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# On the large scale geometry of semisimple groups: vanishing and non-vanishing of $L^p$ -cohomology of Archimedean and non-Archimedean groups

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# Introduction

## 0.1 Résumé de thèse (en français)

Cette thèse se situe entre la théorie de Lie et la géométrie à grande échelle à la Gromov. En une phrase, on s'intéresse à des invariants de nature cohomologique en géométrie à grande échelle associés à des groupes localement compacts, que l'on étudie plus particulièrement dans le cas des groupes de Lie et de certaines généralisations.

La géométrie à grande échelle concerne l'étude des propriétés asymptotiques des espaces métriques. Autrement dit, on s'intéresse aux propriétés qui restent invariantes sous des applications qui oublient la structure topologique locale et n'imposent que des conditions sur les grandes distances, telles que les quasi-isométries ou les équivalences grossières. On peut placer les groupes localement compacts à base dénombrable d'ouverts sous cette loupe, car ils portent des métriques propres invariantes à gauche [Str74] et deux telles métriques restent grossièrement équivalentes [CdlH16, 4.A.6]. Pour ces groupes on peut définir des notions de cohomologie : on parle de *cohomologie continue* à coefficients dans une représentation continue. Ceci est une construction de nature algébrique qui n'a a priori aucun lien avec la géométrie à grande échelle. De façon assez surprenante, elle donne des invariants à grande échelle si on choisit la représentation adéquate. Pour un groupe localement compact à base dénombrable d'ouverts  $G$  et un réel  $p > 1$ , la cohomologie associée à la représentation régulière sur l'espace  $L^p(G)$ , que l'on appelle *cohomologie  $L^p$  continue* et l'on note  $H_{\text{ct}}^*(G, L^p(G))$ , est un invariant de quasi-isométrie en chaque degré et pour chaque  $p > 1$ . C'est le personnage principal de cette thèse.

Les groupes localement compacts sur lesquels on veut calculer cet invariant sont les groupes de Lie et certaines de leurs généralisations. Les groupes de Lie les plus populaires sont les *groupes de Lie réels semi-simples* en partie à cause de leur riche structure (contrairement aux groupes de Lie résolubles, plus difficiles à dompter) mais aussi à cause de leurs actions sur leurs espaces symétriques riemanniens. Une première généralisation de cette situation consiste à regarder les points des groupes algébriques semi-simples sur des corps locaux. Les corps locaux se divisent en deux groupes : archimédiens, c'est-à-dire les nombres réels et les complexes, et non archimédiens. Les groupes de Lie semi-simples non archimédiens sont notre deuxième famille de groupes localement compacts sur lesquels on aimerait faire des calculs de cohomologie  $L^p$ . Les géométries sur lesquelles ils agissent sont leurs *immeubles de Bruhat-Tits*, qui sont des structures simpliciales jouant le rôle des espaces symétriques dans ce nouveau contexte. Une idée générale est que les groupes de Lie réels et non archimédiens ne doivent pas être si différents les uns des autres.

La notion d'*immeuble* est en fait bien plus générale. Un immeuble est un complexe simplicial obtenu en recollant plusieurs fois un même complexe simplicial, qui représente la combinatoire d'un *groupe de Coxeter* fixé, sous des conditions d'incidence inspirées des espaces symétriques. Les groupes de Coxeter apparaissant dans les immeubles de Bruhat-Tits sont dits *affines*, c'est-à-dire que leurs géométries associées sont des pavages d'un espace euclidien. La théorie de Kac-Moody est un moyen de construire des exemples de groupes (discrets et) localement compacts comme des groupes d'automorphismes d'immeubles. En particulier, on obtient des nouveaux exemples de groupes si le groupe de Coxeter de départ n'est ni fini ni affine. Ces *groupes de Kac-Moody non affines* sont une troisième famille de groupes localement compacts associés à la théorie de Lie auxquels on s'intéressera. Cette fois-ci on s'est éloigné davantage des situations classiques : on peut s'attendre à des comportements nouveaux et exotiques.

Cette thèse concerne des calculs de cohomologie  $L^p$  pour les groupes et les espaces présentés ci-dessus. Notre motivation principale est la question suivante due à Gromov : il prédit un comportement classique de la cohomologie  $L^p$  des groupes semi-simples sur des corps locaux (archimédiens ou pas).

**Questions.** Soit  $G$  un groupe semi-simple de rang  $r \geq 2$  sur un corps local.

- (1) A-t-on  $H_{\text{ct}}^l(G, L^p(G)) = \{0\}$  pour tout  $l = 1, \dots, r-1$  et  $p > 1$  ?
- (2) A-t-on  $H_{\text{ct}}^r(G, L^p(G)) \neq \{0\}$  au moins pour certaines valeurs de  $p$  ?
- (3) Est-ce que l'espace  $H_{\text{ct}}^r(G, L^p(G))$  est séparé pour tout  $p > 1$  ?

Dans les pages qui suivent on présentera les résultats principaux de cette thèse (pour des préliminaires, voir Chapitre 1). Trois sujets sont abordés. Chacun d'entre eux correspond à un article. Les deux premiers concernent les questions qu'on vient d'énoncer, traitant respectivement les questions (1) et (2). Le troisième sujet est indépendant car il concerne des groupes non linéaires. On termine ce résumé en présentant des questions et des projets.

Le premier sujet est l'annulation en degré 2 pour des groupes semi-simples de rang  $\geq 3$  sur des corps locaux, ce que l'on montre dans la plupart des cas [LN23]. Le deuxième est la non annulation en degré maximal pour des immeubles affines et pour des immeubles plus généraux [LN22]. Dans le même cercle d'idées, on obtient des inégalités pour la dimension conforme des immeubles hyperboliques au sens de Gromov. Le troisième sujet concerne la cohomologie  $L^2$  de généralisations des groupes semi-simples et des applications à l'équivalence mesurable [LN21].

### 0.1.1 Annulation en degré 2

Un des résultats principaux de cette thèse concerne la question (1) : on montre l'annulation de la cohomologie  $L^p$  en degré 2 pour la plupart des groupes de Lie semi-simples de rang au moins 3 sur des corps locaux.

On commencera par survoler des contributions antérieures pointant vers une réponse affirmative à cette question. La plupart d'entre elles concerne le cas du degré 1. Puis, on donne un énoncé plus précis de notre résultat et on ébauche sa preuve.

**Le cas du degré 1 pour les groupes de Lie** L'annulation en degré 1 pour tout  $p > 1$  pour des groupes semi-simples *réels* de rang au moins 2 a été montrée

par Pansu en 1999 en utilisant la cohomologie  $L^p$  de de Rham (d'abord en tant que prépublication, puis apparu dans [Pan07]). En fait il a démontré un résultat plus fort.

**Théorème 0.1.1.** [Pan07, Théorème 1] *Soit  $M$  une variété homogène. On a une trichotomie :*

- soit le groupe d'isométries de  $M$  est une extension compacte d'un groupe de Lie résoluble unimodulaire,
- soit  $M$  est quasi-isométrique à un espace homogène de courbure sectionnelle strictement négative,
- autrement  $L^p H_{\text{dR}}^1(M) = \{0\}$  pour tout  $p > 1$ .

Ceci donne une réponse affirmative à la question de Gromov dans le cas des groupes réels semi-simples.

Bader, Furman, Gelfand et Monod ont montré l'annulation du premier groupe de cohomologie continue des groupes de Lie *simples* de rang au moins 2 sur des corps locaux (archimédiens ou pas) à coefficients dans des représentations isométriques sur des espaces de la forme  $L^p(X, \mu)$ , où  $(X, \mu)$  est un espace de Borel standard et  $p > 1$  [BFGM07]. Leurs techniques marchent aussi pour des groupes semi-simples, mais où chaque facteur simple est de rang au moins 2.

Finalement, de Cornulier et Tessera étendent la trichotomie de Pansu aux groupes semi-simples sur des corps de caractéristique zéro en adaptant ses arguments à la cohomologie continue des groupes [dCT11]. Leur trichotomie n'est certes valide que pour des groupes sur des corps de caractéristique zéro, mais leurs arguments pour l'annulation pour des groupes semi-simples de rang au moins 2 sont valides aussi en caractéristique positive.

**Théorème 0.1.2.** [dCT11, Theorem 1] *Soit  $G$  un groupe de Lie semi-simple de rang  $r \geq 2$  sur un corps local. On a  $H_{\text{ct}}^1(G, L^p(G)) = \{0\}$  pour tout  $p > 1$ .*

Les démonstrations des contributions présentées ci-dessus ne sont pas équivalentes, mais partagent certains arguments dynamiques. Notamment, le phénomène de Mautner apparaît dans une de ses plusieurs formes dans tous ces travaux.

Cet énoncé est valable aussi pour les versions correspondantes de cohomologie  $L^p$  des géométries sur lesquelles les groupes semi-simples agissent : les espaces symétriques et les immeubles de Bruhat-Tits.

**Le cas du degré 1 pour les immeubles affines** Les immeubles de Bruhat-Tits sont des immeubles affines sur lesquels les groupes semi-simples non archimédiens admettent des actions propres et cocompactes. Tous les immeubles affines de dimension  $\geq 3$  proviennent de telles actions. D'autre part, il y a des immeubles affines exotiques de dimension 2, dont le groupe complet d'automorphismes est petit (par exemple, discret ou trivial). Lécureux, de la Salle et Witzel montrent que l'annulation de la cohomologie  $L^p$  en degré 1 persiste pour certains de ces immeubles.

**Théorème 0.1.3.** [LdlSW20, Theorem A] *Soit  $X$  un immeuble de type  $\tilde{A}_2$  localement fini. On a  $L^p H^1(X) = \{0\}$  pour tout  $p > 1$ .*

La preuve de ce résultat est significativement différente de celle dans le cas des groupes de Lie puisqu'on ne peut pas invoquer d'arguments dynamiques tels que le phénomène de Mautner. À la place, on utilise l'harmonicité et les marches aléatoires.

**Degrés supérieurs** Bourdon et Rémy montrent l'annulation de la cohomologie  $L^p$  de certains groupes de Lie réels simples de rang supérieur en degrés supérieurs [BR20]. Plus précisément, pour certains groupes de Lie simples qu'ils appellent *admissibles* ils prouvent le résultat suivant : pour plusieurs degrés  $k$  il existe des constantes  $p(k) > 1$  (qui dépendent du groupe) telles que l'on a l'annulation de la cohomologie  $L^p$  en degré  $k$  pour tout  $1 < p < p(k)$ . La dualité de Poincaré leur permet d'étendre ce résultat pour  $p$  assez grand, au moins pour des degrés assez élevés (en particulier cet argument de dualité ne concerne pas les degrés au-dessous du rang). Leur méthodes consistent à montrer une version convenable de la suite spectrale de Hochschild-Serre pour la cohomologie  $L^p$  et d'invoquer la description de la cohomologie  $L^p$  de l'espace hyperbolique réel par Pansu [Pan08]. En fait, leur preuve s'applique aussi aux groupes réels simples non admissibles, mais leurs conditions sur la constante  $p(k)$  sont bien plus restrictives (dans ce cas la constante  $p(k)$  tend vers 1 à  $k$  fixé lorsque le rang tend vers l'infini).

Initialement on voulait montrer un énoncé analogue à celui de Bourdon et Rémy dans le cas non archimédien. Il s'est avéré que nos méthodes s'appliquent aussi au cas réel, mais seulement pour des valeurs de  $p$  assez grandes. Le résultat qu'on démontre est le suivant.

**Théorème 0.1.4.** ([LN23, 0.1], voir Theorem 2.0.1 dans le texte) Soit  $F$  un corps local (archimédien ou pas) et soit  $G$  un des groupes suivants :

- $SL(4, D)$ , où  $D$  est une algèbre centrale à division sur  $F$ ,
- un groupe de Lie simple sur  $F$  de rang  $r \geq 4$  qui n'est pas de type  $D_4$  ou de type exceptionnel,
- ou un groupe de Lie semi-simple, non simple sur  $F$  de rang  $r \geq 3$ .

Alors il existe une constante  $p(G) \geq 1$  telle que pour tout  $p > p(G)$  :

$$H_{\text{ct}}^2(G, L^p(G)) = \{0\}.$$

De plus, quand  $F$  est non archimédien on a  $p(G) = 1$ .

Dans le reste de cette sous-section, on esquisse la preuve de ce théorème. Trois types d'arguments sont utilisés : algébriques, dynamiques et combinatoires.

**Invariance par quasi-isométrie et suite spectrale** La stratégie initiale est celle de [BR20].

D'abord, on utilise l'invariance par quasi-isométrie de la cohomologie  $L^p$  pour identifier  $H_{\text{ct}}^2(G, L^p(G))$  à  $H_{\text{ct}}^2(P, L^p(P))$ , où  $P$  est un sous-groupe parabolique maximal de  $G$ . Ce sous-groupe parabolique admet une décomposition de Lévi,  $P = MSU$ . La version de la suite spectrale de Hochschild-Serre pour la cohomologie  $L^p$  dans [BR20] nous permet d'obtenir un isomorphisme linéaire :

$$H_{\text{ct}}^2(P, L^p(P)) = H_{\text{ct}}^1(M, L^p(M, V_p))$$

avec le premier groupe de cohomologie continue  $H_{\text{ct}}^1(M, L^p(M, V_p))$  du facteur de Lévi  $M$  à coefficients dans l'espace  $L^p$  à valeurs banachiques  $L^p(M, V_p)$ . L'espace de Banach  $V_p$  est en fait  $H_{\text{ct}}^1(SU, L^p(SU))$ . Le problème technique principal vient du fait que le  $M$ -module continu  $L^p(M, V_p)$  est de croissance exponentielle, avec un taux de croissance exponentiel qui est trop grand pour appliquer la propriété (T) renforcée de Lafforgue (au moins avec les constantes actuelles) et obtenir ainsi l'annulation voulue.

**Arguments dynamiques** Notre substitut à la propriété (T) de Lafforgue est de passer à un sous-groupe résoluble  $R$ , cocompact dans  $M$  et *non unimodulaire* afin de créer des contractions grâce à sa fonction modulaire  $\Delta_R$  et adapter des techniques dynamiques de [dCT11] pour énoncer une version non isométrique du phénomène de Mautner.

Plus précisément, si on note  $\pi$  la représentation de  $R$  sur  $L^p(R, V_p)$  et  $\pi_0$  la représentation de  $R$  sur  $V_p$ , on a pour chaque  $g \in R$  :

$$\|\|\pi(g)\|\|_{L^p(R, V_p)}^p = \Delta_R(g)^{-1} \|\|\pi_0(g)\|\|_{V_p}^p.$$

L'idée est que bien que  $\|\|\pi_0(g)\|\|_{V_p}$  peut grandir très vite, on espère contrer sa croissance en utilisant  $\Delta_R(g)$  et ainsi trouver des éléments qui contractent la norme  $L^p$ . La conclusion de cette analyse est que l'annulation de l'espace  $H_{\text{ct}}^1(R, L^p(R, V_p))$  est entraînée par la présence de deux éléments qui contractent la norme  $L^p$  dans deux directions bien choisies. Ceci n'est pas difficile à faire dans le cas d'un groupe semi-simple, non simple.

Le plus dur est de montrer l'existence de telles contractions dans le cas d'un groupe simple. Ceci est une sorte de compétition entre la dilatation exponentielle de la représentation  $\pi_0$  sur  $V_p$  et la contraction exponentielle de  $\Delta_R$  dans certaines directions. Pour montrer que  $\Delta_R$  gagne cette compétition pour les groupes de l'énoncé 0.1.4, on contrôle d'abord les normes d'opérateur de  $\pi_0$  par un terme qui peut être écrit explicitement en fonction des données combinatoires associées au groupe de départ. Cette étape utilise l'hyperbolicité du groupe de Heintze  $H = S \times U$ , plus précisément elle utilise des bornes précises sur la distorsion exponentielle de  $U$  à l'intérieur de  $H$ .

**Combinatoire et théorie de Lie** En utilisant la classification des groupes semi-simples sur des corps locaux, on réduit le problème d'existence d'éléments contractants à une étude combinatoire au cas par cas de systèmes de racines avec des multiplicités. La présence des multiplicités dans cette étude nous force à revenir à des présentations classiques des groupes de Lie simples.

Le point principal de la partie combinatoire de cette preuve est que, pour les familles infinies de systèmes de racines ( $A_r, B_r, C_r, BC_r$  et  $D_r$ ) il existe toujours un choix de sous-groupe parabolique maximal tel que notre contrôle (du logarithme) de la dilatation exponentielle croît *linéairement* avec le rang et (le logarithme de) la contraction exponentielle de la fonction modulaire est *quadratique* en fonction du rang (au moins pour ces directions bien choisies). Cette heuristique, qui est a priori de caractère asymptotique, commence à fonctionner assez tôt : à partir de  $r \geq 3$  pour  $A_r$ , de  $r \geq 4$  pour  $B_r, C_r$  et  $BC_r$  et de  $r \geq 5$  pour  $D_r$ .

### 0.1.2 Non annulation en degré maximal et dimension conforme des immeubles hyperboliques

On traite maintenant la question (2), c'est-à-dire, la non annulation en degré égal au rang. On explique l'évolution de ce phénomène, en commençant par les situations les plus classiques et en se tournant vers les plus exotiques.

**Non annulation pour les groupes de Lie réels** Dans notre formulation de la question (2), Bourdon et Rémy donnent une réponse affirmative à cette question pour les groupes semi-simples réels [BR21].

**Théorème 0.1.5.** [BR21, Theorem A] Soit  $G$  un groupe de Lie semi-simple réel de rang  $r \geq 2$ . Pour  $p$  assez grand on a :

$$\overline{H}_{\text{ct}}^r(G, L^p(G)) \neq \{0\}.$$

Par ailleurs, dans [BR20] ils montrent aussi l'annulation de  $\overline{H}_{\text{ct}}^r(G, L^p(G))$  pour  $p > 1$  assez petit pour des groupes admissibles. Il y a toujours un intervalle en  $p$  pour lequel le comportement de  $H_{\text{ct}}^r(G, L^p(G))$  reste inconnu.

**Non annulation pour des immeubles affines** On montre la non annulation de la cohomologie  $L^p$  en degré égal au rang pour des immeubles affines localement finis. La situation est maintenant plus simple : la non annulation est valable pour tout  $p > 1$ . Plus précisément :

**Théorème 0.1.6.** ([LN22, 1.2], voir Theorem 3.0.2 dans le texte) Soit  $X$  un immeuble affine localement fini de dimension  $n$ . On a  $\ell^p \overline{H}^n(X) \neq \{0\}$  pour tout  $p > 1$ . En particulier, tout groupe semi-simple  $G$  de rang  $n$  sur un corps local non archimédien satisfait  $\overline{H}_{\text{ct}}^n(G, L^p(G)) \neq \{0\}$  pour tout  $p > 1$ .

Le résultat 0.1.6 est montré en utilisant une formule d'harmonicit   à la Steinberg sur l'homologie localement finie en degr   maximal de l'immeuble et en montrant la convergence de sa norme  $\ell^p$  pour tout  $p > 1$ .

**Immeubles non affines et dimension cohomologique virtuelle** On quitte le monde des groupes simples sur des corps locaux et on s'attaque    des immeubles localement finis arbitraires.

La premi  re question    se poser dans ce contexte est quelle est notre nouvelle d  finition de *rang* d'un immeuble, vu que l'on n'a plus une action d'un groupe semi-simple. Si on a une action d'un groupe  $G$  sur l'immeuble, on pourrait parler de la *dimension cohomologique virtuelle*  $\text{vcd}(G)$  de ce groupe. Si on n'a pas de groupe d'automorphismes, on peut toujours parler du plus grand entier pour lequel la cohomologie  $\ell^p$  est non triviale, au moins pour certaines valeurs de  $p > 1$ . On montre que cet dernier entier coince toujours avec la dimension cohomologique virtuelle  $\text{vcd}_{\mathbb{R}}(W)$  sur  $\mathbb{R}$  du groupe de Weyl  $(W, S)$  d'un appartement de l'immeuble.

On note  $e_{\mathbf{q}}(W)$  le taux de croissance logarithmique pond  r   du syst  me de Coxeter  $(W, S)$ . Rappelons que, par l'alternative de Tits forte pour les groupes de Coxeter, un groupe de Coxeter non affine  $W$  est de croissance exponentielle, et ainsi  $e_{\mathbf{q}}(W) > 0$ . Le r  sultat pr  cis est le suivant.

**Th  or  me 0.1.7.** ([LN22, 1.4], voir Theorem 3.3.8 dans le texte) Soit  $X$  la r  alisation de Davis d'un immeuble de groupe de Weyl  $(W, S)$  et de vecteur d'  paisseur finie  $\mathbf{q} + 1$ , o    $\mathbf{q} \geq 2$ . Soit  $d = \text{vcd}_{\mathbb{R}}(W)$ . Alors :

- pour tout  $1 < p < 1 + e_{\mathbf{q}}(W)^{-1}$  on a  $\ell^p \overline{H}^d(X) \neq \{0\}$ ,
- pour tout  $k > d$  et  $p > 1$  on a  $\ell^p H^k(X) = \{0\}$ .

La preuve consiste à utiliser la *réalisation de Bestvina* de l'immeuble, qui est définie de façon à ce que sa dimension soit  $d = \text{vcd}_{\mathbb{R}}(W)$ . L'affirmation concernant la non annulation en ce degré a la même preuve que celle du Théorème 0.1.6. L'affirmation concernant l'annulation provient de l'invariance par quasi-isométries de la cohomologie  $\ell^p$ .

**Exposants critiques pour quelques immeubles non affines** Intéressons-nous maintenant à la dépendance en  $p$  de la non annulation : on peut se demander si la borne supérieure  $1 + e_{\mathbf{q}}(W)^{-1}$  de non annulation de cohomologie  $\ell^p$  (réduite) du Théorème 0.1.7 est optimale. On montre que c'est le cas, sous l'hypothèse où les appartements sont des complexes simpliciaux proches des variétés orientables.

**Théorème 0.1.8.** (*[LN22, 1.1], voir Theorem 3.2.6 dans le texte*) Soit  $(W, S)$  un complexe de Coxeter tel que son complexe de Davis  $\Sigma$  est une pseudovariété orientable de dimension  $n$ . Soit  $X$  la réalisation de Davis d'un immeuble de groupe de Weyl  $(W, S)$  et de vecteur épaisseur fini  $\mathbf{q} + 1$ , où  $\mathbf{q} \geq 2$ . Alors on a :

$$1 + e_{\mathbf{q}}(W)^{-1} = \sup\{p > 1 \mid \ell^p \overline{H}^n(X) \neq \{0\}\}.$$

L'annulation montrée dans ce théorème découle du fait que moyennner sur les préimages d'une rétraction réduit la norme  $\ell^p$ , et que notre formule de Steinberg est déjà moyennée sur ces préimages. De ce théorème et de l'invariance de la cohomologie  $\ell^p$  par quasi-isométries on déduit que le taux de croissance  $e_{\mathbf{q}}(W)$  est un nouveau invariant de quasi-isométrie pour les immeubles de Davis localement finis dont les appartements sont des pseudovariétés orientables.

On a constaté plus tard que l'exposant critique pour la cohomologie  $\ell^p$  réduite en degré maximal apparaissant dans le Théorème 0.1.8 a la même expression que la dimension conforme d'un immeuble fuchsien [Bou00].

**Dimension conforme des immeubles hyperboliques** On élabore sur cette dernière remarque : intéressons-nous au cas des immeubles (localement finis) hyperboliques au sens de Gromov. On se demande notamment s'il y a un lien dans un cadre plus général entre l'exposant critique du Théorème 0.1.8 et la dimension conforme.

Nos techniques en degré maximal donnent une première borne inférieure de la dimension conforme d'un immeuble hyperbolique au sens de Gromov dont les appartements sont des pseudovariétés. Clais avait déjà montré des bornes plus fines pour certains immeubles Gromov-hyperboliques [Cla17]. Ceci nous a motivé à "dualiser" nos techniques en homologie  $\ell^p$  en degré maximal et les adapter au premier groupe de cohomologie  $\ell^p$ . Il n'est pas difficile de montrer que pour tout immeuble hyperbolique au sens de Gromov  $X$  de vecteur épaisseur fini  $\mathbf{q} + 1$ , avec  $\mathbf{q} \geq 2$ , il existe une métrique visuelle  $d_{\mathbf{q}}$  sur le bord d'un appartement  $\Sigma$  tel que

$$\text{Confdim}(\partial X) \leq \text{Hausdim}(\partial \Sigma, d_{\mathbf{q}})(1 + e_{\mathbf{q}}(W)^{-1}).$$

Ceci est la borne supérieure de Clais. Nos méthodes en utilisant la cohomologie  $\ell^p$  en degré 1 étendent la borne inférieure de Clais à tout immeuble hyperbolique au sens de Gromov.

**Théorème 0.1.9.** (*[LN22, 1.5], voir Theorem 3.0.5 dans le texte*) Soit  $(W, S)$  un système de Coxeter hyperbolique au sens de Gromov,  $\Sigma$  le complexe de Davis de  $(W, S)$  et  $X$  la réalisation de Davis d'un immeuble de groupe de Weyl  $(W, S)$  et de vecteur épaisseur  $\mathbf{q} + 1$ , avec  $\mathbf{q} \geq 2$ . On a :

$$\text{Confdim}(\partial\Sigma)(1 + e_{\mathbf{q}}(W)^{-1}) \leq \text{Confdim}(\partial X).$$

L'idée de la preuve est, comme en degré maximal, de moyennner sur les préimages des rétractions et estimer la convergence de certaines normes  $\ell^p$ . L'ingrédient additionnel est d'utiliser la relation entre dimension conforme et le premier groupe de cohomologie  $\ell^p$  par des propriétés de séparation d'espaces fonctionnels au bord [BK15, 3.8].

### 0.1.3 Groupes simples de présentation finie et équivalence mesurable

Le dernier sujet de cette thèse est indépendant des deux derniers. Il concerne le problème de distinguer des groupes modulo des relations d'équivalence en utilisant des invariants cohomologiques. Ici on s'intéresse à distinguer des *groupes simples de présentation finie*. À cette date, on connaît peu d'exemples de tels groupes, parmi lesquels on trouve : les groupes de Higman-Thompson et des variantes [SWZ19], les groupes de Burger-Mozes agissant sur des produits d'arbres [BM00] et les réseaux (non uniformes) de Kac-Moody non affines sur des corps finis agissant sur des produits d'immeubles [CR06]. Une tâche intéressante est de séparer certains de ces groupes du point de vue de la quasi-isométrie ou de l'équivalence mesurable.

**Classes d'équivalence mesurable des groupes simples** On connaît déjà des exemples de familles infinies de groupes simples de présentation finie qui sont deux à deux non quasi-isométriques : ceci a été fait d'abord pour certains réseaux de Kac-Moody non affines [CR06] et puis pour des variantes à la Röwer-Nekrashevych du groupe de Thompson [SWZ19]. On montre l'analogie de ces résultats pour l'équivalence mesurable.

**Théorème 0.1.10.** (*[LN21, 1.1], voir Theorem 4.0.1 dans le texte*) Il y a une quantité infinie de classes d'équivalence mesurable contenant des groupes simples de présentation finie avec la propriété (T) de Kazhdan. Les groupes en question sont des réseaux de Kac-Moody sur des corps finis avec des groupes de Weyl non affines bien choisis.

Ceci est fait en étudiant l'annulation et la non annulation de leurs nombres de Betti  $\ell^2$ . En effet, Gaboriau a montré que la suite des nombres de Betti  $\ell^2$  (modulo proportionnalité sur la suite) est un invariant d'équivalence mesurable pour les groupes dénombrables [Gab02]. L'annulation d'un nombre de Betti  $\ell^2$  est un invariant de quasi-isométrie (et même d'équivalence grossière [SS18]) et ainsi on récupère une famille infinie de groupes simples de présentation finie deux à deux non quasi-isométriques comme dans [CR09] ou [SWZ19].

**Des groupes discrets aux groupes topologiques** Pour montrer ce résultat, la philosophie générale est de voir nos candidats discrets  $\Lambda$  comme des réseaux à l'intérieur de groupes topologiques ambiants  $L$  et d'utiliser la structure plus riche

de  $L$  pour obtenir de l'information sur  $\Lambda$ . Dans notre cas, Petersen a montré que les suites des nombres de Betti  $\ell^2$  d'un réseau  $\Lambda$  et de son groupe localement compact à base dénombrable d'ouverts ambiant  $L$  sont proportionnelles [KPV15].

Il reste à comprendre le groupe  $L$  et sa suite de nombres de Betti  $L^2$ . Nos candidats sont des groupes de Kac-Moody discrets sur des corps locaux. Un groupe de Kac-Moody discret  $\Lambda$  agit sur un immeuble, et si on note  $G$  sa complétion dans le groupe d'automorphismes de l'immeuble (que l'on appelle *groupe de Kac-Moody complet*), alors  $\Lambda$  est un réseau dans  $L := G \times G$ .

En utilisant un nouveau point de vue sur la méthode de Garland, Dymara et Januszkiewicz obtiennent une formule pour les nombres de Betti  $\ell^2$  d'un groupe de Kac-Moody complet  $G$  [DJ02]. Ce que cette formule dit est que l'annulation ou la non annulation d'un nombre de Betti  $\ell^2$  correspond à des conditions de topologie combinatoire sur le groupe de Weyl  $W$  de l'immeuble. On étudie la combinatoire du groupe de Weyl et on donne un critère pour prescrire la non annulation d'un nombre de Betti  $\ell^2$  en degré élevé. En jouant avec une famille infinie bien choisie de groupes de Coxeter, on construit une famille dénombrable de réseaux de Kac-Moody avec des suites de nombres de Betti  $\ell^2$  non proportionnelles.

#### 0.1.4 Perspectives et questions

On liste des perspectives de recherche future. L'ordre dans lequel les items suivants sont listés correspond à leur proximité aux sujets présentés ci-dessus.

**Annulation en degrés supérieurs pour des groupes semi-simples** Une direction naturelle que l'on pourrait suivre est de montrer l'annulation de cohomologie  $L^p$  en degrés  $\geq 3$  pour des groupes semi-simples de rang au moins 4. Notre méthode pour montrer l'annulation en degré 2 rencontre un obstacle majeur : on ne peut utiliser la suite spectrale de Hochschild-Serre que si les espaces de cohomologie  $L^p$  du radical résoluble sont séparés en tout degré. Puisque dans notre cas on travaillait avec un parabolique maximal et que la cohomologie  $L^p$  du radical résoluble du parabolique est concentrée en degré 1 (au moins pour  $p$  assez grand) et que l'on sait comment montrer qu'une cohomologie en degré 1 est séparée, on a pu exploiter la suite spectrale. Le problème est que pour des degrés supérieurs, on voudrait faire apparaître des espaces de cohomologie en degrés  $> 1$  et on devrait montrer que ces espaces sont séparés avant d'utiliser la suite spectrale.

À cette date on a deux méthodes capables de faire ceci. D'une part, Pansu a montré que pour la plupart des valeurs de  $p$  la cohomologie  $L^p$  de l'espace hyperbolique réel est séparée en tout degré en identifiant ces espaces de cohomologie à des espaces fonctionnels sur le bord [Pan08].

D'autre part, on peut utiliser des méthodes hilbertiennes pour  $L^2$ , combinées à l'*interpolation* de Riesz-Thorin pour passer à  $L^p$ . Ceci a été fait pour des variétés dans [Loh98] et pour des immeubles dans [DJ02]. Cette méthode a des limitations sur les valeurs de  $p$  car elle implique une sorte de stabilité : le comportement pour  $L^p$  est le même que pour  $L^2$ . Or il est connu dans certaines situations que le comportement de la cohomologie  $L^p$  diffère de la cohomologie  $L^2$  si on s'éloigne trop de  $p = 2$ . Par exemple, pour un immeuble fuchsien cocompact triangulaire  $X$ , la formule de [DJ02] donne de l'annulation de  $\ell^p H^1(X)$  pour les valeurs de  $p$  à laquelle

elle s'applique, cependant, pour  $p > \text{Confdim}(\partial X)$  on a  $\ell^p H^1(X) \neq \{0\}$ .

Pour résumer, on veut dire qu'avec nos techniques actuelles, la prochaine étape dans l'étude de la question de Gromov (1) est d'attaquer d'abord la question (3) et plus généralement d'obtenir des critères pour garantir que des espaces de cohomologie sont séparés en degrés  $> 1$ .

**Dimension conforme des groupes de Coxeter et des immeubles hyperboliques** Concernant nos bornes sur la dimension conforme d'un immeuble hyperbolique  $X$ , on peut s'intéresser à la relation entre  $\text{Confdim}(\partial X)$  et ses bornes supérieures ou inférieures. On ne devrait pas avoir égalité en toute généralité pour un immeuble hyperbolique au sens de Gromov quelconque, mais on pourrait chercher un critère géométrique/combinatoire sur le groupe de Weyl associé qui garantirait l'égalité (ou l'inégalité stricte !).

**Exposants critiques pour des groupes de Lie réels** On peut aussi se demander si le Théorème 0.1.8, sur notre exposant critique pour des immeubles modélés sur des pseudovariétés orientables a un analogue pour des groupes réels. En effet, dans [BR21] Bourdon et Rémy montrent la non annulation en degré égal au rang pas seulement pour des groupes de Lie réels semi-simples, mais aussi pour une classe plus large de groupes de Lie réels résolubles  $R$  de la forme  $R = \mathbb{R}^r \ltimes N$  où  $N$  est un groupe de Lie nilpotent avec des conditions supplémentaires. On peut considérer l'exposant critique  $\inf\{p > 1, L^p H_{\text{dR}}^r(R) \neq \{0\}\}$  (proposé par de Cornulier [BR21, 0.4 remark 4]), qui est invariant par quasi-isométries (et par équivalence grossière). Est-ce que cet invariant coïncide avec un autre invariant numérique de quasi-isométrie de nature géométrique qui peut être défini sans faire référence à la cohomologie ?

**Invariants d'équivalence mesurable à partir de la cohomologie  $L^p$**  Il est naturel de se demander si l'invariance projective de la suite de nombres de Betti  $\ell^2$  par équivalence mesurable prouvée par Gaboriau a un analogue pour  $p \neq 2$ . On n'a certainement pas une théorie des nombres de Betti  $\ell^p$ , donc ce n'est pas de cette façon que l'on devrait formuler une adaptation. Le résultat de Gaboriau implique que l'annulation de l'espace de cohomologie  $L^2$  réduite en un certain degré est un invariant d'équivalence mesurable. La question naturelle est donc : pour  $p > 1$  et  $k \in \mathbb{N}$  fixés, est-ce que l'annulation de la cohomologie  $L^p$  réduite en degré  $k$  est un invariant d'équivalence mesurable ? (ou peut-être il faut changer l'équivalence mesurable par une version plus quantitative, qui pourrait dépendre de  $p$ ).

**Cohomologie  $L^p$  en degré 1 des groupes de Kac-Moody complets non affines** La question suivante m'a été communiquée par Jean Lécureux. La cohomologie bornée en degré 2 et la cohomologie  $L^p$  en degré 1 partagent des propriétés semblables. Caprace et Fujiwara ont montré que la cohomologie bornée en degré 2 des groupes de Kac-Moody complets non affines est non triviale. Est-ce que leurs groupes de cohomologie  $L^p$  en degré 1 sont non triviaux aussi pour  $p$  assez grand ?

### 0.1.5 Structure de la thèse

Ce manuscrit est divisé en quatre chapitres (on ne compte pas cette introduction comme un chapitre) et est écrit en anglais.

Le premier chapitre introduit les notions qui seront utilisées par la suite. On définit la cohomologie continue des groupes, la cohomologie  $L^p$  et les immeubles. Après ceci, on compile des résultats connus (la plupart du temps sans preuves) autour de la cohomologie  $L^p$  des groupes semi-simples. On s'intéresse d'abord aux résultats sur la cohomologie  $L^p$  des groupes simples de rang 1 et par extension des espaces hyperboliques au sens de Gromov, puis aux résultats sur la cohomologie continue et cohomologie  $L^p$  des groupes de rang supérieur.

Les trois autres chapitres sont des reproductions de prépublications et d'articles. Le deuxième reproduit la prépublication [LN23]. Le résultat principal est l'annulation de la cohomologie  $L^p$  en degré 2 pour la plupart des groupes semi-simples de rang au moins 3 sur des corps locaux. Le troisième reproduit la prépublication [LN22]. On étudie l'homologie  $\ell^p$  en degré maximal des immeubles, sa relation avec d'autres invariants tels que la dimension cohomologique virtuelle des groupes de Coxeter et la dimension conforme des immeubles hyperboliques au sens de Gromov. Le quatrième reproduit l'article [LN21], où, à l'aide des nombres de Betti  $L^2$  de groupes agissant sur des produits d'immeubles, on explicite une famille infinie de groupes simples de présentation finie non mesurablement équivalents.

## 0.2 Summary of the thesis

This thesis lies between Lie theory and Gromov's large scale geometry. In short, we deal with large scale geometric invariants of cohomological nature associated to locally compact groups and study them in the case of Lie groups and of some of their generalizations.

Large scale geometry deals with the study of asymptotic properties of metric spaces. By this we mean properties that remain invariant under maps that forget the local topological structure and that impose conditions only on large distances, such as quasi-isometries or coarse equivalences. The study of locally compact second countable groups fits into this framework as they carry left-invariant proper metrics [Str74] and two such metrics remain coarsely equivalent [CdH16, 4.A.6]. For these groups one can define continuous cohomology with values in continuous representations. This is an algebraic construction that has a priori no relation with large scale geometry. Surprisingly, it gives a large scale invariant if one chooses the correct representation, which is the right regular representation on  $L^p$ -functions on the group, for any real number  $p > 1$ . The associated cohomological invariant, that we call *continuous  $L^p$ -cohomology* and denote by  $H_{\text{ct}}^*(G, L^p(G))$  (for some locally compact second countable group  $G$  and some  $p > 1$ ), is the main object of interest of this thesis.

The locally compact groups on which we want to compute this invariant are Lie groups and some of their generalizations. The most popular family of Lie groups are *real semisimple Lie groups* in view of their rich structure theory (as opposed to solvable Lie groups, that remain wild) and their actions on their corresponding Riemannian symmetric spaces. A first generalization of this situation consists in looking at the points of semisimple algebraic groups over local fields. Local fields split into two categories: Archimedean, which are real and complex numbers, and non-Archimedean. Semisimple algebraic groups over non-Archimedean local fields are the second family of locally compact groups we are interested in. The geometries on which they act are *Bruhat-Tits buildings*, these are simplicial structures that play the role of symmetric spaces in this new setting. The philosophy is that real semisimple Lie groups and their non-Archimedean counterparts should not be too different.

The notion of *building* is in fact much more general. A building is obtained by gluing many times the same simplicial complex, representing the combinatorics of a fixed *Coxeter group*, under incidence relations that are reminiscent of symmetric spaces. In particular, Coxeter groups arising from non-Archimedean semisimple groups are said to be *affine* as their associated geometries are tilings of the real affine space. Kac-Moody theory is a way to construct examples of (discrete and) locally compact groups as automorphism groups of buildings, starting from combinatorial data such as an arbitrary Coxeter group. In particular, we obtain new examples if the Coxeter group we start from is not affine. These *non-affine Kac-Moody groups* are a third family of exotic Lie-related locally compact groups that we will be interested in this thesis. These groups stray far away from classical Lie theory, in the sense that they exhibit new properties that their classical counterparts do not have.

This thesis deals with computations of  $L^p$ -cohomology for the groups and spaces we have just introduced. Our main motivation is the following question by Gromov:

he predicts a classical behaviour of (unreduced)  $L^p$ -cohomology of semisimple groups over local fields for every  $p > 1$  [Gro93]. Here a local field can be Archimedean or not.

**Questions.** *Let  $G$  be a semisimple group of rank  $r \geq 2$  over a local field.*

- (1) *Do we have  $H_{\text{ct}}^l(G, L^p(G)) = \{0\}$  for all  $l = 1, \dots, r - 1$  and  $p > 1$ ?*
- (2) *Do we have  $H_{\text{ct}}^r(G, L^p(G)) \neq \{0\}$  at least for some values of  $p$ ?*
- (3) *Is the space  $H_{\text{ct}}^r(G, L^p(G))$  Hausdorff for all  $p > 1$ ?*

In the following pages we present the main results of this thesis (for preliminaries, see Chapter 1). Three topics are discussed. Each of these corresponds to an article. The first two are relevant to these questions, dealing respectively with questions (1) and (2). The third topic is independent as it deals with non-linear groups. We conclude by presenting questions and future projects.

The first topic is vanishing in degree 2 for semisimple groups of rank  $\geq 3$  over local fields, which we show in most cases [LN23]. The second is non-vanishing in top degree for affine buildings and more general types of buildings [LN22]. Here we also discuss conformal dimension of hyperbolic buildings. The third deals with  $L^2$ -cohomology of generalizations of semisimple groups and applications to measure equivalence [LN21].

### 0.2.1 Vanishing in degree 2

One of the main results of this thesis concerns question (1): we show vanishing of  $L^p$ -cohomology in degree 2 for most semisimple Lie groups of rank  $\geq 3$  over local fields.

We first recall previous contributions towards a positive answer to this question. Many of these concern the case of degree 1. Then, we state a more precise version of the result and outline the main arguments of its proof.

**Degree 1 case for Lie groups** Vanishing in degree 1 for all  $p > 1$  for real semisimple groups of rank  $\geq 2$  was first proven by Pansu in 1999 via de Rham  $L^p$ -cohomology (first unpublished, then appeared in [Pan07]). Actually, he showed a stronger statement.

**Theorem 0.2.1.** *[Pan07, Théorème 1] Let  $M$  be a homogeneous manifold. Either:*

- *the isometry group of  $M$  is a compact extension of a solvable unimodular Lie group,*
- *or  $M$  is quasi-isometric to a homogeneous space of strictly negative sectional curvature,*
- *otherwise  $L^p H_{\text{dR}}^1(M) = \{0\}$  for all  $p > 1$ .*

Later, Bader, Furman, Gelander and Monod proved vanishing of the first continuous cohomology of a (real or non-Archimedean) simple group  $G$  of rank  $\geq 2$  acting by isometries on some  $L^p(X, \mu)$ , where  $(X, \mu)$  is a standard Borel space and  $p > 1$  [BFGM07]. Their techniques also work for semisimple groups, but with each simple factor of rank  $\geq 2$ . Lastly, de Cornulier and Tessera extended Pansu's trichotomy to semisimple groups over fields of characteristic zero via continuous group cohomology [dCT11]. Their trichotomy is valid only for groups of characteristic 0,

but their argument for vanishing for semisimple groups of higher rank also works in positive characteristic.

**Theorem 0.2.2.** [*dCT11, Theorem 1*] *Let  $G$  be a semisimple Lie group of rank  $r \geq 2$  over a local field. We have  $H_{\text{ct}}^1(G, L^p(G)) = \{0\}$  for all  $p > 1$ .*

The proofs of the above mentioned works are not equivalent but rely on similar dynamical arguments. Namely, one of the recurring arguments is Mautner's phenomenon in one of its many forms.

This statement also holds for the corresponding versions of  $L^p$ -cohomology on the geometries on which semisimple groups act, namely, symmetric spaces and Bruhat-Tits buildings.

**Degree 1 case for affine buildings** Bruhat-Tits buildings are affine buildings on which non-Archimedean semisimple groups act naturally. All affine buildings of dimension  $\geq 3$  arise from such actions. On the other hand, there are exotic affine buildings of dimension 2 with small full automorphism groups (for instance discrete). Lécureux, de la Salle and Witzel show that vanishing of  $L^p$ -cohomology in degree 1 persists for some of these buildings.

**Theorem 0.2.3.** [*LdlSW20, Theorem A*] *Let  $X$  be a locally finite  $\tilde{A}_2$ -building. Then we have  $\ell^p H^1(X) = \{0\}$  for all  $p > 1$ .*

The proof is significantly different from the Lie group case as here one cannot rely on dynamical arguments such as Mautner's phenomenon. Instead, what is used is harmonicity and random walks.

**Higher degrees** Bourdon and Rémy deal with vanishing of  $L^p$ -cohomology of real simple Lie groups in higher degrees [BR20]. They show that for some real simple Lie groups they call *admissible*, there are constants  $p(k) > 1$  (depending on the group) for every degree  $k$  such that there is vanishing of  $L^p$ -cohomology in degree  $k$  for all  $1 < p < p(k)$ . Poincaré duality allows them to extend this result to large values of  $p$ , at least for large values of  $k$  (in particular, this duality argument does not concern degrees below the rank). Their methods consist in proving a suitable version of the Hochschild-Serre spectral sequence for  $L^p$ -cohomology and invoke Pansu's description of the  $L^p$ -cohomology of the real hyperbolic space [Pan08]. In fact their proof also applies to non-admissible real simple Lie groups, but the conditions on the constant  $p(k)$  are much more restrictive (in this case the constant  $p(k)$  tends to 1 for fixed  $k$  when the rank goes to infinity).

Initially, we wanted to prove a statement similar to that of Bourdon and Rémy in the non-Archimedean case. It turned out that our methods also apply to the real case, but only for large values of  $p$ . The result we prove is the following.

**Theorem 0.2.4.** ([LN23, 0.1], see Theorem 2.0.1 in the text) *Let  $F$  be a local field and suppose that  $G$  is either:*

- $\text{SL}(4, D)$ , where  $D$  is a finite dimensional central division algebra over  $F$ ,
- a simple Lie group over  $F$  of rank  $r \geq 4$  that is not of type  $D_4$  and is not of exceptional type,

- or a semisimple, non-simple Lie group over  $F$  of rank  $r \geq 3$ .  
Then there exists a constant  $p(G) \geq 1$  such that for all  $p > p(G)$ :

$$H_{\text{ct}}^2(G, L^p(G)) = \{0\}.$$

Moreover, when  $F$  is non-Archimedean we have  $p(G) = 1$ .

In the rest of this subsection we outline the proof of this theorem. Three types of arguments are present: algebraic, dynamical and combinatorial.

**Quasi-isometric invariance and spectral sequences** The initial strategy is that of [BR20].

First, we use quasi-isometric invariance in order to identify  $H_{\text{ct}}^2(G, L^p(G))$  to  $H_{\text{ct}}^2(P, L^p(P))$ , where  $P$  is a maximal parabolic subgroup. This parabolic subgroup has a Levi decomposition  $P = MSU$ . The version of the Hochschild-Serre spectral sequence from [BR20] allows us to obtain a linear identification

$$H_{\text{ct}}^2(P, L^p(P)) = H_{\text{ct}}^1(M, L^p(M, V_p))$$

with the first continuous cohomology group  $H_{\text{ct}}^1(M, L^p(M, V_p))$  of the Levi factor  $M$  with coefficients in some Banach-valued  $L^p$ -space  $L^p(M, V_p)$ . The Banach space  $V_p$  is in fact  $H_{\text{ct}}^1(SU, L^p(SU))$ . The main technical problem comes from the fact that the continuous  $M$ -module  $L^p(M, V_p)$  has relatively large *exponential growth*, forbidding us to (directly) invoke Lafforgue's strong property (T) and obtain the desired vanishing.

**Dynamical arguments** The next step is to pass to a cocompact, non-unimodular, solvable group  $R$  in order to create contractions thanks to its modular function  $\Delta_R$  and adapt techniques from [dCT11] to state some non-isometric version of Mautner's phenomenon for the  $R$ -module  $(\pi, L^p(R, V_p))$ .

More precisely, if we denote by  $\pi$  the representation of  $R$  on  $L^p(R, V_p)$  and by  $\pi_0$  the representation of  $R$  on  $V_p$ , we have for every  $g \in R$ :

$$\|\pi(g)\|_{L^p(R, V_p)}^p = \Delta_R(g)^{-1} \|\pi_0(g)\|_{V_p}^p.$$

The idea is that even though  $\|\pi_0(g)\|_{V_p}$  can grow really fast, we hope to counter its growth using  $\Delta_R(g)$  and thus find elements that contract the  $L^p$ -norm. The upshot is that vanishing of  $H_{\text{ct}}^1(R, L^p(R, V_p))$  follows from the presence of two elements contracting the  $L^p$ -norm in well-chosen directions. This is not hard to do in the semisimple, non-simple case.

The non-trivial part consists in showing the existence of such contractions in the simple case. This is a battle between the exponential dilation of the representation  $\pi_0$  on  $V_p$  and the exponential contraction of  $\Delta_R$  in certain directions. To show that the contraction of  $\Delta_R$  wins this battle for the groups in the statement of Theorem 0.2.4, we first control some operator norms of  $\pi_0$  by some term that can be written explicitly in combinatorial terms. This step uses the hyperbolicity of the Heintze group  $H = S \times U$ , and in particular, precise bounds of the exponential distortion of  $U$  inside  $H$ .

**Combinatorics and Lie theory** Using the classification of semisimple Lie groups over local fields, we reduce the existence problem of contracting elements to a combinatorial case-by-case study of root systems with multiplicities. The presence of multiplicities forces us to present our combinatorial study in terms of classical presentations of simple Lie groups.

The main point in the combinatorial part of the proof is that, for the infinite families of root systems  $(A_r, B_r, C_r, BC_r$  and  $D_r)$  there exists always a choice of maximal parabolic subgroup such that our control of the exponential dilation grows *linearly* in the rank and the exponential contraction of the modular function grows *quadratically* in the rank (at least in well-chosen directions). This a priori asymptotic heuristic works quite fast: starting from  $r \geq 3$  for  $A_r$ , from  $r \geq 4$  for  $B_r, C_r$  and  $BC_r$  and from  $r \geq 5$  for  $D_r$ . Our current estimates do not seem to create contractions for exceptional groups.

## 0.2.2 Non-vanishing in top degree and conformal dimension of buildings

We now turn to question (2), that is, non-vanishing in degree equal to the rank. We will explain the evolution of this phenomenon, starting from the more classical settings and moving towards the more exotic ones.

**Non-vanishing for real Lie groups** In the formulation we give, Bourdon and Rémy give a positive answer to question (2) for real semisimple groups [BR21].

**Theorem 0.2.5.** [BR21, Theorem A] *Let  $G$  be a semisimple real Lie group of rank  $r \geq 2$ . Then for  $p$  large enough we have:*

$$\overline{H}_{\text{ct}}^r(G, L^p(G)) \neq \{0\}.$$

In fact in [BR20], they also show that for admissible groups there is vanishing of  $H_{\text{ct}}^r(G, L^p(G))$  for  $p > 1$  small enough. There is still an interval of values of  $p$  for which the behaviour of  $H_{\text{ct}}^r(G, L^p(G))$  remains unknown.

**Non-vanishing for affine buildings** We show that non-vanishing in degree equal to the rank also holds for general locally finite affine buildings. The situation is now simpler: non-vanishing holds for all  $p > 1$ . More precisely:

**Theorem 0.2.6.** ([LN22, 1.2], see Theorem 3.0.2 in the text) *Let  $X$  be a locally finite affine building of dimension  $n$ . We have  $\ell^p \overline{H}^n(X) \neq \{0\}$  for all  $p > 1$ . In particular, any semisimple Lie group  $G$  of rank  $n$  over a non-Archimedean local field satisfies  $\overline{H}_{\text{ct}}^n(G, L^p(G)) \neq \{0\}$  for all  $p > 1$ .*

Theorem 0.2.6 is proved by using some Steinberg harmonicity formula on top dimensional locally finite homology of the building and showing convergence of its  $\ell^p$ -norm for all  $p > 1$ .

**Non-affine buildings and virtual cohomological dimension** We now leave the world of simple groups over local fields and deal with arbitrary (locally finite) buildings.

The first question we should ask ourselves in this setting is what is our new definition of *rank* of a building, as we do not have anymore an action of a semisimple group. If we had a nice action of a group  $G$  on the building, we could talk about the *virtual cohomological dimension*  $\text{vcd}(G)$  of that group. If there is no automorphism group, we can still look at the largest integer for which the  $\ell^p$ -cohomology of the building is nonzero, at least for some  $p > 1$ . We show that the latter always coincides with the virtual cohomological dimension  $\text{vcd}_{\mathbb{R}}(W)$  over  $\mathbb{R}$  of the Weyl group  $(W, S)$  of any apartment of the building.

We denote by  $e_{\mathbf{q}}(W)$  the weighted logarithmic growth-rate of the Coxeter system  $(W, S)$ . Recall that, by the strong Tits' alternative for Coxeter groups, a non-affine Coxeter group  $W$  has exponential growth, and hence  $e_{\mathbf{q}}(W) > 0$ . The precise result we show is the following.

**Theorem 0.2.7.** (*[LN22, 1.4], see Theorem 3.3.8 in the text*) *Let  $X$  be the Davis realization of some building with Weyl group  $(W, S)$  and finite thickness vector  $\mathbf{q}+1$ , where  $\mathbf{q} \geq 2$ . Let  $d = \text{vcd}_{\mathbb{R}}(W)$ . Then:*

- *for all  $1 < p < 1 + e_{\mathbf{q}}(W)^{-1}$  we have  $\ell^p \overline{H^d}(X) \neq \{0\}$ ,*
- *for all  $k > d$  and  $p > 1$  we have  $\ell^p H^k(X) = \{0\}$ .*

The proof consists in using the *Bestvina realization* of the building which is defined so that its dimension is  $d = \text{vcd}_{\mathbb{R}}(W)$ . The non-vanishing assertion now has the same proof as Theorem 0.2.6. The vanishing assertion follows from quasi-isometric invariance of  $\ell^p$ -cohomology.

**Critical exponents for some non-affine buildings** In view of Theorem 0.2.7, we may ask if the upper bound  $1 + e_{\mathbf{q}}(W)^{-1}$  for non-vanishing is optimal. We show that it is the case, under the assumption that our apartments are simplicial complexes that are close to orientable manifolds.

**Theorem 0.2.8.** (*[LN22, 1.1], see Theorem 3.2.6 in the text*) *Let  $(W, S)$  be a Coxeter system such that its associated Davis complex  $\Sigma$  is an orientable pseudomanifold of dimension  $n$ . Let  $X$  be the Davis realization of some building with Weyl group  $(W, S)$  and finite thickness vector  $\mathbf{q} + 1$ , where  $\mathbf{q} \geq 2$ . Then we have:*

$$1 + e_{\mathbf{q}}(W)^{-1} = \sup\{p > 1 \mid \ell^p \overline{H^n}(X) \neq \{0\}\}.$$

The vanishing assertion in this theorem follows from the fact that averaging over preimages of a retraction reduces the  $\ell^p$ -norm, and that our Steinberg formula is already averaged over these preimages. It also follows from Theorem 0.2.8 and quasi-isometric invariance of reduced  $\ell^p$ -cohomology that the growth rate  $e_{\mathbf{q}}(W)$  is a new quasi-isometry invariant among locally finite Davis buildings whose apartments are orientable pseudomanifolds.

We realized later that the critical exponent for top-dimensional reduced  $\ell^p$ -cohomology appearing in Theorem 0.2.8 has the same expression as the conformal dimension of a Fuchsian building [Bou00].

**Conformal dimension of hyperbolic buildings** We now focus on this very last remark and turn to Gromov-hyperbolic (locally finite) buildings. We wonder if there is a relation in a more general setting between the critical exponent appearing in Theorem 0.2.8 and conformal dimension.

Our techniques in top degree yield a first lower bound of the conformal dimension of Gromov-hyperbolic buildings modeled on pseudomanifold apartments. Clais had already showed a sharper lower bound for some particular Gromov-hyperbolic buildings [Cla17]. This motivated us to "dualize" our techniques on top-dimensional  $\ell^p$ -homology and work on the first  $\ell^p$ -cohomology instead. It is not hard to show that for any Gromov-hyperbolic (Davis) building  $X$  of finite thickness vector  $\mathbf{q} + 1$ , with  $\mathbf{q} \geq 2$ , there exists some visual metric  $d_{\mathbf{q}}$  on the boundary  $\partial\Sigma$  of an apartment  $\Sigma$  such that:

$$\text{Confdim}(\partial X) \leq \text{Hausdim}(\partial\Sigma, d_{\mathbf{q}})(1 + e_{\mathbf{q}}(W)^{-1}).$$

This is Clais' upper bound. Our methods on the first  $\ell^p$ -cohomology group generalize Clais' lower bound to any Gromov-hyperbolic building.

**Theorem 0.2.9.** (*[LN22, 1.5], see Theorem 3.0.5 in the text*) *Let  $(W, S)$  be a Gromov-hyperbolic Coxeter system,  $\Sigma$  the Davis complex of  $(W, S)$  and  $X$  the Davis realization of a building with Weyl group  $(W, S)$  and thickness vector  $\mathbf{q} + 1$ , with  $\mathbf{q} \geq 2$ . We have:*

$$\text{Confdim}(\partial\Sigma)(1 + e_{\mathbf{q}}(W)^{-1}) \leq \text{Confdim}(\partial X).$$

The idea of the proof is, as in top degree, to average over preimages of retractions and estimate convergence of  $\ell^p$ -norms. The additional ingredient is to use the relation between conformal dimension and the first  $\ell^p$ -cohomology groups in terms of separation properties on the boundary [BK15, 3.8].

### 0.2.3 Finitely presented simple groups and measure equivalence

The last topic of this thesis is independent from the previous two. It deals with the problem of distinguishing groups up to equivalence relations using cohomological invariants. Here we are interested in infinite *finitely presented simple groups*. To the date, there are few known examples of such groups, among which: Higman-Thompson groups and variants [SWZ19], Burger-Mozes groups acting cocompactly on products of trees [BM00] and non-affine (non-uniform) Kac-Moody lattices over finite fields [CR06]. An interesting task is to separate some of them from the point of view of quasi-isometry or measure equivalence.

**Measure equivalence classes of simple groups** There are already examples of infinite families of finitely presented simple groups that are pairwise not quasi-isometric: this was first shown for some Kac-Moody lattices [CR10] and later for some Röver-Nekrashevych variants of Thompson's group [SWZ19]. We show the measure equivalence analogue of this result.

**Theorem 0.2.10.** (*[LN21, 1.1], see Theorem 4.0.1 in the text*) *There are infinitely many measure equivalence classes containing finitely presented, Kazhdan, simple groups. These groups are Kac-Moody lattices over finite fields with well-chosen non-affine Weyl groups.*

This is done by studying vanishing and non-vanishing of their  $\ell^2$ -Betti numbers. Indeed, Gaboriau showed that the sequence of  $\ell^2$ -Betti numbers (up to proportionality on the sequence) is a measure equivalence invariant of countable groups [Gab02]. Vanishing of an  $\ell^2$ -Betti number is a quasi-invariant (even a coarse equivalence invariant) and thus we recover an infinite family of pairwise non-quasi-isometric finitely presented simple groups as in [CR09] or [SWZ19].

**From discrete to topological groups** The general philosophy is to view our discrete candidates  $\Lambda$  as lattices sitting inside an ambient topological group  $L$  and to use the richer structure of  $L$ . In our case, Petersen showed that the sequences of  $\ell^2$ -Betti numbers of a lattice  $\Lambda$  and of its ambient locally compact second countable group  $L$  are proportional.

It remains to understand the group  $L$  and its sequence of  $\ell^2$ -Betti numbers. Our candidates are discrete Kac-Moody groups over local fields. A discrete Kac-Moody group  $\Lambda$  acts on a building, and if we denote by  $G$  its completion inside the automorphism group of the building, then  $\Lambda$  is a lattice inside  $L = G \times G$ .

Using a new point of view on Garland's method, Dymara and Januszkiewicz obtain a formula of  $\ell^2$ -Betti numbers of a complete Kac-Moody group  $G$  [DJ02]. What this formula says is that vanishing or non-vanishing of some  $\ell^2$ -Betti number of  $G$  boils down to topological combinatorics on the Weyl group  $W$  of the building. We look closer at the combinatorics of the Weyl group and give a criterion to prescribe non-vanishing of an  $\ell^2$ -Betti number of  $G$  in high degree. By playing with an infinite family of well-chosen Coxeter groups, we construct a countable family of Kac-Moody lattices with non-proportional sequences of  $\ell^2$ -Betti numbers.

## 0.2.4 Perspectives

We list perspectives for further research. The order in which they are presented corresponds to how close they are to the works above.

**Vanishing in higher degrees for semisimple groups** A natural direction one could follow is proving vanishing of  $L^p$ -cohomology in degrees  $\geq 3$  for semisimple groups of rank  $\geq 4$ . Our method for vanishing in degree 2 finds a major obstacle: the Hochschild-Serre spectral sequence requires that the  $L^p$ -cohomology of the solvable factor be Hausdorff in every degree. Since we knew that the  $L^p$ -cohomology of the solvable factor was concentrated in degree 1 and since we know when  $L^p$ -cohomology in degree 1 is Hausdorff, we could use the spectral sequence for treating the case of degree 2. The problem is that for higher degrees, we need to know how to show that cohomologies are Hausdorff (without showing they are trivial) in higher degrees  $> 1$  if we want to use the spectral sequence.

To the date there are two methods that have been able to do this. Pansu showed that for the real hyperbolic space many  $L^p$ -cohomology spaces are Hausdorff in degrees  $\geq 2$  by identifying the cohomology space to a rather complicated functional space on the *boundary* (but still Hausdorff).

A second method consists in using hilbertian methods to show that  $L^2$ -cohomology is Hausdorff and then use Riesz-Thorin *interpolation* to extend this to  $L^p$ . This has been done for manifolds in [Loh98] and for buildings in [DJ02]. This method has

limitations as it implies some sort of stability in  $p$ : the behaviour for  $L^p$  is the same as for  $L^2$ . There are examples where the behaviour of  $L^p$ -cohomology is different to that of  $L^2$ -cohomology if  $p$  is far away from  $p = 2$ . For a triangle Fuchsian building  $X$  the formula shown in [DJ02] implies vanishing in degree 1 (and non-vanishing in degree 2) for every  $p$  for which it holds, but it is known that for  $p > \text{Confdim}(\partial X)$  we have non-vanishing in degree 1 (and vanishing of reduced cohomology in degree 2).

To sum up, we are saying that, with our current techniques, the next step to attack Gromov's question (1) is tackling question (3) and more generally finding criteria to guarantee that cohomologies in degrees  $> 1$  are Hausdorff.

**Conformal dimension of Coxeter groups and of buildings** Concerning our bounds for conformal dimension, it would be interesting to see if there is equality between  $\text{Confdim}(\partial X)$  and its upper and lower bounds. Equality may not always hold for a general Gromov-hyperbolic building, but it would be nice to find some geometric/combinatorial criterion on the Coxeter group that guarantees equality (or strict inequality!).

**Critical exponents for real Lie groups** We may also ask if Theorem 0.2.8, on our critical exponent for buildings modeled on orientable pseudomanifolds, has an analogue for real groups. Indeed, [BR21] shows non-vanishing in degree equal to the rank not only for semisimple groups but also for more general types of solvable real Lie groups  $R$  of the form  $R = \mathbb{R}^r \rtimes N$  where  $N$  is a nilpotent group with additional conditions. One may look at the critical exponent  $\inf\{p > 1, L^p H_{\text{dR}}^r(R) \neq \{0\}\}$  (proposed by de Cornulier [BR21, 0.4 remark 4]), which is invariant under quasi-isometry (and coarse equivalence). Does it coincide with another geometric numerical quasi-isometry invariant that can be formulated without invoking cohomology?

**Measure equivalence invariants from  $L^p$ -cohomology** It is natural to wonder whether Gaboriau's projective invariance of the sequence  $\ell^2$ -Betti numbers under measure equivalence (ME) has an analogue for  $p \neq 2$ . We certainly do not have a theory of  $\ell^p$ -Betti numbers, so we cannot adapt this result directly in this way. Gaboriau's result implies that vanishing of reduced  $L^2$ -cohomology in any degree is a ME invariant. The natural question is: for fixed  $p > 1$  and  $k \in \mathbb{N}$ , is vanishing of reduced  $L^p$ -cohomology in degree  $k$  invariant under ME? (or maybe we have to change ME with a more quantitative version, perhaps depending on  $p$ ).

**$L^p$ -cohomology in degree 1 of non-affine complete Kac-Moody groups** The following question was shared to me by Jean Lécureux. Bounded cohomology in degree 2 and  $L^p$ -cohomology in degree 1 share similar results. Since Caprace and Fujiwara showed that non-affine complete Kac-Moody groups have nonzero bounded cohomology in degree 2, do they also have nonzero  $L^p$ -cohomology in degree 1 for  $p$  large enough?

## 0.2.5 Structure of the thesis

This manuscript is divided into four chapters (we are not counting the present introduction as a chapter).

The first chapter introduces the concepts that will be needed later. In it, we first define group cohomology,  $L^p$ -cohomology and buildings. We then compile known results (most of the time without proofs) concerning  $L^p$ -cohomology of semisimple groups. We first deal with results on the  $L^p$ -cohomology of simple groups of rank 1 and by extension of Gromov-hyperbolic spaces, and then we deal with results on continuous cohomology and  $L^p$ -cohomology of groups of rank at least 2.

The next three chapters are reproductions of preprints and articles. The second reproduces the preprint [LN23]. Its main result is vanishing of  $L^p$ -cohomology in degree 2 for most semisimple groups over local fields of split rank at least 3. The third reproduces the preprint [LN22]. In it, we study top degree  $\ell^p$ -homology of buildings, its relation with other invariants such as virtual cohomological dimension of Coxeter groups and conformal dimension of Gromov-hyperbolic buildings. The fourth reproduces the article [LN21], where, using  $L^2$ -Betti numbers of groups acting on products of buildings, we exhibit a first family of finitely presented simple groups that lie in infinitely many measure equivalence classes.

# Chapter 1

## $L^p$ -cohomology, semisimple groups and buildings

### 1.1 Cohomology

We begin by introducing the cohomological invariants we will work with. The most important for us is  $L^p$ -cohomology. This is an invariant that can be defined in different settings: we will define it for simplicial complexes, manifolds, locally compact second countable groups and metric measured spaces. We are interested mostly in groups, so the definition for locally compact second countable groups, that we call *continuous group  $L^p$ -cohomology*, will be our main character. In fact this version of  $L^p$ -cohomology is just a particular case of *continuous cohomology*. This is a general procedure to associate cohomology spaces to a continuous representation of a locally compact second countable group.

This section will begin by introducing continuous cohomology and some of the advantages that continuous group  $L^p$ -cohomology inherits from it. We will then introduce the many definitions of  $L^p$ -cohomology, discuss when these versions coincide through comparison theorems and state its main feature (in contrast with continuous cohomology with values in an arbitrary representation), quasi-isometric invariance for every  $p > 1$ . We close this section by introducing  $\ell^2$ -Betti numbers, which are invariants that can be defined from  $L^2$ -cohomology and that will also be of interest for us in view of their applications to measure equivalence.

#### 1.1.1 Continuous group cohomology

We will only deal with continuous group cohomology with coefficients in Banach spaces. This is defined by applying classical homological constructions in a Banach setting. Since the category of Banach spaces is not abelian, one cannot use the whole toolbox coming from homological algebra. More precisely, the image of a continuous linear map between Banach spaces is not necessarily closed, so cokernels do not live in this category and many problems arise, such as cohomology not always being Hausdorff.

Nevertheless, Hochschild developed a theory, called *relative homological algebra* that allows to partially recover results from classical homological algebra for this type of cohomology. In short this theory says that one can recover classical results

for continuous cohomology under strong topological assumptions, though sometimes we lose the topologies in the conclusions.

### 1.1.1.1 Continuous group cohomology

We first define continuous cohomology of a locally compact second countable group with coefficients in a continuous representation following [BW00, IX].

Let  $G$  be a locally compact second countable (that we abbreviate from now on as *lsc*) group. Let  $(\rho, V)$  be a *continuous representation* of  $G$  (we also use *continuous  $G$ -module* as terminology) i.e. a morphism  $\rho : G \rightarrow B(V)$  such that the map  $G \times V \rightarrow V$  defined by  $(g, v) \mapsto \rho(g)v$  is continuous, where  $V$  is some locally convex topological vector space and  $B(V)$  denotes continuous invertible operators on  $V$ . Here  $V$  will always be at least a Fréchet space.

For  $k \in \mathbb{N}$ , we define the space  $C^k(G, V)$  of  $k$ -cochains as the set of continuous maps from  $G^{k+1}$  to  $V$ . Since  $G$  is  $\sigma$ -compact, the space  $C^k(G, V)$  equipped with the topology of uniform convergence on compact subsets is a Fréchet space.

We define the differential  $d_k : C^k(G, V) \rightarrow C^{k+1}(G, V)$  by the following formula on  $k$ -cochains:

$$(d_k c)(g_0, \dots, g_{k+1}) = \sum_{i=0}^{k+1} (-1)^i c(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_{k+1}).$$

The collection  $d$  of differentials satisfies  $d_{k+1} \circ d_k = 0$ . The following sequence is exact:

$$0 \rightarrow V \xrightarrow{\epsilon} C^0(G, V) \xrightarrow{d_0} C^1(G, V) \xrightarrow{d_1} \dots \xrightarrow{d_{k-1}} C^k(G, V) \xrightarrow{d_k} \dots$$

where  $\epsilon : V \rightarrow C^0(G, V)$  is the inclusion of constant maps. The space  $C^k(G, V)$  can be viewed as a continuous  $G$ -module, by endowing it with the action:

$$(g.c)(g_0, \dots, g_k) = \rho(g)(c(g^{-1}g_0, \dots, g^{-1}g_k)).$$

We consider the space  $C^k(G, V)^G$  of invariants in  $C^k(G, V)$  with respect to this action. Notice that this is just the set of continuous  $G$ -equivariant maps from  $G^{k+1}$  to  $V$ , when endowing  $G^{k+1}$  with the diagonal action by left translation on each factor and  $V$  with the action given by  $\rho$ . The differential  $d_k$  maps  $C^k(G, V)^G$  into  $C^{k+1}(G, V)^G$ . We call  $\ker(d_k|_{C^k(G, V)^G})$  the space of  *$k$ -cocycles* and denote it by  $Z^k(G, \rho)$ , we call  $\text{Im}(d_{k-1}|_{C^{k-1}(G, V)^G})$  the space of  *$k$ -coboundaries* and denote it by  $B^k(G, \rho)$ .

**Definition 1.1.1.** The  *$k$ -th continuous cohomology space* (resp.  *$k$ -th reduced continuous cohomology space*) of  $G$  with coefficients in  $(\rho, V)$  is the topological vector space:

$$H_{\text{ct}}^k(G, \rho) := Z^k(G, \rho) / B^k(G, \rho) \quad (\text{resp. } \overline{H}_{\text{ct}}^k(G, \rho) := Z^k(G, \rho) / \overline{B^k(G, \rho)}).$$

The space  $\overline{H}_{\text{ct}}^k(G, \rho)$  is the largest Hausdorff quotient of  $H_{\text{ct}}^k(G, \rho)$ . In particular these two spaces coincide exactly when  $H_{\text{ct}}^k(G, \rho)$  is Hausdorff.

For this definition we used a particular resolution of the representation  $(\rho, V)$ . As usual in homological algebra, one wants to show that one could have given this

definition with any other resolution of  $\rho$ . This is possible if we assume strong topological assumptions on our resolutions (we need *relatively injective strong resolutions*, see [BR20, 2.1] for a definition). This is usually how one identifies continuous group cohomology to other types of cohomology: in particular, this is how one proves that continuous  $L^p$ -cohomology coincides with other versions of  $L^p$ -cohomology.

**Non-homogeneous cochains** For concrete applications and particularly in degree 1, it is sometimes useful to view elements of  $C^k(G, V)^G$  not as maps from  $G^{k+1}$  to  $V$ , but as maps from  $G^k$  to  $V$ .

For  $k = 1$ , this gives the classical geometric interpretation of the first continuous cohomology space. We can identify  $Z^1(G, V)$  with the space of continuous maps  $b : G \rightarrow V$  satisfying the cocycle relation  $b(gh) = b(g) + \rho(g)b(h)$ . This space can in turn be identified with the space of continuous affine actions of  $G$  on  $V$  with linear part  $\rho$ , via the map  $b \mapsto A_b$  for  $b \in Z^1(G, V)$ , where  $A_b(g) : V \rightarrow V, v \mapsto \rho(g)v + b(g)$  for  $g \in G$ . In a similar way, we may identify  $B^1(G, V)$  with the space of maps  $b : G \rightarrow V$  of the form  $g \mapsto b(g) = v - \rho(g)v$  for some  $v \in V$ . These maps correspond to continuous affine actions of  $G$  on  $V$  having a fixed point.

In particular,  $H_{\text{ct}}^1(G, \rho) = \{0\}$  means that every action by affine isometries on the space  $V$  with linear part  $\rho$  has a fixed point.

### 1.1.1.2 Induction of representations

A central concept in representation theory is induction of representations. The setting is the following. Let  $G$  be a lsc group and  $H$  a closed subgroup in  $G$ . If we are given a continuous representation  $(\rho, V)$  of  $H$ , the question is first how to cook a continuous representation of  $G$  starting from  $\rho$  and second what relation do we have between the cohomologies of these two representations.

Concerning the first question, we may define the *induced representation*  $\text{Ind}_H^G(\rho)$  of  $\rho$  from  $H$  to  $G$  as the space of continuous functions  $f : G \rightarrow V$  such that  $f(hg) = \rho(h)(f(g))$  for all  $g \in G$  and  $h \in H$ . It is a Fréchet space and a continuous  $G$ -module for the right-translation action.

**Proposition 1.1.2.** (*Shapiro's lemma*) [BW00, IX.2.3] *Let  $H$  be a closed subgroup of a lsc group  $G$ . Let  $(\rho, V)$  be a continuous  $H$ -module. Assume that the fibration of  $G$  by  $H$  admits a local cross section. Then:*

$$H_{\text{ct}}^*(H, \rho) = H_{\text{ct}}^*(G, \text{Ind}_H^G(\rho))$$

This type of induction allows one to study representations of a cocompact lattice  $H = \Gamma$  through the representations of the ambient lsc group  $G$ . We will not use this result here because, as we will see later, continuous  $L^p$ -cohomology satisfies a stronger statement: it is invariant under quasi-isometries.

*Remark.* 1. One can induce representations using other functional spaces (such as locally  $p$ -integrable functions for some  $p > 1$  or smooth functions) and state a Shapiro lemma for those induced representations [Bla79].

2. Shalom defines induced representations using  $L^2$ -functions and shows that Shapiro's lemma holds in degree 1 also for non-cocompact lattices, under some  $L^2$ -integrability assumption on the finite measure fundamental domain [Sha00, 3.3].

### 1.1.1.3 Hochschild-Serre spectral sequence

We also have the *Hochschild-Serre spectral sequence* to decompose the cohomology of a semi-direct product in terms of the cohomologies of its factors. This is the continuous group-theoretic analogue of the more classical Leray-Serre spectral sequence. The main difference is that in our new setting we will require stronger topological assumptions (namely, that the cohomology of one of the factors is Hausdorff in every degree) and we may only obtain linear isomorphisms (not necessarily continuous) between vector spaces.

The setting is the following: let  $P$  be a lcsc group such that  $P = R \ltimes H$  where  $R$  and  $H$  are closed subgroups. Let  $(\rho, V)$  be a continuous  $P$ -module. We want to understand the cohomology spaces  $H_{\text{ct}}^*(P, \rho)$  in terms of  $R$  and  $H$ .

The space  $C^k(H, V)$  is a continuous  $R$ -module under the action given by:

$$(g \cdot c)(x_0, \dots, x_k) = \rho(g)(c(g^{-1}x_0g, \dots, g^{-1}x_kg)).$$

for  $c \in C^k(H, V)$  and  $g \in R$ . The quotient  $H_{\text{ct}}^k(H, \rho|_H)$  inherits a natural  $R$ -action, we may now view it as a continuous  $R$ -module. The Hochschild-Serre spectral sequence relates the cohomologies  $H_{\text{ct}}^*(P, \rho)$  to the cohomologies  $H_{\text{ct}}^*(R, H_{\text{ct}}^*(H, \rho|_H))$ .

**Theorem 1.1.3.** *[BW00, IX.4.1] Let  $P$  be a lcsc group and  $(\rho, V)$  a continuous  $P$ -module on a Banach space. Suppose that  $P = R \ltimes H$  where  $R$  and  $H$  are two closed subgroups of  $P$ , such that  $C^*(H, \rho|_H)$  is homotopically equivalent to a complex of Banach spaces and such that the cohomology spaces  $H_{\text{ct}}^*(H, \rho|_H)$  are Hausdorff. Then there exists a spectral sequence  $(E_r)$  abutting to  $H_{\text{ct}}^*(P, \rho)$  in which:*

$$E_2^{k,l} = H_{\text{ct}}^k(R, H_{\text{ct}}^l(H, \rho|_H)).$$

### 1.1.2 $L^p$ -cohomology: definitions and large-scale invariance

We will now introduce our main object of interest.  $L^p$ -cohomology is an invariant that can be defined for different objects: one can talk about simplicial  $\ell^p$ -cohomology of a simplicial complex, de Rham  $L^p$ -cohomology of a manifold, continuous  $L^p$ -cohomology of a locally compact second countable group or asymptotic  $L^p$ -cohomology of a metric measure space.

Historically, the first to be defined was de Rham  $L^p$ -cohomology by Gol'dshtein, Kuz'minov, and Shvedov in [GKS86]. Pansu introduced asymptotic  $L^p$ -cohomology first in degree 1 [Pan89a] (and later in all degrees [Pan95]) and compared it to its de Rham version in the uniformly contractible case, this was the first evidence of the large-scale geometric nature of  $L^p$ -cohomology. Simplicial  $\ell^p$ -cohomology was introduced by Gromov in his book [Gro93].

We will first define these different notions and point out their advantages in each case: continuous group  $L^p$ -cohomology allows us to use relative homological algebra, simplicial  $\ell^p$ -cohomology is the easiest to define and manipulate, de Rham  $L^p$ -cohomology satisfies Poincaré duality and asymptotic  $L^p$ -cohomology brings quasi-isometric invariance (we will give more attention to this point later in the text). We will then state comparison theorems that allow one to pass from one version to another when some space has more than one of these structures. We end this section by discussing one of the main features of  $L^p$ -cohomology: it gives quasi-isometry invariants for every  $p > 1$  and  $k \in \mathbb{N}$ .

### 1.1.2.1 Different versions of $L^p$ -cohomology

We present four definitions of  $L^p$ -cohomology in different settings and some properties that are specific for each one of them.

**Continuous group  $L^p$ -cohomology** Let  $G$  be a locally compact second countable group endowed with a left-invariant Haar measure  $\mu_G$ . We will be interested in the Banach space  $V = L^p(G)$  of  $p$ -integrable functions with respect to the measure  $\mu_G$  for  $1 < p < \infty$ . The representation  $\rho_p$  of  $V$  we are interested in is the *right regular representation* of  $G$  on  $L^p(G)$ , defined by right translation on the argument of an  $L^p$ -function:  $(\rho_p(g)f)(x) = f(xg)$ . This defines a continuous representation of  $G$ .

**Definition 1.1.4.** The *continuous group  $L^p$ -cohomology*  $H_{\text{ct}}^*(G, L^p(G))$  of  $G$  is the continuous cohomology of  $G$  with coefficients in the representation  $\rho_p$ .

Notice that  $\rho_p$  is an isometric representation if and only if the measure  $\mu_G$  is also right-invariant, that is, if and only if  $G$  is unimodular. In this case  $\rho_p$  is also continuously conjugate to the similarly defined left regular representation  $\lambda_p$  via the continuous linear map  $L^p(G) \rightarrow L^p(G)$  sending  $f$  to  $\check{f} : x \mapsto f(x^{-1})$ .

**Simplicial  $\ell^p$ -homology and  $\ell^p$ -cohomology** We first recall the two main points of view on simplicial complexes. For more details see [Dav08, Appendix A.2].

An *abstract simplicial complex* is a set  $X$  whose elements are finite subsets (that we call *simplices*) of some given set  $X^{(0)}$  (the set of *vertices*) such that for every simplex  $\sigma$  of  $X$ , every subset of  $\sigma$  is also a simplex of  $X$ . For  $k \in \mathbb{N}$ , a simplex of cardinal  $k + 1$  is called a  $k$ -simplex and we denote the set of all  $k$ -simplices by  $X^{(k)}$ .

On the other hand, one can define the *geometric realization* of an abstract simplicial complex. For every  $k \in \mathbb{N}$  and every  $k$ -simplex  $\sigma$ , we consider a copy of the standard closed  $k$ -dimensional simplex  $\Delta^k$  and we identify the vertices in  $\sigma$  to the vertices of  $\Delta^k$ . The geometric realization  $\tilde{X}$  of  $X$  is obtained by gluing each of these copies following the inclusion relation of simplices. The resulting set is endowed with the weak topology (that is, a subset of  $\tilde{X}$  is open if and only if, for every  $k \in \mathbb{N}$ , its intersection with any of these copies of  $\Delta^k$  is open in the topology of  $\Delta^k$ ).

The geometric realization is the object we are interested in, to which we will refer most of the time by the term *simplicial complex*. We will not distinguish  $\tilde{X}$  and  $X$  and we will just write  $X$ .

We now introduce simplicial  $\ell^p$ -homology and simplicial  $\ell^p$ -cohomology of simplicial complexes of bounded geometry following [Bou16b].

**Definition 1.1.5.** Let  $X$  be a simplicial complex equipped with a metric  $d$  (metrizing its topology) so that  $(X, d)$  is a length space. We say that such a simplicial complex  $(X, d)$  has *bounded geometry* if:

- (i) there exists a constant  $C > 0$  such that every simplex of  $X$  has diameter  $\leq C$ ,
- (ii) there is a function  $N : [0, \infty[ \rightarrow \mathbb{N}$  such that for all  $r > 0$ , every ball of radius  $r$  contains at most  $N(r)$  simplices of  $X$ .

For the rest of this paragraph, let  $X$  be a simplicial complex of bounded geometry. For  $1 < p < \infty$ , we define:

$$\ell^p C_k(X) = \left\{ \sum_{\sigma \in X^{(k)}} a_\sigma \sigma, (a_\sigma)_{\sigma \in X^{(k)}} \in \ell^p(X^{(k)}) \right\},$$

$$\ell^p C^k(X) = \{ \omega : X^{(k)} \rightarrow \mathbb{R}, \omega \in \ell^p(X^{(k)}) \}.$$

The standard boundary operators  $\partial_k$  and  $d_k$  are defined as usual (after choosing an ordering on the vertices of  $X$ , so that simplices of  $X$  are oriented). Since the complex  $X$  has bounded geometry, they define bounded operators:

$$\partial_k : \ell^p C_k(X) \rightarrow \ell^p C_{k-1}(X), \quad d_k : \ell^p C^k(X) \rightarrow \ell^p C^{k+1}(X).$$

These two operators are related by the simplicial version of Stokes' theorem: for  $c \in \ell^p C^k(X)$  and  $\sigma \in \ell^p C_{k+1}(X)$  we have  $d_k c(\sigma) = c(\partial_{k+1} \sigma)$ .

**Definition 1.1.6.** Denote  $\ell^p Z_k(X) := \ker \partial_k$  and  $\ell^p B_k(X) := \text{Im } \partial_{k+1}$ . The  $k$ -th simplicial  $\ell^p$ -homology space of  $X$  (resp.  $k$ -th reduced simplicial  $\ell^p$ -homology space) is the space:

$$\ell^p H_k(X) := \ell^p Z_k(X) / \ell^p B_k(X) \quad (\text{resp. } \ell^p \overline{H}_k(X) := \ell^p Z_k(X) / \overline{\ell^p B_k(X)}).$$

Denote  $\ell^p Z^k(X) := \ker d_k$  and  $\ell^p B^k(X) := \text{Im } d_{k-1}$ . The  $k$ -th simplicial  $\ell^p$ -cohomology space of  $X$  (resp.  $k$ -th reduced simplicial  $\ell^p$ -cohomology space) is the space:

$$\ell^p H^k(X) = \ell^p Z^k(X) / \ell^p B^k(X) \quad (\text{resp. } \ell^p \overline{H}^k(X) = \ell^p Z^k(X) / \overline{\ell^p B^k(X)}).$$

The closures are considered with respect to the topology of the  $\ell^p$ -norm. The quotient spaces are endowed with the corresponding quotient topology. Reduced homology and cohomology spaces are thus Banach spaces.

If  $X$  is a finite simplicial complex, then these spaces correspond to classical real homology and real cohomology spaces  $H_k(X, \mathbb{R})$  and  $H^k(X, \mathbb{R})$ . Thus we obtain new invariants only when the complex  $X$  is non-compact.

We have a duality result relating  $\ell^p$ -homology and  $\ell^p$ -cohomology spaces but only for the reduced versions [Bou16b].

**Proposition 1.1.7.** For  $p, r \in ]1, +\infty[$  such that  $p^{-1} + r^{-1} = 1$  and  $k \in \mathbb{N}$ , the space  $\ell^p \overline{H}^k(X)$  is canonically isomorphic to the dual of  $\ell^r \overline{H}_k(X)$ . Similarly,  $\ell^r \overline{H}_k(X)$  is canonically isomorphic to the dual of  $\ell^p \overline{H}^k(X)$ .

For simply connected simplicial complexes, integration over loops allows one to represent  $\ell^p$ -cohomology in degree 1 in a simpler way.

**Proposition 1.1.8.** Let  $X$  be a simply connected simplicial complex of bounded geometry. Then for all  $p > 1$ :

$$\begin{aligned} \ell^p H^1(X) &= \{ f : X^{(0)} \rightarrow \mathbb{R}, df \in \ell^p(X^{(1)}) \} / (\ell^p(X^{(0)}) + \mathbb{R}), \\ \ell^p \overline{H}^1(X) &= \{ f : X^{(0)} \rightarrow \mathbb{R}, df \in \ell^p(X^{(1)}) \} / \overline{(\ell^p(X^{(0)}) + \mathbb{R})}. \end{aligned}$$

**de Rham  $L^p$ -cohomology** We define de Rham  $L^p$ -cohomology following [BR21].

Let  $M$  be a Riemannian manifold. We denote by  $dx$  the Riemannian volume of  $M$  and for  $x \in M$  we denote  $|v|_x$  the Riemannian length of a vector  $v \in T_x M$ . For  $k \in \mathbb{N}$ , let  $\Omega^k(M)$  denote the space of  $C^\infty$  differential  $k$ -forms on  $M$ . We define the  $L^p$ -norm of  $\omega \in \Omega^k(M)$  as:

$$\|\omega\|_{L^p}^p = \int_M |\omega|_x^p dx$$

where  $|\omega|_x = \sup\{|\omega_x(X_1, \dots, X_k)|_x : X_i \in T_x M, |X_i|_x = 1\}$ . We denote by  $L^p\Omega^k(M)$  the norm completion of the normed space  $\{\omega \in \Omega^k(M), \|\omega\|_{L^p} < \infty\}$ . The differential of some  $\omega \in L^p\Omega^k(M)$  may not have finite  $L^p$ -norm, hence the spaces  $L^p\Omega^k(M)$  with exterior derivatives do not form a cochain complex. This is why we let for  $\omega \in \Omega^k(M)$ :

$$\|\omega\|_{\Omega^{p,k}} = \|\omega\|_{L^p} + \|d\omega\|_{L^p}$$

and we define the space  $\Omega^{p,k}(M)$  to be the norm completion of the normed space  $\{\omega \in \Omega^k(M), \|\omega\|_{\Omega^{p,k}} < \infty\}$ . The space  $\Omega^{p,k}(M)$  is a Banach space. The exterior derivative extends to a bounded operator  $d_k : \Omega^{p,k}(M) \rightarrow \Omega^{p,k+1}(M)$  satisfying  $d \circ d = 0$ .

**Definition 1.1.9.** We define the  $k$ -th de Rham  $L^p$ -cohomology space  $L^p H_{\text{dR}}^k(M)$  to be the  $k$ -th cohomology space of the cochain complex:

$$\Omega^{p,0}(M) \xrightarrow{d_0} \Omega^{p,1}(M) \xrightarrow{d_1} \Omega^{p,2}(M) \xrightarrow{d_2} \dots$$

The  $k$ -th reduced de Rham  $L^p$ -cohomology space  $L^p \overline{H}_{\text{dR}}^k(M)$  is the largest Hausdorff quotient of  $L^p H_{\text{dR}}^k(M)$ .

We shall state two important results concerning de Rham  $L^p$ -cohomology: regularization of forms and Poincaré duality.

**Theorem 1.1.10.** [GT06, 12.8] *Let  $M$  be a Riemannian manifold. Every cohomology class in  $L^p H_{\text{dR}}^k(M)$  can be represented by a smooth form.*

In particular, this implies that for a compact Riemannian manifold, de Rham  $L^p$ -cohomology coincides with the usual de Rham cohomology.

**Theorem 1.1.11.** [GT10] *Let  $M$  be a complete oriented Riemannian manifold and dimension  $n$  and let  $p, q > 1$  be such that  $p^{-1} + q^{-1} = 1$ . For  $0 \leq k \leq n$ , the Banach space  $L^p \overline{H}_{\text{dR}}^k(M)$  is isometric to the dual of  $L^q \overline{H}_{\text{dR}}^{n-k}(M)$ . The duality is given by:*

$$\begin{aligned} L^p \overline{H}_{\text{dR}}^k(M) \times L^q \overline{H}_{\text{dR}}^{n-k}(M) &\rightarrow \mathbb{R} \\ ([\omega], [\theta]) &\mapsto \int_M \omega \wedge \theta \end{aligned}$$

**Asymptotic  $L^p$ -cohomology** We introduce asymptotic  $L^p$ -cohomology following [Pan95] and [BR21]. First we introduce some vocabulary.

**Definition 1.1.12.** We say that a metric space  $(X, d)$  with a Borel measure  $\mu$  (we call this a *metric measure space*) is of *bounded geometry* if there exist two functions  $v, V : (0, +\infty) \rightarrow (0, +\infty)$  such that for every ball  $B(x, r) \subseteq X$  of radius  $r$  we have:

$$v(r) \leq \mu(B(x, r)) \leq V(r).$$

Let  $(X, d, \mu)$  be a metric space measure space of bounded geometry. For  $k \in \mathbb{N}$  and  $t > 0$  and define  $\Delta_t^k = \{(x_i)_i \in X^{k+1} : d(x_i, x_j) < t, \forall i \neq j\}$ . Now let  $L^p AS^k(X)$  to be the space of (classes of) functions  $u : X^{k+1} \rightarrow \mathbb{R}$  such that for every  $t > 0$  one has:

$$N_t(u)^p := \int_{\Delta_t^k} |u(x_0, \dots, x_k)|^p d\mu(x_0) \dots d\mu(x_k) < \infty.$$

The space  $L^p AS^k(X)$  equipped with the family of semi-norms  $N_t$  for  $t > 0$  is a Fréchet space for every  $k \in \mathbb{N}$ . These form a cochain complex where the differentials  $d_k : L^p AS^k(X) \rightarrow L^p AS^{k+1}(X)$  are the classical simplicial differentials.

**Definition 1.1.13.** We define the  $k$ -th *asymptotic  $L^p$ -cohomology space*  $L^p H_{AS}^k(X)$  to be the  $k$ -th cohomology space of the cochain complex:

$$L^p AS^0(X) \xrightarrow{d_0} L^p AS^1(X) \xrightarrow{d_1} L^p AS^2(X) \xrightarrow{d_2} \dots$$

The  $k$ -th *reduced asymptotic  $L^p$ -cohomology space*  $L^p \overline{H}_{AS}^k(X)$  is the largest Hausdorff quotient of  $L^p H_{AS}^k(X)$ .

### 1.1.2.2 Comparison theorems

We list results relating the different versions of  $L^p$ -cohomology that we introduced.

We start with the  $L^p$ -analogue of de Rham's theorem due to Pansu [Pan95] (see [Seq22, 1.1] and [Gen14, 4.7] for further elaborations). Recall that the classical de Rham's theorem states that de Rham cohomology of a manifold can be computed via simplicial cohomology of a triangulation.

**Theorem 1.1.14.** *Let  $M$  be a Riemannian manifold that admits a continuous triangulation  $X$  of bounded geometry and such that every simplex of  $X$  is bi-Lipschitz homeomorphic to the standard simplex of the same dimension, where the Lipschitz constant does not depend on the simplex. Then for every  $p > 1$  we have an isomorphism of graded topological vector spaces:*

$$L^p H_{dR}^*(M) = \ell^p H^*(X).$$

Asymptotic  $L^p$ -cohomology is hard to compute directly. The next result says that we can compute it using continuous  $L^p$ -cohomology (which can still be hard to compute, but at least we have more homological tools).

**Theorem 1.1.15.** [SS18] [BR20, 3.6] *Let  $G$  be a locally compact second countable group endowed with a left-invariant proper metric. Then for every  $p > 1$  we have an isomorphism of graded topological vector spaces:*

$$H_{ct}^*(G, L^p(G)) = L^p H_{AS}^*(G).$$

Notice that both of our previous results do not require strong hypothesis on our objects. In some sense we can say that on the one hand we have de Rham and simplicial  $L^p$ -cohomology and on the other hand we have asymptotic and continuous  $L^p$ -cohomology. The next results say that we should draw a line between these two sides. Indeed, as we will see later, asymptotic (and hence also continuous)  $L^p$ -cohomology is always invariant under quasi-isometries (and even coarse equivalences). Quasi-isometries do not take into account the local topology of our spaces, but de Rham and simplicial  $L^p$ -cohomology do. Hence if we want to show that these two sides coincide, it is necessary to suppose that there is nothing at local or bounded scale, so that all that remains is of asymptotic nature. The good hypothesis is *uniform contractibility*.

**Definition 1.1.16.** We say that a metric space  $(X, d)$  is *uniformly contractible* if  $X$  is contractible and if there exists a function  $\Phi : [0, +\infty) \rightarrow [0, +\infty)$  such that for every  $x \in X$  and  $r > 0$ , the closed ball  $B(x, r)$  is contractible inside  $B(x, \Phi(r))$ .

The next result is due to Pansu [Pan95]. It has perhaps the most involved proof among these comparison theorems. It relies on a bicomplex lemma (used twice) to identify each of these versions to an  $L^p$ -version of Čech cohomology.

**Theorem 1.1.17.** [Pan95] *Let  $M$  be a uniformly contractible Riemannian manifold of bounded geometry. Then for every  $p > 1$  we have an isomorphism of graded topological vector spaces:*

$$L^p H_{\text{AS}}^*(M) = L^p H_{\text{dR}}^*(M).$$

We end this section by stating a standard result that relates continuous group  $L^p$ -cohomology and simplicial  $L^p$ -cohomology in the case of a group action. Note that a contractible space endowed with a cocompact group action is automatically uniformly contractible.

**Proposition 1.1.18.** *Let  $G$  be a locally compact second countable group acting properly and cocompactly on a contractible simplicial complex  $X$  of bounded geometry. Then for every  $p > 1$  we have an isomorphism of graded topological vector spaces:*

$$H_{\text{ct}}^*(G, L^p(G)) = \ell^p H^*(X).$$

The proof of this proposition is simple: it consists in defining equivariant cochains and showing that they give a relatively injective strong resolution of  $L^p(G)$ , hence the associated cohomology coincides with continuous  $L^p$ -cohomology. Then one should unfold these equivariant cochains (that are defined on a compact fundamental domain) and show that they are just classical  $L^p$ -cochains.

### 1.1.2.3 Quasi-isometric invariance

The main property that distinguishes  $L^p$ -cohomology from continuous cohomology with values in an arbitrary representation is quasi-isometric invariance. We begin with a brief discussion on quasi-isometries and coarse equivalences.

Large-scale geometry refers to the study of geometric features that depend only on asymptotic behaviour of geometric objects. By this we mean that we are interested in properties that remain invariant under bounded perturbations or rescaling

of a metric. Bi-Lipschitz equivalence ignores the importance of rescaling, but is still sensitive to bounded perturbations. We relax the definition of bi-Lipschitz equivalence by allowing some additive constant to obtain the notion of *quasi-isometry*.

**Definition 1.1.19.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric spaces. We say that a function  $f : X \rightarrow Y$  is a *quasi-isometric embedding* if there exist two constants  $C > 1$  and  $D > 0$  such that for all  $x, y \in X$ :

$$C^{-1}d_X(x, y) - D \leq d_Y(f(x), f(y)) \leq Cd_X(x, y) + D$$

We say that  $f : X \rightarrow Y$  is a *quasi-isometry* if there exists  $g : Y \rightarrow X$  such that both  $f$  and  $g$  are quasi-isometric embeddings and  $g \circ f$  is at bounded distance from the identity, that is, there exists  $C > 0$  such that for all  $x \in X$   $d_X(g \circ f(x), x) \leq C$ .

Many of the features that were historically studied in the framework of quasi-isometries are in fact invariant under a coarser relation, called coarse equivalence, that becomes increasingly popular.

**Definition 1.1.20.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric spaces. We say that a function  $f : X \rightarrow Y$  is a *coarse embedding* if there exist two non-decreasing functions  $\rho_-, \rho_+ : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\lim_{t \rightarrow +\infty} \rho_-(t) = \lim_{t \rightarrow +\infty} \rho_+(t) = \infty$  and such that for all  $x, y \in X$  we have:

$$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y)).$$

We say that  $f : X \rightarrow Y$  is a *coarse equivalence* if there exists  $g : Y \rightarrow X$  such that both  $f$  and  $g$  are coarse embeddings and  $g \circ f$  is at bounded distance from the identity.

In this thesis we deal mostly with classical metric spaces that enjoy good properties. In particular they are often length spaces (see [BH99, I.3.1] for a definition). In this setting, a coarse equivalence is automatically a quasi-isometry.

**Proposition 1.1.21.** [DK18, 8.26] *Let  $(X, d_X)$  and  $(Y, d_Y)$  be two geodesic metric spaces. Let  $f : X \rightarrow Y$  be a coarse equivalence. Then  $f : X \rightarrow Y$  is a quasi-isometry.*

The main feature of  $L^p$ -cohomology is that it is invariant under quasi-isometries in many situations. In fact, it is also invariant under coarse equivalences (see [Pan95], notice that the definition of quasi-isometry there is in fact what we call here a coarse equivalence). This is shown first for asymptotic  $L^p$ -cohomology and then one uses comparison theorems (from the previous subsection) to show that other versions are also invariant.

**Theorem 1.1.22.** [Pan95] *Let  $M_1$  and  $M_2$  be two metric measure spaces of bounded geometry. Suppose that  $F : M_1 \rightarrow M_2$  is a coarse equivalence. Then for all  $p > 1$ ,  $F$  induces canonically an isomorphism of graded topological vector spaces:*

$$F^* : L^p H_{\text{AS}}^*(M_2) \xrightarrow{\cong} L^p H_{\text{AS}}^*(M_1).$$

This result can also be shown directly in the simplicial setting. This was first noticed by Gromov in [Gro93] (see [BP03] for a complete proof).

**Theorem 1.1.23.** *Let  $X_1$  and  $X_2$  be two uniformly contractible simplicial complexes of bounded geometry. Let  $F : X_1 \rightarrow X_2$  be a quasi-isometry. Then for all  $p > 1$ ,  $F$  induces canonically an isomorphism of graded topological vector spaces:*

$$F^* : \ell^p H^*(X_2) \xrightarrow{\cong} \ell^p H^*(X_1).$$

### 1.1.3 $\ell^2$ -Betti numbers, $L^2$ -Betti numbers and applications

$\ell^2$ -Betti numbers of discrete groups were first defined by Cheeger and Gromov in [CG86]. Later, Lück gave another point of view by developing a dimension theory for modules on von Neumann algebras with finite trace. For him,  $\ell^2$ -Betti numbers are just the Murray-von Neumann dimensions of the  $L^2$ -cohomology spaces [Lüc02]. Petersen extends some of Lück's work to modules on von Neumann algebras with *semifinite* trace, which allowed him to give a definition of  $L^2$ -Betti numbers for general *unimodular* lcsc groups [Pet13], [KPV15].

Let  $G$  be a unimodular lcsc group. The *von Neumann algebra*  $LG$  of the group  $G$  is the weak operator closure of the operators  $\lambda(\gamma)$  for  $\gamma \in \Gamma$  in the space of bounded operators on  $L^2(G)$ , where  $\lambda$  denotes the left regular representation on  $L^2(G)$  [Lüc02, I.1.1]. We say that a vector space  $V$  is a *right  $LG$ -module* if  $V$  admits a right action of the algebra  $LG$ .

We denote by  $\dim_{(G,\mu)} M$  the *Murray-von Neumann dimension* of some right  $LG$ -module  $M$ , associated to some natural semifinite trace on  $LG$ , as defined in [KPV15, Appendix A].

It remains to see that for a unimodular lcsc group  $G$ , the spaces  $H_{\text{ct}}^k(G, L^2(G))$  and  $\overline{H}_{\text{ct}}^k(G, L^2(G))$  are right  $LG$ -modules for any  $k \in \mathbb{N}$  (here  $L^2(G)$  denotes the *left* regular representation on  $L^2(G)$ , since  $G$  is unimodular this is equivalent to the right regular representation). Essentially, this corresponds to the fact that  $L^2(G)$  is naturally a right  $LG$ -module using the *right* regular representation and all the algebraic machinery that is used to define  $H_{\text{ct}}^k(G, L^2(G))$  and  $\overline{H}_{\text{ct}}^k(G, L^2(G))$  from  $L^2(G)$  may only alter the left action, but respects the right action.

**Definition 1.1.24.** Let  $G$  be a unimodular locally compact second countable group endowed with a Haar measure  $\mu$ . For  $k \in \mathbb{N}$ , we define the  *$k$ -th  $L^2$ -Betti number* of  $G$  to be:

$$\beta^k(G, \mu) := \dim_{(G,\mu)} H_{\text{ct}}^k(G, L^2(G)).$$

Changing the Haar measure only rescales the whole sequence of  $L^2$ -Betti numbers, so after fixing a Haar measure we may only write  $\beta^k(G)$  instead of  $\beta^k(G, \mu)$ .

*Remark.* One can also define  $\overline{\beta}^k(G, \mu) := \dim_{(G,\mu)} \overline{H}_{\text{ct}}^k(G, L^2(G))$ . In fact these two definitions define the same object for unimodular locally compact second countable groups [KPV15, 2.2].

**Applications** We give some applications of  $L^2$ -Betti numbers to the problem of distinguishing groups up to different equivalence relations.

We have already seen that  $L^p$ -cohomology is invariant under quasi-isometry and coarse equivalence for every  $p > 1$ , so by choosing  $p = 2$ , we obtain the following consequence for  $L^2$ -Betti numbers.

**Theorem 1.1.25.** (*Sauer-Schrödl*) [SS18, Theorem 1] *Let  $G$  and  $H$  be two unimodular lcsc groups. Suppose that  $G$  and  $H$  are coarsely equivalent. Then for every  $k \in \mathbb{N}$  we have:  $\beta^k(G) = 0$  if and only if  $\beta^k(H) = 0$ .*

One can study groups up to other equivalence relations, such as *measure equivalence*. We introduce it as the measurable analogue of quasi-isometry in view of Gromov's following characterization of quasi-isometry.

**Theorem 1.1.26.** (Gromov) [Gro93, 0.2.C.2'] *Two finitely generated groups  $\Gamma$  and  $\Lambda$  are quasi-isometric if and only if there exist commuting actions of  $\Gamma$  and  $\Lambda$  on some locally compact space  $X$  that are continuous, proper and cocompact.*

We now define measure equivalence for countable discrete groups by replacing topological conditions in the previous statement by measurable conditions.

**Definition 1.1.27.** Two countable groups  $\Gamma$  and  $\Lambda$  are *measure equivalent* if there exist commuting actions of  $\Gamma$  and  $\Lambda$  on some standard (infinite) measure space  $(\Omega, m)$  that are measure preserving, free, and with finite measure fundamental domain.

We call the number  $[\Gamma : \Lambda]_\Omega = \frac{m(\Omega/\Lambda)}{m(\Omega/\Gamma)}$  the *index* of the coupling  $\Omega$ . An example of two measure equivalent groups are two lattices  $\Gamma$  and  $\Lambda$  inside the same locally compact group  $G$ .

The following statement explains the relation between measure equivalence and  $\ell^2$ -Betti numbers.

**Theorem 1.1.28.** (Gaboriau)[Gab02, 6.3] *Let  $\Gamma$  and  $\Lambda$  be two measure equivalent countable discrete groups. Then for every  $k \in \mathbb{N}$  we have:*

$$\beta^k(\Gamma) = [\Gamma : \Lambda]_\Omega \cdot \beta^k(\Lambda).$$

*In particular, the sequence of  $\ell^2$ -Betti numbers of a discrete countable group is invariant under measure equivalence up to proportionality on the sequence.*

In particular, the sequences of  $\ell^2$ -Betti numbers of two lattices inside the same lsc group are proportional. This corollary of Gaboriau's criterion may also be obtained using the following result, which relates  $\ell^2$ -Betti numbers of lattices to the  $L^2$ -Betti numbers of the ambient locally compact group.

**Theorem 1.1.29.** (Kyed-Petersen-Vaes) [KPV15, Theorem B] *Let  $G$  be a unimodular lsc group and  $H$  a closed unimodular subgroup of  $G$  of finite covolume. After fixing Haar measures on  $G$  and  $H$ , we have for all  $k \in \mathbb{N}$ :*

$$\beta^k(G) = \frac{1}{\text{covol}_G(H)} \beta^k(H).$$

These last two results are the ones we will use in [LN21] to distinguish finitely presented simple groups up to measure equivalence.

## 1.2 Buildings

We now turn to the objects on which we will compute  $L^p$ -cohomology, that is, semisimple groups over local fields, their geometries and generalizations. In fact we will focus mostly on the non-Archimedean case and its corresponding generalizations, as the tools we use for studying real semisimple groups are standard and can be found in references such as [Hel01] or [Kna96]. The generalizations we are talking about are *buildings*.

Buildings were introduced by Tits, first as incidence geometries to show simplicity of some finite groups and later as non-Archimedean analogues of symmetric

spaces. They are not only tools to study groups that we have already met, as they can also be used to define new groups with exotic properties.

The main goal of this section is to introduce buildings. We start with a short discussion on Coxeter systems. We will then define a building as a purely combinatorial object to which we associate metric realizations. We close this section by introducing the main examples of groups and buildings that we will work with: these are semisimple groups over non-Archimedean local fields acting on their Bruhat-Tits buildings and (complete) Kac-Moody groups acting on their corresponding Kac-Moody buildings.

### 1.2.1 Coxeter systems

Before introducing buildings, we define their building blocks, namely, Coxeter systems.

**Definition 1.2.1.** A *Coxeter system*  $(W, S)$  is the data of a *finite* set  $S$  and a group given by generators and relations:

$$W = \langle s \in S \mid (st)^{m_{st}} = 1, s, t \in S \rangle,$$

with  $m_{ss} = 1$  and  $m_{st} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$  for  $s \neq t$ . The matrix  $M = (m_{st})_{s,t \in S}$  is called the *Coxeter matrix* of  $(W, S)$ . For a subset  $T \subseteq S$ , we denote by  $W_T = \langle s \in T \rangle \subseteq W$  the *standard parabolic subgroup of type  $T$*  of  $W$ .

We may separate Coxeter systems in three categories: *spherical*, *affine* and *non-affine*. Let  $(W, S)$  be a Coxeter system. We say that  $(W, S)$  is *spherical* if  $W$  is finite. We say that  $(W, S)$  is *affine* if  $W$  is infinite and  $S$  acts on some Euclidean space as orthogonal reflections with respect to hyperplanes (sometimes it is also asked for them to be *irreducible*, meaning that they cannot be written as a product of two non-trivial Coxeter systems, here we do not make this hypothesis). The term *non-affine* Coxeter system refers to any Coxeter system that is neither spherical nor affine.

Both spherical and affine Coxeter systems are completely classified [Bou68]. As we will see soon after introducing examples of buildings, we will be interested mostly in infinite Coxeter systems. Among these, there is a sharp dichotomy opposing affine and non-affine Coxeter systems, which can be stated as a strong version of the classical Tits' alternative.

**Theorem 1.2.2.** [Dav08, 17.2.1] *Let  $(W, S)$  be an infinite Coxeter system. Then  $W$  is either affine (and hence virtually abelian and of polynomial growth) or  $W$  contains a finite index subgroup that maps onto the free group on two generators (and hence is of exponential growth).*

This dichotomy will also be reflected on their corresponding buildings. Affine Coxeter groups lead to classical Lie theory associated to simple algebraic groups over non-Archimedean local fields. The non-affine case leads to new, exotic buildings and to Kac-Moody groups outside the scope of classical algebraic group theory.

## 1.2.2 Buildings

We will now define buildings first as purely combinatorial objects to which we will associate later topological or metric realizations. In fact we will work with more than one metric realization for a same combinatorial structure. To some extent, the main realization for us is the *Davis realization*, because it is the easiest way to check the  $CAT(0)$ -property.

### 1.2.2.1 Combinatorial buildings

We define buildings from a purely combinatorial point of view following [Ron89].

A *chamber system* over a set  $S$  is a set  $\mathcal{C}$  together with a family of equivalence relations on  $\mathcal{C}$  indexed by  $S$ . The elements of  $\mathcal{C}$  are called *chambers*. Two chambers are *s-equivalent* if they are equivalent under the relation corresponding to  $s$  and *s-adjacent* if they are s-equivalent and not equal.

A *gallery*  $\gamma$  in  $\mathcal{C}$  is a finite sequence of chambers  $(c_0, \dots, c_k)$  such that  $c_{i-1}$  is *s-adjacent* to  $c_i$  for all  $1 \leq i \leq k$ . The gallery is said to have *type*  $s_1 \dots s_k$  (as a word on the free monoid on the alphabet  $S$ ) if  $c_{i-1}$  is  $s_i$ -adjacent to  $c_i$  for every  $i$ .

**Definition 1.2.3.** Let  $(W, S)$  be a Coxeter system. A (*combinatorial*) *building of type*  $(W, S)$  is a chamber system  $\mathcal{C}$  over  $S$  such that:

- (i) for all  $s \in S$ , each  $s$ -equivalence class contains at least two chambers and,
- (ii) there exists a  $W$ -valued *distance function*  $d_W : \mathcal{C} \times \mathcal{C} \rightarrow W$ , that is, a map satisfying that: for each  $w \in W$ , if  $w = s_1 \dots s_k$  is a reduced word for  $w$  in the alphabet  $S$  (that is  $s_i \in S$  for every  $i$  and the integer  $k$  is minimal), then two chambers  $c$  and  $c'$  can be joined by a gallery of type  $s_1 \dots s_k$  if and only if  $d_W(c, c') = w$ .

The group  $W$  is called the *Weyl group* of the building  $\mathcal{C}$  and the integer  $|S| - 1$  is the *rank* of  $\mathcal{C}$ .

**Example 1.2.4.** Let  $(W, S)$  be a Coxeter system. Then the chamber system  $\mathcal{C} = W$ , endowed with the relations  $w \sim_s w'$  when  $w = w's$  for each  $s \in S$  and the  $W$ -valued distance  $d_W : W \times W \rightarrow W$  defined by  $d_W(w, w') = w^{-1}w'$ , is a building of type  $(W, S)$  called the *abstract Coxeter complex of  $W$* .

A building  $\mathcal{C}$  of type  $(W, S)$  has *finite thickness* if for all  $s \in S$ , each  $s$ -equivalence class is finite. In this case, we say that  $\mathcal{C}$  is *regular* if for each  $s \in S$ , each  $s$ -equivalence class has the same number of elements  $q_s + 1$ . We denote by  $\mathbf{q}$  the vector containing the  $q_s$ 's as coordinates. We say  $\mathcal{C}$  that *has constant thickness* if all the  $q_s$ 's have the same value  $q$ . In what follows all of our buildings will always be regular. We say that a building is *spherical*, *affine* or *non-affine* if its Weyl group is respectively spherical, affine or non-affine.

### 1.2.2.2 Metric realizations of buildings

We now define geometric realizations of combinatorial buildings in a relatively general setting following [Dav08, Chapter 18]. This is because in what follows we will be interested in three different metric realizations of buildings. The classical Tits realization is easy to define but unpractical outside the affine case (often not locally finite). The well-known Davis realization carries a  $CAT(0)$ -metric and is perhaps

the most important metric realization of a building. We introduce also the less known Bestvina realization as it is well-suited for computing cohomology.

**General metric realizations** A simplicial complex is a topological space obtained by gluing simplices. Essentially, a metric realization of a building is obtained by gluing the same "local picture" following conditions imposed by the chamber system. The following definition formalizes what we mean by local picture.

A *mirror structure* on a CW complex  $K$  consists of an index set  $S$  and a family of subcomplexes  $(K_s)_{s \in S}$ . The subspaces  $K_s$  are the *mirrors* of  $K$ . In this case, we say that  $K$  is a *mirrored CW complex over  $S$* . We set  $K_\emptyset = K$  and for any nonempty subset  $T \subseteq S$ ,

$$K_T = \bigcap_{t \in T} K_t.$$

For  $x \in K$ , we set  $S(x) = \{s \in S, x \in K_s\}$ .

**Definition 1.2.5.** Let  $\mathcal{C}$  be a (combinatorial) building of type  $(W, S)$  and  $K$  a mirrored CW complex over  $S$ . The  $K$ -*realization* of  $\mathcal{C}$  is the space:

$$X_K = (\mathcal{C} \times K) / \sim,$$

where  $[(c, x)] \sim [(c', x')] if and only if  $x = x' \in K$  and  $d_W(c, c') \in W_{S(x)}$ .$

If  $(K, d)$  is a geodesic metric space, we can extend  $d$  to  $X_K$  by declaring that all translates of  $K$  are isometric and then by defining a piecewise length metric on  $X_K$ . If moreover each  $K_s$  is a proper metric space, then  $(X_K, d)$  is a geodesic metric space [AB08, Corollary 12.28]. The space  $(X_K, d)$  has bounded geometry if and only if the mirror structure satisfies that for every subset  $T \subseteq S$  generating an infinite subgroup  $W_T$ , we have  $K_T = \emptyset$ .

We now introduce some particular metric realizations of buildings.

**Tits buildings** Let  $\mathcal{C}$  be a building of type  $(W, S)$ . Here we choose  $K = \Delta^{|S|-1}$  to be the standard simplex of dimension  $|S| - 1$ . We endow  $\Delta^{|S|-1}$  with a mirror structure by letting  $K_s$  be the codimension 1 faces of  $\Delta^{|S|-1}$ . The space  $X_\Delta = (\mathcal{C} \times K) / \sim$  is called the *simplicial* or *Tits realization* of  $\mathcal{C}$ .

If  $\mathcal{C}$  has finite thickness, then  $X_\Delta$  is a simplicial complex of bounded geometry if and only if  $(W, S)$  is a finite, irreducible affine or *compact hyperbolic* Coxeter system (in the sense of [Bou68, p.133, exercice 14]). The latter are the only infinite, non-affine Coxeter systems such that every proper parabolic subgroup of  $(W, S)$  is finite. These exist, but there is only a finite number of them and they all have rank  $\leq 5$  [Bou68, p.133, exercice 15]. For affine Coxeter systems that are not irreducible, it is better to consider the polysimplicial complex obtained as the product of the Tits realizations of each irreducible affine subsystem.

The Tits realization is classically used as a definition of building. This is because historically the first buildings to be introduced and studied were spherical and affine buildings (for which the Tits realization gives already locally finite complexes for finite thickness buildings). At this time not too many questions were asked outside of this scope, so there was no need to introduce further realizations. The classical definition of a building is the following. A *building* is a polysimplicial complex  $X$

obtained by gluing subcomplexes called *apartments*, under three conditions:

- (i) There exists a Coxeter system  $(W, S) = (W_1 \times \dots \times W_r, S_1 \sqcup \dots \sqcup S_r)$ , where each  $(W_i, S_i)$  is irreducible, and each apartment is a copy of the polysimplicial Coxeter complex  $\Sigma = \Sigma_1 \times \dots \times \Sigma_r$  where each  $\Sigma_i = (W_i \times \Delta^{|S_i|-1}) / \sim$ .
- (ii) Any two polysimplices in  $X$  lie in a common apartment.
- (iii) For two apartments  $A, B$  in  $X$ , there exists a polysimplicial automorphism that carries  $A$  onto  $B$  and fixes  $A \cap B$ .

The Solomon-Tits theorem states that the Tits realization of a spherical building with irreducible Weyl group  $(W, S)$  is homotopy equivalent to a bouquet of spheres of dimension  $|S| - 1$ , where the number of spheres is the number of apartments containing a fixed chamber [AB08, 4.73].

**Davis buildings** We now define the Davis realization  $X_D$  of a building. The main reason to introduce this realization is that for any finite thickness combinatorial building it is locally finite (unlike the Tits realization) and in this case it can always be endowed with a  $CAT(0)$ -metric [Dav94]. We give two different (but equivalent) constructions of  $X_D$ .

We first follow [DJ02], where the Davis realization is constructed by taking out simplices from the Tits realization. Let  $X$  be the Tits realization of a *non-spherical* regular building  $\mathcal{C}$  of type  $(W, S)$ . Recall that the link of a simplex  $\sigma$  in a simplicial complex is the set:

$$\text{Lk}(\sigma) = \{\tau \text{ simplex}, \sigma \text{ and } \tau \text{ are disjoint faces of the same maximal simplex}\}.$$

**Definition 1.2.6.** Let  $X'$  be the first barycentric subdivision of  $X$ . The *Davis complex*  $X_D$  of  $X$  is the subcomplex of  $X'$  generated by the barycenters of simplices of  $X$  with compact links.

From the algebraic topology viewpoint, the space  $X_D$  is similar to  $X$  in the sense that it is a deformation retract of  $X$  [DJ02, 1.4]. The complex  $X_D$  is locally finite, so it is locally compact. Note that with this definition  $X_D$  is a simplicial complex, but it is not necessarily purely dimensional, that is, its maximal simplices may not have the same dimension.

The intersections  $\Delta \cap X_D$  are isomorphic for any chamber  $\Delta$  in  $X$  (because they are all translates of a given chamber by elements of the Weyl group for some apartment). We call such an intersection  $D$  a *Davis chamber* of  $X_D$ .

The main feature of the Davis complex is that it carries a  $CAT(0)$ -metric.

**Theorem 1.2.7.** (*Moussong*) [Dav08, 18.3.1] *There exists a piecewise Euclidean metric on the Davis chamber  $D$  such that its natural extension to  $X_D$  (by transporting the metric to any Davis chamber via the Weyl group) is  $CAT(0)$ .*

This implies that not only that  $X_D$  is contractible, but also that  $X$  is contractible when  $(W, S)$  is non-spherical. Recall that when  $(W, S)$  is spherical, the Solomon-Tits theorem says that  $X$  is homotopy equivalent to a bouquet of spheres.

We now describe a second construction of the Davis realization. As the notation  $X_D$  suggests, the space  $X_D$  can also be constructed from the underlying combinatorial building  $\mathcal{C}$  and a Davis chamber  $D$ , after seeing that  $D$  can be endowed with the structure of a mirrored space [Dav08, Chapter 18].

**Definition 1.2.8.** The *Davis chamber*  $D = D(W, S)$  of  $(W, S)$  is the (geometric realization of the) barycentric subdivision of the poset of all finite parabolic subgroups  $W_T = \langle s \in T \rangle$  for  $T \subseteq S$  in  $W$  ordered by inclusion. For  $s \in S$ , define  $D_s$  to be the subcomplex of  $D$  corresponding to the poset of all finite parabolic subgroups of  $W$  containing  $W_{\{s\}}$ .

Thus  $D$  is a mirrored space over  $S$  and both definitions of  $X_D$  agree:

$$X_D = (\mathcal{C} \times D) / \sim .$$

**Bestvina buildings** We introduce the Bestvina realization of a building following [Bes93]. The dimension of the Davis realization of a building gives an upper bound for the virtual cohomological dimension of its Weyl group, but in general there is no equality. The Bestvina chamber is a topological construction that associates to every Coxeter system  $(W, S)$  a finite contractible CW complex whose dimension coincides with the virtual cohomological dimension  $\text{vcd}(W) = \text{vcd}_{\mathbb{Z}}(W)$  of  $W$  (see [LN22, 3.3.2] for a definition). This last property makes it suitable in some contexts for cohomology.

Denote by  $\mathcal{F}$  the poset of subsets  $T \subseteq S$  such that  $W_T = \langle T \rangle$  is a finite parabolic subgroup of  $W$  ordered with respect to inclusion. For any maximal element  $F \in \mathcal{F}$ , define  $P_F$  to be a point. Assuming that  $P_{F'}$  has been constructed for every  $F' \supset F$ , define  $P_F$  to be a contractible polyhedron containing  $\bigcup_{F' \supset F} P_{F'}$  of the least possible dimension. Most of the time, we have  $\dim(P_F) = \dim(\bigcup_{F' \supset F} P_{F'}) + 1$  and  $P_F$  is just a cone on  $\bigcup_{F' \supset F} P_{F'}$ , but in some situations (e.g. when there is a unique  $F' \in \mathcal{F}$  such that  $F' \supset F$  and  $|F'| = |F| + 1$ ) we have  $\dim(P_F) = \dim(\bigcup_{F' \supset F} P_{F'})$ . We then define the *Bestvina chamber*  $B$  to be  $P_{\emptyset}$  [Bes93].

The Bestvina chamber has a natural mirrored structure given by the subcomplexes  $B_s = P_{\{s\}}$ ,  $s \in S$ . We can thus define the *Bestvina realization*  $X_B$  of a building  $\mathcal{C}$ , as the space:

$$X_B := (\mathcal{C} \times B) / \sim .$$

It is not known if this realization carries a  $CAT(0)$ -metric as the Davis realization, but for our purposes we may switch between these two versions as they are quasi-isometric locally finite simplicial complexes.

### 1.2.3 *BN*-pairs

So far we have introduced buildings, but no examples yet. We will now introduce the most classical examples of buildings. The construction is always the same: we start with the group we are interested in, to which we attach a building thanks to a standard procedure using a *BN-pair*.

**Definition 1.2.9.** Let  $G$  be a group. A *BN-pair* in  $G$  is the data of a pair of subgroups  $B$  and  $N$  in  $G$  and of a set  $S$  of classes modulo  $T := B \cap N$  in  $N$  satisfying:

- (T1)  $G = \langle B \cup N \rangle$  and  $B \cap N \triangleleft N$ ,
- (T2) elements in  $S$  are of order 2 and generate the group  $W := N/T$ ,
- (T3)  $sBw \subset BwB \cup BswB$  for  $s \in S$  and  $w \in W$ ,
- (T4) for  $s \in S$ , we have  $sBs \notin B$ .

The group  $W$  is called the *Weyl group* of the BN-pair defined by  $(B, N)$ . One can show that  $(W, S)$  is a Coxeter system [Bou68, IV.2.4 Théorème 2].

The general procedure to construct a building from a BN-pair is given by the following theorem.

**Theorem 1.2.10.** [Ron89, 5.3] *Let  $(B, N)$  be a BN-pair in  $G$ . Then the set of left cosets  $G/B$  defines a combinatorial building of type  $(W, S)$ , where  $W = N/(B \cap N)$ , with  $s$ -adjacency given by:*

$$gB \sim_s hB \iff g^{-1}h \in B\langle s \rangle B.$$

For  $g, h \in G$ , the  $W$ -valued distance between their cosets  $gB$  and  $hB$  is given by:

$$d_W(gB, hB) = w \iff g^{-1}h \in BwB.$$

Notice that in the preceding construction, the action by left translation of  $G$  on the building  $G/B$  is transitive, it is even transitive on the pairs of chambers at fixed  $W$ -distance and the stabilizer of any apartment acts transitively on the chambers it contains (such an action is said to be *strongly transitive*). Hence all of the examples we will present have large automorphism groups. Conversely, every building  $\mathcal{C}$  admitting a strongly transitive and type preserving (that is, preserves the adjacency relations) action of a group  $G$  comes from a BN-pair in  $G$ , where the subgroup  $B$  is the stabilizer of a chamber  $c \in \mathcal{C}$  and  $N$  is the stabilizer of an apartment containing  $c$  [Ron89, 5.2].

On the other hand, there are many examples of buildings with trivial full automorphism group. A first family of such buildings are trees with different valences in each vertex. In [BP07], there are examples of buildings of dimension 2 with affine Weyl group and trivial full automorphism group.

We now turn to examples of groups with a BN-pair: we start with the more classical situations and we go towards the more exotic ones.

### 1.2.3.1 Spherical buildings and semisimple algebraic groups

We first recall some definitions from algebraic group theory, for more details see [Bor91], [BT65] or [Mar91].

Let  $F$  be a field and  $G$  be the  $F$ -points of a semisimple algebraic group  $\mathbb{G}$  defined over  $F$  and let  $\mathfrak{g}$  denote its Lie algebra. We suppose that  $\mathbb{G}$  is *isotropic* over  $F$ , that is,  $G$  contains a non-trivial non-central  $F$ -split torus and we let  $S$  be a maximal  $F$ -split torus in  $G$  (the dimension of such a torus is the *split rank* of  $G$  over  $F$ ). The following idea is fundamental in classical Lie theory: we diagonalize the adjoint action  $\text{Ad}(s)$  of each element  $s \in S$  acting on the Lie algebra  $\mathfrak{g}$ . Since  $S$  is abelian, all of these endomorphisms are diagonalizable in a common basis of eigenvectors, we call *roots* the nontrivial characters  $\alpha : S \rightarrow F^\times$  such that the space

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall s \in S, \text{Ad}(s)X = \alpha(s)X\}$$

is nonzero. We denote by  $\Phi = \Phi(S, G)$  the set of roots, that we call the *root system* of  $G$ . The isotropy condition on  $G$  ensures that  $\Phi$  is nonempty. The set  $\Phi$  is an *abstract root system* and as such, it carries a Weyl group (which is finite) and we can choose a subset of positive roots that we denote by  $\Phi_+$  [Bor91, 14.7].

We now consider different subgroups of  $G$ , from which the  $BN$ -pair will be constructed. Let  $Z = Z_G(S)$  be the centralizer of  $S$  in  $G$  and  $N = N_G(S)$  be the normalizer of  $S$  in  $G$ . Let  $U^+$  be the subgroup of  $G$  with Lie algebra  $\mathfrak{u}^+ = \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha$ , it is normalized by  $Z$  [Bor91, 21.10]. We set  $B_0 = Z \ltimes U^+$ , it satisfies  $B_0 \cap N = Z$ .

**Theorem 1.2.11.** [Bor91, 21.15] *The pair of subgroups  $(B_0, N)$  is a  $BN$ -pair inside  $G$ , its Weyl group  $W_0 = N/(B_0 \cap N) = N/Z$  is finite and coincides with the Weyl group of the root system  $\Phi$ .*

The fact that this is a  $BN$ -pair is a way to sum up many previously known results on the structure theory of algebraic groups, such as the Bruhat decomposition of the group  $G = \bigsqcup_{w \in W} BwB$  [Bou68, IV.2.3] or properties of parabolic subgroups of  $G$  [Bou68, IV.2.6]. The historical reason to introduce it was that it allowed to run a uniform proof of abstract simplicity of simple isotropic algebraic groups modulo their centers (for such a simplicity criterion see [Bou68, IV.2.7]).

The building associated to this  $BN$ -pair is called the *spherical building of the semisimple algebraic group  $G$* . Historically, these were the first buildings to be studied. Essentially all spherical buildings of rank  $\geq 3$  arise from this construction [AB08, 9.1].

### 1.2.3.2 Affine buildings and Bruhat-Tits theory

Iwahori and Matsumoto showed that some  $p$ -adic semisimple groups carry a second  $BN$ -pair, this time with an affine Weyl group [IM65]. This inspired Bruhat and Tits to initiate a systematic study of reductive groups over non-Archimedean local fields, by extending Iwahori and Matsumoto's results to any such group: this is the celebrated Bruhat-Tits theory.

Let  $F$  be a non-Archimedean local field and  $G$  be the  $F$ -points of a semisimple algebraic group  $\mathbb{G}$  defined over  $F$ . The group  $G$  is lcsc and totally disconnected when endowed with the analytic topology coming from  $F$ . The fundamental theorem of Bruhat-Tits theory is that these groups admit good actions on some affine buildings. The following statement is reconstructed from different parts of [Tit79].

**Theorem 1.2.12.** *Let  $F$  be a non-Archimedean local field and  $G$  be the  $F$ -points of a simply connected simple algebraic group defined over  $F$  of split rank  $r \geq 1$ . Let  $N$  be the normalizer of a maximal split torus in  $G$ . There exists a compact open subgroup  $B$  of  $G$  (for the analytic topology) such that  $(B, N)$  is a  $BN$ -pair with irreducible affine Weyl group  $(W, S)$  and  $|S| = r + 1$ .*

The associated affine building is called the *Bruhat-Tits building* of the simple group  $G$  and the group  $B$  is called an *Iwahori subgroup* of  $G$ . Recall that for buildings coming from  $BN$ -pairs, the group  $B$  is the stabilizer of a chamber  $c$  and the group  $N$  is the stabilizer of an apartment containing  $c$ . This building has constant thickness  $q + 1$ , where  $q$  is the cardinal of the residual field of  $F$ .

We view it as a metric object via its Tits realization  $X$ , which coincides with the Davis realization and hence it inherits a  $CAT(0)$ -metric. Since  $B$  is compact, the group  $G$  acts properly and cocompactly on the contractible simplicial complex  $X$ , and hence  $H_{\text{ct}}^*(G, L^p(G)) = \ell^p H^*(X)$  for all  $p > 1$ . In this thesis we will make

computations on both sides of this equality: in [LN22] we compute the right-hand side and in [LN23] we compute the left-hand side.

Bruhat-Tits buildings are the non-Archimedean analogues of symmetric spaces of non-compact type: apartments are isometric embeddings of the Euclidean space into the building in question and thus correspond to maximal flats in the real case. The axioms that apartments satisfy (that we introduced for the Tits realization) encode properties of maximal flats in symmetric spaces.

Any locally finite affine building of dimension  $\geq 3$  (as a simplicial complex) arises as the Bruhat-Tits building of some semisimple algebraic group over a non-Archimedean local field [Tit86]. Recall that there are many exotic affine buildings of dimension 2 that do not come from this construction as their full automorphism group is trivial [BP07].

### 1.2.3.3 Kac-Moody groups

So far our examples of buildings had spherical or affine Coxeter systems. Kac-Moody theory is a generalization of Lie theory that gives examples of buildings with (usually) non-affine Coxeter systems and large automorphism groups. Because of the Strong Tits' alternative for Coxeter systems (see Theorem 1.2.2 here), one can expect that the groups acting on these buildings satisfy exotic properties: we will see that it is the case. We introduce Kac-Moody groups following the presentation of [DJ02, Appendix TKM].

**Definition 1.2.13.** A *Kac-Moody datum* is the data  $(I, \Lambda, (\alpha_i)_{i \in I}, (h_i)_{i \in I}, A)$  of:

1. A finite set  $I$ .
2. A finitely generated abelian free group  $\Lambda$ .
3. Elements  $\alpha_i \in \Lambda, i \in I$ .
4. Elements  $h_i \in \Lambda^\vee = \text{Hom}(\Lambda, \mathbb{Z}), i \in I$ .
5. A *generalized Cartan matrix*  $(A_{ij})_{i,j \in I}$  given by  $A_{ij} = \langle \alpha_i, h_j \rangle$ , satisfying

$$A_{ii} = 2, \text{ if } i \neq j \text{ then } A_{ij} \leq 0 \text{ and } A_{ij} = 0 \text{ if and only if } A_{ji} = 0.$$

From a Kac-Moody datum (or merely from a generalized Cartan matrix) one can define a Coxeter matrix  $M = (m_{ij})_{i,j \in I}$  as follows:

$m_{ii} = 1$  and for  $i \neq j, m_{ij} = 2, 3, 4, 6$  or  $\infty$  as  $A_{ij}A_{ji} = 0, 1, 2, 3$  or is  $\geq 4$ , respectively.

We consider the Coxeter group  $W$  associated to this matrix:

$$W = \langle r_i \mid (r_i r_j)^{m_{ij}} = 1, \text{ for } m_{ij} \neq \infty \rangle.$$

If a Kac-Moody datum is fixed, Tits defines a group functor associating to each field  $k$  a group  $\Lambda(k)$  [Tit87]. The group  $\Lambda(k)$  has two BN-pairs  $(B_+, N)$  and  $(B_-, N)$  such that their Weyl groups  $N/(B_\pm \cap N)$  are isomorphic to the group  $W$  coming from the generalized Cartan matrix.

These BN-pairs define two buildings  $X_+$  and  $X_-$  of thickness  $|k| + 1$  and Weyl group  $W$ , such that  $\Lambda(k)$  acts transitively on their sets of chambers [DJ02, Appendix TKM]. These buildings are simplicially isomorphic. Denote by  $G_\pm$  the completion of  $\Lambda(k)$  in  $\text{Aut}(X_\pm)$  with respect to the compact-open topology.

The following theorem summarizes some properties that make these groups interesting for us.

**Theorem 1.2.14.** *Let  $\Lambda$  be a Kac-Moody group over  $\mathbb{F}_q$  with Weyl group  $W$ . Then  $\Lambda$  is finitely generated. Moreover:*

1. [CR10] *The covolume of  $\Lambda$  in  $G_+ \times G_-$  (diagonally injected) is  $W(\frac{1}{q})$ , where*

$$W(t) = \sum_{w \in W} t^{l(w)}.$$

*In particular for  $q > |I|$ , the group  $\Lambda$  is a lattice in  $G_+ \times G_-$ .*

2. [CR06] *If  $W$  is non-affine, irreducible and  $\Lambda$  is a lattice in  $G_+ \times G_-$ , then  $\Lambda/Z(\Lambda)$  is simple, where  $Z(\Lambda)$  is the center of  $\Lambda$ .*

3. [AM97] *If  $q \geq 4$  and all the entries of the Coxeter matrix are finite (i.e. the Weyl group is 2-spherical), then  $\Lambda$  is finitely presented.*

If we start with an affine Weyl group, the groups  $G_+$  and  $G_-$  are linear. Hence so is the lattice  $\Lambda$  (and finitely presented by item 3). By Mal'cev's lemma, a finitely generated linear group is residually finite, hence it admits many finite index normal subgroups. On the other hand, in the non-affine case,  $\Lambda$  can be chosen to be simple, which is a strong negation of residual finiteness, hence of linearity.

### 1.3 $L^p$ -cohomology of Gromov-hyperbolic spaces

In this section we compile some results on the  $L^p$ -cohomology of Gromov-hyperbolic spaces. The spaces we will consider will be most of the time simplicial complexes, manifolds or discrete groups.

Again, we pay particular attention to the case of degree 1, where a numerical invariant known as *conformal dimension* plays the role of a critical exponent. Indeed, for simplicial complexes with positive Cheeger constant,  $\ell^p$ -cohomology in degree 1 grows with  $p$  [LdlSW20, 10.1], so we may look at the infimal value of  $p$  for which these spaces are nonzero. For some hyperbolic spaces, this value turns out to be the conformal dimension. In more general settings this can fail, but we can still view conformal dimension as a critical exponent for  $\ell^p$ -cohomology in degree 1.

Pansu has intensively studied de Rham  $L^p$ -cohomology of hyperbolic manifolds and more precisely non-compact symmetric spaces of rank 1. We close the section by reviewing some of his results and techniques.

#### 1.3.1 Hyperbolicity and quasi-conformal analysis

In this section we quickly recall some basic notions on Gromov-hyperbolicity and on metrics on boundaries of Gromov-hyperbolic spaces. For more details on Gromov-hyperbolic spaces, see [GdlH90], [BH99, III.H], [BS00]. For more details on the quasi-conformal structure of the boundary and quasi-conformal analysis, see [MT10], [Hei01], [Väi84].

##### 1.3.1.1 The boundary of a Gromov-hyperbolic space

**Definition 1.3.1.** Let  $\delta > 0$ . A geodesic metric space  $X$  is  $\delta$ -hyperbolic if for every geodesic triangle in  $X$ , the union of the  $\delta$ -neighborhoods of any two sides of the triangle cover the third side. We say that  $X$  is *Gromov-hyperbolic* (or just *hyperbolic*) if  $X$  is  $\delta$ -hyperbolic for some  $\delta > 0$ .

Recall that Gromov-hyperbolicity is invariant under quasi-isometries [GdlH90, 5.2.12]. We say that a lsc group is *Gromov-hyperbolic* (or just *hyperbolic*) if it admits a compact generating set such that its associated Cayley graph is hyperbolic. Notice that this graph is not necessarily locally finite. In fact, a locally compact group is hyperbolic if and only if it has a proper cocompact isometric action on a *proper* geodesic hyperbolic space [CdCMT15, 2.6].

**Example 1.3.2.** Some examples of Gromov-hyperbolic spaces are: simplicial trees, manifolds of strictly negative sectional curvature (in particular Heintze groups and non-compact symmetric spaces of rank 1) and Fuchsian buildings.

A finite thickness (Davis) building with Weyl group  $(W, S)$  is Gromov-hyperbolic if and only if  $W$  is hyperbolic [Dav08, 18.3.9]. If the Davis complex of a Coxeter group is a hyperbolic manifold, then its dimension is at most 30 [Dav08, 12.6.7]. Nevertheless, there are examples of families of hyperbolic Coxeter groups  $(W_n)_{n \in \mathbb{N}}$  such that their Davis complexes are orientable pseudomanifolds of dimension  $n$  [JS03]. Hence we have Gromov-hyperbolic buildings of arbitrarily high dimension.

We can characterize hyperbolicity on a metric space  $(X, d)$  using the Gromov product. For  $x, y \in X$ , define the *Gromov product* based at  $o \in X$  to be:

$$(x|y)_o = \frac{1}{2}(d(x, o) + d(y, o) - d(x, y)).$$

The result is that the geodesic metric space  $(X, d)$  is Gromov-hyperbolic if and only if there exists  $\delta > 0$  such that for all  $x, y, z \in X$  we have:

$$(x|z)_o \geq \min\{(x|y)_o, (x|z)_o\} - \delta.$$

The advantage of this characterization is that  $(X, d)$  need not be geodesic.

We define now the boundary of a Gromov-hyperbolic space.

**Definition 1.3.3.** Let  $X$  be a proper, Gromov-hyperbolic, geodesic metric space. Fix a point  $o \in X$ . We call *boundary at infinity* of  $X$  the set  $\partial X$  of equivalence classes of geodesic rays emanating from  $o$ , up to finite Hausdorff distance.

This definition does not depend on the choice of the basepoint  $o \in X$ . Equivalently, the space  $\partial X$  may be defined as the set of equivalence classes of sequences of points  $(x_n)_{n \in \mathbb{N}}$  in  $X$  such that  $(x_i|x_j)_o \rightarrow \infty$  when  $i, j \rightarrow \infty$ , up to the relation:

$$(x_n)_n \sim (y_n)_n \iff (x_i|y_j)_o \rightarrow \infty \text{ when } i, j \rightarrow \infty.$$

We may define a topology on  $\partial X$  using the Gromov product. We first extend the Gromov-product to points in the boundary. For  $\xi, \eta \in \partial X$  define:

$$(\xi|\eta)_0 = \sup \liminf_{i, j \rightarrow \infty} (x_i|y_j)_o$$

where the supremum is taken over representatives  $(x_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$  of the classes  $\xi$  and  $\eta$ . A basis of neighborhoods of a point  $\xi \in \partial X$  is given by, for  $n \in \mathbb{N}$ :

$$\{\eta \in \partial X, (\xi|\eta)_0 \geq n\}.$$

Endowed with this topology,  $\partial X$  becomes a compact topological space.

### 1.3.1.2 Metrics on the boundary and conformal dimension

We are interested in metrizing the topology of  $\partial X$ . In fact the definition of the basis of neighborhoods already gives us a hint on how to define a metric on  $\partial X$ .

**Proposition 1.3.4.** [BH99, p.435] *Let  $X$  be a proper Gromov-hyperbolic metric space and  $o \in X$  a basepoint. There exists  $\lambda_0 > 1$  small enough such that for every  $1 < \lambda \leq \lambda_0$ , there exists a metric  $d_\lambda$  on  $\partial X$  metrizing its topology and  $C > 0$  such that:*

$$C^{-1}\lambda^{-(\xi|\eta)_o} \leq d_\lambda(\xi, \eta) \leq C\lambda^{-(\xi|\eta)_o},$$

for all  $\xi, \eta \in \partial X$ . Such a metric is called a visual metric of parameter  $\lambda$ . The compact metric space  $\partial X$  is bounded and complete.

If a proper Gromov-hyperbolic space  $X$  carries a proper, cocompact and isometric action of a group, then visual metrics are *Ahlfors-regular*, that is, there exists a constant  $C > 1$  such that for every  $r < \text{diam}(\partial X, d_\lambda)$  and every ball  $B(r)$  of radius  $r$  we have:

$$C^{-1}r^Q \leq \mu(B(r)) \leq Cr^Q,$$

where  $Q$  is the Hausdorff dimension of  $(\partial X, d_\lambda)$  and  $\mu$  is the  $Q$ -Hausdorff measure of  $(\partial X, d_\lambda)$  [Coo93, 7.4].

The choice of a visual metric on the boundary is not canonical, but at least they are all pairwise *quasi-symmetric*.

**Definition 1.3.5.** We say that a homeomorphism  $f : Z_1 \rightarrow Z_2$  between two metric spaces  $(Z_1, d_1)$  and  $(Z_2, d_2)$  is a *quasi-symmetry* if there exists a homeomorphism  $\Psi : [0, \infty) \rightarrow [0, \infty)$  such that for all  $a, b, o \in Z_1$  and  $t \geq 0$ :

$$d_1(a, o) \leq td_1(b, o) \implies d_2(f(a), f(o)) \leq \Psi(t)d_2(f(b), f(o)).$$

In other words, the image of an annulus (that is, the difference of two balls of same center and different radius) by a quasi-symmetry is contained in an annulus of the target space with uniformly controlled ratio of radii. We say that two metrics  $d_1, d_2$  on the same space  $Z$  are *quasi-symmetric* if the identity map  $(Z, d_1) \rightarrow (Z, d_2)$  is a quasi-symmetry.

The following proposition says that quasi-isometries may be read at the boundary, and that instead of working with a precise metric on the boundary it is more natural to work with a family of metrics on it (that includes visual metrics).

**Proposition 1.3.6.** [BS00, 6.5] *Let  $X$  and  $Y$  be two proper geodesic Gromov-hyperbolic metric spaces, endowed with visual metrics. A quasi-isometry  $f : X \rightarrow Y$  induces a quasi-symmetric homeomorphism  $\partial f : \partial X \rightarrow \partial Y$ .*

Under mild assumptions the converse holds: every quasi-symmetric homeomorphism  $\partial X \rightarrow \partial Y$  extends to a quasi-isometry  $X \rightarrow Y$  [MT10, 3.2.13] [BS00, 7.4, 8.2]. In particular the converse holds for discrete hyperbolic groups [Pau96].

**Conformal gauge** In view of these results, we may look at the set of metrics on the boundary that are quasi-symmetric to visual metrics. In practice, it is good to look only at those that are Ahlfors-regular, but to guarantee that such metrics exist we need to impose some mild conditions on the boundary [MT10, 7.1]. We first define those conditions.

Let  $(Z, d)$  be a compact metric space. We say that  $(Z, d)$  is *uniformly perfect* if there exists a constant  $C > 1$  such that for every ball  $B(z, r)$  of radius  $r < \text{diam}(Z, d)$  we have  $B(z, r) \setminus B(z, \frac{r}{C}) \neq \emptyset$ . We say that a measure  $\mu$  on  $(Z, d)$  is *doubling* if there exists a constant  $D \geq 1$  such that for every ball  $B(z, r)$  of radius  $r < \text{diam}(Z, d)$  we have:

$$0 < \mu(B(z, 2r)) \leq D\mu(B(z, r)) < \infty.$$

We give the following definition.

**Definition 1.3.7.** Let  $X$  be a proper Gromov-hyperbolic metric space such that  $\partial X$  is uniformly perfect and carries a doubling measure. The (*Ahlfors-regular*) *conformal gauge* of  $\partial X$ , denoted  $\mathcal{J}(\partial X)$ , is the set of all Ahlfors-regular metrics  $d$  on  $\partial X$  that are quasi-symmetric to a visual metric on  $\partial X$ .

We may give another interpretation of the conformal gauge in terms of *shadows*.

**Definition 1.3.8.** Let  $X$  be a proper geodesic hyperbolic metric space and fix  $R > 0$ . The *shadow cast by  $x \in X$  on  $\partial X$* , denoted  $S(x)$ , is the set of points  $\xi \in \partial X$  such that there exists a geodesic ray from the origin  $o \in X$  converging to  $\xi$  that intersects the ball of radius  $R$  centered at  $x$ .

**Proposition 1.3.9.** (*Sullivan's shadow lemma*) [Bou95, 1.6.2 and 1.6.3] Let  $(X, d)$  be a proper geodesic Gromov-hyperbolic metric space with uniformly perfect boundary, endowed with a visual metric. There exists  $C_1 > 0$  such that for every  $x \in X$ , there exist two balls  $B(r_1)$  and  $B(r_2)$  in  $\partial X$ , with  $r_1 > 0$  and  $\frac{r_2}{r_1} \leq C_1$  such that:

$$B(r_1) \subseteq S(x) \subseteq B(r_2).$$

Conversely, there exists  $C_2 > 0$  such that for any ball  $B(r)$  of  $\partial X$ , there exist two points  $x, y \in X$  with  $d(x, y) \leq C_2$  such that:

$$S(x) \subseteq B(r) \subseteq S(y).$$

In other words, shadows are similar to balls of any visual metric. This is preserved by quasi-symmetries, and we may characterize the conformal gauge as the set of Ahlfors-regular metrics whose balls are similar to shadows, in the sense that they satisfy the statement of the Shadow lemma.

**Conformal dimension and examples** One may wonder about the right metric notion of dimension for  $\partial X$ . We would like a dimension that is invariant under quasi-isometries. A metric space carries a *Hausdorff dimension*, so we may look at the dimension of  $(\partial X, d_\lambda)$  with some visual metric  $d_\lambda$  of parameter  $\lambda > 1$ . This is not invariant under quasi-symmetries on the boundary. Indeed, if  $d$  is a metric on  $\partial X$  and  $0 < a < 1$  then the snowflake metric  $d^a$  satisfies:

$$\text{Hausdim}(\partial X, d^a) = \frac{1}{a} \text{Hausdim}(\partial X, d).$$

Hence when  $\lambda \rightarrow 1$ , we have  $\text{Hausdim}(\partial X, d_\lambda) \rightarrow \infty$ , so this is not the good invariant to look at. Instead of looking at large dimensions, we look at the infimal dimension.

**Definition 1.3.10.** Let  $X$  be a proper Gromov-hyperbolic space with uniformly perfect boundary. The (*Ahlfors-regular*) *conformal dimension* of  $\partial X$  is

$$\text{Confdim}(\partial X) = \inf\{\text{Hausdim}(\partial X, d), d \in \mathcal{J}(\partial X)\}.$$

$\text{Confdim}(\partial X)$  is invariant under quasi-isometries. We will now compute it in some examples.

Let  $T$  be a locally finite tree. Its boundary  $\partial T$  is a Cantor space. We have  $\text{Confdim}(\partial T) = 0$ . Indeed, for all  $x, y, z, o \in T$  the Gromov product satisfies:  $(x|z)_o \geq \min\{(x|y)_o, (x|z)_o\}$ . Hence the formula  $d_\lambda(\xi, \eta) = \lambda^{-(\xi|\eta)_o}$  for  $\xi, \eta \in \partial T$  defines a visual metric on  $\partial T$  for any  $\lambda > 1$  (because it satisfies the ultrametric inequality). By taking  $\lambda \rightarrow +\infty$ , we obtain that  $\text{Confdim}(\partial T) \leq \text{Hausdim}(\partial T, d_\lambda) \rightarrow 0$ .

It is hard to compute conformal dimension in more general settings. More precisely, it is hard to find metrics minimizing the Hausdorff dimension inside  $\mathcal{J}(\partial X)$ . Carnot groups and boundaries of Fuchsian buildings are examples of spaces for which such a metric is known to exist and its dimension has been computed. We will now elaborate on these two examples.

A *Carnot group* is a nilpotent Lie group  $N$  endowed with an automorphism  $\alpha$  such that its Lie algebra  $\mathfrak{n}$  decomposes as  $\mathfrak{n} = \bigoplus_{i \geq 1} \mathfrak{v}^i$  where  $\mathfrak{v}^1$  is a linear subspace of  $\mathfrak{n}$  and  $\mathfrak{v}^{i+1} = [\mathfrak{v}^1, \mathfrak{v}^i]$  for all  $i \geq 1$  and the derivative of  $\alpha$  acts as  $i \cdot \text{Id}$  on each  $\mathfrak{v}^i$ . We are interested in the Heintze groups (i.e. negatively curved homogeneous manifolds) of the form  $H = \mathbb{R} \times_\alpha N$ , where  $N$  is a Carnot group with automorphism  $\alpha$  [Pan89c]. The boundary  $\partial H$  of such a Heintze group carries a *Carnot-Carathéodory metric*  $d_{CC}$ . The conformal dimension of  $\partial H$  is:

$$\text{Confdim}(\partial H) = \text{Hausdim}(\partial H, d_{CC}) = \sum_{i \geq 1} i \dim_{\mathbb{R}} \mathfrak{n}^i$$

[Pan89b, 5.5]. The rank 1 symmetric spaces  $\mathbb{H}_{\mathbb{K}}^n$  (where  $\mathbb{K} = \mathbb{R}, \mathbb{C}$ , quaternions or octonions) are particular cases of such Heintze groups, their conformal dimension is:

$$\text{Confdim}(\partial \mathbb{H}_{\mathbb{K}}^n) = \text{Hausdim}(\partial \mathbb{H}_{\mathbb{K}}^n, d_{CC}) = nk + k - 2.$$

and  $k = \dim_{\mathbb{R}} \mathbb{K}$ .

Conformal dimension of Fuchsian buildings can be computed explicitly in general: if  $X$  denotes the Davis realization of a Fuchsian building with Weyl group  $(W, S)$  and thickness vector  $\mathbf{q} + 1$ , with  $\mathbf{q} \geq 2$ , there exists a metric  $d_{\mathbf{q}}$  in the conformal gauge of  $\partial X$  such that:

$$\text{Confdim}(\partial X) = \text{Hausdim}(\partial X, d_{\mathbf{q}}) = 1 + e_{\mathbf{q}}(W)^{-1},$$

where  $e_{\mathbf{q}}(W) = \limsup_n \frac{1}{n} |\{w \in W, \mathbf{q}_w \leq e^n\}|$  is the weighted logarithmic growth rate of  $(W, S)$  and for a reduced word  $w = s_1 \dots s_n$  in  $(W, S)$ , we define  $\mathbf{q}_w$  to be the product  $q_{s_1} \dots q_{s_n}$  [Bou00, 2.1].

Conformal dimension remains unknown for other Gromov-hyperbolic buildings. After stating Theorem 1.3.16 we will see that using  $\ell^p$ -cohomology in high degree we can obtain a first lower bound for their conformal dimension. Part of the work of this thesis consists in improving this lower bound via  $\ell^p$ -cohomology in degree 1 [LN22, 0.5].

### 1.3.2 Conformal dimension as a critical exponent

In this section we review some results relating  $L^p$ -cohomology to boundaries of Gromov-hyperbolic spaces and more precisely to conformal dimension.

#### 1.3.2.1 Degree 1: Besov spaces

For Gromov-hyperbolic spaces, one can read the first  $L^p$ -cohomology spaces as function spaces on the boundary. This is one of the foundational results of the theory. This was first shown by Pansu in [Pan89a] in the context of de Rham  $L^p$ -cohomology of homogeneous Riemannian manifolds of (strictly) negative sectional curvature.

Let  $(Z, d)$  be a metric space, let  $Q$  denote its Hausdorff dimension and  $\mu$  its  $Q$ -Hausdorff measure. Let  $u$  be a real valued measurable function on  $Z$  and define the *Besov norm* of  $u$ :

$$\|u\|_{B_p}^p = \int_{Z \times Z} \frac{|u(x) - u(y)|^p}{d(x, y)^{2Q}} d\mu(x) d\mu(y).$$

We define the *Besov space*  $B_p(Z, d)$  to be the Banach space of measurable functions  $u$  such that  $\|u\|_{B_p} < \infty$  modulo constant functions [BP03].

**Theorem 1.3.11.** (*Pansu*) [Pan89a, 5.2] *Let  $N$  be a Carnot group with automorphism  $\alpha$  endowed with its Carnot-Carathéodory metric  $d_{CC}$ . Let  $H = \mathbb{R} \rtimes_{\alpha} N$  be the associated negatively curved Heintze group. Then for all  $p > 1$  there is an isomorphism of Banach spaces:*

$$L^p H_{dR}^1(H) = B_p(N, d).$$

A similar result holds for simplicial  $\ell^p$ -cohomology. This is the content of the following statement. It applies to all discrete Gromov-hyperbolic groups, but does not require to have a group action in general.

**Theorem 1.3.12.** (*Bourdon-Pajot*) [BP03, 0.1] *Let  $X$  be a Gromov-hyperbolic simplicial complex of bounded geometry and  $d_0$  a visual metric on  $\partial X$ . Suppose that  $(\partial X, d_0)$  is uniformly perfect and carries a doubling measure. Let  $d$  be any Ahlfors-regular metric on  $\partial X$  quasi-symmetric to  $d_0$ . Then for all  $p > 1$  there is an isomorphism of Banach spaces:*

$$\ell^p H^1(X) = B_p(\partial X, d).$$

The isomorphism  $\ell^p H^1(X) \rightarrow B_p(\partial X, d)$  is given by taking the limit of a function following a geodesic ray. More precisely, for  $f : X^{(0)} \rightarrow \mathbb{R}$  such that  $\|df\|_p < \infty$ , we have that for  $\mu$ -almost every  $\xi \in \partial X$ , if  $\xi = (x_n)_{n \in \mathbb{N}}$ , the sequence  $(f(x_n))_{n \in \mathbb{N}}$  converges to a limit  $f_{\infty}(\xi)$  (where  $\mu$  denotes the  $Q$ -Hausdorff measure of  $(\partial X, d)$  and  $Q$  is its Hausdorff dimension).

The inverse map is given by averaging over shadows. More precisely, given a function  $u \in B_p(\partial X, d)$ , one defines a function  $f : X^{(0)} \rightarrow \mathbb{R}$  defined by:

$$f(x) = \frac{1}{\mu(S(x))} \int_{S(x)} u(\xi) d\mu(\xi),$$

where  $S(x)$  is the shadow casted by  $x \in X^{(0)}$ . This idea is originally due to Elek [Ele97]. We view this process as some sort of Poisson transform.

For any Ahlfors-regular metric  $d$  on  $\partial X$  and  $p > \text{Hausdim}(\partial X, d)$ , the Besov space  $B_p(\partial X, d)$  contains Lipschitz functions (for  $d$ ) and hence it is non-trivial. Hence Theorem 1.3.12 implies the following result:

**Corollary 1.3.13.** *Let  $X$  be a Gromov-hyperbolic simplicial complex as in 1.3.12. Then for  $p > \text{Confdim}(\partial X)$  we have  $\ell^p H^1(X) \neq \{0\}$ .*

One can ask if the condition on  $p > 1$  of the previous corollary is optimal. It is the case if there is a *Loewner* metric in the conformal gauge of the boundary. This is a rather restrictive condition that we will not define here (see [Hei01] for a definition), in particular such a metric minimizes the Hausdorff dimension inside the conformal gauge.

**Theorem 1.3.14.** *(Bourdon-Pajot) [BP03, 0.3] Let  $X$  be a Gromov-hyperbolic simplicial complex as in 1.3.12. Suppose that there exists a Loewner metric  $d \in \mathcal{J}(\partial X)$ . Let  $Q = \text{Confdim}(\partial X) = \text{Hausdim}(\partial X, d)$ . Then:*

$$\ell^p H^1(X) \neq \{0\} \text{ if and only if } p > Q.$$

In the setting of the previous theorem, conformal dimension is a critical exponent for  $\ell^p$ -cohomology in the sense that:

$$\text{Confdim}(\partial X) = \inf\{p > 1, \ell^p H^1(X) \neq \{0\}\}.$$

We can ask if a similar result persists without a Loewner metric. This is the content of the next theorem. Before stating it, we introduce some notation.

Let  $\ell^p H_{\text{cont}}^1(X)$  be the subspace of  $\ell^p H^1(X)$  consisting of  $\ell^p$ -cohomology classes of functions  $f : X^{(0)} \rightarrow \mathbb{R}$  whose extensions to the boundary (by taking radial limits)  $f_\infty : X^{(0)} \cup \partial X \rightarrow \mathbb{R}$  are continuous. We denote by

$$A_p(\partial X) := \{u : \partial X \rightarrow \mathbb{R} \mid u = f_\infty \text{ for } [f] \in \ell^p H_{\text{cont}}^1(X)\}$$

the space of continuous functions on  $\partial X$  that may be achieved as limits of  $\ell^p$ -cohomology classes in degree 1.

We introduce some vocabulary. We say that a Gromov-hyperbolic space  $X$  is *non-degenerate* if every  $x \in X$  lies at uniformly bounded distance from the three sides of some ideal geodesic triangle. We say that a metric space  $(Z, d)$  is *approximately self-similar* if there exists  $L \geq 1$  such that for every ball  $B(z, r) \subseteq Z$  of radius  $r < \text{diam} Z$ , there exists an open set  $U \subseteq Z$  that is  $L$ -bi-Lipschitz homeomorphic to the rescaled ball  $(B(z, r), \frac{1}{r}d)$ .

The following statement relates simplicial  $\ell^p$ -cohomology and conformal dimension in a more general setting.

**Theorem 1.3.15.** *(Bourdon-Kleiner) [BK15, 3.8] Let  $X$  be a non-degenerate, Gromov-hyperbolic simplicial complex of bounded geometry. Suppose that  $\partial X$  is connected and approximately self-similar. Then:*

$$A_p(\partial X) \text{ separates points of } \partial X \text{ if and only if } p > \text{Confdim}(\partial X).$$

The conditions hold in particular for discrete hyperbolic groups with connected boundary, but we do not need a group action in general. In fact the theorem also holds for any discrete hyperbolic group, without the connectedness assumption, see the remark after [LN22, 3.4.3].

### 1.3.2.2 Conformal dimension and $\ell^p$ -cohomology in degrees $\geq 2$

The prototype of a Gromov-hyperbolic space is a simplicial tree. The simplicial  $\ell^p$ -cohomology of such a tree is nonzero in degree 1 for every  $p > 1$  (one can construct by hand some cohomology classes or invoke the results from the previous section) and it is trivial in every other degree. In particular, it is trivial in degrees  $\geq 2$ .

The following result shows that in fact, for large values of  $p$ , most Gromov-hyperbolic simplicial complexes exhibit the same behaviour as a tree.

**Theorem 1.3.16.** *(Bourdon) [Bou16a] Let  $X$  be a non-degenerate Gromov-hyperbolic uniformly contractible simplicial complex of bounded geometry. Then for every  $k \geq 2$  and every  $p > \frac{\text{Confdim}(\partial X)}{k-1}$  we have  $\ell^p H^k(X) = \{0\}$ .*

Notice that for smaller values of  $p$  the  $\ell^p$ -cohomology could be nonzero. We will see in the next section that sometimes (in particular, for the real hyperbolic space) the conditions on  $p$  in this theorem are optimal.

In this thesis we will show that for a (Davis) building  $X$  of finite thickness  $\mathbf{q} + 1$  and Weyl group  $W$ , we have  $\ell^p \overline{H^d}(X) \neq \{0\}$  for  $1 < p < 1 + e_{\mathbf{q}}(W)^{-1}$ , where  $d = \text{vcd}_{\mathbb{R}}(W)$  [LN22, 0.4]. Hence if moreover  $W$  is hyperbolic, then  $X$  is also hyperbolic and combining this result with Theorem 1.3.16 we obtain:

$$\frac{\text{Confdim}(\partial X)}{\text{vcd}_{\mathbb{R}}(W) - 1} \geq 1 + e_{\mathbf{q}}(W)^{-1}.$$

In [Cla17], Clais obtained a similar but in fact stronger inequality for some families of Gromov-hyperbolic buildings constructed from graphs:

$$\frac{\text{Confdim}(\partial X)}{\text{Confdim}(\partial W)} \geq 1 + e_{\mathbf{q}}(W)^{-1}.$$

Indeed, we have  $\text{vcd}_{\mathbb{R}}(W) - 1 \leq \text{Topdim}(\partial W) \leq \text{Confdim}(\partial W)$  [BM91]. In this thesis we will use the first  $\ell^p$ -cohomology spaces to extend Clais' inequality to all Gromov-hyperbolic buildings of finite thickness [LN22, 0.5].

### 1.3.3 $L^p$ -cohomology of pinched Riemannian manifolds

The previous sections focused on simplicial  $\ell^p$ -cohomology of Gromov-hyperbolic spaces. We will now discuss results by Pansu concerning de Rham  $L^p$ -cohomology of pinched Riemannian manifolds, not limited to degree 1. We will not really use this later in the thesis, but we include it to give a more complete picture. We mostly follow [Pan08] and [BR21, 1].

We say that a negatively curved Riemannian manifold is  $\delta$ -pinched for  $\delta < 0$ , if there exists  $a > 0$  such that its sectional curvature  $K$  satisfies:  $-a \leq K \leq \delta a$ .

The main result is that, under contraction hypotheses on  $L^p$ -differential  $k$ -forms (that are satisfied by pinched negatively curved manifolds for some values of  $k$  and  $p$ ), one can identify  $L^p$ -cohomology in degree  $k$  to a function space on the boundary, which in particular is Hausdorff. The same hypotheses imply vanishing of  $L^p$ -cohomology in the previous degree.

Let  $M$  be a Riemannian manifold,  $\varphi : M \rightarrow M$  be a  $C^\infty$  map and  $k \in \mathbb{N}$ . The pullback  $\varphi^* : \Omega^k(M) \rightarrow \Omega^k(M)$  of differential forms may induce bounded operators

$\varphi^* : L^p\Omega^k(M) \rightarrow L^p\Omega^k(M)$ , if it does we denote by  $\|\varphi^*\|_{L^p\Omega^k}$  its operator norm. If it does not, we set  $\|\varphi^*\|_{L^p\Omega^k} = +\infty$ .

We now introduce the contraction hypothesis from Pansu [Pan08].

**Definition 1.3.17.** Let  $M$  be a Riemannian manifold and  $\xi$  be a  $C^\infty$  complete unit vector field on  $M$ . Denote by  $\varphi_t$  its flow. Let  $p > 1$  and  $k \in \mathbb{N}^*$ . We say that  $\xi$  is  $(k, p)$ -contracting if there exist  $C > 0$  and  $\eta > 0$  such that for all  $t \geq 0$  we have:

$$\|\varphi_t^*\|_{L^p\Omega^k \cap \ker \iota_\xi} \leq Ce^{-\eta t}.$$

We say that  $\xi$  is  $(k, p)$ -dilating if  $-\xi$  is  $(k, p)$ -contracting.

We now state Pansu's criterion for vanishing and reducedness of  $L^p$ -cohomology under contraction conditions. Stating properly the identification in higher degrees requires defining currents on manifolds. We will only give a simplified version (see [BR21, 1.3] for a precise statement and further details).

**Theorem 1.3.18.** (Pansu) [Pan08, 10] Let  $M$  be a Riemannian manifold,  $\xi$  be a  $C^\infty$  complete unit vector field on  $M$  and  $k \in \mathbb{N}^*$ . Suppose that  $\xi$  is  $(k-1, p)$ -contracting and  $(k, p)$ -contracting. Then:

- $L^p H_{\text{dR}}^k(M) = \{0\}$ ,
- $L^p H_{\text{dR}}^{k+1}(M)$  can be identified to some space of currents on  $M$  that are invariant by the flow of  $\xi$ . In particular  $L^p H_{\text{dR}}^{k+1}(M)$  is Hausdorff.

This identification is what Pansu calls "Künneth formula" for  $L^p$ -cohomology. To the knowledge of the author, this is one of the only known methods to prove that  $L^p$ -cohomology spaces are Hausdorff in degrees  $> 1$ , without showing that they are trivial (the other being studying  $L^2$ -cohomology combined with Riesz-Thorin interpolation, see [Loh98] for the Riemannian setting, [DJ02] for buildings).

Both of these results rely on some homotopy operators  $B_t^k : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$  defined by:

$$B_t^k(\omega) = \int_0^t \varphi_s^*(\iota_\xi \omega) ds.$$

If  $\xi$  is  $(k-1, p)$ -contracting, taking the limit  $t \rightarrow +\infty$  shows that there is a limit operator  $B_\infty^k : \Omega^{p,k}(M) \rightarrow \Omega^{p,k-1}(M)$  that informally satisfies, for a closed form  $\omega \in \Omega^{p,k}$ , the relation  $dB_\infty^k(\omega) = \lim_{t \rightarrow \infty} \varphi_t^*(\omega) - \omega$ . If  $\xi$  is moreover  $(k, p)$ -contracting, then  $\lim_{t \rightarrow \infty} \varphi_t^* \omega = 0$  and  $dB_\infty^k(\omega) = -\omega$ , showing the vanishing result. In degree  $(k+1)$ , we have to understand the limit  $\lim_{t \rightarrow \infty} \varphi_t^*(\omega)$  (that we may define by the formula  $dB_\infty^{k+1}(\omega) + \omega$ ) by defining a space of currents to which it belongs.

The next result is a non-vanishing criterion.

**Theorem 1.3.19.** (Pansu) [Pan11, 32] Let  $M$  be a Riemannian manifold,  $\xi$  be a  $C^\infty$  complete vector field on  $M$  and  $k \in \mathbb{N}^*$ . Suppose that  $\xi$  is  $(k-1, p)$ -contracting,  $(k, p)$ -contracting, and  $(k+1, p)$ -dilating. Then  $L^p H_{\text{dR}}^{k+1}(M)$  is non-trivial.

Now everything boils down to verifying when these conditions hold. The main observation is that pinching of sectional curvature gives a sufficient condition.

**Theorem 1.3.20.** [Pan08, Proposition 5] Let  $M$  be a simply connected, complete Riemannian manifold of dimension  $n$  with sectional curvature  $K$  satisfying  $-1 \leq$

$K \leq \delta$  for some  $\delta < 0$ . Let  $\xi$  be the gradient of a Busemann function on  $M$ . For  $k = 0, 1, \dots, n-1$  and

$$1 < p < 1 + \frac{n-k-1}{k}\sqrt{-\delta}, \quad (\text{resp. } p > 1 + \frac{n-k-1}{k\sqrt{-\delta}})$$

the vector field  $\xi$  is  $(k, p)$ -contracting (resp.  $(k, p)$ -dilating). In particular:

- $L^p H_{\text{dR}}^k(M)$  is Hausdorff for  $1 < p < 1 + \frac{n-k}{k-1}\sqrt{-\delta}$ .
- $L^p H_{\text{dR}}^k(M) = \{0\}$  for  $1 < p < 1 + \frac{n-k-1}{k}\sqrt{-\delta}$ .

Poincaré duality for de Rham  $L^p$ -cohomology allows one to deal with large values of  $p$ . In this way, Pansu computes the  $L^p$ -cohomology of the real hyperbolic space  $\mathbb{H}_{\mathbb{R}}^n$  in all degrees for all but finitely many values of  $p$ . Indeed,  $\mathbb{H}_{\mathbb{R}}^n$  has constant sectional curvature, so we may choose  $\delta = -1$  in the previous theorem and hence we know exactly when the vector field  $\xi$  is  $(k, p)$ -contracting or  $(k, p)$ -dilating in terms of  $k$  and  $p$ , except for the critical values  $p = 1 + \frac{n-k-1}{k} = \frac{n-1}{k}$ . In fact, Pansu also shows that for  $p = 1 + \frac{n-(k-1)-1}{k-1} = \frac{n-1}{k-1}$  the space  $L^p H_{\text{dR}}^k(\mathbb{H}_{\mathbb{R}}^n)$  is non-Hausdorff [Pan08, Théorème B.1]. We obtain:

**Proposition 1.3.21.** *Let  $M = \mathbb{H}_{\mathbb{R}}^n$  with  $n \geq 2$ . Let  $p > 1$  and  $k = 1, \dots, n$ . We have:*

- $L^p H_{\text{dR}}^0(M) = L^p H_{\text{dR}}^n(M) = \{0\}$ .
- $L^p H_{\text{dR}}^k(M) \neq \{0\}$  if and only if  $\frac{n-1}{k} < p \leq \frac{n-1}{k-1}$  (with  $\frac{n-1}{k-1} = \infty$  if  $k = 1$ ).
- $L^p H_{\text{dR}}^k(M)$  is Hausdorff if and only if  $p \neq \frac{n-1}{k-1}$  (void condition if  $k = 1$ ).

This is one of the only non-trivial full computations of  $L^p$ -cohomology known to date. Notice that for every  $p > 1$ , apart for some finite number of values, the cohomology is Hausdorff in every degree and there exists one unique degree  $k$  for which  $L^p H_{\text{dR}}^k(\mathbb{H}_{\mathbb{R}}^n) \neq \{0\}$ . These are the conditions that allowed Bourdon and Rémy to use the Hochschild-Serre spectral sequence to compute  $L^p$ -cohomology of some real simple Lie groups [BR20].

Unfortunately, the real hyperbolic space appears to be the only pinched negatively curved manifold for which these methods are optimal. The situation for the complex hyperbolic plane  $\mathbb{H}_{\mathbb{C}}^2$  is already much subtler. This is a  $(-\frac{1}{4})$ -pinched negatively curved manifold and the previous methods show vanishing (or reducedness) for both small and large values of  $p > 1$ . There is a strip of intermediate values of  $p$  for which the behaviour of  $L^p$ -cohomology remains unknown. In particular it is not known if, as in the real hyperbolic space, for fixed  $p > 1$  there is a unique  $k$  such that  $L^p H_{\text{dR}}^k(\mathbb{H}_{\mathbb{C}}^2) \neq \{0\}$ . In [Pan11], Pansu shows that the  $L^p$ -cohomology of symmetric spaces of rank 1 is also reduced on some parts of this strip.

## 1.4 Vanishing of cohomology below the rank

In this section we compile different results of similar nature, that deal with vanishing of continuous cohomology (often with unitary coefficients) or  $L^p$ -cohomology for simple or semisimple groups in degrees below the rank. Here a semisimple group (or a semisimple Lie group) can mean either a real semisimple Lie group or the points of a semisimple algebraic group over a non-Archimedean local field. Our main

motivation is the following question by Gromov: he predicts a classical behaviour of (unreduced)  $L^p$ -cohomology of semisimple groups over local fields for every  $p > 1$  [Gro93, p.253].

**Questions.** (Gromov) Let  $G$  be a semisimple group of rank  $r \geq 2$  over a local field.

- (1) Do we have  $H_{\text{ct}}^l(G, L^p(G)) = \{0\}$  for all  $l = 1, \dots, r-1$  and  $p > 1$ ?
- (2) Do we have  $H_{\text{ct}}^r(G, L^p(G)) \neq \{0\}$  at least for some values of  $p$ ?
- (3) Is the space  $H_{\text{ct}}^r(G, L^p(G))$  Hausdorff for all  $p > 1$ ?

Gromov's intuition for this question is that a symmetric space of non-compact type, rank  $r$  and without Euclidean factor is similar to a product of  $r$  non-compact symmetric spaces of rank 1. If we had some sort of Künneth formula for direct products in  $L^p$ -cohomology, question (1) would hold at least for such a product.

This is not the only reason to think that such result is true. Indeed, at the time there were already results due to Garland and Casselman that show vanishing in degrees below the rank for group cohomology of simple higher rank groups with values in *unitary* representations (hence for  $L^2$ ).

We can also ask the corresponding question for unitary coefficients, but this should only be asked for *simple* Lie groups. Indeed, the group  $G = \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$  is of higher rank and we will see later that  $H_{\text{ct}}^1(G, L^p(G)) = \{0\}$  for every  $p > 1$ , but on the other hand this group does not have Property (T), and hence it does not satisfy  $H_{\text{ct}}^1(G, \pi) = \{0\}$  for every unitary representation  $\pi$  of  $G$ .

Many contributions towards an affirmative answer of question (1) deal with the case of degree 1. Most of this section will be dedicated to these contributions, both in the unitary and in the  $L^p$ -cohomology setting.

### 1.4.1 Kazhdan's property (T) and higher rank simple groups

We first discuss vanishing of continuous group cohomology in degree 1 for unitary coefficients: this is Kazhdan's Property (T). Kazhdan's Property (T) was introduced by Kazhdan in [Kaz67]. His original motivation was to find a sufficient condition to guarantee that some countable group is finitely generated. The examples he had in mind were lattices in simple Lie groups, which were not known to be finitely generated to that date. He shows they are, by proving that higher rank simple groups have property (T) and that this property passes to lattices.

After giving a definition of Property (T) and reformulating it in cohomological terms, we may give the classical proof that higher rank simple groups have Property (T). We will end discussing a more recent variant of Property (T) known as *Strong Property (T)*.

#### 1.4.1.1 Definition and cohomological reformulation

Our exposition will be quite short, so we may refer to [BdlHV08] or [Mar91, III] for a much more complete exposition on the subject. We start with one of the most classical definitions of Property (T).

**Definition 1.4.1.** Let  $G$  be a locally compact group. We say that  $G$  has *Kazhdan's property (T)* when for every continuous unitary representation  $(\pi, H)$ , if  $\pi$  has almost

invariant vectors, that is, for every compact subset  $Q \subseteq G$  and  $\varepsilon > 0$  there exists a unit vector  $\xi \in H$  such that

$$\sup_{x \in Q} \|\pi(x)\xi - \xi\|_H < \varepsilon,$$

then  $\pi$  has a nonzero  $G$ -invariant vector.

We may interpret this definition in cohomological terms. Indeed, one can show that if a unitary representation  $\pi$  does not have almost invariant vectors, the space of 1-coboundaries  $B^1(G, \pi)$  is closed in  $Z^1(G, \pi)$ , and hence  $H_{\text{ct}}^1(G, \pi)$  is Hausdorff. The converse holds when  $G$  is  $\sigma$ -compact and  $\pi$  does not have invariant vectors. Hence a  $\sigma$ -compact group has Property (T) if and only if  $H_{\text{ct}}^1(G, \pi)$  is Hausdorff for every unitary representation  $\pi$  of  $G$  without  $G$ -invariant vectors.

The celebrated Delorme-Guichardet theorem states that for  $\sigma$ -compact locally compact groups, Property (T) implies Property (FH) [BdlHV08, 2.12.4]. A group satisfies Property (FH) if every continuous action by affine isometries on a Hilbert space has a fixed point. This turns out to be equivalent to vanishing of continuous cohomology in degree 1 for every unitary representation [BdlHV08, 2.2.10].

Later Shalom gave another characterization of Property (T). He proved that for compactly generated groups, Property (T) can be characterized by the vanishing of the first *reduced* cohomology space for every unitary representation [BdlHV08, 3.2.1]. We sum up the previous theorems in the following statement.

**Theorem 1.4.2.** *Let  $G$  be a compactly generated locally compact group. The following conditions are equivalent:*

- (i)  $G$  has Property (T),
- (ii)  $H_{\text{ct}}^1(G, \pi) = \overline{H_{\text{ct}}^1(G, \pi)}$  for every unitary representation  $\pi$  of  $G$  without invariant vectors,
- (iii) Property (FH):  $H_{\text{ct}}^1(G, \pi) = \{0\}$  for every unitary representation  $\pi$  of  $G$ ,
- (iv) Shalom's criterion:  $\overline{H_{\text{ct}}^1(G, \pi)} = \{0\}$  for every unitary representation  $\pi$  of  $G$ .

#### 1.4.1.2 Property (T) for higher rank simple groups

The most classical examples of groups with Property (T) are higher rank simple groups. In fact, one of Kazhdan's original motivation to introduce Property (T) was to show that lattices in Lie groups are finitely generated.

**Theorem 1.4.3.** *(Kazhdan '67) Let  $G$  be a connected higher rank simple real algebraic group. Then  $G$  (and any lattice in  $G$ ) has Kazhdan's Property (T).*

We will give some details on the proof of this theorem. Namely, we will give a proof of the *Howe-Moore property* for higher rank real simple groups.

**Theorem 1.4.4.** *(Howe-Moore) Let  $G$  be a connected higher rank real simple algebraic group with finite center and let  $(\pi, H)$  be a unitary representation of  $G$ . If  $\pi$  has no invariant vectors, then the matrix coefficients of  $\pi$  vanish at infinity, that is, for every  $v, w \in H$  we have:*

$$\langle \pi(g)v, w \rangle \rightarrow 0 \text{ when } g \in G, g \rightarrow \infty.$$

The idea behind the proof of Theorem 1.4.4 is a general strategy that will appear again later in this thesis: create and propagate invariance. More precisely: if a matrix coefficient does not vanish at infinity, one can first create a vector that is invariant under some element in a root group of  $G$  and then one propagates this invariance to all of  $G$  using *Mautner's phenomenon*. This is a general dynamical argument that allows one to propagate invariance in many situations and that we will use later.

**Lemma 1.4.5.** (*Mautner's lemma [Mar91, II.3.3]*) *Let  $G$  be a locally compact group and let  $(\pi, H)$  be a unitary representation of  $G$ . Let  $g_n, s_n, s'_n$  be three sequences of elements in  $G$  such that  $g_n \rightarrow g \in G$  and  $s_n g_n s'_n \rightarrow e$ . If a vector  $v \in H$  is  $\pi(s_n)$ -invariant and  $\pi(s'_n)$ -invariant for every  $n \in \mathbb{N}$ , then  $v$  is  $\pi(g)$ -invariant.*

The proof of Mautner's phenomenon is a straightforward computation using matrix coefficients. Let us begin the proof of Theorem 1.4.4.

*Proof of Theorem 1.4.4.* Let  $G$  be a higher rank simple real algebraic group with finite center and  $(\pi, H)$  a unitary representation of  $G$ . Suppose that there exists a matrix coefficient not vanishing at infinity, that is, there exist  $v, w \in H$  and a sequence  $(g_n)$  such that  $g_n \rightarrow \infty$  but  $\langle \pi(g_n)v, w \rangle \not\rightarrow 0$ . Up to taking a subsequence we may suppose that  $\langle \pi(g_n)v, w \rangle \rightarrow c$ , with  $c \neq 0$ . Using the Cartan decomposition  $G = KA^+K$ , where  $K$  is a maximal compact subgroup of  $G$  and  $A^+$  a Weyl chamber, we may write  $g_n = k_n a_n k'_n$ , with  $k_n, k'_n \in K$  and  $a_n \in A^+$ . Since  $g_n \rightarrow \infty$ , there exists a simple root in the root system  $\alpha$  of  $G$  such that  $\alpha(a_n) \rightarrow \infty$ . Up to taking a subsequence, we may suppose that  $k_n \rightarrow k \in K$  and  $k'_n \rightarrow k' \in K$ .

After replacing  $v$  and  $w$  by  $\pi(k)v$  and  $\pi(k'^{-1})w$  we obtain  $\langle \pi(a_n)v, w \rangle \rightarrow c$ . By weak-compactness of the unit ball of  $H$ , we have that  $\pi(a_n)v$  converges weakly to some vector  $v_0 \in H$ . We will show that  $v_0$  is nonzero, invariant by the root group  $U_\alpha$  associated to  $\alpha$  and in fact  $G$ -invariant. By weak convergence:

$$\langle v_0, w \rangle = \lim_n \langle \pi(a_n)v, w \rangle = c \neq 0.$$

Hence  $v_0 \neq 0$ . Since  $\pi(a_n)v \rightarrow v_0$ , for any  $u \in U_\alpha$  we have  $\pi(u a_n)v \rightarrow \pi(u)v_0$  and:

$$\pi(a_n)(\pi(a_n^{-1} u a_n)v - v) \rightarrow \pi(u)v_0 - v_0.$$

This implies that:

$$\|\pi(u)v_0 - v_0\| \leq \limsup_n \|\pi(a_n)(\pi(a_n^{-1} u a_n)v - v)\| = \limsup_n \|(\pi(a_n^{-1} u a_n)v - v)\| = 0,$$

because  $a_n^{-1} u a_n \rightarrow e$  as  $u$  lives in the root group  $U_\alpha$  and  $\alpha(a_n) \rightarrow \infty$ . Hence  $v_0$  is  $\pi(U_\alpha)$ -invariant. This is the end of the "create invariance" step.

Now we have to propagate this invariance by using Mautner's phenomenon repeatedly. First, by the Jacobson-Morozov theorem, there exists a closed subgroup  $S$  of  $G$  containing  $U_\alpha$  and with Lie algebra  $\mathfrak{sl}_2(\mathbb{R})$ . We may extend our invariance to the subgroup  $S$  thanks to Mautner's lemma and the following matrix computation in  $\mathrm{SL}_2(\mathbb{R})$ . Fix  $t = p/q \in \mathbb{Q}^*$  and let:

$$u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, s_n = (u_s^{-p})^n, s'_n = (u_s^q)^n, g = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, g_n = \begin{pmatrix} t & 0 \\ \frac{t-1}{snp} & t^{-1} \end{pmatrix}.$$

We have that  $g_n \rightarrow g$  and a computation shows that  $s_n g_n s'_n = \begin{pmatrix} 1 & 0 \\ \frac{t-1}{snp} & 1 \end{pmatrix} \rightarrow e$ . By Mautner's lemma,  $v$  is invariant under the matrix  $g$  and hence by the group  $S$ .

Let  $E \in S$  be the element corresponding to the matrix  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  under this isomorphism. Then its adjoint action on the Lie algebra  $\mathfrak{g} = \text{Lie}(G)$  can be diagonalized as  $\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{R}} \mathfrak{g}_\lambda$  where  $\mathfrak{g}_\lambda = \{X \in \mathfrak{g}, \text{ad}(g)X = \lambda X\}$ . Consider the Lie algebras  $\mathfrak{g}^+ = \bigoplus_{\lambda > 0} \mathfrak{g}_\lambda$  and  $\mathfrak{g}^- = \bigoplus_{\lambda < 0} \mathfrak{g}_\lambda$  and consider the Lie subalgebra  $\mathfrak{n}$  generated by  $\mathfrak{g}^+ \cup \mathfrak{g}^-$ . Since  $[\mathfrak{g}_0, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_\mu$ , we have that  $\mathfrak{n}$  is an ideal. But the Lie algebra  $\mathfrak{g}$  is simple, hence either  $\mathfrak{n} = \{0\}$  or  $\mathfrak{n} = \mathfrak{g}$ . The first case is not possible as it would imply that  $\mathfrak{g} = \mathfrak{g}_0$  and hence  $E$  would lie in the center of  $\mathfrak{g}$ , contradicting the fact that  $G$  has finite center.

Mautner's lemma allows one to extend the invariance to the subgroups  $G^+$  and  $G^-$  that integrate the Lie subalgebras  $\mathfrak{g}^+$  and  $\mathfrak{g}^-$ . But  $\mathfrak{g}^+$  and  $\mathfrak{g}^-$  generate  $\mathfrak{g}$ , so  $G^+$  and  $G^-$  generate  $G$  and hence  $v$  is a  $G$ -invariant vector. This proves the Howe-Moore property for  $G$ .  $\square$

We sketch the proof of Theorem 1.4.3 in the case of  $G = \text{SL}_n(k)$  with  $n \geq 3$ . First,  $G$  contains a closed subgroup  $Q$  isomorphic to  $\text{SL}_2(k) \times k^2$ . If  $(\pi, H)$  is a unitary representation of  $G$  almost having invariant vectors, then the restriction  $\pi|_Q$  to  $Q$  also has almost invariant vectors. One can show that such a representation of  $Q$  has a  $\pi(k^2)$ -invariant vector  $v \in H$  [BdlHV08, 1.4.13], this phenomenon is called *relative property (T)* of the pair  $(\text{SL}_2(k) \times k^2, k^2)$ . Since  $k^2$  is non-compact, this means that the matrix coefficient  $g \mapsto \langle \pi(g)v, v \rangle$  does not vanish at infinity, hence the Howe-Moore property for  $G$  implies that  $\pi$  has an invariant vector.

*Remark.* Bader, Furman, Gelfand and Monod use similar techniques to show vanishing of the first continuous cohomology space of higher rank simple groups acting isometrically on Banach spaces of the form  $L^p(X, \mu)$ , where  $X$  is a standard Borel space,  $\mu$  a  $\sigma$ -finite measure and  $p > 1$  [BFGM07, Theorem B]. In particular, this implies vanishing of continuous  $L^p$ -cohomology in degree 1 of higher rank *simple* groups for all  $p > 1$ .

### 1.4.1.3 Lafforgue's Strong Property (T)

Higher rank simple groups satisfy a much stronger rigidity property, known as *Lafforgue's Strong Property (T)*, which implies the fixed point property for many representations on Hilbert spaces. This was introduced in [Laf08] with  $K$ -theoretic applications in mind. Lafforgue formulates it as a property of a group on a class of topological vector spaces, so one can talk about Strong Property (T) in contexts other than for Hilbert spaces (in particular for classes of Banach spaces).

What is new here is that the representations in question do not need to be isometric: one allows them to have *small* exponential growth. We will not give a definition of Strong Property (T), but we will record its consequences in terms of fixed point properties, after some definitions.

A *length function* on a locally compact group  $G$  is a positive continuous function satisfying  $l(g^{-1}) = l(g)$  and  $l(g_1 g_2) \leq l(g_1) + l(g_2)$ . We say that a representation  $(\rho, V)$  of  $G$  on a Banach space  $V$  has *exponential growth* if there exists a proper length function  $l$  on  $G$  and  $\alpha > 0$  such that  $\|\rho(g)\|_V \gtrsim e^{\alpha l(g)}$  for all  $g \in G$ .

**Theorem 1.4.6.** (Lafforgue) [Laf08, 2.1] Let  $G$  be a real simple Lie group containing  $\mathrm{SL}_3(\mathbb{R})$ . For every proper length function  $l$  there exists  $\alpha > 0$  such that for every representation  $(\pi, H)$  of  $G$  satisfying  $\|\pi(g)\|_H \lesssim e^{\alpha l(g)}$  for all  $g \in G$ , where  $H$  is a Hilbert space, we have  $H_{\mathrm{ct}}^1(G, \pi) = \{0\}$ .

**Theorem 1.4.7.** (Lafforgue) [Laf09, 0.3] Let  $F$  be a non-Archimedean local field and  $G$  be a simple algebraic group over  $F$  containing  $\mathrm{SL}_3(F)$ . For every proper length function  $l$  there exists  $\alpha > 0$  such that for every representation  $(\rho, V)$  of  $G$  satisfying  $\|\rho(g)\|_V \lesssim e^{\alpha l(g)}$  for all  $g \in G$ , where  $V$  is a Banach space  $V$  of type  $> 1$ , we have  $H_{\mathrm{ct}}^1(G, \rho) = \{0\}$ .

*Remark.* 1. Any simple algebraic group over a non-Archimedean local field  $F$  contains a copy of a group isogenous to  $\mathrm{SL}_3(F)$  or  $\mathrm{Sp}_4(F)$ . Liao proved Banach Strong Property (T) for  $\mathrm{Sp}_4(F)$ , hence Theorem 1.4.7 holds for all higher rank simple groups over non-Archimedean local fields.

2. Notice that the statement in the non-Archimedean case is much stronger than in the real case. For  $G = \mathrm{SL}_3(\mathbb{R})$ , de la Salle extended the class of Banach spaces for which Theorem 1.4.6 holds [dlS14].

3. A (non-trivial) finite dimensional irreducible representation of a (non-compact) simple Lie group has exponential growth, which is equal to the module of its highest weight. In particular, since highest weights lie in the weight lattice, Lafforgue's result applies at most to a finite number of finite dimensional irreducible representations.

**The case of affine buildings** Let  $F$  be a non-Archimedean local field and  $G$  a simple algebraic group over  $F$  containing  $\mathrm{SL}_3(F)$ . The spaces  $L^p(G)$  for  $p > 1$  are all Banach spaces of type  $> 1$  and the right regular representation on  $L^p(G)$  is isometric, hence Theorem 1.4.6 applies and gives another proof of  $H_{\mathrm{ct}}^1(G, L^p(G)) = \{0\}$  for every  $p > 1$ . In particular, the group  $G$  acts properly and cocompactly on its Bruhat-Tits building  $X$ , and thus we obtain  $\ell^p H^1(X) = \{0\}$  for every  $p > 1$ .

Bruhat-Tits buildings are the main source of affine buildings. All locally finite affine buildings of dimension  $\geq 3$  arise in this way. On the other hand, there are exotic affine buildings of dimension 2 with small full automorphism groups (for instance discrete). Lécureux, de la Salle and Witzel use techniques similar to those from [Laf08] to show that vanishing of  $\ell^p$ -cohomology in degree 1 persists for some of these buildings.

**Theorem 1.4.8.** [LdlSW20, Theorem A] Let  $X$  be a locally finite  $\tilde{A}_2$ -building. Then we have  $\ell^p H^1(X) = \{0\}$  for all  $p > 1$ .

## 1.4.2 Garland's method

We now deal with vanishing of group cohomology (still with unitary coefficients) in degrees below the rank, not restricted to the case of degree 1.

Before Gromov, Serre had already formulated a first conjecture of this kind in the non-Archimedean case: let  $\Gamma$  be a cocompact lattice of the points a simple algebraic group of rank  $r$  over a non-Archimedean local field, do we have  $H^i(\Gamma, \mathbb{R}) = \{0\}$  for  $0 < i < r$ ? Garland proved this conjecture in [Gar73] using geometric methods (that we call today *Garland's method*) under the assumption that the residual field is large enough. In [Cas74], Casselman reproved Garland's result using representation

theoretic machinery and took out this assumption. The precise statement (from Casselman) is the following.

**Theorem 1.4.9.** (*Garland '73, Casselman '74*) *Let  $F$  be a non-Archimedean local field and  $G$  be the  $F$ -points of a simple algebraic group of split rank  $r$ . Let  $\Gamma$  be a cocompact lattice in  $G$ . For every  $i = 1, \dots, r - 1$  and every finite-dimensional representation  $\pi$  of  $\Gamma$  we have:*

$$H^i(\Gamma, \pi) = \{0\}.$$

Garland's method consists mainly of two steps: reformulating the problem in terms of a simplicial version of Hodge theory and obtaining estimates on the Laplacian on links of simplices to show vanishing. We will now present Garland's ideas from the point of view of Ballmann and Świątkowski [BS97] (which works not only for finite-dimensional representations but also for unitary representations of the cocompact lattice).

First, let  $\Gamma$  be a (discrete) group acting properly on a simplicial complex of bounded geometry  $X$  and let  $(\rho, V)$  be a unitary representation of  $\Gamma$ . Suppose that  $X$  is purely  $n$ -dimensional, that is, every simplex lies inside an  $n$ -dimensional simplex (which is the case for affine buildings). One works with the  $\Gamma$ -equivariant cohomology  $L^2H^*(X, \rho)$  of  $X$ . One starts defining:

$$C^k(X, \rho) = \{c : X^{(k)} \rightarrow V \mid c(\gamma \cdot \sigma) = \rho(\gamma)c(\sigma) \text{ for } \sigma \in X^{(k)}, \gamma \in \Gamma\}.$$

and the usual simplicial differential in every degree  $d : C^k(X, \rho) \rightarrow C^{k+1}(X, \rho)$ . Using the scalar product of the Hilbert space  $V$ , we can define a scalar product  $(\cdot, \cdot)$  on a subspace of  $C^k(X, \rho)$  that we will denote by  $L^k(X, \rho)$ . In fact  $L^k(X, \rho)$  is the subspace of  $c \in C^k(X, \rho)$  such that  $(c, c) < \infty$ . The space  $(L^k(X, \rho), (\cdot, \cdot))$  is a Hilbert space and  $d : L^k(X, \rho) \rightarrow L^{k+1}(X, \rho)$  is a bounded operator for every  $k \in \mathbb{N}$ .

**Definition 1.4.10.** We define  $L^2$ -cohomology  $L^2H^*(X, \rho)$  of  $X$  with respect to  $\rho$  to be the cohomology of the complex obtained in this way. In a similar fashion we can define  $L^2\overline{H}^*(X, \rho)$  the *reduced  $L^2$ -cohomology of  $X$  with respect to  $\rho$*  as the largest Hausdorff quotient of  $L^2H^*(X, \rho)$ .

If moreover  $\Gamma$  acts cocompactly, then  $L^k(X, \rho) = C^k(X, \rho)$  for every  $k \in \mathbb{N}$  and  $L^*(X, \rho)$  gives a resolution of the representation  $\rho$ , hence the spaces  $L^2H^*(X, \rho)$  compute the group cohomology  $H^*(\Gamma, \rho)$  of  $\Gamma$ .

We may consider the formal adjoint  $\delta$  of the operator  $d$ , that is  $\delta : L^{k+1}(X, \rho) \rightarrow L^k(X, \rho)$  such that:

$$(dc, c') = (c, \delta c'),$$

for every  $c \in L^k(X, \rho)$  and  $c' \in L^{k+1}(X, \rho)$ .

The main object of simplicial Hodge theory is the *Laplacian*  $\Delta : L^k(X, \rho) \rightarrow L^k(X, \rho)$ , defined as  $\Delta = \delta d + d\delta$ . We say that a cochain  $c$  is *harmonic* if  $\Delta c = 0$ . We denote by  $\mathcal{H}^k(X, \rho)$  the subspace of  $L^k(X, \rho)$  of harmonic cochains. The main reason to introduce this theory is that it gives many decomposition results. The most important one is the *Hodge decomposition theorem* (which is proven in this simplicial setting by some formal computations).

**Proposition 1.4.11.** *For every  $k \in \mathbb{N}$ , we have orthogonal decompositions:*

$$\ker(d|_{L^k(X, \rho)}) = \mathcal{H}^k(X, \rho) \oplus \overline{\operatorname{Im}(d|_{L^{k+1}(X, \rho)})}.$$

*In particular harmonic cochains compute reduced cohomology:*

$$L^2 \overline{H}^k(X, \rho) = \mathcal{H}^k(X, \rho).$$

The second idea from Garland is to use local estimates on the links of simplices. Recall that the link  $X_\tau$  of a simplex  $\tau \in X^{(k)}$  is the set of all simplices  $\sigma$  in  $X$  that are disjoint from  $\tau$  but such that  $\tau \cup \sigma$  is a simplex in  $X$ . In our setting, if  $\tau \in X^{(k)}$ , then  $X_\tau$  is a purely  $(n - k - 1)$ -dimensional finite simplicial complex. We view it as some sort of combinatorial neighborhood of  $\tau$ . Garland looks at the Laplacian  $\Delta_\tau$  on  $X_\tau$ . The following result says that if we can show that the spectral gap of  $\Delta_\tau$  on 0-cochains is large enough for every  $\tau$ , then  $\mathcal{H}^k(X, \rho)$  vanishes.

**Theorem 1.4.12.** *[BS97, 2.5] Let  $0 < k < n$ . Suppose that  $X_\tau$  is connected and that there exists  $\varepsilon > 0$  such that  $\kappa_\tau \geq \frac{k(n-k)}{k+1} + \varepsilon$  for all  $\tau \in X^{(k-1)}$ , where  $\kappa_\tau$  is the minimal nonzero eigenvalue of  $\Delta_\tau$  on 0-cochains. Then  $\mathcal{H}^k(X, \rho) = \{0\}$ .*

We will not give the proof of this theorem. In the case of degree 1 ( $k = 1$ ), estimates on the constant  $\kappa_\tau$  were obtained in previous work by Feit and Higman. Hence this theorem gives us another proof of property (T) for higher rank non-Archimedean simple groups. The rest of Garland's proof of Theorem 1.4.9 consists in obtaining such estimates for  $k \geq 2$  when the thickness is large enough.

### 1.4.3 $L^p$ -cohomology of semisimple groups

We now compile results on semisimple groups that are particular to  $L^p$ -cohomology and that point towards a positive answer of Gromov's question on vanishing of  $L^p$ -cohomology for all  $p > 1$  in degrees below the rank.

#### 1.4.3.1 Degree 1

Gromov's question has a positive answer in degree 1. This was first proved in the real case by Pansu in 1999 via de Rham  $L^p$ -cohomology (first unpublished, then appeared in [Pan07]). Actually, he showed a stronger statement.

**Theorem 1.4.13.** *[Pan07, Théorème 1] Let  $M$  be a homogeneous manifold. Either:*

- *the isometry group of  $M$  is a compact extension of a solvable unimodular Lie group,*
- *or  $M$  is quasi-isometric to a homogeneous space of strictly negative sectional curvature,*
- *otherwise  $L^p H_{\text{dR}}^1(M) = \{0\}$  for all  $p > 1$ .*

Later, de Cornulier and Tessera extended Pansu's trichotomy to semisimple groups over fields of characteristic zero via continuous group cohomology [dCT11]. Their trichotomy is valid only for groups of characteristic 0, but their argument for vanishing of  $L^p$ -cohomology of semisimple groups of higher rank also works in positive characteristic. This is the statement we will be interested in here.

**Theorem 1.4.14.** [dCT11, Theorem 1] Let  $G$  be a semisimple group of rank  $r \geq 2$  over a local field. We have  $H_{\text{ct}}^1(G, L^p(G)) = \{0\}$  for all  $p > 1$ .

We will sketch a proof of this result following [dCT11], but we will slightly adapt the final part of it. First, we need the correct space to work with: this is the space of  $p$ -Dirichlet functions or formal coboundaries.

**Definition 1.4.15.** Let  $G$  be a locally compact second countable group and let  $\mu$  be a left Haar measure on  $G$ . Denote by  $\rho(g)$  the action by right translation by  $g \in G$  on the space of measurable functions on  $G$ . The space of  $p$ -Dirichlet functions  $D^p(G)$  is the space of measurable functions  $u : G \rightarrow \mathbb{R}$  such that  $b_u(g) = u - \rho(g)u \in L^p(G)$  for every  $g \in G$  and such that the map  $b_u : G \rightarrow L^p(G), g \rightarrow b_u(g)$  is continuous.

The space  $D^p(G)$  contains  $L^p(G)$  and contains constant functions. The following proposition can be viewed as an analogue of the fact that simplicial  $\ell^p$ -cohomology in degree 1 of a simply connected simplicial complex can be computed using functions on vertices whose differentials are  $\ell^p$ , rather than functions on edges. Under this identification, the coboundaries are  $\ell^p$ -functions on vertices and constant functions.

**Proposition 1.4.16.** [Tes09, 5.1] For every  $p > 1$  we have:

$$H_{\text{ct}}^1(G, L^p(G)) \simeq D^p(G)/(L^p(G) + \mathbb{R}).$$

The isomorphism is induced by the map  $u \mapsto b_u$ . Hence a second interpretation of this result is that in every cohomology class of  $H_{\text{ct}}^1(G, L^p(G))$  there exists a cocycle of the form  $b_u$  for some  $u \in D^p(G)$ . The cocycle  $b_u$  looks like a coboundary, but is not one, which is why we call them formal coboundaries.

Now the general strategy behind the proof of Theorem 1.4.14 is the same as the one used for proving property (T) for higher rank simple groups: create and propagate invariance. We refer to [dCT11] for their proofs.

The first step applies only to *non-unimodular* groups. Recall that for  $f \in L^p(G)$  and  $g \in G$ , we have  $\|\rho(g)f\|_p = \Delta_G(g)^{-1/p}\|f\|_p$ . Hence elements  $g \in G$  such that  $\Delta_G(g) > 1$  are exactly the ones that contract the  $L^p$ -norm.

**Proposition 1.4.17.** [dCT11, Lemma 2.1] Let  $\xi \in G$  be such that  $\Delta_G(\xi) > 1$ . For every  $u \in D^p(G)$  there exists  $u_\infty \in D^p(G)$  such that  $u - u_\infty \in L^p(G)$  and  $\rho(\xi)u_\infty = u_\infty$ . In other words:

$$H_{\text{ct}}^1(G, L^p(G)) \simeq D^p(G)^\xi/\mathbb{R}.$$

In particular  $H_{\text{ct}}^1(G, L^p(G))$  is Hausdorff.

The second step is once again Mautner's phenomenon. We could use the same statement as the one introduced before, but it is easier to have a contracting version.

**Proposition 1.4.18.** [dCT11, Lemma 2.4] Let  $\xi \in G$  be such that  $\Delta_G(\xi) > 1$ . Define:

$$P_\xi = \{h \in G, \text{ the sequence } (\xi^{-n}h\xi^n)_{n>0} \text{ is bounded}\}.$$

If some  $u \in D^p(G)$  is  $\rho(\xi)$ -invariant, then  $u$  is  $\rho(P_\xi)$ -invariant.

*Proof of Theorem 1.4.14.* Let  $G$  be a semisimple group of rank  $r \geq 2$  and  $G = KAN$  be an Iwasawa decomposition. By quasi-isometric invariance of  $L^p$ -cohomology, we may only look at the non-unimodular solvable group  $R = A \ltimes N$ . Let  $\sigma_1, \sigma_2 \in A^*$  be two different simple roots of the root system of  $G$  (this is where we use the higher rank hypothesis). Then we may choose two coweights  $\xi_1, \xi_2 \in A$  such that  $|\sigma_i(\xi_i)| < 1$  and for  $i = 1, 2$  we have  $\sigma(\xi_i) = 1$  for every simple root  $\sigma \neq \sigma_i$ . This choice is made so that for  $i = 1, 2$ , we have  $\Delta_G(\xi_i) > 1$  and  $P_{\xi_i} \supset U_\sigma$  for every simple root  $\sigma \neq \sigma_i$ .

Let  $u \in D^p(G)$ . By 1.4.17, we can suppose that  $u$  is  $\rho(\xi_1)$ -invariant. Now by 1.4.18,  $u$  is  $\rho(P_{\xi_1})$ -invariant. But  $\xi_1 \in A$  and  $A$  is abelian, so  $A \subset P_{\xi_1}$  and  $u$  is  $\rho(A)$ -invariant. Again by 1.4.18,  $u$  is  $P_{\xi_2}$ -invariant. But  $P_{\xi_1} \cup P_{\xi_2}$  contains the union of all root groups of simple roots, which generates  $N$ . Hence  $u$  is  $\rho(N)$ -invariant. Hence  $u$  is constant. This concludes the proof.  $\square$

### 1.4.3.2 Higher degrees

In [BR20], Bourdon and Rémy initiate systematically the study of  $L^p$ -cohomology of higher rank real simple Lie groups in higher degrees. They show vanishing of  $L^p$ -cohomology for some real simple Lie groups in many degrees and for some values of  $p > 1$ . The goal of this short paragraph is to make this last sentence more precise.

The strategy of Bourdon and Rémy to study  $L^p$ -cohomology of a real simple (or semisimple) Lie group  $G$  is the following. First, quasi-isometric invariance allows one to pass to a maximal parabolic subgroup  $P$ . A parabolic subgroup has a Levi decomposition  $P = M \ltimes AN$ , where  $M$  is semisimple and  $AN$  is the solvable radical of  $P$ . The main idea is to use the Hochschild-Serre spectral sequence on this decomposition. Since we chose a maximal parabolic subgroup, the group  $AN$  is Gromov-hyperbolic and there is one Gromov-hyperbolic space whose  $L^p$ -cohomology is particularly well-understood: the real hyperbolic space. This motivates the following definition.

**Definition 1.4.19.** We say that a real simple Lie group is *admissible* if it admits a maximal parabolic subgroup  $P$  whose solvable radical is quasi-isometric to the real hyperbolic space  $\mathbb{H}_{\mathbb{R}}^n$  for some  $n \geq 2$ .

Using Pansu's description of vanishings of  $L^p$ -cohomology of the real hyperbolic space, Bourdon and Rémy show the following result for real admissible simple Lie groups.

**Theorem 1.4.20.** [BR20, 1.4] *Let  $G$  be an admissible real simple Lie group and let  $d + 1$  be the dimension of the corresponding real hyperbolic space. Let  $D$  be the dimension of the symmetric space associated to  $G$ . Then:*

$$H_{\text{ct}}^k(G, L^p(G)) = \{0\} \text{ for } k \leq \frac{d}{p} \text{ and for } k \geq \frac{d}{p} + D - d + 1.$$

The integer  $d$  is often larger than the real rank of  $G$ , but certainly smaller than  $D/2$ . This result shows vanishing of  $L^p$ -cohomology for small values of  $p > 1$  in degrees  $< d$  and for large values of  $p > 1$  in degrees close to the dimension  $D$ .

Initially, one of the goals of this thesis was to show vanishing of the second  $L^p$ -cohomology space for most non-Archimedean simple groups of rank at least 3 using

the same strategy. It turned out that our methods also applied to real groups and to semisimple non-simple groups.

For a non-Archimedean semisimple group  $G$ , the solvable radical  $AN$  of a maximal parabolic subgroup  $P = M \ltimes AN$  is quasi-isometric to a locally finite tree.  $L^p$ -cohomology of trees is well-understood. The difference between the real and the non-Archimedean case comes from the fact that for small values of  $p > 1$ , the first integer  $k$  such that  $L^p H_{\text{dR}}^k(\mathbb{H}_{\mathbb{R}}^n) \neq \{0\}$  is large, allowing the spectral sequence to give directly vanishing in many degrees. On the other hand, if  $T$  is a locally finite tree,  $\ell^p H^1(T) \neq \{0\}$  for every  $p > 1$ , and so this term will always appear in the spectral sequence, forbidding it to give vanishing directly. Nevertheless, the spectral sequence still gives the linear isomorphism:

$$H_{\text{ct}}^2(P, L^p(P)) = H_{\text{ct}}^1\left(M, L^p\left(M, H_{\text{ct}}^1(AN, L^p(AN))\right)\right),$$

which is the starting point of [LN23]. Notice that this isomorphism also works for real semisimple groups (not necessarily admissible) for  $p > \text{Confdim}(\partial(AN))$ .

## Chapter 2

# Vanishing in degree 2 for most higher rank semisimple Lie groups

### Introduction

$L^p$ -cohomology is a natural quasi-isometry invariant introduced first in [GKS86] and popularized by Gromov in [Gro93]. It is a rather fine one in the sense that it gives quasi-isometry invariants for every  $p > 1$  and thus uncountably many (and a priori independent) quasi-isometry invariants. It can be defined in different settings: we may talk about simplicial  $\ell^p$ -cohomology for simplicial complexes, de Rham  $L^p$ -cohomology for manifolds, asymptotic  $L^p$ -cohomology for metric spaces or continuous group  $L^p$ -cohomology for locally compact second countable groups. Comparison theorems give criteria to guarantee when these different versions coincide (usually by comparing them to asymptotic  $L^p$ -cohomology).

In this article we deal with continuous group  $L^p$ -cohomology  $H_{\text{ct}}^*(G, L^p(G))$  of a locally compact second countable group  $G$ , endowed with a *left* invariant Haar measure, with coefficients in the *right* regular representation on  $L^p(G)$ . See Section 2.1.2 for precise definitions. The main technical advantage of this version of  $L^p$ -cohomology is that we dispose of more algebraic machinery, namely, we may use the Hochschild-Serre spectral sequence for semi-direct products [BW00, IX].

The first locally compact groups we may probably think about are Lie groups. Here the term Lie group can refer to either a real or a non-Archimedean Lie group. The main motivation for this article is the following question by Gromov: he predicts a classical behaviour of  $L^p$ -cohomology of semisimple Lie groups for every  $p > 1$ , in degrees at most equal to the rank [Gro93].

**Questions.** *Let  $G$  be a semisimple Lie group of rank  $r \geq 2$  over a local field  $F$ .*

- (1) *Do we have  $H_{\text{ct}}^l(G, L^p(G)) = \{0\}$  for all  $l = 1, \dots, r - 1$  and  $p > 1$ ?*
- (2) *Do we have  $H_{\text{ct}}^r(G, L^p(G)) \neq \{0\}$  at least for some values of  $p$ ?*
- (3) *Is the space  $H_{\text{ct}}^r(G, L^p(G))$  Hausdorff for all  $p > 1$ ?*

Question (2) has been addressed in [BR21] for real groups and [LN22] for non-Archimedean groups. This article deals with question (1). Let us review some known results towards a positive answer to this question. Most of these results concern vanishing in degree 1 for all  $p > 1$  for groups of rank  $\geq 2$ .

First, Pansu proved this result in the real case in some unpublished notes in 1999. In there, he showed a trichotomy for a homogeneous manifold  $M$ : either the isometry group of  $M$  is a compact extension of a solvable unimodular Lie group, or  $M$  is quasi-isometric to a homogeneous space of strictly negative sectional curvature, otherwise  $L^p H_{\text{dR}}^1(M) = \{0\}$  for all  $p > 1$ . Later Bader, Furman, Gelander and Monod proved vanishing of the first continuous cohomology of a (real or non-Archimedean) simple Lie group  $G$  of rank  $\geq 2$  acting by isometries on some  $L^p(X, \mu)$ , where  $(X, \mu)$  is a standard Borel space and  $p > 1$  [BFGM07]. In fact their result also applies to semisimple groups whose simple factors have rank  $\geq 2$ . Later, Cornulier and Tessera extended Pansu's trichotomy to semisimple Lie groups over fields of characteristic zero [dCT11] (their trichotomy is stated only for groups of characteristic 0 but their argument for vanishing for semisimple Lie groups of higher rank also works in positive characteristic). The present work is heavily influenced by [dCT11].

Another impressive contribution is the one of Lafforgue in [Laf08]. Motivated by obtaining obstructions to adapt his own proof of the Baum-Connes isomorphism for hyperbolic groups to the case of  $\text{SL}(3, \mathbb{Z})$ , he defines a strong rigidity condition, known today as Lafforgue's Strong Property  $(T)$ , that (if proven for a large enough class of Banach spaces) implies an affirmative answer to Gromov's question in degree 1. He showed that groups containing  $\text{SL}(3, F)$  have this property, where  $F = \mathbb{R}$  or a non-Archimedean local field. The main novelty for us is that this condition implies vanishing of the first continuous cohomology group for non-isometric actions (more precisely, for representations of small exponential growth on uniformly convex Banach spaces).

Bourdon and Rémy deal with vanishing of  $L^p$ -cohomology of real simple Lie groups in higher degrees [BR20]. They show that for some real simple Lie groups they call admissible, there are constants  $p(k) > 1$  for every degree  $k$  such that there is vanishing of  $L^p$ -cohomology in degree  $k$  for all  $1 < p < p(k)$ . Poincaré duality allows them to extend this result to large values of  $p$ , at least for large values of  $k$  (in particular, this duality argument does not concern degrees below the rank). Their methods consist in proving a suitable version of the Hochschild-Serre spectral sequence for  $L^p$ -cohomology and invoke Pansu's description of the  $L^p$ -cohomology of the real hyperbolic space [Pan08]. In fact their proof also applies to (non-admissible) real Lie groups, but the conditions on the constant  $p(k)$  are much more restrictive (in this case the constant  $p(k)$  tends to 1 for fixed  $k$  when the rank goes to infinity).

Initially, we wanted to prove a statement similar to that of Bourdon and Rémy in the non-Archimedean case. It turned out that our methods also apply to the real case, but only for large values of  $p$ . The result we prove is the following.

**Theorem 2.0.1.** *(see Theorems 2.2.10 and 2.4.2 in the text) Let  $F$  be a local field and suppose that  $G$  is either:*

- $\text{SL}(4, D)$ , where  $D$  is a finite dimensional central division algebra over  $F$ ,
- a simple Lie group over  $F$  of rank  $r \geq 4$  that is not of type  $D_4$  and is not of exceptional type,
- or a semisimple, non-simple Lie group over  $F$  of rank  $r \geq 3$ .

*Then there exists a constant  $p(G) \geq 1$  such that for all  $p > p(G)$ :*

$$H_{\text{ct}}^2(G, L^p(G)) = \{0\}.$$

*Moreover, when  $F$  is non-Archimedean we have  $p(G) = 1$ .*

Combining our results with [LN22, 1.2], we obtain a complete description of the vanishings of  $L^p$ -cohomology for  $p > 1$  and in all degrees for  $\mathrm{SL}(4, F)$ , where  $F$  is a non-Archimedean local field. This gives a positive answer to Gromov's questions (1) and (2) for this group.

**Corollary 2.0.2.** *For  $G = \mathrm{SL}(4, F)$ , where  $F$  is a non-Archimedean local field, we have for any  $p > 1$ :*

$$H_{\mathrm{ct}}^k(G, L^p(G)) \neq \{0\} \text{ if and only if } k = 3.$$

For real admissible simple Lie groups (those for which the results in [BR20] apply) our result is complementary to that of [BR20]. In degree 2, their result gives vanishing for small values of  $p$ , say for  $1 < p \leq q(G)$  for some constant  $q(G)$ . A priori, there is an interval  $(q(G), p(G)]$  for which none of our results show vanishing. The funny thing is that, for most admissible simple Lie groups, this interval disappears when the rank is large enough.

**Corollary 2.0.3.** *(see Corollary 2.4.3 in the text) Let  $G$  be an admissible real simple Lie group of rank  $r \geq 8$  that is not of type  $B_r$ . Then we have for all  $p > 1$*

$$H_{\mathrm{ct}}^2(G, L^p(G)) = \{0\}.$$

We outline the proof of Theorem 2.0.1. First, we use quasi-isometric invariance in order to identify  $H_{\mathrm{ct}}^2(G, L^p(G))$  to  $H_{\mathrm{ct}}^2(P, L^p(P))$ , where  $P$  is a maximal parabolic subgroup. This parabolic subgroup has a Levi decomposition  $P = MSU$ . The version of the Hochschild-Serre spectral sequence from [BR20] allows us to identify the space  $H_{\mathrm{ct}}^2(P, L^p(P))$  (at least as an abstract vector space) to the first continuous cohomology group  $H_{\mathrm{ct}}^1(M, L^p(M, V_p))$  of the Levi factor  $M$  with coefficients in some Banach-valued  $L^p$ -space  $L^p(M, V_p)$ . The Banach space  $V_p$  is in fact  $H_{\mathrm{ct}}^1(SU, L^p(SU))$ . The main technical problem comes from the fact that the continuous  $M$ -module  $L^p(M, V_p)$  has relatively large exponential growth, forbidding us to (directly) invoke Lafforgue's strong property  $(T)$  and obtain the desired vanishing.

We amend this by passing to a cocompact, non-unimodular, solvable group  $R$  and creating contractions thanks to its modular function  $\Delta_R$ . We then adapt techniques from [dCT11] to show some non-isometric version of Mautner's phenomenon for the  $R$ -module  $L^p(R, V_p)$ . The upshot is that vanishing of  $H_{\mathrm{ct}}^1(R, L^p(R, V_p))$  follows from the presence of enough contractions. Verifying this condition can be done easily in the semisimple, non-simple case (see Theorem 2.2.10 in the text).

The non-trivial part consists in showing the existence of such contractions in the simple case. This is a battle between the exponential dilation of a maximal torus  $A$  in  $M$  acting on  $V_p$  and the exponential contraction of  $\Delta_R$  in certain directions. To show that the contraction of  $\Delta_R$  wins this battle for the groups in the statement of Theorem 2.0.1, we first control the operator norms of the action of  $A$  on  $V_p$  by some term that can be written explicitly in combinatorial terms. This step requires large-scale geometric considerations, as our bound depends on the Hausdorff dimension of the Carnot-Carathéodory metric on  $U$ . Then, using the classification of semisimple Lie groups over local fields, we reduce the existence problem of contracting elements to a combinatorial case-by-case study of root systems with multiplicities. The main point in the combinatorial part of the proof is that, for the infinite families of root

systems  $(A_r, B_r, C_r, BC_r$  and  $D_r)$  there exists always a choice of maximal parabolic subgroup such that our control of the exponential dilation grows linearly in the rank and the exponential contraction of the modular function grows quadratically in the rank (at least in well-chosen directions). This a priori asymptotic heuristic works quite fast: starting from  $r \geq 3$  for  $A_r$ , from  $r \geq 4$  for  $B_r, C_r$  and  $BC_r$  and from  $r \geq 5$  for  $D_r$ . Our current estimates do not seem to create contractions for exceptional groups.

We do not exclude the possibility that this result could be obtained using Lafforgue's Strong Property  $(T)$ . Indeed, Lafforgue extends his results from  $M = \mathrm{SL}(3, F)$  to a higher rank semisimple Lie group  $G$  containing  $M$  by restricting a representation of  $G$  to  $M$  and using that  $M$  has strong property  $(T)$ . This does not optimize the constants in the exponential growth of the representation as they are the same as for  $\mathrm{SL}(3, F)$ . We expect that showing Lafforgue's strong property  $(T)$  directly in  $G$  improves the constants so that we can treat the representation appearing after applying the spectral sequence, at least starting from a certain rank. Nevertheless, our constants seem to be slightly better since our method already works when the Levi factor  $M$  is  $\mathrm{SL}(3, F)$  inside  $G = \mathrm{SL}(4, F)$ .

The article is organized as follows. Section 1 contains standard preliminaries concerning algebraic groups, continuous group cohomology and  $L^p$ -cohomology. We also recall the Hochschild-Serre spectral sequence for  $L^p$ -cohomology as given in [BR20]. Section 2 studies the first continuous cohomology group with particular coefficients in a Banach-valued  $L^p$ -space. We show that this space can be identified to a more classical functional space, on which we can use Mautner's phenomenon to state a vanishing criterion. We directly apply this criterion to semisimple, non-simple groups of rank  $\geq 3$ . Sections 3 and 4 are the technical heart of this article. In Section 3 we study amenable hyperbolic groups with homotheties and obtain our concrete estimate for cohomology growth. Section 4 explains how to find contracting elements using root systems. We use the classification of simple Lie groups over local fields, study in detail each group to search for contractions and sum up our results in tables. This proves Theorem 2.0.1. We also prove Corollary 2.0.3 as a consequence of our combinatorial study.

## 2.1 Continuous cohomology of Lie groups

This section establishes the setting of the article. We also fix notation for subsequent sections. It contains standard preliminaries of algebraic flavor, more precisely, algebraic groups and continuous cohomology. We also recall the spectral sequence for continuous group  $L^p$ -cohomology.

### 2.1.1 Lie theoretic notions and Heintze groups

These paragraphs collect the necessary Lie theoretic vocabulary and fix notations for subsequent sections. We follow standard references on algebraic groups, such as [Bor91] or [Mar91, Chapter 0].

**Roots and root spaces** Let  $G$  be a semisimple Lie group over an Archimedean or non-Archimedean local field  $F$  of split rank  $r \geq 1$ . Denote by  $\mathfrak{g}$  its Lie algebra.

Fix a maximal  $F$ -split torus  $S$  in  $G$  and denote by  $X(S)$  the group of  $F$ -characters of  $S$ . Multiplication in  $X(S)$  will be denoted additively. The *restricted root system*  $\Phi = \Phi_F(S, G)$  is the set of nonzero  $F$ -characters  $\alpha \in X(S)$  such that the space

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall s \in S, \text{Ad}(s)X = \alpha(s)X\}$$

is nonzero [Bor91, 21.1]. An element  $\alpha \in \Phi$  is called a *root relative to  $F$*  or a *root*. The set  $\Phi$  is a root system in the usual sense [Mar91, 0.26]. The integer  $\dim \mathfrak{g}_\alpha$  will be called the *multiplicity* of the root  $\alpha$ , and will be denoted  $m(\alpha)$ .

Recall that the root system  $\Phi$  is said to be *reduced* if for all  $\alpha \in \Phi$ , the element  $2\alpha$  is not a root. The only family of non-reduced root systems consists of root systems of type  $BC_n$  for  $n \geq 2$ . The space  $\mathfrak{g}_\alpha$  is a commutative Lie subalgebra when  $2\alpha \notin \Phi$  (this follows from  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ ), but it may not be a Lie subalgebra when  $\alpha, 2\alpha \in \Phi$ . We amend this by considering the space  $\mathfrak{g}_{(\alpha)} := \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ , which is always a Lie subalgebra of  $\mathfrak{g}$  for  $\alpha \in \Phi$  [Bor91, 21.7]. If we define  $\Phi_{nd}$  to be the set of roots  $\alpha \in \Phi$  such that  $\alpha/2$  is not a root, then the *root space decomposition* of  $\mathfrak{g}$  may be written:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi_{nd}} \mathfrak{g}_{(\alpha)}.$$

where each summand is a Lie subalgebra [Bor91, 21.7].

Let  $U_{(\alpha)}$  be the unique unipotent closed Zariski-connected Lie subgroup with Lie algebra  $\mathfrak{g}_{(\alpha)}$  normalized by the centralizer  $Z(S)$  of the torus  $S$  [Bor91, 21.9 (i)].

If  $\alpha \in \Phi$  and  $2\alpha \notin \Phi$ , then  $U_\alpha := U_{(\alpha)}$  is abelian and  $F$ -isomorphic (as an algebraic group) to  $\mathfrak{g}_\alpha$  [Bor91, 21.20]. If  $\theta_\alpha : \mathfrak{g}_\alpha \rightarrow U_\alpha$  denotes such an isomorphism, then the action by conjugation of some  $s \in S$  on  $U_\alpha$  becomes the homothety of ratio  $\alpha(s)$  on  $\mathfrak{g}_\alpha$ , that is:  $s\theta_\alpha(X)s^{-1} = \theta_\alpha(\alpha(s)X)$  for all  $X \in \mathfrak{g}_\alpha$  [BT65, 3.17].

If  $\alpha, 2\alpha \in \Phi$  then  $U_{(\alpha)}$  is metabelian, its center is  $U_{2\alpha}$  [Bor91, 21.10 (2)] and is  $F$ -isomorphic (as a variety) to the product  $U_{(\alpha)}/U_{2\alpha} \times U_{2\alpha}$  [Bor91, 21.20 Proof of (i)]. The quotient  $U_\alpha := U_{(\alpha)}/U_{2\alpha}$  is  $F$ -isomorphic (as a variety) to  $\mathfrak{g}_\alpha$ . If  $\theta_\alpha : \mathfrak{g}_\alpha \rightarrow U_\alpha$  denotes this isomorphism, then the action by conjugation of some  $s \in S$  on  $U_\alpha$  becomes again the homothety of ratio  $\alpha(s)$  on  $\mathfrak{g}_\alpha$  [BT65, 3.17]. Notice that under these identifications, the action of  $s \in S$  on  $U_{(\alpha)}$ , seen as the product  $U_\alpha \times U_{2\alpha}$ , is a homothety of ratio  $\alpha(s)$  on  $U_\alpha$  and a homothety of ratio  $2\alpha(s)$  on  $U_{2\alpha}$ .

In the real case, the isomorphism  $\theta_\alpha$  is just the exponential mapping.

**Parabolic subgroups and Levi decomposition** We fix a choice of simple roots  $\Sigma$  inside  $\Phi$ , denote by  $\Phi^+$  the corresponding positive roots. For any  $\alpha \in \Phi$  and  $\sigma \in \Sigma$ , set  $n_\sigma(\alpha) \in \mathbb{Z}$  so that

$$\alpha = \sum_{\sigma \in \Sigma} n_\sigma(\alpha)\sigma.$$

For  $I \subseteq \Sigma$ , denote by  $\Phi_I$  the set of roots in  $\Phi$  which are linear combinations of simple roots in  $I$  ( $\Phi_I$  is a root system in its own right) and  $\Phi_I^+ = \Phi_I \cap \Phi^+$ . In what follows we will introduce many notations for subgroups of  $G$  depending on a subset  $I$  of  $\Sigma$ . If  $I$  is in subscript it means that the corresponding subgroup contains in some way the roots in  $I$ , if  $I$  is in superscript it means that the corresponding subgroup avoids in some way the roots in  $I$ .

For  $I \subsetneq \Sigma$ , the *standard parabolic subgroup*  $P_I$  of type  $I$  is the subgroup of  $G$  with Lie algebra

$$\mathfrak{p}_I = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi_I} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^+ \setminus \Phi_I^+} \mathfrak{g}_\alpha$$

[BT65, 4.2]. Let  $S^I$  denote the  $F$ -split subtorus of  $S$  of rank  $(r - |I|)$  defined by  $S^I = (\bigcap_{\gamma \in I} \text{Ker } \gamma)^0$ . The group  $P_I$  admits a *Levi decomposition*  $P_I = Z_I \ltimes U^I$ , where  $Z_I = Z_G(S^I)$  denotes the centralizer of the torus  $S^I$  in  $G$  and  $U^I$  is the unipotent radical of  $P_I$  [BT65, 4.2]. We will briefly present some properties of both factors in this decomposition.

The group  $Z_I = Z_G(S^I)$  is called the *Levi factor* of  $P_I$  and has Lie algebra  $\mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi_I} \mathfrak{g}_\alpha$ . It is reductive [BT65, 2.15 d)], and so it may be decomposed as the almost direct product  $M_I T^I$  where  $M_I = [Z_I, Z_I]$  is a semisimple group of split rank  $|I|$  and  $T^I$  is the connected center of  $Z_I$ . The group  $T^I$  is defined over  $F$  [BT65, 2.15 a)] and is the almost direct product of  $S^I$  with a (compact) anisotropic subtorus defined over  $F$  [BT65, 1.8]. The decomposition  $P_I = (M_I T^I) \ltimes U^I$  is sometimes called the *Langlands decomposition* of  $P_I$ .

*Remark.* The action by conjugation of  $M_I$  on  $U^I$  preserves the volume of  $U^I$ . Indeed, volume dilation of this action defines a character of  $M_I$ , but  $M_I$  has no nontrivial characters because it is a semisimple group.

On the other hand, the group  $U^I$  has Lie algebra  $\mathfrak{u}^I = \bigoplus_{\alpha \in \Phi^+ \setminus \Phi_I^+} \mathfrak{g}_\alpha$  and is  $F$ -isomorphic (as a variety) to the direct product of the corresponding root groups [Bor91, 21.9 (ii)]:

$$\prod_{\alpha \in \Phi_{nd}^+ \setminus (\Phi_I^+)_{nd}} U_{(\alpha)} = \prod_{\alpha \in \Phi^+ \setminus \Phi_I^+} U_\alpha.$$

**Heintze groups and Iwasawa decomposition** We are mostly concerned with (proper) *maximal* parabolic subgroups, that is, the case where  $I = \Sigma \setminus \{\gamma\}$  for some  $\gamma \in \Sigma$ . In this case,  $S_\gamma := S^I \simeq F^*$  and  $\Phi_{n_\gamma > 0} := \Phi^+ \setminus \Phi_{\Sigma \setminus \{\gamma\}}^+$  is exactly the set  $\{\alpha \in \Phi, n_\gamma(\alpha) > 0\}$ . Conjugation by some  $s \in S_\gamma$  on  $x \in U_\alpha$ , where  $x = \theta_\alpha(X)$  and  $\theta_\alpha$  is our  $S$ -equivariant identification of  $U_\alpha$  with  $\mathfrak{g}_\alpha$ , becomes:

$$s.x := sxs^{-1} = \theta_\alpha(\gamma(s)^{n_\gamma(\alpha)} X).$$

Thus a fixed  $s \in S_\gamma$  either contracts or dilates all the root groups  $U_\alpha$  for  $\alpha \in \Phi_{n_\gamma > 0}$  (depending on the sign of  $\log |\gamma(s)|$ ). In the terminology of [CdCMT15],  $S_\gamma$  acts on  $U_\gamma := U^I$  by confining automorphisms (the notation  $U_\gamma$  is ambiguous as it may also refer to the root group  $U_\gamma$  integrating  $\mathfrak{g}_\gamma$ , but in practice we will never use this notation in this sense).

**Definition 2.1.1.** The group  $H_\gamma := S_\gamma \ltimes U_\gamma$  is the *Heintze group* associated to the simple root  $\gamma$ .

By [CdCMT15, 4.6], the solvable group  $H_\gamma$  is an amenable, non-unimodular, Gromov-hyperbolic locally compact group.

On the other hand, the semisimple part  $M^\gamma := M_I$  of the group  $P_I$  admits an *Iwasawa decomposition*  $K^\gamma A^\gamma N^\gamma$ , where  $K^\gamma$  is a maximal compact subgroup

of  $M^\gamma$ , the group  $A^\gamma$  is a maximal  $F$ -split torus in  $M^\gamma$  (hence a group-theoretic supplementary of  $S_\gamma$  in  $S$ ) and  $N^\gamma$  is isomorphic to the product  $\prod_{\alpha \in \Phi_I^+} U_\alpha$  and has Lie algebra  $\bigoplus_{\alpha \in \Phi_I^+} \mathfrak{g}_\alpha$  (see [Hel01, IX. 1.3] for the real case and [Mac71, 2.6.11] for the non-Archimedean case).

**Definition 2.1.2.** We write  $R^\gamma := A^\gamma N^\gamma$ .

The main object of study of the subsequent sections is the semi-direct product  $R^\gamma \ltimes H_\gamma$ . By this we mean that we will study the group  $H_\gamma$  and the action by conjugation of the group  $R^\gamma$  on  $H_\gamma$ . To sum up the relations between the different groups in this section:

$$G \simeq_{\text{qi}} P_I \simeq_{\text{qi}} M^\gamma \ltimes H_\gamma \simeq_{\text{qi}} R^\gamma \ltimes H_\gamma,$$

where the metrics on each group are just the metrics induced from  $G$ . Moreover, all of these quasi-isometries are cocompact inclusions.

### 2.1.2 Continuous group cohomology

We define continuous cohomology of a locally compact second countable group following [BW00, IX].

Let  $G$  be a locally compact second countable group. Then  $G$  is a countable union of compact sets (for instance, because  $G$  carries a proper metric [Str74]). Let  $(\rho, V)$  be a continuous representation of  $G$  (we also use continuous  $G$ -module as terminology) i.e. a morphism  $\rho : G \rightarrow B(V)$  such that the map  $G \times V \rightarrow V$  defined by  $(g, v) \mapsto \rho(g)v$  is continuous, where  $V$  is some locally convex topological vector space and  $B(V)$  denotes continuous invertible operators on  $V$ . Here  $V$  will always be at least a Fréchet space.

For  $k \in \mathbb{N}$ , we define the space  $C^k(G, V)$  of  $k$ -cochains as the set of continuous maps from  $G^{k+1}$  to  $V$ . Since  $G$  is a countable union of compact sets, the space  $C^k(G, V)$  equipped with the topology of uniform convergence on compact subsets is a Fréchet space.

We define the differential  $d_k : C^k(G, V) \rightarrow C^{k+1}(G, V)$  of a  $k$ -cochain  $c$  as:

$$(d_k c)(g_0, \dots, g_{k+1}) = \sum_{i=0}^{k+1} (-1)^i c(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_{k+1}).$$

It satisfies  $d_{k+1} \circ d_k = 0$ .

The space  $C^k(G, V)$  can be viewed as a continuous  $G$ -module, by endowing it with the action:

$$(g.c)(g_0, \dots, g_k) = \rho(g)(c(g^{-1}g_0, \dots, g^{-1}g_k)).$$

We consider the space  $C^k(G, V)^G$  of invariants in  $C^k(G, V)$  with respect to this action. Notice that this is just the set of continuous  $G$ -equivariant maps from  $G^{k+1}$  to  $V$ , when endowing  $G^{k+1}$  with the diagonal action by left translation on each factor and  $V$  with the action given by  $\rho$ . The differential  $d_k$  maps  $C^k(G, V)^G$  into  $C^{k+1}(G, V)^G$ . We call  $\ker(d_k|_{C^k(G, V)^G})$  the space of  $k$ -cocycles and denote it by

$Z^k(G, \rho)$ , we call  $\text{Im}(d_{k-1}|_{C^{k-1}(G, V)^G})$  the space of  $k$ -coboundaries and denote it  $B^k(G, \rho)$ .

In homological algebra, group cohomology with values in a representation is defined as the cohomology of the complex of invariants of any resolution of the given representation. In this article we will not use other resolutions apart from the one we already defined (even though we invoke a result by [BR20] obtained using other resolutions), so the following definition of continuous group cohomology is enough for our purposes.

**Definition 2.1.3.** The  $k$ -th continuous cohomology group (resp.  $k$ -th reduced continuous cohomology group) of  $G$  with coefficients in  $(\rho, V)$  is the topological vector space:

$$H_{\text{ct}}^k(G, \rho) := Z^k(G, \rho)/B^k(G, \rho) \quad (\text{resp. } \overline{H}_{\text{ct}}^k(G, \rho) := Z^k(G, \rho)/\overline{B^k(G, \rho)}).$$

The space  $\overline{H}_{\text{ct}}^k(G, \rho)$  is the biggest Hausdorff quotient of  $H_{\text{ct}}^k(G, \rho)$ . In particular these two spaces coincide exactly when  $H_{\text{ct}}^k(G, \rho)$  is Hausdorff.

**Non-homogeneous cochains** For concrete applications and particularly in degree 1, it is sometimes useful to view elements of  $C^k(G, V)^G$  not as maps from  $G^{k+1}$  to  $V$ , but as maps from  $G^k$  to  $V$ .

For  $k = 1$ , this gives the classical geometric interpretation of the first continuous cohomology group. We can identify  $Z^1(G, V)$  with the space of continuous maps  $b : G \rightarrow V$  satisfying the cocycle relation  $b(gh) = b(g) + \rho(g)b(h)$ . This space can in turn be identified with the space of continuous affine actions of  $G$  on  $V$  with linear part  $\rho$ , via the map  $b \mapsto A_b$  for  $b \in Z^1(G, V)$ , where  $A_b(g) : V \rightarrow V, v \mapsto \rho(g)v + b(g)$  for  $g \in G$ . In a similar way, we may identify  $B^1(G, V)$  with the space of maps of the form  $b(g) = v - \rho(g)v$  for some  $v \in V$ . These maps correspond to continuous affine actions of  $G$  on  $V$  having a fixed point. If the group  $G$  is compactly generated,  $S$  is a compact generating set and  $(V, \|\cdot\|_V)$  is a Banach space, then  $Z^1(G, V)$  is a Banach space with norm  $\|b\| = \sup_{x \in S} \|b(x)\|_V$  for  $b \in Z^1(G, V)$ .

**Continuous group  $L^p$ -cohomology** Let  $G$  be a locally compact second countable group endowed with a left-invariant Haar measure  $\mu_G$ . In this article we will be interested in the Banach space  $V = L^p(G)$  of  $p$ -integrable functions with respect to the measure  $\mu_G$  for  $1 < p < \infty$ . The representation  $\rho_p$  of  $V$  we are interested in is the *right regular representation* of  $G$ , defined by right translation on the argument of an  $L^p$ -function:  $\rho_p(g)f(x) = f(xg)$ . This defines a continuous representation of  $G$ . Notice that it is an isometric representation if and only if the measure  $\mu_G$  is also right-invariant, that is, if and only if  $G$  is unimodular. In this case  $\rho_p$  is also continuously conjugate to the similarly defined left regular representation  $\lambda_p$  via the continuous linear map  $L^p(G) \rightarrow L^p(G)$  sending  $f$  to  $\check{f}(x) = f(x^{-1})$ .

The representation  $\rho_p$  will appear for several different groups, we will not need to specify the action as it will be enough to specify the corresponding vector space. In order to avoid (even more) cumbersome notation,  $H_{\text{ct}}^k(G, L^p(G))$  will just mean  $H_{\text{ct}}^k(G, \rho_p)$ .

The most important feature of  $L^p$ -cohomology, in contrast to cohomology with respect to an arbitrary representation, is that it is a quasi-isometry invariant for

any degree  $k \in \mathbb{N}$  and  $1 < p < \infty$ . This is a phenomenon that has been shown for different types of  $L^p$ -cohomology by different people: see [Gro93, p. 219] for a sketch of proof in the simplicial  $\ell^p$ -cohomology setting, [BP03] for a more detailed proof in the same context, [Pan95] for a proof for de Rham  $L^p$ -cohomology. Here we are interested in quasi-isometric invariance of continuous group  $L^p$ -cohomology. It was first proven by Elek in [Ele98] for finitely generated groups by comparing it with asymptotic  $L^p$ -cohomology. In [SS18], the same idea is used to show that vanishing of an  $\ell^2$ -Betti number is a coarse equivalence invariant. In [BR20], quasi-isometric invariance of continuous group  $L^p$ -cohomology is proven for general locally compact second countable groups.

**Theorem 2.1.4.** [BR20, Theorem 1.1] *Let  $G_1$  and  $G_2$  be locally compact second countable groups, equipped with left-invariant proper metrics  $d_1$  and  $d_2$ . Every quasi-isometry  $F : (G_1, d_1) \rightarrow (G_2, d_2)$  induces canonically an isomorphism of graded topological vector spaces:*

$$F^* : H_{\text{ct}}^*(G_2, L^p(G_2)) \rightarrow H_{\text{ct}}^*(G_1, L^p(G_1))$$

for every  $1 < p < \infty$ . The same holds for reduced cohomology.

*Remark.* 1. The idea of both [SS18] and [BR20] consists in comparing continuous group  $L^p$ -cohomology to asymptotic  $L^p$ -cohomology. The latter is a coarse equivalence invariant [Pan95] and hence, even if not stated explicitly, continuous group  $L^p$ -cohomology is also invariant under coarse equivalences.

2. Continuous cohomology with coefficients in the left regular representation  $\lambda_{p,G}$  is not invariant under quasi-isometries when at least one of the two groups is not unimodular (when both are unimodular, it is same as considering the right regular representation). For instance, the groups  $G = \text{SL}(2, \mathbb{R})$  and  $B < \text{SL}_2(\mathbb{R})$  of upper triangular matrices are quasi-isometric. One has  $\overline{H}_{\text{ct}}^1(G, \lambda_{p,G}) \neq \{0\}$  for all  $p > 1$  ( $G$  is unimodular, so  $\overline{H}_{\text{ct}}^1(G, \lambda_{p,G}) = \overline{H}_{\text{ct}}^1(G, \rho_{p,G})$  and  $G$  is quasi-isometric to the real hyperbolic plane  $\mathbb{H}_{\mathbb{R}}^2$ , therefore using comparison theorems between continuous group  $L^p$ -cohomology, asymptotic  $L^p$ -cohomology and de Rham  $L^p$ -cohomology [BR21, Theorems 6.5 and 6.7] we have  $\overline{H}_{\text{ct}}^1(G, \rho_{p,G}) = L^p H_{\text{dR}}^1(\mathbb{H}_{\mathbb{R}}^2)$  and the latter is nonzero for all  $p > 1$  [Pan89a, 5.2]). On the other hand  $\overline{H}_{\text{ct}}^1(B, \lambda_{2,B}) = \{0\}$  [Del77].

### 2.1.3 Spectral sequence reduction

Let  $G$  be a semisimple Lie group over a local field  $F$  of split rank  $r$ . The idea of this section is to use quasi-isometric invariance of continuous group  $L^p$ -cohomology [BR20, Theorem 1.1] and the Hochschild-Serre spectral sequence. This reduces the computation of the second  $L^p$ -cohomology group of  $G$  to the first continuous group cohomology of a semisimple factor of a well-chosen parabolic subgroup with values in a more exotic  $L^p$ -module. The following version of the spectral sequence is well-suited to compute  $L^p$ -cohomology:

**Theorem 2.1.5.** [BR20, Corollary 5.4] *Let  $P$  be a locally compact second countable group. Suppose that  $P = Q \times S$  where  $Q$  and  $S$  are two closed subgroups of  $P$ , such that  $C^*(S, L^p(S))$  is homotopically equivalent to a complex of Banach spaces*

and every cohomology space  $H_{\text{ct}}^k(S, L^p(S))$  is Hausdorff. Then there exists a spectral sequence  $(E_r)$  abutting to  $H_{\text{ct}}^*(P, L^p(P))$  in which:

$$E_2^{k,l} = H_{\text{ct}}^k(Q, L^p(Q, H_{\text{ct}}^l(S, L^p(S)))).$$

We keep the same notations as in the previous section. Let  $\gamma \in \Sigma$  be a simple root of  $\Phi = \Phi(S, G)$ . The group  $G$  is quasi-isometric to the semi-direct product  $R^\gamma \ltimes H_\gamma$ . The spectral sequence will turn out to be useful on this semi-direct product because the  $L^p$ -cohomology of  $H_\gamma$  is sufficiently well-understood.

**Corollary 2.1.6.** *For any simple root  $\gamma \in \Sigma$ , any  $p > \max\{\text{Confdim}(\partial H_\gamma), 1\}$  and any integer  $k \geq 1$ , we have linear isomorphisms:*

$$H_{\text{ct}}^k(G, L^p(G)) = H_{\text{ct}}^{k-1}(R^\gamma, L^p(R^\gamma, H_{\text{ct}}^1(H_\gamma, L^p(H_\gamma)))).$$

The  $R^\gamma$ -action of  $g \in R^\gamma$  on  $b : H_\gamma \rightarrow L^p(H_\gamma)$  is defined by:

$$(\pi_0(g)b)(h)(x) = b(g^{-1}hg)(g^{-1}xg).$$

The  $R^\gamma$ -action  $\pi$  on  $f \in L^p(R^\gamma, H_{\text{ct}}^1(H_\gamma, L^p(H_\gamma)))$  is defined for  $g, h \in R^\gamma$  by:

$$(\pi(g)f)(h) = \pi_0(g)(f(hg)).$$

*Proof.* Quasi-isometric invariance of continuous group  $L^p$ -cohomology [BR20, 1.1] gives topological isomorphisms:

$$H_{\text{ct}}^k(G, L^p(G)) = H_{\text{ct}}^k(R^\gamma \ltimes H_\gamma, L^p(R^\gamma \ltimes H_\gamma)).$$

We apply Theorem 2.1.5 to the semi-direct product  $R^\gamma \ltimes H_\gamma$ . When  $F$  is non-Archimedean, the group  $H_\gamma$  is quasi-isometric to a tree [dCT11, 4.6], so its  $L^p$ -cohomology is Hausdorff and concentrated in degree 1 for all  $p > 1$ . In the real case, if  $p > \text{Confdim}(\partial H_\gamma)$  then  $H_{\text{ct}}^1(H_\gamma, L^p(H_\gamma)) \neq \{0\}$  [dCT11, Theorem 1], is Hausdorff [Tes08, 11.9] and moreover  $H_{\text{ct}}^k(H_\gamma, L^p(H_\gamma)) = \{0\}$  for  $k \geq 2$  [Bou16a, Corollaire B]. Thus the Hochschild-Serre spectral sequence collapses in the  $E_2$  page and gives the desired linear isomorphisms.  $\square$

*Remark.* This is the only part of the proof where we need  $p$  to be large in the real case. The rest of the proof works uniformly for all  $p > 1$ , both in the real and in the non-Archimedean case. If one would like to study the second  $L^p$ -cohomology group of real simple Lie groups for smaller values of  $p$ , the spectral sequence will give us other isomorphisms. For instance, if  $H_\gamma$  is the real hyperbolic space of dimension  $d$ , Bourdon and Rémy use [Pan08] and obtain vanishing for  $p \leq (d-1)/2$  [BR20, 1.4], and for  $(d-1)/2 < p \leq d-1$ , we obtain

$$H_{\text{ct}}^2(G, L^p(G)) = H_{\text{ct}}^0(R^\gamma, L^p(R^\gamma, H_{\text{ct}}^2(H_\gamma, L^p(H_\gamma)))) = L^p(R^\gamma, L^p H_{\text{DR}}^2(H_\gamma))^{R^\gamma}.$$

Adapting the techniques we will introduce further on continuous  $L^p$ -cohomology to de Rham  $L^p$ -cohomology may also show that this space vanishes for many groups. We do not do it because in practice, most of the vanishings obtained in this manner are contained in Corollary 2.0.3 (except for type  $B_r$  and some low rank cases).

## 2.2 Contracting automorphisms and cohomology in degree 1 of Banach-valued $L^p$ -spaces

In this section, we adapt techniques from [dCT11, Section 2] to show vanishing of the right hand side of Corollary 2.1.6 when  $k = 2$ , under contraction hypotheses that will be verified in subsequent sections.

The setting for the section is the following (except for the Lie theoretic statements at the very end). Let  $G$  be a locally compact second countable group endowed with a left-invariant Haar measure  $\mu$  and  $(\pi_0, V)$  be a continuous  $G$ -module on some separable Banach space  $V$ . Let  $\pi$  denote the action on measurable functions  $f : G \rightarrow V$  defined by:

$$(\pi(g)f)(h) = \pi_0(g)(f(hg)),$$

for all  $g, h \in G$ . For  $p > 1$ , we consider the Banach space  $L^p(G, V)$  of Bochner  $p$ -integrable functions, that is, the set of measurable functions  $f : G \rightarrow V$  such that:

$$\|f\|_p^p = \int_G \|f(g)\|_V^p d\mu(g) < \infty.$$

We may also denote by  $\pi_p$  the restriction of  $\pi$  to the space  $L^p(G, V)$ .

### 2.2.1 Operator norms and the modular function

Our definition of the modular function  $\Delta_G$  of  $G$  is given by the following formula: for any measurable set  $E$  of  $G$  and  $g \in G$  we have  $\mu(Eg) = \Delta_G(g)\mu(E)$ , or alternatively, for any continuous compactly supported function  $\phi$  on  $G$  we have:

$$\int_G \phi(hg^{-1})d\mu(h) = \Delta_G(g) \int_G \phi(h)d\mu(h).$$

If  $(E, \|\cdot\|_E)$  is any Banach space and  $A : E \rightarrow E$  is some bounded linear operator, we define the operator norm of  $A$  by:

$$\|A\|_E = \sup_{v \in E, v \neq 0} \frac{\|Av\|_E}{\|v\|_E}.$$

We first obtain the following identity on the operator norms of  $\pi$ . It is central in our reasoning because it shows that even though the operator norm of  $\pi_0$  can be really big, we can hope to compensate it using the modular function.

**Proposition 2.2.1.** *For all  $g \in G$  we have:*

$$\|\pi(g)\|_{L^p(G, V)} = \Delta_G(g)^{-1/p} \|\pi_0(g)\|_V.$$

*Proof.* The proof follows from the definition of the modular function:

$$\begin{aligned} \|\pi(g)f\|_{L^p(G, V)}^p &= \int_G \|\pi_0(g)(f(hg))\|_V^p d\mu(h) \\ &= \Delta_G(g)^{-1} \int_G \|\pi_0(g)(f(h))\|_V^p d\mu(h) \\ &\leq \Delta_G(g)^{-1} \|\pi_0(g)\|_V^p \|f\|_{L^p(G, V)}^p \end{aligned}$$

For the reverse inequality, fix  $\varepsilon > 0$  and let  $v \in V$  be such that  $\|\pi_0(g)v\|_V \geq (\|\pi_0(g)\| - \varepsilon)\|v\|_V$ . Fix some compact  $K \subset G$  such that  $\mu_G(K) = 1$  and we may define  $f_v \in L^p(G, V)$  by  $f_v(x) = v$  for  $x \in K$  and  $f(x) = 0$  for  $x \in G \setminus K$ . We have  $\|f_v\|_{L^p(G, V)} = \|v\|_V$  and

$$\begin{aligned} \|\pi(g)f_v\|_{L^p(G, V)}^p &= \Delta_G(g)^{-1} \int_K \|\pi_0(g)v\|_V^p d\mu(h) \\ &\geq \Delta_G(g)^{-1} (\|\pi_0(g)\| - \varepsilon)^p \|f_v\|_{L^p(G, V)}^p. \end{aligned}$$

□

### 2.2.2 Contracting automorphisms and cohomology

We introduce the main object of study of this section, namely,  $\pi_p$ -contracting elements. This subsection is devoted to showing some complementary results that are not necessary for the proof of Theorem 2.0.1 (though we will need Lemma 2.2.4 for some cases) but that highlight the importance of contractions.

**Definition 2.2.2.** We say that  $\xi \in G$  is  $\pi_p$ -contracting if

$$\lim_{n \rightarrow +\infty} \|\pi(\xi^n)\|_{L^p(G, V)} = 0,$$

and  $\pi_p$ -bounded if  $\xi$  generates a non-relatively compact semigroup and

$$\sup_{n > 0} \|\pi(\xi^n)\|_{L^p(G, V)} < \infty.$$

A non-compact group  $G$  satisfies a linear Sobolev inequality on  $L^p(G)$  if and only if  $G$  is not simultaneously amenable and unimodular. The first result we show is a generalization of the fact that a non-unimodular group satisfies a linear Sobolev inequality on  $L^p(G)$  [Tes08, 11.9], but this time for the representation  $\pi_p$ .

**Lemma 2.2.3.** *Suppose that there exists a  $\pi_p$ -contracting element  $\xi \in G$ . Then the space  $H_{\text{ct}}^1(G, \pi_p)$  is Hausdorff, that is:*

$$H_{\text{ct}}^1(G, \pi_p) = \overline{H}_{\text{ct}}^1(G, \pi_p).$$

*Proof.* Up to changing  $\xi$  by some power, we may suppose that  $\|\pi(\xi)\|_{L^p(G, V)} < 1$ . Since  $G$  is  $\sigma$ -compact, we may write it as an increasing countable union of compact subsets  $(Q_k)_k$ . For  $k$  large enough we have that  $\xi \in Q_k$  and hence for  $f \in L^p(G, V)$ :

$$\|f\|_{p, Q_k} = \sup_{g \in Q_k} \|f - \pi(g)f\|_p \geq \|f - \pi(\xi)f\|_p \geq \|f\|_p - \|\pi(\xi)f\|_p \geq C\|f\|_p$$

where  $C = 1 - \|\pi(\xi)\|_{L^p(G, V)} > 0$ . Therefore  $B^1(G, \pi_p)$  is closed in  $Z^1(G, \pi_p)$ . □

**Lemma 2.2.4.** *Suppose that there exists a  $\pi_p$ -bounded element  $\xi \in G$ . If  $f \in L^p(G, V)$  is  $\pi(\xi)$ -invariant, then  $f = 0$ .*

*Proof.* Let  $X$  be any compact subset of  $G$  and denote by  $\|f\|_{X,p}$  the  $L^p$ -norm of  $f1_X$ . Since  $\xi$  generates a non-relatively compact semigroup and the action of  $G$  on itself is proper, we may take a subsequence  $(n_k)_{k \in \mathbb{N}}$  so that the translates  $(X\xi^{n_k})_{k>0}$  are disjoint. Since  $\xi$  is  $\pi$ -bounded, let  $C = \sup_{n>0} \|\pi(\xi^n)\|_{L^p(G,V)} < \infty$ . We have:

$$\begin{aligned} \|f\|_{X,p}^p &= \|\pi(\xi^n)f\|_{X,p}^p = \int_X \|\pi_0(\xi^n)(f(h\xi^n))\|_V^p d\mu(h) \\ &= \Delta_G(\xi)^{-n} \int_{X\xi^n} \|\pi_0(\xi^n)(f(h))\|_V^p d\mu(h) \\ &\leq \Delta_G(\xi)^{-n} \|\pi_0(\xi^n)\|_V^p \int_{X\xi^n} \|f(h)\|_V^p d\mu(h) \\ &= \|\pi(\xi^n)\|_{L^p(G,V)}^p \|f\|_{X\xi^n,p}^p \leq C \|f\|_{X\xi^n,p}^p. \end{aligned}$$

But  $\sum_{k>0} \|f\|_{X\xi^{n_k},p} \leq \|f\|_{G,p} < \infty$ , hence  $\|f\|_{X\xi^{n_k},p} \rightarrow 0$  when  $k \rightarrow +\infty$ . Therefore the previous inequality implies that  $\|f\|_{X,p} = 0$  for any compact  $X \subset G$ , which gives  $f = 0$  almost everywhere.  $\square$

Before working towards a criterion for vanishing in degree 1, we point out that the previous lemma gives an easy criterion for vanishing in degree 0.

**Corollary 2.2.5.** *Suppose that there exists some  $\pi_p$ -bounded element  $\xi \in G$ . Then:*

$$H_{\text{ct}}^0(G, \pi_p) = L^p(G, V)^G = \{0\}.$$

*Proof.* Every  $f \in L^p(G, V)^G$  is  $\pi(\xi)$ -invariant, hence  $f = 0$  by Lemma 2.2.4.  $\square$

### 2.2.3 Mautner's phenomenon and vanishing criterion

Our next goal will be to give a criterion for vanishing in degree 1 for the representation  $\pi_p$  using  $\pi_p$ -contracting elements. Whenever such an element exists, the following proposition allows us to create invariance for any cocycle in  $Z^1(G, \pi_p)$ .

**Proposition 2.2.6.** *Let  $\xi \in G$  be  $\pi_p$ -contracting and  $b \in Z^1(G, \pi_p)$ . There exists an element  $c \in Z^1(G, \pi_p)$  such that  $b - c \in B^1(G, \pi_p)$  and  $c(\xi) = 0$ .*

*Proof.* We first claim that the sequence  $(b(\xi^n))_{n>0}$  is Cauchy in  $L^p(G, V)$ . To see this, first notice that this sequence is bounded, as the cocycle relation yields:

$$b(\xi^n) = \sum_{i=0}^{n-1} \pi(\xi^i)b(\xi)$$

and hence:

$$\|b(\xi^n)\|_p \leq \left( \sum_{i=0}^{n-1} \|\pi(\xi^i)\|_{L^p(G,V)} \right) \|b(\xi)\|_p \leq C \|b(\xi)\|_p.$$

where  $C = \sum_{i=0}^{\infty} \|\pi(\xi^i)\|_{L^p(G,V)} < \infty$  converges as  $\xi$  is  $\pi_p$ -contracting and the operator norm is submultiplicative. Now let  $n \geq m \geq 0$ . From the cocycle relation we have:

$$b(\xi^n) - b(\xi^m) = \pi(\xi^m)b(\xi^{n-m})$$

and hence:

$$\|b(\xi^n) - b(\xi^m)\|_p \leq \|\pi(\xi^m)\|_{L^p(G,V)} \|b(\xi^{n-m})\|_p \leq C \|\pi(\xi^m)\|_{L^p(G,V)} \|b(\xi)\|_p.$$

Hence the sequence  $(b(\xi^n))_{n>0}$  is Cauchy in  $L^p(G, V)$  and converges to a limit function  $f \in L^p(G, V)$ . The formula  $c(g) = b(g) - f + \pi(g)f$  for  $g \in G$  defines a cocycle  $c$  such that  $b - c \in B^1(G, \pi_p)$ . We now show that  $c(\xi) = 0$ . We first see that:

$$\|c(\xi^n)\|_p \leq \|b(\xi^n) - f\|_p + \|\pi(\xi^n)f\|_p$$

and hence  $\|c(\xi^n)\|_p \rightarrow 0$  as  $n \rightarrow +\infty$ . From the cocycle relation we have:

$$c(\xi^n) = c(\xi) + \pi(\xi)c(\xi^{n-1}).$$

Since  $\|c(\xi^n)\|_p \rightarrow 0$  and  $\|\pi(\xi)c(\xi^{n-1})\|_p \rightarrow 0$  we have that  $c(\xi) = 0$ .  $\square$

The next step is propagating the invariance created by this proposition. This involves some non-isometric variant of Mautner's phenomenon (recall that the classical Mautner's lemma concerns unitary, hence isometric, representations [Mar91, II. 3.2]). We formulate this variant in terms of the following dynamical interpretation of the Levi decomposition [Pra77, 2.2].

**Definition 2.2.7.** Let  $\xi \in G$ . We define:

$$\begin{aligned} U_\xi &= \{h \in G, \xi^{-n}h\xi^n \xrightarrow{n \rightarrow +\infty} 1_G\}, \\ P_\xi &= \{h \in G, \text{the sequence } (\xi^{-n}h\xi^n)_{n>0} \text{ is bounded}\}, \\ Z_\xi &= \{h \in G, \xi^{-1}h\xi = h\}. \end{aligned}$$

The sets  $P_\xi$ ,  $U_\xi$  and  $Z_\xi$  are subgroups of  $G$ . The subgroup  $P_\xi$  contains both  $U_\xi$  and  $Z_\xi$  and these satisfy  $U_\xi \cap Z_\xi = \{1_G\}$ . When  $G$  is a semisimple Lie group and  $\xi$  an element of a maximal split torus of  $G$ , we have that  $P_\xi$  is a parabolic subgroup of  $G$  and  $U_\xi$  is its unipotent radical [Pra77, 2.4 (i)], hence the Levi decomposition of  $P_\xi$  may be written as  $P_\xi = Z_\xi \times U_\xi$ .

**Proposition 2.2.8.** (*Mautner's phenomenon*) Let  $\xi \in G$  and let  $b \in Z^1(G, \pi_p)$  be such that  $b(\xi) = 0$ .

1. (*Classical Mautner's lemma*) If  $\xi$  is  $\pi_p$ -bounded, then  $b(h) = 0$  for all  $h \in U_\xi$ .
2. (*Contracting version*) If  $\xi$  is  $\pi_p$ -contracting, then  $b(h) = 0$  for all  $h \in P_\xi$ .
3. (*Commuting version*) If  $\xi$  is  $\pi_p$ -bounded, then  $b(h) = 0$  for all  $h \in Z_\xi$ .

*Proof.* Take  $h \in G$ . The cocycle relation implies:

$$b(\xi^{-1}h\xi) = b(\xi^{-1}h) = \pi(\xi^{-1})b(h).$$

Hence for all  $n > 0$  we obtain:

$$\|b(h)\|_p \leq \|\pi(\xi^n)\|_{L^p(G,V)} \|b(\xi^{-n}h\xi^n)\|_p.$$

1. Suppose that  $h \in U_\xi$ . Since  $\xi$  is  $\pi_p$ -bounded, there exists  $C > 0$  such that  $\|b(h)\|_p \leq C \|b(\xi^{-n}h\xi^n)\|_p$ . Since  $\xi^{-n}h\xi^n \xrightarrow{n \rightarrow +\infty} 1_G$ , we have  $\|b(\xi^{-n}h\xi^n)\|_p \xrightarrow{n \rightarrow +\infty} 0$ . Thus  $b(h) = 0$ .

2. Suppose that  $h \in P_\xi$ . The sequence  $(\xi^{-n}h\xi^n)_n$  is bounded, so by continuity of  $g \mapsto b(g)$  the term  $\|b(\xi^{-n}h\xi^n)\|_p$  is bounded. Since  $\xi$  is  $\pi_p$ -contracting, we have  $\lim_{n \rightarrow \infty} \|\pi(\xi^n)\|_{L^p(G, V)} = 0$  and hence  $b(h) = 0$ .

3. Suppose that  $h \in Z_\xi$ . Hence the function  $b(h) \in L^p(G, V)$  is  $\pi(\xi)$ -invariant. Since  $\xi$  is  $\pi_p$ -bounded, Lemma 2.2.4 gives  $b(h) = 0$ .  $\square$

*Remark.* 1. In what follows, we will use the contracting version most of the time. For groups of lower rank, we may not always dispose of enough contracting elements. In this case the commuting version can be very practical.

2. For semisimple Lie groups, we have  $P_\xi = Z_\xi \times U_\xi$ . Hence in this case point 2 holds even when  $\xi$  is only a  $\pi_p$ -bounded element, thanks to points 1 and 3.

We come back to the setting of Corollary 2.1.6: let  $G$  be a semisimple Lie group,  $\gamma$  a simple root of its restricted root system,  $P_{\Sigma \setminus \{\gamma\}} = M^\gamma \times H_\gamma$  the Levi decomposition of its corresponding maximal parabolic subgroup,  $M^\gamma = K^\gamma A^\gamma N^\gamma$  be the Iwasawa decomposition of  $M^\gamma$  and let  $R^\gamma = A^\gamma \times N^\gamma$ . Recall:

$$V_p := H_{\text{ct}}^1(H_\gamma, L^p(H_\gamma)).$$

We apply the results of this section to the group  $R^\gamma$  acting on  $V_p$  via  $\pi_0$  and on  $L^p(R^\gamma, V_p)$  via  $\pi$ , where  $\pi_0$  and  $\pi$  are given by Corollary 2.1.6.

**Theorem 2.2.9.** *Suppose that there exists some  $\pi_p$ -contracting  $\xi \in A^\gamma$  and that for each simple root  $\sigma \in \Sigma \setminus \{\gamma\}$  there exists some  $\pi_p$ -bounded  $\xi_\sigma \in A^\gamma$  such that  $U_{\xi_\sigma} \cup Z_{\xi_\sigma}$  contains the root group  $U_\sigma$ . Then:*

$$H_{\text{ct}}^1(R^\gamma, L^p(R^\gamma, V_p)) = \{0\}.$$

*Proof.* Let  $b \in Z^1(R^\gamma, \pi_p)$ . By Proposition 2.2.6, the cocycle  $b$  can be chosen (without changing its cohomology class) so that  $b(\xi) = 0$ . Since the group  $A^\gamma$  is abelian, the group  $P_\xi$  contains  $A^\gamma$  and thus by Proposition 2.2.8 point 2, we have  $b(h) = 0$  for  $h \in A^\gamma$ . This implies that  $b(\xi_\sigma) = 0$  for all  $\sigma \in \Sigma \setminus \{\gamma\}$ . By Proposition 2.2.8 points 1 and 3, we have that  $b(h) = 0$  for all  $h$  the sets  $(U_{\xi_\sigma} \cup Z_{\xi_\sigma}) \cap R^\gamma$ , which contain the root subgroups  $U_\sigma$  for all  $\sigma \in \Sigma \setminus \{\gamma\}$ . The groups  $U_\sigma$  for  $\sigma \in \Sigma \setminus \{\gamma\}$  generate the group  $N^\gamma$  [Bor91, 21.9 (ii)], therefore  $b(h) = 0$  for all  $h \in N^\gamma$ . Thus  $b = 0$ . We conclude that  $H_{\text{ct}}^1(R^\gamma, L^p(R^\gamma, V_p)) = \{0\}$  for every  $p > 1$ .  $\square$

## 2.2.4 Vanishing for semisimple, non-simple groups of rank $\geq 3$

In this section we apply our vanishing criterion to semisimple, non-simple Lie groups. The main point is that commutation makes some operator norms of  $\pi_0$  to be equal to 1, hence we may reason directly as in [dCT11].

**Theorem 2.2.10.** *Let  $G$  be a semisimple, non-simple Lie group of rank  $r \geq 3$  over a local field  $F$ . Then there exists a simple root  $\gamma \in \Sigma$  such that for all  $p > \max\{\text{Confdim}(\partial H_\gamma), 1\}$  we have:*

$$H_{\text{ct}}^2(G, L^p(G)) = \{0\}.$$

*Proof.* Let  $G_1, \dots, G_k$  be the simple factors of the group  $G$ . We split the proof in two cases. First suppose that there exists one factor  $G_i$  of rank 1, associated to a simple root  $\gamma \in \Sigma$ . The maximal parabolic subgroup  $P_\gamma$  decomposes as  $P_\gamma = M^\gamma \ltimes H_\gamma$  and in this case  $M^\gamma$  and  $H_\gamma$  live in different factors, hence  $M^\gamma$  and  $H_\gamma$  commute, so  $P_\gamma = M^\gamma \times H_\gamma$ . In particular, the action by conjugation by some element  $g \in R^\gamma$  on  $H_\gamma$  is trivial, and hence Proposition 2.2.1 gives:

$$\|\pi(g)\|_{L^p(R^\gamma, V_p)} = \Delta_{R^\gamma}(g)^{-1}.$$

Since  $r \geq 3$ , the rank of  $M^\gamma$ , is at least 2 which means that the split torus  $A^\gamma$  is of dimension at least 2. It is thus enough to consider two coweights  $\xi_1, \xi_2 \in A^\gamma$  associated to two distinct simple roots  $\sigma_1, \sigma_2 \in \Sigma \setminus \{\gamma\}$  satisfying  $\sigma_i(\xi_i) < 1$ . In this way  $\Delta_{R^\gamma}(\xi_i)^{-1} < 1$  and hence  $\xi_1, \xi_2$  are  $\pi_p$ -contracting for all  $p > 1$ . The subgroup  $P_{\xi_1}$  contains all the root groups  $U_\sigma$  for  $\sigma \in \Sigma \setminus \{\gamma, \sigma_1\}$  and  $P_{\xi_2}$  contains all the root groups  $U_\sigma$  for  $\sigma \in \Sigma \setminus \{\gamma, \sigma_2\}$ . Hence the conditions of Theorem 2.2.9 are satisfied and we showed:

$$H_{\text{ct}}^1(R^\gamma, L^p(R^\gamma, V_p)) = \{0\}$$

for all  $p > 1$ . Then the spectral sequence in Corollary 2.1.6 gives the desired vanishing for  $p > \text{Confdim}(\partial H_\gamma)$ .

Suppose now that all factors have rank  $\geq 2$ . We may pick any simple root  $\gamma$  in the root system of  $G_1$ . The parabolic subgroup  $P_\gamma$  decomposes as  $P_\gamma = M^\gamma \ltimes H_\gamma$ , where the Levi factor  $M^\gamma$  decomposes again in simple factors  $M_1, \dots, M_k$ , with  $M_i \subseteq G_i$  for all  $i$ . The main point is that the factor  $M_2$  has rank  $\geq 2$  and commutes with  $H_\gamma$ . Our hypothesis implies that there are at least two simple roots  $\sigma_1, \sigma_2$  in the root system of  $G_2$ . Hence we may consider again coweights associated to these two simple roots and reason as before to conclude that  $H_{\text{ct}}^1(R^\gamma, L^p(R^\gamma, V_p)) = \{0\}$  for all  $p > 1$ . Again the spectral sequence gives the desired vanishing for  $p > \text{Confdim}(\partial H_\gamma)$ . This second part of the proof is independent of the choice of root  $\gamma$  and hence we can choose  $\gamma$  in order to minimize  $\text{Confdim}(\partial H_\gamma)$ .  $\square$

## 2.3 Growth estimates and Heintze groups

In Theorem 2.2.9 we proved that under the presence of enough contractions, we can show vanishing of cohomology. One would like to have a criterion to guarantee the existence of contracting elements for the  $R^\gamma$ -module  $L^p(R^\gamma, H_{\text{ct}}^1(H_\gamma, L^p(H_\gamma)))$  appearing in Corollary 2.1.6. Thanks to Proposition 2.2.1, the only mysterious quantity that remains to study is the operator norm of the action  $\pi_0$  of  $R^\gamma$  on the space  $H_{\text{ct}}^1(H_\gamma, L^p(H_\gamma))$ . In this section we find an upper bound of this norm that can be computed explicitly in combinatorial terms.

### 2.3.1 Amenable hyperbolic groups with homotheties

To find an upper bound on the operator norms of the action  $\pi_0$  of  $R^\gamma$  on the space  $H_{\text{ct}}^1(H_\gamma, L^p(H_\gamma))$ , we will first obtain some preliminary results in the more general setting of amenable hyperbolic groups and contracting automorphisms developed in [CdCMT15], with the supplementary condition that we will contract using a homothety.

Let  $U$  be a locally compact second countable group. By [Str74], the group  $U$  admits a proper left invariant metric  $d$  metrizing its topology. Suppose moreover that  $(U, d)$  has a (non-trivial) *homothety*  $\alpha \in \text{Aut}(U)$ , that is, a group automorphism such that for some  $\lambda > 1$  we have:

$$d(\alpha(x), \alpha(y)) = \lambda^{-1}d(x, y).$$

By [CdCMT15, 6.5], the group  $U$  is nilpotent and unimodular. Denote by  $\Omega$  the compact unit ball of  $(U, d)$  centered at  $e_U$  and write  $\|x\| = d_U(x, e)$ . Notice that  $\bigcap_{k \geq 0} \alpha^k(\Omega) = \{e_U\}$  and  $\bigcup_{k \leq 0} \alpha^k(\Omega) = U$ .

The set  $S = \{\alpha^{\pm 1}\} \times \Omega$  generates the group  $H = \langle \alpha \rangle \rtimes U$ . The group  $H$  is amenable and Gromov-hyperbolic when endowed with the word metric  $|\cdot|_S$  with respect to  $S$ . Denote by  $B_S(n)$  the closed ball of radius  $n$  centered at  $e_H$ .

The following lemmata come from [CdCMT15] and allow us to estimate distortion of  $(U, d)$  inside  $(H, |\cdot|_S)$ . Notice in particular that only the upper bound needs to have a homothety.

**Lemma 2.3.1.** [CdCMT15, Lemma 4.7] *For  $m \geq 0$ , we have:*

$$\alpha^{-m}(\Omega) \subseteq B_S(2m + 1) \cap U.$$

**Lemma 2.3.2.** [CdCMT15, Lemma 4.8] *There exists  $C > 0$  such that for all  $x \in U$ , there exist  $i, j \in \mathbb{N}$  with  $j \leq C$  and  $x_1, \dots, x_j \in \Omega$  such that  $x = \alpha^{-i}x_1 \dots x_j\alpha^i$  and  $2i \leq |x|_S + C$ .*

**Proposition 2.3.3.** *There exists  $C > 1$  such that for all  $x \in U$ :*

$$C^{-1}\lambda^{\frac{1}{2}|x|_S} \leq \|x\| \leq C\lambda^{\frac{1}{2}|x|_S}.$$

*Proof.* Let  $m$  be such that  $x \in \alpha^{-m}(\Omega) \setminus \alpha^{-m+1}(\Omega)$ . Hence  $|x|_S \leq 2m + 1$  and:

$$\|x\| \geq \lambda^{m-1} \geq \lambda^{\frac{1}{2}(|x|_S - 3)}.$$

Write  $x = \alpha^{-i}x_1 \dots x_j\alpha^i$  as a word in the alphabet  $S$  given by Lemma 2.3.2. Then as an element of  $U$  we have  $x = \alpha^{-i}(x_1 \dots x_j)$  and so:

$$\|x\| = \lambda^i \|x_1 \dots x_j\| \lesssim \lambda^i \lesssim \lambda^{\frac{1}{2}|x|_S}.$$

□

**Lemma 2.3.4.** *Let  $Q$  denote the Hausdorff dimension of the metric space  $(U, d)$ .*

1. *The Haar measure on  $U$  is equivalent to the  $Q$ -Hausdorff measure of  $(U, d)$ .*
2. *Let  $\Delta_H$  denote the modular function of  $H$ . Then we have:  $\Delta_H(\alpha) = \lambda^Q$ .*

*Proof.* 1. Fix some left-invariant Haar measure  $\mu$  on  $U$ . Then the measures  $\alpha_*^k \mu$  on  $U$  are also left invariant for  $k \in \mathbb{Z}$ . Such measures are unique up to positive scalar, so there exists some  $s \in \mathbb{R}_+^*$  such that  $\alpha_*^k \mu = s^k \mu$  for all  $k \in \mathbb{Z}$ . We now show that  $\mu$  is an Ahlfors  $Q$ -regular measure. For this, let  $r > 0$  and  $k \in \mathbb{Z}$  so that  $\lambda^k \leq r < \lambda^{k+1}$ . We have:

$$\mu(B(\lambda^k)) \leq \mu(B(r)) < \mu(B(\lambda^{k+1})).$$

Since  $\alpha$  is a homothety, we have  $B(\lambda^k) = \alpha^{-k}\Omega$  and thus

$$\mu(B(\lambda^k)) = \alpha_*^k \mu(\Omega) = s^k \mu(\Omega).$$

We have  $\mu(\Omega) < \infty$  since  $\mu$  is a Radon measure (and  $\Omega$  is compact, because  $d$  is a proper metric) and  $\mu(\Omega) > 0$  because  $\Omega$  contains an open set as  $d$  metrizes the topology of  $U$ . From this we deduce that there exists  $C > 0$  such that:

$$C^{-1} r^{\frac{\log s}{\log \lambda}} \leq \mu(B(r)) \leq C r^{\frac{\log s}{\log \lambda}}.$$

This implies that  $\mu$  is Ahlfors  $(\frac{\log s}{\log \lambda})$ -regular and that the Hausdorff dimension  $Q$  of the metric space  $(U, d)$  must satisfy  $Q = \frac{\log s}{\log \lambda}$ , that is  $s = \lambda^Q$ .

2. Since the groups  $U$  and  $\langle \alpha \rangle \simeq \mathbb{Z}$  are unimodular, the modular function of the semi-direct product  $H = \langle \alpha \rangle \ltimes U$  is computed by the following formula. For any measurable set  $E$  in  $U$  we have:

$$\mu(\alpha^{-1}(E)) = \Delta_H(\alpha)\mu(E).$$

For  $E = \Omega$  we have  $0 < \mu(\Omega) < \infty$  and:

$$\Delta_H(\alpha)\mu(\Omega) = \mu(\alpha^{-1}(\Omega)) = \alpha_*\mu(\Omega) = \lambda^Q \mu(\Omega),$$

since in the proof of 1. we showed that  $s = \lambda^Q$ . Hence  $\Delta_H(\alpha) = \lambda^Q$ .  $\square$

### 2.3.2 Growth estimates for Heintze groups

We can now obtain an upper bound on the operator norms of the action  $\pi_0$  of  $R^\gamma$  on the space  $H_{\text{ct}}^1(H_\gamma, L^p(H_\gamma))$ . In fact we will only do it for the restricted action of the torus  $A^\gamma$ , but this will be enough for our purposes. We come back to the usual Lie theoretic setting and use the usual notation. Our first task is to show that we can apply the results from the previous subsection. Fix  $s \in S_\gamma$  such that  $\lambda := |\gamma(s^{-1})| > 1$  for the rest of the section.

**Proposition 2.3.5.** *There exists a proper, left-invariant metric  $d_\gamma$  on  $U_\gamma$  so that conjugation by  $s$  is a homothety of ratio  $\lambda^{-1}$  on the metric space  $(U_\gamma, d_\gamma)$ .*

*Proof.* In the real case,  $U_\gamma$  is a Carnot group for the gradation on its Lie algebra  $\text{Lie}(U_\gamma) = \bigoplus_{\alpha \in \Phi^+, n_\gamma(\alpha) > 0} \mathfrak{g}_\alpha$  given by

$$\mathfrak{g}_k := \bigoplus_{\alpha \in \Phi^+, n_\gamma(\alpha) = k} \mathfrak{g}_\alpha$$

for  $k > 0$ . This means that  $\text{Lie}(U_\gamma)$  is generated by the subspace  $\mathfrak{g}_1$ . This is shown for instance in [Yam93], see there Lemma 3.8 and the discussion at the end of Section 3.4. Since  $U_\gamma$  is a Carnot group, it carries a *Carnot-Carathéodory metric* [Pan89c, p. 3], which satisfies the conditions of the Proposition [Pan89c, p. 4].

Suppose now that the local field  $F$  is non-Archimedean, consider its ring of integers  $\mathcal{O}$  and let  $\Omega$  be the compact-open subgroup  $\prod_{\alpha \in \Phi_{n_\gamma > 0}} \theta_\alpha(\mathcal{O})$ . Notice that  $\bigcap_{k \geq 0} s^k \cdot \Omega = \{e_{U_\gamma}\}$  and  $\bigcup_{k \leq 0} s^k \cdot \Omega = U_\gamma$ . These two conditions allow us to define a

valuation  $v(x) = \sup\{k \in \mathbb{Z}, x \in s^k \cdot \Omega\}$  for  $x \in U_\gamma$ . Since  $\Omega$  is a subgroup, it satisfies the inequality  $v(xy) \geq \min\{v(x), v(y)\}$  for all  $x, y \in U_\gamma$ . We may define the norm

$$\|x\|_\gamma := \lambda^{-v(x)},$$

on  $U_\gamma$  and the distance  $d_\gamma(x, y) := \|x^{-1}y\|_\gamma$  for all  $x, y \in U_\gamma$ . The distance  $d_\gamma$  on  $U_\gamma$  satisfies the ultrametric inequality, is left-invariant and satisfies:

$$d_\gamma(s.x, s.y) = \lambda^{-1}d_\gamma(x, y)$$

for all  $x, y \in U_\gamma$ . □

*Remark.* 1. The construction in the non-Archimedean case also works for a group  $U$  with compact neutral component  $U^0$  and with a compacting automorphism  $\alpha$ , meaning that there exists some compact subset  $\Omega$  such that for any compact subset  $K \subseteq U$ , there exists  $k_0$  such that for all  $k \geq k_0$ , we have  $\alpha^k(K) \subseteq \Omega$ . By [dCT11, 4.5.ii], we may choose  $\Omega$  to be a compact-open subgroup such that  $\alpha^k(\Omega) \subseteq \Omega$  for all  $k > 0$ . If  $L = \bigcap_{k \geq 0} \alpha^k(\Omega)$ , then our proof defines a  $U$ -invariant distance on  $U/L$  that satisfies the ultrametric inequality and for which the automorphism induced by  $\alpha$  on  $U/L$  is a homothety.

2. We can also consider this construction when  $U_\gamma$  is a real Lie group by replacing  $\Omega$  with a product of compact intervals containing 0. A minor problem arises from the fact that in this case  $\Omega$  is not a subgroup, so the corresponding valuation will only satisfy  $v(xy) \geq \min\{v(x), v(y)\} - C$  for some constant  $C > 0$ . Thus the formula  $\|x\|_\gamma = e^{-v(x)}$  defines only a left-invariant quasi-metric  $d_\gamma$ . This can be salvaged by considering a power  $d_\gamma^a$  of  $d_\gamma$  for  $0 < a < 1$  sufficiently small. The main problem is that we do not have a good control of the Hausdorff dimension of this metric (it can be really big), as opposed to the Carnot-Carathéodory metric, which has minimal dimension in the conformal gauge of  $\partial H_\gamma$ .

Suppose again that  $F$  is a (real or non-Archimedean) local field. We will consider a function that is Lipschitz-equivalent to the metric  $d_\gamma$  and easier for computations. To construct it, we first identify  $S_\gamma$ -equivariantly each factor  $U_\alpha$  to  $\mathfrak{g}_\alpha$  endowed with some vector space norm  $|\cdot|_\alpha$  (in the non-Archimedean case, we choose these norms to be compatible with the absolute value  $|\cdot|$  on  $F$  such that  $\text{Hausdim}(F, |\cdot|) = 1$  in the sense that  $|xv|_\alpha = |x||v|_\alpha$  for all  $x \in F$  and  $v \in \mathfrak{g}_\alpha$ ). We decompose an element  $x \in U_\gamma$  in coordinates  $x = (x_\alpha)_{\alpha \in \Phi_{n_\gamma > 0}}$ , where  $x_\alpha \in U_\alpha$ , and define:

$$N_\gamma(x) := \max\{|x_\alpha|^{1/n_\gamma(\alpha)}, \alpha \in \Phi_{n_\gamma > 0}\}.$$

In this way  $N_\gamma(s.x) = \max\{|\alpha(s)x_\alpha|^{1/n_\gamma(\alpha)}, \alpha \in \Phi_{n_\gamma > 0}\} = \lambda^{-1}N_\gamma(x)$  for all  $x \in U_\gamma$ . Indeed, as  $s \in S_\gamma = \bigcap_{\sigma \in \Sigma \setminus \{\gamma\}} \ker \sigma$ , we have  $|\alpha(s)| = |\gamma(s)|^{n_\gamma(\alpha)}$  and  $|\alpha(s)|^{1/n_\gamma(\alpha)} = |\gamma(s)|$  for all  $\alpha \in \Phi_{n_\gamma > 0}$ . As the function  $N_\gamma$  is positive on  $U_\gamma \setminus \{1\}$  and  $s$  acts on it as a homothety of ratio  $\lambda^{-1}$ , the left-invariant function  $N_\gamma(x, y) = N_\gamma(x^{-1}y)$  is Lipschitz-equivalent to the metric  $d_\gamma$ .

Using the function  $N_\gamma$  we may compute the Hausdorff dimension  $Q_\gamma$  of the metric space  $(U_\gamma, d_\gamma)$ . Indeed, we have:

$$Q_\gamma = \sum_{\alpha \in \Phi_{n_\gamma > 0}} \text{Hausdim}(\mathfrak{g}_\alpha, |\cdot|_\alpha^{1/n_\gamma(\alpha)}) = \sum_{\alpha \in \Phi_{n_\gamma > 0}} n_\gamma(\alpha)m(\alpha).$$

For  $g \in A^\gamma$  define:

$$\|g\|_\gamma := \max_{\alpha \in \Phi_{n_\gamma > 0}} (|\alpha(g)|^{1/n_\gamma(\alpha)}).$$

**Lemma 2.3.6.** *There exists  $C > 0$  such that for all  $g \in A^\gamma$  and all  $x, y \in U_\gamma$  we have:*

$$d_\gamma(g.x, g.y) \leq C\|g\|_\gamma d_\gamma(x, y)$$

*Proof.* Directly from the definitions of  $N_\gamma$  and  $\|g\|_\gamma$  it follows that for all  $g \in A^\gamma$  and  $x \in U_\gamma$  we have  $N_\gamma(g.x) \leq \|g\|_\gamma N_\gamma(x)$ . Since  $N_\gamma$  and  $d_\gamma$  are Lipschitz-equivalent, there exists  $c > 0$  such that  $(1/c)d_\gamma \leq N_\gamma \leq cd_\gamma$  and hence

$$d_\gamma(g.x, g.y) \leq cN_\gamma(g.x, g.y) \leq c\|g\|_\gamma N_\gamma(x, y) \leq c^2\|g\|_\gamma d_\gamma(x, y).$$

□

The main result of the section is the following:

**Theorem 2.3.7.** *There exists a constant  $C = C(p, \gamma) > 0$  such that for all  $g \in A^\gamma$ :*

$$\|\pi_0(g)\|_{V_p}^p \leq C\|g^{-1}\|_\gamma^{Q_\gamma}.$$

*Remark.* This inequality is optimal (up to the constant  $C > 1$ ) in the case of  $G = \mathrm{SL}(r+1, \mathbb{R})$  with simple root  $\gamma = e_1 - e_2$ . Indeed, we have  $M_{\Sigma \setminus \{\gamma\}} = \mathrm{SL}(r, \mathbb{R})$  acting on  $U_\gamma = \mathbb{R}^r$  by the natural linear action and  $Q_\gamma = r$ . The space  $V_p$  may be identified (as a continuous  $M_{\Sigma \setminus \{\gamma\}}$ -module) with the Besov space  $B_{p,p}^{r/p}(\mathbb{R}^r)$ , that is, the Banach space of measurable functions  $u : \mathbb{R}^r \rightarrow \mathbb{R}$  such that

$$\|u\|_{B_p}^p = \int_{\mathbb{R}^r \times \mathbb{R}^r} \frac{|u(x) - u(x')|^p}{d(x, x')^{2r}} dx dx' < \infty,$$

modulo constant functions [Pan89a, 5.2]. For  $g = \mathrm{diag}(2^{1-r}, 2, \dots, 2) \in M_{\Sigma \setminus \{\gamma\}}$ , we construct the function  $u \in B_{p,p}^{r/p}(\mathbb{R}^r)$  such that  $u(x) = 1$  on the cube  $Q_1 = [1, 2]^r$  and  $u(x) = 0$  on the cube  $Q_2 = [-2, -1] \times [1, 2]^{r-1}$  disjoint from  $Q_1$  and restricting the previous integral to  $Q_1 \times Q_2$  we may see that  $\|\pi_0(g^k)u\|_{B_p}^p \gtrsim \|g^{-k}\|^r \|u\|_{B_p}^p$ .

After combining the inequality in Theorem 2.3.7 with Proposition 2.2.1, we obtain the following corollary.

**Corollary 2.3.8.** *There exists a constant  $C = C(p, \gamma) > 0$  such that for all  $g \in A^\gamma$ :*

$$\|\pi_p(g)\|_{V_p}^p \leq C\Delta_{R^\gamma}(g)^{-1}\|g^{-1}\|_\gamma^{Q_\gamma}.$$

*In particular:*

*any  $g \in A^\gamma$  such that  $\Delta_{R^\gamma}(g)^{-1}\|g^{-1}\|_\gamma^{Q_\gamma} < 1$  is  $\pi_p$ -contracting for every  $p > 1$ ,*

*any  $g \in A^\gamma$  such that  $\Delta_{R^\gamma}(g)^{-1}\|g^{-1}\|_\gamma^{Q_\gamma} \leq 1$  is  $\pi_p$ -bounded for every  $p > 1$ .*

In order to prove Theorem 2.3.7 we apply results from the previous section to the group  $U = U_\gamma$  with its left-invariant metric  $d = d_\gamma$  and the automorphism  $\alpha$  is conjugation by some non-trivial  $s \in S_\gamma$  such that  $\lambda^{-1} := |\gamma(s)| < 1$ . The main idea of the proof is to use the cocycle relation at each step of a word as in Lemma 2.3.2 and compute the contribution of each step using Lemma 2.3.4.

*Proof of theorem 2.3.7.* Here we replace  $H_\gamma$  with its cocompact subgroup  $\langle s \rangle \rtimes U_\gamma$ . Let  $\rho_{H_\gamma}$  be the right regular representation of  $H_\gamma$  on  $L^p(H_\gamma)$ . As  $H_\gamma$  is compactly generated, with compact generating set  $S = \{s^{\pm 1}\} \times \Omega$ , we may consider  $Z^1(H_\gamma, L^p(H_\gamma))$  as a Banach space with norm  $\|b\|_{S,p} = \sup_{x \in S} \|b(x)\|_p$ . Let  $b \in Z^1(H_\gamma, L^p(H_\gamma))$ . Since the action by conjugation of  $R^\gamma$  on  $H_\gamma$  preserves the volume, we have:

$$\|\pi_0(g)b\|_{S,p} = \sup_{x \in S} \|b(g^{-1}xg)\|_p.$$

Fix some  $x \in S$ . We suppose that  $x \in \{s\} \times \Omega$ , the situation being similar for  $x \in \{s^{-1}\} \times \Omega$ . Write  $x = ys$ , with  $y \in \Omega$ . Lemma 2.3.2 says that there exist  $C > 0$  and  $n, j \in \mathbb{N}$  with  $j \leq C$  and  $x_1, \dots, x_j \in \Omega$  such that

$$g^{-1}yg = s^{-n}x_1 \dots x_j s^n,$$

and  $2n \leq |g^{-1}yg|_S + C$ .

Let  $s_i \in S$  be the  $i$ -th term (in the alphabet  $S$ ), from left to right, of the word

$$g^{-1}xg = s^{-n}x_1 \dots x_j s^{n+1}.$$

More precisely,

$$s_i = \begin{cases} s^{-1} & \text{for } 1 \leq i \leq n, \\ x_{i-n} & \text{for } n+1 \leq i \leq n+j, \\ s & \text{for } n+j+1 \leq i \leq 2n+j+1. \end{cases}$$

Let  $h_i = s_1 \dots s_i$  for all  $i > 0$  and  $h_0 = 1$ . Iterating the cocycle relation for  $b$  yields:

$$b(g^{-1}xg) = \sum_{i=1}^{2n+j+1} \rho_{H_\gamma}(h_{i-1})b(s_i).$$

From here we obtain:

$$\|b(g^{-1}xg)\|_p \leq \sum_{i=1}^{2n+j+1} \|\rho_{H_\gamma}(h_{i-1})b(s_i)\|_p = \sum_{i=1}^{2n+j+1} \Delta_{H_\gamma}(h_{i-1})^{-1/p} \|b(s_i)\|_p.$$

Using Lemma 2.3.4, we have that:

$$\Delta_{H_\gamma}(s_i)^{-1} = \begin{cases} \lambda^{Q_\gamma} & \text{for } 1 \leq i \leq n, \\ 1 & \text{for } n+1 \leq i \leq n+j, \\ \lambda^{-Q_\gamma} & \text{for } n+j+1 \leq i \leq 2n+j+1. \end{cases}$$

Thus:

$$\Delta_{H_\gamma}(h_i)^{-1} = \begin{cases} \lambda^{iQ_\gamma} & \text{for } 1 \leq i \leq n, \\ 1 & \text{for } n+1 \leq i \leq n+j, \\ \lambda^{(2n+j+1-i)Q_\gamma} & \text{for } n+j+1 \leq i \leq 2n+j+1. \end{cases}$$

Hence:

$$\begin{aligned} \|b(g^{-1}xg)\|_p &\leq \sum_{k=0}^n \lambda^{kQ_\gamma/p} \|b\|_{S,p} + C\lambda^{nQ_\gamma/p} \|b\|_{S,p} + \sum_{k=0}^n \lambda^{kQ_\gamma/p} \|b\|_{S,p} \\ &\leq C_1 \lambda^{nQ_\gamma/p} \|b\|_{S,p}. \end{aligned}$$

where  $C_1 = 2\frac{1}{1-\lambda^{-Q_\gamma/p}} + C$ . Using  $2n \leq |g^{-1}yg|_S + C$  and Proposition 2.3.3 we obtain

$$\lambda^n \leq \lambda^{\frac{1}{2}|g^{-1}yg|_S + C} \leq C_2 \|g^{-1}yg\|_\gamma.$$

Since  $g \in A^\gamma$ , Lemma 2.3.6 says that  $\|g^{-1}yg\|_\gamma \leq C_3 \|g^{-1}\|_\gamma \|y\|_\gamma$ . We have that  $y \in \Omega$ , so  $\|y\|_\gamma \leq 1$  and  $\lambda^n \leq C_4 \|g^{-1}\|_\gamma$ . From this we may conclude that:

$$\|\pi_0(g)b\|_{S,p} = \sup_{x \in S} \|b(g^{-1}xg)\|_p \leq C_5 \|g^{-1}\|_\gamma^{Q_\gamma/p} \|b\|_{S,p}.$$

Hence  $\|\pi_0(g)\|_{Z^1(H_\gamma, L^p(H_\gamma))} \leq C_5 \|g^{-1}\|_\gamma^{Q_\gamma/p}$  and the operator induced on the quotient Banach space  $V_p = H_{\text{ct}}^1(H_\gamma, L^p(H_\gamma))$  has smaller operator norm:

$$\|\pi_0(g)\|_{V_p} \leq C_5 \|g^{-1}\|_\gamma^{Q_\gamma/p}.$$

□

## 2.4 Existence of contracting elements for simple groups

In this section we want to show existence of  $\pi_p$ -contracting elements as in our vanishing criterion (Theorem 2.2.9) via our estimate (Corollary 2.3.8) for simple groups appearing in the statement of Theorem 2.0.1. We first translate the condition of Theorem 2.2.9 in combinatorial terms as an inequality depending only on a root system and the multiplicities of the roots. Then, using the classification of simple Lie groups over local fields in its more classical form (in the sense that we give a full list of the groups in the classification), we obtain Theorem 2.0.1 thanks to a case-by-case verification of the inequality. We sum up these results in tables. Taking into account multiplicities explains why we cannot stand only by the root system and why we go back to classical presentations.

We conclude this section by obtaining Corollary 2.0.3 as a byproduct of this combinatorial study and by comparing our results with those from [BR20].

### 2.4.1 Combinatorial reformulation

Corollary 2.3.8 says that to guarantee the existence of  $\pi_p$ -contracting (resp.  $\pi_p$ -bounded) elements, it is enough to compute the term  $\Delta_{R^\gamma}(g)^{-1} \|g^{-1}\|_\gamma^{Q_\gamma}$  for  $g \in A^\gamma$  and see that this term is  $< 1$  (resp.  $\leq 1$ ). In practice,  $\pi_p$ -bounded elements that are not  $\pi_p$ -contracting appear only in some exceptional low rank cases.

Our next goal is to write the expression  $\Delta_{R^\gamma}(g)^{-1} \|g^{-1}\|_\gamma^{Q_\gamma}$  in terms of root systems. We may begin with  $\Delta_{R^\gamma}(g)$ . For  $g \in A^\gamma$ , the modular function of  $R^\gamma$  can be expressed as:

$$\Delta_{R^\gamma}(g)^{-1} = \prod_{\alpha \in (\Phi_{\Sigma \setminus \{\gamma\}})^+} |\alpha(g)|^{m(\alpha)}.$$

Choose some simple root  $\sigma \in \Sigma, \sigma \neq \gamma$ . Let  $g \in A^\gamma \cap \bigcap_{\tau \in \Sigma \setminus \{\gamma, \sigma\}} \ker \tau$  such that  $|\sigma(g)| > 1$ . This choice is made so that the subgroup  $P_g$  of  $R^\gamma$  is big. Indeed,  $g$  commutes with  $U_\tau$  for all  $\tau \in \Sigma \setminus \{\gamma, \sigma\}$ , therefore  $P_g \supset U_\tau$  for all  $\tau \in \Sigma \setminus \{\gamma, \sigma\}$ . The modular function of  $g$  may be expressed as follows:

$$\Delta_{R^\gamma}(g)^{-1} = \prod_{\alpha \in \Phi_{n_\sigma > 0} \cap (\Phi_{\Sigma \setminus \{\gamma\}})^+} |\sigma(g)|^{n_\sigma(\alpha)m(\alpha)}.$$

On the other hand, since  $g \in \bigcap_{\tau \in \Sigma \setminus \{\gamma, \sigma\}} \ker \tau$ , we have that:

$$|\alpha(g)| = |\gamma(g)|^{n_\gamma(\alpha)} |\sigma(g)|^{n_\sigma(\alpha)}$$

for all  $\alpha \in \Phi^+$ , so that:

$$\|g\|_\gamma = |\gamma(g)| \max_{\alpha \in \Phi_{n_\gamma > 0}} \{|\sigma(g)|^{n_\sigma(\alpha)/n_\gamma(\alpha)}\}.$$

Thus the condition  $\Delta_{R^\gamma}(g) \|g\|_\gamma^{Q_\gamma} < 1$  is equivalent to:

$$\sum_{\alpha \in \Phi_{n_\sigma > 0} \cap (\Phi_{\Sigma \setminus \{\gamma\}})^+} n_\sigma(\alpha)m(\alpha) > Q_\gamma \left( \max_{\alpha \in \Phi_{n_\gamma > 0}} \left\{ \frac{n_\sigma(\alpha)}{n_\gamma(\alpha)} \right\} + \frac{\log |\gamma(g)|}{\log |\sigma(g)|} \right).$$

Since  $g \in A^\gamma$ , the terms  $\log |\tau(g)|$  for  $\tau \in \Sigma$  satisfy a non trivial linear relation. Moreover,  $g \in \bigcap_{\tau \in \Sigma \setminus \{\gamma, \sigma\}} \ker \tau$ , so that  $\log |\gamma(g)|$  depends linearly on  $\log |\sigma(g)|$  and thus  $\frac{\log |\gamma(g)|}{\log |\sigma(g)|} =: C_\sigma$  is a constant that depends only on the root system  $\Phi$  and not on the choice of  $g$ . We proved the following criterion:

**Proposition 2.4.1.** *Let  $\sigma_1, \sigma_2 \in \Sigma \setminus \{\gamma\}$  be two distinct simple roots satisfying:*

$$\sum_{\alpha \in \Phi_{n_{\sigma_i} > 0} \cap (\Phi_{\Sigma \setminus \{\gamma\}})^+} n_{\sigma_i}(\alpha)m(\alpha) \geq Q_\gamma \cdot \left( \max_{\alpha \in \Phi_{n_\gamma > 0}} \left\{ \frac{n_{\sigma_i}(\alpha)}{n_\gamma(\alpha)} \right\} + C_{\sigma_i} \right)$$

for  $i = 1, 2$ , with at least one of the two satisfying strict inequality. Then hypotheses of Theorem 2.2.9 are satisfied.

*Remark.* The left-hand side and the constant  $Q_\gamma$  depend on the root system  $\Phi$  and the multiplicities of the roots. On the other hand, the second factor of the right-hand side depends only on  $\Phi$ .

## 2.4.2 Vanishing in degree 2 for classical simple groups

We start a case-by-case study of simple Lie groups over a local field  $F$  to find those that satisfy the conditions of Proposition 2.4.1. We restrict ourselves to the so-called classical groups, leaving the exceptional groups aside.

**Classification of classical simple groups over local fields** Let  $F$  be a local field. We recall some parts of the classification of (absolutely) simple Lie groups over  $F$  up to isogeny (see [Tit66] for the classification in the more general algebraic group setting, [Hel01] for the real case and [Tit79] for the non-Archimedean case). We do

not need the full power of the classification via Satake diagrams as we only use its phrasing in classical terms. This means that we look at the associated ordinary Dynkin diagrams and take into account multiplicities. The classification result is that, apart for groups of exceptional type, the only sources of families of simple Lie groups over  $F$  are the groups  $\mathrm{SL}_n$  and, in the terminology of [Bor91, 23.8], groups preserving  $(\epsilon, \sigma)$ -hermitian forms. The latter split into symmetric, skew-symmetric, hermitian and skew-hermitian forms and allow to deal with them in a uniform setting. This classification result applies both to the real and the non-Archimedean setting, though the restrictions on the parameters of the forms are different in each case (in most cases, the non-Archimedean case is more restrictive, except for  $\mathrm{SL}_n$  and the skew-hermitian case).

We now list the families of all (isogeny classes of) classical absolutely simple Lie groups over  $F$  of split rank  $r$ . Our list is based on [Tit79, 4.4], though that list only concerns the non-Archimedean case. For the real groups we mention, we compare each situation with the list in [Hel01, X, p.532-534].

- The special linear group  $\mathrm{SL}_{r+1}(D)$  over  $D$ , where  $D$  is a  $d^2$ -dimensional central division  $F$ -algebra. In the real case,  $d = 1$  or  $d = 2$ , corresponding to  $\mathrm{SL}_{r+1}(\mathbb{R})$  and  $\mathrm{SL}_{r+1}(\mathbb{H})$ . In the non-Archimedean case there is no restriction on  $d$ .

- The special unitary group  $\mathrm{SU}(h)$  of a hermitian form  $h$  in  $n$  variables and Witt index  $r$  over a quadratic extension of  $F$ . In the real case there is no restriction on  $n$  and  $r$ , giving the groups  $\mathrm{SU}(r, n-r)$ . In the non-Archimedean case,  $2r \leq n \leq 2r+2$ .

- The special orthogonal group  $\mathrm{SO}(q)$  of a quadratic form  $q$  in  $n$  variables and Witt index  $r$ . In the real case there is no restriction on  $n$  and  $r$ , giving the groups  $\mathrm{SO}_{r, n-r}(\mathbb{R})$ . In the non-Archimedean case,  $2r \leq n \leq 2r+4$ .

- The symplectic group  $\mathrm{Sp}_{2r}(F)$ .

- The special unitary group  $\mathrm{SU}(\tilde{h})$  of a quaternion hermitian form  $\tilde{h}$  in  $n$  variables and Witt index  $r$ . In the real case there is no restriction on  $n$  and  $r$ , giving the groups  $\mathrm{Sp}_{2r, 2(n-r)}(\mathbb{R})$ . In the non-Archimedean case, the Witt index is always maximal, that is,  $n = 2r$  or  $2r+1$ .

- The special orthogonal group  $\mathrm{SO}(\tilde{q})$  of a quaternion skew-hermitian form  $\tilde{q}$  in  $n$  variables and Witt index  $r$ . In the real case we have  $n = 2r$  or  $2r+1$ , giving the groups  $\mathrm{SO}^*(2n)$ . In the non-Archimedean case,  $2r \leq n \leq 2r+3$ .

**Statement of the vanishing theorem** The result we will prove is the following:

**Theorem 2.4.2.** *Let  $G$  be  $\mathrm{SL}(4, D)$  or one of the simple Lie groups listed above with split rank  $r \geq 4$ . If  $G$  is not of type  $D_4$ , there exists a simple root  $\gamma$  such that  $H_{\mathrm{ct}}^2(G, L^p(G)) = \{0\}$  for all  $p > \max\{1, \mathrm{Confdim}(\partial H_\gamma)\}$ .*

In order to prove this theorem, we need to verify the conditions of Proposition 2.4.1. For this, we need to choose a simple root  $\gamma$  to construct our parabolic subgroup and then two simple roots  $\sigma_1, \sigma_2 \in \Sigma \setminus \{\gamma\}$  to construct two contractions. Each simple root  $\gamma \in \Sigma$  partitions the set of positive roots  $\Phi^+$  into two disjoint sets,  $\Phi_{n_\gamma > 0}$  and  $\Phi_{\Sigma \setminus \{\gamma\}}$ . The idea is to choose  $\gamma$  so that  $|\Phi_{n_\gamma > 0}|$  is as small as possible and  $|\Phi_{\Sigma \setminus \{\gamma\}}|$  is as big as possible (in the corresponding Dynkin diagram, this is done by choosing an extremal vertex).

	$A_r$	$B_r$	$C_r$	$BC_r$	$D_r$
$V = X(S) \otimes \mathbb{R}$	$\{e_1 + e_2 + \dots + e_{r+1} = 0\}$	$\mathbb{R}^r$	$\mathbb{R}^r$	$\mathbb{R}^r$	$\mathbb{R}^r$
Choice for $\gamma$	$\tau_1 = e_1 - e_2$	$\tau_1 = e_1 - e_2$	$\tau_1 = e_1 - e_2$	$\tau_1 = e_1 - e_2$	$\tau_1 = e_1 - e_2$
$n_\gamma(\alpha)$ for $\alpha \in \Phi_{n_\gamma > 0}$	1	1	$n_\gamma(2e_1) = 2$ , else 1	$n_\gamma(2e_1) = 2$ , else 1	1
Equation for $A^\gamma$	$\sum_{i=1}^r (r+1-i)\tau_i = 0$	$\sum_{i=1}^r \tau_i = 0$	$\sum_{i=1}^{r-1} 2\tau_i + \tau_r = 0$	$\sum_{i=1}^r \tau_i = 0$	$\sum_{i=1}^{r-2} 2\tau_i + \tau_{r-1} + \tau_r = 0$
Choice for $\sigma_1$	$\tau_2 = e_2 - e_3$	$\tau_r = e_r$	$\tau_r = 2e_r$	$\tau_r = e_r$	$\tau_r = e_{r-1} + e_r$
$C_{\sigma_1}$	$-(r-1)/r$	-1	-1/2	-1	-1/2
$\max_{\alpha \in \Phi_{n_\gamma > 0}} \left\{ \frac{n_{\sigma_1}(\alpha)}{n_\gamma(\alpha)} \right\}$	1	2	1	2	1
$D_{\sigma_1}$	1/r	1	1/2	1	1/2
Choice for $\sigma_2$	$\tau_3 = e_3 - e_4$	$\tau_{r-1} = e_{r-1} - e_r$	$\tau_{r-1} = e_{r-1} - e_r$	$\tau_{r-1} = e_{r-1} - e_r$	$\tau_{r-1} = e_{r-1} - e_r$
$C_{\sigma_2}$	$-(r-2)/r$	-1	-1	-1	-1/2
$\max_{\alpha \in \Phi_{n_\gamma > 0}} \left\{ \frac{n_{\sigma_2}(\alpha)}{n_\gamma(\alpha)} \right\}$	1	2	2	2	1
$D_{\sigma_2}$	2/r	1	1	1	1/2

Table 2.1: Root systems and the constant  $D_{\sigma_i}$

	$SL_{r+1}(D)$	$SU(h)$	$SO(q)$	$Sp_{2r}(F)$	$SU(\tilde{h})$	$SO(\tilde{q})$
$e_i - e_j$	$d^2$	2	1	1	4	4
$e_i + e_j$	0	2	1	1	4	4
$e_i$	0	$2(n-2r)$	$n-2r$	0	$4(n-2r)$	$4(n-2r)$
$2e_i$	0	1	0	1	3	1

Table 2.2: Multiplicities of roots in terms of classical presentations

**Root systems** We will start by computing the quantities in the inequality of Proposition 2.4.1 that depend only on the root system  $\Phi$ . This is the constant:

$$D_{\sigma_i} = \max_{\alpha \in \Phi_{n_\gamma > 0}} \left\{ \frac{n_{\sigma_i}(\alpha)}{n_\gamma(\alpha)} \right\} + C_{\sigma_i}.$$

The constant  $D_{\sigma_i}$  is computed in Table 2.1 for the infinite families of root systems.

We now explain the contents of Table 2.1. The columns are indexed by the five infinite families of root systems:  $A_r, B_r, C_r, BC_r$  and  $D_r$ .

In the first line, we describe the classical choice for the real vector space  $V = X(S) \otimes \mathbb{R}$  so that the root system  $\Phi$  is described as in [Bou68, VI, Planches, I-IV] with simple roots  $\tau_1, \dots, \tau_r$ . In the second line we describe our choice of simple root  $\gamma \in \Sigma$ , written in terms of the natural description in coordinates of each root system. The third line computes the constants  $n_\gamma(\alpha)$  for  $\alpha \in \Phi_{n_\gamma > 0}$ . The fourth line describes the linear equation we choose so that the ambient space  $X(A^\gamma) \otimes \mathbb{R}$  of the sub-root system  $\Phi_{\Sigma \setminus \{\gamma\}}$  of  $A^\gamma$  sits naturally inside the ambient space  $V$  of  $\Phi$ .

In the next part of Table 2.1, we first list our choices for  $\sigma_1$  and  $\sigma_2$ . We then compute the preliminary constant  $C_{\sigma_i}$  thanks to the equation for  $A^\gamma$  that we computed before. The computation of  $\max_{\alpha \in \Phi_{n_\gamma > 0}} \left\{ \frac{n_{\sigma_i}(\alpha)}{n_\gamma(\alpha)} \right\}$  requires to first compute  $n_{\sigma_i}(\alpha)$  for  $\alpha \in \Phi_{n_\gamma > 0}$  (this can be found in [Bou68, VI, Planches I-IV]), we only recover the final result in Table 2.1. The computation of  $D_{\sigma_i}$  amounts to add the two last rows.

**Multiplicities** The multiplicities of the roots in each of the six families listed above are represented in Table 2.2 and can be found in [Bor91, Chapter V, 23].

	$SL_{r+1}(D)$	$SU(h)$	$SO(q)$	$Sp_{2r}(F)$	$SU(h)$	$SO(\bar{q})$
Root system	$A_r$	$C_r$ if $n = 2r$ $BC_r$ if $n > 2r$	$D_r$ if $n = 2r$ $B_r$ if $n > 2r$	$C_r$	$C_r$ if $n = 2r$ $BC_r$ if $n > 2r$	$C_r$ if $n = 2r$ $BC_r$ if $n > 2r$
$Q_\gamma$	$rd^2$	$2n - 2$	$n - 2$	$2r$	$4n - 2$	$4n - 6$
$D_{\sigma_1}$	$1/r$	$1/2$ 1	$1/2$ 1	$1/2$	$1/2$ 1	$1/2$ 1
$M_1$	$(r-1)d^2$	$(r-1)^2$ $2(r-1)(n-r-1)$	$(r-1)(r-2)/2$ $(r-1)(n-r-2)$	$r(r-1)/2$	$(r-1)(2r-1)$ $2(r-1)(2n-2r-1)$	$(r-1)(2r-3)$ $2(r-1)(2n-2r-3)$
LHS for $\sigma_1$	$r(r-1)d^2$	$2(r-1)(n-r-1)$	$(r-1)(n-r-2)$	$r(r-1)$	$2(r-1)(2n-2r-1)$	$2(r-1)(2n-2r-3)$
Inequality for $\sigma_1$	$r \geq 3$ (equality for $r = 2$ )	$r \geq 4$ if $n - 2r = 0, 1$ $r \geq 3$ if $n - 2r \geq 2$	$r \geq 5$ (equality $r = 4$ ) if $n = 2r$ $r \geq 4$ if $n - 2r = 1, 2$ $r \geq 3$ if $n - 2r \geq 3$	$r \geq 4$ (equality $r = 3$ )	$r \geq 4$ if $n = 2r$ $r \geq 3$ if $n - 2r \geq 1$	$r \geq 4$ if $n - 2r = 0, 1$ $r \geq 3$ if $n - 2r \geq 2$
$D_{\sigma_2}$	$2/r$	1	$1/2$ 1	1	1	1
$M_2$	$2(r-2)d^2$	$2(n-r)(r-2)$	$(r-1)(r-2)/2$ $(r-2)(n-r-1)$	$(r-2)(r+1)$	$2(r-2)(2n-2r+1)$	$2(r-2)(2n-2r-1)$
LHS for $\sigma_2$	$r(r-2)d^2$	$2(r-2)(n-r)$	$(r-2)(n-r-1)$	$(r-2)(r+1)$	$2(r-2)(2n-2r+1)$	$2(r-2)(2n-2r-1)$
Inequality for $\sigma_2$	$r \geq 4$ (equality $r = 3$ )	$r \geq 4$	$r \geq 5$ (equality $r = 4$ ) if $n = 2r$ , $r \geq 4$ otherwise	$r \geq 4$	$r \geq 4$	$r \geq 4$

Table 2.3: Computation of  $Q_\gamma$  and  $\sum_{\alpha \in \Phi_{n\sigma_i > 0} \cap (\Phi_{\Sigma \setminus \{\gamma\}})^+} n_{\sigma_i}(\alpha) m(\alpha)$

We rewrite the inequality appearing in Proposition 2.4.1 as follows:

$$D_{\sigma_i}^{-1} \sum_{\alpha \in \Phi_{n\sigma_i > 0} \cap (\Phi_{\Sigma \setminus \{\gamma\}})^+} n_{\sigma_i}(\alpha) m(\alpha) > Q_\gamma.$$

In Table 2.3, we compute the two sides of this inequality using the multiplicities appearing in Table 2.2 for the choices of  $\gamma, \sigma_1$  and  $\sigma_2$  made in Table 2.1. More precisely, we first compute the dimension  $Q_\gamma = \text{Hausdim}(U_\gamma, |\cdot|_\gamma) = \sum_{\alpha \in \Phi_{n\gamma > 0}} n_\gamma(\alpha) m(\alpha)$  and then the term appearing in the left hand side  $M_i = \sum_{\alpha \in \Phi_{n\sigma_i > 0} \cap (\Phi_{\Sigma \setminus \{\gamma\}})^+} n_{\sigma_i}(\alpha) m(\alpha)$  for our choices of roots  $\sigma_1$  and  $\sigma_2$ . The line called LHS for  $\sigma_i$  contains the term  $D_{\sigma_i}^{-1} M_i = D_{\sigma_i}^{-1} \sum_{\alpha \in \Phi_{n\sigma_i > 0} \cap (\Phi_{\Sigma \setminus \{\gamma\}})^+} n_{\sigma_i}(\alpha) m(\alpha)$ . For groups preserving some type of form,  $n$  denotes the number of variables of the form and the  $F$ -rank  $r$  of the group coincides with its Witt index. In the line "Inequality for  $\sigma_i$ " we recover the conditions on  $r$  so that the condition in Proposition 2.4.1 is satisfied for  $\sigma_1$  or  $\sigma_2$ . Items having two lines correspond to quantities that change depending on the root system of the group (the only line that does not follow this is the line "Inequality for  $\sigma_i$ " where conditions on the rank are slightly more complicated).

*Proof of theorem 2.4.2.* By looking at Table 2.3, we see that the inequality of Proposition 2.4.1 is satisfied for both roots  $\sigma_1$  and  $\sigma_2$  for  $r \geq 4$  in all cases, except for  $SO(q)$  where  $q$  is a quadratic form in 8 variables and Witt index 4. For  $G = SL(4, F)$ , there is strict inequality for  $\sigma_1$  and equality for  $\sigma_2$ . Hence we can apply Theorem 2.2.9 and conclude using Corollary 2.1.6.  $\square$

### 2.4.3 Uniform vanishing for admissible simple real Lie groups

In [BR20], Bourdon and Rémy obtain vanishing of  $L^p$ -cohomology of many real simple Lie groups in many degrees, for values of  $p$  depending on the degree in question. The groups for which their result applies are the simple Lie groups for which there exists some maximal parabolic subgroup such that its solvable radical

is isometric to some real hyperbolic space of dimension  $d$ , for some  $d \geq 2$ . They call these groups *admissible*. We state their vanishing result for degree 2.

**Theorem 2.4.3.** [BR20, Theorem 1.4] *Let  $G$  be an admissible simple real Lie group. Let  $d$  be the dimension of the real hyperbolic space associated to  $G$ . Then*

$$H_{\text{ct}}^2(G, L^p(G)) = \{0\} \text{ for } p \leq \frac{d-1}{2}.$$

Their result gives vanishing of the second  $L^p$ -cohomology group for small values of  $p > 1$ . On the other hand, when  $F = \mathbb{R}$  or  $\mathbb{C}$ , Theorem 2.4.2 gives vanishing of the second  $L^p$ -cohomology group for large values of  $p$ . In this section, we address the question of when these two results combined give vanishing for all  $p > 1$ .

In combinatorial terms, the admissibility condition amounts to ask that there exists some simple root  $\sigma$  such that the coefficients  $n_\sigma(\alpha)$  are 1 for all roots  $\alpha \in \Phi_{n_\sigma > 0}$ . Such a simple root is called a *good* root. In this case the group  $H_\sigma$  is the real hyperbolic space of dimension  $d$ , with  $d-1 = Q_\sigma$ . Then the previous cited theorem gives vanishing for  $p \leq \max\{Q_\sigma, \sigma \text{ good root}\}/2$ . Theorem 2.4.2 gives the desired vanishing for  $p > Q_\gamma$ , where  $\gamma$  is our choice of simple root, as in Table 2.1. Then the condition we need to guarantee that  $H_{\text{ct}}^2(G, L^p(G)) = \{0\}$  for all  $p > 1$  is

$$Q_\gamma \leq \max\{Q_\sigma, \sigma \text{ good root}\}/2.$$

Our choice of simple root  $\gamma$  was made so that  $Q_\gamma$  is small, in particular in all our choices  $Q_\gamma$  grows linearly with the rank  $r$  of the group  $G$ . The previous inequality has chances to be satisfied for many groups of large rank, since there exists often some good root  $\sigma$  such that  $Q_\sigma$  grows quadratically with  $r$ . More precisely, such a root exists in the root systems  $A_r$ ,  $C_r$  and  $D_r$  but not in  $B_r$ . The following corollary is obtained using the computation of  $Q_\gamma$  present in Table 2.3 and the computation of  $\max\{Q_\sigma, \sigma \text{ good root}\}$  that can be found in the tables [BR20, p. 1319 and 1320].

**Corollary 2.4.4.** *Let  $G$  be one of the following admissible simple Lie groups:  $\text{SL}_r(\mathbb{R})$ ,  $\text{SL}_r(\mathbb{C})$ ,  $\text{SL}_r(\mathbb{H})$ ,  $\text{Sp}_{2r}(\mathbb{R})$ ,  $\text{Sp}_{2r}(\mathbb{C})$  with  $r \geq 7$ , or  $\text{SU}_{r,r}(\mathbb{R})$ ,  $\text{Sp}_{2r,2r}(\mathbb{R})$ ,  $\text{SO}_{r,r}(\mathbb{R})$ ,  $\text{SO}_{2r}(\mathbb{C})$ ,  $\text{SO}^*(4r)$  with  $r \geq 8$ . Then:*

$$H_{\text{ct}}^2(G, L^p(G)) = \{0\} \text{ for all } p > 1.$$

*Remark.* The only classical families of admissible simple real Lie groups missing in this corollary are  $\text{SO}_{2r+1}(\mathbb{C})$  and  $\text{SO}_{r,n-r}(\mathbb{R})$  with  $n > 2r$ . These are the admissible groups with (restricted) root system  $B_r$ . We cannot obtain vanishing for all  $p > 1$  for these groups by complementing our results with those from [BR20] because our choice of  $\gamma$  is in fact the only good root in the root system  $B_r$  and it is not possible to have  $Q_\gamma \leq Q_\sigma/2$ .

## Chapter 3

# Top degree $\ell^p$ -cohomology and conformal dimension of buildings

### Introduction

Buildings were introduced by Tits first as incidence geometries allowing to show simplicity of some families of groups. Later generalizations turned them into non-Archimedean analogues of symmetric spaces. A building is obtained by patching together under some incidence conditions, many copies of a same simplicial complex, which is constructed from a Coxeter system and that we call its abstract Coxeter complex.

Buildings may be viewed as metric spaces, and as such we may study some of their quasi-isometry invariants. Here we are interested in their  $\ell^p$ -cohomology. Many variants of  $\ell^p$ -cohomology exist today: de Rham  $L^p$ -cohomology, simplicial  $\ell^p$ -cohomology or (continuous) group  $\ell^p$ -cohomology are some of them. These are quasi-isometry invariants popularized by Gromov in [Gro93] and intensively studied in hyperbolic settings.

A first intuition on  $\ell^p$ -cohomology is that its dependence on  $p$  should behave like the dependence on  $\alpha$  of  $\alpha$ -Hausdorff measures for a given metric space. Indeed, the first  $\ell^p$ -cohomology group increases as  $p$  grows to infinity, thus there exists a critical  $p$  for which it starts to be nonzero. This critical exponent is a numerical quasi-isometry invariant first introduced by Pansu [Pan89a] which can be thought of as some sort of Hausdorff dimension. In higher degrees the situation may be more subtle, as suggested by the description of the vanishings of  $L^p$ -cohomology of the real hyperbolic space  $\mathbb{H}^n$  for  $n \geq 3$  [Pan08].

We are interested in top dimensional simplicial  $\ell^p$ -cohomology of buildings. In maximal degree, it is easier to study simplicial  $\ell^p$ -homology than  $\ell^p$ -cohomology, because the former is not a quotient space. Then duality allows us to recover reduced  $\ell^p$ -cohomology. Top degree  $\ell^p$ -homology has the same behaviour as the first  $\ell^p$ -cohomology group: it increases with  $p$  so there is again a critical exponent for which this space starts to be nonzero. In the case where the Davis realization  $\Sigma_D$  of the Coxeter complex of the building is an orientable pseudomanifold, we compute this critical exponent in terms of the combinatorial data describing the building: its Weyl group  $(W, S)$  and its thickness vector  $\mathbf{q} + 1$ . For  $w \in W$ , let  $q_w$  be the number

of galleries of type  $w$  starting from a given chamber. We call

$$e_{\mathbf{q}}(W) = \limsup_n \frac{1}{n} \log |\{w \in W, q_w \leq e^n\}|$$

the weighted growth rate of combinatorial balls in the building  $X$  (see Section 2.3).

**Theorem 3.0.1.** *(See Theorem 3.2.6 in the text) Let  $(W, S)$  be a Coxeter system such that  $\Sigma_D(W, S)$  is an orientable pseudomanifold. Let  $X_D$  be the Davis realization of a regular building of type  $(W, S)$  and thickness vector  $\mathbf{q} + 1$ , with  $\mathbf{q} \geq 2$ . Let  $n = \dim X_D$ . Then we have:*

$$\begin{aligned} 1 + e_{\mathbf{q}}(W) &= \inf\{p > 1 \mid \ell^p H_n(X_D) \neq 0\}, \\ 1 + e_{\mathbf{q}}(W)^{-1} &= \sup\{p > 1 \mid \ell^p \overline{H}^n(X_D) \neq 0\}. \end{aligned}$$

This result is an extension to  $\ell^p$ -cohomology of [Dym04], with essentially the same proof (the main idea can be traced back at least to [Gro93, p. 221]). We obtain two corollaries from it. The first concerns  $\ell^p$ -cohomology of affine buildings and semisimple Lie groups over non-Archimedean local fields.

**Corollary 3.0.2.** *(See Corollary 3.2.7.1 in the text) Let  $X$  be an affine building of dimension  $n$  and finite thickness. We have  $\ell^p \overline{H}^n(X) \neq \{0\}$  for all  $p > 1$ . In particular, any semisimple Lie group  $G$  of rank  $n$  over a non-Archimedean local field satisfies  $\overline{H}_{\text{ct}}^n(G, L^p(G)) \neq \{0\}$  for all  $p > 1$ .*

In [Gro93, 253], Gromov asked: for a semisimple Lie group  $G$  of rank  $n$  over a local field, do we have  $H_{\text{ct}}^n(G, L^p(G)) \neq \{0\}$  at least for some  $1 < p < \infty$  (where  $L^p(G)$  denotes the right regular representation)? In the real case, this non-vanishing is shown for large  $p > 1$  in [BR21], the behaviour of  $H_{\text{ct}}^n(G, L^p(G))$  when  $p > 1$  is close to 1 is still unknown. In the non-Archimedean case, Gromov showed that  $\overline{H}_{\text{ct}}^n(G, L^p(G)) \neq \{0\}$  for  $1 < p \leq 2$  [Gro93, p. 255] and expected this non-vanishing to hold for all  $1 < p < \infty$ , which is the content of the previous corollary.

Our second corollary is the description of the set of  $p$ 's for which the second  $\ell^p$ -cohomology group of a cocompact Fuchsian building  $X$  vanishes. The same question for the first  $\ell^p$ -cohomology group was already solved in [BP03]: the infimal  $p$  for which this space starts to be nonzero is the conformal dimension  $\text{Confdim}(\partial X)$  of the boundary  $\partial X$  (for the definition of  $\text{Confdim}(\partial X)$  see 3.4.2). Our theorem implies that this is still the case for the second  $\ell^p$ -cohomology group. In [Bou00], Bourdon shows that for these buildings one has  $\text{Confdim}(\partial X) = 1 + e_{\mathbf{q}}(W)^{-1}$ . Moreover [Bou16a], for  $p > \text{Confdim}(\partial X)$ , we have  $\ell^p H^2(X) = 0$ . By putting all of these results together, we obtain the following description of vanishings of  $\ell^p$ -cohomology of cocompact Fuchsian buildings in terms of  $p$  in degrees 1 and 2.

**Corollary 3.0.3.** *(See Corollary 3.2.8 in the text) Let  $X$  be the Davis realization of a cocompact Fuchsian building of type  $(W, S)$  and thickness vector  $\mathbf{q} + 1$ , with  $\mathbf{q} \geq 2$ .*

- *For  $p < \text{Confdim}(\partial X) = 1 + e_{\mathbf{q}}(W)^{-1}$ , we have  $\ell^p H^1(X) = 0$  and  $\ell^p \overline{H}^2(X) \neq 0$ .*
- *For  $p > \text{Confdim}(\partial X) = 1 + e_{\mathbf{q}}(W)^{-1}$ , we have  $\ell^p H^1(X) \neq 0$  and  $\ell^p H^2(X) = 0$ .*

The critical exponent for  $\ell^p$ -homology  $1 + e_{\mathbf{q}}(W)$  is finite when  $\mathbf{q} \geq 2$ , so in top dimension we have non-vanishing of  $\ell^p$ -homology for large  $p$ . We may ask if

this holds without the pseudomanifold assumption on  $\Sigma_D$ . The problem is that the dimension of the Davis realization has no direct relation with cohomology and there are examples where we have uniform vanishing for all  $p > 1$  in degree  $\dim \Sigma_D$ . This happens because of the local topology of the Davis chamber. This local problem is solved by considering a different realization  $\Sigma_B$ , called the Bestvina realization, whose dimension is always equal to the virtual cohomological dimension  $\text{vcd}_{\mathbb{R}}(W)$  of  $W$  over  $\mathbb{R}$  (which may be strictly smaller than the dimension of  $\Sigma_D$ ). The Bestvina realization has poor metric properties but it is well suited for cohomological computations. We show non-vanishing of  $\ell^p$ -homology in degree  $\dim X_B = \text{vcd}_{\mathbb{R}}(W)$  for large  $p$ .

**Theorem 3.0.4.** *(See Theorem 3.3.7 in the text) Let  $X_B$  be the Bestvina realization of a regular building of type  $(W, S)$  and thickness vector  $\mathbf{q} + 1$ , with  $\mathbf{q} \geq 2$ . Let  $d = \dim X_B = \text{vcd}_{\mathbb{R}}(W)$ . For all  $p > 1 + e_{\mathbf{q}}(W)$ , we have  $\ell^p H_d(X_B) \neq 0$ .*

The Davis and Bestvina realizations are quasi-isometric, so this non-vanishing result also holds for the Davis realization. We do not know how to compute the infimal  $p$  for which top degree  $\ell^p$ -homology starts to be nonzero in this more general setting.

These ideas and particularly the formula of the critical exponent in top dimension for  $\ell^p$ -cohomology led us to a further study of conformal dimension of Gromov-hyperbolic buildings. Few is known about these conformal dimensions. The only known exact computation is  $\text{Confdim}(\partial X) = 1 + e_{\mathbf{q}}(W)^{-1}$  [Bou00] for a Fuchsian building  $X$  of type  $(W, S)$ . Our previous non-vanishing result can be compared with [Bou16a, Théorème A], to obtain the inequality:

$$\frac{\text{Confdim}(\partial X)}{\text{vcd}_{\mathbb{R}}(W) - 1} \geq 1 + e_{\mathbf{q}}(W)^{-1}.$$

We may compare this to some bounds obtained by Clais for the conformal dimension of hyperbolic buildings arising from right-angled Coxeter groups [Cla17]. For these buildings, Clais' lower bound of  $\text{Confdim}(\partial X)$  is sharper than our inequality obtained using top-dimensional  $\ell^p$ -cohomology. We adapt some of our techniques to the first  $\ell^p$ -cohomology groups and, combined with a result from [BK15] relating conformal dimension and the first  $\ell^p$ -cohomology groups, we generalize Clais' bounds to arbitrary Gromov-hyperbolic buildings.

**Theorem 3.0.5.** *(See Theorem 3.4.8 in the text) Let  $(W, S)$  be a Gromov-hyperbolic Coxeter system,  $\Sigma$  the Davis complex of  $(W, S)$  and  $X$  the Davis realization of a building with Weyl group  $(W, S)$  and thickness vector  $\mathbf{q} + 1$ . Let  $d_{\mathbf{q}}$  be a visual metric on  $\partial \Sigma$  induced by a combinatorial distance  $|\cdot - \cdot|_{\mathbf{q}}$  on the dual graph of  $X$ . We have:*

$$\text{Confdim}(\partial \Sigma)(1 + e_{\mathbf{q}}(W)^{-1}) \leq \text{Confdim}(\partial X) \leq \text{Hausdim}(\partial \Sigma, d_{\mathbf{q}})(1 + e_{\mathbf{q}}(W)^{-1}).$$

The main idea that holds the different sections together is the following: there are natural maps  $\rho$  retracting a building  $X$  onto its corresponding Coxeter complex  $\Sigma$ . Naturally, there is a pushforward  $\rho_* : C_k^{\text{lf}}(X) \rightarrow C_k^{\text{lf}}(\Sigma)$  sending formal chains from the building to the Coxeter complex. The point is that we can also define a pullback  $\rho^* : C_k^{\text{lf}}(\Sigma) \rightarrow C_k^{\text{lf}}(X)$  on formal chains that commutes with the boundary operator.

This is done using a harmonicity formula, which is basically a dual version of the Steinberg representation. Then we use convexity estimates (in particular Jensen's inequality) and growth rates to decide on finiteness of  $\ell^p$ -norms.

Similar ideas can be applied to the first  $\ell^p$ -cohomology group. Indeed, cochains can be seen as functions  $f : X^{(0)} \rightarrow \mathbb{R}$  on vertices of  $X$  whose simplicial differential is  $\ell^p$ . Here the pullback  $\rho^*$  is naturally defined as precomposition by  $\rho$  and pushforward  $\rho_*$  of functions is defined by taking averages on preimages of  $\rho$ . The main question here is: given  $f : \Sigma \rightarrow \mathbb{R}$  with a control on the infimal  $p > 1$  such that  $\|df\|_p < \infty$ , what can be said about an  $r > 1$  such that

$$\|d(\rho^* f)\|_r^r = \sum_{\sigma \in \Sigma^{(1)}} |\rho^{-1}(\sigma)| |df(\sigma)|^r < \infty?$$

Again, convexity gives a partial answer to this question. More precisely, we use a pressure like function inspired on thermodynamical formalism, that yields the lower bound of Theorem 3.0.5.

The article is organized as follows. Section 2 contains most of the preliminaries. We first introduce simplicial  $\ell^p$ -homology and  $\ell^p$ -cohomology. Then we recall some elementary building theory: combinatorial buildings, their geometric realizations and growth rates. We introduce the maps induced on formal chains by a retraction. Finally, we give a brief discussion on the Davis realization of a building. All other three sections are independent from each other. Section 3 introduces the family of buildings whose apartments are orientable pseudomanifolds and contains the computation of the critical exponent on top dimensional  $\ell^p$ -homology. Section 4 reviews [Bes93] in order to give an  $\mathbb{R}$ -acyclic version of the Bestvina chamber. Then we show non-vanishing of  $\ell^p$ -homology in top degree for the Bestvina realization. Section 5 first introduces the notion of conformal dimension of Gromov-hyperbolic spaces and discusses its connection with  $\ell^p$ -cohomology. Subsequently we prove our generalization of Clais' bounds.

**Conventions.** In what follows  $p$  is a real number  $> 1$  and  $(W, S)$  denotes a Coxeter system where  $W$  is infinite and  $S$  is finite. An affine Coxeter system is the direct product of finite Coxeter systems and at least one irreducible affine Coxeter system. Any other infinite Coxeter system is said to be non-affine. By the strong Tits alternative for Coxeter groups, this is the same as saying that affine Coxeter systems are infinite Coxeter systems of polynomial growth and non-affine Coxeter systems are those of exponential growth.

## 3.1 $\ell^p$ -cohomology and buildings

We first introduce  $\ell^p$ -homology and  $\ell^p$ -cohomology for complexes of bounded geometry and review some fundamental properties. We then review some combinatorial theory of buildings and adopt a uniform point of view on their geometric realizations (as presented in [Dav08, Chapter 18]).

### 3.1.1 $\ell^p$ -homology and $\ell^p$ -cohomology

We define  $\ell^p$ -homology and  $\ell^p$ -cohomology as presented in [Bou16b].

First, let  $X$  be a simplicial complex equipped with a metric so that it becomes a length space. We say that such a complex  $X$  has *bounded geometry* if:

- (i) there exists a constant  $C > 0$  such that every simplex of  $X$  has diameter  $\leq C$ ,
- (ii) there is a function  $N : [0, \infty[ \rightarrow \mathbb{N}$  such that for all  $r > 0$ , every ball of radius  $r$  contains at most  $N(r)$  simplices of  $X$ .

Now we define  $\ell^p$ -homology and  $\ell^p$ -cohomology for complexes of bounded geometry. For  $k \in \mathbb{N}$ , we denote by  $X^{(k)}$  the set of  $k$ -simplices of  $X$ . For  $1 < p < \infty$ , we define:

$$\ell^p C_k(X) = \left\{ \sum_{\sigma \in X^{(k)}} a_\sigma \sigma, (a_\sigma)_{\sigma \in X^{(k)}} \in \ell^p(X^{(k)}) \right\},$$

$$\ell^p C^k(X) = \{ \omega : X^{(k)} \rightarrow \mathbb{R}, \omega \in \ell^p(X^{(k)}) \}.$$

The standard boundary operators  $\partial_k$  and  $d_k$  are defined as usual (after choosing an ordering of the vertices of  $X$  or equivalently an orientation on simplices of  $X$ ). Because the complex  $X$  has bounded geometry, they define bounded operators:

$$\partial_k : \ell^p C_k(X) \rightarrow \ell^p C_{k-1}(X), \quad d_k : \ell^p C^k(X) \rightarrow \ell^p C^{k+1}(X).$$

These two operators are related by the simplicial version of Stokes' theorem: for  $c \in \ell^p C^k(X)$  and  $\sigma \in \ell^p C_{k+1}(X)$  we have  $d_k c(\sigma) = c(\partial_{k+1} \sigma)$ .

**Definition 3.1.1.** Denote  $\ell^p Z_k(X) := \ker \partial_k$  and  $\ell^p B_k(X) := \text{Im } \partial_{k+1}$ . The  $k$ -th  $\ell^p$ -homology group of  $X$  (resp.  $k$ -th reduced  $\ell^p$ -homology group) is the space:

$$\ell^p H_k(X) := \ell^p Z_k(X) / \ell^p B_k(X) \quad (\text{resp. } \ell^p \overline{H}_k(X) := \ell^p Z_k(X) / \overline{\ell^p B_k(X)}).$$

Denote  $\ell^p Z^k(X) := \ker d_k$  and  $\ell^p B^k(X) := \text{Im } d_{k-1}$ . The  $k$ -th  $\ell^p$ -cohomology group of  $X$  (resp.  $k$ -th reduced  $\ell^p$ -cohomology group) is the space:

$$\ell^p H^k(X) = \ell^p Z^k(X) / \ell^p B^k(X) \quad (\text{resp. } \ell^p \overline{H}^k(X) = \ell^p Z^k(X) / \overline{\ell^p B^k(X)}).$$

The closures are considered with respect to the topology of the  $\ell^p$ -norm. The quotient spaces are endowed with the corresponding quotient topology. Reduced homology and cohomology groups are thus Banach spaces. The reduced versions are the greatest Hausdorff quotients of their non-reduced versions, thus a non-reduced homology or cohomology group is never Hausdorff unless it is equal to its corresponding reduced version.

If  $X$  is a finite simplicial complex, then these spaces correspond to classical homology and cohomology groups  $H_k(X)$  and  $H^k(X)$ . Thus we obtain new invariants only when the complex  $X$  is non-compact.

We will be mostly interested in the Davis realization of buildings. Since these are always contractible, all of these spaces are trivial for  $k > 0$  if the building has finite thickness and its Weyl group is finite.

We review some elementary but important facts on  $\ell^p$ -homology and cohomology. First we have a duality result relating  $\ell^p$ -homology groups and  $\ell^p$ -cohomology groups but only for the reduced versions [Bou16b].

**Proposition 3.1.2.** For  $p, r \in ]1, +\infty[$  such that  $p^{-1} + r^{-1} = 1$  and  $k \in \mathbb{N}$ , the space  $\ell^p \overline{H}^k(X)$  is canonically isomorphic to the dual of  $\ell^r \overline{H}_k(X)$ . Similarly,  $\ell^r \overline{H}_k(X)$  is canonically isomorphic to the dual of  $\ell^p \overline{H}^k(X)$ .

Second,  $\ell^p$ -cohomology is invariant by quasi-isometries. Before stating the result, we introduce the following condition. A complex of bounded geometry  $(X, d)$  is said to be *uniformly  $\mathbb{R}$ -acyclic* if  $X$  is  $\mathbb{R}$ -acyclic and if there is a function  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that every cycle supported in a ball  $B(x, r)$  is the boundary of a chain supported in  $B(x, \Phi(r))$ . The following result is due to M. Gromov [Gro93], see [BP03] for a complete proof.

**Theorem 3.1.3.** *Let  $X$  and  $Y$  be two uniformly  $\mathbb{R}$ -acyclic complexes of bounded geometry. If  $X$  and  $Y$  are quasi-isometric, then the spaces  $\ell^p H_k(X)$  and  $\ell^p H_k(Y)$  are topologically isomorphic and the spaces  $\ell^p H^k(X)$  and  $\ell^p H^k(Y)$  are topologically isomorphic. This is also true for reduced homology and cohomology groups.*

*Remark.* The original statement needs "uniformly contractible" as a hypothesis, but by inspecting the proof in [BP03] we see uniformly  $\mathbb{R}$ -acyclic is enough. We need to loosen this hypothesis since we will use the Bestvina complex of a Coxeter system, which is constructed a priori only as an  $\mathbb{R}$ -acyclic complex. It can be made contractible, but it may lose the property that its dimension computes a cohomological dimension when the latter is equal to 2 because of the Eilenberg-Ganea problem [Dav08, p. 154], [Bes93].

### 3.1.2 Chamber systems and geometric realizations

We review some basic combinatorial building theory following [Ron89]. Then we define geometric realizations of buildings (following [AB08, Chapter 12] or [Dav08]) and prove an  $\mathbb{R}$ -acyclicity criterion which applies to both the Davis realization and the Bestvina realization of buildings.

A *chamber system* over a set  $S$  is a set  $\mathcal{C}$  together with a family of equivalence relations on  $\mathcal{C}$  indexed by  $S$ . The elements of  $\mathcal{C}$  are called *chambers*. Two chambers are *s-equivalent* if they are equivalent under the relation corresponding to  $s$  and *s-adjacent* if they are  $s$ -equivalent and not equal.

A *gallery*  $\gamma$  in  $\mathcal{C}$  is a finite sequence of chambers  $(c_0, \dots, c_k)$  such that  $c_{i-1}$  is  $s$ -adjacent to  $c_i$  for all  $1 \leq i \leq k$ . The gallery is said to have *type*  $s_1 \dots s_k$  (as a word on the free monoid on the alphabet  $S$ ) if  $c_{i-1}$  is  $s_i$ -adjacent to  $c_i$  for every  $i$ . If each  $s_j$  belongs to a subset  $T \subseteq S$ , then we say  $\gamma$  is a *T-gallery*. A chamber system is *connected* (or *T-connected*) if any two chambers can be joined by a gallery (or *T-gallery*). The *T-connected* components of a chamber system  $\mathcal{C}$  are called its *T-residues*.

**Definition 3.1.4.** Let  $(W, S)$  be a Coxeter system. A (*combinatorial*) *building of type*  $(W, S)$  is a chamber system  $\mathcal{C}$  over  $S$  such that:

- (i) for all  $s \in S$ , each  $s$ -equivalence class contains at least two chambers and,
- (ii) there exists a *W-valued distance function*  $d_W : \mathcal{C} \times \mathcal{C} \rightarrow W$ , that is, a map satisfying that: for each  $w \in W$ , if  $w = s_1 \dots s_k$  is a reduced word for  $w$  in the alphabet  $S$  (that is  $s_i \in S$  for every  $i$  and  $k$  is minimal), then two chambers  $c$  and  $c'$  can be joined by a gallery of type  $s_1 \dots s_k$  if and only if  $d_W(c, c') = w$ .

The group  $W$  is called the *Weyl group* of the building  $\mathcal{C}$ .

**Example 3.1.5.** Let  $(W, S)$  be a Coxeter system. Then the chamber system  $\mathcal{C} = W$ , endowed with the relations  $w \sim_s w'$  when  $w = sw'$  for each  $s \in S$  and the  $W$ -valued

distance  $d_W : W \times W \rightarrow W$  defined by  $d_W(w, w') = w^{-1}w'$ , is a building of type  $(W, S)$  called the *abstract Coxeter complex* of  $W$ .

A building  $\mathcal{C}$  of type  $(W, S)$  has *finite thickness* if for all  $s \in S$ , each  $s$ -equivalence class is finite. In this case, we say that  $\mathcal{C}$  is *regular* if for each  $s \in S$ , each  $s$ -equivalence class has the same number of elements  $q_s + 1$ . In this case we denote by  $\mathbf{q}$  the vector containing the  $q_s$ 's as coordinates. We say  $\mathcal{C}$  has *constant thickness* if all the  $q_s$ 's have the same value  $q$ . In the next sections our buildings will always be regular.

We now define geometric realizations of combinatorial buildings in a relatively general setting. This is because in what follows we will be interested in two realizations of buildings: the well-known Davis realization and the less known Bestvina realization. A *mirror structure* on a CW complex  $K$  consists of an index set  $S$  and a family of subcomplexes  $(K_s)_{s \in S}$ . The subspaces  $K_s$  are the *mirrors* of  $K$ . In this case, we say that  $K$  is a *mirrored CW complex over  $S$* . We set  $K_\emptyset = K$  and for any nonempty subset  $T \subseteq S$ ,

$$K_T = \bigcap_{t \in T} K_t.$$

For  $x \in K$ , we set  $S(x) = \{s \in S, x \in K_s\}$ .

**Definition 3.1.6.** Let  $\mathcal{C}$  be a (combinatorial) building of type  $(W, S)$  and  $K$  a mirrored CW complex over  $S$ . The  $K$ -*realization* of  $\mathcal{C}$  is the space:

$$X_K = (\mathcal{C} \times K) / \sim,$$

where  $[(c, x)] \sim [(c', x')]$  if and only if  $x = x'$  and  $d_W(c, c') \in W_{S(x)}$ . For simplicity, a simplex  $[(c, \sigma)]$  in  $X_K$  may be written  $c.\sigma$  or  $c\sigma$ .

Let  $\mathcal{C}$  be a regular building of type  $(W, S)$  and finite thickness. If  $(K, d)$  is a geodesic metric space, we can extend  $d$  to  $X_K$  by declaring that all translates of  $K$  are isometric and then defining a piecewise length metric on  $X_K$ . If moreover each  $K_s$  is a proper metric space, then  $(X_K, d)$  is a geodesic metric space [AB08, Corollary 12.28]. The space  $(X_K, d)$  has bounded geometry if and only if the mirror structure satisfies that for every subset  $T \subseteq S$  generating an infinite subgroup  $W_T$ , we have  $K_T = \emptyset$ .

**Example 3.1.7.** Let  $\mathcal{C}$  be a building of type  $(W, S)$ . We can choose  $K = \Delta^{|S|-1}$  to be the standard simplex of dimension  $|S| - 1$ . We endow  $\Delta^{|S|-1}$  with a mirror structure by letting  $K_s$  be the codimension 1 faces of  $\Delta^{|S|-1}$ . The space  $X_\Delta = (\mathcal{C} \times K) / \sim$  is called the *simplicial realization* of  $\mathcal{C}$ . If  $(W, S)$  is not affine, this realization is not locally finite most of the time.

Recall that an orientation on a simplex corresponds to a total order on its vertices. If one is given orientations on simplices of  $K$ , one has a natural orientation on all simplices of  $X_K$  by declaring that all translates  $c.\sigma$  of a simplex  $\sigma$  in  $K$  by  $c \in \mathcal{C}$  have the same orientation. Once simplices have an orientation, we can talk of homology for geometric realizations of buildings.

We now give a criterion for the  $K$ -realization of a building to be uniformly  $\mathbb{R}$ -acyclic, starting from data on the mirrored complex  $K$ . When  $K$  is the Davis chamber, the Davis realization  $X$  is known to be  $CAT(0)$ , thus uniformly contractible.

The main motivation to state this result here is to apply it to the Bestvina complex, which is not necessarily contractible. This is the same proof as [Dav08, 8.2.8] but for buildings. First, set  $K^\emptyset = \emptyset$  and for any nonempty subset  $T \subseteq S$ ,

$$K^T = \bigcup_{t \in T} K_t.$$

**Proposition 3.1.8.** *Let  $\mathcal{C}$  be a building of type  $(W, S)$  and finite thickness and let  $X$  be its  $K$ -realization. Suppose that  $X$  is locally finite and that  $K$  is a geodesic metric space. If  $K$  and  $K_T$  are  $\mathbb{R}$ -acyclic for every  $T \subseteq S$  generating a finite parabolic subgroup of  $W$ , then  $X$  is uniformly  $\mathbb{R}$ -acyclic.*

*Proof.* Since  $\mathcal{C}$  has finite thickness, the set of chambers of  $\mathcal{C}$  is countable. We can order them  $c_0, c_1, \dots, c_n, \dots$  so that  $c := c_0$  and  $l(c_k) \leq l(c_{k+1})$ , where  $l(x) = l_S(d_W(c, x))$  is the length of the  $W$ -distance with respect to  $c$ . We set:

$$P_n = \bigcup_{i=0}^n c_i K \subseteq X.$$

We will show that  $P_n$  is  $\mathbb{R}$ -acyclic by induction. Since  $P_n = P_{n-1} \cup c_n K$  and  $c_n K$  is  $\mathbb{R}$ -acyclic by hypothesis, it is enough to show that the intersection  $P_{n-1} \cap c_n K$  is  $\mathbb{R}$ -acyclic.

The main step is showing the building analogue of [Dav08, 8.1.1]. For  $w \in W$ , we set  $\text{In}(w) = \{s \in S, l(ws) < l(w)\}$ . For all  $i \geq 0$ , set  $w_i = d_W(c, c_i)$ . Our goal is to show the equality  $P_{n-1} \cap c_n K = c_n K^{\text{In}(w_n)}$ . Let  $\Sigma$  be an apartment containing both  $c$  and  $c_n$ . The inclusion  $P_{n-1} \cap c_n K \supseteq c_n K^{\text{In}(w_n)}$  follows from [Dav08, 8.1.1] applied to the apartment  $\Sigma$ . For the converse inclusion, we show that for all  $i < n$ , we have  $c_i K \cap c_n K \subseteq c_n K^{\text{In}(w_n)}$ . Indeed, let  $\rho : X \rightarrow \Sigma$  be a retraction onto the apartment  $\Sigma$ , which we identify with the  $K$ -realization of the abstract Coxeter complex of  $(W, S)$ . Since  $c_n K$  is contained in  $\Sigma$ , the set  $c_i K \cap c_n K$  is fixed under  $\rho$ . Thus:

$$\begin{aligned} c_i K \cap c_n K &= \rho(c_i K \cap c_n K) \\ &\subseteq \rho(c_i K) \cap \rho(c_n K) \\ &= w_i K \cap w_n K. \end{aligned}$$

Now,  $l(c_i) \leq l(c_n)$  means that  $l(w_i) \leq l(w_n)$  and the proof of [Dav08, 8.1.1] shows that  $w_i K \cap w_n K \subseteq w_n K^{\text{In}(w_n)} = c_n K^{\text{In}(w_n)}$ . This shows the converse inclusion and thus we have that  $P_{n-1} \cap c_n K = c_n K^{\text{In}(w_n)}$ .

Since  $W_{\text{In}(w)}$  is a finite parabolic subgroup of  $W$  [Dav08, 4.7.2], our hypothesis implies that  $c_n K^{\text{In}(w_n)}$  is  $\mathbb{R}$ -acyclic. Indeed, if  $W_T$  is a finite parabolic subgroup, then  $K^T$  is the union of  $\mathbb{R}$ -acyclic spaces (the  $K_t$ 's for  $t \in T$ ) whose intersections are all  $\mathbb{R}$ -acyclic (the  $K_{T'}$ 's for  $T' \subseteq T$ ) by hypothesis.

Thus Mayer Vietoris says that  $H_*(P_n, \mathbb{R}) = H_*(P_{n-1}, \mathbb{R})$  and by induction  $H_*(P_n, \mathbb{R}) = H_*(cK, \mathbb{R}) = 0$ , so  $P_n$  is  $\mathbb{R}$ -acyclic for all  $n$ . Since  $X$  is the increasing union of the  $P_n$ , the space  $X$  is  $\mathbb{R}$ -acyclic.

Now suppose that  $K$  is a geodesic metric space and endow  $X$  with the corresponding piecewise length metric. The space  $X$  is uniformly  $\mathbb{R}$ -acyclic since any

cycle supported in a ball  $B(x, r)$  is contained, up to changing the chamber  $c$  so that  $x \in cK$ , in a certain  $P_n$ , which is  $\mathbb{R}$ -acyclic. This  $P_n$  can be chosen so that  $P_n \subseteq B(x, Cr + D)$  where  $C, D > 0$  are two constants. Since every chamber is an isometric copy of  $K$ , these constants may depend on  $r$  but are independent on the center  $x \in X$ .  $\square$

We can define locally finite  $k$ -chains on  $X_K$  as (possibly infinite) formal sums of  $k$ -simplices in  $X_K$  with coefficients in  $\mathbb{R}$ , such that for every compact  $C$  in  $X_K$  only finitely many simplices intersecting  $C$  have non-zero coefficients. Denote by  $C_k^{\text{lf}}(X_K; \mathbb{R})$  their space and define the boundary operators as usual. Notice that the boundary operators only act on the second variable, so if  $c \cdot \sigma$  is a  $k$ -simplex in  $X_K$ , we have:

$$\partial_k(c \cdot \sigma) = c \cdot (\partial_k \sigma).$$

The *locally finite homology*  $H_k^{\text{lf}}(X_K, \mathbb{R})$  of  $X_K$  is the homology of the corresponding complex. In practice, our choice of  $K$  guarantees that the complex  $X_K$  is locally finite, so we can take sums over all simplices in  $X_K$  with no restrictions on the coefficients.

### 3.1.3 Growth function and growth rate

We come back to a slightly more general setting to define the growth function of a Coxeter system  $(W, S)$ . Its corresponding growth rate is a numerical invariant that we will be compare to other numerical invariants.

For  $w \in W$  denote by  $\bar{w}$  the conjugacy class of  $w$ . Let  $\bar{S} = \{\bar{s}, s \in S\}$  and let  $j : \bar{S} \rightarrow \{1, \dots, N\}$  be a bijection. For each  $\bar{s} \in \bar{S}$  introduce a variable  $t_{j(\bar{s})}$ . Let  $w \in W$  and choose a reduced expression  $s_1 \dots s_k$  of  $w$ . The monomial

$$t_w = t_{j(\bar{s}_1)} \dots t_{j(\bar{s}_k)}$$

is independent of the reduced expression  $s_1 \dots s_k$  of  $w$ . This follows from their characterization as minimal galleries from 1 to  $w$ . Indeed, a minimal gallery from 1 to  $w$  crosses the walls separating 1 and  $w$  exactly once, and only the order in which these walls are crossed depends on the minimal gallery.

**Definition 3.1.9.** The *growth function*  $W(\mathbf{t})$  of  $(W, S)$  is the series:

$$W(\mathbf{t}) = W(t_1, \dots, t_N) = \sum_{w \in W} t_w.$$

For  $x \in \mathbb{R}$ , we denote by  $\mathbf{t}^x$  the vector  $(t_1^x, \dots, t_N^x)$ . For a fixed  $\mathbf{t} \in \mathbb{R}^N$ , we study the convergence of the series  $W(\mathbf{t}^{-x})$  in function of  $x$ .

**Proposition 3.1.10.** *The series  $W(\mathbf{t}^{-x})$  converges if  $x > e_{\mathbf{t}}(W)$  and diverges if  $x < e_{\mathbf{t}}(W)$ , where*

$$e_{\mathbf{t}}(W) = \limsup_n \frac{1}{n} \log |\{w \in W, t_w \leq e^n\}|.$$

*Proof.* Set  $Q_n(\mathbf{t}) = \{w \in W, t_w \leq e^n\}$ . The power series  $\sum_n |Q_n(\mathbf{t})|e^{-xn}$  converges for  $x > e_{\mathbf{t}}(W)$  and diverges for  $x < e_{\mathbf{t}}(W)$ . On the other hand we have:

$$\sum_n |Q_n(\mathbf{t})|e^{-xn} = \sum_{w \in W} \sum_{\substack{n \in \mathbb{N} \\ e^n \geq t_w}} e^{-xn}$$

and the term  $\sum_{e^n \geq t_w} e^{-xn}$  is equivalent to  $(t_w)^{-x} \frac{1}{1-e^{-x}}$ , so the previous series converges if and only if  $W(\mathbf{t}^{-x})$  converges.  $\square$

*Remark.* If the  $t_i$ 's are all equal to some real number  $t$ , then the series  $W(t)$  depends only on one variable and may be expressed as

$$W(t) = \sum_{k \geq 0} c_k(W)t^k,$$

where  $c_k(W) = |\{w \in W, l(w) = k\}|$ . In this case the radius of convergence of the series  $W(t^{-x})$  is given by

$$e_t(W) = \frac{1}{\log t} \limsup_n \frac{1}{n} \log c_n(W).$$

The number  $e(W) = \limsup_n \frac{1}{n} \log c_n(W)$  is also called the *growth rate* of  $(W, S)$ . Since  $W$  is finitely generated,  $e(W)$  is always finite.

In practice, if we consider a regular building of finite thickness  $\mathbf{q} + 1$ , the vector  $\mathbf{t}$  will be  $\mathbf{q}$ . The numbers  $e_{\mathbf{q}}(W)$  correspond to growth rates of balls in the building for a combinatorial metric taking into account the thickness  $\mathbf{q} + 1$ .

If  $\mathbf{t} = (t_1, \dots, t_N)$  and  $\mathbf{t}' = (t'_1, \dots, t'_N)$  are two vectors with  $1 \leq t_i < t'_i$  for all  $i$ , then  $e_{\mathbf{t}}(W) \geq e_{\mathbf{t}'}(W)$ . Indeed, there exists  $\alpha > 1$  such that  $t_i^\alpha < t'_i$  for all  $i$ . Then:

$$Q_n(\mathbf{t}') \subseteq Q_n(\mathbf{t}^\alpha) \subseteq Q_n(\mathbf{t})$$

Since  $q_w^\alpha \leq e^n \iff q_w \leq e^{n/\alpha}$ , we have that  $Q_n(\mathbf{t}^\alpha) = Q_{n/\alpha}(\mathbf{t})$ . Thus:

$$e_{\mathbf{t}'}(W) \leq e_{\mathbf{t}^\alpha}(W) = \frac{1}{\alpha} e_{\mathbf{t}}(W) < e_{\mathbf{t}}(W).$$

This implies for instance that if  $\mathbf{t} > 1$ , then  $e_{\mathbf{t}}(W)$  is comparable to  $e(W)$ . Indeed, if  $t_{\min} = \min_{1 \leq i \leq N} t_i$  and  $t_{\max} = \max_{1 \leq i \leq N} t_i$ , then we have:

$$\frac{1}{\log t_{\max}} e(W) \leq e_{\mathbf{t}}(W) \leq \frac{1}{\log t_{\min}} e(W).$$

In particular, the growth rate  $e_{\mathbf{t}}(W)$  is finite for  $\mathbf{t} > 1$ .

### 3.1.4 Retractions

We now introduce our main tool: retractions. These are maps first defined at a combinatorial level, which allow to extend  $W$ -invariant metrics from a Coxeter complex to the whole building. We will use the maps they induce at a homological level.

Let  $\mathcal{C}$  be a regular building of type  $(W, S)$  and choose a chamber  $c \in \mathcal{C}$ . The map  $\rho_c : \mathcal{C} \rightarrow W$  defined by  $\rho_c(c') = d_W(c, c')$  is the *retraction onto the abstract Coxeter complex of  $W$  centered in  $c$* . Retractions are naturally defined on  $K$ -realizations of  $\mathcal{C}$  by acting on the first variable. The set

$$S(c, w) = \rho_c^{-1}(w) = \{c' \in \mathcal{C}, d_W(c, c') = w\}$$

is called the  *$W$ -sphere of radius  $w$  centered in  $c$* .

Since  $\mathcal{C}$  is assumed to be regular, denote by  $q_s + 1$  the cardinal of  $s$ -equivalence classes for each  $s \in S$ . If  $\bar{s}_i = \bar{s}_j$  then  $q_{s_i} = q_{s_j}$  [Dav08, 18.1.16]. Denote by  $\mathbf{q}$  the vector containing the  $q_s$ 's. For  $w \in W$ , let  $w = s_1 \dots s_k$  be a reduced expression for  $w$  as a word in  $S$ . By the same argument as in the previous section, the number

$$q_w = q_{s_1} \dots q_{s_k}$$

is independent of the decomposition  $w = s_1 \dots s_k$ . If  $\mathcal{C}$  has constant thickness equal to  $q + 1$ , we have  $q_w = q^{l(w)}$ .

**Lemma 3.1.11.** [Dav08, 18.1.17] *For all  $w \in W$ , we have  $|S(c, w)| = q_w$ .*

Let  $(W, S)$  be a Coxeter system. Let  $X$  be the  $K$ -realization of a building  $\mathcal{C}$  of type  $(W, S)$  and  $\Sigma$  the  $K$ -realization of the Coxeter complex of  $(W, S)$ . Choose an orientation on simplices of  $K$  so that all simplices in  $X$  and  $\Sigma$  have an orientation.

A retraction  $\rho : X \rightarrow \Sigma$  centered on a chamber defines two maps between chains on  $X$  and chains on  $\Sigma$ . First, we can consider the *pushforward of  $\rho$* , retracting chains on  $X$  onto chains on  $\Sigma$ :

$$\begin{aligned} \rho_* : C_k^{\text{lf}}(X) &\rightarrow C_k^{\text{lf}}(\Sigma) \\ (c, \sigma) &\mapsto (\rho(c), \sigma) \end{aligned}$$

Second, we have the *pullback of  $\rho$* , extending chains on  $\Sigma$  into chains on  $X$ :

$$\begin{aligned} \rho^* : C_k^{\text{lf}}(\Sigma) &\rightarrow C_k^{\text{lf}}(X) \\ (w, \sigma) &\mapsto \sum_{c' \in \rho^{-1}(w)} \frac{1}{q_w} (c', \sigma) \end{aligned}$$

Notice that  $\rho_* \circ \rho^* = \text{Id}_{C_k^{\text{lf}}(\Sigma)}$  and that  $\rho^* \circ \rho_*$  corresponds to averaging chains over  $W$ -spheres in  $X$ .

**Proposition 3.1.12.** *The maps  $\rho_*$  and  $\rho^*$  commute with boundary operators.*

*The map in homology induced by  $\rho^* : H_k^{\text{lf}}(\Sigma) \rightarrow H_k^{\text{lf}}(X)$  is injective.*

*The map in homology induced by  $\rho_* : H_k^{\text{lf}}(X) \rightarrow H_k^{\text{lf}}(\Sigma)$  is surjective.*

*Proof.* The maps  $\rho_*$  and  $\rho^*$  act only on the first variable of  $X_K = (\mathcal{C} \times K) / \sim$  while the boundary operators act only on the second variable. Injectivity and surjectivity of the maps induced in homology follow from the identity  $\rho_* \circ \rho^* = \text{Id}_{C_k^{\text{lf}}(\Sigma)}$ .  $\square$

### 3.1.5 Davis complex and Davis chamber

We now define the Davis realization  $X_D$  of a building. The main reason to introduce this realization is that it can always be endowed with a  $CAT(0)$ -metric [Dav94]. We give two different (but equivalent) constructions of  $X_D$ . We first follow [DJ02].

Let  $X$  be the simplicial realization of a non-spherical regular building  $\mathcal{C}$  of type  $(W, S)$ . Recall that the space  $X$  is endowed with the weak topology, that is, a subset of  $X$  is open if and only if its intersection with any  $k$ -simplex of  $X$  is open in the topology coming from the standard simplex in  $\mathbb{R}^k$  for every  $k \in \mathbb{N}$  (in particular a subcomplex of  $X$  is compact if and only if  $X$  is a finite union of simplices). The space  $X$  does not enjoy good topological properties as it is often not locally finite and group actions on  $X$  are rarely proper. The following construction gives a simplicial complex  $X_D$  that is topologically similar to  $X$  but that enjoys better properties.

Recall that the subcomplex generated by a subset of vertices  $V \subseteq Y^{(0)}$  in a simplicial complex  $Y$  is the subcomplex of simplices of  $Y$  whose vertices lie in  $V$ . Recall that the link of a simplex  $\sigma$  in a simplicial complex is the set:

$$\text{Lk}(\sigma) = \{\tau \text{ simplex}, \sigma \text{ and } \tau \text{ are disjoint faces of the same maximal simplex}\}.$$

**Definition 3.1.13.** Let  $X'$  be the first barycentric subdivision of  $X$ . The *Davis complex*  $X_D$  of  $X$  is the subcomplex of  $X'$  generated by the barycenters of simplices of  $X$  with compact links.

From the algebraic topology viewpoint, the space  $X_D$  is similar to  $X$  in the sense that it is a deformation retract of  $X$  [DJ02, 1.4]. The complex  $X_D$  is locally finite, so it is locally compact. Since  $X_D$  can be endowed with a  $CAT(0)$ -metric, both  $X$  and  $X_D$  are contractible. Note that with this definition  $X_D$  is a simplicial complex, but it is not necessarily purely dimensional, that is, its maximal simplices may not have the same dimension.

The intersections  $\Delta \cap X_D$  are isomorphic for any chamber  $\Delta$  in  $X$  (because they are all translates of a given chamber by elements of the Weyl group for some apartment). We call such an intersection  $D$  a *Davis chamber* of  $X_D$ .

As our notation suggests, the space  $X_D$  can also be constructed from the combinatorial building  $\mathcal{C}$  and a Davis chamber  $D$ , after seeing it can be endowed with the structure of a mirrored space [Dav08, Chapter 18].

**Definition 3.1.14.** The *Davis chamber*  $D = D(W, S)$  of  $(W, S)$  is the (geometric realization of the) barycentric subdivision of the poset of all finite parabolic subgroups  $W_T = \langle s \in T \rangle$  for  $T \subseteq S$  in  $W$  ordered by inclusion. For  $s \in S$ , define  $D_s$  to be the subcomplex of  $D$  corresponding to the poset of all finite parabolic subgroups of  $W$  containing  $W_{\{s\}}$ .

Thus  $D$  is a mirrored space over  $S$  and both definitions of  $X_D$  agree:

$$X_D = (\mathcal{C} \times D) / \sim .$$

Notice that for  $T \subseteq S$  generating a finite subgroup  $W_T$ , the subcomplex  $D_T$  is the cone on the poset of all finite parabolic subgroups of  $W$  strictly containing  $W_T$ , thus  $D_T$  is contractible.

We will now define an orientation on the simplices of  $D$ . All simplices in  $\Delta^{|S|-1}$  have the natural orientation given by  $\mathbb{R}^{|S|-1}$ . Any simplex of its barycentric subdivision inherits a natural orientation. Now we just consider the restriction of this orientation to  $D$ . This determines an orientation for every simplex in  $X_D$ .

## 3.2 Exponent of top $\ell^p$ -homology for buildings of type PM

The first  $\ell^p$ -cohomology group increases as  $p$  grows to infinity, thus there exists a critical  $p$  for which it starts to be nonzero. This is a quasi-isometry invariant first introduced by Pansu [Pan89a]. Top degree  $\ell^p$ -homology has the same behaviour as  $p$  grows, so there is again a critical exponent for which this space starts to be nonzero. We show that in the particular case of buildings of type PM, this critical exponent can be computed in terms of the Weyl group and the thickness.

### 3.2.1 The nerve and Coxeter groups of type PM

In this paragraph we define Coxeter groups of type PM. These are Coxeter groups whose nerve satisfies some natural topological conditions. They can be seen as Coxeter systems whose corresponding Davis complex is close to an orientable manifold. We first introduce the nerve of a Coxeter system.

**Definition 3.2.1.** The *nerve*  $L = L(W, S)$  of a Coxeter system  $(W, S)$  is the poset of all nonempty subsets  $T \subseteq S$  such that the parabolic subgroup  $W_T$  of  $W$  is finite.

Notice that the Davis chamber  $D$  of  $(W, S)$  is the cone on the barycentric subdivision of  $L$ , with apex the vertex corresponding to the empty set. If we view a Davis chamber  $D \subseteq \Delta$  inside a simplicial chamber  $\Delta$ , we may also see the barycentric subdivision  $L'$  as the intersection  $D \cap \partial\Delta$ .

Now we introduce some natural topological conditions on simplicial complexes. A Coxeter group will be of type PM if its nerve satisfies these conditions.

**Definition 3.2.2.** Let  $Y$  be a locally finite purely  $n$ -dimensional simplicial complex. The space  $Y$  is a *pseudomanifold* if every simplex of codimension 1 is contained in exactly two maximal simplices. We say that  $Y$  is *orientable* if there exists an orientation on the simplices of  $Y$  so that the sum of all of its maximal simplices is an  $n$ -cycle in  $H_n^{\text{lf}}(Y)$ . We say that  $Y$  is *gallery connected* if any two maximal simplices in  $Y$  can be joined by a finite sequence of maximal simplices such that any two consecutive maximal simplices share a face of codimension 1.

**Definition 3.2.3.** A Coxeter system  $(W, S)$  is *of type PM* if its nerve  $L$  is an orientable, gallery connected pseudomanifold. A building is *of type PM* if its Weyl group is of type PM.

For a Coxeter system  $(W, S)$  of type PM, the Davis realization of the Coxeter complex  $\Sigma_D$  is a contractible orientable pseudomanifold.

*Remark.* Due to our orientation choices, the canonical  $n$ -cycle on  $\Sigma_D$  will not be the sum of all chambers, but the *alternate* sum of all chambers

$$\sum_{w \in W} (-1)^{l(w)} w.D.$$

**Example 3.2.4.** Affine and compact hyperbolic Coxeter groups (in the sense of [Bou68, p.133, exercice 14]) are of type PM. Indeed, in these cases the Davis chamber  $D$  is just (the barycentric subdivision of) a simplex, and its nerve is a triangulation of a sphere.

Coxeter groups generated by reflections on codimension 1 faces of a right-angled polyhedron  $P$ , as defined in [DJ02, p. 614], are of type PM (e.g. the tiling of the hyperbolic plane by compact regular pentagons). The nerve is simplicially isomorphic to the dual of the boundary of the polyhedron  $P$ , so it is again a triangulation of a sphere.

In [JS03], Januszkiewicz and Świątkowski exhibit an infinite family of Gromov-hyperbolic Coxeter groups of type  $PM$  whose Davis complexes have unbounded dimension. However, they also show that if such a group satisfies Poincaré duality, the dimension of its Davis complex is at most 61.

### 3.2.2 Computation of the critical exponent

In this section we will suppose the Coxeter system  $(W, S)$  is of type PM. Let  $X_D$  be the Davis realization of a regular building  $\mathcal{C}$  of type  $(W, S)$  and thickness  $\mathbf{q} + 1$ . This proof is essentially the same as that of [Dym04] for  $\ell^2$ -cohomology.

**Proposition 3.2.5.** *Let  $X_D$  be the Davis realization of a regular building  $\mathcal{C}$  and  $n = \dim X_D$ . Let  $\rho : X_D \rightarrow \Sigma_D$  be a retraction onto the abstract Coxeter complex. Let  $\eta \in C_n^{\text{lf}}(X_D, \mathbb{R})$ . Then for all  $p > 1$*

$$\|\rho^* \rho_*(\eta)\|_p \leq \|\eta\|_p.$$

*Proof.* Write  $\eta = \sum_{c \in \mathcal{C}, \sigma \in D^n} a_{c, \sigma} c \cdot \sigma$ . The operator  $\rho^* \rho_*$  averages over  $W$ -spheres, thus Jensen's inequality yields the result:

$$\begin{aligned} \|\rho^* \rho_*(\eta)\|_p^p &= \sum_{w \in W} \sum_{c \in \rho^{-1}(w)} \sum_{\sigma \in D^{(n)}} \frac{1}{q_w^p} \left| \sum_{c' \in \rho^{-1}(w)} a_{c', \sigma} \right|^p \\ &\leq \sum_{w \in W} \sum_{c \in \rho^{-1}(w)} \sum_{\sigma \in D^{(n)}} \frac{1}{q_w} \sum_{c' \in \rho^{-1}(w)} |a_{c', \sigma}|^p \\ &= \sum_{w \in W} \sum_{c' \in \rho^{-1}(w)} \sum_{\sigma \in D^{(n)}} |a_{c', \sigma}|^p = \|\eta\|_p^p. \end{aligned}$$

□

**Theorem 3.2.6.** *Let  $(W, S)$  be a Coxeter system of type  $PM$ . Let  $X_D$  be the Davis realization of a regular building of type  $(W, S)$  and thickness  $\mathbf{q} + 1$ , with  $\mathbf{q} \geq 2$ . Let  $n = \dim X_D$ . Then we have:*

$$\begin{aligned} 1 + e_{\mathbf{q}}(W) &= \inf\{p > 1 \mid \ell^p H_n(X_D) \neq 0\}, \\ 1 + e_{\mathbf{q}}(W)^{-1} &= \sup\{p > 1 \mid \ell^p \overline{H}^n(X_D) \neq 0\}. \end{aligned}$$

*Proof.* Since the Davis realization of the Coxeter complex  $\Sigma_D$  is an orientable, gallery connected pseudomanifold, the  $n$ -chain  $\tau = \sum_{w \in W} (-1)^{l(w)} w \cdot D$  is the only non-trivial locally finite  $n$ -cycle on  $\Sigma_D$  up to a constant. Indeed, the pseudomanifold condition implies that if some  $n$ -cycle  $\tau' = \sum_{\sigma} a_{\sigma} \sigma$  satisfies  $a_{\sigma_0} = 0$  for some

$n$ -simplex  $\sigma_0 \subseteq D$ , then for every  $n$ -simplex  $\sigma$  adjacent to  $\sigma_0$  we have  $a_\sigma = 0$  (as  $\sigma$  is the only  $n$ -simplex that can contribute to the boundary operator at the codimension 1 face  $\sigma \cap \sigma_0$ ). Gallery connectedness implies that  $\tau' = 0$ . Hence  $\dim_{\mathbb{R}} H_{\text{lf}}^n(\Sigma_D) \leq 1$ . Orientability implies that  $H_{\text{lf}}^n(\Sigma_D) = \mathbb{R}\tau$ .

Thus  $\rho^*(\tau)$  is a non-trivial locally finite cycle on  $X_D$ . We compute its  $\ell^p$ -norm:

$$\|\rho^*(\tau)\|_p^p = \sum_{w \in W} \sum_{c \in \rho^{-1}(w)} \frac{1}{q_w^p} = \sum_{w \in W} q_w^{1-p} = W(\mathbf{q}^{1-p}).$$

By 3.1.10, this converges for  $p > 1 + e_{\mathbf{q}}(W)$ , so that  $\ell^p H_n(X_D) \neq 0$  for  $p > 1 + e_{\mathbf{q}}(W)$ .

Let  $\eta \in \ell^p H_n(X_D)$  be a nonzero  $\ell^p$ -cycle. We can choose to center the retraction  $\rho$  on a chamber of  $X_D$  where  $\eta$  is nonzero. Because of this choice,  $\rho_*(\eta)$  is a nonzero cycle on  $\Sigma_D$ , hence multiple of  $\tau$ , so we can assume  $\rho_*(\eta) = \tau$ . Then by proposition 3.2.5

$$\|\rho^*(\tau)\|_p = \|\rho^* \rho_*(\eta)\|_p \leq \|\eta\|_p < \infty.$$

But  $\|\rho^*(\tau)\|_p$  diverges when  $p < 1 + e_{\mathbf{q}}(W)$ . Thus we have  $\ell^p H_n(X_D) = 0$  for  $p < 1 + e_{\mathbf{q}}(W)$ .

Results on reduced cohomology follow since in top degree we have  $\ell^p H_n(X_D) = \ell^p \overline{H}_n(X_D) = \ell^r \overline{H}^n(X_D)$  where  $p^{-1} + r^{-1} = 1$ . For such  $p$  and  $r$  the condition  $1 < p \leq 1 + e_{\mathbf{q}}(W)$  is equivalent to  $r \geq 1 + e_{\mathbf{q}}(W)^{-1}$ .  $\square$

Because of the thickness assumption, the number  $e_{\mathbf{q}}(W)$  is finite and comparable to  $e(W)$ . By the strong Tits' alternative for Coxeter groups [Dav08, 17.2.1], an infinite Coxeter group  $W$  is affine if and only if  $W$  has subexponential growth, which in turn is equivalent to  $e_{\mathbf{q}}(W) = 0$ . We obtain the following results for  $\ell^p$ -cohomology:

**Corollary 3.2.7.** *Let  $X_D$  be the Davis realization of a regular building of type  $(W, S)$  and thickness  $\mathbf{q} + 1$ , with  $\mathbf{q} \geq 2$  and  $\dim X_D = n$ .*

1. *If the group  $W$  is affine,  $\ell^p \overline{H}^n(X_D) \neq \{0\}$  for all  $p > 1$ . In particular, any semisimple Lie group  $G$  of rank  $n$  over a non-Archimedean local field satisfies  $\overline{H}_{\text{ct}}^n(G, L^p(G)) \neq \{0\}$  for all  $p > 1$ .*
2. *If the Weyl group  $(W, S)$  is of type  $PM$  and non-affine, then  $0 < e_{\mathbf{q}}(W) < \infty$  so we have  $\ell^p \overline{H}^n(X_D) \neq \{0\}$  for all  $p < 1 + e_{\mathbf{q}}(W)^{-1}$  and  $\ell^p \overline{H}^n(X_D) = \{0\}$  for all  $p > 1 + e_{\mathbf{q}}(W)^{-1}$ .*

The critical exponents  $1 + e_{\mathbf{q}}(W)$  and  $1 + e_{\mathbf{q}}(W)^{-1}$  are quasi-isometry invariants for regular buildings of type  $PM$  with finite thickness. This follows either from quasi-isometric invariance of reduced  $\ell^p$ -cohomology or from the fact that the number  $e_{\mathbf{q}}(W)$  is the growth rate of balls on the dual graph of the building endowed with Coxeter word length as distance.

### 3.2.3 Application to cocompact Fuchsian buildings

As an application, the previous theorem completes the computation of all possible vanishings of  $\ell^p$ -cohomology for cocompact Fuchsian buildings. A building  $X$  of type  $(W, S)$  is said to be *cocompact Fuchsian* if  $W$  is a Fuchsian Coxeter group that acts properly and cocompactly on the hyperbolic plane  $\mathbb{H}^2$ . In particular, the Davis realization of  $(W, S)$  is a tiling of  $\mathbb{H}^2$  and thus these buildings are of type  $PM$ .

First we recall what is known in degree 1. In [Bou00], Bourdon computes the conformal dimension (see Section 6) of these buildings by the formula  $\text{Confdim}(\partial X) = 1 + e_{\mathbf{q}}(W)^{-1}$ . The proof of this result relies on constructing a visual metric on the boundary  $\partial X$  with the right parameter and Hausdorff dimension. Equipped with this metric,  $\partial X$  is Loewner and by [BP03, Théorème 0.3], we obtain that  $\ell^p H^1(X) \neq 0$  if and only if  $p > \text{Confdim}(\partial X) = 1 + e_{\mathbf{q}}(W)^{-1}$ . The vanishing part of this statement is specific to these buildings, while the non-vanishing part holds for any Gromov-hyperbolic complex with bounded geometry (see Proposition 3.4.4).

In [Bou16a], Bourdon obtained that for  $p > \text{Confdim}(\partial X)$  we have  $\ell^p H^2(X) = 0$ . Since  $\text{Confdim}(\partial X) = 1 + e_{\mathbf{q}}(W)^{-1}$ , Theorem 3.2.6 (in particular its non-vanishing statement) implies that Bourdon's vanishing is optimal in this case. We sum these results in the following statement.

**Corollary 3.2.8.** *Let  $X$  be the Davis realization of a cocompact Fuchsian building of type  $(W, S)$  and thickness  $\mathbf{q} + 1$ , with  $\mathbf{q} \geq 2$ .*

- *For  $p < \text{Confdim}(\partial X) = 1 + e_{\mathbf{q}}(W)^{-1}$ , we have  $\ell^p H^1(X) = 0$  and  $\ell^p \overline{H^2}(X) \neq 0$ .*
- *For  $p > \text{Confdim}(\partial X) = 1 + e_{\mathbf{q}}(W)^{-1}$ , we have  $\ell^p H^1(X) \neq 0$  and  $\ell^p H^2(X) = 0$ .*

### 3.3 Non-vanishing of $\ell^p$ -homology and virtual cohomological dimension

The main result of this section is that for large  $p$ , the  $\ell^p$ -homology of a building with Weyl group  $W$  does not vanish in degree equal to the virtual cohomological dimension  $\text{vcd}_{\mathbb{R}}(W)$  of  $W$  over  $\mathbb{R}$ . This is a generalization of the non-vanishing assertion shown for buildings of type  $PM$  in the previous section. We begin by giving a quick review on the notion of virtual cohomological dimension for Coxeter groups. Then we introduce the Bestvina chamber and the Bestvina realization of a building following [Bes93] to obtain this non-vanishing result.

#### 3.3.1 Virtual cohomological dimension

First let  $\Gamma$  be any discrete group and  $R$  be a PID. The *cohomological dimension*  $\text{cd}_R(\Gamma)$  of  $\Gamma$  over  $R$  is the least  $n$  such that the trivial  $R\Gamma$ -module  $R$  has a projective resolution of length  $n$ :

$$0 \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow R \rightarrow 0$$

(and is  $\infty$  if there is no such integer). Usually we denote  $\text{cd}(\Gamma) := \text{cd}_{\mathbb{Z}}(\Gamma)$ . One can show that this number corresponds indeed to some sort of dimension in a more intuitive way:

$$\text{cd}_R(\Gamma) = \sup\{n, H^n(\Gamma; M) \neq 0 \text{ for some } R\Gamma\text{-module } M\}.$$

The group  $\Gamma$  is said to be *of type  $FP_R$*  if there exists a finite length projective resolution of  $R$  by finitely generated projective  $R\Gamma$ -modules. In this case, the cohomological dimension of  $\Gamma$  over  $R$  can be computed as follows:

$$\text{cd}_R(\Gamma) = \sup\{n, H^n(\Gamma; R\Gamma) \neq 0\}.$$

If  $\text{cd}(\Gamma) < \infty$ , then  $\Gamma$  is torsion-free. A Coxeter group is never torsion-free, so cohomological dimension is not directly relevant for such groups since it is always infinite. A more interesting invariant for Coxeter groups can be obtained as follows.

First we have the following result by Serre [Ser71].

**Theorem 3.3.1.** *Suppose  $G$  is a torsion-free group and that  $\Gamma$  is a subgroup of finite index. Then  $\text{cd}_R G = \text{cd}_R \Gamma$ .*

Since the intersection of two subgroups of finite index is still of finite index, this theorem allows us to define virtual cohomological dimension as follows.

**Definition 3.3.2.** Let  $G$  be a group having a torsion-free subgroup of finite index. We define the *virtual cohomological dimension*  $\text{vcd}_R(G)$  of  $G$  over  $R$  as the cohomological dimension  $\text{cd}_R(\Gamma)$  of any of its torsion-free subgroups of finite index.

Let  $(W, S)$  be a finitely generated Coxeter system. First, Tits showed that  $W$  is a linear group in characteristic 0, so by Selberg's lemma it admits a torsion-free subgroup  $\Gamma$  of finite index. Second, such a group  $\Gamma$  is of type  $FP_R$ , for any PID  $R$ . This is because  $\Gamma$  acts freely and cocompactly on the Davis apartment  $\Sigma$  of  $W$ , which is contractible, so the spaces  $C_i(\Sigma; R)$  give the desired projective resolution. This implies:

$$\text{cd}_R(\Gamma) = \sup\{n, H^n(\Gamma; R\Gamma) \neq 0\}.$$

Serre's theorem tells us that this integer does not depend on the choice of a torsion-free subgroup  $\Gamma$  of finite-index in  $W$ , but we can see this directly without invoking this result. Indeed, both  $W$  and  $\Gamma$  act properly discontinuously and cocompactly on the Davis apartment  $\Sigma$ , so  $H^n(\Gamma; R\Gamma) = H_c^n(\Sigma; R) = H^n(W; RW)$ . Thus the virtual cohomological dimension of  $W$  can be computed as follows:

$$\text{vcd}_R(W) = \sup\{n, H^n(W; RW) = H_c^n(\Sigma; R) \neq 0\}.$$

### 3.3.2 Bestvina chamber

The dimension of the Davis realization of a building gives an upper bound for the virtual cohomological dimension of its Weyl group, but in general there is no equality. The Bestvina chamber is a topological construction that associates to every Coxeter system  $(W, S)$  a finite acyclic CW complex whose dimension coincides with  $\text{vcd}(W) = \text{vcd}_{\mathbb{Z}}(W)$ . One can consider variants where the complex is  $\mathbb{F}$ -acyclic for a given field  $\mathbb{F}$ , and in this case the dimension of the complex is  $\text{vcd}_{\mathbb{F}}(W)$ . Since we are interested in  $\ell^p$ -cohomology, we will choose  $\mathbb{F} = \mathbb{R}$ .

In this section we review [Bes93] for the construction of the Bestvina chamber and the computation of its dimension, since it will be useful for non-vanishing of  $\ell^p$ -homology.

First, Bestvina describes an inductive construction of the Davis chamber. Denote by  $\mathcal{F}$  the poset of subsets  $T \subseteq S$  such that  $W_T = \langle T \rangle$  is a finite parabolic subgroup of  $W$  ordered with respect to inclusion. For any maximal element  $F \in \mathcal{F}$ , define  $P_F$  to be a point. Assuming that  $P_{F'}$  has been constructed for every  $F' \supset F$ , define  $P_F$  to be a cone on  $\bigcup_{F' \supset F} P_{F'}$ . The complex  $P_{\emptyset}$  is the Davis chamber  $D$  of  $W$  and the complexes  $P_{\{s\}}$  are mirrors of  $D$ .

Now we construct an  $\mathbb{R}$ -acyclic polyhedron  $B_{\mathbb{R}} = B_{\mathbb{R}}(W)$  as follows. For any maximal element  $F \in \mathcal{F}$ , define  $P_F$  to be a point. Assuming that  $P_{F'}$  has been constructed for every  $F' \supset F$ , define  $P_F$  to be an  $\mathbb{R}$ -acyclic polyhedron containing  $\bigcup_{F' \supset F} P_{F'}$  of the least possible dimension. Most of the time, we have  $\dim(P_F) = \dim(\bigcup_{F' \supset F} P_{F'}) + 1$  and  $P_F$  is just a cone on  $\bigcup_{F' \supset F} P_{F'}$ , but in some situations (e.g. when there is a unique  $F' \in \mathcal{F}$  such that  $F' \supset F$  and  $|F'| = |F| + 1$ ) we have  $\dim(P_F) = \dim(\bigcup_{F' \supset F} P_{F'})$ . We then define the *Bestvina chamber*  $B_{\mathbb{R}}$  to be  $P_{\emptyset}$ .

The Bestvina chamber has a natural mirrored structure given by the subcomplexes  $B_s = P_{\{s\}}$ ,  $s \in S$ . We can thus define the *Bestvina realization*  $X_B$  of a building  $\mathcal{C}$ , as the space:

$$X_B = (\mathcal{C} \times B_{\mathbb{R}}) / \sim .$$

The following lemma gives a more general criterion to guarantee that  $\dim(P_F) = \dim(\bigcup_{F' \supset F} P_{F'})$ . We give the proof for completion and also because the hypotheses are slightly different from those in [Bes93].

**Lemma 3.3.3.** *Let  $L$  be a compact  $n$ -dimensional polyhedron with  $H_n(L; \mathbb{R}) = 0$ . Then  $L$  embeds in a compact  $\mathbb{R}$ -acyclic  $n$ -dimensional polyhedron as a subpolyhedron.*

*Proof.* The lemma is true for  $n = 1$ . For  $n \geq 2$  it is enough to show it for  $L$   $(n - 2)$ -connected. Indeed, one can define successively  $L_{-1} = L$  and  $L_i$  to be  $L_{i-1}$  with the cone on its  $i$ -skeleton attached, we then replace  $L$  by  $L_{n-2}$ . This does not modify  $H_n(L; \mathbb{R})$ .

Now suppose  $L$  is  $(n - 2)$ -connected. The only homology group of  $L$  with real coefficients that may not vanish is  $H_{n-1}(L; \mathbb{R}) = H_{n-1}(L) \otimes \mathbb{R}$ . The Hurewicz homomorphism gives  $H_{n-1}(L) \simeq \pi_{n-1}^{ab}(L)$  (which is just  $\pi_{n-1}(L)$  for  $n \geq 3$ ). Thus the group  $\pi_{n-1}^{ab}(L)$  is (abelian and) finitely generated, but not necessarily free. Choose a basis  $[f_1], \dots, [f_k]$  of the maximal free abelian subgroup of  $\pi_{n-1}^{ab}(L)$ . For each  $i = 1, \dots, k$ , we attach an  $n$ -cell to  $L$  along the image of the map  $f_i : S^{n-1} \rightarrow L$ . Call the resulting space  $L'$ . The homology group  $H_{n-1}(L')$  is a torsion group and therefore  $H_{n-1}(L'; \mathbb{R}) = 0$ . This procedure does not affect other homology groups, thus  $L'$  is  $\mathbb{R}$ -acyclic.  $\square$

To show non-vanishing of  $\ell^p$ -homology of a building, we will use the cycle constructed by Bestvina in the proof of  $\text{vcd}_{\mathbb{R}}(W) = \dim B_{\mathbb{R}}$ . We write its construction for completion.

**Theorem 3.3.4.** *Let  $d = \dim B$ . Then the space  $H_d^{\text{lf}}(\Sigma_B; \mathbb{R})$  contains a non-zero locally-finite cycle with bounded coefficients.*

*Proof.* Denote by  $\mathcal{F}$  the poset of subsets  $T \subseteq S$  such that  $W_T = \langle T \rangle$  is a finite subgroup of  $W$  ordered by inclusion. Let  $F_0$  be a maximal element in  $\mathcal{F}$  such that  $\dim P_{F_0} = d$ . Therefore  $U = \bigcup_{F \in \mathcal{F}, F \supset F_0} P_F$  has dimension  $d - 1$  and  $P_{F_0}$  is a cone on  $U$ . By lemma 3.3.7, we have  $H_{d-1}(U; \mathbb{R}) \neq 0$ . Since  $P_{F_0}$  is  $\mathbb{R}$ -acyclic, we also have  $H_d(P_{F_0}, U; \mathbb{R}) \neq 0$ , so choose  $\tau_0$  a non-trivial cycle in this space. We see this element as a chain on  $B_{\mathbb{R}}$  whose boundary lies in  $U$  and such that its stabilizer in  $W$  (with respect to its action on  $\Sigma_B = (W \times B_{\mathbb{R}}) / \sim$ ) is  $W_{F_0}$ .

Consider the set  $S_0 = \bigcup_{F \in \mathcal{F}, F \supset F_0} F \setminus F_0 \subseteq S$  and its corresponding parabolic subgroup  $W_{S_0}$  in  $W$ . Then the chain

$$\tau = \sum_{w \in W_{S_0}} (-1)^{l(w)} w \cdot \tau_0$$

is a nonzero locally finite cycle in  $H_d^{\text{lf}}(\Sigma_B; \mathbb{R})$ . Indeed, it is a cycle since the only place where the boundary of  $\tau$  may be nonzero is in the translates of  $U$  by  $W_{S_0}$ , and on each of these translates the boundary of  $\tau$  gives twice the same contribution but with opposite sign. The coefficients of  $\tau$  are bounded since it is obtained by placing copies of  $\tau_0$  in different chambers.  $\square$

**Corollary 3.3.5.** *We have  $\text{vcd}_{\mathbb{R}}(W) = \dim B_{\mathbb{R}}$ .*

*Proof.* Let  $d = \dim B_{\mathbb{R}}$ . The group  $W$  acts properly and cocompactly on the space  $\Sigma_B$ , so we have  $H^*(W, \mathbb{R}W) = H_c^*(\Sigma_B, \mathbb{R})$ . Since  $\text{vcd}_{\mathbb{R}}(W)$  is the greatest  $n$  such that  $H^n(W, \mathbb{R}W)$  is non-zero, we have  $\text{vcd}_{\mathbb{R}}(W) \leq d$ .

The natural pairing  $H_d^{\text{lf}}(\Sigma_B; \mathbb{R}) \otimes H_c^d(\Sigma_B; \mathbb{R}) \rightarrow \mathbb{R}$  is non-degenerate, so in particular  $H^d(W, \mathbb{R}W) = H_c^d(\Sigma_B; \mathbb{R}) \neq 0$ . Thus  $\text{vcd}_{\mathbb{R}}(W) = d$ .  $\square$

Unlike the Davis realization, the Bestvina realization may not preserve the underlying combinatorial structure of the building. For instance, the Bestvina complex of any finite Coxeter group is a point. In [Bes93], the example of a 1-dimensional Bestvina chamber in which the mirrors cover the whole chamber is treated. The Bestvina realization of the corresponding Coxeter group is a tree and the chambers are tripods which overlap on segments. Overlapping in top dimension is avoided exactly when  $B = P_{\emptyset}$  is a cone on its mirrors. Even though the combinatorial structure of the building may not be respected by the Bestvina complex, the Davis and Bestvina realization of the same building are still quasi-isometric.

**Proposition 3.3.6.** *Let  $\mathcal{C}$  be a combinatorial regular building of finite thickness and denote by  $X_K$  its  $K$ -realization. The complexes  $X_D$  and  $X_B$  are quasi-isometric locally finite uniformly  $\mathbb{R}$ -acyclic complexes.*

*Proof.* Endowed with its  $CAT(0)$ -metric, the Davis realization  $X_D$  of a building  $\mathcal{C}$  is a uniformly  $\mathbb{R}$ -acyclic (even contractible) complex of bounded geometry. On the other hand, the Bestvina realization  $X_B$  of  $\mathcal{C}$  may be endowed with a length metric by defining a metric on the Bestvina chamber. By proposition 3.1.8, the space  $X_B$  is uniformly  $\mathbb{R}$ -acyclic.

Since  $W$  acts properly and cocompactly on both Coxeter complexes  $\Sigma_D$  and  $\Sigma_B$ , if  $f : \Sigma_D \rightarrow \Sigma_B$  is a map sending a simplex in the chamber  $w.D$  to some point in the simplex of  $w.B$  of the same type, then  $f$  is a quasi-isometry. This map can be defined in the same way between the corresponding buildings, thus we get a map  $F : X_D \rightarrow X_B$  that restricts to a quasi-isometry on each apartment (whose constants do not depend on the choice of an apartment). Since retractions do not increase distances [AB08, Proposition 12.18], a geodesic between two points is contained in an apartment for both realizations, thus  $F$  is a quasi-isometric embedding. Quasi-surjectivity of  $F$  follows since the same apartment system covers both buildings.  $\square$

### 3.3.3 Non-vanishing of $\ell^p$ -homology

Now we show non-vanishing of top dimensional  $\ell^p$ -homology for large  $p$  using the same idea as in the proof of Theorem 3.2.6.

**Theorem 3.3.7.** *Let  $X_B$  be the Bestvina realization of a regular building of type  $(W, S)$  and thickness  $\mathbf{q} + 1$ , with  $\mathbf{q} \geq 2$ . Let  $d = \dim X_B$ . For all  $p > 1 + e_{\mathbf{q}}(W)$ , we have  $\ell^p H_d(X_B) \neq 0$ .*

*Proof.* Let

$$\tau = \sum_{w \in W_\tau} (-1)^{l(w)} w \cdot \tau_0$$

be the nonzero locally finite cycle in  $H_d^{\text{lf}}(X_B; \mathbb{R})$  given by theorem 3.3.4, for some well chosen non-trivial relative cycle  $\tau_0$  and let  $W_\tau$  be the parabolic subgroup of  $W$  corresponding to the support of  $\tau$ . The element  $\rho^*(\tau)$  is a nonzero locally finite cycle in  $X_B$  and

$$\|\rho^*(\tau)\|_p^p \leq \sum_{w \in W_\tau} \sum_{c \in \rho^{-1}(w)} \frac{\|\tau_0\|_p^p}{q_w^p} = \|\tau_0\|_p^p \sum_{w \in W_\tau} q_w^{1-p} = \|\tau_0\|_p^p W_\tau(\mathbf{q}^{1-p}).$$

By proposition 3.1.10, the series  $W_\tau(\mathbf{q}^{1-p})$  converges for  $p > 1 + e_{\mathbf{q}}(W_\tau)$ . The number  $e_{\mathbf{q}}(W_\tau)$  is finite since  $\mathbf{q} \geq 2$ . Thus  $\rho^*(\tau) \in \ell^p H_d(X_B)$  for  $p > 1 + e_{\mathbf{q}}(W_\tau)$ .  $\square$

Since  $\ell^p$ -cohomology is invariant by quasi-isometries (see Theorem 3.1.3), Proposition 3.3.6 implies that this result persists on the  $\ell^p$ -cohomology of the Davis realization of a building.

**Corollary 3.3.8.** *Let  $X_D$  be the Davis realization of a regular building of type  $(W, S)$  and thickness  $\mathbf{q} + 1$ , with  $\mathbf{q} \geq 2$ . Let  $d = \text{vcd}_{\mathbb{R}}(W)$ . Then we have:*

*$\ell^p H^k(X_D) = 0$  for all  $k > d$  and  $p > 1$ ,*

*$\ell^p \overline{H}^d(X_D) \neq 0$  for all  $1 < p < 1 + e_{\mathbf{q}}(W)^{-1}$ .*

*Remark.* Non-vanishing of the  $\ell^p$ -homology of the Davis realization in degree  $d = \text{vcd}_{\mathbb{R}}(W)$  can be obtained directly without invoking the Bestvina realization. Indeed, we have  $d = \max\{n, H_n(D, D^T; \mathbb{R}) \neq 0 \text{ for some spherical } T\}$  [Dav08, Corollary 8.5.5]. If we pick a non-trivial element in  $H_d(D, D^T; \mathbb{R})$ , we can extend it to a cycle  $\tau \in Z_d^{\text{lf}}(\Sigma; \mathbb{R})$  with bounded coefficients and non-trivial in  $H_d^{\text{lf}}(\Sigma; \mathbb{R})$ , as in the proof of 3.3.4. Then the proof of 3.3.7 says that the  $\ell^p$ -norm of the cycle  $\rho^*(\tau)$  on  $X_D$  converges for  $p > 1 + e_{\mathbf{q}}(W)$ . If  $\rho^*(\tau)$  was trivial in  $\ell^p H_d(X_D)$ , then using that  $\rho_*$  commutes with  $\partial$ , we see that  $\tau$  would be trivial inside  $H_d^{\text{lf}}(\Sigma, \mathbb{R})$ .

The vanishing part of the corollary may seem trivial without using the Bestvina complex, but it is not if we do not have a big automorphism group acting on the building. This is the reason why we introduce the Bestvina complex.

## 3.4 Conformal dimension of Gromov-hyperbolic buildings

In [Cla17], Clais obtains bounds for the conformal dimension of Gromov-hyperbolic buildings (of constant thickness) coming from right-angled Coxeter groups. These are formulated in terms of the conformal dimension of an apartment, the parameter of a visual metric and the combinatorial data of the building. Here, we generalize these bounds to arbitrary Gromov-hyperbolic buildings of finite thickness. Most of the proof remains the same, but instead of using combinatorial modulus techniques, we use a separation property of the first  $\ell^p$ -cohomology of hyperbolic groups.

### 3.4.1 Conformal dimension and $\ell^p$ -cohomology

In this subsection let  $X$  be a contractible, Gromov-hyperbolic simplicial complex with bounded geometry, let  $\partial X$  be its Gromov boundary and  $(\cdot|\cdot)_o$  be the Gromov product on  $X$  with basepoint  $o \in X$ .

A metric  $d$  on  $\partial X$  is *visual* if there exist constants  $\lambda > 1$  and  $C \geq 1$  such that for every  $\xi, \eta \in \partial X$  we have:

$$C^{-1}\lambda^{-(\xi|\eta)_o} \leq d(\xi, \eta) \leq C\lambda^{-(\xi|\eta)_o}.$$

The number  $\lambda > 1$  is the *parameter* of the metric  $d$ . It was shown by Gromov that such a metric always exists for  $\lambda > 1$  close enough to 1 [BH99, p. 435]. Moreover, by a theorem due to Coornaert [Coo93, 5.4 and 7.4], such a metric is *Ahlfors-regular*, that is, there exist constants  $C \geq 1$ ,  $Q > 0$  and a measure  $\mu$  on  $\partial X$  such that for every ball  $B(r) \subseteq (\partial X, d)$  of radius  $r < \text{diam}(\partial X, d)$ , we have:

$$C^{-1}r^Q \leq \mu(B(r)) \leq Cr^Q.$$

This implies that  $\mu$  is equivalent to the  $Q$ -Hausdorff measure of  $(\partial X, d)$  and that  $Q$  is the Hausdorff dimension of  $(\partial X, d)$ .

We define the *shadow* of a ball  $B(x, R) \subset X$  as the subset of  $\partial X$ :

$$O(x, R) = \{\xi \in \partial X, [1, \xi[ \cap B(x, R) \neq \emptyset\}.$$

We are ready to define the *conformal gauge*  $\mathcal{J}(\partial X)$  of  $\partial X$  as the set of metrics on  $\partial X$  whose balls are similar to shadows of balls in  $X$ . More precisely:

**Definition 3.4.1.** [Hei01, Chapter 15] A metric  $d$  on  $\partial X$  is in the *conformal gauge*  $\mathcal{J}(\partial X)$  of  $\partial X$  if it is Ahlfors-regular and if it satisfies the following two complementary conditions for every  $R > 0$  large enough:

(i) There is an increasing function  $\varphi : [1, +\infty[ \rightarrow [0, +\infty[$  such that for every pair  $B_1 \subseteq B_2$  of balls of radii  $r_1$  and  $r_2$ , there are shadows  $O(x_1, R)$  and  $O(x_2, R)$  satisfying  $O(x_1, R) \subseteq B_1 \subseteq B_2 \subseteq O(x_2, R)$  and  $|x_1 - x_2| \leq \varphi(\frac{r_2}{r_1})$ .

(ii) There is an increasing function  $\psi : [0, +\infty[ \rightarrow [1, +\infty[$  such that for every pair  $O(x_1, R) \subseteq O(x_2, R)$  of shadows, there are balls  $B_1$  and  $B_2$  of radii  $r_1$  and  $r_2$ , satisfying  $B_1 \subseteq O(x_1, R) \subseteq O(x_2, R) \subseteq B_2$  and  $\frac{r_2}{r_1} \leq \psi(|x_1 - x_2|)$ .

The conformal gauge is a complete quasi-isometry invariant of  $X$ . Indeed, every quasi-isometry  $\phi : X \rightarrow Y$  defines a quasi-symmetric homeomorphism  $\partial\phi : \partial X \rightarrow \partial Y$ , that is, a homeomorphism such that the map sending  $d \in \mathcal{J}(\partial Y)$  to the metric  $d(\partial\phi(\cdot), \partial\phi(\cdot))$  on  $\partial X$  is a bijection from  $\mathcal{J}(\partial Y)$  to  $\mathcal{J}(\partial X)$ . Conversely, every homeomorphism from  $\partial X$  to  $\partial Y$  with this property is the extension on the boundary of a quasi-isometry from  $X$  to  $Y$  [BS00].

**Definition 3.4.2.** The (*Ahlfors-regular*) *conformal dimension* of  $\partial X$  is:

$$\text{Confdim}(\partial X) = \inf\{\text{Hausdim}(\partial X, d); d \in \mathcal{J}(\partial X)\}.$$

In what follows we use the following characterization of the first  $\ell^p$ -cohomology group for contractible complexes of bounded geometry, obtained by integration over 1-cycles:

$$\ell^p H^1(X) = \{f : X^{(0)} \rightarrow \mathbb{R}, df \in \ell^p(X^{(1)})\} / \ell^p(X^{(0)}) + \mathbb{R}.$$

Now we present a characterization of the conformal dimension in terms of functions separating points of the boundary. Define the following subspace of  $\ell^p H^1(X)$ :

$$\ell^p H_{\text{cont}}^1(X) = \{[f] \in \ell^p H^1(X) \mid f \text{ extends continuously to } X^{(0)} \cup \partial X\}$$

and for  $[f] \in \ell^p H_{\text{cont}}^1(X)$ , denote by  $f_\infty$  the extension of  $f$  to the boundary (by radial limit). We define the space of limit functions of  $\ell^p H_{\text{cont}}^1(X)$  by:

$$A_p(\partial X) = \{u : \partial X \rightarrow \mathbb{R} \mid u = f_\infty \text{ for some } [f] \in \ell^p H_{\text{cont}}^1(X)\}.$$

We state the main result we are going to use later. We will only need it in the case of Gromov-hyperbolic groups, so we state it only in this setting in order to have simple hypotheses.

**Theorem 3.4.3.** *[BK15, 3.8] Let  $\Gamma$  be a Gromov-hyperbolic group with connected boundary  $\partial\Gamma$ . Then  $A_p(\partial\Gamma)$  separates points in  $\partial\Gamma$  if and only if  $p > \text{Confdim}(\partial\Gamma)$ .*

*Remark.* This result holds for any Gromov-hyperbolic group  $\Gamma$ , without connectedness assumption. Indeed, connected components of  $\partial\Gamma$  are in bijection with ends of the group  $\Gamma$ . Combining Stallings' theorem on ends of groups [Sta68] and Dunwoody's accessibility theorem [Dun85], we obtain that if  $\Gamma$  is finitely presented and has more than one end, then  $\Gamma$  splits as the fundamental group of a finite graph of groups with finite edge groups, such that vertex groups  $(\Gamma_i)_{i \in I}$  are either finite or 1-ended. It is known that the  $\Gamma_i$  are quasiconvex, thus hyperbolic [Bow98, 1.2], and that any connected component of  $\partial\Gamma$  can be identified either with the boundary  $\partial\Gamma_i$  for some 1-ended vertex group  $\Gamma_i$  or with a point in the boundary of a tree [CM22, 2.4].

With this we can show that  $A_p(\partial\Gamma)$  separates points in  $\partial\Gamma$  if and only if  $A_p(\partial\Gamma_i)$  separates points in  $\partial\Gamma_i$  for every 1-ended  $\Gamma_i$ . The direct implication is trivial. For the converse, it is enough to show that we can separate two points in two different connected components by some element in  $A_p(\partial\Gamma)$ . Denote by  $Z$  the tree of spaces on which  $\Gamma$  acts by deck transformations,  $T$  its associated Bass-Serre tree and  $p : Z \rightarrow T$  the natural retraction. If  $x, y \in \partial Z$  lie in two different connected components, there exists some edge  $e$  of the tree such that any geodesic  $\gamma$  going from  $x$  to  $y$  passes through the edge space  $p^{-1}(e)$ . The complement of  $p^{-1}(e)$  in  $Z$  has two connected components  $Z_x$  and  $Z_y$  (because  $T$  is a tree). Let  $f$  be the characteristic function of vertices of  $Z_x$ . Its differential is nonzero at most on edges of the space  $p^{-1}(e)$ , which is bounded. Thus  $f \in \ell^p H^1(Z)$ , the limit function  $f_\infty$  is in  $A_p(\partial\Gamma)$  and separates  $x$  and  $y$ .

By 3.4.3,  $A_p(\partial\Gamma_i)$  separates points in  $\partial\Gamma_i$  for every 1-ended  $\Gamma_i$  if and only if  $p > \max_{i \in I} \text{Confdim}(\partial\Gamma_i)$ . In [Car11, 6.2], Carrasco shows that either all the  $\Gamma_i$ 's are finite and  $\text{Confdim}(\partial\Gamma) = 0$  or there is at least one 1-ended  $\Gamma_i$  and  $\text{Confdim}(\partial\Gamma) = \max_{i \in I} \text{Confdim}(\partial\Gamma_i)$ .

We finish this subsection by recalling a construction due to Elek of functions on  $X^{(0)}$  starting from functions on  $\partial X$  [Ele97]. This also shows non-vanishing of the first  $\ell^p$ -cohomology of hyperbolic simplicial complexes for large  $p > 1$ . Given a metric  $d \in \mathcal{J}(\partial X)$  and a  $d$ -Lipschitz function  $u : \partial X \rightarrow \mathbb{R}$ , we define a function  $\Phi(u) : X^{(0)} \rightarrow \mathbb{R}$  as follows. Fix  $R > 0$  large enough and for every  $x \in X^{(0)}$  choose some  $\xi_x \in O_X(x, R)$ . We set  $\Phi(u)(x) = u(\xi_x)$  for every  $x \in X^{(0)}$ . In particular, we have  $[\Phi(u)] \in \ell^p H_{\text{cont}}^1(X)$  and  $(\Phi(u))_\infty = u$ .

**Proposition 3.4.4.** [Bou16b, 2.3] For  $p > \text{Hausdim}(\partial X, d)$ , we have  $\|d\Phi(u)\|_p < \infty$ . Moreover, if  $u$  is non-constant,  $[\Phi(u)] \neq 0$  in  $\ell^p H^1(X)$ . In particular,  $\ell^p H^1(X) \neq 0$  for all  $p > \text{Confdim}(\partial X, d)$ .

We can think of this construction as a Poisson transform defined by integrating over shadows with respect to a probability measure. Here we chose a Dirac mass on an element of the shadow, this choice is not important because the function we are integrating is Lipschitz.

### 3.4.2 Upper bound of the conformal dimension of hyperbolic buildings

Let  $X$  be the Davis realization of a building of type  $(W, S)$  and thickness  $\mathbf{q} + 1$ . Denote by  $\Sigma$  the Davis complex of  $(W, S)$ . In this section we assume that  $W$  is a Gromov-hyperbolic Coxeter group, so that  $X$  and  $\Sigma$  are Gromov-hyperbolic spaces (and in fact they can be endowed with a  $CAT(-1)$ -metric by [Mou88]). The previous section applies in this context.

First we obtain an upper bound for  $\text{Confdim}(\partial X)$ . To do this, we bound the Hausdorff dimension of the visual metric associated to a natural distance defined using the underlying combinatorial chamber system  $\mathcal{C}$ .

We call  $\mathcal{G}$  the dual graph of  $X$ , that is,  $\mathcal{G}$  is the graph where vertices are chambers of  $X$ , and two vertices are joined by an edge when their corresponding chambers are adjacent. First, notice that  $\mathcal{G}$  and  $X$  are quasi-isometric, so that the boundaries  $\partial\mathcal{G}$  and  $\partial X$  are homeomorphic. This homeomorphism is also quasi-symmetric, so we can identify their conformal gauges.

We define a combinatorial distance on  $\mathcal{G}$  using both the  $W$ -distance and the thickness of  $X$ . Namely, for  $x, y \in \mathcal{G}^{(0)}$ , set  $|x - y|_{\mathbf{q}} = \log q_{d_W(x, y)}$ . In other words, if  $s_1 \dots s_l$  is a reduced expression for  $w = d_W(x, y)$ , then  $|x - y|_{\mathbf{q}} = \log q_{s_1} + \dots + \log q_{s_l}$ . If the thickness is constant and equal to  $q + 1$ , this is just  $|x - y|_q = l(w) \log q$ . This distance on  $\mathcal{G}$  induces a visual metric  $d_{\mathbf{q}}$  on  $\partial\mathcal{G}$  of parameter  $\lambda(\mathbf{q})$ . We fix an origin  $o \in \mathcal{G}^{(0)}$  and write  $|x|_{\mathbf{q}} := |x - o|_{\mathbf{q}}$ .

The distance  $|\cdot - \cdot|$  naturally restricts to the dual graph of  $\Sigma$ , and induces a visual metric on  $\partial\Sigma$  with the same parameter  $\lambda(\mathbf{q})$ , which we also denote by  $d_{\mathbf{q}}$  and which is also the restriction of the visual metric  $d_{\mathbf{q}}$  on  $\partial X$  to the boundary  $\partial\Sigma$ . From [Coo93, Corollaire 7.6], we know that

$$\text{Hausdim}(\partial\Sigma, d_{\mathbf{q}}) = \frac{e_{\mathbf{q}}(W)}{\log \lambda(\mathbf{q})}.$$

The following upper bound for the Hausdorff dimension of  $(\partial\mathcal{G}, d_{\mathbf{q}})$  can be seen as a thickened version of Coornaert's formula.

**Proposition 3.4.5.** *The visual metric  $d$  satisfies:*

$$\text{Hausdim}(\partial\mathcal{G}, d_{\mathbf{q}}) \leq \text{Hausdim}(\partial\Sigma, d_{\mathbf{q}})(1 + e_{\mathbf{q}}(W)^{-1}).$$

*Proof.* Consider the combinatorial sphere  $\mathcal{G}_n = \{x \in \mathcal{G}^{(0)} \mid n \leq |x|_{\mathbf{q}} < n + 1\}$ . Fix  $R > 0$  bigger than the diameter of all simplices in  $\mathcal{G}$ . Cover the boundary by the shadows  $O_{\mathcal{G}}(x, R)$  for  $x \in \mathcal{G}_n$ . Since the metric  $d$  lies in the conformal gauge of  $\partial\mathcal{G}$ ,

the shadows  $O_{\mathcal{G}}(x, R)$  are similar to balls  $B(\xi, r)$  in  $(\partial\mathcal{G}, d_{\mathbf{q}})$  of radius  $r = \lambda(\mathbf{q})^{-|x|_{\mathbf{q}}}$ . Denote by  $\mu_{\alpha}$  the  $\alpha$ -Hausdorff measure of  $(\partial\mathcal{G}, d_{\mathbf{q}})$ . Thus, there is a constant  $C \geq 1$  such that for every  $x \in \mathcal{G}^{(0)}$ :

$$C^{-1}\lambda(\mathbf{q})^{-|x|_{\mathbf{q}}\alpha} \leq \mu_{\alpha}(O_{\mathcal{G}}(x, R)) \leq C\lambda(\mathbf{q})^{-|x|_{\mathbf{q}}\alpha}.$$

For fixed  $\alpha$ , we compute the  $\alpha$ -Hausdorff measure of the cover  $\{O_{\mathcal{G}}(x, R), x \in \mathcal{G}_n\}$  and estimate its limit when  $n \rightarrow \infty$ . We set  $W_n = \{w \in W \mid n \leq \log q_w < n+1\}$ . Notice that  $e^n|W_n| \leq |\mathcal{G}_n| \leq e^{n+1}|W_n|$ . Thus, the  $\alpha$ -Hausdorff measure of the cover is comparable to:

$$\sum_{w \in W_n} q_w \lambda(\mathbf{q})^{-|w|_{\mathbf{q}}\alpha} = \sum_{w \in W_n} q_w^{1 - \log \lambda(\mathbf{q})\alpha}$$

By proposition 3.1.10, this sum tends to 0 when  $\log \lambda(\mathbf{q})\alpha - 1 > e_{\mathbf{q}}(W)$  (because the associated series on  $n$  converges). This means that  $\mu_{\alpha}(\partial X) = 0$  when  $\alpha > \frac{1}{\log \lambda(\mathbf{q})}(1 + e_{\mathbf{q}}(W))$ . This term is just  $\text{Hausdim}(\partial\Sigma, d_{\mathbf{q}})(1 + e_{\mathbf{q}}(W)^{-1})$  by Coornaert's formula.  $\square$

*Remark.* When the thickness is constant and equal to  $q + 1$ , we can erase the dependence on  $q$  of the parameter  $\lambda(q)$ . Indeed, we may choose the combinatorial distance on  $\mathcal{G}$  defined by  $|x - y| = l(d_W(x, y))$ . Call  $\lambda$  (resp.  $\lambda(q)$ ) the parameter of the visual metric  $d$  (resp.  $d_q$ ) associated to the combinatorial distance  $|\cdot - \cdot|$  (resp.  $|\cdot - \cdot|_q$ ). These parameters are related by

$$\lambda(q) = \lambda^{1/\log q},$$

and the bound in the proposition can be expressed as:

$$\text{Hausdim}(\partial\mathcal{G}, d) \leq \text{Hausdim}(\partial\Sigma, d)\left(1 + \frac{\log q}{e(W)}\right) = \frac{e(W)}{\log \lambda}\left(1 + \frac{\log q}{e(W)}\right),$$

where  $e(W)$  is the growth rate of the group  $(W, S)$  with respect to word length.

### 3.4.3 Lower bound of the conformal dimension of hyperbolic buildings

In this section we obtain a lower bound for the conformal dimension  $\text{Confdim}(\partial X)$  in terms of the conformal dimension  $\text{Confdim}(\partial\Sigma)$  and the growth of the Weyl group  $(W, S)$ , weighted by the thickness  $\mathbf{q} + 1$ . We keep the same notations from the previous section.

*Remark.* Theorem 3.3.8 combined with [Bou16a, Théorème A] already gives a lower bound for  $\text{Confdim}(\partial X)$ :

$$\frac{\text{Confdim}(\partial X)}{\text{vcd}_{\mathbb{R}}(W) - 1} \geq 1 + e_{\mathbf{q}}(W)^{-1}.$$

By [BM91], we know that  $\text{vcd}_{\mathbb{R}}(W) - 1 \leq \text{Topdim}(\partial\Sigma) \leq \text{Confdim}(\partial\Sigma)$ . In this section, we obtain a sharper inequality by replacing  $\text{vcd}_{\mathbb{R}}(W) - 1$  by  $\text{Confdim}(\partial\Sigma)$ .

Fix a retraction  $\rho : X \rightarrow \Sigma$  onto an apartment  $A$ . We identify the apartment  $A$  to the abstract Coxeter complex  $\Sigma$  and the boundary  $\partial A$  to  $\partial\Sigma$ . For a map  $f : \Sigma^{(0)} \rightarrow \mathbb{R}$ , we define its pullback by the retraction  $\rho$  by  $\rho^*f = f \circ \rho$ .

In a similar way, we can pushforward functions on  $X^{(0)}$  through the retraction  $\rho$ , by averaging over preimages. For a map  $h : X^{(0)} \rightarrow \mathbb{R}$ , we define for  $x \in \Sigma^{(0)}$ ,

$$(\rho_*h)(x) = \frac{1}{|\rho^{-1}(x)|} \sum_{y \in \rho^{-1}(x)} h(y).$$

We want to define a pushforward for functions on the boundary. For this, let  $\mathcal{A}_o$  be the set of apartments of  $X$  containing an origin  $o$  lying in the interior of the central Davis chamber of the retraction  $\rho$ . We endow  $\mathcal{A}_o$  with a probability measure  $\nu$  as in [Bou00, Section 2.2.3], the following proposition sums up its main properties.

**Proposition 3.4.6.** [Bou00, 2.2.3 and 2.2.4] *For  $y \in X^{(0)}$ , let  $\mathcal{A}_y$  be the set of apartments containing  $o$  and  $y$ . There exists a Borel probability measure  $\nu$  on  $\mathcal{A}_o$  such that:*

$$\nu(\mathcal{A}_y) = |\rho^{-1}(\rho(y))|^{-1}.$$

We view an apartment in  $\mathcal{A}_o$  as an isometric embedding  $p : \Sigma \rightarrow X$ . This induces a map  $\partial p : \partial\Sigma \rightarrow \partial X$ . Given  $d \in \mathcal{J}(\partial X)$ , we denote by  $\text{Lip}(\partial X, d)$  the space of Lipschitz functions on  $(\partial X, d)$ . For  $u \in \text{Lip}(\partial X, d)$ , define the function  $\partial\rho_*(u)$  on  $\partial\Sigma$  by:

$$\partial\rho_*(u)(\eta) = \int_{\mathcal{A}_o} u(\partial p(\eta)) d\nu(p).$$

The function  $\partial\rho_*(u)$  is continuous and the pushforward  $\partial\rho_*$  factors through  $\rho_*$ :

**Lemma 3.4.7.** *The following diagram commutes:*

$$\begin{array}{ccc} \ell^p H_{\text{cont}}^1(X) & \xleftarrow{\Phi} & \text{Lip}(\partial X, d) \\ \rho_* \downarrow & & \partial\rho_* \downarrow \\ \ell^p H_{\text{cont}}^1(\Sigma) & \xrightarrow{(\cdot)_\infty} & \mathcal{C}(\partial\Sigma) \end{array}$$

*Proof.* Let  $v \in \text{Lip}(\partial X, d)$ . For  $y \in X^{(0)}$ , let  $\mathcal{A}_y$  be the set of apartments containing  $o$  and  $y$ . Notice that the set  $\mathcal{A}_o$  is the disjoint union of the sets  $\mathcal{A}_y$  for  $y$  ranging over a given fiber of the retraction  $\rho$  and recall that  $\nu(\mathcal{A}_y) = |\rho^{-1}(\rho(y))|^{-1}$  [Bou00, Lemme 2.2.4]. Thus:

$$\rho_*\Phi(v)(x) = \frac{1}{|\rho^{-1}(x)|} \sum_{y \in \rho^{-1}(x)} \Phi(v)(y) = \int_{\mathcal{A}_o} \Phi(v)(p(x)) d\nu(p).$$

Since  $v \in \text{Lip}(\partial X, d)$ , we know that for  $\xi \in \partial X$ , we have  $\lim_{y \rightarrow \xi} \Phi(v)(y) = v(\xi)$ . From this and the dominated convergence theorem we first have for  $\eta \in \partial\Sigma$ :

$$\lim_{x \rightarrow \eta} \rho_*\Phi(v)(x) = \int_{\mathcal{A}_o} \lim_{x \rightarrow \eta} \Phi(v)(p(x)) d\nu(p) = \int_{\mathcal{A}_o} v(\partial p(\eta)) d\nu(p) = \partial\rho_*(u)(\eta).$$

□

We are ready to state the main result of this section.

**Theorem 3.4.8.** *Let  $(W, S)$  be a Gromov-hyperbolic Coxeter system,  $\Sigma$  the Davis complex of  $(W, S)$  and  $X$  the Davis realization of a building with Weyl group  $(W, S)$  and thickness  $\mathbf{q}+1$ . Let  $d_{\mathbf{q}}$  be a visual metric on  $\partial\Sigma$  induced by the distance  $|\cdot - \cdot|_{\mathbf{q}}$ . We have:*

$$\text{Confdim}(\partial\Sigma)(1 + e_{\mathbf{q}}(W)^{-1}) \leq \text{Confdim}(\partial X) \leq \text{Hausdim}(\partial\Sigma, d_{\mathbf{q}})(1 + e_{\mathbf{q}}(W)^{-1}).$$

*Remark.* 1. When  $X$  is a Fuchsian building, the boundary  $\partial\Sigma$  is a circle and thus  $\text{Confdim}(\partial\Sigma) = 1$ . In [Bou00], Bourdon constructs a visual metric  $d_{\mathbf{q}}$  with parameter  $\lambda(\mathbf{q}) = \exp e_{\mathbf{q}}(W)$ , so that Coornaert's formula gives  $\text{Hausdim}(\partial\Sigma, d_{\mathbf{q}}) = 1$ . Therefore, the inequalities of the theorem are optimal for these buildings and give a slightly different proof from that of [Bou00] of the equality:

$$\text{Confdim}(\partial X) = 1 + e_{\mathbf{q}}(W)^{-1}.$$

2. In the case of constant thickness the inequalities can be expressed as:

$$\text{Confdim}(\partial\Sigma)(1 + \frac{\log q}{e(W)}) \leq \text{Confdim}(\partial X) \leq \text{Hausdim}(\partial\Sigma, d)(1 + \frac{\log q}{e(W)})$$

where  $d$  is the visual metric on the boundary of  $\Sigma$  equipped with the length function. These are the inequalities obtained by Clais in [Cla17] for some hyperbolic buildings coming from right-angled Coxeter groups. In particular, they imply that  $\text{Confdim}(\partial X)$  is comparable to  $\log q$ .

*Proof.* The upper bound was already shown in the previous proposition.

We show the lower bound. Choose a metric  $d \in \mathcal{J}(\partial X)$  and a family  $\mathcal{V}$  of Lipschitz functions (for  $d$ ) on  $\partial X$  such that the family  $\mathcal{U} = \partial\rho_*\mathcal{V}$  of averages of functions in  $\mathcal{V}$  separates points in  $\partial\Sigma$ . In the remark after 3.4.3 we saw that  $\text{Confdim}(\partial\Sigma)$  is the infimum of all  $p$ 's such that  $A_p(\partial\Sigma)$  separates points in  $\partial\Sigma$ . By Lemma 3.4.7, the limit functions of the family  $\rho_*\Phi(\mathcal{V})$  are the functions in  $\mathcal{U}$ . Thus, for every  $\varepsilon > 0$ , there exists  $v \in \mathcal{V}$  such that the function  $f = \rho_*\Phi(v)$  satisfies:

$$\inf\{p \geq 0, \|df\|_p < \infty\} \geq \text{Confdim}(\partial\Sigma) - \varepsilon.$$

Now we consider  $\rho^*(f)$ , its differential satisfies for each edge  $[x, y] \in \Sigma^{(1)}$ :

$$d(\rho^*f)([x, y]) = df(\rho([x, y])).$$

Thus, the  $\ell^r$ -norm of its differential is given by:

$$\|d(\rho^*f)\|_r^r = \sum_{\sigma \in \Sigma^{(1)}} |\rho^{-1}(\sigma)| \|df(\sigma)\|^r.$$

Define the function:

$$P_f(s) = \inf\{r > 0, \sum_{\sigma \in \Sigma^{(1)}} |\rho^{-1}(\sigma)|^s \|df(\sigma)\|^r < \infty\}$$

and  $P_f(s) = +\infty$  if the corresponding set is empty.

**Claim.** We have:  $(1 + e_{\mathbf{q}}(W)^{-1})(\text{Confdim}(\partial\Sigma) - \varepsilon) \leq P_f(1) \leq \text{Hausdim}(\partial X, d)$ .

The theorem follows from this claim. Indeed, since it holds for any metric  $d \in \mathcal{J}(\partial X)$  and for any  $\varepsilon > 0$ , we obtain the lower bound of the theorem. We now prove separately each of the two inequalities in this claim.

*Proof of the upper bound of the Claim.* We know that  $\|d(\Phi(v))\|_p < \infty$  for  $p > \text{Hausdim}(\partial X, d)$  and by definition of  $f$  we know that  $\rho^* f = \rho^* \rho_*(\Phi(v))$ . By Jensen's inequality, averaging a function on  $X^{(0)}$  over fibers of  $\rho$  reduces the  $\ell^p$ -norm of the differential, that is:

**Lemma 3.4.9.** For any  $h : X^{(0)} \rightarrow \mathbb{R}$  and  $p > 1$  we have  $\|d(\rho^* \rho_* h)\|_p \leq \|dh\|_p$ .

*Proof.* Denote by  $E$  the operator  $\rho^* \rho_*$ . This operator can also be defined on functions  $X^{(1)} \rightarrow \mathbb{R}$ . We first show that the coboundary operator  $d$  commutes with  $E$ . For  $x \in X^{(0)}$  we have  $Eh(x) = \frac{1}{|\rho^{-1}(\rho(x))|} \sum_{y \in \rho^{-1}(\rho(x))} h(y)$ . Let  $\sigma = [x, x'] \in X^{(1)}$ . Notice that:

$$|\rho^{-1}(\rho(\sigma))| = |\rho^{-1}(\rho(x))| |\{\tau \in \rho^{-1}(\rho(\sigma)), x \in \tau\}|.$$

Thus we have:

$$\begin{aligned} d(Eh)(\sigma) &= \frac{1}{|\rho^{-1}(\rho(x))|} \sum_{y \in \rho^{-1}(\rho(x))} h(y) - \frac{1}{|\rho^{-1}(\rho(x'))|} \sum_{y' \in \rho^{-1}(\rho(x'))} h(y') \\ &= \frac{1}{|\rho^{-1}(\rho(\sigma))|} \sum_{[y, y'] \in \rho^{-1}(\rho(\sigma))} (h(y) - h(y')) = E(dh)(\sigma). \end{aligned}$$

Now Jensen's inequality, used as in Proposition 3.2.5, gives directly the result:

$$\|d(Eh)\|_p = \|E(dh)\|_p \leq \|dh\|_p.$$

□

Thus we have  $\|d(\rho^* f)\|_p \leq \|d(\Phi(v))\|_p < \infty$  for  $p > \text{Hausdim}(\partial X, d)$ .

*Proof of the lower bound of the Claim.* Our goal is to obtain a lower bound for  $P_f(1)$ . This will be done using the following convexity lemma, also used in [Cla17].

**Lemma 3.4.10.** The function  $P_f$  is convex on  $\mathbb{R}$ .

*Proof.* We have to check that for  $t \in [0, 1]$ , one has:

$$P_f(ta + (1-t)b) \leq tP_f(a) + (1-t)P_f(b).$$

If  $P_f(a) = +\infty$  or  $P_f(b) = +\infty$ , there is nothing to prove. Otherwise, we have to check that for  $t \in ]0, 1[$  and for every  $\varepsilon > 0$ , one has:

$$\sum_{\sigma \in \Sigma^{(1)}} |\rho^{-1}(\sigma)|^{ta+(1-t)b} |df(\sigma)|^{tP_f(a)+(1-t)P_f(b)+\varepsilon} < \infty.$$

By definition of the function  $P_f$ , we know that the series

$$\sum_{\sigma \in \Sigma^{(1)}} |\rho^{-1}(\sigma)|^a |df(\sigma)|^{P_f(a)+\varepsilon} \quad \text{and} \quad \sum_{\sigma \in \Sigma^{(1)}} |\rho^{-1}(\sigma)|^b |df(\sigma)|^{P_f(b)+\varepsilon}$$

converge. Now the lemma follows from Hölder's inequality:

$$\begin{aligned}
& \sum_{\sigma \in \Sigma^{(1)}} |\rho^{-1}(\sigma)|^{ta+(1-t)b} |df(\sigma)|^{tP_f(a)+(1-t)P_f(b)+\varepsilon} \\
&= \sum_{\sigma \in \Sigma^{(1)}} (|\rho^{-1}(\sigma)|^{ta} |df(\sigma)|^{tP_f(a)+t\varepsilon}) (|\rho^{-1}(\sigma)|^{(1-t)b} |df(\sigma)|^{(1-t)P_f(b)+(1-t)\varepsilon}) \\
&\leq \left( \sum_{\sigma \in \Sigma^{(1)}} |\rho^{-1}(\sigma)|^a |df(\sigma)|^{P_f(a)+\varepsilon} \right)^t \left( \sum_{\sigma \in \Sigma^{(1)}} |\rho^{-1}(\sigma)|^b |df(\sigma)|^{P_f(b)+\varepsilon} \right)^{1-t} \\
&< \infty.
\end{aligned}$$

□

*Proof of the lower bound of the claim (concluded).* We define:

$$s_0 = \sup\{s < 0, \sum_{\sigma \in \Sigma^{(1)}} |\rho^{-1}(\sigma)|^s < \infty\} < 0.$$

This number is defined so that for all  $s < s_0$ , we have  $P_f(s) = 0$ . Since  $P_f$  is continuous, we have  $P_f(s_0) = 0$ . If  $\sigma \in \Sigma^{(1)}$  and if  $w$  denotes the  $W$ -distance from the origin  $o$  to the closest chamber containing  $\sigma$ , then  $|\rho^{-1}(\sigma)| = q_w$ , so by Proposition 3.1.10 the sum appearing in the definition of  $s_0$  converges for  $s < -e_{\mathbf{q}}(W)$  and diverges for  $s > -e_{\mathbf{q}}(W)$ . Thus  $s_0 = -e_{\mathbf{q}}(W)$ .

We already showed that  $P_f(1) < \infty$  and  $P_f(s_0) < \infty$ , thus  $P_f(s) < \infty$  for all  $s \leq 1$ . Moreover, recall that  $f$  was chosen so that  $P_f(0) \geq \text{Confdim}(\partial\Sigma) - \varepsilon$ .

By convexity of the function  $P_f$ , we have for  $t < 0$ :

$$P_f(ts_0) \geq (1-t)P_f(0) + tP_f(s_0) = (1-t)P_f(0).$$

In particular, for  $t = s_0^{-1}$ , we obtain:

$$P_f(1) \geq (1 + e_{\mathbf{q}}(W)^{-1})P_f(0) \geq (1 + e_{\mathbf{q}}(W)^{-1})(\text{Confdim}(\partial\Sigma) - \varepsilon).$$

□

## Chapter 4

# Finitely presented simple groups and measure equivalence

### Introduction

Infinite finitely presented simple groups are rare in geometric group theory. To this date, few examples are known: Burger-Mozes groups acting on products of trees [BM00], non-affine irreducible Kac-Moody lattices over a finite field acting on twin buildings [CR06] and variants of Thompson groups [Rö99].

Naturally, we study these groups not up to isomorphism but up to equivalence relations that are relevant to the theory, like quasi-isometry or measure equivalence. Some results have been obtained on the quasi-isometry side. All Burger-Mozes groups are quasi-isometric since they act properly and cocompactly on products of trees, which are bi-Lipschitz equivalent. It is shown in [CR10] that there is an infinite family of non-affine irreducible Kac-Moody lattices that are pairwise not quasi-isometric. Using cohomological finiteness properties, the same result is shown in [SWZ19] for Röver-Nekrashevych variants of Thompson groups.

We obtain the same result as in [CR10] and in [SWZ19] but for measure equivalence. Namely:

**Theorem 4.0.1.** *There are infinitely many measure equivalence classes containing finitely presented, Kazhdan, simple groups. These groups are Kac-Moody lattices over finite fields with well-chosen non-affine Weyl groups.*

Theorem 4.0.1 is proven by studying the sequence of  $\ell^2$ -Betti numbers  $(\beta^k)_{k \in \mathbb{N}}$  of some of these lattices. Indeed, it was shown by Gaboriau that the sequence of  $\ell^2$ -Betti numbers of discrete countable groups is invariant by measure equivalence up to proportionality [Gab02, 6.3]. More precisely, we find for every  $n \in \mathbb{N}$ , a finitely presented, Kazhdan, simple (Kac-Moody) group  $\Lambda_n$  such that  $\beta^{d(n)}(\Lambda_n) > 0$  and  $\beta^k(\Lambda_n) = 0$  for  $k > d(n)$ , where  $d(n) \rightarrow \infty$ . Vanishing of  $\beta^k$  is a coarse equivalence invariant for any  $k \in \mathbb{N}$  [SS18], so this more precise statement proves the analogue of Theorem 4.0.1 for coarse equivalence, in particular we recover the corresponding result of [CR10] concerning quasi-isometry classes. The result concerning coarse equivalence is not new, as it is implicit in [CR10]: they state that their result may alternatively be obtained using asymptotic dimension, which is a coarse equivalence invariant.

Petersen provides a first family of finitely generated simple (Kac-Moody) groups having non proportional sequences of  $\ell^2$ -Betti numbers [Pet13, 6.8] using an example by Dymara and Januszkiewicz [DJ02, 8.9]. Unfortunately, the groups in question are not Kazhdan and it is not known if they are finitely presented. A conjecture [AG19] says that such groups should never be finitely presented, due to the presence of  $\infty$ 's in the Coxeter diagram of the Weyl group.

Our proof follows the strategy outlined by Petersen but for families of Kac-Moody groups enjoying better properties, we describe the strategy in two steps. First, we use Dymara and Januszkiewicz's formula for  $L^2$ -Betti numbers of locally compact groups acting on some buildings ([DJ02, Section 8], here Theorem 4.1.9, see also [GRO22] for weaker bounds on the thickness). These groups include (products of) complete Kac-Moody groups, which can be seen as the ambient spaces of Kac-Moody lattices. Second, we use a result by Petersen [Pet13, 5.9] relating the sequence of  $\ell^2$ -Betti numbers of a lattice to the sequence of  $L^2$ -Betti numbers of the ambient topological group. This allows us to prescribe vanishing and non-vanishing of some  $\ell^2$ -Betti numbers of Kac-Moody lattices.

The formula for  $L^2$ -Betti numbers of complete Kac-Moody groups is explicit but still hard to manipulate. Roughly, the formula splits into two parts: a topological one and a representation theoretic one. As explained in [DJ02], the behaviour of the representation theoretic part is well understood. Thus, the difficulty comes from understanding the topological part. More precisely, one has to compute the cohomology of some subcomplexes of the Davis chamber which encode the combinatorial complexity of the groups we consider. The Davis chamber, as explained later, can be constructed entirely from the Weyl group of the building. This reduces our study of  $L^2$ -Betti numbers to a purely combinatorial study of Coxeter groups. Indeed, this reduces the proof of Theorem 4.0.1 to finding a sequence of 2-spherical, non-affine Coxeter systems with unbounded virtual cohomological dimension. We compute the cohomology of some of these subcomplexes, precisely enough to give a non-vanishing criterion of an  $L^2$ -Betti number (in high degree) for groups acting on buildings having a certain Weyl group. The condition we require for the Weyl group is that the canonical generating set may be partitioned into two parts, such that one part generates a finite Coxeter group and the other generates an affine Coxeter group. This is enough to obtain Theorem 4.0.1.

We give explicit families of groups as in Theorem 4.0.1 where we compute all of these cohomology spaces, thus simplifying the formula for  $L^2$ -Betti numbers of the corresponding complete Kac-Moody groups and Kac-Moody lattices. This formula can be stated explicitly by following the proof, we do not do it because what matters to us is to determine for which degrees the  $\ell^2$ -Betti numbers vanish.

Now we present the contents of the sections.

Section 2 introduces the necessary background for the rest of the paper. We first define the classes of simplicial complexes and groups discussed in [DJ02], as well as some combinatorial properties of the Davis chamber. We introduce Kac-Moody groups and list the results making them interesting for us. We then state the main theorems from [DJ02], especially the formula for  $L^2$ -Betti numbers of groups acting on buildings.

Section 3 deals with the cohomology of some of the subcomplexes  $D_\sigma$  appearing in the formula of  $L^2$ -Betti numbers given in Section 2. Using nerves, we first give a

vanishing criterion for this cohomology, slightly simplifying this formula. We then give a non-vanishing criterion under a condition on the Weyl group, and apply it to Kac-Moody lattices. This proves Theorem 4.0.1.

Section 4 gives concrete examples of Coxeter groups where results from Section 3 compute all  $L^2$ -Betti numbers of the corresponding complete and discrete Kac-Moody groups.

Section 5 addresses cohomological finiteness properties of the simple Kac-Moody lattices we study. Our arguments point out that it should not be possible to show the results of [CR10] using cohomological finiteness criteria as in [SWZ19].

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## 4.1 $L^2$ -Betti numbers of groups acting on buildings

This section is mainly a review of some parts of [DJ02]. First, we introduce the classes of simplicial complexes of [DJ02, Section 1]. We refer to Chapter 1, Section 1.1.2.1 for more details on simplicial complexes.

Let  $X$  be a *purely  $n$ -dimensional* countable simplicial complex (every simplex is a face of an  $n$ -dimensional simplex). Top dimensional simplices in  $X$  will be called *alcoves*. Let  $\text{Aut}(X)$  be the group of simplicial automorphisms equipped with the compact-open topology and  $G$  be a closed subgroup of  $\text{Aut}(X)$ . We consider the following properties on the pair  $(X, G)$ .

**B1** 0-dimensional links in  $X$  are finite.

**B2** Links of dimension  $\geq 1$  in  $X$  are gallery-connected: for any two alcoves in such a link, there exists a path of alcoves connecting them, such that two consecutive elements meet on a face of codimension 1.

**B3** All the links in  $X$  are either finite or contractible (including  $X$  itself).

The three properties listed above deal with the space  $X$  only. The following condition is the only one that considers both the group  $G$  and the space  $X$ . Roughly it requires the group to have the good size: it is big enough to act transitively on alcoves, and small enough to have a fundamental domain of maximal dimension.

**B4** The group  $G$  acts transitively on the set of alcoves of  $X$  and the quotient map  $X \rightarrow X/G$  restricts on an isomorphism on each alcove.

The next two properties are spectral conditions on the Laplacian that are fundamental in [DJ02] to prove the results presented in Section 2.3. Recall that the Laplacian  $\Delta f$  of a complex-valued function  $f$  on the vertices of a locally finite graph is defined by the formula:

$$\Delta f(x) = f(x) - \frac{1}{|N(x)|} \sum_{y \in N(x)} f(y),$$

where  $N(x)$  denotes the (finite) set of neighbours of the vertex  $x$ .

**B $_\delta$**  Links of dimension 1 are compact and the nonzero eigenvalues of the Laplacian on 1-dimensional links are  $\geq 1 - \delta$ .

$\mathcal{B}_\delta^*$  For every 1-dimensional link in  $X$ , the spectrum of the Laplacian (acting on square integrable functions for the measure  $\mu(\{x\}) = |N(x)|$ ) is a subset of  $\{0\} \cup [1 - \delta, +\infty[$ .

We now begin a brief discussion on buildings, since they are the main source of examples of spaces satisfying (most of) these conditions. Two standard references for buildings are [Bro89] and [Ron89]. They are simplicial complexes obtained by gluing subcomplexes, called *apartments*, under two incidence conditions: any two simplices lie in an apartment and their position is independent of the apartment. Each apartment is a copy of the same Coxeter complex, a purely dimensional simplicial complex with a simply transitive action (on its set of alcoves) of a Coxeter group called the *Weyl group* of the building. The number of alcoves containing a given face of codimension 1 is called the *thickness* (of that face) and for buildings we want it to be  $\geq 3$  for all such faces. The buildings we are interested in have finite thickness, they satisfy  $\mathcal{B}1$ .

#### 4.1.1 Davis complex and Davis chamber

Let  $X$  be a simplicial complex satisfying  $\mathcal{B}3$ .

**Definition 4.1.1.** Let  $X'$  be the first barycentric subdivision of  $X$ . The *Davis complex*  $X_D$  of  $X$  is the subcomplex of  $X'$  generated by the barycenters of simplices of  $X$  with compact links.

This definition is interesting for two reasons. The first, is that  $X_D$  is a deformation retract of  $X$  [DJ02, 1.4] which has the same automorphism group  $\text{Aut}(X)$ , but the action of the latter becomes proper over  $X_D$ . The second reason stems from buildings, and is summarized in the following proposition.

**Proposition 4.1.2.** *Let  $X$  be a building. Then the link of every simplex is a building.*

*Suppose  $X$  is a non-compact building. Then  $X_D$  can be endowed with a  $CAT(0)$  metric. In particular,  $X$  and  $X_D$  are contractible.  $\square$*

The first assertion is [Bro89, IV.1 Prop 3] and the second one can be found in [Dav94]. In particular, this shows that buildings satisfy  $\mathcal{B}2$  and  $\mathcal{B}3$ .

Property  $\mathcal{B}4$  is not always satisfied in this setting since one can consider buildings without automorphisms. However, if  $G$  is a group with  $BN$ -pair and  $X$  is the building constructed from it, then the pair  $(X, G)$  satisfies  $\mathcal{B}4$ .

If a pair  $(X, G)$  satisfies  $\mathcal{B}3$  and  $\mathcal{B}4$ , the intersections  $D = X_D \cap \Delta$  are simplicially isomorphic for any top dimensional simplex  $\Delta$ . We call such an intersection a *Davis chamber* of  $X$  and we denote it  $D$ . We can see  $D$  as a cone over  $D \cap \partial\Delta$  with apex the barycenter of  $\Delta$ , thus  $D$  is contractible.

In the case of buildings, the Davis chamber may also be constructed from its Weyl group. The construction is equivalent to the previous one if the building comes from a  $BN$ -pair.

Let  $(W, S)$  be a Coxeter system with  $|S| = n + 1$  and let  $\Delta$  be the standard simplex of dimension  $n$ . We associate to each codimension 1 face of  $\Delta$  a generator  $s \in S$ . This choice determines for each face  $\sigma$  in  $\Delta$  a parabolic subgroup  $W_\sigma$  of  $W$ , where  $W_\sigma = W_J = \langle J \rangle$  and  $J \subseteq S$  is the set of generators corresponding to codimension 1 faces containing  $\sigma$ .

**Definition 4.1.3.** Let  $\Delta'$  be the first barycentric subdivision of  $\Delta$ . We define the *Davis chamber* of  $W$  as the subcomplex  $D = D_W$  of  $\Delta'$  generated by the barycenters of faces  $\sigma$  in  $\Delta$  whose corresponding parabolic subgroup  $W_\sigma$  is finite.

The next definition measures in a certain sense how many simplices we must remove from  $\Delta$  to obtain  $D$ .

**Definition 4.1.4.** Let  $(W, S)$  be a Coxeter system. We say  $(W, S)$  is *k-spherical* if for all  $J \subseteq S$  with  $|J| \leq k$ , the parabolic subgroup  $W_J = \langle J \rangle$  of  $W$  is finite.

This definition will appear often as a hypothesis on the Weyl group for the results we will obtain. We will discuss it in more detail in Section 5.

If  $\sigma$  and  $\tau$  are faces of  $\Delta$ , then  $\sigma \subseteq \tau$  is equivalent to  $W_\sigma \supseteq W_\tau$ . To describe all faces  $\sigma$  of  $\Delta$  whose corresponding subgroup  $W_\sigma$  is finite, we have to identify those corresponding to maximal finite parabolic subgroups of  $W$ .

If  $\sigma$  is a face of codimension  $k$  of  $\Delta$ , then the subcomplex of  $\Delta'$  generated by the barycenters of faces containing  $\sigma$  is a (simplicial subdivision of a) cube of dimension  $k$ . Each maximal finite parabolic subgroup  $W_J$  ( $J \subseteq S$ ) corresponds to a cube of dimension  $|J|$  in the Davis chamber.

The Davis chamber  $D$  is then obtained as follows. We start with the disjoint union of these cubes, and then we glue some of their faces: if  $W_I$  and  $W_J$  are two maximal finite parabolic subgroups of  $W$ , the intersection of their corresponding cubes in  $\Delta'$  is the cube corresponding to  $W_{I \cap J}$ .

This gives an equivalent construction of the Davis chamber that is independent of  $\Delta$ . From each parabolic subgroup  $W_J$  ( $J \subseteq S$ ) of  $W$  we define the flag complex of parabolic subgroups  $\{W_{J'}, J' \subseteq J\}$  contained in  $W_J$  ordered by inclusion. We construct  $D$  as the union of all flag complexes of maximal finite parabolic subgroups of  $W$ , where we glue the complexes corresponding to  $W_I$  and  $W_J$  over the flag complex of  $W_{I \cap J}$ .

For  $\sigma \subseteq \Delta$ , let  $\Delta_\sigma$  be the union of faces not containing  $\sigma$  and  $D_\sigma = D \cap \Delta_\sigma$ . Notice  $\Delta_\sigma$  is always a union of  $(n-1)$ -dimensional simplices of  $\Delta$  and that  $\Delta_\sigma = \partial\Delta$  if and only if  $\sigma = \Delta$ . More precisely, if  $\sigma \subseteq \Delta$  corresponds to the parabolic subgroup  $W_J$ , then  $\Delta_\sigma$  is exactly the union of codimension 1 faces of  $\Delta$  corresponding to the generators  $s_j$  for  $j \in S \setminus J$ .

## 4.1.2 Kac-Moody groups

The family of simple groups we want to exhibit in Theorem 4.0.1 are Kac-Moody groups. We introduce them following the presentation of [DJ02, Appendix TKM] and then list the properties that make them interesting to us.

**Definition 4.1.5.** A *Kac-Moody datum* is the data  $(I, \Lambda, (\alpha_i)_{i \in I}, (h_i)_{i \in I}, A)$  of:

1. A finite set  $I$ .
2. A finitely generated abelian free group  $\Lambda$ .
3. Elements  $\alpha_i \in \Lambda, i \in I$ .
4. Elements  $h_i \in \Lambda^\vee = \text{Hom}(\Lambda, \mathbb{Z}), i \in I$ .
5. A *generalized Cartan matrix*  $(A_{ij})_{i,j \in I}$  given by  $A_{ij} = \langle \alpha_i, h_j \rangle$ , satisfying

$$A_{ii} = 2, \text{ if } i \neq j \text{ then } A_{ij} \leq 0 \text{ and } A_{ij} = 0 \text{ if and only if } A_{ji} = 0.$$

From a Kac-Moody datum (or merely from a generalized Cartan matrix) one can define a Coxeter matrix  $M = (m_{ij})_{i,j \in I}$  as follows:

$m_{ii} = 1$  and for  $i \neq j$ ,  $m_{ij} = 2, 3, 4, 6$  or  $\infty$  as  $A_{ij}A_{ji} = 0, 1, 2, 3$  or is  $\geq 4$ , respectively.

We consider the Coxeter group  $W$  associated to this matrix:

$$W = \langle r_i \mid (r_i r_j)^{m_{ij}} = 1, \text{ for } m_{ij} \neq \infty \rangle.$$

If a Kac-Moody datum is fixed, Tits defines a group functor associating to each field (or commutative ring in general)  $k$  a group  $\Lambda(k)$  [Tit87]. The group  $\Lambda(k)$  has two BN-pairs  $(B_+, N)$  and  $(B_-, N)$  such that their Weyl groups  $B_{\pm}/(B_{\pm} \cap N)$  are isomorphic to the group  $W$  coming from the generalized Cartan matrix.

These BN-pairs define two buildings  $X_+$  and  $X_-$  of thickness  $|k| + 1$  and Weyl group  $W$  (therefore the dimensions of these buildings is  $|I| - 1$ ), such that  $\Lambda(k)$  acts transitively on their sets of chambers [DJ02, Appendix TKM]. These buildings are simplicially isomorphic, we denote them  $X$  when it is not necessary to distinguish them. Denote by  $G_{\pm}$  the completion of  $\Lambda(k)$  in  $\text{Aut}(X_{\pm})$  with respect to the compact-open topology.

The following theorem summarizes the properties that justify the study of Kac-Moody groups for our purposes.

**Theorem 4.1.6.** *Let  $\Lambda$  be a Kac-Moody group over  $\mathbb{F}_q$  with Weyl group  $W$ . Then  $\Lambda$  is finitely generated. Moreover:*

1. *The covolume of  $\Lambda$  in  $G_+ \times G_-$  (diagonally injected) is  $W(\frac{1}{q})$ , where*

$$W(t) = \sum_{w \in W} t^{l(w)}.$$

*In particular for  $q > |I|$ , the group  $\Lambda$  is a lattice in  $G_+ \times G_-$ .*

2. *If  $W$  is non-affine, irreducible and  $\Lambda$  is a lattice in  $G_+ \times G_-$ , then  $\Lambda/Z(\Lambda)$  is simple, where  $Z(\Lambda)$  is the center of  $\Lambda$ .*
3. *If  $q \geq 4$  and all the entries of the Coxeter matrix are finite (i.e. the Weyl group is 2-spherical), then  $\Lambda$  is finitely presented.*

*Proof.* Assertions 1 and 2 are respectively Proposition 2 and Theorem 20 in [CR09]. Assertion 3 is a simplified version of the main corollary in [AM97].  $\square$

*Remark.* In what follows we will systematically consider center-free Kac-Moody groups. This can always be done without further assumptions on the generalized Cartan matrix by choosing the adjoint root datum, where  $\Lambda$  is generated by the  $\alpha_i$ 's.

### 4.1.3 Cohomology and $L^2$ -Betti numbers of groups acting on buildings

Dymara and Januszkiewicz state their results for classes  $\mathcal{B}t$  and  $\mathcal{B}+$  of pairs  $(X, G)$  groups acting on simplicial complexes. The class  $\mathcal{B}t$  are pairs  $(X, G)$  satisfying  $\mathcal{B}1-4$  and  $\mathcal{B}_{\frac{13}{28^n}}^*$ , the class  $\mathcal{B}+$  are pairs  $(X, G)$  satisfying  $\mathcal{B}1-4$  and  $\mathcal{B}_{\frac{13}{28^n}}$ . For groups with a BN-pair and their associated buildings, the class  $\mathcal{B}t$  corresponds to large minimal

thickness and  $\mathcal{B}+$  corresponds to large minimal thickness and finiteness of all entries of the Coxeter matrix [DJ02, 1.7], that is, 2-sphericity of its Weyl group.

In particular, a complete Kac-Moody group over a finite field  $\mathbb{F}_q$  is in  $\mathcal{B}t$  for large  $q$ , and is in  $\mathcal{B}+$  if all the entries of its Coxeter matrix are finite.

We now mention three important results of [DJ02]. The initial motivation of [DJ02] is to find examples of Kazhdan groups. The first theorem we state addresses this question and gives a criterion for the vanishing of the first and also higher cohomology groups for pairs in  $\mathcal{B}+$  with a finiteness condition.

**Theorem 4.1.7.** [DJ02, 5.2] *Let  $(X, G)$  be in  $\mathcal{B}+$ . Suppose the links of  $X$  of dimensions  $1, \dots, k$  are compact. Then for any unitary representation  $(\rho, V)$  of  $G$ , we have:*

$$H_{ct}^i(G, \rho) = 0 \quad \text{for } i = 1, \dots, k.$$

For buildings of finite thickness, compactness of all links of dimension  $\leq k$  is equivalent to having a  $(k+1)$ -spherical Weyl group. This combinatorial condition is necessary for cohomological vanishing: one can consider the group  $D_\infty$  acting simplicially on a tree  $X$  with edge set  $E$ , the induced action on  $L^2(E)$  does not have a fixed point. Thus we have a space  $X$  not verifying the condition of the theorem (the only link of dimension 1 is  $X = Lk(\emptyset)$ ) and  $H_{ct}^1(G, L^2(E)) \neq 0$ .

The second theorem we mention is a formula for cohomology spaces of groups in  $\mathcal{B}+$  with values in unitary representations.

**Theorem 4.1.8.** [DJ02, 7.1] *Suppose either that the pair  $(X, G)$  is in  $\mathcal{B}+$  and  $(\rho, V)$  is a unitary representation of  $G$ , or that, the pair  $(X, G)$  is in  $\mathcal{B}t$  and  $(\rho, V)$  is a subrepresentation of  $\bigoplus^\infty L^2(G)$ . Then*

$$H_{ct}^*(G, \rho) = \bigoplus_{\sigma \subseteq \Delta} \tilde{H}^{*-1}(D_\sigma; V^\sigma).$$

We draw the attention on the right hand side depending on the classical cohomology groups of the spaces  $D_\sigma$  defined at the end of Section 2.1. Theorem 4.1.7 is proven in [DJ02, Section 7] from this formula by noticing that the cohomology  $H^*(D_\sigma)$  is the cohomology  $H^*(\mathcal{U})$  of a simple covering  $\mathcal{U}$  of  $D_\sigma$ . The combinatorial condition of Theorem 4.1.7 on the Weyl group implies vanishing of  $H^*(\mathcal{U})$  in low degrees for all  $\sigma \subseteq \Delta$ . The proofs given in Section 3 are in the same spirit: under combinatorial conditions on the Weyl group, we compute  $H^*(D_\sigma)$  for some  $\sigma \subseteq \Delta$ . The difference is that instead of considering all simplices  $\sigma \subseteq \Delta$  and obtaining a partial description of the cohomology of  $D_\sigma$ , we pick particular  $\sigma \subseteq \Delta$  for which we can fully describe the spaces  $H^*(D_\sigma)$ .

The last theorem we mention is the starting point for this paper, though the expression we present is obtained directly from the previous theorem. It is a formula for  $L^2$ -Betti numbers, as defined in [Sau17, 4.1], of groups acting on buildings of finite thickness in  $\mathcal{B}t$ . As said before, groups in this class include complete Kac-Moody groups over finite fields of large cardinality.

**Theorem 4.1.9.** [DJ02, 8.5] *Let  $(X, G)$  be a building in  $\mathcal{B}t$  of thickness  $q+1$ . Then the  $L^2$ -Betti numbers of  $G$  are given by*

$$\beta^k(G) = \sum_{\sigma \subseteq \Delta} \dim_{\mathbb{C}} \tilde{H}^{k-1}(D_\sigma; \mathbb{C}) \cdot \dim_G L^2(G)^\sigma.$$

Moreover, the sum can be taken for  $\sigma$  with compact links.

The fundamental observation by Dymara and Januszkiewicz is that for  $q > 2^n$ ,  $\dim_G L^2(G)^\sigma > 0$  for all  $\sigma \subseteq \Delta$  with compact links [DJ02, p. 612]. Thus, for  $q$  large enough, the problem of determining whether  $\beta^k(G)$  is zero or not reduces to determining if there exists  $\sigma \subseteq \Delta$  such that  $\tilde{H}^{k-1}(D_\sigma; \mathbb{C}) \neq 0$  or not. In the end, this reduces to the study of the combinatorics of the Weyl group. In the next two sections we study the topology of  $D_\sigma$  via the combinatorics of the Weyl group of the building.

*Remark.* 1. The techniques of [DJ02] are reformulated in [Kas11] in terms of angles between subspaces. This interpretation allowed Grinbaum-Reizis and Oppenheim [GRO22] to recover the previous results from [DJ02] under weaker thickness bounds. For (thick) affine buildings, the thickness bound disappears and thus they recover a classical theorem by Casselman [Cas74] using geometric methods. This does not improve the main statement of this article, but allows to slightly extend the range of our examples.

2. Dymara and Januszkiewicz apply their results to Kac-Moody groups whose Weyl group  $W_P$  is the right-angled Coxeter group defined by the intersection relations of the faces of codimension 1 of a polytope  $P$  of dimension  $n$  [DJ02, 8.9]. Most of the associated Kac-Moody lattices are non-affine and irreducible, hence simple. When  $P$  is dual to a triangulation of a sphere (up to considering a barycentric subdivision), it is shown that all  $L^2$ -Betti numbers of the completions of such groups vanish except in degree  $n$ . As stated by [Pet13], this gives a first example of an infinite family of finitely generated simple groups with non-proportional sequences of  $\ell^2$ -Betti numbers. Petersen says such groups should often be finitely presented. Unfortunately, the Coxeter matrices of their Weyl groups have  $\infty$  entries, so it is not known whether these groups are finitely presented or not, and a conjecture [AG19] says that such groups should never be finitely presented. The examples we will give here are finitely presented in view of Theorem 4.1.6, Assertion 3.

## 4.2 Cohomology of subcomplexes of the Davis chamber

In this section we elaborate on the formula of Theorem 4.1.9 for  $L^2$ -Betti numbers of groups acting on buildings. We study the contribution of the topological part of this formula to obtain first a vanishing criterion, slightly simplifying the formula, and then a non-vanishing criterion. We then apply this non-vanishing criterion to Kac-Moody lattices and prove Theorem 4.0.1.

### 4.2.1 Cohomology of nerves: vanishing and non-vanishing

In what follows a contractible space is non-empty. Let  $X$  be a finite simplicial complex covered by a finite family of subcomplexes  $\mathcal{U} = \{X_i, i \in I\}$ . The *nerve*  $N(\mathcal{U})$  of the cover  $\mathcal{U}$  is the simplicial complex whose  $k$ -simplices are the subsets  $J \subseteq I$  with  $|J| = k + 1$  such that the intersection  $\bigcap_{j \in J} X_j$  is non-empty. When  $J \subseteq J'$  we have an inclusion of the corresponding simplices. The following result, known as the Nerve Lemma, allows us to compute cohomology using nerves.

**Proposition 4.2.1.** [Hat02, Corollary 4G.3] *Let  $X$  be a finite simplicial complex covered by a finite family of subcomplexes  $\mathcal{U} = \{X_i, i \in I\}$  such that for all nonempty  $J \subseteq I$ , the intersection  $\bigcap_{j \in J} X_j$  is either empty or contractible. Then  $X$  is homotopically equivalent to the nerve  $N(\mathcal{U})$ .*

Vanishing results in this article will be obtained using the following particular case of the Nerve lemma.

**Lemma 4.2.2.** *Let  $X$  be a finite simplicial complex covered by  $\{X_i, i \in I\}$  a finite family of subcomplexes such that  $\bigcap_{j \in J} X_j$  is contractible for all nonempty  $J \subseteq I$ . Then  $X = \bigcup_{i \in I} X_i$  is contractible.*

*Proof.* By Proposition 4.2.1, the space  $X$  is homotopically equivalent to the nerve of the cover  $\{X_i, i \in I\}$ , which is just a simplex.  $\square$

We recall our setting. Let  $(W, S)$  be a Coxeter system and  $\Delta$  be a simplex of dimension  $|S| - 1$ . Let  $D$  be the Davis chamber of  $(W, S)$ . We defined for  $\sigma \subseteq \Delta$ , the subset  $\Delta_\sigma$  to be the union of the faces of  $\Delta$  not containing  $\sigma$  and  $D_\sigma = D \cap \Delta_\sigma$ . Notice that  $\Delta_\sigma$  is always a union of codimension 1 faces of  $\Delta$ . More precisely, if  $\Delta_s$  is the codimension 1 face of  $\Delta$  corresponding to the generator  $s \in S$  and  $\sigma = \bigcap_{s \in J} \Delta_s$ , then  $\Delta_\sigma = \bigcup_{s \in J^c} \Delta_s$  and  $D_\sigma = \bigcup_{s \in J^c} D_s$ . We may apply the Nerve lemma to the union  $\bigcup_{s \in J^c} D_s$  in view of the following remark.

*Remark.* If  $\sigma$  is the simplex in  $\Delta$  corresponding to the parabolic subgroup  $W_J$ , then  $D \cap \sigma$  is the geometric realization of the flag complex of finite parabolic subgroups containing  $W_J$ . Hence, if  $D \cap \sigma$  is non-empty, then  $D \cap \sigma$  is a cone with apex the barycenter of  $\sigma$  and thus  $D \cap \sigma$  is contractible. Notice that  $D \cap \sigma$  is non-empty if and only if  $W_J$  is finite.

The previous lemma takes the following form in our setting.

**Corollary 4.2.3.** *For  $\sigma \subseteq \Delta$ , we write  $\sigma = \bigcap_{s \in J} \Delta_s$  for some  $J \subseteq S$ , so that  $D_\sigma = \bigcup_{s \in J^c} D_s$ . If  $W_{J^c} = \langle J^c \rangle$  is finite (or equivalently, if  $\bigcap_{s \in J^c} D_s$  is non-empty), then  $D_\sigma$  is contractible.*

*Proof.* If  $\bigcap_{s \in J^c} D_s$  is non-empty, then every sub-intersection  $\bigcap_{s \in J'} D_s$  is non-empty for  $J' \subseteq J^c$ , hence contractible by the previous remark. Now the result follows from Lemma 4.2.2.  $\square$

We wish to compute spaces  $\tilde{H}^*(D_\sigma)$  for  $\sigma \subseteq \Delta$  that appear in the formula of Theorem 4.1.9. The sum ranges over  $\sigma$  with compact links, that is, over  $\sigma$  with finite corresponding parabolic subgroups  $W_\sigma$ . What we said allows us to focus on a smaller class of simplices  $\sigma$ . Thus, the sum in the formula of Theorem 4.1.9 reduces to the following.

**Proposition 4.2.4.** *Let  $(X, G)$  be a building in  $\mathcal{B}t$  of thickness  $q + 1$ . Then the  $L^2$ -Betti numbers of  $G$  are given by*

$$\beta^k(G) = \sum_{\sigma \subseteq \Delta} \dim_{\mathbb{C}} \tilde{H}^{k-1}(D_\sigma; \mathbb{C}) \cdot \dim_G L^2(G)^\sigma.$$

*Moreover, the sum can be taken over simplices  $\sigma$  corresponding to finite parabolic subgroups  $W_J$  such that  $W_{J^c}$  is infinite.*

*Proof.* The proof of Theorem 4.1.9 in [DJ02, 8.5] already shows we can restrict ourselves to those  $\sigma$  whose corresponding parabolic subgroup  $W_J$  is finite (the argument is that  $L^2(G)^\sigma \subseteq L^2(G)^{G_\sigma}$  and if the link of  $\sigma$  is non-compact, then  $L^2(G)^{G_\sigma} = \{0\}$ ). Now suppose  $\sigma \subseteq \Delta$  corresponds to a finite parabolic subgroup  $W_J$  and that the parabolic subgroup  $W_{J^c}$  is also finite. Then by Corollary 4.2.3, the space  $D_\sigma$  is contractible, so its cohomology does not contribute to the sum in any degree.  $\square$

We now turn to non-vanishing phenomena. In order to show Theorem 4.0.1, we will use the following non-vanishing criterion obtained using Proposition 4.2.1.

**Corollary 4.2.5.** *For  $\sigma \subseteq \Delta$ , we write  $\sigma = \bigcap_{s \in J} \Delta_s$  for some  $J \subseteq S$ , so that  $D_\sigma = \bigcup_{s \in J^c} D_s$ . If  $W_{J^c} = \langle J^c \rangle$  is infinite, but every proper parabolic subgroup of  $W_{J^c}$  is finite (or equivalently, if  $\bigcap_{s \in J'} D_s \neq \emptyset$  for all nonempty  $J' \subsetneq J^c$  but  $\bigcap_{s \in J^c} D_s = \emptyset$ ), then  $D_\sigma$  is homotopy equivalent to a sphere of dimension  $(|I| - 2)$ .*

*Proof.* The nerve of the cover  $\{X_i, i \in I\}$  is the boundary of a simplex of dimension  $(|I| - 1)$ , which homotopically is a sphere of dimension  $(|I| - 2)$ . Proposition 4.2.1 gives the result.  $\square$

*Remark.* Infinite irreducible Coxeter groups such that every proper parabolic subgroup is finite are classified: they are exactly affine and compact hyperbolic Coxeter groups [Bou68, p.133, exercice 14]. Compact hyperbolic Coxeter groups have rank  $\leq 5$  [Bou68, p.133, exercice 15.c], so in higher rank the only examples are affine Coxeter groups [Bou68, p.100, Proposition 10].

The following result summarizes the main idea, that is, we have non-vanishing of an  $L^2$ -Betti number in high degree for groups acting on buildings whose Weyl group is obtained as a perturbation of an affine Coxeter group by a finite Coxeter group.

**Theorem 4.2.6.** *Let  $(X, G)$  be a building in  $\mathcal{B}t$  of thickness  $q + 1$  and irreducible Weyl group  $(W, S)$ . Suppose  $S$  admits a partition  $S = J_{\text{sph}} \sqcup J_{\text{aff}}$  such that  $W_{\text{sph}} = \langle J_{\text{sph}} \rangle$  is finite and  $W_{\text{aff}} = \langle J_{\text{aff}} \rangle$  is an infinite affine Coxeter group. Put  $|J_{\text{aff}}| = n + 1$ . Then, for  $q$  large enough,*

$$\beta^n(G) > 0.$$

*Proof.* Let  $\sigma$  be the simplex corresponding to the subgroup  $W_{\text{sph}}$ . Hence,  $D_\sigma = \bigcup_{s \in J_{\text{aff}}} D_s$  and thus by 4.2.5, the space  $\tilde{H}^*(D_\sigma)$  is non-zero in degree  $n - 1$ . Recall that for  $q > 2^{|S|}$ , we have  $\dim_G L^2(G)^\sigma > 0$  [DJ02, p. 612]. Therefore by Theorem 4.1.9 we have

$$\beta^n(G) \geq \dim \tilde{H}^{n-1}(D_\sigma) \dim_G L^2(G)^\sigma > 0.$$

$\square$

## 4.2.2 Application to measure equivalence of Kac-Moody lattices

We now apply the previous result to  $\ell^2$ -Betti numbers of Kac-Moody groups. Recall from Section 2.2 that we can suppose our Kac-Moody groups to be center-free without adding conditions on the generalized Cartan matrix.

**Corollary 4.2.7.** *Let  $\Lambda$  be a center-free Kac-Moody group over a finite field  $\mathbb{F}_q$  with irreducible Weyl group  $(W, S)$  as in Theorem 4.2.6, we set  $S = J_{\text{sph}} \sqcup J_{\text{aff}}$  with  $J_{\text{sph}} \neq \emptyset$ . Suppose also that its Coxeter matrix has finite entries. Put  $|J_{\text{aff}}| = n + 1$ . For  $q$  large enough,*

$$\beta^k(\Lambda) \begin{cases} = 0 & \text{if } k \geq 2|S| - 1 \\ > 0 & \text{if } k = 2n \end{cases}$$

Moreover,  $\Lambda$  is infinite finitely presented, Kazhdan and simple.

*Proof.* Let  $G$  be the complete Kac-Moody group associated to  $\Lambda$ . Theorem 4.2.6 applies to  $G$ , so  $\beta^n(G) > 0$  for large  $q$ . Let  $X$  be the building coming from the BN-pair of  $G$ . The group  $G$  acts properly cocompactly on the Davis complex  $X_D$ . This complex has dimension  $\leq |S| - 1$ , thus [DJ02, 3.4] gives  $H_{ct}^k(G, \rho) = 0$  for  $k \geq |S|$  any quasi-complete representation  $(\rho, V)$ . Hence the Künneth formula for  $L^2$ -Betti numbers [Pet13, 6.5] gives for large  $q$

$$\beta^k(G \times G) \begin{cases} = 0 & \text{if } k \geq 2|S| - 1 \\ > 0 & \text{if } k = 2n \end{cases}$$

For  $q > n + 1$ , the group  $\Lambda$  is a lattice in  $G \times G$  by Theorem 4.1.6 Assertion 1. By [Pet13, 5.9], the sequences of  $L^2$ -Betti numbers of  $\Lambda$  and  $G \times G$  are proportional.

Since the entries of the Coxeter matrix of  $(W, S)$  are finite, Theorem 4.1.7 tells us  $G$  has property (T) (for the same bound on  $q$ ), thus so does  $G \times G$  and any lattice in  $G \times G$ . The group  $\Lambda$  is finitely presented and simple in view of Theorem 4.1.6 assertions 2, 3 and because we assumed  $Z(\Lambda) = \{0\}$ .  $\square$

The previous corollary gives a control on the vanishing of  $\ell^2$ -Betti numbers for simple Kac-Moody lattices. We can now prove the theorem stated in the introduction using Gaboriau's projective invariance of  $\ell^2$ -Betti numbers by measure equivalence [Gab02, 6.3].

*Proof of Theorem 4.0.1.* It suffices to take an affine diagram with  $n + 1$  generators  $s_1, \dots, s_{n+1}$ , add a generator  $s_0$ , declare that it does not commute with at least one  $s_i, i \geq 1$  and that the products  $s_0 s_i$  have order  $\leq 6$ . Thus the Coxeter system  $(W, S)$ , where  $S_{\text{sph}} = \{s_0\}$ ,  $S_{\text{aff}} = \{s_1, \dots, s_{n+1}\}$ ,  $S = S_{\text{sph}} \sqcup S_{\text{aff}}$  and  $W = \langle S \rangle$ , satisfies the conditions of the previous corollary with  $|S| = n + 2$ .

Let  $\Lambda_n$  be a Kac-Moody group over  $\mathbb{F}_q$  with Weyl group  $(W, S)$  with  $q$  as in the corollary (its Coxeter matrix comes from a generalized Cartan matrix because of the hypothesis on the order of the products  $s_0 s_i$ ). Then

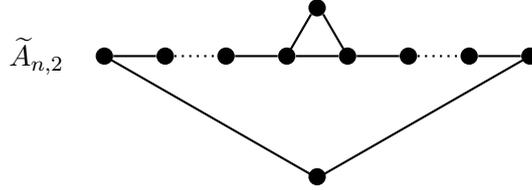
$$\beta^k(\Lambda_n) \begin{cases} = 0 & \text{if } k \geq 2n + 3, \\ > 0 & \text{if } k = 2n. \end{cases}$$

Hence the groups  $(\Lambda_{2n})$  have non-proportional sequences of  $\ell^2$ -Betti numbers, hence they are pairwise non measure equivalent in view of [Gab02, 6.3].  $\square$

### 4.3 An explicit family of Coxeter diagrams

In this section we deal with a concrete family of Coxeter diagrams having a decomposition as in Theorem 4.2.6. The first aim is to exhibit a concrete family as before.

However, in this example we can say more: the previous arguments compute all  $L^2$ -Betti numbers of a pair  $(X, G)$  in  $\mathcal{B}t$  with Weyl groups corresponding to these particular diagrams. Let  $(W, S)$  be the Coxeter system defined by the diagram  $\tilde{A}_{n,2}$  (with  $n + 1$  generators and  $n \geq 3$ ) as below:

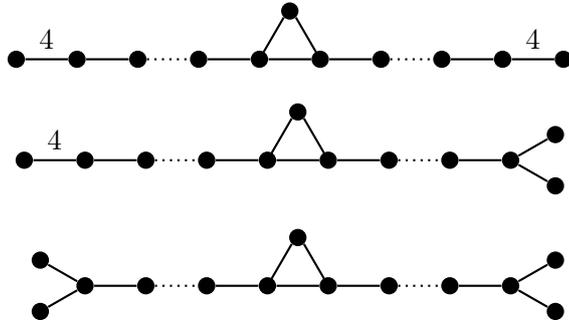


### 4.3.1 Maximal finite parabolic subgroups

Let  $s_1, \dots, s_n$  be the generators corresponding to the affine subgroup of type  $\tilde{A}_{n-1}$  in  $W$  and  $s_0$  be the remaining generator, so that  $\langle s_0, s_1, s_2 \rangle$  is an infinite affine Coxeter group of type  $\tilde{A}_2$ . To obtain finite parabolic subgroups one has to consider subsets  $J \subset S$  that do not contain these three generators simultaneously.

For simplicity, call  $W_i = \langle S \setminus \{s_i\} \rangle$  and for  $i \neq j$  call  $W_{i,j} = \langle S \setminus \{s_i, s_j\} \rangle$ . The subgroups  $W_1$  and  $W_2$  are of type  $A_{n-1}$ , thus finite. Therefore, they are maximal finite parabolic subgroups of  $W$ . The subgroup  $W_0$  is of type  $\tilde{A}_{n-1}$ , thus infinite. It is affine, so every proper parabolic subgroup of  $W_0$  is finite. Therefore the parabolic subgroups  $W_{0,i}$  for  $1 \leq i \leq n$  are maximal finite.

*Remark.* We can proceed in the same way for Kac-Moody groups with Weyl groups coming from the same alteration of an affine Coxeter diagram.



In each case, the combinatorics of maximal finite parabolic subgroups is the same, therefore the Davis chamber we obtain is simplicially isomorphic to that of a Coxeter system of type  $\tilde{A}_{n,2}$  and hence the following results also apply to Coxeter systems having these diagrams.

### 4.3.2 Cohomology of subcomplexes

The following result completes the computation of the cohomology of  $D_\sigma$  for all  $\sigma \subseteq \Delta$ . More precisely, the simplices  $\sigma$  appearing in the following theorem are exactly those whose corresponding parabolic subgroup  $W_J$  of  $W$  is finite but with  $W_{J^c}$  infinite. Let  $\Delta_i$  be the face of codimension 1 of  $\Delta$  corresponding to the generator  $s_i$  of  $W$  and  $D_i = D \cap \Delta_i$ .

**Theorem 4.3.1.** *Let  $D$  be the Davis chamber of  $(W, S)$  of type  $\tilde{A}_{n,2}$ .*

1. *For  $\sigma = \Delta_0$ , the space  $D_\sigma = \bigcup_{j \neq 0} D_j$  has the cohomology of an  $(n-2)$ -dimensional sphere.*
2. *For  $\sigma = \Delta$ , the space  $D_\sigma = D \cap \partial\Delta$  has the cohomology of an  $(n-2)$ -dimensional sphere.*
- Let  $\tau \subset \Delta$  be the simplex such that  $\Delta_\tau = \Delta_0 \cup \Delta_1 \cup \Delta_2$ .
3. *The space  $D_\tau$  has the cohomology of the circle.*
4. *For  $\tau \subsetneq \sigma \subsetneq \Delta$ , the space  $D_\sigma$  is contractible.*

*Proof.* Assertions 1 and 3 follow from 4.2.5 since  $W_{\Delta_0}$  and  $\langle s_0, s_1, s_2 \rangle$  are infinite affine Coxeter groups of type  $\tilde{A}_{n-1}$  and  $\tilde{A}_2$ . It remains to prove assertions 2 and 4, that is, the cases  $\tau \subsetneq \sigma \subseteq \Delta$ . Denote  $A_I = D_1 \cup D_2 \cup \bigcup_{k \in I} D_k$ , for the corresponding nonempty  $I \subseteq \{3, \dots, n\}$  so that  $D_\sigma = D_0 \cup A_I$ . Denote  $D_{ij} = D_i \cap D_j$  and  $D_{ijk} = D_i \cap D_j \cap D_k$ . Our first goal is to prove that the intersection  $D_0 \cap A_I$  is contractible, write it as the following union:

$$D_0 \cap A_I = D_{0,1} \cup D_{0,2} \cup \bigcup_{l \in I} D_{0,l}.$$

First the union  $A'_I = \bigcup_{l \in I} D_{0,l}$  is contractible because of 4.2.2 (the group  $\langle s_0, s_i | i \in I \rangle$  is finite, so the same argument as in the proof of 4.2.3 works). Again by 4.2.2, the intersections  $A'_I \cap D_{0,1} = \bigcup_{l \in I} D_{0,1,l}$  and  $A'_I \cap D_{0,2} = \bigcup_{l \in I} D_{0,2,l}$  are contractible (the groups  $\langle s_0, s_1, s_i | i \in I \rangle \subseteq W_1$  and  $\langle s_0, s_2, s_i | i \in I \rangle \subseteq W_2$  are finite). Notice that  $D_{0,1} \cap D_{0,2} = (\Delta_0 \cap \Delta_1 \cap \Delta_2) \cap D = \emptyset$ . By the Nerve lemma,  $D_0 \cap A_I$  is homotopy equivalent to the nerve of the cover  $\mathcal{U} = \{D_{0,1}, D_{0,2}, A'_I\}$ , which is the barycentric subdivision of a simplex of dimension 1. Thus the intersection  $D_0 \cap A_I$  is contractible.

Therefore Mayer-Vietoris tells us that  $D_0 \cup A_I$  has the same cohomology as  $A_I$ . If  $I = \{3, \dots, n\}$ , we have  $A_I = D_{\Delta_0}$  so from the first assertion we know  $A_I$  has the cohomology of an  $(n-2)$ -dimensional sphere. Thus  $D_\Delta = D \cap \partial\Delta = D_0 \cup A_I$  has the cohomology of an  $(n-2)$ -dimensional sphere. If  $I \subsetneq \{3, \dots, n\}$ , then  $A_I$  is contractible because of 4.2.2, hence the union  $D_\sigma = D_0 \cup A_I$  is contractible.  $\square$

We now recover the corresponding results for  $L^2$ -Betti numbers of groups in  $\mathcal{Bt}$  with Weyl group  $\tilde{A}_{n,2}$ .

**Corollary 4.3.2.** *Let  $(X, G)$  be a building in  $\mathcal{Bt}$  of thickness  $q+1$  and Weyl group of type  $\tilde{A}_{n,2}$  with  $n \geq 3$ . Normalize the Haar measure  $\mu$  on  $G$  so that the stabilizer  $G_\Delta$  of an alcove  $\Delta$  has measure 1. Then we have:*

$$\beta^k(G) = \begin{cases} \dim_G L^2(G)^\Delta + \dim_G L^2(G)^{\Delta_0} & k = n-1, \\ \dim_G L^2(G)^\tau & k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The sum in the formula 4.1.9 runs over  $\sigma \subseteq \Delta$  such that its corresponding parabolic subgroup  $W_J, J \subseteq S$  is finite and such that  $W_{J^c}$  is not finite. Such simplices are exactly the ones treated in the theorem. Their non-vanishing cohomology groups give the result.  $\square$

Thus, by the same arguments as in the previous section, the Künneth formula gives the following more precise statement for Kac-Moody lattices  $\Lambda$  having Weyl group of type  $\tilde{A}_{n,2}$ . Moreover, the theorems we use are quantitative: they give us an explicit formula for  $\beta^k(\Lambda)$ . We do not give the formulas since the only information that matters to us is in which degrees  $\beta^k(\Lambda)$  vanishes and in which it does not vanish for large  $q$ .

**Corollary 4.3.3.** *Let  $\Lambda$  be a center-free Kac-Moody group over a finite field  $\mathbb{F}_q$  with Weyl group of type  $\tilde{A}_{n,2}$ . For  $q$  large enough,*

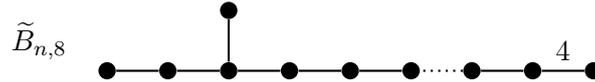
$$\beta^k(\Lambda) \begin{cases} > 0 & k \in \{4, n+1, 2n-2\}, \\ = 0 & \text{otherwise.} \end{cases}$$

Moreover,  $\Lambda$  is infinite finitely presented, Kazhdan and simple.

## 4.4 Sphericity and cohomological finiteness

In this section we discuss connections between  $n$ -sphericity and finiteness properties  $F_n$  and  $FP_n$  of a Kac-Moody group (over a finite field). Theorem 4.1.6 3, says that for large  $q$ , the 2-sphericity condition implies property  $F_2$ , that is, finite presentation. The converse is still a conjecture, but it has at least been proven in particular cases [AG19]. Much less is known for higher finiteness properties. Abramenko obtained some partial results in this direction [Abr04].

Discrete Kac-Moody groups with Weyl group of type  $\tilde{A}_{n,2}$  are finitely presented (at least for  $q > 6$ ) since they are 2-spherical, but they are not 3-spherical because of the subdiagram of type  $\tilde{A}_2$  that we introduced. We can ask if it is possible to obtain stronger finiteness properties for non-affine Kac-Moody groups. Here we present a family of non-affine Coxeter diagrams that are 8-spherical but not 9-spherical. We call  $\tilde{B}_{n,8}$  the Coxeter diagram with  $n+1$  generators as below:



The Davis chamber of a Coxeter group of type  $\tilde{B}_{n,8}$  is similar to that of  $\tilde{A}_{n,2}$ . It contains a subdiagram of type  $\tilde{E}_8$  when  $n \geq 9$ . If we remove the generator  $s_0$  at the left, we obtain an affine subdiagram of type  $\tilde{B}_{n-1}$  with generators  $s_1, \dots, s_n$ . If we take out any other generator  $s_i$  ( $i = 1, \dots, 8$ ) of  $\tilde{E}_8$ , we obtain a maximal finite parabolic subgroup. Hence the Davis chamber consists of an affine part ( $n$  cubes of dimension  $n-1$ ) and 8 cubes of dimension  $n$ . The result for the diagram  $\tilde{A}_{n,2}$  remains valid for  $\tilde{B}_{n,8}$ .

**Theorem 4.4.1.** *Let  $D$  be the Davis chamber of  $(W, S)$  of type  $\tilde{B}_{n,8}$ .*

1. For  $\sigma = \Delta_0$ ,  $D_\sigma = \bigcup_{j \neq 0} D_{s_j}$  has the cohomology of an  $(n-2)$ -dimensional sphere.
2. For  $\sigma = \Delta$ ,  $D_\sigma = D \cap \partial \Delta$  has the cohomology of an  $(n-2)$ -dimensional sphere.
3. Let  $\tau \subset \Delta$  be the simplex such that  $\Delta_\tau = \bigcup_{i=0}^8 \Delta_i$ . The space  $D_\tau$  has the cohomology of a 7-dimensional sphere.
4. For  $\tau \subsetneq \sigma \subsetneq \Delta$ ,  $D_\sigma$  is contractible.

*Proof.* The proof is almost the same as for  $\tilde{A}_{n,2}$ : assertions 1. and 3. follow from 4.2.5 while 2. and 4. need some adjustment. We keep the same notations as in the other proof. The only difference is that in order to show  $D_0 \cup A_I$  has the same cohomology as  $A_I$  for some non-empty  $I \subseteq \{9, \dots, n\}$ , one has to show that the nerve of some cover of  $D_0 \cap A_I$  by 9 elements is the boundary of a simplex of dimension 8 minus one face of codimension 1, showing that  $D_0 \cap A_I$  is contractible.  $\square$

Thus we obtain the same corollaries for Kac-Moody groups as in the previous section. The complete Kac-Moody groups with Weyl group of type  $\tilde{B}_{n,8}$  have non-vanishing  $L^2$ -Betti numbers (for large thickness) exactly in degrees  $n - 1$  and 8. At the level of discrete Kac-Moody groups, we obtain an infinite family of infinite finitely presented, Kazhdan, 8-spherical, simple groups that are pairwise non-measure equivalent. By the Künneth formula, their  $\ell^2$ -Betti numbers vanish except in degrees  $16, n + 7$  and  $2n - 2$ .

One cannot have better sphericity properties for a non-affine irreducible Coxeter diagram. The following proposition has to be stated somewhere. The author did not find a reference, so we give the proof here.

**Proposition 4.4.2.** *A 9-spherical irreducible Coxeter group is either finite or affine.*

*Proof.* The proof consists of ruling out all possibilities by looking at the classification of finite and affine Coxeter groups. More precisely, we look at two families of integers: the valencies of vertices of the Coxeter diagram as a graph and the labelling of edges of the diagram.

Let  $(W, S)$  be a 9-spherical irreducible Coxeter system such that  $|S| \geq 9$ . Its Coxeter diagram is connected by irreducibility. If the valency of every vertex of the diagram is  $\leq 2$ , then the Coxeter diagram of  $W$  without labelling is of type  $A_n$  or  $\tilde{A}_n$  with  $n \geq 9$ . If we label edges of a diagram of type  $A_n$  by numbers  $\geq 4$ , the 5-sphericity of  $W$  rules out all possibilities, except having extremal edges labelled by 4: the possible diagrams for  $W$  are  $A_n, B_n = C_n$  or  $\tilde{C}_n$ . If we label an edge of a diagram of type  $\tilde{A}_n$  by a number  $\geq 4$ , the associated group is not 5-spherical, hence the only possibility for  $W$  in this case is to have a diagram of type  $\tilde{A}_n$ .

If not, there exists a vertex  $y$  of valency  $\geq 3$ . The valency of  $y$  cannot be  $\geq 4$  since this would give directly an infinite subgroup of rank 5. From now on  $y$  has valency 3. Call  $y_1, y_2$  and  $y_3$  the three neighbors of  $y$ . None of these vertices can have valency  $\geq 3$  since this would give rise to an infinite subgroup of rank 6. The three neighbors cannot have valency  $\geq 2$  simultaneously since the Coxeter graph would contain a subgraph of type  $\tilde{E}_6$  that corresponds to an infinite subgroup of rank 7. Thus we can suppose  $y_3$  has valency 1.

If both  $y_1$  and  $y_2$  have valency 2, then there is a subdiagram of type  $\tilde{E}_7$  or of type  $\tilde{E}_8$ , which respectively correspond to infinite subgroups of rank 8 and 9. Again, 9-sphericity rules out these possibilities.

We may assume  $y_2$  and  $y_3$  have valency 1, and  $y_1$  has valency  $\geq 2$ . This implies  $W$  is of type  $\tilde{B}_n$  or  $\tilde{D}_n$ . Indeed, the graph of  $W$  has to contain  $\tilde{B}_n$  or  $\tilde{D}_n$  as a subgraph, with possible vertices of higher valency. If such a vertex exists, then again there is a subdiagram of type  $\tilde{E}_7$  or of type  $\tilde{E}_8$ .  $\square$

*Remark.* In view of this proposition, Theorem 4.1.7 does not tell us anything about cohomological vanishing in degrees  $\geq 8$  for groups with BN-pair acting on a building

of finite thickness with non-affine Weyl group. When the Weyl group is affine, the same vanishing result was obtained by Garland in [Gar73].

*Remark.* In [SWZ19], it is used that properties  $F_n$  are invariant by quasi-isometries [Alo94]. They construct finitely presented simple groups that are  $F_{n-1}$  but not  $F_n$  for each  $n$ . This gives an infinite family of infinite finitely presented simple groups that are pairwise not quasi-isometric.

If one could prove that property  $F_n$  implies  $n$ -sphericity for a Kac-Moody group, then the previous proposition shows that every non-affine Kac-Moody group over a finite field is at most  $F_8$ . Thus the method of [SWZ19] could not work for non-affine Kac-Moody groups if the previous conjecture is true.

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**Titre :** Sur la géométrie à grande échelle des groupes semi-simples : annulation et non annulation de la cohomologie  $L^p$  des groupes archimédiens et non archimédiens

**Mots clés :** Cohomologie  $L^p$ , groupes de Lie, immeubles, quasi-isométrie, équivalence mesurable

**Résumé :** Cette thèse s'intéresse à des invariants de géométrie à grande échelle en théorie de Lie. On étudie notamment un invariant de nature cohomologique : la cohomologie  $L^p$  (où  $p > 1$ ), qui donne des invariants de quasi-isométrie dans divers contextes et généralise la cohomologie  $L^2$ . On s'intéresse à la calculer dans le cas des groupes semi-simples sur des corps locaux (archimédiens ou pas) et pour différents types d'immeubles. Ce manuscrit comporte quatre chapitres. Le premier introduit les concepts de théorie de Lie, cohomologie des groupes et cohomologie  $L^p$  nécessaires dans les chapitres suivants. Le deuxième reproduit la prépublication [LN23]. Le résultat princi-

pal est l'annulation de la cohomologie  $L^p$  en degré 2 pour la plupart des groupes semi-simples de rang au moins 3 sur des corps locaux. Le troisième reproduit la prépublication [LN22]. On étudie l'homologie  $\ell^p$  en degré maximal des immeubles, sa relation avec d'autres invariants tels que la dimension cohomologique virtuelle des groupes de Coxeter et la dimension conforme des immeubles hyperboliques au sens de Gromov. Le quatrième reproduit l'article [LN21], où, à l'aide des nombres de Betti  $L^2$  de groupes agissant sur des produits d'immeubles, on explicite une famille infinie de groupes simples de présentation finie non mesurablement équivalents.

**Title :** On the large scale geometry of semisimple groups: vanishing and non-vanishing of  $L^p$ -cohomology of Archimedean and non-Archimedean groups

**Keywords :**  $L^p$ -cohomology, Lie groups, buildings, quasi-isometry, measure equivalence

**Abstract :** This thesis deals with large scale geometric invariants in Lie theory. More precisely, we study an invariant of cohomological nature called  $L^p$ -cohomology (where  $p > 1$ ) that gives quasi-isometry invariants in different settings and generalizes  $L^2$ -cohomology. We are interested in computing  $L^p$ -cohomology for semisimple groups over local fields (both Archimedean and non-Archimedean) and different types of buildings. The manuscript is divided into four chapters. The first introduces the concepts on Lie theory, group cohomology and  $L^p$ -cohomology that will be needed in subsequent chapters. The second reproduces the preprint [LN23]. Its main result is

vanishing of  $L^p$ -cohomology in degree 2 for most semisimple groups over local fields of split rank at least 3. The third reproduces the preprint [LN22]. In it, we study top degree  $\ell^p$ -homology of buildings, its relation with other invariants such as virtual cohomological dimension of Coxeter groups and conformal dimension of Gromov-hyperbolic buildings. The fourth reproduces the article [LN21], where, using  $L^2$ -Betti numbers of groups acting on products of buildings, we exhibit a first family of finitely presented simple groups that lie in infinitely many measure equivalence classes.