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Volterra Processes in Finance

Habilitation à Diriger des Recherches, le 5 Mars 2024

EDUARDO ABI JABER

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List of publications²³

21. Abi Jaber, E., & Gérard, L.A. *Signature volatility models: pricing and hedging with Fourier*.
20. Abi Jaber, E., & Li, S. *Volatility models in practice: Rough, Path-dependent or Markovian?*
19. Abi Jaber, E., Neuman, E. & Voss, M. *Equilibrium in Functional Stochastic Games with Mean-Field Interaction*, Submitted to [Annals of Applied Probability](#).
18. Abi Jaber, E., & De Carvalho, N. *Reconciling rough volatility with jumps*, Submitted to [SIAM Journal on Financial Mathematics](#).
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16. Abi Jaber, E. & Neuman, E. *Optimal Liquidation with Signals: the General Propagator Case*, Submitted to [Mathematical Finance](#). [Jupyter Notebook](#)
15. Abi Jaber, E. & Villeneuve, S. *Gaussian Agency problems with memory and Linear Contracts*, [Finance & Stochastics](#), to appear.
14. Abi Jaber, E., Illand, C., & Li, S. *The quintic Ornstein-Uhlenbeck volatility model that jointly calibrates SPX & VIX smiles*, [Risk Magazine, Cutting Edge section](#), 2023. [Jupyter Notebook](#)
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12. Abi Jaber, E., Miller, E., & Pham, H. *Markowitz portfolio selection for multivariate affine and quadratic Volterra models*, [SIAM journal on Financial Mathematics](#), 12(1), 369-409, 2021. [Jupyter Notebook](#)
11. Abi Jaber, E. *Weak existence and uniqueness for affine stochastic Volterra equations with L1-kernels*, [Bernoulli](#), 27(3), 1583-1615, 2021.
10. Abi Jaber, E. *The Laplace transform of the integrated Volterra Wishart process*, [Mathematical Finance](#), 32(1), 309-348, 2022.
9. Abi Jaber, E., Miller, E., & Pham, H. *Integral operator Riccati equations arising in stochastic Volterra control problems*, [SIAM journal on Control and Optimization](#), 59(2), 1581-1603, 2021.
8. Abi Jaber, E., Miller, E., & Pham, H. *Linear–Quadratic control for a class of stochastic Volterra equations: solvability and approximation*, [Annals of Applied Probability](#), 31(5), 2244-2274, 2021.
7. Abi Jaber, E., Cuchiero, C., Larsson, M., & Pulido, S. *A weak solution theory for stochastic Volterra equations of convolution type*, [Annals of Applied Probability](#), 31(6), 2924-2952, 2021.
6. Abi Jaber, E. *Lifting the Heston model*, [Quantitative Finance](#), 1-19, 2018.
5. Abi Jaber, E., & El Euch, O. *Markovian structure of the Volterra Heston model*, [Statistics & Probability Letters](#), Volume 149, Pages 63-72, 2019.
4. Abi Jaber, E., & El Euch, O. *Multifactor approximation of rough volatility models*, [SIAM Journal on Financial Mathematics](#), 10(2), 309-349, 2019.
3. Abi Jaber, E., Larsson, M., & Pulido, S. *Affine Volterra processes*, [Annals of Applied Probability](#), Volume 29, Number 5, 3155-3200, 2019.
2. Abi Jaber, E. *Stochastic invariance of closed sets for jump-diffusions with non-Lipschitz coefficients*, [Electronic Communications in Probability](#), 22, paper no. 53, 15 pp, 2017.
1. Abi Jaber, E., Bouchard, B., & Illand, C. *Stochastic invariance of closed sets with non-Lipschitz coefficients*, [Stochastic Processes and their Applications](#), Volume 129, Issue 5, Pages 1726-1748, 2019.

²Publications 1 to 6 were part of my PhD thesis and will not be discussed in the present manuscript.

³Underlined authors: PhD students I am supervising or I have collaborated with.

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Chapter 1

Introduction

This thesis presents a comprehensive summary of my research work conducted since my PhD defense in October 2018. It is structured into three inter-connected parts, each delving into specific aspects of my research findings:

- Stochastic Volterra equations: theory and applications,
- Stochastic control and games with Volterra processes,
- Joint SPX-VIX modeling.

Context

Over the past few decades, the world's financial markets have become increasingly complex, with a rise in the number and diversity of market actors as well as a range of financial products. Despite losses caused by successive financial crises, such as the 2008 crisis and the COVID-19 pandemic, this trend continues unabated. Technological advances and increased capacity for massive data storage have contributed to accelerating and developing the use of high-frequency algorithmic trading strategies. In this context, financial valuation, hedging, and risk management require more precise models and very fast calculation methods.

However, the increasing complexity of the models used makes their analysis more difficult and requires the development of suitable mathematical tools and new numerical methods. In this context, my research aims to develop more flexible models that allow for more realistic representations, capturing interdependencies and temporal persistence, while preserving mathematical tractability. My main strategy consists in identifying universal structures by embedding the problem in a “very large” or even infinite dimension, in order to recover tractability. In many cases, the problem becomes linear and/or simpler to solve in infinite dimensions. This phenomenon of the blessing of dimensionality appears in several types of problems, such as Volterra processes, mean-field games, Deep Learning, signatures... All of these topics are part of my recent and ongoing research. In a nutshell, I will highlight in this thesis the blessing of dimensionality and put it in use in different contexts.

Stochastic Volterra equations: theory and applications

Several phenomena in the fields of finance and economics exhibit memory characteristics and strong intertemporal dependence, which cannot be adequately modeled by stochastic processes such as Brownian motion and the Poisson process, both characterized by independent increments. Stochastic

Volterra processes are a class of processes that extend standard Brownian motion and Poisson processes to include memory; fractional Brownian motion and Hawkes processes are a special case.

In recent years, I specialized in the mathematical study, the numerical analysis and the practical implementation of these equations, which fall outside the Markovian and semi-martingale frameworks. Together with my co-authors we have developed new analytical and probabilistic techniques for studying existence, uniqueness, and the characterization of the law of such processes. In addition, we have identified several lifts in infinite dimension that allow to recover the Markovian and semi-martingale character. This in turns allows to clarify simpler structures in infinite dimension, obtain explicit formulas and develop suitable Markovian approximation schemes in a wide variety of practical problems in finance.

Chapter 2 provides general weak existence, uniqueness and stability results for stochastic convolution equations with jumps under mild regularity assumptions, allowing for non-Lipschitz coefficients and singular kernels. In Section 2.1, we identify a good sample path space for solutions to stochastic Volterra with jumps and we develop a weak solution and stability theory, for kernels in L^2_{loc} . In Section 2.2, under an affine specification, we show how such theory can be extended to the case of kernels belonging only to L^1_{loc} even in the presence of a Brownian component, and establish in addition weak uniqueness.

Chapter 3 illustrates the flexibility and tractability of quadratic Gaussian Volterra processes for multi-dimensional modeling of non-Markovian stochastic short rate and stochastic covariance models in finance. In Section 3.2, we provide an analytic expression for the characteristic function of the log-price in models with a certain Volterra quadratic structure for the stochastic variance process. This nests for instance the fractional Stein-Stein model.

Stochastic control and games with Volterra processes

Chapter 4 provides an exhaustive treatment of solvability and approximation of linear-quadratic control problems for a class of stochastic Volterra equations of convolution type, whose kernels are Laplace transforms of certain signed matrix measures which are not necessarily finite.

Chapter 5 explicitly solves two intricate control problems in finance under a Volterra framework:

- Markowitz portfolio allocation problem for multivariate quadratic Volterra models,
- Optimal liquidation with general transient impact and trading signals.

Both models considered in Chapter 5 involve non-convolution kernels, they fall outside the scope of the framework developed in Chapter 4. Yet, we are still able to solve both problems in terms of explicit operator solutions similar to the one that appear in Chapter 3 by making educated quadratic Ansatz on an adjusted forward process. The related operator Riccati equations that appear can be solved explicitly exploiting again the structure of the solution that has been identified earlier in Chapter 3 in an un-controlled case.

Chapter 6 pursues the study of controlled dynamics with Volterra processes but in the context of stochastic games:

- A Stackelberg game: a Principal-Agent problem à la Holmström and Milgrom but where the usual Brownian motion for modeling the revenues is replaced by a Gaussian Volterra process. What is striking is that we are able to explicitly solve the problem for this class and show that the optimal contract is still linear in the terminal value of the revenue, just like in Holmstrom-Milgrom framework, with a recommended deterministic effort for the agent.
- A class of finite-player and mean-field games that we solve in terms of explicit operator formulas, similar to the ones that appeared in Chapters 3 and 5. However, in contrast with these chapters, there is an important methodological difference, we are able to develop a general direct approach to solve the game directly, without having to guess any particular Ansatz and verify that is indeed the solution.

Joint SPX-VIX modeling

Chapter 7 introduces the first one factor Markovian stochastic volatility model that is able to jointly calibrate SPX and VIX smiles: the Quintic Ornstein-Uhlenbeck model. It has only 6 effective parameters and an input curve that allows to match certain term structures and it is remarkably tractable. More interestingly, we show that it outperforms its rough counterpart, by a clear margin, on all dates and market conditions, on more than 10 years of data, contrary to widespread beliefs.

Chapter 8 reconciles rough volatility with jumps models and provides an interpretation of negative H regimes. In Chapter 2, we saw how one could extend fractional processes based on Brownian motion to negative Hurst indices $H \in (-1/2, 1/2]$. One of the aims of Chapter 8 is to give an interpretation of the region $H \in (-\infty, -1/2]$ for which such processes can no longer be defined. In addition, Chapter 7 shows that calibrated values of the parameter H in the Quintic Ornstein-Uhlenbeck model are negative on average, which also motivates the theoretical understanding of negative H regimes.

Perspectives

Chapter 9 presents some perspectives and ongoing work.

Part I

Stochastic Volterra equations: theory and applications

Chapter 2

A weak solution theory for stochastic convolution equations with jumps

Summary

This chapter provides general weak existence, uniqueness and stability results for stochastic convolution equations with jumps under mild regularity assumptions, allowing for non-Lipschitz coefficients and singular kernels.

Based on:

- [1] Abi Jaber, E., Cuchiero, C., Larsson, M., & Pulido, S. *A weak solution theory for stochastic Volterra equations of convolution type*, [Annals of Applied Probability](#), 31(6), 2924-2952, 2021.
- [2] Abi Jaber, E. *Weak existence and uniqueness for affine stochastic Volterra equations with L1-kernels*, [Bernoulli](#), 27(3), 1583-1615, 2021.

A stochastic Volterra equation of convolution type is a stochastic equation of the form

$$X_t = g_0(t) + \int_{[0,t)} K(t-s)dZ_s, \quad (2.0.1)$$

where X is the d -dimensional process to be solved for, g_0 is a given function, K is a given $d \times k$ matrix-valued convolution kernel, and Z is a k -dimensional Itô semimartingale whose characteristics are given functions of X . The solution concept is described in detail below.

In particular, conditions are needed to ensure that the stochastic integral on the right-hand side of (2.0.1) is well-defined. Indeed, let us consider two simple cases with $d = k = 1$ and $g_0 \equiv 0$:

- Z is a Poisson process N with jump times $(T_i)_{i \in \mathbb{N}}$, then,

$$X_t = \int_{[0,t)} K(t-s)dN_s = \sum_{T_i < t} K(t-T_i).$$

It is clear in this case that X is well defined as long as the kernel $K \in L^1_{loc}$. But this means that K can be singular at 0, which shows that the sample paths of X reach ∞ at each jump time. Said differently, the sample paths of the process X do not belong to the usual càglàd (or càdlàg) spaces used for the study of standard stochastic differential equations with jumps.

- Z is a Brownian motion W , then, the stochastic convolution

$$X_t = \int_0^t K(t-s) dW_s,$$

is well-defined, as an Itô stochastic integral for each t , as long as $K \in L^2_{loc}$. Existence of a version of X with continuous sample paths can be obtained under additional assumptions on K .

In Section 2.1, we identify a good sample path space for solutions to (2.0.1) and we develop a weak solution and stability theory, for kernels in L^2_{loc} . In Section 2.2, under an affine specification of the characteristics of Z , we show how such theory can be extended to the case of kernels belonging only to L^1_{loc} even in the presence of a Brownian component, and establish in addition weak uniqueness.

2.1 Weak solution and stability theory [11]

Solutions of (2.0.1) are neither semimartingales nor Markov processes in general. Classically, they are constructed using Picard iteration, but only under Lipschitz or near-Lipschitz assumptions. Alternatively, one can use scaling limits of Hawkes-type processes to generate continuous solutions for well-chosen kernels and affine characteristics [131]. Yet another approach is to use projections of Markovian solutions to certain degenerate stochastic partial differential equations [6, 38, 68, 67, 151].

We will also use approximation by jumps, but not via scaling limits of Hawkes processes, nor infinite dimensional lifts. Instead we work with a priori L^p estimates for solutions of (2.0.1), combined with a novel ‘‘Volterra’’ martingale problem in \mathbb{R}^d that allows us to pass to weak limits in (2.0.1). In view of the irregular path behavior that occurs, in particular, in the presence of jumps, this identifies L^p spaces as a natural environment for the weak convergence analysis. With this approach we obtained in [11]:

- existence of weak solutions for singular kernels, non-Lipschitz coefficients and general jump behavior;
- strong existence and pathwise uniqueness under suitable Lipschitz conditions (but still singular kernels and jumps);
- convergence and stability theorems in the spirit of classical martingale problem theory, allowing for instance to study scaling limits of nonlinear Hawkes processes and to approximate stochastic Volterra processes by Markovian semimartingales;
- path regularity under certain additional conditions on the kernel and the characteristics.

Let us now describe the solution concept for (2.0.1). For $p \in [2, \infty)$ we denote by $L^p_{loc} = L^p_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ the space of locally p -integrable functions from \mathbb{R}_+ to \mathbb{R}^n , where the dimension n of the image space will depend on the context. Let $d, k \in \mathbb{N}$ and consider the following data:

1. an *initial condition* $g_0: \mathbb{R}_+ \rightarrow \mathbb{R}^d$ in L^p_{loc} ,
2. a *convolution kernel* $K: \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times k}$ in L^p_{loc} ,
3. a *characteristic triplet* (b, a, ν) of measurable maps $b: \mathbb{R}^d \rightarrow \mathbb{R}^k$ and $a: \mathbb{R}^d \rightarrow \mathbb{S}^k_+$ as well as a kernel $\nu(x, d\zeta)$ from \mathbb{R}^d into \mathbb{R}^k such that $\nu(x, \{0\}) = 0$ for all $x \in \mathbb{R}^d$ and, for some $c \in \mathbb{R}_+$,

$$|b(x)| + |a(x)| + \int_{\mathbb{R}^k} (1 \wedge |\zeta|^2) \nu(x, d\zeta) \leq c(1 + |x|^p), \quad x \in \mathbb{R}^d.$$

Given this data, we can now state the following key definition.

Definition 2.1. A weak L^p solution of (2.0.1) for the data (g_0, K, b, a, ν) is an \mathbb{R}^d -valued predictable process X , defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, that has trajectories in L^p_{loc} and satisfies

$$X_t = g_0(t) + \int_{[0,t)} K(t-s) dZ_s \quad \mathbb{P} \otimes dt\text{-a.e.}$$

for some \mathbb{R}^k -valued Itô semimartingale Z with $Z_0 = 0$ whose differential characteristics with respect to the Lebesgue measure (and with respect to some given truncation function) are $b(X)$, $a(X)$, $\nu(X, d\zeta)$. For convenience we often refer to the pair (X, Z) as a weak L^p solution.

Throughout, we assume $\int_{\mathbb{R}^k} |\zeta|^2 \nu(x, d\zeta) < \infty$ for all $x \in \mathbb{R}^d$ so we can use the “truncation function” $\chi(\zeta) = \zeta$. The characteristics of Z are therefore understood with respect to this function. We can now state our main result on existence of weak L^p solutions.

Theorem 2.2. Let $d, k \in \mathbb{N}$, $p \in [2, \infty)$, and consider data (g_0, K, b, a, ν) as in 1–3. Assume b , a , and $x \mapsto |\zeta|^2 \nu(x, d\zeta)$ are continuous. In addition, assume there exist a constant $\eta \in (0, 1)$, a locally bounded function $c_K: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and a constant c_{LG} such that

$$\int_0^T \frac{|K(t)|^p}{t^{\eta p}} dt + \int_0^T \int_0^T \frac{|K(t) - K(s)|^p}{|t-s|^{1+\eta p}} ds dt \leq c_K(T), \quad T \geq 0, \quad (2.1.1)$$

and

$$\begin{aligned} |b(x)|^2 + |a(x)| + \int_{\mathbb{R}^k} |\zeta|^2 \nu(x, d\zeta) \\ + \left(\int_{\mathbb{R}^k} |\zeta|^p \nu(x, d\zeta) \right)^{2/p} \leq c_{\text{LG}}(1 + |x|^2), \quad x \in \mathbb{R}^d. \end{aligned} \quad (2.1.2)$$

Then there is a weak L^p solution (X, Z) of (2.0.1) for the data (g_0, K, b, a, ν) .

Let us mention that several kernels of interest satisfy (2.1.1) such as the fractional kernel $K(t) = t^{\gamma-1}$ with $\gamma > \frac{1}{2}$ (which is singular when $\gamma < 1$) and locally Lipschitz kernels among others.

Let us give a brief overview of the proof of Theorem 2.2. The proof of Theorem 2.2 is based on approximation and weak convergence of laws on suitable function spaces. The semimartingale Z has trajectories in the Skorokhod space $D = D(\mathbb{R}_+, \mathbb{R}^k)$ of càdlàg functions. Weak convergence in D is a classical tool used, for example, to obtain weak solutions of stochastic differential equations with jumps (see, e.g., Ethier and Kurtz [81]). However, as explained in the introduction of this chapter, the trajectories of X need not be càdlàg, only locally p -integrable. Thus it is natural to regard X as a random element of the Polish space $L^p_{\text{loc}} = L^p_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^d)$. It is in this space—or rather, the product space $L^p_{\text{loc}} \times D$ —that our weak convergence analysis takes place.

Relative compactness in L^p is characterized by the Kolmogorov–Riesz–Fréchet theorem; see e.g. Brezis [45, Theorem 4.26]. A more convenient criterion in our context uses the Sobolev–Slobodeckij norms, defined for any measurable function $f: \mathbb{R}_+ \rightarrow \mathbb{R}^d$ by

$$\|f\|_{W^{\eta,p}(0,T)} = \left(\int_0^T |f(t)|^p dt + \int_0^T \int_0^T \frac{|f(t) - f(s)|^p}{|t-s|^{1+\eta p}} ds dt \right)^{1/p},$$

where $p \geq 1$, $\eta \in (0, 1)$, $T \geq 0$ are parameters. The relation between these norms and L^p spaces is somewhat analogous to the relation between Hölder norms and spaces of continuous functions. In particular, balls with respect to $\|\cdot\|_{W^{\eta,p}(0,T)}$ are relatively compact in $L^p(0, T)$; see e.g. Flandoli and Gatarek [84, Theorem 2.1]. The following a priori estimate clarifies the role of the conditions (2.1.1) and (2.1.2) in Theorem 2.2, and is the key tool that allows us to obtain convergent sequences of approximate L^p solutions.

Theorem 2.3. Let $d, k \in \mathbb{N}$, $p \in [2, \infty)$, and consider data (g_0, K, b, a, ν) as in 1–3. Assume there exists a constant c_{LG} such that (2.1.2) holds. Then any weak L^p solution X of (2.0.1) for the data

(g_0, K, b, a, ν) satisfies

$$\mathbb{E}[\|X\|_{L^p(0,T)}^p] \leq c,$$

where $c < \infty$ only depends on $d, k, p, c_{\text{LG}}, T, \|g_0\|_{L^p(0,T)}$, and, L^p -continuously, on $K|_{[0,T]}$. If in addition there exist a constant $\eta \in (0, 1)$ and a locally bounded function $c_K: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that (2.1.1) holds, then

$$\mathbb{E}[\|X - g_0\|_{W^{\eta,p}(0,T)}^p] \leq c, \quad (2.1.3)$$

where $c < \infty$ only depends on $d, k, p, \eta, c_K, c_{\text{LG}}, T$.

An immediate corollary is the following tightness result.

Corollary 2.4. Fix $d, k, p, \eta, c_K, c_{\text{LG}}$ as in Theorem 2.3, and let $G_0 \subset L^p_{\text{loc}}$ be relatively compact. Let \mathcal{X} be the set of all weak L^p solutions X of (2.0.1) as g_0 ranges through G_0 , K ranges through all kernels that satisfy (2.1.1) with the given η and c_K , and (b, a, ν) ranges through all characteristic triplets that satisfy (2.1.2) with the given c_{LG} . Then \mathcal{X} is tight, in the sense that the family $\{\text{Law}(X): X \in \mathcal{X}\}$ is tight in $\mathcal{P}(L^p_{\text{loc}})$.

Proof. Fix $T \in \mathbb{R}_+$ and let c be the constant in (2.1.3). For any $m > 0$, Markov's inequality gives

$$\sup_{X \in \mathcal{X}} \mathbb{P}(\|X - g_0\|_{W^{\eta,p}(0,T)} > m) \leq \frac{c}{m^p}.$$

The balls $\{f: \|f\|_{W^{\eta,p}(0,T)} \leq m\}$ are relatively compact in $L^p(0, T)$, so the above estimate implies that the family $\{(X - g_0)|_{[0,T]}: X \in \mathcal{X}\}$ is tight in $L^p(0, T)$. Since T was arbitrary, it follows that $\mathcal{X}_0 = \{X - g_0: X \in \mathcal{X}\}$ is tight in L^p_{loc} . Since G_0 is relatively compact, $G_0 + \mathcal{X}_0$ is tight as well, and it contains \mathcal{X} . Thus \mathcal{X} is tight. \square

The second main ingredient in the proof of Theorem 2.2 relies on a reformulation of (2.0.1) as a certain martingale problem which is equivalent to the weak L^p solutions of (2.0.1). This point of view is useful because it leads to the following stability result, which under appropriate conditions asserts that the weak limit of a sequence of solutions is again a solution. Recall that D denotes the Skorokhod space of càdlàg functions from \mathbb{R}_+ to \mathbb{R}^k .

Theorem 2.5. Let $d, k \in \mathbb{N}$, $p \in [2, \infty)$. For each $n \in \mathbb{N}$, let (X^n, Z^n) be a weak L^p solution of (2.0.1) given data $(g_0^n, K^n, b^n, a^n, \nu^n)$ as in 1–3. Assume the triplets (b^n, a^n, ν^n) all satisfy (2.1.2) with a common constant c_{LG} . Assume also, for some (g_0, K, b, a, ν) and limiting process (X, Z) , that

- $g_0^n \rightarrow g_0$ in L^p_{loc} ,
- $K^n \rightarrow K$ in L^p_{loc} ,
- $(b^n, a^n, \nu^n) \rightarrow (b, a, \nu)$ in the sense that $A^n f \rightarrow Af$ locally uniformly on $\mathbb{R}^d \times \mathbb{R}^k$ for every $f \in C_c^2(\mathbb{R}^k)$, where Af is defined in terms of the characteristic triplet by

$$\begin{aligned} Af(x, z) &= b(x)^\top \nabla f(z) + \frac{1}{2} \text{Tr}(a(x) \nabla^2 f(z)) \\ &\quad + \int_{\mathbb{R}^k} (f(z + \zeta) - f(z) - \zeta^\top \nabla f(z)) \nu(x, d\zeta), \end{aligned}$$

and $A^n f$ is defined analogously,

- $(X^n, Z^n) \Rightarrow (X, Z)$ in $L^p_{\text{loc}} \times D$.

Then (X, Z) is a weak L^p solution of (2.0.1) for the data (g_0, K, b, a, ν) .

It is important to appreciate that no pointwise convergence of characteristic triplets is required in Theorem 2.5. For example, it may happen that $a^n = 0$ for all n , but the limiting triplet has $a \neq 0$. This is because diffusion can be approximated by small jumps, and we indeed make use of this in a crucial manner.

Moreover, in the paper Abi Jaber et al. [11], we show that by combining the tightness and stability results with an approximation scheme for the characteristic triplet, we reduce the existence question to the pure jump case where Z is piecewise constant with bounded jump intensity. A solution X can then be constructed directly. In addition, pathwise uniqueness is established under suitable Lipschitz conditions. Finally, as applications, we show how our results can be applied to scaling limits of Hawkes processes and approximations of solutions of (2.0.1) by means of finite-dimensional systems of Markovian SDEs.

2.2 Existence and uniqueness for the L^1 -case under affine characteristics [2]

As mentioned in the introduction of this chapter, if the driving semimartingale Z in (2.0.1) includes a Brownian part, then, the kernel K needs to be at least in L^2_{loc} . Interestingly, under affine characteristics of Z in X , the formulation of the equation in (2.0.1) can be extended to the L^1_{loc} -setting. The idea is the following, assume for simplicity that $a(x) = x$ and both b and ν are 0 so that

$$X_t = g_0(t) + \int_0^t K(t-s)dZ_s$$

and Z has characteristics $(0, \int_0^\cdot X_s ds, 0)$ that are absolutely continuous with respect to the Lebesgue measure, so that the differential characteristics (with respect to dt) of Z are given by $(0, X, 0)$. Here, we can think of Z as $Z = \int_0^\cdot \sqrt{X_s} dW_s$. Then, by an application of stochastic Fubini's theorem, the dynamics of the integrated process $\bar{X} := \int_0^\cdot X_s ds$ are given by

$$\bar{X}_t = G_0(t) + \int_0^t K(t-s)Z_s ds,$$

with $G_0 = \int_0^\cdot g_0(s) ds$ and Z has characteristics $(0, \bar{X}, 0)$. The advantage of such formulation is that now the integral that appears is against the Lebesgue measure and makes sense for L^1_{loc} kernel. This motivates the study of stochastic Volterra equation with locally L^1 -kernels K in the form

$$\bar{X}_t = G_0(t) + \int_0^t K(t-s)Z_s ds, \quad t \geq 0, \quad (2.2.1)$$

for a given function $G_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$ where Z is a real-valued semimartingale, starting from zero, with affine characteristics in \bar{X}

$$(b\bar{X}, c\bar{X}, \nu(d\zeta)\bar{X}),$$

with $b \in \mathbb{R}$, $c \geq 0$, ν a nonnegative measure on \mathbb{R}_+ such that $\int_{\mathbb{R}_+} \zeta^2 \nu(d\zeta) < \infty$, with respect to the 'truncation function' $\chi(\zeta) = \zeta$. For L^2 -kernels this formulation agrees with (2.0.1), where Z is a semimartingale but whose characteristics are absolutely continuous with respect to the Lebesgue measure. In the L^1 setting, \bar{X} may fail to be absolutely continuous with respect to the Lebesgue measure. For this reason, the study falls beyond the scope of Section 2.1.

Our motivation for studying such convolution equations is twofold. Stochastic Volterra equations with kernels that are locally in L^1 but not in L^2 with $c > 0$ and $\nu \equiv 0$ arise as scaling limits of branching processes in population genetics, see Dawson and Fleischmann [76], Mytnik and Salisbury [151], and self-exciting Hawkes processes in mathematical finance, see Jusselin and Rosenbaum [133]. The L^1 -framework allows for instance to make sense of fractional dynamics, inspired by the fractional

Brownian motion, for negative Hurst indices $H \in (-1/2, 0)$. In both of the motivating cases, one can compute the Laplace transform of \bar{X} , modulo a deterministic Riccati–Volterra equation of the form

$$\psi(t) = \int_0^t K_H(t-s) \left(\frac{1}{2} \psi^2(s) - 1 \right) ds,$$

either by using the dual process of the *catalytic super-Brownian motion*, see Dawson and Fleischmann [76, Equations (4.2.1)-(4.2.2)], or by exploiting the affine structure of the approximating Hawkes processes, see Jusselin and Rosenbaum [133, Theorem 3.4]. Both constructions provide solutions to (2.2.1), but do not yield uniqueness. Establishing weak uniqueness is one of the main motivation of my work in [2].

In [2], we provide a generic treatment of the limiting macroscopic equation (2.2.1) and we allow for (infinite activity) jumps in Z . For instance, Hawkes processes can be recovered by setting $c = 0$ and $\nu = \delta_1$. The strategy we adopt is based on approximations using stochastic Volterra equations with L^2 kernels, whose existence and uniqueness theory has been established in [11, 10] and the references therein. By doing so, we avoid the infinite-dimensional analysis used for super-processes, we also circumvent the need to study scaling limits of Hawkes processes, allowing for more generality in the choice of kernels K and input functions G_0 . Along the way, we derive a general stability result that encompasses the motivating example with Hawkes processes. Most importantly, we establish weak uniqueness using a duality argument on the Fourier–Laplace transform of \bar{X} via a deterministic Riccati–Volterra integral equation. In particular, this expression extends the one obtained for affine Volterra processes with L^2 -kernels in Abi Jaber, Larsson, and Pulido [10], Cuchiero and Teichmann [68]. We also illustrate the applicability of our results on a class of hyper-rough Volterra Heston models with a Hurst index $H \in (-1/2, 1/2]$ and jumps complementing the results of Abi Jaber et al. [10], El Euch and Rosenbaum [80], Jusselin and Rosenbaum [133]. Such models have recently known a growing interest to account for rough volatility [99].

Modeling with Quadratic Gaussian Volterra processes

Summary

This chapter illustrates the flexibility and tractability of quadratic Gaussian Volterra processes for multi-dimensional modeling of non-Markovian stochastic short rate and stochastic covariance models.

Based on:

[3] Abi Jaber, E. *The Laplace transform of the integrated Volterra Wishart process*, [Mathematical Finance](#), 32(1), 309-348, 2022.

[4] Abi Jaber, E. *The characteristic function of Gaussian stochastic volatility models: an analytic expression*, [Finance & Stochastics](#), 26, 733-769, 2022.

In this chapter, we are essentially interested in studying models with a certain quadratic structure in Gaussian Volterra processes. In particular, the $d \times d$ Volterra Wishart process XX^\top where X is the $d \times m$ -matrix valued Volterra Gaussian process

$$X_t = g_0(t) + \int_0^t K(t, s) dW_s,$$

for some given input curve $g_0 : [0, T] \rightarrow \mathbb{R}^{d \times m}$, suitable kernel $K : [0, T]^2 \rightarrow \mathbb{R}^{d \times d}$ and $d \times m$ -matrix Brownian motion W , for a fixed time horizon $T > 0$.

The introduction of the kernel K allows for flexibility in financial modeling as illustrated in the two following examples. First, one can consider asymmetric (possibly negative) quadratic short rates of the form

$$r_t = \text{tr}(X_t^\top Q X_t) + \xi(t)$$

where $Q \in \mathbb{S}_+^d$, ξ is an input curve used for matching market term structures and tr stands for the trace operator. The kernel K allows for richer autocorrelation structures than the one generated with the conventional Hull and White [128] and Cox, Ingersoll, and Ross [66] models. Second, for $d = m$, one can build stochastic covariance models for d -assets $S = (S^1, \dots, S^d)$ by considering the following dynamics for the stock prices:

$$dS_t = \text{diag}(S_t) X_t dB_t \tag{3.0.1}$$

where B is d -dimensional and correlated with W . Then, the instantaneous covariance between the assets is stochastic and given by $\frac{d(\log S)_t}{dt} = X_t X_t^\top \in \mathbb{S}_+^d$. When $d = m = 1$, one recovers the

Volterra version of the Stein and Stein [171] or Schöbel and Zhu [168] model. Here, singular kernels K satisfying $\lim_{s \uparrow t} |K(t, s)| = \infty$, allow to take into account roughness of the sample paths of the volatility, as documented in Bennedsen et al. [35], Gatheral et al. [99]. As an illustrative example for $d = m = 1$, one could consider the Riemann-Liouville fractional Brownian motion

$$X_t = \frac{1}{\Gamma(H + 1/2)} \int_0^t (t - s)^{H-1/2} dW_s,$$

either with $H \in (0, 1/2)$ to reproduce roughness when modeling the variance process, or with $H \in (1/2, 1)$ to account for long memory in short rate models.

In order to keep the model tractable, one needs to come up with fast pricing and calibration techniques. The main objective of the papers [3, 4] is to show that these models remain highly tractable, despite the inherent non-markovianity and non-semimartingality due to the introduction of the kernel K . Section 3.1 derives the Laplace transform of the multi-dimensional Volterra Wishart process. In Section 3.2, we provide an analytic expression for the characteristic function of the log-price in models of the form (3.0.1), in a one-dimensional setting. The extension of the formulas to the multi-dimension are straightforward, and will later be used in Chapter 5 to solve Markowitz investment problem in a multivariate setting for a large class of models that encompass the dynamics (3.0.1).

3.1 The Laplace transform of the Volterra Wishart process [3]

Integrated quantities of the form $\int_0^t X_s X_s^\top ds$ play a key role for pricing zero-coupon bonds and options on covariance risk. Let us first sketch a very simple way of recovering their Laplace transform. To fix ideas, we consider the 1-dimensional setting with $K : [0, T] \rightarrow \mathbb{R}$, W a standard Brownian motion and

$$X_t = X_0 + \int_0^t K(t, s) dW_s.$$

where $\int_0^T \int_0^T |K(t, s)|^2 dt ds < \infty$. Define the conditional mean process g_t and conditional covariance function Σ_t by:

$$g_t(s) := \mathbb{E} \left[X_s \mid \mathcal{F}_t \right] \mathbf{1}_{s \geq t} = \left(X_0 + \int_0^t K(s, u) dW_u \right) \mathbf{1}_{s \geq t},$$

$$\Sigma_t(s, u) := \int_t^{s \wedge u} K(s, r) K(u, r) dr, \quad t \leq s, u \leq T.$$

Our aim is to compute the conditional Laplace transform of the integrated squared process

$$L_{t,T} = \mathbb{E} \left[\exp \left(w \int_t^T X_s^2 ds \right) \mid \mathcal{F}_t \right], \quad w \leq 0.$$

Fix $t \in [0, T]$, the idea is to exploit Gaussianity in 4 simple steps:

1. Switch from dynamic to static: Fix n and consider a mesh $t = t_0 < t_1 < \dots < t_n = T$ and write

$$w \int_t^T X_s^2 ds \approx w \frac{(T-t)}{n} \sum_{i=1}^n X_{t_i^n}^2 = w \frac{(T-t)}{n} \text{Tr}(X_n X_n^\top),$$

where $X_n = (X_{t_1^n}, \dots, X_{t_n^n})^\top$.

2. Since X is a Gaussian process: conditional on \mathcal{F}_t , X_n is Gaussian with mean vector $\mathbf{g}_t^n = (g_t(t_1^n), \dots, g_t(t_n^n))^\top$ and covariance matrix $\Sigma_t^n = (\Sigma_t(t_i^n, t_j^n))_{1 \leq i, j \leq n}$.

3. It follows that conditional on \mathcal{F}_t , $X_n X_n^\top$ follows a Wishart distribution with explicit Laplace transform:

$$L_{t,T}^n = \mathbb{E} \left[\exp \left(w \frac{(T-t)}{n} \text{Tr} (X_n X_n^\top) \right) \mid \mathcal{F}_t \right] = \exp (\phi_{t,T}^n + (\mathbf{g}_t^n)^\top \Psi_{t,T}^n \mathbf{g}_t^n), \quad (3.1.1)$$

where

$$\Psi_{t,T}^n = w \frac{(T-t)}{n} \left(I_n - 2w \frac{(T-t)}{n} \Sigma_t^n \right)^{-1}, \quad \phi_{t,T}^n = -\frac{1}{2} \log \text{Det} (I_n - 2w \frac{(T-t)}{n} \Sigma_t^n).$$

4. Sending $n \rightarrow \infty$, one expects $L_{t,T}^n \rightarrow L_{t,T}$ so that

$$L_{t,T}^n = \exp (\phi_{t,T}^n + (\mathbf{g}_t^n)^\top \Psi_{t,T}^n \mathbf{g}_t^n) \rightarrow \exp (\phi_{t,T} + \langle g_t, \Psi_{t,T} g_t \rangle_{L^2}),$$

where

$$\begin{aligned} \Psi_{t,T}^n &= w \frac{(T-t)}{n} \left(I_n - 2w \frac{(T-t)}{n} \Sigma_t^n \right)^{-1} \rightarrow w (\text{id} - 2w \Sigma_t)^{-1} := \Psi_{t,T} \\ \phi_{t,T}^n &= -\frac{1}{2} \log \det (I_n - 2w \frac{(T-t)}{n} \Sigma_t^n) \rightarrow -\frac{1}{2} \log \det (\text{id} - 2w \Sigma_t) := \phi_{t,T} \end{aligned}$$

where

- $\langle f, g \rangle_{L^2} = \int_0^T f(s)g(s)ds$
- $\forall K \in L^2([0, T]^2, \mathbb{R})$, \mathbf{K} is the integral operator on $L^2([0, T])$ induced by the kernel K ($\mathbf{K}g)(s) = \int_0^T K(s, u)g(u)du$.

The objects appearing in the limit depend on the infinite dimensional linear covariance operator Σ and its associated determinant, the so-called Fredholm determinant that first appeared in [94]. The statement and derivation are made precise in [3] in a multidimensional setting under mild assumptions on the kernel K : for $w \in \mathbb{S}_+^d$, using the integral operator Σ_t induced by the covariance kernel Σ_t , i.e. $(\Sigma_t f)(s) = \int_0^T \Sigma_t(s, u)f(u)du$ for $f \in L^2([0, T], \mathbb{R}^{d \times m})$, the Laplace transform reads

$$\mathbb{E} \left[\exp \left(- \int_t^T \text{tr} (w X_s X_s^\top) ds \right) \mid \mathcal{F}_t \right] = \frac{\exp \left(- \langle g_t, \sqrt{w} (\text{id} + 2\sqrt{w} \Sigma_t \sqrt{w})^{-1} \sqrt{w} g_t \rangle_{L_t^2} \right)}{\det (\text{id} + 2\sqrt{w} \Sigma_t \sqrt{w})^{m/2}}$$

where $\langle f, g \rangle_{L_t^2} = \int_t^T \text{tr} (f(s)^\top g(s)) ds$ and \det stands for the Fredholm determinant.

Two crucial remarks can be made at this stage:

- The Laplace transform is exponentially quadratic in the forward process $(g_t)_{t \leq T}$. This has to be contrasted with affine Volterra processes introduced in [10] where the Laplace transform is exponentially linear in $(g_t)_{t \leq T}$. In general, such quadratic representation cannot be recovered from that of finite dimensional affine Volterra processes.
- Differentiating $\Psi_t := \sqrt{w} (\text{id} + 2\sqrt{w} \Sigma_t \sqrt{w})^{-1} \sqrt{w}$ yields that Ψ solves an operator Riccati equation of the form

$$\dot{\Psi}_t = -2\Psi \dot{\Sigma}_t \Psi.$$

These two crucial points disentangle the underlying quadratic structure behind such non-Markovian models and open the door for making good Ansatz in an uncontrolled setting but also in control problems sharing a similar linear-quadratic structure in Volterra processes. Furthermore, the Ansatz usually lead to some type of Riccati equations, using the second point above, we now know how to generate explicit operator solutions to such Riccati equations. This will be the main key behind the derivation of the characteristic function of the log-price in Volterra Stein-Stein models in Section 3.2. More interestingly, the same idea can be exploited to its full potential for solving non-Markovian

optimal control problems: the Markowitz problem in a multivariate Volterra Stein-Stein model and the optimal portfolio liquidation problem with a general transient impact kernel in Chapter 5 below.

Although explicit, the expression for the Laplace transform is not known in closed form, except for certain cases. We provide in [3] two approximation procedures either by closed form solutions of conventional Wishart distributions or finite dimensional matrix Riccati equations stemming from conventional linear-quadratic models. We show how these approximations can then be used to price bonds with possible default risk, or options on covariance in multivariate (rough) volatility models by Laplace transform techniques.

Practitioner's corner. Let W^H denote a fractional Brownian motion with Hurst index $H \in (0, 1)$. We can get the following straightforward approximation via closed form Wishart marginals:

$$L^n(H) := \mathbb{E} \left[\exp \left(-\frac{1}{n} \sum_{i=1}^n (W_{t_i}^H)^2 \right) \right] \rightarrow \mathbb{E} \left[\exp \left(-\int_0^1 (W_s^H)^2 ds \right) \right] := L(H)$$

To test the formulas numerically, we proceed as follows. First, we determine the reference value of the right hand side for several values of H . For $H = 1/2$, the exact value is $L(1/2) = \cosh(\sqrt{2})^{-1/2}$. For $H \in \{0.1, 0.3, 0.7, 0.9\}$, we run a Monte-Carlo simulation to estimate $L(H)$ with the trapezoidal rule with a 95% confidence interval and 10^6 sample paths with 10^3 time steps for each sample path. Second, for each value of H , we compute $L^n(H)$ as in (3.1.1), for several values of n with the left Riemann sum and the Gauss-Legendre quadrature. The results are collected in Figure 3.1 below. We observe that the Gauss-Legendre quadrature performs better than the left Riemann sum rule, especially for higher values of H . When $H \geq 0.5$, even with $n = 10$, $L^n(H)$ with the Gauss-Legendre rule falls already within the 95% confidence interval of the Monte-Carlo simulation. Other quadrature rules can be used to get a better approximation of the Fredholm determinant, depending on the kernel, see for instance Bornemann [43].

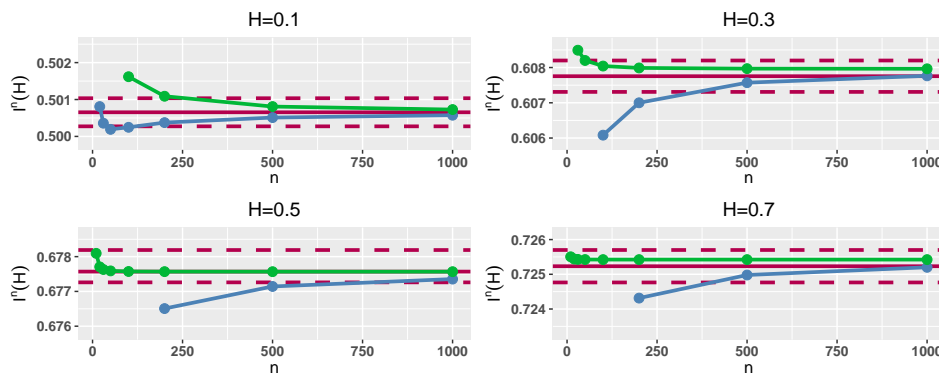


FIGURE 3.1: Convergence of $L^n(H)$ with the Riemann sum (blue) and the Gauss-Legendre quadrature (green) towards the benchmark MC value $L(H)$ (red) for different values of (H, n) . The dashed lines delimit confidence intervals of the Monte-Carlo simulation.

3.2 The characteristic function of the Volterra Stein-Stein model [4]

For $T > 0$, we will consider the following generalized version of the Stein-Stein model:

$$\begin{aligned} dS_t &= S_t X_t dB_t, \quad S_0 > 0, \\ X_t &= g_0(t) + \int_0^t K(t,s) \kappa X_s ds + \int_0^t K(t,s) \nu dW_s, \end{aligned}$$

with $B = \rho W + \sqrt{1 - \rho^2} W^\perp$, $\rho \in [-1, 1]$, $\kappa, \nu \in \mathbb{R}$, g_0 a suitable deterministic input curve, $K : [0, T]^2 \rightarrow \mathbb{R}$ a measurable kernel and (W, W^\perp) a two-dimensional Brownian motion.

For suitable $u, w \in \mathbb{C}$, we provide in [4] the following analytical expression for the conditional joint Fourier-Laplace transform of the log-price and the integrated variance:

$$\mathbb{E} \left[\exp \left(u \log \frac{S_T}{S_t} + w \int_t^T X_s^2 ds \right) \middle| \mathcal{F}_t \right] = \frac{\exp(\langle g_t, \Psi_t g_t \rangle_{L^2})}{\det(\Phi_t)^{1/2}}, \quad (3.2.1)$$

with $\langle f, h \rangle_{L^2} = \int_0^T f(s)h(s)ds$, \det the Fredholm [94] determinant, g_t the adjusted conditional mean given by

$$g_t(s) = 1_{t \leq s} \mathbb{E} \left[X_s - \int_t^T K(s,r) \kappa X_r dr \middle| \mathcal{F}_t \right], \quad s, t \leq T;$$

and Ψ_t a linear operator acting on $L^2([0, T], \mathbb{R})$ defined by

$$\Psi_t = (\text{id} - b\mathbf{K}^*)^{-1} a (\text{id} - 2a\tilde{\Sigma}_t)^{-1} (\text{id} - b\mathbf{K})^{-1}, \quad t \leq T, \quad (3.2.2)$$

where \mathbf{K} denotes the integral operator induced by K , \mathbf{K}^* the adjoint operator, id denotes the identity operator, i.e. $(\text{id}f) = f$ for all $f \in L^2([0, T], \mathbb{C})$,

$$a = w + \frac{1}{2}(u^2 - u), \quad b = \kappa + \rho\nu u,$$

and $\tilde{\Sigma}_t$ the adjusted covariance integral operator defined by

$$\tilde{\Sigma}_t = (\text{id} - b\mathbf{K})^{-1} \Sigma_t (\text{id} - b\mathbf{K}^*)^{-1},$$

with Σ_t defined as the integral operator associated with the covariance kernel

$$\Sigma_t(s, u) = \nu^2 \int_t^T K(s, z) K(u, z) dz, \quad t \leq s, u \leq T,$$

and finally Φ is defined by

$$\Phi_t = \text{id} - 2a\tilde{\Sigma}_t.$$

The proof exploits the ideas that we have identified in the previous section: We adopt a dynamical approach to derive the conditional characteristic function (3.2.1) via Itô's formula on a quadratic Ansatz on the adjusted conditional mean process $(g_t(s))_{t \leq s}$ combined with explicit operator solutions to the underlying operator Riccati equations that appear.

From the numerical perspective, we show that the expression (3.2.1) lends itself to approximation by closed form solutions using finite dimensional matrices after a straightforward discretization of the

operators in the form

$$\mathbb{E} \left[\exp \left(u \log \frac{S_T}{S_0} + w \int_0^T X_s^2 ds \right) \right] \approx \frac{\exp \left(\frac{T}{n} g_n^\top \Psi_0^n g_n \right)}{\text{Det}(\Phi_0^n)^{1/2}}$$

where $g_n \in \mathbb{R}^n$ and $\Phi_0^n, \Psi_0^n \in \mathbb{R}^{n \times n}$ are entirely determined by $(g_0, K, \nu, \kappa, u, w)$ and Det is the standard determinant of a matrix.

Practitioner's Corner. We illustrate the applicability of these formulas on an option pricing and calibration example by Fourier inversion techniques in a (rough) fractional Stein–Stein model.

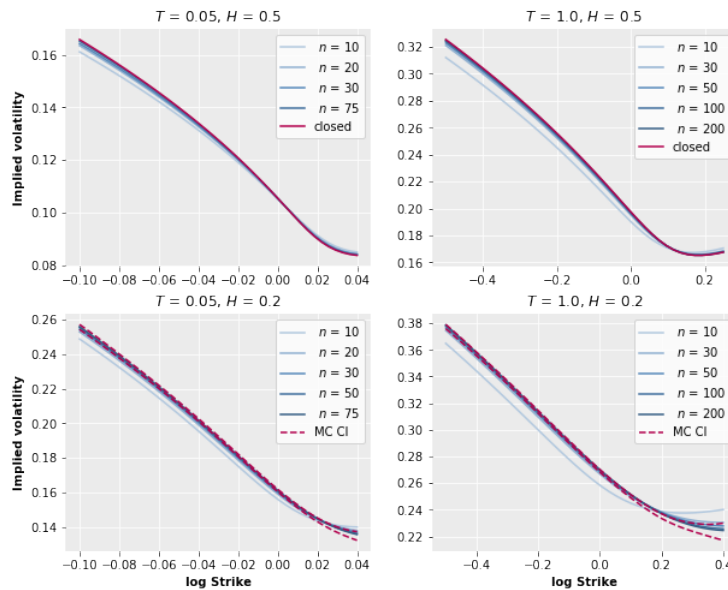


FIGURE 3.2: Convergence of the implied volatility slices for short ($T = 0.05$ year) and long maturities ($T = 1$ year) of the operator discretization towards: (i) the explicit solution of the conventional Stein–Stein model ($H = 0.5$ upper graphs); (ii) the 95% Monte-Carlo confidence intervals ($H = 0.2$ lower graphs). The parameters are $X_0 = \theta = 0.1$, $\kappa = 0$, $\nu = 0.25$ and $\rho = -0.7$.

For real market data, we calibrate the fractional Stein–Stein model to: (i) the at-the-money skew. Keeping the parameters $X_0 = 0.44$, $\theta = 0.3$, $\kappa = 0$ fixed, the calibrated parameters are given by

$$\hat{\nu} = 0.5231458, \quad \hat{\rho} = -0.9436174 \quad \text{and} \quad \hat{H} = 0.2234273. \quad (3.2.3)$$

This power-law like behaviour of the at-the-money skew observed on the market is captured by the fractional Stein–Stein model as illustrated on Figure 3.3 with only three parameters. (ii) the implied volatility surface of the S&P across several maturities for in Figure 3.4.

Both calibration lead to $\hat{H} < 0.5$ indicating that the rough regime of the fractional Stein–Stein model is coherent with the observations on the market.



FIGURE 3.3: Term structure of the at-the-money skew for the S&P index on June 20, 2018 (red dots) and for the rough Stein–Stein model with calibrated parameters (3.2.3) (blue circles with dashed line).

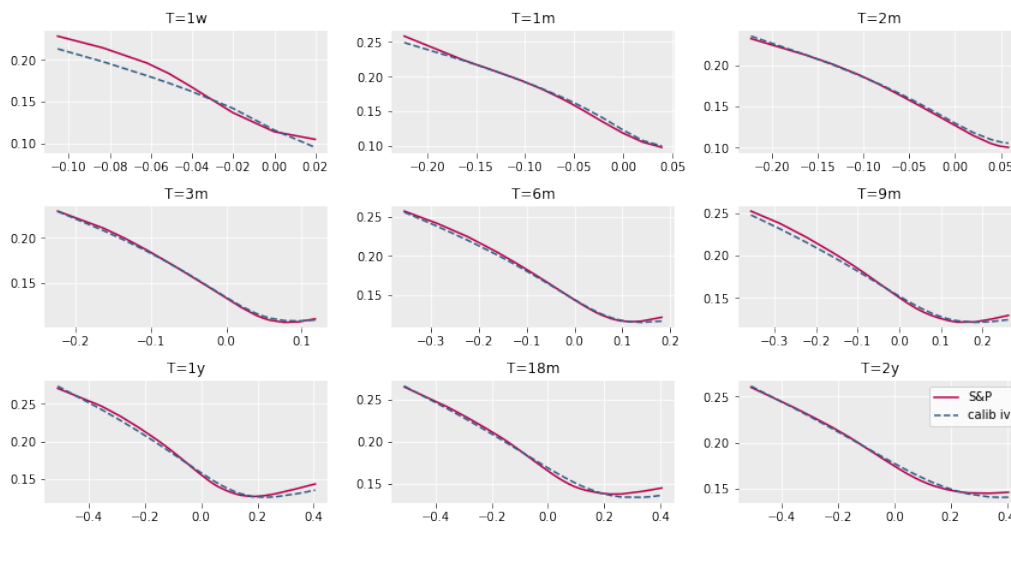


FIGURE 3.4: The implied volatility surface of the S&P index (red) and the calibrated fractional Stein–Stein model (blue) with parameters: $\hat{X}_0 = 0.113$, $\hat{\theta} = -0.044$, $\hat{\kappa} = -8.9e - 5$, $\hat{\nu} = 0.176$, $\hat{\rho} = -0.704$, and $\hat{H} = 0.279$.

Related Literature. Conventional Wishart processes initiated by Bru [47] and introduced in finance by Gouriéroux and Sufana [104] have been intensively applied, together with their variants, in term structure and stochastic covariance modeling, see for instance Alfonsi [19], Buraschi et al. [48], Cuchiero et al. [69, 70], Da Fonseca et al. [73, 74], Gouriéroux et al. [105], Muhle-Karbe et al. [150]. Conventional linear quadratic models have been characterized in Chen et al. [55], Cheng and Scaillet [57]. Volterra Wishart processes have been recently studied in Cuchiero and Teichmann [67], Yue and Huang [177]. Applications of certain quadratic Gaussian processes can be found in Benth and Rohde [37], Corcuera et al. [65], Harms and Stefanovits [122], Kleptsyna et al. [136]. Gaussian stochastic volatility models have been treated in Gulisashvili [113], Gulisashvili et al. [114]. Gaussian quadratic volatility models have been already considered several times in the context of non-Markovian and rough volatility literature [67, 114, 121, 125] but there has been no derivation of the analytic form of the characteristic function.

Part II

Stochastic control and games with Volterra processes

Linear-quadratic control of Volterra processes for a class of convolution kernels

Summary

This chapter provides an exhaustive treatment of solvability and approximation of linear-quadratic control problems for a class of stochastic Volterra equations of convolution type, whose kernels are Laplace transforms of certain signed matrix measures which are not necessarily finite.

Based on:

[14] Abi Jaber, E., Miller, E., & Pham, H. *Linear-Quadratic control for a class of stochastic Volterra equations: solvability and approximation*, [Annals of Applied Probability](#), 31(5), 2244-2274, 2021.

[13] Abi Jaber, E., Miller, E., & Pham, H. *Integral operator Riccati equations arising in stochastic Volterra control problems*, [SIAM journal on Control and Optimization](#), 59(2), 1581-1603, 2021.

In [14, 13], we address the optimal control of d -dimensional stochastic Volterra equations of the form:

$$X_t^\alpha = g_0(t) + \int_0^t K(t-s) \left(b(s, X_s^\alpha, \alpha_s) ds + \sigma(s, X_s^\alpha, \alpha_s) dW_s \right), \quad (4.0.1)$$

where g_0 is a deterministic function and K is a (convolution) matrix-valued kernel of the form

$$K(t) = \int_{\mathbb{R}_+} e^{-\theta t} \mu(d\theta), \quad t > 0,$$

for some signed matrix measure μ . Our framework covers the case of the fractional kernel $K(t) = t^{H-1/2}/\Gamma(H+1/2)$ with $H \leq 1/2$, arising from the Mandelbrot-Van Ness representation of the fractional Brownian motion with Hurst index H . We mainly focus on the case where the coefficients b and σ are in linear form with respect to the state and control arguments, and the cost to be minimized is of linear-quadratic form.

Since the (controlled) stochastic Volterra process (4.0.1) is neither Markovian nor a semimartingale, it is natural to consider Markovian lifts for which suitable stochastic tools and control methods apply. Inspired by the Markovian representation of fractional Brownian motion introduced in [49], and more recently generalized to several un-controlled stochastic Volterra equations in [6, 68, 122], we establish the correspondence of the initial problem with a lifted Markovian controlled system

$(Y_t^\alpha)_{t \in [0, T]}$ taking its values in the possibly infinite-dimensional Banach space $L^1(\mu)$. Next, in the LQ case, i.e., when b, σ are of linear form, and the cost function is linear-quadratic, we prove by means of a refined martingale verification argument combined with a squares completion technique, that the value function is of quadratic form while the optimal control is in linear feedback form with respect to these lifted state variables. The coefficients of the quadratic and linear form of the value function and optimal control are expressed in terms of a non-standard system of integral operator Riccati equations whose solvability (existence and uniqueness) is proved in [13].

A second important feature of our approach is to provide a natural approximation of such solution by a suitable discretization of the measure μ , leading to conventional finite-dimensional LQ control problems, which involve standard matrix Riccati equations that can be numerically implemented.

Notations. For a Banach space \mathcal{B} , $L^2([0, T], \mathcal{B})$ denotes the space of measurable and square integrable functions from $[0, T]$ to \mathcal{B} .

For any $d \times d_1$ -matrix valued measure μ_1 on \mathbb{R}_+ , we denote by $|\mu_1|$ its total variation, which is a scalar nonnegative measure, refer to [110, Section 3.5] for more details. The space $L^1(\mu_1)$ consists of μ_1 -a.e. equivalence classes of $|\mu_1|$ -integrable functions $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}^{d_1}$ endowed with the norm $\|\varphi\|_{L^1(\mu_1)} = \int_{\mathbb{R}_+} |\mu_1|(d\theta) |\varphi(\theta)|$, where we identify the function φ with its class of equivalence. For any such φ the integral

$$\int_{\mathbb{R}_+} \mu_1(d\theta) \varphi(\theta)$$

is well defined by virtue of the inequality

$$\left| \int_{\mathbb{R}_+} \mu_1(d\theta) \varphi(\theta) \right| \leq \int_{\mathbb{R}_+} |\mu_1|(d\theta) |\varphi(\theta)|,$$

see [110, Theorem 5.6]. If μ_2 is a $d \times d_2$ -matrix valued measure, the space $L^1(\mu_1 \otimes \mu_2)$ consists of $\mu_1 \otimes \mu_2$ -a.e. equivalence classes of $|\mu_1| \otimes |\mu_2|$ -integrable functions $\Phi : \mathbb{R}_+^2 \rightarrow \mathbb{R}^{d \times d}$ endowed with the norm $\|\Phi\|_{L^1(\mu_1 \otimes \mu_2)} = \int_{\mathbb{R}_+^2} |\mu_1|(d\theta) |\Phi(\theta, \tau)| |\mu_2|(d\tau) < \infty$. For any such Φ , the integral

$$\int_{\mathbb{R}_+^2} \mu_1(d\theta)^\top \Phi(\theta, \tau) \mu_2(d\tau)$$

is again well defined by virtue of [110, Theorem 5.6]. Both $(L^1(\mu_1), \|\cdot\|_{L^1(\mu_1)})$ and $(L^1(\mu_1 \otimes \mu_2), \|\cdot\|_{L^1(\mu_1 \otimes \mu_2)})$ are Banach spaces, see [164, Theorem 3.11]. We also denote by $L^\infty(\mu_1)$ the set of measurable functions $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}^{d_1}$, which are bounded μ_1 -a.e., and by $L^\infty(\mu_1 \otimes \mu_2)$ the set of measurable functions $\Phi : \mathbb{R}_+^2 \rightarrow \mathbb{R}^{d \times d}$, which are bounded $\mu_1 \otimes \mu_2$ -a.e., that we endow with their usual norms $\|\psi\|_{L^\infty(\mu_1)}$ and $\|\Phi\|_{L^\infty(\mu_1 \otimes \mu_2)}$.

4.1 Formulation of the problem [14, 13]

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space supporting a one dimensional Brownian motion W . Fix $T > 0$ and $d, d', m \in \mathbb{N}$. We consider a controlled d -dimensional stochastic Volterra equation of the form (4.0.1) where α is an element of the admissible set

$$\mathcal{A} = \left\{ \alpha : \Omega \times [0, T] \rightarrow \mathbb{R}^m \text{ progressively measurable such that } \sup_{0 \leq t \leq T} \mathbb{E} [|\alpha_t|^4] < \infty \right\},$$

$g_0 : [0, T] \rightarrow \mathbb{R}^d$ is a measurable function, $K : [0, T] \rightarrow \mathbb{R}^{d \times d'}$ is a measurable kernel, and $b, \sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^{d'}$ are of affine form:

$$\begin{aligned} b(t, x, a) &= \beta(t) + Bx + Ca, \\ \sigma(t, x, a) &= \gamma(t) + Dx + Fa, \end{aligned}$$

where $B, D \in \mathbb{R}^{d' \times d}$, $C, F \in \mathbb{R}^{d' \times m}$, and $\beta, \gamma : [0, T] \rightarrow \mathbb{R}^{d'}$ are measurable functions. We are chiefly interested in the case where K is the Laplace transform

$$K(t) = \int_{\mathbb{R}_+} e^{-\theta t} \mu(d\theta), \quad t > 0, \quad (4.1.1)$$

of a signed $d \times d'$ -measure μ satisfying

$$\int_{\mathbb{R}_+} (1 \wedge \theta^{-1/2}) |\mu|(d\theta) < \infty, \quad (4.1.2)$$

and

$$\theta \mapsto \int_0^T e^{-\theta u} u^{\zeta-1} du \in L^1(|\mu|), \quad \text{for some } \zeta \in (0, 1/4), \quad (4.1.3)$$

where $|\mu|$ denotes the total variation of μ . While condition (4.1.2) does not exclude $\mu_{ij}(\mathbb{R}_+) = \pm\infty$ for some $i \leq d$, $j \leq d'$, or equivalently a singularity of the kernel K at 0, it does ensure that $K \in L^2([0, T], \mathbb{R}^{d \times d'})$ and that $|\mu|$ is σ -finite. The former implies that the stochastic convolution

$$t \mapsto \int_0^t K(t-s) \xi_s dW_s$$

is well defined as an Itô integral, for every $t \leq T$, for any progressively measurable process ξ such that

$$\sup_{t \leq T} \mathbb{E} [|\xi_t|^2] < \infty.$$

Condition (4.1.3) would yield the existence of a continuous version.

We can now make precise the concept of solution to the controlled equation (4.0.1). By a solution to (4.0.1), we mean an \mathbb{F} -adapted process X^α with continuous sample paths such that (4.0.1) holds for all $t \leq T$, \mathbb{P} -almost surely. Under (4.1.1)-(4.1.2)-(4.1.3), assuming that β, γ are measurable and bounded, one can show that the controlled stochastic Volterra equation (4.0.1) admits a unique continuous solution X^α , for any continuous input curve g_0 , and any admissible control $\alpha \in \mathcal{A}$. Furthermore, it holds that

$$\sup_{0 \leq t \leq T} \mathbb{E} [|X_t^\alpha|^4] < \infty. \quad (4.1.4)$$

Remark 4.1. Notice that due to the possible singularity of the kernel K , and in contrast with standard stochastic differential equations, the solution X^α to the controlled stochastic Volterra equation does not satisfy in general the usual square integrability condition of the form: $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^\alpha|^2] < \infty$. For this reason, we impose the stronger condition $\sup_{t \leq T} \mathbb{E}[|\alpha_t|^4] < \infty$ for the set of admissible controls \mathcal{A} , which will turn out to be crucial for the martingale verification result.

We consider a cost functional given by

$$J(\alpha) = \mathbb{E} \left[\int_0^T f(X_s^\alpha, \alpha_s) ds \right],$$

where the running cost f has the following quadratic form

$$f(x, \alpha) = x^\top Q x + \alpha^\top N \alpha + 2x^\top L,$$

for some $Q \in \mathbb{S}_+^d$, $N \in \mathbb{S}_+^m$ and $L \in \mathbb{R}^d$. Here \mathbb{S}_+^d denotes the set of d -dimensional nonnegative symmetric matrices. Note that by virtue of (4.1.4), $J(\alpha)$ is well defined for any $\alpha \in \mathcal{A}$. The aim is to solve

$$V_0 = \inf_{\alpha \in \mathcal{A}} J(\alpha). \quad (4.1.5)$$

Before going further, let us mention several kernels of interest that satisfy (4.1.1)–(4.1.2)–(4.1.3).

Example 4.2. 1. *Smooth kernels:* if $|\mu_{ij}(\mathbb{R}_+)| < \infty$, for every $i = 1, \dots, d, j = 1, \dots, d'$, then (4.1.2)–(4.1.3) are satisfied and K is infinitely differentiable on $[0, T]$. This is the case, for instance, when $\mu(d\theta) = \sum_{i=1}^n c_i^n \delta_{\theta_i^n}(d\theta)$, for some $c_i^n \in \mathbb{R}^{d \times d'}$ and $\theta_i^n \in \mathbb{R}_+$, $i = 1, \dots, n$, which corresponds to

$$K(t) = \sum_{i=1}^n c_i^n e^{-\theta_i^n t}.$$

2. *The fractional kernel* ($d = d' = 1$)

$$K_H(t) = \frac{t^{H-1/2}}{\Gamma(H+1/2)}, \quad (4.1.6)$$

for some $H \in (0, 1/2)$, which is the Laplace transform of

$$\mu_H(d\theta) = \frac{\theta^{-H-1/2}}{\Gamma(H+1/2)\Gamma(1/2-H)} d\theta, \quad (4.1.7)$$

and more generally the Gamma kernel $K(t) = K_H(t)e^{-\zeta t}$ for $H \in (0, 1/2)$ and $\zeta \in \mathbb{R}$ for which

$$\mu(d\theta) = \frac{(\theta - \zeta)^{-H-1/2} \mathbf{1}_{(\zeta, \infty)}(\theta)}{\Gamma(H+1/2)\Gamma(1/2-H)} d\theta.$$

3. If K_1 and K_2 satisfy (4.1.1), then so does $K_1 + K_2$ and $K_1 K_2$ with the respective measures $\mu_1 + \mu_2$ and $\mu_1 * \mu_2$. When μ_1, μ_2 satisfy (4.1.2)–(4.1.3), it is clear that $\mu_1 + \mu_2$ also satisfies (4.1.2)–(4.1.3). Condition (4.1.2) is satisfied for the convolution $\mu_1 * \mu_2$ provided $\int_{[1, \infty)^2} (\theta + \tau)^{-1/2} \mu_1(d\theta) \mu_2(d\tau) < \infty$, which is the case for instance if either $\mu_1(\mathbb{R}_+)$ or $\mu_2(\mathbb{R}_+)$ are finite. Similarly for (4.1.3).

4. If K is a completely monotone kernel, i.e. K is infinitely differentiable on $(0, \infty)$ such that $(-1)^n K^{(n)}(t)$ is nonnegative for each $t > 0$, then, by Bernstein's theorem, there exists a non-negative measure μ such that (4.1.1) holds, see [110, Theorem 5.2.5].

4.1.1 Markovian representation

The solution X^α of (4.0.1) is in general neither Markovian nor a semimartingale as illustrated by the Riemann–Liouville fractional Brownian motion

$$t \mapsto \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} dW_s, \quad H \in (0, 1/2],$$

which is Markovian and a martingale only for $H = 1/2$. Markovian representations of fractional Brownian motion have been introduced in [49], and more recently generalized to several un-controlled stochastic Volterra equations for kernels of the form (4.1.1), see [6, Section 4]; [68, Section 5.1]; [122]. Inspired by these approaches, we establish, by means of stochastic Fubini's theorem, the correspondence of (4.0.1) with a possibly infinite dimensional Markovian controlled system of the form

$$\begin{cases} dY_t^\alpha(\theta) &= \left(-\theta Y_t^\alpha(\theta) + \tilde{b} \left(t, \int_{\mathbb{R}_+} \mu(d\tau) Y_t^\alpha(\tau), \alpha_t \right) \right) dt \\ &\quad + \tilde{\sigma} \left(t, \int_{\mathbb{R}_+} \mu(d\tau) Y_t^\alpha(\tau), \alpha_t \right) dW_t \\ Y_0^\alpha(\theta) &= 0, \end{cases} \quad (4.1.8)$$

where the coefficients $\tilde{b} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^{d'}$, $\tilde{\sigma} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^{d'}$ are defined by

$$\begin{aligned} \tilde{b}(t, x, a) &= \tilde{\beta}(t) + Bx + Ca, \\ \tilde{\sigma}(t, x, a) &= \tilde{\gamma}(t) + Dx + Fa, \end{aligned}$$

with

$$\tilde{\beta} = \beta + Bg_0 \quad \text{and} \quad \tilde{\gamma} = \gamma + Dg_0.$$

Theorem 4.3. *Let g_0, β, γ be bounded functions on $[0, T]$ and K be given as in (4.1.1) such that (4.1.2)-(4.1.3) hold. Fix $\alpha \in \mathcal{A}$. Assume that there exists a progressively measurable process X^α that solves (4.0.1), \mathbb{P} -a.s., for each $t \leq T$, and that (4.1.4) holds. Then, for each $t \leq T$, X_t^α admits the representation*

$$X_t^\alpha = g_0(t) + \int_{\mathbb{R}_+} \mu(d\theta) Y_t^\alpha(\theta), \quad (4.1.9)$$

where, for each $\theta \in \mathbb{R}_+$,

$$Y_t^\alpha(\theta) = \int_0^t e^{-\theta(t-s)} b(s, X_s^\alpha, \alpha_s) ds + \int_0^t e^{-\theta(t-s)} \sigma(s, X_s^\alpha, \alpha_s) dW_s.$$

In particular, Y^α can be chosen to have continuous sample paths in $L^1(\mu)$, satisfying

$$\begin{aligned} \sup_{t \leq T} \mathbb{E} \left[\|Y_t^\alpha\|_{L^1(\mu)}^4 \right] &< \infty, \\ \sup_{t \leq T} \sup_{\theta \in \mathbb{R}_+} |Y_t^\alpha(\theta)| &< \infty, \end{aligned} \quad (4.1.10)$$

and for each $\theta \in \mathbb{R}_+$, $t \mapsto Y_t^\alpha(\theta)$ solves (4.1.8). Conversely, assume that there exists a solution Y^α to (4.1.8) that is continuous in $L^1(\mu)$, i.e., such that

$$\begin{aligned} Y_t^\alpha(\theta) &= \int_0^t e^{-\theta(t-s)} \tilde{b} \left(s, \int_{\mathbb{R}_+} \mu(d\tau) Y_s^\alpha(\tau), \alpha_s \right) ds \\ &\quad + \int_0^t e^{-\theta(t-s)} \tilde{\sigma} \left(s, \int_{\mathbb{R}_+} \mu(d\tau) Y_s^\alpha(\tau), \alpha_s \right) dW_s, \quad \mathbb{P} \otimes \mu - a.e. \end{aligned} \quad (4.1.11)$$

for each $t \leq T$, and that (4.1.10) holds. Then, the process X^α given by (4.1.9) is a continuous solution to (4.0.1) such that (4.1.4) holds.

The following remark justifies our choice for carrying the analysis in the space $L^1(\mu)$.

Remark 4.4. Inspecting the coefficients in (4.1.8), one observes that they are well-defined if $Y_t^\alpha \in L^1(\mu)$ for all $t \leq T$. One might be tempted to look for solutions in the Hilbert space $L^2(\mu)$, but since μ is not always a finite measure (as in (4.1.7) for instance), $L^2(\mu)$ is not necessarily included in $L^1(\mu)$.

Remark 4.5. An alternative lift approach, in the spirit of [6, 79, 119, 130, 172], consists in introducing the double-indexed (controlled) processes

$$G_t^\alpha(u) = \mathbb{E} \left[X_u^\alpha - \int_t^u K(u-s) b(s, X_s^\alpha, \alpha_s) ds \mid \mathcal{F}_t \right], \quad 0 \leq t \leq u \leq T.$$

The control problem can then be reformulated in terms of the infinite dimensional controlled Markov process $\{G_t^\alpha(\cdot), t \in [0, T]\}$ with Itô dynamics

$$dG_t^\alpha(u) = K(u-t) (b(t, X_t^\alpha, \alpha_t) dt + \sigma(t, X_t^\alpha, \alpha_t) dW_t), \quad 0 \leq t < u \leq T.$$

This lift will be considered in the two upcoming chapters.

4.1.2 Formal derivation of the solution

Thanks to Theorem 4.3, the possibly non-Markovian initial problem can be formally recast as a degenerate infinite dimensional Markovian problem in $L^1(\mu)$ on the state variables Y^α given by

(4.1.8). To see this, we define the mean-reverting operator A^{mr} acting on measurable functions $\varphi \in L^1(\mu)$ by

$$(A^{mr}\varphi)(\theta) = -\theta\varphi(\theta), \quad \theta \in \mathbb{R}_+,$$

and consider the dual pairing

$$\langle \varphi, \psi \rangle_\mu = \int_{\mathbb{R}_+} \varphi(\theta)^\top \mu(d\theta)^\top \psi(\theta), \quad (\varphi, \psi) \in L^1(\mu) \times L^\infty(\mu^\top).$$

For any matrix-valued kernel G , we denote by \mathbf{G} the integral operator induced by G , defined by:

$$(\mathbf{G}\phi)(\theta) = \int_{\mathbb{R}_+} G(\theta, \theta') \mu(d\theta') \phi(\theta').$$

Notice that when $G \in L^\infty(\mu \otimes \mu)$, the operator \mathbf{G} is well-defined on $L^1(\mu)$, and we have $\mathbf{G}\phi \in L^\infty(\mu^\top)$ for $\phi \in L^1(\mu)$. In this case, $\langle \phi, \mathbf{G}\psi \rangle_\mu$ is well defined for all $\varphi, \psi \in L^1(\mu)$. When $G \in L^1(\mu \otimes \mu)$, the operator \mathbf{G} is well-defined on $L^\infty(\mu)$, and we have $\mathbf{G}\phi \in L^1(\mu^\top)$, for $\phi \in L^\infty(\mu)$. In this case $\langle \mathbf{G}\phi, \psi \rangle_{\mu^\top}$ is well defined for all $\varphi, \psi \in L^\infty(\mu)$.

To fix ideas we set $g_0 = \beta = \gamma \equiv 0$ and $L = 0$. Noting that relation (4.1.11) is the mild form of the linear controlled dynamics in $L^1(\mu)$,

$$dY_t^\alpha = (A^{mr}Y_t^\alpha + \mathbf{B}Y_t^\alpha + C\alpha_t) dt + (\mathbf{D}Y_t^\alpha + F\alpha_t) dW_t, \quad Y_0^\alpha = 0,$$

we see that the optimization problem (4.1.5) can be reformulated as a Markovian problem in $L^1(\mu)$ with cost functional,

$$J(\alpha) = \mathbb{E} \left[\int_0^T (\langle Y_s^\alpha, \mathbf{Q}Y_s^\alpha \rangle_\mu + \alpha_s^\top N \alpha_s) ds \right], \quad (4.1.12)$$

where, by a slight abuse of notations, C and F denote the respective constant operators from \mathbb{R}^m into $L^\infty(\mu)$ induced by the matrices C and F :

$$(Ca)(\theta) = Ca, \quad (Fa)(\theta) = Fa, \quad \theta \in \mathbb{R}_+, \quad a \in \mathbb{R}^m.$$

Their adjoint operators C^*, F^* from $L^1(\mu^\top)$ into \mathbb{R}^m take the form

$$C^*g = C^\top \int_{\mathbb{R}_+} \mu(d\theta)^\top g(\theta), \quad F^*g = F^\top \int_{\mathbb{R}_+} \mu(d\theta)^\top g(\theta), \quad g \in L^1(\mu^\top).$$

Given the linear-quadratic structure of the problem, standard results in finite-dimensional stochastic control theory, see [176, Chapter 6], as well as in Hilbert spaces, see [83, 126], suggest that the optimal value process V^α associated to the functional (4.1.12) should be of linear-quadratic form

$$V_t^{\alpha^*} = \langle Y_t^{\alpha^*}, \mathbf{\Gamma}_t Y_t^{\alpha^*} \rangle_\mu,$$

with an optimal feedback control α^* satisfying

$$\alpha_t^* = -(N + F^* \mathbf{\Gamma}_t F)^{-1} (C^* \mathbf{\Gamma}_t + F^* \mathbf{\Gamma}_t \mathbf{D}) Y_t^{\alpha^*}, \quad 0 \leq t \leq T,$$

where $\mathbf{\Gamma}_t$ is a symmetric operator from $L^1(\mu)$ into $L^\infty(\mu^\top)$, and solves the operator Riccati equation:

$$\begin{cases} \mathbf{\Gamma}_T = \mathbf{0} \\ \dot{\mathbf{\Gamma}}_t = -\mathbf{\Gamma}_t A^{mr} - (\mathbf{\Gamma}_t A^{mr})^* - \mathbf{Q} - \mathbf{D}^* \mathbf{\Gamma}_t \mathbf{D} - \mathbf{B}^* \mathbf{\Gamma}_t - (\mathbf{B}^* \mathbf{\Gamma}_t)^* \\ \quad + (C^* \mathbf{\Gamma}_t + F^* \mathbf{\Gamma}_t \mathbf{D})^* (N + F^* \mathbf{\Gamma}_t F)^{-1} (C^* \mathbf{\Gamma}_t + F^* \mathbf{\Gamma}_t \mathbf{D}), \quad t \in [0, T]. \end{cases}$$

In particular, when Γ is an integral operator, this formally induces the following Riccati equation for the associated (symmetric) kernel Γ valued in $L^1(\mu \otimes \mu)$:

$$\begin{cases} \Gamma_T(\theta, \tau) &= 0 \\ \dot{\Gamma}_t(\theta, \tau) &= (\theta + \tau)\Gamma_t(\theta, \tau) - Q - D^\top \int_{\mathbb{R}_+^2} \mu(d\theta')^\top \Gamma_t(\theta', \tau') \mu(d\tau') D \\ &\quad - B^\top \int_{\mathbb{R}_+} \mu(d\theta')^\top \Gamma_t(\theta', \tau) - \int_{\mathbb{R}_+} \Gamma_t(\theta, \tau') \mu(d\tau') B + S_t(\theta)^\top \hat{N}_t^{-1} S_t(\tau), \end{cases}$$

where

$$\begin{aligned} S_t(\tau) &= C^\top \int_{\mathbb{R}_+} \mu(d\theta)^\top \Gamma_t(\theta, \tau) + F^\top \int_{\mathbb{R}_+^2} \mu(d\theta')^\top \Gamma_t(\theta', \tau') \mu(d\tau') D \\ \hat{N}_t &= N + F^\top \int_{\mathbb{R}_+^2} \mu(d\theta)^\top \Gamma_t(\theta, \tau) \mu(d\tau) F, \end{aligned}$$

and provides an optimal control in the form

$$\alpha_t^* = -\hat{N}_t^{-1} \int_{\mathbb{R}_+} S_t(\theta) \mu(d\theta) Y_t^{\alpha^*}(\theta), \quad 0 \leq t \leq T.$$

Although the aforementioned infinite dimensional results provide formal expressions for the solution of the problem, they cannot be directly applied, since they concern Hilbert spaces. Here the infinite dimensional controlled process Y^α takes its values in the non reflexive Banach space $(L^1(\mu), \|\cdot\|_{L^1(\mu)})$. The rigorous derivation of the solution is the first main objective of the chapter. Our second goal is to show how to obtain an analytic finite-dimensional approximation of the original control problem after a suitable discretization of the operator Riccati equation.

4.2 Main results

4.2.1 Solvability: optimal control and value function

Let $\alpha \in \mathcal{A}$. Given the linear-quadratic structure of the problem and the formal analysis of Section 4.1.2, it is natural to consider a candidate optimal value process $(V_t^\alpha)_{t \leq T}$ of linear-quadratic form in the state variable Y^α given by (4.1.11), that is

$$V_t^\alpha = \int_{\mathbb{R}_+^2} Y_t^\alpha(\theta)^\top \mu(d\theta)^\top \Gamma_t(\theta, \tau) \mu(d\tau) Y_t^\alpha(\tau) + 2 \int_{\mathbb{R}_+} \Lambda_t(\theta)^\top \mu(d\theta) Y_t^\alpha(\theta) + \chi_t, \quad (4.2.1)$$

where the functions $t \mapsto \Gamma_t, \Lambda_t, \chi_t$ are solutions, in a suitable sense, of the following system of Riccati equations:

$$\begin{cases} \dot{\Gamma}_t(\theta, \tau) &= (\theta + \tau)\Gamma_t(\theta, \tau) - \mathcal{R}_1(\Gamma_t)(\theta, \tau), & \Gamma_T(\theta, \tau) &= 0 \\ \dot{\Lambda}_t(\theta) &= \theta\Lambda_t(\theta) - \mathcal{R}_2(t, \Gamma_t, \Lambda_t)(\theta), & \Lambda_T(\theta) &= 0 \\ \dot{\chi}_t &= -\mathcal{R}_3(t, \Gamma_t, \Lambda_t), & \chi_T &= 0, \end{cases} \quad (4.2.2)$$

where we defined

$$\begin{aligned} \mathcal{R}_1(\Gamma)(\theta, \tau) &= Q + D^\top \int_{\mathbb{R}_+^2} \mu(d\theta')^\top \Gamma(\theta', \tau') \mu(d\tau') D + B^\top \int_{\mathbb{R}_+} \mu(d\theta')^\top \Gamma(\theta', \tau) \\ &\quad + \int_{\mathbb{R}_+} \Gamma(\theta, \tau') \mu(d\tau') B - S(\Gamma)(\theta)^\top \hat{N}^{-1}(\Gamma) S(\Gamma)(\tau) \end{aligned} \quad (4.2.3)$$

$$\begin{aligned} \mathcal{R}_2(t, \Gamma, \Lambda)(\theta) &= L + Qg_0(t) + B^\top \int_{\mathbb{R}_+} \mu(d\theta')^\top \Lambda(\theta') + \int_{\mathbb{R}_+} \Gamma(\theta, \tau') \mu(d\tau') \tilde{\beta}(t) \\ &\quad + D^\top \int_{\mathbb{R}_+^2} \mu(d\theta')^\top \Gamma(\theta', \tau') \mu(d\tau') \tilde{\gamma}(t) - S(\Gamma)(\theta)^\top \hat{N}(\Gamma)^{-1} h(t, \Gamma, \Lambda) \end{aligned} \quad (4.2.4)$$

$$\begin{aligned} \mathcal{R}_3(t, \Gamma, \Lambda) &= g_0(t)^\top Qg_0(t) + 2L^\top g_0(t) + \tilde{\gamma}(t)^\top \int_{\mathbb{R}_+^2} \mu(d\theta')^\top \Gamma(\theta', \tau') \mu(d\tau') \tilde{\gamma}(t) \\ &\quad + 2\tilde{\beta}(t)^\top \int_{\mathbb{R}_+} \mu(d\theta')^\top \Lambda(\theta') - h(t, \Gamma, \Lambda)^\top \hat{N}(\Gamma)^{-1} h(t, \Gamma, \Lambda), \end{aligned} \quad (4.2.5)$$

with

$$\begin{aligned} S(\Gamma)(\tau) &= C^\top \int_{\mathbb{R}_+} \mu(d\theta)^\top \Gamma(\theta, \tau) + F^\top \int_{\mathbb{R}_+^2} \mu(d\theta')^\top \Gamma(\theta', \tau') \mu(d\tau') D \\ \hat{N}(\Gamma) &= N + F^\top \int_{\mathbb{R}_+^2} \mu(d\theta)^\top \Gamma(\theta, \tau) \mu(d\tau) F \\ h(t, \Gamma, \Lambda) &= C^\top \int_{\mathbb{R}_+} \mu(d\theta)^\top \Lambda(\theta) + F^\top \int_{\mathbb{R}_+^2} \mu(d\theta)^\top \Gamma(\theta, \tau) \mu(d\tau) \tilde{\gamma}(t). \end{aligned} \quad (4.2.6)$$

The two following definitions specify the concept of solution to the system (4.2.2).

Definition 4.6. Let $\Gamma : \mathbb{R}_+^2 \rightarrow \mathbb{R}^{d \times d}$ such that $\Gamma \in L^\infty(\mu \otimes \mu)$. We say that Γ is symmetric if

$$\Gamma(\theta, \tau) = \Gamma(\tau, \theta)^\top, \quad \mu \otimes \mu - a.e.$$

and nonnegative if

$$\int_{\mathbb{R}_+^2} \varphi(\theta)^\top \mu(d\theta)^\top \Gamma(\theta, \tau) \mu(d\tau) \varphi(\tau) \geq 0, \quad \text{for all } \varphi \in L^1(\mu).$$

We denote by $\mathcal{S}_+^d(\mu \otimes \mu)$ the set of all symmetric and nonnegative $\Gamma \in L^\infty(\mu \otimes \mu)$.

Remark 4.7. The integral operator $\mathbf{\Gamma}$ associated to a symmetric kernel $\Gamma \in L^\infty(\mu \otimes \mu)$ is symmetric, in the sense that

$$\langle \varphi, \mathbf{\Gamma} \psi \rangle_\mu = \langle \psi, \mathbf{\Gamma} \varphi \rangle_\mu, \quad \varphi, \psi \in L^1(\mu).$$

Moreover, the nonnegativity of Γ translates into

$$\langle \varphi, \mathbf{\Gamma} \varphi \rangle_\mu \geq 0, \quad \varphi \in L^1(\mu).$$

Definition 4.8. By a solution to the system (4.2.2), we mean a triplet $(\Gamma, \Lambda, \chi) \in C([0, T], L^1(\mu \otimes \mu)) \times C([0, T], L^1(\mu^\top)) \times C([0, T], \mathbb{R})$ such that

$$\Gamma_t(\theta, \tau) = \int_t^T e^{-(\theta+\tau)(s-t)} \mathcal{R}_1(\Gamma_s)(\theta, \tau) ds, \quad 0 \leq t \leq T, \quad \mu \otimes \mu - a.e. \quad (4.2.7)$$

$$\Lambda_t(\theta) = \int_t^T e^{-\theta(s-t)} \mathcal{R}_2(s, \Gamma_s, \Lambda_s)(\theta) ds, \quad 0 \leq t \leq T, \quad \mu - a.e. \quad (4.2.8)$$

$$\chi_t = \int_t^T \mathcal{R}_3(s, \Gamma_s, \Lambda_s) ds, \quad 0 \leq t \leq T, \quad (4.2.9)$$

where \mathcal{R}_1 , \mathcal{R}_2 and \mathcal{R}_3 are defined respectively by (4.2.3), (4.2.4) and (4.2.5). In particular $\hat{N}(\Gamma_t)$ given by (4.2.6) is invertible for all $t \leq T$.

The existence and uniqueness of a solution to the Riccati system is proved in [14], and is stated in the next theorem.

Theorem 4.9. *Let g_0, β, γ be bounded functions on $[0, T]$. Assume that μ satisfies (4.1.2) and that*

$$Q \in \mathbb{S}_+^d, \quad N - \lambda I_m \in \mathbb{S}_+^m, \quad (4.2.10)$$

for some $\lambda > 0$. Then, there exists a unique triplet $(\Gamma, \Lambda, \chi) \in C([0, T], L^1(\mu \otimes \mu)) \times C([0, T], L^1(\mu^\top)) \times C([0, T], \mathbb{R})$ to the system of Riccati equation (4.2.2) such that (4.2.7), (4.2.8), (4.2.9) hold and $\Gamma_t \in \mathbb{S}_+^d(\mu \otimes \mu)$, for all $t \leq T$. Furthermore, there exists some positive constant $M > 0$ such that

$$\int_{\mathbb{R}_+} |\mu|(d\tau) |\Gamma_t(\theta, \tau)| \leq M, \quad \text{for } \mu\text{-almost every } \theta, \quad 0 \leq t \leq T.$$

Our second main result addresses the solvability of the problem (4.1.5). Theorem 4.10 establishes the existence of an optimal feedback control of linear form and provides an explicit expression for the value function in terms of the solution to the Riccati equation.

Theorem 4.10. *Let β, γ be bounded functions on $[0, T]$ and g_0 be continuous. Fix K, μ as in (4.1.1)–(4.1.2)–(4.1.3). Under (4.2.10), let (Γ, Λ, χ) be the solution to the system of Riccati equation (4.2.2) produced by Theorem 4.9. Then, there exists an admissible control $\alpha^* \in \mathcal{A}$ with corresponding controlled trajectory Y^{α^*} as in (4.1.11) such that*

$$\alpha_t^* = -\hat{N}(\Gamma_t)^{-1} \left(h(t, \Gamma_t, \Lambda_t) + \int_{\mathbb{R}_+} S(\Gamma_t)(\theta) \mu(d\theta) Y_t^{\alpha^*}(\theta) \right) \quad (4.2.11)$$

for all $t \leq T$. Furthermore, α^* is an admissible optimal control, in the sense that

$$\inf_{\alpha \in \mathcal{A}} J(\alpha) = J(\alpha^*),$$

Y^{α^*} is the optimally controlled trajectory of the state variable and $V_t^{\alpha^*}$ given by (4.2.1) is the optimal value process of the problem, that is

$$V_t^{\alpha^*} = \inf_{\alpha \in \mathcal{A}_t(\alpha^*)} \mathbb{E} \left[\int_t^T f(X_s^\alpha, \alpha_s) ds \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T, \quad (4.2.12)$$

where $\mathcal{A}_t(\alpha) = \{\alpha' \in \mathcal{A} : \alpha'_s = \alpha_s, s \leq t\}$.

Remark 4.11. From (4.2.12), it follows that at initial time $t = 0$, the optimal value V_0 is equal to $V_0 = V_0^{\alpha^*} = \chi_0$, hence

$$V_0 = \int_0^T \mathcal{R}_3(t, \Gamma_t, \Lambda_t) dt.$$

In particular, for a constant initial condition $g_0(t) \equiv X_0$ for some $X_0 \in \mathbb{R}^d$, we have

$$V_0 = X_0^\top \Psi(T) X_0 + \Phi(T) X_0 + \xi(T),$$

for suitable functions Ψ, Φ, ξ , which corresponds to the usual linear–quadratic form in X_0 . However, because of the possible non-Markovianity of the problem, for $t > 0$, the optimal value $V_t^{\alpha^*}$ is not necessarily linear–quadratic in $X_t^{\alpha^*}$ as in the standard case.

Remark 4.12. Conventional linear–quadratic models, see for instance [176, Chapter 7], are naturally nested in our framework. Indeed, they are recovered by setting $d = d'$ and $\mu = \delta_0 I_d$, which corresponds

to $K(t) \equiv I_d$. In this case, the Riccati equations for $\Gamma(0, 0)$, $\Lambda(0)$, χ reduce to the conventional matrix Riccati equations and $Y^\alpha = X^\alpha$ so that we recover the usual expression for the optimal control (4.2.11) and the value function

$$\begin{aligned}\alpha_t^* &= -\hat{N}(\Gamma_t(0))^{-1} \left(h(t, \Gamma_t(0, 0), \Lambda_t(0)) + S(\Gamma_t)(0)X_t^{\alpha^*} \right), \\ V_t &= X_t^\top \Gamma_t(0, 0)X_t + 2X_t^\top \Lambda_t(0) + \chi_t.\end{aligned}$$

Conventional linear–quadratic models can also be recovered by considering a kernel which is a weighted sum of exponentials as detailed in the following example. This will turn out to be of crucial importance in the next section.

Example 4.13. We set $d = d' = m = 1$ and

$$K^n(t) = \sum_{i=1}^n c_i^n e^{-\theta_i^n t}, \quad (4.2.13)$$

for some $n \in \mathbb{N}$, $c_i^n \in \mathbb{R}$, $\theta_i^n \geq 0$, $i = 1, \dots, n$. This corresponds to (4.1.1) with $\mu(d\theta) = \sum_{i=1}^n c_i^n \delta_{\theta_i^n}(d\theta)$ and Theorem 4.3 gives the representation

$$X_t^{n,\alpha} = g_0^n(t) + \sum_{i=1}^n c_i^n Y_t^{n,i,\alpha}, \quad (4.2.14)$$

where $Y^{n,i,\alpha} := Y^\alpha(\theta_i^n)$ are such that

$$\begin{aligned}dY_t^{n,i,\alpha} &= \left[-\theta_i^n Y_t^{n,i,\alpha} + \tilde{b}\left(t, \sum_{j=1}^n c_j^n Y_t^{n,j,\alpha}, \alpha_t\right) \right] dt + \tilde{\sigma}\left(t, \sum_{j=1}^n c_j^n Y_t^{n,j,\alpha}, \alpha_t\right) dW_t \\ Y_0^{n,i,\alpha} &= 0, \quad i = 1, \dots, n.\end{aligned} \quad (4.2.15)$$

Whence, the problem reduces to a conventional linear-quadratic control for the finite-dimensional controlled system $(Y^{n,i,\alpha})_{1 \leq i \leq n}$. In particular, the system of Riccati (4.2.2) reduces to a standard one in finite-dimension. For instance the equation for Γ reduces to the standard $n \times n$ -matrix Riccati equation

$$\begin{cases} \dot{\Gamma}_t^n &= -Q^n - (B^n)^\top \Gamma_t^n - \Gamma_t^n B^n - (D^n)^\top \Gamma_t^n D^n \\ &\quad + ((F^n)^\top \Gamma_t^n D^n + (C^n)^\top \Gamma_t^n)^\top (N^n + (F^n)^\top \Gamma_t^n F^n)^{-1} ((F^n)^\top \Gamma_t^n D^n + (C^n)^\top \Gamma_t^n) \\ \Gamma_T^n &= 0, \end{cases} \quad (4.2.16)$$

where the coefficients $(B^n, C^n, D^n, F^n, N^n, Q^n) \in \mathbb{R}^{n \times n} \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}_+^n$ are defined by

$$\begin{aligned}B_{i,j}^n &= Bc_i^n - \theta_i^n \delta_{ij}, & D_{i,j}^n &= Dc_i^n, \\ C_i^n &= Cc_i^n, & F_i^n &= Fc_i^n, \\ Q_{i,j}^n &= Q, & N^n &= N,\end{aligned}$$

for all $1 \leq i, j \leq n$.

4.2.2 Stability and approximation by conventional LQ problems

Another main result of the paper [14] concerns the approximation of the possibly non-Markovian control problem by sequences of finite dimensional Markovian ones, which is of crucial importance for numerical implementations. The main idea comes from the approximation of the measure μ , appearing in (4.1.1), by simpler measures μ^n , or equivalently approximating K by simpler kernels K^n

given by

$$K^n(t) = \int_{\mathbb{R}_+} e^{-\theta t} \mu^n(d\theta), \quad t > 0. \quad (4.2.17)$$

We also authorize the approximation of the input curve g_0 . By substituting (K, g_0) with (K^n, g_0^n) , the approximating problem reads

$$V_0^n = \inf_{\alpha \in \mathcal{A}} J^n(\alpha)$$

where

$$J^n(\alpha) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T ((X_s^{n,\alpha})^\top Q X_s^{n,\alpha} + 2L_s^\top X_s^{n,\alpha} + \alpha_s^\top N \alpha_s) ds \right],$$

$$X_{t|n, \alpha} = g_0^n(t) + \int_0^t K^n(t-s) b(s, X_s^{n,\alpha}, \alpha_s) ds + \int_0^t K^n(t-s) \sigma(s, X_s^{n,\alpha}, \alpha_s) dW_s.$$

The following theorem establishes the stability of stochastic Volterra linear–quadratic control problems.

Theorem 4.14. *Let β, γ be bounded and measurable functions on $[0, T]$ and g_0 be continuous. Assume that μ satisfies (4.1.2)–(4.1.3) and let K be as in (4.1.1). Let $(g_0^n)_{n \geq 1}$ be a sequence of continuous functions and $(K^n)_{n \geq 1}$ be a sequence of kernels of the form (4.2.17) with respective measures μ^n satisfying (4.1.2)–(4.1.3), for each $n \in \mathbb{N}$. Assume (4.2.10) and that Q is invertible. Denote by V^* and V^{n*} the respective optimal value processes given by Theorem 4.10 for the respective inputs (g_0, K) and (g_0^n, K^n) , for $n \geq 1$. If*

$$\|K^n - K\|_{L^2(0,T)} \rightarrow 0 \quad \text{and} \quad \|g_0^n - g_0\|_{L^2(0,T)} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

then,

$$V_0^{n*} \rightarrow V_0^*, \quad \text{as } n \rightarrow \infty,$$

with a rate of convergence given by

$$|V_0^* - V_0^{n*}| \leq c \left(\|g_0^n - g_0\|_{L^2(0,T)} + \|K^n - K\|_{L^2(0,T)} \right),$$

for some positive constant c independent of n .

Combined with Example 4.13, Theorem 4.14 provides an approximation of linear–quadratic stochastic Volterra optimal control problems by conventional Markovian linear–quadratic models in finite dimension. To ease notations we restrict to the case $d = d' = m = 1$, for higher dimension matrices need to be replaced by tensors in what follows. The idea is to approximate μ by a discrete measure μ^n as follows. Fix $n \geq 1$ and $(\eta_i^n)_{0 \leq i \leq n}$ a partition of \mathbb{R}_+ . Let $\mu^n(d\theta) = \sum_{i=1}^n c_i^n \delta_{\theta_i^n}(d\theta)$ with

$$c_i^n = \int_{\eta_{i-1}^n}^{\eta_i^n} \mu(dx) \quad \text{and} \quad \theta_i^n = \frac{1}{c_i^n} \int_{\eta_{i-1}^n}^{\eta_i^n} \theta \mu(d\theta), \quad i = 1, \dots, n. \quad (4.2.18)$$

Then, for a suitable choice of the partition $(\eta_i^n)_{0 \leq i \leq n}$, we obtain the convergence

$$\|K^n - K\|_{L^2(0,T)} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

where K^n is given by (4.2.13), see for instance [7, Proposition 3.3 and Remark 3.4]. In particular, with the fractional kernel K_H given by (4.1.6), an even n , and the geometric partition $\eta_i^n = r_n^{i-n/2}$,

$i = 0, \dots, n$, for some $r_n > 1$, the coefficients (4.2.18) with μ_H as in (4.1.7) are explicitly given by

$$c_i^n = \frac{(r_n^{1-\alpha} - 1)r_n^{(\alpha-1)(1+n/2)}}{\Gamma(\alpha)\Gamma(2-\alpha)} r_n^{(1-\alpha)i} \quad \text{and} \quad x_i^n = \frac{1 - \alpha r_n^{2-\alpha} - 1}{2 - \alpha r_n^{1-\alpha} - 1} r_n^{i-1-n/2}, \quad i = 1, \dots, n,$$

where $\alpha := H + 1/2$. If the sequence $(r_n)_{n \geq 1}$ satisfies

$$r_n \downarrow 1 \quad \text{and} \quad n \ln r_n \rightarrow \infty, \quad \text{as } n \rightarrow \infty,$$

then,

$$\|K^n - K_H\|_{L^2(0,T)} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

see [1, Lemma A.3]. In practice, the free parameter r_n can be chosen by minimizing the L^2 norm between K^n and K_H , for instance if $n = 20$, setting $r_{20} = 2.5$ yields very good approximations for the un-controlled stochastic Volterra equation, see [1] for a more detailed practical study. For each n , the approximate control problem is a conventional linear quadratic one in finite dimension for the state variables (4.2.15) with the standard $n \times n$ matrix Riccati equation (4.2.16). This allows to numerically solve the Riccati equations and simulate the process $X^{n,\alpha}$ given by (4.2.14), leading to computation of the value function V_0^{n*} and the optimal control α^n as in (4.2.11) with μ replaced by μ^n .

Related literature. The optimal control of stochastic Volterra equations has been considered in [175] by maximum principle method leading to a characterization of the solution in terms of a backward stochastic Volterra equation for the adjoint process. In [18], the authors also use the maximum principle together with Malliavin calculus to obtain a corresponding adjoint equation as a standard backward SDE. Although the kernel considered in these aforementioned papers is not restricted to be of convolution type, the required conditions do not allow singularity of K at zero, hence excluding the case of a fractional kernel with parameter $H < 1/2$. More recently, an extended Bellman equation has been derived in [119] for the associated controlled Volterra equation. The solution to the LQ control problem with controlled drift and additive noise has been obtained in [135] when the noise is a fractional Brownian motion with Hurst parameter $H > 1/2$, and in [78] when the noise is a general Gaussian process with an optimal control expressed as the sum of the well-known linear feedback control for the associated deterministic linear-quadratic control problem and the prediction of the response of the system to the future noise process. Recently, the paper [173] investigated LQ problem of stochastic Volterra equations by providing characterizations of optimal control in terms of some forward-backward system, but leaving aside their solvability, and under some coefficients assumptions that preclude singular kernels such as the fractional kernel with parameter $H < 1/2$. A related infinite-dimensional Riccati equation appeared in [20] for the minimization problem of an energy functional defined in terms of a non-singular (i.e. $K(0) < \infty$) completely monotone kernel. We stress that, although there exists several results for LQ control problems in infinite-dimension, and even for Volterra processes (see [40]), they cannot be applied in our Banach-space context as they only concern Hilbert spaces. Finally, we mention that some financial problems such as optimal execution with transient Market impact [20] and hedging in the presence of Market impact with fractional Brownian motion [27] can be formulated as controlled Volterra problems of linear-quadratic type, we refer to Chapter 5.

Chapter 5

Portfolio optimization and liquidation in a Volterra framework

Summary

In this chapter, we explicitly solve two intricate control problems in finance under a Volterra framework:

- Markowitz portfolio allocation problem for multivariate quadratic Volterra models,
- Optimal liquidation with general transient impact and trading signals.

Based on:

[12] Abi Jaber, E., Miller, E., & Pham, H. *Markowitz portfolio selection for multivariate affine and quadratic Volterra models*, [SIAM journal on Financial Mathematics](#), 12(1), 369-409, 2021.

[Jupyter Notebook](#)

[8] Abi Jaber, E. & Neuman, E. *Optimal Liquidation with Signals: the General Propagator Case*, Submitted to *Mathematical Finance*. [Jupyter Notebook](#)

Both models considered in this chapter involve non-convolution kernels, they fall outside the scope of the framework developed in the previous chapter. Yet, we are still able to solve both problems in terms of explicit operator solutions similar to the one that appear in Chapter 3 by making educated quadratic Ansatz on an adjusted forward process. The related operator Riccati equations that appear can be solved explicitly exploiting again the structure of the solution that has been identified earlier in Chapter 3 in an un-controlled case.

5.1 Markowitz portfolio allocation problem for multivariate quadratic Volterra models [12]

The Markowitz [147] mean-variance portfolio selection problem is the cornerstone of modern portfolio allocation theory. Investment decisions rules are made according to a trade-off between return and risk, and the use of Markowitz efficient portfolio strategies in the financial industry has become quite popular mainly due to its natural and intuitive formulation. A vast volume of research has been devoted over the last decades to extend Markowitz problem from static to continuous-time setting, first in Black-Scholes and complete markets ([178]), and then to consider more general frameworks with random coefficients and multiple assets, see e.g. [142], [60], or more recently [129] for taking into account model uncertainty on the assets correlation.

In the direction of more realistic modeling of asset prices, recent research on portfolio optimization consider fractional and rough models: [89, 29, 120] consider fractional Ornstein-Uhlenbeck and Heston stochastic volatility models for power utility function criterion, and the work [118] studies the Markowitz problem in a Volterra Heston model. These developments have been carried out in the mono-asset case. However, investment in multi-assets by taking into account the correlation risk is an importance feature in portfolio choice in financial markets. The basic goal of our work in [12] is to enrich the literature on portfolio selection:

- (i) by introducing a class of multivariate Volterra models, which captures stylized facts of financial assets, namely various rough volatility patterns across assets, (possibly random) correlation between stocks, and leverage effects, i.e., correlation between a stock and its volatility.
- (ii) by keeping the model tractable for explicit computations of the optimal Markowitz portfolio strategy, which can be a quite challenging task in multivariate non-Markovian settings.

Main contributions. In the work [12], we study the continuous-time Markowitz problem in a multivariate setting with a focus on two classes: (i) affine Volterra models as in [10] that include multivariate rough Heston models, (ii) quadratic Volterra models, which are new class of Volterra models, and embrace multivariate rough Stein-Stein models, and rough Wishart type covariance matrix models, in the spirit of (3.0.1). We provide:

- **A generic verification result** for the corresponding mean-variance problem, which is formulated in an incomplete non-Markovian and non-semimartingale framework with unbounded random coefficients of the volatility and market price of risk, and under general filtration. This result expresses the solution to the Markowitz problem in terms of a Riccati backward stochastic differential equation (BSDE) by checking in particular the admissibility condition of the optimal control. We stress that related existing verification results in the literature (see [142], [132], [60], [169]) cannot be applied directly to our setting, and we shall discuss more in detail this point in Section 5.1.2.
- **Explicit solutions** to the Riccati BSDE in two concrete specifications of multivariate Volterra models exploiting the representation of the solution in terms of a Laplace transform:
 1. **the affine case:** the optimal Markowitz strategy is expressed in terms of multivariate Riccati-Volterra equations which naturally extends the one obtained in [118]. We point out that the martingale distortion arguments used in [118] for the univariate Volterra Heston model, do not apply in higher dimensions, unless the correlation structure is highly degenerate.
 2. **the quadratic case:** our major result is to derive analytic expressions for the optimal investment strategy by explicitly solving operator Riccati equations. This gives new explicit formulae for rough Stein-Stein and Wishart type covariance models. These analytic expressions can be efficiently implemented: the integral operators can be approximated by closed form expressions involving finite dimensional matrices and the underlying processes can be simulated by the celebrated Cholesky decomposition algorithm.
- **Numerical simulations** of the optimal Markowitz strategy in a two-asset rough Stein-Stein model to illustrate our results.¹ We depict the impact of some parameters onto the optimal investment when one asset is rough, and the other smooth (in the sense of the Hurst index of their volatility), and show in particular that for positively correlated assets, the optimal strategy is to “buy rough, sell smooth”, which is consistent with the empirical backtesting in Glasserman and He [102].

¹The code of our implementation can be found at the following [link](#).

Notations. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions, we denote by

$$\begin{aligned} L_{\mathbb{F}}^{\infty}([0, T], \mathbb{R}^d) &= \{Y : \Omega \times [0, T] \mapsto \mathbb{R}^d, \mathbb{F}\text{-prog. measurable and bounded a.s.}\} \\ L_{\mathbb{F}}^p([0, T], \mathbb{R}^d) &= \left\{ Y : \Omega \times [0, T] \mapsto \mathbb{R}^d, \mathbb{F}\text{-prog. measurable s.t. } \mathbb{E} \left[\int_0^T |Y_s|^p ds \right] < \infty \right\} \\ S_{\mathbb{F}}^{\infty}([0, T], \mathbb{R}^d) &= \left\{ Y : \Omega \times [0, T] \mapsto \mathbb{R}^d, \mathbb{F}\text{-prog. measurable s.t. } \sup_{t \leq T} |Y_t(w)| < \infty \text{ a.s.} \right\}. \end{aligned}$$

Here $|\cdot|$ denotes the Euclidian norm on \mathbb{R}^d . Classically, for $p \in [1, \infty)$, we define $L_{\mathbb{F}}^{p,loc}([0, T], \mathbb{R}^d)$ as the set of progressive processes Y for which there exists a sequence of increasing stopping times $\tau_n \uparrow \infty$ such that the stopped processes Y^{τ_n} are in $L_{\mathbb{F}}^p([0, T], \mathbb{R}^d)$ for every $n \geq 1$, and we recall that it consists of all progressive processes Y s.t. $\int_0^T |Y_t|^p dt < \infty$, a.s. To unclutter notation, we write $L_{\mathbb{F}}^{p,loc}([0, T])$ instead of $L_{\mathbb{F}}^{p,loc}([0, T], \mathbb{R}^d)$ when the context is clear.

5.1.1 Formulation of the problem

Fix $T > 0$, $d, N \in \mathbb{N}$. We consider a financial market on $[0, T]$ on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with a non-risky asset S^0 , $dS_t^0 = S_t^0 r(t) dt$, with a deterministic short rate $r : \mathbb{R}_+ \rightarrow \mathbb{R}$, and d risky assets with dynamics

$$dS_t = \text{diag}(S_t) [(r(t)\mathbf{1}_d + \sigma_t \lambda_t) dt + \sigma_t dB_t], \quad (5.1.1)$$

driven by a d -dimensional Brownian motion B , with a $d \times d$ -matrix valued stochastic volatility process σ and a \mathbb{R}^d -valued continuous stochastic process λ , called *market price of risk*. Here $\mathbf{1}_d$ denotes the vector in \mathbb{R}^d with all components equal to 1. The market is typically incomplete, in the sense that the dynamics of the continuous volatility process σ is driven by an N -dimensional process $W = (W^1, \dots, W^N)^\top$ defined by:

$$W_t^k = C_k^\top B_t + \sqrt{1 - C_k^\top C_k} B_t^{\perp, k}, \quad k = 1, \dots, N, \quad (5.1.2)$$

where $C_k \in \mathbb{R}^d$ s.t. $C_k^\top C_k \leq 1$, and $B^\perp = (B^{\perp, 1}, \dots, B^{\perp, N})^\top$ is an N -dimensional Brownian motion independent of B . Note that $d\langle W^k \rangle_t = dt$ but W^k and W^j can be correlated, hence W is not necessarily a Brownian motion. Observe that processes λ and σ are \mathbb{F} -adapted, possibly unbounded, but not necessarily adapted to the filtration generated by W . We point out that \mathbb{F} may be strictly larger than the augmented filtration generated by B and B^\perp as we shall deal with weak solutions to stochastic Volterra equations.

Remark 5.1. In our applications, we will be chiefly interested in the case where λ_t is linear in σ_t , and where the dynamics of the matrix-valued process σ is governed by a Volterra equation of the form

$$\sigma_t = g_0(t) + \int_0^t \mu(t, s, \omega) ds + \int_0^t \chi(t, s, \omega) dW_s.$$

The class of models that we developed includes in particular the case of Volterra Heston model when $d = 1$ with $\lambda_t = \theta \sigma_t$, for some constant θ , as studied in [118], and the case of Wishart process for the covariance matrix process $V_t = \sigma_t \sigma_t^\top$, as studied in [60].

Mean-variance optimization problem. Let π_t denote the vector of the amounts invested in the risky assets S at time t in a self-financing strategy and set $\alpha = \sigma^\top \pi$. Then, the dynamics of the wealth X^α of the portfolio we seek to optimize is given by

$$dX_t^\alpha = (r(t)X_t^\alpha + \alpha_t^\top \lambda_t) dt + \alpha_t^\top dB_t, \quad t \geq 0, \quad X_0^\alpha = x_0 \in \mathbb{R}. \quad (5.1.3)$$

By a solution to (5.1.3), we mean an \mathbb{F} -adapted continuous process X^α satisfying (5.1.3) on $[0, T]$ \mathbb{P} -a.s. and such that

$$\mathbb{E}\left[\sup_{t \leq T} |X_t^\alpha|^2\right] < \infty. \quad (5.1.4)$$

The set of admissible investment strategies is naturally defined by

$$\mathcal{A} = \{\alpha \in L_{\mathbb{F}}^{2,loc}([0, T], \mathbb{R}^d) \text{ such that (5.1.3) has a solution satisfying (5.1.4)}\}.$$

The Markowitz portfolio selection problem in continuous-time consists in solving the following constrained problem

$$V(m) := \inf_{\alpha \in \mathcal{A}} \{\text{Var}(X_T) : \text{s.t. } \mathbb{E}[X_T] = m\}. \quad (5.1.5)$$

given some expected return value $m \in \mathbb{R}$, where $\text{Var}(X_T) = \mathbb{E}[(X_T - \mathbb{E}[X_T])^2]$ stands for the variance.

5.1.2 A generic verification result

In this section, we establish a generic verification result for the optimization problem (5.1.5) given the solution of a certain Riccati BSDE. We stress that our mean-variance problem deals with incomplete markets with unbounded random coefficients σ and λ , so that existing results cannot be applied directly to our setting: Lim [142] presents a general methodology to solve the MV problem for the wealth process (5.1.3) in an incomplete market without assuming any particular dynamics on σ nor that the excess return is proportional to σ . However, a nondegeneracy assumption is made on $\sigma\sigma^\top$, see Lim [142, Assumption (A.1)]. The main verification result in Lim [142, Proposition 3.3], based on a completion of squares argument, states that if a solution to a certain (nonlinear) Riccati BSDE exists, then the MV is solvable. The difficulty resides in proving the existence of solutions to such nonlinear BSDEs (see also Lim and Zhou [143] for similar results in complete markets).

Here, we assume that the excess return is proportional to σ (instead of the nondegeneracy condition) and state a verification result in terms of solutions of Riccati BSDEs (completion of squares, ie LQ problem with random coefficients).

Theorem 5.2 below, can be seen as unifying framework for the aforementioned results. We define $C \in \mathbb{R}^{N \times d}$ by

$$C = (C_1, \dots, C_N)^\top,$$

where we recall that the vectors $C_i \in \mathbb{R}^d$ come from the correlation structure (5.1.2). We will use the matrix norm $|A| = \text{tr}(A^\top A)$ in the subsequent theorem.

Theorem 5.2. *Assume that there exists a solution triplet $(\Gamma, Z^1, Z^2) \in \mathbb{S}_{\mathbb{F}}^\infty([0, T], \mathbb{R}) \times L_{\mathbb{F}}^{2,loc}([0, T], \mathbb{R}^d) \times L_{\mathbb{F}}^{2,loc}([0, T], \mathbb{R}^N)$ to the Riccati BSDE*

$$\begin{cases} d\Gamma_t &= \Gamma_t \left[(-2r(t) + |\lambda_t + Z_t^1 + CZ_t^2|^2) dt + (Z_t^1)^\top dB_t + (Z_t^2)^\top dW_t \right], \\ \Gamma_T &= 1, \end{cases} \quad (5.1.6)$$

such that

$$(H1) \quad 0 < \Gamma_0 < e^{2 \int_0^T r(s) ds}, \text{ and } \Gamma_t > 0, \text{ for all } t \leq T,$$

$$(H2)$$

$$\mathbb{E} \left[\exp \left(a(p) \int_0^T (|\lambda_s|^2 + |Z_s^1|^2 + |Z_s^2|^2) ds \right) \right] < \infty,$$

for some $p > 2$ and a constant $a(p)$ given by

$$a(p) = \max \left[p(3 + |C|), 3(8p^2 - 2p)(1 + |C|^2) \right].$$

Then, the optimal investment strategy for the Markowitz problem (5.1.5) is given by the admissible control

$$\alpha_t^* = -(\lambda_t + Z_t^1 + CZ_t^2)(X_t^{\alpha^*} - \xi^* e^{-\int_t^T r(s)ds}),$$

where

$$\xi^* = \frac{m - \Gamma_0 e^{-\int_0^T r(t)dt} x_0}{1 - \Gamma_0 e^{-2\int_0^T r(t)dt}}. \quad (5.1.7)$$

Furthermore, the value of (5.1.5) for the optimal wealth process $X^* = X^{\alpha^*}$ is

$$V(m) = \text{Var}(X_T^*) = \Gamma_0 \frac{|x_0 - m e^{-\int_0^T r(t)dt}|^2}{1 - \Gamma_0 e^{-2\int_0^T r(t)dt}}. \quad (5.1.8)$$

In the sequel, we will provide concrete specifications of multivariate stochastic Volterra models for which the solution to the non-linear Riccati BSDE (5.1.6) can be computed in closed and semi-closed forms, while satisfying conditions (H1) and (H2). The key idea is to observe that, first, if such solution exists, then, it admits the following representation as a Laplace transform:

$$\Gamma_t = \mathbb{E} \left[\exp \left(\int_t^T (2r(s) - |\lambda_s + Z_s^1 + CZ_s^2|^2) ds \right) \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$

In the special case where λ is deterministic, then the solution to (5.1.6) trivially exists with $Z^1 = Z^2 = 0$, and condition (H1) and (H2) are obviously satisfied when λ is nonzero and bounded. In the general case where λ is an (unbounded) stochastic process, the admissibility of the optimal control is obtained under finiteness of a certain exponential moment of the solution triplet (Γ, Z^1, Z^2) and the risk premium λ as precised in (H2). Such estimate is crucial to deal with the unbounded random coefficients in (5.1.3), see for instance Han and Wong [118], Shen et al. [170], Shen [169] where similar conditions appear. If the coefficients are bounded, such condition is not needed, see Lim [142, Lemma 3.1].

Our main interest is to find specific dynamics for the volatility σ and for the market price of risk λ such that the Laplace transform can be computed in (semi)-explicit form. We shall consider models as mentioned in Remark 5.1, where all the randomness in λ comes from the process W driving σ , and for which we naturally expect that $Z^1 = 0$. We solve more specifically this problem for two classes of models:

1. Multivariate affine Volterra models of Heston type. This extends the results of Han and Wong [118] to the multi dimensional case and provides semi-closed formulas. This part will not be discussed here, we refer to the paper [12] for the full treatment.
2. Multivariate quadratic Volterra models of Stein-Stein and Wishart type in Section 5.1.3 for which we derive new closed-form solutions, which is detailed below.

5.1.3 Multivariate quadratic Volterra models

The model. In this section, we assume that the components of the stochastic volatility matrix σ in (5.1.1) are given by $\sigma^{ij} = \gamma_{ij}^\top Y$, where $\gamma_{ij} \in \mathbb{R}^N$ and $Y = (Y^1, \dots, Y^N)^\top$ is the following

N -dimensional Volterra Ornstein–Uhlenbeck process

$$Y_t = g_0(t) + \int_0^t K(t, s)DY_s ds + \int_0^t K(t, s)\eta dW_s, \quad (5.1.9)$$

where $D, \eta \in \mathbb{R}^{N \times N}$, $g_0 : \mathbb{R}_+ \rightarrow \mathbb{R}^N$ is locally bounded, W is a N -dimensional process as in (5.1.2), i.e.,

$$W_t^k = C_k^\top B_t + \sqrt{1 - C_k^\top C_k} B_t^{\perp, k}, \quad (5.1.10)$$

where $C_k \in \mathbb{R}^d$, such that $C_k^\top C_k \leq 1$, $k = 1, \dots, N$, and $K : [0, T]^2 \rightarrow \mathbb{R}^{N \times N}$ is a Volterra kernel in L^2 . We stress that the process W is not necessarily a N -dimensional Brownian motion due to the possible correlations.

Furthermore, the risk premium is assumed to be in the form

$$\lambda_t = \Theta Y_t, \quad t \leq T,$$

for some $\Theta \in \mathbb{R}^{d \times N}$, so that the dynamics for the stock prices (5.1.1) reads as

$$dS_t^i = S_t^i \left(r(t) + \sum_{k, \ell=1}^N \sum_{j=1}^d \gamma_{ij}^\ell \Theta^{jk} Y_t^\ell Y_t^k \right) dt + S_t^i \sum_{j=1}^d \gamma_{ij}^\top Y_t dB_t^j, \quad i = 1, \dots, d. \quad (5.1.11)$$

The appellation quadratic reflects the quadratic dependence of the drift and the covariance matrix of $\log S$ in Y . Such models nest as special cases the Volterra extensions of the celebrated Stein and Stein [171] or Schöbel and Zhu [168] model and certain Wishart models of Bru [47].

Remark 5.3. Note that with (5.1.10), there are no restrictions on the correlations between Y^i and the stocks S^i in (5.1.9) and (5.1.11), in contrast with the correlation structure in the multivariate Volterra Heston model. Moreover, the quadratic models allow us to deal with correlated stocks in contrast with the multivariate Heston model where no correlation between the driving Brownian motion of the assets S^i and S^j is allowed in order to keep the affine structure.

The explicit solution. We provide an explicit solution for the Markowitz problem for quadratic Volterra models, and our main result is stated in Theorem 5.5 below. Exploiting the quadratic structure of (5.1.9)–(5.1.11), recall Chapter 3, we provide an explicit solution to the Riccati BSDE in Lemma 5.4 below, in terms of the following family of linear operators $(\Psi_t)_{0 \leq t \leq T}$ acting on $L^2([0, T], \mathbb{R}^N)$:

$$\Psi_t = -(\text{id} - \hat{K})^{-*} \Theta^\top (\text{id} + 2\Theta \tilde{\Sigma}_t \Theta^\top)^{-1} \Theta (\text{id} - \hat{K})^{-1}, \quad 0 \leq t \leq T, \quad (5.1.12)$$

where $F^{-*} := (F^{-1})^*$, and \hat{K} is the integral operator induced by the kernel $\hat{K} = K(D - 2\eta C^\top \Theta)$ and $\tilde{\Sigma}_t$ the integral operator defined by

$$\tilde{\Sigma}_t = (\text{id} - \hat{K})^{-1} \Sigma_t (\text{id} - \hat{K})^{-*}, \quad t \in [0, T],$$

with Σ_t defined as the integral operator associated to the kernel

$$\Sigma_t(s, u) = \int_t^{s \wedge u} K(s, z) \eta (U - 2C^\top C) \eta^\top K(u, z)^\top dz, \quad t \in [0, T],$$

where $U = \frac{d(W)_t}{dt} = (1_{i=j} + 1_{i \neq j} (C_i)^\top C_j)_{1 \leq i, j \leq N}$.

For this, denote by g the process

$$g_t(s) = \mathbf{1}_{t \leq s} \left(g_0(s) + \int_0^t K(s, u)DY_u du + \int_0^t K(s, u)\eta dW_u \right). \quad (5.1.13)$$

One notes that for each, $s \leq T$, $(g_t(s))_{t \leq s}$ is the adjusted forward process

$$g_t(s) = \mathbb{E} \left[Y_s - \int_t^s K(s, u) DY_u du \mid \mathcal{F}_t \right], \quad s \geq t.$$

Lemma 5.4. *Assume that $(U - 2C^\top C) \in \mathbb{S}_+^N$. Let Ψ be the operator defined in (5.1.12). Then, the process (Γ, Z^1, Z^2) defined by*

$$\begin{cases} \Gamma_t &= \exp(\phi_t + \langle g_t, \Psi_t g_t \rangle_{L^2}), \\ Z_t^1 &= 0, \\ Z_t^2 &= 2((\Psi_t \mathbf{K} \eta)^* g_t)(t), \end{cases} \quad (5.1.14)$$

where g is given by (5.1.13) and ϕ is a deterministic function explicitly given in terms of Ψ , is a $\mathbb{S}_\mathbb{F}^\infty([0, T], \mathbb{R}) \times L_\mathbb{F}^2([0, T], \mathbb{R}^d) \times L_\mathbb{F}^2([0, T], \mathbb{R}^N)$ -valued solution to the Riccati-BSDE (5.1.6).

From Theorem 5.2, we can now explicitly solve the Markowitz problem (5.1.5) in the quadratic Volterra model (5.1.9), (5.1.10) and (5.1.11).

Theorem 5.5. *Modulo a certain moment condition that has been verified in [8], the optimal investment strategy for the Markowitz problem (5.1.5) is given by the admissible control*

$$\alpha_t^* = - \left((\Theta + 2C[\Psi_t \mathbf{K} \eta]^*) g_t \right)(t) \left(X_t^{\alpha^*} - \xi^* e^{-\int_t^T r(s) ds} \right),$$

where ξ^* is defined in (5.1.7), and the optimal value is given by (5.1.8) with Γ_0 as in (5.1.14).

The following corollary treats the standard Markovian and semimartingale case for $K = I_N$ and shows how to recover the well-known formulae in terms of the usual matrix Riccati equations in the spirit of [60].

Corollary 5.6. *Set $K(t, s) = I_N 1_{s \leq t}$ and $g_0(t) \equiv Y_0$ for some $Y_0 \in \mathbb{R}^N$. Then, the solution to the Riccati BSDE can be re-written in the form*

$$\Gamma_t = \exp(\phi_t + Y_t^\top P_t Y_t), \quad \text{and} \quad Z_t^2 = 2\eta^\top P_t Y_t,$$

where $P : [0, T] \mapsto \mathbb{R}^{N \times N}$ and ϕ solve the conventional system of $N \times N$ -matrix Riccati equations

$$\begin{cases} \dot{P}_t &= \Theta^\top \Theta + P_t(2\eta C^\top \Theta - D) + (2\eta C^\top \Theta - D)^\top P_t + 2P_t(\eta(U - 2C^\top C)\eta^\top)P_t, \\ P_T &= 0, \\ \dot{\phi}_t &= -2r(t) - \text{tr}(P_t \eta U \eta^\top), \quad t \in [0, T], \\ \phi_T &= 0. \end{cases}$$

Furthermore, the optimal control reads

$$\alpha_t^* = - \left(\Theta + 2C(D\eta)^\top P_t Y_t \right) \left(X_t^{\alpha^*} - \xi^* e^{-\int_t^T r(s) ds} \right).$$

Practitioner's corner. We proceed with a numerical experiment for a rough Stein-Stein model with two assets. We illustrate the results of Section 5.1.3 on a special case of the two dimensional rough Stein-Stein model. Our main motivation for considering the multivariate rough Stein-Stein model is to study the 'buy rough sell smooth' strategy of Glasserman and He [102] that was backtested empirically: this strategy consisting in buying the roughest assets while shorting on the smoothest ones was shown to be profitable. Our numerical results below provide new insights on the strategy by showing that the correlation between stocks plays a key role:

- $\rho < 0$: In the case of negatively correlated assets it is natural to expect the following strategy: pick both assets in order to be protected from volatility and benefit from the drift.

- $\rho > 0$: when the two stocks are positively correlated with $\rho > 0$, there is no minimization of variance through diversification by going long in both assets. Thus in the case a positively correlated assets, it is natural to expect the emergence of a starker choice between the assets. In the $\rho > 0$ case, see on Figure 5.1, we observe a *buy rough sell smooth* strategy as the one empirically found in Glasserman and He [102].

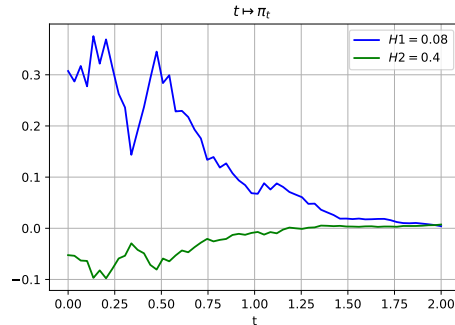


FIGURE 5.1: $\rho = 0.7$, when the two assets are positively correlated we recover *the buy rough sell smooth* strategy as it is described in [102]. (the parameters are: $H_1 = 0.08$, $H_2 = 0.4$, $T = 2.1$, $\eta_1 = \eta_2 = 1$, $c_i = -0.7$.)

5.2 Optimal liquidation with general transient impact and trading signals [8]

Price impact refers to the empirical fact that the execution of a large order affects the risky asset's price in an adverse and persistent manner leading to less favourable prices. Propagator models are a central tool in describing this phenomena mathematically. More precisely, if the trader's holdings in a risky asset is denoted by $Q = \{Q_t\}_{t \geq 0}$, then the asset price S_t is given by,

$$S_t = S_0 + \int_0^t G(t-s) dQ_s + M_t,$$

where M is a martingale and the price impact kernel G is called a propagator. It can be shown both from theoretical arguments such as market efficiency paradox and empirically that $G(t)$ must decay for large values of t , therefore the integral on the right-hand-side of the above equation is referred to as transient price impact (see e.g. Bouchaud et al. [44, Chapter 13]). The two extreme cases where G is Dirac's delta and when $G = 1$ are referred to as temporary price impact and permanent price impact, respectively. They are core features in the well known Almgren-Chriss model [22, 23], up to a multiplicative constant.

Considering the adverse effect of the price impact on the execution price, a trader who wishes to minimize her trading costs has to split her order into a sequence of smaller orders which are executed over a finite time horizon. At the same time, the trader also has an incentive to execute these split orders rapidly because she does not want to carry the risk of an adverse price move far away from her initial decision price. This trade-off between price impact and market risk is usually translated into a stochastic optimal control problem where the trader aims to minimize a risk-cost functional over a suitable class of execution strategies, see [52, 97, 103, 112, 141, 152] among others. In practice however, apart from focusing on the trade-off between price impact and market risk, many traders and trading algorithms also strive for using short term price predictors in their dynamic order execution schedules. Most of such documented predictors relate to order book dynamics as discussed in [139, 140, 144, 61]. From the modelling point of view, incorporating signals into execution problems translates into taking into consideration a non-martingale price process, which changes the problem significantly. The resulting optimal strategies in this setting are often random and in particular signal-adaptive, in

contrast to deterministic strategies, which are typically obtained in the martingale price case [46, 34]. Results on optimal trading with signals but without a transient price impact component (i.e. $G = 0$) were derived in [51, 140, 33].

Results on optimal liquidation problems with a general class of price impact kernels are scarce as the associated stochastic control problem is non-Markovian and often singular (except for the exponential kernel case which makes the problem Markovian [155, 145, 108, 56, 167, 153]). Indeed the transient price impact term and hence the asset execution price encode the entire trajectory of the agent's trading. A first contribution towards solving this problem was made by Gatheral et al. [98], who solved the deterministic case without signals and without a risk-aversion term. They minimised the following energy functional over left-continuous and adapted strategies $Q = \{Q_t\}_{t \geq 0}$ with a fuel constraint, i.e. $Q_{T+} = 0$

$$C(Q) = \int_{[0, T]} \int_{[0, T]} G(|t - s|) dQ_s dQ_t.$$

Here $C(Q)$ represents the trader's transaction costs and Q as before, is the trader's holdings in the risky asset. Under the assumption that the convolution kernel G is non-constant, nonincreasing, convex and integrable, a necessary and sufficient first order condition in the form of a Fredholm equation was derived in [98].

Our main objective in the paper [8] is to solve a general class of liquidation problems in the presence of linear transient price impact, which is induced by a nonnegative-definite Volterra-type propagator, along with taking into account a progressively measurable signal. We formulate these problems as a minimization of revenue-risk aversion functionals over a class of absolutely continuous and signal-adaptive strategies. Our solution to these problems solves an open problem put forward in [140] and also significantly extends the deterministic theory of Alfonsi and Schied [20].

Let $T > 0$ denote a finite deterministic time horizon and fix a progressively measurable processes $P = (P_t)_{0 \leq t \leq T}$ satisfying

$$\mathbb{E} \left[\int_0^T P_s^2 ds \right] < \infty.$$

We consider a trader with an initial position of $q > 0$ shares in a risky asset. The number of shares the trader holds at time $t \in [0, T]$ is prescribed as

$$Q_t^u = q - \int_0^t u_s ds,$$

where $(u_s)_{s \in [0, T]}$ denotes the trading speed which is chosen from the set of admissible strategies

$$\mathcal{A} := \left\{ u : u \text{ progressively measurable s.t. } \mathbb{E} \left[\int_0^T u_s^2 ds \right] < \infty \right\}.$$

We assume that the trader's trading activity causes price impact on the risky asset's execution price. We consider a Volterra kernel $G : [0, T]^2 \rightarrow \mathbb{R}_+$, that is $G(t, s) = 0$ for $s \geq t$, within a certain class of square-integrable admissible kernels. Then, we introduce the actual price S^u in which the orders are executed along a certain admissible strategy u :

$$S_t^u := P_t - \lambda u_t - Z_t^u, \quad 0 \leq t \leq T,$$

where P plays the role of the unaffected price of the risky asset and

$$Z_t^u = h_0(t) + \int_0^t G(t, s) u_s ds, \quad 0 \leq t \leq T,$$

for some square integrable deterministic function $h_0 : [0, T] \rightarrow \mathbb{R}$.

Specifically, the trader's transaction not only instantaneously affects the execution price in (5.2.3) in an adverse manner through a linear temporary price impact $\lambda > 0$ à la Almgren and Chriss [23]; it also induces a longer lasting price distortion Z^u because of the linear transient price impact (see e.g. Gatheral et al. [98]).

We now suppose that the trader's optimal trading objective is to unwind her initial position $q > 0$ in the presence of temporary and transient price impact, along with taking into account the asset's general price, through maximizing the performance functional

$$J(u) := \mathbb{E} \left[\int_0^T (P_t - Z_t^u) u_t dt - \lambda \int_0^T u_t^2 dt + Q_T^u P_T - \phi \int_0^T (Q_t^u)^2 dt - \varrho (Q_T^u)^2 \right], \quad (5.2.5)$$

via her selling rate $u \in \mathcal{A}$. The first three terms in (5.2.5) represent the trader's terminal wealth; that is, her final cash position including the accrued trading costs which are induced by temporary and transient price impact as prescribed in (5.2.3), as well as her remaining final risky asset position's book value. The fourth and fifth terms in (5.2.5) implement a penalty $\phi \geq 0$ and $\varrho \geq 0$ on her running and terminal inventory, respectively. Also observe that $J(u) < \infty$ for any admissible strategy $u \in \mathcal{A}$.

Our aim is to find the optimal strategy u^* that maximizes the trader's performance functional:

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u).$$

Our main results in [8] show that, remarkably, the problem can be solved explicitly despite the path-dependency of the model. More precisely, we show that the optimal strategy u^* is explicitly given by the solution to a linear Volterra equation of the form

$$u_t^* = a_t + \int_0^t B(t, s) u_s^* ds,$$

where $\{a_t\}_{t \in [0, T]}$ is a stochastic process that depends linearly on the price process P and B is a deterministic kernel. Both a and B are given explicitly in terms of the inputs of the model and of the price impact kernel G and share some similarities with the operator formulas that have already appeared in the previous Section, we refer to Section 6.2 below for a more general framework sharing the same structure. Such expressions lend themselves naturally to numerical discretization schemes as shown in the accompanying notebook.

Practitioner's corner. we provide an efficient numerical discretization scheme for the optimal trading speed u^* in (4.2.11). We then illustrate numerically the effect of the transient impact kernel G and the signal on the optimal trading speed. For simplicity, we will fix throughout this section the penalization on the running inventory to zero, i.e. $\phi = 0$ in (5.2.5). The code of our implementation can be found at https://colab.research.google.com/drive/1VQasI92YhdBC0wnn_LxMkx_45VyK1yQ.

In Figure 5.2 we illustrate the solution for $T = 10$, a fractional kernel $t^{-\alpha}$ with parameter α and a positive Ornstein-Uhlenbeck signal. We notice that the monotonicity with respect to the α parameter in the fractional kernel is preserved (see in the left panels). In this scenario, since the signal is positive the agent is first buying in order to make a quick profit and then selling her inventory in order to close the position. We observe that larger values of α allow the trader to buy more inventory at the beginning of the trade.

Related literature. Our results significantly improve the results of [153] as we allow for a general Volterra propagator instead of an exponential one. This turns the stochastic control problem to become non-Markovian as the state variables (e.g. the execution price) depend on the entire trading trajectory, unlike the exponential kernel case where the transient price impact could be regraded as a mean-reverting state variable hence the problem become Markovian (see Lemma 5.3 in [153]). We also generalise the price process dynamics in [153], which was assumed to be a semimartingale, while here we assume that it is a progressively measurable process. Our main results also substantially generalise the results of Alfonsi and Schied [20]. Finally, our work is also related to a recent paper by Forde et al. [85], where a specific example of an optimal liquidation problem with power-law transient price

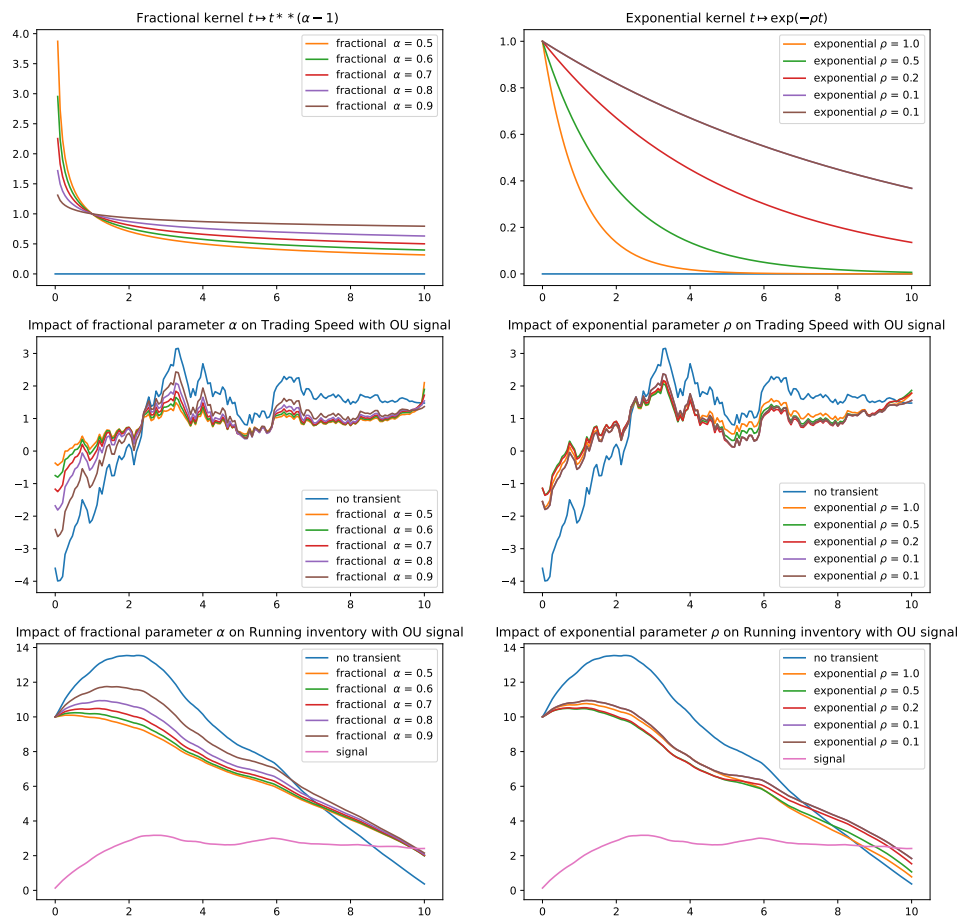


FIGURE 5.2: Impact of parameters of the kernels on the optimal trading speed and inventory with Ornstein-Uhlenbeck signal for the parameters $h_0 \equiv 0, q_0 = 10, T = 10, \lambda = 0.5, \varrho = 2, \phi = 0$; for the OU signal: $I_0 = 2, \gamma = 0.3, \sigma = 0.5$. First column: Fractional kernel; Second column: Exponential kernel.

impact, a Gaussian signal, and without a risk-aversion term was studied. In the main result of [85], a first order condition for the solution was derived in terms of Fredholm integral equations of the first kind. Then, examples for explicit solutions were worked out for a specific choice of signals, which are convolution of fractional kernels with respect to Brownian motion.

Stochastic games in a Volterra framework

Summary

In this chapter, we pursue the study of controlled dynamics with Volterra processes but in the context of stochastic games.

Based on:

[9] Abi Jaber, E. & Villeneuve, S. *Gaussian Agency problems with memory and Linear Contracts*, Finance & Stochastics, to appear.

[17] Abi Jaber, E., Neuman, E. & Voss, M. *Equilibrium in Functional Stochastic Games with Mean-Field Interaction*, Submitted to Annals of Applied Probability.

We consider:

- A Stackelberg game [9]: a Principal-Agent problem à la Holmström and Milgrom but where the usual Brownian motion for modeling the revenues is replaced by a Gaussian Volterra process. What is striking is that we are able to solve explicitly the problem for this class and show that the optimal contract is still linear in the terminal value of the revenue, just like in Holmstrom-Milgrom framework, with a recommended deterministic effort for the agent.
- A class of finite-player and mean-field games [17] that we solve in terms of explicit operator formulas, similar to the ones that appeared in Chapters 3 and 5. However, in contrast with these chapters, there is an important methodological difference, we are able to develop a general direct approach to solve the game directly, without having to guess any particular Ansatz and verify that is indeed the solution.

6.1 Gaussian Agency problems with memory and linear contracts [9]

The extensive literature analyzing the dynamic principal-agent problem has shown that it is important but difficult to design the optimal shape of contracts in a tractable way. Indeed, optimal contracts in dynamic agency problems are generally defined as complex functionals of a stream of contractible variables, such as revenues. Moreover, as first identified by Rogerson [161] (see also [138]) theoretical contracts exhibit memory, even in the most commonly used models that assume uncorrelated shocks, which unfortunately prevents them from matching real-world practices (see Bolton and Dewatripont [39]). In addition, firms' revenues empirically show long memory and we lack a theoretical framework

that justifies the signing of simple tractable contracts in an environment with inter-temporal links across time periods.

In a Brownian setting, the breakthrough paper by Holmström and Milgrom [124] (HM) shows that the optimal contract is linear in profits under some specific assumptions: the agent exerts effort continuously, principal and agent have CARA utilities, the agent bears a pecuniary cost of effort and finally the outcomes generated in the absence of effort are modeled by a fully observable Brownian motion. Nevertheless, as pointed by Schättler and Sung [166], it appears to be very difficult to find general conditions that ensure the optimal contract is linear in end-of-period outcomes. One can read there *at the current level of understanding of the principal-agent problem, one probably has to be content with presenting examples and counterexamples, possibly understanding some classes of problems in depth*. In the paper [9], we study in details the optimality of linear contracts in end-of-period outcomes in a rich class of Volterra models.

We consider a risk-averse investor, who owns a project and signs a fixed-term contract with a risk-averse manager, the latter being necessary to operate a project. Time is continuous and the time horizon is $T > 0$. In the absence of effort, the stochastic output process $(X_t)_{t \leq T}$ of the project evolves up to time T as

$$X_t = g_0(t) + \int_0^t K(t, s) dB_s, \quad (6.1.1)$$

where B is a standard one-dimensional Brownian motion, $g_0 : [0, T] \rightarrow \mathbb{R}$ is a measurable deterministic input function, $K : [0, T]^2 \rightarrow \mathbb{R}$ is a measurable Volterra Kernel, i.e. $K(t, s) = 0$ for $s \geq t$ such that

$$\sup_{t \leq T} \int_0^t K^2(t, s) ds < \infty.$$

An important observation about this framework relates to assumptions about the asymmetry in information, which has been interpreted by [124] as a distinction between linear optimal contracts in outcomes X and those in accounts B . We denote by \mathbb{F}^B the augmented filtration generated by $(B_t)_{t \leq T}$ and \mathbb{F}^X the one generated by the output process $(X_t)_{t \leq T}$. It readily follows from (6.1.1) that $\mathbb{F}^X \subset \mathbb{F}^B$. Hereafter, we assume that the agent has better information than the principal about the project in the sense that he has access to the full information \mathbb{F}^B while the principal observes only some aggregated information generated by the output \mathbb{F}^X . In general, these two filtrations do not coincide even in one-dimensional models as shown in the following example corresponding to a situation where the principal observes the output in a discretionary way.

Discrete observations of a Brownian motion: Assume $X_t = f(t)B_t$ where f is a bounded function on $[0, T]$. Observe that X is a Volterra process with $K(t, s) = f(t)1_{s \leq t}$. Consider a subdivision $0 < t_1 < \dots < t_n = T$ of the interval $[0, T]$ and let f be the function defined as a linear combination of unit impulses

$$f(t) = \sum_{i=1}^N 1_{t_i}(t).$$

The output process is purely discontinuous with $X_{t_i} = B_{t_i}$ and $X_t = 0$ for $t \neq t_i$ and may correspond to a situation where the principal performs audits at regular intervals. Therefore, \mathbb{F}^X is strictly included in \mathbb{F}^B . We deduce that, even in a situation where the principal knows the agent is not exerting effort, the principal has a coarser information than the agent. In Volterra Gaussian models, we must therefore be careful that there may be asymmetric information between the principal and the agent regardless of the agency problem we introduce below.

Agency problem: We assume that the agent can exert a continuous effort $(a_t)_{t \leq T}$ that modifies the probability distribution of X as follows

$$X_t = g_0(t) + \int_0^t K(t, s)(a_s ds + dB_s^a),$$

where B^a is also a Brownian motion. As is customary in the agency theory literature, while the output process X is observable by both players, the effort is the agent's private information. The agent's cost for exercising some effort level a is modeled through a strictly convex C^2 function $k(a)$ satisfying $k(0) = 0$. To alleviate the exposition, we will assume hereafter that the effort cost function is quadratic,

$$k(a) = \kappa \frac{a^2}{2}, \quad \text{for some } \kappa > 0. \quad (6.1.2)$$

Hereafter and in accordance with the paper of Holmström and Milgrom [124], we model the preferences of the principal and the agent with CARA utility functions that are given respectively by

$$U_P(x) := -\exp(-\gamma_P x) \quad \text{and} \quad U_A(x) := -\exp(-\gamma_A x), \quad \forall x \in \mathbb{R}.$$

In the beginning of the relationship, principal and agent agree on a contract of maturity T . To foster incentives, the contract specifies a payment at time T which is modeled by a random variable ξ that is supposed to be \mathcal{F}_T^X measurable. We assume that both players can fully commit to the contract and that the agent has a reservation utility level $R_0 = U_A(y_0) < 0$ below which he will refuse the contract. The latter inequality is referred to the participation constraint of the agent who has the option to reject a contract and enjoy a utility of autarky R_0 .

Description of the probabilistic background: For completeness, we recall the rigorous formulation of the agency problem in order to make understandable the first-order conditions that we will give in the next section. Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \leq T}, \mathbb{P}_0)$ be a filtered probability space on which a \mathbb{F} -Brownian motion $B := (B_t)_{t \leq T}$ is defined with natural (completed) filtration $\mathbb{F}^B := (\mathcal{F}_t^B)_{t \leq T}$.

The firm's output or cash-flows observed by the principal are given by a stochastic process X with dynamics under \mathbb{P}_0 ,

$$X_t = g_0(t) + \int_0^t K(t, s) dB_s,$$

The impact of the agent's effort is modeled as a change of probability measure which changes the drift of the driving Brownian process. More precisely, agent's admissible actions are given by the following set

$$\mathcal{A} = \left\{ (a_t)_{t \leq T} \text{ } \mathbb{F}\text{-progressively measurable: there exists } A > 0 \text{ s.t. } \int_0^T a_s^2 ds \leq A, \mathbb{P}_0 \text{- a.s.} \right\}.$$

Observe that the set of admissible actions \mathcal{A} is not empty because it contains bounded actions. Clearly, any admissible process $a \in \mathcal{A}$ satisfies the Novikov's criterion

$$\mathbb{E}^0 \left[\exp \left(\frac{1}{2} \int_0^T a_s^2 ds \right) \right] < +\infty,$$

where the notation \mathbb{E}^0 denotes the expectation under \mathbb{P}_0 .

This ensures that the process $\left(\exp \left(\int_0^t a_s dB_s - \frac{1}{2} \int_0^t a_s^2 ds \right) \right)_{0 \leq t \leq T}$ is a martingale under \mathbb{P}^0 . We can therefore define a family of equivalent probability measures \mathbb{P}_a by

$$\frac{d\mathbb{P}_a}{d\mathbb{P}_0} = \exp \left(\int_0^T a_s dB_s - \frac{1}{2} \int_0^T a_s^2 ds \right),$$

where a ranges through \mathcal{A} . Hereafter, we will denote \mathbb{E}^a the expectation under \mathbb{P}^a . Under \mathbb{P}_a , the process $B^a = B - \int_0^\cdot a_s ds$ is a \mathbb{F} -Brownian motion by Girsanov theorem and X evolves as

$$X_t = g_0(t) + \int_0^t K(t, s) (a_s ds + dB_s^a). \quad (6.1.3)$$

Because, the effort is unobservable, the principal only observes the trajectory of the output process X , the deterministic curve g_0 but not the last two terms of the decomposition (6.1.3) separately.

Importantly, in the case of general Volterra processes, this moral hazard model leads to a novel simple setting where we have persistence of past efforts on the output variation. It is well-documented that the persistence of effort can have a significant impact on the future outcomes and that it is difficult to provide a setting where the optimal contract with correlated-in-time efforts has a nice and explicit form. This is mainly due to the way one models the complex relationship between past efforts and current outcomes. The strength of the Volterra class is that it provides a simple and effective way to introduce persistence of the agent's efforts in dynamic contracting, while allowing an explicit description of the optimal contract.

The Principal-agent problem: It is well known that principal-agent relationships can be viewed as a Stackelberg game. The principal moves first by offering a contract that consists in a compensation ξ , which belongs to the set of \mathcal{F}_T^X measurable random variables, to the agent. The latter then reacts by choosing an effort policy based on the information available at each date inducing a probability measure \mathbb{P}_a . For any given contract ξ , let $V_0^A(\xi)$ denote the agent's utility at time 0 which is defined as

$$V_0^A(\xi) := \sup_a \mathbb{E}^a \left(U_A \left(\xi - \int_0^T k(a_s) ds \right) \right) \quad (6.1.4)$$

recall the definition of k in (6.1.2). As common in agency problems, we define the concept of incentive-compatible contracts.

Definition 6.1. A contract ξ is said to be *incentive compatible* if V_0^A is finite and if there exists an effort policy $a^*(\xi) \in \mathcal{A}$ that maximizes (6.1.4), i.e.

$$V_0^A(\xi) = \mathbb{E}^{a^*(\xi)} \left(U_A \left(\xi - \int_0^T k(a_s^*(\xi)) ds \right) \right).$$

It is critical to understand what incentive-compatible contracts are, as these are the ones for which the principal can enforce desirable efforts. As common in the literature, we will focus on a class Ξ of contracts ξ that are incentive-compatible (IC). Before defining rigorously the class of IC contracts Ξ we will focus on, we clarify the principal's problem. By offering an incentive-compatible contract $\xi \in \Xi$, the principal will be able to anticipate the optimal effort level $a^*(\xi)$. Hence, she will propose an incentive-compatible contract that maximizes the expected value of her CARA preference. Then, her aim is to solve

$$V_0^P := \sup_{\xi \in \Xi} \mathbb{E}^{a^*(\xi)} [U_P(X_T - \xi)], \quad (6.1.5)$$

under the participation constraint $\mathbb{E}^{a^*(\xi)} \left(U_A \left(\xi - \int_0^T k(a_s^*(\xi)) ds \right) \right) \geq R_0$.

Our main result is given by the following theorem which shows that the problem (6.1.5) admits an optimal contract which is linear in end-of-period outcomes. The result of the Holmstrom-Milgrom model thus extends to all Gaussian Volterra processes, even though these may exhibit very different statistical properties. Following Schättler and Sung [165], we introduce the class of contracts we will focus on. Let us define

$$f^*(z) := \frac{\gamma_A}{2} |z|^2 + \inf_{a \in \mathbb{R}} \{k(a) - az\} = \frac{\kappa\gamma_A - 1}{2\kappa} z^2,$$

and consider the following class Ξ of Incentive Compatible contracts,

$$\Xi = \{ \xi = Y_T^{(y,\beta)} : \mathcal{F}_T^X\text{-mes, where } y \geq y_0, \beta = (\beta_t)_{t \leq T} \in \mathcal{A} \text{ and } Y_T^{(y,\beta)} = y + \int_0^T f^*(\beta_s) ds + \int_0^T \beta_s dB_s \}.$$

We stress that the following class of contracts is not restrictive since any integrable contract ξ can be written as the terminal value of a process $Y^{(y,\beta)}$ as in the set Ξ . Such approach is now well-know in the semimartingale framework, see [72].

We have:

Theorem 6.2. *The optimal contract ξ^* that maximizes the principal problem (6.1.5) is linear in end-of-period profits X_T and is given by*

$$\xi^* = y_0 - \frac{\gamma_P + 1/\kappa}{\gamma_A + \gamma_P + 1/\kappa} g_0(T) + \frac{\kappa\gamma_A - 1}{2\kappa} \int_0^T (\beta_s^*)^2 ds + \frac{\gamma_P + 1/\kappa}{\gamma_A + \gamma_P + 1/\kappa} X_T,$$

and the optimal level of recommended effort a^* that maximizes the agent's problem (6.1.4) is deterministic and given by $a^* = \frac{\beta^*}{\kappa}$ with

$$\beta_t^* = \frac{\gamma_P + 1/\kappa}{\gamma_A + \gamma_P + 1/\kappa} K(T, t), \quad t \leq T.$$

Similarly to HM, the optimal compensation is made up of a deterministic base salary

$$y_0 - \frac{\gamma_P + 1/\kappa}{\gamma_A + \gamma_P + 1/\kappa} g_0(T) + \frac{\kappa\gamma_A + 1}{2\kappa} \int_0^T (\beta_s^*)^2 ds$$

and a random compensation to foster incentives $\frac{\gamma_P + 1/\kappa}{\gamma_A + \gamma_P + 1/\kappa} X_T$. One of the striking results is, when agents have CARA preferences, the incentive part of the optimal contract, through the performance-based bonus coefficient $\frac{\gamma_P + 1/\kappa}{\gamma_A + \gamma_P + 1/\kappa}$, is common to all one-dimensional Volterra Gaussian models and thus independent of the output dynamics, even though they have very different statistical properties. Only the base salary is industry-specific depending on the output dynamic through the Volterra kernel K . The optimal effort level is deterministic and firm-specific and can, depending on the choice of the Volterra kernel, exhibit interesting behaviors. For instance, for the mean-reverting dynamics, *i.e.* $K(t, s) = e^{-\lambda(t-s)} 1_{s < t}$, the optimal effort is increasing if the mean-reverting intensity λ is positive. The closer one gets to contract maturity, the more work the agent has to do. The intuition is that the optimal effort should compensate for the natural tendency of the process to revert to its long-term average. The closer the contract is to maturity, the greater the effort should be to allow X to deviate from its long-term average and thus allow the principal to benefit from a greater profit. When the mean-reverting intensity is negative, the effort must be greater at the beginning of the contract in order to give the necessary impetus to the process to diverge towards large positive values. Once this momentum is established, it is less effective to ask the agent to work.

6.2 N -player and mean-field games with memory [17]

In [17], we consider a class of finite-player and mean-field games that we solve in terms of explicit operator formulas, similar to the ones that appeared in Chapters 3 and 5. However, in contrast with these chapters, there is an important methodological difference, we are able to develop a general direct approach to solve the game directly, without having to guess any particular Ansatz and verify that is indeed the solution.

Specifically, we consider N -player stochastic games in which each agent i has an objective functional of the form

$$J^i(u^i) := \mathbb{E} \left[-\langle \bar{u}, \mathbf{A}_1 \bar{u} \rangle_{L^2} - \langle u^i, \mathbf{A}_2 u^i \rangle_{L^2} - \langle u^i, (\mathbf{A}_3 + \mathbf{A}_3^*) \bar{u} \rangle_{L^2} \right. \\ \left. + \langle b^i, u^i \rangle_{L^2} + \langle b^0, \bar{u} \rangle_{L^2} + c^i \right],$$

where u^i represents the agent's control and $\bar{u} = N^{-1} \sum_{i=1}^N u^i$ captures a mean-field interaction between all agents. Here, the symbols \mathbf{A}_i , $i = 1, 2, 3$ denote non-anticipative linear operators on $L^2([0, T], \mathbb{R})$; $(b_s^i)_{s \in [0, T]}$, $(b_s^0)_{s \in [0, T]}$ are stochastic processes; and c^i is a random variable. The inner product is defined in the usual sense as $\langle f, g \rangle_{L^2} := \int_0^T f(s)g(s)ds$ for $f, g \in L^2([0, T], \mathbb{R})$.

We highlight in [17, Section 3], the versatility of the objective functional in (6.2.1), by showing that it nests several important, challenging and diverse examples of dynamical stochastic games beyond the

Markovian and semimartingale case. In particular, we introduce a general class of Linear-Quadratic Stochastic Volterra game whose objective functional is shown to be equivalent to (6.2.1). Then, we show how such framework accommodates and extends and solves three major examples that appeared in the literature. An inter-bank lending and borrowing model with delay in the control which was studied in Carmona et al. [50], Fouque and Zhang [90]. Advertising models Gozzi and Marinelli [107] with mean-field effect and delay in the state. Multiplayer price impact games with general propagator in the spirit of Abi Jaber and Neuman [8], Neuman and Voß [154].

To simplify the presentation, we only present here the solution to the corresponding mean-field game in what follows. Let $(\Omega, \mathcal{F}_T, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ denote a filtered probability space. We use the notation $\mathbb{E}_t = \mathbb{E}[\cdot | \mathcal{F}_t]$ to represent the conditional expectation with respect to \mathcal{F}_t . Moreover, let c denote an \mathcal{F}_T -measurable random variable and let β, β^0 denote \mathbb{F} -progressively measurable processes such that β^0 is independent of β . We set

$$b := \beta + \beta^0 \quad (6.2.2)$$

and denote by $\mathbb{F}^0 := (\mathcal{F}_t^0)_{0 \leq t \leq T}$ the filtration generated by β^0 satisfying the usual conditions.

Assumption 6.2.1. We assume that for all $t \leq T$, \mathcal{F}_T^0 and \mathcal{F}_t are conditionally independent given \mathcal{F}_t^0 .

We also define the sets of admissible processes

$$\mathcal{U} := \left\{ v : \mathbb{F}\text{-progressively measurable s.t. } \int_0^T \mathbb{E}[v_s^2] ds < \infty \right\},$$

$$\mathcal{U}^0 := \left\{ v : \mathbb{F}^0\text{-progressively measurable s.t. } \int_0^T \mathbb{E}[v_s^2] ds < \infty \right\} \subset \mathcal{U}.$$

The paradigm of the infinite-player mean field game limit of the finite-player game from (6.2.1) is the following: First, we consider a generic player who seeks to implement a strategy $v \in \mathcal{U}$ in order to maximize for a fixed $\mu \in \mathcal{U}^0$ the objective function

$$J(v; \mu) := \mathbb{E}[-\langle \mu, \mathbf{A}_1 \mu \rangle_{L^2} - \langle v, \mathbf{A}_2 v \rangle_{L^2} - \langle v, \mathbf{A}_3 + \mathbf{A}_3^* \mu \rangle_{L^2} + \langle b, v \rangle_{L^2} + \langle b^0, \mu \rangle_{L^2} + c], \quad (6.2.3)$$

with same operators $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ as well as \mathbb{F} -progressively measurable process b^0 . Then, we determine the \mathbb{F}^0 -measurable process μ such that a mean field game equilibrium with common noise β^0 is obtained in the following sense.

Definition 6.3. A pair $(\hat{v}, \hat{\mu}) \in \mathcal{U} \times \mathcal{U}^0$ is called a mean field game equilibrium if the control \hat{v} solves the optimization problem

$$J(\hat{v}; \hat{\mu}) \rightarrow \max_{v \in \mathcal{U}} \quad (6.2.4)$$

under the consistency condition

$$\mathbb{E}[\hat{v}_t | \mathcal{F}_T^0] = \hat{\mu}_t, \quad \Omega \times [0, T] \text{ almost everywhere.} \quad (6.2.5)$$

Remark 6.4. Note that due to Assumption 6.2.1, it holds in (6.2.5) that

$$\mathbb{E}[\hat{v}_t | \mathcal{F}_T^0] = \mathbb{E}[\hat{v}_t | \mathcal{F}_t^0]$$

for all $t \in [0, T]$. Indeed, for all $\xi \in \mathcal{F}_T^0$ it follows that

$$\mathbb{E}[\xi \mathbb{E}[\hat{v}_t | \mathcal{F}_t^0]] = \mathbb{E}[\mathbb{E}[\xi | \mathcal{F}_t^0] \mathbb{E}[\hat{v}_t | \mathcal{F}_t^0]] = \mathbb{E}[\mathbb{E}[\xi \hat{v}_t | \mathcal{F}_t^0]] = \mathbb{E}[\xi \hat{v}_t],$$

where the second equality is obtained from the conditional independence given by Assumption 6.2.1.

Remark 6.5. Observe that the process b defined in (6.2.2) in the generic player's objective functional in (6.2.3) encodes two different sources of noise. Specifically, β^0 can be interpreted as representing

some common noise affecting the whole system whereas β represents the generic player's independent individual source of noise; as well as other random or deterministic factors, which are idiosyncratic to the generic player. In particular, the generic player in the mean field game can be thought of as a representative player chosen from a population of heterogeneous players who are allowed to have their own β , which are not necessarily (statistically) identical.

Using calculus of variation to write the first-order conditions of the optimization problems in (6.2.4), leads to the following non-standard stochastic (adaptive) Fredholm equation for $\hat{\mu}$ with both forward and backward components:

$$\hat{\mu}_t = f_t - \int_0^t K(t, r) \hat{\mu}_r dr - \int_t^T L(r, t) \mathbb{E}_t \hat{\mu}_r dr, \quad t \leq T, \quad (6.2.6)$$

where f is progressively measurable, K, L are deterministic kernels determined by the inputs of the model $(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, b^i, b^0, c^i)$. One of our main innovations is that we derive an explicit solutions to (6.2.6) using an novel approach. Then, we use the solution to (6.2.6) to disentangle and solve the optimisation problem of each player and hence to derive explicitly the Nash equilibrium. In addition to solving (6.2.6), we derive a stability result for (6.2.6). This stability result is the crucial ingredient for linking the N -player game to the mean-field game: (i) by deriving the convergence of the finite-player game towards the mean-field limit, (ii) and establishing an ϵ -Nash equilibrium for the N -player game using the mean-field game.

To state the mean field game equilibrium, it is convenient to introduce the following two solution maps F and G of two associated Fredholm equations:

$$F(t, x) := ((\text{id} - \tilde{\mathbf{B}})^{-1} \tilde{a}^x)(t) \quad G(t, x) := ((\text{id} - \hat{\mathbf{B}})^{-1} \hat{a}^x)(t) \quad t \leq T, \quad (6.2.7)$$

where

$$\begin{aligned} \tilde{a}_t^x &:= \frac{1}{2\lambda} \left(x_t - \langle 1_{t \leq \cdot} \hat{\mathbf{A}}_2(\cdot, t), \tilde{\mathbf{D}}_t^{-1} 1_{t \leq \cdot} \mathbb{E}_t[x \cdot] \rangle_{L^2} \right), \\ \tilde{B}(t, s) &:= 1_{\{s \leq t\}} \frac{1}{2\lambda} \left(\langle 1_{t \leq \cdot} \hat{\mathbf{A}}_2(\cdot, t), \tilde{\mathbf{D}}_t^{-1} 1_{t \leq \cdot} \hat{\mathbf{A}}_2(\cdot, s) \rangle_{L^2} - \hat{\mathbf{A}}_2(t, s) \right), \\ \tilde{\mathbf{D}}_t &:= 2\lambda \text{id} + (\hat{\mathbf{A}}_2)_t + (\hat{\mathbf{A}}_2^*)_t; \end{aligned} \quad (6.2.8)$$

and

$$\begin{aligned} \hat{a}_t^x &:= \frac{1}{2\lambda} \left(x_t - \langle 1_{t \leq \cdot} \hat{\mathbf{A}}_2(\cdot, t), \hat{\mathbf{D}}_t^{-1} 1_{t \leq \cdot} \mathbb{E}_t[x \cdot] \rangle_{L^2} \right), \\ \hat{B}(t, s) &:= 1_{\{s \leq t\}} \frac{1}{2\lambda} \left(\langle 1_{t \leq \cdot} \hat{\mathbf{A}}_2(\cdot, t), \hat{\mathbf{D}}_t^{-1} 1_{t \leq \cdot} (A_3(\cdot, s) + \hat{\mathbf{A}}_2(\cdot, s)) \rangle_{L^2} \right. \\ &\quad \left. - (A_3(t, s) + \hat{\mathbf{A}}_2(t, s)) \right), \\ \hat{\mathbf{D}}_t &:= 2\lambda \text{id} + (A_3)_t + (A_3^*)_t + (\hat{\mathbf{A}}_2)_t + (\hat{\mathbf{A}}_2^*)_t. \end{aligned} \quad (6.2.9)$$

We are now ready to provide the solution to the mean field game:

Theorem 6.6. *Assume that the linear operators $\tilde{\mathbf{D}}_t$ and $\hat{\mathbf{D}}_t$ in (6.2.8) and (6.2.9) are invertible for all $t \in [0, T]$. Then, the unique mean field game equilibrium $(\hat{v}, \hat{\mu}) \in \mathcal{U} \times \mathcal{U}^0$ in the sense of Definition 6.3 is given by*

$$\begin{aligned} \hat{v}_t &= F(t, \beta + \beta^0 - (A_3 + A_3^*)\hat{\mu}), \\ \hat{\mu}_t &= G(t, \mathbb{E}[\beta] + \beta^0) \end{aligned}$$

for all $t \in [0, T]$, where F and G are defined in (6.2.7).

The framework developed in the paper [17] is unique in that the solvability, stability, and consistency of finite-player games with mean-field interaction boil down to the study of the stochastic Fredholm equation (6.2.6). To our knowledge our approach gives the first canonical method for deriving explicitly Nash equilibrium to this general class of stochastic games. The closest result appeared in Huang

et al. [127], where the authors derived a first order conditions for a special case of (6.2.1) in terms of a system of stochastic integral equation but did not derive solution to the system. We also refer to Bensoussan et al. [36] where first order condition for linear-quadratic stochastic games with delays in the state and the control were derived. We show that we can derive explicit solutions to this class of games as a corollary of our main results.

Part III

Joint SPX-VIX Modeling

The Quintic Ornstein-Uhlenbeck model that jointly calibrates SPX and VIX smiles

Summary

In this chapter, we introduce the first one factor Markovian stochastic volatility model that is able to jointly calibrate SPX and VIX smiles: the Quintic Ornstein-Uhlenbeck model. It has only 6 effective parameters and an input curve that allows to match certain term structures and it is remarkably tractable. More interestingly, we show that it outperforms its rough counterpart, on all dates and market conditions, on more than 10 years of data.

Based on:

[15] Abi Jaber, E., Illand, C., & Li, S. *Joint SPX-VIX calibration with Gaussian polynomial volatility models: deep pricing with quantization hints*, Submitted to Mathematical Finance.

[16] Abi Jaber, E., Illand, C., & Li, S. *The quintic Ornstein-Uhlenbeck volatility model that jointly calibrates SPX & VIX smiles*, *Risk Magazine*, to appear, 2023. [Jupyter Notebook](#)

Launched in 1993 by the CBOE, the VIX has become one of the most widely followed volatility index. It represents an estimation of the S&P 500 index (SPX) expected volatility over a one-month period. More precisely, the VIX is calculated by aggregating weighted prices of SPX puts and calls over a wide range of strikes and maturities [53]. By construction, the VIX expresses an interpolation between several points of the SPX implied volatility term structure. Thus, the task of modeling and pricing VIX options for a given maturity T naturally requires some consistency with SPX options maturing up to one month ahead of T . Furthermore, computing the implied volatility of VIX options using Black's formula requires VIX futures that also need to be priced consistently.

By joint SPX-VIX calibration problem, we mean the calibration of a model across several maturities to European call and put options on SPX and VIX together with VIX futures. Such joint calibration turns out to be quite challenging for several reasons: multitude of instruments to be calibrated (SPX and VIX call/put options, VIX futures) across several maturities (to stay consistent with the construction of the VIX), characterized by low levels of implied volatilities of the VIX with an upward slope, in contrast with the important at-the-money (ATM) SPX skew that becomes more pronounced for smaller maturities.

In recent years, substantial progress has been made in developing relatively sophisticated stochastic models that achieve decent joint fits by exploiting a wide variety of mathematical tools such as jump processes [26, 63, 137, 156, 159], rough volatility [41, 100, 163], path-dependent volatility [116] and

multiple-factors [88, 106, 116, 162].¹ However, examples of illustrated fits of these models are usually partial: in some cases VIX futures are not calibrated; in other cases VIX derivatives are calibrated up to maturity slice T , while the SPX derivatives for maturity slices $T + \Delta$ for $\Delta \in (0, 1 \text{ month})$ are missing. Although different in their mathematical nature, these models share in common the fact that they allow for 1) large price movements of the SPX on very short time scales with some forms of spikes in the ‘instantaneous’ volatility process due to a large ‘vol-of-vol’, and 2) fast mean reversions towards relatively low volatility regimes. We believe these are the two crucial ingredients for the joint calibration problem.

The aforementioned literature generally agrees that *conventional one-factor continuous Markovian stochastic volatility models* are not able to achieve a decent joint calibration. Our main motivations can be stated as follows:

*Can joint calibration be achieved **without** appealing to multiple-factors, jumps, roughness or path-dependency?*

*Is joint calibration **possible** with conventional one-factor continuous Markovian models?*

In a nutshell, we show in the papers [15, 16] that the answer to both questions is a resounding: *Yes*. By performing joint calibration on daily SPX-VIX implied volatility surface data between 2012 and 2022 using a large class of models, we identify for the first time a *conventional one-factor Markovian continuous stochastic volatility model* that is able to achieve remarkable fits for a wide range of maturity slices $[T_s, T_e]$ for VIX implied volatility surface and of maturity slices $[T_s, T_e + 1 \text{ month}]$ for SPX implied volatility surface, together with the term structure of VIX futures. What is even more remarkable is that our *conventional one-factor Markovian continuous stochastic volatility model*, dubbed the Quintic Ornstein-Uhlenbeck model, outperforms its rough and non-rough path-dependent counterparts with the same number of parameters: 6 effective parameters that govern the dynamics of the model in addition to the usual input curve that allows to match certain term structures.

7.1 The Quintic Ornstein-Uhlenbeck model [16]

The dynamics of the stock price S , with no interest nor dividends, is given by

$$\begin{aligned} \frac{dS_t}{S_t} &= \sigma_t dB_t, \\ \sigma_t &= \sqrt{\xi_0(t)} \frac{p(X_t)}{\sqrt{\mathbb{E}[p(X_t)^2]}}, \quad p(x) = \alpha_0 + \alpha_1 x + \alpha_3 x^3 + \alpha_5 x^5, \\ X_t &= \varepsilon^{H-1/2} \int_0^t e^{-(1/2-H)\varepsilon^{-1}(t-s)} dW_s, \end{aligned}$$

with $B = \rho W + \sqrt{1 - \rho^2} W^\perp$, (W, W^\perp) a two-dimensional Brownian motion on a risk-neutral filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$, $\rho \in [-1, 1]$, non-negative coefficients $\alpha_0, \alpha_1, \alpha_3, \alpha_5 \geq 0$ ($\alpha_2 = \alpha_4 = 0$), $\varepsilon > 0$, $H \in (-\infty, 1/2]$ and an input curve $\xi_0 \in L^2([0, T], \mathbb{R}_+)$ for any $T > 0$, allowing the model to match certain term-structures observed on the market. For instance, the normalization $\sqrt{\mathbb{E}[p(X_t)^2]}$ allows ξ_0 to match the market forward variance curve since

$$\mathbb{E} \left[\int_0^t \sigma_s^2 ds \right] = \int_0^t \xi_0(s) ds, \quad t \geq 0.$$

The process X driving the volatility is an Ornstein-Uhlenbeck process with a fast mean reversion of order $(1/2 - H)\varepsilon^{-1}$ and a large vol-of-vol of order $\varepsilon^{H-1/2}$ for small values of ε , that is

$$dX_t = -(1/2 - H)\varepsilon^{-1} X_t dt + \varepsilon^{H-1/2} dW_t.$$

¹We mention also techniques involving optimal transport [115] and randomization of the parameters [111].

Such parametrizations are reminiscent of the fast regimes extensively studied by Fouque et al. [92], see also [91, Section 3.6], which corresponds to the case $H = 0$. They can also be linked to more complex models such as jump models [149, 5] for $H \leq -1/2$; and rough volatility models [15], for which $H \in (0, 1/2)$ would play the role of the Hurst index, see also Chapter 8. Letting the parameter $H \in (-\infty, 1/2]$ free in our model introduces more flexibility and leads to better fits than in the aforementioned models. Another advantage of such parametrization is to stabilize the calibrated value of H through time as opposed to calibrating directly on mean reversion and vol-of-vol parameters which are less stable through time, see [15, Figure 3].

Taking p a polynomial of degree five allows us to reproduce the upward slope of the VIX smile. Restricting the coefficients α to be non-negative (with $\alpha_2 = \alpha_4 = 0$) the sign of the at-the-money skew to be the same as ρ , see [15] for more details, as well as ensuring the martingale property of S , whenever $\rho \leq 0$ and $\alpha_5 > 0$.

We fix $\varepsilon = 1/52$ to further reduce the parameters, which gives 6 calibratable parameters:

$$\Theta := \{\alpha_0, \alpha_1, \alpha_3, \alpha_5, \rho, H\},$$

plus the input curve $\xi_0(\cdot)$. Numerical experiments show no significant adverse impact on the joint calibration quality by narrowing the number of parameters.

We show that the model is tractable as it offers an explicit expression for the VIX squared which is again polynomial in the driving Ornstein-Uhlenbeck factor, leading to efficient VIX derivative pricing by integrating directly against a Gaussian density. Simulation of the volatility process is exact so that pricing SPX products can be done efficiently and accurately by standard Monte Carlo techniques with suitable antithetic and control variates. We also provide a notebook with our implementation here: https://colab.research.google.com/drive/14nh9civ_wgQv283eshBWnr146w7Xsbi5?usp=sharing.

For the first time in the literature, remarkable joint fits of SPX and VIX volatility surfaces and VIX futures are achieved between 1 week and beyond 1 year. Although it is challenging, but possible, for another model to achieve similar fits, it would be very difficult to do so with a simpler continuous model than our quintic Ornstein-Uhlenbeck volatility model.

An explicit expression for the VIX. One major advantage of our model is an explicit expression of the VIX. In continuous time, the VIX can be expressed as

$$\text{VIX}_T^2 = -\frac{2}{\Delta} \mathbb{E} [\log(S_{T+\Delta}/S_T) \mid \mathcal{F}_T] \times 100^2 = \frac{100^2}{\Delta} \int_T^{T+\Delta} \xi_T(u) du, \quad (7.1.1)$$

with $\Delta = 30$ days and $\xi_T(u) := \mathbb{E} [\sigma_u^2 \mid \mathcal{F}_T]$ the forward variance process which can be computed explicitly in our model as follows. First, we fix $T \leq u$ and rewrite X as

$$X_u = X_T e^{-(1/2-H)\varepsilon^{-1}(u-T)} + \varepsilon^{H-1/2} \int_T^u e^{-(1/2-H)\varepsilon^{-1}(u-s)} dW_s =: Z_T^u + G_T^u,$$

then, setting

$$g(u) = \mathbb{E}[p(X_u)^2],$$

we have that

$$\xi_T(u) = \mathbb{E} [\sigma_u^2 \mid \mathcal{F}_T] = \frac{\xi_0(u)}{g(u)} \mathbb{E} \left[\left(\sum_{k=0}^5 \alpha_k X_u^k \right)^2 \mid \mathcal{F}_T \right] = \frac{\xi_0(u)}{g(u)} \mathbb{E} \left[\sum_{k=0}^{10} (\alpha * \alpha)_k X_u^k \mid \mathcal{F}_T \right],$$

where $(\alpha * \alpha)_k = \sum_{j=0}^k \alpha_j \alpha_{k-j}$ is the discrete convolution. Using the Binomial expansion, we can further develop the expression for $\xi_T(u)$ in terms of Z^u and G^u to get

$$\xi_T(u) = \frac{\xi_0(u)}{g(u)} \sum_{k=0}^{10} \sum_{i=0}^k (\alpha * \alpha)_k \binom{k}{i} \left(X_T e^{-(1/2-H)\varepsilon^{-1}(u-T)} \right)^i \mathbb{E} [(G_T^u)^{k-i}], \quad (7.1.2)$$

where we used the fact that Z_T^u is \mathcal{F}_T -measurable and that G_T^u is independent of \mathcal{F}_T , with $\binom{k}{i} = k!/((k-i)!i!)$ the binomial coefficient. Furthermore, G_T^u is a Gaussian random variable with mean 0 and variance $\frac{\varepsilon^{2H}}{1-2H}(1 - e^{-(1-2H)\varepsilon^{-1}(u-T)})$. Recall that for a Gaussian variable $Y \sim \mathcal{N}(0, \sigma_Y^2)$, its moments $\mathbb{E}[Y^p]$ for $p \in \mathbb{N}$ can be computed explicitly:

$$\mathbb{E}[Y^p] = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \sigma_Y^p (p-1)!! & \text{if } p \text{ is even} \end{cases}$$

with $p!!$ the double factorial. Therefore all moments of $\mathbb{E}[(G_T^u)^i]$ are given explicitly.

Going back to (7.1.1) and plugging the expression (7.1.2), the explicit expression of the VIX_T^2 turns out to be polynomial in X_T :

$$\begin{aligned} \text{VIX}_T^2 &= \frac{100^2}{\Delta} \sum_{k=0}^{10} \sum_{i=0}^k (\alpha * \alpha)_k \binom{k}{i} \int_T^{T+\Delta} \frac{\xi_0(u)}{g(u)} \mathbb{E}[(G_T^u)^{k-i}] e^{-(1/2-H)\varepsilon^{-1}(u-T)i} du X_T^i \\ &= \frac{100^2}{\Delta} \sum_{i=0}^{10} \sum_{k=i}^{10} \left((\alpha * \alpha)_k \binom{k}{i} \int_T^{T+\Delta} \frac{\xi_0(u)}{g(u)} \mathbb{E}[(G_T^u)^{k-i}] e^{-(1/2-H)\varepsilon^{-1}(u-T)i} du \right) X_T^i \\ &= \frac{100^2}{\Delta} \sum_{i=0}^{10} \beta_i X_T^i, \end{aligned} \tag{7.1.3}$$

where

$$\beta_i = \sum_{k=i}^{10} (\alpha * \alpha)_k \binom{k}{i} \int_T^{T+\Delta} \frac{\xi_0(u)}{g(u)} \mathbb{E}[(G_T^u)^{k-i}] \left(e^{-(1/2-H)\varepsilon^{-1}(u-T)i} \right) du.$$

The integral inside β_i can be easily computed, at least numerically for a variety of choices for $\xi_0(\cdot)$.

Pricing VIX derivatives. Thanks to the closed expression of (7.1.3), VIX_T^2 is a polynomial in X_T that we denote by $h(X_T)$. Since X_T is Gaussian with mean 0 and variance $\sigma_{X_T}^2 = \frac{\varepsilon^{2H}}{1-2H}(1 - e^{-(1-2H)\varepsilon^{-1}T})$, pricing VIX derivatives with payoff function Φ is immediate by integrating directly against the standard Gaussian density:

$$\mathbb{E}[\Phi(\text{VIX}_T)] = \mathbb{E}\left[\Phi\left(\sqrt{h(X_T)}\right)\right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi\left(\sqrt{h(\sigma_{X_T}x)}\right) e^{-x^2/2} dx. \tag{7.1.4}$$

Example 7.1. To price VIX future prices, set $\Phi(v) = v$ and to price VIX vanilla call price, set $\Phi(v) = (v - K)^+$. This integral (7.1.4) can be computed efficiently using a variety of quadrature techniques. The Gaussian quadrature with 400 nodes seems to be more than enough to price accurately VIX call and future prices.

Practitioner's corner. Figure 7.1 shows the joint fit on the 23 October 2017, with calibrated parameters $\rho = -0.6843$, $H = -0.0358$, $(\alpha_0, \alpha_1, \alpha_3, \alpha_5) = (0.5907, 1, 0.2893, 0.0549)$:

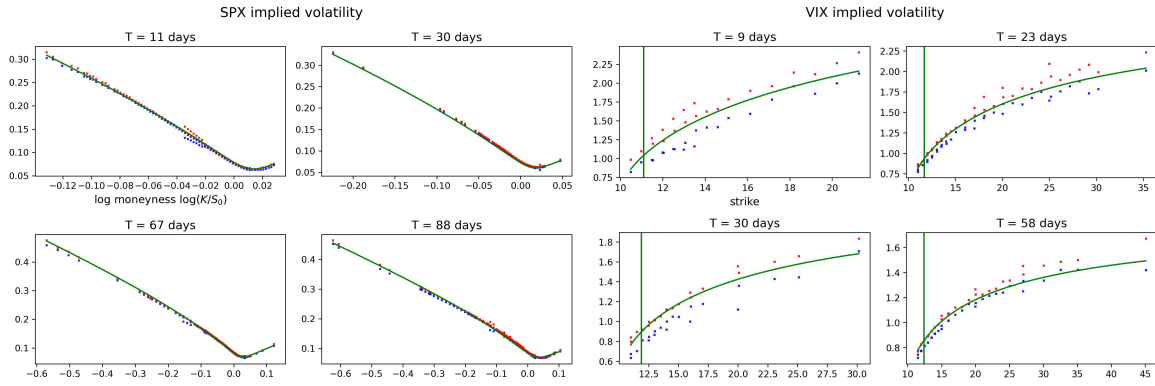


FIGURE 7.1: SPX–VIX smiles (bid/ask in blue/red) and VIX futures (vertical black lines) jointly calibrated with our model (full green lines) for 23 October 2017.

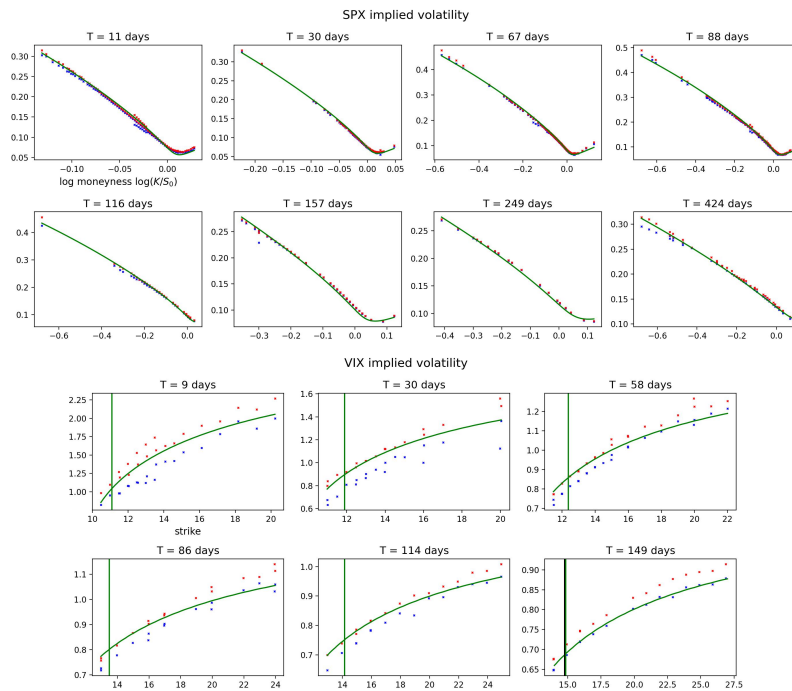


FIGURE 7.2: SPX–VIX smiles (bid/ask in blue/red) and VIX futures (vertical black lines) jointly calibrated with our model for time dependent H (full green lines) for 23 October 2017.

7.2 Extensive empirical study and comparison with non-Markovian and rough models [15]

In order to study the robustness of the quintic OU model in replicating the stylized facts of SPX-VIX smiles and compare it to other models such as rough models, we have put in place in [15] a generic method that ensures a fair comparison on more than 10 years of data.

More precisely, our methodology and contributions in [15] are summarized follows:

Gaussian polynomial volatility models. First, we introduce a general class of *Gaussian polynomial volatility* models, which nests the Quintic Ornstein-Uhlenbeck model, in which the SPX spot

price takes the form

$$\frac{dS_t}{S_t} = \sigma_t \left(\rho dW_t + \sqrt{1 - \rho^2} dW_t^\perp \right),$$

where (W, W^\perp) is two-dimensional Brownian motion. The SPX spot price S is correlated with the volatility process σ which is, up to a normalizing deterministic term, defined as a polynomial function $p(X)$ of a Gaussian Volterra process X in the form

$$X_t = \int_0^t K(t-s) dW_s,$$

for a locally square-integrable kernel K . The choice of the kernel introduces a good deal of flexibility in the modeling of the volatility process, such as rough volatility [4, 7, 10, 30, 35, 80, 99, 100] for singular fractional kernels of the form $K(t) \sim t^{H-1/2}$ with $0 < H \leq 1/2$, or the log-modulated kernel that extends the fractional kernel for the case $H = 0$, see [32]; path-dependent models with non-singular kernel such as the shifted fractional kernel $K(t) \sim (t + \varepsilon)^{H-1/2}$; exponential kernels $K(t) \sim e^{-\lambda t}$ for which X is a (Markovian) Ornstein-Uhlenbeck process or weighted sums of exponentials [1, 6, 68, 122], refer to Table 7.1 below. We will compare the performance of these different kernels on the joint calibration problem. Although it is difficult to decouple the impact of the different input parameters of the model, it turns out, that the choice of K has a major impact on the ATM-skew of the implied volatility of the SPX and the level of the implied volatility of the VIX. While the choice of the polynomial function p has a prominent impact on the shape of the VIX smile. Taking p a polynomial of order 5 (and higher) allows us to reproduce the upward slope of the VIX smile.

Generic, fast and accurate pricing via quantization and Neural Networks. Second, in order to ensure a fair comparison between the calibrated models with different kernels across 10 years of daily joint implied volatility surfaces, we develop a generic unified method that applies to any *Gaussian polynomial volatility model* for pricing SPX and VIX derivatives in an efficient and accurate fashion. The method is based on functional quantization and Neural Networks. The tractability of the quantization approach highly relies on the Gaussian nature of X combined with the polynomial form of the volatility process σ . More precisely:

- **Fast pricing of VIX derivatives via Quantization:** we develop a functional quantization approach for computing VIX derivatives in our class of *Gaussian polynomial volatility models*. When computing expectations in the form of $\mathbb{E}[F(Y)]$ where no closed form solution is available, a fast alternative to Monte Carlo is quantization. The idea is to approximate the random variable Y with a discrete random variable \hat{Y} to compute efficiently the (conditional) expectations of suitable functionals of Y . Quantization was first developed in the 1950's for signal processing [101, 109] and more recently has been studied for applications in numerical probability [157] and mathematical finance [158, 160]. We will exploit the Gaussian nature of the process X to develop a functional quantization approach.

A first attempt to use functional quantization for VIX futures in the context of the rough Bergomi model appears in [42]. Unfortunately, the method is not precise enough in practice, especially for the fractional kernel with small values of H even with a lot of quantization trajectories, see [42, Figure 3] where the number of quantized points were pushed as far as $N = 1,000,000$ but the approximated values for VIX futures are still well-off the correct values. It is well known that the convergence of the quantization for fractional processes is very slow of order $1/(\log N)^H$, see [77].

Using a crucial *moment-matching trick*, we are able to make functional quantization usable in practice by achieving very accurate results for both VIX future prices and VIX option smile with only a couple of hundreds quantization points, even for fractional processes with very low values of H .

- **Fast pricing of SPX options via Neural Networks with Quantization hints:**

In a first step, we extend the previous quantization ideas to quantize SPX. However, the quantization is more delicate whenever $\rho \neq 0$ since it involves the quantization of the stochastic Itô integral $\int_0^t \sigma_s dW_s$. It is well-known since the work of Wong and Zakai [174] that the

approximation $\int_0^t \sigma_s d\widehat{W}_s$, where \widehat{W} is some smooth approximation of the Brownian motion, will converge towards the Stratonovich stochastic integral defined by

$$\int_0^t \sigma_s \circ dW_s := \int_0^t \sigma_s dW_s + \frac{1}{2} \langle \sigma, W \rangle_t.$$

This is an issue whenever the process σ is not a semimartingale and has infinite quadratic variation, which is the case for the fractional kernel with $H < 1/2$: the quadratic covariation $\langle \sigma, W \rangle$ explodes.

To solve this issue, we subtract a diverging term, in order to recover convergence, in the spirit of renormalization theory [117] and the approach in [31, Theorem 1.3], combined with another *moment-matching* trick. Once again the *moment-matching* trick is used to improve the accuracy.

Unfortunately, quantization results for SPX degrade (at a slower rate) as H goes to zero for the SPX derivatives. We therefore develop an approach with Neural Networks acting as a corrector to the quantization points for the SPX. The Neural Networks approach in our paper has a low input dimension (strikes and the input curve are not part of the Neural Networks' input) and preserves the interpretability by directly modelling the joint density of $\log(S)$ and σ . It also improves the SPX derivative pricing to a similar amplitude to that of Monte Carlo simulation, while being extremely fast.

Extensive empirical study. Our final contribution is an extensive empirical joint calibration study. A total of 1,422 days of SPX and VIX joint implied volatility surfaces between August 2011 to September 2022 were calibrated. Interestingly, the Quintic OU model, which is a *conventional one-factor Markovian continuous stochastic volatility model* outperforms, in all market conditions, its rough and non-rough path-dependent counterparts, with the same number of calibrated parameters. A possible explanation for this performance lies in the unconstrained values of H that can be pushed below zero once calibrated, something not possible for the rough fractional kernels.

More precisely, the class of model we consider is the following: we define the class of Gaussian polynomial volatility models under a risk-neutral measure as follows. We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ satisfying the usual conditions and supporting a two-dimensional Brownian motion (W, W^\perp) . For $\rho \in [-1, 1]$, we set

$$B = \rho W + \sqrt{1 - \rho^2} W^\perp,$$

which is again a Brownian motion. The dynamics of the stock price S are assumed to follow a stochastic volatility model such that the volatility process σ is given by a polynomial (possibly of infinite degree) of a Gaussian Volterra process X defined by the relations:

$$\begin{aligned} \frac{dS_t}{S_t} &= \sigma_t dB_t, \quad S_0 > 0, \\ \sigma_t &= \sqrt{\xi_0(t)} \frac{p(X_t)}{\sqrt{\mathbb{E}[p(X_t)^2]}}, \quad p(x) = \sum_{k=0}^M \alpha_k x^k, \\ X_t &= \int_0^t K(t-s) dW_s, \end{aligned}$$

for some $M \in \mathbb{N}$ possibly infinite, real coefficients $(\alpha_k)_{k=0, \dots, M}$, a non-negative square-integrable kernel $K \in L^2([0, T], \mathbb{R}_+)$ and input curve $\xi_0 \in L^2([0, T], \mathbb{R}_+)$ for any $T > 0$, with the convention that $0/0 = 1$. In particular, X is a Gaussian process such that $\mathbb{E}[X_t^2] = \int_0^t K(s)^2 ds < \infty$, for all $t \geq 0$. But X is not necessarily Markovian or a semi-martingale. We will be chiefly interested in the performance of our class of model for the joint SPX-VIX calibration problem for four kernels summarized in Table 7.1.

Kernel	$K(t)$	Domain of H	Semi-martingale	Markovian
Fractional K^{frac}	$t^{H-1/2}$	$(0, 1/2]$	✗	✗
Log-modulated K^{log}	$t^{H-1/2}(\theta \log(1/t) \vee 1)^{-\beta}$	$[0, 1/2]$	✗	✗
Shifted fractional K^{shift}	$(t + \varepsilon)^{H-1/2}$	$(-\infty, 1/2]$	✓	✗
Exponential K^{exp}	$\varepsilon^{H-1/2} e^{-(1/2-H)\varepsilon^{-1}t}$	$(-\infty, 1/2]$	✓	✓

TABLE 7.1: The different kernels K considered in this paper and the properties of their corresponding process $X_t = \int_0^t K(t-s)dW_s$; $\varepsilon > 0$, $\theta > 0$ and $\beta > 1$.

One major advantage of our class of Gaussian polynomial volatility models is an explicit expression of the VIX, a bit more involved than the Markovian setting but still explicit in a Gaussian process, which allows to implement the functional quantization procedure! For details, refer to [15].

Practitioner's corner. We carried out joint calibration on SPX and VIX implied volatilities, together with VIX futures using all four kernels in Table 7.1 for every 2nd day between August 2011 to September 2022. That is a total of 1,422 days of SPX and VIX joint implied volatility surfaces. The VIX is calibrated up to maturity $T = 2$ months, and the SPX is calibrated up to maturity $T + \Delta$, i.e. 3 months. Market data was purchased from the CBOE website <https://datashop.cboe.com/>.

To speed up the joint calibration, we applied functional quantization for fast pricing of VIX derivatives and functional quantization with Neural Networks for fast pricing of SPX derivatives.

Based on empirical results, the exponential kernel K^{exp} produces the best joint fit compared to the other kernels while being the simplest (semi-martingale and Markovian). For SPX maturities up to 3 months and VIX maturities up to 2 months, the exponential kernel K^{exp} can achieve remarkable fits, as shown in Figure 7.1 of implied volatility surfaces dated 23 October 2017, with calibrated parameters $\rho = -0.6997$, $H = -0.06939$, $(\alpha_0, \alpha_1, \alpha_3, \alpha_5) = (0.82695, 0.84388, 0.55012, 0.03271)$.

The historical time series of joint calibration rooted mean square error (RMSE) in figure 7.3 show that the exponential kernel K^{exp} outperforms other kernels for all market conditions for both SPX and VIX fit.

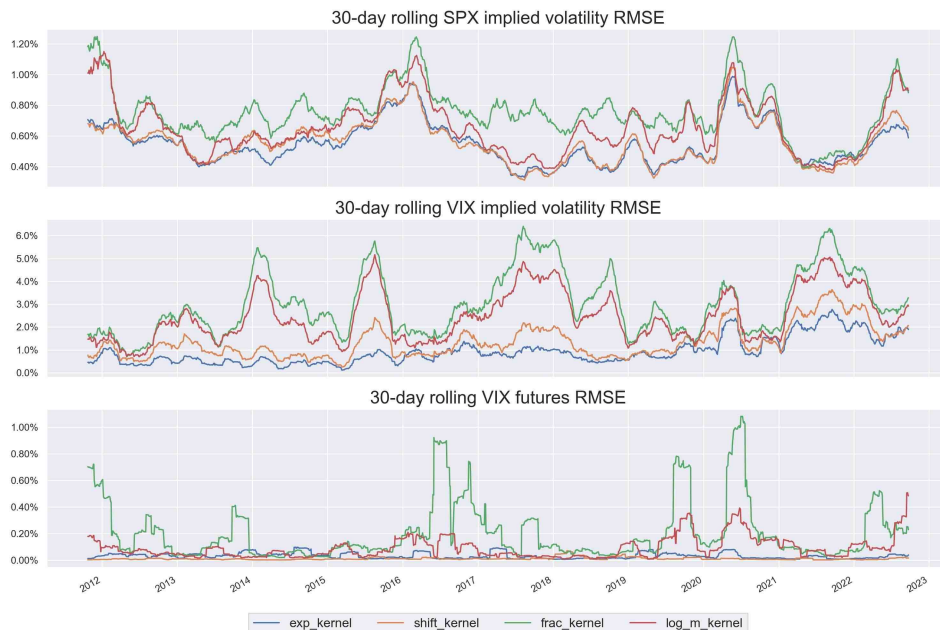


FIGURE 7.3: RMSE across different kernels: the exponential kernel K^{exp} outperforms other kernels in all market conditions.

The evolution of jointly calibrated parameters H and ρ also appear to be stable over time in the case of the exponential kernel as shown in Figure 7.4. This further validates the robustness of the exponential kernel K^{exp} to jointly fit SPX and VIX implied volatilities. Notice that H and ρ also appear to be negatively correlated to one another. We observe that ρ is far from being saturated to -1 and H is on average very small and dip below zero from time to time. The parameters ρ and H for the shifted fractional kernel K^{shift} display a similar trend.

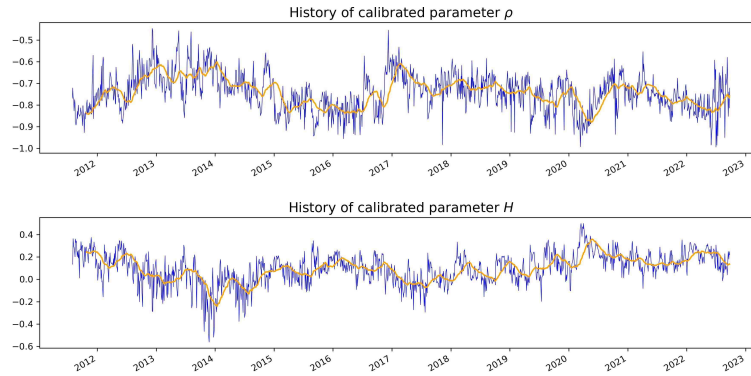


FIGURE 7.4: Evolution of the calibrated parameters ρ and H under the exponential kernel K^{exp} , the blue line is the actual value of the calibrated parameters in time, the orange line is the 30-day moving average.

Finally, we comment on the underperformance of the fractional kernel $K^{frac}(t) = t^{H-1/2}$, with $H \in (0, 1/2]$, which is extensively used in recent literature on rough volatility [30, 80]. Separate calibration of SPX/VIX appears to be satisfactory, however there are inconsistencies in the value of H between the two indices. In order to produce the steep VIX ATM skew and lower level of VIX implied volatility, the calibrated H is very close to zero (similar to that of quadratic rough Heston model in [163] where $H = 0.01$). This is problematic for the SPX due to the ‘vanishing skew’ phenomena as $H \rightarrow 0$, observed in [87] that also plagues models such as the rough Bergomi model.

Despite pushing ρ to the boundary value -1 in most days (which should increase the SPX ATM skew in stochastic volatility models) as shown in Figure 7.5, the joint calibrated SPX ATM skew is too flat compared to the market data. The VIX implied volatility produced by the model is generally too high and does not have enough ATM skew. One can try improving the VIX fit by pushing H closer to zero, but this will further flatten the SPX ATM skew.

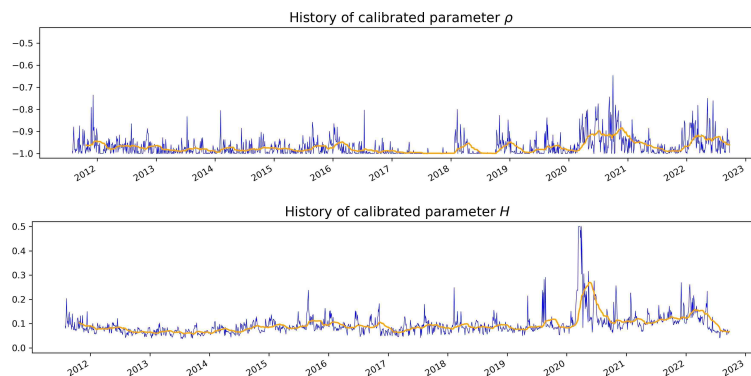


FIGURE 7.5: Fractional kernel K^{frac} : Evolution of the calibrated parameters ρ and H under the fractional kernel K^{frac} , the blue line is the actual value of the calibrated parameters in time, the orange line is the 30-day moving average. Note how ρ is saturated at -1 in most days, but still not enough to capture the SPX ATM skew. H is unable to descent to near zero due to the “vanishing” skew phenomena.

Reconciling rough volatility with jumps [5]

Summary

This chapter reconciles rough volatility with jumps models and provides an interpretation of negative H regimes.

Based on: [5] Abi Jaber, E., & De Carvalho, N. *Reconciling rough volatility with jumps*, Submitted to SIAM Journal on Financial Mathematics.

In Chapter 2, we saw how one could extend fractional processes based on Brownian motion to negative Hurst indices $H \in (-1/2, 1/2]$. One of the aims of this chapter is to give an interpretation of the region $H \in (-\infty, -1/2]$ for which such processes can no longer be defined. In addition, in Chapter 7 we saw that calibrated values of the parameter H in the Quintic Ornstein-Uhlenbeck model are negative on average, which also motivates a deeper study of negative H regimes.

Since the 1987 financial crash, financial option markets have exhibited a notable implied volatility skew, especially for short-term maturities. This skew reflects the market's expectation of significant price movements on very short time scales in the underlying asset, which poses a challenge to traditional continuous models based on standard Brownian motion. To address this issue, the literature has developed several classes of models that capture the skewness in implied volatilities. Three prominent approaches are:

- conventional one-factor stochastic volatility models boosted with large mean-reversion and vol-of-vol. This class of models have been justified by several empirical studies that have identified the presence of very fast mean-reversion in the S&P volatility time series [21, 58, 92, 93] and by the fact that they are able to correct conventional models to reproduce the behavior of the at-the-money (ATM) skew for short maturities [149];
- jump diffusion models, especially the class of affine jump-diffusions for which valuation problems become (semi-)explicit using Fourier inversion techniques, see [75]. Such class of models incorporates occasional and large jumps to explain the skew observed implicitly on option markets, see [64], and [25] for an empirical analysis of the impact of adding jumps to stochastic volatility diffusion on the implied volatility surface;
- rough volatility models, where the volatility process is driven by variants of the Riemann-Liouville fractional Brownian motion

$$W_t^H = \frac{1}{\Gamma(H + 1/2)} \int_0^t (t-s)^{H-1/2} dW_s, \quad t \geq 0,$$

with W a standard Brownian motion and $H \in (0, 1/2)$ the Hurst index. Such models are able to reproduce the roughness of the spot variance's trajectories measured empirically [99, 35] together with the explosive behavior of the ATM-skew [24, 30, 80, 95, 4].

So far, in the mathematical finance community, jump diffusion models and rough volatility models have often been treated as distinct approaches, and, in some cases, they have even been opposed to each other, see for instance [30, Section 5.3.1]. However, on the one side, connections between rough volatility models and fast mean-reverting factors have been established in [1, 6, 7]. On the other side, jump models have been related to fast regimes stochastic volatility models in [149, 148]. In parallel, from the empirical point of view, it can be very challenging for the human eye and for statistical estimators to distinguish between roughness, fast mean-reversions and jump-like behavior, as shown in [7, 62, 96].

The above suggests that rough volatility and jump models may not be that different after all. Our main motivation in [5] was to establish for the first time in the literature a connection between rough volatility and jump models through conventional volatility models with fast mean-reverting regimes.

We aim to reconcile these two classes of models through the use of the celebrated conventional Heston model [123] but with a parametric specification which encodes a trade-off between a fast mean-reversion and a large vol-of-vol. We define the *reversionary Heston model* as follows:

$$dS_t^\epsilon = S_t^\epsilon \sqrt{V_t^\epsilon} \left(\rho dW_t + \sqrt{1 - \rho^2} dW_t^\perp \right), \quad S_0^\epsilon = S_0, \quad (8.0.1)$$

$$dV_t^\epsilon = \left(\epsilon^{H - \frac{1}{2}} \theta - \epsilon^{-1} (V_t^\epsilon - V_0) \right) dt + \epsilon^{H - \frac{1}{2}} \xi \sqrt{V_t^\epsilon} dW_t, \quad V_0^\epsilon = V_0, \quad (8.0.2)$$

where (W, W^\perp) is a two-dimensional Brownian motion, $\theta \geq 0$, $S_0, \xi, V_0 > 0$, $\rho \in [-1, 1]$. The two crucial parameters here are the *reversionary time-scale* $\epsilon > 0$ and $H \in \mathbb{R}$. Such parametrizations nest as special cases the fast regimes extensively studied by Fouque et al. [92], Feng et al. [82], see also [91, Section 3.6], which correspond to the case $H = 0$; and also the regimes studied in [149, 148] for the case $H = -1/2$. Letting the parameter $H \in (\infty, 1/2]$ free in (8.0.2) introduces more flexibility in practice and leads to better fits with stable calibrated parameters across time as shown in Chapter 7. In theory, it allows for a better understanding of the impact of the scaling in H on the limiting behavior of the model as $\epsilon \rightarrow 0$ as highlighted in the paper.

In a nutshell, we show that:

1. for $H > -1/2$, the *reversionary Heston model* can be constructed as a proxy of rough and hyper-rough Heston models where $H \in (-1/2, 1/2]$ plays the role of the Hurst index,
2. for $H \leq -1/2$, as $\epsilon \rightarrow 0$, the *reversionary Heston model* converges towards Lévy jump processes of Normal Inverse Gaussian type with distinct regimes for $H = -1/2$ and $H < -1/2$ respectively,
3. the *reversionary Heston model* is capable of generating implied volatility surfaces and at-the-money (ATM) skews similar to the ones generated by rough, hyper-rough and jump models, and comes arbitrarily close to the ATM skew scaling as $\tau^{-0.5}$ for small τ that characterizes the market, contrary to widespread understanding.

Our results allow for a reconciliation between rough and jump models as they suggest that jump models and (hyper-)rough volatility models are complementary, and do not overlap. For $H > -1/2$, the *reversionary Heston model* can be interpreted as a proxy of rough and hyper-rough volatility models, while for $H \leq -1/2$, it can be interpreted as a proxy of jump models. Jump models actually start at $H = -1/2$ (and below), the first value for which hyper-rough volatility models can no-longer be defined.

More precisely, our argument is structured as follows. First, we show how the reversionary Heston model (8.0.1)-(8.0.2) can be obtained as a Markovian and semimartingale proxy of rough and hyper-rough Heston models [80, 133] with Hurst index $H \in (-1/2, 1/2)$. This is achieved using the resolvent

of the first kind of the shifted fractional kernel. Second, we derive the joint conditional characteristic functional of the log-price $\log S^\epsilon$ and the integrated variance $\bar{V}^\epsilon := \int_0^\cdot V_s^\epsilon ds$ in the model (8.0.1)–(8.0.2) in terms of a solution to a system of time-dependent Riccati ordinary differential equations. Compared to the literature, we provide a novel and concise proof for the existence and uniqueness of a global solution to such Riccati equations using the variation of constant formulas. Finally, we establish the convergence of the log-price and the integrated variance $(\log S^\epsilon, \bar{V}^\epsilon)$ in the reversionary Heston model (8.0.1)–(8.0.2) towards a Lévy jump process (X, Y) , as ϵ goes to 0. More precisely, we show that the limit (X, Y) belongs to the class of Normal Inverse Gaussian - Inverse Gaussian (NIG-IG) processes which we construct from its Lévy exponent and we connect such class to first hitting-time representations in the same spirit of Barndorff-Nielsen [28]. Our main results provide the convergence of the finite-dimensional distributions of the joint process $(\log S^\epsilon, \bar{V}^\epsilon)$ through the study of the limiting behavior of the Riccati equations and hence the characteristic functional. Interestingly, the limiting behavior disentangles three different asymptotic regimes based on the values of H . The convergence of the integrated variance process is even strengthened to a functional weak convergence on the Skorokhod space of càdlàg paths on $[0, T]$ endowed with the M_1 topology. We stress that the usual J_1 topology is not useful here, since jump processes cannot be obtained as limits of continuous processes in the J_1 topology.

Practitioner’s corner. We illustrate numerically our theoretical findings that suggests that for $H > -1/2$ the reversionary model can be interpreted as a proxy of rough and hyper-rough volatility models, while for $H \leq -1/2$ it can be interpreted as a proxy of jump models. The resulting At-The-Money (ATM) skews between 1 week and 1 year for the rough Heston model and the reversionary Heston model are shown on Figure 8.1. The graphs show that the reversionary Heston model seems to be able to generate similar shapes of the implied volatility surfaces of rough and hyper-rough models and very steep skews even in the hyper rough regimes $H \leq 0$. In Figure 8.2, we plot the convergence of the smiles and the skew of the reversionary Heston model $(\log S^\epsilon, \bar{V}^\epsilon)$ for the case $H = -1/2$ towards the Normal Inverse Gaussian model. Similar to Figure 8.1, the graphs show that the fast parametrizations introduced in the Heston model are able to reproduce very steep skews for the implied volatility surface.

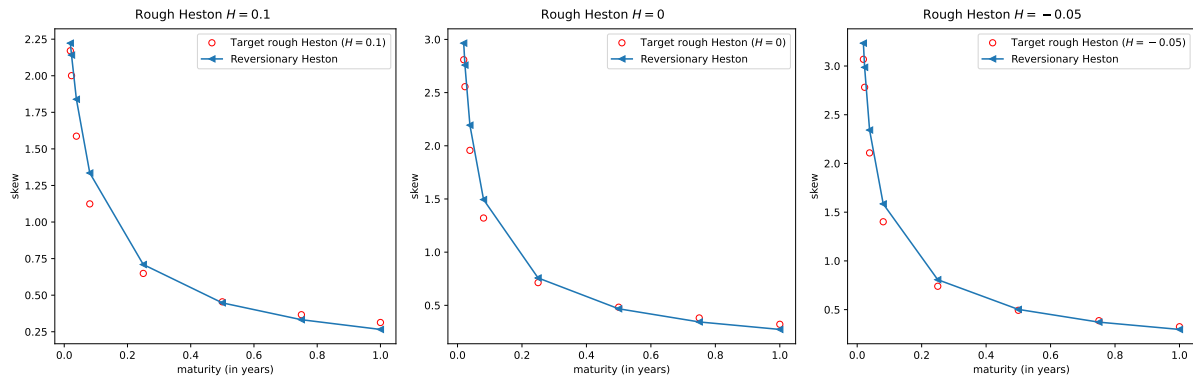


FIGURE 8.1: Resulting ATM skew $\{|\partial_k \sigma_{implicit}(k, T)|_{k=0}\}_T$ comparison between target rough Heston and reversionary Heston for different maturities from one week to one year.

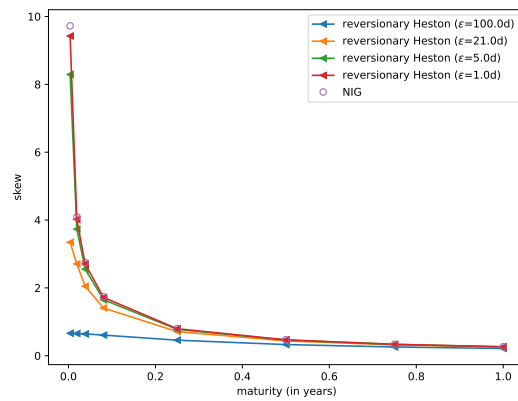


FIGURE 8.2: ATM skew of reversionary Heston and its asymptotic NIG law in the regime $H = -0.5$. Parameters are: $S_0 = 100$, $\rho = -0.7$, $\theta = 0.3$, $\xi = 0.8$, $V_0 = 0.3$ and the reversionary time-scale is varied from one hundred days to one day.

Related Literature. Convergence of the reversionary Heston models towards jump processes: our results clarify and extend the results of [149, 148], derived for the case $H = -1/2$, that establish and make clear the precise limiting connection between the Heston log-price process and the normal inverse-Gaussian (NIG) process of [28]. Connections between the long time behavior of the Heston log-price process and NIG distribution were first exposed in [86, 134] and were the main motivations behind the work of Mechkov [149].

Relevance of fast regimes in practice have been shown in Fouque et al. [92] and Feng et al. [82].

Part IV

Perspectives

Perspectives

Stochastic Volterra equations

A part of my current research agenda naturally builds on the first two parts of this thesis on the theoretical study of Volterra equations and their applications in both uncontrolled and controlled environments to solve complex problems exhibiting memory and intertemporal dependence. Specifically, my program aims to:

- Address the open problem of strong uniqueness for these equations with singular kernels such as the fractional kernel and non-Lipschitz coefficients.
- Study Volterra equations with non-convolution kernels, including existence and uniqueness with non-convolution kernels, as well as Riccati-Volterra with non-convolution kernels. One work in this direction has already begun in collaboration with Aurélien Alfonsi and Guillaume Szulda. Establishing invariance and viability properties of stochastic Volterra equations (as well as for the Riccati-Volterra equation) with non-convolution kernels is not well understood and requires new techniques. The techniques developed in [10] highly rely on the resolvent of the first kind of the kernel, which is no longer a valid object for non-convolution kernels.
- Introduce and develop the class of Polynomial Volterra processes, for which moments can be computed using a new system of integral equations (first work in progress in collaboration with Christa Cuchiero, Luca Pelizzari, Sergio Pulido and Sara Svaluto-Ferro). Exploit such method in practice: e.g. expansions of densities via moments.
- Develop a finer modeling of volatility and correlations in a multivariate framework.
- Develop a more parsimonious modeling of energy markets. In this regard, I am co-supervising Nathan De Carvalho's thesis at ENGIE, in collaboration with Huyên Pham (Université de Paris).
- Study Principal-Agent problems in more complex and realistic dynamics in non-Markovian settings beyond the Gaussian setting that has been treated here.
- Solve optimal execution and liquidation problems for multiple assets with persistent (cross)-market impact using propagator models. Analytic solutions (in terms of infinite-dimensional operators) can be obtained, building on the results in this thesis.
- Explore the direct approach developed in Section 6.2 and extend it to attack different types of mean-field games under aggregated and individual constraints on state variables and controls through a generalization of the KKT conditions, as well as in cooperative games and McKean-Vlasov type optimal control problems. Several practical applications will be developed, such as optimal trading with storage batteries, a first work is in progress with Nathan De Carvalho and Huyên Pham.

Towards the use of signature in mathematical finance

One of my other main objectives is to explore the concept of "signature"¹ and more particularly its property of linearization, both practical and theoretical, for the semi-explicit resolution of problems characterizing the law of certain rich dynamic models and control problems beyond the linear-quadratic framework.

Different types of solutions will be considered. Heuristically, these solutions are based on the key idea of functional linearization of a path via the signature. This linearization property would allow the development of analytical formulas for optimal strategies in models with very rich dynamics that go beyond the quadratic linear framework. We expect to derive these solutions through a system of non-standard ordinary differential equations of Riccati type in infinite dimensions with values in a tensor algebra. These equations fall outside the classical framework, and so far there are no existence and uniqueness results for this type of equation, they have recently appeared in [71]. I plan to study the theory of existence of these equations and to develop and test different numerical schemes for an efficient numerical resolution of the problem by projecting/truncating the signature in finite dimension. I also aim to obtain convergence and stability results for these methods, and if possible, to exhibit convergence rates.

To avoid the curse of dimensionality related to the exponential growth of the number of terms of the signature, I wish to explore new directions of dimensionality reduction. In addition, I plan to use deep learning methods coupled with the signature to construct signals and improve the predictive performance of supervised and unsupervised learning models. Several types of applications will be considered. I plan to build more realistic models to model the stochastic correlations between assets that would depend on the entire trajectory, and I will be interested in portfolio allocation problems and signature computation in these new non-Markovian and non-semimartingale models.

This work will build on ongoing work with Louis-Amand Gérard's for his PhD, which focuses on practical applications of signature and deep learning methods for portfolio allocation and risk management. Finally, I am exploring these different techniques with Stéphane Crépey and Botao Li (postdoc student) for generating realistic market trajectories as part of a project recently started.

Empirical performance and evaluation of stochastic volatility models

I would like to go beyond the third part of this thesis and put in place a general framework for comparing the performance of different volatility models on real market data, in terms of calibration performance, hedging, prediction and stability of calibrated parameters. Part of these points are currently being addressed for the second half of Shaun Li's PhD.

¹a mathematical concept, initially introduced by Chen [54], consisting of the (infinite) sequence of iterated integrals of a path, which plays a crucial role in the theory of rough paths [146] and has recently gained popularity and attracted attention from the machine learning community [59].

Bibliography

- [1] Eduardo Abi Jaber. Lifting the Heston model. *Quantitative Finance*, 19(12):1995–2013, 2019.
- [2] Eduardo Abi Jaber. Weak existence and uniqueness for affine stochastic Volterra equations with L^1 -kernels. *Bernoulli*, 27(3):1583–1615, 2021.
- [3] Eduardo Abi Jaber. The Laplace transform of the integrated Volterra Wishart process. *Mathematical Finance*, 32(1):309–348, 2022.
- [4] Eduardo Abi Jaber. The characteristic function of Gaussian stochastic volatility models: an analytic expression. *Finance and Stochastics*, 26(4):733–769, 2022.
- [5] Eduardo Abi Jaber and Nathan De Carvalho. Reconciling rough volatility with jumps. *Available at SSRN 4387574*, 2023.
- [6] Eduardo Abi Jaber and Omar El Euch. Markovian structure of the Volterra Heston model. *Statistics & Probability Letters*, 149:63–72, 2019.
- [7] Eduardo Abi Jaber and Omar El Euch. Multifactor approximation of rough volatility models. *SIAM Journal on Financial Mathematics*, 10(2):309–349, 2019.
- [8] Eduardo Abi Jaber and Eyal Neuman. Optimal liquidation with signals: the general propagator case. *arXiv preprint arXiv:2211.00447*, 2022.
- [9] Eduardo Abi Jaber and Stéphane Villeneuve. Gaussian agency problems with memory and linear contracts. *Available at SSRN 4226543*, 2022.
- [10] Eduardo Abi Jaber, Martin Larsson, and Sergio Pulido. Affine Volterra processes. *The Annals of Applied Probability*, 29(5):3155–3200, 2019.
- [11] Eduardo Abi Jaber, Christa Cuchiero, Martin Larsson, and Sergio Pulido. A weak solution theory for stochastic Volterra equations of convolution type. *The Annals of Applied Probability*, 31(6):2924–2952, 2021.
- [12] Eduardo Abi Jaber, Enzo Miller, and Huyên Pham. Markowitz portfolio selection for multivariate affine and quadratic Volterra models. *SIAM J. Finan. Math.*, 12(1):369–409, 2021.
- [13] Eduardo Abi Jaber, Enzo Miller, and Huyên Pham. Integral operator Riccati equations arising in stochastic Volterra control problems. *SIAM Journal on Control and Optimization*, 59(2):1581–1603, 2021.
- [14] Eduardo Abi Jaber, Enzo Miller, and Huyên Pham. Linear-Quadratic control for a class of stochastic Volterra equations: solvability and approximation. *The Annals of Applied Probability*, 31(5):2244–2274, 2021.

- [15] Eduardo Abi Jaber, Camille Illand, and Shaun Li. Joint SPX-VIX calibration with Gaussian polynomial volatility models: deep pricing with quantization hints. *arXiv preprint arXiv:2212.08297*, 2022.
- [16] Eduardo Abi Jaber, Camille Illand, and Shaun Li. The quintic Ornstein-Uhlenbeck volatility model that jointly calibrates SPX & VIX smiles. *Risk Magazine*, to appear, 2023.
- [17] Eduardo Abi Jaber, Eyal Neuman, and Moritz Voss. Equilibrium in functional stochastic games with mean-field interaction. *Working paper*, 2023.
- [18] Nacira Agram and Bernt Øksendal. Malliavin calculus and optimal control of stochastic Volterra equations. *Journal of Optimization Theory and Applications*, 167(3):1070–1094, 2015.
- [19] Aurélien Alfonsi. *Affine diffusions and related processes: simulation, theory and applications*, volume 6. Springer, 2015.
- [20] Aurélien Alfonsi and Alexander Schied. Capacitary measures for completely monotone kernels via singular control. *SIAM Journal on Control and Optimization*, 51(2):1758–1780, 2013.
- [21] Sassan Alizadeh, Michael W Brandt, and Francis X Diebold. Range-based estimation of stochastic volatility models. *The Journal of Finance*, 57(3):1047–1091, 2002.
- [22] R. Almgren and N. Chriss. Value under liquidation. *Risk*, 12:61–63, 1999.
- [23] R. Almgren and N. Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3(2): 5–39, 2000.
- [24] Elisa Alòs, Jorge A. León, and Josep Vives. On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility. *Finance and Stochastics*, 11(4):571–589, aug 2007. doi: 10.1007/s00780-007-0049-1. URL <https://doi.org/10.1007/2Fs00780-007-0049-1>.
- [25] Gurdip Bakshi, Charles Cao, and Zhiwu Chen. Empirical performance of alternative option pricing models. *The Journal of Finance*, 52(5):2003–2049, 1997.
- [26] Jan Baldeaux and Alexander Badran. Consistent modelling of vix and equity derivatives using a 3/2 plus jumps model. *Applied Mathematical Finance*, 21(4):299–312, 2014.
- [27] Peter Bank, H Mete Soner, and Moritz Voß. Hedging with temporary price impact. *Mathematics and financial economics*, 11(2):215–239, 2017.
- [28] Ole Barndorff-Nielsen. Normal inverse gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of Statistics*, 24(1):1–13, 1997.
- [29] Nicole Bäuerle and Sascha Desmettre. Portfolio optimization in fractional and rough Heston models. *SIAM Journal on Financial Mathematics*, 11(1):240–273, 2020.
- [30] Christian Bayer, Peter Friz, and Jim Gatheral. Pricing under rough volatility. *Quantitative Finance*, 16(6):887–904, 2016.
- [31] Christian Bayer, Peter K Friz, Paul Gassiat, Jorg Martin, and Benjamin Stemper. A regularity structure for rough volatility. *Mathematical Finance*, 30(3):782–832, 2020.
- [32] Christian Bayer, Fabian A Harang, and Paolo Pigato. Log-modulated rough stochastic volatility models. *SIAM Journal on Financial Mathematics*, 12(3):1257–1284, 2021.
- [33] C. Belak, J. Muhle-Karbe, and K. Ou. Liquidation in target zone models. *Market Microstructure and Liquidity*, 2019. URL <https://doi.org/10.1142/S2382626619500102>.
- [34] C. Bellani, D. Brigo, A. Done, and E. Neuman. Optimal trading: The importance of being adaptive. *International Journal of Financial Engineering*, 08(04):2050022, 2021. doi: 10.1142/S242478632050022X. URL <https://doi.org/10.1142/S242478632050022X>.

- [35] Mikkel Bennedsen, Asger Lunde, and Mikko S Pakkanen. Decoupling the Short- and Long-Term Behavior of Stochastic Volatility. *Journal of Financial Econometrics*, 01 2021.
- [36] Alain Bensoussan, MHM Chau, Y Lai, and Sheung Chi Phillip Yam. Linear-quadratic mean field stackelberg games with state and control delays. *SIAM Journal on Control and Optimization*, 55(4):2748–2781, 2017.
- [37] Fred Espen Benth and Victor Rohde. On non-negative modeling with CARMA processes. *Journal of Mathematical Analysis and Applications*, 2018.
- [38] Fred Espen Benth, Nils Detering, and Paul Kruehner. Stochastic Volterra integral equations and a class of first order stochastic partial differential equations. *arXiv preprint arXiv:1903.05045*, 2019.
- [39] Patrick Bolton and Mathias Dewatripont. *Contract theory*. MIT press, 2005.
- [40] Stefano Bonaccorsi, Fulvia Confortola, and Elisa Mastrogiacomo. Optimal control for stochastic Volterra equations with completely monotone kernels. *SIAM Journal on Control and Optimization*, 50(2):748–789, 2012.
- [41] Alessandro Bondi, Sergio Pulido, and Simone Scotti. The rough hawkes Heston stochastic volatility model. *arXiv preprint arXiv:2210.12393*, 2022.
- [42] Ofelia Bonesini, Giorgia Callegaro, and Antoine Jacquier. Functional quantization of rough volatility and applications to the vix. *arXiv preprint arXiv:2104.04233*, 2021.
- [43] Folkmar Bornemann. On the numerical evaluation of Fredholm determinants. *Mathematics of Computation*, 79(270):871–915, 2010.
- [44] J-P. Bouchaud, J. Bonart, J. Donier, and M. Gould. *Trades, Quotes and Prices: Financial Markets Under the Microscope*. Cambridge University Press, 2018. doi: 10.1017/9781316659335.
- [45] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science & Business Media, 2010.
- [46] D. Brigo and C. Piat. Static vs adapted optimal execution strategies in two benchmark trading models. In K. Glau, D. Linders, M. Scherer, L. Schneider, and R. Zagst, editors, *Innovations in Insurance, Risk- and Asset Management*, pages 239–274. World Scientific Publishing, Munich, 2018.
- [47] Marie-France Bru. Wishart processes. *Journal of Theoretical Probability*, 4(4):725–751, 1991.
- [48] Andrea Buraschi, Paolo Porchia, and Fabio Trojani. Correlation risk and optimal portfolio choice. *The Journal of Finance*, 65(1):393–420, 2010.
- [49] Philippe Carmona and Laure Coutin. Fractional brownian motion and the markov property. *Electronic Communications in Probability*, 3:95–107, 1998.
- [50] René Carmona, Jean-Pierre Fouque, Seyyed Mostafa Mousavi, and Li-Hsien Sun. Systemic risk and stochastic games with delay. *Journal of Optimization Theory and Applications*, 179(2): 366–399, 2018.
- [51] Á. Cartea and S. Jaimungal. Incorporating order-flow into optimal execution. *Mathematics and Financial Economics*, 10(3):339–364, 2016. ISSN 1862-9660. doi: 10.1007/s11579-016-0162-z. URL <http://dx.doi.org/10.1007/s11579-016-0162-z>.
- [52] Á. Cartea, S. Jaimungal, and J. Penalva. *Algorithmic and High-Frequency Trading (Mathematics, Finance and Risk)*. Cambridge University Press, 1 edition, October 2015. ISBN 1107091144. URL <http://www.amazon.com/exec/obidos/redirect?tag=citeulike07-20&path=ASIN/1107091144>.

- [53] CBOE. Volatility index methodology: Cboe volatility index. *White paper*. URL https://cdn.cboe.com/api/global/us_indices/governance/Volatility_Index_Methodology_Cboe_Volatility_Index.pdf.
- [54] Kuo-Tsai Chen. Integration of paths, geometric invariants and a generalized baker-hausdorff formula. *Annals of Mathematics*, pages 163–178, 1957.
- [55] Li Chen, Damir Filipović, and H Vincent Poor. Quadratic term structure models for risk-free and defaultable rates. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, 14(4):515–536, 2004.
- [56] Y. Chen, U. Horst, and H.H. Tran. Portfolio liquidation under transient price impact - theoretical solution and implementation with 100 NASDAQ stocks. Preprint available on arXiv:1912.06426, 2019.
- [57] Peng Cheng and Olivier Scaillet. Linear-quadratic jump-diffusion modeling. *Mathematical Finance*, 17(4):575–598, 2007.
- [58] Mikhail Chernov, A Ronald Gallant, Eric Ghysels, and George Tauchen. Alternative models for stock price dynamics. *Journal of Econometrics*, 116(1-2):225–257, 2003.
- [59] Ilya Chevyrev and Andrey Kormilitzin. A primer on the signature method in machine learning. *arXiv preprint arXiv:1603.03788*, 2016.
- [60] Mei Choi Chiu and Hoi Ying Wong. Mean-variance portfolio selection with correlation risk. *Journal of Computational and Applied Mathematics*, 263:432–444, 2014.
- [61] R. Cont, A. Kukanov, and S. Stoikov. The price impact of order book events. *Journal of Financial Econometrics*, 12(1):47–88, 2014.
- [62] Rama Cont and Purba Das. Rough volatility: fact or artefact? *arXiv preprint arXiv:2203.13820*, 2022.
- [63] Rama Cont and Thomas Kokholm. A consistent pricing model for index options and volatility derivatives. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, 23(2):248–274, 2013.
- [64] Rama Cont and Peter Tankov. *Financial modelling with jump processes*. Chapman and Hall/CRC, 2003.
- [65] José Manuel Corcuera, Gergely Farkas, Wim Schoutens, and Esko Valkeila. A short rate model using ambit processes. In *Malliavin Calculus and Stochastic Analysis*, pages 525–553. Springer, 2013.
- [66] John C Cox, Jonathan Jr E Ingersoll, and Stephen A Ross. A theory of the term structure of interest rates. In *Theory of Valuation*, pages 129–164. World Scientific, 2005.
- [67] Christa Cuchiero and Josef Teichmann. Markovian lifts of positive semidefinite affine Volterra-type processes. *Decisions in Economics and Finance*, 42(2):407–448, 2019.
- [68] Christa Cuchiero and Josef Teichmann. Generalized Feller processes and Markovian lifts of stochastic Volterra processes: the affine case. *Journal of Evolution Equations*, pages 1–48, 2020.
- [69] Christa Cuchiero, Damir Filipović, Eberhard Mayerhofer, and Josef Teichmann. Affine processes on positive semidefinite matrices. *Ann. Appl. Probab.*, 21(2):397–463, 2011. ISSN 1050-5164. doi: 10.1214/10-AAP710. URL <http://dx.doi.org/10.1214/10-AAP710>.
- [70] Christa Cuchiero, Claudio Fontana, and Alessandro Gnoatto. Affine multiple yield curve models. *Mathematical Finance*, 2016.

- [71] Christa Cuchiero, Sara Svaluto-Ferro, and Josef Teichmann. Signature sdes from an affine and polynomial perspective. *arXiv preprint arXiv:2302.01362*, 2023.
- [72] Jakša Cvitanić, Dylan Possamaï, and Nizar Touzi. Dynamic programming approach to principal-agent problems. *Finance and Stochastics*, 22(1):1–37, 2018.
- [73] José Da Fonseca, Martino Grasselli, and Claudio Tebaldi. Option pricing when correlations are stochastic: an analytical framework. *Review of Derivatives Research*, 10(2):151–180, 2007.
- [74] José Da Fonseca, Martino Grasselli, and Claudio Tebaldi. A multifactor volatility Heston model. *Quantitative Finance*, 8(6):591–604, 2008.
- [75] James Darrel Duffie, Jun Pan, and Kenneth Singleton. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68(6):1343–1376, 2002.
- [76] Donald A Dawson and Klaus Fleischmann. A super-Brownian motion with a single point catalyst. *Stochastic Processes and their Applications*, 49(1):3–40, 1994.
- [77] Steffen Dereich and Michael Scheutzow. High resolution quantization and entropy coding for fractional brownian motion. *Electronic Journal of Probability*, 11:700–722, 2006.
- [78] Tyrone E Duncan and Bozenna Pasik-Duncan. Linear-quadratic fractional Gaussian control. *SIAM Journal on Control and Optimization*, 51(6):4504–4519, 2013.
- [79] Omar El Euch and Mathieu Rosenbaum. Perfect hedging in rough Heston models. *The Annals of Applied Probability*, 28(6):3813–3856, 2018. doi: 10.1214/18-AAP1408. URL <https://doi.org/10.1214/18-AAP1408>.
- [80] Omar El Euch and Mathieu Rosenbaum. The characteristic function of rough Heston models. *Mathematical Finance*, 29(1):3–38, 2019.
- [81] Stewart N Ethier and Thomas G Kurtz. *Markov Processes: Characterization and Convergence*. Wiley Series in Probability and Statistics. Wiley, 2005.
- [82] Jin Feng, Marting Forde, and Jean-Pierre Fouque. Short maturity asymptotics for a fast mean-reverting Heston stochastic volatility model. *SIAM journal on Financial Mathematics*, 1, 2010.
- [83] Franco Flandoli. Direct solution of a Riccati equation arising in a stochastic control problem with control and observation on the boundary. *Applied Mathematics and Optimization*, 14(1):107–129, 1986.
- [84] Franco Flandoli and Dariusz Gatarek. Martingale and stationary solutions for stochastic Navier-stokes equations. *Probability Theory and Related Fields*, 102(3):367–391, 1995.
- [85] M. Forde, L. Sánchez-Betancourt, and B. Smith. Optimal trade execution for gaussian signals with power-law resilience. *Quantitative Finance*, 22(3):585–596, 2022. doi: 10.1080/14697688.2021.1950919. URL <https://doi.org/10.1080/14697688.2021.1950919>.
- [86] Martin Forde and Antoine Jacquier. The large-maturity smile for the Heston model. *Finance and Stochastics*, 15(4):755–780, 2011.
- [87] Martin Forde, Masaaki Fukasawa, Stefan Gerhold, and Benjamin Smith. The rough bergomi model as $h \rightarrow 0$ -skew flattening/blow up and non-gaussian rough volatility. *preprint*, 2020.
- [88] J-P Fouque and Yuri F Saporito. Heston stochastic vol-of-vol model for joint calibration of vix and s&p 500 options. *Quantitative Finance*, 18(6):1003–1016, 2018.
- [89] Jean-Pierre Fouque and Ruiming Hu. Optimal portfolio under fast mean-reverting fractional stochastic environment. *SIAM Journal on Financial Mathematics*, 6(2):564–601, 2018.
- [90] Jean-Pierre Fouque and Zhaoyu Zhang. Mean field game with delay: a toy model. *Risks*, 6(3):90, 2018.

- [91] Jean-Pierre Fouque, George Papanicolaou, and K Ronnie Sircar. *Derivatives in financial markets with stochastic volatility*. Cambridge University Press, 2000.
- [92] Jean-Pierre Fouque, George Papanicolaou, Ronnie Sircar, and Knut Solna. Multiscale stochastic volatility asymptotics. *Multiscale Modeling & Simulation*, 2(1):22–42, 2003.
- [93] Jean-Pierre Fouque, George Papanicolaou, Ronnie Sircar, and Knut Solna. Short time-scale in s&p500 volatility. *Journal of Computational Finance*, 6(4):1–24, 2003.
- [94] Ivar Fredholm. Sur une classe d'équations fonctionnelles. *Acta mathematica*, 27(1):365–390, 1903.
- [95] Masaaki Fukasawa. Asymptotic analysis for stochastic volatility: martingale expansion. *Finance and Stochastics*, 15:635–654, 2011.
- [96] Matthieu Garcin and Martino Grasselli. Long versus short time scales: the rough dilemma and beyond. *Decisions in economics and finance*, 45(1):257–278, 2022.
- [97] J. Gatheral and A. Schied. Dynamical models of market impact and algorithms for order execution. In Jean-Pierre Fouque and Joseph Langsam, editors, *Handbook on Systemic Risk*, pages 579–602. Cambridge University Press, 2013.
- [98] J. Gatheral, A. Schied, and A. Slynko. Transient linear price impact and Fredholm integral equations. *Math. Finance*, 22:445–474, 2012.
- [99] Jim Gatheral, Thibault Jaisson, and Mathieu Rosenbaum. Volatility is rough. *Quantitative finance*, 18(6):933–949, 2018.
- [100] Jim Gatheral, Paul Jusselin, and Mathieu Rosenbaum. The quadratic rough heston model and the joint s&p 500/vix smile calibration problem. *arXiv preprint arXiv:2001.01789*, 2020.
- [101] Allen Gersho and Robert M Gray. *Vector quantization and signal compression*, volume 159. Springer Science & Business Media, 2012.
- [102] Paul Glasserman and Pu He. Buy rough, sell smooth. *Quantitative Finance*, 20(3):363–378, 2020.
- [103] S. Gökay, A. Roch, and H.M. Soner. Liquidity models in continuous and discrete time. In Giulia di Nunno and Bern Øksendal, editors, *Advanced Mathematical Methods for Finance*, pages 333–366. Springer-Verlag, 2011.
- [104] Christian Gouriéroux and Razvan Sufana. Wishart quadratic term structure models. *Les Cahiers du CREF of HEC Montreal Working Paper*, (03-10), 2003.
- [105] Christian Gouriéroux, Joann Jasiak, and Razvan Sufana. The Wishart autoregressive process of multivariate stochastic volatility. *Journal of Econometrics*, 150(2):167–181, 2009.
- [106] Stéphane Goutte, Amine Ismail, and Huyèn Pham. Regime-switching stochastic volatility model: estimation and calibration to vix options. *Applied Mathematical Finance*, 24(1):38–75, 2017.
- [107] Fausto Gozzi and Carlo Marinelli. Stochastic optimal control of delay equations arising in advertising models. stochastic partial differential equations and applications-vii, 133-148. *Lect. Notes Pure Appl. Math*, 245, 2005.
- [108] P. Graewe and U. Horst. Optimal trade execution with instantaneous price impact and stochastic resilience. *SIAM Journal on Control and Optimization*, 55(6):3707–3725, 2017. doi: 10.1137/16M1105463. URL <https://doi.org/10.1137/16M1105463>.
- [109] Siegfried Graf and Harald Luschgy. Foundations of quantization for probability distributions. *Foundations Of Quantization For Probability Distributions*, 1730:1–+, 2000.

- [110] Gustaf Gripenberg, Stig-Olof Londen, and Olof Staffans. *Volterra integral and functional equations*, volume 34 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1990. ISBN 0-521-37289-5. doi: 10.1017/CBO9780511662805. URL <http://dx.doi.org/10.1017/CBO9780511662805>.
- [111] Lech A Grzelak. On randomization of affine diffusion processes with application to pricing of options on vix and s&p 500. *arXiv preprint arXiv:2208.12518*, 2022.
- [112] O. Guéant. *The Financial Mathematics of Market Liquidity*. New York: Chapman and Hall/CRC, 2016.
- [113] Archil Gulisashvili. Large deviation principle for Volterra type fractional stochastic volatility models. *SIAM Journal on Financial Mathematics*, 9(3):1102–1136, 2018.
- [114] Archil Gulisashvili, Frederi Viens, and Xin Zhang. Extreme-strike asymptotics for general gaussian stochastic volatility models. *Annals of Finance*, 15(1):59–101, 2019.
- [115] Ivan Guo, Gregoire Loeper, Jan Obloj, and Shiyi Wang. Joint modeling and calibration of spx and vix by optimal transport. *SIAM Journal on Financial Mathematics*, 13(1):1–31, 2022.
- [116] Julien Guyon and Jordan Lekeufack. Volatility is (mostly) path-dependent. *Volatility Is (Mostly) Path-Dependent (July 27, 2022)*, 2022.
- [117] M Hairer. A theory of regularity structures. *Inventiones Mathematicae*, 198(2):269, 2014.
- [118] Bingyan Han and Hoi Ying Wong. Mean-variance portfolio selection under Volterra Heston model. *arXiv preprint arXiv:1904.12442*, 2019.
- [119] Bingyan Han and Hoi Ying Wong. Time-consistent feedback strategies with Volterra processes. *arXiv preprint arXiv:1907.11378*, 2019.
- [120] Bingyan Han and Hoi Ying Wong. Merton’s portfolio problem under Volterra Heston model. *Finance Research Letters*, page 101580, 2020.
- [121] Philipp Harms and David Stefanovits. Affine representations of fractional processes with applications in mathematical finance. *Stochastic Processes and their Applications*, 129(4):1185 – 1228, 2019. ISSN 0304-4149. doi: <https://doi.org/10.1016/j.spa.2018.04.010>.
- [122] Philipp Harms and David Stefanovits. Affine representations of fractional processes with applications in mathematical finance. *Stochastic Processes and their Applications*, 129(4):1185–1228, 2019.
- [123] Steven L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, 6(2):327–343, 1993. ISSN 08939454, 14657368. URL <http://www.jstor.org/stable/2962057>.
- [124] Bengt Holmström and Paul Milgrom. Aggregation and linearity in the provision of intertemporal incentives. *Econometrica: Journal of the Econometric Society*, pages 303–328, 1987.
- [125] Blanka Horvath, Antoine Jacquier, and Chloe Lacombe. Asymptotic behaviour of randomised fractional volatility models. *Available at SSRN 3013658*, 2017.
- [126] Ying Hu and Shanjian Tang. Stochastic LQ and associated Riccati equation of PDEs driven by state-and control-dependent White noise. *arXiv preprint arXiv:1809.05308*, 2018.
- [127] Jianhui Huang, Xun Li, and Tianxiao Wang. Mean-field linear-quadratic-gaussian (lqg) games for stochastic integral systems. *IEEE Transactions on Automatic Control*, 61(9):2670–2675, 2015.
- [128] John Hull and Alan White. Pricing interest-rate-derivative securities. *The review of financial studies*, 3(4):573–592, 1990.

- [129] Amine Ismail and Huy en Pham. Robust Markowitz mean-variance portfolio selection under ambiguous covariance matrix. *Mathematical Finance*, 29(1):174–207, 2019.
- [130] Antoine Jacquier and Mugad Oumgari. Deep PPDEs for rough local stochastic volatility. *arXiv:1906.02551*, 2019.
- [131] Thibault Jaisson and Mathieu Rosenbaum. Rough fractional diffusions as scaling limits of nearly unstable heavy tailed Hawkes processes. *The Annals of Applied Probability*, 26(5):2860–2882, 2016.
- [132] Monique Jeanblanc, Martin Schweizer, and Michael Mania. Mean-variance hedging via stochastic control and BSDEs for general semimartingales. *Annals of Applied Probability*, 22(6):2388–2428, 2012.
- [133] Paul Jusselin and Mathieu Rosenbaum. No-arbitrage implies power-law market impact and rough volatility. *Mathematical Finance*, 30(4):1309–1336, 2020.
- [134] Martin Keller-Ressel. Moment explosions and long-term behavior of affine stochastic volatility models. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, 21(1):73–98, 2011.
- [135] Marina Kleptsyna, Alain Le Breton, and Michel Viot. About the linear quadratic regulator problem under a fractional Brownian perturbation,. *ESAIM Probab. Stat*, 9:161–170, 2003.
- [136] ML Kleptsyna, A Le Breton, and M Viot. New formulas concerning Laplace transforms of quadratic forms for general Gaussian sequences. *International Journal of Stochastic Analysis*, 15(4):309–325, 2002.
- [137] Thomas Kokholm and Martin Stisen. Joint pricing of vix and SPX options with stochastic volatility and jump models. *The Journal of Risk Finance*, 2015.
- [138] Jean-Jacques Laffont and David Martimort. *The theory of incentives*. Princeton university press, 2009.
- [139] C. A. Lehalle and O. Mounjid. Limit Order Strategic Placement with Adverse Selection Risk and the Role of Latency, October 2016. URL <http://arxiv.org/abs/1610.00261>.
- [140] C. A. Lehalle and E. Neuman. Incorporating signals into optimal trading. *Finance and Stochastics*, 23(2):275–311, 2019. doi: 10.1007/s00780-019-00382-7. URL <https://doi.org/10.1007/s00780-019-00382-7>.
- [141] C. A. Lehalle, S. Laruelle, R. Burgot, S. Pelin, and M. Lasnier. *Market Microstructure in Practice*. World Scientific publishing, 2013. URL <http://www.worldscientific.com/worldscibooks/10.1142/8967>.
- [142] Andrew EB Lim. Quadratic hedging and mean-variance portfolio selection with random parameters in an incomplete market. *Mathematics of Operations Research*, 29(1):132–161, 2004.
- [143] Andrew EB Lim and Xun Yu Zhou. Mean-variance portfolio selection with random parameters in a complete market. *Mathematics of Operations Research*, 27(1):101–120, 2002.
- [144] A. Lipton, U. Pesavento, and M. G. Sotiropoulos. Trade arrival dynamics and quote imbalance in a limit order book, December 2013. URL <http://arxiv.org/abs/1312.0514>.
- [145] C. Lorenz and A. Schied. Drift dependence of optimal trade execution strategies under transient price impact. *Finance Stoch.*, 17(4):743–770, 2013. ISSN 0949-2984. doi: 10.1007/s00780-013-0211-x. URL <http://dx.doi.org/10.1007/s00780-013-0211-x>.
- [146] Terry J Lyons, Michael Caruana, and Thierry L evy. *Differential equations driven by rough paths*. Springer, 2007.
- [147] Harry Markowitz. Portfolio selection. *Journal of Finance*, pages 77–91, 1952.

- [148] Ryan McCrickerd. *On spatially irregular ordinary differential equations and a pathwise volatility modelling framework*. Mathematics PhD Theses, Imperial College London, URL: <https://doi.org/10.25560/92202>, 2021.
- [149] Serguei Mechkov. Fast-reversion limit of the Heston model. *Available at SSRN 2418631*, 2015.
- [150] Johannes Muhle-Karbe, Oliver Pfaffel, and Robert Stelzer. Option pricing in multivariate stochastic volatility models of OU type. *SIAM Journal on Financial Mathematics*, 3(1):66–94, 2012.
- [151] Leonid Mytnik and Thomas S Salisbury. Uniqueness for Volterra-type stochastic integral equations. *arXiv preprint arXiv:1502.05513*, 2015.
- [152] E. Neuman and A. Schied. Optimal portfolio liquidation in target zone models and catalytic superprocesses. *Finance and Stochastics*, 20:495–509, 2016.
- [153] E. Neuman and M. Voß. Optimal signal-adaptive trading with temporary and transient price impact. *SIAM Journal on Financial Mathematics*, 13(2):551–575, 2022.
- [154] E. Neuman and M. Voß. Trading with the Crowd. To appear in *Mathematical Finance*, 2023. URL <https://arxiv.org/abs/2106.09267>.
- [155] A. A. Obizhaeva and J. Wang. Optimal trading strategy and supply/demand dynamics. *Journal of Financial Markets*, 16(1):1 – 32, 2013. ISSN 1386-4181. doi: <http://dx.doi.org/10.1016/j.finmar.2012.09.001>. URL <http://www.sciencedirect.com/science/article/pii/S1386418112000328>.
- [156] Claudio Pacati, Gabriele Pompa, and Roberto Renò. Smiling twice: the Heston++ model. *Journal of Banking & Finance*, 96:185–206, 2018.
- [157] Gilles Pagès and Jacques Printems. Optimal quadratic quantization for numerics: the gaussian case. 2003.
- [158] Gilles Pagès and Jacques Printems. Functional quantization for numerics with an application to option pricing. 2005.
- [159] Andrew Papanicolaou and Ronnie Sircar. A regime-switching heston model for vix and s&p 500 implied volatilities. *Quantitative Finance*, 14(10):1811–1827, 2014.
- [160] Huyên Pham and Jacques Printems. Optimal quantization methods and applications to numerical problems in finance. In *Handbook of computational and numerical methods in finance*, pages 253–297. Springer, 2004.
- [161] William P Rogerson. Repeated moral hazard. *Econometrica: Journal of the Econometric Society*, pages 69–76, 1985.
- [162] Sigurd Emil Rømer. Empirical analysis of rough and classical stochastic volatility models to the spx and vix markets. *Quantitative Finance*, 22(10):1805–1838, 2022.
- [163] Mathieu Rosenbaum and Jianfei Zhang. Deep calibration of the quadratic rough heston model. *arXiv preprint arXiv:2107.01611*, 2021.
- [164] Walter Rudin. *Real and complex analysis*. Tata McGraw-hill education, 2006.
- [165] Heinz Schättler and Jaeyoung Sung. The first-order approach to the continuous-time principal-agent problem with exponential utility. *Journal of Economic Theory*, 61(2):331–371, 1993.
- [166] Heinz Schättler and Jaeyoung Sung. On optimal sharing rules in discrete and continuous-time principal-agent problems with exponential utility. *Journal of Economic Dynamic and Control*, 21(2):551–574, 1997.

-
- [167] A. Schied, E. Strehle, and T. Zhang. A hot-potato game under transient price impact: the continuous-time limit. *working paper*, 2015.
- [168] Rainer Schöbel and Jianwei Zhu. Stochastic volatility with an Ornstein–Uhlenbeck process: an extension. *Review of Finance*, 3(1):23–46, 1999.
- [169] Yang Shen. Mean–variance portfolio selection in a complete market with unbounded random coefficients. *Automatica*, 55:165–175, 2015.
- [170] Yang Shen, Xin Zhang, and Tak Kuen Siu. Mean–variance portfolio selection under a constant elasticity of variance model. *Operations Research Letters*, 42(5):337–342, 2014.
- [171] Elias M Stein and Jeremy C Stein. Stock price distributions with stochastic volatility: an analytic approach. *The Review of Financial Studies*, 4(4):727–752, 1991.
- [172] Frederi Viens and Jianfeng Zhang. A martingale approach for fractional Brownian motions and related path dependent PDEs. *Annals of Applied Probability*, to appear, 2018.
- [173] Tianxiao Wang. Linear quadratic control problems of stochastic Volterra integral equations. *ESAIM: Control, Optimisation and Calculus of Variations*, 24:1849–1879, 2018.
- [174] Eugene Wong and Moshe Zakai. On the convergence of ordinary integrals to stochastic integrals. *The Annals of Mathematical Statistics*, 36(5):1560–1564, 1965.
- [175] Jiongmin Yong. Backward stochastic Volterra integral equations and some related problems. *Stochastic Processes and their Applications*, 116(5):779–795, 2006.
- [176] Jiongmin Yong and Xun Yu Zhou. *Stochastic controls: Hamiltonian systems and HJB equations*, volume 43. Springer Verlag, 1999.
- [177] Jia Yue and Nan-jing Huang. Fractional Wishart processes and ε -fractional Wishart processes with applications. *Computers & Mathematics with Applications*, 75(8):2955–2977, 2018.
- [178] Xun Yu Zhou and Duan Li. Continuous-time mean-variance portfolio selection: a stochastic LQ framework. *Applied Mathematics & Optimization*, 42:19–33, 2000.

Title : Volterra Processes in Finance

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Abstract : Empirical studies indicate the presence of memory and strong inter-temporal dependence across various phenomena in the fields of finance and economics. The Brownian motion and Poisson processes, characterized by independent increments, are not suitable for modeling such phenomena.

We will consider Stochastic Volterra processes : a class of processes which extends the standard Brownian motion and Poisson processes to include memory; the fractional Brownian motion and Hawkes

processes constitute a special case.

First, we develop the mathematical tools needed to deal with these stochastic Volterra integral equations that go beyond the standard stochastic calculus theory of Markovian processes and semimartingales. Second, we explore the modeling flexibility of such equations in introducing memory in a broad range of problem in finance and economy including : volatility modeling, portfolio allocation, optimal execution, principal agency, mean-field games . . .