'_N)\Big) : D(v_N - v'_N) \\ + \int \Big(\nu(\rho_N) - \nu(\rho'_N)\Big)|D(v'_N)|^{p-2}D(v'_N) : D(v_N - v'_N)$$

By using the convexity of the $L^p\big(\nu(\rho_N)\,\mathrm{d}x\big)$ norm exactly as in Lemma 3.3.7, we see that the first integral in the righthand side above is nonnegative. Therefore, we deduce from equation (3.3.49) that

$$\frac{1}{N}\|v_N - v'_N\|_{H^k} \leq \big\|\nu(\rho_N) - \nu(\rho'_N)\big\|_{L^\infty}\|D(v'_N)\|_{L^\infty}^{p-1}\|D(v_N) - D(v'_N)\|_{L^1} + \|\rho_N - \rho'_N\|_{L^2}\|v_N - v'_N\|_{L^2}.$$

By fixing the exponent k so that $k > 1 + d/2$, we may benefit from the Sobolev embedding $H^k \subset W^{1,\infty}$. Moreover, we use the fact that the function ν is, by assumption, $\bar\gamma$-Hölder, where $\bar\gamma = \min\{\gamma, 1\}$. The combination of these two facts yields the inequality

$$\frac{1}{N}\|v_N - v'_N\|_{H^k} \lesssim \|\rho_N - \rho'_N\|_{L^\infty}^{\bar\gamma}\|D(v'_N)\|_{H^k}^{p-1} + \|\rho_N - \rho'_N\|_{L^2}.$$

Plugging this inequality in (3.3.48) and using the fact that the operator E_N is in fact bounded in $H^{-1} \longrightarrow L^2$ shows that the map $F_N : L^2 \longrightarrow L^2$ is $\bar\gamma$-Hölder, and therefore continuous. The Cauchy-Peano applied in the *finite dimensional* space

$$X_N := \ker(\mathrm{Id} - E_N) \qquad \text{endowed with the norm } \|\,.\,\|_{L^2}$$

consequently provides the existence of a local solution $\rho_N \in C^1([0, T_N[; X_N)$ for the approximate problem.

STEP 3: lifespan of the approximate solutions. The next step is to prove that the lifespan T_N of the approximate solution ρ_N can be extended to reach $T_N = +\infty$. For this, we note that by performing a simple energy estimate in the transport equation in (3.3.45), we obtain conservation of the L^2 norms: since $\rho_N = E_N\rho_N$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\rho_N|^2 = -\int \rho_N E_N \operatorname{div}(\rho_N u_N) = -\int \rho_N \operatorname{div}(\rho_N u_N) = 0.$$

and so

$$\|\rho_N(t)\|_{L^2} = \|E_N S_n \rho_0\|_{L^2} \leq \|S_n\rho_0\|_{L^2}. \tag{3.3.50}$$

We deduce that the approximate solutions always have a bounded time derivative: $\|\partial_t\rho_N\|_{L^2} \leq \max_{\|r\|_{L^2}\leq 1}\|F(r)\|_{L^2}$. In particular, this means that the Cauchy-Peano solution $\rho_N : [0, T_N[\longrightarrow L^2$ necessarily has a limit at time T_N^-, as in that case the sequence

$$\rho_N\left(T_N - \frac{1}{j}\right) = \rho_N(0) + \int_0^{T_N - \frac{1}{j}} F(\rho_N)\,\mathrm{d}t$$

is Cauchy as $j \to +\infty$. By using the Cauchy-Peano theorem again with initial value $\rho_N(T_N^-)$, we construct an (continuous and non-necessarily unique) extension of the ODE solution on $[0, \bar{T_N}[$, which we continue to call ρ_N. This extension is C^1 with respect to time by virtue of the equation: $\partial_t \rho_N = F_N(\rho_N) \in C(L^2)$. We infer that the maximal time T_N^* beyond which there exists no extension of the solution must in fact be $T_N^* = +\infty$, and hence deduce the existence (but not uniqueness) of a global ODE solution $\rho_N : \mathbb{R}_+ \longrightarrow L^2$.

STEP 4: uniform bounds and weak convergence. We now prepare to take the limit $N \to +\infty$ in the approximate system (3.3.45). With this in mind, we use Proposition 3.3.4 and (3.3.50), and write the following uniform estimates (recall that $q < 2$).

$$(\rho_N) \subset L^\infty(L^2) \subset L^\infty(L^q) \qquad \text{and} \qquad (v_N) \subset L^\infty(W^{1,\beta}) \subset L^\infty(L^{q'}). \tag{3.3.51}$$

Concerning the velocity fields $u_N = S_n v_N$, the presence of the regularization operator S_n implies that the u_n are trigonometric polynomials of uniform degree (at most $C2^n$), and so $(u_N) \subset L^\infty(H^\ell)$ for any $\ell \in \mathbb{Z}$. In addition, by using the transport equation to trade space regularity for time regularity, we see that, provided $\ell > d/2$ so that $H^\ell \subset L^\infty$,

$$\partial_t \rho_N = -\operatorname{div}(\rho_N u_N) \subset L^\infty(H^{-1}), \tag{3.3.52}$$

so that we get, for every finite $T > 0$, the uniform bound $\rho_N \subset W_T^{1,\infty}(H^{-1})$. Ascoli's theorem lets us deduce strong convergence of the densities to some limit

$$\rho_N \underset{N \to +\infty}{\longrightarrow} \rho_n \qquad \text{in } C_T(H^{-2}). \tag{3.3.53}$$

Concerning the sequences (v_N) and (u_N), the uniform bounds above provide weak-$(*)$ convergence

$$v_N \overset{*}{\rightharpoonup} v_n \qquad \text{in } L^\infty(W^{1,\beta} \cap L^{q'}) \tag{3.3.54}$$

and, for all $\ell \in \mathbb{Z}$,

$$u_N \overset{*}{\rightharpoonup} u_n \qquad \text{in } L^\infty(H^\ell). \tag{3.3.55}$$

The high regularity the velocity field has transfers to the density, since the initial datum $\rho_N(0) = E_N S_n \rho_0$ is uniformly bounded with respect to N in H^ℓ. By invoking Theorem 3.2.3, we have, provided that $\ell > 1 + d/2$ so that $H^\ell \subset W^{1,\infty}$,

$$\|\rho_N\|_{L_T^\infty(H^\ell)} \leq \|E_N S_n \rho_0\|_{H^\ell} \exp\left(C \int_0^T \|u_N\|_{H^\ell} \, \mathrm{d}t\right).$$

Because of the inequality $\|E_N S_n \rho_0\|_{H^\ell} \leq \|S_n \rho_0\|_{H^\ell}$ and the uniform bounds on the sequence (u_N), we deduce from the above the estimate $(\rho_N) \subset L_T^\infty(H^\ell)$ for all sufficiently large ℓ. In particular, the interpolation inequality

$$\|\rho_N - \rho_n\|_{L^2} \leq \|\rho_N - \rho_n\|_{H^{-2}}^{1/2} \|\rho_N - \rho_n\|_{H^2}^{1/2}$$

shows that the convergence of the densities (ρ_N) is in fact strong:

$$\rho_N \longrightarrow \rho_n \qquad \text{in } C_T(L^2). \tag{3.3.56}$$

Finally, these high regularity bounds $(\rho_N), (u_N) \subset L_T^\infty(H^\ell)$ show that, by using the transport equation as in (3.3.52) to trade space for time regularity, we have $(\rho_N) \subset W_T^{1,\infty}(H^{\ell-1})$.

It remains to make sure that the functions (ρ_n, u_n, v_n) are solutions of the target problem (3.3.43).

STEP 5: the transport equation. We first check that ρ_n is a solution of the transport equation with velocity field u_n, which is absolutely straightforward given the strong convergence (3.3.56) of

the densities. Indeed, thanks to the uniform bounds $(u_N) \subset L^\infty(H^\ell)$, we have weak convergence of the product

$$\rho_N u_N \rightharpoonup \rho_n u_n \qquad \text{in } L^\infty(L^2).$$

Likewise, we have strong convergence of the initial density defined in (3.3.46),

$$E_N S_n \rho_0 \longrightarrow S_n \rho_0 \qquad \text{in } H^\ell.$$

This implies that ρ_n is indeed a solution of the transport equation with the appropriate initial datum:

$$\begin{cases} \partial_t \rho_n + \operatorname{div}(\rho_n u_n) = 0 \\ \rho_n(0) = S_n \rho_0. \end{cases}$$

STEP 6: the Stokes equation. In order to take the limit $N \to +\infty$ in the Stokes equation, we resort to the continuity properties of the inverse Stokes map, as in Proposition 3.3.6. Let us explain. We note $\Psi_N : L^q \longrightarrow W^{1,\beta}$ the map that associates any $\rho_N \in L^q$ to the weak solution v_N of the Stokes problem (3.3.47). By the remarks we made in STEP 1 above, this map is well-defined. Now, as we have already noticed, the presence of the elliptic summand $-\frac{1}{N}\Delta^k$ does not change any of the convexity or coercivity properties required to solve the Stokes problem. More than that, a quick glance at the proof of Proposition shows that the minor modifications (*e.g.* the addition of a nonnegative term to the functionals X_n) of the exact same argument yield a similar "continuity" property: assume that we have strong convergence

$$\rho_N \underset{N\to+\infty}{\longrightarrow} \rho_n \qquad \text{in } L^q. \tag{3.3.57}$$

Then we may deduce the weak convergence

$$\Psi_N(\rho_N) \underset{N\to+\infty}{\rightharpoonup} \Psi(\rho_n) \qquad \text{in } W^{1,\beta}.$$

Because $q < 2$; the strong $C_T(L^2)$ convergence (3.3.56) of the sequence (ρ_N) implies that the convergence (3.3.57) must take place at (almost) every time $t \in [0, T[$, so that v_n is indeed a solution of the elliptic problem associated to ρ_n at almost every time:

$$\begin{cases} -\operatorname{div}\left(\nu(\rho_n)|D(v_n)|^{p-2}D(v_n)\right) + \nabla \pi_n = \rho_n g \\ \operatorname{div}(v_n) = 0. \end{cases}$$

This ends the proof of Proposition 3.3.9.

□

3.3.5 Strong convergence of the densities

As explained in the previous subsection, Proposition 3.3.9 provides a sequence of approximate solutions (ρ_n, u_n, v_n) which solve problem (3.3.43). We now begin to study the limit $n \to +\infty$ by focusing on the densities (ρ_n), on which we establish strong convergence.

Proposition 3.3.11. *Let (ρ_n, u_n, v_n) be the sequence of approximate solutions as given in 3.3.9. We have, up to an extraction, strong and pointwise convergence of the densities, namely*

$$\rho_n \longrightarrow \rho \qquad \text{in } L^r_{\text{loc}}(L^q) \text{ and a.e. on } \mathbb{R}_+ \times \mathbb{T}^d.$$

for every $1 < r < +\infty$. In addition, still up to an extraction, we have weak-$(*)$ convergence of the velocities $u_n \rightharpoonup^* u$ in $L^\infty(W^{1,\beta})$ and the function ρ is a weak solution of the transport equation:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \rho(0) = \rho_0. \end{cases} \tag{3.3.58}$$

Proof. For the proof of Proposition 3.3.11, we use ideas from DiPerna-Lions theory of transport equations, which is expected to function on our problem because the our *a priori* estimates show that the velocity field is expected to have regularity $u \in L^\infty(W^{1,\beta})$, where β is as defined in Proposition 3.3.1. Indeed, assumptions (3.0.2) or (3.0.3) show that we must have $1 < \beta < +\infty$ so that Theorem 3.1.9 applied to (3.3.8) gives

$$\|\nabla u\|_{L^\beta} = 2\|\nabla(-\Delta)^{-1}\operatorname{div}(D(u))\|_{L^\beta} \lesssim \|D(u)\|_{L^\beta}$$

First of all, we notice that due to the form of the approximate system (3.3.43), all the bounds of Proposition 3.3.9 in fact hold for the approximate solutions. Namely, we have the following uniform estimates:

$$(\rho_n)_n \subset L^\infty(L^q), \qquad \text{and} \qquad \big(1/\rho_n\big)_n \subset L^\infty(L^\sigma), \tag{3.3.59}$$

as well as

$$(u_n)_n \subset L^\infty(W^{1,\beta}) \subset L^\infty(L^{q'}) \qquad \text{and} \qquad (v_n)_n \subset L^\infty(W^{1,\beta}) \subset L^\infty(L^{q'}). \tag{3.3.60}$$

One of the main points of DiPerna-Lions theory is to deal with low regularity solutions by using renormalization functions: more precisely, we say that a function $\eta \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is *admissible* if $\eta' > 0$ everywhere. For one such function η, we may multiply the first equation in the approximate system (3.3.43) by $\eta'(\rho_n)$ and obtain a new solution of the transport equation:

$$\begin{cases} \partial_t \eta(\rho_n) + \operatorname{div}\big(\eta(\rho_n)u_n\big) = 0 \\ \eta(\rho_n(0)) = \eta(S_n\rho_0). \end{cases} \tag{3.3.61}$$

This equation is much easier to handle: indeed the sequence $\eta(\rho_n)$ is bounded in the space $L^\infty(L^\infty)$. Furthermore, thanks to the fact that $\eta' > 0$, no information is lost, as (3.3.61) implies that ρ_n is a solution of the approximate transport equation in (3.3.43).

Now, thanks to the remark that $\big(\eta(\rho_n)\big) \subset L^\infty(L^\infty)$, we deduce weak-$(*)$ convergence in that space up to an extraction: there exists a $g \in L^\infty(L^\infty)$ such that

$$\eta(\rho_n) \overset{*}{\rightharpoonup} g \qquad \text{in } L^\infty(L^\infty).$$

In addition, the uniform bounds of (3.3.60) also provide, up to an extraction, the weak-$(*)$ convergence

$$u_n \overset{*}{\rightharpoonup} u \qquad \text{in } L^\infty(W^{1,\beta} \cap L^{q'}).$$

We will prove that g is a solution of the transport equation with velocity field u. For this, we need to obtain strong compactness on the sequence $\big(\eta(\rho_n)\big)$ in order to take the limit in the product $\eta(\rho_n)u_n$. Note that we have the uniform bound $\big(\eta(\rho_n)\big) \subset L^\infty(L^{q'})$, so that by exploiting equation (3.3.61) and the embedding $L^{q'} \subset B^0_{q',\infty}$, we get

$$\begin{aligned} \partial_t \eta(\rho_n) &= -\operatorname{div}\big(\eta(\rho_n)\big) \\ &\subset L^\infty(B^{-1}_{q,\infty}) \subset L^\infty(B^{-1-d/q'}_{\infty,\infty}). \end{aligned}$$

The last inclusion above is an application of Proposition 3.1.5. The above therefore shows that the sequence $\big(\eta(\rho_n)\big)$ is uniformly bounded in the space $W^{1,\infty}_T(B^{-1-d/q'}_{\infty,\infty})$ for every finite $T > 0$, in addition to being already bounded in the space $L^\infty(L^\infty) \subset L^\infty(B^0_{\infty,\infty})$. Therefore, an interpolation argument shows that, for a small enough $\theta \in]0,1[$, we may write an inclusion in a Hölder-Besov space:

$$L^\infty(B^0_{\infty,\infty}) \cap W^{1,\infty}_T(B^{-1-d/q'}_{\infty,\infty}) \subset C^{0,\theta}_T(B^{-s/2}_{\infty,\infty}),$$

where the regularity exponent $-s/2 < 0$ may be taken as close to zero as desired by taking $\theta > 0$ as close to zero as necessary. In particular, the Ascoli theorem provides strong convergence

$$\eta(\rho_n) \longrightarrow g \qquad \text{in } L^\infty_T(B^{-s}_{\infty,\infty}). \tag{3.3.62}$$

In order to use this strong convergence and prove that g is a solution of the transport equation with velocity field u, we state and prove the following product lemma.

Lemma 3.3.12. *Assume that $0 < s < 1$. The function product $(f,h) \mapsto fh$ is continuous in the $B^1_{\beta,\infty} \times B^{-s}_{\infty,\infty} \longrightarrow B^{-s}_{\beta,\infty}$ topology.*

Proof (of the lemma). This is a direct application of the Bony decomposition. We write the product fh as a sum

$$fh = \mathcal{T}_f(h) + \mathcal{T}_h(f) + \mathcal{R}(f,h).$$

On the one hand, the first paraproduct may be evaluated in the following way: for $j \geq -1$,

$$\begin{aligned}
\|\Delta_j \mathcal{T}_f(h)\|_{L^\beta} &\lesssim \sum_{|j-m|\leq 4} \|S_{m-1}f\|_{L^\beta} \|\Delta_m h\|_{L^\infty} \\
&\lesssim 2^{js} \|f\|_{L^\beta} \|h\|_{B^{-s}_{\infty,\infty}} \\
&\lesssim 2^{js} \|f\|_{B^1_{\beta,\infty}} \|h\|_{B^{-s}_{\infty,\infty}},
\end{aligned}$$

so that we have $\|\mathcal{T}_f(h)\|_{B^{-s}_{\beta,\infty}} \lesssim \|f\|_{B^1_{\beta,\infty}} \|h\|_{B^{-s}_{\infty,\infty}}$. Likewise, the second paraproduct is bounded in the same manner: for any $j \geq -1$,

$$\begin{aligned}
\|\Delta_j \mathcal{T}_h(f)\|_{L^\beta} &\lesssim \sum_{|j-m|\leq 4} \|S_{m-1}h\|_{L^\infty} \|\Delta_m f\|_{L^\beta} \\
&\lesssim \frac{1}{s} 2^{js} \|h\|_{B^{-s}_{\infty,\infty}} 2^{-j} \|f\|_{B^1_{\beta,\infty}} \\
&\lesssim 2^{(1-s)j} \|f\|_{B^1_{\beta,\infty}} \|h\|_{B^{-s}_{\infty,\infty}},
\end{aligned}$$

and this gives the inequality $\|\mathcal{T}_h(f)\|_{B^{1-s}_{\beta,\infty}}$. Finally, the remainder term can be bounded by using Proposition 3.1.10. We have

$$\|\mathcal{R}(f,h)\|_{B^{1-s}_{\beta,\infty}} \lesssim \|f\|_{B^1_{\beta,\infty}} \|h\|_{B^{-s}_{\infty,\infty}}.$$

The combination of all three inequalities proves the lemma. □

Let us apply Lemma 3.3.12 in order to study the convergence of the product $\eta(\rho_n)u_n$ as we let $n \to +\infty$. For any function $\phi \in \mathcal{D}([0,T[\times\mathbb{T}^d)$, we have

$$\big|\langle \eta(\rho_n)u_n - gu, \phi \rangle\big| \lesssim \|\eta(\rho_n) - g\|_{L^\infty_T(B^{-s}_{\beta,\infty})} \|u_n\|_{L^\infty_T(B^1_{\beta,\infty})} \|\phi\|_{L^1_T(B^{s-1}_{\beta',1})} + \big|\langle u_n - u, g\phi \rangle\big|.$$

Firstly, the strong convergence property (3.3.65) shows that the first summand in the previous upper bound tends to zero as $n \to +\infty$. Secondly, we know that u_n converges to u in the weak-$(*)$ topology of $L^\infty(L^{q'})$, while the function $g\phi$ is an element of $L^1(L^\infty)$, and so belongs to the predual space $L^1(L^q)$ (recall that $1 \leq q < d$). This means that the bracket in the above inequality also converges to zero. Finally, the initial data for the transport equation (3.3.61) converges strongly: for any $r < +\infty$, dominated convergence yields

$$\eta(S_n\rho_0) \longrightarrow \eta(\rho_0) \qquad \text{in } L^r$$

for all $r < +\infty$. This shows that the limit g is a weak solution of the initial value problem

$$\begin{cases} \partial_t g + \operatorname{div}(gu) = 0 \\ g(0) = \eta(\rho_0). \end{cases} \tag{3.3.63}$$

We will show that in fact $g = \eta(\rho)$, and thereby deduce that ρ is a solution of the transport equation.

By performing the same steps by replacing the function η by η^2, we see that the functions $\eta(\rho_n)^2$ have a limit

$$\eta(\rho_n)^2 \rightharpoonup h \qquad \text{in } L^\infty(L^\infty)$$

which is a solution of the initial value problem

$$\begin{cases} \partial_t h + \operatorname{div}(hu) = 0 \\ h(0) = \eta(\rho_0)^2. \end{cases} \tag{3.3.64}$$

Now, let us show that g^2 is also a solution of the initial value problem (3.3.64). We wish to multiply the transport equation (3.3.63) by g. In order to make sure we can do this, we first go through a regularization procedure: consider a mollification sequence $(\psi_\epsilon)_{\epsilon>0}$ and set $g_\epsilon = \psi_\epsilon * g$. Then g_ϵ solves the equation

$$\partial_t g_\epsilon + u \cdot \nabla g_\epsilon = \big[u \cdot \nabla, \psi_\epsilon * \big] g,$$

where, in the above, the brackets $[A, B] = AB - BA$ represent a commutator of operators, where $\psi_\epsilon * : f \mapsto \psi_\epsilon * f$ is the convolution operator and where $u \cdot \nabla$ must be understood in the weak sense $u \cdot \nabla = \operatorname{div}(fu)$. Multiplying the above by g_ϵ gives

$$\partial_t(g_\epsilon^2) + \operatorname{div}(g_\epsilon^2 u) = g_\epsilon \big[u \cdot \nabla, \psi_\epsilon * \big] g.$$

To check that g^2 also is a solution of the transport equation (3.3.64), we only must check that the righthand side tends to zero as $\epsilon \to 0^+$. This is the case, thanks to Lemma 3.2.2, which provides

$$\big[u \cdot \nabla, \psi_\epsilon * \big] g \longrightarrow 0 \qquad \text{in } L^1_T(L^\beta).$$

Consequently, $g^2 \in L^\infty(L^\infty)$ and $h \in L^\infty(L^\infty)$ solve the same initial value problem (3.3.64). In addition, the transport equation with a $L^\infty(W^{1,\beta})$ coefficient is well-posed in that space, by Theorem 3.2.1, so that we must have $h = g^2$ and therefore

$$\eta(\rho_n)^2 \overset{*}{\rightharpoonup} g^2 \qquad \text{in } L^\infty(L^\infty).$$

Let us prove that the weak convergence above implies strong convergence of the $\eta(\rho_n)$. By fixing any $T > 0$ and using $\mathbb{1}_{[0,T]} \in L^1(L^1)$ as a test function in the weak-$(*)$ convergence, we see that

$$\|\eta(\rho_n)\|^2_{L^2_T(L^2)} = \langle \eta(\rho_n)^2, \mathbb{1}_{[0,T]} \rangle \longrightarrow \langle g^2, \mathbb{1}_{[0,T]} \rangle = \|g\|^2_{L^2_T(L^2)}.$$

In other words, the sequence $\big(\eta(\rho_n)\big)$ converges weakly-$(*)$ in $L^\infty(L^\infty)$, and therefore weakly in the *Hilbert space* $L^2_T(L^2)$, and additionally the Hilbert space norms converge. This implies convergence in the norm topology of $L^2_T(L^2)$, and so

$$\eta(\rho_n) \longrightarrow g \qquad \text{in } L^2_{\text{loc}}(L^2), \tag{3.3.65}$$

and, up to extracting again, the convergence is also true almost everywhere in $\mathbb{R}_+ \times \mathbb{T}^d$.

We now show that the pointwise convergence of the $\eta(\rho_n)$ implies convergence almost everywhere of the ρ_n. For this, we resort to an appropriate choice of admissible functions η and the notion of

convergence in measure (see Definition A.7.1). We first prove that the sequence (ρ_n) is convergent in measure on every $[0,T]\times\mathbb{T}^d$ by resorting to a Cauchy criterion (see Proposition A.7.2). Fix a $\epsilon>0$ and two indices $m>n\geq 1$. Moreover, recall the functions η_k defined in (3.3.28). Then, by decomposing the product $[0,T]\times\mathbb{T}^d$ according to whether $|\rho_n|,|\rho_m|\leq k$ or not, we have

$$\begin{aligned}\big\{|\rho_n-\rho_m|\geq\epsilon\big\}=\big\{|\rho_n-\rho_m|\geq\epsilon\big\}\cap\big\{|\rho_n|\leq k\text{ and }|\rho_m|\leq k\big\}\\ \sqcup\big\{|\rho_n-\rho_m|\geq\epsilon\big\}\cap\big\{|\rho_n|>k\text{ or }|\rho_m|>k\big\}.\end{aligned}$$

On the one hand, the Markov inequality provides the bound

$$\begin{aligned}\mathrm{meas}\big\{|\rho_n|>k\text{ or }|\rho_m|>k\big\}&\lesssim\frac{1}{k^q}\int_0^T\int\Big(|\rho_m|^q+|\rho_n|^q\Big)\,\mathrm{d}x\,\mathrm{d}t\\&\lesssim T\frac{\|\rho_0\|_{L^q}^q}{k^q}.\end{aligned}$$

On the other hand, on the set where both $|\rho_m|$ and $|\rho_n|$ are smaller than k, we know that these functions are equal to, respectively, $\eta_k(\rho_m)$ and $\eta_k(\rho_n)$. We deduce that

$$\mathrm{meas}\big\{|\rho_n-\rho_m|\geq\epsilon\big\}\lesssim\mathrm{meas}\big\{|\eta_k(\rho_n)-\eta_k(\rho_m)|\geq\epsilon\big\}+T\frac{\|\rho_0\|_{L^q}^q}{k^q}.$$

Now, we had deduced from (3.3.65) that, for any given k, the sequence $\big(\eta_k(\rho_n)\big)_n$ converges almost everywhere (to a limit g_k). Proposition A.7.3 then shows that the sequence converges in measure, and in particular it must fulfill the Cauchy criterion of Proposition A.7.2, so that the measure in the inequality immediately above converges to zero as $m>n\geq N\to+\infty$. By taking the limit superior, we infer that

$$\varlimsup_{m>n\geq N\to+\infty}\mathrm{meas}\big\{|\rho_n-\rho_m|\geq\epsilon\big\}\lesssim T\frac{\|\rho_0\|_{L^q}^q}{k^q}.$$

Since this upper bound can be made as small as desired by taking k as large as necessary, we see that the limit superior must be zero, and so the sequence (ρ_n) must converge in measure. Proposition A.7.3 then asserts that the convergence is also true almost everywhere on $[0,T]\times\mathbb{T}^d$ up to taking an extraction. By taking T as having integral values $T=M\to+\infty$, we see by means of diagonal extraction that the convergence can in fact be assumed hold almost everywhere on $\mathbb{R}_+\times\mathbb{T}^d$. Let f be the limit of this convergence:

$$\rho_n\longrightarrow f\qquad\text{a.e. on }\mathbb{R}_+\times\mathbb{T}^d.$$

It is a consequence of Proposition A.7.4 that in fact $f=\rho$, so that the ρ_n converge to ρ almost everywhere on $\mathbb{R}_+\times\mathbb{T}^d$.

Finally, let us show that the convergence of (ρ_n) takes place in the norm topology of $L^r(L^q)$. According to Proposition A.7.5, it is enough to show convergence of the norms $\|\rho_n\|_{L^r_T(L^q)}$ for every $T>0$, as the space $L^r_T(L^q)$ is uniformly convex (since $1<r<+\infty$). On the one hand, we have, since the flows of the smooth u_n (which are trigonometric polynomials) preserve the Lebesgue norms

$$\|\rho_n(t)\|_{L^q}=\|\rho_n(0)\|_{L^q}=\|S_n\rho_0\|_{L^q}\underset{n\to+\infty}{\longrightarrow}\|\rho_0\|_{L^q}.\tag{3.3.66}$$

If we show that the preservation of the Lebesgue norms transfers to the limit $\|\rho(t)\|_{L^q}=\|\rho_0\|_{L^q}$, then the preceding equation is enough to apply Proposition A.7.5. Instead of working directly with the density ρ, we do this by means of the almost everywhere convergence $\rho_n\longrightarrow\rho$ that we have shown just above, and apply it to the "renormalized" equation (3.3.61). Consider a function

η_k as in (3.3.28) and note that, as $\eta_k(\rho_n)$ is a solution of the initial value problem (3.3.61) with $\eta = \eta_k$, we must have

$$\big\|\eta_k(\rho_n(t))\big\|_{L^q} = \|\eta_k(S_n\rho_0)\|_{L^q} \qquad \text{for a.e. } t \in \mathbb{R}_+.$$

However, the convergence $S_n\rho_0 \longrightarrow \rho_0$ in L^q and the almost everywhere convergence of the ρ_n implies that, by dominated convergence, and up to taking an extraction,

$$\|\eta_k(S_n\rho_0)\|_{L^q} \longrightarrow \big\|\eta_k(\rho_0)\big\|_{L^q} \qquad \text{as } n \to +\infty$$

and

$$\big\|\eta_k(\rho_n(t))\big\|_{L^q} \underset{n\to+\infty}{\longrightarrow} \big\|\eta_k(\rho(t))\big\|_{L^q} \qquad \text{for a.e. } t \in \mathbb{R}_+.$$

Uniqueness of the pointwise limit then shows that

$$\big\|\eta_k(\rho(t))\big\|_{L^q} = \big\|\eta_k(\rho_0)\big\|_{L^q} \qquad \text{for a.e. } t \in \mathbb{R}_+.$$

Lastly, by our choice of η_k, we note that $|\eta_k(r)| \leq |r|$, and $\eta_k(r) \to r$ pointwise as $k \to +\infty$, so that dominated convergence yields

$$\big\|\eta_k(\rho)\big\|_{L^r_T(L^q)} \longrightarrow \|\rho\|_{L^r(L^q)} \qquad \text{as } k \to +\infty$$

and

$$\big\|\eta_k(\rho_0)\big\|_{L^q} \longrightarrow \|\rho_0\|_{L^q} \qquad \text{as } k \to +\infty,$$

so that we do indeed have preservation of the Lebesgue norms $\|\rho(t)\|_{L^q} = \|\rho_0\|_{L^q}$ and (3.3.66) allows us to deduce convergence of the norms $\|\rho_n\|_{L^r_T(L^q)} \longrightarrow \|\rho\|_{L^r_T(L^q)}$. As explained above, this is enough to invoke Proposition A.7.5 and thereby end the proof.

□

3.3.6 Proof of the main Theorem

Here we combine all the elements proved above and show the existence of a weak solution of the non-Newtonian Stokes-Transport problem (3.0.1). Proposition 3.3.9 provides the existence of a family (ρ_n, u_n, v_n) of functions which solve the approximate problem (3.3.43), and for which the *a priori* estimates also hold, see (3.3.59) and (3.3.60). In addition, Proposition 3.3.11 shows that the densities strongly converge (up to an extraction)

$$\rho_n \longrightarrow \rho \qquad \text{in } L^r_{\text{loc}}(L^q), \tag{3.3.67}$$

where ρ is a solution of transport equation (3.3.58) with initial datum ρ_0. Concerning the velocities, we have the weak convergences

$$\begin{matrix} v_n \rightharpoonup^* v \\ u_n \rightharpoonup^* u \end{matrix} \qquad \text{in } L^\infty(W^{1,\beta}).$$

All that remains to show is then that $u = v$ and that v is, at almost every time $t \in \mathbb{R}_+$, a weak solution of the Stokes equation, namely

$$\begin{cases} -\operatorname{div}\big(\nu(\rho)|D(v)|^{p-2}D(v)\big) + \nabla\pi = \rho g \\ \operatorname{div}(v) = 0. \end{cases}$$

Let us rephrase the problem: if $\Psi : L^q \longrightarrow W^{1,\beta}$ is the inverse map introduced in Proposition 3.3.4. We know that $v_n(t) = \Psi\big(\rho_n(t)\big)$ for almost every time $t \in \mathbb{R}_+$ and we have to show that

$u = v = \Psi\big(\rho(t)\big)$. This will be done by resorting to the continuity property of the map Ψ from Proposition 3.3.6. Consider a function $\phi \in L^1(L^q)$. Then,

$$\begin{aligned}\big\langle v_n, \phi \big\rangle_{L^1(L^{q'})\times L^\infty(L^q)} &= \int_0^{+\infty} \int \Psi(\rho_n)\phi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^{+\infty} \langle \Psi(\rho_n), \phi\rangle_{L^{q'}\times L^q}\,\mathrm{d}t.\end{aligned}$$

However, Proposition 3.3.6 shows that Ψ is continuous with respect to the $L^q \longrightarrow W_w^{1,\beta} \subset L_w^{q'}$ topology. In addition, (3.3.67) insures that strong convergence

$$\rho_n(t) \longrightarrow \rho(t) \qquad \text{in } L^q$$

occurs for almost every time $t \in \mathbb{R}_+$, hence convergence for the brackets

$$\big\langle \Psi\big(\rho_n(t)\big), \phi(t)\big\rangle_{L^{q'}\times L^q} \longrightarrow \big\langle \Psi\big(\rho(t)\big), \phi(t)\big\rangle_{L^{q'}\times L^q}$$

also occurs at almost every time $t \in \mathbb{R}_+$. By dominated convergence, we deduce convergence in the whole space-time bracket, and hence $\Psi(\rho_n) \longrightarrow \Psi(\rho)$. In other words, v is indeed a solution of the Stokes problem for almost every time.

The last thing left is to check that $u = v$. Recall that we have $u_n = S_n v_n$, so that the Fourier transforms $\widehat{u_n}$ and $\widehat{v_n}$ must coincide on any ball $B(0,R)$, provided that n is taken large enough $n \geq C\log(R)$. Consequently, the limit Fourier transforms $\widehat{u}$ and $\widehat{v}$ must also agree on every ball, and so $u = v$. We have finished proving Theorem 3.0.2.

Moreover, if we alo have $\rho_0 \in L^q \cap L^r$ for $r \in [1, +\infty]$, then we got conservation of Lebesgue norms, and since we have renormalized solutions, then arguing as above leads, for every $t \in \mathbb{R}_+$:

$$\|\rho(t)\|_{L^r} = \|\rho_0\|_{L^r},$$

and this ends the proof.

Ce que j'aime dans les mathématiques appliquées, c'est qu'elles ont pour ambition de donner du monde des systèmes une représentation qui permette de comprendre et d'agir. Et, de toutes les représentations, la représentation mathématique, lorsqu'elle est possible, est celle qui est la plus souple et la meilleure. Du coup, ce qui m'intéresse, c'est de savoir jusqu'où on peut aller dans ce domaine de la modélisation des systèmes, c'est d'atteindre les limites.

Jacques-Louis Lions

4

Hemodialysis: optimization of chemical species concentrations

Here, we present results obtained as part of a numerical study of a hemodialysis problem. The aim was to determine the diffusion coefficient of the porous membrane present in the fibers of a dialyzer, based on a set of medical data supplied by the Gabriel Montpied University Hospital in Clermont-Ferrand.

4.1 Numerical calibration of the porosity of a dialyser

Hemodialysis is a process for rebalancing the concentrations of chemical species in the blood, and the deletion of some extra fluid for a patient with end-stage renal disease by using a medical device enabling chemical interactions between an external fluid, the dialysate, and blood called a dialyser [110]. The balance between calcium intake and loss during haemodialysis treatment affects the mineral metabolism and can be the cause of bone disorders and cardiovascular calcifications [103]. It is then of high importance to predict calcium exchange during hemodialysis. Establishing a mathematical and numerical model for the concentration of calcium in the blood at the exit of a dialyser was the main subject of the work [7].

As in [7], in order to model the chemical exchanges during a hemodialysis session, we firstly consider the flow of blood and dialysate inside a fiber in a dialyse filter. More exactly, Navier-Stokes equations are employed to model the flow of the dialysate, a non-Newtonian model modelize the flow of the blood, and Darcy's equation is considered to tackle the flow through the porous membrane between the blood and the dialysate. We keep the same approach as in [7] by considering a radially symmetric flow in one fiber and we take into account the geometric anisotropy of the fiber to obtain a simpler model by neglecting the terms of order ε. We recall that ε is an adimensional quantity obtained as the fraction of the radius R of the fiber and its length L. A typical example for these dimensions is $R = 2.3 \times 10^{-4}$ m and $L = 2.3 \times 10^{-1}$ m. The radius of the internal cylinder in which the blood flows is $R_1 = 10^{-4}$ m and the thickness of the membrane is typically 0.4×10^{-4} m. Setting $R_2 = 1.4 \times 10^{-4}$ m the dialysate fluid flows outside the cylinder of radius R_2 and inside the fiber. The solution (U_x, U_r) of this model is then expressed in a two-dimensional domain which is nothing else than half of the longitudinal section of a hollow fiber. Since we supposed that the flow are radially symmetric in the cylindrical fiber, this two-dimensional model is

a good representation of the three-dimensional setting. The two dimensional domain in which we finally state the system of partial differential equations modeling the flow and the concentrations of the chemical species in blood and dialysate is the set $\Omega = \Omega_b \cup \Omega_m \cup \Omega_d$ illustrated in Figure 4.1.

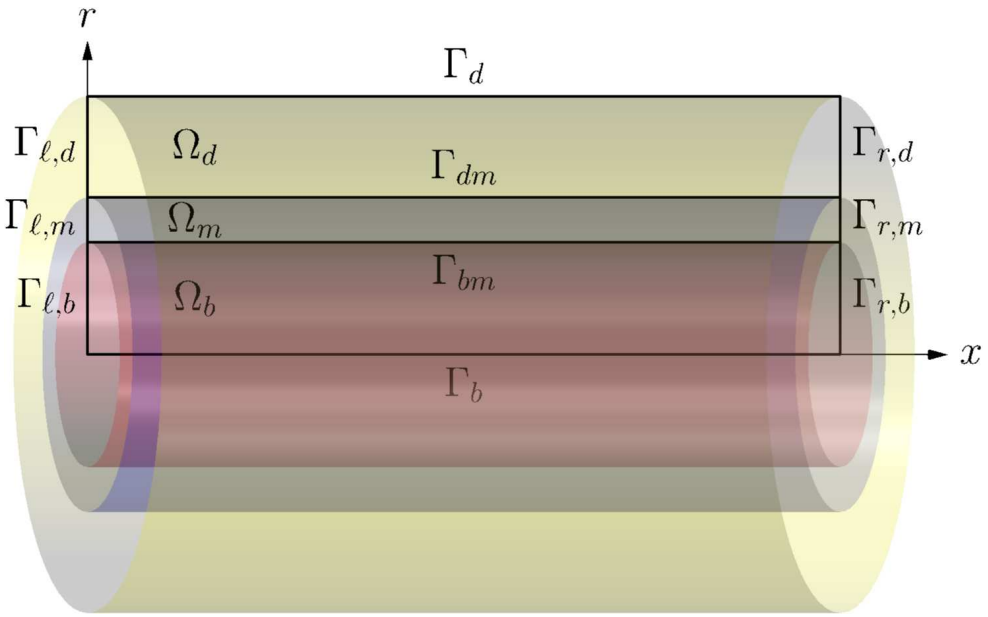


Figure 4.1: Schematic representation of a hollow fiber. Using the radial symmetry of the fiber, the computation are carried in $\Omega = \Omega_b \cup \Omega_m \cup \Omega_d$.

The fluid flow (U_x, U_r) computed in Ω acts as an advective vector field in a reaction-diffusion-advection system involving five chemical species. As in [7] we consider only five interacting chemical species: Calcium (species number 1), Albumin (species number 2), Calcium-Albumin (species number 3), Citrate (species number 4) and Calcium-Citrate (species number 5). We denote by $c_i(t, x, r)$ the concentration of the species i at the time t and at position (x, r) and by $\boldsymbol{c}$ the vector $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{pmatrix}$. The system of equations governing the evolution in time of the concentrations c_i inside the domain Ω are the following

$$\partial_t c_i + S_i \left(U_x \partial_x c_i + U_r \partial_r c_i\right) - \frac{1}{r}\partial_r \left(r D_i \partial_r c_i\right) - \partial_x \left(D_i \partial_x c_i\right) = F_i(\boldsymbol{c}) \text{ in } (0,T) \times \Omega \tag{4.1.1a}$$

$$c_i = \overline{c_{i,\text{ input}}} \text{ on } (0,T) \times (\Gamma_{\ell,b} \cup \Gamma_{r,d}) \tag{4.1.1b}$$

$$\partial_{\boldsymbol{n}} c_i = 0 \text{ on } (0,T) \times \Gamma_n \tag{4.1.1c}$$

$$c_i(0) = c_{i,0} \text{ in } \Omega, \tag{4.1.1d}$$

where D_i represents the diffusion coefficient for the species i, S_i is a sewing coefficient used to tackle the fact that some chemical species (Albumin and Calcium Albumin) does not cross the membrane, *i.e.*, for $i \in \{2,\ 3\}$

$$S_i(x) = \begin{cases} 1, \text{ if } x \in \Omega_b \\ 0, \text{ if } x \in \Omega \setminus \Omega_b \end{cases}$$

and $S_i = 1$ for $i \in \{1,\ 4,\ 5\}$. We also denote

$$\Gamma_n = \Gamma_{\ell,d} \cup \Gamma_{\ell,m} \cup \Gamma_{r,m} \cup \Gamma_{r,b} \cup \Gamma_b \cup \Gamma_d. \tag{4.1.2}$$

The function F in the right hand of equation (4.1.1a) models the interactions between the five

chemical species and is given by:

$$F(\boldsymbol{c}) = \begin{pmatrix} F_1(c_1,\dots,c_5) \\ F_2(c_1,\dots,c_5) \\ F_3(c_1,\dots,c_5) \\ F_4(c_1,\dots,c_5) \\ F_5(c_1,\dots,c_5) \end{pmatrix} = \begin{pmatrix} k_2c_3 + k_4c_5 - k_1c_1c_2 - k_3c_1c_4 \\ k_2c_3 - k_1c_1c_2 \\ -k_2c_3 + k_1c_1c_2 \\ k_4c_5 - k_3c_1c_4 \\ -k_4c_5 + k_3c_1c_4 \end{pmatrix}, \tag{4.1.3}$$

where k_i are some reaction constants. These reaction constants are experimentally known and, for their values see, for instance, [7] and the references therein. Finally, the quantity $\overline{\boldsymbol{c}_{\text{input}}} = (\overline{\boldsymbol{c}_{i,\text{ input}}})_{1\leq i\leq 5}$ represents the measured concentrations of the considered five species in blood and in dialysate at the entrance of the dialyser and the initial data $c_{i,0}$ in (4.1.1d) verifies $c_{i,0} = \overline{c_{i,\text{ input}}}$ on $\Gamma_{\ell,b}\cup\Gamma_{r,d}$ and is constant in Ω_b and Ω_d.

The qualitative results of the numerical simulations presented in [7] are in concordance with the clinical observations. Nevertheless, some quantities describing the model are not very well known, and these errors in the parameters can be the source of errors in the results when compared to the clinical measurements. An example of such badly known parameters are the diffusion coefficients D_i in the porous membrane for different chemical species. The aim of this work is to optimize the coefficients D_i such that the concentrations in the blood at the exit of the dialyser of the five considered chemical species predicted by the model better fit to the corresponding clinical measurements. More exactly, the following data were recorded during a clinical trial at CHU Gabriel Monpied in Clermont-Ferrand for 22 patients:

- flow rates of the blood ($q_{b,\text{input}}$ and, respectively, $q_{b,\text{output}}$) and of dialysate ($q_{d,\text{input}}$ and, respectively, $q_{d,\text{output}}$) at the entrance and, respectively at the exit of the dialyser;
- concentrations of the five considered chemical species $\overline{\boldsymbol{c}_{\text{input}}} = (\overline{c_{i,\text{ input}}})_{1\leq i\leq 5}$ in the blood at the entrance of the dialyser (*i.e.* on boundary $\Gamma_{\ell,b}$) and in the dialysate at the entrance of the dialyser (*i.e.* on boundary $\Gamma_{\ell,b}$);
- concentrations of the five considered chemical species $\overline{\boldsymbol{c}_{\text{output}}} = (\overline{c_{i,\text{ output}}})_{1\leq i\leq 5}$ in the blood at the exit of the dialyser (*i.e.* on boundary $\Gamma_{r,b}$);
- rate of hematocrites before and after the dialyser.

The remaining part of this section is organized as follows. In subsection4.1.1 we propose a Newton method for solving directly the non-transient system associated to (4.1.1). In subsection 4.1.2 we consider the optimization problem of the minimization of a functional which associates to the diffusion coefficients into the dialyser's membrane the relative error between the concentrations c_i in the blood at the exit of the dialyser obtained as solutions of a variational formulation and the experimental data recorded from a patient. The aim is to validate the method proposed in subsection 4.1.1 and subsection 4.1.2 by comparing the results of the numerical simulations to the experimental data from 22 patients. Finally, in subsection 4.1.4 we state some conclusions and propose some perspectives.

4.1.1 Stationnary solution of the reaction-convection-diffusion system

Since the values of interest of the concentrations $\boldsymbol{c}$ in (4.1.1) correspond to its stationary solution, we can consider the non-transient system associated to (4.1.1). Also, a scaling of the appearing quantities is necessary in order to tackle the particular geometry of the domain and the magnitudes of the diffusion coefficients. Moreover, there are two different situations:

- for $i \in \{1, 4, 5\}$ which corresponds to Calcium, Citrate and Calcium-Citrate, respectively, the non-transient system is:

$$U_x \partial_x c_i + U_r \partial_r c_i - \frac{1}{\mathcal{P}e} \frac{1}{r} \partial_r \left(r D_i \partial_r c_i \right) - \frac{\varepsilon^2}{\mathcal{P}e} \partial_x \left(D_i \partial_x c_i \right) = \widetilde{F}_i(\boldsymbol{c}) \text{ in } \Omega \tag{4.1.4a}$$

$$c_i = \overline{c_{i,\text{ input}}} \text{ on } \Gamma_{\ell,b} \cup \Gamma_{r,d} \tag{4.1.4b}$$

$$\partial_{\boldsymbol{n}} c_i = 0 \text{ on } \Gamma_b \cup \Gamma_{r,b} \cup \Gamma_{\ell,d}; \tag{4.1.4c}$$

- since the Albumin and the Calcium-Albumin does not cross the membrane, the equation representing species $i \in \{2,\ 3\}$ holds only in Ω_b:

$$U_x \partial_x c_i + U_r \partial_r c_i - \frac{1}{\mathcal{P}e} \frac{1}{r} \partial_r \left(r D_i \partial_r c_i \right) - \frac{\varepsilon^2}{\mathcal{P}e} \partial_x \left(D_i \partial_x c_i \right) = \widetilde{F}_i(\boldsymbol{c}) \text{ in } \Omega_b \tag{4.1.5a}$$

$$c_i = \overline{c_{i,\text{ input}}} \text{ on } \Gamma_{\ell,b} \tag{4.1.5b}$$

$$\partial_{\boldsymbol{n}} c_i = 0 \text{ on } \Gamma_b \cup \Gamma_{r,b} \tag{4.1.5c}$$

$$D_i \partial_r c_i = c_i U_r \text{ on } \Gamma_{bm}. \tag{4.1.5d}$$

We refer to [7] for details concerning this re-scaling procedure and the expressions of the constants ε and $\mathcal{P}e$. Moreover, as explained in [7], the rescaling procedure require to accordingly rewrite the expression of the nonlinear term appearing in (4.1.4a) and (4.1.5a) as follows:

$$\widetilde{F}(\boldsymbol{c}) = \widetilde{F}(c_1, \ldots, c_5) = \frac{1}{\mathcal{F}d} \begin{pmatrix} c_3 + \delta_2 c_5 - \delta_1 c_1 c_2 - \delta_3 c_1 c_4 \\ c_3 - \delta_1 c_1 c_2 \\ -c_3 + \delta_1 c_1 c_2 \\ \delta_2 c_5 - \delta_3 c_1 c_4 \\ -\delta_2 c_5 + \delta_3 c_1 c_4 \end{pmatrix}, \tag{4.1.6}$$

where δ_1, δ_2, δ_3 and $\mathcal{F}d$ are some new constants.

We approach the solution of the stationnary reaction-advection-diffusion system using a Newton method in a variational form. More exactly we first write the variational formulation associated to (4.1.4): for $i \in \{1,\ 4,\ 5\}$, find $c_i \in C_i$ solutions to:

$$\begin{aligned} &\iint_\Omega \left(U_x \partial_x c_i + U_r \partial_r c_i \right) r \varphi_i \, \mathrm{d}x \, \mathrm{d}r + \frac{1}{\mathcal{P}e} \iint_\Omega r D_i \partial_r c_i \partial_r \varphi_i \, \mathrm{d}x \, \mathrm{d}r - \frac{\varepsilon^2}{\mathcal{P}e} \int_{\Gamma_{r,d}} r D_i \partial_x c_i \varphi_i \, \mathrm{d}r \\ &\quad + \frac{\varepsilon^2}{\mathcal{P}e} \int_{\Gamma_{\ell,b}} r D_i \partial_x c_i \varphi_i \, \mathrm{d}r + \frac{\varepsilon^2}{\mathcal{P}e} \iint_\Omega r D_i \partial_x c_i \partial_x \varphi_i \, \mathrm{d}x \, \mathrm{d}r = \iint_\Omega r \widetilde{F}_i(\boldsymbol{c}) \varphi_i \, \mathrm{d}x \, \mathrm{d}r, \end{aligned} \tag{4.1.7}$$

for every $\varphi_i \in C_i$, where C_i are Sobolev spaces taking into account the boundary conditions in (4.1.4). Similarly, we can write a variational formulation associated to (4.1.5). The main difference with respect to the variational formulation (4.1.7) is that the solutions to (4.1.5) are supported in Ω_b and not in Ω and are null in Ω_m and Ω_d. Another difference is the presence of a boundary term on Γ_{bm} due to the boundary condition. More exactly, for $i \in \{2, 3\}$ find $c_i \in C_i$ solution to:

$$\begin{aligned} &\iint_{\Omega_b} \left(U_x \partial_x c_i + U_r \partial_r c_i \right) r \varphi_i \, \mathrm{d}x \, \mathrm{d}r + \frac{1}{\mathcal{P}e} \iint_{\Omega_b} r D_i \partial_r c_i \partial_r \varphi_i \, \mathrm{d}x \, \mathrm{d}r - \int_{\Gamma_{bm}} \frac{U_r}{\mathcal{P}e} c_i \varphi_i \mathrm{d}r \\ &\quad + \frac{\varepsilon^2}{\mathcal{P}e} \int_{\Gamma_{\ell,b}} r D_i \partial_x c_i \varphi_i \, \mathrm{d}r + \frac{\varepsilon^2}{\mathcal{P}e} \iint_{\Omega_b} r D_i \partial_x c_i \partial_x \varphi_i \, \mathrm{d}x \, \mathrm{d}r = \iint_{\Omega_b} r \widetilde{F}_i(\boldsymbol{c}) \varphi_i \, \mathrm{d}x \, \mathrm{d}r, \end{aligned} \tag{4.1.8}$$

for every $\varphi_i \in C_i$, where C_i are Sobolev spaces taking into account the boundary conditions in (4.1.5). Remark that we can gather together (4.1.7)-(4.1.8) as: find $\boldsymbol{c} = (c_i)_{1 \leq i \leq 5} \in \boldsymbol{C}$ solution to

$$\mathcal{F}(\boldsymbol{c}, \boldsymbol{\varphi}) = \boldsymbol{0}, \tag{4.1.9}$$

for every $\boldsymbol{\varphi} = (\varphi_i)_{1\leq i\leq 5} \in \boldsymbol{C} = \overset{5}{\underset{i=1}{\times}} C_i$ with $\mathcal{F} : \boldsymbol{C} \times \boldsymbol{C} \to \mathbb{R}^5$ regrouping the integral terms in the five equalities (4.1.7)-(4.1.8). Since the application $\mathcal{F}$ is nonlinear, we solve (4.1.9) using the Newton method. In this purpose, we need to compute the gradient $\nabla\mathcal{F}$ of $\mathcal{F}$. More exactly, we have

$$\nabla\mathcal{F}(\boldsymbol{c},\boldsymbol{\varphi})\overline{\boldsymbol{c}} = \lim_{h\to 0} \frac{\mathcal{F}(\boldsymbol{c}+h\overline{\boldsymbol{c}},\boldsymbol{\varphi}) - \mathcal{F}(\boldsymbol{c},\boldsymbol{\varphi})}{h} = \mathcal{L}(\overline{\boldsymbol{c}},\boldsymbol{\varphi}) + \mathcal{N}(\boldsymbol{c},\overline{\boldsymbol{c}},\boldsymbol{\varphi}), \tag{4.1.10}$$

with $\mathcal{L}$ grouping the linear terms in the right-hand side of (4.1.4)–(4.1.5) and $\mathcal{N}$ gathering the remaining nonlinear terms, *i.e.*,

$$\mathcal{N}(\boldsymbol{c},\overline{\boldsymbol{c}},\boldsymbol{\varphi}) = \left(\iint_\Omega r(\nabla\widetilde{F}(\boldsymbol{c})\overline{\boldsymbol{c}})_i\varphi_i \,\mathrm{d}x\,\mathrm{d}r \right)_{1\leq i\leq 5}.$$

We give the detail of the variational Newton method employed to approach the solution of (4.1.9) in Algorithm (2). From a numerical analysis perspective, we approach all the variational formulations by the finite elements method. More exactly we consider structured regular triangulation $\mathcal{T}_h$ of Ω formed by triangles of diameter h and we approach C_i by its finite dimensional subspace C_h formed by the continuous function which restriction to any triangle of $\mathcal{T}_h$ is a first degree polynomial. All the numerical simulations were done using FreeFEM [108].

Algorithm 2 Newton method in variational form for equation (4.1.9).

Require: $\mathcal{F}$, $\boldsymbol{c_0}$, *tol*, $n_{\max}$ ▷ $n_{\max}$ is the maximal number of iterations

$n \leftarrow 0$.

Compute $\boldsymbol{c_{n+1}} \in \boldsymbol{C}$ solution to

$$\nabla\mathcal{F}(\boldsymbol{c_n},\boldsymbol{\varphi})\boldsymbol{c_{n+1}} = \nabla\mathcal{F}(\boldsymbol{c_n},\boldsymbol{\varphi})\boldsymbol{c_n} - \mathcal{F}(\boldsymbol{c_n},\boldsymbol{\varphi}) \qquad \text{for every } \boldsymbol{\varphi} \in \boldsymbol{C}.$$

while $n \leq n_{\max}$ and $\|\boldsymbol{c_{n+1}} - \boldsymbol{c_n}\| >$ tol **do** ▷ The tolerance tol is taken equal to 10^{-4}

$n \leftarrow n+1$

Compute $\boldsymbol{c_{n+1}} \in \boldsymbol{C}$ solution to

$$\nabla\mathcal{F}(\boldsymbol{c_n},\boldsymbol{\varphi})\boldsymbol{c_{n+1}} = \nabla\mathcal{F}(\boldsymbol{c_n},\boldsymbol{\varphi})\boldsymbol{c_n} - \mathcal{F}(\boldsymbol{c_n},\boldsymbol{\varphi}) \qquad \text{for every } \boldsymbol{\varphi} \in \boldsymbol{C}.$$

end while

if $\|\boldsymbol{c_{n+1}} - \boldsymbol{c_n}\| \leq$ tol **then**

The algorithm converged.

The solution of the variational problem (4.1.9) is approximated by $\boldsymbol{c_{n+1}}$.

end if

4.1.2 Optimization algorithm for intramembrane coefficients

Let us recall that D_i is the diffusion coefficient of the species i in water (or in blood). We know that the corresponding diffusion coefficient in the dialyser membrane, which we denote D_i^m is a fraction of D_i (see, for instance, [81], were the diffusion coefficients are considered five times smaller into the membrane than in blood), i.e., $D_i^m = \alpha_i D_i$ with $\alpha_i \in [0,1]$. Since the Albumin does not cross the membrane, these coefficients for Albumin and Calcium-Albumin are equal to 0: $\alpha_i = 0$ for $i \in \{2,3\}$.

We assume that we know, from measurements, the concentrations $\overline{\boldsymbol{c}_{\text{input}}} = (\overline{c_{i,\text{ input}}})_{1\leq i\leq 5}$ in blood and dialysate at entrance of the dialyser and the concentrations $\overline{\boldsymbol{c}_{\text{output}}} = (\overline{c_{i,\text{ output}}})_{1\leq i\leq 5}$ in blood at the exit of the dialyser for one patient. Since the concentrations $c_{i,\text{ input}}$ are constant on $\Gamma_{\ell,b}$ and on $\Gamma_{r,d}$, the vector $\overline{\boldsymbol{c}_{\text{input}}}$ is in $\mathbb{R}^{10}$, and, since we are interested only on the concentrations in blood at the exit of the dialyser, $\overline{\boldsymbol{c}_{\text{output}}} \in \mathbb{R}^5$.

The operator $\mathcal{B}_{\boldsymbol{\alpha}} : \mathbb{R}^{10} \to \mathbb{R}^5$ take as input the known concentrations $\overline{\boldsymbol{c}_{\text{input}}}$ in blood and dialysate at entrance of the dialyser and its output are the concentrations $c_{i,\ \text{output}}$ at the exit of the dialyser numerically computed by the procedure discussed in Section 4.1.1 for some diffusion coefficients $\boldsymbol{\alpha} = (\alpha_i)_{1\leq i\leq 5}$. More exactly, for a solution $\boldsymbol{c} \in \boldsymbol{C}$, we define

$$c_{i,\ \text{output}} = \frac{2R^2}{R_1^2} \int_{\Gamma_{r,b}} r c_i dr.$$

Since the Albumin and Calcium-Albumin does not cross the membrane, we set $\alpha_2 = \alpha_3 = 0$. Moreover, the diffusion coefficients of Citrate and Calcium-Citrate are very close. Therefore, we define the admisible set for the coefficients α_i by

$$\mathcal{A} = \left\{ \boldsymbol{\alpha} = (\alpha_i)_{1\leq i\leq 5} \in \mathbb{R}^5 \ \mid \ \alpha_2 = \alpha_3 = 0,\ \alpha_4 = \alpha_5 \text{ and } \alpha_i \in (0,1] \text{ for } i \in \{1,\ 4,\ 5\} \right\}.$$

We then compute the optimal coefficients $\boldsymbol{\alpha}$ by solving the following problem

$$\min_{\boldsymbol{\alpha}\in\mathcal{A}} J(\boldsymbol{\alpha})$$

where

$$J(\boldsymbol{\alpha}) = \sum_{i=1}^{5} \frac{|\mathcal{B}_{\boldsymbol{\alpha},i}(\overline{\boldsymbol{c}_{\text{input}}}) - \overline{c_{i,\ \text{output}}}|^2}{|\overline{c_{i,\ \text{output}}}|^2}. \tag{4.1.11}$$

The quantity $\mathcal{B}_{\boldsymbol{\alpha},i}(\overline{\boldsymbol{c}_{\text{input}}})$ denotes the i-th component of the vector $\mathcal{B}_{\boldsymbol{\alpha}}(\overline{\boldsymbol{c}_{\text{input}}})$. Considering the constraints present in $\mathcal{A}$, we need only to optimize coefficients α_1 and α_4. Hence, we finally minimize over $[0,1]^2$ the functional $\mathcal{J} : \mathbb{R}^2 \to [0,\infty)$ defined for every $\boldsymbol{\beta} = (\beta_1,\ \beta_2)$ by $\mathcal{J}(\boldsymbol{\beta}) = J(\boldsymbol{\alpha})$ with $\boldsymbol{\alpha} = (\beta_1, 0, 0, \beta_2, \beta_2)$.

In order to compute the minima of $\mathcal{J}$ we use a gradient descent method combined to a projection method. Since the operator $\mathcal{B}_{\boldsymbol{\alpha}}$ is nonlinear we should numerically approach the gradient of $\mathcal{J}$. More exactly we choose a small value of h (for the numerical simulations discussed in the next section we take $h = 0.02$ which is also the size of the triangles in the triangulation) and then approach the gradient of $\mathcal{J}$ in $\boldsymbol{\beta}$ in directions $\boldsymbol{i} = (1,0)$ and $\boldsymbol{j} = (0,1)$ by

$$\nabla\mathcal{J}(\boldsymbol{\beta})\boldsymbol{i} \approx \frac{\mathcal{J}(\boldsymbol{\beta} + h\boldsymbol{i}) - \mathcal{J}(\boldsymbol{\beta})}{h} \qquad \nabla\mathcal{J}(\boldsymbol{\beta})\boldsymbol{j} \approx \frac{\mathcal{J}(\boldsymbol{\beta} + h\boldsymbol{j}) - \mathcal{J}(\boldsymbol{\beta})}{h}.$$

The gradient vector $\nabla\mathcal{J}(\boldsymbol{\beta})$ is then approached by

$$\nabla\mathcal{J}(\boldsymbol{\beta}) = \left(\nabla\mathcal{J}(\boldsymbol{\beta})\boldsymbol{i},\quad \nabla\mathcal{J}(\boldsymbol{\beta})\boldsymbol{j}\right). \tag{4.1.12}$$

In Algorithm 3 we described the gradient descent with adaptive step (see, for instance, [47, Section 8.4] for several gradient descent methods) which we employ to optimize the diffusion coefficients into the membrane. Remark that at each step of this algorithm there are at least four calls to the Newton method described in Algorithm 2.

4.1.3 Comparison to experimental data

In this section we compare the results obtained by the numerical simulation of the solutions of (4.1.9) to the measurements from 22 patients gathered during the clinical trial. These measurements correspond to the boundary conditions on $\Gamma_{\ell,b}$, $\Gamma_{r,d}$ and $\Gamma_{r,b}$ and provides essentially the quantities $\overline{\boldsymbol{c}_{\text{input}}}$ and $\overline{\boldsymbol{c}_{\text{output}}}$.

A first step consists in validating the Newton method in variational form proposed in Section 4.1.1 and its numerical dicretization using P_1 finite elements. More exactly we consider a set of boundary data (constant functions for every chemical species on $\Gamma_{\ell,b}$ and $\Gamma_{r,d}$) corresponding to the

Algorithm 3 Gradient descent algorithm with adaptive step for the minimization of $\mathcal{J}$.

Require: $\mathcal{J}$, $\boldsymbol{\beta_0}$, tol, $n_{\max}$. ▷ $n_{\max}$ is the maximal number of iterations
Require: initial_step ▷ initial_step is the initial descent step
Require: tol. ▷ tol is a small value
  $n \leftarrow 0$.
  Compute $\mathcal{J}(\boldsymbol{\beta_n})$. ▷ Use Newton algorithm described in Algorithm 2
  Compute $\nabla\mathcal{J}(\boldsymbol{\beta_n})$. ▷ Two more calls of Newton method
  s $\leftarrow$ initial_step
  while $\mathcal{J}(\mathcal{P}(\boldsymbol{\beta_n} - \mathrm{s}\cdot\nabla\mathcal{J}(\boldsymbol{\beta_n}))) \geq \mathcal{J}(\boldsymbol{\beta_n})$ and $\|\mathcal{P}(\boldsymbol{\beta_n} - \mathrm{s}\cdot\nabla\mathcal{J}(\boldsymbol{\beta_n})) - \boldsymbol{\beta_n}\| >$ tol **do**
    s $\leftarrow$ s/2. ▷ $\mathcal{P}(\boldsymbol{\beta})$ is the projection of $\boldsymbol{\beta}$ on $[0,1]^2$
  end while
  while $\|\mathcal{P}(\boldsymbol{\beta_n} - \mathrm{s}\cdot\nabla\mathcal{J}(\boldsymbol{\beta_n})) - \boldsymbol{\beta_n}\| >$ tol and $n < n_{\max}$ **do**
    $\boldsymbol{\beta_{n+1}} \leftarrow \mathcal{P}(\boldsymbol{\beta_n} - \mathrm{s}\cdot\nabla\mathcal{J}(\boldsymbol{\beta_n}))$.
    $n \leftarrow n+1$.
    Compute $\nabla\mathcal{J}(\boldsymbol{\beta_n})$.
    s $\leftarrow$ initial_step.
    while $\mathcal{J}(\mathcal{P}(\boldsymbol{\beta_n} - \mathrm{s}\cdot\nabla\mathcal{J}(\boldsymbol{\beta_n}))) \geq \mathcal{J}(\boldsymbol{\beta_n})$ and $\mathrm{s}\cdot\|\mathcal{P}(\boldsymbol{\beta_n} - \mathrm{s}\cdot\nabla\mathcal{J}(\boldsymbol{\beta_n})) - \boldsymbol{\beta_n}\| >$ tol **do**
      s $\leftarrow$ s/2.
    end while
  end while
  The approximation of the minima of $\mathcal{J}$ is $\boldsymbol{\beta_n}$.

experimental measurements for the patient number 1 in the clinical trial. These values are reported in Table 4.1. We also consider that the diffusion coefficient of Calcium and the diffusion coefficient of Citrate are five times smaller than the corresponding diffusion coefficients in water, which corresponds to take $\boldsymbol{\beta} = (0.2, 0.2)$. For this data, eight iterates are sufficient in the Newton method described in Algorithm 2 to reach a precision 10^{-6}. In Figure 4.2 we represent the evolution of the norm between two successive iterations in the Newton method. Similar rate of convergence were observed for all the data considered in this work. We also check that the solution $\boldsymbol{c}$ obtained by this iterative method coincides with the solution obtained once the time-stepping method employed in [7] converged to the stationary solution. An illustration of numerical solution of problem (4.1.9) for the boundary data in Table 4.1 is given in Figure 4.3. In this figure we choose to display only the concentrations of Calcium (c_1) and Citrate (c_4). These solutions have the correct form. Nevertheless the concentration of Calcium in blood at the exit of the dialyser is smaller than expected and the concentration of Citrate is more important than in Table 4.1. Therefore, we assume than the diffusion coefficient $\boldsymbol{\beta} = (0.2, 0.2)$ have its both components too low.

Boundary value	c_1	c_3	c_3	c_4	c_5
on $\Gamma_{\ell,b}$	0.11	3.71602	0.0577928	5.03048	1.37152
on $\Gamma_{r,d}$	1.25	0	0	0	0
on $\Gamma_{r,b}$	0.96	3.577187	0.48553	0.144108	0.342892

Table 4.1: Concentrations (in $\mathrm{mol}\cdot\mathrm{m}^{-3}$) of the five chemical species in blood (boundary $\Gamma_{\ell,b}$) and dialysate (boundary $\Gamma_{r,d}$) at the entrance of the dialyser and in blood at the exit of dialyser (boundary $\Gamma_{r,b}$) for the patient number 1.

Once the method for the computation of the concentrations $\boldsymbol{c}$ at the exit of the dialyser is validated, we apply the procedure described in Section 4.1.2 to optimize the values of the diffusion coefficients $\boldsymbol{\beta}$ for the boundary input and output data recorded for the patient number 1 and reported in Table 4.1. We choose $\boldsymbol{\beta_0} = (0.2, 0.2)$ and then we employ Algorithm 3 to minimize the functional

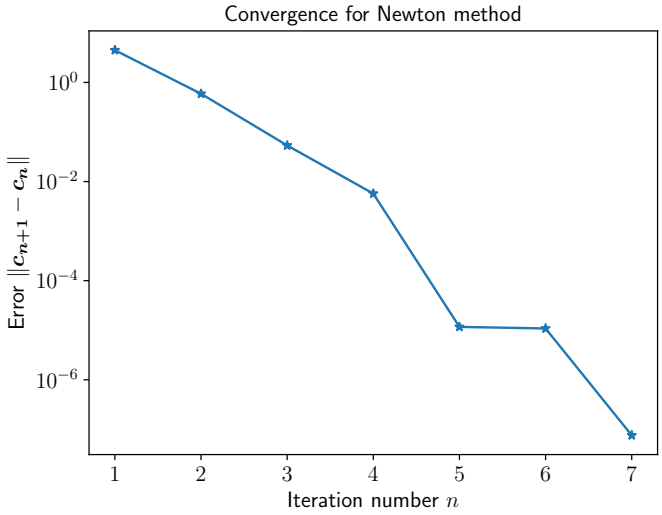


Figure 4.2: Evolution of the error in Newton algorithm described in Algorithm 2 for boundary data in Table 4.1.

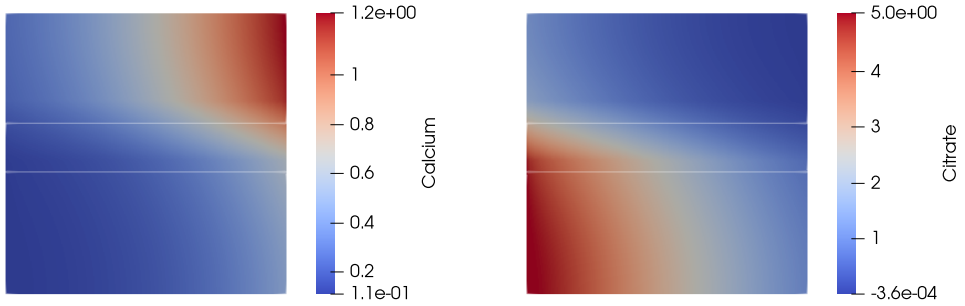


Figure 4.3: Components c_1 (corresponding to Calcium) and c_4 (corresponding to Citrate) of the numerical approximation of the solution $\boldsymbol{c}$ computed using Newton's method for the initial data in Table 4.1 and $\boldsymbol{\beta} = (0.2, 0.2)$.

$\mathcal{J}$. We repeat the experiment with $\boldsymbol{\beta_0} = (0.4, 0.8)$. The successive values of $\mathcal{J}(\boldsymbol{\beta_n})$ in this gradient descent algorithm are displayed in Figure 4.4. Starting from $\boldsymbol{\beta_0} = (0.2, 0.2)$ the Algorithm 3 converge to the local minimum $(0.985295, 0.792707)$ and starting from $\boldsymbol{\beta_0} = (0.4, 0.8)$ a local minimum of the functional $\mathcal{J}$ is reached around $(0.522323, 0.814407)$. As seen in Figure 4.4, the values of J in these local minima are very close. In Figure 4.5 we display the concentration of Calcium and the concentration of Citrate obtained as the solution of (4.1.9) for boundary data in Table 4.1 and $\boldsymbol{\beta} = (0.985295, 0.792707)$. Since the diffusion coefficients were optimized to fit these data the concentrations in blood at the exit of the dialyser obtained from the simulations are very close to the experimental data.

The simplest way to compare the results of the numerical simulations to the experimental data is to look at the values of the functional $\mathcal{J}$ given by (4.1.11). Indeed, the functional $\mathcal{J}$ is nothing else that a relative error between $\mathcal{B}_{\boldsymbol{\alpha}}(\overline{\boldsymbol{c}_{\text{input}}})$ and $\overline{\boldsymbol{c}_{\text{output}}}$. In Figure 4.6 we compare the values of $\mathcal{J}$ for the diffusion coefficients $\boldsymbol{\beta} = (0.522323, 0.814407)$ and $\boldsymbol{\beta} = (0.985295, 0.792707)$, obtained by optimizing $\mathcal{J}$ for the data corresponding to the patient 1, for all the patients included in the clinical study. We removed from out analysis the patients number 14 and 19 for which we suspect

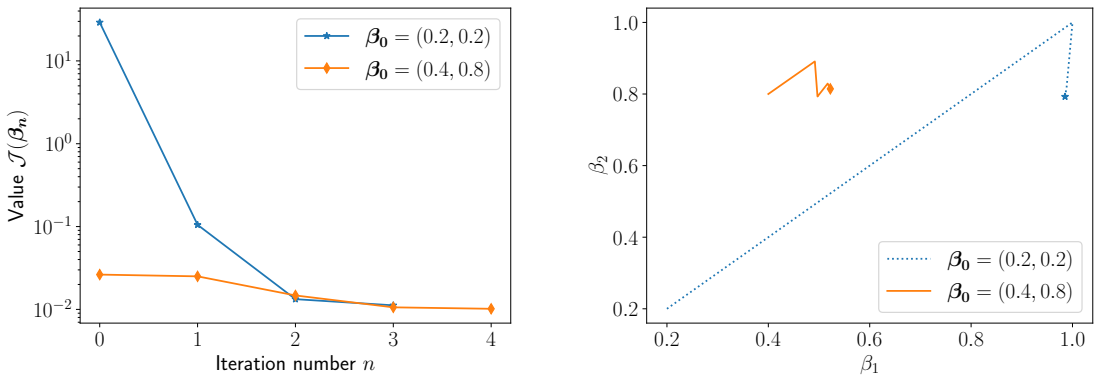


Figure 4.4: Evolution of $\mathcal{J}(\boldsymbol{\beta_n})$ along the iterates of the gradient descent method described in Algorithm 3. (b) Values $\boldsymbol{\beta_n}$ along the iterates of the Algorithm 3 for two different initializations.

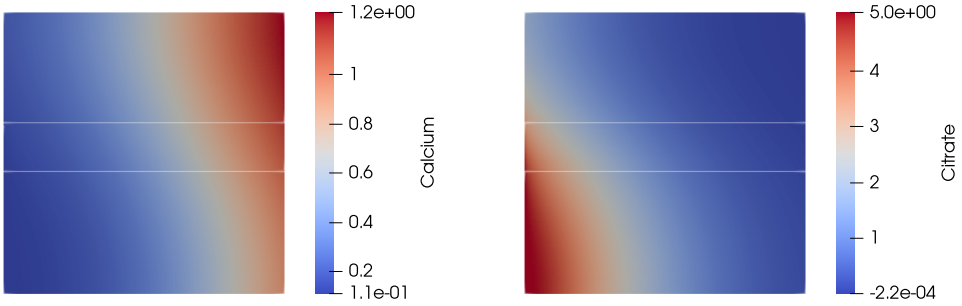


Figure 4.5: Components c_1 (corresponding to Calcium) and c_4 (corresponding to Citrate) of the numerical approximation of the solution $\boldsymbol{c}$ computed using Newton's method for the initial data in Table 4.1 and $\boldsymbol{\beta} = (0.2, 0.2)$.

some experimental errors. By the continuous line we represent the average of the values of $\mathcal{J}(\boldsymbol{\beta})$ for the two considered values of $\boldsymbol{\beta}$. We observe that the value of this average is around 0.5 and that is slightly smaller for $\boldsymbol{\beta} = (0.522323, 0.814407)$. These values are much smaller that the one obtained for $\boldsymbol{\beta} = (0.2, 0.2)$ for which the average value of $\mathcal{J}(\boldsymbol{\beta})$ is close to 86. Nevertheless, the values of $\mathcal{J}(\boldsymbol{\beta})$ are much smaller for the patient number one that for the other patients in the study.

4.1.4 Conclusion and perspectives

This work proposed two main developments with respect to the previous study [7]. The first novelty is that we propose to directly compute the equilibrium solution by solving directly, by a Newton method, the non-transient system associated to the reaction-convection-diffusion inside a dialyser's fiber. In this way, the computations are much faster and more precise than solving in time the initial evolution system considered in [7]. This gain in the computational time allows us to optimize the diffusion coefficients into the membrane in order to fit the data for one patient, which is the second novelty of this paper.

Since the functional $\mathcal{J}$ (associating to the diffusion coefficients into membrane the relative error between the numerical solution and the corresponding experimental data) is not necessary convex, the minimization problem has not an unique solution. The optimized diffusion coefficients provide

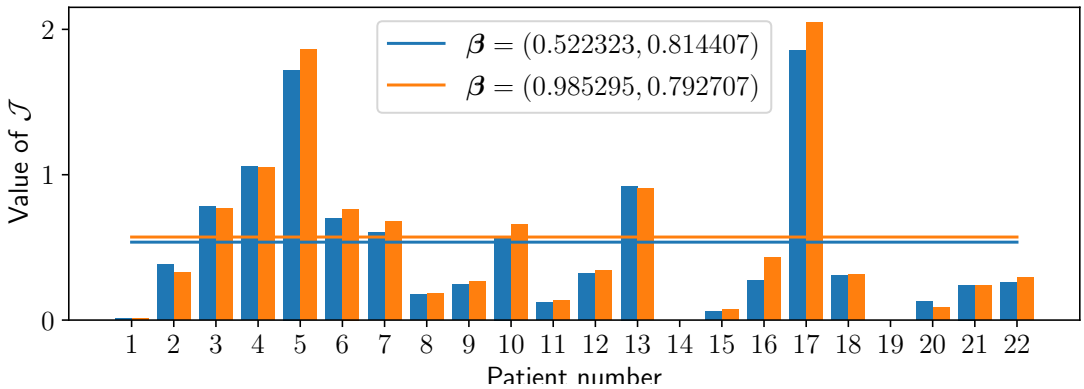


Figure 4.6: Values of $\mathcal{J}(\boldsymbol{\beta})$ for $\boldsymbol{\beta} = (0.522323, 0.814407)$ and $\boldsymbol{\beta} = (0.985295, 0.792707)$ and data from patients in the study.

better results that the coefficients from the literature. Nevertheless, since there is a big difference between the results for the data used to optimize the coefficients and all the other data (recorded from 22 patients), we conclude that the bad known diffusion coefficients cannot entirely explain the differences between the numerical results and the experimental data. A better understanding to exactly what terms into the mathematical model correspond the experimental data and a corresponding adaptation of the model is necessary to further ameliorate the results.

4.2 FreeFEM algorithm code

In this section, we present the FreeFEM code associated with algorithm 3.

```
1
2  NoUseOfWait = true;
3  include "hydro.edp" // This program is used to simulate flow in the dialyzer
4  NoUseOfWait = false;
5
6  // Pe is an adimensional number called the Peclet number
7  Pe = R^2 * V / (L * D);
8
9  // Fd is the coefficient used to scale the equation of the flow to the level of chemical
       ↪ interactions
10 Fd = V / (L * k2);
11
12 // BMD is a macro used to define functions whith different values in Blood, Membrane and
       ↪ Dialysate
13
14 macro BMD(b, m, d)((0 <= y && y <= 1*R1/R) ? b : 0)
15                  + ((1*R1/R < y && y < R2/R) ? m : 0)
16                  + ((R2/R <= y && y <= 1) ? d : 0)// EOM BMD
17
18 real cht = 1, chte = 1;
19
20 for (int ii = 0; ii <= 4; ii++){
21     bloodcon[ii] = 1 / cht * bloodcon[ii];
22     dialcon[ii] = 1 / cht * dialcon[ii];
```

```
23    }
24
25  /*
26   * Here we define initial concentrations in blood, membrane and dialysate
27   */
28  real difCa = getARGV("-difCa",1.0);
29  real difCi = getARGV("-difCi",1.0);
30
31  func ce1 = BMD(bloodcon[0], dialcon[0], dialcon[0]); // concentration of Calcium
32  func ce2 = BMD(bloodcon[1], dialcon[1], dialcon[1]); // concentration of free Albumin sites
33  func ce3 = BMD(bloodcon[2], dialcon[2], dialcon[2]); // concentration of Calcium-Albumin
        ↪ sites
34  func ce4 = BMD(bloodcon[3], dialcon[3], dialcon[3]); // concentration of Citrate
35  func ce5 = BMD(bloodcon[4], dialcon[4], dialcon[4]); // concentration of Calcium-Citrate
36
37  func D1 = BMD(16.6, difCa*16.6, 16.6); // diffusion coefficent of Calcium
38  func D2 = BMD(0.877, 1e-14, 1e-14); // diffusion coefficient of free Albumin sites
39  func D3 = BMD(0.877, 1e-14, 1e-14); // diffusion coefficient of Calcium-Albumin sites
40  func D4 = BMD(7.67, difCi*7.67, 7.67); // diffusion coefficient of Citrate
41  func D5 = BMD(7.67, difCi*7.67, 7.67); // diffusion coefficient of Ca-Citrate Calcium-Citrate
42
43  func S1 = BMD(1.0, 1.0, 1.0); // transport coefficient for Calcium
44  func S2 = BMD(1.0, 0, 0); // transport coefficient for free Albumin sites
45  func S3 = BMD(1.0, 0, 0); // transport coefficient for Calcium-Albumin sites
46  func S4 = BMD(1.0, 1.0, 1.0); // transport coefficient for Citrate
47  func S5 = BMD(1.0, 1.0, 1.0); // transport coefficient for Calcium-Citrate
48
49
50  // ** Description of the boundary of the domain (0, 1) x (0, 1)
51  border C0(t = 0, 1) {x = t; y = 0; label = 10;}
52  border C1(t = 0, R1 / R) {x = 1; y = t; label = 11;}
53  border C2(t = R1 / R, R2 / R) {x = 1; y = t; label = 12;}
54  border C3(t = R2 / R, 1) {x = 1; y = t; label = 13;}
55  border C4(t = 1, 0) {x = t; y = 1; label = 14;}
56  border C5(t = 1, R2 / R) {x = 0; y = t; label = 15;}
57  border C6(t = R2 / R, R1 / R) {x = 0; y = t; label = 16;}
58  border C7(t = R1 /R, 0) {x = 0; y = t; label = 17;}
59
60  border C8(t = 1, 0) {x = t; y = R1 / R; label = 18;}
61
62
63  mesh Th = buildmesh(C0(nn) + C1(nn * R1 / R) + C2(nn * (R2 - R1) / R) + C3(nn * (R -
        ↪  R2) / R)
64                      + C4(nn) + C5(nn * (R - R2) / R) + C6(nn * (R2 - R1) / R) + C7(nn
                            ↪ * R1 / R));
65
66
67  mesh ThD = buildmesh(C0(nn) + C1(nn * R1 / R) + C8(nn) + C7(nn * R1 / R));
68
69  // P1 finite elements in space
70
```

```
71  fespace VhC(Th,P1);
72  fespace VhD(ThD, P1);
73
74  VhC ux = VX / V, uy = VY / (eps * V);
75
76  /* ---- DIFCONV ----
77     DIFCONV is a macro for the terms appearing in the variational formulation
78     of the diffusion and convection equation for one chemical species
79     * c the concentration for the step n
80     * phi test function
81     * cold the concentration for the step n - 1
82     * ce is a function for the boundary conditions
83     * Di the diffusion coefficient
84     * Si transport or (glomerular) Sieving coefficient
85  */
86
87  macro DIFCONV(c, phi, ce, Di, Si)int2d(Th)((Di * eps^2 / Pe) * dx(c) * y * dx(phi))
88                    - int1d(Th, 13)((Di * eps^2 / Pe) * dx(c) * y * phi)
89                    + int1d(Th, 17)((Di * eps^2 / Pe) * dx(c) * y * phi)
90                    + int2d(Th)(Di / Pe * y * dy(c) * dy(phi))
91                    + int2d(Th)((ux * dx(c) + Si * uy * dy(c)) * y * phi)
92                    // EOM DIFCONV
93  macro MDIFCONV(c, phi, ce, Di, Si)int2d(Th)((Di * eps^2 / Pe) * dx(c) * y * dx(phi))
94    +int1d(Th, 13)((Di * eps^2 / Pe) * dx(c) * y * phi)
95    - int1d(Th, 17)((Di * eps^2 / Pe) * dx(c) * y * phi)
96    - int2d(Th)(Di / Pe * y * dy(c) * dy(phi))
97    - int2d(Th)((ux * dx(c) + Si * uy * dy(c)) * y * phi)
98    // EOM MDIFCONV
99
100 VhC cinc1, cinc2, cinc3, cinc4, cinc5; // concentrations increments
101 VhC phi1, phi2, phi3, phi4, phi5; // test functions
102 VhC c1, c2, c3, c4, c5; // concentrations to find
103
104 real al = R1 / R;
105
106 problem INCREMENT([cinc1,cinc2, cinc3, cinc4, cinc5],[phi1, phi2, phi3, phi4, phi5], solver=
        ↪ sparsesolver)=
107   DIFCONV(cinc1, phi1, ce1, D1, S1)
108   +DIFCONV(cinc2, phi2, ce2, D2, S2)
109   +DIFCONV(cinc3, phi3, ce3, D3, S3)
110   +DIFCONV(cinc4, phi4, ce4, D4, S4)
111   +DIFCONV(cinc5, phi5, ce5, D5, S5)
112   + int2d(Th)(1./Fd * delta1 * c2 * cinc1 * y * phi1)
113   + int2d(Th)(1./Fd * delta3 * c4 * cinc1 * y * phi1)
114   + int2d(Th)(1./Fd * delta1 * c1 * cinc2 * y * phi1)
115   - int2d(Th)(1./Fd * cinc3 * y * phi1)
116   + int2d(Th)(1./Fd * delta3 * c1 * cinc4 * y * phi1)
117   - int2d(Th)(1./Fd * delta2 * cinc5 * y * phi1)
118   + int2d(Th)(1./Fd * delta1 * c2 * cinc1 * y * phi2)
119   + int2d(Th)(1./Fd * delta1 * c1 * cinc2 * y * phi2)
120   - int2d(Th)(1./Fd * cinc3 * y * phi2)
```

```
  - int2d(Th)(1./Fd * delta1 * c2 * cinc1 * y * phi3)
  - int2d(Th)(1./Fd * delta1 * c1 * cinc2 * y * phi3)
  + int2d(Th)(1./Fd * cinc3 * y * phi3)
  + int2d(Th)(1./Fd * delta3 * c4 * cinc1 * y * phi4)
  + int2d(Th)(1./Fd * delta3 * c1 * cinc4 * y * phi4)
  - int2d(Th)(1./Fd * delta2 * cinc5 * y * phi4)
  - int2d(Th)(1./Fd * delta3 * c4 * cinc1 * y * phi5)
  - int2d(Th)(1./Fd * delta3 * c1 * cinc4 * y * phi5)
  + int2d(Th)(1./Fd * delta2 * cinc5 * y * phi5)

  + int2d(Th)(exp(-1000*(y-R1/R+0.02)^2) * al / Pe * cinc2 * uy * phi2)
  + int2d(Th)(exp(-1000*(y-R1/R+0.02)^2) * al / Pe * cinc3 * uy * phi3)
  - int2d(Th)(exp(-1000*(y-R1/R+0.02)^2) * al * D2 / Pe * dy(c2) * phi2)
  - int2d(Th)(exp(-1000*(y-R1/R+0.02)^2) * al * D3 / Pe * dy(c3) * phi3)

  -MDIFCONV(c1, phi1, ce1, D1, S1)
  -MDIFCONV(c2, phi2, ce2, D2, S2)
  -MDIFCONV(c3, phi3, ce3, D3, S3)
  -MDIFCONV(c4, phi4, ce4, D4, S4)
  -MDIFCONV(c5, phi5, ce5, D5, S5)
  - int2d(Th)(1. / Fd * delta1 * c1 * c2 * y * phi1)
  - int2d(Th)(1. / Fd * delta3 * c1 * c4 * y * phi1)
  + int2d(Th)(1. / Fd * c3 * y * phi1)
  + int2d(Th)(1. / Fd * delta2 * c5 * y * phi1)
  - int2d(Th)(1. / Fd * delta1 * c1 * c2 * y * phi2)
  + int2d(Th)(1. / Fd * c3 * y * phi2)
  + int2d(Th)(1. / Fd * delta1 * c1 * c2 * y * phi3)
  - int2d(Th)(1. / Fd * c3 * y * phi3)
  - int2d(Th)(1. / Fd * delta3 * c1 * c4 * y * phi4)
  + int2d(Th)(1. / Fd * delta2 * c5 * y * phi4)
  + int2d(Th)(1. / Fd * delta3 * c1 * c4 * y * phi5)
  - int2d(Th)(1. / Fd * delta2 * c5 * y * phi5)
  + on(13, 17, cinc1 = 0)
  + on(13, 17, cinc4 = 0)
  + on(13, 17, cinc5 = 0)
  + on(13, 17, cinc2 = 0)
  + on(13, 17, cinc3 = 0);


/*
* Initializing the parameters used in the loop
*/

real pasi = 0.5 ; // initial step
real h = 0.02 ; // gradient discretization step
real epsilon = 0.001 ; // convergence step
macro NEWTON(IND)
  c1 = ce1;
  c2 = ce2;
  c3 = ce3;
  c4 = ce4;
```

```
  c5 = ce5;

  i = 0;

  errnewton = 1e10;
  mbo1[i] = 2 * R^2 / R1^2 * int1d(Th, 11)(c1 * y);
  mbo2[i] = 2 * R^2 / R1^2 * int1d(Th, 11)(c2 * y);
  mbo3[i] = 2 * R^2 / R1^2 * int1d(Th, 11)(c3 * y);
  mbo4[i] = 2 * R^2 / R1^2 * int1d(Th, 11)(c4 * y);
  mbo5[i] = 2 * R^2 / R1^2 * int1d(Th, 11)(c5 * y);
  while (i < nmax && errnewton > 1e-6){

    INCREMENT ;

    cinc2 = (0 <= y && y <= R1/R) ? cinc2 : 0.0;
    cinc3 = (0 <= y && y <= R1/R) ? cinc3 : 0.0;
    c1 = c1 - cinc1;
    c2 = c2 - cinc2;
    c3 = c3 - cinc3;
    c4 = c4 - cinc4;
    c5 = c5 - cinc5;
    i += 1;
    mbo1[i] = 2 * R^2 / R1^2 * int1d(Th, 11)(c1 * y);
    mbo2[i] = 2 * R^2 / R1^2 * int1d(Th, 11)(c2 * y);
    mbo3[i] = 2 * R^2 / R1^2 * int1d(Th, 11)(c3 * y);
    mbo4[i] = 2 * R^2 / R1^2 * int1d(Th, 11)(c4 * y);
    mbo5[i] = 2 * R^2 / R1^2 * int1d(Th, 11)(c5 * y);

    errnewton = ((mbo1[i]-mbo1[i-1])^2+
                 (mbo2[i]-mbo2[i-1])^2+
                 (mbo3[i]-mbo3[i-1])^2+
                 (mbo4[i]-mbo4[i-1])^2+
                 (mbo5[i]-mbo5[i-1])^2)^0.5;
    err[i] = errnewton;
  }
tableau#IND[0] = cht*mbo1[i];
tableau#IND[1] = cht*chte*mbo2[i];
tableau#IND[2] = cht*chte*mbo3[i];
tableau#IND[3] = cht*mbo4[i];
tableau#IND[4] = cht*mbo5[i];

// END OF MACRO

/*
*Initializing arrays using Newton methods
*/

real[int] tableau1(5);
real[int] tableau2(5);
real[int] tableau3(5);
real[int] tableau4(5);
```

```
real[int] tableau5(5);
real[int] tableau6(5);



int n = 0 ;
bool convergence = false ;
real difCaO ; // optimal diffusion coefficient for calcium to be determined
real difCiO ; // optimal diffusion coefficient for citrate to be determined
real pas = pasi;

int i = 0; // iterator

int nmax = 1000;
real[int] mbo1(nmax);
real[int] mbo2(nmax);
real[int] mbo3(nmax);
real[int] mbo4(nmax);
real[int] mbo5(nmax);
real[int] err(nmax);
real errnewton;

while (!convergence){
  pas = pasi;
  difCa = difCa + h ;
  NEWTON(1)

  difCa = difCa - h ;
  difCi = difCi + h;
  NEWTON(2)

  difCi = difCi - h ;
  NEWTON(4)

  real[int] test1(5);
  test1 = tableau1-cible;
  test1 = test1 ./ cible;
  real[int] test2(5);
  test2 = tableau2-cible;
  test2 = test2 ./ cible;
  real[int] test4(5);
  test4 = tableau4-cible;
  test4 = test4 ./ cible;
  real gradCa = ((test1.l2)^2 - (test4.l2)^2)/h;
  real gradCi = ((test2.l2)^2 - (test4.l2)^2)/h;

  real difCab = difCa;
  real difCib = difCi;

  difCa = difCa - pas*gradCa;
  difCi = difCi - pas*gradCi;
```

```
  if (difCa > 1){
    difCa = 1;
  }
  if (difCi > 1){
    difCi = 1;
  }
  NEWTON(5);

  real[int] test5(5);
  test5 = tableau5 - cible;
  test5 = test5 ./ cible;
  real a = test5.l2 ;
  real b = test4.l2 ;

  int j = 1 ;

  while (a > b && pas > epsilon){
    difCa = difCab;
    difCi = difCib;
    pas = 0.5 * pas ;

    difCa = difCa - pas*gradCa;
    difCi = difCi - pas*gradCi;
    if (difCa > 1){
      difCa = 1;
    }
    if (difCi > 1){
      difCi = 1;
    }
    NEWTON(5)

    test5 = tableau5 - cible;
    test5 = test5 ./ cible;
    a = test5.l2 ;
    j += 1;
  }
  if (pas > epsilon && abs(a - b)>epsilon){
    n += 1;
 }
 else {
   convergence = true;
 }
}
```

Tout au loin mes yeux voyant chaque résultat, réussite revenir à sa source, son nid son gîte, dans l'obligation parfaite,
Tout au loin les heures, les mois, les ans – au loin les commerces, les traités, les établissements, dans leurs moindres détails,
Tout au loin la vie quotidienne, la parole, les ustensiles, la politique, les richesses, les personnes;
Tout au loin nous-mêmes, moi mes feuilles mes chants, pleins de foi, d'admiration,
Ressemblant à un père qui va vers son propre père, il conduit ses enfants par la main.

Walt Whitman, *Savantisme* (Feuilles d'herbe).

Some tools from functional analysis

A.1 Some classical results: miscellaneous

In this appendix, we show some of the classic results we use throughout the dissertation.

A.1.1 Aubin-Lions theorem

In this section, we present a result that we use regularly in this dissertation, and which allows us to obtain compactness.

Theorem A.1.1 (Aubin-J.L.Lions). *Let us consider three Banach space X_1, X, and X_2 such that X is continuously embedded into X_2, and X_1 is compactly embedded into X. We consider, for $1 \leq p, r \leq +\infty$, the set $Y_{p,r}$ given by:*

$$Y_{p,r} := \left\{ u \in L^p((0,T), X_1) \,/\, \frac{du}{dt} \in L^r((0,T), X_2) \right\},$$

which is a Banach space. Then:

1. *If $p < +\infty$, $Y_{p,r}$ is compactly embedded into $L^p((0,T), X)$.*
2. *Otherwise if $p = +\infty$ and $r > 1$, $Y_{p,r}$ is compactly embedded into $C((0,T), X)$.*

The proof of this theorem is not trivial. Basically, it boils down to showing that we can extract a Cauchy sequence in the space in which we aim to show the wished compact embbedding, via the following interpolation lemma :

Lemma A.1.2 (J.L.Lions). *Let us consider three Banach space X_1, X, and X_2 such that X is continuously embedded into X_2, and X_1 is compactly embedded into X. Then, for every $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$, such that for all $u \in X_1$:*

$$\|u\|_X \leq \varepsilon \|u\|_{X_1} + C(\varepsilon) \|u\|_{X_2}.$$

Proof. See [132, Lemme 5.1.]. □

The preceding lemma then shows that it is in fact sufficient to construct a Cauchy sequence in $L^p((0,T),X_2)$ if $p < +\infty$ and in $C((0,T),X_2)$ if $p = +\infty$. To construct such a sequence, we decompose a sequence of elements of X and show that it converges in the right space, by applying Ascoli's theorem. To show that it is Cauchy, we proceed by diagonal extraction. We refer to [31, Theorem 2.5.16.] for a detailed proof.

A.1.2 De Rham theorem

One of the main difficulties encountered a priori in fluid mechanics is determining the pressure term, which is an unknown. In the case of the whole space, pressure can be expressed as a function of the velocity field, as long as it is possible to define a Leray projector: to see this, formally take the divergence on the incompressible system, and invert the Laplacian. We refer the interested reader to [52] for a detailed discussion of this subject.

In the case of a bounded domain, this seems a priori more delicate. Having said that, it is possible to determine the pressure using the structure of functional spaces, via De-Rham's theorem.

Theorem A.1.3 (De Rham)**.** *Let us consider a Lipschitz domain Ω of $\mathbb{R}^N$ and $f \in H^{-1}(\Omega)$ such that for every $\varphi \in \mathcal{D}_\sigma(\Omega)$, we have:*

$$\langle f, \varphi \rangle_{-1,1} = 0.$$

Then there exists $\pi \in L^2(\Omega)$, unique up to an additive constant, such that:

$$f = \nabla \pi \quad \text{in } H^{-1}(\Omega).$$

Proof. See [31, Theorem IV.2.4.]. □

Depending on the regularity available, the previous theorem is not always applicable. There are many variants of this result, depending on the regularity assumptions on f, which lead to different regularities for π. However, the following result is generally applicable.

Theorem A.1.4 (De Rham (distributions))**.** *Let Ω be a domain of $\mathbb{R}^N$ and $f \in \mathcal{D}'(\Omega)$ such that for every $\varphi \in \mathcal{D}_\sigma(\Omega)$, we have:*

$$\langle f, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = 0.$$

Then there exists $\pi \in \mathcal{D}'(\Omega)$, unique up to an additive constant, such that:

$$f = \nabla \pi \quad \text{in } \mathcal{D}'(\Omega).$$

Proof. See [162]. □

A.2 Divergence free spaces

In this section we introduce function spaces with zero divergence in the sense of distributions, which appear naturally in mathematical analysis in fluid mechanics. We pose for $\Omega \subset \mathbb{R}^N$ an open subset:

$$\mathcal{D}_\sigma(\Omega) := \{\varphi \in \mathcal{D}(\Omega) \ / \ \operatorname{div}(\varphi) = 0\},$$

and we sometimes note $\mathcal{D}_\sigma(\Omega) = C^\infty_{0,\sigma}(\Omega)$. For $k \geq 0$, $p \geq 1$, we define the space $W^{k,p}_\sigma(\Omega)$ as being the completion of $\mathcal{D}_\sigma(\Omega)$ into $W^{k,p}(\Omega)$ for the induced norm, $W^{k,p}_{0,\sigma}(\Omega)$, as well as the completion of

$\mathcal{D}_\sigma(\Omega)$ in $W_0^{k,p}(\Omega)$ and denote $W_\sigma^{-1,q}(\Omega)$ the dual of the latter. Naturally, if we argue by density, the usual injections are still valid in these spaces.

We observe, reasoning by density, that these spaces correspond, taking L^2 as the pivot space, to the subspace of $W_\sigma^{k,p}(\Omega)$ (respectively of $W_{0,\sigma}^{k,p}(\Omega)$ having zero divergence in the weak sense, since we can then write for $u_n \in \mathcal{D}(\Omega)$ which converges to $u \in L^2_\sigma(\Omega)$, for $\varphi \in \mathcal{D}(\Omega)$:

$$\int_\Omega u_n \cdot \nabla\varphi \, dx = -\int_\Omega \operatorname{div}(u_n)\varphi \, dx = 0 \underset{n\to+\infty}{\longrightarrow} \int_\Omega u \cdot \nabla\varphi \, dx = 0,$$

hence the assertion.

A.3 Deformation spaces

As we saw when modeling fluid mechanics equations, the naturally occurring tensor is the strain rate tensor (its linear counterpart), which corresponds mathematically to the symmetrized gradient. While in the case of Newtonian flow, this easily becomes a gradient, this is no longer the case in most of the other cases we consider. In fact, as we shall see in this section, if we consider spaces of functions with deformations in L^p, this coincides in most cases with a usual Sobolev space.

Definition A.3.1 (Space of functions with deformations in L^p)**.** *Let us consider $\Omega \subseteq \mathbb{R}^N$ and $1 \leq p < +\infty$. Then the space $Y_p(\Omega)$, defined as*

$$Y_p(\Omega) := \{u \in L^p(\Omega) / \ D(u) \in L^p(\Omega)\}$$

is called the space of functions with deformations in L^p. In the case $p = 1$, we talk about the Lebesgue deformations space, and we denote this last $LD(\Omega)$.

The following result holds.

Proposition A.3.2. *Let us consider $\Omega \subseteq \mathbb{R}^N$ and $1 \leq p < +\infty$. Then:*

(i) The space $Y_p(\Omega)$ is a Banach space.

(ii) The space $Y_p(\Omega)$ is a local space,in the sense that it satisfies

$$\forall \varphi \in C_0^\infty(\Omega), \ \forall u \in Y_p(\Omega), \ \varphi u \in Y_p(\Omega).$$

(iii) If Ω has a C^1 boundary, then $C^\infty(\Omega) \cap Y_p(\Omega)$ is dense into $Y_p(\Omega)$.

Proof. We refer the interested reader to [68, Section 6.7.]. □

The following theorem is a major result.

Theorem A.3.3 (Korn inequality)**.** *Let Ω be a domain of $\mathbb{R}^N$ and $1 < p < +\infty$. Then, if $u \in Y_p(\Omega)$, we have that $u \in W^{1,p}_{\text{loc}}(\Omega)$.*

Moreover, if Ω is bounded with C^2 boundary, we have that if $u \in Y_p(\Omega)$, then $u \in W^{1,p}(\Omega)$. More precisely, the following inequality holds:

$$\exists C_p > 0, \ \|\nabla u\|_{L^p(\Omega)} \leq C_p \left(\|u\|^p_{L^p(\Omega)} + \|D(u)\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} := C_p \|u\|_{Y_p(\Omega)}.$$

Proof. The proof is highly technical in the general case, then we refer the reader to [68, Theorem 7.71.]. □

An interesting implication of this last result will not have escaped the reader, namely that in the case of a regular bounded domain and for $1 < p < +\infty$, the space of functions with deformations L^p is none other than a Sobolev space.

Corollary A.3.4. *Let us consider Ω a bounded domain of $\mathbb{R}^N$ with C^2 boundary, and $1 < p < +\infty$. Then:*

$$Y_p(\Omega) = W^{1,p}(\Omega).$$

Proof. We first have, from triangle inequality, that $Y_p(\Omega) \subseteq W^{1,p}(\Omega)$. Korn inequality leads to the inverse inclusion and the result follows. □

Remark A.3.5. *It follows from Corollary A.3.4 that the notions of trace, as well as embbeddings, of Sobolev spaces are valid on $Y_p(\Omega)$ spaces. For a reason that we shall now explain, however, it is pertinent to question the validity of this assertion in the case of the space $LD(\Omega)$ corresponding to the case $p = 1$.*

Let us point out that Korn inequality as presented in Theorem A.3.3 does not hold in the borderline case $p = 1$. We refer to [149, 57] to some counterexamples. There are also many variants and generalizations of Korn inequality, depending on the spaces considered. In particular, Korn inequality is valid on certain classes of fractional Sobolev spaces (see, for example, [107]), weighted spaces (see [38]), or in a fruitful framework for compressible fluid mechanics (see, for example, [86, Proposition 2.1.]).

A.3.1 Case $p = 1$, Lebesgue deformations space

As we listed earlier, it is relevant to study the case of space $LD(\Omega)$ in a way that is unique to it. Indeed, since Korn inequality is not valid for $p = 1$, we don't have $LD(\Omega) = W^{1,1}(\Omega)$ and, consequently, there is no reason to conclude that the results of injections or traces, for example, can be transposed just as easily to this framework. It is possible to show that this is in fact the case, as the following proposition establishes.

Proposition A.3.6 (Properties of $LD(\Omega)$)**.** *Let us consider Ω an open subset of $\mathbb{R}^N$. Then, the following statements hold.*

(i) If $\partial\Omega$ is of class C^1, then there exists $\gamma_0 \in \mathcal{L}(LD(\Omega), L^1(\partial\Omega))$ which is onto. This application is called the trace and coincides with the usual trace operator when defined in the classical sense, namely on $LD(\Omega) \cap C(\overline{\Omega})$ or $W^{1,1}(\Omega)$.

(ii) If $\partial\Omega$ is Lipschitz, then $LD(\Omega)$ is compactly embbedded into $L^q(\Omega)$ for $q < \frac{N}{N-1}$, and continuously embbedded when $q \leq \frac{N}{N-1}$.

Proof. See [68, Theorem 6.92.,Theorem 6.95.,Theorem 6.99.]. □

A.3.2 Bounded deformations space

One question that then arises is what happens to functions with bounded variations. If this is not the case, what about the properties of $BV(\Omega)$ space? Are they, in a prescribed sense, substitutable for the space of functions with bounded deformations? This is the subject of this subsection.

Definition A.3.7 (Bounded deformations space)**.** *Let Ω be an open subset of $\mathbb{R}^N$. Then, the space*

$$BD(\Omega) := \left\{ u \in L^1(\Omega) / \; VD(u,\Omega) < +\infty \right\}$$

where

$$VD(u,\Omega) := \sup \left\{ k \in \mathbb{R} / \; k = \int_\Omega D(u) : \varphi \, dx, \; \varphi \in C_0^1(\Omega), \; \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\}$$

is called the space of bounded deformations functions.

As a reminder, let us emphasize that the space $\mathcal{M}^1(\Omega)$ is the space of bounded measures, that is:

$$\mathcal{M}^1(\Omega) := \left\{ \varphi \in C_0^0(\Omega) / \; \exists K \subset c_0(\Omega), \; \varphi \in C_0^0(\Omega), \, \mathrm{supp}(\varphi) \subset K \Rightarrow |\langle \mu, \varphi \rangle| \leq C \|\varphi\|_{L^\infty(\Omega)} \right\}^* \tag{A.3.1}$$

where in the previous expression, $c_0(\Omega)$ denotes the space of the compact subsets of Ω, and the constant C does not depend of K (otherwise it would define the space $\mathcal{M}^1(\Omega)$). We recall the following definition.

Definition A.3.8 (Tight convergence of measures)**.** *Let us consider an open subset Ω of $\mathbb{R}^N$, and $(\mu_n)_{n\in\mathbb{N}}$ a sequence of $\mathcal{M}^1(\Omega)$. Then, we say that $(\mu_n)_{n\in\mathbb{N}}$* **converges tightly** *toward $\mu \in \mathcal{M}^1(\Omega)$ if:*

$$\forall \varphi \in C_b(\Omega), \; |\langle \mu_n - \mu, \varphi \rangle| \underset{n\to+\infty}{\longrightarrow} 0.$$

Replacing $C_b(\Omega)$ test functions by $C_0^0(\Omega)$ tests functions in the previous definition leads to the notion of **vague convergence** in $\mathcal{M}^1(\Omega)$.

Let us now present, without proof, the results specific to the expectations we have mentioned.

Proposition A.3.9 (Properties of $BD(\Omega)$)**.** *We consider an open subset Ω of $\mathbb{R}^N$. Then, we have the following properties.*

(i) The space $C^\infty(\Omega) \cap W^{1,1}(\Omega)$ is dense into $BD(\Omega)$ for the tight topology, that is to say that for every $u \in BD(\Omega)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ belonging to $C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that:

$$\begin{cases} u_n \underset{n\to+\infty}{\longrightarrow} u \text{ in } L^1(\Omega), \\ D(u_n) \underset{n\to+\infty}{\rightharpoonup} D(u) \text{ vaguely in } \mathcal{M}^1(\Omega), \\ \displaystyle\int_\Omega |D(u_n)| \, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega |D(u)| \, dx. \end{cases}$$

(ii) If $\partial\Omega$ is Lipschitz, then $BD(\Omega)$ is continuously embbedded into $L^p(\Omega)$ for every $1 \leq p \leq \frac{N}{N-1}$, and compactly embbedded for $1 \leq p < \frac{N}{N-1}$.

(iii) There is no Korn type inequality between $BV(\Omega)$ and $BD(\Omega)$, which means that $BV(\Omega) \neq BD(\Omega)$.

(iv) If $\partial\Omega$ is C^1, then there exists $\gamma_0 \in L(BD(\Omega), L^1(\partial\Omega))$, which is called the trace operator over $BD(\Omega)$. Moreover, γ_0 coincide with the usual notion of traceo ver $W^{1,1}(\Omega)$.

(v) The trace γ_0 is not continuous with respect to the weak topology, but it is with respect to the tight one, in the following sense: if $(u_n)_{n\in\mathbb{N}}$ is a sequence of $BD(\Omega)$ which is tightely

convergent toward $u \in BD(\Omega)$ in the sense that it satisfies:

$$\begin{cases} u_n \underset{n\to+\infty}{\longrightarrow} u \text{ in } L^1(\Omega), \\ \displaystyle\int_\Omega |D(u_n)|\, dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega |D(u)|\, dx, \end{cases}$$

then we have $\gamma_0(u_n) \underset{n\to+\infty}{\longrightarrow} \gamma_0(u)$ in $L^1(\partial\Omega)$.

Proof. We refer to [68, Theorem 6.104.,Corollary 6.105.,Theorem 6.106.,Theorem 6.108., Theorem 6.111.]. □

A.4 Kuratowski convergence

This section is devoted to convergence in the Kuratowski sense (sometimes also known as Painlevé-Kuratowski), which is a mode of convergence in set-valued analysis. Formally, the idea is to make use of the distance between an element of a metric space and a subset or sequence of subsets of it.

Definition A.4.1 (Kuratowski convergence)**.** *Let (X, d) be a metric space and $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of X. For every subset A of X and $x \in X$, we denote $d(x, A) = \inf_{y\in A} d(x, y)$ the distance of x from the subset A. Then, the convergence modes in the sense of Kuratowski are given as follows.*

(i) The upper limit of $(A_n)_{n\in\mathbb{N}}$ is given by:

$$\overline{\lim_{n\to+\infty}} A_n := \left\{ x \in X \;/\; \underline{\lim_{n\to+\infty}} d(x, A_n) = 0 \right\}.$$

(ii) The lower limit of $(A_n)_{n\in\mathbb{N}}$ is given by:

$$\underline{\lim_{n\to+\infty}} A_n := \left\{ x \in X \;/\; \overline{\lim_{n\to+\infty}} d(x, A_n) = 0 \right\}.$$

(iii) If $\underline{\lim}_{n\to+\infty} A_n = \overline{\lim}_{n\to+\infty} A_n$, we say that the limit exists and we denote this last by:

$$\lim_{n\to+\infty} A_n = \underline{\lim_{n\to+\infty}} A_n = \overline{\lim_{n\to+\infty}} A_n.$$

Note that the distance application to a fixed subspace A is Lipschitz continuous, and we have the triangular inequality for all $(x, y) \in X^2$:

$$d(x, A) \leq d(x, y) + d(y, A),$$

and, reversing the roles, it comes:

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

In particular, if $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements of X converging strongly to $x \in X$, it follows that $d(x_n, A) \underset{n\to+\infty}{\longrightarrow} d(x, A)$, and, consequently, if $x \in A$, then $\lim_{n\to\infty} x_n \in A$.

Theorem A.4.2 (Mrowka)**.** *Let X be a metric space and $(A_n)_{n\in\mathbb{N}}$ a sequence of subsets of X. Then, there exists a subsequence $(A_{n_k})_{n_k\in\mathbb{N}}$ of $(A_n)_{n\in\mathbb{N}}$ which convergence in the sense of Kuratowski.*

Proof. See [21, Theorem 5.2.11.]. □

Theorem A.4.2 (which is even valid on Hausdorff spaces) thus ensures that a sequence of sets always converges in Kuratowski's sense, up to an extraction. To rephrase, we can expect a sequence of graphs to converge to a graph, even if it means extracting. Having in mind a regularized solution of a threshold fluid, for example, this leads to the fact that if we have strong convergence, hence regularity on the solution, the sequence of graphs constructed in a Banach space converges to a graph described by a multivalued function at the origin. But what about weak convergence? The following theorem provides an answer to this question.

Theorem A.4.3. *Let us consider a Banach space X, and also a sequence $(x_n)_{n\in\mathbb{N}}$ of X^* which weakly-* converges toward $x \in X^*$. Then the graph of x denoted* $\mathrm{Gr}(x)$ *is the Kuratowski limit of the sequence of the graphs of x_n denoted* $\mathrm{Gr}(x_n)$.

Proof. See [21, Theorem 5.2.19. and Theorem 5.2.17.]. □

Therefore, using a Riesz representation theorem, we deduce that in L^2 (by [32, Proposition III.12.]), graphs described by u_n elements that converge weakly to u do converge to the u graph. Note that Kuratowski convergence on Hausdorff spaces is more delicate, even when considering lower semi-continuous functions. In this case, we need to strengthen the assumptions (see [21, Theorem 5.3.5.]).

For more details on Kuratowski convergence properties, we refer the reader to [21, Chapter 5 and Chapter 6] for a detailed description of this convergence mode and applications to multivalued functions, to [58, Appendix B] for a concise presentation of related properties, and to [124] for a detailed presentation of set-valued analysis.

A.5 Orlicz spaces

Let us look very briefly at the definition and some elementary properties of spaces generalizing L^p spaces. First of all, it is useful to recall a few notions.

Definition A.5.1 (N-function)**.** *An N-function is a function $\Phi : [0,+\infty) \to [0,+\infty)$ such that there exists $\phi : [0,+\infty) \to [0,+\infty)$ satisfying*

$$\forall t \in [0,+\infty),\ \Phi(t) = \int_0^t \phi(s)\, ds,$$

where ϕ verifies:

(i) $\phi(0) = 0$,

(ii) $\forall t \in (0,+\infty),\ \phi(t) > 0$,

(iii) $\phi(t) \underset{t\to+\infty}{\longrightarrow} +\infty$,

(iv) ϕ *is nondecreasing,*

(v) ϕ *is right continuous, namely it satisfies* $\phi(t) \underset{\substack{t\to s\\ t>s}}{\longrightarrow} \phi(s)$.

We then have the following result.

Proposition A.5.2. *Le Φ be an N-function. Then Φ is a continuous increasing convex function. Moreover, it satisfies:*

(i) $\frac{\Phi(t)}{t} \underset{t\to 0}{\longrightarrow} 0$.

(ii) $\frac{\Phi(t)}{t} \underset{t\to +\infty}{\longrightarrow} +\infty$.

(iii) The function $t \mapsto \frac{\Phi(t)}{t}$ is increasing over $(0, +\infty)$.

The proof of Proposition A.5.2 is rather direct and left to the reader.

Example A.5.3. *Let us mention some examples of N-functions.*

- $\Phi : t \mapsto t^p$, $1 < p < +\infty$.
- $\Phi : t \mapsto (1+t)\log(1+t) - t$.
- $\Phi : t \mapsto e^t - t - 1$.

Definition A.5.4 (Equivalence of N-functions)**.** *Let Φ_1 and Φ_2 be two N-functions. Then we say that Φ_2 dominates Φ_1 near infinity if there exists $t_0 \in [0, +\infty)$ and a constant $k > 0$ such that:*

$$\forall t \geq t_0,\ \Phi_1(t) \leq \Phi_2(kt).$$

If $t_0 = 0$, then we say that the dominance is global. When Φ_1 and Φ_2 dominates each other near infinity (respectively globally), then we say that they are equivalent near infinity (respectively globally).

Definition A.5.5 (Conditions over N-functions)**.** *Let Φ be an N-function. Then, we say that Φ satisfies:*

- *The Δ_2-condition near infinity if there exists $k > 0$ and $t_0 \in [0, +\infty)$ such that*

$$\forall t \geq t_0,\ \Phi(2t) \leq k\Phi(t).$$

- *The ∇_2-condition near infinity if there exists $k > 0$ and $t_0 \in [0, +\infty)$ such that*

$$\Phi\left(\frac{t}{2}\right) \leq 2^{-(1+k)}\Phi(t).$$

In both of cases above, when $t_0 = 0$ we say that Φ satifies the condition globally.

We mentioned that Orlicz spaces are a generalization of L^p spaces. Without going into the future role of N-functions in the definition of these spaces, let us just say that it is natural to want to introduce a duality similar to that existing in L^p spaces. This motivates the following definition and results.

Theorem A.5.6 (Complementarity of N-functions)**.** *Let us consider an N-functions Φ associated to ϕ, this last satisfying assumptions (i)-(v) of Definition A.5.1. Then we define the function $\phi^* : [0, +\infty) \to [0, +\infty)$ as being:*

$$\forall t \in [0, +\infty),\ \phi^*(t) = \sup\left\{s \in [0, +\infty) / \ \phi(s) \leq t\right\}.$$

Then, ϕ^* satisfies assumptions (i)-(v) from Definition A.5.1, and so it defines an N-function Φ^*, called the complementary N-function of Φ, given by:

$$\Phi^*(t) = \int_0^t \phi^*(s)\, ds.$$

Moreover, we have:

(i) (Young inequality) $\forall (s,t) \in [0,+\infty)^2,\ st \leq \Phi(s) + \Phi^*(t)$.

(ii) (Legendre transform formulation) $\forall t \in [0,+\infty),\ \Phi^*(t) = \max_{s \geq 0} (ts - \Phi(s))$.

(iii) Φ satisfies the Δ_2-condition if and only if Φ^* satisfies the ∇_2-condition.

Proof. The proof can be found in [3, Sections 8.4-8.6] and [154, Chapter II, Theorem 3]. □

We are now able to give the main definition of this section.

Definition A.5.7 (Orlicz space)**.** *Let Ω be a domain of $\mathbb{R}^N$ and Φ be an N-function. The Orlicz class $K^\Phi(\Omega)$ is the set of all Lebesgue measurable functions u satisfying:*

$$\int_\Omega \Phi(|u|)\, dx < +\infty.$$

The Orlicz space associated to Φ, denoted $L^\Phi(\Omega)$, is the linear hull of $K^\Phi(\Omega)$.

First, let us point out that the letter N in the N-function terminology is not related to the N in $\mathbb{R}^N$, which is the dimension of the space under consideration. Next, the attentive reader will have noticed that we distinguish between the definition of Orlicz class and Orlicz space. This is not without reason: the Orlicz class is not, in general, a vector space, even if it is always a convex set of functions. It is worth asking whether we can establish a condition guaranteeing that the Orlicz class and the Orlicz space coincide.

Proposition A.5.8 (Δ-regularity criterion)**.** *Let Ω be a domain of $\mathbb{R}^N$ and Φ an N-function. Then, the pair (Φ, Ω) is said to be Δ-regular if Φ satisfies a global Δ_2-condition or if it satisfies a Δ_2 condition near infinity and Ω has a finite volume. Moreover, $K^\Phi(\Omega)$ is a vector space if and only if (Φ, Ω) is Δ-regular.*

Proof. See [3, Lemma 8.8]. □

Thus, if the previous criterion is verified, then $K^\Phi(\Omega) = L^\Phi(\Omega)$. Let us now state the first properties of Orlicz spaces.

Theorem A.5.9 (Elementary properties of Orlicz spaces)**.** *Let us assume that Ω is a domain of $\mathbb{R}^N$, and Φ is a given N-function. Then, the space $L^\Phi(\Omega)$, endowed with the* **Luxemburg norm** *defined by*

$$\|u\|_\Phi = \inf \left\{ k \in (0,+\infty) / \ \int_\Omega \Phi\left(\frac{|u|}{k}\right)\, dx \leq 1 \right\}$$

is a Banach space. Moreover, the following Hölder inequality holds:

$$\forall (u,v) \in L^\Phi(\Omega) \times L^{\Phi^*}(\Omega),\ \left| \int_\Omega uv\, dx \right| \leq 2 \|u\|_\Phi \|v\|_{\Phi^*}.$$

Also, if ψ is another N-function, then the continuous embbedding $L^\Phi(\Omega) \hookrightarrow L^\psi(\Omega)$ holds if and only if one of the following assumptions holds:

(i) Φ *dominates* ψ *globally in sense of Definition A.5.4.*

(ii) Φ *dominates* ψ *near infinity and* $|\Omega| < +\infty$.

Proof. We refer to [3, 8.10-8.12]. □

To close this section, we express a simplified duality result in Orlicz spaces, which is nevertheless sufficient for our purposes. We refer to [3, Chapter 8] for more details in a general context.

Theorem A.5.10 (Duality in Orlicz spaces)**.** *Let us consider* Ω *a domain of* $\mathbb{R}^N$ *and an N-function* Φ*. Then, if the couple* (Φ, Ω) *is* Δ*-regular, the dual space of* $L^{\Phi}(\Omega)$ *can be identified to* $L^{\Phi^*}(\Omega)$*. Moreover, the space* $L^{\Phi}(\Omega)$ *is reflexive if and only if both of the pairs* (Φ, Ω) *and* (Φ^*, Ω) *are* Δ*-regular.*

Proof. This is an immediate corollary of [3, Lemma 8.17,Theorem 8.19,Theorem 8.20] and Proposition A.5.8. □

A.6 Useful inequalities

Theorem A.6.1 (Gagliardo-Nirenberg inequality on bounded Lipschitz domain)**.** *Assume that* Ω *is a bounded domain in* $\mathbb{R}^N$ *with Lipschitz boundary. Moreover, assume that there exists a couple* $(q, r) \in [1, +\infty]$*,* $\theta \in [0, 1]$ *and* $(l, k) \in \mathbb{N}^2$ *such that:*

$$\begin{cases} \frac{1}{p} = \frac{k}{N} + \left(\frac{1}{r} - \frac{l}{N}\right)\theta + \frac{1-\theta}{q} \\ \\ \qquad \frac{k}{l} \leq \theta \leq 1. \end{cases}$$

Then, there exists $C := C(k, l, N, r, q, \theta, \Omega) > 0$ *such that the following inequality holds:*

$$\|\nabla^k u\|_{L^p(\Omega)} \leq C \|u\|^{\theta}_{W^{l,r}(\Omega)} \|u\|^{1-\theta}_{L^q(\Omega)}.$$

Proof. See e.g. [147] or [93]. □

Lemma A.6.2. *Assume that* $u \in L^6(\Omega)$*. Then, for all* $r \in (0, 3)$*, the following inequality holds:*

$$\|u\|^r_{L^2(\Omega)} \leq \left(\frac{3-r}{3}\right) \|u\|^{\frac{4r}{3-r}}_{L^{\frac{3}{2}}(\Omega)} + \left(\frac{r}{3}\right) \|u\|^2_{L^6(\Omega)}. \tag{A.6.1}$$

Proof of Lemma A.6.2. First, we write for $s \in (0, 2)$:

$$\|u\|^2_{L^2(\Omega)} := \int_{\Omega} |u|^s |u|^{2-s} \, dx. \tag{A.6.2}$$

Now, passing to the power $\frac{r}{2}$ and using Hölder's inequality in (A.6.2) leads to:

$$\|u\|^r_{L^2(\Omega)} \leq \left(\int_{\Omega} |u|^{sp} \, dx\right)^{\frac{r}{2p}} \left(\int_{\Omega} |u|^{(2-s)q} \, dx\right)^{\frac{r}{2q}}. \tag{A.6.3}$$

Finally, we apply Young's inequality into (A.6.3) to obtain:

$$\|u\|_{L^2(\Omega)}^r \leq \frac{1}{a}\left(\int_\Omega |u|^{sp}\,dx\right)^{\frac{ar}{p}} + \frac{1}{b}\left(\int_\Omega |u|^{(2-s)q}\,dx\right)^{\frac{br}{q}}. \tag{A.6.4}$$

Fixing successively $p = \frac{3}{2s}$, $s = \frac{4}{3}$, and $b = \frac{3}{r}$ leads to (A.6.1), that is the result is proved. We point out that it is necessary to have $r \in (0,3)$ in order to satisfy the necessary conditions in the inequalities used above:

$$\begin{cases} s \in \left(0, \frac{3}{2}\right), \\ r > 0, \\ \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{a} + \frac{1}{b} = 1. \end{cases}$$

□

Lemma A.6.3 (Nirenberg-Strauss inequality)**.** *Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, then there exists a constant $C > 0$ which depends of N and Ω such that for all $u \in W_0^{1,\frac{N}{N-1}}(\Omega)$ the following inequality holds:*

$$\|u\|_{L^{\frac{N}{N-1}}(\Omega)} \leq C\|D(u)\|_{L^1(\Omega)}. \tag{A.6.5}$$

Proof. We refer to [163, Theorem 1]. □

A.7 Miscellaneous convergence results

In this section, we have gathered a number of eclectic results from functional analysis and measure theory which we will use in Chapter 3. We start by presenting a notion that is neighbor to almost everywhere convergence: convergence in measure.

Definition A.7.1 (Convergence in measure)**.** *Consider $T > 0$ and a family of measurable functions $f, f_n : [0,T] \times \mathbb{T}^d \longrightarrow \mathbb{R}$, for $n \geq 1$. We say that (f_n) converges in measure to f if and only if for all $\epsilon > 0$,*

$$\text{meas}\big\{|f_n - f| \geq \epsilon\big\} \longrightarrow 0 \qquad \text{as } n \to +\infty.$$

Remark that convergence in measure is associated to a metric d, defined by

$$d(f,g) := \sum_{n=1} \frac{1}{2^n} \min\Big(1, \text{meas}\big\{|f-g| \geq 2^{-n}\big\}\Big).$$

The next proposition states that the topology of the convergence in measure is complete.

Proposition A.7.2 (Cauchy criterion)**.** *Consider $T > 0$ and a family of measurable functions $f_n : [0,T] \times \mathbb{T}^d \longrightarrow \mathbb{R}$. Then the sequence (f_n) converges in measure to a measurable function f if and only if the following Cauchy criterion is satisfied: for all $\epsilon, \epsilon' > 0$, there exists a rank $N \geq 1$ such that, for all $m > n \geq N$ we have*

$$\text{meas}\big\{|f_n - f_m| \geq \epsilon\big\} \leq \epsilon'.$$

In other words, the sequence (f_n) converges in measure if and only if, for all $\epsilon > 0$, we have

$$\text{meas}\big\{|f_n - f_m| \geq \epsilon\big\} \longrightarrow 0 \qquad \text{as } m > n \geq N \to +\infty.$$

Proof. See [92, Theorem 2.30.]. □

Finally, we gather a few results concerning the links existing between convergence in measure and convergence almost everywhere.

Proposition A.7.3. *Consider $T > 0$ and a family of measurable functions $f_n : [0,T] \times \mathbb{T}^d \longrightarrow \mathbb{R}$, for $n \geq 1$. Then the following statements hold:*

(i) If the sequence (f_n) converges almost everywhere on $[0,T] \times \mathbb{T}^d$, then the sequence converges in measure.

(ii) Conversely, if the sequence (f_n) converges in measure, then there is an extracted sequence which converges almost everywhere on $[0,T] \times \mathbb{T}^d$.

Proof. We refer to [76, Chapter V.13.]. □

A second part of this subsection is concerned with weak convergence, and its relation to almost everywhere or strong convergence.

Proposition A.7.4. *Consider $1 < q, r < +\infty$, and a sequence of functions $f_n : \mathbb{R}_+ \times \mathbb{T}^d \longrightarrow \mathbb{R}$ such that there is convergence*

$$f_n \underset{n\to+\infty}{\rightharpoonup} f \qquad \text{in } L^r(L^q)$$
$$f_n \underset{n\to+\infty}{\longrightarrow} g \qquad \text{a.e.}$$

The both limits must be equal $f = g$.

Proof. The proof is an application of Mazur's lemma (see [32, Corollary 3.8.]). The weak convergence (in the reflexive space $L^r(L^q)$) provides the existence of a sequence of convex combinations of the f_n which converge strongly in that space. More precisely, there exists a set of coefficients $\lambda_n(k) > 0$, with $n \leq k \leq A_n$, $a_n \geq n$, and $n \geq 1$, such that $\sum_k \lambda_n(k) = 1$ and

$$R_n := \sum_{k=n}^{A_n} f_k \longrightarrow f \qquad \text{in } L^r(L^q).$$

Now consider a point $(t,x) \in \mathbb{R}_+ \times \mathbb{T}^d$ such that the convergence $f_n(t,x) \longrightarrow g(t,x)$ holds. Then, by comparing R_n and g at that point, we obtain

$$\begin{aligned} \big|R_n(t,x) - g(t,x)\big| &\leq \sum_{k=n}^{A_n} \lambda_n(k) \big|\rho_k(t,x) - g(t,x)\big| \\ &\leq \sup_{k\geq n} \big|\rho_k(t,x) - g(t,x)\big| \underset{n\to}{\longrightarrow} 0. \end{aligned}$$

We deduce that $R_n \longrightarrow g$ almost everywhere. Since the sequence (R_n) converges almost everywhere to f up to an extraction, uniqueness of the pointwise limit gives $f = g$. □

Proposition A.7.5. *Consider a uniformly convex Banach space X and a sequence (f_n) of X functions such that we have*

$$f_n \rightharpoonup f \qquad \text{in } X$$

for some $f \in X$ and

$$\|f\|_X \longrightarrow \|f\|_X.$$

Then the sequence converges in the norm topology of X.

Proof. See [176, Lemma 3.1.6.]. In the specific case of L^p spaces, it can be proved using the uniform convexity of such spaces (Clarkson's inequalities), see [31, Proposition II.2.32.] for details. □

A.8 About the closed range theorem

In this section, we focus on the closed range theorem, whose statement we recall.

Theorem A.8.1 (Closed range Theorem)**.** *Let us consider X and Y two real Banach spaces, and let $A : X \to Y$ be a linear operator such that its domain $D(A)$ is dense in X. Then, the following statements are equivalent.*

(i) $R(A)$ *is closed.*

(ii) $R(A^*)$ *is closed.*

(iii) $R(A) = N(A^*)^\perp$.

(iv) $R(A^*) = N(A)^\perp$.

Proof. See [32, Theorem 2.19.]. □

This theorem is very useful in practice, not least because of its many corollaries, which are often immediate. Take, for example, the following result.

Corollary A.8.2. *Let H_1 and H_2 be two Hilbert spaces and $A : H_1 \to H_2'$ be a continuous linear operator. Then, the following statements are equivalent:*

(i) $R(A) = H_2'$.

(ii) $R(A^*)$ *is closed and* A^* *is onto.*

(iii) $\exists C_b > 0,\ \forall u \in H_2,\ \|u\|_{H_2} \leq C_b \|A^*(u)\|_{H_1'}$.

(iv) $\exists L_A \in \mathcal{L}(H_2', H_1),\ \forall v \in H_2',\ A(L_A(v)) = v$ *and* $\|L_A\|_{\text{op}} \leq C_b$.

Proof. See [32, Theorem 2.20. and Theorem 2.21.]. □

Let us just point out that Corollary A.8.2 can be used in order to prove Lax-Milgram theorem (see [27, Theorem 4.1.6.]). Sometimes, the existence of a continuous linear operator L_A as given in the previous result is called a **lifting** of A.

La lecture de tous les bons livres est comme une conversation avec les plus honnêtes gens des siècles passés, qui en ont été les auteurs, et même une conversation étudiée en laquelle ils ne nous découvrent que les meilleures de leurs pensées.

René Descartes, *Discours de la méthode.*

B

Some results about quasilinear parabolic equations

In this appendix, we recall some important results about quasilinear parabolic equations. We focus on the results used throughout our presentation, but we encourage the interested reader to refer to [132, 71, 5] for a more precise view of the theory.

Let us start with an essential result concerning the existence of solutions to the following equation:

$$\begin{cases} \partial_t u + A(u) = f \\ u_{|t=0} = u_0. \end{cases} \tag{B.0.1}$$

Theorem B.0.1 (J.L.Lions)**.** *Let us consider a real reflexive Banach space X densely and compactly embbedded into an Hilbert space H and such that*

$$X \subset H \subset X^*$$

where H is continuously embbedded into X^*, which denotes the topological dual of X. Moreover, we consider an hemicontinuous monotone operator $A : X \to X^*$ satisfying for some $1 < p < +\infty$:

(i) For every $u \in X$, there exists a constant $C > 0$ such that $\|A(u)\|_{X^*} \leq C\|u\|_X$.

(ii) For every $u \in X$, there exists $C > 0$ such that $\langle A(u), u\rangle \geq C|u|_X^p$, where $|\cdot|_X$ is a seminorm over X such that there exists $(\alpha, \beta) > 0$ satisfying

$$|u|_X + \alpha\|u\|_H \geq \beta\|u\|_X.$$

Then, for every $(u_0, f) \in H \times L^{\frac{p}{p-1}}((0,T), X^*)$, there exists a solution $u \in L^p((0,T), X)$ to (B.0.1).

Proof. We refer to [132, Théorème 1.2.bis]. □

The previous theorem allows us to prove the existence of solutions for a very large class of quasilinear equations. It should be pointed out that, although it is presented in a parabolic framework, it is also valid in an elliptic one, and also finds an alternative in the hyperbolic case (see [132,

Chapitre 2, Section 6] by the same author). We will now focus on one of its most notable consequences, namely the existence of solutions for the parabolic p-Laplacian.

$$\begin{cases} \partial_t u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f & \text{in } (0,T)\times\Omega \\ u = 0 & \text{on } (0,T)\times\partial\Omega \\ u_{|t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{B.0.2}$$

Corollary B.0.2. *Let us consider Ω a domain of $\mathbb{R}^N$ with Lipschitz boundary, and a real number $p \in (1,+\infty)$. We define the Banach space:*

$$X = \begin{cases} W_0^{1,p}(\Omega) & \text{if } p \geq 2 \\ W_0^{1,p}(\Omega)\cap L^2(\Omega) & \text{otherwise.} \end{cases} \tag{B.0.3}$$

Then, for every u_0 belonging to $L^2(\Omega)$, and every f belonging to $L^{\frac{p}{p-1}}((0,T),X^*)$, there exists $u \in L^p((0,T),X)$ solution to (B.0.2).

Once again, the reader should find the proof (and much more) into [132]. Let us point out that the proof implies an energy estimate. With the intention of being as clear as possible for the reader, we will give a class of p-growth systems, encompassing (B.0.2), for which it is possible to have an accurate energy estimate.

$$\begin{cases} \partial_t u - \operatorname{div}\left(a(t,x,u,\nabla u)\right) = f & \text{in } (0,T)\times\Omega \\ u = 0 & \text{on } (0,T)\times\partial\Omega \\ u_{|t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{B.0.4}$$

where $f \in L^{\left(\frac{2(N+p)}{p-1}\right)}(Q_T)$, and $a : Q_T\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ is measurable. Moreover, we assume that there exists $(C_0,C_1,c_0,c_1) > 0$, $\gamma \in \left[p, \frac{(N+2)p}{N}\right)$, and two non negative functions $(\psi_0,\psi_1) \in L^{\left(\frac{2(N+p)}{p}\right)}(Q_T)\times L^{\left(\frac{2\gamma(N+p)}{p(\gamma-1)}\right)}(Q_T)$ such that both of the following statements hold for every $(t,x,s,\xi) \in Q_T\times\mathbb{R}\times\mathbb{R}^N$:

(P1) $C_0|\xi|^p - c_0|s|^\gamma - \psi_0(t,x) \leq a(t,x,s,\xi)\cdot\xi$.

(P2) $|a(t,x,s,\xi)| \leq C_1|\xi|^{p-1} + c_1|s|^{\frac{\gamma(p-1)}{p}} + \psi_1(t,x)$.

These assumptions can still be generalized, but we have chosen to restrict ourselves to a more "simplified" framework. We have the following result.

Theorem B.0.3 (Di Benedetto). *Let us consider a domain Ω of $\mathbb{R}^N$ with Lipschitz boundary, an initial datum $u_0 \in L^2(\Omega)$, $f \in L^{\frac{p}{p-1}}(Q_T)$ and that assumptions (P1)-(P2) are satisfied. Then, every non negative weak solution u to* (B.0.4) *satisfies the following energy inequality.*

$$\begin{aligned} \|u\|^2_{L^\infty((0,T),L^2(\Omega))} + \|u\|^p_{L^p\left((0,T),W_0^{1,p}(\Omega)\right)} \leq & \gamma\|u_0\|^2_{L^2(\Omega)} + \gamma\|u\|^2_{L^2(Q_T)} \\ & + \|u\|^\gamma_{L^\gamma(Q_T)} + (T|\Omega|)^{1-\frac{p}{2(N+p)}}. \end{aligned}$$

Proof. Thanks, to homogeneous Dirichlet boundary conditions, we can set $k = 0$ into [71, Chapter V, Proposition 6.2.] and the estimate used in this last holds for both the negative and positive parts considered. Then, from our assumptions, the result follows since a weak solution is also a weak subsolution. □

Two comments on this result are welcome. Firstly, note that this result is a corollary of a more general energy estimate, and that it can only be obtained a priori in the case of homogeneous Dirichlet conditions. The behavior of the gradient of solutions to a quasilinear parabolic equation in a neighbourhood of the boundary is generally very difficult to establish, and is still a hot topic of research. Next, mention is made of non-negative weak solutions, the existence of which needs to be established. Note also that the two solution-dependent terms are bounded by the norms of the f term and the initial data u_0.

As you would expect, the regularity of the solutions of a quasilinear equation is generally directly related, in the case of a homogeneous Dirichlet-type boundary condition, to the regularity of the free data of the system, i.e. the initial data and the right-hand term. Consider, for example, the following equation:

$$\begin{cases} \partial_t u - \operatorname{div}\left((\mu + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\right) = f & \text{in } (0,T)\times\Omega \\ u = 0 & \text{on } (0,T)\times\partial\Omega \\ u_{|t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{B.0.5}$$

for some fixed $\mu > 0$ and $p > 1$. We can then show the following result.

Theorem B.0.4. *Let $\alpha \in (0,1)$, Ω be a domain of $\mathbb{R}^N$ such that $\partial\Omega$ has a $C^{2,\alpha}$ boundary, and consider an initial datum $u_0 \in C^{1+\alpha/2}(\overline{\Omega})$. Then, there exists a weak solution $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q_T})$ of the equation* (B.0.5).

Proof. This is a direct consequence of [172, Theorem 10.9.2.]. □

By the previous proposition, we can give a strong meaning to the solutions and as soon as the free data is C^∞, then so is the weak solution by a bootstrap argument. This last observation can be shown via various methods, notably by demonstrating that these solutions then coincide with mild solutions (they can be shown to generate a nonlinear contraction semi-group), and classical regularity theories can then be applied, we refer the reader, for example, to [176, 15, 6, 161, 175, 153] for further details on this aspect of the theory, which we do not cover in this dissertation.

Finally, as we mentioned in Chapter 2, the case $p = 1$ in the (B.0.5) equation is much more difficult to handle, as in the case of the parabolic p-Laplacian. Here, this is not due to a singularity problem, but in particular to the reflexivity defect of the space L^1.

(...) quand les physiciens nous demandent la solution d'un problème, ce n'est pas une corvée qu'ils nous imposent, c'est nous au contraire qui leur devons des remerciements. Mais ce n'est pas tout; la physique ne nous donne pas seulement l'occasion de résoudre des problèmes; elle nous aide à en trouver les moyens, et cela de deux manières. Elle nous fait pressentir la solution; elle nous suggère des raisonnements.

Henri Poincaré, *La valeur de la Science.*

C

Introduction to continuum mechanics for incompressible fluid flows

In this final appendix, we turn our attention to the physics associated with the equations discussed in this dissertation. More specifically, we present the mechanics of continuous media and how these lead to the models considered.

The genesis of classical mechanics can be found as far back as antiquity, in the writings of the polymath Aristotle, for example, and took off with the structure introduced by Isaac Newton in 1687 (see [146]). In classical mechanics, it is customary to consider a body as an (infinite) set of weighted points. In general, this is thought of in the sense that a body is conceived as the limit when $N \to +\infty$ of a set made up of N points, which amounts to saying that we can represent a body as being a countable set of points. If we give ourselves a representation in $\mathbb{R}^d$, this does make sense, if we keep in mind, for example, the density of $\mathbb{Q}^d$ in the latter for the Euclidean norm. However, while this framework is relatively well suited to an undeformable solid, for example, it is less so for a body undergoing deformation. What is more, some properties are not natural: for example, we may have in mind the relative density of a part of the body under consideration. Let us imagine, in a two-dimensional setting, a body Ω approached by a sequence of weighted particles denoted $(x_n, m_n)_{n\geq 1}$, where for $i \in \mathbb{N}\backslash\{0\}$, x_i denotes the position of the i-th particle (in $\mathbb{R}^2$) and m_i its mass. Let us assume that Ω is a polygon, and also that ω is another polygon strictly included in the latter. The Wallace-Bolyai-Gerwien theorem (see [104, Theorem 16.3.]) allows us to return to considering two triangles. Let us consider that the first six points define the points that describe the two triangles. Then at a fixed $N > 6$, even if we have to rearrange the terms, we can suppose that the first $6 \leq k < N$ terms of the sequence are inside ω and the others outside. Then, at a fixed $N > 6$, the density of ω in Ω (more precisely, of their approximations), could be obtained by intuitively writing the quotient of the densities of ω and Ω,

$$\overline{\rho}_{\omega,\Omega} = \frac{|\Omega| \sum_{i=1}^{k} x_i}{|\omega| \sum_{i=1}^{N} x_i}.$$

But what sense would this make when we consider an infinite number of particles in ω, and, a fortiori, in Ω? If convergence is possible, it clearly depends heavily on how we choose to approach the respective weights of ω and Ω. To do this, consider a Ω body composed of particles that all have the same mass (let us say constant and equal to 1). If we take $N = 2k$, which means, roughly

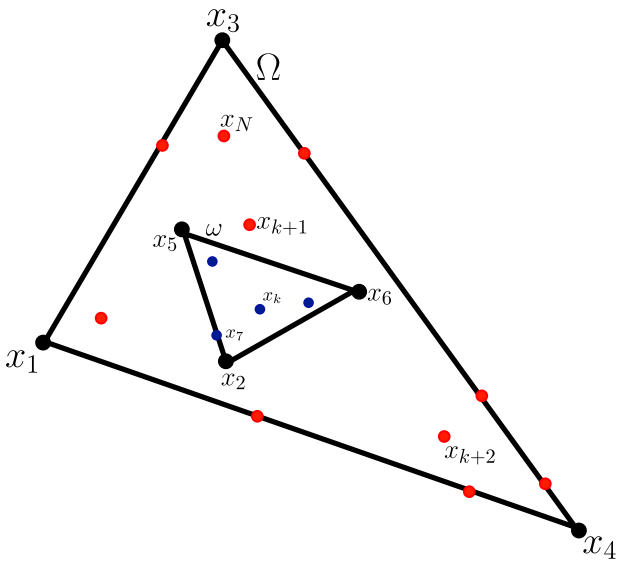


Figure C.1: Relative density taken as the quotient limit

speaking that at each step (grower to 6) there are as many particles outside ω as inside, then the relative density is constant and equal to $\frac{|\Omega|}{2|\omega|}$. But it is possible to approach ω and Ω differently, and we would then obtain a different relative density at the limit, which makes it clear that the way we compute has no meaning in this context, at least in the sense of the previous definition.

Note that the previous example is not valid in a three-dimensional framework (Dehn theorem see [104, Theorem 16.4., Corollary 16.3.]), and is very difficult to adapt to smooth domains (Laczkovich theorem).

Although it is still possible to give a satisfactory meaning to relative density via classical mechanics, this example highlights the difficulty of using it for certain notions, particularly when considering materials undergoing strong deformation. To be more precise, we have simply pointed out that in our case, taking the limit of the quotient is sometimes different from taking the quotient of the limits, and that the first way of proceeding is not correct. In fact, showing that macroscopic models are consistent with microscopic phenomena is a very active area of research, involving a variety of complex (scaling limit) techniques. A precise analysis of the models and concepts under consideration is then required. In the context of our subject, we wish to find a simplified study framework that allows greater flexibility in the computations, i.e. that we need to establish a continuous framework, in which laws similar to those of classical mechanics can be established, but which can be adapted to deforming bodies, such as fluids. In fact, a more general framework needs to be defined, one that links the different approaches. An absolutely general framework is currently non-existent in physics, but it remains possible to link the general principles of classical mechanics and continuum mechanics.

The following approach stems from twentieth-century research into Hilbert's Sixth Problem, whose aim was to provide a precise axiomatic of physics. Thus, the aim is to provide a mathematical framework, based in particular on logic, making it possible to provide a mathematical theory adapted to the physics studied. This is sometimes referred to as rational mechanics. Here, we will present just a few notions and mention some well-known results, omitting notions that are of no major use to our presentation. For a detailed presentation of the results, we refer the interested reader to [167], from which we have drawn the essential elements presented below.

C.1 Notion of body

As we suggested earlier, continuum mechanics is no longer concerned with a set of points, each with a mass, but with a whole, to which it is possible to associate a mass. For the theory to be fully coherent, it is then necessary to have a means of measuring the mass of a body. It is worth noting that, while we do nott a priori consider a body to be a countable set of points, the fact remains that we will intuitively use the possibility of choosing a point in a body when constructing the various tools we are about to present. In this sense, the established theory considers the **axiom of choice**.

The starting point of our presentation is therefore to define a set of elements $\mathbb{U}_{\text{bod}}$, generically noted Ω_i for $i \in I$, which we will call a body. In addition, we are going to give $\mathbb{U}_{\text{bod}}$ a Boolean algebra structure. Then, we provide $\mathbb{U}_{\text{bod}}$ with a partial order relation noted $\prec$, verifying the following relations. The reader may have in mind the relation $\prec$ as the inclusion in set theory $\subset$. However, we will see that while this is what we will be using in practice, and rightly so, it does not provide a mathematically correct framework for defining general theory. Let us begin by some useful notions.

Definition C.1.1. *Let us consider $(\mathbb{U}_{\text{bod}}, \prec)$, which is partially ordered. Then, we define:*

1. *For $(\Omega_1, \Omega_2) \in \mathbb{U}_{\text{bod}}$ the join of Ω_1 and Ω_2 is the set Ω_3 defined such that*

$$[(\Omega_1 \prec \Omega_3) \ \& \ (\Omega_2 \prec \Omega_3)] \ \ \& \ \ [\forall \Omega \in \mathbb{U}_{\text{bod}}, \ (\Omega_1 \prec \Omega) \ \& \ (\Omega_2 \prec \Omega) \Rightarrow \Omega_3 \prec \Omega] \, .$$

   The join of Ω_1 and Ω_2 is written $\Omega_1 \vee \Omega_2$

2. *For $(\Omega_1, \Omega_2) \in \mathbb{U}_{\text{bod}}$ the meet of Ω_1 and Ω_2 is the set Ω_3 defined such that*

$$[(\Omega_3 \prec \Omega_1) \ \& \ (\Omega_3 \prec \Omega_2)] \ \ \& \ \ [\forall \Omega \in \mathbb{U}_{\text{bod}}, \ (\Omega \prec \Omega_1) \ \& \ (\Omega \prec \Omega_2) \Rightarrow \Omega \prec \Omega_3] \, .$$

   The meet of Ω_1 and Ω_2 is written $\Omega_1 \wedge \Omega_2$.

Intuitively, the reader can visualize the join as the union of sets $\cup$ and the meeting as the intersection $\cap$. Again, although we will return to considering these laws later, they are sometimes insufficient to describe a desirable theoretical framework. We now define the axioms for the set of bodies.

Axiomatic C.1.2. *Let $\mathbb{U}_{\text{bod}}$ be a set, whose elements are called bodies. We provide $\mathbb{U}_{\text{bod}}$ with a partial order relation noted $\prec$, verifying the following relations:*

(B1) $\exists (0, \infty) \in \mathbb{U}_{\text{bod}}, \ \forall \Omega \in \mathbb{U}_{\text{bod}}, \ 0 \prec \Omega, \ \Omega \prec \infty.$

(B2) $\forall \Omega \in \mathbb{U}_{\text{bod}}, \ \Omega \prec \Omega$

(B3) $\forall (\Omega_1, \Omega_2) \in \mathbb{U}_{\text{bod}}, \ (\Omega_1 \prec \Omega_2) \ \& \ (\Omega_2 \prec \Omega_1) \Rightarrow (\Omega_1 = \Omega_2).$

(B4) $\forall (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{U}_{\text{bod}}, \ (\Omega_1 \prec \Omega_2) \ \& \ (\Omega_2 \prec \Omega_3) \Rightarrow (\Omega_1 \prec \Omega_3).$

(B5) $\forall (\Omega_1, \Omega_2) \in \mathbb{U}_{\text{bod}}, \ \Omega_1 \wedge \Omega_2 \in \mathbb{U}_{\text{bod}}.$

(B6) $\forall \Omega \in \mathbb{U}_{\text{bod}}, \ \exists \Omega^e \in \mathbb{U}_{\text{bod}}, \ \Omega \wedge \Omega^e = 0, \ \Omega \vee \Omega^e = \infty.$

(B7) $\forall (\Omega_1, \Omega_2) \in \mathbb{U}_{\text{bod}}, \ \Omega_1^e \wedge \Omega_2 = 0 \Rightarrow \Omega_2 \prec \Omega_1.$

It is possible to show that the set $(\mathbb{U}_{\text{bod}}, \prec, \wedge, \vee)$ satisfying the axioms (B1)–(B7) has a Boolean algebra structure (see [167, (I.2-36)–(I.2-40)]). From now on, we will simply write $\mathbb{U}_{\text{bod}}$ to designate $(\mathbb{U}_{\text{bod}}, \prec, \wedge, \vee)$ satisfying the axioms (B1)–(B7). An important concept concerns bodies that are separate, in the sense that they are not in contact. For this purpose, we introduce the following definition.

Definition C.1.3 (Separate bodies)**.** *We will say that two bodies Ω_1 and Ω_2 are separated if they verify*

$$\Omega_1 \wedge \Omega_2 = 0.$$

Example C.1.4. *Let us take a look at some of the possibilities for $\mathbb{U}_{\text{bod}}$.*

- *A first possibility is to consider a set E and U_n the set of subsets of E (not to be confused with 2^E). Then we can set $\prec = \subset$, $\wedge = \cap$, $\vee = \cup$, $0 = \emptyset$, $\infty = E$, and for every $\Omega \in \mathbb{U}_{\text{bod}}$, $\Omega^e = \Omega^c$.*

- *A second possibility is to consider a topological space X, and $\mathbb{U}_{\text{bod}}$ as the set of regularly open sets of X, i.e. the set of elements of its topology satisfying*

$$\Omega = \overset{\circ}{\overline{\Omega}}$$

Then the axioms (B1)-(B7) are generally not satisfied for the operations $\cap$ and $\cup$. It is therefore appropriate to consider the operations:

$$\forall \Omega_1, \Omega_2 \in \mathbb{U}_{\text{bod}},\ \Omega_1 \wedge \Omega_2 = \overset{\circ}{\overline{\Omega_1 \cap \Omega_2}},\ \Omega_1 \vee \Omega_2 = \overset{\circ}{\overline{\Omega_1 \cup \Omega_2}}.$$

Note that an example of a set that is open but not regularly open can be given by an open disk in the plane with an interior point removed. It is also possible to construct examples of sets of bodies that do not satisfy all the above-mentioned axioms. For example, by considering the set of regularly open bodies of a Euclidean space with the topology associated with its norm, having a piecewise smooth boundary. Then it is possible to check that the meet of two bodies does not always exist (see [167, Chapter 1, 3.3]).

C.2 Mass measure

To return to our initial discussion, we are going to give bodies a mass, rather than isolated points. We will refer to this as a massy body. The set of massy bodies forms a non-empty subset of $\mathbb{U}_{\text{bod}}$, but for convenience (and since this will be the case in our talk), we will consider all elements of $\mathbb{U}_{\text{bod}}$ to be massy bodies.

Axiomatic C.2.1. *We assume that there exists $M : \mathbb{U}_{\text{bod}} \to \overline{\mathbb{R}}_+$ satisfying for every separate bodies Ω_1 and Ω_2:*

$$M(\Omega_1 \vee \Omega_2) = M(\Omega_1) + M(\Omega_2),$$

such that M can be extend as a measure over all Borel sets of $\mathbb{U}_{\text{bod}}$, which is called the measure of mass.

We point out that this makes sense since from (B1) and (B5) it follows $M(0) = M(0 \vee 0) = 2M(0) \Rightarrow M(0) = 0$. Also, M is a measure over $\mathbb{U}_{\text{bod}}$ as soon as $\mathbb{U}_{\text{bod}}$ defines a suitable topological space (namely, as soon as we can use $\cap = \wedge$ and $\cup = \vee$).

Definition C.2.2 (Mass of a body). *Let $\Omega \in \mathbb{U}_{\text{bod}}$ be a massy body. Then, the mass of Ω is given by*

$$M(\Omega) = \int_{\Omega} 1 \, dM.$$

The principle of conservation of mass is naturally taken into account by the axiomatic refax:mass. Note, however, that the framework we have presented is specific to a sufficiently large scale, which experimentally corresponds to a mesoscopic or macroscopic scale, since conservation of mass is not always well verified at very small scales during chemical interactions. Note that it is still possible, under these considerations, to return (in a sense) to the framework of classical mechanics by considering the set of $\mathbb{U}_{\text{bod}}$ mass bodies as a set of parts of isolated weighted points. This being said, the identification between $\mathbb{U}_{\text{bod}}$ and all its massive bodies would then be abusive.

C.3 Forces acting on a body

Now that we have given a precise meaning to the notion of body and associated mass, we need to give a meaning to the notion of force applied to a body. In particular, we need to define a suitable axiomatic to preserve the properties of classical mechanics.

Definition C.3.1 (force exerted on a body). *Let $\mathbb{U}_{\text{bod}}$ be a set of massy bodies, namely satisfying axioms (B1)-(B7) and Axiomatic C.2.1. Then, considering an Hilbert space H, we say that $f : \mathbb{U}_{\text{bod}} \times \mathbb{U}_{\text{bod}} \to H$ is a system of forces. The vector $f(\Omega_1, \Omega_2)$ is called the force exerted on Ω_1 over Ω_2.*

In practice, we will take $H = \mathbb{R}^N$, i.e. we will consider an Euclidean space.

C.3.1 Case of separate bodies, Noll axiomatic

We now turn to the definition of the axiomatics associated with a system of forces. First, we will present it in the more intuitive case of separate bodies, then generalize it to the general case.

Axiomatic C.3.2. *We consider $\mathbb{U}_{\text{bod}}$ and H as in Definition C.3.1. Moreover, we only consider separate bodies. More precisely, we assume that for every $(\Omega_1, \Omega_2, \Omega_3) \in \mathbb{U}_{\text{bod}}$ such that for every $(i, j) \in \{1, 3\}$, $i \neq j$, we have $\Omega_i \wedge \Omega_j = 0$.*

(F1) There exists a system of forces $f : \mathbb{U}_{\text{bod}} \times \mathbb{U}_{\text{bod}} \to H$ as given by Definition C.3.1.

(F2) $f(\Omega_1 \vee \Omega_2, \Omega_3) = f(\Omega_1, \Omega_3) + f(\Omega_2, \Omega_3)$.

(F3) $f(\Omega_1, \Omega_2 \vee \Omega_3) = f(\Omega_1, \Omega_2) + f(\Omega_1, \Omega_3)$.

(F4) $f(\cdot, \Omega_1^e)$ *is a vector valued measure over* Ω_1.

The axiom (F2) expresses the fact that the force exerted by two separate bodies Ω_1 and Ω_2 on a third body Ω_3 is equal to the sum of the forces exerted respectively by Ω_1 and by Ω_2 on Ω_3. The axiom (F3) expresses that the distributivity property induced by the axiom (F2) is true according to the two variables considered, i.e. we have distributivity on the left and on the right relatively to the join. At this stage, we even expect commutativity, which is the same as saying that we would have:

$$f(\Omega_1, \Omega_2) = -f(\Omega_2, \Omega_1)$$

that is, in a sense, we are verifying **Newton's third law**. Such a system of forces is said to be *pairwise equilibrated*. We have the following result.

Theorem C.3.3 (Noll-Gurtin-Williams)**.** *A system of forces is pairwise equilibrated if and only if the resultant force $f(\Omega, \Omega^e)$ is additive over separate bodies of $\mathbb{U}_{\text{bod}}$, namely:*

$$\forall (\Omega_1, \Omega_2) \in \mathbb{U}_{\text{bod}},\ \Omega_1 \wedge \Omega_2 = 0 \Rightarrow f(\Omega_1 \vee \Omega_2, (\Omega_1 \vee \Omega_2)^e) = f(\Omega_1, \Omega_1^e) + f(\Omega_2, \Omega_2^e).$$

Now, from Axioms (F2) and (F4), we deduce the following result.

Theorem C.3.4 (Measure relative to two separate bodies)**.** *Let Ω_1 and Ω_2 be two separate bodies of $\mathbb{U}_{\text{bod}}$. Then the function $\omega \mapsto f(\omega, \Omega_2^e)$ defined for $\omega \prec \Omega_1$ is a measure on Ω_1. We then denote:*

$$\forall \omega \in \mathbb{U}_{\text{bod}},\ \omega \prec \Omega_1,\ f(\omega, \Omega_2^e) = \int_\omega 1\, df_{\Omega_2^e}.$$

C.3.2 General case, Rizzo axiomatic

In the previous section, we concentrated on the case of two separate bodies. This being said, in the context of continuum mechanics, we can imagine that two bodies are not always separate, firstly because we can think of two bodies in contact for a fluid, but we can also think of a complex fluid whose elements mix, for example. This raises the question of whether the axioms refax:F1–refax:F4 can be generalized to bodies that are not necessarily separate.

Axiomatic C.3.5. *We consider $\mathbb{U}_{\text{bod}}$ and H as in Definition C.3.1. We assume that for every $(\Omega_1, \Omega_2, \Omega_3) \in \mathbb{U}_{\text{bod}}$, the following statements hold.*

(F1') *There exists a system of forces $f : \mathbb{U}_{\text{bod}} \times \mathbb{U}_{\text{bod}} \to H$ as given by Definition C.3.1.*

(F2') $f(\Omega_1 \vee \Omega_2, \Omega_3) = f(\Omega_1, \Omega_3) + f(\Omega_2, \Omega_3) - f(\Omega_1 \wedge \Omega_2, \Omega_3)$.

(F3') $f(\Omega_1, \Omega_2 \vee \Omega_3) = f(\Omega_1, \Omega_2) + f(\Omega_1, \Omega_3) - f(\Omega_1, \Omega_2 \wedge \Omega_3)$.

(F4') $f(0, \Omega_1) = f(\Omega_1, 0) = 0$.

Some may question the validity of a Newton's third law in this context. The following theorem aims to answer this question.

Theorem C.3.6 (Rizzo)**.** *Let us assume that there exists a system of forces satisfying (F1')-(F4') over a universe of bodies $\mathbb{U}_{\text{bod}}$ and relatively to an Hilbert space H. Then, we have:*

$$\forall (\Omega_1, \Omega_2) \in \mathbb{U}_{\text{bod}},\ f(\Omega_1, \Omega_2) = -f(\Omega_2, \Omega_2),$$

if and only if

$$\forall \Omega \in \mathbb{U}_{\text{bod}},\ f(\Omega, \Omega) = 0.$$

C.4 Motion

Before going any further, let us have a quick heuristic discussion on a notion that will be useful to us later on.

When imagining a flowing fluid, it is common to approach it from two angles. On one hand, we can fix a fluid particle and follow its trajectory, or on the other hand, we can fix a point in the set

of possible trajectories and observe the particles that pass through this last. The first approach is known as the *Lagrangian description* of the flow while the second one is usually refered as its *Eulerian description*. We can express each of them by a flow function (in a mathematical sense) which take its values in different spaces. Indeed, if we denote by Ω_0 the set (namely, an element of $\mathbb{U}_{\text{bod}}$) which is made of the particles that we wish to observe during flow at time $t = 0$, and Ω_t the set of the same particles at time t, then we can write both of these descriptions as follow.

$$(\textbf{Lagrangian description}) \quad \varphi : \left| \begin{array}{l} [0, +\infty) \times \Omega_0 \longrightarrow \Omega_t \\ \\ (t, X) \longmapsto x, \end{array} \right.$$

$$(\textbf{Eulerian description}) \quad \phi : \left| \begin{array}{l} [0, +\infty) \times \Omega_t \longrightarrow \Omega_0 \\ \\ (t, x) \longmapsto X. \end{array} \right.$$

Moreover, we shall assume here that φ and ϕ are smooth, which will simplify our presentation without loss of generality in the physical sense. In order for the above definitions to make sense, i.e. to try to account for a uniqueness in the flow, we have to consider that $\phi = \varphi^{-1}$.

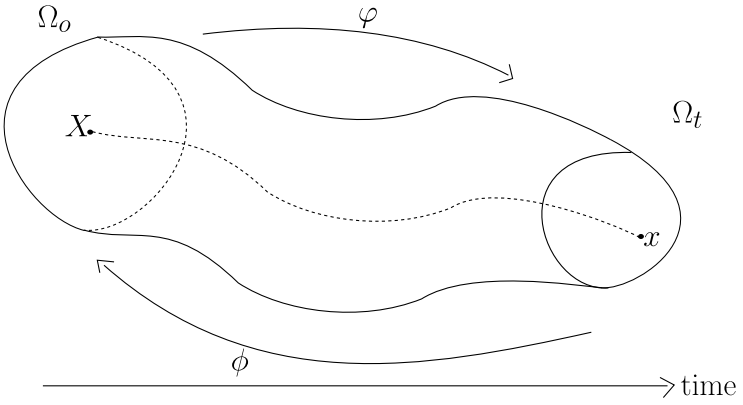


Figure C.2: Lagrangian and Eulerian descriptions of a flow

Heuristically, by defining the fluid velocity field as the derivative of its position, we obtain in Lagrangian description the following definition for the fluid velocity field:

$$\tilde{u}(t, X) = \partial_t \varphi(t, X).$$

As a result, we have the following definition in the Eulerian description:

$$u(t, x) = \partial_t \varphi(t, \phi(t, x)).$$

Thanks to the assumptions we have made, we are able to establish a link between these last two descriptions. In terms of evolution, this is done via the notion of *material derivative*. If we assume that P represents a physical quantity, this is nothing else than the time rate of change of this one. That is, if we denote formally by $\dot{P}(t, .)$ the material derivative of P, we get in Lagrangian description:

$$\dot{P}(t, X) = \partial_t P(t, X),$$

and then, using the chain rule, we should obtain in Eulerian description:

$$\dot{P}(t, x) = \frac{d}{dt} \left(P(t, \phi(t, x)) \right) = \partial_t P(t, \phi(t, x)) + (u \cdot \nabla) P(t, \phi(t, x)),$$

were we use the notation: $((u \cdot \nabla)u)_i := u_j \partial_j u_i$. So, in the sense of material derivative, we define the material acceleration of a particle in Eulerian description given by:

$$a(t,x) := \partial_t u(t,x) + (u \cdot \nabla)u(t,x).$$

Let us return to a more rigorous discussion. First of all, let us note that we have not set a frame of reference here. We could take several: the set of volume elements described by the flow we are studying, i.e. $\cup_{\Omega \in \overline{\Omega}} \Omega$ where $\overline{\Omega} := \{\Omega \in \mathbb{U}_{\text{bod}} \,/\, \exists t \in \mathbb{R}_+,\, \Omega = \Omega_t\}$, the ambient space, say for simplicity the space $\mathbb{R}^N$, or a fixed domain of it. When this frame is sufficiently stable, we speak of a *rigid frame*. This stability translates into the existence of an isometry between the elements of the ambient space at each instant t.

Generally speaking, it is worth asking about angle-preserving transformations, i.e. inner product-preserving tensors.

Proposition C.4.1. *A tensor Q preserves the inner product if and only if it is orthogonal, i.e. if and only if it verifies:*

$$Q^{-1} = Q^T.$$

Typically, this is used to give meaning to frame-invariant transformations, known as *frame-indifference*.

Definition C.4.2 (Motion, shape)**.** *Let $\mathbb{U}_{\text{bod}}$ be the set of bodies, H an Hilbert (or euclidean) space, then a motion is a mapping:*

$$\chi_{\mathbb{U}_{\text{bod}}} : \mathbb{R}_+ \times \mathbb{U}_{\text{bod}} \to 2^H.$$

For $\Omega \in \mathbb{U}_{\text{bod}}$, the space given by $\chi_{\mathbb{U}_{\text{bod}}}(t,\Omega)$ is called the shape of Ω at time t.

So we can see, for example, that the open sets $\Omega_t = \varphi(t,\Omega_0)$ presented above are shapes at time t for the Lagrangian description mapping φ, which describes a motion if we define it by looking at its action on $\mathbb{U}_{\text{bod}}$.

The shape at time t is given by the set of locations in the space considered occupied at time t, these points sometimes being called *substantial points*. Thus, we can always write:

$$\chi_{\mathbb{U}_{\text{bod}}}(t,\Omega) = \{\chi(t,X) \,/\, X \in \Omega\},$$

where $\chi(t,X) = \chi_{\mathbb{U}_{\text{bod}}}(t,\{X\})$. In the following, we will call motion the map $\chi_{\mathbb{U}_{\text{bod}}}$ or one of the applications it describes, given by χ.

Definition C.4.3 (Velocity of a substantial point)**.** *Let χ be a motion. Then, we respectively define the velocity and the acceleration of a substantial point X at time t by:*

$$\tilde{u}(t,X) = \frac{d}{dt}\chi(t,x) \quad \text{and} \quad \tilde{a}(t,X) = \frac{d}{dt}\tilde{u}(t,X) = \frac{d^2}{dt^2}\chi(t,X).$$

More generally, we define its n-th velocity mode by:

$$\tilde{u}_n(t,X) = \frac{d^n}{dt^n}\chi(t,X).$$

Let us note an important fact: we have described bijective motions here, which means that two substantial points, sometimes referred to as particles (and we will confuse these notions later), do not simultaneously occupy the same place at the same instant. This property, which is admitted, is called the *axiom of impenetrability*.

In the following, we will note $\dot{\chi}$ the derivative $\frac{d}{dt}\chi$, as is customary in classical mechanics. With these considerations in mind, we can then define the usual objects of classical mechanics.

Definition C.4.4. *Let $\mathbb{U}_{\text{bod}}$ be a set of massy bodies satisfying the axioms (B1)-(B7), χ be a motion, M be the mass measure given according to the mass function, denoted identically, verifying the axiom C.2.1,f a system of forces, and let the axiom of impenetrability be satisfied. Then we define at time $t \in \mathbb{R}_+$, for $\Omega \in \mathbb{U}_{\text{bod}}$:*

- *The linear moment of Ω relative to the motion χ:*
$$m\left(\Omega, \chi(t)\right) = \int_\Omega \dot{\chi}(t, Y)\, dM(Y);$$
- *The rotational moment of Ω relative to the motion χ and to $x_0 \in \chi(t, \Omega)$:*
$$r_{x_0}\left(\Omega, \chi(t)\right) = \int_\Omega (\chi(t, Y) - x_0) \wedge \dot{\chi}(t, Y)\, dM(Y) = -r_{x_0}\left(\Omega, \chi(t)\right)^T;$$
- *The kinetic energy of Ω relative to the motion χ:*
$$K_E\left(\Omega, \chi(t)\right) = \frac{1}{2} \int_\Omega |\dot{\chi}(t, Y)|^2\, dM(Y);$$
- *The work energy of f_{Ω^e} over Ω relatively to the motion χ:*
$$W\left(\Omega, \chi(t), f_{\Omega^e}\right) = \int_\Omega \dot{\chi}(t, Y) \cdot df_{\Omega^e}(Y);$$
- *The moment (or the* torque*) of f_{Ω^e} on $\omega \prec \Omega$ relatively to the motion χ with respect to $x_0 \in \chi(t, \omega)$:*
$$F_{x_0}\left(\omega, \Omega^e, \chi(t)\right) = \int_\omega (\chi(t, Y) - x_0) \wedge df_{\Omega^e}(Y);$$
The resultant torque being given by $F_{x_0}\left(\Omega, \Omega^e, \chi(t)\right)$.
- *The center of mass of Ω relative to the motion χ:*
$$\overline{p}(t, \Omega) = \frac{1}{M(\Omega)} \int_\Omega \chi(t, Y)\, dM(Y);$$

where we denoted $\chi(t) : Y \mapsto \chi(t, Y)$.

The above definitions cover the usual notions of classical mechanics, adapted to the framework of continuum mechanics. Let us end this discussion with a fact that we mentioned earlier, the change of frame. In fact, it is possible to characterize a spatio-temporal change of frame, i.e. when we want to move from a point (t, x) to a point (t^*, x^*), as long as it defines a desirable isometry. In fact, as we said earlier, preserving angles is tantamount to applying an orthogonal tensor, from which the following result follows.

Theorem C.4.5 (Isometric change of frame)**.** *Let us consider a change of frame from $\mathbb{R}_+ \times H \to \mathbb{R}_+ \times H$, $(t, x) \mapsto (t^*, x^*)$, where:*
$$\begin{cases} x^* = \theta(t, x) \\ t^* = t + \alpha \end{cases} \tag{C.4.1}$$
θ being an isometry and $\alpha \in \mathbb{R}$ is a constant. Then, there exists a function $Q : \mathbb{R}_+ \to O_H(\mathbb{R})$ namely such that for each time t, $Q(t)$ is an orthogonal tensor over H, such that (C.4.1) *is characterized by:*
$$\begin{cases} x^* = \theta(t, x_0) + Q(t)(x - x_0) \\ t^* = t + \alpha \end{cases} \tag{C.4.2}$$
where x_0 is a fixed point in the initial frame. If, moreover, the map $t \mapsto \dot{Q}(t)Q^T(t)$, called the spin, is continuous, such a change of frame is unique.

In the end, as we mentioned earlier, this theorem essentially boils down to the characterization of isometries.

C.4.1 Axioms of mechanics

One of the main ideas of mechanics is that, whatever the observer's point of view, there are elements that remain invariant. This idea is embodied in the following assertions, known as the *axioms of mechanics.*

Axiomatic C.4.6. *Let us consider two different (rigid) frames and a change of frames as given in Theorem C.4.5.*

(Me1) Mass is frame-indifferent, namely if we denote M the mass into the first frame and M^* into the second one, we have:

$$\forall((t,x),(t^*,x^*)) \in (\mathbb{R}_+ \times H)^2,\ M(t,x) = M^*(t^*,x^*).$$

(Me2) Forces are frame-indifferent, namely there exists a function Q as given in Theorem C.4.5 such that for a system of forces f defined over the initial frame, we get $f^* = Qf$, f^* being the system of forces over the final frame.

(Me3) Work is frame-indifferent, namely we have $W^* = W$ over every $\Omega \in \mathbb{U}_{\text{bod}}$ and every motion χ.

This raises the question of whether it is still possible to prove, within this framework, an assertion of the same type as Newton's third law, which has always been the case up to now. This is the subject of the following result.

Theorem C.4.7 (Noll)**.** *The work of a system of forces is indifferent to the frame if and only if this system and its associated torques are both balanced. Consequently, for any pair of separate bodies Ω_1 and Ω_2, we have:*

$$\begin{cases} f(\Omega_1,\Omega_2) = -f(\Omega_2,\Omega_1) \\ F_{x_0}(\Omega_1,\Omega_2) = -F_{x_0}(\Omega_2,\Omega_1). \end{cases}$$

C.4.2 Axioms of inertia

So far, we have defined the basic notions, the frame, and alternative formulations of Newton's third law of classical mechanics. But the notion of inertia, preponderant in classical mechanics, must find its place in our theory, and with it Newton's other two laws. First, we will consider an isolated system $\Sigma \subset \mathbb{U}_{\text{bod}}$, in the sense that we are not looking at the interactions occurring in the set of elements of $\mathbb{U}_{\text{bod}}$, but simply at the forces occurring between elements into Σ. For example, Σ could be the set of substantial points traversed by a domain ω_0 transported by a flow, if these points, initially evolving in a larger domain Ω_0, remain sufficiently close to each other (to form regularly open sets). As usual, we then note Σ^e the join of the set of bodies outside Σ.

Axiomatic C.4.8 (Inertia)**.** *We assume that the two following properties hold.*

(I1) There exists a frame such that, being given a body $\Omega \in \mathbb{U}_{\text{bod}}$, and a mation χ, if the linear momentum $m(\Omega,\chi)$ is constant is constant over $I \subset \mathbb{R}_+$, then $f(\Omega,\Sigma^e) = 0$ in I and conversely. Equivalently, the center of mass of Ω for the motion χ, i.e. $\overline{p}(\Omega,t)$, follows a rectilinear motion with uniform velocity in this frame if and only if Σ^e, the bodies external to the bodies under study, exert no force on Ω. Such a frame of reference is called an inertial frame of reference.

(I2) In an inertial frame, we assume that we always have

$$f(\Omega, \Sigma^e) = -\dot{m}(\Omega, \chi).$$

The two above-mentioned axioms are equivalent to Newton's first and second laws respectively. Note that with our notations and assuming the above axioms, we have for a regular motion χ (at least C^1) and for an integrable function g:

$$\int_\Omega g \, df_{\Sigma^e} = -\int_\Omega g\ddot{\chi} \, dM.$$

That is, the measure of the external forces exerted by Ω is equal to the opposite of the measure of mass weighted by the acceleration function for the motion χ. Once again, we find a formulation analogous to Newton's well-known second law. The reader should not worry about the present sign, because of Theorem C.4.7 (i.e., the equivalent of Newton's third law), it reverses and we immediately find the usual formulation of the second law. In fact, the preceding axioms imply the following property, which fully reflects the laws commonly used in classical mechanics.

Theorem C.4.9 (Euler laws of motion)**.** *In an inertial frame, let us denote by f^{ex} the function being the force exerted on a body into Σ,and similarly by $F^{\text{ex}}_{x_0}$ the resultant torque thus obtained. Then, we have for all $\Omega \in \Sigma$:*

$$f^{\text{ex}}(\Omega) = \dot{m}(\Omega, \chi) \quad \text{and} \quad F^{\text{ex}}_{x_0}(\Omega) = \dot{r}_{x_0}(\Omega, \chi).$$

Namely, this means respectively that the applied force on Ω is equal to the rate of change of the linear momentum, and the applied torque is the rate of change of the rotational momentum.

We draw the reader's attention to the fact that it is quite possible to define a theory analogous to that of classical mechanics for energy (potential energy, total energy, conservation law, etc.). However, we won't be using it here, so we refer the reader to [167, Section I.14., Section I.15.] for a complete presentation of this aspect of continuum mechanics.

C.5 Kinematics of simple incompressible fluids

We can now introduce the section dedicated to fluid flows. From now on, we will be considering motions of regularity C^1 and being global homeomorphisms, which we will sometimes refer to as *placements*. Typically, Lagrangian and Eulerian descriptions constitute such motions. We will also consider $H = \mathbb{R}^N$ from now on.

Applying the Radon-Nikodym theorem to the measure of mass M, it follows that for any motion χ acting on a body Ω, there exists a unique function ρ (which depends on χ) such that we have:

$$M(\Omega) = \int_{\chi(\Omega)} \rho \, dx, \tag{C.5.1}$$

where dx is the Lebesgue measure, i.e. the volume measure. We immediately deduce the following result.

Proposition C.5.1. *For very motion χ defined over Ω, if $g \in L^1(\Omega, dM)$, then $g \in L^1(\chi(\Omega), \rho dx)$ and:*

$$\int_\Omega g \, dM = \int_{\chi(\Omega)} \rho g \, dx.$$

Proof. The proof is an immediate consequence of the definition for a fixed χ. Now, the invariance of mass for a body allows us to deduce that for two motions χ_1 and χ_2, we have the relation:

$$\int_{\chi_1(\Omega)} \rho_1 g \, dx = \int_{\chi_2(\Omega)} \rho_2 g \, dx.$$

Thus, any pair (χ, ρ) defines the same value, by definition. Note that it is always possible to switch from one motion to the other by changing variables. □

C.5.1 Axiomatic and consequences

The aim of this section is to establish how forces work in continuum mechanics, i.e. how they act on bodies. Let us start with a little heuristic discussion specific to fluids. Intuitively, if we consider a body that is a small parallelepiped of fluid, then a force acting on it can be decomposed, by applying the spectral decomposition theorem, into orthogonal forces and forces that are in the planes of the surfaces. The former are called *body forces*, the latter *contact forces*. Orthogonal forces are of two types: either they are directed away from the center of mass, in which case they are called *tension forces*, or they are directed towards the center of mass, in which case they are called *pressure forces*.

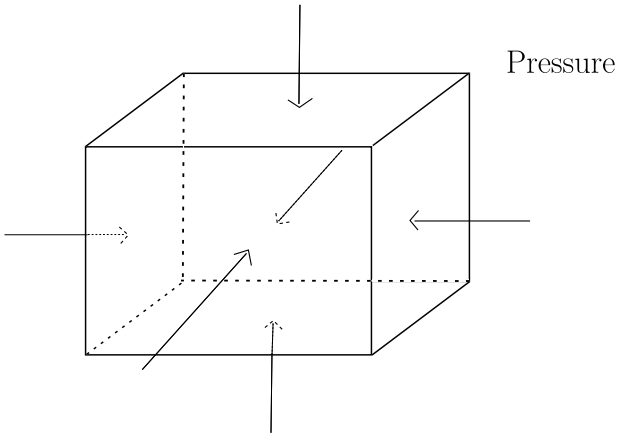


Figure C.3: A representation of pressure forces

Contact forces are generally referred to as *viscosity forces*, from which we distinguish *shear forces*: formally, this refers to the way in which the fluid deforms when forces act in two opposite directions on just two sides of the parallelepiped. When we observe how shear forces act instantaneously, we refer to this as the *shear rate tensor*.

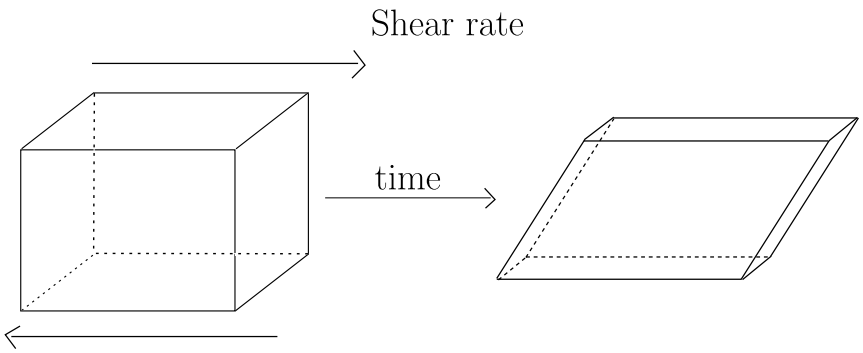


Figure C.4: Shear rate tensor acting over a body which is a small parallelepiped of fluid

Returning to a more rigorous discussion, to be able to establish a good decomposition, we need to be able to estimate the way in which the separate bodies act on each other. This is the purpose of the following axiom.

Axiomatic C.5.2 (Axiom of forces in continuum mechanics). *Let Ω_1 and Ω_2 be two separate bodies the contact area of their shapes for a motion χ being sufficiently small, as is the mass of Ω_1. Then, there exists a positive bounded function $C(\Omega_2)$ such that $C(\Omega_2) \to 0$ when $M(\Omega_2) \to 0$, and a positive constant C such that:*

$$|f(\Omega_1, \Omega_2)| \leq C\,|\partial\chi(\Omega_1) \cap \partial\chi(\Omega_2)| + C(\Omega_2)M(\Omega_1).$$

The preceding axiom is not very intuitive, but it does allow us to express, roughly speaking, the fact that the forces exerted by a body are proportional to their contact surface and their masses.

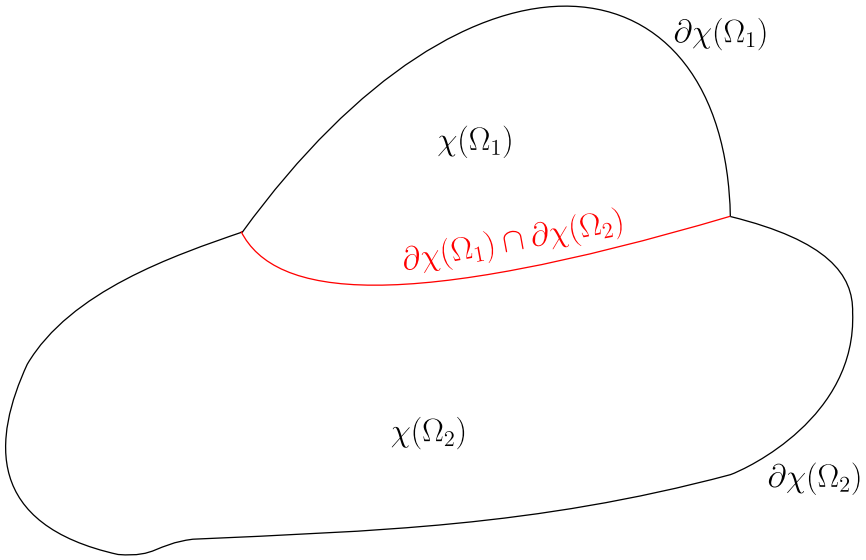


Figure C.5: A contact surface $\partial\chi(\Omega_1) \cap \partial\chi(\Omega_2)$, between two bodies $\chi(\Omega_1)$ and $\chi(\Omega_2)$.

Theorem C.5.3 (Contact and body forces). *Let f be a system of forces over $\mathbb{U}_{\text{bod}}$, then there exists systems of forces f_B and f_C such that $f = f_B + f_C$ and for every Ω_1 and Ω_2 in $\mathbb{U}_{\text{bod}}$, we have:*

$$|f_B(\Omega_1, \Omega_2)| \leq C(\Omega_2)M(\Omega_1) \quad \text{and} \quad |f_C(\Omega_1, \Omega_2)| \leq C\,|\partial\chi(\Omega_1) \cap \partial\chi(\Omega_2)|\,.$$

The systems f_B and f_C are respectively called systems of body and contact forces.

Thus, we know that there is a good characterization of forces in terms of contact and body forces. But how can we characterize contact forces? After all, they act on a surface, so we intuitively expect their integral characterization to be expressed in terms of the Hausdorff measure of the contact surface. The following result confirms that this is indeed the case.

Theorem C.5.4 (Gurtin-Williams). *Let us consider a system of contact forces f_C over $\mathbb{U}_{\text{bod}}$ in the sense of Theorem C.5.3. Then, there exists a density t, called the traction, such that for every bodies Ω_1 and Ω_2 we get:*

$$f_C(\Omega_1, \Omega_2) = \int_{\partial\chi(\Omega_1)\cap\partial\chi(\Omega_2)} t \, d\sigma,$$

where $d\sigma$ is the Hausdorff measure for the hypersurface $\partial\chi(\Omega_1) \cap \partial\chi(\Omega_2)$. Moreover, for every open subset $S \prec \partial\chi(\Omega_1) \cap \partial\chi(\Omega_2)$ the restriction of t over S, denoted similarly, satisfies the same formula, namely:

$$f_C(\Omega_1, \Omega_2)_{|S} = \int_S t \, d\sigma,$$

Finally, if it is true that Theorem C.4.7, i.e. in a formal sense Newton's third law, is verified for bodies, there is nothing to say a priori that it remains true for the contact surfaces induced by them. The following result confirms once again that this is indeed the case.

Theorem C.5.5 (Noll's Action and reaction for contact forces). *If a system of forces is pairwise equilibrated, then so are the body and contact force systems, in particular we have for every bodies Ω_1 and Ω_2 in $\mathbb{U}_{\text{bod}}$:*

$$\begin{cases} f_B(\Omega_1, \Omega_2) = -f_B(\Omega_2, \Omega_1) \\ f_C(\Omega_1, \Omega_2) = -f_C(\Omega_2, \Omega_1). \end{cases}$$

A major consequence of the Theorem C.5.4 is the following.

Theorem C.5.6 (Hamel-Noll). *We assume a system of contact forces f_C as in the Theorem C.5.4, and a traction t associated with it via a motion χ. Then, for any $S \prec \partial\chi(\Omega_1)\cap\partial\chi(\Omega_2)$, the traction t is expressed almost everywhere as a function of $x \in S$ and of a normal vector n to S at x only, i.e. $t = t(x, n)$. Moreover, we have:*

$$t(x, -n) = -t(x, n).$$

The principle that traction can be expressed pointwise and as a function of a vector normal to the surface is often referred to as *Cauchy postulate*. Theorem C.5.6 highlights that in the framework of rational mechanics, this principle is in fact provable, as is the equivalent of Newton's third law.

C.5.2 Deformations

In this section, we study deformations in $\mathbb{R}^N$. This theory is prevalent in fluid mechanics as well as in elasticity theory. We consider orientation-preserving motion, which we will denote φ, and which in spirit can be thought of as the flow associated with the Lagrangian description of a fluid (and a fortiori, the Eulerian description). The $\nabla\varphi$ tensor is called the *deformation gradient*, and orientation preservation translates into $\nabla\varphi > 0$ on a considered body. In the following, without loss of generality, we will take a body $\Omega \in \mathbb{U}_{\text{bod}}$.

We also introduce the displacement u, defined by $u = \varphi - \text{Id}$, and the *displacement gradient* $\nabla u = \nabla\varphi - \text{Id}$. For simplicity's sake, we will regularly write $x^\varphi = \varphi(x)$ for $x \in \Omega$.

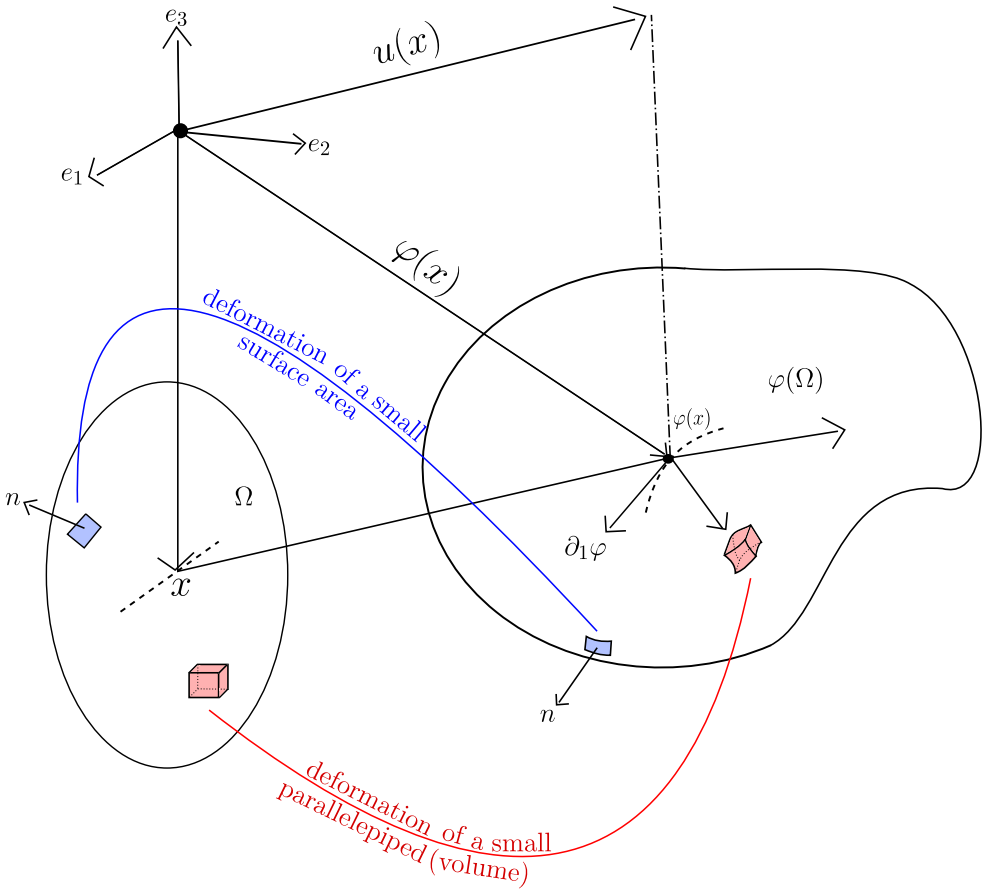


Figure C.6: A deformation in $\mathbb{R}^3$. Here, n denotes a outward unit normal vector.

Now we put forward that, according to Proposition C.4.1, angles are preserved by a function whenever it is expressed as the composite of an orthogonal tensor and a translation. If a motion φ satisfies this condition, i.e. $\varphi = a + Q$, where $a \in \mathbb{R}^N$ and $Q \in O_N(\mathbb{R})$, we say that it is a *rigid deformation*. Note that if we wish to show that how such motions act locally on the distances, we are led to observe that, for $x \in \Omega$ and $y \in \Omega$ such that $x + y \in \Omega$, we can write:

$$\varphi(x + y) - \varphi(x) = \nabla\varphi(x)y + o(|y|),$$

which implies, taking the matrix inner product:

$$|\varphi(x + y) - \varphi(x)|^2 = y^T \nabla\varphi(x)^T \nabla\varphi(x)y + o(|y|^2).$$

This motivates the introduction of the tensor $C = \nabla\varphi^T\nabla\varphi$, called the *right Cauchy-Green strain tensor*. We have the following result.

Theorem C.5.7. *A motion φ is a rigid deformation if and only if the right Cauchy-Green strain tensor satisfies $C = \mathrm{Id}$.*

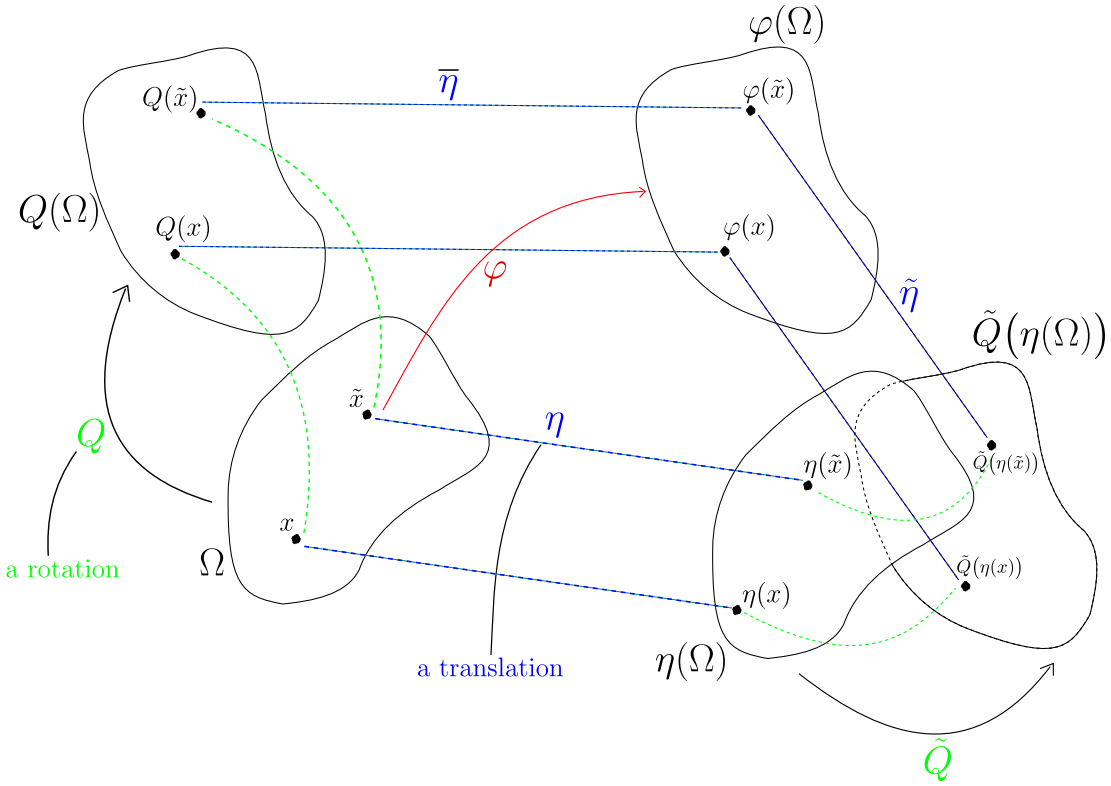


Figure C.7: A rigid deformation φ, which is by definition a translation followed by a rotation (or vice versa). Here, η, $\tilde{\eta}$ and $\overline{\eta}$ are translations, while Q and $\tilde{Q}$ are rotations. We can see that $\varphi = \eta \circ \tilde{Q} \circ \tilde{\eta} = Q \circ \overline{\eta}$. Angles and distances are preserved.

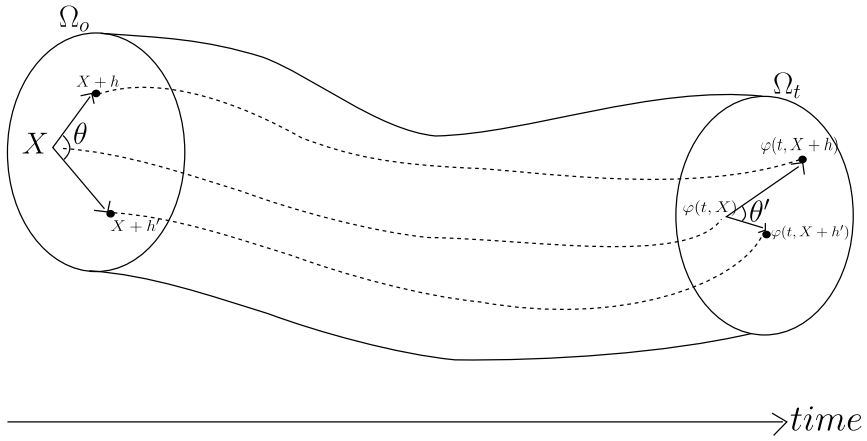


Figure C.8: Example of a fluid undergoing non-rigid deformation, angles are not preserved.

Theorem C.5.7 gives us a very convenient way of quantifying the stiffness defect of a motion, by introducing the tensor:

$$2E = C - \mathrm{Id}.$$

Note that the coefficient 2 in the left-hand side of the equation is introduced so that the tensor E is symmetric. By definition, then, we have:

$$C = \nabla\varphi^T \nabla\varphi = \mathrm{Id} + \nabla u + \nabla u^T + \nabla u \nabla u^T = I + 2E,$$

which then leads to:

$$E(u) = \frac{1}{2}\left(\nabla u + \nabla u^T + \nabla u \nabla u^T\right),$$

which is called the *Green-St Venant strain tensor*. The linearized Green-St Venant strain tensor is called the *strain rate tensor*, and is denoted $D(u)$. It is then the symmetrical gradient, namely:

$$D(u) = \frac{\nabla u + \nabla u^T}{2}.$$

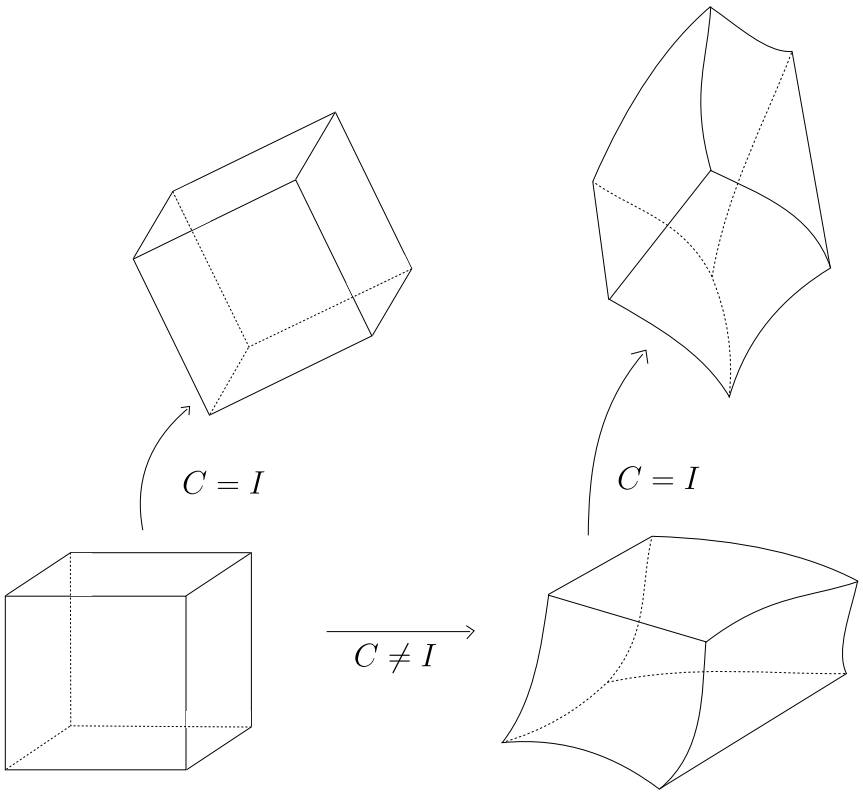


Figure C.9: The right Cauchy-Green tensor and its link with rigid deformation.

C.5.3 Stress tensors

We have already seen how deformations act on a body. Now we need to focus on how we can account for the stresses applied to a deformable material, let alone a fluid. To do this, we need to take up the results set out in Theorem C.5.4 and Theorem C.5.6.

Theorem C.5.8 (Cauchy stress tensor)**.** *Let φ be a motion and t the associated traction as given in Theorem C.5.4 and Theorem C.5.6, such that $t : (x, n) \mapsto t(x, n)$ is continuously differentiable along its first variable and continuous along its second. Then there exists a symetric tensor T on $\mathbb{R}^N$, which depends only on x, and such that:*

$$t(x, n) = T(x)n.$$

The tensor T is called the Cauchy stress tensor.

Note that the Cauchy stress tensor depends on φ in the sense that it can be rewritten (in the spirit of the Lagrangian description), introducing T^φ which acts only on a body:

$$T(x) := T^\varphi(\varphi(x)).$$

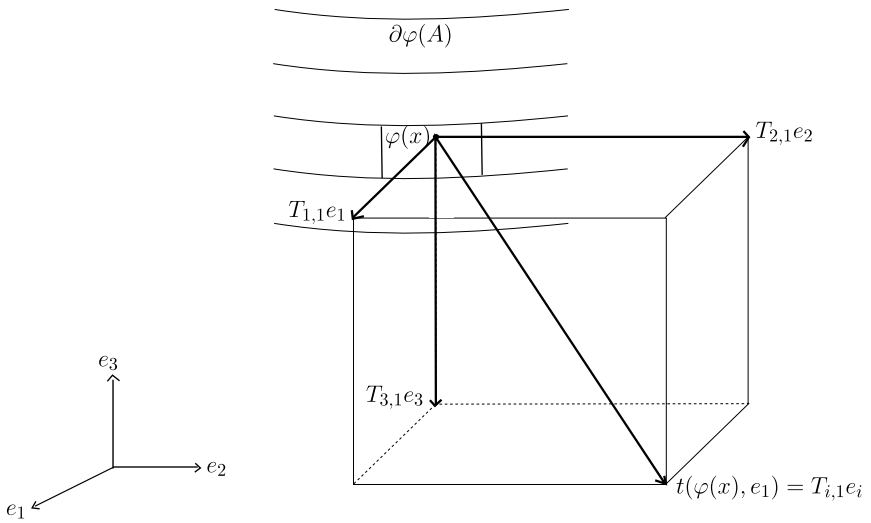


Figure C.10: A representation of the Cauchy stress tensor in $\mathbb{R}^3$

What's more, the Cauchy stress tensor can be used to give a precise decomposition of the forces applied to a small sub-volume, e.g. a parallelepiped, of fluid, which we intuitively described earlier.

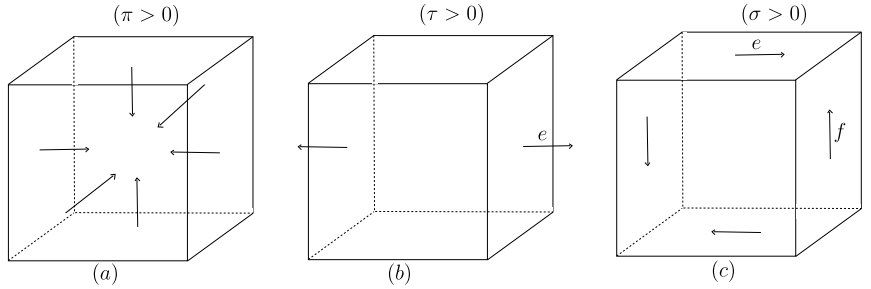


Figure C.11: Special cases from a decomposition of the Cauchy stress tensor. (a) Pressure force, $T^\varphi = -\pi\mathrm{Id}$; (b) Tension following a direction e, $T^\varphi = \tau e \otimes e$: (c) Shear relatively to e and f; $T^\varphi = \sigma(e \otimes f + f \otimes e)$

Nous rappelons le résultat suivant.

Theorem C.5.9 (Reynolds)**.** *Let Ω_t be a regular domain into $\mathbb{R}^N$ and φ be a motion, so that there exists Ω_0 such that $\varphi(\Omega_0) = \Omega_t$. Consider u the velocity field associated with varphi along the boundary, and n an external normal unit vector. Then, for a C^1 function g over Ω_t, one gets:*

$$\frac{d}{dt}\left(\int_{\Omega_t} g\, dx\right) = \int_{\Omega_t} \partial_t g\, dx + \int_{\partial\Omega_t} ug \cdot n\, d\sigma.$$

Corollary C.5.10 (Cauchy first law of motion)**.** *Consider a fluid flowing from a domain $\Omega_0 \in \mathbb{U}_{\text{bod}}$ at time $t = 0$, and let u be its velocity field in Eulerian description. Then, it satisfies for every $(t, x) \in \mathbb{R}_+ \times \Omega_t$, where $\Omega_t = \varphi(t, \Omega_0)$:*

$$\rho\left(\partial_t u + (u \cdot \nabla)u\right) - \operatorname{div}(T) = \rho f_B, \tag{C.5.2}$$

where T is the Cauchy stress tensor and ρ is the density of the fluid.

Proof. It is a matter of using the balance of linear momentum, then the definition of contact and body forces, and we can then apply Theorem C.5.8 and Theorem C.5.9 to get the result, applying in a final step Stokes formula over the surface term since we have:

$$-\int_{\Omega_t} \operatorname{div}(\tau)\, dx = \int_{\partial\Omega_t} T \cdot n\, d\sigma,$$

□

It is also possible to show, using the balance of rotational momentum, that $T = T^T$, which is known as *Cauchy second law of motion*.

C.5.4 Constitutive equations

This brings us to the final section of this chapter. In simple terms, a constitutive equation is a good expression of the Cauchy stress tensor, in the sense that it is a way of writing this tensor for the study of a specific type of material. Since we have established an expression for contact forces using the Cauchy stress tensor, we will henceforth note f for body forces, which we will also sometimes refer to generically as *external forces*.

Let us start by presenting the axiomatics we will be using.

Axiomatic C.5.11. *Consider a fluid whose flow is described by the Lagrangian motion $\varphi : \Omega \to \Omega_t = \varphi(\Omega_0)$.*

(R1) (Principle of determinism) The stresses at a point (t, x), where $x = X^\varphi$ (i.e. x is expressed according to the Eulerian flow description) depend solely on the flow history, i.e. the flow prior to time t. More precisely, if we note:

$$\varphi^t = \left\{y \in \mathbb{U}_{\text{bod}} \;/\; \exists (s, X_0) \in [0, t) \times \Omega_0,\ y = \varphi(s, X_0)\right\}.$$

Then we can write:

$$T(t, x) = T^\varphi(t, X^\varphi) = \mathcal{F}(\varphi^t; t, x), \tag{C.5.3}$$

where $\mathcal{F}$ is a mapping over the set of histories and substantial spatio-temporal point into the symetric tensors space.

(R2) (Principle of local action) Let y be a point belonging to a ring centered at x, of smaller radius $R_{\text{inf}} > 0$. Then the stresses applied to x do not depend on stresses applied to y, nor on the history of y.

(R3) (Principle of material frame-indifference) Let χ be a motion evolving in the same space, and $t^* = t + \alpha$, $\alpha \in \mathbb{R}$. Then there exists a function Q such that for all $t \in \mathbb{R}_+$, $Q(t) \in 0_N(\mathbb{R})$, and verifying:

$$\mathcal{F}(\chi^{t^*}, t^*, x) = Q(t)\mathcal{F}(\varphi^t, t, x)Q(t)^T.$$

Let us comment on these axioms: (R1) asserts that if we look at a fluid particle, i.e. a substantial point, then the stresses punctually associated with it over time depend on the flow history of that particle; (R2) asserts that, in fact, the stresses applied depend only on local history and present, so there is no a priori dependence on distance for constraints, either through particle history or at present time. This is a major difference between classical mechanics and continuum mechanics, since in classical mechanics such an axiom would make the use of stress extremely difficult. The last axiom, (R3), describes the fact that the flow of material is independent of the observer. An equation of the form (C.5.3) is called a *constitutive equation*.

As the title of this section indicates, we are interested in a certain class of fluids, which satisfy the following definition.

Definition C.5.12 (Simple material)**.** *A material is said to be simple if, associated with motion φ, the history φ^t depends only on the associated displacement gradient ∇u.*

Clearly, a simple material is linked a priori to the deformation induced by φ motion, according to what we established earlier. In our framework, we will in fact reinforce the axiom refax:R1, additionally considering the following property.

Axiomatic C.5.13 (Principal of determinism for simple material subject to constraints)**.** *The Cauchy stress tensor is written:*

$$T = \tau + N, \tag{C.5.4}$$

where τ depends on the history of the material through the displacement gradient ∇u, and N accounts for normal stresses.

As we also mentioned in the title, we are focusing on incompressible fluids, so it is worth clarifying what we mean. A fluid, initially studied on a domain Ω_0, is said to be incompressible if the volume of any body defined by the flow history is constant for motion, i.e. noting $\omega_t = \varphi(\omega_0)$, for any $\omega_0 \prec \Omega_0$ if:

$$\frac{d}{dt}\left(\int_{\omega_t} 1 \, dx\right) = 0.$$

Applying the Theorem C.5.9, this is then equivalent to:

$$\int_{\omega_t} \operatorname{div}(u) \, dx = 0.$$

Since this holds for any subdomain ω_0, and a fortiori arguing by regularity over any subdomain ω_t, it follows that for almost all (t, x), we have:

$$\operatorname{div}(u) = 0. \tag{C.5.5}$$

The condition given by (C.5.5) is called the *incompressibility condition*.

The following result establishes the precise structure that the Cauchy stress tensor can have in the case of a simple material.

Theorem C.5.14 (Stress tensor for incompressible simple materials (Noll))**.** *Let φ be a motion acting on an incompressible simple material. Then assuming axioms (R1)–(R3) and C.5.13, we have that the Cauchy stress tensor is expressed as:*

$$T = \tau(\rho, D(u)) - \pi(\rho)\mathrm{Id},$$

where $\pi(\rho)$ describes the vector field of pressure forces, and τ is frame-indifferent and maps tensors into tensors.

At last, we are going to reach our goal. First of all, let us remember that a fluid is a perfectly deformable material medium, so a simple fluid is a perfectly deformable simple material medium. This definition remains vague, but intuitively it means that a fluid is a material which can undergo non-rigid deformations and which can always, under stress, return to a given configuration. To define a fluid rigorously, we need to translate this assertion rigorously, which is done through the use of the *Peer group*[a].

Note that the frame indifference relation expressed in Theorem C.5.14 is referred to as an *isotropy* property. Thus, any simple fluid is an isotropic material.

Finally, note that the conservation of mass, which we have obtained by construction, is expressed in the following form:

$$\frac{d}{dt}M(\omega_t) = 0,$$

which, using the Theorem C.5.9 leads to, for u the :

$$\frac{d}{dt}M(t) = \frac{d}{dt}\left(\int_{\omega_t} \rho \, dx\right) = \int_{\omega_t} \partial_t \rho + \operatorname{div}(\rho u) \, dx = 0.$$

Since the reasoning is valid, as for the incompressibility condition, on any subdomain ω_t of Ω_t, we then obtain the following equation, called *continuity equation*, which is a transport equation and is valid for almost all (t, x), and for all by regularity:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0. \tag{C.5.6}$$

Now, let us specify that we will always assume that the fluid flows in a parabolic pipe, i.e. we will consider that for any $t \in \mathbb{R}_+$, we have $\Omega_t = \Omega_0 := \Omega$. In addition, we need to consider how the fluid flow behaves along the boundary when the latter is a domain (i.e. bounded). We will always assume no slip, which means that the trace of the velocity field is zero on the boundary, and translates into the homogeneous Dirichlet condition:

$$u = 0 \quad \text{on} \quad \mathbb{R}_+ \times \partial\Omega. \tag{C.5.7}$$

For the system of equations to be properly defined, we still need to add initial data for the density ρ and the velocity field in Eulerian description u, i.e. that we consider:

$$\rho_{|t=0} = \rho_0 \quad \text{and} \quad u_{|t=0} = u_0. \tag{C.5.8}$$

Observing that for the fluid flow in Lagrangian description φ, we have the displacement gradient u which coincides with the fluid velocity field in Eulerian description in the Corollary C.5.10, it comes by applying in it the Theorem C.5.14, considering the incompressibility condition (C.5.5), and the continuity equation (C.5.6), then by considering the non-slip condition on the boundary (C.5.7), and finally by fixing initial data as in (C.5.8), that the flow of a simple incompressible fluid is translated by the system:

[a] We refer the reader to [167, Section IV.12.] for more details on this subject. A fluid is then defined as an egalitarian non-solid material.

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ \rho\left(\partial_t u + (u \cdot \nabla)u\right) - \operatorname{div}\left(\tau(\rho, D(u))\right) + \nabla \pi(\rho) = \rho f & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \\ \rho_{|t=0} = \rho_0 & \text{in } \Omega \\ u_{|t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{C.5.9}$$

In the previous case, density is not necessarily constant, so the fluid is said to be non-homogeneous. Otherwise, if the density is assumed to be constant, the fluid is said to be homogeneous, and the system (C.5.9) reduces to:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \operatorname{div}\left(\tau(D(u))\right) + \nabla \pi = f & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \\ u_{|t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{C.5.10}$$

This assumption will often be made throughout the dissertation. Recall that a fluid is said to be Newtonian when the tensor τ is linear relative to the strain rate tensor $D(u)$, non-Newtonian otherwise. For fluids with shear-thinning or shear-thickening behavior, we then recall that the τ tensors obtained verify the following relations, where $\kappa \in \{0, 1\}$ according to considerations on fluid viscosity, which comes down to whether or not we add diffusion to the equation.

- **Carreau**:
$$\tau(M) = \kappa M + (1 + |M|^2)^{\frac{p-2}{2}} M,\ p > 1,$$
- **Carreau-Yasuda**:
$$\tau(M) = \kappa M + (1 + |M|^\alpha)^{\frac{p-2}{\alpha}} M,\ p > 1,\ \alpha > 0,$$
- **Ostwald-De Waele (power law)**:
$$\tau(M) = \kappa M + |M|^{p-2} M,\ p > 1,$$
- **Cross**:
$$\tau(M) = \kappa M + \frac{M}{1 + |M|^{2-p}},\ p > 1,$$
- **Bingham**:
$$\tau(M) = \kappa M + \tilde{\tau}(M), \quad \tilde{\tau}(M) = \begin{cases} \frac{M}{|M|} & \text{if } |M| > 0 \\ [0, 1] & \text{otherwise.} \end{cases}$$
- **Herschel-Bulkley**:
$$\tau(M) = \kappa M + \tilde{\tau}(M), \quad \tilde{\tau}(M) = \begin{cases} \frac{M}{|M|} + |M|^{p-2} M,\ p > 1 & \text{if } |M| > 0 \\ [0, 1] & \text{otherwise.} \end{cases}$$

We refer the reader to [59, 105, 56, 111, 20, 109, 64, 69] for detailed discussions concerning the viscometry and the analysis of such constitutive equations.

L'essence des mathématiques, c'est la liberté.

Georg Cantor.

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Bibliography

[1] Anna Abbatiello and Eduard Feireisl. On a class of generalized solutions to equations describing incompressible viscous fluids. *Ann. Mat. Pura Appl. (4)*, 199(3):1183–1195, 2020.

[2] Gabriel Acosta, Ricardo G. Durán, and Fernando López García. Korn inequality and divergence operator: counterexamples and optimality of weighted estimates. *Proc. Amer. Math. Soc.*, 141(1):217–232, 2013.

[3] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[4] Herbert Amann. *Linear and quasilinear parabolic problems. Vol. 1: Abstract linear theory*, volume 89 of *Monogr. Math., Basel*. Basel: Birkhäuser, 1995.

[5] Herbert Amann. *Linear and quasilinear parabolic problems. Vol. II*, volume 106 of *Monographs in Mathematics*. Birkhäuser/Springer, Cham, 2019. Function spaces.

[6] Fuensanta Andreu-Vaillo, Vicent Caselles, and José M. Mazón. *Parabolic quasilinear equations minimizing linear growth functionals*, volume 223 of *Prog. Math.* Basel: Birkhäuser, 2004.

[7] Julien Aniort, Laurent Chupin, and Nicolae Cîndea. Mathematical model of calcium exchange during haemodialysis using a citrate containing dialysate. *Math. Med. Biol.*, 35(suppl.):87–120, 2018.

[8] S. N. Antontsev, J. I. Díaz, and S. Shmarev. *Energy methods for free boundary problems*, volume 48 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 2002. Applications to nonlinear PDEs and fluid mechanics.

[9] D. G. Aronson and J. Serrin. Local behavior of solutions of quasilinear parabolic equations. *Arch. Ration. Mech. Anal.*, 25:81–122, 1967.

[10] Jean-Pierre Aubin and Arrigo Cellina. *Differential inclusions*, volume 264 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. Set-valued maps and viability theory.

[11] Jean-Pierre Aubin and Halina Frankowska. Controllability and observability of control systems under uncertainty. *Ann. Polon. Math.*, 51:37–76, 1990.

[12] Jean-Pierre Aubin and Hélène Frankowska. *Set-valued analysis*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009. Reprint of the 1990 edition [MR1048347].

[13] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier Analysis and nonlinear Partial Differential Equations*, volume 343 of *Grundlehren der mathematischen Wissenschaften*. Springer Verlag, 2011.

[14] V. Barbu. *Optimal control of variational inequalities*, volume 100 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[15] Viorel Barbu. *Nonlinear differential equations of monotone types in Banach spaces*. Springer Monographs in Mathematics. Springer, New York, 2010.

[16] Viorel Barbu. *Controllability and stabilization of parabolic equations*, volume 90 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2018. Subseries in Control.

[17] Viorel Barbu and Michael Röckner. Stochastic variational inequalities and applications to the total variation flow perturbed by linear multiplicative noise. *Arch. Ration. Mech. Anal.*, 209(3):797–834, 2013.

[18] Viorel Barbu and Aurel Răşcanu. Parabolic variational inequalities with singular inputs. *Differential Integral Equations*, 10(1):67–83, 1997.

[19] H. A. Barnes, J. F. Hutton, and K. Walters. *An introduction to rheology*. Amsterdam etc.: Elsevier, 1989.

[20] Ernst Becker. Simple non-newtonian fluid flows. *Advances in applied mechanics*, 20:177–226, 1980.

[21] Gerald Beer. *Topologies on closed and closed convex sets*, volume 268 of *Math. Appl., Dordr.* Dordrecht: Kluwer Academic Publishers, 1993.

[22] H. Beirão da Veiga. On the Ladyzhenskaya-Smagorinsky turbulence model of the Navier-Stokes equations in smooth domains. The regularity problem. *J. Eur. Math. Soc. (JEMS)*, 11(1):127–167, 2009.

[23] Michel Bernadou and Kamal Hassan. Basis functions for general Hsieh-Clough-Tocher triangles, complete or reduced. *Internat. J. Numer. Methods Engrg.*, 17(5):784–789, 1981.

[24] Luigi C. Berselli and Michael Ružička. Natural second-order regularity for parabolic systems with operators having (p, δ)-structure and depending only on the symmetric gradient. *Calc. Var. Partial Differential Equations*, 61(4):Paper No. 137, 49, 2022.

[25] Luigi C. Berselli and Michael Růžička. Natural second-order regularity for parabolic systems with operators having (p, δ)-structure and depending only on the symmetric gradient. *Calc. Var. Partial Differ. Equ.*, 61(4):49, 2022. Id/No 137.

[26] Lucio Boccardo, Luigi Orsina, and Maria Michaela Porzio. Regularity results and asymptotic behavior for a noncoercive parabolic problem. *J. Evol. Equ.*, 21(2):2195–2211, 2021.

[27] Daniele Boffi, Franco Brezzi, and Michel Fortin. *Mixed finite element methods and applications*, volume 44 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2013.

[28] Michael Böhm. On a nonhomogeneous Bingham fluid. *J. Differential Equations*, 60(2):259–284, 1985.

[29] Jean-Michel Bony. Calcul symbolique et propagation des singularites pour les équations aux dérivées partielles non linéaires. (Symbolic calculus and propagation of singularities for nonlinear partial differential equations). *Ann. Sci. Éc. Norm. Supér. (4)*, 14:209–246, 1981.

[30] F. Boyer. On the penalised HUM approach and its applications to the numerical approximation of null-controls for parabolic problems. *ESAIM, Proc.*, 41:15–58, 2013.

[31] Franck Boyer and Pierre Fabrie. *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*, volume 183 of *Appl. Math. Sci.* New York, NY: Springer, 2013.

[32] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[33] D E Brooks, J W Goodwin, and G V Seaman. Interactions among erythrocytes under shear. *Journal of Applied Physiology*, 28(2):172–177, 1970. PMID: 5413303.

[34] A. Brøndsted and R. T. Rockafellar. On the subdifferentiability of convex functions. *Proceedings of the American Mathematical Society*, 16(4):605–611, 1965.

[35] Miroslav Bulíček, Piotr Gwiazda, Josef Málek, and Agnieszka Świerczewska Gwiazda. On unsteady flows of implicitly constituted incompressible fluids. *SIAM J. Math. Anal.*, 44(4):2756–2801, 2012.

[36] Sun-Sig Byun, Dian K. Palagachev, and Pilsoo Shin. Boundedness of solutions to quasilinear parabolic equations. *J. Differ. Equations*, 261(12):6790–6805, 2016.

[37] Santiago Cano-Casanova, Julián López-Gómez, and Kazuhiro Takimoto. A quasilinear parabolic perturbation of the linear heat equation. *J. Differ. Equations*, 252(1):323–343, 2012.

[38] Kleber Carrapatoso, Jean Dolbeault, Frédéric Hérau, Stéphane Mischler, and Clément Mouhot. Weighted Korn and Poincaré-Korn inequalities in the Euclidean space and associated operators. *Arch. Ration. Mech. Anal.*, 243(3):1565–1596, 2022.

[39] Eduardo Casa and Luis Alberto Fernández. Distributed control of systems governed by a general class of quasilinear elliptic equations. *J. Differ. Equations*, 104(1):20–47, 1993.

[40] Eduardo Casas and Konstantinos Chrysafinos. Analysis and optimal control of some quasilinear parabolic equations. *Math. Control Relat. Fields*, 8(3-4):607–623, 2018.

[41] Eduardo Casas, Luis A. Fernández, and Jiongmin Yong. Optimal control of quasilinear parabolic equations. *Proc. R. Soc. Edinb., Sect. A, Math.*, 125(3):545–565, 1995.

[42] Eduardo Casas, Peter I. Kogut, and Günter Leugering. Approximation of optimal control problems in the coefficient for the p-Laplace equation. I: Convergence result. *SIAM J. Control Optim.*, 54(3):1406–1422, 2016.

[43] Pascal Cherrier and Albert Milani. *Linear and quasi-linear evolution equations in Hilbert spaces*, volume 135 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2012.

[44] Laurent Chupin, Nicolae Cîndea, and Geoffrey Lacour. Variational inequality solutions and finite stopping time for a class of shear-thinning flows. 2023. https://doi.org/10.48550/arXiv.2112.02871.

[45] Andrea Cianchi and Vladimir G. Maz'ya. Second-order regularity for parabolic p-Laplace problems. *J. Geom. Anal.*, 30(2):1565–1583, 2020.

[46] Philippe G. Ciarlet. Interpolation error estimates for the reduced Hsieh-Clough-Tocher triangle. *Math. Comp.*, 32(142):335–344, 1978.

[47] Philippe G. Ciarlet. *Introduction to numerical linear algebra and optimisation*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1989. With the assistance of Bernadette Miara and Jean-Marie Thomas, Translated from the French by A. Buttigieg.

[48] Philippe G. Ciarlet. *Mathematical elasticity. Volume I. Three-dimensional elasticity*, volume 84 of *Class. Appl. Math.* Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), reprint of the 1998 edition edition, 2022.

[49] Nicolae Cindea and Geoffrey Lacour. Null controllability of quasilinear parabolic equations with gradient dependent coefficients. September 2023. https://doi.org/10.48550/arXiv.2304.08022.

[50] Doïna Cioranescu. Sur une classe de fluides non-newtoniens. *Appl. Math. Optim.*, 3(2-3):263–282, 1976/77.

[51] Ioana Cioranescu. *Geometry of Banach spaces, duality mappings and nonlinear problems*, volume 62 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1990.

[52] Dimitri Cobb. Bounded solutions in incompressible hydrodynamics. 2023.

[53] Dimitri Cobb. On the well-posedness of a fractional stokes-transport system. 2023.

[54] Dimitri Cobb and Geoffrey Lacour. Weak solutions of a non-newtonian stokes-transport system (in progress). 2023.

[55] Dimitri A. Cobb. *Étude mathématiques de fluides en interaction avec un champ magnétique*. PhD thesis, Institut Camille Jordan, 2022.

[56] Bernard D. Coleman, H. Markovitz, and Walter Noll. *Viscometric flows of non-newtonian fluids. Theory and experiment*. 1966.

[57] Sergio Conti, Daniel Faraco, and Francesco Maggi. A new approach to counterexamples to L^1 estimates: Korn's inequality, geometric rigidity, and regularity for gradients of separately convex functions. *Arch. Ration. Mech. Anal.*, 175(2):287–300, 2005.

[58] Nicuşor Costea, Alexandru Kristály, and Csaba Varga. *Variational and monotonicity methods in nonsmooth analysis*. Front. Math. Cham: Birkhäuser, 2021.

[59] Philippe Coussot. *Rhéophysique: la matière dans tous ses états*. EDP sciences Les Ulis, 2012.

[60] Anne-Laure Dalibard, Julien Guillod, and Antoine Leblond. Long-time behavior of the stokes-transport system in a channel. 2023.

[61] Raphaël Danchin. Estimates in Besov spaces for transport and transport-diffusion equations with almost Lipschitz coefficients. *Rev. Mat. Iberoam.*, 21(3):863–888, 2005.

[62] Raphaël Danchin. Fourier analysis methods for pdes. 2005.

[63] Klaus Deimling. *Multivalued differential equations*, volume 1 of *De Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, 1992.

[64] Jerome Delhommelle, J. Petravic, and Denis J. Evans. Non-Newtonian behavior in simple fluids. *The Journal of Chemical Physics*, 120(13):6117–6123, 03 2004.

[65] Françoise Demengel. On some nonlinear partial differential equations involving the "1"-Laplacian and critical Sobolev exponent. *ESAIM Control Optim. Calc. Var.*, 4:667–686, 1999.

[66] Françoise Demengel. On some nonlinear equation involving the 1-Laplacian and trace map inequalities. *Nonlinear Anal.*, 48(8, Ser. A: Theory Methods):1151–1163, 2002.

[67] Françoise Demengel. Some existence's results for noncoercive "1-Laplacian" operator. *Asymptot. Anal.*, 43(4):287–322, 2005.

[68] Françoise Demengel and Gilbert Demengel. *Functional spaces for the theory of elliptic partial differential equations.* Universitext. Springer, London; EDP Sciences, Les Ulis, 2012. Translated from the 2007 French original by Reinie Erné.

[69] Abhijit P. Deshpande, J. Murali Krishnan, and P. B. Sunil Kumar, editors. *Rheology of complex fluids. Lectures delivered in the school on 'Rheology of complex fluids', Chennai, India, January 4–9, 2010.* Dordrecht: Springer, 2010.

[70] Jesús I. Díaz, Roland Glowinski, Giovanna Guidoboni, and Taebeom Kim. Qualitative properties and approximation of solutions of Bingham flows: on the stabilization for large time and the geometry of the support. *Revista de la real academia de Ciencias exactas, Fisicas y Naturales*, 104, 2010.

[71] Emmanuele DiBenedetto. *Degenerate parabolic equations.* Universitext. New York, NY: Springer-Verlag, 1993.

[72] Lars Diening, Michael Ružička, and Jörg Wolf. Existence of weak solutions for unsteady motions of generalized Newtonian fluids. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 9(1):1–46, 2010.

[73] Lars Diening, Toni Scharle, and Sebastian Schwarzacher. Regularity for parabolic systems of Uhlenbeck type with Orlicz growth. *J. Math. Anal. Appl.*, 472(1):46–60, 2019.

[74] Leopold Dintenfass. *Blood Viscosity: Hyperviscosity and Hyperviscosaemia.* Kluwer Academic Publisher, 1985.

[75] R. J. DiPerna and P. L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[76] Joseph L. Doob. *Measure theory*, volume 143 of *Grad. Texts Math.* New York: Springer-Verlag, 1994.

[77] G. Duvaut and J.-L. Lions. *Les inéquations en mécanique et en physique.* Travaux et Recherches Mathématiques, No. 21. Dunod, Paris, 1972.

[78] Frank Duzaar and Giuseppe Mingione. Gradient estimates via non-linear potentials. *Am. J. Math.*, 133(4):1093–1149, 2011.

[79] Ivar Ekeland and Roger Témam. *Convex analysis and variational problems.*, volume 28 of *Class. Appl. Math.* Philadelphia, PA: Society for Industrial and Applied Mathematics, unabridged, corrected republication of the 1976 English original edition, 1999.

[80] Tarek M. Elgindi. On the asymptotic stability of stationary solutions of the inviscid incompressible porous medium equation. *Arch. Ration. Mech. Anal.*, 225(2):573–599, 2017.

[81] S. Eloot, J. Vierendeels, and P. Verdonck. Optimisation of solute transport in dialysers using a three-dimensional finite volume model. *Computer Methods in Biomechanics and Biomedical Engineering*, 9(6):363–370, 2006. PMID: 17145670.

[82] Hans Engler, Bernhard Kawohl, and Stephan Luckhaus. Gradient estimates for solutions of parabolic equations and systems. *J. Math. Anal. Appl.*, 147(2):309–329, 1990.

[83] Lawrence C. Evans. The 1-Laplacian, the ∞-Laplacian and differential games. In *Perspectives in nonlinear partial differential equations*, volume 446 of *Contemp. Math.*, pages 245–254. Amer. Math. Soc., Providence, RI, 2007.

[84] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[85] Robin Fåhræus. The suspension stability of the blood. *Physiological Reviews*, 9(2):241–274, 1929.

[86] Eduard Feireisl and Antonín Novotný. *Singular limits in thermodynamics of viscous fluids*. Advances in Mathematical Fluid Mechanics. Birkhäuser/Springer, Cham, 2017. Second edition of [MR2499296].

[87] Yawen Feng, Mikko Parviainen, and Saara Sarsa. On the second-order regularity of solutions to the parabolic p-Laplace equation. *J. Evol. Equ.*, 22(1):17, 2022. Id/No 6.

[88] E. Fernández-Cara, J. Límaco, and I. Marín-Gayte. Theoretical and numerical local null controllability of a quasi-linear parabolic equation in dimensions 2 and 3. *J. Franklin Inst.*, 358(5):2846–2871, 2021.

[89] Enrique Fernández-Cara and Sergio Guerrero. Global Carleman inequalities for parabolic systems and applications to controllability. *SIAM J. Control Optim.*, 45(4):1399–1446, 2006.

[90] Enrique Fernández-Cara, Francisco Guillén, and Rubens R. Ortega. Some theoretical results for viscoplastic and dilatant fluids with variable density. *Nonlinear Anal.*, 28(6):1079–1100, 1997.

[91] Enrique Fernández-Cara, Juan Límaco, Yuri Thamsten, and Denilson Menezes. Local null controllability of a quasi-linear system and related numerical experiments. *ESAIM Control Optim. Calc. Var.*, 29:Paper No. 27, 34, 2023.

[92] Gerald B. Folland. *Real analysis. Modern techniques and their applications.* Pure Appl. Math., Wiley-Intersci. Ser. Texts Monogr. Tracts. New York, NY: Wiley, 2nd ed. edition, 1999.

[93] Avner Friedman. *Partial differential equations of parabolic type.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[94] Avner Friedman. Optimal control for variational inequalities. *SIAM J. Control Optim.*, 24(3):439–451, 1986.

[95] Avner Friedman. Optimal control for parabolic variational inequalities. *SIAM J. Control Optim.*, 25(2):482–497, 1987.

[96] A. V. Fursikov and O. Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[97] Giovanni P. Galdi, Rolf Rannacher, Anne M. Robertson, and Stefan Turek. *Hemodynamical flows*, volume 37 of *Oberwolfach Seminars*. Birkhäuser Verlag, Basel, 2008. Modeling, analysis and simulation, Lectures from the seminar held in Oberwolfach, November 20–26, 2005.

[98] Amrita Ghosh. *Navier-Stokes equations with Navier boundary conditions*. PhD thesis, Université de Pau et des Pays de l'Adour and Universidad del País Vasco, 2018.

[99] Roland Glowinski, Jacques-Louis Lions, and Jiwen He. *Exact and approximate controllability for distributed parameter systems*, volume 117 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2008. A numerical approach.

[100] Roland Glowinski, Jacques-Louis Lions, and Raymond Trémolières. *Numerical analysis of variational inequalities*, volume 8 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam-New York, 1981. Translated from the French.

[101] H. L. Goldsmith, G. R. Cokelet, and P. Gaehtgens. Robin fahraeus: evolution of his concepts in cardiovascular physiology. *American Journal of Physiology-Heart and Circulatory Physiology*, 257(3):H1005–H1015, 1989. PMID: 2675631.

[102] Loukas Grafakos. *Modern Fourier analysis*, volume 250 of *Grad. Texts Math.* New York, NY: Springer, 3rd ed. edition, 2014.

[103] Kidney Disease: Improving Global Outcomes (KDIGO) CKD-MBD Work Group et al. Kdigo clinical practice guideline for the diagnosis, evaluation, prevention, and treatment of chronic kidney disease-mineral and bone disorder (ckd-mbd). *Kidney international. Supplement*, (113):S1, 2009.

[104] Peter M. Gruber. *Convex and discrete geometry*, volume 336 of *Grundlehren Math. Wiss.* Berlin: Springer, 2007.

[105] Etienne Guyon, Jean-Pierre Hulin, Luc Petit, and Catalin D. Mitescu. *Physical hydrodynamics*. Oxford: Oxford University Press, 2nd revised ed. edition, 2015.

[106] Alain Haraux. *Nonlinear evolution equations - global behavior of solutions*, volume 841 of *Lect. Notes Math.* Springer, Cham, 1981.

[107] Davit Harutyunyan and Hayk Mikayelyan. On the fractional Korn inequality in bounded domains: counterexamples to the case $ps < 1$. *Adv. Nonlinear Anal.*, 12(1):Paper No. 20220283, 8, 2023.

[108] F. Hecht. New development in FreeFem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.

[109] D. M. Heyes. Non-Newtonian behaviour of simple liquids. *J. Non-Newton. Fluid Mech.*, 21:137–155, 1986.

[110] Jonathan Himmelfarb and T. Alp Ikizler. Hemodialysis. *New England Journal of Medicine*, 363(19):1833–1845, 2010. PMID: 21047227.

[111] B. E. Hobbs. Deformation of Non-Newtonian materials in simple shear. *Washington DC American Geophysical Union Geophysical Monograph Series*, 16:243–258, January 1972.

[112] Richard M. Höfer. The inertialess limit of particle sedimentation modeled by the Vlasov-Stokes equations. *SIAM J. Math. Anal.*, 50(5):5446–5476, 2018.

[113] Richard M. Höfer. Sedimentation of inertialess particles in Stokes flows. *Commun. Math. Phys.*, 360(1):55–101, 2018.

[114] Richard M. Höfer. Convergence of the method of reflections for particle suspensions in Stokes flows. *J. Differ. Equations*, 297:81–109, 2021.

[115] Richard M. Höfer and Richard Schubert. The influence of Einstein's effective viscosity on sedimentation at very small particle volume fraction. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 38(6):1897–1927, 2021.

[116] Lars Hörmander. *The analysis of linear partial differential operators. III: Pseudo-differential operators.* Class. Math. Berlin: Springer, reprint of the 1994 ed. edition, 2007.

[117] Kazufumi Ito and Karl Kunisch. Optimal control of parabolic variational inequalities. *J. Math. Pures Appl. (9)*, 93(4):329–360, 2010.

[118] Benjamin Jourdain, Tony Lelièvre, and Claude Le Bris. Existence of solution for a micro-macro model of polymeric fluid: The FENE model. *J. Funct. Anal.*, 209(1):162–193, 2004.

[119] Petri Juutinen, Peter Lindqvist, and Juan J. Manfredi. On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation. *SIAM J. Math. Anal.*, 33(3):699–717, 2001.

[120] Tosio Kato. Demicontinuity, hemicontinuity and monotonicity. II. *Bull. Amer. Math. Soc.*, 73:886–889, 1967.

[121] Otared Kavian. *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, volume 13 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Paris, 1993.

[122] Bernd Kawohl and Nickolai Kutev. Global behaviour of solutions to a parabolic mean curvature equation. *Differ. Integral Equ.*, 8(8):1923–1946, 1995.

[123] A. Koshelev. *Regularity problem for quasilinear elliptic and parabolic systems*, volume 1614 of *Lect. Notes Math.* Berlin: Springer-Verlag, 1995.

[124] K. Kuratowski. *Topology. Vol. I. New edition, revised and augmented.* 1966.

[125] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva. *Linear and quasi-linear equations of parabolic type. Translated from the Russian by S. Smith*, volume 23 of *Transl. Math. Monogr.* American Mathematical Society (AMS), Providence, RI, 1968.

[126] Khalid Latrach, M. Aziz Taoudi, and Ahmed Zeghal. Some fixed point theorems of the Schauder and the Krasnosel'skii type and application to nonlinear transport equations. *J. Differential Equations*, 221(1):256–271, 2006.

[127] Claude Le Bris. *Systèmes multi-échelles. Modélisation et simulation*, volume 47 of *Math. Appl. (Berl.)*. Berlin: Springer, 2005.

[128] Antoine Leblond. Well-posedness of the Stokes-transport system in bounded domains and in the infinite strip. *J. Math. Pures Appl. (9)*, 158:120–143, 2022.

[129] Nicolas Lerner. *Carleman inequalities. An introduction and more*, volume 353 of *Grundlehren Math. Wiss.* Cham: Springer, 2019.

[130] John L. Lewis. Regularity of the derivatives of solutions to certain degenerate elliptic equations. *Indiana Univ. Math. J.*, 32(6):849–858, 1983.

[131] Gary M. Lieberman. *Second order parabolic differential equations.* Singapore: World Scientific, 1996.

[132] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris; Gauthier-Villars, Paris, 1969.

[133] J.-L. Lions. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1: Contrôlabilité exacte. (Exact controllability, perturbations and stabilization of distributed systems. Vol. 1: Exact controllability)*, volume 8 of *Rech. Math. Appl.* Paris etc.: Masson, 1988.

[134] J.-L. Lions. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 2: Perturbations. (Exact controllability, perturbations and stabilization of distributed systems. Vol. 2: Perturbations)*, volume 9 of *Rech. Math. Appl.* Paris etc.: Masson, 1988.

[135] J.-L. Lions. Exact controllability, stabilization and perturbations for distributed systems. *SIAM Rev.*, 30(1):1–68, 1988.

[136] J.-L. Lions. The work of G. Stampacchia in variational inequalities. In *Variational analysis and applications*, volume 79 of *Nonconvex Optim. Appl.*, pages 3–30. Springer, New York, 2005. Reprinted from ıt Variational inequalities and complementarity problems [1–24, Wiley, Chichester, 1980; MR0578736].

[137] J.-L. Lions and G. Stampacchia. Variational inequalities. *Comm. Pure Appl. Math.*, 20:493–519, 1967.

[138] Pierre-Louis Lions. *Mathematical topics in fluid mechanics. Volume 1: Incompressible models*. Oxford: Oxford University Press, reprint of the 1996 hardback ed. edition, 2013.

[139] Xu Liu. Insensitizing controls for a class of quasilinear parabolic equations. *J. Differ. Equations*, 253(5):1287–1316, 2012.

[140] Xu Liu and Xu Zhang. On the local controllability of a class of multidimensional quasilinear parabolic equations. *C. R., Math., Acad. Sci. Paris*, 347(23-24):1379–1384, 2009.

[141] J. Málek, J. Nečas, and M. Ružička. On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains: the case $p \geq 2$. *Adv. Differential Equations*, 6(3):257–302, 2001.

[142] Amina Mecherbet and Franck Sueur. A few remarks on the transport-stokes system. 2022.

[143] Arnaud Münch and Diego A. Souza. A mixed formulation for the direct approximation of L^2-weighted controls for the linear heat equation. *Adv. Comput. Math.*, 42(1):85–125, 2016.

[144] Mitsuhiro Nakao and Caisheng Chen. Global existence and gradient estimates for the quasilinear parabolic equations of m-Laplacian type with a nonlinear convection term. *J. Differ. Equations*, 162(1):224–250, 2000.

[145] Mitsuhiro Nakao and Yasuhiro Ohara. Gradient estimates for a quasilinear parabolic equation of the mean curvature type. *J. Math. Soc. Japan*, 48(3):455–466, 1996.

[146] Isaac Newton. *The mathematical principles of natural philosophy. Translated from the the 3rd edition of the* Principia *and annotated by C. R. Leedham-Green*. Cambridge: Cambridge University Press, 2021.

[147] L. Nirenberg. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 13:115–162, 1959.

[148] Herbert Oertel, editor. *Prandtl-essentials of fluid mechanics. With contributions by M. Böhle, D. Etling, U. Müller, K. R. S. Sreenivasan, U. Riedel, and J. Warnatz. Translated by Katherine Mayes.*, volume 158 of *Appl. Math. Sci.* Dordrecht: Springer, 3rd ed. edition, 2010.

[149] Donald Ornstein. A non-equality for differential operators in the L_1 norm. *Arch. Rational Mech. Anal.*, 11:40–49, 1962.

[150] Nikolaos S. Papageorgiou. On the optimal control of strongly nonlinear evolution equations. *J. Math. Anal. Appl.*, 164(1):83–103, 1992.

[151] Maria Michaela Porzio. Regularity and time behavior of the solutions to weak monotone parabolic equations. *J. Evol. Equ.*, 21(4):3849–3889, 2021.

[152] A. R. Pries, D. Neuhaus, and P. Gaehtgens. Blood viscosity in tube flow: dependence on diameter and hematocrit. *American Journal of Physiology-Heart and Circulatory Physiology*, 263(6):H1770–H1778, 1992. PMID: 1481902.

[153] Jan Prüss and Gieri Simonett. *Moving interfaces and quasilinear parabolic evolution equations*, volume 105 of *Monogr. Math., Basel.* Basel: Birkhäuser/Springer, 2016.

[154] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*, volume 146 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker, Inc., New York, 1991.

[155] James C. Robinson, José L. Rodrigo, and Witold Sadowki. *The three-dimensional Navier-Stokes equations.* Number 157 in Cambridge studies in advanced mathematics. Cambridge University Press, 2016.

[156] Tomáš Roubiček. *Nonlinear partial differential equations with applications*, volume 153 of *International Series of Numerical Mathematics.* Birkhäuser Verlag, Basel, 2005.

[157] Thomas J. Sadowski and R. Byron Bird. Non-Newtonian Flow through Porous Media. I. Theoretical. *Transactions of The Society of Rheology*, 9(2):243–250, 09 1965.

[158] Pierre Saramito. *Complex fluids*, volume 79 of *Mathématiques & Applications (Berlin) [Mathematics & Applications].* Springer, Cham, 2016. Modeling and algorithms.

[159] Yoshihiro Sawano. *Theory of Besov spaces*, volume 56 of *Dev. Math.* Singapore: Springer, 2018.

[160] James Serrin. Pathological solutions of elliptic differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 18:385–387, 1964.

[161] R. E. Showalter. *Monotone operators in Banach space and nonlinear partial differential equations*, volume 49 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1997.

[162] J. Simon. Representation of distributions and explicit antiderivatives up to the boundary. In *Progress in partial differential equations: the Metz surveys 2. Proceedings of the conferences given at the University of Metz (France) during the 1992 "Metz Days"*, pages 201–215. Harlow: Longman Scientific & Technical, 1993.

[163] Monty J. Strauss. Variations of Korn's and Sobolev's equalities. In *Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971)*, pages 207–214. Amer. Math. Soc., Providence, R.I., 1973.

[164] Michael Struwe. *Variational methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, fourth edition, 2008. Applications to nonlinear partial differential equations and Hamiltonian systems.

[165] Hans Triebel. *Theory of function spaces*. Mod. Birkhäuser Class. Basel: Birkhäuser, reprint of the 1983 original edition, 2010.

[166] David J. Tritton. *Physical fluid dynamics*. 1977.

[167] Clifford A. III Truesdell. *A first course in rational continuum mechanics. Vol. 1.*, volume 71 of *Pure Appl. Math., Academic Press*. Boston, MA: Academic Press, 2nd ed. edition, 1991.

[168] Marius Tucsnak and George Weiss. *Observation and control for operator semigroups*. Birkhäuser Adv. Texts, Basler Lehrbüch. Basel: Birkhäuser, 2009.

[169] Mingxin Wang. *Nonlinear second order parabolic equations*. CRC Press, 2021.

[170] Michael Wiegner. On the asymptotic behaviour of solutions of nonlinear parabolic equations. *Math. Z.*, 188:3–22, 1984.

[171] Jörg Wolf. Existence of weak solutions to the equations of non-stationary motion of non-Newtonian fluids with shear rate dependent viscosity. *J. Math. Fluid Mech.*, 9(1):104–138, 2007.

[172] Zhuoqun Wu, Jingxue Yin, and Chunpeng Wang. *Elliptic and parabolic equations*. Hackensack, NJ: World Scientific, 2006.

[173] Eberhard Zeidler. *Nonlinear functional analysis and its applications. III: Variational methods and optimization. Transl. from the German by Leo F. Boron*. 1985.

[174] Eberhard Zeidler. *Nonlinear functional analysis and its applications. II/A*. Springer-Verlag, New York, 1990. Linear monotone operators, Translated from the German by the author and Leo F. Boron.

[175] Eberhard Zeidler. *Nonlinear functional analysis and its applications. II/B*. Springer-Verlag, New York, 1990. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.

[176] Songmu Zheng. *Nonlinear evolution equations*, volume 133 of *Chapman Hall/CRC Monogr. Surv. Pure Appl. Math.* Boca Raton, FL: Chapman & Hall/CRC, 2004.