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# Perturbative renormalization of field theories breaking translation invariance

Majdouline Borji

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# Perturbative renormalization of QFTs which are not translation invariant

Thèse de doctorat de l'Institut Polytechnique de Paris  
préparée à l'École Polytechnique

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Thèse présentée et soutenue à Palaiseau, le 24 Octobre 2023, par

**MAJDOULINE BORJI**

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ  
"الْحَمْدُ لِلَّهِ الَّذِي بَفِضِهِ تَمَّ الصَّالِحَاتِ"

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# Résumé

En théorie quantique des champs, la renormalisation est un ensemble de techniques utilisées pour donner un sens à la limite du continu. Une difficulté célèbre à laquelle se heurte la théorie quantique des champs est celle des divergences ultraviolettes. L'idée principale de la renormalisation est d'absorber ces divergences en ajoutant un nombre fini de "contre-termes" au lagrangien de départ. Cette procédure permet de donner un sens aux observables physiques qui ont des valeurs finies. La difficulté à établir une telle procédure vient du problème des divergences enchevêtrées. Considérons l'exemple simple d'un champ scalaire massif  $\phi$  dans l'espace euclidien  $\mathbb{R}^4$  avec l'auto-interaction quartique  $\frac{\lambda}{4!}\phi^4$ . Parmi les approches rigoureuses à ce problème on retrouve les équations de Polchinski, dites équations de flot. Ces équations constituent un système dynamique dont les variables sont les fonctions de corrélation  $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$  à  $l$  boucles et  $n$  points externes. En imposant des conditions aux bords mixtes, c'est-à-dire aux deux échelles  $\Lambda = 0$  et  $\Lambda = \Lambda_0$ , les équations de flot définissent de manière unique les fonctions de corrélation. L'avantage de ces équations est le schéma inductif qu'elles procurent et qui permet de borner inductivement les fonctions de corrélation uniformément par rapport à la coupure ultraviolette  $\Lambda_0$ . La borne obtenue implique la convergence des fonctions de corrélation quand  $\Lambda_0 \rightarrow \infty$ .

L'objectif de cette thèse est de comprendre comment la renormalisation est affectée par la brisure de la symétrie de translation. Dans le contexte de cette thèse, nous nous intéressons principalement aux deux exemples de la théorie scalaire dans  $\mathbb{R}^4$  régularisée par un réseau et celui de la théorie scalaire dans le demi-espace  $\mathbb{R}^+ \times \mathbb{R}^3$  euclidien appelée modèle semi-infini. Ces deux projets sont indépendants l'un de l'autre. Nous considérons en premier le modèle semi-infini en établissant une preuve rigoureuse de la renormalisation de cette théorie en nous basant sur les équations de flot. D'un point de vue théorie de champ, ce problème a été étudié pour la première fois en 1981 par Diehl et Dietrich [1]- [2] et indépendamment par Symanzik [3] en inspectant les divergences des graphes de Feynman à une et deux boucles. Le problème à un nombre arbitraire de boucles est resté longtemps ouvert. Dans cette thèse, nous avons résolu le problème de renormalisation perturbative de cette théorie à tous les ordres en perturbation. La difficulté de ce problème réside dans la brisure de l'invariance par translation par la surface bord du demi-espace qui nécessite de procéder dans l'espace des positions. Cela a pour conséquence que les fonctions de corrélation du système sont des distributions. Pour résoudre

ce problème par les équations de flot, le point de départ est de définir rigoureusement la théorie quantique de champ associée à ce modèle. Cela se base sur l'intégrale de chemins qui est définie à partir de la mesure gaussienne associée au modèle considéré. Une mesure gaussienne de moyenne nulle est déterminée de façon unique par son opérateur de covariance, dont le noyau est donné par le propagateur. Le propagateur est défini mathématiquement comme étant le noyau de la résolvante du Laplacien. L'existence de ce noyau est garantie par le calcul fonctionnel qui nécessite que l'opérateur considéré soit autoadjoint. Dans le cas d'un demi-espace, il existe plusieurs extensions autoadjointes du Laplacien et chacune définit une condition au bord physique ainsi qu'une théorie quantique de champ indépendante. Ces conditions aux bords pour le modèle semi-infini sont du type Dirichlet, Neumann et Robin. Le support des mesures gaussiennes associées aux différents propagateurs Dirichlet, Neumann et Robin est singulier, dans le sens qu'il contient des distributions, pour lesquelles le produit au même point n'est pas défini. Par conséquent, une régularisation par coupure ultraviolette est nécessaire. Celle-ci agit sur le support des mesures gaussiennes en le réduisant à des fonctions indéfiniment dérivables. Ainsi, il devient possible d'introduire l'auto-interaction  $\phi^4$ . La mesure gaussienne combinée à l'interaction permet de définir l'action effective par l'intégrale de chemins. Les propriétés de dérivation de la mesure gaussienne impliquent par la suite l'équation de flot. Une série formelle en nombre de boucles permet de déduire l'équation de flot vérifiée par les "fonctions" de corrélation régularisées. Comme l'invariance par translation est brisée par la surface, les "fonctions" de corrélation dans ce cas sont des distributions. Nous introduisons une classe de fonctions tests sur lesquelles ces distributions agissent et nous bornons uniformément ces distributions par rapport à la coupure ultraviolette, en fixant dans un premier moment des conditions de renormalisation BPHZ. Cela a pour conséquence que les contre-terms sont des fonctions qui dépendent de la position dans le demi-espace.

La deuxième partie de cette thèse est consacrée à une étude détaillée des contre-terms. Nous établissons qu'il est possible de choisir ordre par ordre des conditions de renormalisation pour lesquelles les contre-terms sont des constantes. En outre, ces contre-terms sont donnés par ceux de la théorie invariante par translation et deux contre-terms surface proportionnels à  $\int_S \phi^2$  et  $\int_S \phi \partial_n \phi$  dans le cas de conditions aux bords du type Robin et Neumann. Pour Dirichlet, les contre-terms usuels de la théorie invariante par translation sont suffisants pour rendre la théorie finie.

La dernière partie de cette thèse est dédiée à l'étude de la théorie scalaire massive  $\phi^4$  régularisée par réseau. Nous démontrons qu'elle est renormalisable, et que les symétries euclidiennes sont rétablies dans la limite du continu.

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# Introduction

Quantum field theory was originally developed as a theoretical framework that combines classical field theory, special relativity, and quantum mechanics and has become the general theoretical framework to study physical systems with an infinite (or large) number of degrees of freedom.

A rigorous mathematical analysis of quantum field theories is faced with the problem that path integrals describing systems in field theory are generally not defined. There exists a complete theory of Gaussian measures that apply to free theories. However, for the interacting case, a rigorous mathematical description starts from regularized versions of the theory, where the number of degrees of freedom in space and momentum has been (in some sense) made finite. This is a task common to all regularizations, such as simple momentum cutoff, dimensional regularization and lattice cutoff. An essential task of renormalization is to prove that the correlation functions have uniform limits in the cutoffs. For these limits a sequence of axioms must be satisfied in order to construct Euclidean quantum field theory. These are the Osterwalder-Schrader axioms [4, 5]. In general, convergence is achieved by appropriately adjusting a finite number of bare parameters of the action and by a rescaling of the fields.

The scalar field theory in  $\mathbb{R}^4$  with a  $\phi^4$  interaction is known to be renormalizable in the sense that when the momentum cutoff is sent to infinity, the correlation functions stay finite. Several proofs exist to prove the renormalizability of  $\phi_4^4$  theory. Among them, the perturbative proof of renormalization based on Wilson flow equations performed by Polchinski in the seminal paper [6] in 1984. The flow equation allows to describe theories by an effective action  $L^{\Lambda, \Lambda_0}$ , depending on a scale  $\Lambda$  with  $0 \leq \Lambda \leq \Lambda_0 < \infty$  for Euclidean quantum field theories in the continuum with a momentum cutoff. Here,  $\Lambda$  is a parameter that flows from 0 to  $\Lambda_0$  and  $\Lambda_0$  denotes the ultraviolet cutoff.  $L^{\Lambda, \Lambda_0}$  should satisfy the following conditions:

- At the ultraviolet cutoff  $\Lambda = \Lambda_0$ ,  $L^{\Lambda, \Lambda_0}$  coincides with the bare interaction.
- For  $\Lambda < \Lambda_0$ ,  $L^{\Lambda, \Lambda_0}$  is obtained upon integration of the field degrees of freedom which propagate with momenta  $p$  roughly between  $\Lambda$  and  $\Lambda_0$ .

The flow equation allows to give a rigorous proof of perturbative renormalizability of different quan-

tum field theories. The conventional proof of perturbative renormalizability is based on an approach in terms of Feynman diagrams. In perturbation theory, the correlation functions are represented as a sum of Feynman integrals. A central tool in this approach is the existence of a power-counting theorem, which provides a criterion on the convergence of Feynman integrals by counting suitably defined UV-divergence degrees [7]. The Feynman diagram associated to a Feynman integral, together with all its subdiagrams, are required to have negative UV-divergence degrees to ensure the absolute convergence of the Feynman integral. The second step in this approach consists in formulating a general renormalization prescription [8] to subtract the UV-divergences, order by order in perturbation theory, as for example the BPHZ-subtraction scheme (the Bogoliubov-Parasiuk-Hepp-Zimmermann finite part prescription). It is the combinatorics of counting divergence degrees and of removing divergences which makes the proof cumbersome. A further issue is to show that a theory with some symmetry properties can be renormalized in such a way that the symmetry is preserved, which is highly nontrivial for theories with gauge symmetries. This concerns in particular field theories like Yang-Mills theories [9, 10], QCD and the electroweak standard model. The flow equation allows for a simple transparent rigorous solution of the perturbative renormalization problem without introducing Feynman diagrams. It is based on a tight inductive scheme wherefrom bounds on the regularized Schwinger functions implying renormalizability can be deduced. Renormalizability of a quantum field theory implies that the unregularized (renormalized) correlation functions exist in the sense that they are both IR (in the limit of very small momenta) and UV finite (in the limit of very large momenta). Finite limits are achieved by imposing a finite set of renormalization conditions on a physical scale that is independent of the UV cutoff  $\Lambda_0$ . For simplicity, throughout this thesis we only consider the case in which the field is massive to avoid any IR problems. Proving perturbative renormalizability then amounts to show the existence of the large UV cutoff limit  $\Lambda_0 \rightarrow \infty$ .

The method of flow equations has been extensively used to prove perturbative renormalizability in the context of various quantum field theories [11–13]. The aim of this PhD thesis is to investigate the perturbative renormalizability of quantum (scalar) field theories that break translation invariance basing our approach on the Polchinski flow equations. One may ask if the renormalizability of a given field theory depends only on the interaction introduced and the dimension of space-time, or whether it depends also on the geometrical and topological properties of the considered space-time. In Chapter [7], we analyze the  $\phi^4$  theory in the Euclidean space  $\mathbb{R}^4$  regularized by a lattice which is an example of a regularization scheme that breaks translation invariance. We prove that the theory is renormalizable and that the Euclidean symmetries are restored in the continuum limit. In Chapters [4–5], we analyse a more explicit breaking of translation invariance by the presence of a boundary, by studying the massive scalar field model on a half-space, as a prototype model to study the surface effects on renormalization. The semi-infinite geometry denotes a  $d$ -dimensional half-space bounded in one direction by a  $d - 1$ -dimensional plane of infinite extent, which is the surface  $S$  of the system. In the context of this

thesis, the  $d$ -dimensional semi-infinite geometry is the space  $\mathbb{R}^+ \times \mathbb{R}^{d-1}$ .

The renormalization of the semi-infinite massive scalar field theory was firstly motivated by trying to understand the surface effects on critical behaviour in condensed matter physics. At first sight, it may seem that surface effects are not relevant in systems with short-range interactions. However, experiments showed that local observables are sensitive to the presence of boundaries, in the sense that their behaviour deep inside the bulk and on the surface is considerably different. In the spin polarized low-energy electron diffraction experiment (SPLEED) the surface magnetization of the Heisenberg ferromagnet Ni close to the bulk transition temperature ( $T_c \approx 630K$ ) was measured by Celotta et al. [14] at the Ni (100) surface and also by Alvaredo et al. [15] at the Ni (100) surface. The quantity observed in this experiment is the scattering asymmetry between the scattering with the electron spin parallel and antiparallel to the surface magnetization, which turns out to be proportional to local mean magnetization  $m_1$ . In the SPLEED performed by Alvaredo et al. [15], they showed that  $m_1$  behaves as  $m_1 \sim |T - T_c^b|^{\beta_1}$  with an exponent  $\beta_1 \sim 0.8$ , which is distinctly different from the bulk value  $\beta \sim 0.33$  implying that  $m_1$  vanishes faster than the bulk magnetization  $m_b$ .

From a theoretical standpoint, the semi-infinite criticalities were first sparked by Fisher in 1971 and by Binder and Hohenberg [16] who later presented a Monte-Carlo study [17]. Meanwhile, the semi-infinite mean-field theories were worked out by Lubensky and Rubin [18, 19] who studied a model of ferromagnetically coupled classical spins on a semi-infinite lattice and provided via a mean-field approach a qualitatively correct understanding of the different phases undergone by the system which are: the ordinary, extraordinary, surface and special transitions. Later, the phenomenological theory of scaling [20, 21] was generalized to surfaces, and implied relations between bulk critical exponents and the additional surface critical exponents, needed to describe the singular behaviour of surface related properties. However, the mean field approach is known to yield incorrect results for bulk properties near their critical temperature if the space dimension is below its upper critical value  $d = 4$  and such breakdown is expected to hold for local critical properties. Therefore, the renormalization of the model is necessary if one wants to go beyond the mean-field approximation. The first study from a field theoretical approach was performed by Diehl and Dietrich in [1, 2], in which they studied the critical behavior of the semi-infinite system using renormalization group methods. They considered the ordinary [22] and special transitions [2] which correspond respectively to the Dirichlet and Robin boundary conditions and found that in addition to the usual two bulk counter-terms, an additional surface counter-term is needed to make the two-point function finite in the case of the Dirichlet boundary condition. For the Robin boundary condition, two surface counter-terms are needed. The calculations were performed to two-loop order using dimensional regularization, and the surface counter-terms were obtained by inserting the operators  $\lim_{z \rightarrow 0} \partial_z \phi(z, x)$  where  $x \in \mathbb{R}^3$  in the case of Dirichlet b.c., and  $\phi(0, x)$ ,  $\phi^2(0, x)$  in the case of the Robin b.c., where  $\phi$  is the considered scalar field.

The semi-infinite model was also addressed by Symanzik in his study of the Schrödinger representa-

tion for renormalizable quantum fields [3] in which he discusses the renormalization of surface operators in a different context, but also finds that surface counter-terms are required to make the two-point function finite. So far no rigorous proof of perturbative renormalizability of the semi-infinite model was performed. Let us explain why a conventional proof of renormalization based on the analysis of Feynman diagrams is complicated to achieve in the context of a boundary field theory: Apart from the combinatorial aspect of this method as compared to the flow equations, there is an additional complexity that stems from the breaking of translation invariance. For translationally invariant systems, one has the advantage that in momentum space one has to deal only with functions analytic in the external momenta. Given a Feynman diagram, it is made finite by subtracting the divergences of each subdiagram by applying a Taylor operator in the external momenta of the subdiagram. The renormalized Feynman integral is defined in such a way that the ultraviolet (UV) divergence degrees of all subdiagrams are negative. The convergence of such integrals is ensured by a power counting theorem [7]. Unfortunately, these methods assume a rational structure of the Feynman integrands and hence do not apply to diagrams corresponding to Feynman graphs in position space. The main difficulty encountered when one wants to generalize these methods to position space, is the distributional aspect of the considered objects. It is tempting to work in momentum space as it is the case for the translationally invariant theory, for which the momentum-space representation is characterized by computational simplicity. However, in the case of a space with a surface, the momentum-space representation does not simplify computations in perturbation theory. In this case, the correlators in momentum space have a complicated distributional structure, different from its translationally invariant counter-part which is simply given by a product of a function and a momenta conservation Dirac distribution. Besides the technical difficulties, there are some fundamental problems that explain the complexity of working in a full momentum space. Lubensky and Rubin [18, 19] performed a one-loop order renormalization group analysis of the Dirichlet case in momentum space. However, their approach is complicated for two reasons: First, the calculation scheme is not well-suited for higher-order computations and second their approach is essentially restricted to the Dirichlet case and can not be generalized to Neumann and Robin boundary conditions. The particularity of the Dirichlet case [1] is that it corresponds to the ordinary transition which is described by a fixed point, for which the value of  $c = +\infty$  is known to all orders of perturbation theory and the eigenfunctions of the Dirichlet Hamiltonian do not change under renormalization group transformation. However, the Neumann and Robin cases are characterized by  $c < +\infty$  which is renormalized at each order, and this implies that the eigenfunctions corresponding to the Neumann/Robin Hamiltonian verify at each order of perturbation theory a different Robin boundary condition with a different renormalized Robin parameter. This generates a complicated non-diagonal  $c$ -dependent quadratic term and even a more complicated  $c$ -dependent four point function. In

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<sup>1</sup> $c$  denotes the Robin parameter which in the case of Dirichlet boundary conditions is  $+\infty$ .

position-space or mixed  $pz$ -representation<sup>2</sup>, the renormalized Robin parameter can be simply obtained from the effective action. In view of these reasons, we prefer to work in a  $pz$ -representation for a perturbative analysis of the renormalizability of the semi-infinite model.

The goal of this thesis is to investigate the problem of perturbative renormalization of the semi-infinite model. How does it differ technically from the translationally invariant case? What are the steps needed to prove perturbative renormalizability by the flow equations in a mixed position-momentum space? What are the bounds that lead to perturbative renormalizability and more importantly how are the translationally invariant model and the semi-infinite model related to each other?

Before tackling these questions, we give in chapter 1 an overview of the flow equations. In section 1.1.1 and 1.1.2, we start by introducing the basic ingredients to write rigorously the Euclidean (free) quantum scalar field theory. In section 1.1.3, we write the flow equations and we explain how these are used in establishing a rigorous proof of perturbative renormalizability for the basic example of the massive scalar field theory with a  $\phi^4$  interaction in  $\mathbb{R}^4$ . We conclude this chapter with a detailed summary of the rest of the thesis. In chapter 2, we give a rigorous computation of the propagators in  $\mathbb{R}^+ \times \mathbb{R}^{d-1}$  together with the possible boundary conditions for the semi-infinite model. In chapter 4, we prove the renormalizability of the semi-infinite model using BPHZ renormalization conditions in the mixed position-momentum space. As we will see, this choice of renormalization conditions implies having position dependent counter-terms in the effective action. In chapter 5, we go one step further in studying the renormalizability of the model, by investigating in more details the effect of the presence of the surface on the counter-terms. The main goal of chapter 5 is to prove that for a particular choice of renormalization conditions, the theory is renormalized by the bulk counter-terms and by surface counter-terms proportional to  $\int_S \phi^2$  and  $\int_S \phi \partial_n \phi$  with  $S$  denoting the surface of the half-space. Through out this thesis, the bulk theory denote the scalar field model in  $\mathbb{R}^4$  with a quartic interaction supported on the half-space. We treat this theory in detail in chapter 6 and establish that it is renormalized with the usual counter-terms of the translationally invariant theory, modulo the surface counter-terms  $\int_S \phi^2$  and  $\int_S \phi \partial_n \phi$ . Finally, in chapter 7, we give a rigorous proof of perturbative renormalizability of the massive scalar field  $\phi_4^4$  model regularized by a lattice. The novelty of this work lies in giving a proof of the restoration of the Euclidean symmetries in the continuum limit. We end the thesis by some concluding remarks and by presenting a few future perspectives.

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<sup>2</sup>in which Fourier transforms are taken partially w.r.t. the parallel components  $x$  of the vector  $(x, z)$ .





# Chapter 1

## Background and motivations

### 1.1 Perturbative Renormalization by flow equations: A simple example

In this section, we give an overview of rigorous renormalization theory based on the differential flow equations of the Wilsonian renormalization group. We consider the massive Euclidean  $\phi^4$  theory in  $\mathbb{R}^4$ , to which we will refer also later as the translationally invariant model. The method of the proof using the flow equations is based on the observation that the flow equations give access to a tight inductive scheme. This allows to obtain bounds on the regularized correlation functions which imply renormalizability. The correlation functions are regularized by a UV cutoff  $\Lambda_0$  and a flow parameter  $\Lambda$ . The bounds on these correlation functions are uniform in the cutoff and finite for  $\Lambda \rightarrow 0$ , which basically solves the renormalization problem. The aim of this section is to introduce on one hand the method of the proof of perturbative renormalizability with flow equations. On the other hand, this section provides a background to compare to the case that breaks translation invariance of the semi-infinite scalar field model which is the central object of the next chapters.

#### 1.1.1 A rigorous definition of the free Euclidean scalar field theory

The starting point in defining a Euclidean quantum field theory is to write the corresponding path integral, which can be defined through the theory of Gaussian measures. Denoting by  $\mathcal{S}(\mathbb{R}^4)$  the Schwartz space and by  $\mathcal{S}'(\mathbb{R}^4)$  its dual space, we have the nested triple

$$\mathcal{S}(\mathbb{R}^4) \subset L^2(\mathbb{R}^4) \subset \mathcal{S}'(\mathbb{R}^4).$$

In the sequel, the bracket  $\langle \cdot, \cdot \rangle$  denotes the usual scalar product in  $L^2(\mathbb{R}^4)$ . The aim of this section is to explain the interplay of three notions which are central in writing rigorously a Euclidean quantum field: the self-adjointness of the Hamiltonian, the propagator and the Gaussian measure. We will not

expose the general theory of Gaussian measures which is extensively studied in [23]- [24], but we rather restrict ourselves to the discussion of the example of the scalar field model in  $\mathbb{R}^4$ . Let us introduce first the following definitions:

**Definition 1.** (*Gaussian measure*) A measure  $d\mu$  defined on the nuclear space  $\mathcal{S}'(\mathbb{R}^4)$  is said to be Gaussian if for each finite dimensional subspace  $\mathcal{F} \subset \mathcal{S}(\mathbb{R}^4)$ , the restriction of  $d\mu$  to  $\mathcal{F}$  cylinder sets is Gaussian.

**Definition 2.** (*Covariance operator*) The covariance operator  $C$  on the nuclear space  $\mathcal{S}(\mathbb{R}^4)$  is defined as the continuous bilinear form

$$C : \mathcal{S}(\mathbb{R}^4) \times \mathcal{S}(\mathbb{R}^4) \longrightarrow \mathbb{R} \quad (1.1.1)$$

$$(f, g) \longmapsto \langle f, Cg \rangle, \quad (1.1.2)$$

with the additional property of being non-degenerate (i.e.  $\langle f, Cf \rangle = 0 \Rightarrow f = 0$ ).

The definitions [1] and [2] can be generalized to any nuclear space. Theorem A.4.1. in [25] ensures the existence of a unique Gaussian measure  $d\mu_C$  on the space of tempered distributions  $\mathcal{S}'(\mathbb{R}^4)$  having  $C$  as its covariance operator. The free Hamiltonian is given by the operator

$$-\Delta + m^2$$

with domain  $\mathcal{C}_c^\infty(\mathbb{R}^4)$ . This operator is symmetric but is not self-adjoint. It is though essentially self-adjoint and we denote the closure of the Laplacian by  $\bar{\Delta}$  with domain  $D(\bar{\Delta}) = H^2(\mathbb{R}^4)$ . Therefore, using the functional calculus, the inverse of  $-\bar{\Delta} + m^2$  exists and it is given by

$$(-\bar{\Delta} + m^2)^{-1} := \int_0^\infty dt e^{t\bar{\Delta}} e^{-tm^2}. \quad (1.1.3)$$

We recall the following result from [26], which proves that  $e^{t\bar{\Delta}}$  is an integral operator and gives explicitly its kernel:

**Theorem 1.** Let  $f \in L^\infty(\mathbb{R}^n)$ . If either (i)  $f \in L^2(\mathbb{R}^n)$  or (ii) [1]  $\mathcal{F}^{-1}(f) \in L^1(\mathbb{R}^n)$ , then

$$(f(-i\nabla)\phi)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathcal{F}^{-1}(f)(x-y)\phi(y)dy, \quad (1.1.6)$$

for all  $\phi \in L^2(\mathbb{R}^n)$ . The integral converges for all  $x$  in case (i) and for almost all  $x$  in case (ii).

<sup>1</sup> $\mathcal{F}^{-1}(f)$  is the inverse Fourier transform of  $f$  with the following conventions for the Fourier and inverse Fourier transforms

$$\mathcal{F}(f)(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x)e^{ix \cdot \xi} dx. \quad (1.1.4)$$

$$\mathcal{F}^{-1}(f)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(\xi)e^{-ix \cdot \xi} d\xi. \quad (1.1.5)$$

Using Theorem [1](#) for  $f(x) = e^{-tx^2}$ , we obtain

$$\left(e^{t\bar{\Delta}}\phi\right)(x) = \frac{1}{(2\pi t)^2} \int_{\mathbb{R}^4} e^{-\frac{(x-y)^2}{2t}} \phi(y) dy. \quad (1.1.7)$$

Hence, we deduce that  $(-\bar{\Delta} + m^2)^{-1}$  is an integral operator with a kernel given by

$$(-\bar{\Delta} + m^2)^{-1}(x, y) = \int_0^\infty dt e^{-tm^2} \frac{1}{(2\pi t)^2} e^{-\frac{(x-y)^2}{2t}}. \quad (1.1.8)$$

This also implies that  $(-\bar{\Delta} + m^2)^{-1}$  is a bounded bilinear form on  $\mathcal{S}(\mathbb{R}^4) \times \mathcal{S}(\mathbb{R}^4)$  so Theorem A.4.1. applies. Therefore, there exists a unique Gaussian measure associated to the operator  $(-\bar{\Delta} + m^2)^{-1}$  on  $\mathcal{S}'(\mathbb{R}^4)$ .

### 1.1.2 The regularized flowing propagator

The starting point in writing a Euclidean massive scalar field theory is the Gaussian measure. It is the support of this measure that defines the field. A Gaussian measure with mean zero is uniquely defined by its covariance operator. As we already explained in section [1.1.1](#) the covariance operator of this model is integral and the so-called propagator plays the role of its kernel. In Fourier space, the propagator simply reads

$$C(p) = \frac{1}{p^2 + m^2}.$$

The Gaussian measure associated to this propagator has its support included in  $\mathcal{S}'(\mathbb{R}^4)$  (i.e. the space of tempered distributions)<sup>[2](#)</sup>. The support of the Gaussian measure defines the space to which the field belongs. However, powers of distributions, or more generally products of distributions depending on the same variables are not well-defined [\[28\]](#). This means that the bare interaction, which includes local powers of the field and of its derivatives, cannot be given any mathematical meaning if the field is a distribution. Therefore, the propagator is regularized to restrict the support of the associated Gaussian measure to a subspace of  $\mathcal{C}^2(\mathbb{R}^4)$  (i.e. the space of functions with continuous zeroth, first and second derivatives). We choose the following regularization

$$C^{\Lambda, \Lambda_0}(p) = \frac{1}{p^2 + m^2} (R^{\Lambda_0}(p) - R^\Lambda(p)), \quad (1.1.9)$$

where  $\Lambda_0$  is the UV-cut-off and  $\Lambda$  a flow parameter  $0 \leq \Lambda \leq \Lambda_0$ . For all  $0 \leq \Lambda \leq \Lambda_0$ ,  $R^\Lambda(p)$  is a smooth regularizing function that satisfies the following constraints: It rapidly vanishes for  $p^2 > 2\Lambda^2$ , whereas for  $p^2 < \frac{\Lambda^2}{2}$  the function  $R^\Lambda(p)$  approaches the identity sufficiently fast. Furthermore for  $\Lambda = \Lambda_0$ , the

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<sup>2</sup>For a more precise characterization of this support, we refer to [\[27\]](#).

regularized propagator  $C^{\Lambda, \Lambda_0}$  vanishes. These conditions may still be relaxed and, in any case, there is a large amount of arbitrariness in the choice of  $R^\Lambda$ . A convenient choice is

$$R^\Lambda(p) = e^{-\frac{p^2+m^2}{\Lambda^2}}. \quad (1.1.10)$$

It has non-compact support but is rapidly decaying for large momenta. In the sequel, we will use the regularization (1.1.10). Note that for  $\Lambda \rightarrow 0$  and  $\Lambda_0 \rightarrow +\infty$ , the full propagator of the theory is recovered. The Gaussian measure with covariance (1.1.9) is supported with probability one on  $\mathcal{C}^\infty(\mathbb{R}^4)$  [24, 29]. Let us give a simple argument which proves that the support of the Gaussian measure associated to the regularized flowing propagator is included in  $\mathcal{C}^\infty(\mathbb{R}^4)$ . We define the continuous linear mapping

$$\begin{aligned} \Psi : \mathcal{S}'(\mathbb{R}^4) &\longrightarrow \mathcal{S}'(\mathbb{R}^4) \\ T &\longmapsto \mathcal{F}^{-1}(R^{\Lambda_0}) * T, \end{aligned}$$

where  $\mathcal{F}^{-1}(R^{\Lambda_0})$  is the inverse Fourier transform of  $R^{\Lambda_0}$ . We consider the following measure

$$\mu_{\Lambda_0} = \mu \circ \Psi^{-1}, \quad (1.1.11)$$

where  $\mu$  denotes the Gaussian measure associated with the unregularized propagator  $C$ . It is easy to see that (1.1.11) is again Gaussian with mean zero and its covariance is given by

$$C^{\Lambda_0}(p) = \frac{e^{-\frac{p^2+m^2}{\Lambda^2}}}{p^2 + m^2}. \quad (1.1.12)$$

The support of (1.1.11) then simply reads

$$\{\Psi(T) | T \in \text{supp } \mu\}.$$

Since  $\mathcal{F}^{-1}(R^{\Lambda_0})$  is in  $\mathcal{S}(\mathbb{R}^4)$ , we have for  $T \in \mathcal{S}'(\mathbb{R}^4)$  that  $\mathcal{F}^{-1}(R^{\Lambda_0}) * T$  is in  $\mathcal{C}^\infty(\mathbb{R}^4)$ . This gives directly that the support of (1.1.11) is in  $\mathcal{C}^\infty(\mathbb{R}^4)$ .

### 1.1.3 The flow equations

The theory we consider is the massive Euclidean  $\phi^4$ -theory on  $\mathbb{R}^4$ . This means that we start from the bare interaction

$$L^{\Lambda_0, \Lambda_0}(\phi) = \int_{\mathbb{R}^4} d^4x \left( \frac{\lambda}{4!} \phi^4(x) + a(\Lambda_0) \phi^2(x) - b(\Lambda_0) \phi(x) \Delta \phi(x) + c(\Lambda_0) \phi^4(x) \right). \quad (1.1.13)$$

The first term defines the self-interaction of the field with real coupling constant  $\lambda > 0$  having mass dimension equal to zero. The second part contains the related counterterms, determined according to

a power counting rule: the canonical mass dimension of the field is one so the counterterms allowed in the bare interaction are all local terms of mass dimension  $\leq 4$  that can be formed of the field and of its derivatives respecting the Euclidean symmetries. From the bare interaction and the flowing propagator (1.1.9), we may define Wilson's flowing effective action  $L^{\Lambda, \Lambda_0}$  by integrating out momenta in the region  $\Lambda \leq |p| \leq \Lambda_0$ :

$$e^{-\frac{1}{\hbar}L^{\Lambda, \Lambda_0}(\phi)} := \mathcal{N} \int d\mu_{\Lambda, \Lambda_0}(\Phi) e^{-\frac{1}{\hbar}L^{\Lambda_0, \Lambda_0}(\Phi + \phi)}. \quad (1.1.14)$$

It can be recognized to be the generating functional of the connected amputated correlation functions of the theory with propagator  $C^{\Lambda, \Lambda_0}$  and bare action  $L^{\Lambda_0, \Lambda_0}$  [11]. For the normalization factor  $\mathcal{N}$  to be finite, the theory has to be restricted to finite volume. All subsequent formulae are also valid in the thermodynamic limit since they do not involve the vacuum functional [11].

The fundamental tool for our study of the renormalization problem is the functional flow equation

$$\partial_{\Lambda} (L^{\Lambda, \Lambda_0} + I^{\Lambda, \Lambda_0}) = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \phi}, \dot{C}^{\Lambda, \Lambda_0} \frac{\delta}{\delta \phi} \right\rangle L^{\Lambda, \Lambda_0} - \frac{1}{2} \left\langle \frac{\delta L^{\Lambda, \Lambda_0}}{\delta \phi}, \dot{C}^{\Lambda, \Lambda_0} \frac{\delta L^{\Lambda, \Lambda_0}}{\delta \phi} \right\rangle, \quad (1.1.15)$$

where  $\dot{C}^{\Lambda, \Lambda_0} := \partial_{\Lambda} C^{\Lambda, \Lambda_0}$ . Let us explain briefly the steps leading to (1.1.15): we derive both sides of (1.1.14) with respect to  $\Lambda$  and perform an integration by parts in the functional integral on the RHS using the following property of a Gaussian measure  $\mu_{\Lambda}$  with covariance  $C^{\Lambda}$  [25]:

$$\frac{d}{d\Lambda} \int d\mu_{\Lambda}(\phi) A(\phi) = \frac{1}{2} \int d\mu_{\Lambda}(\phi) \left\langle \frac{\delta}{\delta \phi}, \dot{C}^{\Lambda} \frac{\delta}{\delta \phi} \right\rangle A(\phi), \quad (1.1.16)$$

where  $A(\phi)$  denotes a polynomial formed of local powers of the field  $\phi^n(x)$  and of its derivatives  $(\partial_x \phi(x))^m$  with  $(n, m) \in \mathbb{N}^2$ . In the context of the proof of perturbative renormalization, the objects of interest are the  $n$ -point connected amputated Schwinger functions. These are obtained by first expanding  $L^{\Lambda, \Lambda_0}$  in moments with respect to  $\phi$ :

$$\forall (p_i)_{1 \leq i \leq n} \in \mathbb{R}^4, \quad (2\pi)^{4(n-1)} \delta_{\phi(p_1)} \cdots \delta_{\phi(p_n)} L^{\Lambda, \Lambda_0}|_{\phi=0} = \delta^4(p_1 + \cdots + p_n) \mathcal{Z}_n^{\Lambda, \Lambda_0}(p_1, \cdots, p_n), \quad (1.1.17)$$

then in a formal powers series with respect to  $\hbar$  to select the loop order  $l$ ,

$$\mathcal{Z}_n^{\Lambda, \Lambda_0}(p_1, \cdots, p_n) = \sum_{l=0}^{\infty} \hbar^l \mathcal{Z}_{l,n}^{\Lambda, \Lambda_0}(p_1, \cdots, p_n).$$

Using the functional flow equation (1.1.15), we write the perturbative flow equations for the  $n$ -point correlation functions by identifying coefficients

$$\begin{aligned} \partial_{\Lambda} \partial^w \mathcal{Z}_{l,n}^{\Lambda, \Lambda_0}(p_1, \cdots, p_n) &= \frac{1}{2} \int_{\mathbb{R}^4} \frac{d^4 k}{(2\pi)^4} \partial^w \mathcal{Z}_{l-1, n+2}^{\Lambda, \Lambda_0}(k, p_1, \cdots, p_n, -k) \dot{C}^{\Lambda}(k) \\ &\quad - \frac{1}{2} \sum_{l_1, l_2} \sum_{n_1, n_2} \sum_{w_i} c_w \left[ \partial^{w_1} \mathcal{Z}_{l_1, n_1+1}^{\Lambda, \Lambda_0}(p_1, \cdots, p_{n_1}, p) \partial^{w_3} \dot{C}^{\Lambda}(p) \right. \\ &\quad \left. \times \partial^{w_2} \mathcal{Z}_{l_2, n_2+1}^{\Lambda, \Lambda_0}(-p, p_{n_1+1}, \cdots, p_n) \right]_{rsym} \end{aligned} \quad (1.1.18)$$

$$p = -p_1 - \cdots - p_{n_1} = p_{n_1+1} + \cdots + p_n$$

where the prime on top of the summation imposes the restriction to  $n_1 + n_2 = n$  and  $l_1 + l_2 = l$  and the symbol "rsym" means summation over unordered partitions of length 2 of the set  $(p_1, \dots, p_n)$ , which produce mutually different pairs of (unordered) image subsets.

Here we wrote the equation directly in a form where a number  $|w|$  of momentum derivatives, characterized by a multi index  $w$ , act on both sides. Momentum derivatives of the  $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$  are needed to obtain a closed inductive scheme. We set

$$\partial^w = \frac{\partial^{|w|}}{\partial p_{1,1}^{w_1} \partial p_{1,2}^{w_2} \cdots \partial p_{n,4}^{w_{4,n}}} \quad |w| = w_{1,1} + \dots + w_{n,4}. \quad (1.1.19)$$

Moreover, the combinatorial factor  $c_{\{w_i\}} = w!(w_1!w_2!w_3!)^{-1}$  comes from Leibniz's rule. In the loop order  $l = 0$ , the first term on the RHS is absent.

#### 1.1.4 Boundary conditions

Before bounding the solutions of the system of flow equations, we first need to specify the boundary conditions. Our choice of the bare interaction implies that at  $\Lambda = \Lambda_0$ , we have for all  $n + |w| \geq 5$ ,

$$\partial^w \mathcal{L}_{l,n}^{\Lambda_0,\Lambda_0}(p_1, \dots, p_n) = 0, \quad \forall (p_i)_{1 \leq i \leq n} \in \mathbb{R}^{4n}. \quad (1.1.20)$$

For the relevant terms, which correspond to  $n + |w| \leq 4$ , they are fixed by renormalization conditions imposed for the fully integrated theory at  $\Lambda = 0$ :

$$\mathcal{L}_{4,l}^{0,\Lambda_0}(0, \dots, 0) = \lambda, \quad \mathcal{L}_{2,l}^{0,\Lambda_0}(0, 0) = 0, \quad \partial_{p^2} \mathcal{L}_{2,l}^{0,\Lambda_0}(0, 0) = 0, \quad \forall l \geq 1. \quad (1.1.21)$$

The renormalization point is chosen at zero momentum for simplicity (BPHZ renormalization conditions).

#### 1.1.5 Renormalizability

Perturbative renormalizability of the regularized field theory (1.1.14) amounts to the following: given the coupling constant  $\lambda$  in the bare interaction  $L^{\Lambda_0,\Lambda_0}$ , the coefficients  $a(\Lambda_0)$ ,  $b(\Lambda_0)$  and  $c(\Lambda_0)$  of the counterterms can be adjusted within a loop expansion of the theory,

$$a(\Lambda_0) = \sum_{l=1}^{\infty} \hbar^l a_l(\Lambda_0), \dots, \quad c(\Lambda_0) = \sum_{l=1}^{\infty} \hbar^l c_l(\Lambda_0),$$

in such a way that the limits of the  $n$ -point correlation functions exist when  $\Lambda$  goes to 0 and  $\Lambda_0$  goes to  $+\infty$  in every loop order  $l$  uniformly on compact momentum sets:

$$\forall (p_i)_{1 \leq i \leq n} \in \mathbb{R}^4, \quad n \in \mathbb{N}, \quad l \in \mathbb{N}^*, \quad \mathcal{L}_{l,n}^{0,\infty}(p_1, \dots, p_n) := \lim_{\Lambda \rightarrow 0, \Lambda_0 \rightarrow +\infty} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \dots, p_n). \quad (1.1.22)$$

A simple inductive proof of  $\phi_4^4$  theory regularized by a UV-cutoff has been exposed several times in the literature [11, 12]. It is based on proving the following induction hypotheses:

A) Boundedness in the UV-cutoff<sup>3</sup>

$$\left| \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_n) \right| \leq (\Lambda + m)^{4-n-|w|} \mathcal{P}_1 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right). \quad (1.1.23)$$

B) Convergence in the UV-limit

$$\left| \partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_n) \right| \leq \frac{(\Lambda + m)^{5-n-|w|}}{(\Lambda_0 + m)^2} \mathcal{P}_3 \left( \log \frac{\Lambda_0 + m}{m} \right) \mathcal{P}_4 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right). \quad (1.1.24)$$

The  $\mathcal{P}_i$ , each time they appear denote possibly new, polynomials with nonnegative coefficients of coefficients depending on  $l, n, |w|$  but not on  $\{p_i\}$ ,  $\Lambda$ ,  $\Lambda_0$ . For  $l = 0$ , all polynomials  $\mathcal{P}_i$  reduce to 1. Integration of the bound (1.1.24) over the cutoff  $\Lambda_0$  immediately proves the convergence of all  $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_n)$  for fixed  $\Lambda$  to finite limits when  $\Lambda_0 \rightarrow \infty$ . In particular, one obtains for all  $\tilde{\Lambda}_0 < \Lambda_0$ ,

$$\left| \mathcal{L}_{l,n}^{0, \Lambda_0}(p_1, \dots, p_n) - \mathcal{L}_{l,n}^{0, \tilde{\Lambda}_0}(p_1, \dots, p_n) \right| < \Lambda_0^{-1} m^{5-n} \left( \log \frac{\Lambda_0 + m}{m} \right)^\nu \mathcal{P}_5 \left( \frac{\|\vec{p}_n\|}{m} \right).$$

It is also possible to establish inductively that the connected amputated Schwinger (CAS)  $n$ -point functions are  $\mathcal{C}^\infty$  w.r.t. momenta,  $\Lambda$  and  $\Lambda_0$ . This stems mainly from the regularity properties of the regularized flowing propagator. Thus, due to the Cauchy criterion in  $\mathcal{C}^\infty(\mathbb{R}^+)$  (w.r.t. to  $\Lambda_0$ ) finite limits exist to all loop orders  $l$ .

## Method of the proof

Let us shortly describe the method of the proof of (1.1.23). For a detailed proof we refer to [11, 30]. The inductive scheme starts from  $(l, n) = (0, 4)$  for which the induction hypothesis (1.1.23) clearly holds. Higher orders in loop are generated inductively by integrating successively the flow equations. The order of the induction is as follows:

- Ascend in the loop order  $l$ ,
- For fixed  $l$  ascend in  $n$ ,
- For fixed  $l$  and  $n$  descend with  $w$  down to 0.

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<sup>3</sup>We use the following notation  $\|\vec{p}_n\| := \sup_{1 \leq i \leq n} |p_i|$ .

Inserting the induction hypothesis on the right-hand side of the flow equations, the integrals are bounded in an elementary way using

$$\left| \partial^w \dot{C}^\Lambda(p) \right| \leq (\Lambda + m)^{-3-|w|} \mathcal{P} \left( \frac{|p|}{\Lambda + m} \right), \quad (1.1.25)$$

and we obtain

$$\left| \partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_n) \right| \leq (\Lambda + m)^{3-n-|w|} \mathcal{P}_1 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right). \quad (1.1.26)$$

The next step is to integrate (1.1.26). Because of the mixed boundary conditions of subsection 1.1.4, the relevant and irrelevant are integrated differently:

- *Irrelevant terms:* In this case  $n + |w| > 4$ , the flow equation is integrated from the initial point  $\Lambda_0$  downwards to  $\Lambda$  with the initial condition (1.1.20).
- *Relevant terms:* These terms are characterized by  $n + |w| \leq 4$  and are integrated at the renormalization point from 0 to  $\Lambda$  as follows

$$\partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(0, \dots, 0) = \partial^w \mathcal{L}_{l,n}^{0, \Lambda_0}(0, \dots, 0) + \int_0^\Lambda d\lambda \partial_\lambda \partial^w \mathcal{L}_{l,n}^{\lambda, \Lambda_0}(0, \dots, 0). \quad (1.1.27)$$

Using the renormalization conditions (1.1.21) together with (1.1.26), the bound is obtained at the renormalization point and it is extended to general momenta using the Taylor formula

$$\mathcal{L}_{l,2}^{\Lambda, \Lambda_0}(p, -p) = \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}(0, 0) + \sum_\mu p_\mu \int_0^1 d\lambda \left( \frac{\partial}{\partial p_\mu} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0} \right) (\lambda p, -\lambda p), \quad (1.1.28)$$

with similar formulas where  $\mathcal{L}_{l,2}^{\Lambda, \Lambda_0}(p, -p)$  is replaced by  $\frac{\partial}{\partial p_\mu} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}(p, -p)$  or  $\frac{\partial^2}{\partial p_\mu \partial p_\nu} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}(p, -p)$ . Applying three derivatives to the two-point function or one derivative to the four-point function makes the contribution irrelevant, and then it is integrated downwards from  $\Lambda_0$ . For the four-point function only one step is required. Note that the Euclidean symmetries imply that no renormalization conditions are needed for the terms which are not scalars w.r.t. this symmetry.

## 1.2 Summary of the thesis

In the last section, we saw that the Polchinski flow equations provide a strong tool to establish a rigorous proof of perturbative renormalizability of the massive scalar field model with a  $\phi^4$  interaction on  $\mathbb{R}^4$ , with the advantage of circumventing the difficulties arising from the analysis of the combinatorics of Feynman diagrams. In the method presented in section 1.1, translation invariance plays a major role in the proof of renormalizability in the sense that the correlation functions<sup>4</sup> are smooth w.r.t. to

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<sup>4</sup>after splitting off a global momentum conserving  $\delta$ -function.



the external momenta in Fourier space, and that the propagator together with its regularized version have a simple form. The first problem encountered when one wants to generalize this method of proof to systems breaking translation invariance is the distributional nature of the correlation "functions" in position space. The same problem was encountered in [13] in which the authors studied the massive scalar field theory on a Riemannian manifold without a boundary. In this work, translation invariance is broken by the metric of the considered space-time and therefore the full renormalization procedure was carried out in position space. The perturbative renormalizability of the semi-infinite model treated in this thesis is faced with two difficulties: the first one is technical in the sense that the renormalization problem has to be considered (partially) in position space as was the case in [13], and the second complication is due to the fact that the surface affects the power counting and the renormalizability of the model.

### 1.2.1 Perturbative renormalization of the semi-infinite model

#### The propagators of the semi-infinite model

In the next chapter, we start by considering the operator

$$H = -\Delta + m^2$$

on the Hilbert space  $L^2(\mathbb{R}^+ \times \mathbb{R}^{d-1})$ . Initially, we may define  $H$  on the domain<sup>5</sup>  $\mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R}^{d-1})$ . Defined in this way,  $H$  will be symmetric but not self-adjoint. A first task is to extend the domain of the definition of  $H$  to make it self-adjoint. Since  $H$  is unbounded, by the Hellinger-Toeplitz theorem [31] no self-adjoint extension can act on all vectors in  $L^2(\mathbb{R}^+ \times \mathbb{R}^{d-1})$ . The choice of functions on which  $H$  acts is intimately related to the choice of the boundary conditions on the Green function associated with  $H$ . Once  $H$  is extended to a self-adjoint operator (and assuming that the extended operator is positive), we may use the spectral theorem to define the "Green operator"  $H^{-1}$  as a densely defined self-adjoint operator. We prove that if  $H_\star$  is a self-adjoint extension of  $H$  then the Green operator  $H_\star^{-1}$  can be realized as a kernel  $C_\star((z, x), (z', x'))$ . Since the range of  $H_\star^{-1}$  is precisely the domain of  $H$ , the specification of what functions  $H$  acts on is equivalent to the specification of what functions can result from action by the Green function  $H^{-1}$ . In other words, the problem of defining a Feynman propagator is equivalent to the problem of defining a self-adjoint extension of  $H$ . The positivity of  $H$  on its initial domain  $\mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R}^{d-1})$  implies the existence of self-adjoint extensions of  $H$ . If  $H$  is essentially self-adjoint (i.e. the closure of  $H$  is self-adjoint), only a unique self-adjoint extension exists and no boundary conditions are required. However, in the context of Euclidean space with a boundary,  $H$  is not essentially self-adjoint, which implies that further boundary conditions must be imposed to define a self-adjoint extension of  $H$ . One condition to impose on the self-adjoint extension

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<sup>5</sup> $\mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R}^{d-1})$  is the space of infinitely differentiable functions of compact support in  $\mathbb{R}^+ \times \mathbb{R}^{d-1}$ .

is that it preserves the positivity of  $H$ , but this condition alone need not uniquely determine a self-adjoint extension. In the next chapter, we present a quick review of the von Neumann theory of deficiency indices which provides a strong mathematical tool to determine the self-adjoint extensions of  $H$ . We start by determining the self-adjoint extensions of the Laplacian on the half-line and relate them afterwards to the physical Dirichlet, Neumann and Robin boundary conditions. Then we extend our findings to the study of the Laplacian on the half space  $\mathbb{R}^+ \times \mathbb{R}^{d-1}$ . The second part of the next chapter is devoted to the computation of the propagators (i.e. Green functions) associated with the self-adjoint extensions of the Laplacian by proceeding first on the half-line and then generalizing to the half-space.

### Renormalization of the semi-infinite model

The aim of chapter 4 is to give a rigorous proof of perturbative renormalizability of the semi-infinite model based on the Polchinski flow equations. Since the translation invariance is broken in one direction by the presence of the boundary, we then work in a mixed position-momentum space (i.e.  $pz$ -representation) which consists of taking a partial Fourier transform w.r.t. the variable  $x \in \mathbb{R}^3$ . In this chapter, we expose in detail the steps to carry out the renormalization procedure with the flow equations in the  $pz$ -representation. Since the correlation "functions" are distributions in this representation, we introduce a class of test functions. The trees and their weight factors, presented in chapter 3, provide a key ingredient in obtaining inductive bounds on the correlation distributions that imply renormalizability. In chapter 4, we proceed by imposing BPHZ renormalization conditions. As a consequence of this choice, the effective action will have the most general form that accounts for the loss of translation invariance. More precisely, the counter-terms are functions of the position on the half-line and are not of the same form as the terms appearing in the original Hamiltonian of the system. The central point to be retained from this chapter is that imposing constant renormalization conditions yields position dependent counter-terms. The effect of the surface on the renormalizability of the theory is concealed by the choice of BPHZ renormalization conditions, in the sense that its only manifestation is the dependence of the counter-terms on the position in  $\mathbb{R}^+$ . However, this represents a common feature of all theories that break the translation invariance, even those without a boundary [13].

### Surface counter-terms and Bulk counter-terms

Methods based on the renormalization group [1, 2, 22] proved that for the semi-infinite model, the behaviour on the surface differs considerably from the bulk, in the sense that the critical exponents of this model can not be fully expressed in terms of the bulk critical exponents. For Robin and Neumann boundary conditions, the critical exponents are given in terms of the bulk critical exponents and two independent surface critical exponents. In the case of Dirichlet boundary conditions, there is only one

additional surface critical exponent in addition to the usual bulk critical exponents. This implies that a given bulk universality class splits into different surface universality classes which results in a rich bulk-surface phase diagram. Each of the possible boundary conditions can be associated to a phase of the semi-infinite Ising model. The theory of bulk critical phenomena suggests that the number of independent critical exponents should follow directly from the number of independent renormalization functions (i.e.  $Z$  functions). This correspondence between the critical exponents and counter-terms suggests that some of the counterterms are the same as those which renormalize the translationally invariant theory, while the remaining ones are new counter-terms which result from the presence of the boundary and can be associated to the independent surface critical exponents. The aim of chapters 5 and 6 is to make this correspondence explicit by proving that there exists a particular choice of renormalization conditions for which the counter-terms appearing in the bare interaction are position independent. From a theoretical physics point of view, we aim to establish renormalizability in the restrictive sense that all counter-terms are of the same form as the interactions included in the original Hamiltonian. The strategy is based on the following ideas: all possible propagators in the mixed position momentum space can be decomposed into a sum of two terms. The first term is the propagator of the translationally invariant theory  $C_b$  and the second one is the part that breaks translation invariance  $C_s$ . Inserting this decomposition in the Feynman graph expansion, we obtain graphs involving exclusively  $C_b$  (i.e. bulk graphs), and others involving  $C_s$  or  $C_s$  and  $C_b$  (i.e. surface graphs). The bulk graphs are given by Feynman integrals which are identical to those of the corresponding translationally invariant theory up to the restriction  $z \geq 0$  on  $z$ -integrations. This implies that these graphs can be renormalized using the same counter-terms as for  $\phi_4^4$  in  $\mathbb{R}^4$  with an interaction supported on the half-space which will be the object of study of chapter 6. The remaining surface graphs of our semi-infinite model which do also involve  $C_s$  can be renormalized by adding position independent surface counter-terms. This means that the semi-infinite correlation "functions" can be decomposed into a bulk part, plus a remainder which we call the "surface part". One of the important results of this chapter is that the surface part admits a power counting which is dimensionally better by one scaling dimension as compared to the bulk counterpart. This modified scaling dimension appears in Theorem 9. In chapter 6, we prove that for a particular choice of renormalization conditions the  $\phi_4^4$ -theory in  $\mathbb{R}^4$  with an interaction supported on the half-space is renormalized by adding the usual counter-terms of the translationally invariant theory together with two surface counter-terms proportional to  $\int_S \phi^2$  and  $\int_S \phi \partial_n \phi$ .

### 1.2.2 Perturbative renormalization of the lattice regularized massive $\phi_4^4$ theory

In chapter 7, we investigate the renormalizability of the massive  $\phi_4^4$ -theory in  $\mathbb{R}^4$  regularized by a lattice cut-off, which is another manifestation of the breaking of the translation invariance. However, in this case, the considered quantum field theory (Euclidean space, the propagator and the interaction)

is translation invariant and the loss of translation invariance is induced by the regularization scheme. The proof of perturbative renormalizability of a lattice regularized field theory exists in the literature but it is not direct from the usual power counting theorems. The well known power counting theorems of Weinberg [32], and Hahn, Zimmermann [7] which state sufficient conditions for the convergence of Feynman integrals do not apply in the presence of a lattice cutoff. Reisz [33] has given a generalization of the power counting theorem for a wide class of lattice field theories where a new kind of an ultraviolet divergence degree is used. The existence of a power counting theorem ensures that the combinatorics of subtractions to renormalize a diagram is described by Zimmermann's forest formula [8]. The situation is different for a lattice field theory. Reisz [34] has proved that the counterterms instead of being polynomials are periodic functions in the external momenta, which can be obtained with the help of new operators he introduced, called subtraction operators. The renormalization of lattice regularized  $\phi_4^4$  theory in Polchinski's framework has been addressed in [35]. The paper presents interesting arguments, but it does not aim at mathematical rigour and thus leaves certain mathematical questions unsolved, in particular w.r.t. to  $O(4)$  and translation invariance of the continuum limit. Davoudi and Savage [36] proposed a mechanism for the restoration of rotational symmetry in the continuum limit of lattice field theories on hyper cubic lattices. The approach is based on constructing smeared lattice operators that smoothly evolve into continuum operators with definite angular momentum as the lattice-spacing is reduced. However, this method regards only finite lattices and the full recovery of rotational invariance in the lattice theories requires the suppression of rotational symmetry breaking contributions to the physical quantities not only as a result of short-distance discretization effects, but also as a result of boundary effects of the finite cubic lattice. More precisely, the rotational invariant theory is achieved as the lattice becomes infinitely large, corresponding to an infinitely large number of points in momentum space. In this chapter, the main contribution of our work is to give a proof of Euclidean symmetry restoration for  $\phi_4^4$  lattice regularized field theory on an infinite lattice.

## Chapter 2

# The propagators of the massive scalar field theory in $\mathbb{R}^+ \times \mathbb{R}^{d-1}$

### 2.1 The von Neumann theory of deficiency indices

In this section, we present a short review of von Neumann's theory of deficiency indices, which provides a solid mathematical background to study symmetric operators and their extensions.

In quantum mechanics and quantum field theory, the Hamiltonian of the system is usually described by a "formal" partial differential operator on an appropriate  $L^2$  space. The domain of the Hamiltonian is not specified, but it is easy to find a dense domain on which the Hamiltonian is a well-defined symmetric operator  $H$ . As we explained in section [1.1](#), the operator  $H$  must be self-adjoint in order to define its inverse  $H^{-1}$  through the functional calculus. If the closure  $\overline{H}$  of  $H$  is self-adjoint, then we can use  $\overline{H}$ . However, if  $\overline{H}$  is not self-adjoint, then a natural question to ask is: does  $\overline{H}$  admits self-adjoint extensions<sup>1</sup>? Another problem is encountered in the case in which  $\overline{H}$  has several self-adjoint extensions. One is faced with the problem of which self-adjoint extension must be chosen to generate the dynamics. In the sequel, we will see that the problem of selecting the "right" self-adjoint extension is not a "technical" mathematical problem, but is rather related to the physics of the system being described.

Before stating the principal theorems of von Neumann's theory, let us recall some basic definitions from the spectral analysis of unbounded linear operators:

**Definition 3.** (A closed operator) Given a Hilbert space  $\mathcal{H}$ , the **graph** of the linear transformation  $T$  is the set of pairs

$$\{ \langle \phi, T\phi \rangle \mid \phi \in D(T) \},$$

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<sup>1</sup>see definitions below.

with  $D(T) \subset \mathcal{H}$  is the domain of the operator  $T$ . The graph of  $T$ , denoted by  $\Gamma(T)$ , is a subset of  $\mathcal{H} \times \mathcal{H}$  which is a Hilbert space with inner product

$$(\langle \phi_1, \psi_1 \rangle, \langle \phi_2, \psi_2 \rangle) := (\phi_1, \phi_2) + (\psi_1, \psi_2).$$

$T$  is called a **closed** operator if  $\Gamma(T)$  is a closed subset of  $\mathcal{H} \times \mathcal{H}$ .

**Definition 4.** (*Extension of an operator*) Let  $T$  and  $\tilde{T}$  be operators on  $\mathcal{H}$ . If  $D(T) \subset D(\tilde{T})$  and  $T\phi = \tilde{T}\phi$  for all  $\phi \in D(T)$ , then  $\tilde{T}$  is said to be an **extension** of  $T$ . We write  $T \subset \tilde{T}$ .

**Definition 5.** (*A closable operator*) An operator  $T$  is **closable** if it has a closed extension. Every closable operator has a smallest closed extension, called its **closure**, which is denoted by  $\bar{T}$ .

The notion of adjoint operator exists as well in the unbounded case.

**Definition 6.** (*Adjoint*) Let  $T$  be a densely defined linear operator on a Hilbert space  $\mathcal{H}$ . Let  $D(T^*)$  be the set of  $\phi \in \mathcal{H}$  for which there exists an  $\eta \in \mathcal{H}$  with

$$(T\psi, \phi) = (\psi, \eta) \quad \text{for all} \quad \psi \in D(T). \quad (2.1.1)$$

For each  $\phi \in D(T^*)$ , we define  $T^*\phi = \eta$  and  $T^*$  is called the **adjoint** of  $T$ .

Note that the assumption on the density of the domain of  $T$  ensures that  $\eta$  is uniquely determined by (2.1.1). In contrast to the situation for bounded operators,  $D(T^*)$  is not necessarily dense.

**Definition 7.** (*Symmetric*) Let  $T$  be a densely defined operator on a Hilbert space.  $T$  is **symmetric** if  $T^*$  is an extension of  $T$ . Equivalently,  $T$  is symmetric if and only if

$$(T\phi, \psi) = (\phi, T\psi), \quad \forall \phi, \psi \in D(T).$$

**Definition 8.**  $T$  is called **self-adjoint** if  $T = T^*$ , that is, if and only if  $T$  is symmetric and  $D(T) = D(T^*)$ .

Note that if  $T$  is symmetric, then  $T^*$  is a closed extension of  $T$ , since  $D(T) \subset D(T^*)$  is dense in  $\mathcal{H}$ . Furthermore, the smallest closed extension  $T^{**}$  of  $T$  must be contained in  $T^*$ . Hence, for symmetric operators we have

$$T \subset T^{**} \subset T^*.$$

For closed symmetric operators,

$$T = T^{**} \subset T^*$$

and for self-adjoint operators

$$T = T^{**} = T^*.$$

**Definition 9.** (Essentially self-adjoint) A symmetric operator  $T$  is **essentially self-adjoint** if its closure  $\bar{T}$  is self-adjoint.

**Definition 10.** (Deficiency sub-spaces/indices) Let  $T$  be a symmetric operator. We define

$$\mathcal{K}_+ := \ker(T^* - i), \quad \mathcal{K}_- := \ker(T^* + i).$$

$\mathcal{K}_+$  and  $\mathcal{K}_-$  are called the deficiency sub-spaces of  $T$ . The pair of numbers

$$n_+(T) := \dim \mathcal{K}_+, \quad n_-(T) := \dim \mathcal{K}_-,$$

are called the deficiency indices of  $T$ .

**Definition 11.** (Partial isometry) Given a Hilbert space  $\mathcal{H}$ , a linear transformation  $U : \mathcal{H} \rightarrow \mathcal{H}$  is a partial isometry if and only if the restriction of  $U$  to the set  $I(U) := (\ker U)^\perp$  is an isometry.

The following theorem characterizes the closed symmetric extensions of a closed symmetric operator  $T$ :

**Theorem 2.** Let  $T$  be a closed symmetric operator on a Hilbert space  $\mathcal{H}$ . The closed symmetric extensions of  $T$  are in one-to-one correspondence with the set of partial isometries of  $\mathcal{K}_+$  into  $\mathcal{K}_-$ . If  $U$  is such an isometry with  $I(U) \subseteq \mathcal{K}_+$ , then the corresponding closed symmetric extension  $T_U$  has domain

$$D(T_U) = \{\phi + \phi^+ + U\phi^+ | \phi \in D(T), \phi^+ \in I(U)\}, \quad (2.1.2)$$

and

$$T_U(\phi + \phi^+ + U\phi^+) = T\phi + i\phi_+ - U\phi_+. \quad (2.1.3)$$

If the dimension of  $I(U)$  is finite, then the deficiency indices of  $T_U$  are related to those of  $T$  as follows,

$$n_\pm(T_U) = n_\pm(T) - \dim [I(U)]. \quad (2.1.4)$$

The proof of this theorem can be found in [37]. As a corollary to theorem 2, we have:

**Corollary 1.** Let  $T$  be a closed symmetric operator with deficiency indices  $n_+$  and  $n_-$ . Then,

- i)  $T$  is self-adjoint if and only if  $n_+ = n_- = 0$ .
- ii)  $T$  has self-adjoint extensions if and only if  $n_+ = n_-$ . There is a one-to-one correspondence between self-adjoint extensions of  $T$  and unitary maps from  $\mathcal{K}_+$  to  $\mathcal{K}_-$ .
- iii) If either  $n_+ = 0 \neq n_-$  or  $n_- = 0 \neq n_+$ , then  $T$  has no nontrivial symmetric extensions. Such operators are called maximal symmetric.

There exists a simple and useful criterion for a symmetric operator to have self-adjoint extensions given by von Neumann's theorem. Let us first define the notion of **conjugation**:

**Definition 12.** (*Conjugation*) A conjugation  $C$  is an anti-linear map  $C : \mathcal{H} \rightarrow \mathcal{H}$ <sup>2</sup> that is norm-preserving and satisfies  $C^2 = \mathbf{1}$ .

Now, we have all ingredients to state the von Neumann's theorem:

**Theorem 3.** (*von Neumann's theorem*) Let  $T$  be a symmetric operator and suppose that there exists a conjugation  $C$  with  $C : D(T) \rightarrow D(T)$  and  $CT = TC$ . Then  $T$  has equal deficiency indices and therefore has self-adjoint extensions.

*Proof.* We have  $C^2 = I$  and  $CD(T) \subseteq D(T)$ , which imply that  $CD(T) = D(T)$ . Let  $\phi \in \mathcal{K}_+$  and  $\psi \in D(T)$ , we have

$$\begin{aligned} 0 &= \overline{(\phi_+, (T+i)\psi)} = (C\phi_+, C(T+i)\psi) \\ &= (C\phi_+, (T-i)C\psi). \end{aligned}$$

Since  $C$  takes  $D(T)$  onto  $D(T)$ , we deduce that  $C\phi_+ \in \mathcal{K}_-$ , so  $C : \mathcal{K}_+ \rightarrow \mathcal{K}_-$ . Similar steps lead to  $C : \mathcal{K}_- \rightarrow \mathcal{K}_+$ . Since  $C^2 = \mathbf{1}$ , we obtain

$$\dim \mathcal{K}_+ = \dim \mathcal{K}_- .$$

□

## 2.2 The self-adjoint extensions of the Laplacian on the half-line

In the previous section, we introduced all the ingredients required to study the self-adjoint extensions of the Laplacian on the positive half-line. Let  $T$  be the one-dimensional operator  $-d^2/dx^2$  on  $L^2(\mathbb{R}^+)$  with domain  $\mathcal{C}_c^\infty(\mathbb{R}^+)$ . As we saw in the previous section, von Neumann's theory of deficiency indices applies to *closed* and *symmetric* operators. Therefore, the operator we consider is the closure  $\overline{T}$  of  $T$  with the following graph

$$\Gamma(\overline{T}) = \overline{\Gamma(T)} = \{(f, \Delta f), f \in H_0^2(\mathbb{R}^+)\},$$

where

$$H_0^2(\mathbb{R}^+) = \{f \in H^2(\mathbb{R}^+), f(0) = f'(0) = 0\},$$

and  $H^2$  is the  $W^{2,2}$  Hilbert Sobolev space. In the sequel, we consider the operator  $\overline{T}$  with domain  $H_0^2(\mathbb{R}^+)$ . The complex conjugation commutes with  $\overline{T}$ , which implies by Theorem 3 that the deficiency

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<sup>2</sup> $C(\alpha\phi + \beta\psi) = \overline{\alpha}C\phi + \overline{\beta}C\psi.$



indices of  $\overline{T}$  are equal. The next step is to determine  $\mathcal{K}_+$  and  $\mathcal{K}_-$ , which is achieved by finding the solutions of

$$\overline{T}^* \phi = \pm i \phi \quad (2.2.1)$$

in  $L^2(\mathbb{R}^+)$ . Since  $L^2(\mathbb{R}^+) \subset \mathcal{D}'(\mathbb{R}^+)$ , we deduce that (2.2.1) is solved by finding the weak solutions of (2.2.1). Let us recall the local regularity theorem for Schrödinger's equation:

**Theorem 4.** *Let  $\phi$  be a weak solution of*

$$(-\Delta + V)\phi = E\phi, \quad (2.2.2)$$

where  $V$  is a measurable function and  $E$  is a complex number. If  $V$  is equal to a  $\mathcal{C}^\infty$  function on an open region  $\Omega$ , then  $\phi$  is  $\mathcal{C}^\infty$  in that region too.

The proof can be found in Chapter IX in [37]. It follows from Theorem 4 that the solutions of (2.2.1) are infinitely differentiable, and thus strong solutions. Using elementary ordinary differential equations tools, we deduce the strong solutions of  $-\phi''(x) = +i\phi(x)$  given by

$$e^{-\frac{x}{\sqrt{2}}} e^{i\frac{x}{\sqrt{2}}}, \quad e^{\frac{x}{\sqrt{2}}} e^{-i\frac{x}{\sqrt{2}}}.$$

Similarly, the strong solutions of  $-\phi''(x) = -i\phi(x)$  are

$$e^{\frac{x}{\sqrt{2}}} e^{i\frac{x}{\sqrt{2}}}, \quad e^{-\frac{x}{\sqrt{2}}} e^{-i\frac{x}{\sqrt{2}}}.$$

Since only

$$e^{-\frac{x}{\sqrt{2}}} e^{i\frac{x}{\sqrt{2}}}, \quad e^{-\frac{x}{\sqrt{2}}} e^{-i\frac{x}{\sqrt{2}}}$$

are in  $L^2(\mathbb{R}^+)$ , we deduce that the deficiency indices are in this case  $n_\pm(\overline{T}) = 1$ . Let

$$\phi_+(x) = \frac{1}{2^{\frac{1}{4}}} e^{-\frac{x}{\sqrt{2}}} e^{i\frac{x}{\sqrt{2}}}, \quad \phi_-(x) = \frac{1}{2^{\frac{1}{4}}} e^{-\frac{x}{\sqrt{2}}} e^{-i\frac{x}{\sqrt{2}}}$$

be normalized vectors from  $\mathcal{K}_\pm$ . Then the only partial isometries of  $\mathcal{K}_+$  into  $\mathcal{K}_-$  are the maps  $\phi_+ \mapsto \gamma\phi_-$  where  $|\gamma| = 1$ . By Theorem 2, the only closed symmetric extensions of  $\overline{T}$  are the operators  $T_\gamma$  with domain

$$D(T_\gamma) := \{\phi + \beta\phi_+ + \beta\gamma\phi_- \mid \phi \in H_0^2(\mathbb{R}^+), \beta \in \mathbb{C}\}. \quad (2.2.3)$$

By the last statement of Theorem 2, we have

$$n_\pm(T_\gamma) = n_\pm(\overline{T}) - \dim I(U) = 0.$$

Using i) from Corollary 1, each  $T_\gamma$  is self-adjoint. Now, let us show that the set of domains of the self-adjoint extensions of  $T$  can be parametrized by  $\mathbb{R} \cup \{\infty\}$  with

$$\mathcal{J}_\alpha := \{\phi \mid \phi \in H^2(\mathbb{R}^+), \phi'(0) = \alpha \phi(0)\} \quad (2.2.4)$$

for  $\alpha$  in  $\mathbb{R}$  and

$$\mathcal{F}_\infty := \{ \phi \mid \phi \in H^2(\mathbb{R}^+) , \phi(0) = 0 \} \quad (2.2.5)$$

for  $\alpha = \infty$ . If  $\psi \in D(T_\gamma)$ , then

$$\psi(0) = \phi(0) + \frac{\beta}{2^{\frac{1}{4}}}(1 + \gamma) \quad \text{and} \quad \psi'(0) = \phi'(0) - \frac{\beta}{2^{\frac{1}{5}}} [(1 - i) + \gamma(1 + i)], \quad (2.2.6)$$

so for  $\gamma = -1$  we have  $\psi(0) = 0$ . Otherwise, we obtain

$$\psi'(0) = -\frac{1}{\sqrt{2}} \left[ 1 - \frac{1 - \gamma}{1 + \gamma} i \right] \psi(0) = \alpha \psi(0), \quad \text{with} \quad \alpha = -\frac{1}{\sqrt{2}} \left[ 1 - \frac{1 - \gamma}{1 + \gamma} i \right]. \quad (2.2.7)$$

Note that  $\phi(0)$  and  $\phi'(0)$  vanish since  $\phi \in H_0^2(\mathbb{R}^+)$ . Furthermore, the constant  $\alpha$  in (2.2.7) is in  $\mathbb{R}$  since  $|\gamma| = 1$ . Hence, we obtain

$$D(T_\gamma) \subseteq \mathcal{F}_\alpha \quad \text{for } \gamma \neq -1 \quad \text{and} \quad D(T_{-1}) = \mathcal{F}_\infty. \quad (2.2.8)$$

Conversely, if  $\psi'(0) = \alpha \psi(0)$ , then  $\psi$  can be written  $\psi = \phi + \beta\phi^+ + \beta\gamma\phi^-$  for some  $\beta$  where

$$\gamma = \frac{1 + i(\sqrt{2}\alpha + 1)}{1 - i(\sqrt{2}\alpha + 1)}.$$

### 2.3 The self-adjoint extensions of the Laplacian on the half-space $\mathbb{R}^+ \times \mathbb{R}^{d-1}$

In this section, we consider the Laplacian operator on the half-space  $\mathbb{R}^+ \times \mathbb{R}^{d-1}$  given by

$$H = -\mathbf{1}_{\mathbb{R}^+} \Delta_{d-1} + T \mathbf{1}_{\mathbb{R}^{d-1}}, \quad D(H) := \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R}^{d-1}), \quad (2.3.1)$$

where  $T$  is the one dimensional Laplacian on the half-line of domain  $D(T) = \mathcal{C}_c^\infty(\mathbb{R}^+)$  and  $\Delta_{d-1}$  denotes the  $(d-1)$ -Laplacian of domain  $\mathcal{C}_c^\infty(\mathbb{R}^{d-1})$ . Given a set  $K$ , the operator  $\mathbf{1}_K$  is the identity given by

$$\begin{aligned} \mathbf{1}_{\mathbb{R}^{d-1}} : L^2(K) &\longrightarrow L^2(K) \\ u &\longmapsto u. \end{aligned}$$

The following proposition relates the self-adjoint extensions of  $\overline{H}$  to those of  $\overline{T}$ .

**Proposition 1.** *Let  $H_\lambda$  be a self-adjoint extension of  $\overline{H}$ . There exists a unique self-adjoint extension  $T_\lambda$  of  $T$  such that*

$$H_\lambda = -\mathbf{1}_{\mathbb{R}^+} \overline{\Delta}_{d-1} + T_\lambda \mathbf{1}_{\mathbb{R}^{d-1}} \quad (2.3.2)$$

of domain

$$D(H_\lambda) = \left\{ u(z, x) \in L^2(\mathbb{R}^+ \times \mathbb{R}^3) \mid u(z, \cdot) \in H^2(\mathbb{R}^{d-1}), u(\cdot, x) \in D(T_\lambda) \right\}. \quad (2.3.3)$$

We recall the following theorems which will be needed in the proof of proposition [1](#)

**Theorem 5.** *Let  $A$  and  $B$  be commuting self-adjoint operators. Then  $AB$  is self-adjoint on  $D(A) \cap D(B)$ .*

The following theorem is a particular case of the Kato-Rellich theorem [37](#):

**Theorem 6.** *Suppose  $A$  is self-adjoint and  $B$  is a bounded self-adjoint operator. Then  $A + B$  is self-adjoint on  $D(A)$ .*

The proofs of theorems [5](#) and [6](#) can be found in [38](#).

*Proof.* In the proof of proposition [1](#), we proceed in two steps by proving the following statements:

- A) Given a self-adjoint extension  $T_\lambda$  of  $\bar{T}$ , the operator  $-\mathbb{1}_{\mathbb{R}^+} \bar{\Delta}_{d-1} + T_\lambda \mathbb{1}_{\mathbb{R}^{d-1}}$  is a self-adjoint extension of  $\bar{H}$ .
- B) Given a self-adjoint extension  $H_\lambda$  of  $\bar{H}$ , there exists a unique self-adjoint extension  $T_\lambda$  of  $T$  such that:

$$H_\lambda = -\mathbb{1}_{\mathbb{R}^+} \bar{\Delta}_{d-1} + T_\lambda \mathbb{1}_{\mathbb{R}^{d-1}} \quad (2.3.4)$$

and

$$D(H_\lambda) = \left\{ u(z, x) \in L^2(\mathbb{R}^+ \times \mathbb{R}^3) \mid u(z, \cdot) \in H^2(\mathbb{R}^{d-1}), u(\cdot, x) \in D(T_\lambda) \right\}.$$

First, we prove A). Given a self-adjoint extension  $T_\lambda$  of  $\bar{T}$ , we define the operator

$$H_\lambda := -\mathbb{1}_{\mathbb{R}^+} \bar{\Delta}_{d-1} + T_\lambda \mathbb{1}_{\mathbb{R}^{d-1}}.$$

$H_\lambda$  can be rewritten as

$$\left( -\frac{\bar{\Delta}_{d-1}}{2} - i \right) W_\lambda \left( -\frac{\bar{\Delta}_{d-1}}{2} + i \right) \quad (2.3.5)$$

with

$$W_\lambda := \mathbb{1}_{\mathbb{R}^+} R_{-1}(\bar{\Delta}_{d-1}) + \mathbb{1}_{\mathbb{R}^+} R_1(\bar{\Delta}_{d-1}) + R_{-1}(\bar{\Delta}_{d-1}) T_\lambda R_1(\bar{\Delta}_{d-1}) \quad (2.3.6)$$

and for  $\mu \in \mathbb{R}$

$$R_\mu(\bar{\Delta}_{d-1}) := \left( -\frac{\bar{\Delta}_{d-1}}{2} + i\mu \right)^{-1}. \quad (2.3.7)$$

Note that  $T_\lambda$  and  $\bar{\Delta}_{d-1}$  commute [3](#), which implies the commutation of  $T_\lambda$  and  $R_\mu(\bar{\Delta}_{d-1})$ . Combining this with Theorem [5](#) gives that

$$R_{-1}(\bar{\Delta}_{d-1}) T_\lambda R_1(\bar{\Delta}_{d-1}) \quad (2.3.8)$$

---

<sup>3</sup>We slightly abuse notation, in the sense that the operators commuting are  $T_\lambda \mathbb{1}_{\mathbb{R}^{d-1}}$  and  $\bar{\Delta}_{d-1} \mathbb{1}_{\mathbb{R}^+}$ .

is self-adjoint on  $\{u(z, x) \mid u(z, \cdot) \in L^2(\mathbb{R}^{d-1}), u(\cdot, x) \in D(T_\lambda)\}$ . Remembering that  $R_\mu(\overline{\Delta}_{d-1})$  is bounded, we deduce using Theorem [6](#) that  $W_\lambda$  is self-adjoint on the domain

$$D(W_\lambda) = \left\{ u(z, x) \mid u(z, \cdot) \in L^2(\mathbb{R}^{d-1}), u(\cdot, x) \in D(T_\lambda) \right\}.$$

Again using Theorem [5](#) together with the commutation of  $T_\lambda$  and  $\overline{\Delta}_{d-1}$ , we deduce that  $H_\lambda$  is self-adjoint on

$$D(W_\lambda) \cap \left\{ u(z, x) \mid u(z, \cdot) \in H^2(\mathbb{R}^{d-1}) \right\}. \quad (2.3.9)$$

Now, we prove B): By definition, we have

$$H = -\mathbf{1}_{\mathbb{R}^+} \Delta_{d-1} + T \mathbf{1}_{\mathbb{R}^{d-1}} \quad \text{on} \quad \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R}^{d-1}). \quad (2.3.10)$$

$H$  is closable and the domain of its closure is

$$D(\overline{H}) = \left\{ u(z, x) \in L^2(\mathbb{R}^+ \times \mathbb{R}^{d-1}) \mid u(z, \cdot) \in H^2(\mathbb{R}^{d-1}), u(\cdot, x) \in H_0^2(\mathbb{R}^+) \right\}. \quad (2.3.11)$$

Furthermore, we have

$$\overline{H} = -\overline{\Delta}_{d-1} \mathbf{1}_{\mathbb{R}^+} + \overline{T} \mathbf{1}_{\mathbb{R}^{d-1}} \quad \text{on} \quad D(\overline{H}). \quad (2.3.12)$$

Given a self-adjoint extension  $H_\lambda$  of  $\overline{H}$ , it commutes with  $\mathbf{1}_{\mathbb{R}^+} \overline{\Delta}_{d-1}$ . Hence, following the same steps as in the proof of A), we deduce that the operator  $\mathbf{1}_{\mathbb{R}^+} \overline{\Delta}_{d-1} + H_\lambda$  is self-adjoint of domain  $D(H_\lambda)$ . Hence, we deduce that  $\mathbf{1}_{\mathbb{R}^+} \overline{\Delta}_{d-1} + H_\lambda$  is a self-adjoint extension of  $\overline{T}^d := \overline{T} \mathbf{1}_{\mathbb{R}^{d-1}}$  with domain  $D(\overline{H})$ . Proceeding as for the self-adjoint extensions of  $(\overline{T}, H_0^2(\mathbb{R}^+))$  in section [2.2](#) the self-adjoint extensions of  $\overline{T}^d$  are the operators  $T_\lambda^d$  of domain

$$D(T_\gamma^d) := \left\{ \phi(z, x) + \beta(x)\phi_+(z) + \beta(x)\gamma\phi_-(z) \mid \phi(\cdot, x) \in H_0^2(\mathbb{R}^+); \phi(z, \cdot), \beta \in H^2(\mathbb{R}^{d-1}) \right\}. \quad (2.3.13)$$

Then the domain [\(2.3.13\)](#) can be parametrized by  $\mathbb{R} \cup \{\infty\}$  with

$$D(T_\gamma^d) = \left\{ \psi(z, x) \mid \psi(\cdot, x) \in H^2(\mathbb{R}^+), \psi(z, \cdot) \in H^2(\mathbb{R}^{d-1}), \psi'(0, x) = \alpha \psi(0, x) \right\} \quad (2.3.14)$$

for  $\gamma \neq -1$  and

$$D(T_{-1}^d) = \left\{ \psi(z, x) \mid \psi(\cdot, x) \in H^2(\mathbb{R}^+), \psi(z, \cdot) \in H^2(\mathbb{R}^{d-1}), \psi(0, x) = 0 \right\}. \quad (2.3.15)$$

This proves that for any self-adjoint extension  $H_\lambda$  of  $H$ , there exists a self-adjoint extension  $T_\lambda^d$  such that

$$H_\lambda = -\mathbf{1}_{\mathbb{R}^+} \overline{\Delta}_{d-1} + T_\lambda^d.$$

This ends the proof of proposition [1](#). □

## 2.4 The heat kernels and the propagators

### 2.4.1 The propagators in $\mathbb{R}^+$

In section [2.3](#), we characterized the self-adjoint extensions  $\{T_c\}_{c \in \overline{\mathbb{R}}}$  of  $(\Delta, \mathcal{C}_c^\infty(\mathbb{R}^+))$  of domain

$$D(T_c) = \{\phi \in H^2(\mathbb{R}^+), \quad \phi'(0) = c \phi(0)\}. \quad (2.4.1)$$

In the sequel, we use  $\Delta_D$ ,  $\Delta_N$  and  $\Delta_R$  to denote respectively the Dirichlet, Neumann and Robin Laplacians. Each of these boundary conditions corresponds to a self-adjoint extension with respectively  $c = +\infty$ ,  $c = 0$  and  $c > 0$ . We define

$$H_\star := -\Delta_\star + m^2, \quad \star \in \{D, N, R\}. \quad (2.4.2)$$

The inverse of  $H_\star$  exists and is defined by functional calculus as follows

$$(-\Delta_\star + m^2)^{-1} := \int_{\mathbb{R}^+} d\lambda e^{-\lambda m^2} e^{\lambda \Delta_\star}. \quad (2.4.3)$$

In this section, we prove that the operator [\(2.4.3\)](#) for the different boundary conditions is integral and we present a rigorous computation of the kernels, which allows to define the propagators associated to the Dirichlet, Neumann and Robin boundary conditions. Let us first define the following maps which will be useful in stating the main proposition of this section:

$$\begin{aligned} i_o : L_o^2(\mathbb{R}) &\longrightarrow L^2(\mathbb{R}^+) \\ \phi(z) &\longmapsto \sqrt{2} \chi^+(z) \phi(z), \end{aligned}$$

where  $L_o^2(\mathbb{R})$  is the restriction of  $L^2(\mathbb{R})$  to odd functions and  $\chi^+$  is the characteristic function of the positive line. The map  $i_o$  defines an isometry with the following inverse

$$\begin{aligned} i_o^{-1} : L^2(\mathbb{R}^+) &\longrightarrow L_o^2(\mathbb{R}) \\ \phi(z) &\longmapsto \frac{1}{\sqrt{2}} (\chi^+(z) \phi(z) - \chi^-(z) \phi(-z)). \end{aligned}$$

Similarly, we define

$$\begin{aligned} i_e : L_e^2(\mathbb{R}) &\longrightarrow L^2(\mathbb{R}^+) \\ \phi(z) &\longmapsto \sqrt{2} \chi^+(z) \phi(z), \end{aligned}$$

together with its inverse

$$\begin{aligned} i_e^{-1} : L^2(\mathbb{R}^+) &\longrightarrow L_e^2(\mathbb{R}) \\ \phi(z) &\longmapsto \frac{1}{\sqrt{2}} (\chi^+(z) \phi(z) + \chi^-(z) \phi(-z)), \end{aligned}$$

with  $L_e^2(\mathbb{R})$  denoting the space  $L^2(\mathbb{R})$  restricted to even functions. We have

**Proposition 2.** *Let  $\overline{\Delta}$  be the closure of the Laplacian on  $\mathbb{R}$ . The Dirichlet and Neumann Laplacians are related to  $\overline{\Delta}$  as follows*

$$i_o \overline{\Delta} i_o^{-1} = \Delta_D, \quad i_e \overline{\Delta} i_e^{-1} = \Delta_N. \quad (2.4.4)$$

*Proof.* Given  $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^+)$ , we have

$$i_o \overline{\Delta} i_o^{-1} \phi(x) = \phi''(x), \quad (2.4.5)$$

which implies that  $i_o \overline{\Delta} i_o^{-1}$  is a closed symmetric extension of  $(\Delta, \mathcal{C}_c^\infty(\mathbb{R}^+))$ . Furthermore, we have

$$D(i_o \overline{\Delta} i_o^{-1}) = i_o (H^2(\mathbb{R}) \cap L_o^2(\mathbb{R})) = H_0^1(\mathbb{R}^+) \cap H^2(\mathbb{R}^+).$$

This gives directly (2.4.4) for the Dirichlet case. The Neumann case is treated similarly.  $\square$

The operators  $H_D$ ,  $H_N$  and  $-\overline{\Delta} + m^2$  are invertible and together with (2.4.4) we deduce

$$H_D^{-1} = i_o (-\overline{\Delta} + m^2)^{-1} i_o^{-1}, \quad H_N^{-1} = i_e (-\overline{\Delta} + m^2)^{-1} i_e^{-1}. \quad (2.4.6)$$

The operator  $(-\overline{\Delta} + m^2)^{-1}$  is integral on  $L^2(\mathbb{R})$ , and its kernel is given by (1.1.14)

$$(-\overline{\Delta} + m^2)^{-1}(z, z') = \frac{1}{2m} e^{-m|z-z'|}.$$

Using (2.4.6), we obtain that  $H_D^{-1}$  is also integral on  $L^2(\mathbb{R}^+)$  with the following kernel

$$(-\Delta_D + m^2)^{-1}(z, z') = \frac{1}{2m} \left\{ e^{-m|z-z'|} - e^{-m|z+z'|} \right\}. \quad (2.4.7)$$

We proceed similarly with the Neumann boundary conditions to deduce that  $H_N^{-1}$  is integral on  $L^2(\mathbb{R}^+)$ , and its kernel is given by

$$(-\Delta_N + m^2)^{-1}(z, z') = \frac{1}{2m} \left\{ e^{-m|z-z'|} + e^{-m|z+z'|} \right\}. \quad (2.4.8)$$

Now, we treat the Robin boundary condition. The key idea is to relate the Robin laplacian  $\Delta_R$  to  $\Delta_D$ . Let  $i_c$  be the bijective map given by

$$\begin{aligned} i_c : H_0^1(\mathbb{R}^+) \cap H^2(\mathbb{R}^+) &\longrightarrow D(\Delta_R) \\ \phi(z) &\longmapsto i_c(\phi)(z) = - \int_{\mathbb{R}^+} dw e^{-cw} g(w+z) \end{aligned} \quad (2.4.9)$$

and its inverse  $i_c^{-1}$

$$\begin{aligned} i_c^{-1} : D(\Delta_R) &\longrightarrow H_0^1(\mathbb{R}^+) \cap H^2(\mathbb{R}^+) \\ \phi(z) &\longmapsto i_c^{-1}(\phi)(z) = \phi'(z) - c \phi(z). \end{aligned} \quad (2.4.10)$$

Then  $i_c$  can be continuously extended to  $L^2(\mathbb{R}^+)$ , and we denote its extension by  $\tilde{i}_c$ . The Robin Laplacian is related to the Dirichlet Laplacian through

$$\Delta_R = \tilde{i}_c \Delta_D i_c^{-1}. \quad (2.4.11)$$

This comes from the fact that  $\Delta_R$  and  $\tilde{i}_c \Delta_D i_c^{-1}$  are closed symmetric extensions of  $(\Delta, \mathcal{C}_c^\infty(\mathbb{R}^+))$  together with

$$D(\tilde{i}_c \Delta_D i_c^{-1}) = D(\Delta_R).$$

The operators  $H_D$  and  $H_R$  are invertible and the existence of their inverse is ensured by (2.4.3). For  $f \in \mathcal{C}_c^\infty(\mathbb{R}^+)$ , we have

$$H_R f = \tilde{i}_c H_D i_c^{-1} f = i_c H_D i_c^{-1} f.$$

Hence, we obtain

$$H_R^{-1} f = i_c H_D^{-1} i_c^{-1} f. \quad (2.4.12)$$

Using (2.4.7) together with (2.4.9) and (2.4.10), we deduce

$$i_c H_D^{-1} i_c^{-1} f(z) = \int_{\mathbb{R}^+} dz' \frac{1}{2m} \left( e^{-m|z-z'|} - \frac{c-m}{c+m} e^{-m|z+z'|} \right) f(z'). \quad (2.4.13)$$

By density of  $\mathcal{C}_c^\infty(\mathbb{R}^+)$  in  $L^2(\mathbb{R}^+)$ , we deduce that for all  $g$  in  $L^2(\mathbb{R}^+)$

$$(-\Delta_R + m^2)^{-1} g(z) = \int_{\mathbb{R}^+} dz' \frac{1}{2m} \left( e^{-m|z-z'|} - \frac{c-m}{c+m} e^{-m|z+z'|} \right) g(z'). \quad (2.4.14)$$

### 2.4.2 The propagators in $\mathbb{R}^+ \times \mathbb{R}^{d-1}$

In section 2.3, we characterized the self-adjoint extensions of the Laplacian  $(\Delta, \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R}^{d-1}))$  for  $d \geq 1$ , which are the operators  $\{\Delta_c\}_{c \in \mathbb{R}}$  of domains

$$D(\Delta_\star) = \left\{ \phi(z, x) \mid \phi(z, \cdot) \in H^2(\mathbb{R}^{d-1}), \phi(\cdot, x) \in D(T_\star) \right\}. \quad (2.4.15)$$

Using functional calculus, the operators  $\{-\Delta_\star + m^2\}_{\star \in \{D, N, R\}}$  are invertible, and their inverses are given by

$$(-\Delta_\star + m^2)^{-1} = \int_0^{+\infty} dt e^{-tm^2} e^{t\Delta_\star}. \quad (2.4.16)$$

Using proposition 1, we have

$$-\Delta_\star = -\mathbb{1}_{\mathbb{R}^+} \bar{\Delta}_{d-1} + T_\star \mathbb{1}_{\mathbb{R}^{d-1}}. \quad (2.4.17)$$

The operators  $\mathbb{1}_{\mathbb{R}^+} \bar{\Delta}_{d-1}$  and  $T_\star \mathbb{1}_{\mathbb{R}^{d-1}}$  commute which implies by the Kato-Trotter formula that

$$e^{t\Delta_\star} = e^{t\bar{\Delta}_{d-1} \mathbb{1}_{\mathbb{R}^+}} \cdot e^{-tT_\star \mathbb{1}_{\mathbb{R}^{d-1}}}. \quad (2.4.18)$$

**Lemma 1.** *The heat kernel operator  $e^{t\Delta_c}$  verifies*

$$e^{t\Delta_c} = e^{t\bar{\Delta}_{d-1}} \cdot e^{-tT_c} \quad (2.4.19)$$

*Proof.* We shall prove first that for  $f \in \mathcal{C}_c^\infty(\mathbb{R})$ , we have

$$f(\bar{\Delta}_{d-1}\mathbb{1}_{\mathbb{R}^+}) = f(\bar{\Delta}_{d-1})\mathbb{1}_{\mathbb{R}^+} \quad \text{and} \quad f(T_\star\mathbb{1}_{\mathbb{R}^{d-1}}) = f(T_\star)\mathbb{1}_{\mathbb{R}^{d-1}}. \quad (2.4.20)$$

For  $f \in \mathcal{C}_c^\infty(\mathbb{R})$  there exists  $\tilde{f} \in \mathcal{C}_c^\infty(\mathbb{C})$  with the properties

$$|\bar{\partial}f(z)| \leq C_N |\operatorname{Im} z|^N, \quad \forall N \geq 0, \quad \tilde{f}|_{\mathbb{R}} = f.$$

Here  $\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)$ .  $\tilde{f}$  is called an almost analytic extension of  $f$ . The operator  $\mathbb{1}_{\mathbb{R}^+}\bar{\Delta}_{d-1}$  is self-adjoint on the Hilbert space  $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$ , which implies by the analytic functional calculus [39]

$$f(\mathbb{1}_{\mathbb{R}^+}\bar{\Delta}_{d-1}) = -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z) (z - \mathbb{1}_{\mathbb{R}^+}\bar{\Delta}_{d-1})^{-1} \mu(dz), \quad (2.4.21)$$

where  $\mu(dz) = dx dy$  is the Lebesgue measure on  $\mathbb{C}$ . (2.4.20) is directly deduced from (2.4.21).

Let  $f_n$  be a sequence of uniformly bounded functions in  $\mathcal{C}_c^\infty(\mathbb{R})$  simply converging to  $e^{-|x|}$ . Using functional calculus and remembering that  $-\bar{\Delta}_{d-1}$  is a positive operator, the operators  $f_n(\mathbb{1}_{\mathbb{R}^+}\bar{\Delta}_{d-1})$  and  $f_n(\bar{\Delta}_{d-1})$  strongly converge respectively to  $e^{t\mathbb{1}_{\mathbb{R}^+}\bar{\Delta}_{d-1}}$  and  $e^{t\bar{\Delta}_{d-1}}$ . Combining this with (2.4.20) gives

$$e^{t\mathbb{1}_{\mathbb{R}^+}\bar{\Delta}_{d-1}} = \mathbb{1}_{\mathbb{R}^+} e^{t\bar{\Delta}_{d-1}}. \quad (2.4.22)$$

Proceeding similarly with  $T_\star$  we deduce that

$$e^{-tT_\star\mathbb{1}_{\mathbb{R}^{d-1}}} = e^{-tT_\star}\mathbb{1}_{\mathbb{R}^{d-1}}. \quad (2.4.23)$$

Combining (2.4.18), (2.4.22) and (2.4.23) we deduce (2.4.19).  $\square$

The propagators associated to Dirichlet, Neumann and Robin boundary conditions are then given by

$$(-\Delta_\star + m^2)^{-1}((x, z), (x', z')) = \int_0^\infty dt e^{-tm^2} e^{-tT_\star}(z, z') e^{t\bar{\Delta}_{d-1}}(x, x'). \quad (2.4.24)$$

From chapter [1], we have

$$e^{t\bar{\Delta}_{d-1}}(x, x') = \frac{1}{(2\pi t)^{\frac{d-1}{2}}} e^{-\frac{(x-x')^2}{2t}}. \quad (2.4.25)$$

In the  $pz$ -representation, which corresponds to taking the partial Fourier transformation with respect to the variable  $x \in \mathbb{R}^3$ , (2.4.26) simply reads

$$(-\Delta_\star + m^2)^{-1}(p; z, z') = \int_0^\infty dt e^{-t(p^2+m^2)} e^{-tT_\star}(z, z'). \quad (2.4.26)$$



This formula is similar to (2.4.3) in  $\mathbb{R}^+$  where the mass  $m$  is now replaced by  $\sqrt{p^2 + m^2}$ . Therefore, using (2.4.7)-(2.4.14) we deduce that the Dirichlet, Neumann and Robin propagators are given by

$$C_D(p; z, z') = \frac{1}{2\sqrt{p^2 + m^2}} \left[ e^{-\sqrt{p^2 + m^2}|z-z'|} - e^{-\sqrt{p^2 + m^2}|z+z'|} \right], \quad (2.4.27)$$

$$C_N(p; z, z') = \frac{1}{2\sqrt{p^2 + m^2}} \left[ e^{-\sqrt{p^2 + m^2}|z-z'|} + e^{-\sqrt{p^2 + m^2}|z+z'|} \right], \quad (2.4.28)$$

$$C_R(p; z, z') = \frac{1}{2\sqrt{p^2 + m^2}} \left[ e^{-\sqrt{p^2 + m^2}|z-z'|} + \frac{\sqrt{p^2 + m^2} - c}{\sqrt{p^2 + m^2} + c} e^{-\sqrt{p^2 + m^2}|z+z'|} \right]. \quad (2.4.29)$$

In terms of the heat kernels, (2.4.27)-(2.4.29) can be rewritten as

$$\begin{aligned} C_D((z, x); (z', x')) &= \int_0^\infty d\lambda e^{-\lambda m^2} p_B(\lambda; x, x') \frac{1}{\sqrt{2\pi\lambda}} \left( e^{-\frac{(z-z')^2}{2\lambda}} - e^{-\frac{(z+z')^2}{2\lambda}} \right), \\ C_N((z, x); (z', x')) &= \int_0^\infty d\lambda e^{-\lambda m^2} p_B(\lambda; x, x') \frac{1}{\sqrt{2\pi\lambda}} \left( e^{-\frac{(z-z')^2}{2\lambda}} + e^{-\frac{(z+z')^2}{2\lambda}} \right), \\ C_R((z, x); (z', x')) &= \int_0^\infty d\lambda e^{-\lambda m^2} p_B(\lambda; x, x') p_R(\lambda; z, z'), \end{aligned}$$

where

$$p_B(\lambda; x, x') := \frac{1}{(2\pi\lambda)^{\frac{3}{2}}} e^{-\frac{(x-x')^2}{2\lambda}}, \quad (2.4.30)$$

$$p_R(\lambda; z, z') := p_N(\lambda; z, z') - 2 \int_0^\infty \frac{dw}{\sqrt{2\pi\lambda}} e^{-w} e^{-\frac{(z+z'+\frac{w}{c})^2}{2\lambda}} \quad (2.4.31)$$

and

$$p_N(\lambda; z, z') := \frac{1}{\sqrt{2\pi\lambda}} \left( e^{-\frac{(z-z')^2}{2\lambda}} + e^{-\frac{(z+z')^2}{2\lambda}} \right). \quad (2.4.32)$$

One can easily verify that we have

$$\begin{aligned} C_D(p; 0, z') = C_D(p; z, 0) = 0, \quad \lim_{z \rightarrow 0} \partial_z C_N(p; z, z') = \lim_{z' \rightarrow 0} \partial_{z'} C_N(p; z, z') = 0, \\ \lim_{z \rightarrow 0} \partial_z C_R(p; z, z') = c C_R(p; 0, z'), \quad \lim_{z' \rightarrow 0} \partial_{z'} C_R(p; z, z') = c C_R(p; z, 0), \end{aligned} \quad (2.4.33)$$

where we used that the associated heat kernels verify respectively the Dirichlet, Neumann and Robin boundary conditions.



## Chapter 3

# Trees, forests and weight factors

In this chapter, we present the trees and forests formalism together with their weight factors. We also give some basic properties of these weight factors which will be important later in establishing the proofs of the main results of chapters 4-6. The tree and forest formalism emerges naturally from the structure of the flow equation when considered in position space, in which the correlation "functions" are distributions. Hence, in the next chapters we will introduce a class of test functions to be smeared with the correlation distributions. Since the proof of renormalizability in the flow equation framework is inductive, the class of test functions is restricted to products of heat kernels of the same type that enters in the definition of the propagator. The right hand side of the flow equations for scalar field theories (independently of the considered space-time) consists of two contributions: a part which is linear in the correlation "functions" and a second part which is quadratic. From a Feynman diagrammatic point of view, the linear term is the part in which a new loop is created by the induction, while the quadratic part has more the role of attaching two diagrams without creating an additional loop. Now, the idea behind these trees is to bound a given correlation function with an arbitrary number of loops, by a tree contribution where the loops are contracted in vertices of incidence number 2. One should keep in mind that these loops are not in one-to-one correspondence with the vertices of incidence number 2 of the trees, but the number of loops controls through a bound the number of these vertices. The forest formalism is a novelty which appears when we study the effect of the surface on the renormalizability. In chapter 5, we study the surface correlation distributions which are the part that encapsulate all the UV divergences created by the "surface part" of the semi-infinite model propagators. The flow equation (5.5.80) associated to these objects has a quadratic part, which is factorizable in the sense that it can be bounded by a product of weight factors of trees with an external point on the surface, to which we refer as a surface tree. This collection of surface trees creates a forest.

Before presenting these concepts, let us introduce first the heat kernels that we use throughout this thesis together with their properties. In particular, the bulk heat kernel is needed in the sequel to

define the weight factors of the trees and forests.

### 3.1 The heat kernels and some basic properties

In this section, we define the bulk heat kernel together with the half-line heat kernels. We collect some basic properties of these kernels which we will need in the proof of the main results of chapters [4](#), [5](#) and [6](#). The bulk heat kernel is defined as follows

$$p_B(\tau; z, z') = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(z-z')^2}{2\tau}}, \quad \tau > 0. \quad (3.1.1)$$

In terms of  $p_B$ , the Dirichlet, Neumann and Robin heat kernels read for  $z, z' \geq 0$

$$p_D\left(\frac{1}{\Lambda^2}; z, z'\right) = p_B\left(\frac{1}{\Lambda^2}; z, z'\right) - p_B\left(\frac{1}{\Lambda^2}; z, -z'\right), \quad (3.1.2)$$

$$p_N\left(\frac{1}{\Lambda^2}; z, z'\right) = p_B\left(\frac{1}{\Lambda^2}; z, z'\right) + p_B\left(\frac{1}{\Lambda^2}; z, -z'\right), \quad (3.1.3)$$

$$p_R\left(\frac{1}{\Lambda^2}; z, z'\right) = p_B\left(\frac{1}{\Lambda^2}; z, z'\right) + p_B\left(\frac{1}{\Lambda^2}; z, -z'\right) - 2 \int_0^\infty dw e^{-w} p_B\left(\frac{1}{\Lambda^2}; z, -\frac{w}{c} - z'\right), \quad (3.1.4)$$

$p_B$  verifies the following basic properties:

- (The bulk semi-group property) For  $z_1$  and  $z_2$  in  $\mathbb{R}$

$$\int_{\mathbb{R}} du p_B(\tau_1; z_1, u) p_B(\tau_2; u, z_2) = p_B(\tau_1 + \tau_2; z_1, z_2). \quad (3.1.5)$$

- (The  $\star$  semi-group property) For  $z_1$  and  $z_2$  in  $\mathbb{R}^+$  and  $\star \in \{D, N, R\}$ , we have

$$\int_{\mathbb{R}^+} du p_\star(\tau_1; z_1, u) p_\star(\tau_2; u, z_2) = p_\star(\tau_1 + \tau_2; z_1, z_2). \quad (3.1.6)$$

- (Completeness) For  $z_1$  in  $\mathbb{R}$ , we have

$$\int_{\mathbb{R}} du p_B(\tau_1; z_1, u) = 1. \quad (3.1.7)$$

- For  $z_1$  and  $z_2$  in  $\mathbb{R}^+$ , we have

$$\int_{\mathbb{R}} du p_B(\tau_1; z_1, u) p_B(\tau_2; u, z_2) \leq 2 \int_{\mathbb{R}^+} du p_B(\tau_1; z_1, u) p_B(\tau_2; u, z_2). \quad (3.1.8)$$

- For  $\delta \geq 0$ ,  $\tau_\delta = (1 + \delta)\tau$  and  $z_1, z_2 \in \mathbb{R}^+$ , we have

$$p_B(\tau; z_1, z_2) \leq \sqrt{1 + \delta} p_B(\tau_\delta; z_1, z_2) \quad (3.1.9)$$

and for  $\delta' > \delta$

$$|z_1 - z_2|^r p_B(\tau_\delta; z_1, z_2) \leq C_{\delta, \delta'} \tau^{\frac{r}{2}} p_B(\tau_{\delta'}; z_1, z_2), \quad (3.1.10)$$

where

$$C_{\delta, \delta'} = \sqrt{\frac{1 + \delta'}{1 + \delta}} \left\| x^r e^{-\frac{x^2}{2} \frac{\delta' - \delta}{(1 + \delta)(1 + \delta')}} \right\|_\infty \leq O(1) |\delta - \delta'|^{-\frac{r}{2}}. \quad (3.1.11)$$

- For  $z, z' \in \mathbb{R}^+$ ,  $\tau > 0$  and  $c \geq 0$ , we have

$$p_B(\tau; z, -z') \leq p_B(\tau; z, z'), \quad \int_{\mathbb{R}^+} dw e^{-w} p_B\left(\tau; z, -z' - \frac{w}{c}\right) \leq p_B(\tau; z, z'). \quad (3.1.12)$$

### 3.2 Trees, Forests and Weight factors

As we mentioned in chapter [1](#) in section [1.1](#), the proof of perturbative renormalization with flow equations is based on obtaining inductive bounds uniform in the UV cutoff for the connected amputated Schwinger "functions" of the considered theory. For the semi-infinite model, the bounds on the surface and bulk correlation distributions are specified in terms of weighted trees and forests, which we define in the following, and for which we also derive some properties that will be important later. Our trees basically represent tree level Feynman graphs. However, we stress that this analogy must not be taken literally, the trees and the incidence number of vertices are independent of the detailed form of the  $n$ -point interactions in the theory, the loop order controls the number of vertices of incidence number 2 of the trees and forests via a bound, but there is no one-to-one correspondence between the loop order and the number of these vertices. First, we start with some notations that we will use in the sequel:

- For  $s \geq 1$ , we denote by  $\sigma_s$  the set  $\{1, \dots, s\}$  and for  $i \leq j$  we denote by  $\sigma_{i:j}$  the set  $\{i, \dots, j\}$ .
- Let  $\mathcal{P}_s$  be the set of all the partitions of  $\sigma_s$ . For a partition  $\Pi \in \mathcal{P}_s$ , we write  $\Pi = (\pi_i)_{1 \leq i \leq l_\Pi}$  with  $\pi_i$  denoting an element of the partition  $\Pi$  and  $l_\Pi$  the cardinality of  $\Pi$ .
- For  $\Pi \in \mathcal{P}_s$  such that  $r \in \pi_i$ , we define

$$\pi_i^r := \pi_i \setminus \{r\}, \quad \Pi^r := \left( \bigcup_{j=1, j \neq i}^r \pi_j \right) \cup \pi_i^r. \quad (3.2.1)$$

- Given  $\Pi \in \mathcal{P}_{s+2}$  such that  $\{s+1, s+2\} \in \pi_i$ , we define the reduced sub-partition

$$\pi_i^{s+1, s+2} := \pi_i \setminus \{s+1, s+2\}. \quad (3.2.2)$$

- We denote by  $\mathcal{P}_s^1$  the set of partitions which contain at least one sub-partition of length 1 (i.e.  $\exists \pi_i \subset \Pi, |\pi_i| = 1$ ) and  $\mathcal{P}_s^{1;c}$  its complementary set.
- We denote by  $\tilde{\mathcal{P}}_{2,s}$  the set of partitions of length 2 of the set  $\sigma_s$ . Note that  $\tilde{\mathcal{P}}_{2,s}$  is a subset of  $\mathcal{P}_s$ .

### 3.2.1 Bulk trees, surface trees and forests

- A tree is an undirected graph in which any two vertices are connected by exactly one path. We define the incidence number of a vertex  $z$  of the tree by the number of lines of the tree that have  $z$  as an edge and we denote it by  $c(z)$ . Given a tree  $T$ , we denote by  $\mathcal{V}(T)$  the set of vertices of the tree. The set of external vertices  $\mathcal{V}_e(T)$  of the tree  $T$  is defined as follows

$$\mathcal{V}_e(T) := \{z \in \mathcal{V}(T) \mid c(z) = 1\}. \quad (3.2.3)$$

The set of internal vertices  $\mathcal{V}_i(T)$  is then defined as  $\mathcal{V}(T) \setminus \mathcal{V}_e(T)$ .

- For  $s \geq 2$ , we denote by  $\mathcal{T}^s$  the set of all trees that have a root vertex and  $s - 1$  external vertices. For a tree  $T^s \in \mathcal{T}^s$  we will call  $z_1 \in \mathbb{R}^+$  its root vertex. Denoting by  $\mathcal{V}(T)$  the set of vertices of  $T^s$ , the set of external vertices of the tree  $T^s$  is defined as

$$\mathcal{V}_e(T^s) := \{z \in \mathcal{V}(T^s) \setminus \{z_1\} \mid c(z) = 1\}. \quad (3.2.4)$$

The set of internal vertices is the relative complement of  $\mathcal{V}_e(T^s)$  in  $\mathcal{V}(T^s) \setminus \{z_1\}$ . Note that the root vertex  $z_1$  is a vertex which is neither internal nor external. For simplicity, we use in the sequel the set of points  $Y = \{y_2, \dots, y_s\}$  in  $\mathbb{R}^{s-1}$  to be identified with the external vertices of  $T^s$ . Likewise we call  $Z = \{z_2, \dots, z_{r+1}\}$  the set of internal vertices of  $T^s$  where  $z_i \in \mathbb{R}^+$  and  $r$  is the cardinality of  $\mathcal{V}_i(T^s)$ .

- We denote by  $c_1 = c(z_1)$  the incidence number of the root vertex. We call a line  $p$  an external line of the tree if one of its edges is in  $Y$ . The set of external lines is denoted  $\mathcal{J}$ . The remaining lines are called internal lines of the tree and are denoted by  $\mathcal{F}$ .
- By  $T_l^s$  we denote a tree  $T^s \in \mathcal{T}^s$  satisfying  $v_2 + \delta_{c_1,1} \leq 3l - 2 + s/2$  for  $l \geq 1$  and satisfying  $v_2 = 0$  for  $l = 0$ , where  $v_n$  is the number of vertices having incidence number  $n$ . Then  $\mathcal{T}_l^s$  denotes the set of all trees  $T_l^s$ . We indicate the external vertices and internal vertices of the tree by writing  $T_l^s(z_1, y_{2,s}, \vec{z})$  with  $y_{2,s} = (y_2, \dots, y_s)$  and  $\vec{z} = (z_2, \dots, z_{r+1})$ .
- We also define the set of twice rooted trees denoted as  $\mathcal{T}_l^{s,(12)}$ . The trees  $T_l^{s,(12)} \in \mathcal{T}_l^{s,(12)}$  are defined such that they have two root vertices  $z_1$  and  $z_2$  and  $s - 2$  external vertices. Furthermore, the number of vertices of incidence number 2 of a tree  $T_l^{s,(12)}$  satisfies

$$v_2 + \delta_{c_1,1} + \delta_{c_2,1} \leq 3l - 2 + \frac{s-1}{2}, \quad \forall l \geq 2$$

and  $v_2 \leq 1$  for  $l \leq 1$ . In particular, the set  $\mathcal{T}_l^{1,(12)}$  is given by all trees with two root vertices, no external vertices and all its internal vertices are of incidence number 2 with a total number that verifies  $v_2 \leq 1$  for  $l \leq 1$  and

$$v_2 \leq 3l - 4, \quad \forall l \geq 2.$$

- Given  $\pi$  a subset of  $\sigma_{2:s}$ , we define

$$\mathcal{Y}_\pi(\vec{z}) := \{z \in \vec{z} \mid \exists y_i \in Y_\pi, (z, y_i) \in \mathcal{I}\}. \quad (3.2.5)$$

For a partition  $\Pi = (\pi_1, \pi_2)$  of the set  $\sigma_{2:s}$ , we define

$$\mathcal{T}_{l;\pi_1,\pi_2}^s := \{T_{l;\pi_1,\pi_2}^s(z_1, y_{2,s}, \vec{z}) \in \mathcal{T}_l^s \mid \mathcal{Y}_{\pi_1}(\vec{z}) \cap \mathcal{Y}_{\pi_2}(\vec{z}) = \emptyset\}. \quad (3.2.6)$$

Note that the tree  $T_l^s(z_1, y_{2,s})$  with no internal vertices belongs to  $\mathcal{T}_{l;\pi_1,\pi_2}^s$  for any partition  $(\pi_1, \pi_2)$  of  $\sigma_{2:s}$ .

- For  $s \geq 1$ , we define the set of bulk trees  $\hat{\mathcal{T}}_l^s$  as the set of all trees with  $s$  external vertices, no root vertex [\[1\]](#) and which satisfy  $v_2 \leq 3l - 2 + \frac{s}{2}$  for all  $l \geq 1$ , or  $v_2 = 0$  for  $l = 0$ .
- For a tree  $T_{l-1}^{s+2}(z_1, y_{2,s+2}, \vec{z})$  we define the reduced tree  $\mathcal{R}_{y_i, y_j} T_{l-1}^{s+2}$  to be the unique tree obtained from  $T_{l-1}^{s+2}(z_1, y_{2,s+2}, \vec{z})$  through the following procedure:
  - By taking off the two external vertices  $y_i, y_j$  with the external lines attached to them.
  - By taking off the internal vertices -if any- which have acquired incidence number  $c = 1$  through the previous process, and by also taking off the lines attached to them.
  - If a new vertex of incidence number 1 is created, the second step of the process is repeated.

Note that these steps produce a tree in  $\mathcal{T}_l^s$  for  $s \geq 1$ . For  $s = 0$ , it is clear that the reduced tree is empty. Let us explain briefly why the obtained tree belongs to  $\mathcal{T}_l^s$  for  $s \geq 1$ : the number of external vertices goes from  $s + 1$  to  $s - 1$ , which clearly implies that  $\mathcal{R}_{y_i, y_j} T_{l-1}^{s+2} \in \mathcal{T}^s$ . Now, let us see how the number of vertices of incidence number 2 is affected by the reduction. Each amputation reduces the incidence number of the internal vertex of the amputated leg by one. Hence, the number of vertices of incidence number 2 is at most increased by two at the end of the reduction process. Denoting by  $v'_2$  the number of incidence number 2 of the internal vertices of the reduced tree, we have

$$v'_2 + \delta_{c_1,1} \leq v_2 + 2 + \delta_{c_1,1} \leq 3(l-1) - 2 + \frac{s+2}{2} + 2 = 3l - 2 + \frac{s}{2}.$$

This implies that the reduced tree  $\mathcal{R}_{y_i, y_j} T_{l-1}^{s+2}$  is indeed in  $\mathcal{T}_l^s$  and we write

$$\mathcal{R}_{y_i, y_j} \mathcal{T}_{l-1}^{s+2} \subset \mathcal{T}_l^s. \quad (3.2.7)$$

Note that the reduction steps described above do not affect the set  $\mathcal{Y}_{\sigma_{s+2}^{i,j}}(\vec{z})$ , in the sense that [\[2\]](#) they are not removed by the process. However, the incidence number of these vertices can possibly

<sup>1</sup>All the vertices of incidence number one are external.

<sup>2</sup>we used the notation [\(3.2.2\)](#).

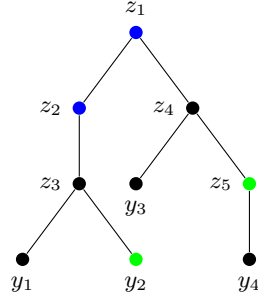


Figure 3.1

be decreased at most by 2. We note that if  $T_{l-1}^{s+2} \in \mathcal{T}_{l-1; \pi_1, \pi_2}^{s+2}$  with  $(\pi_1, \pi_2)$  a partition of  $\sigma_{s+2}$  such that  $\{i, j\} \in \pi_2$ , then  $\mathcal{R}_{y_i, y_j} T_{l-1}^{s+2}$  belongs to  $\mathcal{T}_{l; \pi_1, \pi_2}^{s, i, j}$ . We write

$$\mathcal{R}_{y_i, y_j} \mathcal{T}_{l-1; \pi_1, \pi_2}^{s+2} \subset \mathcal{T}_{l; \pi_1, \pi_2}^{s, i, j}. \quad (3.2.8)$$

- Given a tree  $T^s \in \mathcal{T}^s$  and two vertices  $u$  and  $v$  of the tree, we denote by  $]u, v[$  the set of internal vertices of which consists the path joining  $u$  and  $v$  including  $u$  and  $v$ . To exclude  $u$  or  $v$ , the associated square bracket is opened. We define the distance  $d(u, v)$  as the cardinality of the set  $]u, v[$ . In the sequel, we write  $]u, v[ = \{u_0, \dots, u_q\}$  and we stress the importance of the order on which the internal vertices  $(u_i)_{0 \leq i \leq q}$  appear in  $]u, v[$ , in the sense that going from  $u$  to  $v$  in the tree  $T^s$ , one has to cross first the vertex  $u_0$ , then  $u_1$  etc. Let us illustrate these notions with the example of the tree depicted in Figure 3.1. Clearly, we have  $]z_1, z_2[ = \emptyset$  and  $]z_5, y_2[ = \{z_4, z_1, z_2, z_3\}$  which implies that  $d(z_1, z_2) = 0$  and  $d(z_5, y_2) = 4$ .
- Let  $s \geq 1$ . For  $Y_{\sigma_s} := (y_1, \dots, y_s) \in \mathbb{R}^s$ , we define the set of surface trees  $\mathcal{T}^{s,0}$  to be the set consisting of all trees of  $s+1$  external vertices  $\{y_1, \dots, y_s, 0\}$ . In the sequel, we refer to the external vertex  $0$  as the surface external vertex to distinguish it from the other external vertices.
- By  $T_l^{s,0}$  we denote a surface tree  $T^{s,0} \in \mathcal{T}^{s,0}$  satisfying  $v_2 \leq 3l - 2 + \frac{s+1}{2}$  for  $l \geq 1$  and satisfying  $v_2 = 0$  for  $l = 0$ . Then  $\mathcal{T}_l^{s,0}$  denotes the set of all surface trees  $T_l^{s,0}$ . For a tree  $T_l^{s,0} \in \mathcal{T}_l^{s,0}$ , the set  $\{y_1, \dots, y_s, 0\}$  of points in  $\mathbb{R}$  is identified with its external vertices, and  $\vec{z} = (z_1, \dots, z_r)$  such that  $r \geq 1$  with the set of its internal vertices. We indicate the external vertices and the internal vertices of the tree by writing  $T_l^{s,0}(Y_{\sigma_s}, 0, \vec{z})$ . Note that this definition implies for all  $l' \leq l$ ,  $\mathcal{T}_{l'}^{s,0} \subset \mathcal{T}_l^{s,0}$ .
- For  $s = 1$ , the set of surface trees  $\mathcal{T}_l^{1,0}$  consists of the surface tree  $T_l^{1,0}(y_1, 0)$  with no internal vertices and one surface external vertex attached to the external vertex  $y_1$ .



- Given a partition  $\Pi \in \mathcal{P}_s$  and trees  $T_l^{s_{\pi_i},0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i}) \in \mathcal{T}_l^{s_{\pi_i},0}$ , we define the forest  $W_l^s(\Pi)$  as follows,

$$W_l^s(\Pi) = \bigcup_{i=1}^{l_\Pi} \left\{ T_l^{s_{\pi_i},0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i}) \right\} \quad \text{where } s_{\pi_i} := |\pi_i|, \quad \sum_{i=1}^{l_\Pi} s_{\pi_i} = s \quad \text{and } Y_{\pi_j} = \bigcup_{i \in \pi_j} \{y_i\}.$$

Here  $|\pi_i|$  is the cardinality of the set  $\pi_i$ . We write shortly  $T_l^{s_{\pi_i},0} \equiv T_l^{s_{\pi_i},0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i})$ . Then the set of all forests  $W_l^s(\Pi)$  denoted by  $\mathcal{W}_l^s(\Pi)$ , is defined as:

$$\mathcal{W}_l^s(\Pi) := \bigcup_{i=1}^{l_\Pi} \mathcal{T}_l^{s_{\pi_i},0}. \quad (3.2.9)$$

Note that for the trivial partition  $\Pi_0 = \sigma_s$ , the length of the partition is equal to one. Therefore, the set  $W_l^s(\sigma_s)$  reduces to surface trees  $\mathcal{T}_l^{s,0}$ . We write

$$\mathcal{W}_l^s(\sigma_s) = \mathcal{T}_l^{s,0}. \quad (3.2.10)$$

This implies that each tree  $T_l^{s_{\pi_i},0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i})$  can be identified with a forest in  $\mathcal{W}_l^{s_{\pi_i}}(\sigma_{s_{\pi_i}})$ , where  $\sigma_{s_{\pi_i}} := \bigcup_{k \in \pi_i} \{k\}$ .

- We define the global set of forests  $\mathcal{W}_l^s$  by

$$\mathcal{W}_l^s := \bigcup_{\Pi \in \mathcal{P}_s} \mathcal{W}_l^s(\Pi).$$

To illustrate these concepts, we give some examples of trees and forests for  $s = 3$  and  $l = 2$ . The set of partitions is in this case

$$\mathcal{P}_3 = \left\{ \bigcup_{i=1}^3 \{i\}, \{1\} \cup \{2, 3\}, \{2\} \cup \{1, 3\}, \{3\} \cup \{1, 2\}, \sigma_3 \right\}. \quad (3.2.11)$$

- For the trivial partition  $\Pi_0 = \sigma_3$ , the partition length  $l_\Pi$  is equal to one and therefore the elements of the set  $\mathcal{W}_2^3(\Pi_0)$  are the trees  $T_2^{3,0} \in \mathcal{T}_2^{3,0}$  such that  $v_2 \leq 5$ . For  $v_2 = 3$ , Figure 1 is an example of a surface tree in  $\mathcal{W}_2^3(\Pi_0)$ .

- For the partition  $\Pi_1 = \bigcup_{i=1}^3 \{i\}$ , an element of  $\mathcal{W}_2^3(\Pi_1)$  (i.e. the set of forests of the partition  $\Pi_1$ ) is given by the forest in Figure 2.

This forest is composed of three trees. Each tree has two external vertices. The external vertex  $y_i$  has an index which belongs to the sub-partition  $\{i\}$ . Note that the total number of vertices of incidence number 2 does not exceed 5 (in this case it is equal to 5). Note also that all the internal vertices of a surface tree with only two external vertices are of incidence number 2.

- For the partition  $\Pi_2 = \{1, 2\} \cup \{3\}$ , Figure 3 is an example of a forest in  $\mathcal{W}_2^3(\Pi_2)$  with a total number of vertices of incidence number 2 equal to 4.

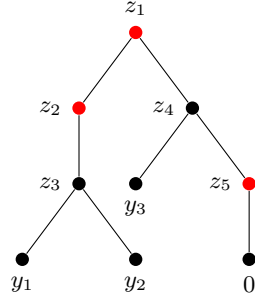


Figure 3.2: Example of a forest  $W_2^3(\Pi_0)$  with  $v_2 = 3$  and  $\vec{z} = (z_1, \dots, z_5)$ . The red color is used for the internal vertices of incidence number 2.

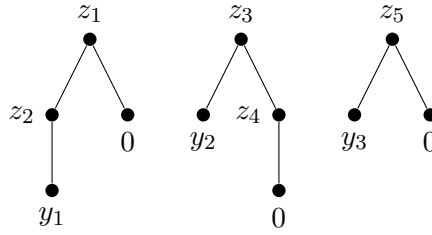


Figure 3.3: Example of a forest  $W_2^3(\Pi_1)$  with  $v_2 = 5$  and  $\vec{z} = (z_1, \dots, z_5)$ .

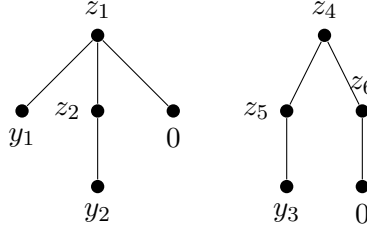


Figure 3.4: Example of a forest  $W_2^3(\Pi_2) \in \mathcal{W}_2^3(\Pi_2)$  with  $v_2 = 4$ .

Similar examples for the forest  $W_2^3(\Pi_3)$  (respectively  $W_2^3(\Pi_4)$ ) for  $\Pi_3 = \{1, 3\} \cup \{2\}$  (resp.  $\Pi_4 = \{2, 3\} \cup \{1\}$ ) can be constructed by replacing in Figure 2 the vertices  $\{y_1, y_2\}$  by  $\{y_1, y_3\}$  and the vertex  $y_3$  by  $y_2$  (resp.  $\{y_1, y_2\}$  by  $\{y_2, y_3\}$  and  $y_3$  by  $y_1$ ).

An example of a forest in the global set of forests  $\mathcal{W}_2^3$  is  $\cup_{i=0}^4 W_2^3(\Pi_i)$ .

- Given  $\Pi = \cup_{i=1}^l \tilde{\pi}_i$  in  $\mathcal{P}_s$ ,  $(\pi_1, \pi_2)$  in  $\tilde{\mathcal{P}}_{2;s}$  and a forest  $W_l^s(\Pi) \in \mathcal{W}_l^s(\Pi)$ , we define

$$\mathcal{Y}_{\pi_k}(W_l^s(\Pi)) = \left\{ T_l^{s, \tilde{\pi}_j, 0} \in W_l^s(\Pi) : \exists i \in \pi_k, y_i \in Y_{\tilde{\pi}_j} \right\}, \quad k \in \{1, 2\}. \quad (3.2.12)$$

We also define

$$\mathcal{W}_{l; \pi_1, \pi_2}^s(\Pi) := \{W_l^s(\Pi) \in \mathcal{W}_l^s(\Pi) : \mathcal{Y}_{\pi_1}(W_l^s(\Pi)) \cap \mathcal{Y}_{\pi_2}(W_l^s(\Pi)) = \emptyset\}. \quad (3.2.13)$$

A forest in  $\mathfrak{W}_{l;\pi_1,\pi_2}^s(\Pi)$  is characterized by the fact that each of its surface trees have their external vertices either in  $Y_{\pi_1}$  or in  $Y_{\pi_2}$ .

### 3.2.2 Some operations on Forests and Trees

#### Reduction

Let  $W_{l-1}^{s+2}(\Pi)$  be a forest in  $\mathfrak{W}_{l-1}^{s+2}(\Pi)$ . In this part, we define and explain the process of reducing the forest  $W_{l-1}^{s+2}(\Pi)$  to a forest in  $\mathfrak{W}_l^s$ .

**Definition 13.** (*Reduced partition*) Let  $s \geq 1$  and  $\Pi$  be in  $\mathcal{P}_{s+2}$ . We denote by  $\pi_i$  and  $\pi_j$  the subpartitions of  $\Pi$  such that  $s+1 \in \pi_i$  and  $s+2 \in \pi_j$ . The reduced partition  $\Pi^{s+1,s+2}$  is defined as follows,

$$\Pi^{s+1,s+2} = \begin{cases} \left\{ \bigcup_{k=1, k \notin \{i,j\}}^{l_\Pi} \pi_k \right\} \cup \pi_i^{s+1} \cup \pi_j^{s+2} & \text{if } i \neq j \\ \left\{ \bigcup_{k=1, k \neq i} \pi_k \right\} \cup \pi_i^{s+1,s+2} & \text{otherwise,} \end{cases}$$

where we used the notations (3.2.1)-(3.2.2). For  $(\pi_1, \pi_2)$  in  $\tilde{\mathcal{P}}_{2;s}$ , we write

$$(\pi_1, \pi_2)^{s+1,s+2} = \begin{cases} \pi_1^{s+1}, \pi_2^{s+2} & \text{if } s+i \in \pi_i \\ \pi_1^{s+1,s+2}, \pi_2 & \text{if } \{s+1, s+2\} \in \pi_1 \\ \pi_1, \pi_2^{s+1,s+2} & \text{if } \{s+1, s+2\} \in \pi_2. \end{cases}$$

**Proposition 3.** (*Reduction process*) Let  $s \geq 1$ . For  $\Pi \in \mathcal{P}_{s+2}$ , we define  $C_{y_{s+1}, y_{s+2}}$  to be the operator which acts on a forest  $W_{l-1}^{s+2}(\Pi) \in \mathfrak{W}_{l-1}^{s+2}(\Pi)$  by removing the two external legs attached to  $y_{s+1}$  and  $y_{s+2}$ . If this operation produces an internal vertex of incidence number one, it is removed until an internal vertex of incidence number  $c(z) \geq 2$  is reached. We have

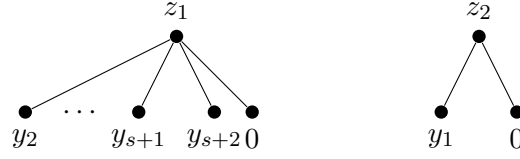
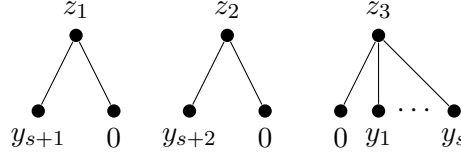
$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) \in \mathfrak{W}_l^s(\Pi^{s+1,s+2}). \quad (3.2.14)$$

For  $(\pi_1, \pi_2) \in \tilde{\mathcal{P}}_{2;s+2}$  and  $W_{l-1}^{s+2}(\Pi)$  in  $\mathfrak{W}_{l-1;\pi_1,\pi_2}^{s+2}(\Pi)$ , we also have

$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) \in \mathfrak{W}_{l;(\pi_1,\pi_2)^{s+1,s+2}}^s(\Pi^{s+1,s+2}). \quad (3.2.15)$$

*Proof.* Let us first prove (3.2.14). The set  $\mathcal{P}_{s+2}$  can be separated into two subsets  $\tilde{\mathcal{P}}_{s+2}$  and  $\tilde{\mathcal{P}}_{s+2}^c$  defined as follows:

- $\tilde{\mathcal{P}}_{s+2}$  is defined as a subset of  $\mathcal{P}_{s+2}$  which contains all the partitions  $\Pi$  for which there exists  $\pi_i \in \Pi$  such that  $\{s+1, s+2\} \in \pi_i$ .
- $\tilde{\mathcal{P}}_{s+2}^c$  is the complementary set of  $\tilde{\mathcal{P}}_{s+2}$ .

Figure 3.5: Example of a forest  $W_{l-1}^{s+2}(\Pi)$  where  $\Pi \in \tilde{\mathcal{P}}_{s+2}$  and  $l_{\Pi} = 2$ .Figure 3.6: Example of a forest  $W_{l-1}^{s+2}(\Pi^c)$  where  $\Pi^c \in \tilde{\mathcal{P}}_{s+2}^c$  and  $l_{\Pi} = 3$ .

Diagrammatically, the global set of forests  $\mathcal{W}_{l-1}^{s+2}$  is partitioned into two subsets: the subset of forests for which  $y_{s+1}$  and  $y_{s+2}$  both belong to the same surface tree and the subset of forests in which  $y_{s+1}$  and  $y_{s+2}$  belong to different surface trees.

The proof of the statement (3.2.14) follows directly from establishing that

$$\forall \Pi \in \tilde{\mathcal{P}}_{s+2} : C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) \in \mathcal{W}_l^s(\Pi^{s+1, s+2}) \quad (3.2.16)$$

and

$$\forall \Pi \in \tilde{\mathcal{P}}_{s+2}^c : C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) \in \mathcal{W}_l^s(\Pi^{s+1, s+2}) . \quad (3.2.17)$$

- First, we prove (3.2.16). Given a partition  $\Pi$  in  $\tilde{\mathcal{P}}_{s+2}$ , there exists a sub-partition  $\pi_i \in \Pi$  such that  $\{s+1, s+2\} \in \pi_i$ . Therefore, we can write in slightly abusive notation

$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) = \bigcup_{k=1, k \neq i}^{l_{\pi}} \left\{ T_{l-1}^{s_k, 0}(Y_{\pi_k}, 0, \vec{z}_{\pi_k}) \right\} \cup \left\{ C_{y_{s+1}, y_{s+2}} T_{l-1}^{s_{\pi_i}, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i}) \right\}, \quad (3.2.18)$$

where the tree  $T_{l-1}^{s_{\pi_i}, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i})$  can be identified with a forest in  $\mathcal{W}_{l-1}^{s_{\pi_i}}(\sigma_{s_{\pi_i}})$ . Deducing (3.2.16) amounts to prove for  $s_{\pi_i} > 2$

$$C_{y_{s+1}, y_{s+2}} T_{l-1}^{s_{\pi_i}, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i}) \in \mathcal{F}_l^{s_{\pi_i}-2, 0}. \quad (3.2.19)$$

If  $s_{\pi_i} = 2$

$$C_{y_{s+1}, y_{s+2}} T_{l-1}^{2, 0}(y_{s+1}, y_{s+2}, 0, \vec{z}_{\pi_i}) = \emptyset \quad (3.2.20)$$

and we have

$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) = \bigcup_{k=1, k \neq i}^{l_{\pi}} \left\{ T_{l-1}^{s_k, 0}(Y_{\pi_k}, 0, \vec{z}_{\pi_k}) \right\}, \quad (3.2.21)$$

which is clearly in  $W_l^s(\Pi^s)$ . To treat the case  $s_{\pi_i} > 2$ , the discussion is simplified by considering the case of the trivial partition  $\Pi = \sigma_{s+2}$  s.t.  $s \geq 1$ . In this case, the set of forests  $\mathcal{W}_{l-1}^{s+2}(\sigma_{s+2})$  is given by all the surface trees  $T_{l-1}^{s+2,0}$  (see (3.2.10)). Let  $J_i$  (resp.  $J_j$ ) be the external line which attaches the internal vertex  $z_i$  (resp.  $z_j$ ) to the external vertex  $y_{s+1}$  (resp.  $y_{s+2}$ ). The operator  $C_{y_{s+1}, y_{s+2}}$  removes the external legs  $J_i$  and  $J_j$  from the forest  $W_{l-1}^{s+2}(\sigma_{s+2})$ , and if one of the internal vertices  $z_i$  and  $z_j$  becomes of incidence number one, it is removed and the process continues until an internal vertex  $z$  of incidence number  $c(z) \geq 2$  is reached. This implies that  $v'_2$  (i.e. the number of vertices of incidence number 2 of the new forest  $C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\sigma_{s+2})$ ) is at most  $v_2 + 2$ , with  $v_2$  the number of vertices of incidence number 2 of  $W_{l-1}^{s+2}(\sigma_{s+2})$ . Therefore,

$$v'_2 \leq v_2 + 2 \leq 3(l-1) - 2 + \frac{s+3}{2} + 2 \leq 3l - 2 + \frac{s+1}{2}.$$

The last point to verify is that the reduction process converges for  $s \geq 1$  in the sense that we have

$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\sigma_{s+2}) \neq \emptyset. \quad (3.2.22)$$

In order to obtain (3.2.22), we need to prove that there exists at least one internal vertex  $\tilde{z}$  such that  $c(\tilde{z}) \geq 2$ . If  $W_{l-1}^{s+2}(\sigma_{s+2})$  has at least one internal vertex such that  $c(z) \geq 4$ , then eventually (3.2.22) holds. If all the internal vertices are of incidence number less than or equal to 3, then since  $s \geq 1$ , the tree  $W_{l-1}^{s+2}(\sigma_{s+2})$  has a number of external vertices greater than or equal to 4 (taking into account the surface external vertex 0 as well). This implies that it has at least two internal vertices  $z$  and  $z'$  such that  $c(z) = c(z') = 3$  which leads directly to (3.2.22). This proves (3.2.16).

- Now, we prove (3.2.17). Take  $\Pi \in \tilde{\mathcal{P}}_{s+2}^c$ , there exist  $\pi_i, \pi_j \in \Pi$  such that  $i \neq j$ ,  $\{s+1\} \in \pi_i$  and  $\{s+2\} \in \pi_j$ . Therefore, we can write

$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) = \bigcup_{k=1, k \neq i, k \neq j}^{l_\Pi} \left\{ T_{l-1}^{s_{\pi_k}, 0}(Y_{\pi_k}, 0, \vec{z}_{\pi_k}) \right\} \cup \left\{ C_{y_{s+1}} T_{l-1}^{s_{\pi_i}, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i}) \right\} \\ \cup \left\{ C_{y_{s+2}} T_{l-1}^{s_{\pi_j}, 0}(Y_{\pi_j}, 0, \vec{z}_{\pi_j}) \right\}, \quad (3.2.23)$$

where the operator  $C_{y_{s+1}}$  acts on the tree  $T_{l-1}^{s_{\pi_i}, 0}$  by removing the external leg to which  $y_{s+1}$  is attached and by removing all the internal vertices which through this process become of incidence number one. Following the same steps of the discussion above, we deduce that  $v'_{2,i}$  the number of vertices of incidence number 2 of the tree  $C_{y_{s+1}} T_{l-1}^{s_{\pi_i}, 0}$  is at most  $v_{2,i} + 1$  and it obeys

$$v'_{2,i} \leq v_{2,i} + 1 \leq 3(l-1) - 2 + \frac{s_{\pi_i} + 1}{2} \leq 3l - 2 + \frac{s_{\pi_i}}{2},$$

---

<sup>3</sup> $v_{2,i}$  is the number of vertices of incidence number 2 of  $T_l^{s_{\pi_i}+1, 0}$ .

which implies that  $C_{y_{s+1}} T_l^{s_{\pi_i}, 0} \in \mathcal{F}_l^{s_{\pi_i}-1, 0}$ .

Here again, we need to verify that the reduction process of the forest  $W_{l-1}^{s+2}$  converges in the sense of (3.2.22). If  $|\pi_i| = |\pi_j| = 1$ , then we have

$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) = \bigcup_{k=1, k \neq i, k \neq j}^{l_\pi} \left\{ T_{l-1}^{s_{\pi_k}, 0}(Y_{\pi_k}, 0, \vec{z}_{\pi_k}) \right\}. \quad (3.2.24)$$

If  $|\pi_i| \geq 2$ , we have

$$C_{y_{s+1}} T_{l-1}^{s_{\pi_i}, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i}) \neq \emptyset.$$

This holds since the tree  $T_{l-1}^{s_{\pi_i}, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i})$  has at least three external vertices which implies that there exists at least one internal vertex such that  $c(z) \geq 3$ , and removing at most one external leg at each step of the reduction process implies that the incidence number of  $z$  is strictly greater than 1 at the end of the process.

Using (3.2.16) and (3.2.17), we deduce for  $W_{l-1}^{s+2}(\Pi)$  in  $\mathcal{W}_{l-1; \pi_1, \pi_2}^{s+2}(\Pi)$

$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) \in \mathcal{W}_l^s(\Pi^{s+1, s+2}). \quad (3.2.25)$$

Note that the reduction steps imply

$$\mathcal{Y}_{\pi_1}(W_{l-1}^{s+2}(\Pi)) = \mathcal{Y}_{\hat{\pi}_1^{s+1, s+2}}(C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi)) \quad (3.2.26)$$

and

$$\mathcal{Y}_{\pi_2}(W_{l-1}^{s+2}(\Pi)) = \mathcal{Y}_{\hat{\pi}_2^{s+1, s+2}}(C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi)), \quad (3.2.27)$$

where

$$\hat{\pi}_i^{s+1, s+2} = \begin{cases} \pi_i^{s+1} & \text{if } s+1 \in \pi_i, s+2 \notin \pi_i \\ \pi_i^{s+2} & \text{if } s+1 \notin \pi_i, s+2 \in \pi_i \\ \pi_i^{s+1, s+2} & \text{if } \{s+1, s+2\} \in \pi_i. \end{cases} \quad (3.2.28)$$

Since

$$\mathcal{Y}_{\pi_1}(W_{l-1}^{s+2}(\Pi)) \cap \mathcal{Y}_{\pi_2}(W_{l-1}^{s+2}(\Pi)) = \emptyset,$$

we deduce that

$$\mathcal{Y}_{\hat{\pi}_1^{s+1, s+2}}(C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi)) \cap \mathcal{Y}_{\hat{\pi}_2^{s+1, s+2}}(C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi)) = \emptyset,$$

which together with (3.2.25) implies (3.2.15).

□

### Fusion

In this part, we define and explain the merging process of a bulk tree with a forest.

**Proposition 4.** *For  $s \geq 2$  and  $l \geq 0$ , we consider the partition  $(\tilde{\pi}_1, \tilde{\pi}_2)$  in  $\tilde{\mathcal{P}}_{2;s}$  such that  $|\tilde{\pi}_i| = s_i$  and  $s_1 + s_2 = s$ . Given a partition  $\Pi$  of the set  $\tilde{\pi}_2 \cup \{s+2\}$ , we define the  $a$ -merging operator  $M_{y_{s+1}, y_{s+2}}^a$  acting on the forest  $W_{l_2}^{s_2+1}(\Pi)$  and the bulk tree  $\hat{T}_{l_1}^{s_1+1}(Y_{\tilde{\pi}_1}, y_{s+1}; \vec{z})$  at the external vertices  $y_{s+1}$  and  $y_{s+2}$  following the steps below:*

- i) Let  $J_{s+1} = (z, y_{s+1})$  and  $J_{s+2} = (z', y_{s+2})$  be the external legs which attach respectively  $y_{s+1}$  to the internal vertex  $z \in \hat{T}_{l_1}^{s_1+1}$  and  $y_{s+2}$  to the internal vertex  $z' \in W_{l_2}^{s_2+1}(\Pi)$ . In the first step of the merging process  $J_{s+1}$  and  $J_{s+2}$  are removed.*
- ii) A new internal line  $(z, z')$  is added.*

Similarly, we define the  $b$ -merging operator  $M_{y_{s+1}, y_{s+2}}^b$  acting on  $W_{l_2}^{s_2+1}(\Pi)$  and  $\hat{T}_{l_1}^{s_1+1}(Y_{\tilde{\pi}_1}, y_{s+1}, \vec{z})$  following the same steps above except for adding an internal vertex of incidence number 2, which replaces the internal line  $(z, z')$  in step ii) by the two internal lines  $(z, u)$  and  $(u, z')$ . Then we claim

$$M_{y_{s+1}, y_{s+2}}^j \left( \hat{T}_{l_1}^{s_1+1}(Y_{\tilde{\pi}_1}, y_{s+1}; \vec{z}), W_{l_2}^{s_2+1}(\Pi) \right) \in \mathcal{W}_l^s(\Pi'), \quad j \in \{a, b\} \quad (3.2.29)$$

where  $\Pi' := \tilde{\pi}_1 \cup \Pi^{s+2}$  and  $l := l_1 + l_2$ .

*Proof.* Let  $\pi_i$  be the sub-partition of  $\Pi$  such that  $s+2 \in \pi_i$ . The merging operators (a) and (b) act only on the tree  $T_{l_2}^{s_{\pi_i}, 0}$  since all trees corresponding to the remaining sub-partitions do not have external vertices on which the merging operators act. Therefore, without loss of generality, we simplify the discussion by considering the case of a partition  $\Pi$  of length one.

The first and second step of the two merging processes create a tree with  $s+1$  external vertices given by the set

$$\{Y_{\tilde{\pi}_1}\} \cup \{Y_{\tilde{\pi}_2}, 0\}.$$

The only difference between the two cases is related to the set of internal vertices, which in case of (a) is given by the union of the internal vertices of the bulk tree  $\hat{T}_{l_1}^{s_1+1}(Y_{\tilde{\pi}_1}, y_{s+1}, \vec{z})$  and the forest  $W_{l_2}^{s_2+1}(\Pi)$ . For (b), a new vertex of incidence number 2 is added, which implies

$$v_{2,a} = v_{2,1} + v_{2,2}, \quad v_{2,b} = v_{2,1} + v_{2,2} + 1, \quad (3.2.30)$$

where  $v_{2,i}$  denotes the number of vertices of incidence number 2 of the surface tree obtained through the merging process  $j \in \{a, b\}$ . Therefore, we obtain

$$v_{2,j} \leq 3(l_1 + l_2) - 4 + \frac{s_1 + s_2 + 3}{2} + 1 = 3l - 2 + \frac{s+1}{2}, \quad j \in \{a, b\}. \quad (3.2.31)$$

---

<sup>4</sup>we used the notation (3.2.1).

This concludes that the surface trees obtained through the merging processes (a) and (b) are indeed in  $\mathcal{W}_l^s(\sigma_s)$ .  $\square$

### 3.2.3 Weight factors

#### The bulk weight factors

Let  $0 < \delta < 1$ . Given a set  $\tau_{2,s} := \{\tau_2, \dots, \tau_s\}$  with  $\tau := \inf_{2 \leq i \leq s} \tau_i$ , a set of external vertices  $y_{2,s} = \{y_2, \dots, y_s\} \in \mathbb{R}^{s-1}$  and a set of internal vertices  $\vec{z} = (z_2, \dots, z_{r+1}) \in \mathbb{R}^r$ , and attributing positive parameters  $\Lambda_{\mathcal{J}} = \{\Lambda_I | I \in \mathcal{J}\}$  to the internal lines, the weight factor  $\mathcal{F}_\delta(\Lambda_{\mathcal{J}}, \tau_{2,s}; T_l^s(z_1, y_{2,s}, \vec{z}))$  of a tree  $T_l^s(z_1, y_{2,s}, \vec{z})$  at scales  $\Lambda_I$  is defined as a product of heat kernels associated with the internal and external lines of the tree. We set

$$\mathcal{F}_\delta(\Lambda_{\mathcal{J}}, \tau_{2,s}; T_l^s(z_1, y_{2,s}, \vec{z})) := \prod_{I \in \mathcal{J}} p_B\left(\frac{1+\delta}{\Lambda_I^2}; I\right) \prod_{J \in \mathcal{J}} p_B(\tau_{J,\delta}; J), \quad (3.2.32)$$

where  $\tau_{J,\delta}$  denotes the entry  $\tau_{i,\delta}$  in  $\tau$  carrying the index of the external coordinate  $y_i$  in which the external line  $J$  ends, and  $\tau_{i,\delta} := (1+\delta)\tau_i$ . For  $I = \{a, b\}$  the notation  $p_B(\frac{1+\delta}{\Lambda_I^2}; I)$  stands for  $p_B(\frac{1+\delta}{\Lambda_I^2}; a, b)$ . We also define the integrated weight factors

$$\mathcal{F}_\delta(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}) := \sup_{\Lambda \leq \Lambda_I \leq \Lambda_0} \int_{(\mathbb{R}^+)^r} d\vec{z} \mathcal{F}_\delta(\Lambda_{\mathcal{J}}, \tau_{2,s}; T_l^s(z_1, y_{2,s}, \vec{z})), \quad (3.2.33)$$

$$\mathcal{F}_\delta^-(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}) := \sup_{\Lambda \leq \Lambda_I \leq \Lambda_0} \int_{(\mathbb{R}^-)^r} d\vec{z} \mathcal{F}_\delta(\Lambda_{\mathcal{J}}, \tau_{2,s}; T_l^s(z_1, y_{2,s}, \vec{z})) \quad (3.2.34)$$

and

$$\mathcal{F}_\delta^\infty(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}) := \sup_{\Lambda \leq \Lambda_I \leq \Lambda_0} \int_{\mathbb{R}^r} d\vec{z} \mathcal{F}_\delta(\Lambda_{\mathcal{J}}, \tau_{2,s}; T_l^s(z_1, y_{2,s}, \vec{z})). \quad (3.2.35)$$

We denote by  $\mathcal{J}_e(\pi_i)$  the set of the indices of the internal vertices of the tree  $T_l^s$  which are attached to the external vertices  $\{y_j\}_{j \in \pi_i}$  and let  $q_i$  be its cardinality. We write shortly  $\vec{z}_{\mathcal{J}_e(\pi_i)} := (z_j)_{j \in \mathcal{J}_e(\pi_i)}$  and  $\vec{z}_i^c := (z_j)_{j \in \mathcal{J}_e^c(\pi_i)}$  with  $\mathcal{J}_e^c(\pi_i)$  denoting the complementary set of  $\mathcal{J}_e(\pi_i)$ . We define the integrated weight factor of a tree  $T_l^s \in \mathcal{T}_{l;\pi_1,\pi_2}^s$  as follows

$$\begin{aligned} & \mathcal{F}_\delta^\infty(\Lambda, \tau_{\pi_1}^+, \tau_{\pi_2}; T_l^s; z_1, y_{2,s}) \\ & := \sup_{\Lambda \leq \Lambda_I \leq \Lambda_0} \int_{(\mathbb{R}^+)^{q_1}} d\vec{z}_{\mathcal{J}_e(\pi_1)} \int_{\mathbb{R}^{r-q_1}} d\vec{z}_1^c \mathcal{F}_\delta(\Lambda_{\mathcal{J}}, \tau_{2,s}; T_l^s(z_1, y_{2,s}, \vec{z})). \end{aligned} \quad (3.2.36)$$

For the trivial partition  $\pi_1 = \sigma_{2,s}$ , we denote by  $\mathcal{J}_e$  the set of the indices of the internal vertices attached to the external vertices  $y_{2,s}$  and by  $q$  its cardinality. We write shortly  $\vec{z}_{\mathcal{J}_e} := (z_i)_{i \in \mathcal{J}_e}$  and  $\vec{z}^c := (z_i)_{i \in \mathcal{J}_e^c}$  and define the integrated weight factors

$$\mathcal{F}_{\delta,+}^\infty(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}) := \sup_{\Lambda \leq \Lambda_I \leq \Lambda_0} \int_{(\mathbb{R}^+)^q} d\vec{z}_{\mathcal{J}_e} \int_{\mathbb{R}^{r-q}} d\vec{z}^c \mathcal{F}_\delta(\Lambda_{\mathcal{J}}, \tau_{2,s}; T_l^s(z_1, y_{2,s}, \vec{z})) \quad (3.2.37)$$



and

$$\mathcal{F}_{\delta;-}^{\infty}(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}) := \sup_{\Lambda \leq \Lambda_l \leq \Lambda_0} \int_{(\mathbb{R}^-)^q} d\vec{z}_{\mathcal{J}_e} \int_{\mathbb{R}^{r-q}} d\vec{z}^c \mathcal{F}_{\delta}(\Lambda_{\mathcal{J}}, \tau_{2,s}; T_l^s(z_1, y_{2,s}, \vec{z})). \quad (3.2.38)$$

(3.2.33)-(3.2.38) depend on  $\Lambda_0$ , but note that their limits for  $\Lambda_0 \rightarrow \infty$  exist, and that typically the sup is expected to be taken for the minimal values of  $\Lambda$  admitted. Therefore we suppress the dependence on  $\Lambda_0$  in the notation. The definitions (3.2.32)-(3.2.34) can be generalized to a bulk tree  $\hat{T}_l^s$ . Finally we introduce the global weight factors which are defined through

$$\mathcal{F}_{s,l;\delta}(\Lambda, \tau_{2,s}, z_1, y_{2,s}) := \sum_{T_l^s \in \mathcal{T}_l^s} \mathcal{F}_{\delta}(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}), \quad (3.2.39)$$

$$\mathcal{F}_{s,l;\delta}^{-}(\Lambda, \tau_{2,s}, z_1, y_{2,s}) := \sum_{T_l^s \in \mathcal{T}_l^s} \mathcal{F}_{\delta}^{-}(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}) \quad (3.2.40)$$

and

$$\mathcal{F}_{s,l;\delta}^{\infty}(\Lambda, \tau_{2,s}, z_1, y_{2,s}) := \sum_{T_l^s \in \mathcal{T}_l^s} \mathcal{F}_{\delta}^{\infty}(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}). \quad (3.2.41)$$

Similarly, we define the global bulk weight factor

$$\hat{\mathcal{F}}_{s,l;\delta}(\Lambda, \tau_{2,s}, y_{2,s}) := \sum_{\hat{T}_l^s \in \hat{\mathcal{T}}_l^s} \mathcal{F}_{\delta}(\Lambda, \tau_{2,s}; \hat{T}_l^s; y_{2,s}). \quad (3.2.42)$$

For  $z_1 \in \mathbb{R}^+$ , we define the global weight factors associated to (3.2.36) and (3.2.37)

$$\mathcal{F}_{s,l;\delta}^{\infty}(\Lambda, \tau_{\pi_1}^+, \tau_{\pi_2}, z_1, y_{2,s}) := \sum_{T_l^s \in \mathcal{T}_{l;\pi_1, \pi_2}^s} \mathcal{F}_{\delta}^{\infty}(\Lambda, \tau_{\pi_1}^+, \tau_{\pi_2}; T_l^s; z_1, y_{2,s}), \quad (3.2.43)$$

$$\mathcal{F}_{s,l;+\delta}^{\infty}(\Lambda, \tau_{2,s}, z_1, y_{2,s}) := \sum_{T_l^s \in \mathcal{T}_l^s} \mathcal{F}_{\delta;+}^{\infty}(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}) \quad (3.2.44)$$

and for  $z_1 \in \mathbb{R}^-$  we define

$$\mathcal{F}_{s,l;-;\delta}^{\infty}(\Lambda, \tau_{2,s}, z_1, y_{2,s}) := \sum_{T_l^s \in \mathcal{T}_l^s} \mathcal{F}_{\delta;-}^{\infty}(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}). \quad (3.2.45)$$

If this does not lead to ambiguity we write shortly

$$\mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s}) \equiv \mathcal{F}_{s,l;\delta}(\Lambda, \tau_{2,s}, z_1, y_{2,s}), \quad \mathcal{F}_{s,l;\delta}^{\Lambda;\infty}(\tau_{2,s}) \equiv \mathcal{F}_{s,l;\delta}^{\infty}(\Lambda, \tau_{2,s}, z_1, y_{2,s}), \quad (3.2.46)$$

$$\mathcal{F}_{s,l;\pm;\delta}^{\Lambda;\infty}(\tau_{2,s}) \equiv \mathcal{F}_{s,l;\pm;\delta}^{\infty}(\Lambda, \tau_{2,s}, z_1, y_{2,s}), \quad \mathcal{F}_{s,l;\delta}^{\Lambda;\infty}(\tau_{\pi_1}^+, \tau_{\pi_2}) \equiv \mathcal{F}_{s,l;\delta}^{\infty}(\Lambda, \tau_{\pi_1}^+, \tau_{\pi_2}; z_1, y_{2,s}). \quad (3.2.47)$$

In complete analogy we define the weight factors and global weight factors for twice rooted trees which we denote as  $\mathcal{F}(\Lambda, \tau_{3,s}; T_l^{s,(12)}; z_1, z_2, y_{3,s})$  resp.  $\mathcal{F}_{s,l}^{(12)}(\Lambda, \tau_{3,s}, z_1, z_2, y_{3,s})$  or  $\mathcal{F}_{s,l}^{12}(\Lambda, \tau_{3,s})$ .

For  $s = 1$  we set  $\mathcal{F}_{1,l;\delta}^{\Lambda} \equiv 1$ .

### The surface weight factors

- In the sequel, we will use the following notations:

$$\tau_{\pi_i} := \{\tau_k \mid k \in \pi_i\}, \quad Y_{\pi_i} := (y_k)_{k \in \pi_i}, \quad \tau_{\pi_i, \delta} := \{(1 + \delta)\tau_k \mid k \in \pi_i\}, \quad \tau := \inf_{1 \leq i \leq s} \tau_i. \quad (3.2.48)$$

- Let  $0 < \delta < 1$  and  $\tau_{1,s} := \{\tau_1, \dots, \tau_s\}$  such that  $\tau > 0$  and let  $Y_{\sigma_s} \in \mathbb{R}^s$  be the set of the external vertices. Given a partition  $\Pi \in \mathcal{P}_s$ , let  $\vec{z}_\Pi = (\vec{z}_{\pi_1}, \dots, \vec{z}_{\pi_{l_\Pi}}) \in (\mathbb{R}^+)^p$ , where each vector  $\vec{z}_{\pi_i}$  consists of the internal vertices of the tree  $T_l^{s\pi_i, 0}$  in the forest  $W_l^s(\Pi)$ . We denote by  $\mathcal{I} = \cup_{i=1}^{l_\Pi} \mathcal{I}_k$  the set of the internal lines of the trees of  $W_l^s(\Pi)$  and by  $\mathcal{J} = \cup_{k=1}^{l_\Pi} \mathcal{J}_k$  the set of the external lines which link an internal vertex to an external vertex belonging to the set  $Y_{\sigma_s}$ . Each set  $\mathcal{I}_k$  (resp.  $\mathcal{J}_k$ ) denotes the internal lines (resp. the external lines) of the tree  $T_l^{s\pi_k, 0}$ . We also use the notation  $\mathcal{J}_k^0 = \{J_k^0 \mid 1 \leq k \leq l_\Pi\}$  to denote the set of surface external lines which link an internal vertex to 0.
- Attributing positive parameters  $\Lambda_{\mathcal{I}} = \{\Lambda_I \mid I \in \mathcal{I}\}$  to the internal lines and  $\tilde{\Lambda} = \{\tilde{\Lambda}_k \mid k \in \mathcal{J}_k^0\}$  to the surface external lines, the weight factor  $\mathcal{F}_\delta^0(\Lambda_{\mathcal{I}}, \tilde{\Lambda}; \tau_{1,s}; W_l^s(\Pi); \vec{z}_\Pi; Y_{\sigma_s})$  of the forest  $W_l^s(\Pi)$  at scales  $\Lambda_I$  and  $\tilde{\Lambda}_k$  is defined as the product of heat kernels associated to the internal and external lines of each tree of the forest. For a sub-partition  $\pi_k \in \Pi$ , we define the weight factor of the tree  $T_l^{s\pi_k, 0}$  as follows:

$$\begin{aligned} \mathcal{F}_\delta^0(\Lambda_{\mathcal{I}_k}, \tilde{\Lambda}_k; \tau_{\pi_k}; T_l^{s\pi_k, 0}; \vec{z}_{\pi_k}; Y_{\pi_k}) \\ := \prod_{I \in \mathcal{I}_k} p_B\left(\frac{1 + \delta}{\Lambda_I^2}; I\right) \prod_{J \in \mathcal{J}_k} p_B((1 + \delta)\tau_J; J) p_B\left(\frac{1 + \delta}{\tilde{\Lambda}_k^2}; J_0^k\right), \end{aligned} \quad (3.2.49)$$

where we used the same notations as in [\(3.2.32\)](#) and  $J_0^k$  denotes the line which links an internal vertex to the external vertex 0 with an attributed positive parameter  $\tilde{\Lambda}_k$ . For a surface tree  $T_l^{1,0}$  with no internal vertices, the surface weight factor reads

$$\mathcal{F}_\delta^0(\tau_{\pi_k}; T_l^{1,0}; y_{\pi_k}) := p_B((1 + \delta)\tau_{\pi_k}; y_{\pi_k}, 0). \quad (3.2.50)$$

The weight factor of the forest  $W_l^s(\Pi)$  is defined for  $\Pi \in \mathcal{P}_s^{1,c}$  as follows:

$$\mathcal{F}_\delta^0(\Lambda_{\mathcal{I}}, \tilde{\Lambda}; \tau_{1,s}; W_l^s(\Pi); \vec{z}_\Pi; Y_{\sigma_s}) := \prod_{\pi_k \in \Pi} \mathcal{F}_\delta^0(\Lambda_{\mathcal{I}_k}, \tilde{\Lambda}_k; \tau_{\pi_k}; T_l^{s\pi_k, 0}; \vec{z}_{\pi_k}; Y_{\pi_k}). \quad (3.2.51)$$

For  $\Pi \in \mathcal{P}_s^1$ , it is given by

$$\begin{aligned} \mathcal{F}_\delta^0(\Lambda_{\mathcal{I}}, \tilde{\Lambda}; \tau_{1,s}; W_l^s(\Pi); \vec{z}_\Pi; Y_{\sigma_s}) &:= \prod_{\pi_k} \mathcal{F}_\delta^0(\Lambda_{\mathcal{I}_k}, \tilde{\Lambda}_k; \tau_{\pi_k}; T_l^{s\pi_k, 0}; \vec{z}_{\pi_k}; Y_{\pi_k}) \\ &\times \prod_{\tilde{\pi}_k} \mathcal{F}_\delta^0(\Lambda_{\mathcal{I}_k}, \tilde{\Lambda}_k; 2\tau_{\tilde{\pi}_k}; T_l^{s\tilde{\pi}_k, 0}; \vec{z}_{\tilde{\pi}_k}; y_{\tilde{\pi}_k}), \end{aligned} \quad (3.2.52)$$

where the product  $\prod_{\tilde{\pi}_k}$  runs over all sub-partitions in  $\Pi$  of length equal to 1.

- We also define the integrated surface weight factor

$$\mathcal{F}_\delta^0(\Lambda, \tau_{1,s}; W_l^s(\Pi); Y_{\sigma_s}) := \sup_{\Lambda \leq \Lambda_I, \tilde{\Lambda}_k \leq \Lambda_0} \int_{\vec{z}_\Pi} \mathcal{F}_\delta^0(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_{1,s}; W_l^s(\Pi); \vec{z}_\Pi; Y_{\sigma_s}), \quad (3.2.53)$$

where  $\int_{\vec{z}_\Pi} := \prod_{i=1}^p \int_0^\infty dz_i$ .

- Given a forest  $W_l^s(\Pi)$  in  $\mathcal{W}_{l;\pi_1,\pi_2}^s(\Pi)$ , we define the corresponding integrated surface weight factor as follows

$$\begin{aligned} & \mathcal{F}_\delta^0(\Lambda, \tau_{\pi_1}^+, \tau_{\pi_2}^-; W_l^s(\Pi); Y_{\sigma_s}) \\ & := \sup_{\Lambda \leq \Lambda_I, \tilde{\Lambda} \leq \Lambda_0} \int_{(\mathbb{R}^+)^{p_1}} d\vec{z}_{\pi_1} \int_{(\mathbb{R}^-)^{p_2}} d\vec{z}_{\pi_2} \mathcal{F}_\delta^0(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_{1,s}; W_l^s(\Pi); \vec{z}_\Pi; Y_{\sigma_s}), \end{aligned} \quad (3.2.54)$$

where  $\int_{\vec{z}_{\pi_i}}$  denotes the integration over the internal vertices of all surface trees in  $\mathcal{Y}_{\pi_i}(W_l^s(\Pi))$ .

- The weight factor associated to a global forest  $W_l^s$  is defined as

$$\mathcal{F}_\delta^0(\Lambda, \tau_{1,s}; W_l^s; Y_{\sigma_s}) := \sum_{\Pi \in \mathcal{P}_s} \mathcal{F}_\delta^0(\Lambda, \tau_{1,s}; W_l^s(\Pi); Y_{\sigma_s}). \quad (3.2.55)$$

Similarly, the weight factor of a global forest  $W_l^s \in \mathcal{W}_{l;\pi_1,\pi_2}^s$  is given by

$$\mathcal{F}_\delta^0(\Lambda, \tau_{\pi_1}^+, \tau_{\pi_2}^-; W_l^s; Y_{\sigma_s}) := \sum_{\Pi \in \mathcal{P}_s} \mathcal{F}_\delta^0(\Lambda, \tau_{\pi_1}^+, \tau_{\pi_2}^-; W_l^s(\Pi); Y_{\sigma_s}). \quad (3.2.56)$$

- We define the global surface weight factors

$$\mathcal{F}_{s,l;\delta}^0(\Lambda, \tau_{1,s}; Y_{\sigma_s}) := \sum_{W_l^s \in \mathcal{W}_l^s} \mathcal{F}_\delta^0(\Lambda, \tau_{1,s}; W_l^s; Y_{\sigma_s}) \quad (3.2.57)$$

and

$$\mathcal{F}_{s,l;\delta}^0(\Lambda, \tau_{\pi_1}^+, \tau_{\pi_2}^-; Y_{\sigma_s}) := \sum_{W_l^s \in \mathcal{W}_{l;\pi_1,\pi_2}^s} \mathcal{F}_\delta^0(\Lambda, \tau_{\pi_1}^+, \tau_{\pi_2}^-; W_l^s; Y_{\sigma_s}). \quad (3.2.58)$$

If it does not lead to ambiguity we write shortly

$$\mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}) \equiv \mathcal{F}_{s,l;\delta}^0(\Lambda, \tau_{1,s}; Y_{\sigma_s}). \quad (3.2.59)$$

and

$$\mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{\pi_1}^+, \tau_{\pi_2}^-) \equiv \mathcal{F}_{s,l;\delta}^0(\Lambda, \tau_{\pi_1}^+, \tau_{\pi_2}^-; Y_{\sigma_s}). \quad (3.2.60)$$

For  $s = 0$  we set  $\mathcal{F}_{0,l;\delta}^{\Lambda,0} \equiv 1$ .

**Remark 1.** • The definitions (3.2.33) and (3.2.53) imply for  $0 \leq \Lambda' \leq \Lambda$

$$\mathfrak{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}) \leq \mathfrak{F}_{s,l;\delta}^{\Lambda',0}(\tau_{1,s}), \quad \mathfrak{F}_{s,l;\delta}^{\Lambda,0}(\tau_{\pi_1}^+, \tau_{\pi_2}^-) \leq \mathfrak{F}_{s,l;\delta}^{\Lambda',0}(\tau_{\pi_1}^+, \tau_{\pi_2}^-). \quad (3.2.61)$$

The same bounds hold for (3.2.46)-(3.2.47).

- Combining the bound (3.1.10) together with the definitions (3.2.32) and (3.2.49), the following bounds hold for all  $0 < \delta < \delta'$  and  $0 \leq \Lambda \leq \Lambda_0$

$$\mathfrak{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}) \leq C'_{s,l} \mathfrak{F}_{s,l;\delta'}^{\Lambda,0}(\tau_{1,s}), \quad \mathfrak{F}_{s,l;\delta}^{\Lambda}(\tau_{1,s}) \leq C_{s,l} \mathfrak{F}_{s,l;\delta'}^{\Lambda}(\tau_{1,s}). \quad (3.2.62)$$

The constants  $C_{s,l}^0$  and  $C_{s,l}$  are explicitly given by

$$C'_{s,l} := \sup_{(\mathcal{J}, \mathcal{F}) \in W_l^s, W_l^s \in W_l^s} C_{\delta, \delta'}^{|\mathcal{J}|+|\mathcal{F}|}, \quad C_{s,l} := \sup_{(\mathcal{J}, \mathcal{F}) \in T_l^s, T_l^s \in \mathcal{T}_l^s} C_{\delta, \delta'}^{|\mathcal{J}|+|\mathcal{F}|},$$

where  $\mathcal{J}$  and  $\mathcal{F}$  are respectively the set of internal and external lines of the tree/forest and  $|\cdot|$  denotes their cardinality. The constant  $C_{\delta, \delta'}$  is given by (3.1.11) for  $r = 0$ .

### 3.2.4 Useful bounds

For the proof of Theorem 9 in chapter 5 and Proposition 9 and Theorem 12 in chapter 6, we need to bound the tree/forest weight factors for reduced forests and for merged trees and forests.

**Lemma 2.** (Reduction) Let  $\tau, \delta > 0, 0 \leq \Lambda \leq \Lambda_0, l \geq 1$  and  $Y_{\sigma_s} \in \mathbb{R}^s$ , we have

$$\int_{\mathbb{R}} du \mathfrak{F}_{s+2, l-1; \delta}^0 \left( \Lambda; \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, u \right) \leq O(1) \Lambda \mathfrak{F}_{s,l;\delta}^0(\Lambda; \tau_{1,s}; Y_{\sigma_s}). \quad (3.2.63)$$

Given a partition  $(\pi_1, \pi_2)$  of  $\sigma_{s+2}$ , we have

$$\int_{\mathbb{R}} du \mathfrak{F}_{s+2, l-1; \delta}^0(\Lambda; \tau_{\pi_1}^+, \tau_{\pi_2}^-; Y_{\sigma_s}, u, u) \leq O(1) \Lambda \mathfrak{F}_{s,l;\delta}^0(\Lambda; \tau_{\pi_1}^+, \tau_{\pi_2}^-; Y_{\sigma_s}), \quad (3.2.64)$$

where  $\tau_{s+1} = \tau_{s+2} = \frac{1}{2\Lambda^2}$  and  $\tilde{\pi}_i := \hat{\pi}_i^{s+1, s+2}$ . The constants  $O(1)$  depend only on  $s$  and  $l$ .

*Proof.* First, we prove (3.2.63). Let us recall the definition of the surface weight factor

$$\begin{aligned} \mathfrak{F}_{s+2, l-1; \delta}^0 \left( \Lambda; \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s+2}} \right) \\ = \sum_{W_{l-1}^{s+2} \in \mathcal{W}_{l-1}^{s+2}} \sum_{\Pi \in \mathcal{P}_{s+2}} \mathfrak{F}_{\delta}^0 \left( \Lambda; \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; W_{l-1}^{s+2}(\Pi); Y_{\sigma_{s+2}} \right), \end{aligned} \quad (3.2.65)$$

<sup>5</sup>we used the notation (3.2.28).

where  $y_{s+1} := u$  and  $y_{s+2} = u$ . The weight factor

$$\mathcal{F}_\delta^0 \left( \Lambda; \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; W_{l-1}^{s+2}(\Pi); Y_{\sigma_{s+2}} \right)$$

is given by (3.2.49)-(3.2.52). Let  $(z_i, u)$  and  $(z_j, u)$  be the external lines attaching respectively the internal vertices  $z_i$  and  $z_j$  to the external vertices  $y_{s+1}$  and  $y_{s+2}$ . Using (3.1.5), we obtain

$$\int_{\mathbb{R}} du p_B \left( \frac{\alpha_1(1+\delta)}{2\Lambda^2}; z_i, u \right) p_B \left( \frac{\alpha_2(1+\delta)}{2\Lambda^2}; z_j, u \right) = p_B \left( \frac{(\alpha_1 + \alpha_2)(1+\delta)}{2\Lambda^2}; z_i, z_j \right) \leq \Lambda. \quad (3.2.66)$$

We recall that a tree of two external vertices (including the surface external vertex 0) corresponds to a sub-partition of length 1 and the surface weight factor associated to these trees differs from a surface tree of three or more external vertices by a factor 2 multiplying the parameter  $\tau_i$  of the corresponding external vertex, as it appears in (3.2.52). Therefore, the constants  $\alpha_1$  and  $\alpha_2$  take either the value 2 or 1 depending on whether the two external vertices at  $u$  belong to a surface tree of only two external vertices or more. Diagrammatically, the effect of the bound (3.2.66) on the forest  $W_{l-1}^{s+2}(\Pi)$  is the removal of the external legs  $(z_i, u)$  and  $(z_j, u)$  by bounding their contribution in the surface weight factor by  $\Lambda$ . Furthermore, the property

$$\int_0^\infty dz p_B \left( \frac{1+\delta}{\Lambda_i^2}; z, z' \right) \leq 1 \quad (3.2.67)$$

implies that all internal vertices which after removing  $(z_i, u)$  and  $(z_j, u)$ , become of incidence number one are removed. These two steps correspond to reducing the forest  $W_{l-1}^{s+2}(\Pi)$  at the external vertices  $(u, u)$ . Therefore, we have

$$\int_{\mathbb{R}} du \mathcal{F}_\delta^0 \left( \Lambda; \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; W_{l-1}^{s+2}(\Pi); Y_{\sigma_s}, u, u \right) \leq \Lambda \mathcal{F}_\delta^0 \left( \Lambda; \tau_{1,s}; C_{u,u} W_{l-1}^{s+2}(\Pi); Y_{\sigma_s} \right). \quad (3.2.68)$$

Proposition 1 gives that  $C_{u,u} W_{l-1}^{s+2}(\Pi) \in \mathcal{W}_l^s(\Pi^{s+1, s+2})$ , where  $\Pi^{s+1, s+2} \in \mathcal{P}_s$  is the reduced partition obtained from  $\Pi$ . Hence, we obtain

$$\begin{aligned} \sum_{W_{l-1}^{s+2} \in \mathcal{W}_{l-1}^{s+2}} \sum_{\Pi \in \mathcal{P}_{s+2}} \mathcal{F}_\delta^0 \left( \Lambda, \tau_{1,s}; C_{u,u} W_{l-1}^{s+2}(\Pi); Y_{\sigma_s} \right) \\ \leq O(1) \sum_{W_l^s \in \mathcal{W}_l^s} \sum_{\Pi \in \mathcal{P}_s} \mathcal{F}_\delta^0 \left( \Lambda, \tau_{1,s}; W_l^s(\Pi); Y_{\sigma_s} \right). \end{aligned} \quad (3.2.69)$$

The constant  $O(1)$  takes into account that the reduction operator is not a one-to-one map, in the sense that the same forest can be obtained by reducing different forests, which implies that some weight factors  $\mathcal{F}_\delta^0(\Lambda, \tau_{1,s}; W_l^s(\Pi); Y_{\sigma_s})$  are possibly summed more than once in (3.2.68). Combining (3.2.68) and (3.2.69) gives the final bound (3.2.63).

If the external vertex  $y_{s+1}$  belongs to the sub-surface tree  $T_l^{1,0}(y_{s+1}, 0)$  with no internal vertex, (3.2.66) reads in this case

$$\int_{\mathbb{R}} du p_B \left( \frac{1+\delta}{2\Lambda^2}; 0, u \right) p_B \left( \frac{1+\delta}{2\Lambda^2}; z_j, u \right) = p_B \left( \frac{1+\delta}{\Lambda^2}; 0, z_j \right) \leq \Lambda. \quad (3.2.70)$$

If  $y_{s+2}$  is an external vertex of  $T_l^{1,0}(y_{s+2}, 0)$ , then  $z_j$  is replaced by 0 in (3.2.70). This corresponds to removing the corresponding sub-surface trees from the forest  $W_{l-1}^{s+2}(\Pi)$ . The bound (3.2.69) is obtained in these two cases following the same line of reasoning as before.

The proof of (3.2.64) is obtained by following the same steps of the proof of (3.2.63) and using (3.2.15).  $\square$

**Lemma 3.** (*Forest-Forest Fusion*) Let  $\delta, \delta' > 0$  and  $1 \leq l_1, l_2 \leq l-1$  such that  $l_1 + l_2 = l$ . Given  $(\tilde{\pi}_1, \tilde{\pi}_2) \in \tilde{\mathcal{P}}_{2,s}$ , we have

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta}^0 \left( \Lambda; \tau_{\tilde{\pi}_1}, \frac{1}{2\Lambda^2}; Y_{\tilde{\pi}_1}, u \right) \mathcal{F}_{s_2+1, l_2; \delta'}^0 \left( \Lambda; \tau_{\tilde{\pi}_2}, \frac{1}{2\Lambda^2}; Y_{\tilde{\pi}_2}, u \right) \\ \leq O(1) \Lambda \mathcal{F}_{s, l; \delta''}^0 \left( \Lambda; \tau_{1,s}; Y_{\sigma_s} \right), \end{aligned} \quad (3.2.71)$$

where  $s_i := |\pi_i|$  and  $\delta'' = \max(\delta, \delta')$ .

Given  $(\pi_1, \pi_2)$  in  $\tilde{\mathcal{P}}_{2,s}$ , we define  $\pi_{ij} = \pi_i \cap \tilde{\pi}_j$ . For

$$(\pi'_{11}, \pi'_{21}) \in \{(\pi_{11} \cup \{s+1\}, \pi_{21}), (\pi_{11}, \pi_{21} \cup \{s+1\})\} \quad (3.2.72)$$

and

$$(\pi'_{12}, \pi'_{22}) \in \{(\pi_{12}, \pi_{22} \cup \{s+2\}), (\pi_{12} \cup \{s+2\}, \pi_{22})\}, \quad (3.2.73)$$

we have

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta}^0 \left( \Lambda; \tau_{\pi'_{11}}^+, \tau_{\pi'_{21}}^-; Y_{\tilde{\pi}_1}, u \right) \mathcal{F}_{s_2+1, l_2; \delta'}^0 \left( \Lambda; \tau_{\pi'_{12}}^+, \tau_{\pi'_{22}}^-; Y_{\tilde{\pi}_2}, u \right) \\ \leq O(1) \Lambda \mathcal{F}_{s, l; \delta''}^0 \left( \Lambda; \tau_{\pi_1}^+, \tau_{\pi_2}^-; Y_{\sigma_s} \right), \end{aligned} \quad (3.2.74)$$

where  $\tau_{s+1} = \tau_{s+2} = \frac{1}{2\Lambda^2}$ . Again the constants  $O(1)$  depend only on  $s$  and  $l$ .

*Proof.* First, we prove (3.2.71). Without loss of generality, we consider the ordered sub-partitions  $\tilde{\pi}_1 := \sigma_{s_1}$  and  $\tilde{\pi}_2 := \sigma_{s_1+1, s}$ . To establish (3.2.71), it is sufficient to bound

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta}^0 \left( \Lambda; \tau_{1, s_1}, \frac{1}{2\Lambda^2}; W_{l_1}^{s_1+1}(\Pi_1); Y_{\sigma_{s_1}}, u \right) \\ \times \mathcal{F}_{s_2+1, l_2; \delta'}^0 \left( \Lambda; \tau_{s_1+1, s}, \frac{1}{2\Lambda^2}; W_{l_2}^{s_2+1}(\Pi_2); Y_{\sigma_{s_1+1: s}}, u \right) \end{aligned} \quad (3.2.75)$$

where  $\Pi_1 \in \tilde{\mathcal{P}}_{s_1+1}$  and  $\Pi_2 \in \tilde{\mathcal{P}}_{s_2+1}$ . The sets  $\tilde{\mathcal{P}}_{s_1+1}$  and  $\tilde{\mathcal{P}}_{s_2+1}$  denote respectively the set of partitions of  $\sigma_{s_1} \cup \{s+1\}$  and  $\sigma_{s_1+1:s} \cup \{s+2\}$ . Using (3.2.62), we bound (3.2.75) by

$$\int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta''}^0 \left( \Lambda; \tau_{1, s_1}, \frac{1}{2\Lambda^2}; W_{l_1}^{s_1+1}(\Pi_1); Y_{\sigma_{s_1}}, u \right) \\ \times \mathcal{F}_{s_2+1, l_2; \delta''}^0 \left( \Lambda; \tau_{s_1+1, s}, \frac{1}{2\Lambda^2}; W_{l_2}^{s_2+1}(\Pi_2); Y_{\sigma_{s_1+1:s}}, u \right), \quad (3.2.76)$$

where  $\delta'' = \max(\delta, \delta')$ .

Let  $\pi_{i,1}$  and  $\pi_{j,2}$  be respectively the sub-partitions in  $\Pi_1$  and  $\Pi_2$  such that  $\{s+1\} \in \pi_{i,1}$  and  $\{s+2\} \in \pi_{j,2}$ . We denote by  $z_i$  and  $z_j$  the internal vertices in the sub-surface trees  $T_{l_1}^{s\pi_{i,1}, 0}(Y_{\pi_{i,1}}, u, 0)$  and  $T_{l_2}^{s\pi_{j,2}, 0}(Y_{\pi_{j,2}}, u, 0)$  in the forests  $W_{l_1}^{s_1+1}(\Pi_1)$  and  $W_{l_2}^{s_2+1}(\Pi_2)$ , which are attached to  $u$ . As we mentioned previously, the bound (3.2.66) amputates the external legs  $(z_i, u)$  and  $(z_j, u)$  and bounds their contribution in (3.2.76) by  $\Lambda$ . Furthermore, (3.2.67) implies that all internal vertices of incidence number 1 are removed. The amputation can possibly create in each tree at most one internal vertex of incidence number 2. Denoting by  $T_{l_1}^{s\pi_{i,1}-1, 0}(Y_{\pi_{i,1}}, 0)$  the surface tree obtained by amputating the external leg  $(z, u)$  from  $T_{l_1}^{s\pi_{i,1}, 0}(Y_{\pi_{i,1}}, u, 0)$ , we deduce

$$\tilde{v}_{2,i}^{s_1} \leq v_{2,i}^{s_1} + 1 \leq 3l_1 - 1 + \frac{s\pi_{i,1}}{2} + \frac{1}{2},$$

where  $\tilde{v}_{2,i}^{s_1}$  and  $v_{2,i}^{s_1}$  denote respectively the number of vertices of incidence number 2 of the surface trees  $T_{l_1}^{s\pi_{i,1}-1, 0}(Y_{\pi_{i,1}}, 0)$  and  $T_{l_1}^{s\pi_{i,1}, 0}(Y_{\pi_{i,1}}, u, 0)$ . Since  $1 \leq l_1 \leq l-1$ , we obtain

$$\tilde{v}_{2,i}^{s_1} \leq 3l - 3 - 1 + \frac{s\pi_{i,1}}{2} + \frac{1}{2} \leq 3l - 2 + \frac{s\pi_{i,1}}{2}. \quad (3.2.77)$$

Proceeding similarly with  $T_{l_2}^{s\pi_{j,2}, 0}(Y_{\pi_{j,2}}, u, 0)$ , we deduce that the number of vertices of the amputated tree obeys

$$\tilde{v}_{2,j}^{s_2} \leq 3l - 2 + \frac{s\pi_{j,2}}{2}. \quad (3.2.78)$$

From (3.2.95) and (3.2.78), we deduce that  $T_{l_i}^{s\pi_{i,1}-1, 0}(Y_{\pi_{i,1}}, 0) \in \mathcal{F}_l^{s\pi_{i,1}-1, 0}$ . Therefore, we obtain that

$$\int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta''}^0 \left( \Lambda; \tau_{\pi_1}, \frac{1}{2\Lambda^2}; W_{l_1}^{s_1+1}(\Pi_1); Y_{\pi_1}, u \right) \\ \times \mathcal{F}_{s_2+1, l_2; \delta''}^0 \left( \Lambda; \tau_{\pi_2}, \frac{1}{2\Lambda^2}; W_{l_2}^{s_2+1}(\Pi_2); Y_{\pi_2}, u \right) \quad (3.2.79)$$

is bounded by

$$\Lambda \mathcal{F}_{\delta''}^0 \left( \Lambda; \tau_{1, s_1}; W_{l_1}^{s_1}(\Pi_1^{s_1+1}); Y_{\sigma_{s_1}} \right) \mathcal{F}_{\delta''}^0 \left( \Lambda; \tau_{s_1+1, s}; W_{l_2}^{s_2}(\Pi_2^{s_2+1}); Y_{\sigma_{s_1+1:s}} \right), \quad (3.2.80)$$

where we used the notation (3.2.1). Note that  $\Pi_1^{s+1} \cup \Pi_2^{s+2} \in \mathcal{P}_s$ , which together with (3.2.80) gives the integrated surface weight factor of the forest  $W_{l_1}^{s_1}(\Pi_1^{s+1}) \cup W_{l_2}^{s_2}(\Pi_2^{s+2})$ . Hence (3.2.80) is bounded by

$$O(1) \Lambda \mathcal{F}_{\delta''}^0(\Lambda; \tau_{1,s}; W_l^s(\Pi); Y_{\sigma_s}),$$

where  $\Pi = \Pi_1^{s+1} \cup \Pi_2^{s+2}$  belongs to  $\mathcal{P}_s$  and we deduce

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta}^{\Lambda; 0} \left( \tau_{1, s_1}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s_1}}, u \right) \mathcal{F}_{s_2+1, l_2; \delta'}^{\Lambda; 0} \left( \tau_{s_1+1, s}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s_1+1, s}}, u \right) \\ \leq O(1) \Lambda \mathcal{F}_{s, l; \delta''}^{\Lambda, 0}(\tau_{1, s}). \end{aligned} \quad (3.2.81)$$

Note that the central point in the proof of (3.2.71) is the fact that the amputations of the external legs  $(z, u)$  and  $(z', u)$  from the forests  $W_{l_1}^{s_1+1}(\Pi_1)$  and  $W_{l_2}^{s_2+1}(\Pi_2)$  produce respectively the forests  $W_{l_1}^{s_1}(\Pi_1^{s+1})$  and  $W_{l_2}^{s_2}(\Pi_2^{s+2})$  such that

$$W_{l_1}^{s_1}(\Pi_1^{s+1}) \cup W_{l_2}^{s_2}(\Pi_2^{s+2}) \in \mathcal{W}_l^s(\Pi).$$

The proof of (3.2.74) can be performed following the same steps of the proof of (3.2.71), but still we need to check that for given forests  $W_{l_1}^{s_1+1}(\Pi_1)$  and  $W_{l_2}^{s_2+1}(\Pi_2)$  respectively in  $\mathcal{W}_{l_1; \pi'_{11}, \pi'_{21}}^{s_1+1}$  and  $\mathcal{W}_{l_2; \pi'_{12}, \pi'_{22}}^{s_2+1}$  we have

$$W_{l_1}^{s_1}(\Pi_1^{s+1}) \cup W_{l_2}^{s_2}(\Pi_2^{s+2}) \in \mathcal{W}_{l; \pi_1, \pi_2}^s(\Pi). \quad (3.2.82)$$

Clearly

$$W_l^s(\Pi) := W_{l_1}^{s_1}(\Pi_1^{s+1}) \cup W_{l_2}^{s_2}(\Pi_2^{s+2}) \in \mathcal{W}_l^s(\Pi).$$

Since the amputations do not affect the location of the external vertices  $y_i \in Y_{\sigma_s}$ , in the sense that their positions in the surface trees of the two forests remain unchanged. This means that there is no surface tree in  $W_l^s(\Pi)$  such that it has at least two external vertices in  $Y_{\pi_1}$  and  $Y_{\pi_2}$ . This implies directly (3.2.82).  $\square$

**Lemma 4.** (*Bulk tree-Forest Fusion*) Let  $\delta, \delta' > 0$  and  $l_1, l_2 \geq 1$  such that  $l_1 + l_2 = l$ . Given  $(\tilde{\pi}_1, \tilde{\pi}_2) \in \tilde{\mathcal{P}}_{2; s}$ , we have

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta}^0 \left( \Lambda; \tau_{\tilde{\pi}_1}, \frac{1}{2\Lambda^2}; Y_{\tilde{\pi}_1}, u \right) \hat{\mathcal{F}}_{s_2+1, l_2; \delta'} \left( \Lambda; \tau_{\tilde{\pi}_2}, \frac{1}{2\Lambda^2}; Y_{\tilde{\pi}_2}, u \right) \\ \leq O(1) \mathcal{F}_{s, l; \delta''}^0(\Lambda; \tau_{1, s}; Y_{\sigma_s}). \end{aligned} \quad (3.2.83)$$

Given another partition  $(\pi_1, \pi_2) \in \tilde{\mathcal{P}}_{2; s}$  and defining  $\pi_{ij} := \pi_i \cap \tilde{\pi}_j$ , we have

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{F}_{\delta}^0 \left( \Lambda; \tau_{\pi_{11}}^+, \tau_{\pi_{12}}^-, Y_{\pi_{11}}, Y_{\pi_{12}}, u \right) \hat{\mathcal{F}}_{\delta'} \left( \Lambda; \tau_{\pi_{12}}, \frac{1}{2\Lambda^2}; Y_{\pi_{12}}, u \right) \\ \leq O(1) \mathcal{F}_{s, l; \delta''}^0(\Lambda; \tau_{\pi_1}^+, \tau_{\pi_2}^-; Y_{\sigma_s}), \end{aligned} \quad (3.2.84)$$



where

$$(\pi'_{11}, \pi'_2) \in \{(\pi_{11} \cup \{s+1\}, \pi_2), (\pi_{11}, \pi_2 \cup \{s+1\})\}. \quad (3.2.85)$$

Similarly, we have

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{F}_{\delta}^0 \left( \Lambda; \tau_{\pi_1}^+, \tau_{\pi_{21}}^-; Y_{\pi_1}, Y_{\pi_{21}}, u \right) \hat{\mathcal{F}}_{\delta'}^- \left( \Lambda; \tau_{\pi_{22}}, \frac{1}{2\Lambda^2}; Y_{\pi_{22}}, u \right) \\ \leq O(1) \mathcal{F}_{s,l;\delta''}^0 \left( \Lambda; \tau_{\pi_1}^+, \tau_{\pi_2}^-; Y_{\sigma_s} \right) \end{aligned} \quad (3.2.86)$$

and

$$(\pi'_1, \pi'_{21}) \in \{(\pi_1 \cup \{s+1\}, \pi_{21}), (\pi_1, \pi_{21} \cup \{s+1\})\}. \quad (3.2.87)$$

In (3.2.83)-(3.2.86),  $\delta'' := \max(\delta, \delta')$ .

*Proof.* Without loss of generality, we again consider the ordered sub-partitions  $\sigma_{s_1}$  and  $\sigma_{s_1+1:s}$ . In order to obtain the bound (3.2.83), it is sufficient to bound for a given  $\Pi_1 \in \tilde{\mathcal{P}}_{s_1+1}$

$$\int_{\mathbb{R}} du \mathcal{F}_{\delta}^0 \left( \Lambda; \tau_{1,s_1}, \frac{1}{2\Lambda^2}; W_{l_1}^{s_1+1}(\Pi_1); Y_{\sigma_{s_1}}, u \right) \mathcal{F}_{\delta'} \left( \Lambda; \tau_{s_1+1,s}, \frac{1}{2\Lambda^2}; \hat{T}_{l_2}^{s_2+1}; Y_{\sigma_{s_1+1:s}}, u \right). \quad (3.2.88)$$

Using the bound (3.2.62), we bound (3.2.88) by

$$\int_{\mathbb{R}} du \mathcal{F}_{\delta''}^0 \left( \Lambda; \tau_{1,s_1}, \frac{1}{2\Lambda^2}; W_{l_1}^{s_1+1}(\Pi_1); Y_{\sigma_{s_1}}, u \right) \mathcal{F}_{\delta''} \left( \Lambda; \tau_{s_1+1,s}, \frac{1}{2\Lambda^2}; \hat{T}_{l_2}^{s_2+1}; Y_{\sigma_{s_1+1:s}}, u \right), \quad (3.2.89)$$

where  $\delta'' := \max(\delta, \delta')$ . Let  $z_i$  and  $z_j$  be respectively the internal vertices attached to  $u$  in  $W_{l_1}^{s_1+1}(\Pi_1)$  and to  $u$  in  $\hat{T}_{l_2}^{s_2+1}$ . Interchanging the integral over  $u$  with the integral over the internal vertices of the forest  $W_{l_1}^{s_1+1}(\Pi_1)$  and the bulk tree  $\hat{T}_{l_2}^{s_2+1}$  in their respective weight factors and using (3.1.5) we deduce

$$\int_{\mathbb{R}} du p_B \left( \frac{\alpha(1+\delta)}{2\Lambda^2}; z_i, u \right) p_B \left( \frac{1+\delta}{2\Lambda^2}; z_j, u \right) = p_B \left( \frac{(\alpha+1)(1+\delta)}{2\Lambda^2}; z_i, z_j \right) \quad (3.2.90)$$

with  $\alpha \in \{1, 2\}$ . Here, we proceed similarly to (3.2.66) to differentiate the surface trees with two external vertices from other surface trees with more than two external vertices. For  $\alpha = 2$ , we keep the integration over  $u$  and write

$$\begin{aligned} \int_{\mathbb{R}} du p_B \left( \frac{1+\delta}{\Lambda^2}; z_i, u \right) p_B \left( \frac{1+\delta}{2\Lambda^2}; z_j, u \right) \\ \leq 2 \int_0^{\infty} du p_B \left( \frac{1+\delta}{\Lambda^2}; z_i, u \right) p_B \left( \frac{1+\delta}{2\Lambda^2}; z_j, u \right). \end{aligned} \quad (3.2.91)$$

Therefore, (3.2.90) and (3.2.91) correspond to the fact that the two external legs attached to  $(z_i, u)$  and  $(z_j, u)$  are removed. If  $\alpha = 1$ , the external lines are replaced by the internal line  $(z_i, z_j)$  and for

$\alpha = 2$  the vertex  $u$  becomes internal with incidence number 2. The first case corresponds to the steps of merging the forest  $W_{l_1}^{s_1+1}(\Pi_1)$  and the bulk tree  $\hat{T}_{l_2}^{s_2+1}$  at the external vertices  $(u, u)$  through the process a). In the second case, the forest and the tree are merged following the merging process b). From Proposition 4 we have

$$M_{u,u}^i \left( W_{l_1}^{s_1+1}(\Pi_1), \hat{T}_{l_2}^{s_2+1} \right) \in \mathfrak{M}_l^s(\Pi'), \quad i \in \{a, b\}$$

where  $\Pi' = \Pi_1^{s_1+1} \cup \sigma_{s_1+1:s}$ . This implies that (3.2.89) is bounded by

$$2 \mathfrak{F}_{\delta''}^0 \left( \Lambda; \tau_{1,s}; W_l^s(\Pi'); Y_{\sigma_s} \right).$$

Therefore we deduce

$$\int_{\mathbb{R}} du \mathfrak{F}_{s_1+1, l_1; \delta}^0 \left( \Lambda; \tau_{\pi_1}, \frac{1}{2\Lambda^2}; Y_{\pi_1}, u \right) \hat{\mathfrak{F}}_{s_2+1, l_2; \delta'} \left( \Lambda; \tau_{\pi_2}, \frac{1}{2\Lambda^2}; Y_{\pi_2}, u \right) \leq O(1) \mathfrak{F}_{s, l; \delta''}^0 \left( \Lambda; \tau_{1,s}; Y_{\sigma_s} \right),$$

where  $O(1)$  is a constant which depends on  $s$  and  $l$ .

Now, we establish (3.2.84). Given  $W_{l_1}^{s_1+1}(\Pi_1)$  in  $\mathfrak{M}_{l_1; \pi'_{11}, \pi'_2}^{s_1+1}$  and  $\hat{T}_{l_2}^{s_2+1} \in \hat{\mathcal{T}}_{l_2}^{s_2+1}$ , let us bound

$$\int_{\mathbb{R}} du \mathfrak{F}_{\delta}^0 \left( \Lambda; \tau_{\pi'_{11}}^+, \tau_{\pi'_2}^-; W_{l_1}^{s_1+1}(\Pi_1); Y_{\pi_{11}}, Y_{\pi_2}, u \right) \mathfrak{F}_{\delta'} \left( \Lambda; \tau_{\pi_{12}}, \frac{1}{2\Lambda^2}; \hat{T}_{l_2}^{s_2+1}; Y_{\pi_{12}}, u \right). \quad (3.2.92)$$

If  $s+1 \in \pi'_{11}$ , then (3.2.92) is bounded by

$$2 \mathfrak{F}_{\delta''}^0 \left( \Lambda; \tau_{\pi'_{11} \cup \bar{\pi}_1}^+, \tau_{\bar{\pi}_2}^-; W_l^s(\Pi'); Y_{\sigma_s} \right)$$

following the same steps as before to establish (3.2.83).

The case which needs a careful treatment is  $s+1 \in \pi'_2$ . Using the same notations as before, the contributions of the external legs  $(z_i, u)$  and  $(z_j, u)$  in the weight factors of  $W_{l_1}^{s_1+1}(\Pi_1)$  and  $\hat{T}_{l_2}^{s_2+1}$  read

$$\int_{\mathbb{R}} du p_B \left( \frac{\alpha(1+\delta)}{2\Lambda^2}; z_i, u \right) p_B \left( \frac{1+\delta}{2\Lambda^2}; z_j, u \right) = p_B \left( \frac{(\alpha+1)(1+\delta)}{2\Lambda^2}; z_i, z_j \right) \quad (3.2.93)$$

with  $\alpha \in \{1, 2\}$ . By definition of the weight factors of the forest  $W_{l_1}^{s_1+1}(\Pi_1)$  and the tree  $\hat{T}_{l_2}^{s_2+1}$ , the internal vertex  $z_i$  is integrated over  $\mathbb{R}^-$  and  $z_j$  is integrated over  $\mathbb{R}^+$ . Hence, for  $\alpha = 1$  we bound (3.2.93) as follows

$$p_B \left( \frac{1+\delta}{\Lambda^2}; z_i, z_j \right) \leq p_B \left( \frac{1+\delta}{\Lambda^2}; z_j, 0 \right). \quad (3.2.94)$$

From a diagrammatic point of view, the bound (3.2.94) implies that the leg  $(z_i, u)$  is amputated from the forest  $W_{l_1}^{s_1+1}(\Pi_1)$  and  $(z_j, u)$  is replaced by  $(z_j, 0)$  in the bulk tree  $\hat{T}_{l_2}^{s_2+1}$ , which produces the surface tree  $T_{l_2}^{s_2, 0}$ . As we explained in the proof of Lemma 3, the amputation of one external leg

in the forest changes the number of vertices of incidence number 2 of the corresponding surface tree  $T_{l_1}^{s_{\tilde{\pi}_i}-1,0}(Y_{\tilde{\pi}_i}, 0, \vec{z}_{\tilde{\pi}_i})$  as follows for  $1 \leq l_1 \leq l-1$

$$\tilde{v}_{2,i}^{s_1} \leq 3l - 3 - 1 + \frac{s_{\tilde{\pi}_i}}{2} + \frac{1}{2} \leq 3l - 2 + \frac{s_{\tilde{\pi}_i}}{2}. \quad (3.2.95)$$

For  $\alpha = 2$ , we keep the integration over  $u$  and we write

$$\begin{aligned} p_B \left( \frac{3(1+\delta)}{2\Lambda^2}; z_i, z_j \right) &\leq p_B \left( \frac{3(1+\delta)}{2\Lambda^2}; z_j, 0 \right) = \int_{\mathbb{R}} du p_B \left( \frac{1+\delta}{\Lambda^2}; 0, u \right) p_B \left( \frac{1+\delta}{2\Lambda^2}; u, z_j \right) \\ &\leq 2 \int_0^\infty du p_B \left( \frac{1+\delta}{\Lambda^2}; 0, u \right) p_B \left( \frac{1+\delta}{2\Lambda^2}; u, z_j \right). \end{aligned} \quad (3.2.96)$$

From a diagrammatic point of view, the bound (3.2.96) implies that the leg  $(z_i, u)$  is amputated from the forest  $W_{l_1}^{s_1+1}(\Pi_1)$  and  $(z_j, u)$  is attached to  $(u, 0)$  in the bulk tree  $\hat{T}_{l_2}^{s_2+1}$ . Since  $\alpha = 2$ , we deduce that the surface tree in  $W_{l_1}^{s_1+1}(\Pi_1)$  which contains the external vertex  $u$ , belongs to  $\mathcal{F}_{l_1}^{2,0}$ . It contains only the external vertex  $u$  and the surface external vertex 0. The amputation of  $(z_i, u)$  lowers the incidence number of  $z_i$  by one and the process of amputation continues until the surface tree  $T_{l_1}^{2,0}$  becomes empty. Concerning the new tree obtained from  $\hat{T}_{l_2}^{s_2+1}$ , it has  $s_2$  external vertices and one surface external vertex. Furthermore, the number of vertices of incidence number 2 of this tree is  $v_2 + 1$ , where  $v_2$  is the number of vertices of incidence number 2 of  $\hat{T}_{l_2}^{s_2+1}$ . Therefore, we have

$$v_2 + 1 \leq 3l_2 - 1 + \frac{s_2 + 1}{2},$$

Remembering that  $l_2 \leq l-1$ , we deduce that

$$v_2 + 1 \leq 3l - 2 + \frac{s_2 + 1}{2}.$$

This proves that the new tree, which we denote by  $T_{l_2}^{s_2,0}$  is a surface tree and we deduce that  $W_{l_1}^{s_1}(\Pi_1^{s_1+1}) \cup T_{l_2}^{s_2,0}$  belongs to  $\mathcal{W}_{l_1; \pi_1, \pi_2}^s(\Pi'')$  with  $\Pi'' = \Pi_1^{s_1+1} \cup \pi_{12}$ . Combining these arguments with the bounds (3.2.94) and (3.2.96), we deduce that (3.2.92) is bounded by

$$O(1) \mathcal{F}_{\delta''}^0(\Lambda; \tau_{\pi_1}^+, \tau_{\pi_2}^-; W_l^s(\Pi''); Y_{\sigma_s}).$$

This gives (3.2.84). The same method of the proof gives also the bound (3.2.86).  $\square$

**Lemma 5.** *Let  $\delta > 0$ ,  $l \geq 0$  and  $s \geq 2$ , we have*

$$\mathcal{F}_{s,l; +; \delta}^\infty(\Lambda, z_1, \tau_{2,s}, y_{2,s}) \leq C_{s,l} \mathcal{F}_{s,l; \delta}(\Lambda, z_1, \tau_{2,s}, y_{2,s}), \quad \forall z_1 \in \mathbb{R}^+ \quad (3.2.97)$$

and

$$\mathcal{F}_{s,l; -; \delta}^\infty(\Lambda, z_1, \tau_{2,s}, y_{2,s}) \leq C_{s,l} \mathcal{F}_{s,l; \delta}^-(\Lambda, z_1, \tau_{2,s}, y_{2,s}), \quad \forall z_1 \in \mathbb{R}^-. \quad (3.2.98)$$

$C_{s,l}$  is a positive constant that depends on  $s$  and  $l$ .

*Proof.* Take  $s \geq 2$  and a tree  $T_l^s(z_1, y_{2,s}, \vec{z}) \in \mathcal{T}_l^s$  with  $\vec{z} = (z_2, \dots, z_{r+1})$  the set of its internal vertices. We recall the definition (3.2.37):

$$\mathcal{F}_{\delta;+}^\infty(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}) := \sup_{\Lambda \leq \Lambda_I \leq \Lambda_0} \int_{\mathbb{R}^{+q}} d\vec{z}_{\mathcal{J}_e} \int_{\mathbb{R}^{r-q}} d\vec{z}_1^c \mathcal{F}_\delta(\Lambda_{\mathcal{J}}, \tau_{2,s}; T_l^s(z_1, y_{2,s}, \vec{z})), \quad (3.2.99)$$

where

$$\mathcal{F}_\delta(\Lambda_{\mathcal{J}}, \tau_{2,s}; T_l^s(z_1, y_{2,s}, \vec{z})) := \prod_{I \in \mathcal{J}} p_B\left(\frac{1+\delta}{\Lambda_I^2}; I\right) \prod_{J \in \mathcal{J}} p_B(\tau_{J,\delta}; J). \quad (3.2.100)$$

We write

$$\begin{aligned} & \int_{\mathbb{R}^{+q}} d\vec{z}_{\mathcal{J}_e} \int_{\mathbb{R}^{r-q}} d\vec{z}^c \prod_{I \in \mathcal{J}} p_B\left(\frac{1+\delta}{\Lambda_I^2}; I\right) \prod_{J \in \mathcal{J}} p_B(\tau_{J,\delta}; J) \\ &= \int_{\mathbb{R}^{+q}} d\vec{z}_{\mathcal{J}_e} \prod_{J \in \mathcal{J}} p_B(\tau_{J,\delta}; J) \sum_{(\pi_1, \pi_2)} \int_{(\mathbb{R}^+)^{r_1}} d\vec{z}_{\pi_1}^c \int_{(\mathbb{R}^-)^{r_2}} d\vec{z}_{\pi_2}^c \prod_{I \in \mathcal{J}} p_B\left(\frac{1+\delta}{\Lambda_I^2}; I\right), \end{aligned} \quad (3.2.101)$$

where the sum runs over all the partitions of length 2 of  $\mathcal{J}_e^c$  and  $r_i := |\pi_i|$ . We perform the change of variable  $z \rightarrow -z$  for the variables integrated over  $\mathbb{R}^-$  and we use the bound

$$p_B(\tau; z, -z') \leq p_B(\tau; z, z'), \quad \forall z, z' \geq 0 \quad (3.2.102)$$

to deduce that

$$\mathcal{F}_{\delta;+}^\infty(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}) \leq 2^{r-q} \mathcal{F}_\delta(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}). \quad (3.2.103)$$

Denoting by  $|\mathcal{V}_i(T_l^s)|$  the cardinality of the set of the internal vertices of the tree  $T_l^s$  and remembering (3.2.39), we deduce

$$\mathcal{F}_{s,l;+;\delta}^\infty(\Lambda, z_1, \tau_{2,s}, y_{2,s}) \leq C_{s,l} \mathcal{F}_{s,l;\delta}(\Lambda, z_1, \tau_{2,s}, y_{2,s}), \quad (3.2.104)$$

where  $C_{s,l} := \max_{T_l^s \in \mathcal{T}_l^s} 2^{|\mathcal{V}_i(T_l^s)|}$ . Performing a change of variable  $z \rightarrow -z$  for the internal vertices  $\vec{z}_{\mathcal{J}_e}$  and summing over the trees  $T_l^s$ , we have

$$\mathcal{F}_{s,l;-;\delta}^\infty(\Lambda, z_1, \tau_{2,s}, y_{2,s}) = \mathcal{F}_{s,l;+;\delta}^\infty(\Lambda, -z_1, \tau_{2,s}, -y_{2,s}). \quad (3.2.105)$$

Using the bound (3.2.104) together with (3.2.105), we deduce

$$\mathcal{F}_{s,l;-;\delta}^\infty(\Lambda, z_1, \tau_{2,s}, y_{2,s}) \leq C_{s,l} \mathcal{F}_{s,l;\delta}(\Lambda, -z_1, \tau_{2,s}, -y_{2,s}). \quad (3.2.106)$$

which after a change of variable  $z \rightarrow -z$  for the internal vertices  $\vec{z}$  of the trees  $T_l^s$  is bounded by

$$C_{s,l} \mathcal{F}_{s,l;\delta}^-(\Lambda, z_1, \tau_{2,s}, y_{2,s}). \quad (3.2.107)$$

This ends the proof of lemma 5. □

## Chapter 4

# Renormalization of the semi-infinite massive $\phi^4$ model: General counter-terms

### 4.1 Introduction

In the present chapter, we establish a rigorous proof of the perturbative renormalizability of the semi-infinite model based on the Polchinski flow equation. This model appeared in 1971 [40] as a simple model that allows to study surface effects alone disentangled from the finite size effects. It is defined starting from the massive  $\phi^4$  model in infinite space, with the difference that it is defined on a half space bounded by a plane. We consider this model with three types of boundary conditions which are Dirichlet, Neumann and Robin boundary conditions (b.c.). From a mathematical point of view, each of these b.c. corresponds to a self-adjoint extension of the Laplacian in  $\mathbb{R}^+ \times \mathbb{R}^3$ . As we explained in chapter 1, the self-adjointness of the Laplacian is a key property to define the propagator of a quantum field theory. Each boundary condition defines a particular propagator and thus a different theory. This chapter is organized as follows: In section 4.2 we define the regularized flowing propagator and present the considered action together with the system of perturbative flow equations satisfied by the connected amputated Schwinger distributions (CAS). Section 4.3 will be devoted to prove some regularity properties of the support of the Gaussian measure associated to the regularized propagator. To establish bounds on the CAS, which are distributions, they have to be folded first with test functions. In section 4.4.1 a suitable class of test functions is introduced that will be used in the bounds on the CAS to be derived. In section 4.4.2 we state the boundary and the renormalization conditions used to integrate the flow equations of the irrelevant and relevant terms respectively. Section 4.5 is the central one of this chapter. We state and prove inductive bounds on the amputated Schwinger distributions folded with the introduced test functions which, being uniform in the cutoff, directly lead to renormalizability.

## 4.2 The Action and the flow equations

We will analyze the perturbative renormalizability of the semi-infinite  $\phi_4^4$  theory with Robin boundary conditions. It will be proved by analyzing the generating functional  $L^{\Lambda, \Lambda_0}$  of connected amputated Schwinger distributions (CAS). The upper indices  $\Lambda_0$  and  $\Lambda$  enter through the regularized propagator. We choose the following regularization

$$C_R^{\Lambda, \Lambda_0}(p; z, z') = \int_{\frac{1}{\Lambda_0^2}}^{\frac{1}{\Lambda^2}} d\lambda e^{-\lambda(p^2+m^2)} p_R(\lambda; z, z'). \quad (4.2.1)$$

Clearly,  $C_R^{\Lambda, \Lambda_0}$  verifies the b.c. (2.4.33). For  $\Lambda \rightarrow 0$  and  $\Lambda_0 \rightarrow \infty$  we recover the unregularized propagator (2.4.29). We denote

$$\dot{C}_R^\Lambda(p; z, z') = \frac{\partial}{\partial \Lambda} C_R^{\Lambda, \Lambda_0}(p; z, z') = \dot{C}^\Lambda(p) p_R\left(\frac{1}{\Lambda^2}; z, z'\right), \quad (4.2.2)$$

where  $\dot{C}^\Lambda(p) = -\frac{2}{\Lambda^3} e^{-\frac{p^2+m^2}{\Lambda^2}}$ . In the sequel, we use the following estimate for the 3-dimensional propagator. Given  $w \in \mathbb{N}^3$  and a polynomial  $\mathcal{P}$ , we have

$$\left| \partial^w \dot{C}^\Lambda(p) \mathcal{P}\left(\frac{p}{\Lambda}\right) \right| \leq (\Lambda + m)^{-3-|w|} \tilde{\mathcal{P}}\left(\frac{|p|}{\Lambda + m}\right), \quad (4.2.3)$$

with  $\tilde{\mathcal{P}}$  a new polynomial with positive coefficients that depend only on the coefficients of  $\mathcal{P}$  and  $w$ . We assume  $0 \leq \Lambda \leq \Lambda_0 < \infty$  so that the flow parameter  $\Lambda$  takes the role of an infrared cutoff, whereas  $\Lambda_0$  is a UV cutoff. The full propagator is recovered for  $\Lambda = 0$  and  $\Lambda_0 \rightarrow \infty$ . For finite  $\Lambda_0$  and in finite volume the positivity and the regularity properties of  $C_R^{\Lambda, \Lambda_0}$  permit to define the theory rigorously from the functional integral

$$e^{-\frac{1}{\hbar}(L^{\Lambda, \Lambda_0}(\phi) + I^{\Lambda, \Lambda_0})} := \int d\mu_{\Lambda, \Lambda_0, R}(\Phi) e^{-\frac{1}{\hbar} L^{\Lambda_0, \Lambda_0}(\Phi + \phi)}, \quad (4.2.4)$$

$$L^{\Lambda, \Lambda_0}(0) = 0,$$

where the factors of  $\hbar$  have been introduced to allow for a consistent loop expansion in the sequel. Here,  $d\mu_{\Lambda, \Lambda_0, R}$  denotes the Gaussian measure with covariance  $\hbar C_R^{\Lambda, \Lambda_0}$ . The test functions  $\phi$  and  $\Phi$  belong to the support of the Gaussian measure  $d\mu_{\Lambda, \Lambda_0, R}$ , which in particular implies that they are in  $\mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$  as we will prove in section 4.5. The normalization factor  $e^{-\frac{1}{\hbar} I^{\Lambda, \Lambda_0}}$  is due to vacuum contributions.

The functional  $L^{\Lambda_0, \Lambda_0}(\phi)$  is the bare interaction of a renormalizable theory including counter-terms, viewed as a formal power series in  $\hbar$ . Since translation invariance is broken in the  $z$ -direction (the semi-line), all counter-terms may be  $z$ -dependent. In general, the constraints on the bare action result from

the symmetry properties of the theory which are imposed, on its field content and on the form of the propagator. It is therefore natural to consider the general bare interaction

$$L^{\Lambda_0, \Lambda_0}(\phi) = \frac{\lambda}{4!} \int_V \phi^4(z, x) + \frac{1}{2} \int_V \left( a^{\Lambda_0}(z) \phi^2(z, x) - b^{\Lambda_0}(z) \phi(z, x) \Delta_x \phi(z, x) \right. \\ \left. - d^{\Lambda_0}(z) \phi(z, x) \partial_z^2 \phi(z, x) + s^{\Lambda_0}(z) \phi(z, x) (\partial_z \phi)(z, x) + \frac{2}{4!} c^{\Lambda_0}(z) \phi^4(z, x) \right).$$

Here we supposed the theory to be symmetric under  $\phi \rightarrow -\phi$ , and we included only relevant terms with respect to (4.2.4) in the sense of the renormalization group. The functions  $a^{\Lambda_0}(z)$ ,  $b^{\Lambda_0}(z)$ ,  $c^{\Lambda_0}(z)$ ,  $d^{\Lambda_0}(z)$  and  $s^{\Lambda_0}(z)$  are supposed to be smooth. As we will explain later, it is a consequence of smooth imposed renormalization conditions together with the regularity of the regularized flowing propagator.

The flow equation (FE) is obtained from (4.2.4) on differentiating w.r.t.  $\Lambda$ . It is obtained following the same steps that led to the flow equation (1.1.15). It is a differential equation for the functional  $L^{\Lambda, \Lambda_0}$ :

$$\partial_\Lambda (L^{\Lambda, \Lambda_0} + I^{\Lambda, \Lambda_0}) = \frac{\hbar}{2} \langle \frac{\delta}{\delta \phi}, \dot{C}_R^\Lambda \frac{\delta}{\delta \phi} \rangle L^{\Lambda, \Lambda_0} - \frac{1}{2} \langle \frac{\delta}{\delta \phi} L^{\Lambda, \Lambda_0}, \dot{C}_R^\Lambda \frac{\delta}{\delta \phi} L^{\Lambda, \Lambda_0} \rangle. \quad (4.2.5)$$

By  $\langle, \rangle$  we refer to the standard inner product in  $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$ .

We may expand the functional  $L^{\Lambda, \Lambda_0}(\phi)$  in a formal power series w.r.t.  $\hbar$ ,

$$L^{\Lambda, \Lambda_0}(\phi) = \sum_{l=0}^{\infty} \hbar^l L_l^{\Lambda, \Lambda_0}(\phi).$$

Corresponding expansions for  $a^{\Lambda_0}(z)$ ,  $b^{\Lambda_0}(z)$ ..., are  $a^{\Lambda_0}(z) = \sum_{l=0}^{\infty} \hbar^l a_l^{\Lambda_0}(z)$ , etc. From  $L_l^{\Lambda, \Lambda_0}(\phi)$  we obtain the CAS distributions of loop order  $l$  as

$$\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}((z_1, x_1), \dots, (z_n, x_n)) := \delta_{\phi(z_1, x_1)} \cdots \delta_{\phi(z_n, x_n)} L_l^{\Lambda, \Lambda_0} |_{\phi=0},$$

where we used the notation  $\delta_{\phi(z,x)} = \delta/\delta\phi(z,x)$ .

Since translation invariance in the  $x$ -directions is preserved, we will use in all what follows a mixed representation, where the Fourier transform to  $p$ -space is performed only with respect to  $x \in \mathbb{R}^3$ . In this representation, we set

$$\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \Phi_n) = \int_0^\infty dz_2 \cdots dz_n \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \phi_2(z_2) \cdots \phi_n(z_n). \quad (4.2.6)$$

Here we denote

$$\Phi_n(z_2, \dots, z_n) := \prod_{i=2}^n \phi_i(z_i)$$

and

$$\delta^{(3)}(p_1 + \cdots + p_n) \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) = (2\pi)^{3(n-1)} \frac{\delta^n}{\delta\phi(z_1, p_1) \cdots \delta\phi(z_n, p_n)} L_l^{\Lambda, \Lambda_0}(\phi) |_{\phi=0}.$$

The  $\delta^{(3)}(p_1 + \dots + p_n)$  appears because of the translation invariance in the  $x$  directions. The FE for the CAS distributions derived from (4.2.5) are [13]

$$\begin{aligned}
& \partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \\
&= \frac{1}{2} \int_z \int_{z'} \int_k \partial^w \mathcal{L}_{l-1, n+2}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \dot{C}_R^\Lambda(k; z, z') \\
&- \frac{1}{2} \int_z \int_{z'} \sum'_{l_1, l_2} \sum'_{n_1, n_2} \sum_{w_i} c_w \left[ \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_R^\Lambda(p; z, z') \right. \\
&\quad \left. \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \right]_{rsym}, \\
& p = -p_1 - \dots - p_{n_1} = p_{n_1+1} + \dots + p_n. \quad (4.2.7)
\end{aligned}$$

Note that we wrote (4.2.7) directly in a form where a number  $|w|$  of momentum derivatives, characterized by a multi-index, act on both sides and we used the shorthand notation (1.1.19). The symbol "rsym" means summation over those permutations of the position-momenta  $(z_1, p_1), \dots, (z_n, p_n)$ , which do not leave invariant the (unordered) subsets  $((z_1, p_1), \dots, (z_{n_1}, p_{n_1}))$  and  $((z_{n_1+1}, p_{n_1+1}), \dots, (z_n, p_n))$ , and therefore, produce mutually different pairs of (unordered) image subsets, and the primes restrict the summations to  $n_1 + n_2 = n$ ,  $l_1 + l_2 = l$ ,  $w_1 + w_2 + w_3 = w$ , respectively.

### 4.3 Regularity of the support of the regularized gaussian measure

The bare interaction  $L^{\Lambda_0, \Lambda_0}$  consists of powers of the field  $\phi$  and of its derivatives. It can not be given any mathematical meaning if the field  $\phi$  is not sufficiently regular, e.g. at least in  $\mathcal{C}^2(\mathbb{R}^+ \times \mathbb{R}^3)$ . In this section, we prove that the field  $\phi$  belongs to  $\mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$ . This result is due to the regularity properties of the support of the regularized Gaussian measure.

For this we will consider the UV-regularized propagator without infrared cut-off

$$C_R^{\Lambda_0}(p; x, y) = \int_{\frac{1}{\Lambda_0}}^{\infty} d\lambda e^{-\lambda(p^2 + m^2)} p_R(\lambda; x, y),$$

assuming that  $\Lambda_0 \geq 1$ ,  $p \in \mathbb{R}^3$  and  $x, y \in \mathbb{R}^+$ . The same arguments work for the Gaussian measure associated to the propagator  $C_R^{\Lambda, \Lambda_0}(p; x, y)$ . The main result of this section is the following proposition:

**Proposition 5.** *Let  $\mu_{\Lambda_0, R}$  be the gaussian measure associated to the propagator  $C_R^{\Lambda_0}$ . The support of  $\mu_{\Lambda_0, R}$  satisfies*

$$\text{supp} \mu_{\Lambda_0, R} \subset \bigcap_{n \geq 1} \left\{ (-\Delta_R + m^2)^{-n} f \mid f \in L^2(\mathbb{R}^+ \times \mathbb{R}^3) \right\}.$$

The proof of Proposition 5 is based on the following corollary of the Minlos theorem [23] and [41]:



**Corollary 2.** *Given a nuclear space  $E$ ,  $\mu$  a measure on  $E'$ , and  $C$  its characteristic function, we introduce a continuous inner product  $(\cdot, \cdot)_0$  on  $E$  and let  $H_0$  be the completion of  $E$  with respect to  $(\cdot, \cdot)_0$ . Suppose that  $C$  is continuous on  $H_0$ . Let  $T$  be a Hilbert-Schmidt operator on  $H_0$  satisfying:*

- (a)  $T$  is one to one (injective map).
- (b)  $E \subset \text{Im}T$  and  $T^{-1}(E)$  is dense in  $H_0$ .
- (c) The map  $T^{-1} : E \rightarrow H_0$  is continuous.

Then the support of  $\mu$  is on  $(T^{-1})^*H'_0 \subset E'$ . The notations  $(T^{-1})^*$  and  $H'_0$  are used for the "adjoint" and the "dual space" in the pairing between  $E$  and  $E'$ .

For a proof of this corollary see [23]- [41].

*Proof.* We apply Corollary 2 to  $E = \mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^3) \cong \mathcal{S}(\mathbb{R}^+) \otimes \mathcal{S}(\mathbb{R}^3)$ . This is a nuclear space which is a tensor product of two nuclear spaces. See [42] for the proof that  $\mathcal{S}(\mathbb{R}^+)$  is a nuclear space. The theorem A.4.1 in [25] implies the existence of the Gaussian measure  $\mu_{\Lambda_0, R}$  with covariance  $C_R^{\Lambda_0}$  with support included in  $\mathcal{S}'(\mathbb{R}^+ \times \mathbb{R}^3)$ . We apply the corollary of Minlos's theorem to the scalar product  $\langle f, g \rangle_n := \langle f, P^{-2n}g \rangle$  where  $\langle \cdot, \cdot \rangle$  is the usual scalar product in  $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$ , and  $P = -\Delta_R + m^2$ .  $P^n$  is a unitary map from  $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$  into  $H_{-n}$ , the completion of  $\mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^3)$  with respect to  $\langle \cdot, \cdot \rangle_n$ . First, we shall verify that the regularized covariance is continuous on  $H_{-n}$  for any  $n \in \mathbb{N}$ , that is

$$\exists C > 0 \text{ such that } \forall f, g \in H_{-n} : \quad \left| \langle f, C_R^{\Lambda_0} g \rangle \right| \leq C \|f\|_{H_{-n}} \|g\|_{H_{-n}} .$$

One can verify that the operators  $C_R^{\Lambda_0}$  and  $(-\Delta_R + m^2)^{-n}$  commute. Since  $(-\Delta_R + m^2)^{-n}$  is self-adjoint, we obtain

$$\langle f, C_R^{\Lambda_0} g \rangle = \left\langle (-\Delta_R + m^2)^{-n} f, (-\Delta_R + m^2)^{2n} C_R^{\Lambda_0} (-\Delta_R + m^2)^{-n} g \right\rangle. \quad (4.3.1)$$

By the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \left| \langle f, C_R^{\Lambda_0} g \rangle \right| &\leq \int \frac{d^3p}{(2\pi)^3} \left\| (-\Delta_R + m^2)^{-n} f \right\|_2 \\ &\quad \times \left\| (-\Delta_R + m^2)^{2n} C_R^{\Lambda_0} (-\Delta_R + m^2)^{-n} g \right\|_2, \end{aligned} \quad (4.3.2)$$

where for  $u \in L^2(\mathbb{R}^+ \times \mathbb{R}^3)$  we write

$$\|u\|_2 := \left( \int_{\mathbb{R}^+} dz |u(z, p)|^2 \right)^{\frac{1}{2}}$$

and

$$\begin{aligned} & \left\| (-\Delta_R + m^2)^{2n} C_R^{\Lambda_0} (-\Delta_R + m^2)^{-n} g \right\|_2^2 \\ & := \int_0^\infty dx \left| \int_0^\infty dy (-\Delta_R + m^2)_x^{2n} C_R^{\Lambda_0}(p; x, y) \int_0^\infty dz (-\Delta_R + m^2)^{-n}(p; y, z) g(z, p) \right|^2. \end{aligned} \quad (4.3.3)$$

Again by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \left| \int_0^\infty dy (-\Delta_R + m^2)_x^{2n} C_R^{\Lambda_0}(p; x, y) \int_0^\infty dz (-\Delta_R + m^2)^{-n}(p; y, z) g(z, p) \right|^2 \\ & \leq \int_0^\infty dy \left| (-\Delta_R + m^2)_x^{2n} C_R^{\Lambda_0}(p; x, y) \right|^2 \left\| (-\Delta_R + m^2)^{-n} g \right\|_{L^2(\mathbb{R}^+)}^2. \end{aligned} \quad (4.3.4)$$

Therefore (4.3.2) is bounded by

$$\left( \int_0^\infty dx \int_0^\infty dy \left| (-\Delta_R + m^2)_x^{2n} C_R^{\Lambda_0}(p; x, y) \right|^2 \right) \left\| (-\Delta_R + m^2)^{-n} g \right\|_{L^2(\mathbb{R}^+)}^2. \quad (4.3.5)$$

We have

$$(-\Delta_R + m^2)_x^{2n} C_R^{\Lambda_0}(p; x, y) = \int_{\frac{1}{\Lambda_0^2}}^\infty d\lambda e^{-\lambda(p^2+m^2)} (-\partial_x^2 + p^2 + m^2)^{2n} p_R(\lambda; x, y). \quad (4.3.6)$$

Using the Leibniz formula we get

$$(-\partial_x^2 + p^2 + m^2)^{2n} p_R(\lambda; x, y) = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (p^2 + m^2)^{2n-k} \partial_x^{2k} p_R(\lambda; x, y). \quad (4.3.7)$$

One can prove by induction

$$\partial_x^{2k} \left( \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(x-y)^2}{2\lambda}} \right) = \lambda^{-k} \sum_{i=0}^k c_i(k) \left( \frac{x-y}{\sqrt{\lambda}} \right)^i \left( \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(x-y)^2}{2\lambda}} \right), \quad (4.3.8)$$

where  $c_i(k) \in \mathbb{R}$ , which implies that

$$\left| \partial_x^{2k} p_B(\lambda; x, y) \right| \leq c_k \lambda^{-k} p_B(2\lambda; x, y).$$

Here  $c_k$  is a positive constant. Therefore we find using (2.4.31) and (2.4.32) that

$$\left| \partial_x^{2k} p_R(\lambda; x, y) \right| \leq c_k \lambda^{-k} \left( p_B(2\lambda; x, y) + p_B(2\lambda; x, -y) + 2 \int_0^\infty dw e^{-w} p_B \left( 2\lambda; x, -\frac{w}{c} - y \right) \right).$$

Using that

$$2 \int_0^\infty \frac{dw}{\sqrt{2\pi\lambda}} e^{-w} e^{-\frac{(x+y+\frac{w}{c})^2}{2\lambda}} = p_N(\lambda; x, y) - p_R(\lambda; x, y),$$

we obtain

$$\begin{aligned} \left| \partial_x^{2k} p_R(\lambda; x, y) \right| &\leq c_k \lambda^{-k} (p_B(2\lambda; x, y) + p_B(2\lambda; x, -y) + p_N(2\lambda; x, y) - p_R(2\lambda; x, y)) \\ &\leq 2c_k \lambda^{-k} p_N(2\lambda; x, y), \end{aligned}$$

and this implies that

$$\left| (-\partial_x^2 + p^2 + m^2)^{2n} p_R(\lambda; x, y) \right| \leq 2C_n \sum_{k=0}^{2n} \lambda^{-k} (p^2 + m^2)^{2n-k} p_N(2\lambda; x, y).$$

where  $C_n := \sup_{1 \leq k \leq n} c_k$ .

Therefore we obtain

$$\left| (-\Delta_R + m^2)^{2n} C_R^{\Lambda_0}(p; x, y) \right| \leq \tilde{C}_n \Lambda_0^{4n} \sum_{k=0}^{2n} \left( \frac{p^2 + m^2}{\Lambda_0^2} \right)^{2n-k} \int_{\frac{1}{\Lambda_0^2}}^{\infty} d\lambda e^{-\lambda(p^2+m^2)} p_N(2\lambda; x, y),$$

where we used that

$$p_R(\tau; x, y) \leq p_N(\tau; x, y) \quad \forall \tau > 0, \forall (x, y) \in (\mathbb{R}^+)^2.$$

$C_n$  and  $\tilde{C}_n$  are suitable positive constants that depend on  $n$ .

We have by Cauchy-Schwarz

$$\begin{aligned} \left| \int_{\frac{1}{\Lambda_0^2}}^{\infty} d\lambda e^{-\lambda(p^2+m^2)} p_N(2\lambda; x, y) \right|^2 &\leq e^{-\frac{p^2+m^2}{\Lambda_0^2}} \int_{\frac{1}{\Lambda_0^2}}^{\infty} d\lambda e^{-\lambda(p^2+m^2)} |p_N(2\lambda; x, y)|^2 \\ &\leq \frac{4\Lambda_0}{\sqrt{2\pi}} e^{-\frac{p^2+m^2}{\Lambda_0^2}} \int_0^{\infty} d\lambda e^{-\lambda(p^2+m^2)} \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(x-y)^2}{2\lambda}} \\ &\leq \frac{4\Lambda_0}{\sqrt{2\pi}} e^{-\frac{p^2+m^2}{\Lambda_0^2}} \frac{e^{-\sqrt{p^2+m^2}|x-y|}}{\sqrt{p^2+m^2}}, \end{aligned}$$

where we used

$$e^{-\frac{(x+y)^2}{2\lambda}} \leq e^{-\frac{(x-y)^2}{2\lambda}}, \quad \forall (x, y) \in (\mathbb{R}^+)^2.$$

Therefore we obtain the following bound for the first factor from (4.3.5)

$$\left( \int_0^{\infty} dx \int_0^{\infty} dy \left| (-\Delta_R + m^2)^{2n} C_R^{\Lambda_0}(p; x, y) \right|^2 \right) \leq \Lambda_0^{8n+1} \mathcal{P} \left( \frac{p^2 + m^2}{\Lambda_0^2} \right) \frac{e^{-\frac{p^2+m^2}{\Lambda_0^2}}}{(p^2 + m^2)^{\frac{3}{2}}}, \quad (4.3.9)$$

where  $\mathcal{P}$  is a suitable polynomial with positive coefficients. All constants were absorbed in the polynomial  $\mathcal{P}$ , and we obtain the final bound for (4.3.1) using again the Cauchy-Schwarz inequality w.r.t. the momenta  $p$ ,

$$\begin{aligned} \left| \langle f, C_R^{\Lambda_0} g \rangle \right| &\leq C_{\Lambda_0} \left\| (-\Delta_R + m^2)^{-n} f \right\|_{L^2(\mathbb{R}^+ \cdot \mathbb{R}^3)} \left\| (-\Delta_R + m^2)^{-n} g \right\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^3)} \\ &= C_{\Lambda_0} \|f\|_{H_{-n}} \cdot \|g\|_{H_{-n}}, \end{aligned} \quad (4.3.10)$$

The constant  $C_{\Lambda_0}$  depends on  $\Lambda_0$ . To obtain (4.3.10), we have performed the  $p$ -integral in (4.3.2) using the bound (4.3.9).

We apply Corollary 2 with  $T = P^{-2} : H_{-n} \rightarrow H_{-n}$ . The operator  $P^{-2}$  is integral with a kernel given by

$$k(p; x, y) := \int_{\mathbb{R}^+} \frac{du}{4(p^2 + m^2)} \left( e^{-\sqrt{p^2+m^2}|x-u|} + \frac{\sqrt{p^2+m^2}-c}{\sqrt{p^2+m^2}+c} e^{-\sqrt{p^2+m^2}|x+u|} \right) \\ \times \left( e^{-\sqrt{p^2+m^2}|y-u|} + \frac{\sqrt{p^2+m^2}-c}{\sqrt{p^2+m^2}+c} e^{-\sqrt{p^2+m^2}|y+u|} \right). \quad (4.3.11)$$

Note that (4.3.11) is bounded as follows

$$k(p; x, y) \leq \int_0^\infty du e^{-\sqrt{p^2+m^2}|x-u|} \frac{e^{-\sqrt{p^2+m^2}|y-u|}}{\left(\sqrt{p^2+m^2}+c\right)^2}.$$

Remembering that  $P^{-n}$  is bounded on  $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$ , we obtain

$$\|k\|_{H_{-n}}^2 = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} dx dy \left| (-\partial_x^2 + p^2 + m^2)^{-n} k(p; x, y) \right|^2 \leq C \|k\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^3)}^2. \quad (4.3.12)$$

Using

$$\|k\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^3)}^2 \leq \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{\left(\sqrt{p^2+m^2}+c\right)^4} \left( \int_0^\infty dx du e^{-2\sqrt{p^2+m^2}|x-u|} \right)^2 \\ \leq C \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2+m^2)^4} < \infty \quad \text{for suitable } C > 0,$$

we deduce that  $T$  is Hilbert-Schmidt on  $H_{-n}$ . Clearly,  $T$  is injective and  $T^{-1}(\mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^3)) \subset \mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^3)$ , which is by definition dense in  $H_{-n}$ . The other assumptions of corollary 2 can be verified easily. Since the dual of  $H_{-n}$  is the space  $H_n$  of functions whose image under  $P^n$  is in  $L^2$ , we obtain the the Gaussian measure  $\mu_{\Lambda_0, R}$  is supported on the set  $\{P^{-n}f, f \in L^2(\mathbb{R}^+ \times \mathbb{R}^3)\}$ . This holds for any  $n \in \mathbb{N}^*$  which implies that  $\mu_{\Lambda_0, R}$  has its support within the set

$$\bigcap_{n \geq 1} \left\{ (-\Delta_R + m^2)^{-n} L^2(\mathbb{R}^+ \times \mathbb{R}^3) \right\}.$$

□

## 4.4 Boundary and renormalization conditions

### 4.4.1 Test functions

Standard proofs of perturbative renormalizability by flow equations use inductive bounds on the  $n$ -point correlation functions. These objects are no more functions if considered in the mixed position-momentum space, but rather tempered distributions. We introduce tempered distributions in  $\mathcal{S}'((\mathbb{R}^+)^n)$  w.r.t. the semi-norms

$$\prod_{i=1}^n \mathcal{N}_2(\phi_i) ,$$

where  $\mathcal{N}_2(\phi) := \sup_{0 \leq \alpha, \beta \leq 2} \|(1 + z^\beta) \partial_z^\alpha \phi(z)\|_\infty$  and  $\partial_z \phi|_{z=0} = \lim_{z \rightarrow 0^+} \partial_z \phi$ . The space  $\mathcal{S}'((\mathbb{R}^+)^n)$  can be regarded as the topological dual of the space

$$\mathcal{S}((\mathbb{R}^+)^n) := \{ \chi^+(z) \phi(z) \mid \phi \in \mathcal{S}(\mathbb{R}^n) \} , \quad (4.4.1)$$

with  $\chi^+$  denoting the characteristic function of the space  $\mathbb{R}^+$ . For an extensive topological study of the space  $\mathcal{S}(\mathbb{R}^+)$  and its dual space  $\mathcal{S}'(\mathbb{R}^+)$ , we refer the reader to [42].

We now introduce test functions against which  $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}$  will be integrated. In the sequel we will bound the CAS folded with test functions of the following form:

Let  $1 \leq s \leq n$ , we define

$$\tau := \inf \tau_{2,s} \text{ where } \tau_{2,s} = (\tau_2, \dots, \tau_s) \text{ with } \tau_i > 0,$$

and similarly  $z_{2,s} = (z_2, \dots, z_s)$ . Given  $y_2, \dots, y_s \in \mathbb{R}^+$ , we define

$$\phi_{\tau_{2,s}, y_{2,s}}^R(z_{2,s}) := \prod_{i=2}^s p_R(\tau_i; z_i, y_i) \prod_{i=s+1}^n \chi^+(z_i) , \quad (4.4.2)$$

where  $\chi^+(z_i)$  is the characteristic function of the semi-line  $\mathbb{R}^+$ . This definition can be generalized by choosing any other subset of  $s$  coordinates among  $z_2, \dots, z_n$ . We also define for  $2 \leq j \leq s$

$$\phi_{\tau_{2,s}, y_{2,s}}^{R,(j)}(z_{2,n}) := p_R^{(1)}(\tau_j; z_j, z_1; y_j) \prod_{i=2, i \neq j}^s p_R(\tau_i; z_i, y_i) \prod_{i=s+1}^n \chi^+(z_i) \quad (4.4.3)$$

with

$$p_R^{(1)}(\tau_j; z_j, z_1; y_j) = p_R(\tau_j; z_j, y_j) - p_R(\tau_j; z_1, y_j) . \quad (4.4.4)$$

Our choice of the test functions is not optimal, in the sense that the proof of renormalizability given in section 4.5 holds also for larger classes of test functions indexed by a strictly positive parameter  $\tau$  such that

$$|\partial_z^\alpha \phi_\tau(z)| \leq \tau^{-\frac{\alpha}{2}} |\phi_\tau(z)| \quad \forall z \in \mathbb{R}^+ , \quad \forall \alpha \geq 0 .$$

We choose a simple example of these functions, which are the Robin heat kernels. To go further one could either prove (in a more functional analysis type of approach) that our test functions are dense in the set of smooth rapidly decaying functions on  $\mathbb{R}^+$  w.r.t. a suitable norm and that  $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \Phi_n)$  are continuous w.r.t. this semi-norm.

#### 4.4.2 Relevant terms and renormalization conditions

First, let us introduce some notations that we will be using through out the rest of this chapter:

$$\int_z := \int_{\mathbb{R}^+} dz, \quad \int_{z_{2,n}} := \int_{(\mathbb{R}^+)^{n-1}} dz_2 \cdots dz_n, \quad \int_p := \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3}, \quad (4.4.5)$$

$$(\vec{z}_n, \vec{p}_n) = (z_1, p_1), \dots, (z_n, p_n), \quad (\vec{z}_{i,j}, \vec{p}_{i,j}) = (z_i, p_i), \dots, (z_j, p_j), \quad \text{for } i < j. \quad (4.4.6)$$

Given test functions  $\phi_i \in \mathcal{S}(\mathbb{R}^+)$ , we define

$$\Phi_4(z_2, z_3, z_4) := \prod_{i=1}^4 \phi_i(z_i).$$

The relevant terms in the bare interaction are fixed by renormalization conditions at the value  $\Lambda = 0$  of the flow parameter, all other boundary terms are fixed at  $\Lambda = \Lambda_0$ . To extract the relevant terms contained in

$$\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; \vec{0}; \phi_2) := \int_{z_2} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \phi_2(z_2) \quad (4.4.7)$$

and

$$\mathcal{L}_{l,4}^{\Lambda,\Lambda_0}(z_1; \vec{0}; \Phi_4) := \int_{z_{2,4}} \mathcal{L}_{l,4}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_4, 0)) \prod_{i=2}^4 \phi_i(z_i), \quad (4.4.8)$$

we use a Taylor expansion of the test functions  $\phi_2$  and  $\Phi_4$ , which gives

$$\begin{aligned} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; \vec{0}; \phi_2) &= a_l^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) - s_l^{\Lambda,\Lambda_0}(z_1) (\partial_{z_1} \phi_2)(z_1) + d_l^{\Lambda,\Lambda_0}(z_1) (\partial_{z_1}^2 \phi_2)(z_1) \\ &\quad + l_{l,2}^{\Lambda,\Lambda_0}(z_1; \phi_2), \end{aligned} \quad (4.4.9)$$

$$\left( \partial_{p^2} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0} \right) (z_1; \vec{0}; \phi_2) = b_l^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) + \left( \partial_{p^2} l_{l,2}^{\Lambda,\Lambda_0} \right) (z_1; \phi_2), \quad (4.4.10)$$

$$\mathcal{L}_{l,4}^{\Lambda,\Lambda_0}(z_1; \vec{0}; \Phi_4) = c_l^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) \phi_3(z_1) \phi_4(z_1) + l_{l,4}^{\Lambda,\Lambda_0}(z_1; \Phi_4). \quad (4.4.11)$$

Then the relevant terms appear as

$$a_l^{\Lambda, \Lambda_0}(z_1) = \int_{z_2} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)), \quad (4.4.12)$$

$$s_l^{\Lambda, \Lambda_0}(z_1) = \int_{z_2} (z_1 - z_2) \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)), \quad (4.4.13)$$

$$d_l^{\Lambda, \Lambda_0}(z_1) = \frac{1}{2} \int_{z_2} (z_1 - z_2)^2 \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)), \quad (4.4.14)$$

$$b_l^{\Lambda, \Lambda_0}(z_1) = \int_{z_2} \partial_{p^2} \left( \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}((z_1, p), (z_2, -p)) \right) \Big|_{p=0}, \quad (4.4.15)$$

$$c_l^{\Lambda, \Lambda_0}(z_1) = \int_{z_{2,4}} \mathcal{L}_{l,4}^{\Lambda, \Lambda_0}((z_1, 0), \dots, (z_4, 0)), \quad (4.4.16)$$

and the remainders  $l_{l,2}^{\Lambda, \Lambda_0}(z_1; \phi_2)$ ,  $(\partial_{p^2} l_{l,2}^{\Lambda, \Lambda_0})(z_1; \phi_2)$  and  $l_{l,4}^{\Lambda, \Lambda_0}(z_1; \Phi_4)$  are given by

$$l_{l,2}^{\Lambda, \Lambda_0}(z_1; \phi_2) = \int_{z_2} \int_0^1 dt \frac{(1-t)^2}{2!} \partial_t^3 \phi_2(tz_2 + (1-t)z_1) \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)), \quad (4.4.17)$$

$$(\partial_{p^2} l_{l,2}^{\Lambda, \Lambda_0})(z_1; \phi_2) = \int_{z_2} \int_0^1 dt \partial_t \phi_2(tz_2 + (1-t)z_1) \partial_{p^2} \left( \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}((z_1, p), (z_2, -p)) \right) \Big|_{p=0},$$

and

$$\begin{aligned} l_{l,4}^{\Lambda, \Lambda_0}(z_1; \Phi_4) &= \int_{z_{2,4}} \mathcal{L}_{l,4}^{\Lambda, \Lambda_0}((z_1, 0), \dots, (z_4, 0)) \left[ \int_0^1 dt \partial_t \phi_2(tz_2 + (1-t)z_1) \phi_3(z_3) \phi_4(z_4) \right. \\ &\quad \left. + \phi_2(z_1) \int_0^1 dt \partial_t \phi_3(tz_3 + (1-t)z_1) \phi_4(z_4) + \phi_2(z_1) \phi_3(z_1) \int_0^1 dt \partial_t \phi_4(tz_4 + (1-t)z_1) \right]. \end{aligned} \quad (4.4.18)$$

**Boundary conditions at  $\Lambda = \Lambda_0$ :**

At  $\Lambda = \Lambda_0$ , we include the counter-terms that are required to make  $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}$  finite as the cutoffs are removed (i.e.  $\Lambda_0 \rightarrow +\infty$ ,  $\Lambda \rightarrow 0$ ). The form of these counter-terms is not known, but in any case they must be local functionals not exceeding a certain dimension. Hence, this aspect of the boundary conditions can be imposed at  $\Lambda = \Lambda_0$ :

$$\begin{aligned} \mathcal{L}_{l,n}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) &= 0, \quad \forall n \geq 5, \quad \mathcal{L}_{0,2}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) = 0, \\ \mathcal{L}_{l,4}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) &= \left( c_l^{\Lambda_0}(z_1) (1 - \delta_{l,0}) + \lambda \delta_{l,0} \right) \prod_{i=2}^4 \delta(z_1 - z_i), \\ \mathcal{L}_{l,2}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) &= \left( a_l^{\Lambda_0}(z_1) + b_l^{\Lambda_0}(z_1) p^2 + s_l^{\Lambda_0}(z_1) \partial_{z_1} - d_l^{\Lambda_0}(z_1) \partial_{z_1}^2 \right) \delta(z_1 - z_2), \quad \forall l \geq 1. \end{aligned} \quad (4.4.19)$$

**Renormalization conditions at  $\Lambda = 0$**

The renormalization conditions are fixed at  $\Lambda = 0$  by imposing

$$a_l^{0, \Lambda_0}(z_1), \quad s_l^{0, \Lambda_0}(z_1), \quad d_l^{0, \Lambda_0}(z_1), \quad b_l^{0, \Lambda_0}(z_1), \quad c_l^{0, \Lambda_0}(z_1)$$

to be smooth functions in  $\mathcal{C}^\infty(\mathbb{R}^+)$ , uniformly bounded w.r.t.  $\Lambda_0$ .

Typically all the renormalization conditions are assumed to be cutoff-independent. The simplest renormalization conditions are BPHZ-renormalization conditions, where we set for all  $z_1 \geq 0$

$$a_l^{0,\Lambda_0}(z_1) \equiv 0, \quad s_l^{0,\Lambda_0}(z_1) \equiv 0, \quad d_l^{0,\Lambda_0}(z_1) \equiv 0, \quad b_l^{0,\Lambda_0}(z_1) \equiv 0, \quad c_l^{0,\Lambda_0}(z_1) \equiv 0. \quad (4.4.20)$$

These will be adopted in the following.

## 4.5 Proof of renormalizability

### 4.5.1 Notations

For  $n \geq 2$  and  $0 \leq r \leq 3$ , we define

$$\mathcal{L}_{l,n;r}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s},y_{2,s}}^R) := \int_{z_{2,n}} (z_1 - z_i)^r \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \prod_{i=2}^s p_R(\tau_i; z_i, y_i), \quad (4.5.1)$$

and for  $n \geq 3$

$$F_{12} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau_{3,s},y_{3,s}}^R) := (z_1 - z_2)^3 \int_{z_{3,n}} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \prod_{i=3}^s p_R(\tau_i; z_i, y_i). \quad (4.5.2)$$

For  $n = 2$  we write

$$F_{12} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1, z_2; p) := (z_1 - z_2)^3 \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)). \quad (4.5.3)$$

### 4.5.2 Results

In this subsection, we gathered the main results of this chapter. The following theorem establishes the uniform boundedness w.r.t. the UV-cutoff  $\Lambda_0$  of the Robin correlation distributions:

**Theorem 7.** (*Boundedness*) *We consider  $0 \leq \Lambda \leq \Lambda_0 < \infty$ ,  $1 \leq s \leq n$ ,  $2 \leq j \leq s$  and  $0 \leq r \leq 3$ . We consider test functions either of the form (4.4.2)-(4.4.3), which are also denoted in shorthand as  $\phi_{\tau_{2,s},y_{2,s}}^R$  resp.  $\phi_{\tau_{2,s},y_{2,s}}^{R,(j)}$ . Adopting (4.4.20) we claim*

$$(A) \left| \partial^w \mathcal{L}_{l,n;r}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s},y_{2,s}}^R) \right| \leq (\Lambda + m)^{4-n-|w|-r} \mathcal{P}_1 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^\Lambda(\tau_{2,s}). \quad (4.5.4)$$

$$(B) \left| F_{12} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau_{3,s},y_{3,s}}^R) \right| \leq (\Lambda + m)^{1-n} \mathcal{P}_3 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_4 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{12}(\Lambda, \tau_{3,s}). \quad (4.5.5)$$



$$\begin{aligned}
(C) \left| \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^{R,(j)}) \right| \\
\leq (\Lambda + m)^{3-n-|w|} \tau_j^{-\frac{1}{2}} \mathcal{P}_5 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_6 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q}_2 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s}) . \quad (4.5.6)
\end{aligned}$$

(D) The functions defined in (4.5.1), (4.5.2) and (4.5.3) together with their momentum derivatives are in  $\mathcal{C}^\infty(\mathbb{R}^+)$  w.r.t.  $z_1$ .

Here and in the following the  $\mathcal{P}_i$  and  $\mathbb{Q}_i$  denote (each time they appear possibly new) polynomials with nonnegative coefficients. The polynomials  $\mathbb{Q}_i$  are reduced to a constant if  $s = 1$ . The coefficients depend on  $l, n, |w|, \delta$  but not on  $\{p_i\}$ ,  $\Lambda$ ,  $\Lambda_0$  and  $z_1$ . For  $l = 0$ , all polynomials  $\mathcal{P}_i$  reduce to constants. In the definition of (3.2.32),  $\delta > 0$  may be chosen arbitrarily small.

**Theorem 8.** (Convergence) Let  $0 \leq \Lambda \leq \Lambda_0 < \infty$ . Using the same notations, conventions and adopting the same renormalization conditions (4.4.20) as in Theorem 7, we have the following bounds

$$\begin{aligned}
\left| \partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n;r}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R) \right| &\leq \frac{(\Lambda + m)^{5-n-|w|-r}}{(\Lambda_0 + m)^2} \tilde{\mathcal{P}}_1 \left( \log \frac{\Lambda_0 + m}{m} \right) \\
&\times \tilde{\mathcal{P}}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \tilde{\mathbb{Q}}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s}) \quad \forall n + |w| + r \geq 4, \quad (4.5.7)
\end{aligned}$$

$$\begin{aligned}
\left| \partial_{\Lambda_0} \partial^w \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}(z_1; p, -p; \phi_{\tau_2, y_2}^R) \right| \\
\leq \frac{(\Lambda + m)^{3-|w|}}{(\Lambda_0 + m)^2} \tilde{\mathcal{P}}_3 \left( \log \frac{\Lambda_0 + m}{m} \right) \tilde{\mathcal{P}}_4 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \tilde{\mathbb{Q}}_2 \left( \frac{\tau_2^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda}(\tau_2), \quad (4.5.8)
\end{aligned}$$

$$\begin{aligned}
\left| \partial_{\Lambda_0} F_{12} \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau_{3,s}, y_{3,s}}^R) \right| \\
\leq \frac{(\Lambda + m)^{2-n}}{(\Lambda_0 + m)^2} \tilde{\mathcal{P}}_5 \left( \log \frac{\Lambda_0 + m}{m} \right) \tilde{\mathcal{P}}_6 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{12}(\Lambda, \tau_{3,s}) \quad \forall n \geq 2, \quad (4.5.9)
\end{aligned}$$

$$\begin{aligned}
\left| \partial_{\Lambda_0} \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^{R,(j)}) \right| &\leq \frac{(\Lambda + m)^{4-n}}{(\Lambda_0 + m)^2} \tau_j^{-\frac{1}{2}} \tilde{\mathcal{P}}_7 \left( \log \frac{\Lambda_0 + m}{m} \right) \\
&\times \tilde{\mathcal{P}}_8 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \tilde{\mathbb{Q}}_3 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s}) \quad \forall n \geq 4. \quad (4.5.10)
\end{aligned}$$

Note that the integration of the bounds (4.5.7) and (4.5.8) over  $\Lambda_0$  immediately proves the convergence of all  $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1, \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R)$  for fixed  $\Lambda$  to finite limits when  $\Lambda_0 \rightarrow \infty$ . In particular, one obtains for

all  $\Lambda'_0 > \Lambda_0$  and  $\vec{p}_n \in \mathbb{R}^{3n}$

$$\begin{aligned} & \left| \mathcal{L}_{l,n}^{0,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R) - \mathcal{L}_{l,n}^{0,\Lambda'_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R) \right| \\ & < \frac{m^{5-n}}{\Lambda_0} \left( \log \frac{\Lambda_0 + m}{m} \right)^\nu \tilde{\mathcal{P}}_2 \left( \frac{\|\vec{p}_n\|}{m} \right) \tilde{\mathcal{Q}}_1 \left( \frac{\tau^{-\frac{1}{2}}}{m} \right) \mathcal{F}_{s,l}(0, \tau_{2,s}, z_1, y_{2,s}) . \end{aligned}$$

Then the Cauchy criterion in  $\mathcal{C}^\infty(\mathbb{R}^+)$  w.r.t. to  $\Lambda_0$  implies the existence of finite limits to all loop orders.

### Remarks 1.

- The bounds (4.5.5) and (4.5.6) are required to close the inductive argument in the subsequent proof. The bound (4.5.4) is the central result of the boundedness Theorem 7 needed later in Theorem 8 to prove the convergence of  $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R)$  in the limits  $\Lambda \rightarrow 0$  and  $\Lambda_0 \rightarrow \infty$ .
- The role of the parameter  $\tau$  as it appears in the proof, is to absorb negative powers of the flow parameter  $\Lambda$  by producing powers of  $\frac{\tau}{\Lambda+m}$  that contribute to the polynomial  $\mathcal{Q}$  at each step of the induction. This preserves the global power counting in terms of  $\Lambda + m$ .
- The parameter  $\delta$  depends on the loop order  $l$  and it verifies  $0 < \delta_l < \delta_{l+1} < \frac{1}{2}$ . For simplicity, we omit the index  $l$ .
- The value of the integral

$$\int_0^\infty dw \delta_w \tag{4.5.11}$$

admits two possible choices, which are 1 and  $\frac{1}{2}$ . These two choices are called respectively, the weak and strong definitions of the Dirac distribution [4.3].

The subsequent proof uses the strong definition of the Dirac distribution. For the weak definition, all the points of Theorem 7 hold except for (D). In the weak convention, the functions defined in (4.5.1), (4.5.2) and (4.5.3) are in  $\mathcal{C}^\infty(\mathbb{R}^{+*})$  w.r.t.  $z_1$  and are not continuous at 0. One can verify that the proof of renormalizability is independent of the chosen convention. This comes from the fact that for a continuous function  $f$ , in both conventions, we have that

$$\int_0^\infty dz \left( \int_0^\infty dz' \delta(z - z') \right) f(z) = \int_0^\infty dz f(z).$$

### 4.5.3 Useful lemmas

Before getting the proof of the theorems 7 and 8, we collect and prove some useful lemmas which are used repeatedly in the inductive proof of the bounds (4.5.4)-(4.5.6):

**Lemma 6.** For all  $t, u, v$  and  $y$  in  $\mathbb{R}^+$ ,  $\tau > 0$  and some constant  $C_{k,\delta} > 0$ , we have

$$\left| \partial_t^k p_B(\tau; tu + (1-t)v, y) \right| \leq C_{k,\delta} \frac{|u-v|^k}{\tau^{\frac{k}{2}}} p_B(\tau_\delta; tu + (1-t)v, y) .$$

*Proof.* One can prove by induction that

$$\partial_t^k p_B(\tau; tu + (1-t)v, y) = \frac{(u-v)^k}{\tau^{\frac{k}{2}}} \mathcal{P}_k \left( \frac{tu + (1-t)v - y}{\sqrt{\tau}} \right) p_B(\tau; tu + (1-t)v, y) ,$$

where  $\mathcal{P}_k$  is a polynomial of degree  $k$  with at least one root for odd  $k$ . Therefore,

$$\begin{aligned} \left| \partial_t^k p_B(\tau; tu + (1-t)v, y) \right| &\leq \left| \mathcal{P}_k \left( \frac{tu + (1-t)v - y}{\sqrt{\tau}} \right) \right| e^{-\frac{(tu+(1-t)v-y)^2}{2(1+\delta)\tau} \cdot \frac{\delta}{1+2\delta}} \\ &\quad \times \frac{|u-v|^k}{\tau^{\frac{k}{2}}} p_B(\tau_\delta; tu + (1-t)v, y) . \end{aligned}$$

The lemma follows directly with  $C_{k,\delta} := \sup_{x \in \mathbb{R}} \left| \mathcal{P}_k(x) e^{-\frac{x^2}{1+\delta} \cdot \frac{\delta}{1+2\delta}} \right|$ .  $\square$

**Corollary 3.** For all  $t, u, v$  and  $y$  in  $\mathbb{R}^+$ ,  $\tau > 0$

$$\left| \partial_t^k p_R(\tau; tu + (1-t)v, y) \right| \leq 4 C_{k,\delta} \frac{|u-v|^k}{\tau^{\frac{k}{2}}} p_B(\tau_\delta; tu + (1-t)v, y) .$$

*Proof.* We recall that

$$p_R(\tau; z, z') = p_B(\tau; z, z') + p_B(\tau; z, -z') - 2 \int_0^\infty dw e^{-w} p_B \left( \tau; z, -\frac{w}{c} - z' \right) .$$

Using Lemma 6 we obtain

$$\begin{aligned} \left| \partial_t^k p_R(\tau; tu + (1-t)v, y) \right| &\leq C_{k,\delta} \frac{|u-v|^k}{\tau^{\frac{k}{2}}} \left\{ p_B(\tau_\delta; tu + (1-t)v, y) \right. \\ &\quad \left. + p_B(\tau_\delta; tu + (1-t)v, -y) + 2 \int_0^\infty dw e^{-w} p_B \left( \tau_\delta; tu + (1-t)v, -y - \frac{w}{c} \right) \right\} . \end{aligned}$$

For  $z, z', w \in \mathbb{R}^+$ , we have

$$e^{-\frac{(z+z'+w/c)^2}{2\tau}} \leq e^{-\frac{(z+z')^2}{2\tau}} \leq e^{-\frac{(z-z')^2}{2\tau}} ,$$

which implies

$$\partial_t^k p_R(\tau; tu + (1-t)v, y) \leq 4 C_{k,\delta} \frac{|u-v|^k}{\tau^{\frac{k}{2}}} p_B(\tau_\delta; tu + (1-t)v, y) .$$

$\square$

**Lemma 7.** For  $0 < \delta < 1$ , we define  $b = 2 \frac{1+2\delta}{1-2\delta}$  and let  $0 < \delta' < 1$  and  $\Lambda_I \geq \Lambda$  such that  $\delta' \Lambda^2 \geq b\tau^{-1}$ . For  $z_1, z_2$  and  $y_2$  in  $\mathbb{R}^+$  and  $\tau > 0$  we have

$$\begin{aligned} |z_1 - z_2| p_B \left( \frac{1+\delta}{\Lambda_I^2}; z_1, z_2 \right) & \int_0^1 dt p_B (\tau\delta'; tz_2 + (1-t)z_1, y_2) \\ & \leq C_\delta \Lambda^{-1} p_B \left( \frac{2}{\Lambda_I^2}; z_1, z_2 \right) p_B ((1+\delta')^3 \tau; z_1, y_2) . \end{aligned} \quad (4.5.12)$$

*Proof.* We have

$$p_B \left( \frac{1+\delta}{\Lambda_I^2}; z_1, z_2 \right) = \sqrt{\frac{1+2\delta}{1+\delta}} p_B \left( \frac{1+2\delta}{\Lambda_I^2}; z_1, z_2 \right) e^{-\frac{\Lambda_I^2(z_1-z_2)^2}{2(1+\delta)} \cdot \frac{\delta}{1+2\delta}} .$$

Since  $\Lambda_I \geq \Lambda$ , we deduce

$$|z_1 - z_2| p_B \left( \frac{1+\delta}{\Lambda_I^2}; z_1, z_2 \right) \leq C_\delta \Lambda^{-1} p_B \left( \frac{1+2\delta}{\Lambda_I^2}; z_1, z_2 \right), \quad (4.5.13)$$

with  $C_\delta := \sqrt{\frac{1+2\delta}{1+\delta}} \left\| x e^{-\frac{x^2}{2(1+\delta)} \cdot \frac{\delta}{1+2\delta}} \right\|_\infty$ .

Now, we bound

$$p_B \left( \frac{1+2\delta}{\Lambda_I^2}; z_1, z_2 \right) \int_0^1 dt p_B (\tau\delta'; tz_2 + (1-t)z_1, y_2) . \quad (4.5.14)$$

For  $0 < \delta' < 1$  and  $\delta' \Lambda_I^2 \geq b\tau^{-1}$ , we write

$$\begin{aligned} & \frac{\Lambda_I^2}{1+2\delta} (z_1 - z_2)^2 + \frac{1}{\tau(1+\delta')} (tz_2 + (1-t)z_1 - y_2)^2 \\ & = \frac{\Lambda_I^2 |z_1 - z_2|^2}{b} + \frac{\Lambda_I^2 |z_1 - z_2|^2}{2} + \frac{1}{\tau(1+\delta')} (tz_2 + (1-t)z_1 - y_2)^2 \\ & \geq \frac{|z_1 - z_2|^2}{\tau\delta'} + \frac{\Lambda_I^2 |z_1 - z_2|^2}{2} + \frac{1}{\tau(1+\delta')^2} (tz_2 + (1-t)z_1 - y_2)^2 . \end{aligned} \quad (4.5.15)$$

For  $0 \leq t \leq 1$ , we have

$$|z_1 - z_2| = |z_1 - tz_2 - (1-t)z_1| + |tz_2 + (1-t)z_1 - y_2| .$$

Therefore, we write

$$\begin{aligned} & \frac{1}{\delta'} |z_1 - z_2|^2 + |tz_2 + (1-t)z_1 - y_2|^2 \\ & \geq \frac{1}{\delta'} |z_1 - tz_2 - (1-t)z_1|^2 + |tz_2 + (1-t)z_1 - y_2|^2 \\ & \geq \frac{1}{1+\delta'} (|z_1 - tz_2 - (1-t)z_1| + |tz_2 + (1-t)z_1 - y_2|)^2 \\ & \geq \frac{|z_1 - y_2|^2}{1+\delta'} . \end{aligned}$$

Combining this with (4.5.15) and

$$\frac{|z_1 - z_2|^2}{\delta'} + \frac{|tz_2 + (1-t)z_1 - y_2|^2}{(1+\delta')^2} \geq \frac{1}{(1+\delta')^2} \left[ \frac{1}{\delta'} |z_1 - z_2|^2 + |tz_2 + (1-t)z_1 - y_2|^2 \right],$$

we deduce that

$$\frac{\Lambda_I^2}{1+2\delta} (z_1 - z_2)^2 + \frac{1}{\tau(1+\delta')} (tz_2 + (1-t)z_1 - y_2)^2 \geq \frac{|z_1 - y_2|^2}{\tau(1+\delta')^3} + \frac{\Lambda_I^2 |z_1 - z_2|^2}{2}.$$

Hence, (4.5.14) is bounded by

$$C_\delta p_B \left( \frac{2}{\Lambda_I^2}; z_1, z_2 \right) p_B \left( (1+\delta')^3 \tau; z_1, y_2 \right),$$

which together with (4.5.13), gives the final bound (4.5.12).  $\square$

#### 4.5.4 Proof of Theorems 7 and 8

##### Proof of Theorem 7

*Proof.* The bounds are proved inductively using the standard inductive scheme which proceeds upwards in  $l$ , for given  $l$  upwards in  $n$ , and for given  $(n, l)$  downwards in  $|w|$  starting from some arbitrary  $|w_{\max}| \geq 3$ . The induction works because the terms on the r.h.s. of the FE are always prior to the one of the l.h.s. in the inductive order. So the bounds (4.5.4)-(4.5.6) may be used as an induction hypothesis on the r.h.s. Once verified in the first induction step, we integrate the FE, where the terms with  $n + |w| + r \geq 5$  are integrated down from  $\Lambda_0$  to  $\Lambda$  because of the boundary conditions (4.4.19), and the terms with  $n + |w| + r \leq 4$  at the renormalization point are integrated upwards from 0 to  $\Lambda$  using (4.4.20). We can write remembering (4.2.6)

$$\partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0} \left( z_1; \vec{0}; \phi_{\tau_{2,s}, y_{2,s}}^R \right) = \partial^w \mathcal{L}_{l,n}^{0, \Lambda_0} \left( z_1; \vec{0}; \phi_{\tau_{2,s}, y_{2,s}}^R \right) + \int_0^\Lambda d\lambda \partial_\lambda \partial^w \mathcal{L}_{l,n}^{\lambda, \Lambda_0} \left( z_1; \vec{0}; \phi_{\tau_{2,s}, y_{2,s}}^R \right). \quad (4.5.16)$$

Once a bound has been obtained at the renormalization point, it is possible to move away from the renormalization point using the integrated Taylor formula

$$\begin{aligned} \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0} \left( z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R \right) &= \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0} \left( z_1; \vec{0}; \phi_{\tau_{2,s}, y_{2,s}}^R \right) \\ &+ \sum_{i=1}^n \sum_{\mu=1}^4 p_{i,\mu} \int_0^1 dt \left( \partial_{p_{i,\mu}} \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0} \right) \left( z_1; t\vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R \right). \end{aligned} \quad (4.5.17)$$

The induction starts with the pair  $(0, 4)$  for which the r.h.s. of the FE vanishes so that

$$\mathcal{L}_{0,4}^{\Lambda, \Lambda_0} \left( (z_1, p_1), \dots, (z_4, p_4) \right) = \lambda \prod_{i=2}^4 \chi^+(z_i) \delta(z_1 - z_i)$$

which implies for the test function  $\phi_{\tau_{2,s},y_{2,s}}^R$  defined in (4.4.2) that

$$\mathcal{L}_{0,4}^{\Lambda,\Lambda_0} \left( z_1; \vec{p}_4; \phi_{\tau_{2,s},y_{2,s}}^R \right) = \lambda \prod_{i=2}^s p_R(\tau_i; z_1, y_i). \quad (4.5.18)$$

Using the bounds (3.1.12) and (3.1.9), we deduce that

$$\prod_{i=2}^s p_R(\tau_i; z_1, y_i) \leq 2^s \prod_{i=2}^s p_B(\tau_i; z_1, y_i) \leq 2^s (1 + \delta)^{\frac{s}{2}} \prod_{i=2}^s p_B(\tau_{i,\delta}; z_1, y_i),$$

which implies that (4.5.18) can be bounded by a tree with no internal vertices and with a root vertex  $z_1$  linked to the external vertices  $y_{2,s}$ , which is in agreement with the bound (4.5.4). The constants are absorbed in the polynomial  $\mathcal{P}_1$ , which is of degree 0 at the tree order.

(I) Bounds on the r.h.s. of the FE:

We want to establish the bounds

$$\begin{aligned} \left| \partial_\Lambda \partial^w \mathcal{L}_{l,n;r}^{\Lambda,\Lambda_0} \left( z_1; \vec{p}_n; \phi_{\tau_{2,s},y_{2,s}}^R \right) \right| &\leq (\Lambda + m)^{3-n-|w|-r} \mathcal{P}_1 \left( \log \frac{\Lambda + m}{m} \right) \\ &\times \mathcal{P}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^\Lambda(\tau_{2,s}), \end{aligned} \quad (4.5.19)$$

$$\begin{aligned} \left| \partial_\Lambda F_{12} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0} (z_1, z_2; \vec{p}_n; \phi_{\tau_{3,s},y_{3,s}}^R) \right| &\leq (\Lambda + m)^{-n} \mathcal{P}_3 \left( \log \frac{\Lambda + m}{m} \right) \\ &\times \mathcal{P}_4 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{12}(\Lambda, \tau_{3,s}) \end{aligned} \quad (4.5.20)$$

and

$$\begin{aligned} \left| \partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0} (z_1; \vec{p}_n; \phi_{\tau_{2,s},y_{2,s}}^{R,(j)}) \right| \\ \leq (\Lambda + m)^{2-n-|w|} \tau_j^{-\frac{1}{2}} \mathcal{P}_5 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_6 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q}_2 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^\Lambda(\tau_{2,s}). \end{aligned} \quad (4.5.21)$$

(A) In this part we consider the case  $r = 0$ .

(A<sub>1</sub>) Let  $R_1$  be the first term on the r.h.s. of the FE

$$R_1 := \int_p \int_{z_2, n, z, z'} \partial^w \mathcal{L}_{n+2,l-1}^{\Lambda,\Lambda_0} ((z_n, \vec{p}_n), (z, p), (z', -p)) \dot{C}_R^\Lambda(p; z, z') \prod_{i=2}^s p_R(\tau_i; z_i, y_i)$$

which can be written as

$$\int_p \dot{C}^\Lambda(p) \int_{\mathbb{R}^+} du \partial^w \mathcal{L}_{n+2,l-1}^{\Lambda,\Lambda_0} \left( z_1; p, -p, \vec{p}_n; \phi_{\tau_{2,s},y_{2,s}}^R \times p_R \left( \frac{1}{2\Lambda^2}; u, \cdot \right) p_R \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right). \quad (4.5.22)$$

In (4.5.22), we used (4.2.2) and the semi-group property (3.1.6) for  $p_R$ . Applying the induction hypothesis together with

$$\Lambda^{-3} \int_p e^{-\frac{p^2+m^2}{\Lambda^2}} \mathcal{P} \left( \frac{|p|}{\Lambda+m} \right) \leq O(1), \quad (4.5.23)$$

we deduce that  $R_1$  is bounded by

$$\begin{aligned} & (\Lambda+m)^{2-n-|w|} \mathcal{P}_1 \left( \log \frac{\Lambda+m}{m} \right) \mathcal{P}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda+m} \right) \mathcal{Q}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda+m} \right) \\ & \times \int_{\mathbb{R}^+} du \int_{\vec{z}} \sum_{T_{l-1}^{s+2}(z_1, y_{2,s}, u, u, \vec{z})} \mathcal{F}_\delta \left( \Lambda, \left\{ \tau_{2,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2} \right\}; T_{l-1}^{s+2}(z_1, y_{2,s}, u, u, \vec{z}) \right). \end{aligned} \quad (4.5.24)$$

In the sequel, the polynomials are mentioned without their lower indices. One should keep in mind that these polynomials may have, each time they appear, different positive coefficients which depend on  $l, n, |w|, \delta$  only and not on  $\vec{p}_n, \Lambda, \Lambda_0, z_1, m$  and  $c$ .

For any contribution to (4.5.24) we denote by  $z', z''$  the vertices in the tree  $T_{l-1}^{s+2}(z_1, y_{2,s}, u, u, \vec{z})$  to which the test functions  $p_B \left( \frac{1+\delta}{2\Lambda^2}; u, \cdot \right)$  and  $p_B \left( \frac{1+\delta}{2\Lambda^2}; \cdot, u \right)$  are attached. Interchanging  $\int_{\vec{z}}$  and  $\int_u$  and performing the integral over  $u$  using (3.1.5), we obtain

$$\int_{\mathbb{R}^+} du p_B \left( \frac{1+\delta}{2\Lambda^2}; z', u \right) p_B \left( \frac{1+\delta}{2\Lambda^2}; u, z'' \right) \leq p_B \left( \frac{1+\delta}{\Lambda^2}; z', z'' \right) \leq \Lambda.$$

The vertices which become of incidence number 1 together with their adjacent line are eliminated and this is justified by

$$\int_{z'} p_B \left( \frac{1+\delta}{\Lambda^2}; z', z'' \right) \leq 1. \quad (4.5.25)$$

Hence, we obtain

$$\int_{\mathbb{R}^+} du \mathcal{F}_\delta \left( \Lambda, \left\{ \tau_{2,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2} \right\}; T_{l-1}^{s+2}; z_1, y_{2,s}, u, u \right) \leq O(1) \Lambda \mathcal{F}_\delta \left( \Lambda, \tau_{2,s}; \mathcal{R}_{u,u} T_{l-1}^{s+2}; z_1, y_{2,s} \right),$$

where  $\mathcal{R}_{u,u} T_{l-1}^{s+2}$  is defined in section 3.2. The constant  $O(1)$  accounts for the fact that the reduction map  $\mathcal{R}_{u,u}$  is not injective<sup>1</sup>. Remembering (3.2.7), we deduce

$$\begin{aligned} |R_1| \leq & (\Lambda+m)^{3-n-|w|} \mathcal{P} \left( \log \frac{\Lambda+m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda+m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda+m} \right) \\ & \times \sum_{T_l^s} \mathcal{F}_\delta \left( \Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s} \right). \end{aligned} \quad (4.5.26)$$

<sup>1</sup>The reduction of two different trees in  $\mathcal{T}_{l-1}^{s+2}$  can possibly yield the same reduced tree  $T_l^s$ .

(A<sub>2</sub>) Let us now consider the second term on the r.h.s. of the FE (4.2.7). Without loss of generality, we restrict our analysis to the following term from the symmetrized sum

$$R_2 := \int_{z_{2,n}} \int_{z,z'} \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_R^\Lambda(p; z, z') \\ \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z_{n_1+1}, p_{n_1+1}), \dots, (z_n, p_n), (z', -p)) \phi_{\tau_{2,s}, y_{2,s}}^R(z_{2,n}),$$

in which the arguments  $(z_i, p_i)$  appear ordered in  $\mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}$  and  $\mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}$ . Proceeding as in (A<sub>1</sub>), we rewrite  $R_2$  using (3.1.5) and the notations (4.4.6) as follows

$$R_2 := \int_{\mathbb{R}^+} du \int_{z_{2,n}} \int_{z,z'} \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((\vec{z}_{n_1}, \vec{p}_{n_1}), (z, p)) p_R \left( \frac{1}{2\Lambda^2}; z, u \right) \partial^{w_3} \dot{C}^\Lambda(p) \\ \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((\vec{z}_{n_1+1, n}, \vec{p}_{n_1+1, n}), (z', -p)) \phi_{\tau_{2,s}, y_{2,s}}^R(z_{2,n}) p_R \left( \frac{1}{2\Lambda^2}; u, z' \right).$$

We define

$$\phi'_{s_1(n)}(z_{2, n_1}) = \begin{cases} \prod_{r=2}^{n_1} p_R(\tau_r; z_r, y_r), & s_1(n) = n_1 - 1 \quad \text{if } s \geq n_1 \\ \prod_{r=2}^s p_R(\tau_r; z_r, y_r), & s_1(n) = s - 1 \quad \text{otherwise} \end{cases}$$

and

$$\phi''_{s_2(n)}(z_{n_1+1, n-1}) = \begin{cases} \prod_{r=n_1+1}^s p_R(\tau_r; z_r, y_r), & s_2(n) = s - n_1 \quad \text{if } n_1 < s < n \\ \prod_{r=n_1+1}^{n-1} p_R(\tau_r; z_r, y_r), & s_2(n) = n - n_1 - 1 \quad \text{if } s = n \\ \prod_{r=n_1+1}^n \chi^+(z_r), & s_2(n) = 0 \quad \text{if } s \leq n_1. \end{cases}$$

Then  $R_2$  can be rewritten as

$$\int_{z_n} \int_u \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0} \left( z_1; \vec{p}_{1, n_1}, p; \phi'_{s_1(n)} \times p_R \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \partial^{w_3} \dot{C}^\Lambda(p) \\ \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0} \left( z_n; -p, \vec{p}_{n_1+1, n}; \phi''_{s_2(n)} \times p_R \left( \frac{1}{2\Lambda^2}; u, z' \right) \right) \phi_n(z_n). \quad (4.5.27)$$

Applying the induction hypothesis to both terms in (4.5.27) and using (4.2.3) we obtain

$$|R_2| \leq (\Lambda + m)^{8-n-|w|-2-3} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \times \int_{z'} \int_u \sum_{T_{l_1}^{s_1(n)+1}, T_{l_2}^{s_2(n)+2}} \mathcal{F}_{\delta_1} \left( \Lambda, \left\{ \tau_{2, s_1(n)+1}, \frac{1}{2\Lambda^2} \right\}; T_{l_1}^{s_1(n)+1}; z_1, y_{2, s_1(n)+1}, u \right) \\ \times \mathcal{F}_{\delta_2} \left( \Lambda, \left\{ \tau_{n_1+1, s(n)}, \frac{1}{2\Lambda^2} \right\}; T_{l_2}^{s_2(n)+2}; z_n, y_{n_1+1, s(n)}, u \right) \phi_n(z_n), \quad (4.5.28)$$



with  $s(n) = s$  if  $s < n$ , otherwise  $s(n) = n - 1$ .

Interchanging the integral over  $u$  with the sum over trees we obtain

$$|R_2| \leq (\Lambda + m)^{3-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \times \sum_{T_l^s(T_{l_1}^{s_1(n)+1}, \hat{T}_{l_2}^{s_2(n)+\delta_{s,n+1}})} \mathcal{F}_{\delta_3}(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}),$$

with the following explanations:

- The integration over the root vertex  $z_n$  converts  $z_n$  into an internal vertex. If  $s = n$ , then  $z_n$  is also attached to the external vertex  $y_n$  and the total number of external vertices is then  $s_2(n) + \delta_{s,n} + 1$ . We denote the obtained tree by  $\hat{T}_{l_2}^{s_2(n)+\delta_{s,n+1}}$ . Therefore, using (3.1.9) we write

$$\int_{z_n} \mathcal{F}_{\delta_2}(\Lambda, \tau_{n_1+1,s(n)}; T_{l_2}^{s_2(n)}; z, y_{n_1+1,s(n)}, u) \phi_n(z_n) \\ \leq \max(1, C_{0,\delta}) \mathcal{F}_{\delta_2} \left( \Lambda, \left\{ \tau_{n_1+1,s(n)}, \frac{1}{2\Lambda^2} \right\}; \hat{T}_{l_2}^{s_2(n)+\delta_{s,n+1}}; z_n, y_{n_1+1,s(n)}, u \right). \quad (4.5.29)$$

The number of vertices of incidence number 2 of  $\hat{T}_{l_2}^{s_2(n)+\delta_{s,n+1}}$  is increased at most by 1 by the integration over  $z_n$ <sup>2</sup>. Hence, we have

$$\hat{v}_{2,2} \leq v_{2,2} + \delta_{c(z_n),1} \leq 3l_2 - 2 + \frac{s_2(n) + 2 + \delta_{s,n}}{2}.$$

- Any contribution in the sum over trees  $T_l^s(T_{l_1}^{s_1(n)+1}, \hat{T}_{l_2}^{s_2(n)+\delta_{s,n+1}})$  is obtained from  $T_{l_1}^{s_1(n)+1}$  and  $\hat{T}_{l_2}^{s_2(n)+\delta_{s,n+1}}$  by joining these two trees via the lines going from the vertices  $z'$  and  $z''$  to  $u$ , where  $z'$  and  $z''$  are the vertices attached to  $u$  in the two trees. Using (3.1.5), we obtain

$$\int_u p_B \left( \frac{1 + \delta_1}{2\Lambda^2}; z', u \right) p_B \left( \frac{1 + \delta_2}{2\Lambda^2}; u, z'' \right) \leq O(1) p_B \left( \frac{1 + \delta_3}{\Lambda^2}; z', z'' \right)$$

so that the new internal line has a parameter in the interval  $[\Lambda, \Lambda_0]$  over which the sup is taken in the definition of  $\mathcal{F}$  and  $\delta_3 := \max(\delta_1, \delta_2)$ .

- The total number of vertices  $v_2$  with incidence number 2 in the new tree obtained after integrating out  $u$  is equal to  $v_{2,1} + \hat{v}_{2,2}$  so that

$$v_2 + \delta_{c_1,1} \leq 3l - 4 + \frac{s_1 + s_2(n) + 2 + \delta_{s,n}}{2} \leq 3l - 3 + \frac{s}{2},$$

<sup>2</sup>if and only if initially  $z_n$  is a root vertex of incidence number one.

which proves that  $T_l^s \left( T_{l_1}^{s_1(n)+1}, \hat{T}_{l_2}^{s_2(n)+\delta_{s,n+1}} \right)$  belongs to  $\mathcal{T}_l^s$ . Using the bound (3.1.9), we therefore conclude that

$$|R_2| \leq (\Lambda + m)^{3-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \times \sum_{T_l^s \in \mathcal{T}_l^s} \mathcal{F}_{\delta_3}(\Lambda, \tau_{2,s}; T_l^s; z_1, y_{2,s}) .$$

(B) We consider now the case  $r \neq 0$ :

For the first term on the r.h.s. of the flow equation (4.2.7) the bounds are established exactly as in (A<sub>1</sub>). For the second term we proceed similarly as in (A<sub>2</sub>). We pick a term on the r.h.s.,

$$\int_u \int_{z_2, n} \int_{z, z'} \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0} ((\vec{z}_{n_1}, \vec{p}_{n_1}), (z, p)) \phi_{\tau_{2,s}, y_{2,s}}^R(z_{2,n}) \partial^{w_3} \dot{C}^\Lambda(p) p_R \left( \frac{1}{2\Lambda^2}; z, u \right) \\ \times (z_1 - z_i)^r \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0} ((z_{n_1+1}, p_{n_1+1}), \dots, (z_n, p_n), (z', -p)) p_R \left( \frac{1}{2\Lambda^2}; u, z' \right) .$$

In the case where  $i \leq n_1$  the proof is the same as for  $r = 0$ , up to inserting the modified induction hypothesis for

$$\partial^{w_1} \mathcal{L}_{l_1, n_1+1; r}^{\Lambda, \Lambda_0} \left( z_1; \vec{p}_n; \phi'_{s_1} \times p_R \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \\ = \int_{z_2, n_1} \int_z (z_1 - z_i)^r \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0} ((\vec{z}_{n_1}, \vec{p}_{n_1}), (z, p)) \phi'_{s_1}(z_{2, n_1}) p_R \left( \frac{1}{2\Lambda^2}; z, u \right) .$$

If  $i > n_1$  we assume without restriction  $i = n$  and proceed again as in (A<sub>2</sub>) to obtain the bound

$$(\Lambda + m)^{3-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \times \int_{z_n} \int_u |z_n - z_1|^r \sum_{T_{l_1}^{s_1(n)+1}, T_{l_2}^{s_2(n)+2}} \mathcal{F}_{\delta_1} \left( \Lambda, \left\{ \tau_{2, s_1(n)}, \frac{1}{2\Lambda^2} \right\}; T_{l_1}^{s_1(n)+1}; z_1, y_{2, s_1(n)}, u \right) \\ \times \mathcal{F}_{\delta_2} \left( \Lambda, \left\{ \tau_{s_1(n)+1, s(n)}, \frac{1}{2\Lambda^2} \right\}; T_{l_2}^{s_2(n)+2}; z_n, u, y_{n_1+1}, \dots, y_{s(n)} \right) \phi_n(z_n) .$$

We bound

$$|z_n - z_1| \leq \sum_{a=1}^q |v_a - v_{a-1}| , \quad (4.5.30)$$

where  $\{v_a\}$  are the positions of the internal vertices in the tree  $T_l^s(T_{l_1}^{s_1(n)+1}, \hat{T}_{l_2}^{s_2(n)+\delta_{s,n+1}})$  defined as in (A<sub>2</sub>), on the path joining  $z_1 = v_0$  and  $z_n = v_q$ . Using the inequality (3.1.10) for  $\tau = \frac{1}{\Lambda^2}$ , we obtain

for  $\delta'$  such that  $\delta_2 < \delta' < \frac{1}{2}$

$$|v_a - v_{a-1}|^r p_B \left( \frac{1 + \delta_2}{\Lambda_l^2}; v_a, v_{a-1} \right) \leq O(1) \Lambda^{-r} p_B \left( \frac{1 + \delta'}{\Lambda_l^2}; v_a, v_{a-1} \right), \quad (4.5.31)$$

and this implies (4.5.19) with a new parameter  $\delta' < \frac{1}{2}$ . Note that the cases  $s = n$  and  $s < n$  are treated as in (A<sub>2</sub>).

The previous reasoning holds as well for  $\partial_\Lambda F_{12} \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau_{3,s}, y_{3,s}}^R)$ , where  $z_2$  takes the role of  $z_n$ . After absorbing all constants in  $\mathcal{P}$ , we obtain

$$\begin{aligned} \left| \partial_\Lambda \partial^w \mathcal{L}_{l,n;r}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R) \right| &\leq (\Lambda + m)^{3-n-|w|-r} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta'}^\Lambda(\tau_{2,s}), \end{aligned} \quad (4.5.32)$$

$$\left| \partial_\Lambda F_{12} \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau_{3,s}, y_{3,s}}^R) \right| \leq (\Lambda + m)^{-n} \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{F}_{s,l;\delta'}^{12}(\Lambda, \tau_{3,s}) \quad (4.5.33)$$

and  $\delta' := \max(\delta_1, \delta_2, \delta_3, \delta)$ . The bounds for (4.4.12)-(4.4.16)

$$|\partial_\Lambda c_l^{\Lambda, \Lambda_0}(z_1)| \leq (\Lambda + m)^{-1} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right), \quad (4.5.34)$$

$$|\partial_\Lambda a_l^{\Lambda, \Lambda_0}(z_1)| \leq (\Lambda + m) \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right), \quad (4.5.35)$$

$$|\partial_\Lambda b_l^{\Lambda, \Lambda_0}(z_1)| \leq (\Lambda + m)^{-1} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right), \quad (4.5.36)$$

$$|\partial_\Lambda d_l^{\Lambda, \Lambda_0}(z_1)| \leq (\Lambda + m)^{-1} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right), \quad (4.5.37)$$

$$|\partial_\Lambda s_l^{\Lambda, \Lambda_0}(z_1)| \leq \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \quad (4.5.38)$$

are obtained on restricting the previous considerations to the case  $s = 1$ , in which all the coordinates  $z_{2,n}$  are integrated over with  $n = 2$  or  $n = 4$  and the momenta  $\vec{p}_n$  set to  $\vec{0}$ .

(C) In this part, we bound  $\partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^{R,(j)})$ . As compared to (A), the only case which requires new analysis is the bound on the second term from the r.h.s. of the FE (5.4.1) in the case  $j > s_1$ . Then we assume without restriction, similarly as in (A), that  $j = s$ . The term to be bounded

corresponding to (4.5.28) is then for  $s \geq n_1$

$$\begin{aligned} & (\Lambda + m)^{3-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ & \quad \times \int_u \sum_{T_{l_1}^{n_1+1}, T_{l_2}^{s_2+1}} \mathcal{F}_\delta \left( \Lambda, \left\{ \tau_{2, n_1}, \frac{1}{2\Lambda^2} \right\}; T_{l_1}^{n_1+1}; z_1, y_{2, n_1}, u \right) \\ & \quad \times e^{-\frac{m^2}{2\Lambda^2}} \int_{z_s} \mathcal{F}_\delta \left( \Lambda, \left\{ \tau_{s_1+1, s-1}, \frac{1}{2\Lambda^2} \right\}; T_{l_2}^{s_2+1}; z_s, y_{s_1+1, s-1}, u \right) |p_R^{(1)}(\tau_s; z_s, z_1; y_s)|, \end{aligned} \quad (4.5.39)$$

where  $s_2 := s - n_1 - 1$ . We factorize  $e^{-\frac{m^2}{2\Lambda^2}}$  from the derivative of the flowing propagator  $\dot{C}^\Lambda(p)$  to be able to use the following bound in the sequel

$$(\Lambda + m) e^{-\frac{m^2}{2\Lambda^2}} \leq C \Lambda, \quad C := \|(1+x)e^{-\frac{x^2}{2}}\|_\infty. \quad (4.5.40)$$

To bound (4.5.39) we telescope the difference  $p_R^{(1)}(\tau_s; z_s, z_1; y_s)$  along the tree<sup>3</sup>  $T_l^s(T_{l_1}^{n_1+1}, T_{l_2}^{s_2+1})$  similarly as in (4.5.30). We then have to bound expressions of the type

$$p_B \left( \frac{1+\delta}{\Lambda_I^2}; v_{a-1}, v_a \right) |p_R(\tau_s; v_a, y_s) - p_R(\tau_s; v_{a-1}, y_s)|, \quad (4.5.41)$$

where  $v_{a-1}$  and  $v_a$  are adjacent internal vertices in  $T_l^s(T_{l_1}^{n_1+1}, T_{l_2}^{s_2+1})$  on the unique path from  $z_1$  to  $y_s$ . Taylor expansion of  $p_R(\tau_s; v_a, y_s)$  gives

$$p_R(\tau_s; v_a, y_s) = p_R(\tau_s; v_{a-1}, y_s) + \int_0^1 dt (\partial_t p_R)(\tau_s; tv_{a-1} + (1-t)v_a, y_s).$$

Lemma 6 implies that for all  $0 < \delta' < 1$ , we have

$$|p_R^{(1)}(\tau_s; v_a, v_{a-1}, y_s)| \leq C_{1,\delta} \frac{|v_a - v_{a-1}|}{\sqrt{\tau_s}} \int_0^1 dt p_B(\tau_{s,\delta'}; tv_{a-1} + (1-t)v_a, y_s).$$

Therefore (4.5.41) is bounded by

$$C_{1,\delta} \tau_s^{-\frac{1}{2}} |v_a - v_{a-1}| p_B \left( \frac{1+\delta}{\Lambda_I^2}; v_{a-1}, v_a \right) \int_0^1 dt p_B(\tau_{s,\delta'}; tv_{a-1} + (1-t)v_a, y_s). \quad (4.5.42)$$

Introducing for  $2\delta < 1$ ,

$$b = 2 \frac{1+2\delta}{1-2\delta},$$

we distinguish between the two cases:

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<sup>3</sup>The tree  $T_l^s(T_{l_1}^{n_1+1}, T_{l_2}^{s_2+1})$  is obtained by joining  $T_{l_1}^{n_1+1}$  and  $T_{l_2}^{s_2+1}$ , by suppressing their respective external lines  $(z', u)$  and  $(z'', u)$  and replace them with the internal line  $(z', z'')$ .

- Case (1):  $\delta' \Lambda^2 \leq b \tau_s^{-1}$

Using (4.5.40), we obtain

$$(\Lambda + m)^{3-n-|w|} e^{-\frac{m^2}{2\Lambda^2}} \leq C \left( \frac{b}{\delta'} \right)^{\frac{1}{2}} (\Lambda + m)^{2-n-|w|} \tau_s^{-\frac{1}{2}}.$$

The tree  $T_l^s(T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1})$  is obtained from the two initial trees by joining them via  $u$  and we bound

$$|p_R^{(1)}(\tau_s; z_s, z_1; y_s)| \leq |p_R(\tau_s; z_s, y_s)| + |p_R(\tau_s; z_1, y_s)|. \quad (4.5.43)$$

Here  $p_R(\tau_s; z_1, y_s)$  is associated to  $\phi_s(z_s) \equiv 1$ , and the integration over  $z_s$  in  $T_l^s(T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1})$  is performed similarly as the integration over  $z_n$  in (A<sub>2</sub>). This implies that for  $\delta' \Lambda^2 \leq b \tau_s^{-1}$ , (4.5.39) is bounded by

$$\left( \frac{b}{\delta'} \right)^{\frac{1}{2}} (\Lambda + m)^{2-n-|w|} \tau_s^{-\frac{1}{2}} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta'}^\Lambda(\tau_{2,s}). \quad (4.5.44)$$

- Case (2):  $\delta' \Lambda^2 \geq b \tau_s^{-1}$

Using Lemma 7 we obtain the following bound

$$\begin{aligned} |v_a - v_{a-1}| p_B \left( \frac{1 + \delta}{\Lambda_I^2}; v_{a-1}, v_a \right) \int_0^1 dt p_B(\tau_{s,\delta'}; t v_{a-1} + (1-t)v_a, y_s) \\ \leq C_\delta \Lambda^{-1} p_B \left( \frac{2}{\Lambda_I^2}; v_{a-1}, v_a \right) p_B((1 + \delta')^3 \tau_s; v_{a-1}, y_s), \end{aligned}$$

which implies that (4.5.42) can be bounded using (3.1.5) by

$$C_\delta C_{1,\delta} \Lambda^{-1} \tau_s^{-\frac{1}{2}} \int_v p_B \left( \frac{1}{\Lambda_I^2}; v_a, v \right) p_B \left( \frac{1}{\Lambda_I^2}; v, v_{a-1} \right) p_B((1 + \delta')^3 \tau_s; v_{a-1}, y_s).$$

Choosing  $\delta'$  such that  $(1 + \delta')^3 = 1 + \delta$ , that is  $\delta' = \frac{\delta}{3} + O(\delta^2)$  and using (3.1.9), the final bound obtained for (4.5.41) reads

$$O(1) \tau_s^{-\frac{1}{2}} \Lambda^{-1} \int_v p_B \left( \frac{1 + \delta}{\Lambda_I^2}; v_a, v \right) p_B \left( \frac{1 + \delta}{\Lambda_I^2}; v, v_{a-1} \right) p_B(\tau_{s,\delta}; v_{a-1}, y_s). \quad (4.5.45)$$

In the second case, the bound (4.5.45) implies that the incidence number of the internal vertex  $v_{a-1}$  has increased by one unit in the tree  $T_l^s(T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1})$  by attaching  $v_{a-1}$  to the external vertex  $y_s$ . Furthermore, we integrate over  $z_s$  in (4.5.39). If in the new tree:

- $z_s$  has  $c(z_s) > 1$ , then  $z_s$  takes the role of an internal vertex of the new tree.
- $z_s$  has  $c(z_s) = 1$  we integrate over  $z_s$  using (4.5.25) so that the vertex  $z_s$  disappears.

Note that the case a) implies that  $v_2$ , the number of vertices of incidence number 2 of the new tree is at most increased by one. Namely,  $v_2$  verifies the following bound

$$v_2 + \delta_{c_1,1} \leq v_{2,1} + v_{2,2} + 1 + \delta_{c(z_s),1} + \delta_{c_1,1} \leq 3l - 4 + \frac{s_1 + s_2 + 2}{2} + 1 = 3l - 2 + \frac{s}{2}. \quad (4.5.46)$$

Hence, the new tree belongs to  $\mathcal{T}_l^s$ . As a consequence of the bounds (4.5.44) and (4.5.45), on replacing again  $s \rightarrow j$  we thus obtain for  $n \geq 2$

$$\begin{aligned} \left| \partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^{R,(j)}) \right| &\leq \left( \left( \frac{b}{\delta} \right)^{\frac{1}{2}} + C'_\delta \right) (\Lambda + m)^{2-n-|w|} \tau_j^{-\frac{1}{2}} \\ &\quad \times \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^\Lambda(\tau_{2,s}) . \end{aligned}$$

All the constants are again absorbed in the polynomial  $\mathcal{Q}$ .

(D) To prove (D) we use (4.5.22) and (4.5.27) to show inductively that

$$\partial_\Lambda \partial^w \mathcal{L}_{l,n;r}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R) \quad (4.5.47)$$

and

$$\partial_\Lambda F_{12} \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau_{3,s}, y_{3,s}}^R) \quad (4.5.48)$$

are  $\mathcal{C}^\infty(\mathbb{R}^+)$  w.r.t.  $z_1$ . For (4.5.48), we will integrate from  $\Lambda$  to  $\Lambda_0$  and for (4.5.47) we integrate from 0 to  $\Lambda$  for  $n + |w| + r \leq 4$ , and from  $\Lambda$  to  $\Lambda_0$  for  $n + |w| + r \geq 5$ . The details of these integrations can be deduced from (II,a) and (II,b).

(II) Integration of the FE:

From the bounds on the derivatives  $\partial_\Lambda \partial^w \mathcal{L}_{n,l;r}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R)$  we verify the induction hypothesis by integrating over  $\Lambda$ . In all cases, we need the bound (3.2.61) which we recall

$$\mathcal{F}_{s,l;\delta}^{\Lambda_2}(\tau_{2,s}) \leq \mathcal{F}_{s,l;\delta}^{\Lambda_1}(\tau_{2,s}) \quad \text{for } \Lambda_1 \leq \Lambda_2.$$

a) Irrelevant terms:

Using the boundary conditions (4.4.19), the integration from  $\Lambda$  to  $\Lambda_0$  for  $n + |w| + r \geq 5$  gives

$$\begin{aligned} \left| \partial^w \mathcal{L}_{l,n;r}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R) \right| \\ \leq (\Lambda + m)^{4-n-|w|-r} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^\Lambda(\tau_{2,s}) , \end{aligned}$$

$$\begin{aligned} \left| \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^{R,(j)}) \right| \\ \leq (\Lambda + m)^{3-n-|w|} \tau_j^{-\frac{1}{2}} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^\Lambda(\tau_{2,s}) , \end{aligned}$$

and

$$\left| F_{12} \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R) \right| \leq (\Lambda + m)^{1-n} \mathcal{P}_3 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_4 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{12}(\Lambda, \tau_{3,s}) .$$

(b) Relevant terms:

(b<sub>1</sub>)  $n = 4, w = 0, r = 0$ : We start from the decomposition [\(4.4.11\)](#)

$$\mathcal{L}_{l,4}^{\Lambda, \Lambda_0}(z_1; \vec{0}; \Phi_{4,s}) = c_l^{\Lambda, \Lambda_0}(z_1) \Phi_{4,s}(z_1, z_1, z_1) + l_{l,4}^{\Lambda, \Lambda_0}(z_1, \Phi_{4,s}) , \quad s \leq 4 , \quad (4.5.49)$$

where

$$c_l^{\Lambda, \Lambda_0}(z_1) := \int_{(\mathbb{R}^+)^3} dz_{2,4} \mathcal{L}_{l,4}^{\Lambda, \Lambda_0}((z_1, 0), \dots, (z_4, 0))$$

and

$$\Phi_{4,s}(z_2, z_3, z_4) = \prod_{i=2}^4 \phi_i(z_i), \quad \phi_i(z_i) = p_R(\tau_i; z_i, y_i) \quad \text{if } i \leq s, \quad \text{otherwise } \phi_i \equiv 1 .$$

From the renormalization conditions [\(4.4.20\)](#), we have for all  $l \geq 1$  and  $z_1 \geq 0$

$$c_l^{0, \Lambda_0}(z_1) \equiv 0 .$$

Integrating [\(4.5.34\)](#) from 0 to  $\Lambda$  at zero momenta then gives

$$|c_l^{\Lambda, \Lambda_0}(z_1)| \leq \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) .$$

We decompose the test function

$$\Phi_{4,s}(z_2, z_3, z_4) = \prod_{i=2}^s p_R(\tau_i; z_i, y_i) = \Phi_{4,s}(z_1, z_1, z_1) + \psi(z_2, z_3, z_4) , \quad (4.5.50)$$

where for  $s = 4$

$$\psi(z_2, z_3, z_4) := \sum_{j=2}^4 \prod_{f=2}^{j-1} p_R(\tau_f; z_1, y_f) p_R^{(1)}(\tau_j; z_j, z_1; y_j) \prod_{i=j+1}^4 p_R(\tau_i; z_i, y_i) = \sum_{j=2}^4 \phi_{\tau_{2,s}, y_{2,s}}^{R,(j)}(z_{2,4}) \quad (4.5.51)$$

where we used the notation [\(4.4.3\)](#). Note that if  $\phi_i \equiv 1$  for some  $i$  which corresponds to  $s < 4$ , then the corresponding contribution to the sum vanishes.

Using [\(4.5.50\)](#) and the linearity of  $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}$  w.r.t. to the test functions, we deduce that

$$\mathcal{L}_{l,4}^{\Lambda, \Lambda_0}(z_1; \vec{0}; \Phi_{4,s}) = c_l^{\Lambda, \Lambda_0}(z_1) \Phi_{4,s}(z_1, z_1, z_1) + \mathcal{L}_{l,4}^{\Lambda, \Lambda_0}(z_1; \vec{0}; \psi) .$$

Therefore we have  $l_{l,4}^{\Lambda,\Lambda_0}(z_1; \Phi_{4,s}) = \mathcal{L}_{l,4}^{\Lambda,\Lambda_0}(z_1; \vec{0}; \psi)$ , and hence the FE (4.2.7) provides

$$\begin{aligned} \partial_\Lambda l_{l,4}^{\Lambda,\Lambda_0}(z_1; \Phi_{4,s}) &= \frac{1}{2} \int_{z_2, z_3, z_4} \psi(z_2, z_3, z_4) \\ &\quad \left[ \int_k \mathcal{L}_{l-1,6}^{\Lambda,\Lambda_0}((z, k), (z_1, 0), \dots, (z_4, 0), (z', -k)) \dot{C}_R^\Lambda(k; z, z') \right. \\ &\quad - \frac{1}{2} \sum_{l_1+l_2=l} \sum_{n_1+n_2=4} \left[ \mathcal{L}_{l_1, n_1+1}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_{n_1}, 0), (z, 0)) \dot{C}_R^\Lambda(0; z, z') \right. \\ &\quad \left. \left. \mathcal{L}_{l_2, n_2+1}^{\Lambda,\Lambda_0}((z', 0), \dots, (z_4, 0)) \right]_{rsym} \right]. \end{aligned} \quad (4.5.52)$$

The r.h.s. is a sum over expressions similar to  $\partial_\Lambda \mathcal{L}_{l,4}^{\Lambda,\Lambda_0}(z_1, \phi_{\tau_2,s}^{R,(j)})$  analysed in part (C). We obtain in the same way as there the bound

$$\left| \partial_\Lambda l_{l,4}^{\Lambda,\Lambda_0}(z_1; \Phi_{4,s}) \right| \leq (\Lambda + m)^{-2} \tau^{-\frac{1}{2}} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^\Lambda(\tau_{2,s}).$$

Integrating from  $\Lambda$  to  $\Lambda_0$  and majorizing  $(\lambda + m)^{-1}$  by  $(\Lambda + m)^{-1}$ , we obtain

$$\left| l_{l,4}^{\Lambda,\Lambda_0}(z_1; \Phi_{4,s}) \right| \leq \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^\Lambda(\tau_{2,s}),$$

which gives the bound for  $l_{l,4}^{\Lambda,\Lambda_0}(z_1, \Phi_{4,s})$ . The bound is extended to general momenta using the Taylor formula (4.5.17).

(b<sub>2</sub>)  $n = 2, r = 0, w = 0$ : We start from the decomposition (4.4.9)

$$\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; \vec{0}; \phi_2) = a_l^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) - s_l^{\Lambda,\Lambda_0}(z_1) \phi_2'(z_1) + d_l^{\Lambda,\Lambda_0}(z_1) \phi_2''(z_1) + l_{l,2}^{\Lambda,\Lambda_0}(z_1; \phi_2), \quad (4.5.53)$$

where  $\phi_2(z_2) := p_R(\tau_2; z_2, y_2)$ . Using the renormalization conditions (4.4.20) and integrating (4.5.35)-(4.5.38) from 0 to  $\Lambda$ , we obtain

$$\left| a_l^{\Lambda,\Lambda_0}(z_1) \right| \leq (\Lambda + m)^2 \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right), \quad \left| s_l^{\Lambda,\Lambda_0}(z_1) \right| \leq (\Lambda + m) \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \quad (4.5.54)$$

and

$$\left| d_l^{\Lambda,\Lambda_0}(z_1) \right| \leq \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right). \quad (4.5.55)$$

Since  $|\phi_2'(z_1)| \leq \tau_2^{-\frac{1}{2}} \phi_2(z_1)$  and  $|\phi_2''(z_1)| \leq \tau_2^{-1} \phi_2(z_1)$ , we obtain

$$\left| s_l^{\Lambda,\Lambda_0}(z_1) \phi_2'(z_1) \right| \leq (\Lambda + m)^2 \left( \frac{\tau_2^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \phi_2(z_1)$$



and

$$\left| d_l^{\Lambda, \Lambda_0}(z_1) \phi_2''(z_1) \right| \leq (\Lambda + m)^2 \left( \frac{\tau_2^{-\frac{1}{2}}}{\Lambda + m} \right)^2 \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \phi_2(z_1) .$$

For the irrelevant part of the two-point function we have

$$\partial_\Lambda l_{2,l}^{\Lambda, \Lambda_0}(z_1; \phi_2) = \int_0^\infty dz_2 \int_0^1 dt \frac{(1-t)^2}{2!} \partial_t^3 \phi_2(tz_2 + (1-t)z_1) \partial_\Lambda \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \quad (4.5.56)$$

$$= \int_0^\infty dz_2 \int_0^1 dt \frac{(1-t)^2}{2!} \frac{\partial_t^3 \phi_2(tz_2 + (1-t)z_1)}{(z_2 - z_1)^3} \partial_\Lambda F_{12} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}(z_1, z_2; 0) . \quad (4.5.57)$$

The bound (4.5.33) for  $n = 2$ ,  $r = 0$  yields

$$\begin{aligned} \left| \partial_\Lambda l_{l,2}^{\Lambda, \Lambda_0}(z_1; \phi_2) \right| &\leq (\Lambda + m)^{-2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \\ &\quad \times \int_{z_2} \mathcal{F}_{1,l}^{(12)}(\Lambda; z_1, z_2) \left| \int_0^1 dt \frac{(1-t)^2}{2!} \frac{\partial_t^3 p_R(\tau_2; tz_2 + (1-t)z_1, y_2)}{(z_2 - z_1)^3} \right| . \end{aligned}$$

Using Lemma 6 we obtain

$$\begin{aligned} \left| \partial_\Lambda l_{l,2}^{\Lambda, \Lambda_0}(z_1, \phi_2) \right| &\leq (\Lambda + m)^{-2} \tau_2^{-\frac{3}{2}} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \\ &\quad \times \int_{z_2} \mathcal{F}_{1,l}^{(12)}(\Lambda; z_1, z_2) \int_0^1 dt p_R(\tau_{2,\delta}; tz_2 + (1-t)z_1, y_2) . \end{aligned}$$

Remembering (3.2.39) and (3.2.46) we have

$$\begin{aligned} \mathcal{F}_{1,l}^{(12)}(\Lambda; z_1, z_2) &= \sum_{T_l^{1,(12)}} \mathcal{F}_{1,l}(\Lambda; T_l^{1,(12)}; z_1, z_2) \\ &= \sum_{n=1}^{3l-4} \sup_{\{\Lambda_{I_\nu} | \Lambda \leq \Lambda_{I_\nu} \leq \Lambda_0\}} \left[ \prod_{1 \leq \nu \leq n} \int_{\tilde{z}_\nu} \right] p_B \left( \frac{1+\delta}{\Lambda_{I_1}^2}; z_1, \tilde{z}_1 \right) \cdots p_B \left( \frac{1+\delta}{\Lambda_{I_n}^2}; \tilde{z}_n, z_2 \right) \\ &\leq \sum_{n=1}^{3l-4} \sup_{\{\Lambda_{I_\nu} | \Lambda \leq \Lambda_{I_\nu} \leq \Lambda_0\}} p_B \left( \frac{1+\delta}{\Lambda_n^2}; z_1, z_2 \right) , \end{aligned} \quad (4.5.58)$$

where  $\Lambda_n := (\sum_{\nu=1}^n \Lambda_{I_\nu}^{-2})^{-\frac{1}{2}}$ . The sum (4.5.58) stems from the fact that the double rooted trees have all their internal vertices with incidence number 2. Their number  $v_2$  is constrained by the relation  $v_2 \leq 3l - 4$ .

Using Lemma 7 together with

$$p_R(\tau_{2,\delta'}; tz_2 + (1-t)z_1, y_2) \leq 2 p_B(\tau_{2,\delta'}; tz_2 + (1-t)z_1, y_2) ,$$

we obtain for  $\delta'\Lambda^2 \geq b\tau_2^{-1}$

$$\left| \partial_\Lambda l_{2,l}^{\Lambda,\Lambda_0}(z_1; \phi_2) \right| \leq \tau_2^{-\frac{3}{2}} (\Lambda + m)^{-2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) p_B((1 + \delta')^3 \tau_2; z_1, y_2) . \quad (4.5.59)$$

As before, we choose  $\delta' = \frac{\delta}{3} + O(\delta^2)$  such that  $(1 + \delta')^3 = 1 + \delta$ , which implies that

$$\left| \partial_\Lambda l_{2,l}^{\Lambda,\Lambda_0}(z_1; \phi_2) \right| \leq \tau_2^{-\frac{3}{2}} (\Lambda + m)^{-2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) p_B(\tau_{2,\delta}; z_1, y_2) . \quad (4.5.60)$$

For  $\delta'\Lambda^2 \leq b\tau^{-1}$  we use

$$\begin{aligned} \partial_\Lambda l_{2,l}^{\Lambda,\Lambda_0}(z_1; \phi_2) &= \partial_\Lambda \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2) - \partial_\Lambda a_l^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) \\ &\quad + \partial_\Lambda s_l^{\Lambda,\Lambda_0}(z_1) \phi_2'(z_1) - \partial_\Lambda d_l^{\Lambda,\Lambda_0}(z_1) \phi_2''(z_1) \end{aligned}$$

and the bounds (4.5.32) and (4.5.35) to (4.5.38) to obtain

$$\left| \partial_\Lambda l_{2,l}^{\Lambda,\Lambda_0}(z_1; \phi_2) \right| \leq (\Lambda + m) \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2) . \quad (4.5.61)$$

Since  $\delta'\Lambda^2 \leq b\tau^{-1}$  we have

$$\begin{aligned} \left| \partial_\Lambda l_{2,l}^{\Lambda,\Lambda_0}(z_1; \phi_2) \right| &\leq \max \left( m^3, \tau^{-\frac{3}{2}} \left( \frac{b}{\delta'} \right)^{\frac{3}{2}} \right) (\Lambda + m)^{-2} \\ &\quad \times \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2) . \end{aligned} \quad (4.5.62)$$

Combining (4.5.60) and (4.5.62) and using the bounds (3.1.9), we obtain for all  $\Lambda \geq 0$ ,

$$\begin{aligned} \left| \partial_\Lambda l_{2,l}^{\Lambda,\Lambda_0}(z_1; \phi_2) \right| &\leq \max \left( m^3, \tau^{-\frac{3}{2}} \left( \frac{b}{\delta'} \right)^{\frac{3}{2}} \right) (\Lambda + m)^{-2} \\ &\quad \times \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2) . \end{aligned} \quad (4.5.63)$$

Integrating from  $\Lambda$  to  $\Lambda_0$  using (3.2.61) gives

$$\begin{aligned} \left| l_{2,l}^{\Lambda,\Lambda_0}(z_1, \phi_2) \right| &\leq \max \left( m^3, \tau^{-\frac{3}{2}} \left( \frac{b}{\delta'} \right)^{\frac{3}{2}} \right) (\Lambda + m)^{-1} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2) \\ &\leq \frac{\max \left( m^3, \tau^{-\frac{3}{2}} \left( \frac{b}{\delta'} \right)^{\frac{3}{2}} \right)}{(\Lambda + m)^3} (\Lambda + m)^2 \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2) \\ &\leq (\Lambda + m)^2 \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \tilde{\mathbb{Q}} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2) , \end{aligned}$$

where  $\tilde{\mathbb{Q}}(x) = (1 + x^3)\mathbb{Q}(x)$ . Again, all the constants were absorbed in the coefficients of  $\mathcal{P}$ . This concludes the proof for  $n = 2, r = 0$  and  $w = 0$ .

To establish the bounds on  $\partial^w \mathcal{L}_{l,2;r}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_2)$  for  $r \in \{1, 2\}$ ,  $w = 0$  and  $r = 0, w = 2$ , we expand the respective test functions as follows,

$$\begin{aligned} \mathcal{L}_{l,2;1}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2) &= -s_l^{\Lambda,\Lambda_0}(z_1)\phi_2(z_1) + d_l^{\Lambda,\Lambda_0}(z_1)(\partial_{z_1}\phi_2)(z_1) \\ &\quad + \int_0^\infty dz_2 \int_0^1 dt (1-t)\partial_t^2 \phi_2(tz_2 + (1-t)z_1)(z_1 - z_2) \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{l,2;2}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2) &= d_l^{\Lambda,\Lambda_0}(z_1)\phi_2(z_1) \\ &\quad + \int_0^\infty dz_2 \int_0^1 dt \partial_t \phi_2(tz_2 + (1-t)z_1)(z_1 - z_2)^2 \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \end{aligned}$$

$$\begin{aligned} \left(\partial_{p^2} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}\right)(z_1; 0, 0; \phi_2) &= b_l^{\Lambda,\Lambda_0}(z_1)\phi_2(z_1) \\ &\quad + \int_0^\infty dz_2 \int_0^1 dt \frac{(1-t)^2}{2!} \partial_t^3 \phi_2(tz_2 + (1-t)z_1) \partial_{p^2} \left(\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p))\right) \Big|_{p=0}. \end{aligned}$$

The bounds on the relevant terms are given in (4.5.54)-(4.5.55), and the integration of the remainders is performed in a similar way to the integration of  $\partial_\Lambda \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; \phi_2)$ . We obtain

$$\begin{aligned} \left|\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; \vec{0}; (z_1 - z_2)\phi_2)\right| &\leq (\Lambda + m) \mathcal{P} \left(\log \frac{\Lambda + m}{m}\right) \mathbb{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2), \\ \left|\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; \vec{0}; (z_1 - z_2)^2 \phi_2)\right| &\leq \mathcal{P} \left(\log \frac{\Lambda + m}{m}\right) \mathbb{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2), \\ \left|\partial_p^2 \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; \vec{0}; \phi_2)\right| &\leq \mathcal{P} \left(\log \frac{\Lambda + m}{m}\right) \mathbb{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2). \end{aligned}$$

The extension to general momenta is performed using the Taylor expansion of  $\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; (z_1 - z_2)^r \phi_2)$  for  $r = 0, 1, 2$  w.r.t. the variable  $p \in \mathbb{R}^3$ .

Finally, note that for

$$\phi_2^{(2)}(z_2) = p_R(\tau; z_2, y_2) - p_R(\tau; z_1, y_2) = \phi_2(z_2) - \phi_2(z_1),$$

we have

$$\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2^{(2)}) = s_l^{\Lambda,\Lambda_0}(z_1)(\partial_{z_1}\phi_2)(z_1) + d_l^{\Lambda,\Lambda_0}(z_1)(\partial_{z_1}^2\phi_2)(z_1) + l_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2). \quad (4.5.64)$$

Proceeding again similarly as before - see (4.5.57), (4.5.60) and (4.5.61) - provides

$$\left|\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2^{(2)})\right| \leq (\Lambda + m) \tau_2^{-\frac{1}{2}} \mathcal{P} \left(\log \frac{\Lambda + m}{m}\right) \mathbb{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2).$$

The extension to general momenta is done by Taylor expansion. This ends the proof of Theorem 7.  $\square$

**Proof of Theorem 8**

*Proof.* The proof of Theorem 8 follows the same steps of the previous proof. The case  $(n, l) = (4, 0)$  evidently satisfies the claim (4.5.7). We integrate the system of flow equations (4.2.7) and derive the individual  $n$ -point folded distributions (4.5.1) w.r.t.  $\Lambda_0$ . We denote the r.h.s. of (4.2.7) by  $\partial^w \mathcal{R}_{l,n}^{\Lambda, \Lambda_0}(\vec{z}_n, \vec{p}_n)$ . The relevant and irrelevant terms are bounded separately:

I) Irrelevant terms:  $n + r + |w| > 4$ : For these terms, we use the boundary conditions (4.4.19) which imply

$$-\partial^w \mathcal{L}_{l,n;r}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R) = \int_{\Lambda}^{\Lambda_0} d\lambda \partial^w \mathcal{R}_{l,n;r}^{\lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R). \quad (4.5.65)$$

Hence, differentiating (4.5.65) w.r.t.  $\Lambda_0$  gives

$$\begin{aligned} -\partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n;r}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R) &= \partial^w \mathcal{R}_{l,n;r}^{\Lambda_0, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R) \\ &\quad + \int_{\Lambda}^{\Lambda_0} d\lambda \partial_{\Lambda_0} \partial^w \mathcal{R}_{l,n;r}^{\lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R). \end{aligned} \quad (4.5.66)$$

For the first term on the r.h.s. of (4.5.66), the only contributing term is the non-linear part on the r.h.s. of (4.2.7) because of the boundary condition (4.4.19). Using Theorem 7, we obtain as before the bound

$$\begin{aligned} &\left| \partial^w \mathcal{R}_{l,n;r}^{\Lambda_0, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R) \right| \\ &\leq (\Lambda_0 + m)^{3-n-|w|-r} \mathcal{P} \left( \log \frac{\Lambda_0 + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s}). \end{aligned}$$

Remembering that  $n + |w| + r > 4$ , we deduce that for all  $0 \leq \Lambda \leq \Lambda_0$  we have

$$\begin{aligned} &\left| \partial^w \mathcal{R}_{l,n;r}^{\Lambda_0, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}^R) \right| \\ &\leq \frac{(\Lambda + m)^{5-n-|w|-r}}{(\Lambda_0 + m)^2} \mathcal{P} \left( \log \frac{\Lambda_0 + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s}). \end{aligned}$$

Now, let us treat the second term on the r.h.s. of (4.5.66). We have

$$\begin{aligned}
\partial_{\Lambda_0} \partial^w \mathcal{R}_{l,n}^{\Lambda, \Lambda_0}((\vec{z}_n, \vec{p}_n)) &= \frac{1}{2} \int_{z, z'} \int_k \partial_{\Lambda_0} \partial^w \mathcal{L}_{l-1, n+2}^{\Lambda, \Lambda_0}((\vec{z}_n, \vec{p}_n), (z, k), (z', -k)) \dot{C}_R^\Lambda(p; z, z') \\
&\quad - \frac{1}{2} \int_{z, z'} \sum'_{l_1, l_2} \sum'_{n_1, n_2} \sum_{w_i} c_{w_i} \left[ \partial_{\Lambda_0} \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((\vec{z}_{n_1}, \vec{p}_{n_1}), (z, p)) \partial^{w_3} \dot{C}_R^\Lambda(p; z, z') \right. \\
&\quad \quad \left. \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), (\vec{z}_{n_1+1, n}, \vec{p}_{n_1+1, n})) \right]_{rsym} \\
&\quad - \frac{1}{2} \int_{z, z'} \sum'_{l_1, l_2} \sum'_{n_1, n_2} \sum_{w_i} c_{w_i} \left[ \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((\vec{z}_{n_1}, \vec{p}_{n_1}), (z, p)) \partial^{w_3} \dot{C}_R^\Lambda(p; z, z') \right. \\
&\quad \quad \left. \times \partial_{\Lambda_0} \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), (\vec{z}_{n_1+1, n}, \vec{p}_{n_1+1, n})) \right]_{rsym}, \\
&\quad p = -p_1 - \dots - p_{n_1} = p_{n_1+1} + \dots + p_n,
\end{aligned}$$

where we used that  $\partial_{\Lambda_0} \dot{C}_R^\Lambda(k; z, z') = 0$  and the notations of (4.2.7). Using Theorem 7 and the induction hypothesis (4.5.7), and following the same steps of the proof of Theorem 7 we get

$$\begin{aligned}
&\left| \partial_{\Lambda_0} \partial^w \mathcal{R}_{l, n; r}^{\Lambda, \Lambda_0} \left( z_1; \vec{p}_n; \phi_{\tau_{2, s}, y_{2, s}}^R \right) \right| \\
&\leq \frac{(\Lambda + m)^{4-n-|w|-r}}{(\Lambda_0 + m)^2} \mathcal{P} \left( \log \frac{\Lambda_0 + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l; \delta}^\Lambda(\tau_{2, s}).
\end{aligned}$$

Integrating from  $\Lambda$  to  $\Lambda_0$  and using (3.2.61) together with

$$(\lambda + m)^{-1} \leq (\Lambda + m)^{-1}, \quad \forall \Lambda \leq \lambda, \quad (4.5.67)$$

we obtain a bound on the second term on the r.h.s. of (4.5.66), which is of the type (4.5.7). (4.5.9) and (4.5.10) are proved following the same steps.

II) Relevant terms:  $(n, r, |w|) = (4, 0, 0)$ ,  $(n, r, |w|) = (2, 0, 0)$  and  $(n, r, |w|) = (2, 0, 2)$ :

The FE equation (4.2.7) differentiated w.r.t.  $\Lambda_0$  provides inductive bounds on the relevant parts in these cases. As before, we integrate from 0 to  $\Lambda$  the following bound

$$|\partial_\lambda \partial_{\Lambda_0} c_l^{\lambda, \Lambda_0}(z_1)| \leq (\Lambda_0 + m)^{-2} \mathcal{P} \left( \log \frac{\Lambda_0 + m}{m} \right). \quad (4.5.68)$$

Remembering that the renormalization conditions (4.4.20) are independent from the UV cutoff  $\Lambda_0$ , we obtain

$$|\partial_{\Lambda_0} c_l^{\Lambda, \Lambda_0}(z_1)| \leq \frac{(\Lambda + m)}{(\Lambda_0 + m)^2} \mathcal{P} \left( \log \frac{\Lambda_0 + m}{m} \right). \quad (4.5.69)$$

Proceeding similarly, we deduce the bounds for the remaining relevant terms

$$|\partial_{\Lambda_0} a_l^{\Lambda, \Lambda_0}(z_1)| \leq \frac{(\Lambda + m)^3}{(\Lambda_0 + m)^2} \mathcal{P} \left( \log \frac{\Lambda_0 + m}{m} \right), \quad (4.5.70)$$

$$|\partial_{\Lambda_0} b_l^{\Lambda, \Lambda_0}(z_1)| \leq \frac{(\Lambda + m)}{(\Lambda_0 + m)^2} \mathcal{P} \left( \log \frac{\Lambda_0 + m}{m} \right), \quad (4.5.71)$$

$$|\partial_{\Lambda_0} d_l^{\Lambda, \Lambda_0}(z_1)| \leq \frac{(\Lambda + m)}{(\Lambda_0 + m)^2} \mathcal{P} \left( \log \frac{\Lambda_0 + m}{m} \right), \quad (4.5.72)$$

$$|\partial_{\Lambda_0} s_l^{\Lambda, \Lambda_0}(z_1)| \leq \frac{(\Lambda + m)^2}{(\Lambda_0 + m)^2} \mathcal{P} \left( \log \frac{\Lambda_0 + m}{m} \right). \quad (4.5.73)$$

In the case  $n = 4$ , we use the decomposition (4.5.49) together with (4.4.3), (4.5.10) and (4.5.69) to deduce the bound

$$\begin{aligned} \left| \partial_{\Lambda_0} \partial^w \mathcal{L}_{l,4}^{\Lambda, \Lambda_0}(z_1; \vec{0}; \phi_{\tau_{2,s}, y_{2,s}}^R) \right| &\leq \frac{(\Lambda + m)}{(\Lambda_0 + m)^2} \tilde{\mathcal{P}}_1 \left( \log \frac{\Lambda_0 + m}{m} \right) \\ &\quad \times \tilde{\mathcal{P}}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \tilde{\mathcal{Q}}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s}). \end{aligned} \quad (4.5.74)$$

For  $n = 2$ , we use the decomposition (4.5.53) and follow the same steps as in part b2) of the proof of Theorem 7. Using (4.5.56) and the bound (4.5.9), we obtain for all  $0 < \delta' < 1$ ,

$$\begin{aligned} \partial_{\Lambda_0} l_{2,l}^{\Lambda, \Lambda_0}(z_1; \phi_2) &= \int_{z_2} \int_0^1 dt \frac{(1-t)^2}{2!} \frac{\partial_t^3 \phi_2(tz_2 + (1-t)z_1)}{(z_2 - z_1)^3} \partial_{\Lambda_0} F_{12} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \\ &\leq (\Lambda_0 + m)^{-2} \mathcal{P} \left( \log \frac{\Lambda_0 + m}{m} \right) \int_{z_2} \mathcal{F}_{1,l;\delta}^{12}(\Lambda; z_1, z_2) \int_0^1 dt p_R(\tau_{2,\delta'}; tz_2 + (1-t)z_1, y_2). \end{aligned}$$

Following the same steps as those used before, we obtain

$$\left| \partial_{\Lambda_0} l_{2,l}^{\Lambda, \Lambda_0}(z_1; \phi) \right| \leq \tau_2^{-\frac{3}{2}} (\Lambda_0 + m)^{-2} \mathcal{P} \left( \log \frac{\Lambda_0 + m}{m} \right) p_B(\tau_{2,\delta}; z_1, y_2) \quad \forall \delta' \Lambda^2 \geq b\tau_2^{-1}, \quad (4.5.75)$$

where we choose as before  $\delta' = \delta/3 + O(\delta^2)$ .

For the case  $\delta' \Lambda^2 \leq b\tau_2^{-1}$ , the bound is obtained as in the proof of Theorem 7 by using the decomposition (4.5.53) which yields

$$\begin{aligned} \partial_{\Lambda} \partial_{\Lambda_0} l_{l,2}^{\Lambda, \Lambda_0}(z_1; \phi_2) &= \partial_{\Lambda} \partial_{\Lambda_0} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}(z_1; \vec{0}; \phi_2) - \partial_{\Lambda} \partial_{\Lambda_0} a_l^{\Lambda, \Lambda_0}(z_1) \phi_2(z_1) \\ &\quad + \partial_{\Lambda} \partial_{\Lambda_0} s_l^{\Lambda, \Lambda_0}(z_1) (\partial_{z_1} \phi_2)(z_1) - \partial_{\Lambda} \partial_{\Lambda_0} d_l^{\Lambda, \Lambda_0}(z_1) (\partial_{z_1}^2 \phi_2)(z_1). \end{aligned}$$

Using the induction hypothesis (4.5.7), we obtain as in (4.5.61)

$$\left| \partial_{\Lambda} \partial_{\Lambda_0} l_{l,2}^{\Lambda, \Lambda_0}(z_1; \phi_2) \right| \leq \frac{(\Lambda + m)^2}{(\Lambda_0 + m)^2} \mathcal{P} \left( \log \frac{\Lambda_0 + m}{m} \right) \mathcal{Q} \left( \frac{\tau_2^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda}(\tau_2).$$

Since  $\delta' \Lambda^2 \leq b\tau_2^{-1}$  we find

$$\begin{aligned} & \left| \partial_\Lambda \partial_{\Lambda_0} l_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) \right| \\ & \leq \max\left(m^4, \tau_2^{-2} \left(\frac{b}{\delta'}\right)^2\right) \frac{(\Lambda + m)^{-2}}{(\Lambda_0 + m)^2} \mathcal{P}\left(\log \frac{\Lambda_0 + m}{m}\right) \mathbb{Q}\left(\frac{\tau_2^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2) . \end{aligned} \quad (4.5.76)$$

Integrating from  $\Lambda$  to  $\Lambda_0$  (with  $\Lambda_0$  large enough), we obtain

$$\partial_{\Lambda_0} l_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) = \int_\Lambda^{\frac{b}{\delta'} \sqrt{\tau_2}} d\lambda \partial_\lambda \partial_{\Lambda_0} l_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) + \int_{\frac{b}{\delta'} \sqrt{\tau_2}}^{\Lambda_0} d\lambda \partial_\lambda \partial_{\Lambda_0} l_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) .$$

Using (4.5.76) we obtain

$$\begin{aligned} & \left| \int_\Lambda^{\frac{b\sqrt{\tau_2}}{\delta'}} d\lambda \partial_\lambda \partial_{\Lambda_0} l_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) \right| \\ & \leq \max\left(m^4, \tau_2^{-2} \left(\frac{b}{\delta'}\right)^2\right) \frac{(\Lambda + m)^{-1}}{(\Lambda_0 + m)^2} \mathcal{P}\left(\log \frac{\Lambda_0 + m}{m}\right) \mathbb{Q}\left(\frac{\tau_2^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2) , \end{aligned}$$

which can be bounded by

$$\frac{(\Lambda + m)^3}{(\Lambda_0 + m)^2} \mathcal{P}\left(\log \frac{\Lambda_0 + m}{m}\right) \mathbb{Q}\left(\frac{\tau_2^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2) . \quad (4.5.77)$$

We have

$$\int_{\frac{b}{\delta'} \tau_2^{-\frac{1}{2}}}^{\Lambda_0} d\lambda \partial_\lambda \partial_{\Lambda_0} l_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) = \partial_{\Lambda_0} l_{2,l}^{\frac{b}{\delta'} \tau_2^{-\frac{1}{2}}, \Lambda_0}(z_1, \phi_2) . \quad (4.5.78)$$

Using (4.5.75) for  $\delta' \Lambda^2 \geq b\tau_2^{-1}$ , we deduce that

$$\left| \int_{\frac{b}{\delta'} \tau_2^{-\frac{1}{2}}}^{\Lambda_0} d\lambda \partial_\lambda \partial_{\Lambda_0} l_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) \right| \leq \frac{(\Lambda + m)^3}{(\Lambda_0 + m)^2} \mathcal{P}\left(\log \frac{\Lambda_0 + m}{m}\right) \left(\frac{\tau_2^{-\frac{1}{2}}}{\Lambda + m}\right)^3 \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2) , \quad (4.5.79)$$

(4.5.77) together with (4.5.79) imply that for all  $0 \leq \Lambda \leq \Lambda_0$ , we have

$$\left| \partial_{\Lambda_0} l_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) \right| \leq \frac{(\Lambda + m)^3}{(\Lambda_0 + m)^2} \mathcal{P}\left(\log \frac{\Lambda_0 + m}{m}\right) \mathbb{Q}\left(\frac{\tau_2^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta}^\Lambda(\tau_2) .$$

This concludes the proof for  $n = 2, r = 0$  and  $w = 0$ . The case  $n = 2, r = 0$  and  $w = 2$  is treated similarly. Extension to general momenta is again achieved via the Taylor formula (4.5.17). Note that

compared to the proof of Theorem [7](#), we don't need extra bounds for the cases  $n = 2$ ,  $r = 2$  and  $n = 2$ ,  $r = 1$  since they are not required to close the inductive scheme. The bounds [\(4.5.70\)](#)-[\(4.5.73\)](#) leading to convergence are obtained using only the FE together with the inductive hypotheses [\(4.5.7\)](#) and [\(4.5.8\)](#) in addition to the bound [\(4.5.4\)](#).

Thus, the proof of Theorem [8](#) is complete. □



## Chapter 5

# Multiplicative renormalization of the semi-infinite model: Surface counter-terms

### 5.1 Introduction

In chapter [4](#), we proved the renormalizability of the semi-infinite model. We considered BPHZ renormalization conditions and found that the semi-infinite model is renormalized by adding five position dependent counter-terms to the bare interaction given by

$$\begin{aligned} L^{\Lambda_0, \Lambda_0}(\phi) = & \frac{\lambda}{4!} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^3} d^3x \phi^4(z, x) \\ & + \frac{1}{2} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^3} d^3x \left( a^{\Lambda_0}(z) \phi^2(z, x) - b^{\Lambda_0}(z) \phi(z, x) \Delta_x \phi(z, x) \right. \\ & \left. - d^{\Lambda_0}(z) \phi(z, x) \partial_z^2 \phi(z, x) - s^{\Lambda_0}(z) \phi(z, x) (\partial_z \phi)(z, x) + \frac{2}{4!} c^{\Lambda_0}(z) \phi^4(z, x) \right). \end{aligned} \quad (5.1.1)$$

In the present chapter, we prove that there exists a particular choice of renormalization conditions for which the counter-terms appearing in the effective action are separated into bulk and surface counter-terms, in the sense that the effective action can be written as the sum of counter-terms which are independent of the considered boundary conditions and others denoted by "surface" counter-terms which absorb the divergences that result from the presence of the surface. More precisely, as it appears in [\(2.4.27\)](#)-[\(2.4.29\)](#) the different propagators of the semi-infinite model can be decomposed into a bulk part  $C_B$  which is similar to the translationally invariant  $\phi_4^4$  theory propagator and a surface part  $C_{S,*}$  which accounts for the presence of the surface and depends on the considered boundary condition. The central idea of this chapter is to decompose the correlation distributions into two

parts: a part which is associated to Feynman graphs with  $C_B$  propagators, which we call the "bulk" correlation distributions, and a second part that corresponds to Feynman graphs with at least a surface propagator  $C_{S,*}$ . These will be denoted by the "surface" correlation distributions and are the object of study of this chapter. The aim of this approach is to be able at the end to prove that the counter-terms are position independent, and furthermore relate them to those of the translationally invariant theory. A motivation to our approach is related to the critical behaviour of the semi-infinite scalar field model which was studied extensively in [1,2,22]. The theory of bulk critical phenomena suggests that the number of independent critical exponents should follow directly from the number of independent renormalization functions (i.e.  $Z$  functions). This correspondence between the critical exponents and counter-terms suggests that some of the counter-terms are the same as those which renormalize the translationally invariant theory, while the remaining ones are new counter-terms which result from the presence of the boundary and can be associated to the independent surface critical exponents.

In this chapter, this correspondence is made explicit but only partially and this for the following two reasons: the first reason concerns the surface critical exponent of the ordinary transition (i.e. Dirichlet b.c.). For Robin and Neumann boundary conditions, we establish that two surface counter-terms are needed to make the semi-infinite model finite which correspond to the two surface critical exponents. For Dirichlet boundary conditions, the theory of critical phenomena implies that a single surface exponent which follows from the anomalous dimension of the derivative  $\partial_n\phi(x,0)$ , characterizes the ordinary transition. Following [22] the counter-term corresponding to the Dirichlet surface critical exponent can be retrieved by renormalizing the theory with the insertion of the operator  $\partial_n\phi(x,0)$ . In this work, we do not consider this insertion and we focus our study on the renormalization of the (non-inserted) connected amputated Schwinger (CAS) distributions of the semi-infinite model. We find that no surface counter-term is needed to renormalize the semi-infinite model with Dirichlet boundary conditions. This follows from the fact that the renormalized non-amputated connected Schwinger Dirichlet  $n$ -point functions with a point on the surface vanish. The second point which is missing regarding the correspondence mentioned above, concerns the bulk critical exponents. The latter are associated to the counter-terms which renormalize the translationally invariant scalar field theory. However, in the context of this work, the bulk counter-terms are defined as those needed to renormalize the  $\phi_4^4$  theory in  $\mathbb{R}^4$  with an interaction restricted to the half-space. In chapter [6], we show that this theory is renormalized by adding the translationally invariant  $\phi_4^4$  theory counter-terms and two surface counter-terms which are independent of the half-space boundary conditions. This will complete the proof of the independence of the counter-terms from the position in the space.

The technique of our proof is based on constructing a solution to the flow equation of the semi-infinite model derived in chapter [4] such that the bare interaction has the following form (in the case

of Robin boundary conditions)

$$\begin{aligned}
L_R^{\Lambda_0, \Lambda_0}(\phi) = & \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^3} d^3x \left( \frac{\lambda}{4!} \phi^4(z, x) + \frac{1}{2} a_B^{\Lambda_0}(z) \phi^2(z, x) - \frac{1}{2} s_B^{\Lambda_0}(z) \phi(z, x) \partial_z \phi(z, x) \right. \\
& - \frac{1}{2} b_B^{\Lambda_0}(z) \phi(z, x) \Delta_x \phi(z, x) - \frac{1}{2} d_B^{\Lambda_0}(z) \phi(z, x) \partial_z^2 \phi(z, x) + \frac{1}{4!} c_B^{\Lambda_0}(z) \phi^4(z, x) \Big) \\
& + \int_{\mathbb{R}^3} d^3x \left( \frac{1}{2} s_R^{\Lambda_0} + c e_R^{\Lambda_0} \right) \phi^2(0, x), \quad (5.1.2)
\end{aligned}$$

where  $c$  denotes the Robin parameter associated to Robin boundary conditions.

The chapter is organized as follows. In section [5.2](#), we review the basic setting by recalling some basic properties of the regularized Robin, Neumann and Dirichlet propagators. Section [5.3](#) is devoted to define the scalar field theory in  $\mathbb{R}^4$  with a quartic self-interaction restricted to the half-space  $\mathbb{R}^+ \times \mathbb{R}^3$ . The CAS of this theory obey the standard flow equations of the  $\phi_4^4$ -theory in  $\mathbb{R}^4$ , with the exception that the  $z$ ,  $z'$  integrations appearing on the RHS of the flow equations are restricted to  $\mathbb{R}^+$  instead of the full space  $\mathbb{R}$ . The bare interaction corresponding to this theory reads

$$\begin{aligned}
L_B^{\Lambda_0, \Lambda_0}(\phi) = & \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^3} d^3x \left( \frac{\lambda}{4!} \phi^4(z, x) + \frac{1}{2} a_B^{\Lambda_0}(z) \phi^2(z, x) - \frac{1}{2} s_B^{\Lambda_0}(z) \phi(z, x) \partial_z \phi(z, x) \right. \\
& \left. - \frac{1}{2} b_B^{\Lambda_0}(z) \phi(z, x) \Delta_x \phi(z, x) - \frac{1}{2} d_B^{\Lambda_0}(z) \phi(z, x) \partial_z^2 \phi(z, x) + \frac{1}{4!} c_B^{\Lambda_0}(z) \phi^4(z, x) \right),
\end{aligned}$$

where  $a_B^{\Lambda_0}(z)$ ,  $s_B^{\Lambda_0}(z)$ ,  $b_B^{\Lambda_0}(z)$ ,  $d_B^{\Lambda_0}(z)$  and  $c_B^{\Lambda_0}(z)$  are the bulk counter-terms which can depend (smoothly) on  $z$  since the interaction breaks translation invariance. In section [5.4](#), we construct the surface correlation distributions  $\mathcal{S}_{l,n;\star}^{\Lambda, \Lambda_0}$  associated to the boundary condition  $\star$ . Section [5.5](#) is the central part of this chapter. We present Theorem [9](#) which contains the power counting for the connected amputated Schwinger distributions (CAS)  $\mathcal{S}_{l,n;\star}^{\Lambda, \Lambda_0}$  as well as their boundedness w.r.t. to  $\Lambda_0$ . Then, Proposition [7](#) proves that the Dirichlet surface correlation distributions can be viewed as the limit of Robin surface correlation distributions when the Robin parameter  $c$  is taken to infinity. Theorem [9](#) together with Proposition [6](#) imply Corollary [4](#) which states that the Dirichlet surface correlation distributions when folded with Dirichlet heat kernels are irrelevant. In section [5.6](#), we explain how the minimal form [\(5.1.2\)](#) of the bare interaction is deduced from Theorem [9](#). First order calculations in perturbation theory [\[44, 45\]](#) suggest that the amputated theory is renormalized differently w.r.t. the non-amputated one in the sense that the tadpole needs more counter-terms depending on whether one of its external points is on the surface or not. We explain this in more detail to all orders of perturbation theory in section [5.7](#). In section [5.8](#), we collected technical lemmas which we use in the proof of Theorem [9](#).

## 5.2 The regularized propagators

We denote by  $\star$  the type of boundary conditions considered. For  $0 \leq \Lambda \leq \Lambda_0$ , we recall the regularized flowing propagator associated to the boundary condition  $\star \in \{D, N, R\}$  given by:

$$C_{\star}^{\Lambda, \Lambda_0}(p; z, z') := \int_{\frac{1}{\Lambda_0}^2}^{\frac{1}{\Lambda^2}} d\lambda p_{\star}(\lambda; z, z') e^{-\lambda(p^2+m^2)}. \quad (5.2.1)$$

(5.2.1) can be rewritten as

$$C_{\star}^{\Lambda, \Lambda_0}(p; z, z') = C_B^{\Lambda, \Lambda_0}(p; z, z') + C_{S, \star}^{\Lambda, \Lambda_0}(p; z, z'), \quad (5.2.2)$$

where

$$C_B^{\Lambda, \Lambda_0}(p; z, z') := \int_{\frac{1}{\Lambda_0}^2}^{\frac{1}{\Lambda^2}} d\lambda p_B(\lambda; z, z') e^{-\lambda(p^2+m^2)} \quad (5.2.3)$$

and

$$C_{S, \star}^{\Lambda, \Lambda_0}(p; z, z') := \int_{\frac{1}{\Lambda_0}^2}^{\frac{1}{\Lambda^2}} d\lambda p_{S, \star}(\lambda; z, z') e^{-\lambda(p^2+m^2)}, \quad (5.2.4)$$

with the surface heat kernel  $p_{S, \star}$  defined as  $p_{\star} - p_B$  and  $p_B$  defined by (3.1.1). In the case of Robin boundary conditions, the surface Robin heat kernel is given by

$$p_{S, R}\left(\frac{1}{\Lambda^2}; z, z'\right) := p_B\left(\frac{1}{\Lambda^2}; z, -z'\right) - 2 \int_0^{\infty} dw e^{-w} p_B\left(\frac{1}{\Lambda^2}; z, -\frac{w}{c} - z'\right). \quad (5.2.5)$$

Note that the Robin heat kernel and  $p_{S, R}$  are uniformly bounded w.r.t. the Robin parameter  $c$ . Namely, we have using (3.1.12)

$$p_R(\tau; z, z') \leq 4 p_B(\tau; z, z'), \quad p_{S, R}(\tau; z, z') \leq 3 p_B(\tau; z, -z'), \quad (5.2.6)$$

for all  $z, z' \geq 0$ ,  $\tau > 0$  and  $c \geq 0$ . Similarly, we also have

$$p_D(\tau; z, z') \leq p_N(\tau; z, z') \leq 2 p_B(\tau; z, z'). \quad (5.2.7)$$

In the sequel, we denote the derivative of the flowing propagators w.r.t.  $\Lambda$  by

$$\dot{C}_{\bullet}^{\Lambda}(p; z, z') = \frac{\partial}{\partial \Lambda} C_{\bullet}^{\Lambda, \Lambda_0}(p; z, z') = \dot{C}^{\Lambda}(p) p_{\bullet}\left(\frac{1}{\Lambda^2}; z, z'\right), \quad (5.2.8)$$

where  $\dot{C}^{\Lambda}(p) = -\frac{2}{\Lambda^3} e^{-\frac{p^2+m^2}{\Lambda^2}}$  and  $\bullet \in \{\star, \{S, \star\}, B\}$  with  $\star \in \{D, N, R\}$ .

## 5.3 The bulk theory on the half-space $\mathbb{R}^+ \times \mathbb{R}^3$

### 5.3.1 The Action and the Flow Equations

We consider the theory of a real scalar field  $\phi$  with mass  $m$  on the four dimensional Euclidean spacetime  $\mathbb{R}^4$ . The regularized flowing propagator is given by (5.2.3). Note that for  $\Lambda \rightarrow 0$  and  $\Lambda_0 \rightarrow \infty$  we recover the unregularized propagator. As in section 1.1.3 we define the theory rigorously from the functional integral

$$e^{-\frac{1}{\hbar}(L_B^{\Lambda, \Lambda_0}(\phi) + I^{\Lambda, \Lambda_0})} := \int d\mu_B^{\Lambda, \Lambda_0}(\Phi) e^{-\frac{1}{\hbar}L_B^{\Lambda_0, \Lambda_0}(\Phi + \phi)}, \quad (5.3.1)$$

$$L_B^{\Lambda, \Lambda_0}(0) = 0.$$

Here, the Gaussian measure  $d\mu_B^{\Lambda, \Lambda_0}$  is of mean zero and covariance  $\hbar C_B^{\Lambda, \Lambda_0}$ . The test function  $\phi$  is in the support of  $d\mu_B^{\Lambda, \Lambda_0}$  which in particular implies that it is in  $\mathcal{C}^\infty(\mathbb{R}^4)$ . This regularity stems from the UV-regularization determined by the cutoff  $\Lambda_0$ .

The functional  $L_B^{\Lambda_0, \Lambda_0}(\phi)$  is the bare interaction of a renormalizable theory including counter-terms, viewed as a formal power series in  $\hbar$ . It contains the tree order interaction and the related counter-terms. The interaction is supported only on the half-space  $\mathbb{R}^+ \times \mathbb{R}^3$  which implies that translation invariance is broken in the  $z$ -direction (the semi-line). This implies that the counter-terms may be  $z$ -dependent. In general, the constraints on the bare action result from the symmetry properties of the theory which are imposed, on its field content and on the form of the propagator. It is therefore natural to consider the general bare interaction

$$L_B^{\Lambda_0, \Lambda_0}(\phi) = \frac{\lambda}{4!} \int_z \int_{\mathbb{R}^3} d^3x \phi^4(z, x) + \frac{1}{2} \int_z \int_{\mathbb{R}^3} d^3x \left( a_B^{\Lambda_0}(z) \phi^2(z, x) - b_B^{\Lambda_0}(z) \phi(z, x) \Delta_x \phi(z, x) \right. \\ \left. - d_B^{\Lambda_0}(z) \phi(z, x) \partial_z^2 \phi(z, x) - s_B^{\Lambda_0}(z) \phi(z, x) (\partial_z \phi)(z, x) + \frac{2}{4!} c_B^{\Lambda_0}(z) \phi^4(z, x) \right).$$

Here we supposed the theory to be symmetric under  $\phi \rightarrow -\phi$ , and we included only relevant terms in the sense of the renormalization group. The functions  $a_B^{\Lambda_0}(z)$ ,  $b_B^{\Lambda_0}(z)$ ,  $c_B^{\Lambda_0}(z)$ ,  $d_B^{\Lambda_0}(z)$  and  $s_B^{\Lambda_0}(z)$  are supposed to be smooth.

The flow equation (FE) is obtained from (5.3.1) on differentiating w.r.t.  $\Lambda$ , see section 1.1.3 and [11, 30, 46]. It is a differential equation for the functional  $L_B^{\Lambda, \Lambda_0}$ :

$$\partial_\Lambda (L_B^{\Lambda, \Lambda_0} + I^{\Lambda, \Lambda_0}) = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \phi}, \dot{C}_B^\Lambda \frac{\delta}{\delta \phi} \right\rangle L_B^{\Lambda, \Lambda_0} - \frac{1}{2} \left\langle \frac{\delta}{\delta \phi} L_B^{\Lambda, \Lambda_0}, \dot{C}_B^\Lambda \frac{\delta}{\delta \phi} L_B^{\Lambda, \Lambda_0} \right\rangle. \quad (5.3.2)$$

We expand the functional  $L_B^{\Lambda, \Lambda_0}(\phi)$  in a formal power series w.r.t.  $\hbar$ ,

$$L_B^{\Lambda, \Lambda_0}(\phi) = \sum_{l=0}^{\infty} \hbar^l L_{l, B}^{\Lambda, \Lambda_0}(\phi).$$

Corresponding expansions for  $a_B^{\Lambda_0}(z)$ ,  $b_B^{\Lambda_0}(z)$ ..., are  $a_B^{\Lambda_0}(z) = \sum_{l=1}^{\infty} \hbar^l a_{l,B}^{\Lambda_0}(z)$ , etc. From  $L_{l,B}^{\Lambda,\Lambda_0}(\phi)$  we obtain the CAS distributions of loop order  $l$  as

$$\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}((z_1, x_1), \dots, (z_n, x_n)) := \delta_{\phi(z_1, x_1)} \cdots \delta_{\phi(z_n, x_n)} L_{l,B}^{\Lambda,\Lambda_0} |_{\phi=0},$$

where we used the notation  $\delta_{\phi(z,x)} = \delta / \delta \phi(z, x)$ .

In the  $pz$ -representation, we set for  $r$ ,  $r_1$  and  $r_2 \in \mathbb{N}^*$

$$\mathcal{D}_{l,n;r}^{\Lambda,\Lambda_0;(i)}(z_1; \vec{p}_n; \Phi_n) := \int_{(\mathbb{R}^+)^{n-1}} dz_{2,n} (z_1 - z_i)^r \mathcal{D}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \phi_2(z_2) \cdots \phi_n(z_n), \quad (5.3.3)$$

$$\begin{aligned} \mathcal{D}_{l,n;r_1,r_2}^{\Lambda,\Lambda_0;(i,j)}(z_1; \vec{p}_n; \Phi_n) &:= \int_{(\mathbb{R}^+)^{n-1}} dz_{2,n} (z_1 - z_i)^{r_1} (z_1 - z_j)^{r_2} \mathcal{D}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \\ &\times \phi_2(z_2) \cdots \phi_n(z_n), \quad r_1 + r_2 = r, \end{aligned} \quad (5.3.4)$$

and for  $r = 0$

$$\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \Phi_n) := \int_{(\mathbb{R}^+)^{n-1}} dz_{2,n} \mathcal{D}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \phi_2(z_2) \cdots \phi_n(z_n). \quad (5.3.5)$$

Here we denote

$$\Phi_n(z_2, \dots, z_n) := \prod_{i=2}^n \phi_i(z_i)$$

and

$$\delta^{(3)}(p_1 + \cdots + p_n) \mathcal{D}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) = (2\pi)^{3(n-1)} \frac{\delta^n}{\delta \phi(z_1, p_1) \cdots \delta \phi(z_n, p_n)} L_{l,B}^{\Lambda,\Lambda_0}(\phi) |_{\phi=0}.$$

The distribution  $\delta^{(3)}(p_1 + \cdots + p_n)$  appears because of translation invariance in the  $x$  directions. The FE for the CAS distributions derived from (5.3.2) are

$$\begin{aligned} &\partial_\Lambda \partial^w \mathcal{D}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \\ &= \frac{1}{2} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^+} dz' \int_k \partial^w \mathcal{D}_{l-1,n+2}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \dot{C}_B^\Lambda(k; z, z') \\ &- \frac{1}{2} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^+} dz' \sum_{l_1, l_2} \sum_{n_1, n_2} \sum_{w_i} c_w \left[ \partial^{w_1} \mathcal{D}_{l_1, n_1+1}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_B^\Lambda(p; z, z') \right. \\ &\quad \left. \times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda,\Lambda_0}((z', -p), \dots, (z_n, p_n)) \right]_{rsym}, \\ &p = -p_1 - \cdots - p_{n_1} = p_{n_1+1} + \cdots + p_n. \end{aligned} \quad (5.3.6)$$

For the notations, see after (1.1.18).

### 5.3.2 Test functions and boundary conditions

The  $n$ -point correlation "functions"  $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$  when considered in the  $pz$ -representation are tempered distributions which belong for fixed  $\vec{p}_n$  to the space  $\mathcal{S}'(\mathbb{R}^{+n})$  w.r.t. the semi-norms

$$\prod_{i=1}^n \mathcal{N}_2(\phi_i) ,$$

where  $\mathcal{N}_2(\phi) := \sup_{0 \leq \alpha, \beta \leq 2} \|(1+z^\beta) \partial_z^\alpha \phi(z)\|_\infty$  and  $\partial_z \phi|_{z=0} = \lim_{z \rightarrow 0^+} \partial_z \phi$ . Our method of proof relies on inductive bounds deduced from the flow equations (5.3.6). The induction restricts our choice of the test functions. To proceed inductively we cannot admit any arbitrary test function in  $\mathcal{S}(\mathbb{R}^{+n})$ . Let us give the set of test functions we will be using in the sequel: For  $2 \leq s \leq n$ , we define

$$\tau := \inf \tau_{2,s}, \quad \text{where } \tau_{2,s} = (\tau_2, \dots, \tau_s) \text{ with } \tau_i > 0 ,$$

and similarly  $z_{2,s} = (z_2, \dots, z_s)$ . Given  $(y_2, \dots, y_s) \in \mathbb{R}^{s-1}$ , we define

$$\phi_{\tau_{2,s}, y_{2,s}}(z_{2,s}) := \prod_{i=2}^s p_B(\tau_i; z_i, y_i) \prod_{i=s+1}^n \chi^+(z_i) , \quad (5.3.7)$$

where  $\chi^+$  is the characteristic function of the semi-line  $\mathbb{R}^+$ . This definition can be generalized by choosing any other subset of  $s-1$  coordinates among  $z_2, \dots, z_n$ . The characteristic functions  $\chi^+$  are introduced in order to be able to extract the relevant terms in the sense of the renormalization group from the full  $n$ -point distributions and to get inductive control of the local counter terms. Following section 4.4.2, the relevant terms contained in

$$\mathcal{D}_{l,2}^{\Lambda,\Lambda_0}(z_1; \vec{0}; \phi_2) := \int_{z_2} \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \phi_2(z_2) \quad (5.3.8)$$

and

$$\mathcal{D}_{l,4}^{\Lambda,\Lambda_0}(z_1; \vec{0}; \Phi_4) := \int_{z_{2,4}} \mathcal{D}_{l,4}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_4, 0)) \prod_{i=2}^4 \phi_i(z_i) , \quad (5.3.9)$$

are extracted by using a Taylor expansion of the test functions  $\phi_2$  and  $\Phi_4$ , which gives

$$\begin{aligned} \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2) &= a_{l,B}^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) - s_{l,B}^{\Lambda,\Lambda_0}(z_1) (\partial_{z_1} \phi_2)(z_1) \\ &\quad - d_{l,B}^{\Lambda,\Lambda_0}(z_1) (\partial_{z_1}^2 \phi_2)(z_1) + l_{l,2;B}^{\Lambda,\Lambda_0}(z_1; \phi_2) , \end{aligned} \quad (5.3.10)$$

$$\left( \partial_{p^2} \mathcal{D}_{l,2}^{\Lambda,\Lambda_0} \right) (z_1; 0, 0; \phi_2) = b_{l,B}^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) + \left( \partial_{p^2} l_{l,2}^{\Lambda,\Lambda_0} \right) (z_1; \phi_2) , \quad (5.3.11)$$

$$\mathcal{D}_{l,4}^{\Lambda,\Lambda_0}(z_1; 0, \dots, 0; \Phi_4) = c_{l,B}^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) \phi_3(z_1) \phi_4(z_1) + l_{l,4;B}^{\Lambda,\Lambda_0}(z_1; \Phi_4) . \quad (5.3.12)$$

Then the relevant terms appear as

$$a_{l,B}^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (5.3.13)$$

$$s_{l,B}^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 (z_1 - z_2) \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (5.3.14)$$

$$d_{l,B}^{\Lambda,\Lambda_0}(z_1) = -\frac{1}{2} \int_0^\infty dz_2 (z_1 - z_2)^2 \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (5.3.15)$$

$$b_{l,B}^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 \partial_{p^2} \left( \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)) \right) \Big|_{p=0}, \quad (5.3.16)$$

$$c_{l,B}^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 dz_3 dz_4 \mathcal{D}_{l,4}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_4, 0)), \quad (5.3.17)$$

and the remainders  $l_{l,2,B}^{\Lambda,\Lambda_0}(z_1; \phi_2)$ ,  $(\partial_{p^2} l_{l,2,B}^{\Lambda,\Lambda_0})(z_1; \phi_2)$  and  $l_{l,4,B}^{\Lambda,\Lambda_0}(z_1; \Phi_4)$  can be written as

$$l_{l,2,B}^{\Lambda,\Lambda_0}(z_1; \phi_2) = \int_0^\infty dz_2 \int_0^1 dt \frac{(1-t)^2}{2!} \partial_t^3 \phi_2(tz_2 + (1-t)z_1) \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (5.3.18)$$

$$(\partial_{p^2} l_{l,2,B}^{\Lambda,\Lambda_0})(z_1; \phi_2) = \int_0^\infty dz_2 \int_0^1 dt \partial_t \phi_2(tz_2 + (1-t)z_1) \partial_{p^2} \left( \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)) \right) \Big|_{p=0}$$

and

$$\begin{aligned} & l_{l,4,B}^{\Lambda,\Lambda_0}(z_1; \Phi_4) \\ &= \int_{(\mathbb{R}^+)^3} dz_{2,4} \mathcal{D}_{l,4}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_4, 0)) \left[ \int_0^1 dt \partial_t \phi_2(tz_2 + (1-t)z_1) \phi_3(z_3) \phi_4(z_4) \right. \\ & \left. + \phi_2(z_1) \int_0^1 dt \partial_t \phi_3(tz_3 + (1-t)z_1) \phi_4(z_4) + \phi_2(z_1) \phi_3(z_1) \int_0^1 dt \partial_t \phi_4(tz_4 + (1-t)z_1) \right]. \quad (5.3.19) \end{aligned}$$

### Boundary conditions at $\Lambda = \Lambda_0$ :

The bare interaction implies that at  $\Lambda = \Lambda_0$

$$\begin{aligned} \mathcal{D}_{l,2}^{\Lambda_0,\Lambda_0}((z_1, p), (z_2, -p)) &= \left( a_{l,B}^{\Lambda_0}(z_1) + b_{l,B}^{\Lambda_0}(z_1) p^2 - s_{l,B}^{\Lambda_0}(z_1) \partial_{z_1} - d_{l,B}^{\Lambda_0}(z_1) \partial_{z_1}^2 \right) \delta(z_1 - z_2), \\ \mathcal{D}_{l,4}^{\Lambda_0,\Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) &= \left( \lambda \delta_{l,0} + c_{l,B}^{\Lambda_0}(z_1) (1 - \delta_{l,0}) \right) \prod_{i=2}^4 \delta(z_1 - z_i). \\ \mathcal{D}_{l,n}^{\Lambda_0,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) &= 0, \quad \forall n \geq 5. \end{aligned} \quad (5.3.20)$$

### Renormalization conditions at $\Lambda = 0$ (BPHZ renormalization conditions):

The renormalization conditions are fixed at  $\Lambda = 0$  by imposing for all  $z_1 \geq 0$

$$a_{l,B}^{0,\Lambda_0}(z_1) \equiv 0, \quad s_{l,B}^{0,\Lambda_0}(z_1) \equiv 0, \quad d_{l,B}^{0,\Lambda_0}(z_1) \equiv 0, \quad b_{l,B}^{0,\Lambda_0}(z_1) \equiv 0, \quad c_{l,B}^{0,\Lambda_0}(z_1) \equiv 0. \quad (5.3.21)$$



These will be adopted in the following. Note that the boundary conditions are invariant under the three-dimensional Euclidean group.

We will need the following result, which we do not prove. The proof can be performed following the same steps as in the proof of Theorem [7](#) in chapter [4](#).

**Proposition 6.** *For  $0 \leq \Lambda \leq \Lambda_0 < \infty$ ,  $1 \leq s \leq n$ ,  $2 \leq i \leq n$  and  $0 \leq r \leq 3$ , we consider test functions of the form  $\phi_{\tau_{2,s}, y_{2,s}}(z_{2,s})$ , which are also denoted in shorthand as  $\phi_{\tau_{2,s}, y_{2,s}}$ .*

Adopting [\(5.3.20\)](#)-[\(5.3.21\)](#) we claim

$$\begin{aligned} & \left| \partial^w \mathcal{D}_{l,n;r}^{\Lambda, \Lambda_0; (i)}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}) \right| \\ & \leq (\Lambda + m)^{4-n-|w|-r} \mathcal{P}_1 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s}), \end{aligned} \quad (5.3.22)$$

$$\begin{aligned} & \left| \partial^w \mathcal{D}_{l,n;r_1,r_2}^{\Lambda, \Lambda_0; (i,j)}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}) \right| \\ & \leq (\Lambda + m)^{4-n-|w|-r_1-r_2} \mathcal{P}'_1 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}'_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q}'_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s}), \end{aligned} \quad (5.3.23)$$

and  $\partial^w \mathcal{D}_{l,n;r}^{\Lambda, \Lambda_0; (i)}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}})$  is continuous w.r.t.  $z_1$ . Here,  $\mathcal{P}_i$  and  $\mathbb{Q}_i$  denote polynomials with non-negative coefficients which depend on  $l, n, |w|, r$ , but not on  $\{p_i\}$ ,  $\Lambda$ ,  $\Lambda_0$  and  $z_1$ . The polynomials  $\mathbb{Q}_i$  are reduced to a constant if  $s = 1$ , and for  $l = 0$  all polynomials  $\mathcal{P}_i$  and  $\mathbb{Q}_i$  reduce to constants.

The weight factors  $\mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s})$  are defined in Section [3.2](#).

## 5.4 The Surface correlation distributions

### 5.4.1 The semi-infinite theory

In this subsection, we recall the flow equation of the semi-infinite massive scalar field model presented in chapter [4](#).

$$\begin{aligned} & \partial_{\Lambda} \partial^w \mathcal{L}_{l,n;\star}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \\ & = \frac{1}{2} \int_z \int_{z'} \int_k \partial^w \mathcal{L}_{l-1, n+2; \star}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \dot{C}_{\star}^{\Lambda}(k; z, z') \\ & - \frac{1}{2} \int_z \int_{z'} \sum_{l_1, l_2}^l \sum_{n_1, n_2}^l \sum_{w_i} c_{w_i} \left[ \partial^{w_1} \mathcal{L}_{l_1, n_1+1; \star}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_{\star}^{\Lambda}(p; z, z') \right. \\ & \quad \left. \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1; \star}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \right]_{rsym}, \\ & p = -p_1 - \dots - p_{n_1} = p_{n_1+1} + \dots + p_n. \end{aligned} \quad (5.4.1)$$

where  $C_{\star}^{\Lambda, \Lambda_0}(p; z, z')$  is defined in (5.2.2) and  $\mathcal{L}_{l, n; \star}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n))$  denote the semi-infinite correlation distributions at loop order  $l$  and with  $n$  external points. The  $\star$  index refers to the type of considered boundary conditions, namely Dirichlet, Neumann and Robin. In chapter 4, we imposed the following mixed boundary conditions:

- At  $\Lambda = \Lambda_0$ :

$$\begin{aligned} \mathcal{L}_{l, 2; \star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) &= \left( a_{l; \star}^{\Lambda_0}(z_1) + b_{l; \star}^{\Lambda_0}(z_1)p^2 - s_{l; \star}^{\Lambda_0}(z_1)\partial_{z_1} - d_{l; \star}^{\Lambda_0}(z_1)\partial_{z_1}^2 \right) \delta(z_1 - z_2), \\ \mathcal{L}_{l, 4; \star}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) &= \lambda \delta_{l, 0} + c_{l; \star}^{\Lambda_0}(z_1)(1 - \delta_{l, 0}) \prod_{i=2}^4 \delta(z_1 - z_i). \\ \mathcal{L}_{l, n; \star}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) &= 0, \quad \forall n \geq 5. \end{aligned} \quad (5.4.2)$$

- At  $\Lambda = 0$ : We impose BPHZ type renormalization conditions. Namely, for all  $z_1 \geq 0$  we set

$$a_{l; \star}^{0, \Lambda_0}(z_1) \equiv 0, \quad s_{l; \star}^{0, \Lambda_0}(z_1) \equiv 0, \quad d_{l; \star}^{0, \Lambda_0}(z_1) \equiv 0, \quad b_{l; \star}^{0, \Lambda_0}(z_1) \equiv 0, \quad c_{l; \star}^{0, \Lambda_0}(z_1) \equiv 0, \quad (5.4.3)$$

where

$$a_{l; \star}^{\Lambda, \Lambda_0}(z_1) = \int_0^\infty dz_2 \mathcal{L}_{l, 2; \star}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)), \quad (5.4.4)$$

$$s_{l; \star}^{\Lambda, \Lambda_0}(z_1) = \int_0^\infty dz_2 (z_1 - z_2) \mathcal{L}_{l, 2; \star}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)), \quad (5.4.5)$$

$$d_{l; \star}^{\Lambda, \Lambda_0}(z_1) = -\frac{1}{2} \int_0^\infty dz_2 (z_1 - z_2)^2 \mathcal{L}_{l, 2; \star}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)), \quad (5.4.6)$$

$$b_{l; \star}^{\Lambda, \Lambda_0}(z_1) = \int_0^\infty dz_2 \partial_{p^2} \left( \mathcal{L}_{l, 2; \star}^{\Lambda, \Lambda_0}((z_1, p), (z_2, -p)) \right) \Big|_{p=0}, \quad (5.4.7)$$

$$c_{l; \star}^{\Lambda, \Lambda_0}(z_1) = \int_0^\infty dz_2 dz_3 dz_4 \mathcal{L}_{l, 4; \star}^{\Lambda, \Lambda_0}((z_1, 0), \dots, (z_4, 0)). \quad (5.4.8)$$

This yielded five position dependent counter-terms which appear in the bare interaction of our semi-infinite model

$$\begin{aligned} L_{\star}^{\Lambda_0, \Lambda_0}(\phi) &= \frac{\lambda}{4!} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^3} d^3x \phi^4(z, x) + \frac{1}{2} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^3} d^3x \left( a_{\star}^{\Lambda_0}(z) \phi^2(z, x) \right. \\ &\quad \left. - b_{\star}^{\Lambda_0}(z) \phi(z, x) \Delta_x \phi(z, x) - d_{\star}^{\Lambda_0}(z) \phi(z, x) \partial_z^2 \phi(z, x) - s_{\star}^{\Lambda_0}(z) \phi(z, x) (\partial_z \phi)(z, x) + \frac{2}{4!} c_{\star}^{\Lambda_0}(z) \phi^4(z, x) \right). \end{aligned} \quad (5.4.9)$$

In chapter 4, we saw that imposing constant renormalization conditions w.r.t. the position  $z$  at the scale  $\Lambda = 0$  is at the expense of obtaining position dependent counter-terms. In this chapter, we would like to reverse the process in the sense that we aim to obtain position independent counter-terms by

transferring the position dependent part to the renormalization conditions. Our strategy is based on extracting the surface counter-terms from the semi-infinite counter-terms by separating the bulk and the surface effects. Concretely, we proceed by subtracting the bulk correlation distributions defined in Section 5.3 from the semi-infinite correlation distributions and study the behaviour of the difference to which we refer as the surface correlation distributions. Namely, we write

$$\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)) := \mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)) - \mathcal{D}_{l,n}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)). \quad (5.4.10)$$

The definition (5.4.10) allows to write the FE verified by  $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ , which we give explicitly in the next subsection.

### 5.4.2 The surface correlation distributions

Before getting to the mathematical definition of the surface correlation distributions, let us give a brief motivation of our approach based on a diagrammatic approach to the renormalization problem of the semi-infinite scalar field model. The propagator associated to the b.c.  $\star$  can be decomposed into a sum of the two contributions given in (5.2.2), where  $C_B^{\Lambda,\Lambda_0}$  is the regularized bulk propagator which is responsible for the singularities arising from coalescing of points and  $C_{S,\star}^{\Lambda,\Lambda_0}$  is the part which is responsible of singularities arising when a point approaches the surface. Therefore, an arbitrary Feynman diagram of the semi-infinite model can be written as the sum of a diagram which contains only bulk internal lines consisting of propagators  $C_B^{\Lambda,\Lambda_0}$  only, and other diagrams which contain at least one surface internal line given by the propagator  $C_{S,\star}^{\Lambda,\Lambda_0}$ . Renormalizing the massive semi-infinite model then amounts to renormalizing the diagrams with only bulk internal lines and those with at least a surface internal line. This approach has the advantage to disentangle the surface divergences from the bulk divergences. From the renormalization group point of view, we proceed similarly by writing (5.4.10) with  $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$  (resp.  $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ ) consisting of all connected amputated diagrams with  $n$  external legs and  $l$  loops involving exclusively  $C_B^{\Lambda,\Lambda_0}$  (resp.  $C_B^{\Lambda,\Lambda_0}$  and at least one  $C_{S,\star}^{\Lambda,\Lambda_0}$ ). Using the flow equations (5.3.6) and (5.4.1), we obtain the flow equation verified by the surface correlation

distributions  $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n))$  :

$$\begin{aligned}
& \partial_\Lambda \partial^w \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)) \\
&= \frac{1}{2} \int_{z,z'} \int_k \partial^w \mathcal{S}_{l-1,n+2;\star}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n), (z, k), (z', -k)) \dot{C}_\star^\Lambda(k; z, z') \\
&+ \frac{1}{2} \int_{z,z'} \int_k \partial^w \mathcal{D}_{l-1,n+2}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n), (z, k), (z', -k)) \dot{C}_{S,\star}^\Lambda(k; z, z') \\
&- \frac{1}{2} \int_{z,z'} \sum_{l_1, l_2}^l \sum_{\pi_1, \pi_2}^{\prime\prime} \sum_{w_i} c_w \\
&\left[ \partial^{w_1} \mathcal{S}_{l_1, n_1+1;\star}^{\Lambda,\Lambda_0}((\vec{z}_{\pi_1}, \vec{p}_{\pi_1}), (z, p)) \partial^{w_3} \dot{C}_\star^\Lambda(p; z, z') \partial^{w_2} \mathcal{S}_{l_2, n_2+1;\star}^{\Lambda,\Lambda_0}((z', -p), (\vec{z}_{\pi_2}, \vec{p}_{\pi_2})) \right. \\
&+ \partial^{w_1} \mathcal{D}_{l_1, n_1+1}^{\Lambda,\Lambda_0}((\vec{z}_{\pi_1}, \vec{p}_{\pi_1}), (z, p)) \partial^{w_3} \dot{C}_\star^\Lambda(p; z, z') \partial^{w_2} \mathcal{S}_{l_2, n_2+1;\star}^{\Lambda,\Lambda_0}((z', -p), (\vec{z}_{\pi_2}, \vec{p}_{\pi_2})) \\
&+ \partial^{w_1} \mathcal{S}_{l_1, n_1+1;\star}^{\Lambda,\Lambda_0}((\vec{z}_{\pi_1}, \vec{p}_{\pi_1}), (z, p)) \partial^{w_3} \dot{C}_\star^\Lambda(p; z, z') \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda,\Lambda_0}((z', -p), (\vec{z}_{\pi_2}, \vec{p}_{\pi_2})) \\
&\left. + \partial^{w_1} \mathcal{D}_{l_1, n_1+1}^{\Lambda,\Lambda_0}((\vec{z}_{\pi_1}, \vec{p}_{\pi_1}), (z, p)) \partial^{w_3} \dot{C}_{S,\star}^\Lambda(p; z, z') \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda,\Lambda_0}((z', -p), (\vec{z}_{\pi_2}, \vec{p}_{\pi_2})) \right], \\
& \quad p = -p_1 - \dots - p_{n_1} = p_{n_1+1} + \dots + p_n, \tag{5.4.11}
\end{aligned}$$

where we used the shorthand notation  $(\vec{z}_{\pi_i}, \vec{p}_{\pi_i}) = ((z_j, p_j))_{j \in \pi_i}$ . The prime restricts the summations to  $l_1 + l_2 = l$  and the double prime to the partitions  $(\pi_1, \pi_2) \in \tilde{\mathcal{P}}_{2;n}$ . For the tree order  $l = 0$  we have

$$\mathcal{S}_{0,4;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) = 0. \tag{5.4.12}$$

The existence of  $\mathcal{S}_{l,n}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n))$  is ensured by [\(5.4.12\)](#) and by the flow equation [\(5.4.11\)](#) through induction in  $n + 2l$  and in  $l$  for fixed  $n + 2l$ .

### 5.4.3 Boundary and renormalization conditions

For  $\phi_1$  and  $\phi_2$  in  $\mathcal{S}(\mathbb{R}^+)$ , the relevant terms are contained in

$$\mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}(0,0) := \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \phi_1(z_1) \phi_2(z_2). \tag{5.4.13}$$

They are extracted from [\(5.4.13\)](#) by performing a Taylor expansion of the test functions  $\phi_1$  and  $\phi_2$  around 0 which gives [□](#)

$$\begin{aligned}
\mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}(0,0) &= s_{l;\star}^{\Lambda,\Lambda_0} \phi_1(0) \phi_2(0) + e_{l;\star}^{\Lambda,\Lambda_0} \phi_1(0) (\partial_n \phi_2)(0) + h_{l;\star}^{\Lambda_0} \phi_2(0) (\partial_n \phi_1)(0) \\
&\quad + l_{l,2;\star}^{\Lambda,\Lambda_0}(\phi_1, \phi_2). \tag{5.4.14}
\end{aligned}$$

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<sup>1</sup> $(\partial_n \phi)(0) = \lim_{z \rightarrow 0} (\partial_z \phi)(z)$

Then the relevant terms  $s_{l;\star}^{\Lambda,\Lambda_0}$ ,  $e_{l;\star}^{\Lambda,\Lambda_0}$  and  $h_{l;\star}^{\Lambda,\Lambda_0}$  are obtained as

$$\begin{aligned} s_{l;\star}^{\Lambda,\Lambda_0} &:= \int_{z_1, z_2} \mathcal{S}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), & e_{l;\star}^{\Lambda,\Lambda_0} &:= \int_{z_1, z_2} z_2 \mathcal{S}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \\ h_{l;\star}^{\Lambda,\Lambda_0} &:= \int_{z_1, z_2} z_1 \mathcal{S}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)). \end{aligned} \quad (5.4.15)$$

Bose symmetry implies that

$$\int_{z_1, z_2} z_2 \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) = \int_{z_1, z_2} z_1 \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (5.4.16)$$

so that the counter-terms  $e_{l;\star}^{\Lambda,\Lambda_0}$  and  $h_{l;\star}^{\Lambda,\Lambda_0}$  are equal to all orders of perturbation theory. The remainder  $l_{l,2;\star}^{\Lambda,\Lambda_0}(\phi_1, \phi_2)$  has the form

$$\begin{aligned} l_{l,2;\star}^{\Lambda,\Lambda_0}(\phi_1, \phi_2) &= \left( \int_{z_1, z_2} z_1 z_2 \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \right) (\partial_n \phi_1)(0) (\partial_n \phi_2)(0) \\ &+ \phi_1(0) \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_2)(tz_2) \\ &+ \phi_2(0) \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \\ &+ (\partial_n \phi_1)(0) \int_{z_1, z_2} z_1 \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_2)(tz_2) \\ &+ (\partial_n \phi_2)(0) \int_{z_1, z_2} z_2 \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \\ &+ \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \left( \int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \right) \times \left( \int_0^1 dt' (1-t') (\partial_{t'}^2 \phi_2)(t'z_2) \right). \end{aligned} \quad (5.4.17)$$

In the sequel, we use the following notations. For  $\star \in \{R, N, D\}$ , we write

$$\begin{aligned} \partial^w \mathcal{S}_{l,n;\star;r_1,r_2}^{\Lambda_0,\Lambda_0}(\vec{p}_n; \phi_{\tau_1,s,y_1,s}) &:= \int_{z_1, \dots, z_n} z_1^{r_1} z_2^{r_2} \partial^w \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i), \\ \partial^w \mathcal{S}_{l,n;\star;r_1,r_2}^{\Lambda_0,\Lambda_0}(\vec{p}_n; \phi_{\tau_1,s,y_1,s}^\star) &:= \int_{z_1, \dots, z_n} z_1^{r_1} z_2^{r_2} \partial^w \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \prod_{i=1}^s p_\star(\tau_i; z_i, y_i). \end{aligned}$$

The boundary conditions imposed on  $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$  are the following:

- At  $\Lambda = \Lambda_0$ , we impose for  $\star \in \{R, N\}$

$$\mathfrak{S}_{l,2;\star}^{\Lambda_0,\Lambda_0}((z_1, p), (z_2, -p)) = s_{l;\star}^{\Lambda_0,\Lambda_0} \delta_{z_1} \delta_{z_2} + e_{l;\star}^{\Lambda_0,\Lambda_0} (\delta_{z_1} \delta'_{z_2} + \delta'_{z_1} \delta_{z_2}) , \quad \forall l \geq 1 , \quad (5.4.18)$$

$$\begin{aligned} \mathfrak{S}_{0,2;\star}^{\Lambda_0,\Lambda_0}((z_1, p), (z_2, -p)) &= 0 , \\ \mathfrak{S}_{l,n;\star}^{\Lambda_0,\Lambda_0}((\vec{z}_n, \vec{p}_n)) &= 0 , \quad \forall n \geq 4, \forall l \geq 0 . \end{aligned} \quad (5.4.19)$$

- At  $\Lambda = 0$ , we fix the renormalization conditions for  $\star \in \{R, N\}$  as

$$s_{l;\star}^{0,\Lambda_0} = 0, \quad e_{l;\star}^{0,\Lambda_0} = 0 . \quad (5.4.20)$$

- For Dirichlet boundary conditions we impose

$$\mathfrak{S}_{l,n;D}^{\Lambda_0,\Lambda_0}((\vec{z}_n, \vec{p}_n)) = 0, \quad \forall n \geq 2, \forall l \geq 0 . \quad (5.4.21)$$

**Remark 2.** - The boundary conditions (5.4.18)-(5.4.21) together with the flow equation (5.4.11) and the tree order (5.4.12) define uniquely the surface correlation distributions

$$\mathfrak{S}_{l,n;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)), \quad \star \in \{D, R, N\} .$$

This can be verified inductively by taking the difference of two solutions of the flow equation which obey the same boundary conditions (5.4.18)-(5.4.21) and by proving to all orders of perturbation theory that this difference vanishes.

- We would like to emphasize w.r.t. (5.4.10) that we do not require any a priori knowledge on the semi-infinite correlation distributions  $\mathfrak{L}_{l,n;\star}^{\Lambda,\Lambda_0}$  to give a meaning to  $\mathfrak{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ . The flow equation (5.4.11) together with the bulk correlation distributions defined in Section 5.3, the tree order (5.4.12) and the boundary conditions (5.4.18)-(5.4.21) are sufficient to define uniquely the surface correlation distributions. The relation (5.4.10) implies the flow equation to be verified by  $\mathfrak{S}_{l,n;\star}^{\Lambda,\Lambda_0}$  such that the sum

$$\mathfrak{D}_{l,n}^{\Lambda,\Lambda_0} + \mathfrak{S}_{l,n;\star}^{\Lambda,\Lambda_0}$$

is a solution to the FE (5.4.1).

## 5.5 Results and Proofs

Our main result is summarized in the following theorem:

**Theorem 9.** (Boundedness) Let  $0 \leq \Lambda \leq \Lambda_0 < \infty$  and  $(r_i, s) \in \mathbb{N}^2$  such that  $0 \leq r_i \leq 4$  and  $0 \leq s \leq n$ . For  $Y_{\sigma_s} \in \mathbb{R}^s$ ,  $\star \in \{R, N\}$  and adopting (5.4.18)-(5.4.20) we claim

$$\begin{aligned} \left| \partial^w \mathcal{S}_{l,n;\star;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s}, y_{1,s}}) \right| &\leq (\Lambda + m)^{3-n-r_1-r_2-|w|} \mathcal{P}_1 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \mathbb{Q}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}), \quad \forall n \geq 2. \end{aligned} \quad (5.5.1)$$

Here and subsequently  $\mathcal{P}_i$  and  $\mathbb{Q}_i$  denote polynomials with non-negative coefficients which depend on  $l, n, |w|, \delta, r_1, r_2$ , but not on  $\{p_i\}$ ,  $\Lambda, \Lambda_0, m$  and  $c$ . The polynomial  $\mathbb{Q}_i$  is reduced to a constant for  $s = 1$ , and for  $l = 0$  all polynomials  $\mathcal{P}_i$  reduce to constants. The parameter  $\delta$  depends on the loop order  $l$  and verifies  $0 < \delta_l \leq \delta_{l+1} < 1$ .

As a consequence of Theorem 9, we have:

**Proposition 7.** For fixed  $0 \leq \Lambda \leq \Lambda_0 < \infty$ ,  $\tau > 0$  and  $(y_1, \dots, y_n) \in (\mathbb{R}^+)^n$ , we have:

$$\mathcal{S}_{l,n;D}^{\Lambda,\Lambda_0} \left( \vec{p}_n; \prod_{i=1}^n p_D(\tau_i; \cdot, y_i) \right) = \lim_{c \rightarrow +\infty} \mathcal{S}_{l,n;R}^{\Lambda,\Lambda_0} \left( \vec{p}_n; \prod_{i=1}^n p_R(\tau_i; \cdot, y_i) \right), \quad (5.5.2)$$

where the parameter  $c$  denotes the Robin parameter.

**Corollary 4.** For Dirichlet boundary conditions, adopting (5.4.21) and the assumptions of Proposition 7 we have

$$\begin{aligned} \left| \mathcal{S}_{l,n;D}^{\Lambda,\Lambda_0} \left( \vec{p}_n; \prod_{i=1}^n p_D(\tau_i; \cdot, y_i) \right) \right| \\ \leq (\Lambda + m)^{3-n} \mathcal{P}_3 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_4 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q}_2 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{n,l;\delta}^{\Lambda,0}(\tau_{1,n}), \quad \forall n \geq 4, \end{aligned} \quad (5.5.3)$$

and for  $n = 2$  we have

$$\begin{aligned} \left| \mathcal{S}_{l,2;D}^{\Lambda,\Lambda_0} \left( p, -p; \prod_{i=1}^2 p_D(\tau_i; \cdot, y_i) \right) \right| \\ \leq (\Lambda + m)^{-1} \tau_1^{-\frac{1}{2}} \tau_2^{-\frac{1}{2}} \mathcal{P}_5 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_6 \left( \frac{|p|}{\Lambda + m} \right) \mathbb{Q}_3 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}). \end{aligned} \quad (5.5.4)$$

**Theorem 10.** (Convergence) Let  $0 \leq \Lambda \leq \Lambda_0 < \infty$ . Using the same notations, conventions and adopting the same renormalization conditions (5.4.18)-(5.4.21) as in Theorem 9 and Proposition 7, we

have the following bounds

$$(A) \left| \partial_{\Lambda_0} \partial^w \mathcal{S}_{l,n;\star;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s},y_{1,s}}) \right| \leq \frac{(\Lambda+m)^{4-n-|w|-r_1-r_2}}{(\Lambda_0+m)^2} \tilde{\mathcal{P}}_1 \left( \log \frac{\Lambda_0+m}{m} \right) \tilde{\mathcal{P}}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda+m} \right) \\ \times \tilde{\mathcal{Q}}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda+m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}), \quad \forall n+|w|+r_1+r_2 \geq 2, \star \in \{N, R\}. \quad (5.5.5)$$

$$(B) \left| \partial_{\Lambda_0} \mathcal{S}_{l,n;D}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n},y_{1,n}}^D) \right| \\ \leq \frac{(\Lambda+m)^{4-n}}{(\Lambda_0+m)^2} \tilde{\mathcal{P}}_1 \left( \log \frac{\Lambda_0+m}{m} \right) \tilde{\mathcal{P}}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda+m} \right) \tilde{\mathcal{Q}}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda+m} \right) \mathcal{F}_{n,l;\delta}^{\Lambda,0}(\tau_{1,n}), \quad \forall n \geq 4. \quad (5.5.6)$$

$$(C) \left| \partial_{\Lambda_0} \mathcal{S}_{l,2;D}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n},y_{1,n}}^D) \right| \\ \leq \tau^{-1} (\Lambda_0+m)^{-2} \tilde{\mathcal{P}}_1 \left( \log \frac{\Lambda_0+m}{m} \right) \tilde{\mathcal{P}}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda+m} \right) \tilde{\mathcal{Q}}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda+m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}). \quad (5.5.7)$$

**Remarks 2.** • There are two differences between the Robin/Neumann case (5.5.1) and the Dirichlet case (5.5.3)-(5.5.4): The boundary conditions (5.4.21) for  $\mathcal{S}_{l,n;D}^{\Lambda,\Lambda_0}$  are imposed at scale  $\Lambda = \Lambda_0$  only, whereas for  $\mathcal{S}_{l,n;\star;r_1,r_2}^{\Lambda,\Lambda_0}$  we impose mixed boundary conditions (5.4.18)-(5.4.20). The second difference concerns the type of test functions considered, which in the case of Dirichlet are product of Dirichlet heat kernels (i.e.  $\prod_{i=1}^n p_D(\tau_i; z_i, y_i)$ ), whereas in the case of Robin and Neumann b.c. the test functions are product of bulk heat kernels and characteristic functions of the semi-lines (i.e.  $\prod_{i=1}^s p_B(\tau_i; z_i, y_i) \prod_{i=s+1}^n \chi^+(z_i)$ ).

- The bounds (5.5.1) and (5.5.3)-(5.5.4) can be established by induction separately using the associated flow equation. For the Dirichlet boundary conditions, the associated flow equation are integrated from  $\Lambda$  to  $\Lambda_0$ . For the Robin/Neumann cases, the flow equation is integrated from 0 to  $\Lambda$  for the relevant terms using the boundary condition (5.4.20) and from  $\Lambda$  to  $\Lambda_0$  for the irrelevant terms using the boundary condition (5.4.18).
- Adopting the boundary conditions (5.3.20)-(5.3.21) together with (5.4.18)-(5.4.20), the distributions  $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$  and  $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$  are uniquely defined as the solutions of the flow equation (5.3.6) and (5.4.11). Furthermore, their sum

$$\mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0} = \mathcal{D}_{l,n}^{\Lambda,\Lambda_0} + \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}, \quad (5.5.8)$$



is the unique solution of the flow equation (5.4.1) such that  $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$  and  $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$  obey respectively (5.3.20)-(5.3.21) and (5.4.18)-(5.4.20). Theorem 1 together with Proposition 6 gives for  $s \geq 1$

$$\begin{aligned} & \left| \partial^w \mathcal{L}_{l,n;r,\star}^{\Lambda,\Lambda_0} \left( \vec{p}_n; \prod_{i=1}^s p_B(\tau_i; \cdot, y_i) \right) \right| \\ & \leq \left\{ (\Lambda + m)^{4-n-|w|-r} \hat{\mathcal{F}}_{s,l;\delta}^{\Lambda}(\tau_{1,s}) + (\Lambda + m)^{3-n-|w|-r} \mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}) \right\} \\ & \quad \times \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right), \quad (5.5.9) \end{aligned}$$

where

$$\begin{aligned} & \partial^w \mathcal{L}_{l,n;r,\star}^{\Lambda,\Lambda_0} \left( \vec{p}_n; \prod_{i=1}^s p_B(\tau_i; \cdot, y_i) \right) \\ & := \int_{z_1, \dots, z_n} (z_1 - z_2)^r \partial^w \mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) \end{aligned}$$

and  $\hat{\mathcal{F}}_{s,l;\delta}^{\Lambda}(\tau_{1,s}) := \int_{z_1} p_B(\tau_1; z_1, y_1) \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s})$ .

The bound (5.5.9) implies that  $\partial^w \mathcal{L}_{l,n;r,\star}^{\Lambda,\Lambda_0}(\vec{p}_n; \prod_{i=1}^s p_B(\tau_i; \cdot, y_i))$  are bounded uniformly w.r.t.  $\Lambda_0$ . It is also possible to deduce a convergence Theorem which implies the existence of the limit  $\Lambda \rightarrow 0$  and  $\Lambda_0 \rightarrow \infty$  for  $\partial^w \mathcal{L}_{l,n;r,\star}^{\Lambda,\Lambda_0}$  which we do not explicit here.

- We do not prove Theorem 10 since there is no novelty in the proof, which is mainly based on combining arguments from the proof of Theorem 9 with the steps of the proof of the convergence Theorem 8 in chapter 4.
- The difference between  $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0} + \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ , and  $\mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0}$  studied in chapter 4, is their distributional structure, in the sense that one can prove inductively using the FE (5.4.11) and the boundary conditions (5.4.18)-(5.4.19) that

$$\begin{aligned} \mathcal{D}_{l,n}^{\Lambda,\Lambda_0} \left( z_1; \vec{0}; \phi_{\tau_{2,s}, y_{2,s}} \right) + \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0} \left( z_1; \vec{0}; \phi_{\tau_{2,s}, y_{2,s}} \right) &= a_{l,n;\star}^{\Lambda,\Lambda_0}(z_1, y_{2,s}, \tau_{2,s}) + b_{l,n;\star}^{\Lambda,\Lambda_0}(y_{2,s}, \tau_{2,s}) \delta_{z_1} \\ & \quad + c_{l,n;\star}^{\Lambda,\Lambda_0}(y_{2,s}, \tau_{2,s}) \delta'_{z_1}, \quad (5.5.10) \end{aligned}$$

where  $a_{l,n;\star}^{\Lambda,\Lambda_0}$  is smooth w.r.t.  $z_1$  and  $a_{l,n;\star}^{\Lambda,\Lambda_0}$ ,  $b_{l,n;\star}^{\Lambda,\Lambda_0}$  and  $c_{l,n;\star}^{\Lambda,\Lambda_0}$  are smooth w.r.t.  $y_{2,s}$  and  $\tau_{2,s}$ . However, the semi-infinite correlation distributions  $\mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0}$  considered in chapter 4 are smooth w.r.t.  $z_1$  which is a consequence of the type of mixed b.c.s imposed on the semi-infinite correlation distributions.

- Note that if the bulk correlation distributions obey the bound

$$\begin{aligned} & \left| \partial^w \mathcal{D}_{l,n;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s},y_{1,s}}) \right| \\ & \leq \left\{ (\Lambda + m)^{3-n-r_1-r_2-|w|} \mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}) + (\Lambda + m)^{4-n-r_1-r_2-|w|} \hat{\mathcal{F}}_{s,l;\delta}^{\Lambda}(\tau_{1,s}) \right\} \\ & \quad \times \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right), \quad \forall n \geq 2, \quad \forall s \geq 1 \end{aligned} \quad (5.5.11)$$

instead of (5.3.22)-(5.3.23), the bound (5.5.1) still holds.

- The bound (5.5.1) holds also for the surface correlation distributions folded with  $\star$  heat kernels (i.e.  $\star \in \{N, R\}$ ), that is

$$\begin{aligned} \left| \partial^w \mathcal{S}_{l,n;\star;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s},y_{1,s}}^\star) \right| & \leq (\Lambda + m)^{3-n-r_1-r_2-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ & \quad \times \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}), \quad \forall n \geq 2, \end{aligned} \quad (5.5.12)$$

where the external points  $y_{1,s}$  belong to  $(\mathbb{R}^+)^s$ . This is a direct consequence of (3.1.3)-(3.1.4) together with the bounds (3.1.12). In particular, the bound (5.5.12) implies that  $\mathcal{S}_{l,n;R;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau,y_{1,s}}^R)$  is uniformly bounded w.r.t. the Robin parameter  $c$ , with the same assumptions on the polynomials as in (5.5.1).

### 5.5.1 Proof of Theorem 9

The bound (5.5.1) and (5.5.3)-(5.5.4) are established inductively using the standard inductive scheme described in the proof of Theorem 7. We proceed in a similar way by bounding first the RHS of the flow equation (5.4.11) and then integrating the irrelevant terms from  $\Lambda$  to  $\Lambda_0$  using the boundary conditions (5.4.18)-(5.4.19) and from 0 to  $\Lambda$  using (5.4.20).

*Proof.* We establish the proof in the case of the Robin boundary conditions. For the Neumann boundary conditions, we proceed similarly. In the sequel, we omit the subscript  $R$  from  $\mathcal{S}_{l,n;R}^{\Lambda,\Lambda_0}$ . The induction starts at the tree order for which we have

$$\mathcal{S}_{0,4}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) = 0,$$

and the bound (5.5.1) obviously holds.

**The right-hand side of the FE**

The bounds that we want to obtain for the RHS of the flow equation (5.4.11) are of the form

$$\begin{aligned} & \left| \partial_\Lambda \partial^w \mathfrak{S}_{l,n;r_1,r_2}^{\Lambda,\Lambda_0} (\vec{p}_n; \phi_{\tau_{1,s},y_{1,s}}) \right| \\ & \leq (\Lambda + m)^{2-n-|w|-r_1-r_2} \mathcal{P}_1 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda,0} (\tau_{1,s}) , \end{aligned} \quad (5.5.13)$$

for all  $n \geq 2$ ,  $0 \leq s \leq n$  and  $0 \leq r_1, r_2 \leq 4$ . In the sequel, we drop the lower indices from the polynomials  $\mathcal{P}_1$ ,  $\mathcal{P}_2$  and  $\mathcal{Q}_1$ . But one should keep in mind that these polynomials, whenever they appear, may have different positive coefficients which depend on  $l, n, |w|, \delta_l$  only and not on  $\{p_i\}$ ,  $\Lambda$ ,  $\Lambda_0$ ,  $\tau_{1,s}$  and the Robin parameter  $c$ .

The bound (5.5.13) is established by bounding each of the terms on the RHS of the FE (5.4.11). We consider first the case  $r_1 = r_2 = 0$ .

- We start by treating the linear terms  $R_1^D$  and  $R_1^S$  given by

$$\begin{aligned} R_1^S := & \int_{z,z'} \int_{z_{1,n}} \int_k \partial^w \mathfrak{S}_{l-1,n+2}^{\Lambda,\Lambda_0} ((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \\ & \times \dot{C}^\Lambda(k) p_R \left( \frac{1}{\Lambda^2}; z, z' \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) \end{aligned} \quad (5.5.14)$$

and

$$\begin{aligned} R_1^D := & \int_{z,z'} \int_{z_{1,n}} \int_k \partial^w \mathfrak{D}_{l-1,n+2}^{\Lambda,\Lambda_0} ((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \\ & \times \dot{C}^\Lambda(k) p_{S,R} \left( \frac{1}{\Lambda^2}; z, z' \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) , \end{aligned} \quad (5.5.15)$$

where  $p_R$  and  $p_{S,R}$  are given by (3.1.4) and (5.2.5). First, we bound  $R_1^S$ . Using the decomposition of the Robin heat kernel (3.1.4), we obtain that  $R_1^S$  can be written as the sum of three contributions such that for each contribution the Robin heat kernel  $p_R$  in  $R_1^S$  is replaced by a term from the decomposition (3.1.4). We analyze first the term

$$\begin{aligned} \tilde{R}_1^S := & \int_{z,z'} \int_{z_{1,n}} \int_k \partial^w \mathfrak{S}_{l-1,n+2}^{\Lambda,\Lambda_0} ((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \\ & \times \dot{C}^\Lambda(k) p_B \left( \frac{1}{\Lambda^2}; z, z' \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) . \end{aligned}$$

Using the semi-group property for the bulk heat kernel (3.1.5),  $\tilde{R}_1^S$  can be rewritten as

$$\int_{\mathbb{R}} du \int_{z, z'} \int_{z_1, n} \int_k \partial^w \mathcal{S}_{l-1, n+2}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \\ \times \dot{C}^\Lambda(k) p_B \left( \frac{1}{2\Lambda^2}; z, u \right) p_B \left( \frac{1}{2\Lambda^2}; z', u \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i).$$

We now insert the induction hypothesis to obtain that  $\tilde{R}_1^S$  is bounded by

$$(\Lambda + m)^{1-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \int_k |\dot{C}^\Lambda(k)| \mathcal{P} \left( \frac{|k|}{\Lambda + m}, \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ \int_{\mathbb{R}} du \mathcal{F}_{s+2, l-1; \delta_1}^{\Lambda, 0} \left( \tau_{1, s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, u \right). \quad (5.5.16)$$

Using

$$\int_k |\dot{C}^\Lambda(k)| \left( \frac{|k|}{\Lambda + m} \right)^\alpha = \int_{\mathbb{R}^3} \frac{d^3 k}{(2\pi)^3} \frac{2}{\Lambda^3} e^{-\frac{k^2 + m^2}{\Lambda^2}} \left( \frac{|k|}{\Lambda + m} \right)^\alpha \leq O(1), \quad \forall \alpha \in \mathbb{N}, \quad (5.5.17)$$

we have

$$|\tilde{R}_1^S| \leq (\Lambda + m)^{1-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ \int_{\mathbb{R}} du \mathcal{F}_{s+2, l-1; \delta_1}^{\Lambda, 0} \left( \tau_{1, s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, u \right). \quad (5.5.18)$$

Applying Lemma 2, we obtain the bound

$$|\tilde{R}_1^S| \leq (\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l; \delta_1}^{\Lambda, 0}(\tau_{1, s}). \quad (5.5.19)$$

The other contributions to  $R_1^S$  are

$$\int_{z, z'} \int_{z_1, n} \int_k \partial^w \mathcal{S}_{l-1, n+2}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \\ \times \dot{C}^\Lambda(k) p_B \left( \frac{1}{\Lambda^2}; z, -z' \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) \quad (5.5.20)$$

and

$$-2 \int_{z, z'} \int_{z_1, n} \int_k \partial^w \mathcal{S}_{l-1, n+2}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \\ \times \dot{C}^\Lambda(k) \int_v e^{-v} p_B \left( \frac{1}{\Lambda^2}; z, -z' - \frac{v}{c} \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i). \quad (5.5.21)$$

These terms can be rewritten using (3.1.5) as

$$\int_k \int_{\mathbb{R}} du \partial^w \mathcal{S}_{l-1, n+2}^{\Lambda, \Lambda_0} \left( \vec{p}_n, k, -k; \phi_{\tau_{1,s}, y_{1,s}} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) p_B \left( \frac{1}{2\Lambda^2}; \cdot, -u \right) \right) \dot{C}^\Lambda(k) \quad (5.5.22)$$

and

$$\begin{aligned} -2 \int_k \int_{\mathbb{R}} du \int_v \partial^w \mathcal{S}_{l-1, n+2}^{\Lambda, \Lambda_0} \left( \vec{p}_n, k, -k; \phi_{\tau_{1,s}, y_{1,s}} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) p_B \left( \frac{1}{2\Lambda^2}; \cdot, -\frac{v}{c} - u \right) \right) \\ \times e^{-v} \dot{C}^\Lambda(k). \end{aligned} \quad (5.5.23)$$

Applying the induction hypothesis together with (5.5.17), we obtain that (5.5.22) is bounded by

$$\begin{aligned} (\Lambda + m)^{1-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ \times \int_{\mathbb{R}} du \mathcal{F}_{s+2, l-1; \delta_1}^{\Lambda, 0} \left( \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, -u \right). \end{aligned} \quad (5.5.24)$$

Similarly, we have the following bound for (5.5.23)

$$\begin{aligned} (\Lambda + m)^{1-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \int_k \dot{C}(k) \mathcal{P} \left( \frac{|k|}{\Lambda + m}, \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ \times \int_{\mathbb{R}} du \int_v e^{-v} \mathcal{F}_{s+2, l-1; \delta_1}^{\Lambda, 0} \left( \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, -u - \frac{v}{c} \right). \end{aligned} \quad (5.5.25)$$

Using the bounds (3.1.12) and remembering the definition (3.2.49) of the surface weight factor, we deduce that

$$\int_{\mathbb{R}} du \int_v e^{-v} \mathcal{F}_{s+2, l-1; \delta_1}^{\Lambda, 0} \left( \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, -u - \frac{v}{c} \right)$$

and

$$\int_{\mathbb{R}} du \mathcal{F}_{s+2, l-1; \delta_1}^{\Lambda, 0} \left( \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, -u \right)$$

are bounded by

$$\int_{\mathbb{R}} du \mathcal{F}_{s+2, l-1; \delta_1}^{\Lambda, 0} \left( \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, u \right).$$

The rest of the proof follows the steps used to obtain the final bound for  $\tilde{R}_1^S$ , which gives

$$|\tilde{R}_1^S| \leq (\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l; \delta_1}^{\Lambda, 0}(\tau_{1,s}). \quad (5.5.26)$$

Now we analyse  $R_1^D$ . This term is independent of the induction hypothesis and will be bounded using only the bound (5.3.22) for  $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$ . Using (3.1.5),  $R_1^D$  can be rewritten as

$$\int_{\mathbb{R}} du \int_{z,z'} \int_{z_{1,n}} \int_k \partial^w \mathcal{D}_{l-1,n+2}^{\Lambda,\Lambda_0} ((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \\ \times \dot{C}^\Lambda(k) p_B \left( \frac{1}{2\Lambda^2}; z, u \right) p_B \left( \frac{1}{2\Lambda^2}; z', -u \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) .$$

The bound (5.3.22) implies that  $R_1^D$  is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \int_{\mathbb{R}} du \int_{z_1} p_B(\tau_1; z_1, y_1) \\ \times \sum_{T_{l-1}^{s+2}(z_1, y_{2,s}, u, -u, \vec{z})} \int_{\vec{z}} \mathcal{F}_{\delta_2} \left( \Lambda; \left\{ \tau_{2,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2} \right\}; T_{l-1}^{s+2}(z_1, y_{2,s}, u, -u, \vec{z}) \right) . \quad (5.5.27)$$

For any contribution to (5.5.27) we denote by  $z', z''$  the vertices in the tree  $T_{l-1}^{s+2}$  to which the test functions  $p_B \left( \frac{1+\delta_2}{2\Lambda^2}; u, \cdot \right)$  and  $p_B \left( \frac{1+\delta_2}{2\Lambda^2}; \cdot, -u \right)$  are attached. Performing the integral over  $u$  we obtain using (3.1.5)

$$\int_{\mathbb{R}} du p_B \left( \frac{1+\delta_2}{2\Lambda^2}; z', u \right) p_B \left( \frac{1+\delta_2}{2\Lambda^2}; -u, z'' \right) = p_B \left( \frac{1+\delta_2}{\Lambda^2}; z', -z'' \right) \\ \leq p_B \left( \frac{1+\delta_2}{\Lambda^2}; z', 0 \right) . \quad (5.5.28)$$

The bound (5.5.28) implies that the legs  $(z, u)$  and  $(z', u)$  are amputated from the tree  $T_{l-1}^{s+2}$  and  $(z', u)$  is replaced by the surface external leg  $(z', 0)$  with the parameter  $\Lambda$ . If  $z''$  is of incidence one, it is removed using  $\int_{z''} p_B \left( (1+\delta_2)/\Lambda^2; z, z'' \right) \leq 1$ , and this operation is iterated until a vertex  $\vec{z}$  such that  $c(\vec{z}) \geq 2$  is reached. This iteration process converges to a non-empty tree since for  $s \geq 1$ , there exists at least one internal vertex of incidence number greater or equal to 3 in the tree  $T_{l-1}^{s+2}$ . The integration over  $z_1$  in (5.5.27) implies that  $z_1$  becomes an internal vertex attached to  $y_1$ . Therefore, the reduction process produces a tree which belongs to  $\mathcal{T}^{s,0}$ . Furthermore,  $v'_2$  which denotes the number of vertices of incidence number 2 of the new tree, is increased at most by 2. This stems from the reduction process which can produce one additional internal vertex such that  $c(z) = 2$  when the vertex  $z''$  is removed, but also from the vertex  $z_1$  which was initially a root vertex. If  $z_1$  has an incidence number equal to one then after introducing the test function  $p_B(\tau_1; z_1, y_1)$ , it becomes internal of incidence number 2. If  $v_2$  is the number of vertices of incidence number 2 of  $T_{l-1}^{s+2}$ , then  $v'_2 \leq v_2 + \delta_{c_1,1} + 1$  which implies

$$v'_2 \leq v_2 + \delta_{c_1,1} + 1 \leq 3(l-1) - 2 + \frac{s+2}{2} + 1 \leq 3l - 2 + \frac{s+1}{2} .$$

This also means that the obtained tree is a surface tree in  $\mathcal{T}_l^{s,0} \equiv \mathcal{W}_l^s(\sigma_s)$ , which can also be seen as the set of forests corresponding to the trivial partition. Therefore,  $R_1^D$  is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{\delta_2}^0(\Lambda; \tau_{1,s}; W_l^s(\sigma_s); Y_{\sigma_s}) \quad (5.5.29)$$

which implies (see [\(3.2.55\)](#))

$$|R_1^D| \leq (\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta_2}^{\Lambda;0}(\tau_{1,s}) . \quad (5.5.30)$$

- In this part, we treat the quadratic terms on the RHS of the flow equations. To simplify the discussion, we analyse the terms from the symmetrized sum in which the arguments  $(z_i, p_i)$  appear ordered in  $(\mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}, \mathcal{S}_{l_2, n_2+1}^{\Lambda, \Lambda_0})$ ,  $(\mathcal{D}_{l_1, n_1+1}^{\Lambda, \Lambda_0}, \mathcal{S}_{l_2, n_2+1}^{\Lambda, \Lambda_0})$  and  $(\mathcal{D}_{l_1, n_1+1}^{\Lambda, \Lambda_0}, \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0})$ . These terms are given by

$$R_2^{SS} := \int_{z_{1,n}} \int_{z, z'} z_1^{r_1} z_2^{r_2} \partial^{w_1} \mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}^\Lambda(p) p_R \left( \frac{1}{\Lambda^2}; z, z' \right) \\ \times \partial^{w_2} \mathcal{S}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) ,$$

$$R_2^{DS} := \int_{z_{1,n}} \int_{z, z'} z_1^{r_1} z_2^{r_2} \partial^{w_1} \mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}^\Lambda(p) p_R \left( \frac{1}{\Lambda^2}; z, z' \right) \\ \times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) ,$$

and

$$R_2^{DD} := \int_{z_{1,n}} \int_{z, z'} z_1^{r_1} z_2^{r_2} \partial^{w_1} \mathcal{D}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}^\Lambda(p) p_{S,R} \left( \frac{1}{\Lambda^2}; z, z' \right) \\ \times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) .$$

First, we treat the case  $(r_1, r_2) = (0, 0)$ .

- We start with the term  $R_2^{DS}$ . The property [\(3.1.6\)](#) implies that  $R_2^{DS}$  can be rewritten as

$$\int_{\mathbb{R}^+} du \int_{z_{1,n}} \int_{z, z'} \partial^{w_1} \mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}^\Lambda(p) \\ \times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) p_R \left( \frac{1}{2\Lambda^2}; z, u \right) p_R \left( \frac{1}{2\Lambda^2}; z', u \right) .$$

Using the decomposition of the Robin heat kernel (3.1.4), we restrict our analysis to the following term only

$$\begin{aligned} \tilde{R}_2^{DS} &:= \int_{\mathbb{R}^+} du \int_{z_1, n} \int_{z, z'} \partial^{w_1} \mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}^\Lambda(p) \\ &\times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) p_B\left(\frac{1}{2\Lambda^2}; z, u\right) p_B\left(\frac{1}{2\Lambda^2}; z', u\right). \end{aligned}$$

The line of reasoning in treating the remaining contributions in  $R_2^{DS}$  is similar to the one used in bounding  $R_1^S$ . We define

$$\phi'_{s_1}(z_{1, n_1}) = \prod_{r=1}^{n_1} \phi_i(z_i), \quad \phi''_{s_2}(z_{n_1+1, n-1}) = \prod_{r=n_1+1}^n \phi_i(z_i), \quad (5.5.31)$$

where

$$\phi_i(z_i) = \begin{cases} p_B(\tau_i; z_i, y_i) & \text{if } i \leq s \\ \chi^+(z_i) & \text{otherwise.} \end{cases} \quad (5.5.32)$$

Note that  $s_1 = n_1$  if  $n_1 \leq s$  and  $s_2 = s - n_1$ . Otherwise, we have  $s_1 = s$  and  $s_2 = 0$ . Therefore,  $\tilde{R}_2^{DS}$  can be rewritten as

$$\begin{aligned} \tilde{R}_2^{DS} &= \int_{\mathbb{R}^+} du \int_{z'} \partial^{w_1} \mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}(\vec{p}_{n_1}, p; \phi'_{s_1} \times p_B\left(\frac{1}{2\Lambda^2}; \cdot, u\right)) \partial^{w_3} \dot{C}^\Lambda(p) \\ &\times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}(z'; -p, \vec{p}_{n_1+1, n}; \phi''_{s_2}) \times p_B\left(\frac{1}{2\Lambda^2}; u, z'\right). \quad (5.5.33) \end{aligned}$$

Applying the induction hypothesis to  $\mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}$  and using the bound (5.3.22) for  $\mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}$ , we obtain that  $\tilde{R}_2^{DS}$  is bounded by

$$\begin{aligned} &(\Lambda + m)^{2-n-|w|} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{P}\left(\frac{\|\vec{p}_n\|}{\Lambda + m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \\ &\times \int_{\mathbb{R}^+} du \mathcal{F}_{s_1+1, l_1; \delta_3}^0\left(\Lambda; \tau_{1, s_1}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s_1}}, u\right) \\ &\times \int_{z'} \mathcal{F}_{s_2, l_2; \delta'_3}(\Lambda; \tau_{s_1+1, s}; z'; Y_{\sigma_{s_1+1:s}}) p_B\left(\frac{1}{2\Lambda^2}; z', u\right). \end{aligned}$$

Since the global weight factor  $\mathcal{F}_{s_2, l_2; \delta'_3}(\Lambda; \tau_{s_1+1, s}; z'; Y_{\sigma_{s_1+1:s}})$  is a sum of the weight factors of all trees  $T_{l_2}^{s_2+1}(z'; \tau_{s_1+1, s}; Y_{\sigma_{s_1+1:s}})$  in  $\mathcal{T}_{l_2}^{s_2+1}$ , we deduce that integrating over  $z'$  gives the global weight factor of the bulk trees  $\hat{\mathcal{T}}_{l_2}^{s_2+1}$ , and therefore we can write

$$\begin{aligned} &\hat{\mathcal{F}}_{s_2+1, l_2; \delta'_3}\left(\Lambda; \tau_{s_1+1, s}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s_1+1:s}}, u\right) \\ &= \sum_{\hat{T}_{l_2}^{s_2+1} \in \hat{\mathcal{T}}_{l_2}^{s_2+1}} \mathcal{F}_{\delta'_3}\left(\Lambda; \tau_{s_1+1, s}, \frac{1}{2\Lambda^2}; \hat{T}_{l_2}^{s_2+1}; Y_{\sigma_{s_1+1:s}}, u\right), \quad (5.5.34) \end{aligned}$$



where

$$\begin{aligned} \mathcal{F}_{\delta'_3} \left( \Lambda; \tau_{s_1+1,s}, \frac{1}{2\Lambda^2}; \hat{T}_{l_2}^{s_2+1}; Y_{\sigma_{s_1+1:s}}, u \right) \\ := \int_{z'} \mathcal{F}_{\delta'_3} \left( \Lambda; \tau_{s_1+1,s}; T_{l_2}^{s_2+1}; z'; Y_{\sigma_{s_1+1:s}} \right) p_B \left( \frac{1}{2\Lambda^2}; z', u \right). \end{aligned} \quad (5.5.35)$$

Applying Lemma 4, we deduce that  $\tilde{R}_2^{DS}$  is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta_3''}^{\Lambda,0}(\tau_{1,s}),$$

where  $\delta_3'' := \max(\delta_3, \delta'_3)$ .

– In this part we bound the term  $R_2^{SS}$ . As for  $R_2^{DS}$ , we only treat the term

$$\begin{aligned} \tilde{R}_2^{SS} &:= \int_{\mathbb{R}^+} du \int_{z_1,n} \int_{z,z'} \partial^{w_1} \mathcal{S}_{l_1,n_1+1}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}^\Lambda(p) \\ &\times \partial^{w_2} \mathcal{S}_{l_2,n_2+1}^{\Lambda,\Lambda_0}((z', -p), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) p_B \left( \frac{1}{2\Lambda^2}; z, u \right) p_B \left( \frac{1}{2\Lambda^2}; z', u \right). \end{aligned}$$

Using the notations (5.5.31)-(5.5.32), we rewrite  $\tilde{R}_2^{SS}$  as follows,

$$\begin{aligned} \tilde{R}_2^{SS} &= \int_{\mathbb{R}^+} du \partial^{w_1} \mathcal{S}_{l_1,n_1+1}^{\Lambda,\Lambda_0}(p_1, \dots, p_{n_1}, p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right)) \partial^{w_3} \dot{C}^\Lambda(p) \\ &\times \partial^{w_2} \mathcal{S}_{l_2,n_2+1}^{\Lambda,\Lambda_0}(-p, p_{n_1+1}, \dots, p_n; \phi''_{s_2} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right)). \end{aligned} \quad (5.5.36)$$

Note that  $\tilde{R}_{2,SS} = 0$  for  $l_i \in \{0, l\}$ . Using the induction hypothesis, we obtain

$$\begin{aligned} |\tilde{R}_2^{SS}| &\leq (\Lambda + m)^{1-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ &\times \int_{\mathbb{R}} du \mathcal{F}_{s_1+1,l_1;\delta_3}^0 \left( \Lambda; \tau_{1,s_1}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s_1}}, u \right) \mathcal{F}_{s_2+1,l_2;\delta_4}^0 \left( \Lambda; \tau_{s_1+1,s}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s_1+1:s}}, u \right). \end{aligned}$$

Applying Lemma 3, we deduce that  $\tilde{R}_2^{SS}$  is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta_5}^{\Lambda,0}(\tau_{1,s}),$$

where  $\delta_5 := \max(\delta_3, \delta_4)$ .

– In this part, we bound the term  $R_2^{DD}$ , which we rewrite using (5.5.31)-(5.5.32) as follows,

$$\int_{z, z'} \partial^{w_1} \mathcal{D}_{l_1, n_1+1}^{\Lambda, \Lambda_0} (z; \vec{p}_{1, n_1}, p; \phi'_{s_1}) \partial^{w_3} \dot{C}^\Lambda(p) p_{S, R} \left( \frac{1}{\Lambda^2}; z, -z' \right) \\ \times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0} (z'; \vec{p}_{n_1+1, n}, -p; \phi''_{s_2}).$$

Using the bounds (5.2.6) and (5.3.22), we obtain that  $R_2^{DD}$  is bounded by

$$(\Lambda + m)^{3-n-|w|} e^{-\frac{m^2}{2\Lambda^2}} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \times \int_{z, z'} \mathcal{F}_{s_1, l_1; \delta'_4} (\Lambda; \tau_{1, s_1}; z; Y_{\sigma_{s_1}}) \mathcal{F}_{s_2, l_2; \delta'_3} (\Lambda; \tau_{s_1+1, s}; z'; Y_{\sigma_{s_1+1: s}}) p_B \left( \frac{1}{\Lambda^2}; z', -z \right). \quad (5.5.37)$$

The bound

$$p_B \left( \frac{1}{\Lambda^2}; z', -z \right) \leq \sqrt{2\pi} \Lambda^{-1} p_B \left( \frac{1}{\Lambda^2}; z', 0 \right) p_B \left( \frac{1}{\Lambda^2}; z, 0 \right) \quad (5.5.38)$$

together with

$$\Lambda^{-\alpha} e^{-\frac{m^2}{2\Lambda^2}} \leq O(1) (\Lambda + m)^{-\alpha} \quad \text{for } \alpha \in \mathbb{N}, \quad (5.5.39)$$

gives

$$e^{-\frac{m^2}{2\Lambda^2}} \int_{z, z'} \mathcal{F}_{s_1, l_1; \delta'_4} (\Lambda; \tau_{1, s_1}; z; Y_{\sigma_{s_1}}) \mathcal{F}_{s_2, l_2; \delta'_3} (\Lambda; \tau_{s_1+1, s}; z'; Y_{\sigma_{s_1+1: s}}) p_B \left( \frac{1}{\Lambda^2}; z', -z \right) \\ \leq O(1) (\Lambda + m)^{-1} \int_z \mathcal{F}_{s_1, l_1; \delta'_4} (\Lambda; \tau_{1, s_1}; z; Y_{\sigma_{s_1}}) p_B \left( \frac{1}{\Lambda^2}; z, 0 \right) \\ \times \int_{z'} \mathcal{F}_{s_2, l_2; \delta'_3} (\Lambda; \tau_{s_1+1, s}; z'; Y_{\sigma_{s_1+1: s}}) p_B \left( \frac{1}{\Lambda^2}; z', 0 \right).$$

The definition of the global surface weight factor (3.2.32)-(3.2.57) implies

$$\int_z \mathcal{F}_{s_1, l_1; \delta'_4} (\Lambda; \tau_{1, s_1}; z; Y_{\sigma_{s_1}}) p_B \left( \frac{1}{\Lambda^2}; z, 0 \right) \\ \times \int_{z'} \mathcal{F}_{s_2, l_2; \delta'_3} (\Lambda; \tau_{s_1+1, s}; z'; Y_{\sigma_{s_1+1: s}}) p_B \left( \frac{1}{\Lambda^2}; z', 0 \right) \\ \leq O(1) \left( \sum_{W_l^s \in \mathcal{W}_l^s} \sum_{\{\Pi \in \mathcal{P}_s, l_\Pi=2\}} \mathcal{F}_{\delta_6}^0 (\Lambda; \tau_{1, s}; W_l^s(\Pi); Y_{\sigma_s}) \right), \quad (5.5.40)$$

where  $\Pi := \Pi_1 \cup \Pi_2$ ,  $\Pi_1 := \{1, \dots, s_1\}$  and  $\Pi_2 := \{s_1 + 1, \dots, s\}$ . In (5.5.40), we also used

the bound [\(3.2.62\)](#) and  $\delta_6 := \max(\delta'_3, \delta'_4)$ . Therefore, we find that  $R_2^{DD}$  is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \times \left( \sum_{W_l^s \in \mathcal{W}_l^s} \sum_{\{\Pi \in \mathcal{P}_s, l_{\Pi}=2\}} \mathcal{F}_{\delta_6}^0(\Lambda; \tau_{1,s}; W_l^s(\Pi); Y_{\sigma_s}) \right). \quad (5.5.41)$$

This shows that  $R_2^{DD}$  is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta_6}^{\Lambda,0}(\tau_{1,s}).$$

- Case  $(r_1, r_2) \neq (0, 0)$ : The linear terms  $R_1^S$  and  $R_1^D$  together with the non-linear term  $R_2^{SS}$  are treated following the same steps as before. The only terms that require a careful analysis are  $R_2^{DS}$  and  $R_2^{DD}$ . To shorten the discussion, we analyze the term  $R_2^{DD}$  only,  $R_2^{DS}$  may be treated using similar arguments. We write

$$z_1^{r_1} = \sum_{\alpha_1+\beta_1=r_1} \binom{r_1}{\alpha_1} (z_1 - z)^{\alpha_1} z^{\beta_1}, \quad z_2^{r_2} = \sum_{\alpha_2+\beta_2=r_2} \binom{r_2}{\alpha_2} (z_2 - z')^{\alpha_2} z'^{\beta_2}.$$

This allows to rewrite  $R_2^{DD}$  for all  $n_1 \geq 2$  as follows

$$\sum_{\alpha_1+\beta_1=r_1, \alpha_2+\beta_2=r_2} \binom{r_1}{\alpha_1} \binom{r_2}{\alpha_2} z^{\beta_1+\beta_2} \partial^{w_1} \mathcal{D}_{l_1, n_1+1; \alpha_1, \alpha_2}^{\Lambda, \Lambda_0; (1,2)}(z; \vec{p}_{1, n_1}, p; \phi'_{s_1}) \partial^{w_3} \dot{C}^{\Lambda}(p) \\ \times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}(z'; \vec{p}_{n_1+1, n}, -p \cdots, p_n; \phi''_{s_2}) p_{S,R} \left( \frac{1}{\Lambda^2}; z, z' \right), \quad (5.5.42)$$

and for  $n_1 = 1$  we have

$$\sum_{\alpha_1+\beta_1=r_1, \alpha_2+\beta_2=r_2} \binom{r_1}{\alpha_1} \binom{r_2}{\alpha_2} z^{\beta_1} z'^{\beta_2} \partial^{w_1} \mathcal{D}_{l_1, 2; \alpha_1}^{\Lambda, \Lambda_0; (1)}(z; p_1, p; \phi'_{s_1}) \partial^{w_3} \dot{C}^{\Lambda}(p) \\ \times \partial^{w_2} \mathcal{D}_{l_2, n; \alpha_2}^{\Lambda, \Lambda_0; (2)}(z'; \vec{p}_{2, n}, -p; \phi''_{s_2}) p_{S,R} \left( \frac{1}{\Lambda^2}; z, z' \right). \quad (5.5.43)$$

Using the bounds [\(5.3.22\)](#)-[\(5.3.23\)](#), we deduce that the summands in [\(5.5.42\)](#)-[\(5.5.43\)](#) are bounded by

$$(\Lambda + m)^{3-n-|w|-\alpha_1-\alpha_2} e^{-\frac{m^2}{2\Lambda^2}} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \times \int_{z, z'} \mathcal{F}_{s_1, l_1; \delta'_1}(\Lambda; \tau_{1, s_1}; z; Y_{\sigma_{s_1}}) \mathcal{F}_{s_2, l_2; \delta'_2}(\Lambda; \tau_{s_1+1, s}; z'; Y_{\sigma_{s_1+1, s}}) \chi_{n_1}^{\Lambda}(z, z'), \quad (5.5.44)$$

where

$$\chi_{n_1}^\Lambda(z, z') = \begin{cases} z^{\beta_1} z'^{\beta_2} p_B\left(\frac{1}{\Lambda^2}; z', -z\right) & \text{if } n_1 = 1, \\ z^{\beta_1 + \beta_2} p_B\left(\frac{1}{\Lambda^2}; z', -z\right) & \text{otherwise.} \end{cases}$$

Using the bound (5.5.38) together with (3.1.10), we deduce for all  $\delta, \tilde{\delta} > 0$

$$\chi_{n_1}^\Lambda(z, z') \leq O(1) \Lambda^{-1-\beta_1-\beta_2} p_B\left(\frac{1+\delta}{\Lambda^2}; z, 0\right) p_B\left(\frac{1+\tilde{\delta}}{\Lambda^2}; z', 0\right), \quad (5.5.45)$$

which implies together with (5.5.39) that (5.5.44) is bounded by

$$\begin{aligned} & (\Lambda + m)^{2-n-|w|-r_1-r_2} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{P}\left(\frac{\|\vec{p}_n\|}{\Lambda + m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \\ & \quad \times \int_z \mathcal{F}_{s_1, l_1; \delta_1''}(\Lambda; \tau_{1, s_1}; z; Y_{\sigma_{s_1}}) p_B\left(\frac{1+\delta_1''}{\Lambda^2}; z, 0\right) \\ & \quad \times \int_{z'} \mathcal{F}_{s_2, l_2; \delta_2''}(\Lambda; \tau_{s_1+1, s}; z'; Y_{\sigma_{s_1+1, s}}) p_B\left(\frac{1+\delta_2''}{\Lambda^2}; z', 0\right). \end{aligned} \quad (5.5.46)$$

This together with the bounds (5.5.40) and (5.5.41) imply the final bound for  $R_2^{DD}$  given by

$$(\Lambda + m)^{2-n-|w|-r_1-r_2} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{P}\left(\frac{\|\vec{p}_n\|}{\Lambda + m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{s, l; \delta_7}^{\Lambda, 0}(\tau_{1, s}), \quad (5.5.47)$$

where  $\delta_7 := \max(\delta_1'', \delta_2'')$ .

Using the bound (3.2.62), we deduce (5.5.13) where  $\delta := \max\{\delta_i, \delta_i', \delta_i'', 1 \leq i \leq 7\}$ .

### Integration of the FEs

- We start by integrating the irrelevant terms for which  $n + |w| + r_1 + r_2 \geq 4$ . In this case, (5.5.13) is integrated from  $\Lambda$  to  $\Lambda_0$  using the boundary condition (5.4.18)-(5.4.19) together with (3.2.61) and we obtain

$$\begin{aligned} & \left| \partial^w \mathcal{S}_{l, n; r_1, r_2}^{\Lambda, \Lambda_0}(\vec{p}_n; \phi_{\tau_{1, s}, y_{1, s}}) \right| \\ & \leq (\Lambda + m)^{3-n-|w|-r_1-r_2} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{P}\left(\frac{\|\vec{p}_n\|}{\Lambda + m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{s, l; \delta}^{\Lambda, 0}(\tau_{1, s}). \end{aligned} \quad (5.5.48)$$

- The relevant terms for which  $n + |w| + r_1 + r_2 \leq 3$  are written

$$\int_0^\infty dz_1 \int_0^\infty dz_2 z_1^{r_1} z_2^{r_2} \partial_\Lambda \mathcal{S}_{l, 2}^{\Lambda, \Lambda_0}((z_1, p), (z_2, -p)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i), \quad (5.5.49)$$

where  $r_1, r_2$  are integers such that  $r_1 + r_2 \leq 1$ , and  $0 \leq s \leq 2$ . We restrict our analysis to the case  $s = 2$ , the case  $s = 1$  can be treated similarly and the case  $s = 0$  will be integrated in the sequel. For  $s = 2$ , the relevant part is extracted from

$$\int_{z_1, z_2} \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, p), (z_2, -p)) \phi_1(z_1) \phi_2(z_2) \quad (5.5.50)$$

by performing a Taylor expansion of  $\phi_1$  and  $\phi_2$  around  $z_i = 0$  at  $p = 0$ , where  $\phi_i(z_i) := p_B(\tau_i; z_i, y_i)$ , using (5.4.14) and (5.4.17). The bound (5.5.13) for  $s = 0$  and  $r_1 + r_2 \leq 1$  gives

$$\left| \partial_\Lambda s_l^{\Lambda, \Lambda_0} \right| \leq \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right), \quad \left| \partial_\Lambda e_l^{\Lambda, \Lambda_0} \right| \leq (\Lambda + m)^{-1} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right). \quad (5.5.51)$$

Integrating (5.5.51) from 0 to  $\Lambda$  and using the renormalization conditions (5.4.20), we have

$$\left| s_l^{\Lambda, \Lambda_0} \right| \leq (\Lambda + m) \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right), \quad \left| e_l^{\Lambda, \Lambda_0} \right| \leq \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right). \quad (5.5.52)$$

Applying Lemma 10 from the Appendix, we have

$$\begin{aligned} & \left| s_l^{\Lambda, \Lambda_0} \phi_1(0) \phi_2(0) + e_l^{\Lambda, \Lambda_0} \{ \phi_1(0) (\partial_n \phi_2)(0) + \phi_2(0) (\partial_n \phi_1)(0) \} \right| \\ & \leq (\Lambda + m) \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}). \end{aligned} \quad (5.5.53)$$

Now, we bound and integrate the remainder  $\partial_\Lambda l_{l,2}^{\Lambda, \Lambda_0}(\phi_1, \phi_2)$ , which is irrelevant as we will see in the sequel, from  $\Lambda$  to  $\Lambda_0$ . We distinguish between the two cases:

–  $\underline{\Lambda \leq 3\sqrt{l}\tau^{-\frac{1}{2}}}$ : Using (5.5.13) and (5.5.53), we deduce that  $\partial_\Lambda l_{l,2}^{\Lambda, \Lambda_0}(\phi_1, \phi_2)$  is bounded by

$$O(1) \max(m^2, \tau^{-1}) (\Lambda + m)^{-2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}). \quad (5.5.54)$$

–  $\underline{\Lambda \geq 3\sqrt{l}\tau^{-\frac{1}{2}}}$ : In the sequel, we restrict our analysis to the integration of the following terms, for which we need to proceed differently.

$$\mathcal{H}_1 := \left( \int_{z_1, z_2} z_1 z_2 \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \right) (\partial_n \phi_1)(0) (\partial_n \phi_2)(0), \quad (5.5.55)$$

$$\mathcal{H}_2 := (\partial_n \phi_2)(0) \int_{z_1, z_2} z_2 \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_1(tz_1)), \quad (5.5.56)$$

and

$$\begin{aligned} \mathcal{H}_3 := & \int_{z_1, z_2} \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \left( \int_0^1 dt (1-t) \partial_t^2 \phi_1(tz_1) \right) \\ & \times \left( \int_0^1 dt' (1-t') \partial_{t'}^2 \phi_2(t'z_2) \right). \end{aligned} \quad (5.5.57)$$

The other terms which also contribute to  $\partial_\Lambda l_{l,2}^{\Lambda, \Lambda_0}(\phi_1, \phi_2)$  can be treated similarly.

- We start first with  $\mathcal{H}_1$  for which the bound (5.5.13) implies that

$$\left| \left( \int_{z_1, z_2} z_1 z_2 \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0} ((z_1, 0), (z_2, 0)) \right) \right| \leq (\Lambda + m)^{-2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right). \quad (5.5.58)$$

Using Lemma 10, we obtain

$$\begin{aligned} & \left| \left( \int_{z_1, z_2} z_1 z_2 \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0} ((z_1, 0), (z_2, 0)) \right) (\partial_n \phi_1)(0) (\partial_n \phi_2)(0) \right| \\ & \leq (\Lambda + m)^{-2} \tau_1^{-\frac{1}{2}} \tau_2^{-\frac{1}{2}} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}). \end{aligned} \quad (5.5.59)$$

- The term  $\mathcal{H}_2$ : we have for  $0 \leq t \leq 1$

$$\phi_i(tz_i) = \frac{1}{\sqrt{2\pi\tau_i}} e^{-\frac{(tz_i - y_i)^2}{2\tau_i}}. \quad (5.5.60)$$

Differentiating (6.4.1) twice w.r.t.  $t$ , we obtain

$$\partial_t^2 (\phi_i(tz_i)) = \frac{1}{t} \left[ -\frac{z_i^2}{\tau_i} + \frac{z_i^2 (tz_i - y_i)^2}{\tau_i^2} \right] p_B \left( \frac{\tau_i}{t^2}; z_i, \frac{y_i}{t} \right), \quad (5.5.61)$$

which implies that the term

$$\int_{z_1, z_2} z_2 \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0} ((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) \partial_t^2 \phi_1(tz_1) \quad (5.5.62)$$

can be rewritten as

$$\begin{aligned} & \sum_{(\alpha, \beta) \in \mathcal{F}_2} c_{\alpha\beta} \frac{y_1^\beta}{\tau_1^{1+\frac{\beta+\alpha}{2}}} \int_0^1 dt t^{\alpha-1} (1-t) \\ & \int_{z_1, z_2} z_2 z_1^{2+\alpha} \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0} ((z_1, 0), (z_2, 0)) p_B \left( \frac{\tau_1}{t^2}; z_1, \frac{y_1}{t} \right), \end{aligned} \quad (5.5.63)$$

where  $\mathcal{F}_2 := \{(0, 0), (\alpha, \beta) \mid \alpha + \beta = 2, (\alpha, \beta) \in \mathbb{N}^2\}$ , and the coefficients  $c_{\alpha\beta} \in \mathbb{R}$  depend only on the exponents  $\alpha$  and  $\beta$ . The bound (5.5.13) implies that the term

$$\int_{z_1, z_2} z_2 z_1^{2+\alpha} \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0} ((z_1, 0), (z_2, 0)) p_B \left( \frac{\tau_1}{t^2}; z_1, \frac{y_1}{t} \right) \quad (5.5.64)$$

is bounded by

$$(\Lambda + m)^{-3-\alpha} e^{-\frac{m^2}{2\Lambda^2}} \mathcal{Q} \left( \frac{t\tau_1^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{1,l;\delta}^0 \left( \Lambda; \frac{\tau_1}{t^2}; \frac{y_1}{t} \right). \quad (5.5.65)$$

From Lemma [11](#), we obtain

$$\left(\frac{y_1}{\sqrt{\tau_1}}\right)^\beta \mathcal{F}_{1,l;\delta}^0\left(\Lambda; \frac{\tau_1}{t^2}; \frac{y_1}{t}\right) \leq O(1) t \left(1 + \frac{\sqrt{\tau_1}}{\Lambda}\right)^\beta \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1), \quad (5.5.66)$$

where  $0 < \delta < \delta'$ . Using [\(5.5.39\)](#) together with [\(5.5.64\)](#) and [\(5.5.65\)](#), we deduce that [\(5.5.63\)](#) is bounded by

$$(\Lambda + m)^{-3} \tau_1^{-1} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathbb{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1). \quad (5.5.67)$$

Lemma [10](#) together with [\(5.8.3\)](#) imply that  $\mathcal{H}_2$  is bounded by

$$(\Lambda + m)^{-3} \tau_1^{-1} \tau_2^{-\frac{1}{2}} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathbb{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta'}^{\Lambda,0}(\tau_{1,2}). \quad (5.5.68)$$

Following similar steps, we obtain

$$\begin{aligned} & \left| (\partial_n \phi_1)(0) \int_{z_1, z_2} z_2 \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_2(tz_1)) \right| \\ & \leq (\Lambda + m)^{-3} \tau_1^{-\frac{1}{2}} \tau_2^{-1} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathbb{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta'}^{\Lambda,0}(\tau_{1,2}) \end{aligned} \quad (5.5.69)$$

and

$$\begin{aligned} & \left| \phi_i(0) \int_{z_1, z_2} \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_j(tz_1)) \right| \\ & \leq (\Lambda + m)^{-2} \tau_j^{-1} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathbb{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta'}^{\Lambda,0}(\tau_{1,2}). \end{aligned} \quad (5.5.70)$$

- The term  $\mathcal{H}_3$ : We start from [\(5.5.57\)](#) and [\(5.5.61\)](#) and write the term  $\mathcal{H}_3$  as follows

$$\begin{aligned} & \sum_{(\alpha, \beta) \in \mathcal{J}_2} \sum_{(\alpha', \beta') \in \mathcal{J}_2} c_{\alpha\beta} c_{\alpha'\beta'} \frac{y_1^\beta}{\tau_1^{1+\frac{\beta+\alpha}{2}}} \frac{y_2^{\beta'}}{\tau_2^{1+\frac{\beta'+\alpha'}{2}}} \int_0^1 dt dt' t^{\alpha-1} (1-t) t'^{\alpha'-1} (1-t') \\ & \int_{z_1, z_2} z_2^{2+\alpha'} z_1^{2+\alpha} \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) p_B\left(\frac{\tau_1}{t^2}; z_1, \frac{y_1}{t}\right) p_B\left(\frac{\tau_2}{t'^2}; z_2, \frac{y_2}{t'}\right). \end{aligned} \quad (5.5.71)$$

The bound [\(5.5.13\)](#) implies that the term

$$\int_{z_1, z_2} z_2^{2+\alpha'} z_1^{2+\alpha} \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) p_B\left(\frac{\tau_1}{t^2}; z_1, \frac{y_1}{t}\right) p_B\left(\frac{\tau_2}{t'^2}; z_2, \frac{y_2}{t'}\right)$$

is bounded by

$$(\Lambda + m)^{-4-\alpha-\alpha'} e^{-\frac{m^2}{2\Lambda^2}} \mathbb{Q} \left( \frac{t\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^0 \left( \Lambda; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; \frac{y_1}{t}, \frac{y_2}{t'} \right).$$

From Lemma [12](#) and [\(5.5.39\)](#), we deduce that [\(5.5.71\)](#) is bounded by

$$\tau_1^{-\frac{1}{2}} \tau_2^{-\frac{1}{2}} (\Lambda + m)^{-2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \tilde{\mathbb{Q}} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta'}^{\Lambda;0}(\tau_{1,2}), \quad (5.5.72)$$

where  $0 < \delta < \delta' < 1$ .

The boundary conditions [\(5.4.18\)](#) together with [\(5.4.17\)](#), [\(5.5.63\)](#) and [\(5.5.71\)](#) imply that

$$l_{l,2;R}^{\Lambda_0,\Lambda_0}(\phi_1, \phi_2) = 0.$$

Integrating from  $\Lambda$  to  $\Lambda_0$ , we obtain for  $\Lambda_0 \geq 3\sqrt{l}\tau^{-\frac{1}{2}}$

$$l_{l,2}^{\Lambda,\Lambda_0}(\phi_1, \phi_2) = \int_{\Lambda}^{3\sqrt{l}\tau^{-\frac{1}{2}}} d\lambda \partial_{\lambda} l_{l,2}^{\lambda,\Lambda_0}(\phi_1, \phi_2) + \int_{3\sqrt{l}\tau^{-\frac{1}{2}}}^{\Lambda_0} d\lambda \partial_{\lambda} l_{l,2}^{\lambda,\Lambda_0}(\phi_1, \phi_2). \quad (5.5.73)$$

Using the bounds [\(5.5.54\)](#), [\(5.5.59\)](#), [\(5.5.68\)](#) and [\(5.5.72\)](#) together with [\(5.5.69\)](#) and [\(5.5.70\)](#), we obtain that the remainder  $l_{l,2}^{\Lambda,\Lambda_0}$  is bounded by

$$(\Lambda + m) \max \left\{ \frac{\tau^{-1}}{(\Lambda + m)^2}, \frac{m^2}{(\Lambda + m)^2} \right\} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \tilde{\mathbb{Q}} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta'}^{\Lambda,0}(\tau_{1,2}). \quad (5.5.74)$$

For  $\Lambda_0 \leq 3\sqrt{l}\tau^{-\frac{1}{2}}$ , we conclude by integrating [\(5.5.54\)](#) from  $\Lambda$  to  $\Lambda_0$  and we deduce

$$\begin{aligned} & \left| \partial^w \mathcal{S}_{l,2;r_1,r_2}^{\Lambda,\Lambda_0}(0; \phi_{\tau_1,s,y_1,s}) \right| \\ & \leq (\Lambda + m)^{1-|w|-r_1-r_2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta'}^{\Lambda,0}(\tau_{1,s}), \end{aligned} \quad (5.5.75)$$

where we used [\(3.2.62\)](#).

- Extension to general momenta: We now extend the bound [\(5.5.75\)](#) to general momenta using the Taylor formula with integral remainder, which reads

$$\partial^w \mathcal{S}_{l,2;r_1,r_2}^{\Lambda,\Lambda_0}(p; \phi_{\tau_1,s,y_1,s}) = \partial^w \mathcal{S}_{l,2;r_1,r_2}^{\Lambda,\Lambda_0}(0; \phi_{\tau_1,s,y_1,s}) + \int_0^1 dt \partial_t \partial^w \mathcal{S}_{l,2;r_1,r_2}^{\Lambda,\Lambda_0}(tp; \phi_{\tau_1,s,y_1,s}). \quad (5.5.76)$$

Applying this formula, the bound of the integrand ( due to the derivative ) yields an additional factor  $(\Lambda + m)^{-1}$  which combines with the momentum produced by the  $t$ -derivative to give a bound as in [\(5.5.1\)](#).

This ends the proof of Theorem [9](#). □



### 5.5.2 Proof of Proposition [7](#)

*Proof.* The proof follows the same inductive scheme used in the proof of Theorem [9](#). For the tree order, we have

$$\mathfrak{S}_{0,n;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) = 0, \quad \forall n \geq 2, \star \in \{D, R, N\}.$$

Clearly, the statement [\(7.3.8\)](#) holds.

A1) We start by verifying inductively the following statement

$$\partial_\Lambda \mathfrak{S}_{l,n;D}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^D) = \lim_{c \rightarrow +\infty} \partial_\Lambda \mathfrak{S}_{l,n;R}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^R). \quad (5.5.77)$$

In the sequel, we use the symbol  $\star$  to denote either Dirichlet or Robin boundary conditions. Given  $\Pi = (\pi_1, \pi_2) \in \mathcal{P}_n$  such that  $|\pi_i| = n_i$  and  $n_1 + n_2 = n$ , we introduce the following notations:

$$\begin{aligned} & \mathfrak{S}_{l_i, n_i+1; \star}^{\Lambda, \Lambda_0}(\vec{p}_{\pi_i}, (-1)^i p; \Phi_{\pi_i}^{\Lambda; \star}; Y_{\pi_i}, u) \\ & := \int_{\vec{z}_{\pi_i}, z} \mathfrak{S}_{l_i, n_i+1; \star}^{\Lambda, \Lambda_0}((\vec{z}_{\pi_i}, \vec{p}_{\pi_i}), (z, p)) \prod_{i \in \pi_i} p_\star(\tau_i; z_i, y_i) p_\star\left(\frac{1}{2\Lambda^2}; z, u\right), \quad i \in \{1, 2\} \end{aligned} \quad (5.5.78)$$

and

$$\begin{aligned} & \mathfrak{S}_{l-1, n+2; \star}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; \star}; Y_{\sigma_n}, u, u) := \int_{\vec{z}_n, z, z'} \mathfrak{S}_{l-1, n+2; \star}^{\Lambda, \Lambda_0}((\vec{z}_n, \vec{p}_n), (z, k), (z', -k)) \\ & \quad \times \phi_{\tau_{1,n}, y_{1,n}}^\star(z_{1,n}) p_\star\left(\frac{1}{2\Lambda^2}; z, u\right) p_\star\left(\frac{1}{2\Lambda^2}; z', u\right), \end{aligned} \quad (5.5.79)$$

where

$$\Phi_{\pi_i}^{\Lambda; \star}(\vec{z}_{\pi_i}, z) = \phi_{\pi_i}^\star(\vec{z}_{\pi_i}) p_\star\left(\frac{1}{2\Lambda^2}; z, u\right) \quad \text{with} \quad \phi_{\pi_i}^\star(\vec{z}_{\pi_i}) := \prod_{i \in \pi_i} p_\star(\tau_i; z_i, y_i),$$

and

$$\Phi_{n+2}^{\Lambda; \star}(z_{1,n}, z, z') = \phi_{\tau_{1,n}, y_{1,n}}^\star(z_{1,n}) p_\star\left(\frac{1}{2\Lambda^2}; z, u\right) p_\star\left(\frac{1}{2\Lambda^2}; z', u\right).$$

We consider the flows equation (5.4.11) folded with the test functions  $\phi_{\tau_{1,n}, y_{1,n}}^*$  given by

$$\begin{aligned}
& \partial_\Lambda \mathcal{S}_{l,n;\star}^{\Lambda, \Lambda_0} \left( \vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^* \right) \\
&= \frac{1}{2} \int_u \int_k \mathcal{S}_{l-1, n+2; \star}^{\Lambda, \Lambda_0} \left( \vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; \star}; Y_{\sigma_n}, u, u \right) \dot{C}^\Lambda(k) \\
&+ \frac{1}{2} \int_{z, z'} \int_k \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0} \left( (\vec{z}_n \cdot \vec{p}_n), (z, k), (z', -k) \right) \dot{C}_{S, \star}^\Lambda(k; z, z') \phi_{\tau_{1,n}, y_{1,n}}^*(z_{1,n}) \\
&- \frac{1}{2} \sum'_{l_1, l_2} \sum''_{\pi_1, \pi_2} \left[ \int_u \left\{ \mathcal{S}_{l_1, n_1+1; \star}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_1}, -p; \Phi_{\pi_1}^{\Lambda; \star}; Y_{\pi_1}, u \right) \dot{C}^\Lambda(p) \right. \right. \\
&\quad \times \mathcal{S}_{l_2, n_2+1; \star}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_2}, p; \Phi_{\pi_2}^{\Lambda; \star}; Y_{\pi_2}, u \right) \\
&+ \mathcal{D}_{l_1, n_1+1}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_1}, -p; \Phi_{\pi_1}^{\Lambda; \star}; Y_{\pi_1}, u \right) \dot{C}^\Lambda(p) \mathcal{S}_{l_2, n_2+1; \star}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_2}, p; \Phi_{\pi_2}^{\Lambda; \star}; Y_{\pi_2}, u \right) \\
&+ \left. \mathcal{S}_{l_1, n_1+1; \star}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_1}, -p; \Phi_{\pi_1}^{\Lambda; \star}; Y_{\pi_1}, u \right) \dot{C}^\Lambda(p) \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_2}, p; \Phi_{\pi_2}^{\Lambda; \star}; Y_{\pi_2}, u \right) \right\} \\
&+ \left. \int_{z, z'} \mathcal{D}_{l_1, n_1+1}^{\Lambda, \Lambda_0} \left( z; -p, \vec{p}_{1, n_1}; \phi_{\pi_1}^* \right) \dot{C}_{S, \star}^\Lambda(p; z, z') \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0} \left( z'; p, \vec{p}_{n_1+1, n}; \phi_{\pi_2}^* \right) \right]_{rsym}, \\
& \quad p = - \sum_{i \in \pi_1} p_i = \sum_{i \in \pi_2} p_i, \tag{5.5.80}
\end{aligned}$$

where we used (3.2.66) and the notations (5.5.78)-(5.5.79). Here, the prime denotes all pairs  $(l_1, l_2)$  such that  $l_1 + l_2 = l$ , and the double prime refers to a summation over  $(\pi_1, \pi_2) \in \tilde{\mathcal{P}}_{2; n}$  with  $n_i := |\pi_i|$ .

Using the induction hypothesis, we obtain

$$\mathcal{S}_{l-1, n+2; D}^{\Lambda, \Lambda_0} \left( \vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; D}; Y_{\sigma_n}, u, u \right) = \lim_{c \rightarrow +\infty} \mathcal{S}_{l-1, n+2; R}^{\Lambda, \Lambda_0} \left( \vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; R}; Y_{\sigma_n}, u, u \right), \tag{5.5.81}$$

$$\begin{aligned}
& \mathcal{S}_{l_i, n_i+1; D}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_i}, (-1)^i p; \Phi_{\pi_i}^{\Lambda; D}; Y_{\pi_i}, u \right) \\
&= \lim_{c \rightarrow +\infty} \mathcal{S}_{l_i, n_i+1; R}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_i}, (-1)^i p; \Phi_{\pi_i}^{\Lambda; R}; Y_{\pi_i}, u \right), \quad i \in \{1, 2\}. \tag{5.5.82}
\end{aligned}$$

For  $\tau_i > 0$  and  $y_i \in \mathbb{R}^+$ , we have

$$\lim_{c \rightarrow +\infty} \mathcal{N}_p \left( p_D(\tau_i; \cdot, y_i) - p_R(\tau_i; \cdot, y_i) \right) = 0, \tag{5.5.83}$$

where for  $\phi$  in  $\mathcal{S}(\mathbb{R}^+)$  the semi-norm  $\mathcal{N}_p$  is given by

$$\mathcal{N}_p(\phi) = \sum_{\alpha, \beta \leq p} \sup_{x \in \mathbb{R}^+} |x^\alpha \partial^\beta \phi(x)|.$$

Remembering that  $\mathcal{D}_{l,n}^{\Lambda, \Lambda_0} \in \mathcal{S}'(\mathbb{R}^{+n})$  and using (5.5.83), we deduce

$$\mathcal{D}_{l_i, n_i+1}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_i}, (-1)^i p; \Phi_{\pi_i}^{\Lambda; D}; Y_{\pi_i}, u \right) = \lim_{c \rightarrow +\infty} \mathcal{D}_{l_i, n_i+1}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_i}, (-1)^i p; \Phi_{\pi_i}^{\Lambda; R}; Y_{\pi_i}, u \right). \tag{5.5.84}$$

We rewrite the term

$$\int_k \int_{z, z'} \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}((\vec{z}_n \cdot \vec{p}_n), (z, k), (z', -k)) \dot{C}_{S, R}^{\Lambda}(k; z, z') \phi_{\tau_{1, n}, y_{1, n}}^R(z_{1, n}) \quad (5.5.85)$$

as follows

$$\begin{aligned} & \int_k \int_u \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; R}; Y_{\sigma_n}, u, u) \dot{C}^{\Lambda}(k) \\ & - \int_k \int_{\mathbb{R}} du \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \phi_{\tau_{1, n}, y_{1, n}; R} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right)) \dot{C}^{\Lambda}(k). \end{aligned} \quad (5.5.86)$$

Following the same steps that led to (5.5.84), we obtain

$$\mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; D}; Y_{\sigma_n}, u, u) = \lim_{c \rightarrow +\infty} \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; R}; Y_{\sigma_n}, u, u) \quad (5.5.87)$$

and

$$\begin{aligned} & \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \phi_{\tau_{1, n}, y_{1, n}; D} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right)) \\ & = \lim_{c \rightarrow +\infty} \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \phi_{\tau_{1, n}, y_{1, n}; R} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right)). \end{aligned} \quad (5.5.88)$$

Part A1) in the proof of Theorem 9 implies that the integrands of each term on the RHS of the FEs (5.5.80), in the case of Robin boundary conditions are bounded independently of the Robin parameter  $c$ , and the Lemmas 24 show that these bounds are integrable w.r.t.  $u$ . We refer the reader to the proof of Theorem 9 for more details.

A2) Integration: Lebesgue's dominated convergence theorem together with (5.5.81)-(5.5.88) and the FES (5.5.80) gives

$$\partial_{\Lambda} \mathcal{S}_{l, n; D}^{\Lambda, \Lambda_0}(\vec{p}_n; \phi_{\tau_{1, n}, y_{1, n}}^D) = \lim_{c \rightarrow +\infty} \partial_{\Lambda} \mathcal{S}_{l, n; R}^{\Lambda, \Lambda_0}(\vec{p}_n; \phi_{\tau_{1, n}, y_{1, n}}^R). \quad (5.5.89)$$

This implies (again by the Lebesgue's dominated convergence theorem and the integrability of the bound (5.5.13) in the proof of Theorem 9)

$$\int_{\Lambda}^{\Lambda_0} d\lambda \partial_{\lambda} \mathcal{S}_{l, n; D}^{\lambda, \Lambda_0}(\vec{p}_n; \phi_{\tau_{1, n}, y_{1, n}}^D) = \lim_{c \rightarrow +\infty} \int_{\Lambda}^{\Lambda_0} d\lambda \partial_{\lambda} \mathcal{S}_{l, n; R}^{\lambda, \Lambda_0}(\vec{p}_n; \phi_{\tau_{1, n}, y_{1, n}}^R). \quad (5.5.90)$$

– Irrelevant terms: These terms are characterized by  $n \geq 4$ . Using the boundary condition

$$\mathcal{S}_{l, n; \star}^{\Lambda_0 \Lambda_0}(\vec{p}_n; \phi_{\tau_{1, n}, y_{1, n}}^{\star}) = 0, \quad \star \in \{D, R\} \quad (5.5.91)$$

together with (5.5.90), we deduce

$$\mathcal{S}_{l, n; D}^{\Lambda, \Lambda_0}(\vec{p}_n; \phi_{\tau_{1, n}, y_{1, n}}^D) = \lim_{c \rightarrow +\infty} \mathcal{S}_{l, n; R}^{\Lambda, \Lambda_0}(\vec{p}_n; \phi_{\tau_{1, n}, y_{1, n}}^R). \quad (5.5.92)$$

– Relevant terms ( $n = 2$ ): We have

$$\begin{aligned} \int_{\Lambda}^{\Lambda_0} d\lambda \partial_{\lambda} \mathfrak{S}_{l,2;R}^{\lambda, \Lambda_0} \left( \vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^R \right) \\ = \mathfrak{S}_{l,2;R}^{\Lambda_0, \Lambda_0} \left( \vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^R \right) - \mathfrak{S}_{l,2;R}^{\Lambda, \Lambda_0} \left( \vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^R \right) \end{aligned} \quad (5.5.93)$$

and

$$\int_{\Lambda}^{\Lambda_0} d\lambda \partial_{\lambda} \mathfrak{S}_{l,2;D}^{\lambda, \Lambda_0} \left( \vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^R \right) = -\mathfrak{S}_{l,2;D}^{\Lambda, \Lambda_0} \left( \vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^D \right). \quad (5.5.94)$$

In (5.5.94), we used the boundary condition (5.4.21) for the Dirichlet case. The boundary condition (5.4.18) implies

$$\begin{aligned} \mathfrak{S}_{l,2;R}^{\Lambda_0, \Lambda_0} \left( \vec{p}_n; \phi_{\tau_{1,2}, y_{1,2}}^R \right) &= s_{l;R}^{\Lambda_0, \Lambda_0} \prod_{i=1}^2 p_R(\tau_i; y_i, 0) \\ &+ e_{l;R}^{\Lambda_0, \Lambda_0} \{ p_R(\tau_1; y_1, 0) \partial_n p_R(\tau_2; y_2, 0) + p_R(\tau_2; y_2, 0) \partial_n p_R(\tau_1; y_1, 0) \}. \end{aligned} \quad (5.5.95)$$

Using

$$|\partial_n p_R(\tau_1; y_1, 0)| \leq O(1) \tau_i^{-\frac{1}{2}} p_B(\tau_{i,\delta}; y_i, 0),$$

and the fact that  $s_{l;R}^{\Lambda_0, \Lambda_0}$  and  $e_{l;R}^{\Lambda_0, \Lambda_0}$  are uniformly bounded w.r.t. the Robin parameter  $c$ , which is implied by the bound given in Theorem 9 for  $s = 1$ ,  $r_1 \in \{0, 1\}$  and  $r_2 = 0$ , we obtain from  $p_R \rightarrow_{c \rightarrow +\infty} p_D$

$$\lim_{c \rightarrow +\infty} \mathfrak{S}_{l,2;R}^{\Lambda_0, \Lambda_0} \left( \vec{p}_n; \phi_{\tau_{1,2}, y_{1,2}}^R \right) = 0. \quad (5.5.96)$$

Therefore, we deduce

$$\mathfrak{S}_{l,2;D}^{\Lambda, \Lambda_0} \left( \vec{p}_n; \phi_{\tau_{1,2}, y_{1,2}}^D \right) = \lim_{c \rightarrow +\infty} \mathfrak{S}_{l,2;R}^{\Lambda, \Lambda_0} \left( \vec{p}_n; \phi_{\tau_{1,2}, y_{1,2}}^R \right). \quad (5.5.97)$$

This ends the proof of (7.3.8). □

### 5.5.3 Proof of Corollary 4

*Proof.* In this part, we prove the bounds (5.5.3) and (5.5.4). As a consequence of Theorem 9, we have for Robin boundary conditions

$$\begin{aligned} \left| \mathfrak{S}_{l,n;R}^{\Lambda, \Lambda_0} \left( \vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^R \right) \right| \\ \leq (\Lambda + m)^{3-n} \mathcal{F}_1 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{F}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{n,l;\delta}^{\Lambda,0}(\tau_{1,n}), \quad \forall n \geq 2. \end{aligned} \quad (5.5.98)$$

Using Theorem 9 and taking the limit  $c \rightarrow +\infty$ , we deduce

$$\begin{aligned} & \left| \mathcal{S}_{l,n;D}^{\Lambda,\Lambda_0} \left( \vec{p}_n; \phi_{\tau,y_1,s}^D \right) \right| \\ & \leq (\Lambda + m)^{3-n} \mathcal{P}_1 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{n,l;\delta}^{\Lambda,0}(\tau_{1,n}), \quad \forall n \geq 2. \end{aligned} \quad (5.5.99)$$

For  $n = 2$ , it is possible to obtain a sharper bound by performing a Taylor expansion around 0 of the test functions  $p_R(\tau_i; \cdot, y_i)$  as follows

$$\begin{aligned} \mathcal{S}_{l,2;R}^{\Lambda,\Lambda_0} \left( \vec{p}_n; \phi_{\tau_1,2,y_1,2}^R \right) &= s_{l;R}^{\Lambda,\Lambda_0} p_R(\tau_1; y_1, 0) p_R(\tau_2; y_2, 0) + e_{l;R}^{\Lambda,\Lambda_0} \{ p_R(\tau_1; y_1, 0) (\partial_n p_R)(\tau_2; y_2, 0) \\ & \quad + p_R(\tau_2; y_2, 0) (\partial_n p_R)(\tau_1; y_1, 0) \} + l_{l,2;R}^{\Lambda,\Lambda_0} (p_R(\tau_1; \cdot, y_1), p_R(\tau_2; \cdot, y_2)). \end{aligned} \quad (5.5.100)$$

Taking the limit  $c \rightarrow \infty$ , we deduce

$$\mathcal{S}_{l,2;D}^{\Lambda,\Lambda_0} \left( \vec{p}_n; \phi_{\tau_1,2,y_1,2}^D \right) = \lim_{c \rightarrow +\infty} \tilde{l}_{l,2;R}^{\Lambda,\Lambda_0} (p_R(\tau_1; \cdot, y_1), p_R(\tau_2; \cdot, y_2)), \quad (5.5.101)$$

where the remainder  $\tilde{l}_{l,2;R}^{\Lambda,\Lambda_0}$  is given by

$$\begin{aligned} & \left( \int_{z_1, z_2} z_1 z_2 \mathcal{S}_{l,2;R}^{\Lambda,\Lambda_0} ((z_1, p), (z_2, -p)) \right) (\partial_n \phi_1)(0) (\partial_n \phi_2)(0) \\ & + \int_{z_1, z_2} \mathcal{S}_{l,2;R}^{\Lambda,\Lambda_0} ((z_1, p), (z_2, -p)) \left( \int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \right) \times \left( \int_0^1 dt' (1-t') (\partial_{t'}^2 \phi_2)(t'z_2) \right) \\ & + (\partial_n \phi_2)(0) \int_{z_1, z_2} z_2 \mathcal{S}_{l,2;R}^{\Lambda,\Lambda_0} ((z_1, p), (z_2, -p)) \int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \\ & + (\partial_n \phi_1)(0) \int_{z_1, z_2} z_1 \mathcal{S}_{l,2;R}^{\Lambda,\Lambda_0} ((z_1, p), (z_2, -p)) \int_0^1 dt (1-t) (\partial_t^2 \phi_2)(tz_2) \end{aligned} \quad (5.5.102)$$

and  $\phi_i(z_i) := p_R(\tau_i; z_i, y_i)$ . These terms can be bounded in a similar way as  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  and  $\mathcal{H}_3$  in the proof of Theorem 9. One should keep in mind that the test functions considered within the proof of Theorem 9 were products of bulk heat kernels. However, the same bounds (5.5.55), (5.5.68) and (5.5.57) which are uniform in  $c$ , hold for Robin type test functions using the bounds (3.1.12). Therefore, we deduce

$$\left| \tilde{l}_{l,2;R}^{\Lambda,\Lambda_0} (\phi_1, \phi_2) \right| \leq (\Lambda + m)^{-2} \tau_1^{-\frac{1}{2}} \tau_2^{-\frac{1}{2}} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}), \quad (5.5.103)$$

which gives

$$\left| \mathcal{S}_{l,2;D}^{\Lambda,\Lambda_0} \left( \vec{p}_n; \phi_{\tau_1,2,y_1,2}^R \right) \right| \leq (\Lambda + m)^{-2} \tau_1^{-\frac{1}{2}} \tau_2^{-\frac{1}{2}} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}), \quad (5.5.104)$$

and this ends the proof of Corollary 4.  $\square$

## 5.6 The minimal form of the bare interaction

In this section, we show that the bare interaction (5.1.2) corresponds to the boundary conditions imposed in Theorem 9 for  $L_\star^{\Lambda, \Lambda_0}$ . Given  $\phi \in \text{supp } \mu_\star^{\Lambda, \Lambda_0}$ , we expand  $L_\star^{\Lambda_0, \Lambda_0}(\phi)$  in powers of the field  $\phi$ :

$$L_\star^{\Lambda_0, \Lambda_0}(\phi) = \sum_{n=1}^{+\infty} \frac{1}{n!} \int_{\vec{z}_n} \int_{\mathbb{R}^{3n}} \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3} \mathcal{L}_{l, n; \star}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \\ \times \delta^{(3)}(p_1 + \dots + p_n) \phi(z_1, p_1) \cdots \phi(z_n, p_n). \quad (5.6.1)$$

Using (5.5.8), we can write

$$L_\star^{\Lambda_0, \Lambda_0}(\phi) = D^{\Lambda_0, \Lambda_0}(\phi) + S_\star^{\Lambda_0, \Lambda_0}(\phi),$$

where

$$D^{\Lambda_0, \Lambda_0}(\phi) := \sum_{n=1}^{+\infty} \frac{1}{n!} \int_{\vec{z}_n} \int_{\mathbb{R}^{3n}} \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3} \mathcal{D}_{l, n}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \delta^{(3)}(p_1 + \dots + p_n) \\ \times \phi(z_1, p_1) \cdots \phi(z_n, p_n) \quad (5.6.2)$$

and

$$S_\star^{\Lambda_0, \Lambda_0}(\phi) = \sum_{n=1}^{+\infty} \frac{1}{n!} \int_{\vec{z}_n} \int_{\mathbb{R}^{3n}} \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3} \mathcal{S}_{l, n; \star}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \delta^{(3)}(p_1 + \dots + p_n) \\ \times \phi(z_1, p_1) \cdots \phi(z_n, p_n). \quad (5.6.3)$$

Proposition 5 in chapter 4 implies that there exists  $f$  in  $L^2(\mathbb{R}^+)$  such that

$$\phi(z, p) = \int_0^\infty dz' C_\star^{\Lambda, \Lambda_0}(p; z, z') f(p, z'),$$

which can be rewritten as

$$\phi(p, z) = \int_0^\infty dz' \int_{\frac{1}{\Lambda^2}}^{\frac{1}{\Lambda_0^2}} d\lambda e^{-\lambda(p^2 + m^2)} p_\star(\lambda; z, z') f(p, z').$$

Therefore, we can write

$$\int_{z_1, \dots, z_n} \mathcal{S}_{l, n; \star}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \phi(z_1, p_1) \cdots \phi(z_n, p_n) \\ = \int_{\vec{z}'_n} \prod_{i=1}^n \int_{\frac{1}{\Lambda^2}}^{\frac{1}{\Lambda_0^2}} d\lambda_i e^{-\lambda_i(p_i^2 + m^2)} f(p_i, z'_i) \mathcal{S}_{l, n; \star}^{\Lambda_0, \Lambda_0}(\vec{p}_n; \phi_{\lambda_{1, n}, z'_{1, n}}^\star). \quad (5.6.4)$$

The boundary conditions (5.4.19) and (5.4.21) imply that

$$S_{\star}^{\Lambda_0, \Lambda_0}(\phi) = \frac{1}{2} \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} \int_{z_1, z_2} \mathcal{S}_{l, 2; \star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) \phi(z_1, p) \phi(z_2, -p), \quad (5.6.5)$$

where  $\mathcal{S}_{l, 2; \star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p))$  is given by (5.4.18) for Robin/Neumann boundary conditions and by (5.4.21) for Dirichlet boundary conditions:

- Robin/Neumann boundary conditions ( $c \geq 0$ ): In this case, we obtain

$$S_{\star}^{\Lambda_0, \Lambda_0}(\phi) = \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} \left( \frac{1}{2} s_{\star}^{\Lambda_0} + c e_{\star}^{\Lambda_0} \right) \phi(0, p) \phi(0, -p), \quad (5.6.6)$$

where

$$s_{\star}^{\Lambda_0} = \int_{z_1, z_2} \mathcal{S}_{l, 2; \star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)), \quad e_{\star}^{\Lambda_0} = \int_{z_1, z_2} z_1 \mathcal{S}_{l, 2; \star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) \quad (5.6.7)$$

and  $\star \in \{R, N\}$ .

- Dirichlet boundary conditions: For Dirichlet boundary conditions, we obtain

$$S_D^{\Lambda_0, \Lambda_0}(\phi) = 0.$$

## 5.7 The Amputated vs the Non-Amputated theory

For quantum field theories on spaces without boundary, the renormalization problem of the amputated and the non-amputated theory is equivalent in the sense that the required counter-terms render finite the amputated and unamputated amplitudes, independently of the location of the external points of the unamputated diagrams [11, 13]. However, first order calculations [45] give clear evidence that this is not the case when one considers the renormalization of the semi-infinite model. The tadpole diverges w.r.t. the UV cutoff, and its renormalization depends on the location of the external points (i.e. if they are on the surface or not). If the two external points are not on the surface, then in addition to the usual mass counter-term only one additional surface counter-term, which diverges linearly in the UV cutoff, is needed. This is not the case when one considers the tadpole with at least one external point on the surface. The latter needs one additional surface counter-term which diverges logarithmically w.r.t. the UV cutoff. This suggests that the amputated and non-amputated diagrams are renormalized differently for the semi-infinite model. In this section, we prove the following proposition which sheds some light on this finding:

**Proposition 8.** Let  $\star \in \{R, N\}$ . We denote by  $C_\star$  the unregularized propagator associated to the boundary condition  $\star$ . For nonvanishing  $s_{l;\star}^{\Lambda_0}$  and  $e_{l;\star}^{\Lambda_0}$ , we have for  $y_2 > 0$

$$\begin{aligned} \lim_{y_1 \rightarrow 0^+} \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, y_1) C_R(p; z_2, y_2) \\ \neq \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, 0) C_R(p; z_2, y_2) \end{aligned} \quad (5.7.1)$$

and

$$\begin{aligned} \lim_{y_1 \rightarrow 0^+} \lim_{y_2 \rightarrow 0^+} \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, y_1) C_R(p; z_2, y_2) \\ \neq \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, 0) C_R(p; z_2, 0) . \end{aligned} \quad (5.7.2)$$

*Proof.* We give the proof of Proposition 8 in the case of Robin boundary conditions. Neumann b.c. can be treated analogously.

The particular choice of the boundary conditions (5.4.18)-(5.4.20) implies

$$\mathcal{S}_{l,2;R}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) = \left( s_R^{\Lambda_0} + e_R^{\Lambda_0} (\partial_{z_1} + \partial_{z_2}) \right) \delta_{z_1} \delta_{z_2}. \quad (5.7.3)$$

Hence, we obtain for  $y_1, y_2 > 0$

$$\begin{aligned} \int_{z_1, z_2} \mathcal{S}_{l,2;R}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, y_1) C_R(p; z_2, y_2) = s_R^{\Lambda_0} C_R(p; 0, y_1) C_R(p; 0, y_2) \\ + e_R^{\Lambda_0} \{ C_R(p; 0, y_1) \partial_n C_R(p; 0, y_2) + C_R(p; 0, y_2) \partial_n C_R(p; 0, y_1) \}. \end{aligned} \quad (5.7.4)$$

Using

$$\lim_{z \rightarrow 0} \lim_{y \rightarrow 0} \partial_z C_R(p; z, y) = -\kappa_p C_R(p; 0, 0), \quad \lim_{y \rightarrow 0} \lim_{z \rightarrow 0} \partial_z C_R(p; z, y) = c C_R(p; 0, 0) \quad (5.7.5)$$

with  $\kappa_p := \sqrt{p^2 + m^2}$ , we deduce

$$\begin{aligned} \int_{z_1, z_2} \mathcal{S}_{l,2;R}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, y_1) C_R(p; z_2, y_2) \\ = \left( s_R^{\Lambda_0} + 2c e_R^{\Lambda_0} \right) C_R(p; 0, y_1) C_R(p; 0, y_2), \quad \forall y_1, y_2 > 0, \end{aligned} \quad (5.7.6)$$

$$\begin{aligned} \int_{z_1, z_2} \mathcal{S}_{l,2;R}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, y_1) C_R(p; z_2, 0) \\ = \left( s_R^{\Lambda_0} + 2c e_R^{\Lambda_0} \right) C_R(p; 0, y_1) C_R(p; 0, 0) - e_R^{\Lambda_0} C_R(p; 0, y_1) (\kappa_p + c) C_R(p; 0, 0), \end{aligned} \quad (5.7.7)$$



$$\begin{aligned} & \int_{z_1, z_2} \mathcal{S}_{l,2}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, 0) C_R(p; z_2, 0) \\ &= \left( s_R^{\Lambda_0} + 2c e_R^{\Lambda_0} \right) C_R(p; 0, 0) C_R(p; 0, 0) - 2 e_R^{\Lambda_0} C_R(p; 0, 0) (\kappa_p + c) C_R(p; 0, 0), \end{aligned} \quad (5.7.8)$$

from which (5.7.1) and (5.7.2) follow directly.  $\square$

**Remark 3.** Denoting  $\sigma_R^{\Lambda_0} = s_R^{\Lambda_0} + 2ce_R^{\Lambda_0}$ , we deduce that (5.7.6) implies that the unamputated two-point function of the semi-infinite model which has two external points in the interior of the bulk, requires only the surface counter-term  $\sigma_R^{\Lambda_0}$  to be renormalized. This is not the case when at least one of the external points is on the surface. From (5.7.7) and (5.7.8), we deduce that  $\sigma_R^{\Lambda_0}$  is not sufficient and the additional surface counter-term  $e_R^{\Lambda_0}$  is required to make the two-point function finite. This generalizes the remarks given in [2, 47] and [45] concerning the tadpole to all loop orders.

## 5.8 Some properties of the surface weight factor for $s = 2$ and $l \geq 1$

In this section, we prove several lemmas that we use in the proof of Theorem 9. These lemmas concern the case  $s = 2$  for which the set of partitions  $\mathcal{P}_2$  simply reads

$$\mathcal{P}_2 := \{\Pi_0, \Pi_1\},$$

where  $\Pi_0 := \sigma_2$ ,  $\Pi_1 = \pi_1 \cup \pi_2$  and  $\pi_i = \{i\}$ .

From the definition (3.2.9), we have

$$\mathcal{W}_l^2(\sigma_2) = \left\{ T_l^{2,0}(Y_{\sigma_2}, 0; \vec{z}) \mid T_l^{2,0} \in \mathcal{T}^{2,0}, \quad v_2 \leq 3l - 1 \right\}$$

and

$$\begin{aligned} \mathcal{W}_l^2(\Pi_1) &= \left\{ T_{l,1}^{1,0}(y_1, 0; \vec{z}) \cup T_{l,2}^{1,0}(y_2, 0; \vec{z}') \mid T_{l,1}^{1,0}, T_{l,2}^{1,0} \in \mathcal{T}_l^{1,0} \right\} \\ &= \mathcal{W}_l^1(\pi_1) \cup \mathcal{W}_l^1(\pi_2), \end{aligned}$$

which implies that the global surface weight factor  $\mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2})$  simply reads

$$\mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}) = \sum_{T_l^{2,0} \in \mathcal{W}_l^2(\sigma_2)} \mathcal{F}_\delta^0(\Lambda, \tau_1, \tau_2; T_l^{2,0}; Y_{\sigma_2}) + \mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_1) \mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_2), \quad (5.8.1)$$

where

$$\mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_i) := \sum_{T_l^{1,0} \in \mathcal{W}_l^1(\pi_i)} \mathcal{F}_\delta^0(\Lambda, 2\tau_i; T_l^{1,0}; y_i). \quad (5.8.2)$$

Note that (5.8.1) implies

$$\mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_1) \mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_2) \leq \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}). \quad (5.8.3)$$

**Lemma 8.** Let  $v$  be the total number of vertices of incidence number 2 of the tree  $T_l^{2,0}$ . For  $p \geq 1$ , we denote by  $\vec{z} = (z_1, \dots, z_p)$  the set of the internal vertices of  $T_l^{2,0}$  and  $(y_1, y_2) \in (\mathbb{R}^+)^2$  its external vertices. For  $\Lambda_{\mathcal{J}} := \{\Lambda_i \mid 1 \leq i \leq v-1\}$ ,  $\tilde{\Lambda} \in [\Lambda, \Lambda_0]$  and  $\tau_1, \tau_2 > 0$ , we have

$$\int_{\vec{z}} \mathcal{F}_{\delta}^0 \left( \Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_1, \tau_2; T_l^{2,0}; \vec{z}; Y_{\sigma_2} \right) \leq \int_0^{\infty} dz p_B(c_{1,\delta}; z, y_1) p_B(c_{2,\delta}; z, y_2) p_B \left( \frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right) \quad (5.8.4)$$

and

$$\int_0^{\infty} dz p_B(c_{1,\delta}; z, y_1) p_B(c_{2,\delta}; z, y_2) p_B \left( \frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right) \leq 2^v \int_{\vec{z}} \mathcal{F}_{\delta}^0 \left( \Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_1, \tau_2; T_l^{2,0}; \vec{z}; Y_{\sigma_2} \right), \quad (5.8.5)$$

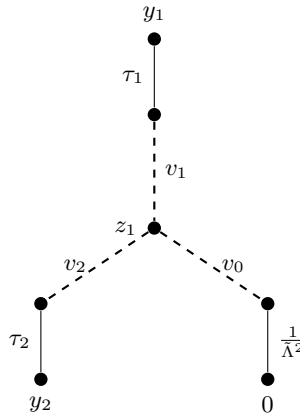
where  $0 \leq v_1, v_2, v_0 \leq v$  such that  $v_1 + v_2 + v_0 = v$ ,  $c_{1,\delta} := (1+\delta)c_1$  and  $c_{2,\delta} := (1+\delta)c_2$ . The parameters  $c_1, c_2$  and  $\tilde{\Lambda}_1$  are given by

$$c_1 = \tau_1 + \left( \sum_{i=1}^{v_1} \frac{1}{\Lambda_i^2} \right) (1 - \delta_{v_1,0}), \quad (5.8.6)$$

$$c_2 = \tau_2 + \left( \sum_{i=v_1+1}^{v_1+v_2-1} \frac{1}{\Lambda_i^2} \right) (1 - \delta_{v_2,0}), \quad (5.8.7)$$

$$\frac{1}{\tilde{\Lambda}_1^2} = \left( \sum_{i=v_1+v_2}^{v-1} \frac{1}{\Lambda_i^2} \right) (1 - \delta_{v,0}) + \frac{1}{\Lambda^2}. \quad (5.8.8)$$

*Proof.* • First, we prove the bound (5.8.4). A tree  $T_l^{2,0}$  in  $\mathcal{W}_l^2(\sigma_2)$  has the following structure



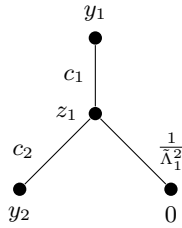
It contains one internal vertex of incidence number 3 and all the other internal vertices are of incidence number 2. We assume that each dashed line contains a number  $v_i \geq 1$  of internal vertices of incidence number 2. The case  $v_i = 0$  can be treated similarly. Remember that  $v_0 + v_1 + v_2 = v$ . Let  $\{z_2, \dots, z_{v_1+1}\}$ ,  $\{z_{v_1+2}, \dots, z_{v_1+v_2}\}$  and  $\{z_{v_1+v_2+1}, \dots, z_v\}$  be respectively the internal vertices on the paths from  $z_1$  to  $y_1$ ,  $z_1$  to  $y_2$  and  $z_1$  to 0. From (3.2.49), the integral surface weight factor of  $T_l^{2,0}$  is then given by

$$\begin{aligned} \int_{\vec{z}} \mathcal{F}_\delta^0 \left( \Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_1, \tau_2; T_l^{2,0}; \vec{z}; Y_{\sigma_2} \right) &= \int_{z_1, \dots, z_v} \prod_{j=2}^{v_1+1} p_B \left( \frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j \right) p_B(\tau_{1,\delta}; z_{v_1+1}, y_1) \\ &\times p_B \left( \frac{1+\delta}{\Lambda_{v_1+1}^2}; z_1, z_{v_1+2} \right) \prod_{j=v_1+3}^{v_1+v_2} p_B \left( \frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j \right) p_B(\tau_{2,\delta}; z_{v_1+v_2}, y_2) \\ &\times p_B \left( \frac{1+\delta}{\Lambda_{v_1+v_2}^2}; z_1, z_{v_1+v_2+1} \right) \prod_{j=v_1+v_2+2}^v p_B \left( \frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j \right) p_B \left( \frac{1+\delta}{\tilde{\Lambda}^2}; z_v, 0 \right). \end{aligned} \quad (5.8.9)$$

Bounding the integral over  $\mathbb{R}^+$  by the integral over  $\mathbb{R}$  and using (3.1.5), we obtain

$$\int_{z_2 \dots z_{v_1+1}} \prod_{j=2}^{v_1+1} p_B \left( \frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j \right) p_B(\tau_{1,\delta}; z_{v_1+1}, y_1) \leq p_B(c_{1,\delta}; z_1, y_1), \quad (5.8.10)$$

where  $c_{1,\delta} = (1+\delta) \left( \tau_1 + \sum_{i=1}^{v_1} \frac{1}{\Lambda_i^2} \right)$ . Proceeding similarly on the paths from  $z_1$  to  $y_2$  and from  $z_1$  to 0, we obtain that the weight factor of a tree in  $\mathcal{W}_l^2(\sigma_2)$  is bounded by the weight factor of the tree



where  $c_1$ ,  $c_2$  and  $\frac{1}{\tilde{\Lambda}_1^2}$  are the new parameters associated respectively to the edges  $(z_1, y_1)$ ,  $(z_1, y_2)$  and  $(z_1, 0)$ . The relations between these new parameters and those of the tree  $T_l^{2,0}$  are given by

$$c_2 = \tau_2 + \left( \sum_{i=1}^{v_2} \frac{1}{\Lambda_{i+v_1}^2} \right) \quad \text{and} \quad \frac{1}{\tilde{\Lambda}_1^2} = \left( \sum_{i=v_1+v_2}^{v-1} \frac{1}{\Lambda_i^2} \right) + \frac{1}{\tilde{\Lambda}^2}.$$

This proves the statement (5.8.4).

- To prove (5.8.5), we assume without loss of generality that  $v_1 \geq 1$ . Using (3.1.8), we find

$$\begin{aligned} p_B(c_{1,\delta}; z_1, y_1) &= \int_{\mathbb{R}^{v_1}} dz_2 \cdots dz_{v_1+1} \prod_{j=2}^{v_1+1} p_B\left(\frac{1+\delta}{\Lambda_{j-2}^2}; z_j, z_{j-1}\right) p_B(\tau_{1,\delta}; z_{v_1+1}, y_1) \\ &\leq 2^{v_1} \int_{(\mathbb{R}^+)^{v_1}} dz_2 \cdots dz_{v_1+1} \prod_{j=2}^{v_1+1} p_B\left(\frac{1+\delta}{\Lambda_{j-2}^2}; z_j, z_{j-1}\right) p_B(\tau_{1,\delta}; z_{v_1+1}, y_1). \end{aligned} \quad (5.8.11)$$

Proceeding similarly for  $p_B(c_{2,\delta}; z_1, y_2)$  and  $p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right)$ , we deduce

$$\begin{aligned} \int_0^\infty dz p_B(c_{1,\delta}; z, y_1) p_B(c_{2,\delta}; z, y_2) p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right) \\ \leq 2^v \int_{\vec{z}} \mathcal{F}_\delta^0\left(\Lambda_{\mathcal{F}}, \tilde{\Lambda}; \tau_1, \tau_2; T_l^{2,0}; \vec{z}; Y_{\sigma_2}\right). \end{aligned} \quad (5.8.12)$$

□

**Lemma 9.** Let  $W_l^2(\Pi_1) := T_{l;1}^{1,0}(y_1, 0; \vec{z}) \cup T_{l;2}^{1,0}(y_2, 0; \vec{z}')$  be a forest in  $\mathcal{W}_l^2(\Pi_1)$  with  $(y_1, y_2) \in (\mathbb{R}^+)^2$  and  $v_{2,1}$  (resp.  $v_{2,2}$ ) the total number of vertices of incidence number 2 of the tree  $T_{l;1}^{1,0}$  (resp.  $T_{l;2}^{1,0}$ ). For  $p, q \geq 1$ , we denote by  $\vec{z} = (z_1, \dots, z_p)$  (resp.  $\vec{z}' = (z'_1, \dots, z'_q)$ ) the set of the internal vertices of  $T_{l;1}^{1,0}$  (resp.  $T_{l;2}^{1,0}$ ). For  $\Lambda_{\mathcal{F}} := \{\Lambda_i, \Lambda'_j \mid 1 \leq i \leq v_{2,1} - 1, 1 \leq j \leq v_{2,2} - 1\}$ ,  $\tilde{\Lambda}_1, \tilde{\Lambda}_2 \in [\Lambda, \Lambda_0]$  and  $\tau_1, \tau_2 > 0$ , we have

$$\int_{\vec{z}} \mathcal{F}_\delta^0\left(\Lambda_{\mathcal{F}}, \tilde{\Lambda}_1, \tilde{\Lambda}_2; \tau_1, \tau_2; W_l^2(\Pi_1); \vec{z}; Y_{\sigma_2}\right) \leq p_B(\tilde{c}_{1,\delta}; y_1, 0) p_B(\tilde{c}_{2,\delta}; y_2, 0) \quad (5.8.13)$$

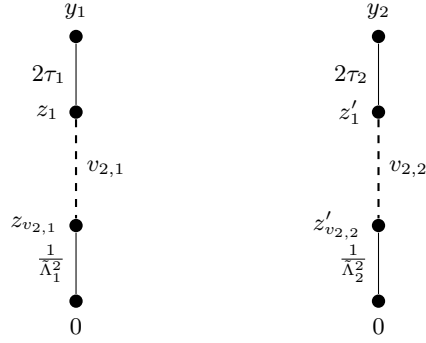
and

$$p_B(\tilde{c}_{1,\delta}; y_1, 0) p_B(\tilde{c}_{2,\delta}; y_2, 0) \leq 2^{v_{2,1}+v_{2,2}} \int_{\vec{z}} \mathcal{F}_\delta^0\left(\Lambda_{\mathcal{F}}, \tilde{\Lambda}_1, \tilde{\Lambda}_2; \tau_1, \tau_2; W_l^2(\Pi_1); \vec{z}; Y_{\sigma_2}\right), \quad (5.8.14)$$

where  $\tilde{c}_1$  and  $\tilde{c}_2$  are given by

$$\begin{aligned} \tilde{c}_1 &= 2\tau_1 + \sum_{i=1}^{v_{2,1}-1} \frac{1}{\Lambda_{i,1}^2} + \frac{1}{\tilde{\Lambda}_1^2}, \\ \tilde{c}_2 &= 2\tau_2 + \sum_{i=1}^{v_{2,2}-1} \frac{1}{\Lambda'_{i,2}{}^2} + \frac{1}{\tilde{\Lambda}_2^2}. \end{aligned}$$

*Proof.* The forest  $W_l^2(\Pi_1)$  is of the following form



The integrated weight factor of this forest reads

$$\begin{aligned} & \int_{\vec{z}} \mathcal{F}_\delta^0 \left( \Lambda_{\mathcal{J}}, \tilde{\Lambda}_1, \tilde{\Lambda}_2; \tau_{1,2}; W_l^2(\Pi_1); \vec{z}; y_1, y_2 \right) \\ &= \int_0^\infty dz_1 \cdots dz_{v_{2,1}} \prod_{j=2}^{v_{2,1}} p_B \left( \frac{1+\delta}{\Lambda_{j-1}^2}; z_j, z_{j-1} \right) p_B(2\tau_{1,\delta}; z_1, y_1) p_B \left( \frac{1+\delta}{\tilde{\Lambda}_1^2}; z_{v_{2,1}}, 0 \right) \\ & \quad \times \int_0^\infty dz'_1 \cdots dz'_{v_{2,2}} \prod_{j=2}^{v_{2,2}} p_B \left( \frac{1+\delta}{\Lambda_{j-1}'^2}; z'_j, z'_{j-1} \right) p_B(2\tau_{2,\delta}; z'_1, y_2) p_B \left( \frac{1+\delta}{\tilde{\Lambda}_2^2}; z'_{v_{2,2}}, 0 \right). \end{aligned}$$

Bounding the integral over  $\mathbb{R}^+$  by the integral over  $\mathbb{R}$  and using (3.1.5), we obtain

$$\int_{\vec{z}} \mathcal{F}^0 \left( \Lambda_{\mathcal{J}}, \tilde{\Lambda}_1, \tilde{\Lambda}_2; \tau_{1,2}; W_l^2(\Pi_1); \vec{z}; y_1, y_2 \right) \leq p_B(\tilde{c}_{1,\delta}; y_1, 0) p_B(\tilde{c}_{2,\delta}; y_2, 0),$$

where for  $v_{2,i} > 1$

$$\tilde{c}_1 = 2\tau_1 + \sum_{i=1}^{v_{2,1}-1} \frac{1}{\Lambda_{i,1}^2} + \frac{1}{\tilde{\Lambda}_1^2}, \quad (5.8.15)$$

$$\tilde{c}_2 = 2\tau_2 + \sum_{i=1}^{v_{2,2}-1} \frac{1}{\Lambda_{i,2}^2} + \frac{1}{\tilde{\Lambda}_2^2}. \quad (5.8.16)$$

If  $v_{2,i} = 1$ , then  $\tilde{c}_i = 2\tau_i + \frac{1}{\tilde{\Lambda}_i^2}$ .

Using again (3.1.8) and proceeding as in (5.8.11), we deduce

$$p_B(\tilde{c}_{1,\delta}; y_1, 0) p_B(\tilde{c}_{2,\delta}; y_2, 0) \leq 2^{v_{2,1}+v_{2,2}} \int_{\vec{z}} \mathcal{F}^0 \left( \Lambda_{\mathcal{J}}, \tilde{\Lambda}_1, \tilde{\Lambda}_2; \tau_{1,2}; W_l^2(\Pi_1); \vec{z}; y_1, y_2 \right). \quad (5.8.17)$$

□

**Lemma 10.** For  $0 \leq \alpha \leq 1$  and  $y_1, y_2 \in \mathbb{R}$ , we have

$$|\partial_n^\alpha \phi_i(0)| \leq C_{0,\delta} \tau_i^{-\frac{\alpha}{2}} \mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_i), \quad \forall 0 < \delta < 1, \quad (5.8.18)$$

where  $\partial_n^\alpha \phi_i(0) = \lim_{z_i \rightarrow 0^+} \partial_{z_i}^\alpha \phi_i(z_i)$  with  $\phi_i(z_i) := p_B(\tau_i; z_i, y_i)$  and  $C_{0,\delta}$  is defined in (3.1.10).

*Proof.* For  $\alpha = 0$ , we have

$$\phi_i(0) = \frac{1}{\sqrt{2\pi\tau_i}} e^{-\frac{y_i^2}{2\tau_i}} \leq \sqrt{2} p_B(2\tau_i; y_i, 0). \quad (5.8.19)$$

For  $\alpha = 1$ , we have  $|\partial_n \phi_i(0)| = \frac{y_i}{\tau_i} \phi_i(0)$ . Using the bound (3.1.10) for  $r = 1$ , we obtain

$$|\partial_n \phi_i(0)| \leq \sqrt{2} C_{0,\delta} \tau_i^{-\frac{1}{2}} p_B(2\tau_i, \delta; y_i, 0). \quad (5.8.20)$$

We consider the surface tree  $T_{l;i}^{1,0}$  which consists of the external vertex  $y_i$  and the surface external vertex 0. We associate to the external line  $(0, y_i)$  the parameter  $2\tau_i$ . The integrated surface weight factor of the tree  $T_{l;i}^{1,0}$  then reads

$$\mathcal{F}_{1,l;\delta}^0(2\tau_i; T_{l;i}^{1,0}; y_i) = p_B(2\tau_i, \delta; y_i, 0). \quad (5.8.21)$$

Since  $T_{l;i}^{1,0} \in \mathcal{W}_l^1(\pi_1)$ , we obtain using (5.8.2)

$$\mathcal{F}_{1,l;\delta}^0(2\tau_i; T_{l;i}^{1,0}; y_i) \leq \mathcal{F}_{1,l;\delta}^0(\tau_i). \quad (5.8.22)$$

Combining the bounds (7.3.11), (5.8.20) and (5.8.22), we deduce

$$|\partial_n^\alpha \phi_i(0)| \leq \sqrt{2} C_{0,\delta} \tau_i^{-\frac{\alpha}{2}} \mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_i), \quad \alpha \in \{0, 1\}. \quad (5.8.23)$$

Furthermore, we also obtain

$$\left| \partial_n^\alpha \phi_1(0) \partial_n^\beta \phi_2(0) \right| \leq 2 C_{0,\delta}^2 \tau_1^{-\frac{\alpha}{2}} \tau_2^{-\frac{\beta}{2}} \prod_{i=1}^2 \mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_i), \quad \alpha, \beta \in \{0, 1\}. \quad (5.8.24)$$

Recalling (5.8.3), we deduce

$$\left| \partial_n^\alpha \phi_1(0) \partial_n^\beta \phi_2(0) \right| \leq 2 C_{0,\delta}^2 \tau_1^{-\frac{\alpha}{2}} \tau_2^{-\frac{\beta}{2}} \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}). \quad (5.8.25)$$

□

**Lemma 11.** For  $0 < t \leq 1$ ,  $\gamma \in \mathbb{N}$ ,  $y_1 \in \mathbb{R}$  and  $0 < \delta < \delta' < 1$ , we have

$$\left( \frac{y_1}{\sqrt{\tau_1}} \right)^\gamma \mathcal{F}_{1,l;\delta}^{\Lambda,0}(t, \tau_1) \leq O(1) t \left( 1 + \frac{\tau_1^{-\frac{1}{2}}}{\Lambda} \right)^\gamma \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1), \quad (5.8.26)$$

where

$$\mathcal{F}_{1,l;\delta}^{\Lambda,0}(t, \tau_1) := \sum_{W_l^1 \in \mathcal{W}_l^1(\pi_1)} \mathcal{F}_\delta^0 \left( \Lambda, \frac{2\tau_1}{t^2}; W_l^1; \frac{y_1}{t} \right) = \sum_{v=0}^{3l-1} \mathcal{F}_\delta^0 \left( \Lambda, \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right), \quad (5.8.27)$$

and  $O(1)$  is a constant which depends on  $\delta, \delta', v, \gamma$  and the loop order  $l$ . We also have

$$\left( \frac{y_1}{\sqrt{\tau_1}} \right)^\gamma \mathcal{F}_{1,l;\delta}^{\Lambda,0}(t, \tau_1^-) \leq O(1) t \left( 1 + \frac{\tau_1^{-\frac{1}{2}}}{\Lambda} \right)^\gamma \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1^-). \quad (5.8.28)$$

*Proof.* For  $s = 1$  the global set of forests  $\mathcal{W}_l^1(\pi_1)$  consists of all surface trees with two external vertices (including the surface external vertex) which have a number of vertices of incidence number 2 less or equal to  $3l - 2$ . We consider the surface tree  $T_l^{1,0} \in \mathcal{W}_l^1(\pi_1)$  with the external vertex  $\frac{y_1}{t}$  and the internal vertices  $\vec{z}_v = (z_1, \dots, z_v)$ . Let  $v$  be the number of its vertices of incidence number 2 and  $\Lambda_{\mathcal{F}} = (\Lambda_i)_{1 \leq i \leq v-1}$ ,  $\Lambda_v$  and  $\tau_1/t^2$  be respectively the parameters associated to the internal lines, the surface external line and the external line of  $T_l^{1,0}$ . Then the integrated surface weight factor of  $T_l^{1,0}$  reads

$$\begin{aligned} \mathcal{F}_\delta^0 \left( \Lambda_{\mathcal{F}}; \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right) \\ = \int_{\vec{z}_v} p_B \left( \frac{2\tau_{1,\delta}}{t^2}; \frac{y_1}{t}, z_1 \right) \prod_{i=1}^{v-1} p_B \left( \frac{1+\delta}{\Lambda_i^2}; z_i, z_{i+1} \right) p_B \left( \frac{1+\delta}{\Lambda_v^2}; z_v, 0 \right). \end{aligned} \quad (5.8.29)$$

If  $y_1 \in \mathbb{R}^-$ , (5.8.29) is bounded by

$$p_B \left( \frac{2\tau_{1,\delta}}{t^2}; \frac{y_1}{t}, 0 \right) \int_{\vec{z}_v} \prod_{i=1}^{v-1} p_B \left( \frac{1+\delta}{\Lambda_i^2}; z_i, z_{i+1} \right) p_B \left( \frac{1+\delta}{\Lambda_v^2}; z_v, 0 \right). \quad (5.8.30)$$

Therefore, we obtain

$$\begin{aligned} \mathcal{F}_\delta^0 \left( \Lambda_{\mathcal{F}}; \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right) \\ \leq O(1) t p_B(2\tau_{1,\delta}; y_1, 0) \leq O(1) t \mathcal{F}_{1,l,\delta}^{\Lambda,0}(\tau_1), \quad \forall T_l^{1,0} \in \mathcal{W}_l^1(\pi_1). \end{aligned} \quad (5.8.31)$$

Using (3.1.10), we deduce

$$\left( \frac{y_1}{\sqrt{\tau_1}} \right)^\gamma \mathcal{F}_{1,l,\delta}^0(t, \tau_1) \leq O(1) t \mathcal{F}_{1,l,\delta'}^{\Lambda,0}(\tau_1), \quad (5.8.32)$$

where  $0 < \delta < \delta' < 1$ .

Now, we treat the case in which  $y_1 \in \mathbb{R}^+$ . Bounding the integral over  $(\mathbb{R}^+)^v$  by the integral over  $\mathbb{R}^v$  in (5.8.29) and using (3.1.5), we obtain

$$\begin{aligned} \mathcal{F}_\delta^0 \left( \Lambda_{\mathcal{F}}; \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right) \\ \leq t p_B \left( 2\tau_{1,\delta} + \sum_{i=1}^v \frac{t^2(1+\delta)}{\Lambda_i^2}; y_1, 0 \right) \leq t p_B \left( 2\tau_{1,\delta} + \sum_{i=1}^v \frac{(1+\delta)}{\Lambda_i^2}; y_1, 0 \right). \end{aligned} \quad (5.8.33)$$

Furthermore, using (3.1.10) we obtain for all  $0 \leq \delta < \delta' < 1$

$$\begin{aligned} \left( \frac{y_1}{\sqrt{\tau_1}} \right)^\gamma p_B \left( 2\tau_{1,\delta} + \sum_{i=1}^v \frac{1+\delta}{\Lambda_i^2}; y_1, 0 \right) \\ \leq C_{\delta,\delta'} \left( 1 + \sum_{i=1}^v \frac{\tau_1^{-\frac{1}{2}}}{\sqrt{2\Lambda_i}} \right)^\gamma p_B \left( 2\tau_{1,\delta'} + \sum_{i=1}^v \frac{1+\delta'}{\Lambda_i^2}; y_1, 0 \right). \end{aligned} \quad (5.8.34)$$

Since  $\Lambda_i, \Lambda_v \geq \Lambda$  for all  $i \in \mathcal{J}$ , we deduce that (5.8.34) is bounded by

$$C_{\delta, \delta'} \max \left( \left( \frac{v}{\sqrt{2}} \right)^\gamma, 1 \right) \left( 1 + \frac{\tau_1^{-\frac{1}{2}}}{\Lambda} \right)^\gamma p_B \left( 2\tau_{1, \delta'} + \sum_{i=1}^v \frac{1 + \delta'}{\Lambda_i^2}; y_1, 0 \right) \quad (5.8.35)$$

Proceeding similarly to (5.8.11), we deduce

$$p_B \left( 2\tau_{1, \delta'} + \sum_{i=1}^v \frac{1 + \delta'}{\Lambda_i^2}; y_1, 0 \right) \leq 2^v \mathcal{F}_{\delta'}^0 \left( \Lambda_{\mathcal{J}}, \Lambda_v; 2\tau_1; T_l^{1,0}; y_1 \right), \quad (5.8.36)$$

which together with (5.8.33) and (5.8.34) imply

$$\left( \frac{y_1}{\sqrt{\tau_1}} \right)^\gamma \mathcal{F}_\delta^0 \left( \Lambda_{\mathcal{J}}; \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right) \leq C t \left( 1 + \frac{\tau_1^{-\frac{1}{2}}}{\Lambda} \right)^\gamma \mathcal{F}_{\delta'}^0 \left( \Lambda_{\mathcal{J}}; 2\tau_1; T_l^{1,0}; y_1 \right), \quad (5.8.37)$$

where  $C := 2^v C_{\delta, \delta'} \max \left( \left( \frac{v}{\sqrt{2}} \right)^\gamma, 1 \right)$ . Using

$$\mathcal{F}_{1, l; \delta}^{\Lambda, 0}(\tau_1) := \sum_{v=0}^{3l-1} \mathcal{F}_\delta^0 \left( \Lambda, 2\tau_1; T_l^{1,0}; y_1 \right), \quad (5.8.38)$$

we deduce

$$\left( \frac{y_1}{\sqrt{\tau_1}} \right)^\gamma \mathcal{F}_{1, l; \delta}^0(t, \tau_1) \leq O(1) t \left( 1 + \frac{\tau_1^{-\frac{1}{2}}}{\Lambda} \right)^\gamma \mathcal{F}_{1, l; \delta'}^{\Lambda, 0}(\tau_1). \quad (5.8.39)$$

To deduce (5.8.28), note that performing a change of variable we have for all  $y_1 \in \mathbb{R}$

$$\mathcal{F}_\delta^0 \left( \Lambda_{\mathcal{J}}; \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right) = \mathcal{F}_\delta^0 \left( \Lambda_{\mathcal{J}}; \frac{2\tau_1^-}{t^2}; T_l^{1,0}; \frac{-y_1}{t} \right), \quad (5.8.40)$$

which together with the bound (5.8.26) gives directly (5.8.28).  $\square$

**Lemma 12.** *Let  $\Lambda \geq 3\sqrt{l}\tau^{-\frac{1}{2}}$ ,  $0 < \delta < \delta' \leq 1$  and  $(y_1, y_2) \in \mathbb{R}^2$ . For  $0 < t, t' \leq 1$ ,  $l \geq 1$  and  $\gamma_1, \gamma_2 \in \mathbb{N}$ , we have*

$$\left( \frac{y_1}{\sqrt{\tau_1}} \right)^{\gamma_1} \left( \frac{y_2}{\sqrt{\tau_2}} \right)^{\gamma_2} \mathcal{F}_{2, l; \delta}^0 \left( \Lambda; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; \frac{y_1}{t}, \frac{y_2}{t'} \right) \leq tt' \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda} \right) \mathcal{F}_{2, l; \delta'}^0 \left( \Lambda; \tau_{1,2}; y_{1,2} \right). \quad (5.8.41)$$

The polynomial  $\mathcal{Q}$  has nonnegative coefficients which are independent of  $\tau_1, \tau_2$  and  $\Lambda$  but depend on  $l, \delta, \delta', \gamma_1$  and  $\gamma_2$ .



*Proof.* Using (5.8.1),  $\mathcal{F}_{2,l;\delta}^{\Lambda,0} \left( \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; \frac{y_1}{t}, \frac{y_2}{t'} \right)$  can be written as follows

$$\begin{aligned} & \sum_{T_l^{2,0} \in \mathcal{W}_l^2(\sigma_2)} \mathcal{F}_\delta^0 \left( \Lambda; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}; \frac{y_1}{t}, \frac{y_2}{t'} \right) \\ & + \left( \sum_{T_l^{1,0} \in \mathcal{W}_l^1(\pi_1)} \mathcal{F}_\delta^0 \left( \Lambda, \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right) \right) \cdot \left( \sum_{T_l^{1,0} \in \mathcal{W}_l^1(\pi_2)} \mathcal{F}_\delta^0 \left( \Lambda, \frac{2\tau_2}{t'^2}; T_l^{1,0}; \frac{y_2}{t'} \right) \right). \end{aligned} \quad (5.8.42)$$

- First, we prove

$$\begin{aligned} & \left( \frac{y_1}{\sqrt{\tau_1}} \right)^{\gamma_1} \left( \frac{y_2}{\sqrt{\tau_2}} \right)^{\gamma_2} \sum_{T_l^{2,0} \in \mathcal{W}_l^2(\sigma_2)} \mathcal{F}^0 \left( \Lambda, \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}; \frac{y_1}{t}, \frac{y_2}{t'} \right) \\ & \leq tt' \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda} \right) \mathcal{F}_{2,l;\delta'}^0 \left( \Lambda; \tau_{1,2}, y_{1,2} \right), \end{aligned} \quad (5.8.43)$$

where  $0 < \delta < \delta' < 1$ . Let us start first with the case  $y_1, y_2 \geq 0$ . Given a surface tree  $T_l^{2,0}$  in  $\mathcal{W}_l^2(\sigma_2)$ , we have by Lemma 8

$$\begin{aligned} & \mathcal{F}_\delta^0 \left( \Lambda_{\mathcal{J}}, \tilde{\Lambda}; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}; \frac{y_1}{t}, \frac{y_2}{t'} \right) \\ & \leq \int_0^\infty dz p_B \left( c_{1,\delta}(t); z, \frac{y_1}{t} \right) p_B \left( c_{2,\delta}(t'); z, \frac{y_2}{t'} \right) p_B \left( \frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right), \end{aligned} \quad (5.8.44)$$

where  $c_{1,\delta}(t) = c_1(t)(1+\delta)$  and  $c_{2,\delta}(t') = c_2(t')(1+\delta)$ . The parameters  $c_1(t)$ ,  $c_2(t')$  and  $\tilde{\Lambda}_1$  are given by (5.8.6) with  $\tau_1 \rightarrow \tau_1/t^2$  and  $\tau_2 \rightarrow \tau_2/t'^2$ . For  $y_1 \leq y_2$ , we write

$$\begin{aligned} & \int_0^\infty dz p_B \left( c_{1,\delta}(t); z, \frac{y_1}{t} \right) p_B \left( c_{2,\delta}(t'); z, \frac{y_2}{t'} \right) p_B \left( \frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right) \\ & = \mathcal{J}(0, y_1) + \mathcal{J}(y_1, y_2) + \mathcal{J}(y_2, +\infty), \end{aligned} \quad (5.8.45)$$

where

$$\mathcal{J}(a, b) := \int_a^b dz p_B \left( c_{1,\delta}(t); z, \frac{y_1}{t} \right) p_B \left( c_{2,\delta}(t'); z, \frac{y_2}{t'} \right) p_B \left( \frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right).$$

- First, we bound  $\mathcal{J}(0, y_1)$ . For  $0 \leq t, t' \leq 1$ , we have

$$p_B \left( c_{1,\delta}(t); z, \frac{y_1}{t} \right) \leq t p_B \left( c_{1,\delta}; tz, y_1 \right), \quad (5.8.46)$$

$$p_B \left( c_{2,\delta}(t'); z, \frac{y_2}{t'} \right) \leq t' p_B \left( c_{2,\delta}; t'z, y_2 \right), \quad (5.8.47)$$

with  $c_i := c_i(1)$ . For  $0 \leq z \leq y_1 \leq y_2$ , we also have

$$p_B(c_{1,\delta}; tz, y_1) \leq t p_B(c_{1,\delta}; z, y_1) , \quad (5.8.48)$$

$$p_B(c_{2,\delta}; t'z, y_2) \leq t' p_B(c_{2,\delta}; z, y_2) . \quad (5.8.49)$$

This implies

$$\mathcal{J}(0, y_1) \leq t t' \int_0^\infty dz p_B(c_{1,\delta}; z, y_1) p_B(c_{2,\delta}; z, y_2) p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right) , \quad (5.8.50)$$

which again by Lemma 8 is bounded by

$$O(1) \int_{\tilde{z}} \mathfrak{F}_\delta^0(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_{\sigma_2}; T_l^{2,0}; \tilde{z}; Y_{\sigma_2}) \leq O(1) \mathfrak{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}) .$$

For  $(\gamma_1, \gamma_2) \neq (0, 0)$ , we need to bound also the following term

$$\left(\frac{y_1}{\sqrt{\tau_1}}\right)^{\gamma_1} \left(\frac{y_2}{\sqrt{\tau_2}}\right)^{\gamma_2} \mathcal{J}(0, y_1) . \quad (5.8.51)$$

Using (5.8.50), (5.8.51) is bounded by

$$t t' \tau^{-\gamma_1-\gamma_2} \sum_{k=0}^{\gamma_1} \sum_{k'=0}^{\gamma_2} \binom{\gamma_1}{k} \binom{\gamma_2}{k'} \int_0^\infty dz |y_1 - z|^k |y_2 - z|^{k'} z^{\gamma_1+\gamma_2-k-k'} \\ \times p_B(c_{1,\delta}; z, y_1) p_B(c_{2,\delta}; z, y_2) p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right) . \quad (5.8.52)$$

Using (3.1.10), we obtain

$$z^{\gamma_1+\gamma_2-k-k'} p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right) \leq C_{\delta,\delta'} \tilde{\Lambda}_1^{-\gamma_1-\gamma_2+k'+k} p_B\left(\frac{1+\delta'}{\tilde{\Lambda}_1^2}; z, 0\right) . \quad (5.8.53)$$

Since  $\Lambda_i, \tilde{\Lambda} \geq \Lambda$  for all  $i \in \mathcal{J}$ , we deduce that

$$\frac{1}{\tilde{\Lambda}_1^2} = \left( \sum_{i=v_1+v_2+1}^v \frac{1}{\tilde{\Lambda}_i^2} \right) (1 - \delta_{v,0}) + \frac{1}{\tilde{\Lambda}^2} \leq \frac{v_0+1}{\Lambda^2} \quad \text{and} \quad c_i \leq \tau_i + \frac{v_i}{\Lambda^2} . \quad (5.8.54)$$

This gives

$$z^{\gamma_1+\gamma_2-k-k'} p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right) \leq O(1) \Lambda^{-\gamma_1-\gamma_2+k'+k} p_B\left(\frac{1+\delta'}{\tilde{\Lambda}_1^2}; z, 0\right) . \quad (5.8.55)$$

Similarly, we have

$$|y_i - z|^{\gamma_i - k} p_B(c_{i,\delta}; y_i, z) \leq C'_{\delta, \delta'} c_i^{\frac{k}{2}} p_B(c_{i,\delta'}; y_i, z), \quad (5.8.56)$$

$$\leq O(1) \tau_i^{\frac{k}{2}} \left(1 + \frac{\tau_i^{-\frac{1}{2}}}{\Lambda}\right)^k p_B(c_{i,\delta'}; y_i, z), \quad (5.8.57)$$

where we used (5.8.54). Whenever it appears,  $O(1)$  denotes a constant which depends on  $\delta$ ,  $\delta'$ ,  $l$  and  $v$ . Combining (5.8.52), (5.8.55) and (5.8.57), we deduce that (5.8.51) is bounded by

$$t t' \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda} \right) \int_0^\infty dz p_B(c_{1,\delta'}; z, y_1) p_B(c_{2,\delta'}; z, y_2) p_B \left( \frac{1 + \delta'}{\tilde{\Lambda}_1^2}; z, 0 \right). \quad (5.8.58)$$

By Lemma 8, we deduce that (5.8.51) is bounded by

$$t t' \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda} \right) \int_{\tilde{z}} \mathcal{F}_\delta^0(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_{\sigma_2}; T_l^{2,0}; \tilde{z}; Y_{\sigma_2}). \quad (5.8.59)$$

– Using the bounds (5.8.46), (5.8.47) and (5.8.49), the term  $\mathcal{J}(y_1, y_2)$  is bounded by

$$t t' \int_{y_1}^{y_2} dz p_B(c_{1,\delta}; tz, y_1) p_B(c_{2,\delta}; z, y_2) p_B \left( \frac{1 + \delta}{\tilde{\Lambda}_1^2}; z, 0 \right). \quad (5.8.60)$$

For  $z \geq y_1$ , we have

$$p_B \left( \frac{1 + \delta}{\tilde{\Lambda}_1^2}; z, 0 \right) \leq p_B \left( \frac{2(1 + \delta)}{\tilde{\Lambda}_1^2}; z, 0 \right) \exp \left( -\frac{y_1^2 \tilde{\Lambda}_1^2}{4(1 + \delta)} \right). \quad (5.8.61)$$

Knowing that  $v_0 \leq 3l - 1$  together with

$$\forall \Lambda_i \in \Lambda_{\mathcal{J}}, \quad \Lambda_i \geq \Lambda, \quad \tilde{\Lambda} \geq \Lambda, \quad (5.8.62)$$

and recalling (5.8.54), we obtain

$$\tilde{\Lambda}_1 \geq \frac{\Lambda}{\sqrt{3l}} \geq \sqrt{3}\tau^{-\frac{1}{2}}. \quad (5.8.63)$$

where we also used  $\Lambda \geq 3\sqrt{l}\tau^{-\frac{1}{2}}$ . This implies

$$\exp \left( -\frac{y_1^2 \tilde{\Lambda}_1^2}{4(1 + \delta)} \right) \leq \exp \left( -\frac{y_1^2 \Lambda^2}{12(1 + \delta)l} \right) \leq \exp \left( -\frac{y_1^2}{2(1 + \delta)\tau_1} \right). \quad (5.8.64)$$

Furthermore, we have

$$p_B(c_{1,\delta}; tz, y_1) \leq \frac{1}{\sqrt{2\pi c_{1,\delta}}}. \quad (5.8.65)$$

Combining (5.8.65) with the fact that  $c_1 \geq \tau_1$ , we deduce

$$\mathcal{F}(y_1, y_2) \leq \int_{\mathbb{R}} dz p_B(c_{2,\delta}; z, y_2) p_B\left(\frac{2(1+\delta)}{\tilde{\Lambda}_1^2}; z, 0\right) p_B(c_{1,\delta}; y_1, 0),$$

and by (3.1.5) we deduce that  $\mathcal{F}(y_1, y_2)$  is bounded by

$$p_B\left(c_{2,\delta} + \frac{2(1+\delta)}{\tilde{\Lambda}_1^2}; y_2, 0\right) p_B(c_{1,\delta}; y_1, 0).$$

Using the property (3.1.10) of the bulk heat kernel together with (5.8.62), we obtain

$$\begin{aligned} & \left(\frac{y_1}{\sqrt{\tau_1}}\right)^{\gamma_1} \left(\frac{y_2}{\sqrt{\tau_2}}\right)^{\gamma_2} p_B\left(c_{2,\delta} + \frac{2(1+\delta)}{\tilde{\Lambda}_1^2}; y_2, 0\right) p_B(c_{1,\delta}; y_1, 0) \\ & \leq \mathbb{Q}\left(\frac{\tau_i^{-\frac{1}{2}}}{\Lambda}\right) p_B\left(c_{2,\delta'} + \frac{2(1+\delta')}{\tilde{\Lambda}_1^2}; y_2, 0\right) p_B(c_{1,\delta'}; y_1, 0), \end{aligned} \quad (5.8.66)$$

where  $0 < \delta < \delta' < 1$ . For  $\Lambda \geq 3\sqrt{l}\tau^{-\frac{1}{2}}$  and  $l \geq 1$ , we have

$$\forall \Lambda_i \in \Lambda_{\mathcal{F}}, \quad \Lambda_i \geq \Lambda \geq \sqrt{3l}\tau_2^{-\frac{1}{2}}, \quad (5.8.67)$$

and this implies

$$\begin{aligned} \frac{1}{\Lambda_{v_1+v_2}^2} & \leq \frac{\tau_2}{3}, \quad \frac{1}{\Lambda_{v_1+v_2-1}^2} \leq \frac{\tau_2}{3}, \\ \frac{1}{\tilde{\Lambda}_1^2} & = \left(\sum_{i=v_1+v_2+1}^v \frac{1}{\Lambda_i^2}\right) (1 - \delta_{v,0}) + \frac{1}{\Lambda^2} \leq \frac{v_0+1}{\Lambda^2} \leq \frac{\tau_2}{3}, \end{aligned} \quad (5.8.68)$$

where again we used that  $v_0 \leq 3l - 1$ . Hence, we have

$$\begin{aligned} c_{2,\delta'} + \frac{2(1+\delta')}{\tilde{\Lambda}_1^2} & = \tau_{2,\delta'} + \sum_{i=1}^{v_2} \frac{1+\delta'}{\Lambda_{i+v_1}^2} + \frac{2(1+\delta')}{\tilde{\Lambda}_1^2} \\ & \leq 2\tau_{2,\delta'} + \sum_{i=1}^{v_2-2} \frac{1+\delta'}{\Lambda_{i+v_1}^2} + \frac{1+\delta'}{\tilde{\Lambda}_1^2}, \end{aligned} \quad (5.8.69)$$

which gives

$$p_B\left(c_{2,\delta'} + \frac{2(1+\delta')}{\tilde{\Lambda}_1^2}; y_2, 0\right) \leq \sqrt{2} p_B\left(2\tau_{2,\delta'} + \sum_{i=1}^{v_2-2} \frac{1+\delta'}{\Lambda_{i+v_1}^2} + \frac{1+\delta'}{\tilde{\Lambda}_1^2}; y_2, 0\right). \quad (5.8.70)$$

(3.1.8) together with Lemma 8 gives

$$\begin{aligned}
& p_B \left( 2\tau_{2,\delta'} + \sum_{i=1}^{v_2-2} \frac{1+\delta'}{\tilde{\Lambda}_{i+v_1}^2} + \frac{1+\delta'}{\tilde{\Lambda}_1^2}; y_2, 0 \right) \\
& \leq O(1) \int_{z_1} \cdots \int_{z_{v_2+v_0-1}} p_B(2\tau_{2,\delta'}; z_1, y_2) \prod_{i=2}^{v_2-1} p_B \left( \frac{1+\delta'}{\tilde{\Lambda}_{v_1+i-1}^2}; z_i, z_{i-1} \right) \\
& \quad \times \prod_{i=0}^{v_0-1} p_B \left( \frac{1+\delta'}{\tilde{\Lambda}_{v_1+v_2+i}^2}; z_{v_2+i}, z_{v_2+i-1} \right) p_B \left( \frac{1+\delta'}{\tilde{\Lambda}^2}; z_{v_2+v_0-1}, 0 \right). \quad (5.8.71)
\end{aligned}$$

The RHS of (5.8.71) corresponds to the integrated surface weight factor of a surface tree  $T_l^{1,0}$  which has an external vertex  $y_2$  and  $v_2 + v_0 - 1$  internal vertices which all are of incidence number 2. This tree belongs to the set of forests  $\mathcal{W}_l^1(\pi_2)$  if and only if

$$v_0 + v_2 - 1 \leq 3l - 1. \quad (5.8.72)$$

Since the tree  $T_l^{2,0} \left( \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; \frac{y_1}{t}, \frac{y_2}{t'} \right)$  is in the forest  $\mathcal{W}_l^2(\sigma_2)$ ,  $v_0$ ,  $v_1$  and  $v_2$  necessarily verify

$$v_0 + v_2 - 1 \leq 3l - 2 - v_1 + \frac{1}{2},$$

which implies (5.8.72). Hence,  $T_l^{1,0}$  belongs to the set  $\mathcal{W}_l^1(\pi_2)$ . From (5.8.70) and (5.8.71), we deduce

$$p_B \left( c_{2,\delta'} + \frac{1+\delta'}{\tilde{\Lambda}^2}; y_2, 0 \right) \leq O(1) \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_2). \quad (5.8.73)$$

Using (5.8.11), we obtain

$$p_B(c_{1,\delta'}; y_1, 0) \leq O(1) \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1). \quad (5.8.74)$$

(5.8.60) and (5.8.66) together with (5.8.73) and (5.8.77) give

$$\left( \frac{y_1}{\sqrt{\tau_1}} \right)^{\gamma_1} \left( \frac{y_2}{\sqrt{\tau_2}} \right)^{\gamma_2} \mathcal{J}(y_1, y_2) \leq t t' \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda} \right) \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1) \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_2), \quad (5.8.75)$$

where all the constants were absorbed in the coefficients of the polynomial  $\mathcal{Q}$ .

– The last term to bound is

$$\mathcal{J}(y_2, +\infty) := \int_{y_2}^{\infty} dz p_B(c_{1,\delta}(t); z, \frac{y_1}{t}) p_B(c_{2,\delta}(t'); z, \frac{y_2}{t'}) p_B \left( \frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right).$$

For  $z \geq y_2 \geq y_1$  we have

$$p_B \left( \frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right) \leq \frac{\tilde{\Lambda}_1}{\sqrt{2\pi}} \exp \left( -\frac{z^2 \tilde{\Lambda}_1^2}{6(1+\delta)} \right) \exp \left( -\frac{y_1^2}{2c_{1,\delta}} \right) \exp \left( -\frac{y_2^2}{2c_{2,\delta}} \right),$$

where we used (5.8.63) and  $c_i \geq \tau_i$ .

Bounding respectively  $p_B(c_{1,\delta}(t); z, \frac{y_1}{t})$  and  $p_B(c_{2,\delta}(t'); z, \frac{y_2}{t'})$  by  $\frac{t}{\sqrt{2\pi c_{1,\delta}}}$  and  $\frac{t'}{\sqrt{2\pi c_{2,\delta}}}$  we deduce that  $\mathcal{J}(y_2, +\infty)$  is bounded by

$$O(1) t t' p_B(c_{1,\delta}; y_1, 0) p_B(c_{2,\delta}; y_2, 0).$$

Using the bound (3.1.10), we find

$$\begin{aligned} \left(\frac{y_1}{\sqrt{\tau_1}}\right)^{\gamma_1} \left(\frac{y_2}{\sqrt{\tau_2}}\right)^{\gamma_2} p_B(c_{2,\delta}; y_2, 0) p_B(c_{1,\delta}; y_1, 0) \\ \leq \mathbb{Q} \left(\frac{\tau_i^{-\frac{1}{2}}}{\Lambda}\right) p_B(c_{2,\delta'}; y_2, 0) p_B(c_{1,\delta'}; y_1, 0), \end{aligned} \quad (5.8.76)$$

where  $0 < \delta < \delta' < 1$ . Using (5.8.11), we deduce

$$p_B(c_{i,\delta'}; y_i, 0) \leq O(1) \mathfrak{F}_{1,l;\delta'}^{\Lambda,0}(\tau_i), \quad (5.8.77)$$

which implies

$$\left(\frac{y_1}{\sqrt{\tau_1}}\right)^{\gamma_1} \left(\frac{y_2}{\sqrt{\tau_2}}\right)^{\gamma_2} \mathcal{J}(y_2, +\infty) \leq t t' \mathbb{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda}\right) \mathfrak{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1) \mathfrak{F}_{1,l;\delta'}^{\Lambda,0}(\tau_2). \quad (5.8.78)$$

Combining (5.8.59), (5.8.75) and (7.2.12) together with (5.8.3) and (5.8.44), we obtain (5.8.43). Now, we treat the case in which the external vertices are negative: Given a surface tree  $T_l^{2,0}$  in  $\mathcal{W}_{2,l}(\sigma_2)$ , we recall

$$\begin{aligned} \int_{\vec{z}} \mathfrak{F}_\delta^0 \left( \Lambda_{\mathcal{J}}, \tilde{\Lambda}; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}; \vec{z}; \frac{y_1}{t}, \frac{y_2}{t'} \right) &= \int_{z_1, \dots, z_v} \prod_{j=2}^{v_1+1} p_B \left( \frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j \right) p_B \left( \frac{\tau_{1,\delta}}{t}; z_{v_1+1}, \frac{y_1}{t} \right) \\ &\times p_B \left( \frac{1+\delta}{\Lambda_{v_1+1}^2}; z_1, z_{v_1+2} \right) \prod_{j=v_1+3}^{v_1+v_2} p_B \left( \frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j \right) p_B \left( \frac{\tau_{2,\delta}}{t'^2}; z_{v_1+v_2}, \frac{y_2}{t'} \right) \\ &\times p_B \left( \frac{1+\delta}{\Lambda_{v_1+v_2}^2}; z_1, z_{v_1+v_2+1} \right) \prod_{j=v_1+v_2+2}^v p_B \left( \frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j \right) p_B \left( \frac{1+\delta}{\tilde{\Lambda}^2}; z_v, 0 \right). \end{aligned} \quad (5.8.79)$$

For  $y_1 \leq 0$  and  $z_{v_1+1} \in \mathbb{R}^+$ , we have

$$p_B \left( \frac{\tau_{1,\delta}}{t^2}; z, \frac{y_1}{t} \right) \leq t p_B(\tau_{1,\delta}; y_1, 0).$$

Hence, we obtain for  $y_1, y_2 \leq 0$  using (3.1.10)

$$\begin{aligned} \left(\frac{y_1}{\sqrt{\tau_1}}\right)^{\gamma_1} \left(\frac{y_2}{\sqrt{\tau_2}}\right)^{\gamma_2} \mathfrak{F}_\delta^0 \left( \Lambda_{\mathcal{J}}, \tilde{\Lambda}; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}; \frac{y_1}{t}, \frac{y_2}{t'} \right) \\ \leq O(1) t t' p_B(\tau_{1,\delta'}; y_1, 0) p_B(\tau_{2,\delta'}; y_2, 0), \end{aligned} \quad (5.8.80)$$

which is bounded by

$$O(1) t t' \mathfrak{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1) \mathfrak{F}_{1,l;\delta'}^{\Lambda,0}(\tau_2). \quad (5.8.81)$$

If  $y_1 \leq 0$  and  $y_2 \geq 0$ , we have

$$\begin{aligned} & \mathfrak{F}_\delta^0 \left( \Lambda_{\mathcal{J}}, \tilde{\Lambda}; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}; \frac{y_1}{t}, \frac{y_2}{t'} \right) \\ & \leq O(1) t p_B(\tau_{1,\delta}; y_1, 0) \int_0^\infty dz p_B \left( c_{2,\delta}(t'); z, \frac{y_2}{t'} \right) p_B \left( \frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right). \end{aligned} \quad (5.8.82)$$

Using (3.1.5) together with (5.8.11) and (5.8.2), we have

$$\int_0^\infty dz p_B \left( c_{2,\delta}(t'); z, \frac{y_2}{t'} \right) p_B \left( \frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right) \leq \mathfrak{F}_{1,l;\delta}^{\Lambda,0}(t', \tau_2). \quad (5.8.83)$$

Combining Lemma 11 with (5.8.82) and (5.8.83) gives

$$\begin{aligned} & \left( \frac{y_1}{\sqrt{\tau_1}} \right)^{\gamma_1} \left( \frac{y_2}{\sqrt{\tau_2}} \right)^{\gamma_2} \mathfrak{F}_\delta^0 \left( \Lambda_{\mathcal{J}}, \tilde{\Lambda}; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}, \frac{y_1}{t}, \frac{y_2}{t'} \right) \\ & \leq O(1) t t' \mathfrak{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda} \right) \mathfrak{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1) \mathfrak{F}_{1,l;\delta'}^{\Lambda,0}(\tau_2). \end{aligned} \quad (5.8.84)$$

Using (5.8.3), we obtain the bound (5.8.43).

- By definition, we have

$$\mathfrak{F}_{1,l;\delta}^{\Lambda,0}(t, \tau_1) = \sum_{T_l^{1,0} \in \mathcal{W}_l^1(\pi_1)} \mathfrak{F}_\delta^0 \left( \Lambda, \frac{\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right). \quad (5.8.85)$$

Applying Lemma 11 to the global surface weight factors  $\mathfrak{F}_{1,l;\delta}^{\Lambda,0}(t, \tau_1)$  and  $\mathfrak{F}_{1,l;\delta}^{\Lambda,0}(t', \tau_2)$  we obtain

$$\begin{aligned} & \left( \frac{y_1}{\sqrt{\tau_1}} \right)^{\gamma_1} \sum_{T_l^{1,0} \in \mathcal{W}_l^1(\pi_1)} \mathfrak{F}_\delta^0 \left( \Lambda, \frac{\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right) \\ & \quad \times \left( \frac{y_2}{\sqrt{\tau_2}} \right)^{\gamma_2} \sum_{T_l^{1,0} \in \mathcal{W}_l^1(\pi_2)} \mathfrak{F}_\delta^0 \left( \Lambda, \frac{\tau_2}{t'^2}; T_l^{1,0}; \frac{y_2}{t'} \right) \\ & \leq O(1) t t' \left( 1 + \frac{\tau_1^{-\frac{1}{2}}}{\Lambda} \right)^{\gamma_1} \left( 1 + \frac{\tau_2^{-\frac{1}{2}}}{\Lambda} \right)^{\gamma_2} \mathfrak{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1) \mathfrak{F}_{1,l;\delta'}^{\Lambda,0}(\tau_2), \end{aligned} \quad (5.8.86)$$

and this together with (5.8.3), (5.8.42) and (5.8.43) conclude the proof of Lemma 12.  $\square$





## Chapter 6

# Renormalization of the bulk theory and the position independence of the counter-terms

### 6.1 Introduction

In this chapter, we study in more detail the renormalizability of the "bulk theory" that we presented very briefly in section 5.3 of chapter 5. This theory describes the massive scalar field in  $\mathbb{R}^4$  with a quartic self-interaction supported on the half-space  $\mathbb{R}^+ \times \mathbb{R}^3$ . The translation invariance in this case is broken by the interaction. In section 5.3, we imposed constant renormalization conditions and as a consequence of the simplicity of this choice, the counter-terms required to render the theory finite can depend on the position in space. However, we aim to prove that the semi-infinite model can be renormalized by position independent counter-terms of the same structure of the terms included in the original Hamiltonian and this was partially established in chapter 5 by studying the surface correlation distributions  $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda}$ . In order to complete our proof, we need to study the remaining part of the semi-infinite correlation distributions, which is described by the bulk correlation distributions  $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$ . In this chapter, we establish that there exists a particular choice of the renormalization conditions such that the counter-terms appearing in the bulk effective action are position independent. Furthermore, we relate these counter-terms to the counter-terms of the translationally invariant  $\phi_4^4$  theory. The method of the proof in this chapter is similar to the method used in chapter 5 in the sense that we construct a unique solution of the bulk flow equations of the following form

$$\prod_{i=1}^n \chi^+(z_i) \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) + \hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)), \quad (6.1.1)$$

where  $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$  denote the correlation distributions of the translationally invariant model and  $\hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$  are distributions in  $\mathcal{S}'((\mathbb{R}^+)^n)$  such that (6.1.1) is a solution of the bulk flow equation (5.3.6). The main advantage of this method is to use the a priori knowledge of the renormalizability of the translationally invariant theory to prove the renormalization of the bulk theory. As a consequence of this approach, the bare interaction has the following form

$$\begin{aligned} \frac{\lambda}{4!} \int_{\mathbb{R}^3} d^3x \int_{\mathbb{R}^+} dz \phi^4(z, x) + \frac{1}{2} \int_{\mathbb{R}^3} d^3x \int_{\mathbb{R}^+} dz \left( a_{\infty}^{\Lambda_0} \phi^2(z, x) - b_{\infty}^{\Lambda_0} \phi(z, x) \Delta \phi(z, x) \right. \\ \left. + \frac{2}{4!} c_{\infty}^{\Lambda_0} \phi^4(z, x) \right) + \frac{1}{2} s_+^{\Lambda_0} \int_{\mathbb{R}^3} d^3x \phi^2(0, x) + e_+^{\Lambda_0} \int_{\mathbb{R}^3} d^3x \phi(0, x) \partial_n \phi(0, x), \quad (6.1.2) \end{aligned}$$

where  $a_{\infty}^{\Lambda_0}$ ,  $b_{\infty}^{\Lambda_0}$  and  $c_{\infty}^{\Lambda_0}$  are the counter-terms of the translationally invariant model. The counter-terms  $s_+^{\Lambda_0}$  and  $e_+^{\Lambda_0}$  are surface counter-terms that result from the breaking of the translation invariance by the support of the interaction. This chapter is organized as follows: we recall briefly in section 6.2 the free massive scalar field theory in  $\mathbb{R}^4$  in the mixed position-momentum space and introduce the  $\phi_4^4$  interactions respectively on the full space  $\mathbb{R}^4$  and restricted to the half-space  $\mathbb{R}^+ \times \mathbb{R}^3$ . Then, we write for each of these two theories the corresponding flow equations. In section 6.3 we present the technique of the proof which is based on constructing the solution of the bulk flow equation using our knowledge of the translationally invariant correlation distributions. In this section, we define the boundary correlation distributions  $\hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$  through their flow equation and the boundary conditions. Section 6.4 gathers the central results of this chapter. In proposition 9 we present a uniform boundedness result w.r.t. the UV cutoff  $\Lambda_0$  of the correlation distributions  $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$  smeared with a product of test functions that are supported partially on  $\mathbb{R}^+$  and partially on  $\mathbb{R}^-$ . This proposition is a key result to establish Theorem 12, which proves that  $\hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$  smeared with a suitable class of test functions obey bounds sharper by one power of  $\Lambda$  than  $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$ . In proposition 10, we bound the correlation distributions  $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$  smeared with a suitable set of test functions that are supported on  $\mathbb{R}^+$ . In section 6.5, we give the proof of the propositions 9 and 10 together with Theorem 12.

## 6.2 The $\phi_4^4$ theory in $\mathbb{R}^4$

### 6.2.1 The action and the regularized propagator

In this section, we consider the massive  $\phi_4^4$ -theory in  $\mathbb{R}^4$  in the  $pz$ -representation. The point of departure is to write the associated regularized path integral uniquely defined by the corresponding Gaussian measure. In the mixed position-momentum space, the regularized flowing propagator is given

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<sup>1</sup>Here, we slightly abuse of notation since we used  $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$  in chapter 4 to denote the semi-infinite correlation distributions. In the sequel, the semi-infinite correlation distributions corresponding to  $\star$  boundary condition are denoted by  $\mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0}$ .

by<sup>2</sup>

$$C_B^{\Lambda, \Lambda_0}(p; z, z') = \int_{\frac{1}{\Lambda^2}}^{\frac{1}{\Lambda_0^2}} d\lambda e^{-\lambda(p^2+m^2)} p_B(\lambda; z, z'). \quad (6.2.1)$$

The derivative w.r.t.  $\Lambda$  of  $C_B^\Lambda$  is given by

$$\dot{C}_B^\Lambda(p; z, z') = \dot{C}^\Lambda(p) p_B\left(\frac{1}{\Lambda^2}; z, z'\right) \quad \text{with} \quad \dot{C}^\Lambda(p) := -\frac{2}{\Lambda^3} e^{-\frac{p^2+m^2}{\Lambda^2}}. \quad (6.2.2)$$

For finite  $\Lambda_0$  and in finite volume the effective action is rigorously defined from the functional integral

$$e^{-\frac{1}{\hbar}(L_a^{\Lambda, \Lambda_0}(\phi) + I^{\Lambda, \Lambda_0})} := \int d\mu_{\Lambda, \Lambda_0, B}(\Phi) e^{-\frac{1}{\hbar} L_a^{\Lambda_0, \Lambda_0}(\Phi + \phi)}, \quad L_a^{\Lambda, \Lambda_0}(0) = 0. \quad (6.2.3)$$

Here, the Gaussian measure  $d\mu_{\Lambda, \Lambda_0, B}$  is of mean zero and covariance  $\hbar C_B^{\Lambda, \Lambda_0}$ . The functional  $L_a^{\Lambda_0, \Lambda_0}(\phi)$  is the bare interaction of a renormalizable theory including counter-terms, viewed as a formal power series in  $\hbar$ . The counter-terms functionals must be local and not exceed a certain dimension. The subscript  $a \in \{\infty, +\}$  is used to distinguish between the translationally invariant  $\phi_4^4$  model and the  $\phi_4^4$  model with an interaction supported on the half space  $\mathbb{R}^+ \times \mathbb{R}^3$ . For the translationally invariant  $\phi_4^4$  theory, the bare interaction reads

$$L_\infty^{\Lambda_0, \Lambda_0}(\phi) = \frac{\lambda}{4!} \int_{\mathbb{R}^3} d^3x \int_{\mathbb{R}} dz \phi^4(z, x) + \frac{1}{2} \int_{\mathbb{R}^3} d^3x \int_{\mathbb{R}} dz \left( a_\infty^{\Lambda_0} \phi^2(z, x) - b_\infty^{\Lambda_0} \phi(z, x) \Delta \phi(z, x) + \frac{2}{4!} c_\infty^{\Lambda_0} \phi^4(z, x) \right). \quad (6.2.4)$$

For the  $\phi_4^4$  model in  $\mathbb{R}^4$  with the interaction restricted to the half-space, translation invariance is broken in the  $z$ -direction (i.e. the semi-line). Therefore, the counter-terms can possibly depend on  $z$ . Therefore, it is natural to consider the general bare interaction

$$L_+^{\Lambda_0, \Lambda_0}(\phi) = \frac{\lambda}{4!} \int_{\mathbb{R}^3} d^3x \int_{\mathbb{R}^+} dz \phi^4(z, x) + \frac{1}{2} \int_{\mathbb{R}^3} d^3x \int_{\mathbb{R}^+} dz \left( a_+^{\Lambda_0}(z) \phi^2(z, x) - b_+^{\Lambda_0}(z) \phi(z, x) \Delta_x \phi(z, x) - d_+^{\Lambda_0}(z) \phi(z, x) \partial_z^2 \phi(z, x) + s_+^{\Lambda_0}(z) \phi(z, x) (\partial_z \phi)(z, x) + \frac{2}{4!} c_+^{\Lambda_0}(z) \phi^4(z, x) \right). \quad (6.2.5)$$

Here we supposed that both theories are symmetric under  $\phi \rightarrow -\phi$ , and we included in (6.2.4)-(6.2.5) only relevant terms in the sense of the renormalization group. The functions  $a_+^{\Lambda_0}(z)$ ,  $b_+^{\Lambda_0}(z)$ ,  $c_+^{\Lambda_0}(z)$ ,  $d_+^{\Lambda_0}(z)$  and  $s_+^{\Lambda_0}(z)$  are supposed to be smooth.

The goal of this chapter is to establish that for a suitable choice of renormalization conditions, the

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<sup>2</sup>We recall the bulk heat kernel  $p_B(\lambda; z, z') := \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(z-z')^2}{2\lambda}}$ .

bare interaction  $L_+^{\Lambda_0, \Lambda_0}$  has the following form

$$L_+^{\Lambda_0, \Lambda_0}(\phi) = \frac{\lambda}{4!} \int_{\mathbb{R}^3} d^3x \int_{\mathbb{R}^+} dz \phi^4(z, x) + \frac{1}{2} \int_{\mathbb{R}^3} d^3x \int_{\mathbb{R}^+} dz \left( a_\infty^{\Lambda_0} \phi^2(z, x) - b_\infty^{\Lambda_0} \phi(z, x) \Delta \phi(z, x) \right. \\ \left. + \frac{2}{4!} c_\infty^{\Lambda_0} \phi^4(z, x) \right) + \frac{1}{2} s_+^{\Lambda_0} \int_{\mathbb{R}^3} d^3x \phi^2(0, x) + e_+^{\Lambda_0} \int_{\mathbb{R}^3} d^3x \phi(0, x) \partial_n \phi(0, x) . \quad (6.2.6)$$

Following the steps of chapters [1](#) and [4](#), we expand the functional  $L_a^{\Lambda, \Lambda_0}(\phi)$  in a formal power series w.r.t.  $\hbar$ ,

$$L_a^{\Lambda, \Lambda_0}(\phi) = \sum_{l=0}^{\infty} \hbar^l L_{l,a}^{\Lambda, \Lambda_0}(\phi) .$$

From  $L_\infty^{\Lambda, \Lambda_0}$ , we obtain the connected amputated Schwinger distributions of loop order  $l$  defined as follows

$$\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}((z_1, x_1), \dots, (z_n, x_n)) := \delta_{\phi(z_1, x_1)} \cdots \delta_{\phi(z_n, x_n)} L_{l, \infty}^{\Lambda, \Lambda_0} |_{\phi=0} , \quad (6.2.7)$$

where we used the notation  $\delta_{\phi(z, x)} = \delta / \delta \phi(z, x)$ . Similarly, we obtain the bulk correlation distributions of loop order  $l$  from  $L_+^{\Lambda, \Lambda_0}$

$$\mathcal{D}_{l,n}^{\Lambda, \Lambda_0}((z_1, x_1), \dots, (z_n, x_n)) := \delta_{\phi(z_1, x_1)} \cdots \delta_{\phi(z_n, x_n)} L_{l, +}^{\Lambda, \Lambda_0} |_{\phi=0} . \quad (6.2.8)$$

In the  $pz$ -representation, [\(6.2.7\)](#) and [\(6.2.8\)](#) read

$$\delta^{(3)}(p_1 + \cdots + p_n) \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \\ := (2\pi)^{3(n-1)} \frac{\delta^n}{\delta \phi(z_1, p_1) \cdots \delta \phi(z_n, p_n)} L_{l, \infty}^{\Lambda, \Lambda_0}(\phi) |_{\phi \equiv 0} \quad (6.2.9)$$

and

$$\delta^{(3)}(p_1 + \cdots + p_n) \mathcal{D}_{l,n}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \\ = (2\pi)^{3(n-1)} \frac{\delta^n}{\delta \phi(z_1, p_1) \cdots \delta \phi(z_n, p_n)} L_{l, +}^{\Lambda, \Lambda_0}(\phi) |_{\phi \equiv 0} . \quad (6.2.10)$$

The Dirac distribution  $\delta^{(3)}(p_1 + \cdots + p_n)$  appears because of translation invariance in the  $x$ -directions.

## 6.2.2 The flow equations

The flow equation (FE) is obtained from [\(6.2.3\)](#) on differentiating w.r.t.  $\Lambda$  as in chapter [1](#):

$$\partial_\Lambda (L_a^{\Lambda, \Lambda_0} + I^{\Lambda, \Lambda_0}) = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \phi}, \dot{C}_B^\Lambda \frac{\delta}{\delta \phi} \right\rangle_a L_a^{\Lambda, \Lambda_0} - \frac{1}{2} \left\langle \frac{\delta}{\delta \phi} L_a^{\Lambda, \Lambda_0}, \dot{C}_B^\Lambda \frac{\delta}{\delta \phi} L_a^{\Lambda, \Lambda_0} \right\rangle_a . \quad (6.2.11)$$

By  $\langle, \rangle_\infty$  (resp.  $\langle, \rangle_B$ ) we denote the standard inner product in  $L^2(\mathbb{R}^4)$  (resp.  $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$ ).

The FE for the CAS distributions derived from (6.2.11) are

$$\begin{aligned}
& \partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \\
&= \frac{1}{2} \int_{\mathbb{R}} dz \int_{\mathbb{R}} dz' \int_k \partial^w \mathcal{L}_{l-1, n+2}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \dot{C}_B^\Lambda(k; z, z') \\
&- \frac{1}{2} \int_{\mathbb{R}} dz \int_{\mathbb{R}} dz' \sum_{l_1, l_2} \sum_{n_1, n_2} \sum_{w_i} c_w \left[ \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_B^\Lambda(p; z, z') \right. \\
&\quad \left. \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \right]_{rsym},
\end{aligned} \tag{6.2.12}$$

and

$$\begin{aligned}
& \partial_\Lambda \partial^w \mathcal{D}_{l,n}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \\
&= \frac{1}{2} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^+} dz' \int_k \partial^w \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \dot{C}_B^\Lambda(k; z, z') \\
&- \frac{1}{2} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^+} dz' \sum_{l_1, l_2} \sum_{n_1, n_2} \sum_{w_i} c_w \left[ \partial^{w_1} \mathcal{D}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_B^\Lambda(p; z, z') \right. \\
&\quad \left. \times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \right]_{rsym},
\end{aligned} \tag{6.2.13}$$

where we used the same notations of chapters 1 and 4. The tree orders  $\mathcal{L}_{0,4}^{\Lambda, \Lambda_0}$  and  $\mathcal{D}_{0,4}^{\Lambda, \Lambda_0}$  are given by

$$\mathcal{L}_{0,4}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) = \lambda \delta(z_1 - z_2) \delta(z_1 - z_3) \delta(z_1 - z_4) \tag{6.2.14}$$

and

$$\mathcal{D}_{0,4}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) = \lambda \prod_{i=1}^4 \chi^+(z_i) \delta(z_1 - z_2) \delta(z_1 - z_3) \delta(z_1 - z_4). \tag{6.2.15}$$

Note that at the tree order, it is clear that  $\mathcal{L}_{0,4}^{\Lambda, \Lambda_0} \in \mathcal{S}'(\mathbb{R}^4)$  and  $\mathcal{D}_{0,4}^{\Lambda, \Lambda_0} \in \mathcal{S}'((\mathbb{R}^+)^4)$ .

### 6.3 The technique of the proof

As previously mentioned, the purpose of this chapter is to prove that for a suitable choice of the renormalization conditions, the counter-terms appearing in the effective action (6.2.5) are independent of the position in space and can be related to the counter-terms of the translationally invariant  $\phi_4^4$ -theory through (6.2.6). Our technique is to start from the a priori knowledge of the translationally

invariant  $\phi_4^4$  model to build a specific solution to the flow equation (6.2.13) that produces the required properties of the bulk counter-terms. More precisely, we study (6.1.1). Before getting to the central results of this chapter that lead to an effective action of the form (6.2.6), we need to impose first the boundary conditions for  $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$ , and then define the correlation distributions  $\hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$ .

### 6.3.1 Boundary conditions for $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$

In chapter 1, we presented briefly the proof of renormalization of the massive  $\phi_4^4$  in  $\mathbb{R}^4$  in momentum space. In order to be consistent with the treatment of the semi-infinite model, we need to consider this theory in the mixed position-momentum space. As we discussed before, one of the main consequences of working in the mixed  $pz$ -space is that the correlation "functions" of the considered theory are distributions. For the translationally invariant model, the corresponding correlation distributions belong to  $\mathcal{S}'(\mathbb{R}^{4n})$ . Following section 1.1.4, we impose mixed boundary conditions:

- At  $\Lambda = \Lambda_0$ , the irrelevant part of the theory is required to be equal to 0. This translates into an effective action of the form (6.2.4) from which we can deduce the following boundary conditions:

$$\mathcal{L}_{l,2}^{\Lambda_0,\Lambda_0}((z_1, p), (z_2, -p)) = \left( a_{l,\infty}^{\Lambda_0} + b_{l,\infty}^{\Lambda_0} (p^2 - \partial_{z_1}^2) \right) \delta(z_1 - z_2), \quad \forall l \geq 1, \quad (6.3.1)$$

$$\mathcal{L}_{0,2}^{\Lambda_0,\Lambda_0}((z_1, p), (z_2, -p)) = 0, \quad (6.3.2)$$

$$\mathcal{L}_{l,4}^{\Lambda_0,\Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) = \left( \lambda \delta_{l,0} + c_{l,\infty}^{\Lambda_0} (1 - \delta_{l,0}) \right) \prod_{i=2}^4 \delta(z_1 - z_i), \quad \forall l \geq 0. \quad (6.3.3)$$

- Let us introduce the following notations that we use in the sequel: Given a partition  $(\pi_1, \pi_2)$  of  $\sigma_{2;4}$ , we define

$$\int_{\bar{z}_{\pi_1}^+} := \prod_{j=1}^{|\pi_1|} \int_{\mathbb{R}^+} dz_j, \quad \int_{\bar{z}_{\pi_2}^-} := \prod_{j=1}^{|\pi_2|} \int_{\mathbb{R}^-} dz_j. \quad (6.3.4)$$

Our candidate of the right<sup>3</sup> "bulk" correlation distributions (6.1.1) contains the following part

$$\prod_{i=1}^n \chi^+(z_i) \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)), \quad (6.3.5)$$

which belongs to  $\mathcal{S}'((\mathbb{R}^+)^{4n})$ . In order to obtain a global bound on the bulk correlation distributions, we need to bound (6.3.5). Hence, the renormalization conditions that we impose must be consistent with the integration of the derivative w.r.t.  $\Lambda$  of (6.3.5) smeared with appropriate test functions. Another constraint on the renormalization conditions is that they must imply (1.1.20)-(1.1.21) to ensure that the counter-terms appearing in (6.3.1)-(6.3.3) are equal to

<sup>3</sup>The one that leads to the bare interaction (6.2.6).

the counter-terms presented in chapter [1](#). Therefore, we consider the following renormalization conditions: At  $\Lambda = 0$ , we impose for all  $z_1 \in \mathbb{R}$  and  $l \geq 1$

$$\int_{\mathbb{R}^+} dz_2 (z_1 - z_2)^r \mathcal{L}_{l,2}^{0,\Lambda_0}((z_1, 0), (z_2, 0)) = 0, \quad \forall 0 \leq r \leq 2, \quad (6.3.6)$$

$$\int_{\mathbb{R}^-} dz_2 (z_1 - z_2)^r \mathcal{L}_{l,2}^{0,\Lambda_0}((z_1, 0), (z_2, 0)) = 0, \quad \forall 0 \leq r \leq 2 \quad (6.3.7)$$

and

$$\int_{(\mathbb{R}^-)^3} dz_{2,4} \mathcal{L}_{l,4}^{0,\Lambda_0}((z_1, 0), \dots, (z_4, 0)) = 0, \quad (6.3.8)$$

$$\int_{(\mathbb{R}^+)^3} dz_{2,4} \mathcal{L}_{l,4}^{0,\Lambda_0}((z_1, 0), \dots, (z_4, 0)) = 0, \quad (6.3.9)$$

$$\int_{\tilde{z}_{\pi_1}^+} \int_{\tilde{z}_{\pi_2}^-} \mathcal{L}_{l,4}^{0,\Lambda_0}((z_1, 0), \dots, (z_4, 0)) = 0, \quad \forall (\pi_1, \pi_2) \in \tilde{\mathcal{P}}_{2;4}. \quad (6.3.10)$$

Note that [\(6.3.6\)](#)-[\(6.3.10\)](#) written in momentum space imply the usual BPHZ renormalization conditions for the translationally invariant  $\phi_4^4$ -theory given in [\(1.1.20\)](#)-[\(1.1.21\)](#).

### 6.3.2 The correlation distributions $\hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$

For  $(n, l) = (4, 0)$ , we define

$$\hat{\mathcal{D}}_{0,4}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) := 0. \quad (6.3.11)$$

For  $(n, 0)$  such that  $n \geq 6$  and  $(n, l)$  such that  $n \geq 2$  and  $l \geq 1$ , the correlation distributions are generated inductively by the following flow equation

$$\begin{aligned} \partial_\Lambda \hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)) &= \frac{1}{2} \int_k \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^+} dz' \hat{\mathcal{D}}_{l-1,n+2}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n), (z, k), (z', -k)) \dot{C}_B^\Lambda(k; z, z') \\ &\quad - \frac{1}{2} \int_k \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^-} dz' \mathcal{L}_{l-1,n+2}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n), (z, k), (z', -k)) \dot{C}_B^\Lambda(k; z, z') \\ &\quad - \frac{1}{2} \int_k \int_{\mathbb{R}^-} dz \int_{\mathbb{R}} dz' \mathcal{L}_{l-1,n+2}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n), (z, k), (z', -k)) \dot{C}_B^\Lambda(k; z, z') \\ &\quad - \frac{1}{2} \sum'_{l_1, l_2} \sum''_{\pi_1, \pi_2} \left\{ \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^+} dz' \left[ \hat{\mathcal{D}}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((\vec{z}_{\pi_1}, \vec{p}_{\pi_1}), (z, p)) \hat{\mathcal{D}}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), (\vec{z}_{\pi_2}, \vec{p}_{\pi_2})) \right. \right. \\ &\quad \quad \quad \left. \left. + \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((\vec{z}_{\pi_1}, \vec{p}_{\pi_1}), (z, p)) \hat{\mathcal{D}}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), (\vec{z}_{\pi_2}, \vec{p}_{\pi_2})) \right. \right. \\ &\quad \quad \quad \left. \left. + \hat{\mathcal{D}}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((\vec{z}_{\pi_1}, \vec{p}_{\pi_1}), (z, p)) \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), (\vec{z}_{\pi_2}, \vec{p}_{\pi_2})) \right] \dot{C}_B^\Lambda(p; z, z') \right. \\ &\quad \left. - \int_{(\mathbb{R}^2)^-} dz dz' \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((\vec{z}_{\pi_1}, \vec{p}_{\pi_1}), (z, p)) \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), (\vec{z}_{\pi_2}, \vec{p}_{\pi_2})) \dot{C}_B^\Lambda(p; z, z') \right\}, \quad (6.3.12) \end{aligned}$$

where we used the same notations as in (5.4.11). The structure of the flow equation (6.3.12) is motivated by the fact that (6.1.1) is required to verify the flow equation (6.2.13) and the tree order (6.2.15). In order to define uniquely  $\hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$ , we need to impose boundary conditions.

### Boundary conditions for $\hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$

At  $\Lambda = \Lambda_0$ , we include the counter-terms that make  $\hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$  finite as the cutoffs are removed. As we already mentioned before, the form of these counter-terms is not known, but they must be local functionals which do exceed a certain dimension. We write:

$$\hat{\mathcal{D}}_{l,2}^{\Lambda_0,\Lambda_0}((z_1, p), (z_2, -p)) = s_l^{\Lambda_0} \delta_{z_1} \delta_{z_2} + e_l^{\Lambda_0} \delta_{z_1} \delta'_{z_2} + h_l^{\Lambda_0} \delta'_{z_1} \delta_{z_2} , \quad (6.3.13)$$

$$\hat{\mathcal{D}}_{0,2}^{\Lambda_0,\Lambda_0}((z_1, p), (z_2, -p)) = 0 , \quad (6.3.14)$$

$$\hat{\mathcal{D}}_{l,n}^{\Lambda_0,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) = 0 , \quad \forall n \geq 4 . \quad (6.3.15)$$

The remaining information concerns only a finite number of relevant terms, for which it is much more convenient to encode it in a boundary condition at  $\Lambda = 0$ . To extract these relevant terms, we perform a Taylor expansion of the test functions  $\phi_1$  and  $\phi_2$  in  $\in \mathcal{S}(\mathbb{R}^+)$  around the origin in

$$\int_{\mathbb{R}^+} dz_1 \int_{\mathbb{R}^+} dz_2 \hat{\mathcal{D}}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \phi_1(z_1) \phi_2(z_2). \quad (6.3.16)$$

Namely, we rewrite (6.3.16) as follows

$$s_{l,+}^{\Lambda,\Lambda_0} \phi_1(0) \phi_2(0) + e_{l,+}^{\Lambda,\Lambda_0} \phi_1(0) \partial_n \phi_2(0) + h_{l,+}^{\Lambda,\Lambda_0} \phi_2(0) \partial_n \phi_1(0) + \hat{l}_{l,2}^{\Lambda,\Lambda_0}(\phi_1, \phi_2) . \quad (6.3.17)$$

Then the relevant terms appear as

$$s_{l,+}^{\Lambda,\Lambda_0} := \int_{\mathbb{R}^+} dz_1 \int_{\mathbb{R}^+} dz_2 \hat{\mathcal{D}}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) , \quad (6.3.18)$$

$$e_{l,+}^{\Lambda,\Lambda_0} := \int_{\mathbb{R}^+} dz_1 \int_{\mathbb{R}^+} dz_2 z_1 \hat{\mathcal{D}}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) , \quad (6.3.19)$$

$$h_{l,+}^{\Lambda,\Lambda_0} := \int_{\mathbb{R}^+} dz_1 \int_{\mathbb{R}^+} dz_2 z_2 \hat{\mathcal{D}}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \quad (6.3.20)$$

and the remainder  $\hat{l}_{l,2}^{\Lambda,\Lambda_0}(\phi_1, \phi_2)$  has the form (5.4.17) with  $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$  replaced by  $\hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$ . At  $\Lambda = 0$ , we impose BPHZ renormalization conditions given by

$$s_{l,+}^{0,\Lambda_0} = e_{l,+}^{0,\Lambda_0} = h_{l,+}^{0,\Lambda_0} = 0 . \quad (6.3.21)$$

Note that the Bose symmetry for  $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$  together with the boundary conditions (6.3.13)-(6.3.15) and (6.3.21) and the flow equation (6.3.12) imply inductively that  $\hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$  is symmetric w.r.t. its arguments. In particular, this implies  $e_{l,+}^{\Lambda,\Lambda_0} = h_{l,+}^{\Lambda,\Lambda_0}$ .



## 6.4 Results

Before stating the main results of this chapter, let us introduce first the test functions against which  $\widehat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$  and  $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$  will be integrated. The family of the test functions we will be considering are of the following form for  $1 \leq s \leq n$ :

$$\phi_{\tau_{i,s},y_{i,s}}(z_{i,n}) := \prod_{j=i}^s p_B(\tau_j; z_j, y_j), \quad i \in \{1, 2\}, \quad (6.4.1)$$

with  $\tau_i > 0$ ,  $z_{i,n} = (z_i, \dots, z_n)$  and  $y_{1,s}$  set of points in  $\mathbb{R}$ . We also use the test functions

$$\phi_{\tau_{2,s},y_{2,s}}^{(j)}(z_{2,n}) := p_B^{(1)}(\tau_j; z_j, z_1; y_j) \prod_{i=2, i \neq j}^s p_B(\tau_i; z_i, y_i), \quad (6.4.2)$$

where

$$p_B^{(1)}(\tau_j; z_j, z_1; y_j) = p_B(\tau_j; z_j, y_j) - p_B(\tau_j; z_1, y_j). \quad (6.4.3)$$

These definitions can be generalized by choosing any other subset of  $s$  coordinates among  $\{z_1, \dots, z_n\}$ . Given a test function  $\Psi_s$  in  $\{\phi_{\tau_{2,s},y_{2,s}}, \phi_{\tau_{2,s},y_{2,s}}^{(j)}\}$ , we define

$$\mathcal{L}_{l,n;r}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \Psi_s) := \int_{\mathbb{R}^{n-1}} dz_{2,n} (z_1 - z_2)^r \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \Psi_s(z_{2,n}). \quad (6.4.4)$$

For  $n \geq 3$  and  $0 \leq r \leq 3$ , we define

$$F_{12} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau_{3,s},y_{3,s}}) := (z_1 - z_2)^3 \int_{\mathbb{R}^{n-2}} dz_{3,n} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \prod_{i=3}^s p_B(\tau_i; z_i, y_i). \quad (6.4.5)$$

For  $n = 2$  we define

$$F_{12} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1, z_2; p) := (z_1 - z_2)^3 \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)). \quad (6.4.6)$$

The uniform boundedness of  $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$  w.r.t. the UV cutoff  $\Lambda_0$  in the  $pz$ -representation can be summarized in the following theorem:

**Theorem 11.** For  $0 \leq \Lambda \leq \Lambda_0 < \infty$ ,  $0 \leq s \leq n$ ,  $0 \leq r_1, r_2 \leq 3$ ,  $z_1 \in \mathbb{R}$  and  $y_{2,s} \in \mathbb{R}^{s-1}$ , we consider test functions of the form  $\phi_{\tau_{2,s},y_{2,s}}(z_{2,s})$ . Adopting (6.3.1)-(6.3.10), we have

$$\begin{aligned} \left| \partial^w \mathcal{L}_{l,n;r}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s},y_{2,s}}) \right| &\leq (\Lambda + m)^{4-n-|w|-r} \mathcal{F}'_1 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{F}'_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \mathcal{Q}'_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda;\infty}(\tau_{2,s}), \quad (6.4.7) \end{aligned}$$

$$\begin{aligned} \left| \partial^w \mathcal{L}_{l,n;r}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s},y_{2,s}}^{(j)}) \right| &\leq (\Lambda + m)^{3-n-|w|-r} \tau_j^{-\frac{1}{2}} \mathcal{P}'_3 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}'_4 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \mathcal{Q}'_2 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda,\infty}(\tau_{2,s}) , \end{aligned} \quad (6.4.8)$$

$$\begin{aligned} \left| \partial^w F_{12} \mathcal{L}_{l,n;r}^{\Lambda,\Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau_{3,s},y_{3,s}}) \right| &\leq (\Lambda + m)^{1-n-|w|-r} \mathcal{P}'_5 \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}'_6 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \mathcal{Q}'_3 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda,\infty;(12)}(\tau_{3,s}) . \end{aligned} \quad (6.4.9)$$

Here and subsequently, the polynomials  $\mathcal{P}'_i$  and  $\mathcal{Q}'_i$  verify the same conditions mentioned in Theorem [1](#) and the parameter  $\delta$  depends on the loop order  $l$  and verifies  $0 < \delta_l \leq \delta_{l+1} < \frac{1}{2}$ .

The proof of this theorem follows the same steps of the proof of Theorem [1](#) in chapter [4](#). Hence, we do not explicit the steps of the proof in this chapter. Let us define

$$\hat{\mathcal{D}}_{l,n;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s},y_{1,s}}) := \int_{(\mathbb{R}^+)^n} dz_{1,n} z_1^{r_1} z_2^{r_2} \hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) . \quad (6.4.10)$$

The central result of this chapter is summarized in the following Theorem:

**Theorem 12.** For  $0 \leq \Lambda \leq \Lambda_0$ ,  $\tau > 0$ ,  $s \geq 0$  and adopting [\(6.3.13\)](#)-[\(6.3.15\)](#) and [\(6.3.21\)](#) we claim

$$\begin{aligned} \left| \partial^w \hat{\mathcal{D}}_{l,n;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s},y_{1,s}}) \right| &\leq (\Lambda + m)^{3-n-|w|-r_1-r_2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}) , \end{aligned} \quad (6.4.11)$$

using the same notations of Theorem [9](#).

The proof of Theorem [12](#) is inductive. However, there are some terms in the the flow equation [\(6.3.12\)](#) for which the induction hypothesis cannot be used. These terms involve the translationally invariant correlation distributions  $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$  integrated over functions supported partially on  $\mathbb{R}^+$  and partially on  $\mathbb{R}^-$ . Upper bounds on them are summarized in proposition [9](#). Let us first introduce the following notation: For  $(\pi_1, \pi_2)$  in  $\tilde{\mathcal{P}}_{2;n}$ , we define

$$\begin{aligned} \partial^w \mathcal{L}_{l,n;r_1,r_2;\pi_1^+,\pi_2^-}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s},y_{1,s}}) &:= \int_{(\mathbb{R}^+)^{q_1}} d\vec{z}_{\pi_1} \int_{(\mathbb{R}^-)^{q_2}} d\vec{z}_{\pi_2} z_1^{r_1} z_2^{r_2} \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) , \end{aligned} \quad (6.4.12)$$

where  $q_i := |\pi_i|$ .

**Proposition 9.** For  $0 \leq \Lambda \leq \Lambda_0 < \infty$ ,  $0 \leq s \leq n$ ,  $0 \leq r_1, r_2 \leq 3$ ,  $z_1 \in \mathbb{R}$  and  $Y_{\sigma_s} \in \mathbb{R}^s$ , we consider test functions of the form  $\phi_{\tau_{1,s}, y_{1,s}}(z_{1,s})$ . Adopting (6.3.1)-(6.3.10) and using the notation (6.4.12), we claim for a given partition  $(\pi_1, \pi_2) \in \tilde{\mathcal{P}}_{2;n}$

$$\begin{aligned} \left| \partial^w \mathcal{L}_{l,n;r_1,r_2;\pi_1^+,\pi_2^-}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s}, y_{1,s}}) \right| &\leq (\Lambda + m)^{3-n-|w|-r_1-r_2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{\pi_{1,s}}^+, \tau_{\pi_{2,s}}^-), \quad \forall n \geq 2, \end{aligned} \quad (6.4.13)$$

where the polynomials  $\mathcal{P}$ ,  $\mathbb{Q}$  and  $\delta$  obey the same conditions mentioned in Theorem 7.

**Remark 4.** Note that the power counting on the RHS of (6.4.13) is improved and this mainly comes from the following bound on the bulk heat kernel for  $z \in \mathbb{R}^+$  and  $z' \in \mathbb{R}^-$

$$p_B \left( \frac{1}{\Lambda^2}; z, z' \right) \leq \sqrt{2\pi} \Lambda^{-1} p_B \left( \frac{1}{\Lambda^2}; z, 0 \right) p_B \left( \frac{1}{\Lambda^2}; z', 0 \right). \quad (6.4.14)$$

In the proof of proposition 9, we need additional bounds that we gather in proposition 10. Before stating the proposition, we need to define for a partition  $(\pi_1, \pi_2)$  of the set  $\sigma_{2;n}$

$$\begin{aligned} &\partial^w \mathcal{L}_{l,n;r;\pi_1^+,\pi_2}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}) \\ &:= \int_{(\mathbb{R}^+)^{q_1}} d\vec{z}_{\pi_1} \int_{\mathbb{R}^{q_2}} d\vec{z}_{\pi_2} (z_1 - z_2)^r \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \prod_{i=2}^s p_B(\tau_i; z_i, y_i) \end{aligned} \quad (6.4.15)$$

and

$$\begin{aligned} &\partial^w \mathcal{L}_{l,n;r;+}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}) \\ &:= \int_{(\mathbb{R}^+)^{n-1}} d\vec{z}_{2,n} (z_1 - z_2)^r \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \prod_{i=2}^s p_B(\tau_i; z_i, y_i). \end{aligned} \quad (6.4.16)$$

**Proposition 10.** For  $0 \leq \Lambda \leq \Lambda_0 < \infty$ ,  $1 \leq s \leq n$ ,  $0 \leq r \leq 3$  and  $Y_{\sigma_{2;s}} \in \mathbb{R}^{s-1}$ , we consider test functions of the form  $\phi_{\tau_{2,s}, y_{2,s}}(z_{2,s})$  and a partition  $(\pi_1, \pi_2)$  of  $\sigma_{2;n}$  such that  $\pi_{i,s} := \pi_i \cap \sigma_{2;s}$ . Adopting (6.3.1)-(6.3.10), we claim for  $z_1 \geq 0$

$$\begin{aligned} \left| \partial^w \mathcal{L}_{l,n;r;\pi_1^+,\pi_2}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}) \right| &\leq (\Lambda + m)^{4-n-|w|-r} \tilde{\mathcal{P}}_1 \left( \log \frac{\Lambda + m}{m} \right) \tilde{\mathcal{P}}_2 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \tilde{\mathbb{Q}}_1 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda,\infty}(\tau_{\pi_{1,s}}^+, \tau_{\pi_{2,s}}) , \end{aligned} \quad (6.4.17)$$

$$\begin{aligned} \left| \partial^w \mathcal{L}_{l,n;r;+}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}) \right| &\leq (\Lambda + m)^{4-n-|w|-r} \tilde{\mathcal{P}}_3 \left( \log \frac{\Lambda + m}{m} \right) \tilde{\mathcal{P}}_4 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \tilde{\mathbb{Q}}_2 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;+;\delta}^{\Lambda,\infty}(\tau_{2,s}) , \end{aligned} \quad (6.4.18)$$

$$\begin{aligned} \left| \partial^w \mathcal{L}_{l,n;\pi_1^+,\pi_2}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s},y_{2,s}}^{(j)}) \right| &\leq (\Lambda + m)^{3-n-|w|} \tau_j^{-\frac{1}{2}} \tilde{\mathcal{P}}_7 \left( \log \frac{\Lambda + m}{m} \right) \tilde{\mathcal{P}}_8 \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \tilde{\mathcal{Q}}_4 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda;\infty}(\tau_{\pi_1^+}, \tau_{\pi_2,s}) , \end{aligned} \quad (6.4.19)$$

$$\begin{aligned} \left| \partial^w \mathcal{L}_{l,n;+}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s},y_{2,s}}^{(j)}) \right| &\leq (\Lambda + m)^{3-n-|w|} \tau_j^{-\frac{1}{2}} \tilde{\mathcal{P}}_9 \left( \log \frac{\Lambda + m}{m} \right) \tilde{\mathcal{P}}_{10} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \tilde{\mathcal{Q}}_5 \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;+;\delta}^{\Lambda;\infty}(\tau_{2,s}) . \end{aligned} \quad (6.4.20)$$

**Remarks 3.** • Note that the boundary conditions (6.3.13)-(6.3.15) and (6.3.21) together with the flow equation (6.3.12) and the tree order (6.3.11) allow to define uniquely  $\hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$ .

• We define

$$\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)) = \hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)) + \prod_{i=1}^n \chi^+(z_i) \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)) . \quad (6.4.21)$$

Combining the bounds (6.4.11) and (6.4.18) we obtain

$$\begin{aligned} \left| \partial^w \mathcal{D}_{l,n;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s},y_{1,s}}) \right| &\leq \left\{ (\Lambda + m)^{3-n-|w|-r_1-r_2} \mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}) + (\Lambda + m)^{4-n-|w|-r_1-r_2} \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{1,s}) \right\} \\ &\quad \times \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P}_- \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) , \quad \forall s \geq 1, \end{aligned} \quad (6.4.22)$$

which is compatible with the proof of Theorem 9 in chapter 5 as we already mentioned in remark 2 at the end of section 5.4.3.

## 6.5 Proofs

All the bounds of Theorem 12 and Propositions 9 and 10 are proved using the same inductive scheme as in the proof of Theorem 1. They all can be established in the same way: first we bound the right-hand side of the corresponding flow equation using the induction hypothesis, and then we integrate over  $\lambda$  using the appropriate boundary conditions.

### 6.5.1 Proof of Proposition 10

*Proof.* The bounds (6.4.17)-(6.4.20) obviously hold in the starting case  $n = 4, l = 0$ .

**The right-hand side**

Let  $z_1 \geq 0$ . The bounds that we want to obtain for the right-hand side of the flow equation are

$$\begin{aligned} \left| \partial_\Lambda \partial^w \mathcal{L}_{l,n;r;\pi_1^+,\pi_2}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s},y_{2,s}}) \right| &\leq (\Lambda + m)^{3-n-|w|-r} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda;\infty}(\tau_{\pi_{1,s}^+}, \tau_{\pi_{2,s}}) . \end{aligned} \quad (6.5.1)$$

Similarly, we want to prove

$$\begin{aligned} \left| \partial_\Lambda \partial^w \mathcal{L}_{l,n;r;+}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s},y_{2,s}}) \right| &\leq (\Lambda + m)^{3-n-|w|-r} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;+;\delta}^{\Lambda;\infty}(\tau_{2,s}) . \end{aligned} \quad (6.5.2)$$

We give the details of the proof of (6.5.1). The bound (6.5.2) is then established similarly.

A) First, we analyze the case  $r = 0$ :

- **The linear term:** This term is given by

$$\begin{aligned} \int_k \int_{(\mathbb{R}^+)^{q_1}} d\vec{z}_{\pi_1} \int_{\mathbb{R}} dz \int_{\mathbb{R}} dz' \int_{\mathbb{R}^{q_2}} d\vec{z}_{\pi_2} \mathcal{L}_{l-1,n+2}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n), (z, k), (z', -k)) \\ \times \dot{C}^\Lambda(k) p_B \left( \frac{1}{\Lambda^2}; z, z' \right) \prod_{i=2}^s p_B(\tau_i; z_i, y_i) . \end{aligned} \quad (6.5.3)$$

Using (3.1.5) and the notation (6.4.15), we rewrite (6.5.3) as follows

$$\int_{\mathbb{R}} du \int_k \mathcal{L}_{l-1,n+2;\pi_1^+,\pi_2'}^{\Lambda,\Lambda_0} \left( z_1; \vec{p}_n, k, -k; \phi_{\tau_{2,s},y_{2,s}} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \dot{C}^\Lambda(k), \quad (6.5.4)$$

where  $\pi_2' = \pi_2 \cup \{s+1, s+2\}$ . Using the induction hypothesis (6.4.18), (6.5.3) is bounded by

$$\begin{aligned} (\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \int_k \dot{C}^\Lambda(k) \mathcal{P} \left( \frac{|k|}{\Lambda + m}, \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \times \int_{\mathbb{R}} du \mathcal{F}_{s+2,l-1;\pi_1,\pi_2';\delta}^\infty \left( \Lambda; \tau_{\pi_{1,s}^+}, \tau_{\pi_{2,s}}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; z_1, y_{2,s}, u, u \right) . \end{aligned} \quad (6.5.5)$$

We have

$$\begin{aligned} \mathcal{F}_{s+2,l-1;\delta}^\infty \left( \Lambda, \tau_{\pi_{1,s}^+}, \tau_{\pi_{2,s}}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}, z_1, y_{2,s} \right) \\ = \sum_{T_{l-1}^{s+2} \in \mathcal{T}_{l-1;\pi_1,\pi_2'}^{s+2}} \mathcal{F}_\delta^\infty \left( \tau_{\pi_{1,s}^+}, \tau_{\pi_{2,s}}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; T_{l-1}^{s+2}; z_1, y_{2,s}, u, u \right) . \end{aligned} \quad (6.5.6)$$

For a tree  $T_{l-1}^{s+2}$  in  $\mathcal{T}_{l-1; \pi_1, \pi_2}^{s+2}$ , we denote by  $z$  and  $z'$  the internal vertices attached to the external vertices  $\{u, u\}$ . The contribution of the external lines  $(z, u)$  and  $(z', u)$  in the summand in (6.5.6) is

$$\int_{\mathbb{R}} du p_B \left( \frac{1+\delta}{2\Lambda^2}; z, u \right) p_B \left( \frac{1+\delta}{2\Lambda^2}; z', u \right) = p_B \left( \frac{1+\delta}{\Lambda^2}; z, z' \right) \leq \frac{\Lambda}{\sqrt{2\pi}}. \quad (6.5.7)$$

Hence, we deduce that the contribution of the heat kernels corresponding to the external legs  $(z, u)$  and  $(z', u)$  in the weight factor is bounded by  $\Lambda$ . From a diagrammatic point of view, this means that the legs  $(z, u)$  and  $(z', u)$  are amputated from the tree  $T_{l-1}^{s+2}$ . If an internal vertex becomes of incidence number one after the amputation, it is removed using

$$\int_{\mathbb{R}} dz p_B \left( \frac{1+\delta}{\Lambda_i^2}; z, \tilde{z} \right) = 1.$$

These steps correspond to the reduction of the tree  $T_{l-1}^{s+2}$  at the external vertices  $\{u, u\}$ . Using (3.2.8), we deduce that the reduced tree is in  $\mathcal{T}_{l; \pi_1, \pi_2}^s$  and we obtain

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{F}_{s+2, l-1; \delta}^{\infty} \left( \Lambda; \tau_{\pi_{1,s}}^+, \tau_{\pi_{2,s}}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; T_{l-1}^{s+2}(z_1; \vec{z}; y_{2,s}, u, u) \right) \\ \leq \frac{\Lambda}{\sqrt{2\pi}} \mathcal{F}_{s, l; \delta}^{\infty} \left( \Lambda; \tau_{\pi_{1,s}}^+, \tau_{\pi_{2,s}}; T_l^s(z_1; \vec{z}'; y_{2,s}) \right). \end{aligned} \quad (6.5.8)$$

Combining (6.5.5), (6.5.8) and the bound (4.5.23), we deduce that (6.5.3) is bounded by

$$(\Lambda + m)^{3-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l; \delta}^{\Lambda; \infty} \left( \tau_{\pi_{1,s}}^+, \tau_{\pi_{2,s}} \right). \quad (6.5.9)$$

- **The quadratic term:** This term denotes the second term on the right-hand of the flow equation (6.2.12). Given a partition  $(\pi'_1, \pi'_2)$  of  $\sigma_{2:n}$ , we analyze the corresponding term in the symmetrized sum given by

$$\begin{aligned} \int_{(\mathbb{R}^+)^{q_1}} d\vec{z}_{\pi_1} \int_{\mathbb{R}^{q_2}} d\vec{z}_{\pi_2} \int_{\mathbb{R}} dz \int_{\mathbb{R}} dz' \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0} \left( (\vec{z}_{\pi'_1}, \vec{p}_{\pi'_1}), (z, p) \right) \partial^{w_3} \dot{C}^{\Lambda}(p) \\ \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0} \left( (\vec{z}_{\pi'_2}, \vec{p}_{\pi'_2}), (z', -p) \right) p_B \left( \frac{1}{\Lambda^2}; z, z' \right) \prod_{i=2}^s p_B(\tau_i; z_i, y_i). \end{aligned} \quad (6.5.10)$$

We define

$$\phi'_{s_1}(\vec{z}_{\pi'_1}) := \prod_{r \in \pi'_{1,s}} p_B(\tau_r; z_r, y_r), \quad \phi''_{s_2}(\vec{z}_{\pi'_2}) := \prod_{r \in \pi'_{2,s}} p_B(\tau_r; z_r, y_r), \quad (6.5.11)$$

where  $\pi'_{i,s} := \pi'_i \cap \sigma_{2:s}$  and  $s_i := |\pi'_{i,s}|$ . Introducing the shorthand notation  $\pi_{ij} = \pi_i \cap \pi'_j$ , we distinguish in the sequel between the following cases:

-  $\pi_{ij} \neq \emptyset$  for all  $i, j \in \{1, 2\}$ : Using (3.1.5), we rewrite (6.5.10) as follows

$$\begin{aligned} & \int_{\mathbb{R}} du \int_{\mathbb{R}^+} dz_q \partial^{w_1} \mathcal{L}_{l_1, n_1+1; \pi_{11}^+, \pi_{21} \cup \{s+1\}}^{\Lambda, \Lambda_0} \left( z_1; \vec{p}_{\pi_1'}, p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \\ & \quad \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1; \pi_{12}^q, \pi_{22} \cup \{s+2\}}^{\Lambda, \Lambda_0} \left( z_q; \vec{p}_{\pi_2'}, -p; \phi''_{s_2, q} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \\ & \quad \times \partial^{w_3} \dot{C}^\Lambda(p) \phi_q(z_q), \end{aligned} \quad (6.5.12)$$

with  $\pi_{12}^q = \pi_{12} \setminus \{q\}$  and

$$\phi''_{s_2, q}(z_{\pi_2'}) := \prod_{r \in \pi_{2, s}^q, r \neq q} p_B(\tau_r; z_r, y_r). \quad (6.5.13)$$

$\phi_q(z_q) = p_B(\tau_q; z_q, y_q)$  if  $q \leq s$ , otherwise  $\phi_q \equiv 1$ . Using the induction hypothesis (6.4.18) and the bound (4.2.3), we deduce that (6.5.12) is bounded by

$$\begin{aligned} & (\Lambda + m)^{3-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ & \quad \times \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta_1}^\infty \left( \Lambda; z_1; \tau_{\pi_{11}, s}^+, \tau_{\pi_{21}, s}, \frac{1}{2\Lambda^2}; Y_{\pi_1', s}, u \right) \\ & \quad \times \int_{\mathbb{R}^+} dz_q \mathcal{F}_{s_2+1, l_2; \delta_2}^\infty \left( \Lambda; z_q; \tau_{\pi_{12}, s}^q, \tau_{\pi_{22}, s}, \frac{1}{2\Lambda^2}; Y_{\pi_2', s}, u \right) \phi_q(z_q) \end{aligned} \quad (6.5.14)$$

Given a pair of trees

$$(T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1}) \in \mathcal{T}_{l_1; \pi_{11}, \pi_{21} \cup \{s+1\}}^{s_1+1} \times \mathcal{T}_{l_2; \pi_{12}^q, \pi_{22} \cup \{s+2\}}^{s_2+1}, \quad (6.5.15)$$

let us bound the following term

$$\begin{aligned} & \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta_1}^\infty \left( \Lambda; z_1; \tau_{\pi_{11}, s}^+, \tau_{\pi_{21}, s}, \frac{1}{2\Lambda^2}; T_{l_1}^{s_1+1}; Y_{\pi_1', s}, u \right) \\ & \quad \times \int_{\mathbb{R}^+} dz_q \mathcal{F}_{s_2+1, l_2; \delta_2}^\infty \left( \Lambda; z_q; \tau_{\pi_{12}, s}^q, \tau_{\pi_{22}, s}, \frac{1}{2\Lambda^2}; T_{l_2}^{s_2+1}; Y_{\pi_2', s}, u \right) \phi_q(z_q). \end{aligned} \quad (6.5.16)$$

Denoting by  $z$  and  $z'$  respectively the internal vertices in  $T_{l_1}^{s_1+1}$  and  $T_{l_2}^{s_2+1}$  attached to  $u$ , we use (3.1.5) together with the bound (3.1.9) to obtain

$$\int_{\mathbb{R}} du p_B \left( \frac{1 + \delta_1}{2\Lambda^2}; z, u \right) p_B \left( \frac{1 + \delta_2}{2\Lambda^2}; z', u \right) \leq O(1) p_B \left( \frac{1 + \delta_1'}{\Lambda^2}; z', z \right), \quad (6.5.17)$$

where  $\delta_1' := \max(\delta_1, \delta_2)$ . Diagrammatically, the contribution  $p_B \left( \frac{1 + \delta_1'}{\Lambda^2}; z', z \right)$  in (6.5.16) means that the external legs  $(z, u)$  and  $(z', u)$  are replaced by the internal line  $(z, z')$ .

The integration over  $z_q$  implies that  $z_q$  is converted into an internal vertex, and if  $q \leq s$  then it is attached to the external vertex  $y_q$ . These steps correspond to the fusion of the trees  $T_{l_1}^{s_1+1}$  and  $T_{l_2}^{s_2+1}$  at the vertex  $u$ . We denote the resulting tree by  $T_l^s \left( T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1} \right)$ . The number of its internal vertices<sup>4</sup> is  $v_{2,1} + v_{2,2} + \delta_{c(z_q),1}$  and we have

$$v_{2,1} + v_{2,2} + \delta_{c(z_q),1} + \delta_{c_1,1} \leq 3l - 4 + \frac{s+2}{2} \leq 3l - 2 + \frac{s}{2}.$$

Hence the tree  $T_l^s \left( T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1} \right)$  belongs to  $\mathcal{T}_l^s$ . Let  $\vec{z}'_{\pi'_{1,s}}$ ,  $\vec{z}'_{\pi'_{2,s}}$  and  $\vec{z}$  be respectively the set of the internal vertices of the trees  $T_{l_1}^{s_1+1}$ ,  $T_{l_2}^{s_2+1}$  and  $T_l^s \left( T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1} \right)$ . If  $q \leq s$ , we have<sup>5</sup>

$$\mathcal{Y}_{\pi_{1,s}}(\vec{z}) = \mathcal{Y}_{\pi_{11,s}}(\vec{z}'_{\pi'_{1,s}}) \cup \mathcal{Y}_{\pi_{12,s}}(\vec{z}'_{\pi'_{2,s}}) \cup \{z_q\}. \quad (6.5.18)$$

Otherwise,

$$\mathcal{Y}_{\pi_{1,s}}(\vec{z}) = \mathcal{Y}_{\pi_{11,s}}(\vec{z}'_{\pi'_{1,s}}) \cup \mathcal{Y}_{\pi_{12,s}}(\vec{z}'_{\pi'_{2,s}}). \quad (6.5.19)$$

We also have

$$\mathcal{Y}_{\pi_{2,s}}(\vec{z}) = \mathcal{Y}_{\pi_{21,s}}(\vec{z}'_{\pi'_{1,s}}) \cup \mathcal{Y}_{\pi_{22,s}}(\vec{z}'_{\pi'_{2,s}}). \quad (6.5.20)$$

Remembering that  $T_{l_1}^{s_1+1} \in \mathcal{T}_{l_1; \pi_{11,s}, \pi_{21,s} \cup \{s+1\}}^{s_1+1}$  and  $T_{l_2}^{s_2+1} \in \mathcal{T}_{l_2; \pi_{12,s}, \pi_{22,s} \cup \{s+2\}}^{s_2+1}$  and using (6.5.18)-(6.5.19), we deduce that

$$\mathcal{Y}_{\pi_{1,s}}(\vec{z}) \cap \mathcal{Y}_{\pi_{2,s}}(\vec{z}) = \emptyset, \quad (6.5.21)$$

which proves that  $T_l^s \left( T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1} \right)$  is indeed in  $\mathcal{T}_{l; \pi_{1,s}, \pi_{2,s}}^s$ . Combining all these arguments together, we deduce that (6.5.16) is bounded by

$$\mathcal{F}_{s,l;\delta}^{\Lambda;\infty} \left( \Lambda; z_1; \tau_{\pi_{1,s}^+}, \tau_{\pi_{2,s}}; T_l^s \left( T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1} \right); y_{2,s} \right). \quad (6.5.22)$$

Using (3.2.62) and (6.5.22), we obtain that (6.5.14) is bounded by

$$(\Lambda + m)^{3-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta'_1}^{\Lambda;\infty} \left( \tau_{\pi_{1,s}^+}, \tau_{\pi_{2,s}} \right). \quad (6.5.23)$$

- Now, let us treat the case  $\pi_{ij} = \emptyset$  for some  $i, j \in \{1, 2\}$ . In order to simplify the discussion, we consider the case  $\pi_{12} = \emptyset$  and rewrite (6.5.10) as follows

$$\begin{aligned} & \int_{\mathbb{R}} du \int_{\mathbb{R}^+} dz_q \partial^{w_1} \mathcal{L}_{l_1, n_1+1; \pi_1^+, \pi_{21}}^{\Lambda, \Lambda_0} \left( z_1; \vec{p}'_{\pi_1}, p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \partial^{w_3} \dot{C}^\Lambda(p) \\ & \quad \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0} \left( z_q; \vec{p}'_{\pi_2}, -p; \phi''_{s_2, q} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \phi_q(z_q). \end{aligned} \quad (6.5.24)$$

<sup>4</sup>  $v_{2,1}$  and  $v_{2,2}$  denote respectively the number of vertices of incidence number 2 of the trees  $T_{l_1}^{s_1+1}$  and  $T_{l_2}^{s_2+1}$ .

<sup>5</sup> we used the notation (3.2.5)



Inserting the induction hypothesis (6.4.18) and using the bound (6.4.7) we deduce that (6.5.24) is bounded by

$$\begin{aligned} & (\Lambda + m)^{3-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ & \quad \times \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta_1}^{\infty} \left( \Lambda; z_1; \tau_{\pi_{1,s}}^+, \tau_{\pi_{21,s}}, \frac{1}{2\Lambda^2}; Y_{\pi_{1,s}}, Y_{\pi_{21,s}}, u \right) \\ & \quad \times \int_{\mathbb{R}^+} dz_q \mathcal{F}_{s_2+1, l_2; \delta_2}^{\infty} \left( \Lambda; z_q; \tau_{\pi_{22,s}}, \frac{1}{2\Lambda^2}; Y_{\pi_{22,s}}, u \right) \phi_q(z_q). \end{aligned} \quad (6.5.25)$$

Following the same steps as before, we find

$$\begin{aligned} & \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta_1}^{\infty} \left( \Lambda; z_1; \tau_{\pi_{1,s}}^+, \tau_{\pi_{21,s}}, \frac{1}{2\Lambda^2}; Y_{\pi_{1,s}}, Y_{\pi_{21,s}}, u \right) \\ & \quad \times \int_{\mathbb{R}^+} dz_q \mathcal{F}_{s_2+1, l_2; \delta_2}^{\infty} \left( \Lambda; z_q; \tau_{\pi_{22,s}}, \frac{1}{2\Lambda^2}; Y_{\pi_{22,s}}, u \right) \phi_q(z_q) \\ & \leq \mathcal{F}_{s, l; \delta_1}^{\Lambda, \infty} \left( \Lambda; z_1; \tau_{\pi_{1,s}}^+, \tau_{\pi_{2,s}}; T_l^s \left( T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1} \right); y_{2,s} \right), \end{aligned} \quad (6.5.26)$$

where the tree  $T_l^s \left( T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1} \right)$  belongs to  $\mathcal{T}_l^s$ . The only point that we need to verify carefully is that the fusion of  $T_{l_1}^{s_1+1}$  and  $T_{l_2}^{s_2+1}$  indeed produces a tree in  $\mathcal{T}_{l; \pi_{1,s}, \pi_{2,s}}^s$ . We have

$$\mathcal{Y}_{\pi_{1,s}}(\vec{z}) = \mathcal{Y}_{\pi_{1,s}}(\vec{z}'_{\pi'_{1,s}}) \quad (6.5.27)$$

and

$$\mathcal{Y}_{\pi_{2,s}}(\vec{z}) = \mathcal{Y}_{\pi_{21,s}}(\vec{z}'_{\pi'_{1,s}}) \cup \mathcal{Y}_{\pi'_{22,s}}(\vec{z}'_{\pi'_{2,s}}). \quad (6.5.28)$$

For  $T_{l_1}^{s_1+1} \in \mathcal{T}_{l_1; \pi_{1,s}, \pi_{21,s}}^{s_1+1}$  and  $T_{l_2}^{s_2+1} \in \mathcal{T}_{l_2}^{s_2+1}$ , we clearly have

$$\mathcal{Y}_{\pi_{1,s}}(\vec{z}) \cap \mathcal{Y}_{\pi_{2,s}}(\vec{z}) = \emptyset, \quad (6.5.29)$$

which implies that  $T_l^s \left( T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1} \right)$  belongs to  $\mathcal{T}_{l; \pi_{1,s}, \pi_{2,s}}^s$ . (6.5.25) together with (6.5.26) give that (6.5.24) is bounded by (6.5.23).

B) Case  $r \neq 0$ : The linear term in this case can be treated similarly to the case  $r = 0$ . The only part that must be treated carefully is the quadratic term. Without loss of generality, we consider a term from the symmetrized sum that corresponds to a partition  $(\pi'_1, \pi'_2)$  of  $\sigma_{2:n}$  such that  $2 \in \pi'_2$  which is given by

$$\begin{aligned} & \int_{\vec{z}_{\pi'_1}^+} \int_{\vec{z}_{\pi'_2}} \int_{\mathbb{R}} dz (z_1 - z_2)^r \int_{\mathbb{R}} dz' \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0} \left( (\vec{z}'_{\pi'_1}, \vec{p}'_{\pi'_1}), (z, p) \right) \partial^{w_3} \dot{C}^{\Lambda}(p) \\ & \quad \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0} \left( (\vec{z}'_{\pi'_2}, \vec{p}'_{\pi'_2}), (z', -p) \right) \prod_{i=2}^s p_B(\tau_i; z_i, y_i) p_B \left( \frac{1}{\Lambda^2}; z, z' \right). \end{aligned} \quad (6.5.30)$$

We write

$$(z_1 - z_2)^r = \sum_{r_1+r_2+r_3=r} \binom{r}{r_1, r_2, r_3} (z_1 - u)^{r_1} (u - z_q)^{r_2} (z_q - z_2)^{r_3} \quad (6.5.31)$$

where the multinomial coefficient is given by

$$\binom{r}{r_1, r_2, r_3} = \frac{r!}{r_1! r_2! r_3!}.$$

For  $\pi_{ij} \neq \emptyset$ , we rewrite (6.5.30) using the notation (6.5.11) together with (3.1.5) as follows

$$\begin{aligned} & \int_{\mathbb{R}} du \int_{\mathbb{R}^+} dz_q (z_1 - u)^{r_1} \partial^{w_1} \mathcal{L}_{l_1, n_1+1; \pi_{11}^+, \pi_{21} \cup \{s+1\}}^{\Lambda, \Lambda_0} \left( z_1; \vec{p}_{\pi_1'}, p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \\ & \times (u - z_q)^{r_2} \partial^{w_2} \mathcal{L}_{l_2, n_2+1; r_3; \pi_{12}^+, \pi_{22} \cup \{s+2\}}^{\Lambda, \Lambda_0} \left( z_q; \vec{p}_{\pi_2'}, -p; \phi''_{s_2, q} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \\ & \times \partial^{w_3} \dot{C}^\Lambda(p) \phi_q(z_q), \end{aligned} \quad (6.5.32)$$

where  $\phi_q$  is defined as in part A). Using the induction hypothesis (6.5.1) for  $r = 0$  and  $r \neq 0$ , we deduce that (6.5.30) is bounded by

$$\begin{aligned} & (\Lambda + m)^{3-n-|w|-r_3} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) e^{-\frac{m^2}{\Lambda^2}} \\ & \times \int_{\mathbb{R}} du |z_1 - u|^{r_1} \mathcal{F}_{s_1+1, l_1; \delta_1}^\infty \left( \Lambda; z_1; \tau_{\pi_{11}, s}^+, \tau_{\pi_{21}, s}, \frac{1}{2\Lambda^2}; Y_{\pi_{1, s}'}, u \right) \\ & \times \int_{\mathbb{R}^+} dz_q |z_q - u|^{r_2} \mathcal{F}_{s_2+1, l_2; \delta_2}^\infty \left( \Lambda; z_q; \tau_{\pi_{12}, s}^+, \tau_{\pi_{22}, s}, \frac{1}{2\Lambda^2}; Y_{\pi_{2, s}'}, u \right) \phi_q(z_q). \end{aligned} \quad (6.5.33)$$

Now, let us analyze the term

$$\int_{\mathbb{R}} du |z_1 - u|^{r_1} \mathcal{F}_{s_1+1, l_1; \delta_1}^\infty \left( \Lambda; z_1; \tau_{\pi_{11}, s}^+, \tau_{\pi_{21}, s}, \frac{1}{2\Lambda^2}; Y_{\pi_{11}, s}, Y_{\pi_{21}, s}, u \right). \quad (6.5.34)$$

Given a tree  $T_{l_1}^{s_1+1}$  in  $\mathcal{T}_{l_1}^{s_1+1}$ , we denote by  $(v_1, \dots, v_q)$  the set of internal vertices on the path joining the root vertex  $z_1$  to the external vertex  $u$ . We have

$$|z_1 - u| \leq |z_1 - v_1| + \sum_{a=1}^{q-1} |v_a - v_{a+1}| + |v_q - u|. \quad (6.5.35)$$

Using (3.1.10) and the fact that  $\Lambda_i \geq \Lambda$  for all  $1 \leq i \leq q$ , we deduce the bound

$$\begin{aligned} & |z_1 - u|^{r_1} \mathcal{F}_{s_1+1, l_1; \delta_1}^\infty \left( \Lambda; \tau_{\pi_{11}}^+, \tau_{\pi_{21}}, \frac{1}{2\Lambda^2}; T_{l_1}^{s_1+1} \left( z_1; \vec{z}; Y_{\pi_{1, s}'}, u \right) \right) \\ & \leq O(1) \Lambda^{-r_1} \mathcal{F}_{s_1+1, l_1; \delta_1}^\infty \left( \Lambda; \tau_{\pi_{11}}^+, \tau_{\pi_{21}}, \frac{1}{2\Lambda^2}; T_{l_1}^{s_1+1} \left( z_1; \vec{z}; Y_{\pi_{2, s}'}, u \right) \right), \end{aligned} \quad (6.5.36)$$

where  $\delta_1 < \delta'_1 < \frac{1}{2}$ . Proceeding similarly, we deduce that

$$\begin{aligned} & |z_q - u|^{r_q} \mathcal{F}_{s_2+1, l_2; \delta_2}^\infty \left( \Lambda; \tau_{\pi_{12, s}}^+, \tau_{\pi_{22, s}}, \frac{1}{2\Lambda^2}; T_{l_2}^{s_2+1} \left( z_q; \vec{z}', Y_{\pi'_{1, s}}, u \right) \right) \\ & \leq O(1) \Lambda^{-r_2} \mathcal{F}_{s_2+1, l_2; \delta'_2}^\infty \left( \Lambda; \tau_{\pi_{12, s}}^+, \tau_{\pi_{22, s}}, \frac{1}{2\Lambda^2}; T_{l_2}^{s_2+1} \left( z_q; \vec{z}', Y_{\pi'_{2, s}}, u \right) \right), \end{aligned} \quad (6.5.37)$$

where  $0 < \delta_2 < \delta'_2 < \frac{1}{2}$ . Using (5.5.39) together with (6.5.36) and (6.5.37) we deduce that  $\tilde{R}_2$  is bounded by

$$\begin{aligned} & (\Lambda + m)^{3-n-|w|-r} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ & \quad \times \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta'_1}^\infty \left( \Lambda; z_1; \tau_{\pi_{11}}^+, \tau_{\pi_{21}}, \frac{1}{2\Lambda^2}; Y_{\pi'_{1, s}}, u \right) \\ & \quad \times \int_{\mathbb{R}^+} dz_q \mathcal{F}_{s_2+1, l_2; \delta'_2}^\infty \left( \Lambda; z'; \tau_{\pi_{12}}^+, \tau_{\pi_{22}}; Y_{\pi'_{2, s}}, u \right) \phi_q(z_q). \end{aligned} \quad (6.5.38)$$

The rest of the proof which leads to the final bound for (6.5.30) is similar to the case  $r = 0$ . The same method holds for  $\pi_{ij} = \emptyset$  for some  $i, j \in \{1, 2\}$ .

- C) The bounds for  $\partial_\Lambda \partial^w \mathcal{L}_{l, n; \pi_1^+, \pi_2}^{\Lambda, \Lambda_0} (z_1; \vec{p}_n; \phi_{\tau_{2, s}, y_{2, s}}^{(j)})$  and  $\partial_\Lambda \partial^w \mathcal{L}_{l, n; +}^{\Lambda, \Lambda_0} (z_1; \vec{p}_n; \phi_{\tau_{2, s}, y_{2, s}}^{(j)})$  are deduced following the same steps of part C) in the proof of Theorem 7 in chapter 4. Then, we obtain

$$\begin{aligned} & \left| \partial_\Lambda \partial^w \mathcal{L}_{l, n; \pi_1^+, \pi_2}^{\Lambda, \Lambda_0} (z_1; \vec{p}_n; \phi_{\tau_{2, s}, y_{2, s}}^{(j)}) \right| \leq (\Lambda + m)^{2-n-|w|} \tau_j^{-\frac{1}{2}} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ & \quad \times \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l; \delta}^{\Lambda, \infty} (\tau_{\pi_1^+, s}, \tau_{\pi_2, s}) \end{aligned} \quad (6.5.39)$$

and

$$\begin{aligned} & \left| \partial_\Lambda \partial^w \mathcal{L}_{l, n; +}^{\Lambda, \Lambda_0} (z_1; \vec{p}_n; \phi_{\tau_{2, s}, y_{2, s}}^{(j)}) \right| \leq (\Lambda + m)^{2-n-|w|} \tau_j^{-\frac{1}{2}} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ & \quad \times \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l; +; \delta}^{\Lambda, \infty} (\tau_{2, s}) . \end{aligned} \quad (6.5.40)$$

## Integration

The integration of the bounds is performed similarly to the integration part in Theorem 7. We refer the reader to the proof of Theorem 7 in chapter 4 for further details.  $\square$

**Remark 5.** Using the reflection symmetry of the translationally invariant theory that is

$$\mathcal{L}_{l, n}^{\Lambda, \Lambda_0} ((-z_1, p_1), (-z_2, p_2) \cdots, (-z_n, p_n)) = \mathcal{L}_{l, n}^{\Lambda, \Lambda_0} ((z_1, p_1), (z_2, p_2) \cdots, (z_n, p_n)),$$

we obtain

$$\begin{aligned} & \int_{(\mathbb{R}^+)^{n-1}} d\vec{z}_{2,n} (z_1 - z_2)^r \mathcal{L}_{l,n}^{\Lambda, \Lambda_0} ((z_1, p_1), (z_2, p_2) \cdots, (z_n, p_n)) \prod_{i=2}^s p_B(\tau_i; z_i, y_i) \\ &= \int_{(\mathbb{R}^-)^{n-1}} d\vec{z}_{2,n} (z_1 + z_2)^r \mathcal{L}_{l,n}^{\Lambda, \Lambda_0} ((-z_1, p_1), (-z_2, p_2) \cdots, (-z_n, p_n)) \prod_{i=2}^s p_B(\tau_i; z_i, -y_i). \end{aligned} \quad (6.5.41)$$

Denoting by  $\mathcal{L}_{l,n;-}^{\Lambda, \Lambda_0}(-z_1; \vec{p}_n; \phi_{\tau_{2,s}, -y_{2,s}})$  the right hand side of (6.5.41) and using the bound (6.4.18), we deduce for  $z_1 \in \mathbb{R}^-$

$$\begin{aligned} \left| \partial^w \mathcal{L}_{l,n;-}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, y_{2,s}}) \right| &\leq (\Lambda + m)^{4-n-|w|-r} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;-;\delta}^{\Lambda; \infty}(\tau_{2,s}). \end{aligned} \quad (6.5.42)$$

## 6.5.2 Proof of Proposition 9

*Proof.* Starting from the tree order

$$\mathcal{L}_{0,4}^{\Lambda, \Lambda_0}((z_1, p_1), \cdots, (z_4, p_4)) = \lambda \prod_{i=2}^4 \delta(z_1 - z_i), \quad (6.5.43)$$

we have for a given partition  $(\pi_1, \pi_2)$  in  $\tilde{\mathcal{P}}_{2;4}$

$$\mathcal{L}_{0,4;\pi_1^+, \pi_2^-}^{\Lambda, \Lambda_0}(\vec{p}_4; \phi_{\tau_{1,s}, y_{1,s}}) = 0, \quad \forall 2 \leq s \leq 4.$$

This implies using the flow equation (5.4.1) for all  $(\pi_1, \pi_2)$  in  $\tilde{\mathcal{P}}_{2;n}$  that we have

$$\mathcal{L}_{0,n;\pi_1^+, \pi_2^-}^{\Lambda, \Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s}, y_{1,s}}) = 0, \quad \forall 2 \leq s \leq n.$$

The bound (6.4.17) obviously holds for the tree order.

### The right-hand side

The bound that we want to obtain for the right-hand side of the flow equation is

$$\begin{aligned} \left| \partial_\Lambda \partial^w \mathcal{L}_{l,n;r_1, r_2; \pi_1^+, \pi_2^-}^{\Lambda, \Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s}, y_{1,s}}) \right| &\leq (\Lambda + m)^{2-n-|w|-r} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\quad \times \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda, 0}(\tau_{\pi_1, s}^+, \tau_{\pi_2, s}^-). \end{aligned} \quad (6.5.44)$$

Case  $r_1 = r_2 = 0$ : first, we treat the linear term

$$\int_k \int_{(\mathbb{R}^+)^{p_1}} d\vec{z}_{\pi_1} \int_{(\mathbb{R}^-)^{p_2}} d\vec{z}_{\pi_2} \int_{\mathbb{R}} dz \int_{\mathbb{R}} dz' \mathcal{L}_{l-1, n+2}^{\Lambda, \Lambda_0} \left( (\vec{z}_n, \vec{p}_n), (z, k), (z', -k) \right) \\ \times \dot{C}^\Lambda(k) p_B \left( \frac{1}{\Lambda^2}; z, z' \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i). \quad (6.5.45)$$

Using (3.1.5) and the notation (6.4.12), we rewrite  $R_1$  as follows

$$\int_{\mathbb{R}} du \int_k \sum_{(\tilde{\pi}_1, \tilde{\pi}_2)}' \mathcal{L}_{l-1, n+2; \tilde{\pi}_1^+, \tilde{\pi}_2^-}^{\Lambda, \Lambda_0} \left( \vec{p}_n, k, -k; \phi_{\tau_1, s, y_1, s} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \dot{C}^\Lambda(k). \quad (6.5.46)$$

The prime on the sum refers to a summation over partitions in the set

$$\{(\pi_1 \cup \mathcal{E}, \pi_2 \cup \mathcal{E}^c) \mid \mathcal{E} \in \mathcal{P}(\{s+1, s+2\})\},$$

where  $\mathcal{P}(\{s+1, s+2\}) := \{\{s+1\}, \{s+2\}, \{s+1, s+2\}\}$ .

Using the induction hypothesis, (6.5.46) is bounded by

$$(\Lambda + m)^{1-n-|w|} \mathcal{F}_1 \left( \log \frac{\Lambda + m}{m} \right) \int_k \dot{C}^\Lambda(k) \mathcal{P} \left( \frac{|k|}{\Lambda + m}, \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \times \sum_{(\tilde{\pi}_1, \tilde{\pi}_2)}' \int_{\mathbb{R}} du \mathcal{F}_{s+2, l-1; \delta}^0(\Lambda; \tau_{\tilde{\pi}_1}^+, \tau_{\tilde{\pi}_2}^-; Y_{\sigma_s}, u, u), \quad (6.5.47)$$

where  $\tau_{s+1} = \tau_{s+2} = \frac{1}{2\Lambda^2}$ . Using the bound (3.2.64) from lemma 2, we deduce the final bound on (6.5.46)

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l; \delta}^{\Lambda, 0}(\tau_{\pi_1, s}^+, \tau_{\pi_2, s}^-). \quad (6.5.48)$$

In this part, we bound the quadratic term by analyzing a given term from the symmetrized sum. Given a partition  $(\pi'_1, \pi'_2)$  of  $\sigma_n$ , we bound the following term

$$\int_{(\mathbb{R}^+)^{p_1}} d\vec{z}_{\pi_1} \int_{(\mathbb{R}^-)^{p_2}} d\vec{z}_{\pi_2} \int_{\mathbb{R}} dz \int_{\mathbb{R}} dz' \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0} \left( (\vec{z}_{\pi'_1}, \vec{p}_{\pi'_1}), (z, p) \right) \partial^{w_3} \dot{C}^\Lambda(p) \\ \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0} \left( (\vec{z}_{\pi'_2}, \vec{p}_{\pi'_2}), (z', -p) \right) p_B \left( \frac{1}{\Lambda^2}; z, z' \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i). \quad (6.5.49)$$

We define  $\pi_{ij} := \pi_i \cap \pi'_j$  and rewrite (6.5.49) using the notation (6.5.11) as follows

$$\sum_{(\tilde{\pi}_{11}, \tilde{\pi}_{21})}' \sum_{(\tilde{\pi}_{12}, \tilde{\pi}_{22})}'' \int_{\mathbb{R}} du \partial^{w_1} \mathcal{L}_{l_1, n_1+1; \tilde{\pi}_{11}^+, \tilde{\pi}_{21}^-}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi'_1}, p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \\ \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1; \tilde{\pi}_{12}^+, \tilde{\pi}_{22}^-}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi'_2}, -p; \phi''_{s_2} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \partial^{w_3} \dot{C}^\Lambda(p). \quad (6.5.50)$$

The prime on the first sum refers to a summation over partitions in

$$\{(\pi_{11} \cup \{s+1\}, \pi_{21}), (\pi_{11}, \pi_{21} \cup \{s+1\})\},$$

and the double prime on the second sum refers to a summation over

$$\{(\pi_{12}, \pi_{22} \cup \{s+2\}), (\pi_{12} \cup \{s+2\}, \pi_{22})\}.$$

- In order to bound (6.5.50), let us first consider the case  $\pi_{ij} \neq \emptyset$  for all  $i, j \in \{1, 2\}$ . Using the induction hypothesis we deduce that (6.5.50) is bounded by

$$\begin{aligned} & (\Lambda + m)^{1-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ & \times \sum'_{(\tilde{\pi}_{11}, \tilde{\pi}_{21})} \sum''_{(\tilde{\pi}_{12}, \tilde{\pi}_{22})} \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta_1}^0 \left( \Lambda; \tau_{\tilde{\pi}_{11}, s}^+, \tau_{\tilde{\pi}_{12}, s}^-; Y_{\pi'_{1, s}}, u \right) \\ & \times \mathcal{F}_{s_2+1, l_2; \delta_2}^0 \left( \Lambda; \tau_{\tilde{\pi}_{12}, s}^+, \tau_{\tilde{\pi}_{22}, s}^-; Y_{\pi'_{2, s}}, u \right). \end{aligned} \quad (6.5.51)$$

Using the bound (3.2.74) from lemma 3, we deduce that (6.5.51) is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l; \delta'_1}^0 \left( \Lambda; \tau_{\pi_{1, s}}^+, \tau_{\pi_{2, s}}^-; Y_{\sigma_s} \right), \quad (6.5.52)$$

where  $\delta'_1 := \max(\delta_1, \delta_2)$ .

- To treat the opposite cases, we restrict our analysis to the summand in (6.5.50)

$$\begin{aligned} & \int_{\mathbb{R}} du \partial^{w_1} \mathcal{L}_{l_1, n_1+1; \tilde{\pi}_{11}^+, \tilde{\pi}_{21}^-}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi'_1}, p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \\ & \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1; \tilde{\pi}_{12}^+, \tilde{\pi}_{22}^-}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi'_2}, -p; \phi''_{s_2} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \partial^{w_3} \dot{C}^\Lambda(p) \end{aligned} \quad (6.5.53)$$

and distinguish between different cases. We separate between these cases since in order to bound each one of them, a different induction hypothesis or bound is required:

- Let us start with the case  $\tilde{\pi}_{21} = \tilde{\pi}_{12} = \emptyset$ , which implies that  $\tilde{\pi}_{ii} = \pi_i = \pi'_i$ . In this case, (6.5.53) reads

$$\begin{aligned} & \int_{\mathbb{R}} du \partial^{w_1} \mathcal{L}_{l_1, n_1+1; +}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi'_1}, p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \\ & \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1; -}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi'_2}, -p; \phi''_{s_2} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \partial^{w_3} \dot{C}^\Lambda(p), \end{aligned} \quad (6.5.54)$$

where

$$\begin{aligned} \partial^{w_1} \mathcal{L}_{l_1, n_1+1;+}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_1'}; p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \\ = \int_{\mathbb{R}^+} dz \partial^{w_1} \mathcal{L}_{l_1, n_1+1;+}^{\Lambda, \Lambda_0} \left( z; \vec{p}_{\pi_1'}, p; \phi'_{s_1} \right) p_B \left( \frac{1}{2\Lambda^2}; z, u \right) \end{aligned} \quad (6.5.55)$$

and

$$\begin{aligned} \partial^{w_2} \mathcal{L}_{l_2, n_2+1;-}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_2'}; p; \phi''_{s_2} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \\ = \int_{\mathbb{R}^-} dz' \partial^{w_2} \mathcal{L}_{l_2, n_2+1;-}^{\Lambda, \Lambda_0} \left( z'; \vec{p}_{\pi_2}, -p; \phi''_{s_2} \right) p_B \left( \frac{1}{2\Lambda^2}; z', u \right). \end{aligned} \quad (6.5.56)$$

Using the bounds (6.4.18) and (6.5.42) together with (3.1.5), we deduce that (6.5.54) is bounded by

$$\begin{aligned} (\Lambda + m)^{3-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) e^{-\frac{m^2}{\Lambda^2}} \\ \times \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^-} dz' \mathcal{F}_{s_1+1, l_1;+; \delta_1}^{\infty} (\Lambda; z; \tau_{\pi_1, s}; Y_{\pi_1, s}) \\ \times \mathcal{F}_{s_2+1, l_2;-; \delta_2}^{\infty} (\Lambda; z'; \tau_{\pi_2, s}; Y_{\pi_2, s}) p_B \left( \frac{1}{\Lambda^2}; z, z' \right). \end{aligned} \quad (6.5.57)$$

Lemma 5 gives

$$\mathcal{F}_{s_1+1, l_1;+; \delta_1}^{\infty} (\Lambda; z; \tau_{\pi_1, s}; Y_{\pi_1, s}) \leq O(1) \mathcal{F}_{s_1+1, l_1; \delta_1} (\Lambda; z; \tau_{\pi_1, s}; Y_{\pi_1, s}) \quad (6.5.58)$$

and

$$\mathcal{F}_{s_2+1, l_2;-; \delta_2}^{\infty} (\Lambda; z'; \tau_{\pi_2, s}; Y_{\pi_2, s}) \leq O(1) \mathcal{F}_{s_2+1, l_2; \delta_2}^- (\Lambda; z'; \tau_{\pi_2, s}; Y_{\pi_2, s}). \quad (6.5.59)$$

Since  $z \in \mathbb{R}^+$  and  $z' \in \mathbb{R}^-$ , we have

$$p_B \left( \frac{1}{\Lambda^2}; z, z' \right) \leq \sqrt{2\pi} \Lambda^{-1} p_B \left( \frac{1}{\Lambda^2}; z, 0 \right) p_B \left( \frac{1}{\Lambda^2}; z', 0 \right). \quad (6.5.60)$$

Furthermore, for a given tree  $T_{l_1}^{s_1+1} \in \mathcal{T}_{l_1}^{s_1+1}$  we have for  $z \in \mathbb{R}^+$

$$\begin{aligned} p_B \left( \frac{1}{\Lambda^2}; z, 0 \right) \mathcal{F}_{\delta_1} \left( \Lambda_{\mathcal{J}_1}; \tau_{\pi_1, s}; T_{l_1}^{s_1+1} (z; \vec{z}_{\pi_1}; Y_{\pi_1, s}) \right) \\ = \mathcal{F}_{\delta_1} \left( \Lambda_{\mathcal{J}_1}, \Lambda; \tau_{\pi_1, s}; T_{l_1}^{s_1, 0} (\vec{z}_{\pi_1}, z; Y_{\pi_1, s}, 0) \right), \end{aligned} \quad (6.5.61)$$

with the following explanations: the surface tree  $T_{l_1}^{s_1, 0}$  is obtained from  $T_{l_1}^{s_1+1}$  by attaching the root vertex  $z$  to the surface external vertex  $0$  with a parameter  $\Lambda$  and converting  $z$  to

an internal vertex. Hence, the number of vertices of incidence number 2 of the new tree is equal to  $v_2 + \delta_{c(z),1}$ , with  $v_2$  denoting the number of vertices of incidence number 2 of the tree  $T_{l_1}^{s_1}$ . Hence, we have

$$v_{2,1} + \delta_{c(z'),1} \leq 3l_1 - 2 + \frac{s_1 + 1}{2} \leq 3l - 2 + \frac{s_1 + 1}{2}, \quad (6.5.62)$$

which implies that  $T_{l_1}^{s_1,0}$  is in  $\mathcal{F}_l^{s_1,0}$ . Proceeding similarly, we deduce for  $z' \in \mathbb{R}^-$

$$\begin{aligned} p_B \left( \frac{1}{\Lambda^2}; z', 0 \right) \mathcal{F}_{\delta_2}^- \left( \Lambda_{\mathcal{J}_2}; \tau_{\pi_{2,s}}; T_{l_2}^{s_2+1} (z'; \vec{z}_{\pi_2}; Y_{\pi_{2,s}}) \right) \\ = \mathcal{F}_{\delta_2}^- \left( \Lambda_{\mathcal{J}_2}, \Lambda; \tau_{\pi_{2,s}}; T_{l_2}^{s_2,0} (\vec{z}_{\pi_2}, z'; Y_{\pi_{2,s}}, 0) \right), \end{aligned} \quad (6.5.63)$$

where  $T_{l_2}^{s_2,0} \in \mathcal{F}_l^{s_2,0}$ . Therefore, combining (6.5.57)-(6.5.61) and (6.5.63) we deduce for  $l \geq 1$

$$\begin{aligned} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^-} dz' \mathcal{F}_{s_1+1, l_1; +; \delta_1}^\infty \left( \Lambda; \tau_{\pi_{1,s}}; T_{l_1}^{s_1} (z; \vec{z}_{\pi_1}; Y_{\pi_{1,s}}) \right) \\ \times \mathcal{F}_{s_2+1, l_2; -; \delta_2}^\infty \left( \Lambda; \tau_{\pi_{2,s}}; T_{l_2}^{s_2} (z'; \vec{z}_{\pi_2}; Y_{\pi_{2,s}}) \right) p_B \left( \frac{1}{\Lambda^2}; z, z' \right) \\ \leq O(1) \Lambda^{-1} \mathcal{F}_{s,l;\delta}^0 \left( \Lambda; \tau_{\pi_{1,s}}^+, \tau_{\pi_{2,s}}^-; W_l^s(\Pi); Y_{\sigma_s} \right), \end{aligned} \quad (6.5.64)$$

where  $\Pi = \pi_{1,s} \cup \pi_{2,s}$  and  $W_l^s(\Pi) = T_{l_1}^{s_1,0} \cup T_{l_2}^{s_2,0}$ . Hence, using (5.5.39) we obtain that (6.5.57) is bounded by

$$\begin{aligned} (\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \times \sum_{W_l^s(\Pi) \in W_{l_1, \pi_{1,s}, \pi_{2,s}}^s(\Pi)} \mathcal{F}_{s,l;\delta}^0 \left( \Lambda; \tau_{\pi_{1,s}}^+, \tau_{\pi_{2,s}}^-; W_l^s(\Pi); Y_{\sigma_s} \right). \end{aligned} \quad (6.5.65)$$

ii) In this part, we analyze the case  $\tilde{\pi}_{22} = \emptyset$  and  $\tilde{\pi}_{ij} \neq \emptyset$  for  $(i, j) \neq (2, 2)$ . In this case, we have  $\pi_2 \subset \pi_1'$  and (6.5.53) has the following form

$$\begin{aligned} \int_{\mathbb{R}} du \partial^{w_1} \mathcal{L}_{l_1, n_1+1; \tilde{\pi}_{11}^+, \pi_2}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_1'}, p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \partial^{w_3} \dot{C}^\Lambda(p) \\ \times \int_{\mathbb{R}^+} dz \partial^{w_2} \mathcal{L}_{l_2, n_2+1; +}^{\Lambda, \Lambda_0} \left( z; \vec{p}_{\pi_2}', -p; \phi''_{s_2} \right) p_B \left( \frac{1}{2\Lambda^2}; z, u \right). \end{aligned} \quad (6.5.66)$$

Using the induction hypothesis and the bound (6.4.18), we deduce that (6.5.66) is bounded



by

$$\begin{aligned}
& (\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\
& \quad \times \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta_1}^0 \left( \Lambda; \tau_{\tilde{\pi}_{11}, s}^+, \tau_{\tilde{\pi}_{2, s}}^-; Y_{\pi'_{1, s}}, u \right) \\
& \quad \times \int_{\mathbb{R}^+} dz' \mathcal{F}_{s_2+1, l_2; +; \delta_2}^\infty \left( \Lambda; \tau_{\pi'_{2, s}}; z', Y_{\pi'_{2, s}} \right) p_B \left( \frac{1}{\Lambda^2}; z', u \right). \quad (6.5.67)
\end{aligned}$$

Using lemma [5](#), we have

$$\mathcal{F}_{s_2+1, l_2; +; \delta_2}^\infty \left( \Lambda; \tau_{\pi'_{2, s}}; z', Y_{\pi'_{2, s}} \right) \leq O(1) \mathcal{F}_{s_2+1, l_2; \delta_2} \left( \Lambda; \tau_{\pi'_{2, s}}; z', Y_{\pi'_{2, s}} \right). \quad (6.5.68)$$

Hence, we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^+} dz' \mathcal{F}_{s_2+1, l_2; +; \delta_2}^\infty \left( \Lambda; \tau_{\pi'_{2, s}}; z', Y_{\pi'_{2, s}} \right) p_B \left( \frac{1}{\Lambda^2}; z', u \right) \\
& \quad \leq O(1) \hat{\mathcal{F}}_{s_2, l_2; \delta_2} \left( \Lambda; \tau_{\pi'_{2, s}}; z', Y_{\pi'_{2, s}} \right), \quad (6.5.69)
\end{aligned}$$

which together with [\(6.5.67\)](#) and [\(3.2.84\)](#) imply that [\(6.5.66\)](#) is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l; \delta'_2}^{\Lambda; 0} \left( \tau_{\pi_{1, s}}^+, \tau_{\pi_{2, s}}^- \right), \quad (6.5.70)$$

where  $\delta'_2 := \max(\delta_1, \delta_2)$ .

- iii) The last case to treat is  $\tilde{\pi}_{12} = \emptyset$  and  $\tilde{\pi}_{ij} \neq \emptyset$  for  $(i, j) \neq (2, 2)$ . In this case, [\(6.5.53\)](#) has the following form

$$\begin{aligned}
& \int_{\mathbb{R}} du \partial^{w_1} \mathcal{L}_{l_1, n_1+1; \tilde{\pi}_{11}^+, \tilde{\pi}_2}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi'_1}, p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \partial^{w_3} \dot{C}^\Lambda(p) \\
& \quad \times \int_{\mathbb{R}^-} dz \partial^{w_2} \mathcal{L}_{l_2, n_2+1; -}^{\Lambda, \Lambda_0} \left( z; \vec{p}_{\pi'_2}, -p; \phi''_{s_2} \right) p_B \left( \frac{1}{2\Lambda^2}; z, u \right). \quad (6.5.71)
\end{aligned}$$

Using the induction hypothesis, the bound [\(6.5.42\)](#) together with the bound [\(3.2.86\)](#) from lemma [5](#) and following the same steps as in (ii), we deduce that [\(6.5.71\)](#) is also bounded by [\(6.5.70\)](#).

The case  $(r_1, r_2) \neq (0, 0)$  can be treated following the same steps used in the proof of Theorem [9](#).

### Integration

- **Irrelevant terms:** We integrate (6.5.44) from  $\Lambda$  to  $\Lambda_0$  downwards with vanishing boundary conditions at  $\Lambda = \Lambda_0$ . Using (3.2.61), we obtain for  $n \geq 4$

$$\begin{aligned} & \left| \partial^w \mathcal{L}_{l,n;r;\pi_1^+,\pi_2^-}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s},y_{1,s}}) \right| \\ & \leq (\Lambda + m)^{3-n-|w|-r} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{\pi_{1,s}^+}, \tau_{\pi_{2,s}^-}) . \end{aligned} \quad (6.5.72)$$

- **Relevant term:** In this part, we integrate the cases that correspond to  $n + |w| + r_1 + r_2 \leq 3$  which are given by

$$\int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} z_1^{r_1} z_2^{r_2} \partial_{\Lambda} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) , \quad (\pi_1, \pi_2) \in \tilde{\mathcal{P}}_{2,2}, \quad (6.5.73)$$

where  $r_1, r_2$  are integers such that  $r_1 + r_2 \leq 1$ , and  $0 \leq s \leq 2$  and . We restrict our analysis to the case  $s = 2$ , the case  $s = 1$  can be treated similarly and the case  $s = 0$  will be integrated in the sequel. For  $s = 2$ , the relevant part is extracted from

$$\int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} \partial_{\Lambda} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)) \phi_1(z_1) \phi_2(z_2) \quad (6.5.74)$$

by performing a Taylor expansion of  $\phi_1$  and  $\phi_2$  around  $z_i = 0$  at  $p = 0$  with  $\phi_i(z_i) := p_B(\tau_i; z_i, y_i)$ . This part of the proof is similar to the integration of the relevant terms in the proof of Theorem 9. However some simplifications occur and we explicit the steps where the treatment is different. We have

$$\begin{aligned} \mathcal{L}_{l,2;\pi_1^+,\pi_2^-}^{\Lambda,\Lambda_0}(0,0) &= s_{l;\pm}^{\Lambda,\Lambda_0} \phi_1(0) \phi_2(0) + e_{l;\pm}^{\Lambda,\Lambda_0} \phi_1(0) (\partial_n \phi_2)(0) + h_{l;\pm}^{\Lambda_0} \phi_2(0) (\partial_n \phi_1)(0) \\ & \quad + l_{l,2;\pm}^{\Lambda,\Lambda_0}(\phi_1, \phi_2) , \end{aligned} \quad (6.5.75)$$

where the relevant terms  $s_{l;\pm}^{\Lambda,\Lambda_0}$ ,  $e_{l;\pm}^{\Lambda,\Lambda_0}$  and  $h_{l;\pm}^{\Lambda,\Lambda_0}$  are given by

$$\begin{aligned} s_{l;\pm}^{\Lambda,\Lambda_0} &:= \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) , \\ e_{l;\pm}^{\Lambda,\Lambda_0} &:= \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} z_2 \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) , \\ h_{l;\pm}^{\Lambda,\Lambda_0} &:= \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} z_1 \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) . \end{aligned} \quad (6.5.76)$$

Bose symmetry implies that

$$\int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} (z_1 - z_2) \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) = 0 , \quad (6.5.77)$$

so that the counter-terms  $e_{l;\pm}^{\Lambda,\Lambda_0}$  and  $h_{l;\pm}^{\Lambda,\Lambda_0}$  are equal to all orders of perturbation theory. The bound (6.5.44) for  $s = 0$  and  $r_1 + r_2 \leq 1$  gives

$$\left| \partial_\Lambda s_{l;\pm}^{\Lambda,\Lambda_0} \right| \leq \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right), \quad \left| \partial_\Lambda e_{l;\pm}^{\Lambda,\Lambda_0} \right| \leq (\Lambda + m)^{-1} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right). \quad (6.5.78)$$

Integrating (6.5.78) from 0 to  $\Lambda$  and using the renormalization conditions (6.3.6)-(6.3.7), we have

$$\left| s_{l;\pm}^{\Lambda,\Lambda_0} \right| \leq (\Lambda + m) \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right), \quad \left| e_{l;\pm}^{\Lambda,\Lambda_0} \right| \leq \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right). \quad (6.5.79)$$

Applying lemma 10, we obtain

$$\begin{aligned} & \left| s_l^{\Lambda,\Lambda_0} \phi_1(0) \phi_2(0) + e_l^{\Lambda,\Lambda_0} \{ \phi_1(0) (\partial_n \phi_2)(0) + \phi_2(0) (\partial_n \phi_1)(0) \} \right| \\ & \leq (\Lambda + m) \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2;l;\delta}^{\Lambda,0} (\tau_{\pi_1}^+, \tau_{\pi_2}^-). \end{aligned} \quad (6.5.80)$$

Now, we bound and integrate the remainder  $\partial_\Lambda l_{l,2;\pm}^{\Lambda,\Lambda_0}(\phi_1, \phi_2)$  from  $\Lambda$  to  $\Lambda_0$  which reads

$$\begin{aligned} \partial_\Lambda l_{l,2;\pm}^{\Lambda,\Lambda_0}(\phi_1, \phi_2) &= \left( \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} z_1 z_2 \partial_\Lambda \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \right) (\partial_n \phi_1)(0) (\partial_n \phi_2)(0) \\ &+ \phi_1(0) \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} \partial_\Lambda \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_2)(tz_2) \\ &+ \phi_2(0) \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} \partial_\Lambda \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \\ &+ (\partial_n \phi_1)(0) \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} z_1 \partial_\Lambda \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_2)(tz_2) \\ &+ (\partial_n \phi_2)(0) \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} z_2 \partial_\Lambda \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \\ &+ \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} \partial_\Lambda \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \left( \int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \right) \\ &\quad \times \left( \int_0^1 dt' (1-t') (\partial_{t'}^2 \phi_2)(t'z_2) \right). \end{aligned} \quad (6.5.81)$$

Following the same steps of the integration of the remainder  $\partial_\Lambda l_{l,2;\pm}^{\Lambda,\Lambda_0}(\phi_1, \phi_2)$  in the proof of Theorem 9 in chapter 5, we obtain

$$\begin{aligned} & \left| \left( \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} \partial_\Lambda \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \right) (\partial_n \phi_1)(0) (\partial_n \phi_2)(0) \right| \\ & \leq (\Lambda + m)^{-2} \tau_1^{-\frac{1}{2}} \tau_2^{-\frac{1}{2}} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2;l;\delta}^{\Lambda,0} (\tau_{\pi_1}^+, \tau_{\pi_2}^-), \end{aligned} \quad (6.5.82)$$

$$\begin{aligned} & \left| \left( \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} z_2 \partial_{\Lambda} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0} ((z_1, 0), (z_2, 0)) \right) \int_0^1 dt (1-t) \partial_t^2 \phi_1(tz_1) \right| \\ & \leq (\Lambda + m)^{-3} \tau_1^{-1} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0} (\tau_{\pi_1}^+, \tau_{\pi_2}^-), \end{aligned} \quad (6.5.83)$$

$$\begin{aligned} & \left| (\partial_n \phi_1)(0) \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} z_2 \partial_{\Lambda} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0} ((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_2(tz_1)) \right| \\ & \leq (\Lambda + m)^{-3} \tau_1^{-\frac{1}{2}} \tau_2^{-1} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0} (\tau_{\pi_1}^+, \tau_{\pi_2}^-) \end{aligned} \quad (6.5.84)$$

and

$$\begin{aligned} & \left| \phi_i(0) \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} \partial_{\Lambda} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0} ((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_j(tz_1)) \right| \\ & \leq (\Lambda + m)^{-2} \tau_j^{-1} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0} (\tau_{\pi_1}^+, \tau_{\pi_2}^-). \end{aligned} \quad (6.5.85)$$

The only term which needs to be treated differently is

$$\begin{aligned} & \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} \partial_{\Lambda} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0} ((z_1, 0), (z_2, 0)) \left( \int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \right) \\ & \quad \times \left( \int_0^1 dt' (1-t') (\partial_{t'}^2 \phi_2)(t'z_2) \right). \end{aligned} \quad (6.5.86)$$

Proceeding as in the proof of Theorem [9](#), we rewrite [\(6.5.86\)](#) as follows

$$\begin{aligned} & \sum_{(\alpha, \beta) \in \mathcal{J}_2} \sum_{(\alpha', \beta') \in \mathcal{J}_2} c_{\alpha\beta} c_{\alpha'\beta'} \frac{y_1^{\beta}}{\tau_1^{1+\frac{\beta+\alpha}{2}}} \frac{y_2^{\beta'}}{\tau_2^{1+\frac{\beta'+\alpha'}{2}}} \int_0^1 dt dt' t^{\alpha-1} (1-t) t'^{\alpha'-1} (1-t') \\ & \int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} z_2^{2+\alpha'} z_1^{2+\alpha} \partial_{\Lambda} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0} ((z_1, 0), (z_2, 0)) p_B \left( \frac{\tau_1}{t^2}; z_1, \frac{y_1}{t} \right) p_B \left( \frac{\tau_2}{t'^2}; z_2, \frac{y_2}{t'} \right). \end{aligned} \quad (6.5.87)$$

The bound [\(6.5.44\)](#) implies that the term

$$\int_{\mathbb{R}^+} dz_{\pi_1} \int_{\mathbb{R}^-} dz_{\pi_2} z_2^{2+\alpha'} z_1^{2+\alpha} \partial_{\Lambda} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0} ((z_1, 0), (z_2, 0)) p_B \left( \frac{\tau_1}{t^2}; z_1, \frac{y_1}{t} \right) p_B \left( \frac{\tau_2}{t'^2}; z_2, \frac{y_2}{t'} \right)$$

is bounded by

$$(\Lambda + m)^{-4-\alpha-\alpha'} e^{-\frac{m^2}{2\Lambda^2}} \mathbb{Q} \left( \frac{t\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^0 \left( \Lambda; \frac{\tau_{\pi_1}^+}{t^2}, \frac{\tau_{\pi_2}^-}{t'^2}; \frac{y_1}{t}, \frac{y_2}{t'} \right). \quad (6.5.88)$$

Using [\(3.2.54\)](#), we have

$$\mathcal{F}_{2,l;\delta}^0 \left( \Lambda; \frac{\tau_{\pi_1}^+}{t^2}, \frac{\tau_{\pi_2}^-}{t'^2}; \frac{y_1}{t}, \frac{y_2}{t'} \right) = \mathcal{F}_{1,l;\delta}^0 \left( \Lambda; \frac{\tau_{\pi_1}}{t^2}; \frac{y_{\pi_1}}{t} \right) \mathcal{F}_{1,l;-\delta}^0 \left( \Lambda; \frac{\tau_{\pi_2}}{t'^2}; \frac{y_{\pi_2}}{t'} \right). \quad (6.5.89)$$

The bound (5.8.28) from lemma 11 together with (6.5.88) gives that (6.5.87) is bounded by

$$\tau_1^{-\frac{1}{2}} \tau_2^{-\frac{1}{2}} (\Lambda + m)^{-2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \tilde{\mathcal{Q}} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta'}^{\Lambda,0} (\tau_{\pi_1}^+, \tau_{\pi_2}^-), \quad (6.5.90)$$

where  $0 < \delta < \delta' < 1$ . Using the bounds (6.5.82)-(6.5.85) and (6.5.90) together with (6.5.81), we obtain that the remainder  $l_{l,2;\pm}^{\Lambda,\Lambda_0}$  is bounded by

$$(\Lambda + m) \max \left\{ \frac{\tau^{-1}}{(\Lambda + m)^2}, \frac{m^2}{(\Lambda + m)^2} \right\} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \tilde{\mathcal{Q}} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta'}^{\Lambda,0} (\tau_{\pi_1}^+, \tau_{\pi_2}^-). \quad (6.5.91)$$

The bound is extended to general momenta using the Taylor formula with integral remainder as in (5.5.76).

□

**Remark 6.** *The simplification that occurs in the integration part of the proof of proposition 9 compared to the proof of Theorem 9 is the distinction between the cases  $\Lambda \geq O(1)\tau^{-\frac{1}{2}}$  and  $\Lambda \leq O(1)\tau^{-\frac{1}{2}}$ , which is not necessary here, since we have*

$$\mathcal{F}_{2,l;\delta'}^{\Lambda,0} (\tau_{\pi_1}^+, \tau_{\pi_2}^-) = \mathcal{F}_{1,l;\delta'}^{\Lambda,0} (\tau_{\pi_1}) \mathcal{F}_{1,l;-\delta'}^{\Lambda,0} (\tau_{\pi_2}). \quad (6.5.92)$$

### 6.5.3 Proof of Theorem 12

The method of the proof of this Theorem is similar to the proof of Theorem 9 in chapter 5, since the right-hand side of the bounds (5.5.1) and (6.4.11) on  $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda}$  and  $\hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$  are similar and the right-hand side of the flow equations (5.4.11) and (6.3.12) have a similar structure. The induction starts at the tree order for which we have

$$\hat{\mathcal{D}}_{0,4}^{\Lambda,\Lambda_0} ((z_1, p_1), \dots, (z_4, p_4)) = 0$$

and the bound (6.4.11) obviously holds.

#### The right-hand side of the FE

The bound that we want to obtain for the RHS of the flow equation (6.3.12) is

$$\begin{aligned} & \left| \partial_\Lambda \partial^w \hat{\mathcal{D}}_{l,n;r_1,r_2}^{\Lambda,\Lambda_0} (\vec{p}_n; \phi_{\tau_{1,s}, y_{1,s}}) \right| \\ & \leq (\Lambda + m)^{2-n-|w|-r_1-r_2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda,0} (\tau_{1,s}), \end{aligned} \quad (6.5.93)$$

for all  $n \geq 2$ ,  $0 \leq s \leq n$  and  $0 \leq r_1, r_2 \leq 4$ .

We proceed as before and bound each term on the RHS of the FE (6.3.12).

- Let us start first with the linear terms given by

$$\begin{aligned} \hat{R}_1 := & \int_k \int_{z, z'} \int_{z_{1,n}} z_1^{r_1} z_2^{r_2} \partial^w \hat{\mathcal{D}}_{l-1, n+2}^{\Lambda, \Lambda_0} ((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \\ & \times \dot{C}^\Lambda(k) p_B \left( \frac{1}{\Lambda^2}; z, z' \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i), \quad (6.5.94) \end{aligned}$$

$$\begin{aligned} R_{1, \pm} := & \int_k \int_{\mathbb{R}^-} dz \int_{\mathbb{R}^+} dz' \int_{z_{1,n}} z_1^{r_1} z_2^{r_2} \partial^w \mathcal{L}_{l-1, n+2}^{\Lambda, \Lambda_0} ((\vec{z}_n, \vec{p}_n), (z, k), (z', -k)) \\ & \times \dot{C}^\Lambda(k) p_B \left( \frac{1}{\Lambda^2}; z, z' \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i), \quad (6.5.95) \end{aligned}$$

and

$$\begin{aligned} R_{1, -} := & \int_k \int_{\mathbb{R}^-} dz \int_{\mathbb{R}} dz' \int_{z_{1,n}} z_1^{r_1} z_2^{r_2} \partial^w \mathcal{L}_{l-1, n+2}^{\Lambda, \Lambda_0} ((\vec{z}_n, \vec{p}_n), (z, k), (z', -k)) \\ & \times \dot{C}^\Lambda(k) p_B \left( \frac{1}{\Lambda^2}; z, z' \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i). \quad (6.5.96) \end{aligned}$$

Using (3.1.5), we rewrite  $\hat{R}_1$  as follows

$$\int_{\mathbb{R}} du \int_k \partial^w \hat{\mathcal{D}}_{l-1, n+2; r_1, r_2}^{\Lambda, \Lambda_0} \left( \vec{p}_n, k, -k; \phi_{\tau_{1,s}, y_{1,s}} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \dot{C}^\Lambda(k).$$

Inserting the induction hypothesis, we obtain that  $\hat{R}_1$  is bounded by

$$\begin{aligned} (\Lambda + m)^{1-n-|w|-r_1-r_2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \int_k |\dot{C}^\Lambda(k)| \mathcal{P} \left( \frac{|k|}{\Lambda + m}, \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ \times \int_{\mathbb{R}} du \mathcal{F}_{s+2, l-1; \delta_1}^{\Lambda, 0} \left( \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, u \right). \quad (6.5.97) \end{aligned}$$

Using (4.5.23) and lemma 2, we obtain the bound

$$|\hat{R}_1| \leq (\Lambda + m)^{2-n-|w|-r_1-r_2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l; \delta_1}^{\Lambda, 0}(\tau_{1,s}). \quad (6.5.98)$$

Using (3.1.5) and the notation (6.4.12), we rewrite  $R_{1, \pm}$  as follows

$$\begin{aligned} \int_k \int_{\mathbb{R}} du \partial^w \mathcal{L}_{l-1, n+2; r_1, r_2; \pi_1^+, \pi_2^-}^{\Lambda, \Lambda_0} \left( \vec{p}_n, k, -k; \phi_{\tau_{1,s}, y_{1,s}} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) p_B \left( \frac{1}{2\Lambda^2}; z, u \right) \right) \\ \times \dot{C}^\Lambda(k), \quad (6.5.99) \end{aligned}$$

where  $\pi_1 = \sigma_{s+1}$  and  $\pi_2 = \{s+2\}$ . Proposition 9 gives that  $R_{1,\pm}$  is bounded by

$$(\Lambda + m)^{1-n-|w|-r_1-r_2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \int_k |\dot{C}^\Lambda(k)| \mathcal{P} \left( \frac{|k|}{\Lambda + m}, \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ \times \int_{\mathbb{R}} du \mathcal{F}_{s+2,l-1;\delta_2}^0(\Lambda; \tau_{\pi_1}^+, \tau_{\pi_2}^-, Y_{\sigma_s}, u), \quad (6.5.100)$$

where  $\tau_{s+1} = \tau_{s+2} = \frac{1}{2\Lambda^2}$ . Using the bound (3.2.64) from lemma 2, we deduce

$$\int_{\mathbb{R}} du \mathcal{F}_{s+2,l-1;\delta_2}^0(\Lambda; \tau_{\pi_1}^+, \tau_{\pi_2}^-, Y_{\sigma_s}, u) \leq O(1) \mathcal{F}_{s,l;\delta_2}^0(\Lambda; \tau_{\pi_1}^+, Y_{\sigma_s}). \quad (6.5.101)$$

Since  $\pi_1 = \sigma_{s+1}$ , we have

$$\mathcal{F}_{s,l;\delta_2}^0(\Lambda; \tau_{\pi_1}^+, Y_{\sigma_s}) = \mathcal{F}_{s,l;\delta_2}^0(\Lambda; \tau_{1,s}; Y_{\sigma_s}). \quad (6.5.102)$$

Combining (4.5.23), (6.5.100) and (6.5.102), we deduce that  $R_{1,\pm}$  is bounded by

$$(\Lambda + m)^{2-n-|w|-r_1-r_2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta_2}^{\Lambda,0}(\tau_{1,s}). \quad (6.5.103)$$

Proceeding similarly, we rewrite  $R_{1,-}$  as follows

$$\int_k \int_{\mathbb{R}} \sum_{(\tilde{\pi}_1, \tilde{\pi}_2)} \partial^w \mathcal{L}_{l-1,n+2;r_1,r_2;\tilde{\pi}_1^+, \tilde{\pi}_2^-}^{\Lambda, \Lambda_0} \left( \vec{p}_n, k, -k; \phi_{\tau_{1,s}, y_{1,s}} \times p_B \left( \frac{1}{2\Lambda^2}, \cdot, u \right) p_B \left( \frac{1}{2\Lambda^2}; z, u \right) \right) \\ \times \dot{C}^\Lambda(k), \quad (6.5.104)$$

where the sum runs over partitions  $(\tilde{\pi}_1, \tilde{\pi}_2)$  in the set

$$\{(\sigma_{s+1}, \{s+2\}), (\sigma_s, \{s+1, s+2\})\}.$$

Using again proposition 9, we obtain that  $R_{1,-}$  is bounded by

$$(\Lambda + m)^{1-n-|w|-r_1-r_2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \int_k |\dot{C}^\Lambda(k)| \mathcal{P} \left( \frac{|k|}{\Lambda + m}, \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ \times \sum_{(\tilde{\pi}_1, \tilde{\pi}_2)} \int_{\mathbb{R}} du \mathcal{F}_{s+2,l-1;\delta_2}^{\Lambda,0}(\Lambda; \tau_{\tilde{\pi}_1}^+, \tau_{\tilde{\pi}_2}^-, Y_{\sigma_s}, u). \quad (6.5.105)$$

where  $\tau_{s+1} = \tau_{s+2} = \frac{1}{2\Lambda^2}$ . The bound (3.2.64) from lemma 2 gives

$$\int_{\mathbb{R}} du \mathcal{F}_{s+2,l-1;\delta_2}^{\Lambda,0}(\Lambda; \tau_{\tilde{\pi}_1}^+, \tau_{\tilde{\pi}_2}^-, Y_{\sigma_s}, u) \leq O(1) \mathcal{F}_{s,l;\delta_2}^0(\Lambda; \tau_{1,s}; Y_{\sigma_s}). \quad (6.5.106)$$

This together with (4.5.23) and (6.5.105) gives that  $R_{1,-}$  is bounded by (6.5.103).

- In this part, we analyze the quadratic part of the flow equation (6.3.12). Given a partition  $(\pi_1, \pi_2)$  in  $\tilde{\mathcal{P}}_{2;n}$ , we bound the following terms

$$\begin{aligned} \hat{R}_{2,DD} := & \int_{z,z'} \int_{z_{1,n}} z_1^{r_1} z_2^{r_2} \hat{\mathcal{D}}_{l_1, n_1+1}^{\Lambda, \Lambda_0} ((\vec{z}_{\pi_1}, \vec{p}_{\pi_1}), (z, p)) \\ & \times \hat{\mathcal{D}}_{l_2, n_2+1}^{\Lambda, \Lambda_0} ((z', -p), (\vec{z}_{\pi_2}, \vec{p}_{\pi_2})) \dot{C}_B^\Lambda(p; z, z') \prod_{i=1}^s p_B(\tau_i; z_i, y_i), \end{aligned} \quad (6.5.107)$$

$$\begin{aligned} \hat{R}_{2,DL} := & \int_{z,z'} \int_{z_{1,n}} z_1^{r_1} z_2^{r_2} \hat{\mathcal{D}}_{l_1, n_1+1}^{\Lambda, \Lambda_0} ((\vec{z}_{\pi_1}, \vec{p}_{\pi_1}), (z, p)) \\ & \times \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0} ((z', -p), (\vec{z}_{\pi_2}, \vec{p}_{\pi_2})) \dot{C}_B^\Lambda(p; z, z') \prod_{i=1}^s p_B(\tau_i; z_i, y_i) \end{aligned} \quad (6.5.108)$$

and

$$\begin{aligned} \hat{R}_{2,LL} := & \int_{\mathbb{R}^-} dz \int_{\mathbb{R}^-} dz' \int_{z_{1,n}} z_1^{r_1} z_2^{r_2} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0} ((\vec{z}_{\pi_1}, \vec{p}_{\pi_1}), (z, p)) \\ & \times \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0} ((z', -p), (\vec{z}_{\pi_2}, \vec{p}_{\pi_2})) \dot{C}_B^\Lambda(p; z, z') \prod_{i=1}^s p_B(\tau_i; z_i, y_i). \end{aligned} \quad (6.5.109)$$

Let us define

$$\phi'_{s_1}(\vec{z}_{\pi_1}) = \prod_{r \in \pi_{1,s}} p_B(\tau_r; z_r, y_r), \quad \phi''_{s_2}(\vec{z}_{\pi_2}) = \prod_{r \in \pi_{2,s}} p_B(\tau_r; z_r, y_r) \quad (6.5.110)$$

with  $\pi_{i,s} = \pi_i \cap \sigma_s$  and  $s_i := |\pi_{i,s}|$ . Using (3.1.5), we rewrite  $\hat{R}_{2,DD}$  as follows

$$\begin{aligned} \hat{R}_{2,DD} := & \int_{\mathbb{R}} du \hat{\mathcal{D}}_{l_1, n_1+1; r_1(1), r_1(2)}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_1}, p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}, \cdot, u \right) \right) \\ & \times \hat{\mathcal{D}}_{l_2, n_2+1; r_2(1), r_2(2)}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_2}, -p; \phi''_{s_2} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \dot{C}^\Lambda(p), \end{aligned} \quad (6.5.111)$$

where

$$r_i(j) := \begin{cases} r_i & \text{if } j \in \pi_i \\ 0 & \text{otherwise.} \end{cases} \quad (6.5.112)$$

Using the induction hypothesis, (6.5.111) is bounded by

$$\begin{aligned} & (\Lambda + m)^{1-n-|w|-r_1-r_2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ & \times \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta_3}^0 \left( \Lambda; \tau_{\pi_{1,s}}, \frac{1}{2\Lambda^2}; Y_{\pi_{1,s}}, u \right) \mathcal{F}_{s_2+1, l_2; \delta_4}^0 \left( \Lambda; \tau_{\pi_{2,s}}, \frac{1}{2\Lambda^2}; Y_{\pi_{2,s}}, u \right), \end{aligned}$$



which is bounded using Lemma [3](#) by

$$(\Lambda + m)^{2-n-|w|-r_1-r_2} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta_5}^{\Lambda,0}(\tau_{1,s})$$

and  $\delta_5 := \max(\delta_3, \delta_4)$ . Now, we bound  $\hat{R}_{2,DL}$ :

- First, we treat the case  $r_1 = r_2 = 0$ : Proceeding as we did for  $\hat{R}_{2,DD}$ , we rewrite  $\hat{R}_{2,DL}$  using [\(3.1.5\)](#) and the notation [\(6.5.110\)](#) as follows

$$\begin{aligned} \hat{R}_{2,DL} := & \int_{\mathbb{R}} du \int_{\mathbb{R}^+} dz' \hat{\mathcal{D}}_{l_1, n_1+1}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_1}; p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}, \cdot, u \right) \right) \\ & \times \mathcal{L}_{l_2, n_2+1; +}^{\Lambda, \Lambda_0} (z'; \vec{p}_{\pi_2}, -p; \phi''_{s_2}) p_B \left( \frac{1}{2\Lambda^2}; z', u \right) \dot{C}^\Lambda(p). \end{aligned} \quad (6.5.113)$$

Using the induction hypothesis together with the bound [\(6.4.18\)](#), we deduce that  $\hat{R}_{2,DL}$  is bounded by

$$\begin{aligned} (\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \times \int_{\mathbb{R}^+} du \mathcal{F}_{s_1+1, l_1; \delta_5}^0 \left( \Lambda; \tau_{\pi_1, s}, \frac{1}{2\Lambda^2}; Y_{\pi_1, s}, u \right) \\ \times \int_{z'} \mathcal{F}_{s_2+1, l_2; +; \delta_6}^\infty (\Lambda; \tau_{\pi_2, s}; z'; Y_{\pi_2, s}) p_B \left( \frac{1}{2\Lambda^2}; z', u \right). \end{aligned}$$

Lemma [5](#) gives for  $z' \in \mathbb{R}^+$

$$\mathcal{F}_{s_2+1, l_2; +; \delta_6}^\infty (\Lambda; \tau_{\pi_2, s}; z'; Y_{\pi_2, s}) \leq O(1) \mathcal{F}_{s_2+1, l_2; \delta_3}' (\Lambda; \tau_{\pi_2, s}; z'; Y_{\pi_2, s}). \quad (6.5.114)$$

Hence, we deduce that  $\hat{R}_{2,DL}$  is bounded by

$$\begin{aligned} (\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \times \int_{\mathbb{R}^+} du \mathcal{F}_{s_1+1, l_1; \delta_5}^0 \left( \Lambda; \tau_{\pi_1, s}, \frac{1}{2\Lambda^2}; Y_{\pi_1, s}, u \right) \\ \times \int_{z'} \mathcal{F}_{s_2+1, l_2; +; \delta_6} (\Lambda; \tau_{\pi_2, s}; z'; Y_{\pi_2, s}) p_B \left( \frac{1}{2\Lambda^2}; z', u \right). \end{aligned} \quad (6.5.115)$$

In the proof of Theorem [9](#) in chapter [5](#), we treated the term  $\tilde{R}_2^{DS}$  which ended up bounded by [\(6.5.115\)](#), for which we had already established that it is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta_4}^{\Lambda,0}(\tau_{1,s}), \quad (6.5.116)$$

where  $\delta'_4 := \max(\delta_5, \delta_6)$ .

The last term we need to consider is  $\hat{R}_{2,LL}$ . Using (3.1.5), we rewrite  $\hat{R}_{2,LL}$  as follows

$$\begin{aligned} \sum'_{\{\mathcal{E}_1, \mathcal{E}_2\}} \int_{\mathbb{R}} du \mathcal{L}_{l_1, n_1+1; (\pi_1 \cup \mathcal{E}_1)^+, \mathcal{E}_1^{c-}}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_1}, p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}; z, u \right) \right) \\ \times \mathcal{L}_{l_2, n_2+1; (\pi_2 \cup \mathcal{E}_2)^+, \mathcal{E}_2^{c-}}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_2}, -p; \phi''_{s_2} \times p_B \left( \frac{1}{2\Lambda^2}; z', u \right) \right) \dot{C}^\Lambda(p), \end{aligned} \quad (6.5.117)$$

where the sum runs over pairs  $\{\mathcal{E}_1, \mathcal{E}_2\}$  such that  $\mathcal{E}_i \in \{\{s+i\}, \emptyset\}$ . The prime on the sum means that the pair  $\{\{s+1\}, \{s+2\}\}$  is not included in the sum. In order to treat (6.5.117), we must distinguish between two cases:

- First, we analyze the case  $\mathcal{E}_1 = \mathcal{E}_2 = \emptyset$  and (6.5.117) reads in this case

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{L}_{l_1, n_1+1; \pi_1^+, \{s+1\}^-}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_1}, p; \phi'_{s_1} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \\ \times \mathcal{L}_{l_2, n_2+1; \pi_2^+, \{s+2\}^-}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_2}, -p; \phi''_{s_2} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \dot{C}^\Lambda(p). \end{aligned} \quad (6.5.118)$$

Using proposition 9, we deduce that (6.5.118) is bounded by

$$\begin{aligned} (\Lambda + m)^{1-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta_1}^0(\Lambda; \tau_{\pi_1}^+, \tau_{s+1}^-; Y_{\pi_1, s}, u) \mathcal{F}_{s_2+1, l_2; \delta_2}^0(\Lambda; \tau_{\pi_2}^+, \tau_{s+2}^-; Y_{\pi_2, s}, u), \end{aligned} \quad (6.5.119)$$

where  $\tau_{s+1} = \tau_{s+2} = \frac{1}{2\Lambda^2}$ . Using lemma 2, we have

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta_1}^0(\Lambda; \tau_{\pi_1}^+, \tau_{s+1}^-; Y_{\pi_1, s}, u) \mathcal{F}_{s_2+1, l_2; \delta_2}^0(\Lambda; \tau_{\pi_2}^+, \tau_{s+2}^-; Y_{\pi_2, s}, u) \\ \leq O(1) \Lambda \mathcal{F}_{s, l; \delta'_1}^0(\Lambda; \tau_{1, s}; Y_{\sigma_s}), \end{aligned} \quad (6.5.120)$$

where  $\delta'_1 := \max(\delta_1, \delta_2)$ . This together with (6.5.119) gives that (6.5.118) is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathbb{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l; \delta'_1}^{\Lambda, 0}(\tau_{1, s}). \quad (6.5.121)$$

- The second case to be treated is  $\mathcal{E}_i \neq \emptyset$ . Without loss of generality, we assume that  $\mathcal{E}_1 \neq \emptyset$  and write

$$\begin{aligned} \int_{\mathbb{R}} du \int_{\mathbb{R}^+} dz \mathcal{L}_{l_1, n_1+1; +}^{\Lambda, \Lambda_0} \left( z; \vec{p}_{\pi_1}, p; \phi'_{s_1} \right) p_B \left( \frac{1}{2\Lambda^2}; z, u \right) \\ \times \mathcal{L}_{l_2, n_2+1; \pi_2^+, \{s+2\}^-}^{\Lambda, \Lambda_0} \left( \vec{p}_{\pi_2}, -p; \phi''_{s_2} \times p_B \left( \frac{1}{2\Lambda^2}; \cdot, u \right) \right) \dot{C}^\Lambda(p). \end{aligned} \quad (6.5.122)$$

Using the bound (6.4.18) and proposition 9, we deduce that (6.5.122) is bounded by

$$\begin{aligned} & (\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ & \quad \times \int_{\mathbb{R}} du \int_{\mathbb{R}^+} dz \mathcal{F}_{s_1+1, l_1; +; \delta_1}^{\infty} (\Lambda, \tau_{\pi_1, s}, z, Y_{\pi_1, s}) p_B \left( \frac{1}{2\Lambda^2}; z, u \right) \\ & \quad \quad \quad \times \mathcal{F}_{s_2+1, l_2; \delta_2}^{\Lambda, 0} \left( \Lambda, \tau_{\pi_2, s}^+, \tau_{s+2}^-, Y_{\pi_2, s}, u \right) \end{aligned} \quad (6.5.123)$$

Using lemma 5, we deduce

$$\mathcal{F}_{s_1+1, l_1; +; \delta_1}^{\infty} (\Lambda, \tau_{\pi_1, s}, z, Y_{\pi_1, s}) \leq O(1) \mathcal{F}_{s_1+1, l_1; \delta_1} (\Lambda, \tau_{\pi_1, s}, z, Y_{\pi_1, s}). \quad (6.5.124)$$

Hence, we obtain

$$\begin{aligned} & \int_{\mathbb{R}} du \int_{\mathbb{R}^+} dz \mathcal{F}_{s_1+1, l_1; +; \delta_1}^{\infty} (\Lambda, \tau_{\pi_1, s}, z, Y_{\pi_1, s}) p_B \left( \frac{1}{2\Lambda^2}; z, u \right) \\ & \quad \quad \quad \times \mathcal{F}_{s_2+1, l_2; \delta_2}^0 \left( \Lambda, \tau_{\pi_2, s}^+, \tau_{s+2}^-, Y_{\pi_2, s}, u \right) \\ & \leq O(1) \int_{\mathbb{R}} du \hat{\mathcal{F}}_{s_1+1, l_1; \delta_1} (\Lambda, \tau_{\pi_1, s}, Y_{\pi_1, s}, u) \mathcal{F}_{s_2+1, l_2; \delta_2}^{\Lambda, 0} \left( \Lambda, \tau_{\pi_2, s}^+, \tau_{s+2}^-, Y_{\pi_2, s}, u \right). \end{aligned} \quad (6.5.125)$$

Using lemma 4 and (6.5.125), we deduce that (6.5.123) is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left( \log \frac{\Lambda + m}{m} \right) \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left( \frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l; \delta}^{\Lambda, 0}(\tau_{1, s}). \quad (6.5.126)$$

- The method used to integrate the bound (6.5.93) is similar to the integration of the bound (5.5.1) in Theorem 9, since both bounds have the same right-hand side. The only difference are the boundary conditions used in the integration of the irrelevant/relevant parts of  $\hat{\mathcal{D}}_{l, n}^{\Lambda, \Lambda_0}$ . These are given in (6.3.13)-(6.3.15) and (6.3.13). For more details, we refer the reader to the proof of Theorem 9.



## Chapter 7

# Renormalization of the lattice regularized $\phi_4^4$ model

In this chapter, we analyse the Euclidean four-dimensional massive  $\phi^4$  theory using lattice regularization. We present a rigorous proof that this quantum field theory is renormalizable, to all orders of the loop expansion based on the flow equations. The lattice regularization is known to break Euclidean symmetry. A key novelty of this work compared to the existing proofs in the literature [33, 34] is a proof to all orders in perturbation theory of the restoration of the Euclidean symmetries. This chapter is organized as follows: In section [7.1], we introduce the lattice regularized correlations functions and their respective flow equations. In section [7.2] we give the steps of proving renormalizability of four-dimensional  $\phi^4$  theory on the lattice by means of the flow equations, following [11]. Renormalizability is stated in terms of uniform bounds on the (coefficient functions of the) solution  $L^{a_0, a}(\phi)$  of the flow equation and its derivative with respect to the lattice cutoff  $a^{-1}$ , with boundary conditions imposed at  $a = \infty$  for the relevant couplings and at  $a = a_0$  for the irrelevant interactions. Sections [7.3] and [7.4] are the central part of this chapter. In section [7.3] we introduce the rotated lattice and we show that the difference  $\mathcal{D}_{l, n}^{a_0, a, O}(p_1, \dots, p_n)$  of the correlation function of arguments defined on the rotated lattice and on the original lattice:

$$\mathcal{D}_{l, n}^{a_0, a, O}(p_1, \dots, p_n) := \mathcal{L}_{l, n}^{a_0, a, O}(Op_1, \dots, Op_n) - \mathcal{L}_{l, n}^{a_0, a}(p_1, \dots, p_n)$$

converges to zero when  $a_0 \rightarrow 0$  and  $a \rightarrow \infty$ . In section [7.4] we give a proof of the existence of the continuum limit in position space in the sense of tempered distributions. We find that the obtained limit is invariant under translations which concludes the restoration of the Euclidean symmetries in the continuum limit.

## 7.1 The flow equations

We consider  $\phi^4$  scalar field theory on the Euclidean space  $\mathbb{R}^4$ . We will formulate our theory with a lattice cutoff in the standard path integral formalism, where the lattice refers to the discretization of space-time. In the following, we introduce general notions of a space-time lattice and the  $\phi^4$  model on the lattice, but only to the extent that is relevant to this paper.

### 7.1.1 Lattice field theory

The four-dimensional hypercubic lattice is a set of sites denoted by

$$\Lambda_{a_0} = a_0 \mathbb{Z}^4,$$

where  $a_0$  denotes the lattice spacing in Euclidean time and spatial directions. One of the first questions in lattice field theory is how to put a model on the lattice once it is defined on the space-time continuum. The question refers both to the framework of classical field theory, i.e. at the level of the classical action, and to quantum field theory. Naturally discretization of space and time implies that differentiation with respect to space and time is to be replaced by a corresponding difference operation.

### 7.1.2 $\phi^4$ scalar field theory on the lattice

Perturbative renormalizability of euclidean  $\phi_4^4$  theory will be established by analysing the generating functional  $L^{a_0, a}$  of connected (free propagator) amputated Schwinger functions (CAS). The upper indices  $a_0$  and  $a$  enter through the regularized propagator

$$C^{a_0, a}(p) = \frac{1}{\hat{p}^2 + m^2} \left( e^{-a_0^2(\hat{p}^2 + m^2)} - e^{-a^2(\hat{p}^2 + m^2)} \right), \quad (7.1.1)$$

where the map  $\hat{p} := (\hat{p}(p_\mu))_{1 \leq \mu \leq 4}$  is defined as follows

$$\begin{aligned} \hat{p} : \quad ]-\frac{\pi}{a_0}, \frac{\pi}{a_0}[ &\rightarrow \quad ]-\frac{2}{a_0}, \frac{2}{a_0}[ \\ p_\mu &\mapsto \frac{2}{a_0} \sin\left(\frac{a_0 p_\mu}{2}\right). \end{aligned} \quad (7.1.2)$$

In the sequel we shall write with slight abuse of notation

$$C^{a_0, a}(\hat{p}) := C^{a_0, a}(p), \quad \hat{p}(p_\mu) := \hat{p}_\mu.$$

Upon removal of the cutoffs, i.e. in the limit  $a_0 \rightarrow 0$ ,  $a \rightarrow \infty$ , we indeed recover the free propagator  $\frac{1}{p^2 + m^2}$ . For the Fourier transform we use the convention

$$\hat{f}(x) = \int_{p, \mathcal{B}_{a_0}} f(p) e^{ip \cdot x} := \int_{]-\frac{\pi}{a_0}, \frac{\pi}{a_0}[}^4 \frac{d^4 p}{(2\pi)^4} f(p) e^{ip \cdot x}. \quad (7.1.3)$$

In the sequel, we use the shorthand notation

$$\int_{p, \mathcal{B}_{a_0}} := \int_{-\frac{\pi}{a_0}, \frac{\pi}{a_0}}^4 \frac{d^4 p}{(2\pi)^4},$$

with  $\mathcal{B}_{a_0} = ]-\frac{\pi}{a_0}, \frac{\pi}{a_0}[^4$  denoting the first Brillouin zone. For the inverse Fourier transform we write

$$f(p) = a_0^4 \sum_{x \in \Lambda_{a_0}} \hat{f}(x) e^{-ip \cdot x}, \quad (7.1.4)$$

so that in position space

$$\hat{C}^{a_0, a}(x, y) = \int_{p, \mathcal{B}_{a_0}} C^{a_0, a}(\hat{p}) e^{ip \cdot (x-y)}.$$

We assume

$$0 \leq a_0 \leq a \leq \infty$$

so that the Wilson flow parameter  $1/a$  takes the role of an IR cutoff, whereas  $1/a_0$  is the UV cutoff. We introduce the convention

$$\hat{\phi}_{a_0}(x) = \int_{p, \mathcal{B}_{a_0}} \phi_{a_0}(p) e^{ip \cdot x}, \quad \frac{\delta}{\delta \hat{\phi}_{a_0}(x)} = \int_{p, \mathcal{B}_{a_0}} \frac{\delta}{\delta \phi_{a_0}(p)} e^{-ip \cdot x}.$$

For our purposes the field  $\hat{\phi}_{a_0}(x)$  is assumed to belong to the Hilbert space  $l_2(\Lambda_{a_0})$  endowed with the inner scalar product

$$\langle f, g \rangle_{l_2(\Lambda_{a_0})} = a_0^4 \sum_{x \in \Lambda_{a_0}} f(x) \overline{g(x)}.$$

Our starting point is the bare action of symmetric  $\phi_4^4$  theory

$$L^{a_0, a_0}(\hat{\phi}_{a_0}) = a_0^4 \sum_{x \in \Lambda_{a_0}} \left\{ \frac{\lambda}{4!} \hat{\phi}_{a_0}^4 + d(a_0) \hat{\phi}_{a_0}^2 + b(a_0) (\hat{\partial}_{\mu, a_0} \hat{\phi}_{a_0})^2 + c(a_0) \hat{\phi}_{a_0}^4 \right\}, \quad (7.1.5)$$

$$d(a_0), c(a_0) = \mathcal{O}(\hbar), \quad b(a_0) = \mathcal{O}(\hbar^2).$$

The differentiation in (7.1.5) is defined by the difference operator

$$\left( \hat{\partial}_{\mu} \hat{\phi}_{a_0} \right) (x) = \frac{\hat{\phi}_{a_0}(x + a_0 e_{\mu}) - \hat{\phi}_{a_0}(x)}{a_0},$$

with  $x \in \Lambda_{a_0}$  and  $e_{\mu}$  the unit vector in the  $\mu^{\text{th}}$  coordinate direction. The first term is formed of the field's self-interaction with real positive coupling constant  $\lambda$  having mass dimension equal to zero. The second part contains the related counter terms, determined according to the following rule. The canonical mass dimension of the field is one, the counter-terms allowed in the bare interaction are all local terms of mass dimension  $\leq 4$  formed out of the field and its derivatives respecting cubic lattice

symmetry. The  $O(4)$  and translation symmetries are violated by the lattice regularization. From the bare action and the flowing propagator, we may define Wilson's flowing effective action  $L^{a_0, a}$  through

$$e^{-\frac{1}{\hbar}(L^{a_0, a}(\hat{\phi}_{a_0}) + I^{a_0, a})} := \int d\mu_{a_0, a}(\Phi) e^{-\frac{1}{\hbar}L^{a_0, a_0}(\Phi + \hat{\phi}_{a_0})}, \quad L^{a_0, a}(0) = 0. \quad (7.1.6)$$

It can be recognized to be the generating functional of the CAS of the theory with propagator  $\hat{C}^{a_0, a}$  and bare action  $L^{a_0, a_0}$ . In (7.1.6),  $d\mu_{a_0, a}(\Phi)$  denotes the Gaussian measure with covariance  $\hbar\hat{C}^{a_0, a}$ . It is proved in [25] that such a measure exists as a lattice approximation of the continuum Gaussian measure.  $I^{a_0, a}$  denotes the field independent so called vacuum contributions. It is finite only in the finite volume approximation. The infinite volume limit is taken only when it has been eliminated [11]. Again, we do not make the finite volume explicit here since it plays no role in our analysis.

The fundamental tool for the study of the renormalization problem is the functional flow equation

$$\partial_{1/a} L^{a_0, a} = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \hat{\phi}_{a_0}}, \left( \partial_{1/a} \hat{C}^{a_0, a} \right) * \frac{\delta}{\delta \hat{\phi}_{a_0}} \right\rangle L^{a_0, a} - \frac{1}{2} \left\langle \frac{\delta L^{a_0, a}}{\delta \hat{\phi}_{a_0}}, \left( \partial_{1/a} \hat{C}^{a_0, a} \right) * \frac{\delta L^{a_0, a}}{\delta \hat{\phi}_{a_0}} \right\rangle \quad (7.1.7)$$

By  $\langle \cdot, \cdot \rangle$  we denote the scalar product in  $l_2(\Lambda_{a_0})$ . (7.1.7) is obtained by deriving both sides of the equation (7.1.6) with respect to  $1/a$  and performing an integration by parts in the functional integral on the RHS using the property (1.1.16) of the lattice Gaussian measure [25], and finally rearranging the powers of  $\hbar$  stemming from  $L^{a_0, a}/\hbar$  and from  $\hbar\partial_{1/a}\hat{C}^{a_0, a}$ . To derive the flow equations verified by the  $n$ -point correlation functions, we first expand  $L^{a_0, a}$  in moments for all  $(p_i)_{1 \leq i \leq n} \in \mathcal{B}_{a_0}$  with respect to  $\phi_{a_0}$ ,

$$(2\pi)^{4(n-1)} \delta_{\phi_{a_0}(p_1)} \cdots \delta_{\phi_{a_0}(p_n)} L^{a_0, a}|_{\phi_{a_0}=0} = \delta_{\left[\frac{2\pi}{a_0}\right]}^4(p_1 + \cdots + p_n) \mathcal{L}_n^{a_0, a}(p_1, \cdots, p_n),$$

where we have written  $\delta_{\phi_{a_0}(p)} = \delta/\delta\phi_{a_0}(p)$  and  $\delta_{\left[\frac{2\pi}{a_0}\right]}^4 := \sum_{k \in \mathbb{Z}^4} \delta_{\frac{2k\pi}{a_0}}^{(4)}$ . We also expand in a formal powers series with respect to  $\hbar$  to select the loop order  $l$ ,

$$\mathcal{L}_n^{a_0, a} = \sum_{l=0}^{\infty} \hbar^l \mathcal{L}_{l, n}^{a_0, a}.$$

From the functional flow equation (7.1.7), we then obtain the perturbative flow equations for the (connected free propagator amputated)  $n$ -point functions by identifying coefficients

$$\begin{aligned} \partial_{1/a} \partial^w \mathcal{L}_{l, n}^{a_0, a}(p_1, \cdots, p_n) &= \frac{1}{2} \int_{k, \mathcal{B}_{a_0}} \partial^w \mathcal{L}_{l-1, n+2}^{a_0, a}(k, p_1, \cdots, p_n, -k) \partial_{1/a} C^{a_0, a}(\hat{k}) \\ &- \frac{1}{2} \sum_{l_1, l_2, n_1, n_2}^i \sum_{w_i}^i c_w \left[ \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{a_0, a}(p_1, \cdots, p_{n_1}, p) \partial^{w_3} \partial_{1/a} C^{a_0, a}(\hat{p}) \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{a_0, a}(-p, p_{n_1+1}, \cdots, p_n) \right]_{rsym} \end{aligned} \quad (7.1.8)$$

$$p \equiv -p_1 - \cdots - p_{n_1} \equiv p_{n_1+1} + \cdots + p_n \quad \left[ \frac{2\pi}{a_0} \right].$$

For the notations, see after (1.1.18) and (1.1.19).



## 7.2 Renormalization of lattice $\phi_4^4$ theory

Perturbative renormalizability of the regularized field theory (7.1.6) amounts to the following: Given a coupling constant  $\lambda$  in the bare interaction (7.1.5), the coefficients  $d(a_0)$ ,  $b(a_0)$  and  $c(a_0)$  of the counter-terms can be adjusted within a loop expansion of the theory,

$$d(a_0) = \sum_{l=1}^{\infty} \hbar^l d_l(a_0), \quad b(a_0) = \sum_{l=2}^{\infty} \hbar^l b_l(a_0), \quad c(a_0) = \sum_{l=1}^{\infty} \hbar^l c_l(a_0)$$

in such a way that the limits of the lattice  $n$ -point CAS functions exist when  $a_0$  goes to 0 and  $a$  goes to  $\infty$  in every loop order  $l$ .

$$\forall (p_i)_{1 \leq i \leq n} \in \mathbb{R}^4, \exists \tilde{a}_0 > 0 \text{ such that uniformly in } \mathcal{B}_{\tilde{a}_0} :$$

$$\mathcal{L}_{l,n}^{0,\infty}(p_1, \dots, p_n) := \lim_{a_0 \rightarrow 0, a_0 \leq \tilde{a}_0} \lim_{a \rightarrow \infty} \mathcal{L}_{l,n}^{a_0,a}(p_1, \dots, p_n), \quad n \in \mathbb{N}, l \in \mathbb{N}^*. \quad (7.2.1)$$

The parameter  $\tilde{a}_0$  guarantees that  $(p_i)_{1 \leq i \leq n} \in \mathcal{B}_{\tilde{a}_0} \subset \mathcal{B}_{a_0}$  for all  $a_0 \leq \tilde{a}_0$  so that they are well defined as arguments of the regularized  $n$ -point functions  $\mathcal{L}_{l,n}^{a_0,a}$ . The lattice breaks Euclidean symmetry and an essential point to the renormalizability of the theory is to prove the restoration of this symmetry. We will analyse the limits  $\mathcal{L}_{l,n}^{0,\infty}(p_1, \dots, p_n)$  and prove in particular their invariance under rotations and translations in sections 7.3 and 7.4.

### 7.2.1 Propagator bounds

The subsequent bounds on the CAS functions will depend essentially on the propagator of the theory we consider. The bare propagator is, apart from the renormalization conditions, the main ingredient which decides what kind of bounds can be achieved. In this subsection we collect the bounds on the propagator and its derivatives that we will need subsequently. From the definition (7.1.1) we directly obtain

$$\partial_{1/a} C^{a_0,a}(\hat{p}) = (-2a^3) e^{-a^2(\hat{p}^2+m^2)}. \quad (7.2.2)$$

One can then prove by induction that

$$\partial^w e^{-a^2 \hat{p}^2} = \prod_{\mu=1}^4 \left( \sum_{k=1}^{w_\mu} a_0^{w_\mu-k} a^k P_{k,\mu} \left( \cos \frac{a_0 p_\mu}{2}, \sin \frac{a_0 p_\mu}{2} \right) \tilde{P}_{k,\mu}(a \hat{p}_\mu) \right) e^{-a^2 \hat{p}^2}. \quad (7.2.3)$$

Here  $P, \tilde{P}$  are real polynomials which we do not specify. Using (7.2.3) together with  $a_0 \leq a$ , we obtain the following bound on the propagator and its derivatives

$$|\partial^w \partial_{1/a} C^{a_0,a}(\hat{p})| \leq a^{|w|+3} \mathcal{P}_1(a|\hat{p}|) e^{-a^2(\hat{p}^2+m^2)}. \quad (7.2.4)$$

Using (7.2.3) and (7.2.14) below one can also show that

$$|\partial^w \partial_{1/a} C^{a_0, a}(\hat{p})| \leq \left(\frac{1}{a} + m\right)^{-|w|-3} \mathcal{P}_2 \left(\frac{a|p|}{1+am}\right). \quad (7.2.5)$$

Both bounds are expressed in terms of suitable polynomials  $\mathcal{P}_1, \mathcal{P}_2$  with nonnegative coefficients. The following lemma shows how to bound integrals of powers of momenta multiplied by the exponential appearing in the regularized propagator

**Lemma 13.**  $\forall \alpha \in \mathbb{N}, \exists C_\alpha > 0$  independent of  $a$  and  $a_0$  such that:

$$a^4 \int_{\mathcal{B}_{a_0}} e^{-a^2 \hat{k}^2} (a|k|)^\alpha dk \leq C_\alpha. \quad (7.2.6)$$

*Proof.* It is sufficient to bound

$$a \int_0^{\frac{\pi}{a_0}} e^{-a^2 \hat{k}^2} (ak)^\alpha dk \quad (7.2.7)$$

uniformly with respect to  $a$  and  $a_0$ . Using that  $\forall x \in [0, \frac{\pi}{2}]$  we have  $\sin x \geq \frac{2}{\pi}x$ , one obtains

$$a \int_0^{\frac{\pi}{a_0}} e^{-a^2 \hat{k}^2} (ak)^\alpha dk \leq a \int_0^{\frac{\pi}{a_0}} e^{-\frac{a^2 k^2}{\pi^2}} (ak)^\alpha dk \leq \int_0^\infty e^{-\frac{u^2}{\pi^2}} u^\alpha du \leq C_\alpha. \quad (7.2.8)$$

□

When studying the restoration of rotational symmetry we will also have to bound differences of derived propagators, where one of them has undergone an arbitrary rotation  $O \in O(4)$ . The following lemma permits to bound these differences:

**Lemma 14.** For all  $w \in \mathbb{N}^4$ , for all  $p \in \mathcal{B}_{\alpha a_0}$  for some  $\alpha > 0$  holds

$$|\partial^w \partial_{1/a} C^{a_0, a}(\hat{p}) - \partial^w \partial_{1/a} C^{a_0, a}(\hat{p}^O)| \leq a_0 \left(\frac{1}{a} + m\right)^{-2-|w|} \mathcal{P} \left(\frac{a|p|}{1+am}\right), \quad (7.2.9)$$

with  $\hat{p}^O := \hat{p}(Op)$ .

*Proof.* If  $|\hat{p}^O| \geq |\hat{p}|$  we write

$$\partial^w \left( e^{-a^2(\hat{p}^2+m^2)} - e^{-a^2((\hat{p}^O)^2+m^2)} \right) = \partial^w \left\{ e^{-a^2(\hat{p}^2+m^2)} \left( 1 - e^{-a^2((\hat{p}^O)^2-\hat{p}^2)} \right) \right\}. \quad (7.2.10)$$

In case  $|\hat{p}^O| \leq |\hat{p}|$ , we factorize instead  $e^{-a^2((\hat{p}^O)^2+m^2)}$  and follow again the subsequent reasoning. By the Leibniz formula, we obtain

$$\partial^w \left\{ e^{-a^2(\hat{p}^2+m^2)} \left( 1 - e^{-a^2((\hat{p}^O)^2-\hat{p}^2)} \right) \right\} = \sum_{w_1+w_2=w} c_{w_i} \partial^{w_1} e^{-a^2(\hat{p}^2+m^2)} \partial^{w_2} \left( 1 - e^{-a^2((\hat{p}^O)^2-\hat{p}^2)} \right)$$

The first factor in each entry in the sum can be bounded as in (7.2.5). As regards the second factor we first consider the exponential without derivatives

$$1 - e^{-a^2[(\hat{p}^O)^2 - \hat{p}^2]}$$

We can rewrite the exponent as

$$a^2 [(\hat{p}^O)^2 - \hat{p}^2] = \frac{2a^2}{a_0^2} \sum_{\mu=1}^4 [\cos(a_0(Op)_\mu) - \cos(a_0 p_\mu)], \quad (7.2.11)$$

$$= 2 \frac{a_0}{a} \sum_{\mu=1}^4 \int_0^1 dt \frac{(1-t)^2}{2!} \left[ [a(Op)_\mu]^3 \cos^{(3)}[t a_0 (Op)_\mu] - [a p_\mu]^3 \cos^{(3)}[t a_0 p_\mu] \right]. \quad (7.2.12)$$

We used a Taylor formula with integrated remainder around 0 for both cosine functions and the fact that the constant and quadratic terms in the difference of the two cosine functions cancel. The statement of the lemma is then a consequence of the following facts:

a)

$$|\partial^w a^2 [(\hat{p}^O)^2 - \hat{p}^2]| \leq \frac{a_0}{a} a^{-|w|} \mathcal{P}(a|p|. \quad (7.2.13)$$

This follows directly from (7.2.11), (7.2.12). The degree of the polynomial  $\mathcal{P}$  can be chosen to be less or equal than 3.

b)

$$\left| e^{-f(x)} - 1 \right| \leq f(x) \quad \text{for } f(x) \geq 0.$$

c)

$$\begin{aligned} & \partial^w e^{-a^2[(\hat{p}^O)^2 - \hat{p}^2]} \\ &= \frac{a_0^2}{a^2} a^{|w|} \mathcal{P} \left( \left\{ a p_\mu, \int_0^1 dt (1-t)^2 \cos^{(3)}(t a_0 p_\mu), \int_0^1 dt (1-t)^2 \cos^{(3)}(t a_0 (Op)_\mu), \frac{a_0}{a} \right\} \right). \end{aligned}$$

This statement follows by induction on  $|w|$  from (7.2.11), (7.2.12). The polynomial  $P$  (whose coefficients are real but may have either sign) is at most of degree  $3|w|$ . The coefficients do not depend on  $a_0$ ,  $a$ ,  $p$ .

d) The inequality

$$e^{-a^2 m^2} \leq \frac{C(n)}{(1+am)^n}, \quad (7.2.14)$$

which holds for any  $n \in \mathbb{N}$  and suitable positive  $C(n)$  can be used to turn powers of  $a$  or of  $a|p|$  into powers of  $a/(1+am)$  or  $a|p|/(1+am)$ .  $\square$

## 7.2.2 Renormalizability

A simple inductive proof of the renormalizability of  $\phi_4^4$  theory, regularized by a UV-cutoff has been exposed several times in the literature [11, 30]. Our proof follows the same line of reasoning. New difficulties arise due to the particular form of the lattice propagator (7.1.1) that breaks Euclidean symmetry. The boundary conditions following from (7.1.5) are

$$\partial^w \mathcal{L}_{l,n}^{a_0, a_0}(p_1, \dots, p_n) = 0, \quad n + |w| > 4 \quad \text{such that } n \neq 2, \quad (7.2.15)$$

$$\partial^w \mathcal{L}_{l,2}^{a_0, a_0}(p, -p) = b_l(a_0) \partial^w \hat{p}^2, \quad \forall |w| \geq 3. \quad (7.2.16)$$

As compared to continuum theory [11], note that the boundary conditions (7.2.16) are not equal to zero. For terms with  $n + |w| \leq 4$ , the boundary conditions are explicitly fixed by ( $a_0$ -independent) renormalization conditions imposed for the fully integrated theory at  $a = \infty$ :

$$\mathcal{L}_{4,l}^{a_0, \infty}(0, \dots, 0) = \lambda, \quad \mathcal{L}_{2,l}^{a_0, \infty}(0, 0) = 0, \quad \partial_{p^2} \mathcal{L}_{2,l}^{a_0, \infty}(0, 0) = 0, \quad \forall l \geq 1. \quad (7.2.17)$$

The renormalization point is chosen at zero momentum for simplicity (BPHZ renormalization conditions). The induction hypotheses to be proven are

**Theorem 13.** *For all  $l \in \mathbb{N}^*$ ,  $n \in \mathbb{N}$ ,  $w$  and for  $0 \leq a_0 \leq a$ ,  $a_0 < \frac{1}{m}$  holds*

A) *Boundedness in the UV-cutoff*

$$\left| \partial^w \mathcal{L}_{l,n}^{a_0, a}(p_1, \dots, p_n) \right| \leq \left( \frac{1}{a} + m \right)^{4-n-|w|} \mathcal{P}_1 \left( \log \frac{1+am}{am} \right) \mathcal{P}_2 \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right). \quad (7.2.18)$$

B) *Convergence in the UV-limit*

$$\left| \partial_{1/a_0} \partial^w \mathcal{L}_{l,n}^{a_0, a}(p_1, \dots, p_n) \right| \leq \frac{\left( \frac{1}{a} + m \right)^{5-n-|w|}}{\left( \frac{1}{a_0} + m \right)^2} \mathcal{P}_3 \left( \log \frac{1+a_0 m}{a_0 m} \right) \mathcal{P}_4 \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right), \quad (7.2.19)$$

where  $(p_i)_{1 \leq i \leq n} \in \mathcal{B}_{a_0}$  and  $p_1 + \dots + p_n \equiv 0 \left[ \frac{2\pi}{a_0} \right]$ . Here and in the following the  $\mathcal{P}$ ,  $\mathcal{P}_i$  denote (each time they appear possibly new) polynomials with nonnegative coefficients. The coefficients depend on  $l, n, |w|$ , but not on  $m, \{p_i\}, a, a_0$ . For  $l = 0$ , all polynomials  $\mathcal{P}_1, \mathcal{P}_3$  reduce to 1.

*Remarks:*

- Theorem [13] is established for  $p_i \in \mathcal{B}_{a_0}$  but it is possible to extend it to  $(p_i)_{1 \leq i \leq n} \in \mathbb{R}^4$ . Since  $\mathcal{L}_{l,n}^{a_0, a}$  is  $\frac{2\pi}{a_0}$ -periodic,  $\mathcal{L}_{l,n}^{a_0, a}(p_1, \dots, p_n)$  such that  $p_i \in \mathcal{B}_{k_i a_0} := \left] -\frac{(2k_i+1)\pi}{a_0}, \frac{(2k_i+1)\pi}{a_0} \right[$ ,  $k_i \in \mathbb{Z}^4$  and  $\sum_{i=1}^n p_i \equiv 0 \left[ \frac{2\pi}{a_0} \right]$ , also verifies the flow equations (7.1.8) with the same boundary conditions, as we will see later, and therefore it verifies Theorem 1. The extension to the boundaries of the extended Brillouin zones  $\mathcal{B}_{k_i, a_0}$  is performed using the continuity of  $\mathcal{L}_{l,n}^{a_0, a}$  w.r.t.  $p_i$  and taking

the limits  $p_i \rightarrow \frac{k_i \pi}{a_0}$  in (7.2.18). The fact that  $\mathcal{L}_{l,n}^{a_0,a}$  is  $\mathcal{C}^\infty$  w.r.t.  $p_i$  and that it is  $2\pi/a_0$ -periodic can be proved inductively using the flow equations and that the propagator and the boundary conditions are  $2\pi/a_0$ -periodic and  $\mathcal{C}^\infty$ . We will not prove it here.

- It is also possible to prove a stronger version of Theorem 13, replacing  $\mathcal{P}\left(\left\{\frac{a|p_i|}{1+am}\right\}\right)$  by  $\mathcal{P}\left(\left\{\frac{a|\hat{p}_i|}{1+am}\right\}\right)$ .
- The statement (7.2.19) implies that for sufficiently small  $a_0$  and suitable  $\nu > 0$

$$\left| \partial_{1/a_0} \partial^w \mathcal{L}_{l,n}^{a_0,a}(p_1, \dots, p_n) \right| \leq a_0^2 \left( \frac{1}{a} + m \right)^{5-n-|w|} \left( \log \frac{1+a_0 m}{a_0 m} \right)^\nu \mathcal{P}_4 \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right). \quad (7.2.20)$$

Integration of the bound (7.2.20) over the lattice cutoff  $1/a_0$  immediately proves the convergence of all  $\mathcal{L}_{l,n}^{a_0,a}(p_1, \dots, p_n)$  for fixed  $a$  to finite limits when  $a_0 \rightarrow 0$ . In particular, one obtains for all  $\hat{a}_0 < a_0$  and  $(p_i)_{1 \leq i \leq n} \in \mathcal{B}_{a_0}$ ,

$$\left| \mathcal{L}_{l,n}^{a_0,\infty}(p_1, \dots, p_n) - \mathcal{L}_{l,n}^{\hat{a}_0,\infty}(p_1, \dots, p_n) \right| < a_0 m^{5-n} \left( \log \frac{1}{a_0 m} \right)^\nu \mathcal{P}_5 \left( \left\{ \frac{|p_i|}{m} \right\} \right). \quad (7.2.21)$$

Thus, due to the Cauchy criterion in  $\mathcal{C}^\infty(\mathbb{R}^+)$  (w.r.t. to  $a_0$ ) finite limits exist to all loop orders  $l$ .

*Proof.* We use the standard inductive scheme, see section 1.1.5. The irrelevant terms are integrated from  $1/a_0$  to  $1/a$  and the relevant terms from 0 to  $1/a$  using the integrated Taylor formula to pass from the renormalization point to arbitrary momenta.

- (A) **Boundedness:** To start the induction, we prove the bound (7.2.18) at the tree level. The classical interaction contains no terms linear or quadratic in the fields. Hence, we have

$$\mathcal{L}_{0,2}^{a_0,a}(p, -p) = 0, \quad \mathcal{L}_{0,4}^{a_0,a}(p_1, \dots, p_4) = \lambda.$$

Since the  $\mathbb{Z}_2$ -symmetry  $\phi \rightarrow -\phi$ , is not broken by the renormalization procedure, we have

$$\mathcal{L}_{l,n}^{a_0,a}(p_1, \dots, p_n) = 0, \quad \forall n \text{ odd}, \forall l \geq 0.$$

Thus, the bound evidently holds for  $n + |w| \leq 2$  at the tree order. For  $n + |w| > 4$  (the irrelevant cases) we proceed inductively ascending in  $n$ . For given  $n$  the various  $w$  dealt with in arbitrary order, by integrating the respective flow equation (7.1.8) from the initial point  $1/a_0$ .

Using the induction hypothesis for  $\mathcal{L}_{0,n_1+1}^{a_0,a}$  and  $\mathcal{L}_{0,n_2+1}^{a_0,a}$ , and (7.2.2), (7.2.5) we obtain a bound for the quadratic part of the r.h.s. of (7.1.8)

$$\begin{aligned} & \left| \partial^{w_1} \mathcal{L}_{0,n_1+1}^{a_0,a}(p_1, \dots, p_{n_1}, p) \partial^{w_3} \partial_{1/a} C^{a_0,a}(\hat{p}) \partial^{w_2} \mathcal{L}_{0,n_2+1}^{a_0,a}(-p, p_{n_1+1}, \dots, p_n) \right| \\ & \leq \left( \frac{1}{a} + m \right)^{4-n-|w|-1} \mathcal{P} \left( \frac{a \|\vec{p}_n\|}{1+am} \right). \quad (7.2.22) \end{aligned}$$

Therefore

$$\left| \partial_\lambda \partial^w \mathcal{L}_{0,n}^{a_0, \frac{1}{\lambda}}(p_1, \dots, p_n) \right| \leq (\lambda + m)^{4-n-|w|-1} \mathcal{P} \left( \frac{\|\vec{p}_n\|}{\lambda + m} \right). \quad (7.2.23)$$

This proves (7.2.18) at the tree order.

To generate inductively the bounds (7.2.18) for higher loop orders, we use them in bounding the r.h.s of the FE (7.1.8), together with the bound (7.2.4) in the linear and in the quadratic term respectively. For the linear term of the r.h.s. of FE, we use the induction hypothesis for  $\partial^w \mathcal{L}_{l-1, n+2}^{a_0, a}$ , and we obtain the upper bound

$$\int_{k, \mathcal{B}_{a_0}} (2a^3) e^{-a^2(\hat{k}^2 + m^2)} \mathcal{P} \left( \frac{a|k|}{1 + am}, \frac{a\|\vec{p}_n\|}{1 + am} \right).$$

Using lemma 13 this can be turned into the bound

$$\int_{k, \mathcal{B}_{a_0}} (2a^3) e^{-a^2(\hat{k}^2 + m^2)} \mathcal{P} \left( \frac{a|k|}{1 + am}, \frac{a\|\vec{p}_n\|}{1 + am} \right) \leq \frac{1}{a} \tilde{\mathcal{P}} \left( \frac{a\|\vec{p}_n\|}{1 + am} \right).$$

Hence, we obtain

$$\begin{aligned} \int_{k, \mathcal{B}_{a_0}} \partial_{1/a} C^{a_0, a}(\hat{k}) \left| \partial^w \mathcal{L}_{l-1, n+2}^{a_0, a}(-k, \dots, k) \right| \\ \leq \left( \frac{1}{a} + m \right)^{4-n-|w|-1} \mathcal{P}_1 \left( \log \frac{am + 1}{am} \right) \mathcal{P}_2 \left( \frac{a\|\vec{p}_n\|}{1 + am} \right). \end{aligned}$$

For the quadratic part of the flow equations (7.1.8), we use the induction hypothesis for  $\partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{a_0, a}$  and  $\partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{a_0, a}$  together with the bound (7.2.3) and we obtain

$$\begin{aligned} \left| \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{a_0, a}(p_1, \dots, p_{n_1}, p) \partial^{w_3} \partial_{1/a} C^{a_0, a}(\hat{p}) \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{a_0, a}(-p, p_{n_1+1}, \dots, p_n) \right| \\ \leq \left( \frac{1}{a} + m \right)^{4-n-|w|-1} \mathcal{P}_1 \left( \log \frac{1 + am}{am} \right) \mathcal{P}_2 \left( \frac{a\|\vec{p}_n\|}{1 + am} \right). \end{aligned}$$

Therefore, we deduce

$$\left| \partial_{1/a} \partial^w \mathcal{L}_{l,n}^{a_0, a}(p_1, \dots, p_n) \right| \leq \left( \frac{1}{a} + m \right)^{4-n-|w|-1} \mathcal{P}_1 \left( \log \frac{am + 1}{am} \right) \mathcal{P}_2 \left( \left\{ \frac{a|p_i|}{1 + am} \right\} \right). \quad (7.2.24)$$

Following the order of the induction stated before, for the irrelevant cases  $n + |w| \geq 5$  the bound (7.2.24) is integrated downwards from  $1/a$  to  $1/a_0$ . For  $n + |w| \geq 5$  such that  $n \neq 2$ , integrating from  $1/a$  to  $1/a_0$  yields

$$\left| \partial^w \mathcal{L}_{l,n}^{a_0, a}(p_1, \dots, p_n) \right| \leq \int_{1/a}^{1/a_0} d\lambda (\lambda + m)^{4-n-|w|-1} \mathcal{P}_1 \left( \log \frac{\lambda + m}{m} \right) \mathcal{P}_2 \left( \left\{ \frac{|p_i|}{\lambda + m} \right\} \right).$$

We now have, see [30]

$$\int_{1/a}^{1/a_0} d\lambda (\lambda + m)^{4-n-|w|-1} \mathcal{P} \left( \log \frac{\lambda + m}{m} \right) < \left( \frac{1}{a} + m \right)^{4-n-|w|} \tilde{\mathcal{P}} \left( \log \frac{1 + am}{am} \right).$$

For the particular case  $(n, |w|) = (2, 2)$ , (7.2.24) is integrated from 0 to  $1/a_0$  at zero momenta,

$$\left| \partial_{p^2} \mathcal{Z}_{l,2}^{a_0, a_0}(0, 0) - \partial_{p^2} \mathcal{Z}_{l,2}^{a_0, \infty}(0, 0) \right| \leq \int_0^{1/a_0} d\lambda (\lambda + m)^{-1} \mathcal{P} \left( \log \frac{\lambda + m}{m} \right) \leq \mathcal{P} \left( \log \frac{1 + a_0 m}{a_0 m} \right).$$

This gives

$$|b_l(a_0)| \leq \mathcal{P} \left( \log \frac{1 + a_0 m}{a_0 m} \right). \quad (7.2.25)$$

It then follows from (7.2.16) that the 2-point function and its derivatives at  $a = a_0$  can be bounded

$$\left| \partial^w \mathcal{Z}_{l,2}^{a_0, a_0}(p, -p) \right| \leq 2|b_l(a_0)| a_0^{|w|-2} C,$$

for some positive constant  $C$  depending on  $|w|$ , which implies for all  $|w| \geq 3$

$$\left| \partial^w \mathcal{Z}_{l,2}^{a_0, a_0}(p, -p) \right| \leq \left( \frac{1}{a_0} + m \right)^{2-|w|} \mathcal{P} \left( \log \frac{1 + a_0 m}{a_0 m} \right) \leq \left( \frac{1}{a} + m \right)^{2-|w|} \mathcal{P} \left( \log \frac{1 + am}{am} \right).$$

Integrating the inductive bound from  $1/a$  to  $1/a_0$  for  $n = 2$ ,  $|w| \geq 3$  then gives

$$\begin{aligned} \left| \partial^w \mathcal{Z}_{l,2}^{a_0, a}(p, -p) \right| &\leq \int_{1/a}^{1/a_0} d\lambda \left| \partial_\lambda \partial^w \mathcal{Z}_{l,2}^{a_0, \frac{1}{\lambda}}(p, -p) \right| + \left| \partial^w \mathcal{Z}_{l,2}^{a_0, a_0}(p, -p) \right| \\ &\leq \left( \frac{1}{a} + m \right)^{2-|w|} \mathcal{P}_1 \left( \log \frac{1 + am}{am} \right) \mathcal{P}_2 \left( \left\{ \frac{|p_i|}{\lambda + m} \right\} \right). \end{aligned}$$

For the relevant terms ( $n + |w| \leq 4$ ), we start with the case  $(n = 2, |w| = 2)$  and continue to  $(n = 2, |w| = 1)$  and  $(n = 2, w = 0)$ . Bounding equation (4.5.16) in absolute value, we obtain using the bound (7.2.24) at vanishing momenta:

$$\begin{aligned} \left| \int_0^{1/a} d\lambda \partial_{\frac{1}{\lambda}} \partial^w \mathcal{Z}_{l,n}^{a_0, \frac{1}{\lambda}}(0, \dots, 0) \right| &\leq \int_0^{1/a} d\lambda (\lambda + m)^{4-n-|w|-1} \mathcal{P} \left( \log \frac{1 + am}{am} \right) \\ &\leq (\lambda + m)^{4-n-|w|} \mathcal{P} \left( \log \frac{1 + am}{am} \right). \end{aligned} \quad (7.2.26)$$

Hence, the assertion (7.2.18) is established at the renormalization point. In each case extension to general momenta is guaranteed by the bounds established before. This concludes the proof of (7.2.18).

(B) Convergence: The bound (7.2.20) follows on applying the same inductive scheme to bound the solutions of the FE, integrated over  $1/a$  and then derived w.r.t.  $1/a_0$ . The proof is analogous to [11, 30], apart from the changes induced by the lattice momenta  $\hat{p}$  which were dealt with in the proof of (7.2.18).

□

## 7.3 Restoration of $O(4)$ symmetry

### 7.3.1 The flow equations

The lattice breaks the rotation and translation symmetries. In order to define the rotated scalar field on the lattice, we consider the rotated lattice

$$\Lambda_{a_0}^O := O\Lambda_{a_0}, \quad O \in O(4).$$

The rotated scalar field  $\hat{\phi}_{a_0}^O$  is defined by

$$\hat{\phi}_{a_0}^O := \hat{\phi}|_{\Lambda_{a_0}^O},$$

where  $\hat{\phi}$  is the continuum scalar field. For our purposes,  $\hat{\phi}_{a_0}^O$  is considered to live in  $l_2(\Lambda_{a_0}^O)$  and the Brillouin zone associated to the rotated lattice  $\Lambda_{a_0}^O$  is

$$\mathfrak{B}_{a_0}^O := O \left( \left[ -\frac{\pi}{a_0}, \frac{\pi}{a_0} \right]^4 \right)$$

The Fourier transform of  $\hat{\phi}_{a_0}^O$  is defined by

$$\phi_{a_0}^O(p) := a_0^4 \sum_{x \in \Lambda_{a_0}^O} e^{-ip \cdot x} \hat{\phi}_{a_0}^O(x).$$

The inverse Fourier transform is defined by

$$\hat{\phi}_{a_0}^O(x) := \int_{\mathfrak{B}_{a_0}^O} \frac{d^4 p}{(2\pi)^4} \phi_{a_0}^O(p) e^{ip \cdot x},$$

such that the Plancherel identity is preserved.

The bare action associated to the rotated field is defined by

$$L_O^{a_0, a_0}(\hat{\phi}_{a_0}^O) := a_0^4 \sum_{x \in \Lambda_{a_0}^O} \left\{ \frac{f}{4!} (\hat{\phi}_{a_0}^O)^4 + d^O(a_0) (\hat{\phi}_{a_0}^O)^2 + b^O(a_0) (\hat{\partial}_{\mu, a_0}^O \hat{\phi}_{a_0}^O)^2 + c^O(a_0) (\hat{\phi}_{a_0}^O)^4 \right\}.$$



The counter terms  $d^O(a_0)$ ,  $b^O(a_0)$  and  $c^O(a_0)$  depend on the rotation matrix  $O$ . The lattice derivative on the rotated lattice is defined as follows for  $\hat{\phi}_{a_0}^O \in l_2(\Lambda_{a_0}^O)$

$$\left(\hat{\partial}_{\mu,a_0}^O \hat{\phi}_{a_0}^O\right)(x) := \frac{\hat{\phi}_{a_0}^O(x + a_0 e_\mu^O) - \hat{\phi}_{a_0}^O(x)}{a_0}, \quad x \in \Lambda_{a_0}^O,$$

where  $e_\mu^O := Oe_\mu$  is the rotated unit vector in the  $\mu^{\text{th}}$  direction.

The flowing propagator is defined by

$$\hat{C}^{O,a_0,a}(x,y) := \int_{\mathfrak{B}_{a_0}^O} \frac{d^4 p}{(2\pi)^4} e^{ip \cdot (x-y)} C^{a_0,a}(p),$$

where  $C^{a_0,a}$  is defined as before

$$C^{a_0,a}(p) := \frac{1}{\hat{p}^2 + m^2} \left( e^{-a_0^2(\hat{p}^2 + m^2)} - e^{-a^2(\hat{p}^2 + m^2)} \right).$$

The lattice momentum  $\hat{p}$  was defined in (7.1.2). The derivation of the FE corresponding to the rotated field follows the same steps as before, starting from the functional integral

$$e^{-\frac{1}{\hbar}(L_O^{a_0,a}(\hat{\phi}_{a_0}^O) + I^{a_0,a})} := \int d\mu_{a_0,a}^O(\Phi) e^{-\frac{1}{\hbar}L_O^{a_0,a}(\hat{\phi}_{a_0}^O + \Phi)} \quad (7.3.1)$$

where  $d\mu_{a_0,a}^O$  is uniquely defined by its covariance operator  $\hat{C}^{O,a_0,a}$  through

$$\int d\mu_{a_0,a}^O(\Phi) e^{\langle \Phi, J \rangle_{l_2(\Lambda_{a_0}^O)}} := e^{\frac{1}{2} \langle J, \hat{C}^{O,a_0,a} J \rangle_{l_2(\Lambda_{a_0}^O)}}, \quad J \in l_2(\Lambda_{a_0}^O).$$

In terms of momenta in  $\mathfrak{B}_{a_0}$ , the propagator  $C^{a_0,a}$  has the following form

$$C^{a_0,a}(Op) = \frac{1}{(\hat{p}^O)^2 + m^2} \left( e^{-a_0^2((\hat{p}^O)^2 + m^2)} - e^{-a^2((\hat{p}^O)^2 + m^2)} \right).$$

The FE are obtained by differentiating (7.3.1) w.r.t.  $1/a$ ,

$$\partial_{1/a} L_O^{a_0,a} = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \hat{\phi}_{a_0}^O}, \hat{C}^{a_0,a} * \frac{\delta}{\delta \hat{\phi}_{a_0}^O} \right\rangle_{l_2(\Lambda_{a_0}^O)} L_O^{a_0,a} - \frac{1}{2} \left\langle \frac{\delta L_O^{a_0,a}}{\delta \hat{\phi}_{a_0}^O}, \hat{C}^{a_0,a} * \frac{\delta L_O^{a_0,a}}{\delta \hat{\phi}_{a_0}^O} \right\rangle_{l_2(\Lambda_{a_0}^O)}. \quad (7.3.2)$$

We expand in a formal power series w.r.t.  $\hbar$  to select the loop order

$$L_O^{a_0,a}(\hat{\phi}_{a_0}^O) = \sum_{l=0}^{+\infty} \hbar^l L_{O,l}^{a_0,a}(\hat{\phi}_{a_0}^O).$$

From  $L_{O,l}^{a_0,a}$  we obtain the CAS of loop order  $l$  in momentum space  $\mathfrak{B}_{a_0}$  as

$$\delta_{\left[\frac{2\pi}{a_0}\right]}^4 (Op_1 + \dots + Op_n) \mathcal{L}_{l,n}^{a_0,a,O}(Op_1, \dots, Op_n) := (2\pi)^{4(n-1)} \delta_{\phi_{a_0}^O(Op_1)} \dots \delta_{\phi_{a_0}^O(Op_n)} L_{O,l}^{a_0,a} |_{\phi_{a_0}^O \equiv 0}.$$

From the functional flow equations (7.1.8), we obtain the perturbative flow equations for the CAS  $n$ -point functions

$$\begin{aligned} & \partial_{1/a} \partial^w \mathcal{L}_{l,n}^{a_0,a,O}(Op_1, \dots, Op_n) \\ &= \frac{1}{2} \int_{k, \mathcal{B}_{a_0}} \partial^w \mathcal{L}_{l-1, n+2}^{a_0,a,O}(Ok, Op_1, \dots, Op_n, -Ok) \partial_{1/a} C^{a_0,a,O}(\hat{k}^O) \\ & - \frac{1}{2} \sum_{l_1, l_2} \sum_{n_1, n_2} \sum_{w_1, w_2, w_3} c_{w_i} \left[ \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{a_0,a,O}(Op_1, \dots, Op_{n_1}, Op) \partial^{w_3} \partial_{1/a} C^{a_0,a,O}(\hat{p}^O) \right. \\ & \left. \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{a_0,a,O}(-Op, \dots, Op_n) \right]_{rsym}, \quad (7.3.3) \end{aligned}$$

$$Op \equiv -Op_1 - \dots - Op_{n_1} \equiv Op_{n_1+1} + \dots + Op_n \left[ \frac{2\pi}{a_0} \right], \quad (p_i)_{1 \leq i \leq n} \in \mathcal{B}_{a_0},$$

where we used the same conventions as in (7.1.8). The uniqueness of  $\mathcal{L}_{l,n}^{a_0,a,O}$  is ensured by imposing the following boundary conditions following from (7.3.1) given by

$$\partial^w \mathcal{L}_{l,n}^{a_0,a_0,O}(Op_1, \dots, Op_n) = 0, \quad n + |w| > 4 \quad \text{such that } n \neq 2, \quad (7.3.4)$$

$$\partial^w \mathcal{L}_{l,2}^{a_0,a_0,O}(p, -p) = b_l^O(a_0) \partial^w (\hat{p}^O)^2, \quad \forall |w| \geq 3. \quad (7.3.5)$$

For terms with  $n + |w| \leq 4$ , the boundary conditions are explicitly fixed by ( $a_0$ -independent) BPHZ renormalization conditions imposed for the fully integrated theory at  $a = \infty$ :

$$\mathcal{L}_{4,l}^{a_0,\infty,O}(0, \dots, 0) = \lambda, \quad \mathcal{L}_{2,l}^{a_0,\infty,O}(0, 0) = 0, \quad \partial_{p^2} \mathcal{L}_{2,l}^{a_0,\infty,O}(0, 0) = 0, \quad \forall l \geq 1. \quad (7.3.6)$$

Following Theorem 13 and assuming (7.3.4)-(7.3.6), the following bound holds for  $l \in \mathbb{N}^*$ ,  $n \in \mathbb{N}$ ,  $w$  and for  $0 \leq a_0 \leq a$ ,  $a_0 < \frac{1}{m}$

$$\left| \partial^w \mathcal{L}_{l,n}^{a_0,a,O}(Op_1, \dots, Op_n) \right| \leq \left( \frac{1}{a} + m \right)^{4-n-|w|} \mathcal{P}_1 \left( \log \frac{1+am}{am} \right) \mathcal{P}_2 \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right). \quad (7.3.7)$$

### 7.3.2 Proof of rotation symmetry restoration

The  $O(4)$ -symmetry is restored for  $a_0 \rightarrow 0$  if and only if  $\forall (p_i)_{1 \leq i \leq n} \in \mathbb{R}^4$ ,  $\forall O \in O(4) \exists \tilde{a}_0 \geq 0$ ,

$$\lim_{a_0 \rightarrow 0, 0 \leq a_0 \leq \tilde{a}_0} \lim_{a \rightarrow \infty} \left( \mathcal{L}_{l,n}^{a_0,a}(p_1, \dots, p_n) - \mathcal{L}_{l,n}^{a_0,a,O}(Op_1, \dots, Op_n) \right) = 0 \quad (7.3.8)$$

Here we introduced the parameter  $\tilde{a}_0$  as in (7.2.1). For  $(p_i)_{1 \leq i \leq n} \in \mathcal{B}_{a_0}$  we thus define

$$\partial^w \mathcal{D}_{l,n}^{a_0,a}(p_1, \dots, p_n) := \partial^w \mathcal{L}_{l,n}^{a_0,a}(p_1, \dots, p_n) - \partial^w \mathcal{L}_{l,n}^{a_0,a,O}(Op_1, \dots, Op_n).$$

From the flow equations (7.1.8) and (7.3.3), we can derive a FE for  $\partial^w \mathcal{D}_{l,n}^{a_0,a}(p_1, \dots, p_n)$  :

$$\begin{aligned}
\partial_{1/a} \partial^w \mathcal{D}_{l,n}^{a_0,a}(p_1, \dots, p_n) &= \frac{1}{2} \int_{k, \mathcal{B}_{a_0}} \partial_{1/a} C^{a_0,a}(\hat{k}) \partial^w \mathcal{D}_{l-1,n+2}^{a_0,a}(k, p_1, \dots, p_n, -k) \\
&+ \frac{1}{2} \int_{k, \mathcal{B}_{a_0}} \partial^w \mathcal{L}_{l-1,n+2}^{a_0,a}(Ok, Op_1, \dots, Op_n, -Ok) \left[ \partial_{1/a} C^{a_0,a}(\hat{k}) - \partial_{1/a} C^{a_0,a}(\hat{k}^O) \right] \\
&- \frac{1}{2} \sum_{\substack{l_1, l_2 \\ n_1, n_2}}' \sum_{w_1, w_2, w_3}' c_{w_i} \left[ \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{a_0,a}(p_1, \dots, p_{n_1}) \partial^{w_3} \partial_{1/a} C^{a_0,a}(\hat{p}) \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{a_0,a}(-p, \dots, p_n) \right. \\
&+ \partial^{w_1} \mathcal{D}_{l_1, n_1+1}^{a_0,a}(p_1, \dots, p_{n_1}) \partial^{w_3} \partial_{1/a} C^{a_0,a}(\hat{p}^O) \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{a_0,a,O}(-Op, \dots, Op_n) \\
&+ \left. \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{a_0,a}(p_1, \dots, p_{n_1}) \partial^{w_3} (\partial_{1/a} C^{a_0,a}(\hat{p}) - \partial_{1/a} C^{a_0,a}(\hat{p}^O)) \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{a_0,a,O}(-Op, \dots, Op_n) \right]_{\text{rsym}} \\
& p_1 + \dots + p_n \equiv 0 \left[ \frac{2\pi}{a_0} \right] \\
& Op_1 + \dots + Op_n \equiv 0 \left[ \frac{2\pi}{a_0} \right], \quad (p_i)_{1 \leq i \leq n} \in \mathcal{B}_{a_0}. \quad (7.3.9)
\end{aligned}$$

Restoration of  $O(4)$ -symmetry, i.e.

$$\lim_{a_0 \rightarrow 0, a \rightarrow \infty} \mathcal{D}_{l,n}^{a_0,a}(p_1, \dots, p_n) = 0$$

follows from the following Theorem:

**Theorem 14.**  $\forall n, \forall w, \forall (p_i)_{1 \leq i \leq n} \in \mathcal{B}_{a_0}$  such that  $\sum_{i=1}^n p_i, \sum_{i=1}^n Op_i \equiv 0 \left[ \frac{2\pi}{a_0} \right]$ ,

$$\left| \partial^w \mathcal{D}_{l,n}^{a_0,a}(p_1, \dots, p_n) \right| \leq a_0 \left( \frac{1}{a} + m \right)^{5-n-|w|} \mathcal{P}_1 \left( \log \frac{1+a_0 m}{a_0 m} \right) \mathcal{P}_2 \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right), \quad (7.3.10)$$

where  $\mathcal{P}_i$  denote polynomials with nonnegative coefficients, that depend, as well as the degree of the polynomials on  $l, n, w$  but not on  $m, \{p_i\}, a, a_0$ .

### 7.3.3 Proof of Theorem 14

*Proof.* We prove (7.3.10) using the inductive scheme indicated previously. The only terms in which (7.3.10) cannot be used as an induction hypothesis are

$$\int_{k, \mathcal{B}_{a_0}} \partial^w \mathcal{L}_{l-1,n+1}^{a_0,a,O}(Ok, Op_1, \dots, p_n, -Ok) \left[ \partial_{1/a} C^{a_0,a}(\hat{k}) - \partial_{1/a} C^{a_0,a}(\hat{k}^O) \right] \quad (7.3.11)$$

and

$$\partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{a_0,a}(p_1, \dots, p_{n_1}) (\partial^{w_3} \partial_{1/a} C^{a_0,a}(\hat{p}) - \partial^{w_3} \partial_{1/a} C^{a_0,a}(\hat{p}^O)) \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{a_0,a,O}(-Op, \dots, Op_n). \quad (7.3.12)$$

Our bound on  $\mathcal{D}_{l,n}^{a_0,a}$  will be verified by proving it for these difference terms.

- We first bound (7.3.11). Using the bound (7.3.7), we obtain

$$\begin{aligned} & \left| \int_{k, \mathcal{B}_{a_0}} \partial^w \mathcal{L}_{l-1, n+1}^{a_0, a, O}(Ok, Op_1, \dots, Op_n, -Ok) \left[ \partial_{1/a} C^{a_0, a}(\hat{k}) - \partial_{1/a} C^{a_0, a}(\hat{k}^O) \right] \right| \\ & \leq \int_{k, \mathcal{B}_{a_0}} 2a^3 \left( \frac{1}{a} + m \right)^{3-n-|w|} \left| e^{-a^2(\hat{k}^2+m^2)} - e^{-a^2((\hat{k}^O)^2+m^2)} \right| \\ & \quad \times \mathcal{P}_1 \left( \log \frac{1+am}{am} \right) \mathcal{P}_2 \left( \frac{a|k|}{1+am}, \left\{ \frac{a|p_i|}{1+am} \right\} \right). \end{aligned}$$

We define

$$\mathcal{J}_{a_0}^O := \left\{ k \in \mathcal{B}_{a_0} : \sum_{\mu=1}^4 \sin^2 \frac{a_0 k_\mu}{2} \leq \sum_{\mu=1}^4 \sin^2 \frac{a_0 (Ok)_\mu}{2} \right\}.$$

We decompose the integral over the Brillouin zone  $\mathcal{B}_{a_0}$  into integrals over  $\mathcal{J}_{a_0}^O$  and  $\mathcal{J}_{a_0}^{Oc}$ ,

$$\begin{aligned} & \int_{k, \mathcal{B}_{a_0}} 2a^3 \left| e^{-a^2(\hat{k}^2+m^2)} - e^{-a^2((\hat{k}^O)^2+m^2)} \right| \mathcal{P} \left( \frac{a|k|}{1+am}, \left\{ \frac{a|p_i|}{1+am} \right\} \right) \\ & = \int_{k, \mathcal{J}_{a_0}^O} 2a^3 e^{-a^2(\hat{k}^2+m^2)} \left| e^{-a^2((\hat{k}^O)^2-\hat{k}^2)} - 1 \right| \mathcal{P} \left( \frac{a|k|}{1+am}, \left\{ \frac{a|p_i|}{1+am} \right\} \right) \\ & \quad + \int_{(k, \mathcal{J}_{a_0}^O)^c} 2a^3 e^{-a^2((\hat{k}^O)^2+m^2)} \left| e^{-a^2(\hat{k}^2-(\hat{k}^O)^2)} - 1 \right| \mathcal{P} \left( \frac{a|k|}{1+am}, \left\{ \frac{a|p_i|}{1+am} \right\} \right). \end{aligned}$$

From the definition of  $\mathcal{J}_{a_0}^O$ , we have

$$\forall k \in \mathcal{J}_{a_0}^O, |\hat{k}| \leq |\hat{k}^O| \quad \forall k \in \mathcal{J}_{a_0}^{Oc}, |\hat{k}| > |\hat{k}^O|$$

which implies that

$$\begin{aligned} & \left| e^{-a^2((\hat{k}^O)^2-\hat{k}^2)} - 1 \right| \leq a^2 \left| (\hat{k}^O)^2 - \hat{k}^2 \right|, \quad \forall k \in \mathcal{J}_{a_0}^O \\ & \left| e^{-a^2(\hat{k}^2-(\hat{k}^O)^2)} - 1 \right| \leq a^2 \left| (\hat{k}^O)^2 - \hat{k}^2 \right|, \quad \forall k \in \mathcal{J}_{a_0}^{Oc} \end{aligned}$$

Using the bound (7.2.13), we obtain

$$a^2 \left| (\hat{k}^O)^2 - \hat{k}^2 \right| \leq \frac{a_0}{a} \mathcal{P}(a|k|). \quad (7.3.13)$$

This gives the following bound

$$\begin{aligned} & \int_{k, \mathcal{J}_{a_0}^O} a^3 e^{-a^2(\hat{k}^2+m^2)} \left| e^{-a^2((\hat{k}^O)^2-\hat{k}^2)} - 1 \right| \mathcal{P} \left( \frac{a|k|}{1+am}, \left\{ \frac{a|p_i|}{1+am} \right\} \right) \\ & \leq \frac{a_0}{a} \int_{k, \mathcal{B}_{a_0}} a^3 e^{-a^2(\hat{k}^2+m^2)} \mathcal{P} \left( a|k|, \left\{ \frac{a|p_i|}{1+am} \right\} \right) \leq \frac{a_0}{a} \tilde{\mathcal{P}} \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right), \quad (7.3.14) \end{aligned}$$

where the last inequality follows from lemma [13](#). Similarly, we obtain for the second integral supported on  $\mathcal{J}_{a_0}^{O^c}$

$$\begin{aligned} & \int_{k, \mathcal{J}_{a_0}^{O^c}} a^3 e^{-a^2((\hat{k}^O)^2+m^2)} \left| e^{-a^2(\hat{k}^2-(\hat{k}^O)^2)} - 1 \right| \mathcal{P} \left( \frac{a|k|}{1+am}, \left\{ \frac{a|p_i|}{1+am} \right\} \right) \\ & \leq \frac{a_0}{a} \int_{k, \mathcal{J}_{a_0}^{O^c}} a^3 e^{-a^2((\hat{k}^O)^2+m^2)} \mathcal{P} \left( a|k|, \left\{ \frac{a|p_i|}{1+am} \right\} \right). \end{aligned}$$

Performing the change of variables  $k \rightarrow Ok$  yields

$$\begin{aligned} & \int_{k, \mathcal{J}_{a_0}^{O^c}} a^3 e^{-a^2((\hat{k}^O)^2+m^2)} \mathcal{P} \left( a|k|, \left\{ \frac{a|p_i|}{1+am} \right\} \right) \tag{7.3.15} \\ & = \int_{k, O(\mathcal{J}_{a_0}^{O^c})} a^3 e^{-a^2(\hat{k}^2+m^2)} \mathcal{P} \left( a|O^{-1}k|, \left\{ \frac{a|p_i|}{1+am} \right\} \right) \\ & \leq \int_{k, \mathcal{B}_{\alpha a_0}} a^3 e^{-a^2(\hat{k}^2+m^2)} \mathcal{P} \left( a|k|, \left\{ \frac{a|p_i|}{1+am} \right\} \right) \leq \tilde{\mathcal{P}} \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right), \end{aligned}$$

where  $\alpha$  is a parameter strictly less than 1 such that  $O\mathcal{B}_{a_0} \subset \mathcal{B}_{\alpha a_0}$ , and the last inequality follows again from lemma [13](#). Combining [\(7.3.14\)](#) and [\(7.3.15\)](#) the first difference term is bounded as follows

$$\begin{aligned} & \left| \int_{k, \mathcal{B}_{a_0}} \partial^w \mathcal{L}_{l-1, n+1}^{a_0, a, O}(Ok, Op_1, \dots, Op_n, -Ok) \left[ \partial_{1/a} C^{a_0, a}(\hat{k}) - \partial_{1/a} C^{a_0, a}(\hat{k}^O) \right] \right| \\ & \leq a_0 \left( \frac{1}{a} + m \right)^{4-n-|w|} \mathcal{P}_1 \left( \log \frac{1+a_0m}{a_0m} \right) \mathcal{P}_2 \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right). \tag{7.3.16} \end{aligned}$$

- The second step is to bound [\(7.3.12\)](#). For this step we use lemma [14](#). Using [\(7.2.18\)](#) for  $\partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{a_0, a}$  and  $\partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{a_0, a, O}$  we obtain

$$\begin{aligned} & \left| \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{a_0, a} \left( \partial^{w_3} \partial_{1/a} C^{a_0, a}(\hat{p}) - \partial^{w_3} \partial_{1/a} C^{a_0, a}(\hat{p}^O) \right) \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{a_0, a, O} \right| \\ & \leq a_0 \left( \frac{1}{a} + m \right)^{4-n-|w|} \mathcal{P}_1 \left( \log \frac{1+a_0m}{a_0m} \right) \mathcal{P}_2 \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right). \end{aligned}$$

Using the induction bound on  $\partial^{w_i} \mathcal{D}_{l_i, n_i+1}^{a_0, a}$  and the bound [\(7.2.5\)](#), we deduce that

$$\begin{aligned} & \left| \partial^{w_j} \mathcal{L}_{l_j, n_j+1}^{a_0, a} \partial^{w_3} \partial_{1/a} C^{a_0, a}(\hat{p}) \partial^{w_i} \mathcal{D}_{l_i, n_i+1}^{a_0, a} \right| \\ & \leq a_0 \left( \frac{1}{a} + m \right)^{4-n-|w|} \mathcal{P}_1 \left( \log \frac{1+a_0m}{a_0m} \right) \mathcal{P}_2 \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right). \end{aligned}$$

Combining all the previous estimates of each term of the r.h.s. of the FE (7.3.9), we obtain

$$\left| \partial_{1/a} \partial^w \mathcal{D}_{l,n}^{a_0,a}(p_1, \dots, p_n) \right| \leq a_0 \left( \frac{1}{a} + m \right)^{4-n-|w|} \mathcal{P}_1 \left( \log \frac{1+a_0m}{a_0m} \right) \mathcal{P}_2 \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right). \quad (7.3.17)$$

In particular, we have

$$\left| \partial_{1/a} \partial^w \mathcal{D}_{l,2}^{a_0,a}(0,0) \right| \leq a_0 \left( \frac{1}{a} + m \right)^{2-|w|} e^{-\frac{a^2m^2}{2}} \mathcal{P}_1 \left( \log \frac{1+a_0m}{a_0m} \right). \quad (7.3.18)$$

- After these preparation steps, we integrate the flow equations (7.3.9):

C1) For the irrelevant terms, because of the boundary conditions

$$\begin{aligned} \partial^w \mathcal{D}_{l,n}^{a_0,a_0}(p_1, \dots, p_n) &= 0, \quad \forall n + |w| \geq 5 \ (n \neq 2) \\ \partial^w \mathcal{D}_{l,2}^{a_0,a_0}(p, -p) &= b_l^O(a_0) \partial^w (\hat{p}^O)^2 - b_l(a_0) \partial^w \hat{p}^2, \quad \forall |w| \geq 3, \end{aligned}$$

we integrate from  $1/a_0$  to  $1/a$ . We exclude for the moment  $(n, |w|) \in \{(4, 1); (2, 3)\}$  which have to be treated as relevant in this case.

$\forall n + |w| > 5$ , such that  $n \neq 2$  we have

$$\begin{aligned} \left| \partial^w \mathcal{D}_{l,n}^{a_0,a}(p_1, \dots, p_n) \right| &\leq \int_{1/a}^{1/a_0} d\lambda \left| \partial_\lambda \partial^w \mathcal{D}_{l,n}^{a_0, \frac{1}{\lambda}}(p_1, \dots, p_n) \right| \\ &\leq a_0 \mathcal{P}_1 \left( \log \frac{1+a_0m}{a_0m} \right) \mathcal{P}_2 \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right) \int_{1/a}^{1/a_0} d\lambda (\lambda + m)^{5-n-|w|-1} \\ &\leq a_0 \left( \frac{1}{a} + m \right)^{5-n-|w|} \mathcal{P}_1 \left( \log \frac{1+a_0m}{a_0m} \right) \mathcal{P}_2 \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right). \end{aligned} \quad (7.3.19)$$

For  $n = 2$  and  $|w| \geq 4$ , the boundary conditions are not equal to zero. Therefore,

$$\left| \partial^w \mathcal{D}_{l,2}^{a_0,a}(p, -p) \right| \leq \int_{1/a}^{1/a_0} d\lambda \left| \partial_\lambda \partial^w \mathcal{D}_{l,2}^{a_0, \frac{1}{\lambda}}(p, -p) \right| + \left| \partial^w \mathcal{D}_{l,2}^{a_0,a_0}(p, -p) \right|.$$

We recall that

$$\partial^w \mathcal{D}_{l,2}^{a_0,a_0}(p, -p) = b_l^O(a_0) \partial^w ((\hat{p}^O)^2 - \hat{p}^2) - \partial_{p^2} \mathcal{D}_{l,2}^{a_0,a_0}(0,0) \partial^w (\hat{p})^2$$

with

$$\partial_{p^2} \mathcal{D}_{l,2}^{a_0,a_0}(0,0) := b_l(a_0) - b_l^O(a_0).$$

Integrating (7.3.18) from  $+\infty$  to  $1/a$  and using the boundary conditions (7.2.17)-(7.3.6), we deduce that

$$\left| \partial^w \mathcal{D}_{l,2}^{a_0,a}(0,0) \right| \leq a_0 \left( \frac{1}{a} + m \right)^{3-|w|} e^{-\frac{a^2m^2}{2}} \mathcal{P}_1 \left( \log \frac{1+a_0m}{a_0m} \right). \quad (7.3.20)$$

Remembering that  $a_0 \leq a$  we have

$$\left| \partial_{p^2} \mathcal{D}_{l,2}^{a_0,a_0}(0,0) \right| \leq (1+am) e^{-\frac{a^2 m^2}{2}} \mathcal{P}_1 \left( \log \frac{1+a_0 m}{a_0 m} \right). \quad (7.3.21)$$

Using (7.1.2), we deduce

$$|\partial^w(\hat{p}^2)| \leq O(1) a_0^{|w|-2}, \quad (7.3.22)$$

with  $O(1)$  a positive constant independent of  $p$  and  $a_0$ . Hence, combining (7.3.21) with (7.3.22) and using the bound (7.2.14) together with  $a_0 \leq a$ , we deduce for  $|w| \geq 3$

$$\left| \partial_{p^2} \mathcal{D}_{l,2}^{a_0,a_0}(0,0) \partial^w(\hat{p}^2) \right| \leq a_0 \left( \frac{1}{a} + m \right)^{3-|w|} \mathcal{P} \left( \log \frac{1+a_0 m}{a_0 m} \right). \quad (7.3.23)$$

From the bound (7.3.7), we obtain

$$\partial_{p^2} \mathcal{L}_{l,2}^{a_0,a_0,O}(0,0) = 2b_l^O(a_0) \leq \mathcal{P} \left( \log \frac{1+a_0 m}{a_0 m} \right). \quad (7.3.24)$$

Using (7.2.11) and (7.2.12) together with (7.3.19), (7.3.23) and (7.3.24) we obtain

$$\left| \partial^w \mathcal{D}_{l,2}^{a_0,a}(p,-p) \right| \leq a_0 \left( \frac{1}{a} + m \right)^{3-|w|} \mathcal{P}_1 \left( \log \frac{1+a_0 m}{a_0 m} \right) \mathcal{P}_2 \left( \frac{a|p|}{1+am} \right). \quad (7.3.25)$$

C2) For the cases  $n + |w| \leq 5$ , the claim (7.3.10) has to be deduced from the respective integrated flow equation (7.3.9) at the renormalization point followed by an extension to general momenta using the Taylor Formula (??) for  $\mathcal{D}_{l,2}^{a_0,a}$ . We proceed in the order of the induction starting with the cases  $(n = 2, |w| = 3)$ ,  $(n = 2, |w| = 2)$  and going down in  $|w|$ . The integral in

$$\partial^w \mathcal{D}_{l,n}^{a_0,a}(0, \dots, 0) = \partial^w \mathcal{D}_{l,n}^{a_0,\infty}(0, \dots, 0) + \int_0^{1/a} d\lambda \partial^w \mathcal{D}_{l,n}^{a_0,1/\lambda}(0, \dots, 0) \quad (7.3.26)$$

is bounded using (7.3.17) at vanishing momenta:

$$\begin{aligned} \left| \int_0^{1/a} d\lambda \partial^w \mathcal{D}_{l,n}^{a_0,1/\lambda}(0, \dots, 0) \right| &\leq a_0 \int_0^{1/a} d\lambda (\lambda + m)^{5-n-|w|-1} \mathcal{P} \left( \log \frac{1+a_0 m}{a_0 m} \right) \\ &\leq a_0 \left( \frac{1}{a} + m \right)^{5-n-|w|} \mathcal{P} \left( \log \frac{1+a_0 m}{a_0 m} \right). \end{aligned}$$

Hence, the assertion is established at the renormalization point. In each case extension to general momenta is guaranteed by bounds established before. This concludes the proof of Theorem 14.  $\square$

## 7.4 Translation invariance

### 7.4.1 Some properties of the Schwartz space

We recall the definition of the Schwartz space

$$\mathcal{S}(\mathbb{R}^{4n}) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^{4n}) \mid \forall (\alpha, \beta) \in \mathbb{N}^{4n} \times \mathbb{N}^{4n}, \sup_{x \in \mathbb{R}^{4n}} |x^\alpha D^\beta f(x)| < +\infty \right\}$$

The Schwartz space is a Fréchet space endowed with a topology induced by the filtrant family of semi-norms

$$\mathcal{N}_p(\cdot) = \sum_{|\alpha|, |\beta| \leq p} \|\cdot\|_{\alpha, \beta}, \quad p \in \mathbb{N}$$

where

$$\|f\|_{\alpha, \beta} := \sup_{x \in \mathbb{R}^{4n}} |x^\alpha D^\beta f(x)|$$

**Lemma 15.** *Let  $f \in \mathcal{S}(\mathbb{R}^{4n})$  and  $\mathcal{P}_r$  a polynomial of degree  $r$ , we have the following bound*

$$|\mathcal{P}_r(x_1, \dots, x_n) f(x_1, \dots, x_n)| \leq \left( \prod_{i=1}^n \frac{1}{(1 + |x_i|)^s} \right) \mathcal{N}_{s+r}(f), \quad \forall s \in \mathbb{N}$$

The proof of Lemma 15 which we do not reproduce here uses the definition of Schwartz functions and will be useful in the sequel. For more details about the properties of Schwartz space and tempered distributions, we refer the reader to [48].

### 7.4.2 Translation invariance

The lattice breaks Euclidean translation invariance. In this section, we prove that the continuum limit restores translation invariance.

The regularized (CAS)  $n$ -point functions in position space are tempered distributions that we define through their Fourier transform, that is for  $f \in \mathcal{S}(\mathbb{R}^{4n})$

$$\langle \mathcal{L}_{l,n,\Lambda_{a_0}}^{a_0,a}, f \rangle_{\mathcal{S}', \mathcal{S}} := \int_{\mathcal{B}_{a_0}^n} \frac{d^4 p_1 \cdots d^4 p_n}{(2\pi)^{4n}} \mathcal{L}_{l,n}^{a_0,a}(p_1, \dots, p_n) \delta_{\left[\frac{2\pi}{a_0}\right]}^{(4)}(p_1 + \cdots + p_n) \mathcal{F}^{-1}(f)(p_1, \dots, p_n),$$

where

$$\delta_{\left[\frac{2\pi}{a_0}\right]}^{(4)}(p_1 + \cdots + p_n) := \sum_{k \in \mathbb{Z}^4} \delta^{(4)}\left(p_1 + \cdots + p_n - \frac{2k\pi}{a_0}\right)$$

accounts for the invariance of  $\mathcal{L}_{l,n}^{a_0,a}$  under lattice translations and  $\mathcal{F}^{-1}(f)$  is the inverse Fourier transform of  $f$ .  $\mathcal{L}_{l,n,\Lambda_{a_0}}^{a_0,a}$  is well defined as a tempered distribution since

$$\mathcal{L}_{l,n}^{a_0,a}(p_1, \dots, p_n) \delta_{\left[\frac{2\pi}{a_0}\right]}^{(4)}(p_1 + \cdots + p_n)$$



is a  $\frac{2\pi}{a_0}$ -periodic distribution [48].

Similarly, we define the renormalized (CAS)  $n$ -point functions in position space

$$\langle \mathcal{L}_{l,n,x}^{0,\infty}, f \rangle_{S',S} := \int_{\mathbb{R}^{4n}} \frac{d^4 p_1 \cdots d^4 p_n}{(2\pi)^{4n}} \mathcal{L}_{l,n}^{0,\infty}(p_1, \dots, p_n) \delta^{(4)}(p_1 + \cdots + p_n) \mathcal{F}^{-1}(f)(p_1, \dots, p_n)$$

$\mathcal{L}_{l,n,x}^{0,\infty}$  denotes the continuum limit position space (CAS)  $n$ -point function. It is a tempered distribution for which the translation by a vector  $c \in \mathbb{R}^4$  is defined as

$$\langle \tau_c \mathcal{L}_{l,n,x}^{0,\infty}, f \rangle_{S',S} := \langle \mathcal{L}_{l,n,x}^{0,\infty}, \tau_{-c} f \rangle_{S',S}, \quad \forall f \in \mathcal{S}(\mathbb{R}^{4n})$$

and

$$(\tau_{-c} f)(p_1, \dots, p_n) := f(p_1 + c, \dots, p_n + c)$$

Therefore,

$$\begin{aligned} & \langle \tau_c \mathcal{L}_{l,n,x}^{0,\infty}, f \rangle_{S',S} \\ &= \int_{\mathbb{R}^{4n}} \prod_{i=1}^n \frac{d^4 p_i}{(2\pi)^{4n}} \mathcal{L}_{l,n}^{0,\infty}(p_1, \dots, p_n) \delta^{(4)}(p_1 + \cdots + p_n) e^{-i(p_1 + \cdots + p_n) \cdot c} \mathcal{F}^{-1}(f)(p_1, \dots, p_n) \end{aligned}$$

which implies

$$\langle \tau_c \mathcal{L}_{l,n,x}^{0,\infty}, f \rangle_{S',S} = \langle \mathcal{L}_{l,n,x}^{0,\infty}, f \rangle_{S',S}, \quad \forall f \in \mathcal{S}(\mathbb{R}^{4n}).$$

The continuum limit of the UV cutoff regularized theory is clearly invariant under translations. Thus, proving the translation invariance of the continuum limit of the lattice regularized theory amounts to establishing the following convergence result

**Theorem 15.** *Let  $f \in \mathcal{S}(\mathbb{R}^{4n})$ ,*

$$\langle \mathcal{L}_{l,n,\Lambda_{a_0}}^{a_0,a}, f \rangle_{S',S} \longrightarrow \langle \mathcal{L}_{l,n,x}^{0,\infty}, f \rangle_{S',S} \quad \text{for } a_0 \rightarrow 0, a \rightarrow \infty \quad (7.4.1)$$

The proof of Theorem [15] relies on the following lemma

**Lemma 16.** *Let  $f \in \mathcal{S}(\mathbb{R}^{4n})$ ,*

$$\langle \delta_{\left[\frac{2\pi}{a_0}\right]}^{(4)}(p_1 + \cdots + p_n), f \rangle_{S',S} \longrightarrow \langle \delta^{(4)}(p_1 + \cdots + p_n), f \rangle_{S',S} \quad \text{when } a_0 \rightarrow 0 \quad (7.4.2)$$

### 7.4.3 Proof of lemma [16]

*Proof.* Let  $f \in \mathcal{S}(\mathbb{R}^{4n})$ , using lemma [15], one can verify that

$$\left| \langle \delta^{(4)}(p_1 + \cdots + p_n), f \rangle_{S',S} \right| \leq C \mathcal{N}_{5(n-1)}(f)$$

which proves that  $\delta^4(p_1 + \cdots + p_n)$  is indeed a tempered distribution. We have that

$$\begin{aligned} \langle \delta_{\left[\frac{2\pi}{a_0}\right]}^{(4)}, f \rangle_{\mathcal{S}', \mathcal{S}} &= \int_{\mathbb{R}^{4n}} d^4 p_1 \cdots d^4 p_n \delta_{\left[\frac{2\pi}{a_0}\right]}^{(4)}(p_1 + \cdots + p_n) f(p_1 + \cdots + p_n) \\ &= \int_{\mathbb{R}^{4(n-1)}} d^4 p_2 \cdots d^4 p_n \sum_{k \in \mathbb{Z}^4} f\left(\frac{2k\pi}{a_0} - \sum_{i=2}^n p_i + \cdots + p_n\right). \end{aligned}$$

We write

$$\sum_{k \in \mathbb{Z}^4} f\left(\frac{2k\pi}{a_0} - \sum_{i=2}^n p_i, \cdots, p_n\right) = f\left(-\sum_{i=2}^n p_i, \cdots, p_n\right) + \sum_{k \in \mathbb{Z}^{4,*}} f\left(\frac{2k\pi}{a_0} - \sum_{i=2}^n p_i, \cdots, p_n\right).$$

Since  $f \in \mathcal{S}(\mathbb{R}^{4n})$ , we have the following bound for any  $k \in \mathbb{Z}^{4,*}$ ,

$$\left| f\left(\frac{2k\pi}{a_0} - \sum_{i=2}^n p_i, p_2, \cdots, p_n\right) \right| \leq \frac{1}{\left(\sum_{i=2}^n |p_i|^2 + \left|\frac{2k\pi}{a_0} - \sum_{i=2}^n p_i\right|^2\right)^4} \prod_{i=2}^n \frac{1}{(1 + |p_i|)^5} N_s(f),$$

with  $s = 5(n-1) + 8$  and

$$N_s(f) = \sup_{p_i \in \mathbb{R}^4} \sup_{|\alpha| \leq 13n} |p_1|^{\alpha_1} \cdots |p_n|^{\alpha_n} |f(p_1 + \cdots + p_n)|.$$

Using

$$\frac{1}{|a|^2 + |b|^2} \leq \frac{2}{|a+b|^2}, \quad \forall a, b \in \mathbb{R}^{p,*},$$

we obtain

$$\left| f\left(\frac{2k\pi}{a_0} - \sum_{i=2}^n p_i, p_2, \cdots, p_n\right) \right| \leq C \left(\frac{a_0}{|k|}\right)^8 \prod_{i=2}^n \frac{1}{(1 + |p_i|)^5} N_s(f).$$

Since

$$\sum_{k \in \mathbb{Z}^{4,*}} \frac{1}{|k|^8} \leq \sum_{k_i \in \mathbb{Z}^*} \prod_{i=1}^4 \frac{1}{|k_i|^2} = \left(\frac{\pi^2}{3}\right)^4 < +\infty,$$

we deduce

$$\begin{aligned} \langle \delta_{\left[\frac{2\pi}{a_0}\right]}^{(4)}, f \rangle_{\mathcal{S}', \mathcal{S}} &= \int_{\mathbb{R}^{4(n-1)}} d^4 p_1 \cdots d^4 p_n f\left(-\sum_{i=2}^n p_i + \cdots + p_n\right) + C a_0^8 N_s(f) \\ &= \langle \delta^{(4)}(p_1 + \cdots + p_n), f \rangle_{\mathcal{S}', \mathcal{S}} + C a_0^8 N_s(f) \end{aligned}$$

together with the useful bound

$$\left| \langle \delta_{\left[\frac{2\pi}{a_0}\right]}^{(4)}, f \rangle_{\mathcal{S}', \mathcal{S}} \right| \leq C(1 + a_0^8) N_s(f). \quad (7.4.3)$$

This establishes that for  $a_0 \rightarrow 0$  we have

$$\langle \delta_{\left[\frac{2\pi}{a_0}\right]}^{(4)}(p_1 + \cdots + p_n), f \rangle_{\mathcal{S}', \mathcal{S}} \rightarrow_{a_0 \rightarrow 0} \langle \delta^{(4)}(p_1 + \cdots + p_n), f \rangle_{\mathcal{S}', \mathcal{S}}.$$

□

### 7.4.4 Proof of Theorem 15

*Proof.* We recall the boundedness inequality (7.2.18) for the (CAS)  $n$ -point functions. For all  $(p_i)_{1 \leq i \leq n} \in \mathcal{B}_{a_0}$  such that  $\sum_{i=1}^n p_i \equiv 0 \left[ \frac{2\pi}{a_0} \right]$ , we have

$$\left| \partial^w \mathcal{L}_{l,n}^{a_0,a}(p_1, \dots, p_n) \right| \leq \left( \frac{1}{a} + m \right)^{4-n-|w|} \mathcal{P}_1 \left( \log \frac{1+am}{am} \right) \mathcal{P}_2 \left( \left\{ \frac{a|p_i|}{1+am} \right\} \right)$$

This proves that  $\mathcal{L}_{l,n}^{a_0,a}$  are  $\mathcal{C}^\infty$  w.r.t. to the momenta and are at most of polynomial growth. Therefore,

$$\forall f \in \mathcal{S}(\mathbb{R}^{4n}), \quad \mathcal{L}_{l,n}^{a_0,a}(p_1, \dots, p_n) f \in \mathcal{S}(\mathbb{R}^{4n})$$

Taking the limit in the boundedness inequality (7.2.18), the same reasoning applies to  $\mathcal{L}_{l,n}^{0,\infty}(p_1, \dots, p_n)$  to prove that

$$\forall f \in \mathcal{S}(\mathbb{R}^{4n}), \quad \mathcal{L}_{l,n}^{0,\infty}(p_1, \dots, p_n) f \in \mathcal{S}(\mathbb{R}^{4n}).$$

We write

$$\begin{aligned} g_{a_0,a}(p_1, \dots, p_n) &:= \mathcal{L}_{l,n}^{a_0,a}(p_1, \dots, p_n) f(p_1, \dots, p_n), \\ g(p_1, \dots, p_n) &:= \mathcal{L}_{l,n}^{0,\infty}(p_1, \dots, p_n) f(p_1, \dots, p_n). \end{aligned}$$

Using (7.4.3), we obtain

$$\left| \left\langle \delta_{\left[ \frac{2\pi}{a_0} \right]}^{(4)}, g_{a_0,a} - g \right\rangle \right| \leq C(1 + a_0^8) N_s(g_{a_0,a} - g).$$

Taking the limit  $\hat{a}_0 \rightarrow 0$  in (7.2.21) we find

$$\left| \mathcal{L}_{l,n}^{a_0,\infty}(p_1, \dots, p_n) - \mathcal{L}_{l,n}^{0,\infty}(p_1, \dots, p_n) \right| \leq a_0 m^{5-n} \left( \log \frac{1}{a_0 m} \right)^\nu \mathcal{P} \left( \left\{ \frac{|p_i|}{m} \right\} \right),$$

where  $\nu$  is the same constant as in (7.2.21). Therefore, for any polynomial  $\mathcal{Q}$  with nonnegative coefficients we obtain

$$\begin{aligned} \left| \mathcal{Q}(\{|p_i|\}) \left( \mathcal{L}_{l,n}^{a_0,\infty}(p_1, \dots, p_n) - \mathcal{L}_{l,n}^{0,\infty}(p_1, \dots, p_n) \right) f \right| \\ \leq a_0 m^{5-n} \left( \log \frac{1}{a_0 m} \right)^\nu \tilde{\mathcal{P}} \left( \left\{ \frac{|p_i|}{m} \right\} \right) |f(p_1, \dots, p_n)|. \end{aligned}$$

Thus,

$$N_s(g_{a_0,a} - g) \leq a_0 m^{5-n} \left( \log \frac{1}{a_0 m} \right)^\nu N_r(f)$$

which implies

$$\left\langle \delta_{\left[ \frac{2\pi}{a_0} \right]}^{(4)}, g_{a_0,a} - g \right\rangle \rightarrow_{a_0 \rightarrow 0, a \rightarrow \infty} 0$$

Lemma 16 gives that

$$\langle \delta_{\left[\frac{2\pi}{a_0}\right]}^{(4)} - \delta^{(4)}, g \rangle \rightarrow_{a_0 \rightarrow 0, a \rightarrow \infty} 0,$$

so that

$$\langle \delta_{\left[\frac{2\pi}{a_0}\right]}^{(4)}, g_{a_0, a} \rangle \rightarrow_{a_0 \rightarrow 0, a \rightarrow \infty} \langle \delta^{(4)}, g \rangle.$$

Hence, for all  $f \in \mathcal{S}(\mathbb{R}^{4n})$  we have

$$\langle \mathcal{L}_{l, n, \Lambda_{a_0}}^{a_0, a}, f \rangle_{\mathcal{S}', \mathcal{S}} \longrightarrow \langle \mathcal{L}_{l, n, x}^{0, \infty}, f \rangle_{\mathcal{S}', \mathcal{S}} \quad \text{for } a_0 \rightarrow 0, a \rightarrow \infty.$$

□

## Concluding remarks

We have presented an alternative proof of the perturbative renormalizability of massive lattice regularized  $\phi_4^4$ -theory. The starting point were the bounds (7.2.18)-(7.2.20) which prove the existence of the continuum limit. In the flow equation formalism, they serve at the same time as induction hypotheses for the inductive proof. Bounds of this sort have been established rigorously for all theories of physical interest, including gauge theories [9]. In this context it is also interesting to study the difference

$$\mathcal{L}_{l, n, \Lambda_0}^{a_0, a} - \mathcal{L}_{l, n, a_0}^{a_0, a}$$

where  $\mathcal{L}_{l, n, \Lambda_0}^{a_0, a}$  denotes the momentum space regularized correlation functions and  $\mathcal{L}_{l, n, a_0}^{a_0, a}$  denotes the lattice regularized correlation functions. The UV-cutoff can be related to the lattice parameter by  $\Lambda_0 = 1/a_0$ , similarly for the corresponding flowing parameters  $\Lambda = 1/a$ . The study of this difference by flow equations should allow to prove that in the limit  $a_0 \rightarrow 0$  and  $a \rightarrow \infty$ , the difference vanishes, implying consistency, that is the two regularization schemes converge to the same limit. This would be an alternative way to prove that the continuum limit of the lattice regularization is given by the  $O(4)$ -symmetric correlation functions. The approach presented in this chapter could also be generalized to massless lattice regularized theories [49]. In this case the appearing infrared singularities have to be controlled in a similar way as it has been done for theories with momentum cutoff regularization [9]. A particularly interesting subject is the extension to gauge theories since the lattice regularization respects a priori gauge invariance. It seems however that analyzing the flow equations still requires a gauge fixing procedure. In any case the important issue is to prove that the continuum limit respects the continuum Ward identities for suitable renormalization conditions.

# Conclusion and future perspectives

In this thesis, we studied the renormalization problem of theories with no translation invariance in the context of three models: the semi-infinite model, the  $\phi_4^4$  theory with an interaction supported on the half-space and the lattice regularized  $\phi_4^4$  theory. First, we considered the renormalization problem of the semi-infinite (massive) scalar field model as well as the properties of the counter-terms that appear in the effective action associated to this theory. We started with a general approach to this problem by imposing constant renormalization conditions (more precisely BPHZ renormalization conditions) which implies that the theory is renormalizable by adding five position dependent counter-terms to the effective action given by

$$L^{\Lambda_0, \Lambda_0}(\phi) = \frac{\lambda}{4!} \int_V \phi^4(z, x) + \frac{1}{2} \int_V \left( a^{\Lambda_0}(z) \phi^2(z, x) - b^{\Lambda_0}(z) \phi(z, x) \Delta_x \phi(z, x) \right. \\ \left. - d^{\Lambda_0}(z) \phi(z, x) \partial_z^2 \phi(z, x) + s^{\Lambda_0}(z) \phi(z, x) (\partial_z \phi)(z, x) + \frac{2}{4!} c^{\Lambda_0}(z) \phi^4(z, x) \right).$$

Then, we established that there exists a particular choice of the renormalization conditions for which the counter-terms are independent of the position in space and moreover are proportional to terms, which have the same structure as those appearing in the original Hamiltonian. We proceeded by constructing a solution of the flow equation of the semi-infinite model in two steps:

- The propagator of the semi-infinite model is a sum of two contributions: the usual translationally invariant  $\phi_4^4$  model propagator and a "surface" part which is singular only if at least one of the arguments of the propagator approaches the surface. The key idea of the method presented in this thesis is to analyze the correlation distributions generated by each of these two parts. We considered the "surface" correlation distributions and established that these are bounded uniformly with respect to the UV cutoff. For a suitable choice of renormalization conditions, these correlation distributions are renormalized by adding the following two surface counter-terms

$$s_\star^{\Lambda_0} \int_{\mathbb{R}^3} d^3x \phi^2(0, x) + 2e_\star^{\Lambda_0} \int_{\mathbb{R}^3} d^3x \phi(0, x) (\partial_n \phi)(0, x), \quad \star \in \{R, N\}.$$

For the Dirichlet case, no surface counter-term is needed. The proof of this statement is based on the a priori knowledge of bounds on the bulk correlation distributions smeared with an ap-

appropriate set of test functions. The bounds that were used initially are obtained by considering BPHZ renormalization conditions, which again implies position dependent counter-terms.

- The second step consisted in exploring in more detail the structure of the counter-terms that renormalize the bulk correlation distributions. One would expect from a naive approach to this problem that these correlation distributions are renormalized by the usual counter-terms of the translationally invariant  $\phi_4^4$  theory as they share the same free theory. The only difference stems from the fact that the bulk correlation distributions correspond to a scalar field theory in  $\mathbb{R}^4$  with an interaction supported on the half-space  $\mathbb{R}^+ \times \mathbb{R}^3$ , which is another manifestation of the breaking of translation invariance. For this model, we proceeded in chapter [6](#) by using our knowledge on the renormalizability of the translationally invariant  $\phi_4^4$  theory. The key idea is to construct a solution to the bulk flow equation of the following form

$$\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)) := \prod_{i=1}^n \chi^+(z_i) \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)) + \hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)), \quad (.0.4)$$

where  $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$  are the translationally invariant  $\phi_4^4$  theory correlation distributions and  $\hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}$  is a difference term that it is renormalized by

$$s^{\Lambda_0} \int_{\mathbb{R}^3} d^3x \phi^2(0, x) + 2e^{\Lambda_0} \int_{\mathbb{R}^3} d^3x \phi(0, x) (\partial_n \phi)(0, x).$$

The bound on  $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$  is compatible with the inductive bound on  $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ .

As a conclusion to all the steps summarized above, the  $n$ -point correlation distributions

$$\prod_{i=1}^n \chi^+(z_i) \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)) + \hat{\mathcal{D}}_{l,n}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)) + \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n))$$

is the solution of the semi-infinite flow equation that imply the following bare interaction

$$L_{\star}^{\Lambda_0,\Lambda_0}(\phi) = \frac{\lambda}{4!} \int_V \phi^4(z, x) + \frac{1}{2} \int_V \left( a_{\infty}^{\Lambda_0} \phi^2(z, x) - b_{\infty}^{\Lambda_0} \phi(z, x) \Delta \phi(z, x) + \frac{2}{4!} c_{\infty}^{\Lambda_0} \phi^4(z, x) \right) \\ + \frac{1}{2} (s^{\Lambda_0} + s_{\star}^{\Lambda_0}) \int_{\mathbb{R}^3} d^3x \phi^2(0, x) + (e^{\Lambda_0} + e_{\star}^{\Lambda_0}) \int_{\mathbb{R}^3} d^3x \phi \partial_n \phi, \quad \star \in \{R, N\}.$$

Now, let us discuss to which extent the method developed here can be used in the case of other boundary field theories. The tools we presented here can be adapted to prove the renormalizability of scalar field theories in more general geometries. Let us mention few examples:

- A film geometry in which the volume  $V$  of the system is bounded by two  $d - 1$  dimensional parallel plates of infinite extent which are separated by a distance  $L$ . The surface consists of two

disconnected pieces: the planes  $S_0 = \{(x, z) \in \mathbb{R}^d | z = 0\}$  and  $S_L = \{(x, z) \in \mathbb{R}^d | z = L\}$ . The volume is  $V := V_L = \{(x, z) \in \mathbb{R}^d | 0 \leq z \leq L\}$ . For  $d = 4$  and for Robin, Neumann and Dirichlet boundary conditions imposed on the plates  $S_0$  and  $S_L$ , the theory is expected to be renormalized by the usual bulk counter-terms, whereas the surface singularities can be absorbed by the counter-terms of the semi-infinite theory for each plate. For periodic boundary conditions, the volume  $V_L$  has the topology of  $S_1 \times \mathbb{R}^3$ , where  $S_1$  denotes the unit circle. The UV singularities of this theory can be renormalized by adding the translationally invariant  $\phi_4^4$  theory counter-terms only.

- A defect plane for which the volume is the entire space  $\mathbb{R}^d$ . The translational invariance is in this case broken by a defect plane, DP, of infinite extent in  $d - 1$  dimensions, rather than by a surface.
- Edges and curvature effects: The real systems are bounded by surfaces which are not necessarily planar. Furthermore, even if all surfaces are planar, the system will have edges and corners. The problem of the renormalization of such systems was approached for the first time by Cardy in 1983 [50]. To investigate the critical behaviour, Cardy considered a wedge-shaped geometry bounded by two semi-infinite plates meeting at an angle  $\theta$ . An edge contribution must be added to the Hamiltonian. In addition to the bulk and surface singularities, there exist some singularities which are localized at the edge. An interesting problem would be to study the renormalizability of such model to all orders of perturbation theory and for more general boundary conditions.
- Another problem concerns a rigorous study of the renormalization of the scalar field theories in curved spacetimes in the presence of boundaries. Symanzik [3] explored a simple version of this problem in 1981 by considering systems bounded by smooth  $d - 1$ -dimensional surfaces which are of the form  $z_S(x_S) = 0$ . He showed that besides the bulk and surface counter-terms necessary for absorbing the UV singularities in the case of a flat surface, additional curvature-dependent ones are required. Their form follows from a power-counting argument. In the case of the Dirichlet boundary condition, these curvature-dependent counter-terms do not contribute.

The common complexity between the problems mentioned above is the exact knowledge of the propagator. We believe that our method holds for establishing renormalizability in the previous cases as long as we know at least sharp bounds on the propagator(s).

Let us conclude with a broader issue, which concerns the use of the flow equations in exploring rigorously the critical behaviour of systems and establishing rigorous results with respect to their phase diagrams. A first step then would be the study of the Callan-Symanzik equation in the context of the Polchinski flow equation.





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**Titre :** Renormalisation perturbative des théories de champ brisant l'invariance par translation

**Mots clés :** équations de flot, espace semi-infini, réseau, renormalisation, propagateurs, contre-terme

**Résumé :** L'objectif de cette thèse est de comprendre comment la renormalisation est affectée par la brisure de la symétrie de translation à travers les deux exemples de la théorie scalaire dans  $\mathbb{R}^4$  régularisée par un réseau et celui de la théorie scalaire dans le demi-espace  $\mathbb{R}^+ \times \mathbb{R}^3$  euclidien appelée modèle semi-infini. Nous considérons en premier le modèle semi-infini en établissant une preuve rigoureuse de la renormalisation de cette théorie en nous basant sur les équations de flot. Dans un premier temps, nous établissons la renormalisation perturbative de cette théorie en fixant des conditions de renormalisation BPHZ. Cela a pour conséquence que les contre-terme sont des fonctions qui dépendent de la position dans le demi-espace. La deuxième partie de cette thèse est consacrée à une étude détaillée des

contre-terme. Nous établissons qu'il est possible de choisir ordre par ordre des conditions de renormalisation pour lesquelles les contre-terme sont des constantes. En outre, ces contre-terme sont donnés par ceux de la théorie invariante par translation et deux contre-terme surface proportionnels à  $\int_S \phi^2$  et  $\int_S \phi \partial_n \phi$  dans le cas de conditions aux bords du type Robin et Neumann. Pour Dirichlet, les contre-terme usuels de la théorie invariante par translation sont suffisants pour rendre la théorie finie. La dernière partie de cette thèse est dédiée à l'étude de la théorie scalaire massive  $\phi^4$  régularisée par réseau. Nous démontrons qu'elle est renormalisable, et que les symétries euclidiennes sont retablies dans la limite du continu.

**Title :** Perturbative renormalization of QFTs which are not translation invariant

**Keywords :** Flow equations, renormalization, semi-infinite model, lattice, counter-terms, propagators

**Abstract :** This thesis focuses on the study of the perturbative renormalization of theories that break translation invariance using the Polchinski flow equations. We study in particular the scalar field theory in  $\mathbb{R}^4$  regularized by a lattice and the scalar field theory in the semi-infinite geometry  $\mathbb{R}^+ \times \mathbb{R}^3$ . First, we prove the renormalizability of the semi-infinite model using BPHZ renormalization conditions in the mixed position-momentum space. However, this choice of renormalization conditions implies having position dependent counter-terms in the effective action. We go one step further in studying the renormalizability of the model, by investigating in more details the effect of the presence of the surface on the counter-terms

and we prove that for a particular choice of renormalization conditions, the theory is renormalized by the usual counter-terms of the translationally invariant theory and by surface counter-terms proportional to  $\int_S \phi^2$  and  $\int_S \phi \partial_n \phi$  with  $S$  denoting the surface of the half-space in the case of Neumann and Robin boundary conditions. For the Dirichlet case, we establish that the theory is renormalized using only the bulk counter-terms. We end the thesis by giving a rigorous proof of renormalizability of the massive scalar field  $\phi^4$  model regularized by a lattice. The novelty of this work lies in giving a proof of the restoration of the Euclidean symmetries in the continuum limit.