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Stable sheaves on hyper-Kähler manifolds
Faisceaux stables sur variétés hyper-Kähleriennes

Thèse de doctorat de l'université Paris-Saclay et de l'università di
Roma Tor vergata

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Tor-Vergata, sous la direction de **Emanuele MACRÌ**, Professeur, la co-direction de
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Thèse soutenue à Rome, le 6 novembre 2023, par

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Titre: Faisceaux stables sur variétés hyper-Kähleriennes

Mots clés: Géométrie algébrique, Variétés hyper-Kähleriennes, Espaces de modules, Faisceaux cohérent, Catégories dérivées.

Résumé: Dans cette thèse, on étudie les faisceaux sur les variétés hyper-Kähleriennes, avec objectif final de résoudre une conjecture de longue date due à Markman et O'Grady. Plus précisément, on démontre qu'on peut réaliser OG10 en tant qu'espace de modules de fibrés stables sur une variété hyper-Kählerienne de dimension quatre. On commence par étudier les faisceaux stables et, de manière plus générale, les complexes stables sur les surfaces K3. Il est admis, grâce au travail de nombreuses personnes, que les espaces de modules de tels complexes stables sont des variétés projectives hyper-Kähleriennes. Dans le premier chapitre, on propose une nouvelle démonstration conceptuelle de ce fait, en exploitant les puissantes techniques de wall-crossing rendues possibles par la théorie des conditions de stabilité de Bridgeland.

Le reste de la thèse est consacré aux faisceaux sur les variétés hyper-Kähleriennes de dimensions supérieures. Il s'agit d'une théorie relativement récente, qui a débuté avec le travail révolutionnaire de Taelman sur l'algèbre LLV et les travaux

d'O'Grady sur les faisceaux modulaires. Le deuxième chapitre de cette thèse est consacré à une revue des principales propriétés de la cohomologie d'une variété hyper-Kählerienne, en mettant l'accent sur le travail de Taelman.

La théorie a ensuite été développée indépendamment par Beckmann et Markman, qui ont introduit une classe de faisceaux qu'on appelle maintenant atomiques. On passe en revue leurs travaux, ainsi que ceux d'O'Grady, dans le chapitre trois. Dans le reste de la thèse, on considère deux nouveaux exemples de faisceaux atomiques stables sur une variété hyper-Kählerienne de dimension quatre. Ils sont obtenus en appliquant des équivalences dérivées aux surfaces lagrangiennes. Dans les deux cas, leurs espaces de modules sont des variétés symplectiques de dimension dix (éventuellement singulières) et sont birationnels à OG10. Dans l'un de ces cas, on est en mesure de prouver la régularité et donc d'obtenir une variété hyper-Kählerienne de type OG10 en tant qu'espace de modules de faisceaux atomiques stables.

Title: Stable sheaves on hyper-Kähler manifolds

Keywords: Algebraic Geometry, Hyper-Kähler manifolds, Moduli spaces, Coherent sheaves, Derived Categories.

Abstract: In this thesis we study sheaves on hyper-Kähler manifolds, with final goal to solve a long standing conjecture due to Markman and O'Grady. Namely, we show that we can realize OG10 as a moduli space of stable bundles on a hyper-Kähler fourfold. We begin with stable sheaves and, more in general, stable complexes on K3 surfaces. It is a celebrated result, due to the work of many people, that moduli spaces parametrizing such stable complexes are projective hyper-Kähler manifolds. In the first chapter we provide a new conceptual proof of this fact, exploiting the powerful wall-crossing techniques made possible by the theory of Bridgeland stability conditions.

The rest of the thesis is devoted to sheaves on higher dimensional hyper-Kähler manifolds. This is a relatively recent theory, which started with the breakthrough work by Taelman on the LLV alge-

bra, and O'Grady's works on modular sheaves. The second chapter of this thesis is devoted to a review of the main properties of the cohomology of a hyper-Kähler manifold, especially on Taelman's work.

The theory was later developed independently by Beckmann and Markman, who introduced a class of sheaves which we now call atomic. We review their works, together with O'Grady's, in chapter three. In the rest of the thesis we consider two new examples of stable atomic sheaves on a hyper-Kähler fourfold. They are obtained by applying derived equivalences to Lagrangian surfaces. In both cases, their moduli spaces are ten dimensional (possibly singular) symplectic varieties, and are birational to OG10. In one of these cases, we are able to prove smoothness and therefore obtain a hyper-Kähler manifold of type OG10 as a moduli space of stable atomic sheaves.

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Introduction

0.1 Hyper-Kähler manifolds

We say that a compact Kähler manifold X is hyper-Kähler (or simply HK) if it is simply connected and

$$H^0(X, \Omega_X^2) = \mathbb{C}\sigma,$$

where σ is a holomorphic symplectic form on X . It would be more appropriate to call such manifolds irreducible holomorphic symplectic, but in fact, thanks to Yau's Theorem, this condition is equivalent to the existence of a hyper-Kähler metric on X .

Hyper-Kähler manifolds have attracted a lot of interest in recent years. Historically, the main motivation to study them comes from the Beauville-Bogomolov decomposition Theorem, which implies that hyper-Kähler manifolds form building blocks for Kähler manifolds with trivial canonical bundle.

Theorem 0.1.1 ([12, Théorème 1]). *Let Y be a compact Kähler manifold with trivial canonical bundle. There exists a finite étale cover*

$$T \times \prod_i V_i \times \prod X_i \rightarrow Y$$

where T is a complex torus, each V_i is a Calabi-Yau manifold and each X_i is hyper-Kähler.

The presence of a symplectic form forces the dimension of a hyper-Kähler manifold X to be even. In dimension two, the definition agrees with that of a K3 surface, but in higher dimension it is much harder to find examples. The first was provided by Beauville [12] and Fujiki [36], and it is the Hilbert scheme of points $S^{[n]}$ on a K3 surface S . We say that a hyper-Kähler manifold in the same deformation class as $S^{[n]}$ is of type K3^[n]. In [12] is shown that, if A is an abelian surface, the fiber $K_n(A)$ of the albanese morphism

$$A^{[n]} \rightarrow A$$

is a HK as well. The corresponding deformation class is called generalized Kummer and denoted by Kum_n . Besides these two classes of examples, which exist in every even dimension, there are only two sporadic examples, first constructed by O'Grady [91, 92]. They have dimension respectively six and ten, and their deformation classes are called OG6 and OG10.

In contrast with the lack of examples, the general theory is rich and well developed, and it culminates in the Global Torelli Theorem [79, 113]. In close analogy to the case of K3 surfaces, also for high dimensional hyper-Kähler manifolds much of the geometry is encoded by the (integral) second cohomology group. It is equipped with a non-degenerate quadratic form of signature $(3, b_2(X) - 3)$

$$q : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}$$

called Beauville-Bogomolov-Fujiki (or BBF) form. If X is of dimension $2n$, the BBF form satisfies the relation

$$\int_X \alpha^{2n} = (2n - 1)!! c_X q(\alpha)^n, \text{ for every } \alpha \in H^2(X, \mathbb{Z}) \quad (1)$$

where $c_X > 0$ is a rational constant, which depends only on the deformation class of X . In all known examples, the isomorphism class of this lattice has been computed. For example, in the case of a K3 surface S , we have

$$(H^2(S, \mathbb{Z}), q) \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}, \quad (2)$$

where U is the hyperbolic plane, and $E_8(-1)$ is the negative unimodular lattice of rank eight.

The natural Hodge structure on $H^2(X, \mathbb{Z})$ is compatible with the BBF form q , and therefore it is completely determined by the subspace $H^{2,0}(X)$. This gives rise to a period map which is valued in the period domain

$$\{\sigma \in \mathbb{P}(H^2(X, \mathbb{C})) \mid q(\sigma) = 0, q(\sigma, \bar{\sigma}) > 0\}.$$

The Global Torelli Theorem can be stated in terms of this period map, see [50] for more details.

0.2 Moduli spaces of sheaves on K3 surfaces

The most powerful method we have to construct hyper-Kähler manifolds is the theory of moduli spaces of sheaves on K3 surfaces. Let S be a projective K3 surface, and consider its Mukai lattice $\tilde{H}(S, \mathbb{Z})$. This is the lattice

$$\tilde{H}(S, \mathbb{Z}) := H^*(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}).$$

together with its Hodge structure and a non-degenerate pairing called Mukai pairing, see Section 2.2.1 for the precise definition.

Just as the geometry of S is encoded by the second cohomology group $H^2(S, \mathbb{Z})$ together with the intersection form and the Hodge structure, so the properties of the derived category of coherent sheaves $D^b(S)$ are captured by the Mukai lattice.

In order to construct moduli spaces, we have to choose a discrete (topological) invariant. There are several equivalent choices, but for various reasons the Mukai vector is best suited for K3 surfaces. If E is a coherent sheaf on S , its Mukai vector is the algebraic class

$$v(E) := \text{ch}(E) \cup \text{td}(S)^{\frac{1}{2}} \in \tilde{H}_{\text{alg}}(S, \mathbb{Z}).$$

Having fixed a class $\mathbf{v} \in \tilde{H}_{\text{alg}}(S, \mathbb{Z})$, we can consider the set

$$\{E \in \text{Coh}(S) \mid v(E) = \mathbf{v}\} / \cong$$

of isomorphism classes of sheaves fixed Mukai vector \mathbf{v} . This set in general is not countable, and the goal of moduli theory is to put a reasonable geometric structure on it. We would like the resulting space to be (at least) a scheme of finite type. Most of the times this will not be the case, and to solve this issue we can impose a stability condition.

The classical choice, and the best suited to the construction of moduli spaces, is Gieseker stability.

Definition 0.2.1. Let H be a polarization on S . A coherent sheaf E over S is Gieseker H -semistable if it is pure and

$$p_H(F, m) \leq p_H(E, m)$$

for every proper subsheaf $F \subset E$, where $p_H(E, m)$ is the reduced Hilbert polynomial. It is stable if the strict inequality holds.

For every polarization H there is a projective scheme $M_{S,H}(\mathbf{v})$ over \mathbb{C} , whose closed points are

$$\{E \in \text{Coh}(S) \mid E \text{ is } H\text{-semistable, and } v(E) = \mathbf{v}\} / \cong,$$

see [48, Theorem 4.3.7]. If \mathbf{v} is primitive, and H is generic, i.e. it is in the complement of a union of hyperplanes in the ample cone, then $M_{S,H}(\mathbf{v})$ is smooth and parameterizes stable sheaves. The tangent space to a closed point $[E] \in M_{S,H}(\mathbf{v})$ is identified with $\text{Ext}^1(E, E)$.

The first insights in the rich geometry of $M_{S,H}(\mathbf{v})$ are due to Mukai [87]. He noted that the pairing

$$\text{Ext}^1(E, E) \times \text{Ext}^1(E, E) \rightarrow \mathbb{C}, \quad (a, b) \mapsto \text{Tr}(a \circ b)$$

induces a holomorphic two form on $M_{S,H}(\mathbf{v})$, which is non-degenerate by Serre duality. Therefore, the moduli space $M_{S,H}(\mathbf{v})$ is a smooth projective holomorphic symplectic variety.

In the groundbreaking works [22, 23], Bridgeland introduced a notion of stability conditions on $D^b(S)$, allowing for the construction of moduli spaces $M_{S,\sigma}(\mathbf{v})$ of stable complexes up to quasi-isomorphisms. Most of the properties of moduli of sheaves carry over to the derived setting as we recall in Theorem 2.3.2, but moduli spaces of stable complexes are much more flexible than their classical counterparts. In Chapter 2 we take full advantage of this flexibility to reprove the following classical result.

Theorem 0.2.2 (Theorem 2.1.1). *Let S be a K3 surface. Assume that $\mathbf{v} \in H_1^*(S, \mathbb{Z})$ is a primitive vector and let $\sigma \in \text{Stab}^\dagger(S)$ be a \mathbf{v} -generic stability condition on S . Then:*

- (1) *The moduli space $M_{\text{alg}}S, \sigma(\mathbf{v})$ is non-empty if and only if $\mathbf{v}^2 \geq -2$. In this case, it is a smooth projective HK variety of dimension $\mathbf{v}^2 + 2$, deformation-equivalent to a Hilbert scheme of points on a K3 surface.*
- (2) *If $\mathbf{v}^2 \geq 0$, then there exists a natural Hodge isometry*

$$\theta_{\mathbf{v}}^{-1}: H^2(M_{S,\sigma}(\mathbf{v}), \mathbb{Z}) \xrightarrow{\sim} \begin{cases} \mathbf{v}^\perp & \text{if } \mathbf{v}^2 > 0 \\ \mathbf{v}^\perp / \mathbb{Z}\mathbf{v} & \text{if } \mathbf{v}^2 = 0, \end{cases}$$

where the orthogonal is taken in $H^*(S, \mathbb{Z})$.

The analogous result for Gieseker stable sheaf is a celebrated theorem due to the work of many people: Beauville [12], Mukai [88, 89], Kuleshov [63], O'Grady [96], Huybrechts [44], with the final statement appearing in Yoshioka's paper [118]. The idea of the proof is simple: with a sequence of derived equivalences, wall-crossing and deformations, we show that the moduli space $M_\sigma(\mathbf{v})$ is deformation invariant to the Hilbert scheme of points on a (different) K3 surface.

We can ask what happens if we allow the Mukai vector \mathbf{v} to be a multiple

$$\mathbf{v} = m\mathbf{v}_0, \text{ with } m > 1$$

of a primitive vector $\mathbf{v}_0 \in H^*(S, \mathbb{Z})$. Then, the moduli space $M_{S,H}(\mathbf{v})$ will be always singular, due to the presence of strictly semistable sheaves, but the singularities can be understood. If $m = 2$ and $\mathbf{v}_0^2 = 2$, O'Grady [92] showed that there is a symplectic resolution

$$\widetilde{M}_{S,H}(\mathbf{v}) \rightarrow M_{S,H}(\mathbf{v}),$$

which means that the pullback of the symplectic form extends to a global holomorphic two form on $\widetilde{M}_{S,H}(\mathbf{v})$. In this case, the smooth space $\widetilde{M}_{S,H}(\mathbf{v})$ is a ten dimensional HK manifold of type OG10. In all other cases there is no symplectic resolution, by [60, Theorem B]. The other exceptional example OG6 is constructed in a similar way, starting from an abelian surface instead of a K3 surface.

0.3 Sheaves on higher dimensional HK manifolds

It is natural to wonder if Theorem 2.1.1 generalizes to moduli spaces on higher dimensional HK manifolds. Historically the first to observe that this could be a fruitful direction was Kobayashi. In [61] he noted that some moduli spaces of vector bundles on a hyper-Kähler manifold are naturally equipped with a symplectic form on their smooth locus.

Unfortunately, the theory in higher dimensions turns out to be much more difficult than on K3 surfaces, and it is still in its infancy. Here we list a few difficulties:

- On a K3 surface S , Serre duality implies that for every sheaf $F \in \text{Coh}(S)$ we have

$$\text{ext}^0(F, F) = \text{ext}^2(F, F).$$

Since $\chi(E, E)$ is a numerical invariant, this implies that on the stable locus of a moduli space of sheaves the dimension $\text{ext}^1(F, F)$ is constant. In higher dimension, there are also other Ext groups responsible for the Euler characteristic, which make smoothness of the stable locus much harder to control.

- One of the key steps in the proof of Theorem 2.1.1 is the deformation of the base K3 surface to an elliptic one. Indeed on a K3 surface, every simple sheaf deforms sideways provided its first Chern class remains algebraic [88, Theorem 1.17]. This is far from being the case if the dimension is bigger than two: a necessary condition for a sheaf to deform sideways is that the entire Chern character stays algebraic.
- The proof of the irreducibility result for surfaces [60, Theorem 4.1] also crucially uses the dimension two, and it does not generalize to higher dimensions. As of the present moment, there is no argument available to prove irreducibility of moduli spaces of sheaves on high dimensional HK.

Therefore, just considering stable sheaves does not cut it, we need to find a class of sheaves with good enough properties that resemble those of sheaves on K3 surfaces. A first candidate for this class was introduced by Verbitsky in the form of hyperholomorphic sheaves.

Definition 0.3.1. Let X be a HK manifold and $\omega \in H^{1,1}(X, \mathbb{R})$ a Kähler class. A sheaf F on X is called hyperholomorphic if it is μ_ω -stable and $c_1(F)$ and $c_2(F)$ remain of Hodge type along the twistor line $\mathcal{X}_\omega \rightarrow \mathbb{P}_\omega^1$ generated by ω . It is called projectively hyperholomorphic if the endomorphism sheaf $\mathcal{E}nd(F)$ is hyperholomorphic.

In [112] he noted that hyperholomorphicity is a sufficient condition to deform

a stable bundle sideways along the twistor deformation.¹

Theorem 0.3.2 ([112, Theorem 3.19]). *Let $\omega \in H^{1,1}(X, \mathbb{R})$ be a Kähler class, and let E be a (possibly twisted) ω -slope vector bundle. Assume that $\Delta(E)$ remains of type $(2, 2)$ along the twistor line $\mathcal{X}_\omega \rightarrow \mathbb{P}_\omega^1$ spanned by ω . Then there exists a flat deformation \mathcal{E} on \mathcal{X}_ω , such that \mathcal{E}_t is stable for every $t \in \mathbb{P}_\omega^1$.*

For us the most important feature of (projectively) hyperholomorphic bundles is that their moduli spaces is naturally equipped with a symplectic structure, as Verbitsky first noted. The symplectic form can be given pointwise as follows. Let E be a projectively hyperholomorphic vector bundle on X . There is a natural map

$$H^i(X, \mathcal{E}nd(E)) \rightarrow H^{i+2}(X, \mathcal{E}nd(E))$$

induced by cup product with the class $\bar{\sigma} \in H^2(X, \mathcal{O}_X)$.

Theorem 0.3.3 ([108, Theorem 4.2A]). *Let E be a slope stable projectively hyperholomorphic vector bundle on a HK manifold X . For every i the map above*

$$H^i(X, \mathcal{E}nd(E)) \rightarrow H^{i+2}(X, \mathcal{E}nd(E))$$

is an isomorphism.

This generalizes also the work of Kobayashi [61]. In Section 4.8.2 we show that this in fact gives rise to a symplectic form on the smooth locus of a moduli space of stable projectively hyperholomorphic bundles. In Section 4.8 we will prove a result about the smoothness of moduli spaces of hyperholomorphic sheaves, see Theorem 4.8.8.

After [111] hyperholomorphic vector bundles were not really considered much, especially because of the lack of examples. Indeed, apart from the tangent bundle and its relatives, it is extremely difficult to construct examples. The more recent theory of modular and atomic sheaves is helpful in this direction.

0.4 Modular and atomic sheaves

In a quest to generalize to higher dimensions the properties of stable bundles on elliptic K3 surfaces, O’Grady [93] introduced the notion of modular sheaves. A torsion-free sheaf F on a hyper-Kähler manifold X is called modular if there exists a constant $d(F) \in \mathbb{Q}$ such that

$$\int_X \Delta(F) \cup \omega^{2n-2} = d(F)(2n-3)!!q(\omega)^{n-1},$$

¹This was later generalized by Markman [78, Corollary 6.11] to stable reflexive sheaves.

for every $\omega \in H^2(X, \mathbb{Q})$, see Definition 4.3.7 and Remark 4.3.8. This property is satisfied, for example, if $\Delta(F)$ remains of type $(2, 2)$ along all deformations of X .

The most important feature of modularity is its relationship with slope stability. One of the main results in [94], which we recall in Section 4.7 is that, for modular sheaves, slope stability works just as it does for sheaves on surfaces. Namely, for every choice of the Mukai vector, the ample cone has a wall and chamber decomposition with similar properties of that on K3 surfaces. Furthermore, if F is a sheaf on a HK with a Lagrangian fibration

$$\pi : X \rightarrow \mathbb{P}^n,$$

slope stability with respect to polarizations in one of this chamber is intimately related to the stability of the restriction F_t to a general fiber, see Section 4.7.2.

O’Grady [93, 94] constructs rigid stable modular vector bundles on every HK of K3^[n]-type, and shows that they are also unique. That is, the moduli space of stable *locally free* sheaves with certain invariant is a single reduced point, but it might be that there are components which parametrizes sheaves none of which is locally free. Nevertheless, this is the only instance in which a partial irreducibility result is known. We recall this construction and others in Section 4.5.

Atomic objects

Motivated by O’Grady’s results, Markman [77] studied a similar, more categorical notion. The starting point is the observation that, if the discriminant $\Delta(E)$ of a stable bundle E remains of type $(2, 2)$ on every commutative deformation of X , then E is modular. We can ask what happens when we look at non-commutative deformations.

Recall that a first-order deformation of $D^b(X)$ can be seen as an element of the group

$$\mathrm{HT}^2(X) = H^2(X, \mathcal{O}_X) \oplus H^1(X, T_X) \oplus H^0(X, \bigwedge^2 T_X),$$

for details see Section 4.2.1. The cohomological obstruction map for an object $E \in D^b(X)$ is the map

$$\chi_E^H : \mathrm{HT}^2(X) \rightarrow H^*(X, \mathbb{C}), \quad \eta \mapsto \eta \lrcorner v(E)$$

given by contraction with the Mukai vector of E , see Definition 4.2.11. As the name suggests, if

$$\chi_E^H(\eta) = 0,$$

then $v(E)$ remains algebraic along the deformation induced by η . There is almost no object with $\chi_E^H \equiv 0$, so the smallest possible rank for χ_E^H for interesting objects

is one. Objects with cohomological obstruction map of rank one are called atomic and have been studied independently by Beckmann in [15]. It turns out that they enjoy several good properties.

First of all, as could be expected, an atomic torsion-free sheaf is modular as we show in Proposition 4.3.11, but the reverse implication is far from being true. Most importantly though, as we review in Section 4.3, an atomic object $E \in \mathrm{D}^b(X)$ possess an extended Mukai vector

$$\tilde{v}(E) \in \tilde{H}(X, \mathbb{Q}),$$

which lives in the extended Mukai lattice. It was first introduced by Verbitsky [107], as an analogue of the classical Mukai lattice of a K3 surface, and in the breakthrough work by Taelman [103] it was shown to be a derived invariant. This is the rational vector space

$$\tilde{H}(X, \mathbb{Q}) := \mathbb{Q}\alpha \oplus H^2(X, \mathbb{Q}) \oplus \mathbb{Q}\beta,$$

equipped with the quadratic form \tilde{q} , obtained by extending the BBF form on $H^2(X, \mathbb{Q})$ by declaring that α and β are orthogonal to $H^2(X, \mathbb{Q})$, isotropic and $\tilde{q}(\alpha, \beta) = -1$. We can also equip it with a Hodge structure obtained by declaring

$$\tilde{H}(X, \mathbb{Q})^{2,0} = H^{2,0}(X, \mathbb{C})$$

and imposing compatibility with \tilde{q} .

The main feature of the extended Mukai lattice is the fact that is preserved under derived equivalences. Precisely, if

$$\Phi : \mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$$

is a derived equivalence, then there exists an induced Hodge isometry

$$\Phi^{\tilde{H}} : \tilde{H}(X, \mathbb{Q}) \simeq \tilde{H}(Y, \mathbb{Q}).$$

This is compatible, in some sense which is made precise in Theorem 3.6.8, with the usual isometry

$$\Phi^H : H^*(X, \mathbb{Q}) \simeq H^*(Y, \mathbb{Q})$$

induced in cohomology. This turns out to be extremely useful.

Partly because the extended Mukai lattice has small dimension compared to the entire cohomology $H^*(X, \mathbb{Q})$. Hence it is much easier to compute the action of derived equivalences, as we do for instance in Proposition 5.2.9.

But mostly because the extended Mukai vector of an atomic object $E \in \mathrm{D}^b(X)$ is preserved (up to constants) under $\Phi^{\tilde{H}}$:

$$\Phi^{\tilde{H}}(\langle \tilde{v}(E) \rangle) = \langle \tilde{v}(\Phi(E)) \rangle \subset \tilde{H}(X, \mathbb{Q}).$$

In particular the notion of atomicity is invariant under derived equivalences, and this is fundamental to find new examples of modular bundles.

Indeed, one can find an atomic object in $D^b(X)$ for which the condition of being atomic is easily checked (in our case it will be the structure sheaf of a Lagrangian submanifold) and then map it to a vector bundle via a derived equivalence.

0.5 Main results

The main goal of this thesis is to investigate the following question, posed by Markman and O’Grady.

Question: Can we realize a HK manifold of type OG10 as a moduli space of sheaves on a hyper-Kähler manifold of type K3^[2]?

In Chapter 6 we manage to give an affirmative answer to it. The base HK fourfold we consider is the Beauville-Mukai system

$$M := M_S(0, H, -1) \rightarrow |H|,$$

where (S, H) is a general polarized K3 surface of degree two. The reason for this is that there exists a Poincaré sheaf

$$\mathcal{U} \in \text{Coh}(M \times_{|H|} M),$$

which induces an autoequivalence

$$\Phi_{\mathcal{U}} : D^b(M) \simeq D^b(M).$$

As we show in Proposition 5.2.6, this equivalence transforms (the structure sheaves of) Lagrangian surfaces finite over \mathbb{P}^2 into locally-free sheaves. Our first result, which is the main result of [21], is the construction of a stable atomic vector bundle, which is not rigid and whose moduli space is close to OG10.

Theorem 0.5.1 (Theorem 5.7.5 and Proposition 5.8.2). *Let $M = M_S(0, H, -1)$ be the Beauville Mukai system above. There exist a stable, atomic vector bundle F on M with Mukai vector*

$$v(F) = 5 \left(1 - \frac{3}{4} \mathbf{q}_2 + \frac{9}{32} \mathbf{pt} \right).$$

The $\text{Ext}^1(F, F)$ is ten dimensional. The Yoneda pairing is skew-symmetric and induces an isomorphism

$$\bigwedge^2 \text{Ext}^1(F, F) \xrightarrow{\sim} \text{Ext}^2(F, F).$$

In particular, its deformation functor is smooth. Moreover, there is a component \mathfrak{M} of the moduli space of semistable sheaves with Mukai vector $v(F)$ which is birational to OG10.

The bundle F is obtained by applying the equivalence $\Phi_{\mathcal{U}}$ to a reducible Lagrangian surface $Z \subset M$. The surface Z has two components, and for this reason the resulting bundle will not be stable. To make it stable we apply one more autoequivalence, as described in Section 5.7.

In order to prove this we also prove a number of general results on the structure of moduli of atomic bundles. This is done in Section 4.8. The most important of these is Theorem 4.8.8, which is the only way that we have to tackle the singularities of these moduli spaces. It can be reformulated as follows.

Theorem 0.5.2 (Theorem 4.8.8). *Let $M_{X,H}(\mathbf{v})$ be a moduli space of semistable projectively hyperholomorphic sheaves. Assume that it parametrizes only slope stable vector bundles. Then its singular locus is either empty or pure of codimension one.*

The assumption of parametrizing locally free sheaves is important in the proof, although one could imagine that with more work it could be extended to all sheaves. For this reason we are not able to prove smoothness of the whole component \mathfrak{M} . Indeed, the semistable deformations of the bundle F could very well be non locally free, and as of now we have no way of understanding them.

To circumvent this difficulty, and find a moduli space parametrizing only locally free sheaves, we slightly modify the construction. Namely, we consider a cubic fourfold $Y \subset \mathbb{P}^5$, such that its Fano variety of lines is equipped with a Lagrangian fibration

$$F(Y) \rightarrow \mathbb{P}^2.$$

And we also ask for the cubic Y to be general among those, see Section 6.3 for the precise assumptions. In this situation, combining Theorem 6.2.5 with Proposition 6.5.1, we prove that there exists a twisted Poincaré sheaf

$$\mathcal{U} \in \mathrm{Coh}(F(Y) \times_{\mathbb{P}^2} M, p_2^*(\alpha)),$$

where $\alpha \in \mathrm{Br}'(M)$ is a Brauer class. This enjoys similar properties to the untwisted Poincaré sheaf. Namely, the twisted Fourier Mukai transform

$$\Phi_{\mathcal{U}} : \mathrm{D}^b(F(Y)) \simeq \mathrm{D}^b(M)$$

is an equivalence. Moreover it maps Lagrangian surfaces which are finite over the base to vector bundles.

In this case, the surfaces we are interested in are the surfaces of lines of hyperplane sections $F(Y_H) \subset F(Y)$. In Section 6.3 we show that they are indeed finite

over \mathbb{P}^2 . Therefore we produce, via the equivalence $\Phi_{\mathcal{U}}$, a twisted vector bundle on M . Now the surfaces are integral, so this bundle is immediately slope stable, as we show in Section 6.5. Surprisingly, in this case we manage to understand also its semistable degenerations, enough to obtain a positive answer to the question above.

Theorem 0.5.3 (Theorem 6.6.3). *There is a connected component \mathcal{M} of a moduli space of twisted semi stable sheaves on M , which is a smooth projective HK manifold of type OG10.*

The fundamental geometrical step for analyzing the semistable degeneration is understanding the compactified Picard group of the surface $F(Y_H)$. Indeed, through the equivalence $\Phi_{\mathcal{U}}$, degenerating the bundle is equivalent to degenerating the structure sheaf of the surface $F(Y_H) \subset F(Y)$.

Theorem 0.5.4 (Theorem 6.4.1). *For any hyperplane section $Y_H \subset Y$, the moduli space $\overline{\text{Pic}}^0(F(Y_H))$ parametrizes only Cohen-Macaulay sheaves.*

The property of being CM is the technical condition that ensures that the image under $\Phi_{\mathcal{U}}$ is a vector bundle. This theorem is proved in Section 6.4, and relies heavily on the work [66]. Hence the proof of Theorem 6.6.3 is not completely internal to moduli theory, it still requires a geometric input.

To conclude this quick summary we highlight that, although apparently irrelevant, the framework of atomic and modular sheaves was central in our arguments. For example, in order to prove stability of the bundles we consider, we rely heavily on the notion of suitable polarization Definition 4.7.8. This notion was introduced by O’Grady [94] in the context of modular sheaves, and it allows to prove stability by analyzing the stability of the restriction to a general fiber. In turn, to prove that our bundles are modular, the notion of atomicity comes into play. Indeed as anticipated, for Lagrangians submanifold atomicity is an easy condition to check, and is preserved under derived equivalences.

Further work

The construction of Theorem 0.5.3 seems to be a good indication that this could be a fruitful direction to find new examples of HK manifolds. For example, one could try to replicate this strategy to investigate moduli spaces of sheaves on HK manifolds of type OG10. In particular, let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold, and

$$J_Y \rightarrow |\mathcal{O}_Y(1)| \text{ and } J_Y^t \rightarrow |\mathcal{O}_Y(1)|$$

be respectively the intermediate Jacobian fibration (constructed in [66]) and its twisted version (constructed in [115]). Then, it is conjectured that there exists a

(possibly twisted) Poincaré-like sheaf

$$\mathcal{U} \in \text{Coh}(J_Y^t \times_{\mathbb{P}^n} J_Y, p_2^*(\alpha)),$$

where $\alpha \in \text{Br}'(J_Y)$. By Poincaré-like we simply mean that the associated Fourier-Mukai transform is an equivalence, and that, for every $t \in \mathbb{P}^n$, the restriction

$$\mathcal{U}|_t \in \text{Coh}(X_t \times Y_t, \alpha_t)$$

is a CM sheaf. The goal now would be to find a Lagrangian subvariety

$$Z \subset J_Y^t$$

with the following properties:

- (1) Its structure sheaf $\mathcal{O}_Z \in \text{D}^b(J_Y^t)$ is an atomic object. That is, it satisfies the conditions of Theorem 4.6.2.
- (2) There is a component of the Hilbert scheme of the embedded deformations of $Z \subset J_Y^t$ which is a projective space P . Moreover, we ask that every deformation $Z' \in P$ is integral.
- (3) Lastly, we want that for every $Z' \in P$, the closure of the set of line bundles

$$\overline{\text{Pic}^0(Z')} \subset \text{Pic}^0(Z')^=$$

parametrizes only Cohen-Macaulay sheaves.

If all of these conditions were satisfied, then the arguments of this thesis would produce a moduli space of twisted bundles on J_Y which is a smooth projective symplectic manifold. If it were also HK, it would most likely be a new example, because we would expect the second Betti number to be 25.

Of course, this is very far away from current research. Already finding a Lagrangian surface which satisfies (1) would be extremely interesting. We could produce possibly twisted vector bundles on J_Y , and perhaps study the degenerations by different means.

In a different direction, we could try to study objects supported on Lagrangians which have high rank. There is an explicit condition for when such an object is atomic, see Theorem 4.6.6. Studying such objects on the known Lagrangian surfaces could produce new examples of singular HK manifolds.

0.6 Structure of the thesis

In Chapter 2 we give a new proof of Theorem 2.1.1. The proof is a reformulation in modern language of the classical proof by O'Grady and Yoshioka. This is the content of the paper [19].

In Chapter 3 we review some results on the cohomology of hyper-Kähler manifolds. In particular the LLV algebra and its interactions with the derived category, which first appeared in the work of Taelman [103]. With the exception of Section 3.7, everything in this chapter is not original work. The first few sections are the content of the paper [20]

In Chapter 4 we review the theory of atomic and modular sheaves following [94, 93, 77, 17, 15]. With the exception of Section 4.3 where we prove some numerical properties of atomic sheaves, and Section 4.8 where we prove Theorem 0.5.2, everything else is not original work.

In Chapter 5 we construct our first example of atomic bundle, and we prove Theorem 0.5.1. This chapter also contains the main technical result we use to prove local freeness, which is Proposition 5.2.6. This is the content of the preprint [21].

In Chapter 6 deal with the twisted case and we prove Theorem 0.5.3, which gives a positive answer to the question above.

0.7 Notation and conventions

Unless otherwise specified, all functors are derived. We use O’Grady’s normalization for the Fujiki constant c_X , which means that it satisfies (1). With the exception of Chapter 2 where X is used to denote a K3 surface, we will always denote a K3 surface by the letter S , while X will be reserved for higher dimensional HK manifolds. A moduli space of stable sheaves on a projective variety (X, h) with fixed invariants \mathbf{v} will be usually denoted by $M_{X,h}(\mathbf{v})$. When it will not cause confusion we will drop the X from the notation.

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Chapter 1

Résumé en Français

1.1 Variétés hyper-Kähleriennes

On dit qu'une variété Kähler compacte X est hyper-Kählerienne (ou simplement HK) si elle est simplement connexe et

$$H^0(X, \Omega_X^2) = \mathbb{C}\sigma,$$

où σ est une forme symplectique holomorphe sur X . Il serait plus approprié d'appeler de telles variétés symplectiques holomorphes irréductibles, mais en fait, grâce au Théorème de Yau, cette condition est équivalente à l'existence d'une métrique hyper-Kähler sur X .

Les variétés hyper-Kähleriennes ont suscité beaucoup d'intérêt ces dernières années. Historiquement, la principale motivation pour les étudier provient du Théorème de décomposition Beauville-Bogomolov, qui implique que les variétés hyper-Kähler forment des éléments constitutifs pour les variétés Kähleriennes avec fibré canonique trivial.

Theorem 1.1.1 ([12, Théorème 1]). *Soit Y une variété Kähler compacte avec fibré canonique trivial. Il existe un revêtement fini étale*

$$T \times \prod_i V_i \times \prod X_i \rightarrow Y$$

où T est un tore complexe, chaque V_i est une variété de Calabi-Yau et chaque X_i est hyper-Kählerienne.

La présence d'une forme symplectique impose que la dimension d'une variété hyper-Kähler X soit paire. En dimension deux, la définition correspond à celle d'une surface K3, mais en dimension supérieure, il est beaucoup plus difficile de trouver des exemples. Le premier a été fourni par Beauville [12] et Fujiki [36], et

c'est le schéma de Hilbert des points $S^{[n]}$ sur une surface K3 S . On dit qu'une variété hyper-Kähler dans la même classe de déformation que $S^{[n]}$ est de type K3^[n]. Dans [12], il est montré que, si A est une surface abélienne, la fibre $K_n(A)$ du morphisme d'Albanese

$$A^{[n]} \rightarrow A$$

est aussi hyper-Kähler. La classe de déformation correspondante est appelée Kummer généralisé et notée Kum_n . Outre ces deux classes d'exemples, qui existent dans chaque dimension paire, il n'y a que deux exemples sporadiques, d'abord construits par O'Grady [91, 92]. Ils ont respectivement une dimension six et dix, et leurs classes de déformation sont appelées OG6 et OG10.

Contrairement au manque d'exemples, la théorie générale est riche et bien développée, et elle aboutit au Théorème de Torelli Global [79, 113]. En analogie étroite avec le cas des surfaces K3, également pour les variétés hyper-Kähler de grande dimension, une grande partie de la géométrie est encodée par le groupe de cohomologie de degré deux (entier). Il est équipé d'une forme quadratique non dégénérée de signature $(3, b_2(X) - 3)$

$$q : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}$$

appelée forme de Beauville-Bogomolov-Fujiki (ou BBF). Si X est de dimension $2n$, la forme BBF satisfait la relation

$$\int_X \alpha^{2n} = (2n - 1)!! c_X q(\alpha)^n, \text{ pour tout } \alpha \in H^2(X, \mathbb{Z}) \quad (1.1)$$

où $c_X > 0$ est une constante rationnelle, qui dépend uniquement de la classe de déformation de X . Dans tous les exemples connus, la classe d'isomorphisme de ce réseau a été calculée. Par exemple, dans le cas d'une surface K3 S , on a

$$(H^2(S, \mathbb{Z}), q) \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}, \quad (1.2)$$

où U est le plan hyperbolique, et $E_8(-1)$ est le réseau unimodulaire négatif de rang huit.

La structure de Hodge naturelle sur $H^2(X, \mathbb{Z})$ est compatible avec la forme BBF q , et donc elle est complètement déterminée par le sous-espace $H^{2,0}(X)$. Cela donne lieu à une application de période qui prend ses valeurs dans le domaine des périodes

$$\{\sigma \in \mathbb{P}(H^2(X, \mathbb{C})) \mid q(\sigma) = 0, q(\sigma, \bar{\sigma}) > 0\}.$$

Le Théorème de Torelli Global peut être formulé en termes de cette application de période, voir [50] pour plus de détails.

1.2 Espaces de modules de faisceaux stables sur les surfaces K3

La méthode la plus puissante que nous ayons pour construire des variétés hyper-Kähler est la théorie des espaces de modules des faisceaux sur les surfaces K3. Soit S une surface projective K3, et considérons son réseau de Mukai $\tilde{H}(S, \mathbb{Z})$. Il s'agit du réseau

$$\tilde{H}(S, \mathbb{Z}) := H^*(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}).$$

accompagné de sa structure de Hodge et d'un appariement non dégénéré appelé appariement de Mukai.

Tout comme la géométrie de S est encodée par le groupe de cohomologie $H^2(S, \mathbb{Z})$ avec la forme d'intersection et la structure de Hodge, les propriétés de la catégorie dérivée des faisceaux cohérents $D^b(S)$ sont capturées par le réseau de Mukai.

Pour construire des espaces de modules, nous devons choisir un invariant discret (topologique). Il existe plusieurs choix équivalents, mais pour diverses raisons, le vecteur de Mukai est le mieux adapté aux surfaces K3. Si E est un faisceau cohérent sur S , son vecteur de Mukai est la classe algébrique

$$v(E) := \text{ch}(E) \cup \text{td}(S)^{\frac{1}{2}} \in \tilde{H}_{\text{alg}}(S, \mathbb{Z}).$$

Ayant fixé une classe $\mathbf{v} \in \tilde{H}_{\text{alg}}(S, \mathbb{Z})$, nous pouvons considérer l'ensemble

$$\{E \in \text{Coh}(S) \mid v(E) = \mathbf{v}\} / \cong$$

des classes d'isomorphisme de faisceaux ayant le vecteur de Mukai fixé \mathbf{v} . Cet ensemble n'est en général pas dénombrable, et le but de la théorie des modules est de lui donner une structure géométrique raisonnable. Nous aimerions que l'espace résultant soit (au moins) un schéma de type fini. La plupart du temps, cela ne sera pas le cas, et pour résoudre ce problème, nous pouvons imposer une condition de stabilité.

Le choix classique, et le mieux adapté à la construction d'espaces de modules, est la stabilité de Gieseker.

Definition 1.2.1. Soit H une polarisation sur S . Un faisceau cohérent E sur S est H -semi-stable de Gieseker s'il est pur et

$$p_H(F, m) \leq p_H(E, m)$$

pour tout sous-faisceau propre $F \subset E$, où $p_H(E, m)$ est le polynôme de Hilbert réduit. Il est stable si l'inégalité stricte est vérifiée.

Pour chaque polarisation H , il existe un schéma projectif $M_{S,H}(\mathbf{v})$ sur \mathbb{C} , dont les points fermés sont

$$\{E \in \text{Coh}(S) \mid E \text{ est } H\text{-semi-stable, et } v(E) = \mathbf{v}\} / \cong,$$

voir [48, Théorème 4.3.7]. Si \mathbf{v} est primitif, et H est générique, c'est-à-dire qu'il est dans le complément d'une union d'hyperplans dans le cône ample, alors $M_{S,H}(\mathbf{v})$ est lisse et paramétrise des faisceaux stables. L'espace tangent en un point fermé $[E] \in M_{S,H}(\mathbf{v})$ est identifié avec $\text{Ext}^1(E, E)$.

Les premières observations dans la riche géométrie de $M_{S,H}(\mathbf{v})$ sont dues à Mukai [87]. Il a remarqué que l'appariement

$$\text{Ext}^1(E, E) \times \text{Ext}^1(E, E) \rightarrow \mathbb{C}, \quad (a, b) \mapsto \text{Tr}(a \circ b)$$

induit une deux-forme holomorphe sur $M_{S,H}(\mathbf{v})$, qui est non dégénérée par dualité de Serre. Par conséquent, l'espace de modules $M_{S,H}(\mathbf{v})$ est une variété symplectique holomorphe projective lisse.

Dans les travaux révolutionnaires [22, 23], Bridgeland a introduit une notion de conditions de stabilité sur $D^b(S)$, permettant la construction d'espaces de modules $M_{S,\sigma}(\mathbf{v})$ de complexes stables jusqu'à quasi-isomorphisme. La plupart des propriétés des espaces de modules de faisceaux se transfèrent au cadre dérivé, mais les espaces de modules de complexes stables sont beaucoup plus flexibles que leurs homologues classiques. Nous tirons pleinement parti de cette flexibilité pour démontrer à nouveau le résultat classique suivant.

Theorem 1.2.2. *Soit S une surface K3. Supposons que $\mathbf{v} \in H^*(S, \mathbb{Z})$ soit un vecteur primitif et que $\sigma \in \text{Stab}^\dagger(S)$ soit une condition de stabilité \mathbf{v} -générique sur S . Alors:*

- (1) *L'espace de modules $M_{\text{alg}}S, \sigma(\mathbf{v})$ est non vide si et seulement si $\mathbf{v}^2 \geq -2$. Dans ce cas, il s'agit d'une variété hyper-Kähler (hyperkählérienne) lisse projective de dimension $\mathbf{v}^2 + 2$, déformation-équivalente à un schéma de Hilbert de points sur une surface K3.*
- (2) *Si $\mathbf{v}^2 \geq 0$, alors il existe un isomorphisme de Hodge naturel*

$$\theta_{\mathbf{v}}^{-1}: H^2(M_{S,\sigma}(\mathbf{v}), \mathbb{Z}) \xrightarrow{\sim} \begin{cases} \mathbf{v}^\perp & \text{si } \mathbf{v}^2 > 0 \\ \mathbf{v}^\perp / \mathbb{Z}\mathbf{v} & \text{si } \mathbf{v}^2 = 0, \end{cases}$$

où l'orthogonal est pris dans $H^*(S, \mathbb{Z})$.

Le résultat analogue pour les faisceaux stables de Gieseker est un théorème célèbre dû au travail de plusieurs personnes : Beauville [12], Mukai [88, 89], Kuleshov [63], O'Grady [96], Huybrechts [44], avec l'énoncé final apparaissant

dans l'article de Yoshioka [118]. L'idée de la démonstration est simple : avec une séquence d'équivalences dérivées, de sauts de murs (*wall-crossing*) et de déformations, nous montrons que l'espace de modules $M_\sigma(\mathbf{v})$ est invariant par déformation vers le schéma de Hilbert de points sur une surface K3 (différente).

Nous pouvons nous demander ce qui se passe si nous autorisons le vecteur de Mukai \mathbf{v} à être un multiple

$$\mathbf{v} = m\mathbf{v}_0, \text{ avec } m > 1$$

d'un vecteur primitif $\mathbf{v}_0 \in H^*(S, \mathbb{Z})$. Alors, l'espace de modules $M_{S,H}(\mathbf{v})$ sera toujours singulier, en raison de la présence de faisceaux strictement semi-stables, mais les singularités peuvent être comprises. Si $m = 2$ et $\mathbf{v}_0^2 = 2$, O'Grady [92] a montré qu'il existe une résolution symplectique

$$\widetilde{M}_{S,H}(\mathbf{v}) \rightarrow M_{S,H}(\mathbf{v}),$$

ce qui signifie que le relevé de la forme symplectique s'étend à une deux-forme holomorphe globale sur $\widetilde{M}_{S,H}(\mathbf{v})$. Dans ce cas, l'espace lisse $\widetilde{M}_{S,H}(\mathbf{v})$ est une variété hyper-Kähler de dimension dix de type OG10. Dans tous les autres cas, il n'y a pas de résolution symplectique, par [60, Théorème B]. L'autre exemple exceptionnel OG6 est construit de manière similaire, en partant d'une surface abélienne au lieu d'une surface K3.

1.3 Faisceaux sur des variétés HK de dimensions supérieures

Il est naturel de se demander si le Theorem 1.2.2 généralise aux espaces de modules sur des variétés HK de dimensions supérieures. Historiquement, le premier à observer que cela pourrait être une direction fructueuse fut Kobayashi. Dans [61], il nota que certains espaces de modules de fibrés vectoriels sur une variété hyper-Kählérienne sont naturellement équipés d'une forme symplectique sur leur lieu lisse.

Malheureusement, la théorie en dimensions supérieures se révèle bien plus difficile qu'en surfaces K3, et elle en est encore à ses débuts. Voici quelques difficultés:

- Sur une surface K3 S , la dualité de Serre implique que pour chaque faisceau $F \in \text{Coh}(S)$, nous avons

$$\text{ext}^0(F, F) = \text{ext}^2(F, F).$$

Puisque $\chi(E, E)$ est un invariant numérique, cela implique que sur le lieu stable d'un espace de modules de faisceaux, la dimension $\text{ext}^1(F, F)$ est constante. En dimension supérieure, il existe également d'autres groupes Ext

responsables de la caractéristique d'Euler, ce qui rend le contrôle de la régularité du lieu stable beaucoup plus difficile.

- Une des étapes clés dans la démonstration du Theorem 1.2.2 est la déformation de la surface K3 vers une surface elliptique. En effet, sur une surface K3, chaque faisceau simple se déforme latéralement à condition que sa première classe de Chern reste algébrique [88, Théorème 1.17]. Ce n'est de loin pas le cas si la dimension est supérieure à deux : une condition nécessaire pour qu'un faisceau se déforme latéralement est que l'ensemble du caractère de Chern reste algébrique.
- La démonstration du résultat d'irréductibilité pour les surfaces [60, Théorème 4.1] utilise également de manière cruciale la dimension deux, et elle ne se généralise pas aux dimensions supérieures. À l'heure actuelle, aucun argument n'est disponible pour prouver l'irréductibilité des espaces de modules de faisceaux sur des variétés HK de dimensions élevées.

Par conséquent, se limiter à considérer uniquement les faisceaux stables ne suffit pas, nous devons trouver une classe de faisceaux ayant des propriétés assez bonnes pour ressembler à celles des faisceaux sur les surfaces K3. Un premier candidat pour cette classe a été introduit par Verbitsky sous la forme de faisceaux hyperholomorphes.

Definition 1.3.1. Soit X une variété HK et $\omega \in H^{1,1}(X, \mathbb{R})$ une classe Kählérienne. Un faisceau F sur X est appelé hyperholomorphe si son discriminant

$$\Delta(F) := -2 \operatorname{rk}(E) \operatorname{ch}_2(E) + c_1(E)^2 \in H^4(X, \mathbb{Q})$$

reste de type $(2, 2)$ le long de la ligne du twistor $\mathcal{X}_\omega \rightarrow \mathbb{P}_\omega^1$ générée par ω . Il est appelé projectivement hyperholomorphe si le faisceau d'endomorphismes $\mathcal{E}nd(F)$ est hyperholomorphe.

Dans [112], il nota que l'hyperholomorphie est une condition suffisante pour déformer latéralement un fibré stable le long de la déformation du twistor.¹

Theorem 1.3.2 ([112, Théorème 3.19]). *Soit $\omega \in H^{1,1}(X, \mathbb{R})$ une classe Kählérienne, et soit E un fibré vectoriel ω -pente (éventuellement tordu). Supposons que $\Delta(E)$ reste de type $(2, 2)$ le long de la ligne du twistor $\mathcal{X}_\omega \rightarrow \mathbb{P}_\omega^1$ engendrée par ω . Alors il existe une déformation plate \mathcal{E} sur \mathcal{X}_ω , telle que \mathcal{E}_t soit stable pour tout $t \in \mathbb{P}_\omega^1$.*

Pour nous, la caractéristique la plus importante des faisceaux (projectivement) hyperholomorphes est que leurs espaces de modules sont naturellement équipés

¹Cela a été ensuite généralisé par Markman [78, Corollaire 6.11] aux faisceaux réflexifs stables.

d'une structure symplectique, comme l'a d'abord noté Verbitsky. La forme symplectique peut être donnée ponctuellement comme suit. Soit E un fibré vectoriel projectivement hyperholomorphe sur X . Il existe une application naturelle

$$H^i(X, \mathcal{E}nd(E)) \rightarrow H^{i+2}(X, \mathcal{E}nd(E))$$

induite par le produit cup avec la classe $\bar{\sigma} \in H^2(X, \mathcal{O}_X)$.

Theorem 1.3.3 ([108, Théorème 4.2A]). *Soit E un fibré vectoriel projectivement hyperholomorphe stable en pente sur une variété HK X . Pour chaque i , l'application ci-dessus*

$$H^i(X, \mathcal{E}nd(E)) \rightarrow H^{i+2}(X, \mathcal{E}nd(E))$$

est un isomorphisme.

Cela généralise également le travail de Kobayashi [61]. Nous montrons que cela donne en fait une forme symplectique sur le lieu lisse d'un espace de modules de fibrés projectivement hyperholomorphes stables. Nous aussi prouverons un résultat sur la régularité des espaces de modules de faisceaux hyperholomorphes.

Après [111], les fibrés vectoriels hyperholomorphes n'ont pas été vraiment considérés, notamment en raison du manque d'exemples. En effet, à part le fibré tangent et ses apparentés, il est extrêmement difficile de construire des exemples. La théorie plus récente des faisceaux modulaires et atomiques est utile dans cette direction.

1.4 Faisceaux modulaires et atomiques

Dans une quête pour généraliser aux dimensions supérieures les propriétés des fibrés stables sur les surfaces elliptiques K3, O'Grady [93] a introduit la notion de faisceaux modulaires. Un faisceau libre de torsion F sur une variété HK X est appelé modulaire s'il existe une constante $d(F) \in \mathbb{Q}$ telle que

$$\int_X \Delta(F) \cup \omega^{2n-2} = d(F)(2n-3)!!q(\omega)^{n-1},$$

pour tout $\omega \in H^2(X, \mathbb{Q})$. Cette propriété est satisfaite, par exemple, si $\Delta(F)$ reste de type $(2, 2)$ le long de toutes les déformations de X .

La caractéristique la plus importante de la modularité est sa relation avec la stabilité en pente. L'un des principaux résultats de [94], est que, pour les faisceaux modulaires, la stabilité en pente fonctionne de la même manière que pour les faisceaux sur les surfaces. En d'autres termes, pour chaque choix du vecteur de

Mukai, le cône ample a une décomposition en murs et chambres avec des propriétés similaires à celles sur les surfaces K3. De plus, si F est un faisceau sur une variété HK avec une fibrations lagrangienne

$$\pi : X \rightarrow \mathbb{P}^n,$$

la stabilité en pente par rapport à des polarisations dans l'une de ces chambres est intimement liée à la stabilité de la restriction F_t à une fibre générale.

O'Grady [93, 94] construit des fibrés vectoriels modulaires stables rigides sur chaque variété HK de type K3^[n], et montre qu'ils sont également uniques. Autrement dit, l'espace de modules des faisceaux stables *localement libres* avec certain invariant est un point réduit unique, mais il se pourrait qu'il existe des composantes qui paramétrisent des faisceaux dont aucun n'est localement libre. Néanmoins, c'est le seul cas où un résultat partiel d'irréductibilité est connu. Nous rappelons cette construction et d'autres dans.

Objets atomiques

Motivé par les résultats d'O'Grady, Markman [77] a étudié une notion similaire, mais plus catégorique. Le point de départ est l'observation que si le discriminant $\Delta(E)$ d'un faisceau stable E reste de type (2, 2) sur chaque déformation commutative de X , alors E est modulaire. On peut se demander ce qui se passe lorsque l'on regarde les déformations non commutatives.

Rappelons qu'une déformation de premier ordre de $D^b(X)$ peut être vue comme un élément du groupe

$$\mathrm{HT}^2(X) = H^2(X, \mathcal{O}_X) \oplus H^1(X, T_X) \oplus H^0(X, \bigwedge^2 T_X),$$

pour plus de détails voir. La carte d'obstruction cohomologique pour un objet $E \in D^b(X)$ est l'application

$$\chi_E^H : \mathrm{HT}^2(X) \rightarrow H^*(X, \mathbb{C}), \quad \eta \mapsto \eta \lrcorner v(E)$$

donnée par contraction avec le vecteur de Mukai de E . Comme son nom l'indique, si

$$\chi_E^H(\eta) = 0,$$

alors $v(E)$ reste algébrique le long de la déformation induite par η . Il n'y a presque aucun objet avec $\chi_E^H \equiv 0$, donc le rang le plus petit possible pour χ_E^H pour des objets intéressants est un. Les objets avec une carte d'obstruction cohomologique de rang un sont appelés atomiques et ont été étudiés indépendamment par Beckmann dans [15]. Il s'avère qu'ils possèdent plusieurs bonnes propriétés.

Tout d'abord, comme on pouvait s'y attendre, un faisceau libre de torsion atomique est modulaire, mais l'implication inverse est loin d'être vraie. Surtout, un objet atomique $E \in D^b(X)$ possède un vecteur de Mukai étendu

$$\tilde{v}(E) \in \tilde{H}(X, \mathbb{Q}),$$

qui vit dans le réseau de Mukai étendu. Il a été introduit pour la première fois dans le travail révolutionnaire de Taelman [103] en tant qu'analogue du réseau de Mukai classique d'une surface K3. Il s'agit de l'espace vectoriel rationnel

$$\tilde{H}(X, \mathbb{Q}) := \mathbb{Q}\alpha \oplus H^2(X, \mathbb{Q}) \oplus \mathbb{Q}\beta,$$

équipé de la forme quadratique \tilde{q} , obtenue en étendant la forme BBF sur $H^2(X, \mathbb{Q})$ en déclarant que α et β sont orthogonaux à $H^2(X, \mathbb{Q})$, isotropes et $\tilde{q}(\alpha, \beta) = -1$. On peut également l'équiper d'une structure de Hodge obtenue en déclarant

$$\tilde{H}(X, \mathbb{Q})^{2,0} = H^{2,0}(X, \mathbb{C})$$

et en imposant la compatibilité avec \tilde{q} .

La caractéristique principale du réseau de Mukai étendu est le fait qu'il est préservé sous les équivalences dérivées. Précisément, si

$$\Phi : D^b(X) \simeq D^b(Y)$$

est une équivalence dérivée, alors il existe une isométrie de Hodge induite

$$\Phi^{\tilde{H}} : \tilde{H}(X, \mathbb{Q}) \simeq \tilde{H}(Y, \mathbb{Q}).$$

Cela est compatible, d'une certaine manière, avec l'isométrie habituelle

$$\Phi^H : H^*(X, \mathbb{Q}) \simeq H^*(Y, \mathbb{Q})$$

induite en cohomologie. Cela s'avère extrêmement utile.

En partie parce que le réseau de Mukai étendu a une petite dimension par rapport à la cohomologie entière $H^*(X, \mathbb{Q})$. Il est donc beaucoup plus facile de calculer l'action des équivalences dérivées.

Mais surtout parce que le vecteur de Mukai étendu d'un objet atomique $E \in D^b(X)$ est préservé (à des constantes près) sous $\Phi^{\tilde{H}}$:

$$\Phi^{\tilde{H}}(\langle \tilde{v}(E) \rangle) = \langle \tilde{v}(\Phi(E)) \rangle \subset \tilde{H}(X, \mathbb{Q}).$$

En particulier, la notion d'atomicité est invariante sous les équivalences dérivées, et cela est fondamental pour trouver de nouveaux exemples de faisceaux modulaires.

En effet, on peut trouver un objet atomique dans $D^b(X)$ pour lequel la condition d'atomicité est facilement vérifiée (dans notre cas, ce sera le faisceau structural d'une sous-variété lagrangienne) et ensuite le mapper vers un fibré vectoriel via une équivalence dérivée.

1.5 Résultats principaux

L'objectif principal de cette thèse est d'explorer la question suivante, posée par Markman et O'Grady.

Question : Peut-on réaliser une variété hyper-Kähler (HK) de type OG10 en tant que lieu des modules de faisceaux sur une variété HK de type $K3^{[2]}$?

Dans cette thèse nous parvenons à donner une réponse affirmative à cette question. La variété HK quadratique que nous considérons est le système de Beauville-Mukai

$$M := M_S(0, H, -1) \rightarrow |H|,$$

où (S, H) est une surface K3 polarisée de degré deux générale. La raison en est qu'il existe un faisceau de Poincaré

$$\mathcal{U} \in \text{Coh}(M \times_{|H|} M),$$

qui induit une autoéquivalence

$$\Phi_{\mathcal{U}} : D^b(M) \simeq D^b(M).$$

Cette équivalence transforme (les faisceaux structurels des) surfaces lagrangiennes finies sur \mathbb{P}^2 en faisceaux localement libres. Notre premier résultat, qui est le principal résultat de [21], est la construction d'un fibré vectoriel atomique stable, qui n'est pas rigide et dont l'espace des modules est proche de OG10.

Theorem 1.5.1. *Soit $M = M_S(0, H, -1)$ le système de Beauville-Mukai mentionné ci-dessus. Il existe un fibré vectoriel atomique stable F sur M avec un vecteur de Mukai*

$$v(F) = 5 \left(1 - \frac{3}{4} \mathbf{q}_2 + \frac{9}{32} \mathbf{pt} \right).$$

L'espace $\text{Ext}^1(F, F)$ est de dimension dix. Le couplage de Yoneda est antisymétrique et induit un isomorphisme

$$\bigwedge^2 \text{Ext}^1(F, F) \xrightarrow{\sim} \text{Ext}^2(F, F).$$

En particulier, son foncteur de déformations est lisse. De plus, il existe une composante \mathfrak{M} de l'espace des modules de faisceaux semi-stables avec vecteur de Mukai $v(F)$ qui est birationnelle à OG10.

Le faisceau F est obtenu en appliquant l'équivalence $\Phi_{\mathcal{U}}$ à une surface lagrangienne réductible $Z \subset M$. La surface Z a deux composantes, et c'est pourquoi le fibré résultant ne sera pas stable. Pour le rendre stable, nous appliquons une autre autoéquivalence.

Pour prouver cela, nous démontrons également plusieurs résultats généraux sur la structure des espaces des modules de faisceaux atomiques. Le plus important d'entre eux est Theorem 1.5.2, qui est la seule façon que nous avons de traiter les singularités de ces espaces des modules. Il peut être reformulé comme suit.

Theorem 1.5.2. *Soit $M_{X,H}(\mathbf{v})$ un espace des modules de faisceaux projectivement hyperholomorphes semi-stables. Supposons qu'il paramètre uniquement des fibrés vectoriels stables. Alors son lieu singulier est soit vide, soit de codimension un.*

L'hypothèse de paramétrisation de faisceaux localement libres est importante dans la démonstration, bien que l'on puisse imaginer qu'avec plus de travail, elle pourrait être étendue à tous les faisceaux. Pour cette raison, nous ne sommes pas en mesure de prouver la régularité de toute la composante \mathfrak{M} . En effet, les déformations semi-stables du fibré F pourraient très bien être non localement libres, et à l'heure actuelle, nous ne savons pas comment les comprendre.

Pour contourner cette difficulté, et trouver un espace des modules paramétrant uniquement des faisceaux localement libres, nous modifions légèrement la construction. Nous considérons une quadrique $Y \subset \mathbb{P}^5$, telle que sa variété de Fano des droites est équipée d'une fibration lagrangienne

$$F(Y) \rightarrow \mathbb{P}^2,$$

et nous demandons également que la quadrique Y soit générale parmi celles-ci. Dans cette situation, nous prouvons qu'il existe un faisceau de Poincaré tordu

$$\mathcal{U} \in \text{Coh}(F(Y) \times_{\mathbb{P}^2} M, p_2^*(\alpha)),$$

où $\alpha \in \text{Br}'(M)$ est une classe de Brauer. Cela a des propriétés similaires au faisceau de Poincaré non tordu. En particulier, la transformée de Fourier tordue de Mukai

$$\Phi_{\mathcal{U}} : D^b(F(Y)) \simeq D^b(M)$$

est une équivalence. De plus, il mappe les surfaces lagrangiennes qui sont finies sur la base en fibrés vectoriels.

Dans ce cas, les surfaces qui nous intéressent sont les surfaces des droites des sections hyperplanes $F(Y_H) \subset F(Y)$. Nous aussi montrons qu'elles sont effectivement finies sur \mathbb{P}^2 . Par conséquent, nous produisons, via l'équivalence $\Phi_{\mathcal{U}}$, un fibré vectoriel tordu sur M . Maintenant que les surfaces sont intégrales, ce fibré est immédiatement stable. Étonnamment, dans ce cas, nous parvenons également à comprendre ses dégénérescences semi-stables, suffisamment pour obtenir une réponse positive à la question ci-dessus.

Theorem 1.5.3. *Il existe une composante connectée \mathcal{M} d'un espace des modules de faisceaux tordus semi-stables sur M , qui est une variété HK projective lisse de type OG10.*

L'étape géométrique fondamentale pour analyser la dégénérescence semi-stable est de comprendre le groupe de Picard compactifié de la surface $F(Y_H)$. En effet, grâce à l'équivalence $\Phi_{\mathcal{U}}$, dégénérer le fibré est équivalent à dégénérer le faisceau structural de la surface $F(Y_H) \subset F(Y)$.

Theorem 1.5.4. *Pour toute section hyperplane $Y_H \subset Y$, l'espace des modules $\overline{\text{Pic}}^0(F(Y_H))$ paramètre uniquement des faisceaux de Cohen-Macaulay.*

La propriété d'être CM est la condition technique qui garantit que l'image sous $\Phi_{\mathcal{U}}$ est un fibré vectoriel. Ce théorème repose fortement sur le travail [66]. Ainsi, la preuve de Theorem 1.5.3 n'est pas entièrement interne à la théorie des modules, elle nécessite toujours une contribution géométrique.

Pour conclure ce résumé rapide, soulignons que, bien que cela puisse sembler sans importance, le cadre des faisceaux atomiques et modulaires était central dans nos arguments. Par exemple, pour prouver la stabilité des faisceaux que nous considérons, nous nous appuyons fortement sur la notion de polarisation appropriée. Cette notion a été introduite par O'Grady [94] dans le contexte des faisceaux modulaires, et elle permet de prouver la stabilité en analysant la stabilité de la restriction à une fibre générale. À son tour, pour prouver que nos faisceaux sont modulaires, la notion d'atomicité entre en jeu. En effet, comme anticipé, pour les sous-variétés lagrangiennes, l'atomicité est une condition facile à vérifier et elle est préservée par les équivalences dérivées.

Chapter 2

Moduli spaces of sheaves on K3 Surfaces

2.1 Introduction

Moduli spaces of semistable sheaves on a complex projective K3 surface X are a well studied class of algebraic varieties, and they are among the only known examples of compact hyperkähler (or irreducible holomorphic symplectic) varieties. Classically, we consider the moduli space $M_H(\mathbf{v})$ of Gieseker-stable coherent sheaves with fixed topological invariants, encoded in the Mukai vector $\mathbf{v} \in H_{\text{alg}}^*(X, \mathbb{Z})$. Recall that, given a polarization H , a coherent sheaf E is Gieseker semistable if it is pure and

$$p(F, m) \leq p(E, m)$$

for every proper subsheaf $F \subset E$, where $p(E, m)$ is the reduced Hilbert polynomial. It is stable if the strict inequality holds. The stability condition gives a GIT construction of $M_H(\mathbf{v})$, which is then projective. If \mathbf{v} is primitive, and H is generic, i.e. it is in the complement of a union of hyperplanes in the ample cone, then $M_H(\mathbf{v})$ is smooth and parameterizes stable sheaves.

In [23] and [22] Bridgeland defined the notion of a stability condition on a triangulated category, and constructed stability conditions on the bounded derived category of coherent sheaves $D^b(X)$ on a K3 surface X . These stability conditions form a complex manifold $\text{Stab}(X)$, and given a Mukai vector $\mathbf{v} \in H_{\text{alg}}^*(X, \mathbb{Z})$ there is a set of real-codimension one submanifolds, such that stability of objects with class \mathbf{v} is constant in each chamber, i.e. in each connected component of the complement of the walls. If \mathbf{v} is primitive, we say that a stability condition $\sigma \in \text{Stab}(X)$ is \mathbf{v} -generic if it varies in a chamber for \mathbf{v} . In this case, every σ -semistable object is σ -stable. Moreover, there is a chamber, near the “large volume limit”, where Bridgeland stability recovers Gieseker stability. The connected component

of $\text{Stab}(X)$ containing this chamber is called the *distinguished component* and denoted by $\text{Stab}^\dagger(X)$.

Moduli stacks of semistable complexes were studied by many people: Toda [105], Abramovich-Polishchuk [2], and finally a complete treatment can be found [11, Part II]. If \mathbf{v} is primitive, and $\sigma \in \text{Stab}^\dagger(X)$ is \mathbf{v} -generic there exists a coarse moduli space as an algebraic space, and it parameterizes stable complexes. Moreover, under these assumptions the coarse moduli space is a smooth and proper algebraic space, by results of Inaba [58, 57] and Lieblich [69]. In contrast to the classical Gieseker moduli spaces, these have no obvious GIT construction. Projectivity was shown in [9]. The idea is to use a Fourier-Mukai argument to reduce to the classical case of (twisted) Gieseker stability for which a GIT construction is available.

The goal of this chapter is to give a new proof of the following result.

Theorem 2.1.1. *Let X be a K3 surface. Let $H_{\text{alg}}^*(X, \mathbb{Z})$ be its extended Mukai lattice, together with the Mukai Hodge structure. Assume that $\mathbf{v} \in H_{\text{alg}}^*(X, \mathbb{Z})$ is a primitive vector and let $\sigma \in \text{Stab}^\dagger(X)$ be a \mathbf{v} -generic stability condition on X . Then:*

- (1) *The moduli space $M_\sigma(\mathbf{v})$ is non-empty if and only if $\mathbf{v}^2 \geq -2$. Moreover, it is a smooth projective hyperkähler variety of dimension $\mathbf{v}^2 + 2$, deformation-equivalent to a Hilbert scheme of points on a K3 surface.*
- (2) *If $\mathbf{v}^2 \geq 0$, then there exists a natural Hodge isometry*

$$\theta_{\mathbf{v}}^{-1}: H^2(M_\sigma(\mathbf{v}), \mathbb{Z}) \xrightarrow{\sim} \begin{cases} \mathbf{v}^\perp & \text{if } \mathbf{v}^2 > 0 \\ \mathbf{v}^\perp / \mathbb{Z}\mathbf{v} & \text{if } \mathbf{v}^2 = 0, \end{cases}$$

where the orthogonal is taken in $H^*(X, \mathbb{Z})$.

Here $\theta_{\mathbf{v}}$ is the *Mukai homomorphism*, see Definition 2.3.7 for the precise definition.

Now we briefly explain the idea of the proof, the complete argument will be in Section 2.6. We start with a K3 surface X , a primitive Mukai vector $\mathbf{v} \in H_{\text{alg}}^*(X, \mathbb{Z})$ with $\mathbf{v}^2 \geq -2$ and a generic stability condition σ in the distinguished component $\text{Stab}^\dagger(X)$. The basic geometric input is that Theorem 2.1.1 holds for Hilbert schemes of points on a K3 surface, this is a classical result due to Beauville [12]. So, we want to reduce from our starting moduli space $M_\sigma(\mathbf{v})$ on X , to the Hilbert scheme $\text{Hilb}^n(Y)$ on another K3 surface Y , in such a way that Theorem 2.1.1 remains true at every step. For this reduction we will use the following tools:

- (1) Derived equivalences:
 - Shifts $E \mapsto E[1]$,
 - Tensor product with $L \in \text{Pic}(X)$,

- The spherical twist $ST_{\mathcal{O}_X}$ around the structure sheaf,
 - The Fourier-Mukai transform $\Phi_{\mathcal{E}}$ with kernel the universal family of a fine, two-dimensional, projective moduli space parameterizing Gieseker-stable sheaves.
- (2) Existence of relative stability conditions on a smooth projective family $\mathcal{X} \rightarrow C$ of K3 surfaces over a smooth quasi-projective curve, and existence of the corresponding relative moduli spaces, this is done in [11].
- (3) Wall-crossing for moduli spaces of spherical and isotropic objects on any K3 surface, and for the Hilbert scheme $\text{Hilb}^n(X)$ on a K3 surface X with $\text{Pic}(X) = \mathbb{Z} \cdot H$ with $H^2 = 2k^2(n-1)$.

In the first five sections we show that the statement of Theorem 2.1.1 is invariant under operations of the above type, and in Section 2.6 we combine them to conclude the argument.

The argument goes roughly as follows: we begin with a sequence of autoequivalences of type (1) to modify the Mukai vector \mathbf{v} . This is done for the following reason. We can choose a polarization H on X with $H^2 = 2d$ and the new Mukai vector \mathbf{v}' , so that its Hodge locus in the moduli space of polarized K3 surfaces of degree $2d$ contains a polarized K3 surface (Y, H') with the following properties:

- (1) Its Picard group $\text{Pic}(Y)$ is a hyperbolic plane.
- (2) There is an algebraic class $\mathbf{w} \in H_{\text{alg}}^*(Y, \mathbb{Z})$ such that the moduli space $M := M_{H'}(\mathbf{w})$ is fine, non-empty, and a K3 surface.
- (3) The product of the classes \mathbf{v} and \mathbf{w} is $(\mathbf{v}, \mathbf{w}) = -1$.

Then, we deform to this K3 surface Y , and consider the Fourier-Mukai transform given by the universal family \mathcal{E} of M . The transformed vector is, up to tensoring with line bundles on M , the Mukai vector of the Hilbert scheme.

If $\mathbf{v}^2 = 0$ or -2 we argue as follows. We connect the resulting stability condition on M to the Gieseker chamber with a path. This path meets finitely many walls, so we only need to study wall-crossing at each of them. For this, we prove the following result.

Theorem 2.1.2 (Theorem 2.4.1). *Let X be a K3 surface, \mathbf{v} be a primitive vector, with $\mathbf{v}^2 = -2$ or $\mathbf{v}^2 = 0$. Let \mathcal{W} be a wall for the wall and chamber decomposition for \mathbf{v} , and denote by σ_{\pm} two generic stability conditions, one on each side of the wall, and $\sigma_0 \in \mathcal{W}$ a generic stability condition on the wall.*

- If $\mathbf{v}^2 = -2$, then $M_{\sigma_+}(\mathbf{v}) \neq \emptyset$ implies $M_{\sigma_-}(\mathbf{v}) \neq \emptyset$.
- If $\mathbf{v}^2 = 0$, then there exists a spherical, σ_0 -stable object S such that either $ST_S : M_{\sigma_+}(\mathbf{v}) \rightarrow M_{\sigma_-}(\mathbf{v})$ or $ST_S^{\pm 2} : M_{\sigma_+}(\mathbf{v}) \rightarrow M_{\sigma_-}(\mathbf{v})$ are isomorphism.

By applying the above Theorem finitely many times we complete the proof of Theorem 2.1.1 in the case where $\mathbf{v}^2 = 0$ or -2 .

A similar statement holds for $\mathbf{v}^2 > 0$ but is much more complicated, and it is the main result of [8]. In general, wall-crossing induces birational maps $M_{\sigma_+}(\mathbf{v}) \dashrightarrow$

$M_{\sigma_-}(\mathbf{v})$, which are not necessarily isomorphisms. Moreover, showing that this birational map is defined in codimension one requires a detailed analysis, and it relies on existence of stable complexes.

Instead, for the positive square case we use a different argument. We note that the Picard group $\text{Pic}(M)$ of the Fourier-Mukai partner M is again a hyperbolic plane, hence it has polarizations of any degree. In particular, we can deform to a K3 surface X' with $\text{Pic}(X') = \mathbb{Z} \cdot H'$ with $(H')^2 = 2k^2(n-1)$. Up to changing the Fourier-Mukai partner M with an isomorphic one obtained by wall-crossing via the above theorem, we can assume that the resulting stability condition on X' lies in a domain $V(X') \subset \text{Stab}^\dagger(X')$. This can be characterized as the locus of stability conditions where all the skyscraper sheaves are stable of phase one, see Definition 2.2.12 and Lemma 2.2.14. Under these assumptions, Theorem 2.1.1 will be established in Section 2.5, where we prove the following result.

Theorem 2.1.3 (Corollary 2.5.5). *Let X be a K3 surface with $\text{Pic}(X) = \mathbb{Z} \cdot H$ with $H^2 = 2d$ and $d = k^2(n-1)$ for $k > 1$ integer. There is only one wall for $\mathbf{v} = (1, 0, 1-n)$ in $V(X)$, and the shifted derived dual $\text{RHom}(-, \mathcal{O}_X)[2]$ induces an isomorphism*

$$M_{\sigma_+}(\mathbf{v}) \xrightarrow{\sim} M_{\sigma_-}(\mathbf{v}),$$

where σ_+ and σ_- are two generic stability conditions in the two chambers. In particular, Theorem 2.1.1 holds for both of them.

Structure of the chapter

In Section 2.2 we review the theory of Bridgeland stability conditions on K3 surfaces. We pay particular attention to those results in [22] which rely on existence of stable sheaves. The main purpose of this section is to recall the definition of the distinguished component $\text{Stab}^\dagger(X)$ of stability conditions. We also recall a result by Hartmann: the derived equivalences above preserve the distinguished component $\text{Stab}^\dagger(X)$.

In Section 2.3 we review some aspects of theory of moduli of stable complexes, and hyperkähler varieties.

In Section 2.4 we study the wall-crossing behavior for Mukai vectors \mathbf{v} with $\mathbf{v}^2 \leq 0$, and prove Theorem 2.4.1.

In Section 2.5 we study wall-crossing for the Hilbert scheme on a K3 surface with $\text{Pic}(X) = \mathbb{Z} \cdot H$ and degree $H^2 = 2k^2(n-1)$. In this section we restrict our attention to stability conditions of the form $\sigma_{\alpha H, \beta H}$, with $\alpha > 0$.

In Section 2.6 we complete the proof of Theorem 2.1.1, by reducing to the case of the Hilbert scheme.

2.2 Review: Bridgeland stability conditions

In this section we review the theory of Bridgeland stability on K3 surfaces, as introduced in [22]. The main objective is to define the *distinguished component* $\text{Stab}^\dagger(X)$ of the space of stability conditions and to show that (some) derived equivalences preserve this component. All of the results here are well known, due to Bridgeland and Hartmann, but some proofs in the literature use existence of slope stable spherical sheaves. Here we give a treatment that avoids that problem by slightly modifying the standard definitions.

2.2.1 Basic definitions and results

Let X be a K3 surface, denote by $D^b(X)$ the bounded derived category of coherent sheaves on X , and by

$$H_{\text{alg}}^*(X, \mathbb{Z}) := H^0(X, \mathbb{Z}) \oplus \text{NS}(X) \oplus H^4(X, \mathbb{Z})$$

the algebraic part of the cohomology. It comes equipped with an integral even bilinear form of signature $(2, \rho(X))$, called *Mukai pairing* and defined by:

$$(\mathbf{v}, \mathbf{v}') = \Delta \cdot \Delta' - rs' - r's,$$

where we write $\mathbf{v} = (r, \Delta, s)$ and $\mathbf{v}' = (r', \Delta', s')$. Recall that given an object $E \in D^b(X)$, its *Mukai vector* $v(E) \in H_{\text{alg}}^*(X, \mathbb{Z})$ is defined as

$$v(E) := \text{ch}(E) \cdot \sqrt{\text{td}_X} = (\text{ch}_0(E), \text{ch}_1(E), \text{ch}_2(E) + \text{ch}_0(E)).$$

Definition 2.2.1. A (full, numerical) stability condition is a pair $\sigma = (Z, \mathcal{A})$, where $Z : H_{\text{alg}}^*(X, \mathbb{Z}) \rightarrow \mathbb{C}$ is a group homomorphism (called *central charge*), and $\mathcal{A} \subset D^b(X)$ is a heart of a bounded t-structure, satisfying the following properties:

- (1) For any $0 \neq E \in \mathcal{A}$, the central charge $Z(E)$ lies in the semi-closed upper half-plane

$$\mathbb{H} := \mathbb{R}_{>0} e^{(0,1]i\pi}$$

- (2) Given an object $0 \neq E \in \mathcal{A}$ we define the slope as $\nu_\sigma(E) := \frac{-\Re Z(E)}{\Im Z(E)}$. A non-zero object E is said (semi)stable if for every proper subobject $F \subset E$ the following inequality holds:

$$\nu_\sigma(F) < (\leq) \nu_\sigma(E).$$

Then, every $0 \neq E \in \mathcal{A}$ has a Harder-Narasimhan filtration, i.e. a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

with semistable quotients of decreasing slope.

- (3) Fix a norm $\|*\|$ on $H_{\text{alg}}^*(X, \mathbb{Z})$. Then there is a constant $C > 0$, such that for every semistable object $E \in \mathcal{A}$, we have

$$\|E\| < C|Z(E)|$$

Given a pair (Z, \mathcal{A}) as above, we can extend the notion of stability to the full derived category $D^b(X)$ in the following way. For every $\varphi \in (0, 1]$ define $\mathcal{P}(\varphi)$ as the full subcategory of semistable objects $E \in \mathcal{A}$ with $Z(E) \in \mathbb{R}_{>0}e^{i\varphi}$. Then extend this definition to every $\varphi \in \mathbb{R}$ by the compatibility condition $\mathcal{P}(\varphi + n) = \mathcal{P}(\varphi)[n]$.

Every non-zero object $E \in D^b(X)$ has a Harder-Narasimhan filtration, i.e. a sequence of maps

$$0 = E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n = E,$$

with cones A_i that are semistable of decreasing phases. The phases of the first and last Harder-Narasimhan factors are denoted by $\varphi_\sigma^+(E)$ and $\varphi_\sigma^-(E)$. The category $\mathcal{P}(\varphi)$ is abelian of finite length, so every semistable object has a Jordan-Holder filtration, i.e. a finite filtration with stable cones of the same phase. Two semistable objects with the same associated graded are called S -equivalent.

It is shown in [23, Proposition 5.3] that the data of $(Z, \{\mathcal{P}(\varphi)\}_{\varphi \in \mathbb{R}})$ is equivalent to the data of the heart \mathcal{A} and the central charge Z . The inverse equivalence is given by forming the category $\mathcal{P}(0, 1]$, where $\mathcal{P}((a, b])$ is the full subcategory of $D^b(X)$ with objects

$$\{E \in D^b(X) \mid \varphi^-(E), \varphi^+(E) \in (a, b]\}.$$

The definition was extended in [11] to include openness of stability in families and existence of moduli spaces. Precisely we add the conditions:

- (4) For every scheme T and for every T -perfect complex $E \in D_{T\text{-perf}}(S \times T)$ the set

$$\{t \in T \mid E_t \in \mathcal{P}(\varphi)\}$$

is open.

- (5) for every $\mathbf{v} \in H_{\text{alg}}^*(X, \mathbb{Z})$ and every $\varphi \in \mathbb{R}$ such that $Z(\mathbf{v}) \in \mathbb{R}_{>0}e^{i\pi\varphi}$ the functor

$$T \rightarrow \mathfrak{M}_\sigma(\mathbf{v}, \varphi)(T) := \{E \in D_{T\text{-perf}}(S \times T) \mid E_t \in \mathcal{P}(\varphi) \text{ and } v(E_t) = \mathbf{v}\}$$

is bounded.

Let $\text{Stab}(X)$ be the set of all stability conditions. It has a natural topology induced by a metric, see [23, Section 6] for the precise form of the metric. This topology can be characterized as the coarsest topology that makes the functions

$$\sigma \mapsto Z \text{ and } \sigma \mapsto \varphi_\sigma^\pm(E)$$

continuous, for every $0 \neq E \in D^b(X)$. The main result in [23] is the following.

Theorem 2.2.2 (Bridgeland Deformation Theorem). *The map*

$$\begin{aligned}\pi : \text{Stab}(X) &\rightarrow \text{Hom}(H_{\text{alg}}^*(X, \mathbb{Z}), \mathbb{C}) \\ \sigma = (Z, \mathcal{A}) &\mapsto Z\end{aligned}$$

is a local homeomorphism. In particular, every connected component of $\text{Stab}(X)$ is a complex manifold of dimension $\text{rk}(H_{\text{alg}}^(X, \mathbb{Z}))$.*

Remark 2.2.3 ([23, Lemma 8.2]). There are two natural actions on the space of stability condition: a left action by the group $\text{Aut}(D^b(X))$ of exact autoequivalences of $D^b(X)$, and a right action by $\widetilde{GL}_2^+(\mathbb{R})$, the universal cover of the group $GL_2^+(\mathbb{R})$. Given an autoequivalence Φ and a stability condition $\sigma = (Z, \mathcal{P})$ we set $\Phi(\sigma) := (Z \circ \Phi^{-1}, \mathcal{P}')$, where $\mathcal{P}'(t) := \Phi(\mathcal{P}(t))$. The action of $\widetilde{GL}_2^+(\mathbb{R})$ is given by lifting the right action of $GL_2^+(\mathbb{R})$ on $\text{Hom}(H_{\text{alg}}^*(X, \mathbb{Z}), \mathbb{C})$.

We are interested in knowing how stability varies when we deform σ ; this was first done in [22, Proposition 9.3]. See also [10, Proposition 3.3] and [105, Proposition 2.8].

Proposition 2.2.4. *Fix a class $\mathbf{v} \in H_{\text{alg}}^*(X, \mathbb{Z})$. There exists a locally finite set of real codimension one manifold with boundary, called walls, in $\text{Stab}(X)$, such that when σ varies within a chamber (a connected component of the complement of the set of walls), the set of σ -semistable and σ -stable objects does not change. If \mathbf{v} is primitive and σ varies in a chamber, every semistable object is stable.*

Definition 2.2.5. Let $\mathbf{v} \in H_{\text{alg}}^*(X, \mathbb{Z})$. We say a stability condition σ is \mathbf{v} generic if it is not on a wall for \mathbf{v} . We say a stability condition is generic on a wall if it lies on only one wall.

Remark 2.2.6. If an object E is semistable in a chamber, it continues to be semistable on a wall of that chamber. Indeed, the condition for E to be semistable is given by the equality $\varphi_{\sigma}^+(E) = \varphi_{\sigma}^-(E)$, which is a closed condition on the space of stability conditions.

2.2.2 Construction of stability conditions on K3 surfaces.

We review the construction of stability conditions on K3 surfaces, given in [22]. Let $\omega \in NS(X) \otimes \mathbb{R}$ ample. Recall the definition of the slope of a coherent sheaf $E \in \text{Coh}(X)$,

$$\mu_{\omega}(E) := \begin{cases} \frac{\omega \cdot c_1(E)}{\omega^2 r(E)} & \text{if } r(E) > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $r(E)$ denotes the rank of E .

Definition 2.2.7. A coherent sheaf $E \in \text{Coh}(X)$ is slope (semi)stable if for every proper subsheaf $A \subset E$ we have

$$\mu_\omega(A) < (\leq) \mu_\omega(E/A).$$

For later use we also recall the definition of B -twisted Gieseker stability, for B a rational divisor. Note that if $B = 0$ we recover the classical notion of Gieseker stability.

Definition 2.2.8. Let $\omega, B \in \text{NS}(X)_\mathbb{Q}$, with ω ample. We define the B -twisted Hilbert polynomial of a coherent sheaf E as

$$P(E, m) := \int_X e^{m\omega - B} \cdot v(E).$$

A pure d -dimensional coherent sheaf E is B -twisted Gieseker (semi)stable if, for every proper non trivial subsheaf $F \subset E$ we have

$$\frac{P(F, m)}{\alpha_d(F)} < (\leq) \frac{P(E, m)}{\alpha_d(E)},$$

for $m \gg 0$, where $\alpha_d(E)$ is the degree d coefficient of $P(E, m)$.

Remark 2.2.9. Similarly to Bridgeland stability, both slope stability and Gieseker stability satisfy the existence of Harder-Narasimhan filtrations. That is every non-zero coherent sheaf $E \in \text{Coh}(X)$ has a filtration with slope-semistable (respectively Gieseker semistable) quotients of decreasing slope (respectively decreasing reduced Hilbert polynomial).

Now consider the pair $\sigma_{\omega, B} = (Z_{\omega, B}, \text{Coh}^{\omega, B})$ where

$$Z_{\omega, B}(\mathbf{v}) := (\mathbf{v}, \exp(B + i\omega)),$$

and $\text{Coh}^{\omega, B}(X)$ is the tilt of $\text{Coh}(X)$ with respect to the torsion pair $(\mathcal{T}^\beta, \mathcal{F}^\beta)$, defined as follows

$$\begin{aligned} \mathcal{T}^\beta &:= \{T \in \text{Coh}(X) \mid \text{All HN factors } A_i \text{ of } T \text{ satisfy } \mu_\omega(A_i) > \frac{\omega \cdot B}{\omega^2}\}; \\ \mathcal{F}^\beta &:= \{F \in \text{Coh}(X) \mid \text{All HN factors } A_i \text{ of } F \text{ satisfy } \mu_\omega(A_i) \leq \frac{\omega \cdot B}{\omega^2}\}, \end{aligned}$$

where the Harder-Narasimhan factors are with respect to slope stability.

Definition 2.2.10. An object $E \in D^b(X)$ is spherical if

$$\text{Ext}^i(E, E) = \begin{cases} \mathbb{C} & \text{if } i = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.2.11 ([22, Lemma 6.2]). *The above construction gives a stability condition $\sigma_{\omega,B}$ on $D^b(X)$, provided $Z_{\omega,B}(E) \notin \mathbb{R}_{\leq 0}$ for every spherical torsion-free sheaf E .*

Definition 2.2.12. The set of stability conditions $\sigma_{\omega,B} = (Z_{\omega,B}, \text{Coh}^{\omega,B})$, with $Z_{\omega,B}(E) \notin \mathbb{R}_{\leq 0}$ for every spherical sheaf E is denoted by $V(X)$. We define the *geometric chamber* $U(X)$ as the subset

$$\widetilde{GL}_2^+(\mathbb{R}).V(X) \subseteq \text{Stab}(X)$$

obtained from $V(X)$ via the action of $\widetilde{GL}_2^+(\mathbb{R})$. A stability condition is *geometric* if it belongs to $U(X)$.

To define the distinguished component we need to show that the sets $V(X)$ and $U(X)$ are connected. To show this we follow the proof in [22] and introduce several auxiliary spaces. Using the Mukai pairing form on $H_{\text{alg}}^*(X, \mathbb{Z})$ we can identify the central charge Z of a stability condition with a vector $\Omega_Z \in H_{\text{alg}}^*(X, \mathbb{Z}) \otimes \mathbb{C}$.

Define $\mathcal{P}(X) \subset H_{\text{alg}}^*(X, \mathbb{Z}) \otimes \mathbb{C}$ as the set of vectors Ω whose real and imaginary parts span positive definite two-planes in $H_{\text{alg}}^*(X, \mathbb{Z}) \otimes \mathbb{R}$. Define $\mathcal{P}_0(X)$ as the subset of $\mathcal{P}(X)$ of classes not orthogonal to any spherical class:

$$\mathcal{P}_0(X) := \{\Omega \in \mathcal{P}(X) \mid (\Omega, \delta) \neq 0 \text{ for every } \delta \in \Delta(X)\}$$

where $\Delta(X) := \{\delta \in H_{\text{alg}}^*(X, \mathbb{Z}) \mid \delta^2 = -2\}$. Consider the subset of $\mathcal{P}_0(X)$ given by the vectors Ω obtained by the construction above:

$$\mathcal{K}(X) := \{\Omega \in \mathcal{P}_0(X) \mid \Omega = \exp(B + i\omega) \text{ with } \omega \in \text{Amp}(X) \text{ and } B \in \text{NS}(X) \otimes \mathbb{R}\}$$

The set $\mathcal{P}_0(X)$ has two connected components, we call $\mathcal{P}_0^+(X)$ the one containing $\mathcal{K}(X)$.

Proposition 2.2.13 ([22, Proposition 8.3]). *The set $\mathcal{P}_0(X) \subset \mathcal{N}(X) \otimes \mathbb{C}$ is open, and the restriction*

$$\pi : \pi^{-1}(\mathcal{P}_0(X)) \rightarrow \mathcal{P}_0(X)$$

is a covering map.

To show that $V(X)$ and $U(X)$ are connected, we need to introduce two more subsets:

$$\mathcal{Q}(X) := \{\Omega \in \mathcal{P}(X) \mid (\Omega, \Omega) = 0, (\Omega, \bar{\Omega}) > 0, r(\Omega) = 1\}$$

and

$$\mathcal{L}(X) := \{\Omega \in \mathcal{K}(X) \mid (\Omega, \delta) \notin \mathbb{R}_{\leq 0} \quad \forall \delta \in \Delta^+(X)\},$$

where $\Delta^+(X) := \{\delta \in \Delta(X) \mid r(\delta) > 0\}$, and $r : H_{\text{alg}}^*(X, \mathbb{Z}) \otimes \mathbb{C} \rightarrow H^0(X, \mathbb{C}) \cong \mathbb{C}$ is the first component.

Lemma 2.2.14. *We have*

$$V(X) = \{\sigma \in \text{Stab}(X) \mid \mathcal{O}_x \text{ is stable of phase 1 } \forall x \in X, \text{ and } \pi(\sigma) \in \mathcal{Q}(X)\}$$

and

$$U(X) = \{\sigma \in \text{Stab}(X) \mid \mathcal{O}_x \text{ is stable } \forall x \in X, \text{ and } \pi(\sigma) \in \mathcal{P}_0^+(X)\}$$

Proof. We start by noticing that by [35, Proposition 2.9], we have that if every skyscraper sheaf is stable, then they are all stable of the same phase. Now, without the condition $\pi(\sigma) \in \mathcal{P}_0^+(X)$ the lemma from the proof of [22, Proposition 10.3]. In the third step of the proof, Bridgeland shows, using existence of slope stable sheaves, that if all the skyscraper sheaves are σ -stable, then the central charge is in $\mathcal{P}_0^+(X)$. Since we explicitly ask for the central charge to be in $\mathcal{P}_0^+(X)$, we can skip this step. \square

Lemma 2.2.15. *The restriction $\pi|_{V(X)} : V(X) \rightarrow \mathcal{Q}(X)$ has open image and it is an homeomorphism onto its image.*

Proof. Since every stability condition in $V(X)$ is obtained by tilting, the map π is injective when restricted to $V(X)$. So it is enough to show that it remains a covering on $V(X)$. Notice that Lemma 2.2.14 and [22, Proposition 9.4] imply that $U(X) \subset \text{Stab}(X)$ is an open subset. Since $\pi(U(X)) \subset \mathcal{P}_0(X)$ by definition and $U(X)$ is open, the map π restricted to $U(X)$ is a covering onto its image. Moreover, a stability condition $\sigma \in U(X)$ is determined by its central charge $\pi(\sigma)$ up to even shifts, because the even shifts are the only elements of $\widetilde{GL}_2^+(\mathbb{R})$ that fix the central charge. Let A be a small neighborhood of $\pi(\sigma)$, since π is a cover on $U(X)$, the inverse image $\pi^{-1}(A) \cap U(X)$ is homeomorphic to $A \times \mathbb{Z}$, where the second factors records the shift. Restricting to $\mathcal{Q}(X) \cap A$ we see that $\pi^{-1}(A \cap \mathcal{Q}(X)) \cap V(X)$ is contained in one component of $\pi^{-1}(A) \cap U(X)$, so $\pi|_{V(X)}$ induces an homeomorphism onto its image. \square

It is easy to see that the pairing $(-, \delta)|_{\mathcal{K}(X)} : \mathcal{K}(X) \rightarrow \mathbb{C}$ with any class δ is submersive when restricted to $\mathcal{K}(X)$. In particular the preimage of a real half-line is a locally closed submanifold of real codimension one. These submanifolds are contained in real hyperplanes of $H_{\text{alg}}^*(X, \mathbb{Z}) \otimes \mathbb{C}$. Bridgeland shows that the union of these hyperplanes is locally finite, and it uses this to show the following.

Lemma 2.2.16 ([22, Lemma 11.1]). *The set $\mathcal{L}(X) \subset \mathcal{Q}(X)$ is open and contractible.*

The proof of the following Proposition is the same as [22, Proposition 11.2] with an extra step, but we reproduce the entire proof for readability.

Proposition 2.2.17. *The spaces $V(X)$ and $U(X)$ are connected.*

Proof. First we claim that

$$\mathcal{L}(X) \subseteq \pi(V(X)).$$

Lemma 2.2.15 implies that $\pi(V(X))$ is an open subset of $\mathcal{Q}(X)$ and $\pi|_{V(X)} : V(X) \rightarrow \pi(V(X))$ is an homeomorphism. From Lemma 2.2.16 we get that $\mathcal{L}(X) \cap \pi(V(X))$ is open in $\mathcal{L}(X)$. Since $\mathcal{L}(X)$ is connected, we only need to show that the intersection is closed in $\mathcal{L}(X)$.

Let $\Omega \in \overline{\mathcal{L}(X) \cap \pi(V(X))} \subset \mathcal{L}(X)$. Since π is an homeomorphism restricted to $V(X)$, there exists a stability condition $\sigma \in \overline{V(X)}$ such that $\pi(\sigma) = \Omega$. If σ is not in $V(X)$, Lemma 2.2.14 implies that there is a strictly semistable skyscraper sheaf \mathcal{O}_x ; consider its Jordan-Holder factors A_i . From the definition of the category $\text{Coh}^{\omega, B}(X)$, and the fact that ω is ample, it follows that if $\Im Z(A_i) = 0$ and $r(A_i) = 0$, then $\mathbf{v}(A_i)$ is a multiple of $\mathbf{v}(\mathcal{O}_x)$. This implies that there is a Jordan-Holder factor A with positive rank. Since A has the same phase as \mathcal{O}_x we have $Z(A) \in \mathbb{R}_{<0}$, and we claim that A is spherical. Recall that

$$\Re Z(A) = \frac{1}{2r}((\Delta^2 - 2rs) + r^2\omega^2 - (\Delta - rB)^2),$$

where $\mathbf{v}(A) = (r, \Delta, s)$. We have $\Im Z(A) = (\Delta - rB).\omega = 0$, which implies $(\Delta - rB)^2 \leq 0$ by the Hodge index Theorem. Hence $\Delta^2 - 2rs < 0$, which means A spherical, because it is stable. We found a spherical class $\delta := \mathbf{v}(A) \in \Delta^+(X)$ such that $(\Omega, \delta) < 0$, which contradicts the assumption that $\Omega \in \mathcal{L}(X)$.

To finish the proof, it is enough to show that $V(X)$ is connected. Since π is a homeomorphism restricted to $V(X)$, we need to show that $\pi(V(X))$ is connected. Since $\mathcal{L}(X)$ is connected, it is enough to show that $\mathcal{L}(X)$ is dense in $\pi(V(X))$. If we assume non-emptiness of moduli stable of slope stable spherical sheaves, we have equality, as showed in [22, Proposition 11.2]. If we do not have the non-emptiness, it could happen that for an $\Omega \in \pi(V(X))$ we have $(\Omega, \mathbf{v}(E)) \notin \mathbb{R}_{\leq 0}$ for every spherical torsion-free sheaf E , but $(\Omega, \delta) \in \mathbb{R}_{\leq 0}$ for some spherical class δ for which there are no corresponding sheaves. That is, the difference $\pi(V(X)) - \mathcal{L}(X)$ is contained in a locally finite union of locally closed submanifolds of real codimension one. Hence $\mathcal{L}(X)$ is dense in $\pi(V(X))$ and $V(X)$ is connected because $\mathcal{L}(X)$ is. \square

Definition 2.2.18. Define the distinguished component $\text{Stab}^\dagger(X)$ as the connected component of the preimage $\pi^{-1}(\mathcal{P}_0^+(X)) \subset \text{Stab}(X)$ containing $U(X)$.

Remark 2.2.19. As mentioned above, our definition differs from Bridgeland's original definition in [22]. In ibidem it is defined as the connected component of $\text{Stab}(X)$ containing the geometric chamber $U(X)$, and it is a Theorem that it gets mapped onto $\mathcal{P}_0^+(X)$ via π . The proof requires existence of slope stable sheaves.

In any case, our definition is sufficient to prove existence of slope stable sheaves, because $\text{Stab}^\dagger(X)$ contains the Gieseker chamber. So, once we prove Theorem 2.1.1 with our definition, it will also follow Theorem 2.1.1 for the standard definition of $\text{Stab}^\dagger(X)$.

2.2.3 Equivalences preserving $\text{Stab}^\dagger(X)$.

To conclude this section we want to show that there are enough equivalences between derived categories of K3 surfaces preserving $\text{Stab}^\dagger(X)$. Every result here has already been shown by Hartmann in the appendix of [41]. The idea is simple: since $\text{Stab}^\dagger(X)$ is connected and contains the geometric chamber $U(X)$ it is enough to find a point $\sigma \in \text{Stab}^\dagger(X)$ that goes to the geometric chamber. This is easy to check thanks to the explicit description of $U(X)$ in Lemma 2.2.14. In our argument, there is the extra check that the equivalences preserve the domain $\mathcal{P}_0^+(X)$, which (in the generality needed in this paper) is due to Huybrechts and Stellari [49].

We start by recalling some generalities about Fourier-Mukai equivalences between K3 surfaces. Let H be a polarization on X , let $\mathbf{w} \in H_{\text{alg}}^*(X, \mathbb{Z})$ be a Mukai vector, and consider the moduli space $M := M_H(\mathbf{w})$ of Gieseker stable sheaves of class \mathbf{w} . Assume that it is a smooth projective surface, and that it is fine, i.e. it has a universal family $\mathcal{E} \in \text{Coh}(X \times M)$. We can consider the Fourier-Mukai transform with kernel the universal family:

$$\begin{aligned} \Phi_{\mathcal{E}} : D^b(M) &\rightarrow D^b(X) \\ F &\mapsto q_*(\mathcal{E} \otimes p^*F), \end{aligned}$$

where we denoted by p, q the projections from $X \times M$ on the first and second factor, and where every functor is derived.

Proposition 2.2.20 ([89],[46, Proposition 10.25]). *If M is a smooth projective surface, and it is a fine moduli space, then the functor $\Phi_{\mathcal{E}}$ is an equivalence.*

Recall that if X is a K3 surface the structure sheaf \mathcal{O}_X is a spherical object. In particular we can consider the spherical twist $\text{ST}_{\mathcal{O}_X}$ around \mathcal{O}_X defined as follows.

Definition 2.2.21. Let $S \in D^b(X)$ be a spherical object. The *spherical twist* around S , denoted by $\text{ST}_S(-)$ is defined, for every $E \in D^b(X)$, as the cone of the evaluation map:

$$\text{RHom}(S, E) \otimes S \rightarrow E \rightarrow \text{ST}_S(E) \rightarrow \text{RHom}(S, E) \otimes S[1].$$

Proposition 2.2.22 ([101, Proposition 2.10],[46, Proposition 8.6]). *The spherical twist ST_S around a spherical object S is an equivalence.*

Every Fourier-Mukai equivalence $\Phi_{\mathcal{E}} : D^b(X) \xrightarrow{\sim} D^b(X')$ between derived categories of K3 surfaces induces a map in integral cohomology. It is the cohomological Fourier-Mukai transform with kernel the Mukai vector $v(\mathcal{P})$:

$$\Phi_{\mathcal{E}}^H(x) = q_*(v(\mathcal{P}).p^*(x)),$$

where p and q are the projections $X \times X'$ on the first and second factor. It is well known that it is a Hodge isometry, if we equip the integral cohomology with the following weight two Hodge structure :

$$H^*(X, \mathbb{C}) = H^{2,0}(X) \oplus (H^0(X, \mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X, \mathbb{C})) \oplus H^{0,2}(X).$$

In particular it induces an isometry $\Phi_{\mathcal{E}}^H : H_{\text{alg}}^*(X, \mathbb{Z}) \xrightarrow{\sim} H_{\text{alg}}^*(X', \mathbb{Z})$ between the algebraic parts. Every derived equivalence $\Phi : D^b(X) \xrightarrow{\sim} D^b(X')$ is of Fourier-Mukai type, and the induced isometry does not depend on the kernel. If the kernel is not given, we will denote the induced isometry by Φ^H .

The equivalences we are interested in are the following:

- (1) Tensor product by a line bundle: $E \mapsto E \otimes L$;
- (2) Shift: $E \mapsto E[1]$
- (3) The spherical twist $\text{ST}_{\mathcal{O}_X}$ around \mathcal{O}_X ;
- (4) Fourier-Mukai transforms $\Phi_{\mathcal{E}}$ associated to a fine two dimensional moduli space of Gieseker stable sheaves.

Proposition 2.2.23 ([49, Remark 5.4 and Proposition 5.5]). *The isometries induced by the equivalences of type (1) – (4) preserve the set $\mathcal{P}_0^+(X)$.*

An equivalence $\Phi : D^b(X) \xrightarrow{\sim} D^b(X')$ induces an isomorphism of spaces of stability conditions

$$\begin{aligned} \Phi_* : \text{Stab}(X) &\xrightarrow{\sim} \text{Stab}(X') \\ (Z, \mathcal{P}) &\mapsto (Z \circ \Phi^{-1}, \mathcal{P}') \end{aligned}$$

where $\mathcal{P}'(t) = \Phi(\mathcal{P}(t))$. We say that Φ preserves the distinguished component if

$$\Phi_*(\text{Stab}^\dagger(X)) = \text{Stab}^\dagger(X').$$

Lemma 2.2.24. *Let $\Phi : D^b(X) \xrightarrow{\sim} D^b(X')$ be a derived equivalence of K3 surfaces of type (1) – (4). Assume that there exists $\sigma' = (Z', \mathcal{P}') \in \text{Stab}^\dagger(X')$ such that the objects $\Phi(\mathcal{O}_x)$ are σ' -stable and such that $\Omega_{Z'} \in \mathcal{P}_0^+(X')$. Then Φ preserves the distinguished component.*

Proof. It is enough to show that a point of $\text{Stab}^\dagger(X)$ gets mapped to $\text{Stab}^\dagger(X')$. Consider $(Z, \mathcal{P}) = \sigma := \Phi_*^{-1}(\sigma')$. By assumption the skyscraper sheaves \mathcal{O}_x are all σ -stable. Proposition 2.2.23 shows that the induced isometry in cohomology sends $\mathcal{P}_0^+(X)$ to $\mathcal{P}_0^+(X')$. In particular $\Omega_Z = (\Phi^H)^{-1}(\Omega_{Z'})$ is in $\mathcal{P}_0^+(X)$. Then, Lemma 2.2.14 implies $\sigma \in U(X) \subset \text{Stab}^\dagger(X)$. \square

In order to show that the equivalences we are interested in preserve the distinguished component, we need a standard result about the large volume limit. Let $H \in \text{NS}(X)$ be an ample class, and $B \in \text{NS}_{\mathbb{Q}}(X)$ a rational class. Consider the stability condition $\sigma_{\alpha H, B}$.

Theorem 2.2.25 ([22, Proposition 14.1] and [105, Section 6]). *Let $\mathbf{v} = (r, \Delta, s)$ be a primitive Mukai vector, with either $r > 0$ or $r = 0$ and $\Delta \neq 0$ effective. Then there exists an α_0 such that, for every $\alpha \geq \alpha_0$, an object $E \in D^b(X)$ of class \mathbf{v} is $\sigma_{\alpha H, B}$ -stable if and only if it is a shift of a B -twisted H -Gieseker stable sheaf.*

Corollary 2.2.26 ([41, Lemma 7.2, Propositions 7.5 and 7.6]). *The equivalences of type (1) – (4) preserve the distinguished component.*

Proof. The equivalences of type (1) and (2) send skyscraper sheaves to (shifts of) skyscraper sheaves, so by Lemma 2.2.14 and Proposition 2.2.23 they preserve the geometric chamber $U(X)$ and, a fortiori, the distinguished component.

For the remaining two (3) and (4) we use Lemma 2.2.24: it is enough to find a stability condition $\sigma \in \text{Stab}^{\dagger}(X)$ such that $\Phi(\mathcal{O}_x)$ are σ -stable, and whose central charge satisfies $\Omega_Z \in \mathcal{P}_0^+(X)$. For the spherical twist, notice that $\text{ST}_{\mathcal{O}_X}(\mathcal{O}_x) = \mathfrak{m}_x$, the ideal sheaf of the point x . These are Gieseker stable, so by choosing $\sigma_{\alpha H, B}$ appropriately as in Theorem 2.2.25 we find a $\sigma \in V(X)$ that works. Similarly, if \mathcal{E} is a universal family over a Gieseker moduli space, the objects $\Phi_{\mathcal{E}}(\mathcal{O}_x)$ are Gieseker stable, and again we conclude by Theorem 2.2.25. \square

2.3 Review: Moduli spaces of sheaves

In this section we give a short review on hyperkähler varieties, and basic facts about moduli spaces of stable sheaves and stable complexes.

Definition 2.3.1. A projective hyperkähler variety is a smooth projective complex variety, which is simply connected and such that $H^0(X, \Omega_X^2)$ is one dimensional and spanned by a symplectic 2-form.

On the $H^2(X, \mathbb{Z})$ there is a natural integral quadratic form q_X , called Beauville-Bogomolov-Fujiki (or BBF) form. It is a deformation invariant, and has signature $(3, b_2(X) - 3)$. It satisfies the Fujiki relation

$$\int_X \alpha^n = (2n - 1)!! c_X q_X(\alpha)^n, \quad \alpha \in H^2(X, \mathbb{Z}).$$

The constant c_X is called Fujiki constant, and it is deformation invariant.

Fix $\sigma = (Z, \mathcal{P}) \in \text{Stab}(X)$ a stability condition, a phase $\varphi \in \mathbb{R}$ and a Mukai vector $\mathbf{v} \in H_{\text{alg}}^*(X, \mathbb{Z})$. Consider the moduli stack $\mathfrak{M}_{\sigma}(\mathbf{v}, \varphi)$ of σ -semistable objects

of class \mathbf{v} and phase φ . Its objects over S are S -perfect complexes $\mathcal{E} \in D_{S\text{-perf}}^b(S \times X)$, whose restriction over a closed point $s \in S$ belongs to $\mathcal{P}(\varphi)$ and has class \mathbf{v} . The following is a collection of result by Toda [105], Inaba [57], and Lieblich [69].

Theorem 2.3.2. *Let X be a K3 surface, $\mathbf{v} \in H_{\text{alg}}^*(X, \mathbb{Z})$ and $\sigma \in \text{Stab}^\dagger(X)$. Then $\mathfrak{M}_\sigma(\mathbf{v}, \varphi)$ is an Artin stack of finite type over \mathbb{C} . Denote by $\mathfrak{M}_\sigma^s(\mathbf{v}, \varphi) \subseteq \mathfrak{M}_\sigma(\mathbf{v}, \varphi)$ the open substack parametrizing σ -stable objects. If $\mathfrak{M}_\sigma^s(\mathbf{v}, \varphi) = \mathfrak{M}_\sigma(\mathbf{v}, \varphi)$, then $\mathfrak{M}_\sigma(\mathbf{v}, \varphi)$ is a \mathbb{G}_m -gerbe over its coarse moduli space $M_\sigma(\mathbf{v}, \varphi)$, which is a smooth, proper, symplectic algebraic space with expected dimension $\mathbf{v}^2 + 2$.*

In particular the assumptions are satisfied if \mathbf{v} is primitive and σ is \mathbf{v} -generic. The phase φ is determined by the rest of the data, up to an even integer. Since the corresponding moduli spaces are isomorphic via shifts, from now on we drop the φ from the notation, and denote a moduli space simply by $M_\sigma(\mathbf{v})$.

The moduli spaces $M_\sigma(\mathbf{v})$ are not necessarily fine, but if \mathbf{v} is primitive and $\sigma \in \text{Stab}^\dagger(X)$ is \mathbf{v} -generic, they are equipped with a quasi-universal family unique up to equivalence, by [89, Theorem A.5].

Definition 2.3.3. Let $M = M_\sigma(\mathbf{v}, \varphi)$ be a coarse moduli space.

- (1) A flat family \mathcal{E} on $M \times X$ is called a quasi-family of objects in $\mathfrak{M}_\sigma(\mathbf{v})$ if, for all closed points $m \in M$, there exists an integer $\rho > 0$, and an element $E \in \mathfrak{M}_\sigma(\mathbf{v})(\mathbb{C})$ such that $\mathcal{E}|_{m \times X} \cong E^{\oplus \rho}$. If M is connected ρ is independent of m , and is called the similitude of \mathcal{E} .
- (2) Two quasi-families are equivalent if there exists vector bundles V and V' on M such that $\mathcal{E}' \otimes p_M^* V \cong \mathcal{E} \otimes p_M^* V'$.
- (3) a quasi-family \mathcal{E} is called a quasi-universal family if, for every scheme T and for any quasi-family \mathcal{T} on $M \times X$, there exists a unique morphism $f : M \rightarrow T$ such that $f^* \mathcal{E}$ and \mathcal{T} are equivalent.

Projectivity of the coarse moduli space $M_\sigma(\mathbf{v})$ was proved in [9]. The problem is that in general for moduli spaces of stable complexes there is no obvious GIT construction. Bayer and Macrì constructed a divisor class $l_\sigma \in \text{NS}(M_\sigma(\mathbf{v}))_{\mathbb{R}}$ as follows:

$$C \mapsto l_\sigma \cdot C := \mathfrak{S}\left(-\frac{Z(v(\Phi_{\mathcal{E}}(\mathcal{O}_C)))}{Z(\mathbf{v})}\right),$$

where $C \in M_\sigma(\mathbf{v})$ is a curve, and \mathcal{E} is a quasi-universal family.

Theorem 2.3.4 ([9, Theorem 4.1 and Remark 4.6]). *Let \mathbf{v} be a primitive Mukai vector, $\sigma \in \text{Stab}^\dagger(X)$ a \mathbf{v} -generic stability condition. Then the class l_σ defined above is ample.*

One of the key steps in the proof is the use of [9, Lemma 7.3] to reduce to to the classical case of Gieseker stable sheaves. The same reduction argument also

shows irreducibility of $M_\sigma(\mathbf{v})$. The following statement summarizes the discussion above.

Corollary 2.3.5. *Let X be a K3 surface, $\mathbf{v} \in H_{\text{alg}}^*(X, \mathbb{Z})$ a primitive vector with $\mathbf{v}^2 \geq -2$. Let $\sigma \in \text{Stab}^\dagger(X)$ be a \mathbf{v} -generic stability condition. Then if $M_\sigma(\mathbf{v})$ is non-empty, it is a smooth, projective symplectic variety of dimension $\mathbf{v}^2 + 2$ and it consists of stable objects.*

Remark 2.3.6. There is a subtlety here. To use the arguments in [9, Lemma 7.3] we need to know Theorem 2.1.1 for the case of a primitive vector $\mathbf{v} \in H_{\text{alg}}^*(X, \mathbb{Z})$ with $\mathbf{v}^2 = 0$. We will prove Theorem 2.1.1 for the a Mukai vector with square zero before the positive square case. Any such moduli space will be automatically projective, since it is a two-dimensional, smooth and proper algebraic space. Having this, one can use the projectivity result in [9] for the positive square case.

Definition 2.3.7. Let $\mathbf{v} \in H_{\text{alg}}^*(X, \mathbb{Z})$ a primitive class with $\mathbf{v}^2 > 0$, and let $\sigma \in \text{Stab}^\dagger(X)$ be a \mathbf{v} -generic stability condition, in particular every σ -semistable object of Mukai vector \mathbf{v} is σ -stable. We define the Mukai homomorphism $\theta_{\mathbf{v}} : \mathbf{v}^\perp \rightarrow H^2(M_\sigma(\mathbf{v}), \mathbb{Z})$ by

$$\theta_{\mathbf{v}}(\mathbf{x}) = \frac{1}{\rho} [\Phi_{\mathcal{E}}^H(\mathbf{x}^\vee)]_1 \quad (2.1)$$

where \mathcal{E} is a quasi-universal family of similitude ρ , $[-]_1$ is the component belonging to $H^2(M_\sigma(\mathbf{v}), \mathbb{Z})$, and $(x_0, x_1, x_2)^\vee := (x_0, -x_1, x_2)$. If $\mathbf{v}^2 = 0$, the same formula gives a well defined map $\theta_{\mathbf{v}} : \mathbf{v}^\perp / \mathbb{Z} \cdot \mathbf{v} \rightarrow H^2(M_\sigma(\mathbf{v}), \mathbb{Z})$

It can be shown that it does not depend on the quasi-universal family \mathcal{E} if we restrict to \mathbf{v}^\perp .

Remark 2.3.8. The definition of Mukai homomorphism in [96] and [118] is

$$\frac{1}{\rho} [p_{M_\sigma(\mathbf{v})}^* \text{ch}(\mathcal{E}) p_X^*(\sqrt{\text{td}_X} x^\vee)]_1.$$

This is equivalent to ours. Indeed, recall the definition of the Mukai vector

$$v(\mathcal{E}) = \text{ch}(\mathcal{E}) p_{M_\sigma(\mathbf{v})}^* \left(\sqrt{\text{td}_{M_\sigma(\mathbf{v})}} \right) p_X^* \left(\sqrt{\text{td}_X} \right).$$

The degree two component of Todd class $\text{td}_{M_\sigma(\mathbf{v})}$ is 0, because $M_\sigma(\mathbf{v})$ has trivial canonical bundle, so its square root does not contribute to the degree two component.

Recall that an anti-equivalence is an equivalence from the opposite category $D^b(X)^{\text{op}}$ to $D^b(X')$. Every anti-equivalence is given by a composition of an equivalence and the dualizing functor $\text{RHom}(-, \mathcal{O}_X)$.

Remark 2.3.9. Let $\Phi : D^b S \xrightarrow{\sim} D^b S'$ an equivalence. Assume that it maps the set of σ -stable object of fixed class \mathbf{v} bijectively to the set of σ' -stable objects with class \mathbf{v}' . Then, at the level of moduli spaces the map

$$M_{S,\sigma}(\mathbf{v}) \rightarrow M_{S',\sigma'}(\mathbf{v}'), \quad E \mapsto \Phi(E)$$

is an isomorphism of algebraic varieties. Indeed, for any test scheme T one considers the equivalence

$$\Phi \boxtimes \mathrm{id}_{D^b(T)} : D(S \times T) \xrightarrow{\sim} D(S' \times T).$$

It is T -linear and it preserves T -perfect complexes. It follows from this that the collection $(\Phi \boxtimes \mathrm{id}_{D^b(T)})_T$ is an isomorphism of stacks $\mathfrak{M}_{S,\sigma}(\mathbf{v}) \xrightarrow{\sim} \mathfrak{M}_{S',\sigma'}(\mathbf{v}')$ inducing the map above at the level of moduli spaces.

Proposition 2.3.10. *Let X, X' be two K3 surfaces, \mathbf{v}, \mathbf{v}' two Mukai vectors on X and X' respectively. Let $\sigma \in \mathrm{Stab}^\dagger(X)$ be a \mathbf{v} -generic stability condition on X , and $\sigma' \in \mathrm{Stab}^\dagger(X')$ a \mathbf{v}' -generic stability condition on X' . Assume that there is an (anti)-equivalence $\Phi : D^b(X) \xrightarrow{\sim} D^b(X')$ that induces an isomorphism $M_{X,\sigma}(\mathbf{v}) \xrightarrow{\sim} M_{X',\sigma'}(\mathbf{v}')$. If $\mathbf{v}^2 > 0$ we have a commutative diagram*

$$\begin{array}{ccc} \mathbf{v}^\perp & \xrightarrow{\pm\Phi^H} & \mathbf{v}'^\perp \\ \theta_{\mathbf{v}} \downarrow & & \theta_{\mathbf{v}'} \downarrow \\ H^2(M_{X,\sigma}(\mathbf{v}), \mathbb{Z}) & \xlongequal{\quad} & H^2(M_{X',\sigma'}(\mathbf{v}'), \mathbb{Z}) \end{array}$$

where the sign is $+$ if Φ is an equivalence and $-$ if it is an anti-equivalence. The analogous statement holds if $\mathbf{v}^2 = 0$. In particular, if $\theta_{\mathbf{v}}$ is a Hodge isometry then so it is $\theta_{\mathbf{v}'}$.

Proof. This follows from the same computations of [118, Propositions 2.4 and 2.5]. \square

We conclude this section with two concrete examples. These will be the main geometric input in the proof of Theorem 2.1.1: the Hilbert scheme is the base case to which we want to reduce, and moduli spaces of vector bundles on the fibers of an elliptic K3 surface will be Fourier-Mukai partners.

Example 2.3.11. Let X be a K3 surface, consider the vector $\mathbf{v} = (1, 0, 1 - n)$ with $n \geq 2$. A sheaf with class \mathbf{v} has trivial double dual, and the natural map $\mathcal{F} \rightarrow \mathcal{F}^{\vee\vee} \cong \mathcal{O}_X$ has cokernel of length n . Every such sheaf is torsion free with rank one, so it is Gieseker stable with respect to any polarization H . The natural map

$$M_H(\mathbf{v}) \rightarrow \mathrm{Hilb}^n(X)$$

$$\mathcal{F} \mapsto (\mathcal{F} \rightarrow \mathcal{F}^{\vee\vee}/\mathcal{F})$$

is an isomorphism, where $\text{Hilb}^n(X)$ is the Hilbert scheme of n points. The Mukai homomorphism is compatible with such identification, and for $\text{Hilb}^n(X)$ it is an isometry [12, Proposition 6.6 and Lemma 9.1].

Example 2.3.12. Let X be an elliptic K3 surface, assume that $\text{Pic } X = \mathbb{Z}\Sigma \oplus \mathbb{Z}f$, where f is the class of a fiber, and Σ is the class of a section. The intersection form with respect to this basis is

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix},$$

so the Picard group is a hyperbolic plane. Consider a Mukai vector

$$\mathbf{w} = (0, \alpha f, \beta), \text{ with } \alpha > 0, \beta \neq 0 \text{ and } \gcd(\alpha, \beta) = 1.$$

Let H be a generic polarization, see [99, Definition 2.3]. Since we are assuming $\beta \neq 0$ and $\gcd(\alpha, \beta) = 1$ (that is the class \mathbf{w} is primitive), every H -semistable sheaf with is also H -stable, as explained in [99, Remark 2.6]. It follows from the definition and the Grothendieck-Riemann-Roch Theorem that if E is a slope-stable bundle of rank α and degree β supported on a smooth fiber $C \in |f|$, then it is H -stable as a torsion sheaf on X . So the Gieseker moduli space $M := M_H(\mathbf{w})$ is a smooth, projective, symplectic surface.

2.4 Wall-crossing: Semirigid case

The objective of this section is to show that the statement of Theorem 2.1.1 is preserved under wall-crossing, when the Mukai vector \mathbf{v} is spherical ($\mathbf{v}^2 = -2$) or isotropic ($\mathbf{v}^2 = 0$). The precise setup is the following. We fix a K3 surface X , a primitive Mukai vector with $\mathbf{v}^2 = -2$ or $\mathbf{v}^2 = 0$, and $\mathcal{W} \subset \text{Stab}^\dagger(X)$ a wall for \mathbf{v} . We denote the adjacent chambers with \mathcal{C}_+ and \mathcal{C}_- , we also denote with σ_\pm a generic stability condition in \mathcal{C}_\pm , and with $\sigma_0 = (Z_0, \mathcal{P}_0)$ a generic stability condition on the wall. The following is the main result of this section.

Theorem 2.4.1. *Let X be a K3 surface, \mathbf{v} be a primitive vector, with $\mathbf{v}^2 = -2$ or $\mathbf{v}^2 = 0$. Let \mathcal{W} be a wall for the wall and chamber decomposition for \mathbf{v} .*

- (1) *If $\mathbf{v}^2 = -2$, then $M_{\sigma_+}(\mathbf{v}) \neq \emptyset$ implies $M_{\sigma_-}(\mathbf{v}) \neq \emptyset$.*
- (2) *If $\mathbf{v}^2 = 0$, then there exists a spherical, σ_0 -stable object S such that either ST_S or $\text{ST}_S^{\pm 2}$ induce an isomorphism $M_{\sigma_+} \xrightarrow{\sim} M_{\sigma_-}$.*

This immediately implies the invariance of Theorem 2.1.1 under wall-crossing for spherical and isotropic classes, see Corollary 2.4.11.

2.4.1 Lattice associated to the wall

The key tool to study wall-crossing is a rank two lattice $\mathcal{H}_{\mathcal{W}}$ associated to our setup. It was introduced in [8, Section 5], for the case of a vector with $\mathbf{v}^2 > 0$. In that case, $\mathcal{H}_{\mathcal{W}}$ is always hyperbolic, while if $\mathbf{v}^2 \leq 0$ it can also be negative semi-definite.

Definition 2.4.2. Define the lattice associated to \mathcal{W} as

$$\mathcal{H}_{\mathcal{W}} := \{\mathbf{w} \in H_{\text{alg}}^*(X, \mathbb{Z}) \mid \Im \frac{Z_0(\mathbf{w})}{Z_0(\mathbf{v})} = 0\},$$

where Z_0 is the central charge of a generic stability condition on the wall \mathcal{W} .

Proposition 2.4.3 ([8, Proposition 5.1]). *The lattice $\mathcal{H}_{\mathcal{W}}$ has the following properties.*

- (1) *It is a rank 2 primitive sublattice of $H_{\text{alg}}^*(X, \mathbb{Z})$.*
- (2) *For every σ_+ -stable object E of class \mathbf{v} , the Mukai vectors of its Harder-Narasimhan factors with respect to σ_- are contained in $\mathcal{H}_{\mathcal{W}}$.*
- (3) *If E is σ_0 -semistable of class \mathbf{v} , then its Harder-Narasimhan factors with respect to σ_- have Mukai vectors contained in $\mathcal{H}_{\mathcal{W}}$.*
- (4) *If E is σ_0 -semistable of class $\mathbf{v}(E) \in \mathcal{H}_{\mathcal{W}}$, then its Jordan-Holder factors have Mukai vector in $\mathcal{H}_{\mathcal{W}}$.*

Lemma 2.4.4. *The lattice $\mathcal{H}_{\mathcal{W}}$ is not positive definite.*

Proof. Acting with $\widetilde{GL}_2(\mathbb{R})$ we can assume σ_0 be such that $Z_0(\mathbf{v}) = -1$. Write $Z_0 = (-, \Omega)$, since $\sigma_0 \in \text{Stab}^\dagger(X)$ we have $\Omega \in \mathcal{P}(X)$, in particular $(\Im \Omega)^2 > 0$. By definition, the lattice $\mathcal{H}_{\mathcal{W}}$ is contained in the orthogonal complement to $\Im \Omega$. The Mukai lattice has signature $(2, \rho(X))$, hence the orthogonal to $\Im \Omega$ has signature $(1, \rho(X))$. This implies that $\mathcal{H}_{\mathcal{W}}$ contains classes with negative square, hence the thesis. \square

Remark 2.4.5. Notice that if $\mathcal{H}_{\mathcal{W}}$ were strictly negative definite, there would be at most two spherical classes up to sign, and no isotropic class. It is easy to see that in this case every spherical object with class in $\mathcal{H}_{\mathcal{W}}$ remains stable on the wall. Since we are interested in studying wall-crossing, we can restrict our attention to consider only hyperbolic or negative semi-definite case.

We are going to need a couple of technical lemmas, that we recall here.

Lemma 2.4.6 (Mukai's Lemma, [22, Lemma 5.2]). *Let $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ be a short exact sequence inside a heart $\mathcal{A} \subset D^b(X)$. If $\text{Hom}(A, B) = 0$, then*

$$\text{ext}^1(E, E) \geq \text{ext}^1(A, A) + \text{ext}^1(B, B).$$

Lemma 2.4.7. *Let $\mathcal{W} \subset \text{Stab}(X)$ be a wall for \mathbf{v} , $\sigma_0 \in \mathcal{W}$ a generic stability condition, and σ_+ a stability condition on one of the adjacent chambers. Consider a short exact sequence in \mathcal{A}_{σ_+}*

$$0 \rightarrow S \rightarrow E \rightarrow T^{\oplus a} \rightarrow 0,$$

where S and T are σ_0 -stable of the same phase, and $v(E) = \mathbf{v}$. Assume that $\varphi_{\sigma_+}(S) < \varphi_{\sigma_+}(E) < \varphi_{\sigma_+}(T)$, and $\text{Hom}(T, E) = 0$, then E is σ_+ -stable.

Proof. Assume E is not σ_+ -stable, and consider a σ_+ -stable destabilizing subobject $A \hookrightarrow E$ in \mathcal{A}_{σ_+} . By assumption we have $\varphi_{\sigma_+}(A) \geq \varphi_{\sigma_+}(E) > \varphi_{\sigma_+}(S)$. If $\varphi_{\sigma_+}(A) > \varphi_{\sigma_+}(T)$ we have $\text{Hom}(A, T) = 0$ by stability. Then, the morphism $A \hookrightarrow E$ factors via S , but $\text{Hom}(A, S) = 0$ by stability, which implies that $A = 0$. If $\varphi_{\sigma_+}(A) = \varphi_{\sigma_+}(T)$ the map $A \rightarrow T^{\oplus a}$ is non-zero if and only if $A = T$, because T is also σ_+ -stable. The case $A = T$ contradicts the assumption $\text{Hom}(T, E) = 0$. So the map $A \rightarrow T^{\oplus a}$ is zero, and we conclude as above that $A = 0$. So we must have

$$\varphi_{\sigma_+}(S) < \varphi_{\sigma_+}(A) < \varphi_{\sigma_+}(T).$$

This implies that A is σ_0 -semistable of the same σ_0 -phase as S and T . Since S and T are simple objects in the abelian category of σ_0 -semistable objects of their phase (i.e. they do not have proper subobjects), we see that A is an extension

$$0 \rightarrow S \rightarrow A \rightarrow T^{\oplus b} \rightarrow 0$$

or $A = T^{\oplus b}$ for some $0 < b < a$. The first case contradicts $\varphi_{\sigma_+}(A) \geq \varphi_{\sigma_+}(E)$ and the second one $\text{Hom}(T, E) = 0$. \square

2.4.2 Spherical Mukai vector

Here we prove part (1) of Theorem 2.4.1. Fix a Mukai vector \mathbf{v} with $\mathbf{v}^2 = -2$. Given a σ_+ -stable spherical object E with $v(E) = \mathbf{v}$, we want to construct a σ_- -stable spherical object E' with the same Mukai vector. The idea is to deform the stability condition σ_+ to a generic stability condition σ_0 on the wall \mathcal{W} and take the Jordan-Holder filtration of E . It turns out (Proposition 2.4.8) that E has only two Jordan-Holder factors, although they can appear multiple times. Call this two Jordan-Holder factors S and T , and their classes \mathbf{s} and \mathbf{t} . They are σ_0 -stable spherical objects, so they are σ_- -stable too, since the condition that an object is stable is open in $\text{Stab}(X)$. To construct the desired object E' , we will construct inductively, starting from S and T , a σ_- -stable spherical object with class \mathbf{v}' for every spherical \mathbf{v}' that is a linear combination of \mathbf{s} and \mathbf{t} with positive coefficients. Since E has a Jordan-Holder filtration with factors S and T , its class \mathbf{v} is of that form.

Proposition 2.4.8. *Let $E \in M_{\sigma_+}(\mathbf{v})$. Assume that it is not stable on the wall. Then there are two σ_0 -stable spherical objects that appear as Jordan-Holder factors of E , possibly with multiplicity.*

Proof. Assume that E gets destabilized. From Lemma 2.4.6 it follows that its Jordan-Holder factors with respect to σ_0 are all spherical. Since \mathbf{v} is primitive, it must have at least two different factors S, T , call their classes \mathbf{s}, \mathbf{t} . Since S, T are stable and non isomorphic we have $(s, t) = \text{ext}^1(S, T) \geq 0$. This in turn implies that \mathbf{s} and \mathbf{t} are linearly independent over \mathbb{R} . Indeed, if we could write $s = \lambda t$, then λ would be positive, because S and T have the same σ_0 -phase, hence $(\mathbf{s}, \mathbf{t}) = -2\lambda < 0$. The argument to show that these are the only Jordan-Holder factors is different in the case when $\mathcal{H}_{\mathcal{W}}$ is semi-definite and in the case when it is hyperbolic.

Semi-definite case. From the linear independence it follows that $(\mathbf{s}, \mathbf{t}) = 2$. The spherical classes of σ_0 -stable objects of the same phase as E lie on two parallel half-lines, as shown in Figure 2.1(A). Furthermore, the product of two classes is positive if and only if they lie on different lines. We conclude that, up to shifts, S and T are the only two σ_0 -stable spherical objects with classes in $\mathcal{H}_{\mathcal{W}}$.

Hyperbolic case. In this case we have $m := (\mathbf{s}, \mathbf{t}) \geq 3$. Then, by the following argument from [8], we see again that, up to shifts, S and T are the only two σ_0 -stable spherical objects. Assume \mathbf{r} is the class of another σ_0 -stable spherical object, we can write $\mathbf{r} = x\mathbf{s} + y\mathbf{t}$. We see that

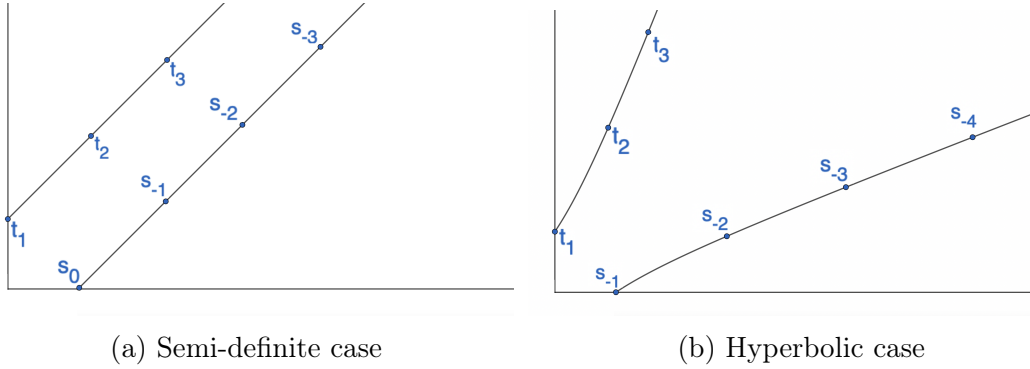
$$\begin{aligned} (\mathbf{s}, \mathbf{r}) \geq 0 &\implies y \geq \frac{2x}{m} \\ (\mathbf{t}, \mathbf{r}) \geq 0 &\implies y \leq \frac{mx}{2} \\ (\mathbf{r}, \mathbf{r}) = -2 &\implies -2x^2 + 2mxy - 2y^2 = -2, \end{aligned}$$

which is easily seen to be contradictory. \square

Remark 2.4.9. Assume that E is σ_+ -stable, spherical and not stable on the wall. Proposition 2.4.8 gives two spherical classes \mathbf{s}, \mathbf{t} . They are a basis for $\mathcal{H}_{\mathcal{W}, \mathbb{R}}$, and the class $\mathbf{v} = v(E)$ is a linear combination of \mathbf{s} and \mathbf{t} with positive coefficients. Writing the quadratic form with respect to the basis $\{\mathbf{s}, \mathbf{t}\}$ we get:

$$\begin{aligned} &-2x^2 + 4xy - 2y^2 \quad \text{in the semi-definite case,} \\ &-2x^2 + 2mxy - 2y^2, \text{ with } m := (\mathbf{s}, \mathbf{t}) > 2 \quad \text{in the hyperbolic case.} \end{aligned}$$

In both cases there are infinitely many spherical classes in the lattice $\langle \mathbf{s}, \mathbf{t} \rangle$ spanned by \mathbf{s} and \mathbf{t} . In the hyperbolic case there are no isotropic classes, because $\sqrt{m^2 - 4}$ is irrational if $m \geq 3$. The spherical classes live on two branches of a hyperbola in the hyperbolic case, and on two parallel lines in the semi-definite case.

Figure 2.1: Spherical classes in $\langle \mathbf{s}, \mathbf{t} \rangle$

Assume without loss of generality that $\varphi_{\sigma_+}(\mathbf{t}) > \varphi_{\sigma_+}(\mathbf{s})$. Consider the spherical classes that are linear combination of \mathbf{s} and \mathbf{t} with positive coefficient, ordered with respect to σ_+ phase. Call \mathbf{t}_i for $i \geq 1$ the classes on the upper branch, and \mathbf{s}_i for $i \leq 0$ the classes on the lower branch, as shown in the Figure 2.1. They can also be defined inductively by

$$\begin{cases} \mathbf{t}_1 = \mathbf{t} \\ \mathbf{t}_2 = \rho_{\mathbf{s}}(\mathbf{s}), \\ \mathbf{t}_{i+1} = -\rho_{\mathbf{t}_i}(\mathbf{t}_{i-1}). \end{cases} \quad \begin{cases} \mathbf{s}_0 = \mathbf{s} \\ \mathbf{s}_{-1} = \rho_{\mathbf{s}}(\mathbf{t}), \\ \mathbf{s}_{-i-1} = -\rho_{\mathbf{s}_{-i}}(\mathbf{s}_{-i+1}). \end{cases}$$

where $\rho_{\mathbf{s}}(\mathbf{v}) := \mathbf{v} + (\mathbf{s}, \mathbf{v})\mathbf{s}$. This is clear in the semi-definite case, because \mathbf{t}_i is the midpoint of the segment $\overline{\mathbf{t}_{i-1}, \mathbf{t}_{i+1}}$, and is also easy to see in the hyperbolic case by writing down the previous reflections in coordinates with respect to \mathbf{s} and \mathbf{t} .

With this we are ready to show the first part of Theorem 2.4.1.

Proof of Theorem 2.4.1(1). Let $E \in M_{\sigma_+}(\mathbf{v})$, we want to show that there exists a σ_- -stable object with Mukai vector \mathbf{v} . Let φ be the phase of E with respect to σ_0 ; we can assume up to shifts that $0 < \varphi \leq 1$. Assume E is not stable on the wall, otherwise we are done. From Proposition 2.4.8 and Remark 2.4.9 we get that $\mathbf{v} = \mathbf{t}_i$ or $\mathbf{v} = \mathbf{s}_{-i}$ for some i . Assume $\mathbf{v} = \mathbf{s}_{-i}$, the other case is analogous. We prove existence of σ_- -stable objects of class \mathbf{s}_{-i} by induction on i . Proposition 2.4.8 implies that there is a σ_0 -stable object S of class $\mathbf{s}_0 = \mathbf{s}$, and a σ_0 -stable object T of class \mathbf{t}_1 . Define S_{-i}^- inductively as

$$S_{-i-1}^- := \begin{cases} \mathrm{ST}_S(T) & \text{if } i = 0, \\ \mathrm{ST}_{S_{-i}^-}(S_{-i+1}^-)[-1] & \text{if } i > 0 \end{cases}$$

By stability of S and T we have a short exact sequence in $\mathcal{P}(\varphi)$

$$0 \rightarrow T \rightarrow \mathrm{ST}_S(T) \rightarrow \mathrm{Ext}^1(S, T) \otimes S \rightarrow 0.$$

Since S, T are simple in the abelian category $\mathcal{P}(\varphi)$ and $\varphi_{\sigma_-}(T) < \varphi_{\sigma_-}(S)$, we can apply Lemma 2.4.7 and conclude that S_{-1}^- is σ_- -stable. Furthermore, if we take σ_- close to the wall, S and T lie in the heart $\mathcal{A}_{\sigma_-} = \mathcal{P}_{\sigma_-}(0, 1]$, and so does S_{-1}^- .

Now, assume by induction that S_{-j}^- is σ_- -stable for every $j \leq i$, and that it lies in the heart \mathcal{A}_{σ_-} . We want to show that the same holds for S_{-i-1}^- . First we claim that $\mathrm{RHom}(S_{-i}^-, S_{-i+1}^-)$ is concentrated in degree zero. Indeed S_{-i}^-, S_{-i+1}^- are two σ_- -stable objects with $\varphi_{\sigma_-}(S_{-i}^-) < \varphi_{\sigma_-}(S_{-i+1}^-)$. Therefore $\mathrm{Hom}^2(S_{-i}^-, S_{-i+1}^-)$ vanishes by stability and Serre duality. From the inductive definition and Serre duality we get

$$\begin{aligned} \mathrm{ext}^1(S_{-i}^-, S_{-i+1}^-) &= \mathrm{ext}^1(S_{-i+1}^-, S_{-i}^-) \\ &= \mathrm{hom}(S_{-i+1}^-, \mathrm{ST}_{S_{-i+1}^-}(S_{-i+2}^-)) \\ &= \mathrm{hom}(\mathrm{ST}_{S_{-i+1}^-}^{-1}(S_{-i+1}^-), S_{-i+2}^-) \\ &= \mathrm{hom}(S_{-i+1}^-[1], S_{-i+2}^-) \end{aligned}$$

which is zero because it is a negative Ext between two objects of a heart.

This shows that we have the exact triangle

$$S_{-i-1}^- \rightarrow \mathrm{Hom}(S_{-i}^-, S_{-i+1}^-) \otimes S_{-i}^- \rightarrow S_{-i+1}^- \rightarrow S_{-i-1}^-[1] \quad (2.2)$$

Taking the long exact sequence of cohomology with values in the heart $\mathcal{A}_{\sigma_-} = \mathcal{P}_{\sigma_-}(0, 1]$, we see that $S_{-i-1}^- \in \mathcal{P}_{\sigma_-}(0, 2]$. Now let F be a σ_- -stable object with bigger phase $\varphi_{\sigma_-}(F) > \varphi_{\sigma_-}(S_{-i-1}^-)$, we want to show that $\mathrm{Hom}(F, S_{-i-1}^-) = 0$, which will prove σ_- -semistability of S_{-i-1}^- .

Applying the functor $\mathrm{Hom}(F, -)$ to the triangle 2.2 we get the exact sequence:

$$\mathrm{Hom}(F, S_{-i+1}^-[-1]) \rightarrow \mathrm{Hom}(F, S_{-i-1}^-) \rightarrow \mathrm{Hom}(S_{-i}^-, S_{-i+1}^-) \otimes \mathrm{Hom}(F, S_{-i}^-). \quad (2.3)$$

Assume that $F \in \mathcal{A}_{\sigma_-}[n]$ with $n > 0$. By induction hypothesis S_{-i+1}^- and S_{-i}^- lie in the heart \mathcal{A}_{σ_-} . Therefore, the first and the third terms vanish because they are negative Ext between objects of a heart, so the middle one does too.

If $F \in \mathcal{A}_{\sigma_-}$ is an object of the heart with bigger σ_- -phase than S_{-i-1}^- , then its class \mathbf{f} lies under the line $\overline{0, s_{-i-1}}$ in $\mathcal{H}_{\mathcal{W}, \mathbb{R}}$, because the ordering of the phases with respect to σ_- is opposite than the one induced by σ_+ . In particular, outside of the closed positive cone, so it has negative square $\mathbf{f}^2 < 0$. Since F is stable, it must be spherical, because $\mathbf{f}^2 \geq -2$. So $F \in \{S_0, \dots, S_{-i}^-\}$ as can be seen in Figure 2.1. If $F \in \{S_0, \dots, S_{-i+1}^-\}$ we conclude by the inductive assumption and the exact sequence (2.3), so the only case to check is $F = S_{-i}^-$. We have

$$\mathrm{Hom}(S_{-i}^-, \mathrm{ST}_{S_{-i}^-}(S_{-i+1}^-)[-1]) = \mathrm{Hom}(S_{-i}^-[2], S_{-i+1}^-)$$

which is again zero because it is a negative Ext between objects of a heart. This shows that S_{-i-1}^- is σ_- -semistable, since $\varphi_{\sigma_-}(T) < \varphi_{\sigma_-}(S_{-i-1}^-) < \varphi_{\sigma_-}(S)$ it also

lies in \mathcal{A}_σ . Now to show that is σ_- -stable, consider its Jordan-Holder filtration. Every factor must be a spherical object of the same phase, by Mukai's Lemma. Since the line connecting the origin to \mathbf{v} meets the lower branch of the hyperbola only in \mathbf{v} , there is only one σ_- -stable spherical object of that phase up to shifts, so the Jordan-Holder filtration is trivial. \square

2.4.3 Isotropic Mukai vector

Now we turn our attention on moduli spaces $M_{\sigma_+}(\mathbf{v})$ with vector \mathbf{v} with $\mathbf{v}^2 = 0$. The lattice $\mathcal{H}_\mathcal{W}$ can be negative semi-definite or hyperbolic, and in the latter case there is only one spherical class up to sign. Indeed, if there were two linearly independent spherical classes, the argument in Remark 2.4.9 would show that there could be no isotropic classes. In contrast to the spherical case, where the proof works the same in both cases, if \mathbf{v} is isotropic the signature of the lattice $\mathcal{H}_\mathcal{W}$ matters. In Proposition 2.4.10 we describe the Jordan-Holder filtration of a stable object E with $v(E) = \mathbf{v}$ with respect to σ_0 . This result is analogous to [22, Theorem 12.1], where Bridgeland studies wall-crossing for $\mathbf{v} = (0, 0, 1)$ and \mathcal{W} a wall on the boundary of the geometric chamber $U(X)$. In fact, assuming non-emptiness of moduli spaces with isotropic vector, Proposition 2.4.10 follows from Bridgeland's result via a Fourier-Mukai argument, as shown in [8, Lemma 8.1].

Proposition 2.4.10. *Keeping notation as above we have:*

- (1) *If $\mathcal{H}_\mathcal{W}$ is semi-definite then there is a smooth rational C curve inside $M_{\sigma_+}(\mathbf{v})$ that becomes σ_- -unstable, and the Jordan-Holder filtration for $E \in C$ with respect to σ_0 is of the form*

$$0 \rightarrow S \rightarrow E \rightarrow T \rightarrow 0,$$

where S and T are two σ_0 -stable spherical objects.

- (2) *If $\mathcal{H}_\mathcal{W}$ is hyperbolic the wall \mathcal{W} is totally semistable, i.e. every object $E \in M_{\sigma_+}(\mathbf{v})$ becomes σ_0 -semistable. The Jordan-Holder filtration of an object $E \in M_{\sigma_+}(\mathbf{v})$ is*

$$0 \rightarrow S^{\oplus a} \rightarrow E \rightarrow F \rightarrow 0 \quad \text{or} \quad 0 \rightarrow F \rightarrow E \rightarrow S^{\oplus a} \rightarrow 0,$$

where S is a σ_0 -stable spherical object, and F is a σ_0 -stable isotropic object. Moreover, in both cases the Jordan-Holder filtration with respect to σ_0 coincides with the Harder-Narasimhan filtration with respect to σ_- .

Proof. We begin by proving part (1). Since $\mathcal{H}_\mathcal{W}$ is a negative semi-definite lattice of rank two, the isotropic classes in $\mathcal{H}_{\mathcal{W}, \mathbb{R}}$ form a one dimensional subspace, which

is the radical of the Mukai pairing. Therefore there is at most one, up to a sign, primitive isotropic class in $\mathcal{H}_{\mathcal{W}}$. Assume that $E \in M_{\sigma_+}(\mathbf{v})$ becomes semistable on \mathcal{W} . From Lemma 2.4.6 it follows that its Jordan-Holder factors are spherical or isotropic, with at most one being isotropic.

Since there is just one primitive isotropic class, this means that all the Jordan-Holder factors are spherical, in particular there are two distinct σ_0 -stable spherical objects S, T . The only isotropic class is $\mathbf{s} + \mathbf{t}$, hence the Jordan-Holder filtration is

$$0 \rightarrow S \rightarrow E \rightarrow T \rightarrow 0,$$

where we assume $\varphi_{\sigma_+}(S) < \varphi_{\sigma_+}(T)$. We have $\text{ext}^1(S, T) = (\mathbf{s}, \mathbf{t}) = 2$, and every non trivial extension gives a σ_+ -stable object by Lemma 2.4.7. So there is a rational curve $\mathbb{P}(\text{Ext}^1(S, T)) \subset M_{\sigma_+}(\mathbf{v})$ of objects that become semistable on the wall. Notice also that the Jordan-Holder filtration with respect to σ_0 coincides with the Harder-Narasimhan filtration with respect to σ_- , because S, T are σ_- -stable with $\varphi_{\sigma_-}(S) > \varphi_{\sigma_-}(T)$.

Now we prove part (2). First we show the second part of the statement, so let E be σ_0 -semistable. Lemma 2.4.6 implies that the only objects that can appear as Jordan-Holder factors are spherical or isotropic, with at most one being isotropic. Furthermore, from the discussion in Remark 2.4.9, we see that if $\mathcal{H}_{\mathcal{W}}$ contains an isotropic class, then it contains at most one spherical class up to a sign. Therefore there is a unique σ_0 -stable spherical object of the same phase as E . Hence all the Jordan-Holder spherical factors are of the form $S^{\oplus a}$. This implies that the Jordan-Holder filtration is of the form

$$0 \rightarrow S^{\oplus a} \rightarrow E \rightarrow F \rightarrow 0 \quad \text{or} \quad 0 \rightarrow F \rightarrow E \rightarrow S^{\oplus a} \rightarrow 0,$$

with F isotropic and σ_0 -stable. Which one it is depends on the ordering of the phases: it is the first one if $\varphi_{\sigma_+}(S) < \varphi_{\sigma_+}(E)$ and the second one if $\varphi_{\sigma_+}(E) > \varphi_{\sigma_+}(S)$. We can assume to be in the first case, the proof of the other is analogous.

As in the previous case, since S and F are σ_0 -stable, they are also σ_- -stable, so the Harder-Narasimhan filtration coincides with the Jordan-Holder filtration on the wall.

To show that the wall is totally semistable we argue as follows. The Jordan-Holder filtration implies that we can write $\mathbf{v} = \mathbf{s} + \mathbf{w}$ where $\mathbf{w} := v(F)$. We have

$$\mathbf{v}^2 = 0 = -2a^2 + 2a(\mathbf{s}, \mathbf{w}),$$

hence $a = (\mathbf{s}, \mathbf{w})$. The spaces $\text{Hom}(S, F)$ and $\text{Hom}(F, S)$ vanish for σ_0 -stability, hence $a = \text{ext}^1(S, F)$. Applying $\text{Hom}(S, -)$ to the Jordan-Holder filtration we see that $\text{hom}(S, E) = a$ and we get the exact sequence:

$$0 \rightarrow \text{Ext}^1(S, E) \rightarrow \text{Ext}^1(S, F) \rightarrow \text{Ext}^2(S, S)^{\oplus a} \rightarrow \text{Ext}^2(S, E). \quad (2.4)$$

By Serre Duality and σ_+ -stability the last space is 0, which implies that

$$\mathrm{Ext}^1(S, F) \cong \mathrm{Ext}^2(S, S)^{\oplus a}$$

because they have the same dimension. Thus $\mathrm{Ext}^1(S, E) = 0$, and $\mathrm{RHom}(S, E) = \mathrm{Hom}(S, E)$. This implies $(\mathbf{s}, \mathbf{v}) = -\mathrm{hom}(S, E) = -a < 0$. In particular, for every object $E' \in M_{\sigma_+}(\mathbf{v})$ there are non zero morphisms $\mathrm{Hom}(S, E') \neq 0$. Hence, every $E' \in M_{\sigma_+}(\mathbf{v})$ is σ_0 -semistable. \square

We can finish the proof of Theorem 2.1.1 of this section.

Proof of Theorem 2.4.1(2). We separate the proof in two cases, depending on the signature of \mathcal{H}_W . If \mathcal{H}_W is negative semi-definite, we want to show that the spherical twist ST_S induces an isomorphism $M_{\sigma_+}(\mathbf{v}) \xrightarrow{\sim} M_{\sigma_-}(\mathbf{v})$, where S is the spherical object of Proposition 2.4.10. If \mathcal{H}_W is hyperbolic, we want to show that $\mathrm{ST}_S^{\pm 2}$ induces an isomorphism $M_{\sigma_+}(\mathbf{v}) \xrightarrow{\sim} M_{\sigma_-}(\mathbf{v})$, where S is the spherical object of Proposition 2.4.10 and the sign depends on the ordering of the phases $\varphi_{\sigma_+}(S)$ and $\varphi_{\sigma_+}(E)$.

Semi-definite case. Consider the destabilizing spherical object S of Proposition 2.4.10. We claim that

- (1) If $E \in M_{\sigma_+}(\mathbf{v})$ remains stable on the wall, then $\mathrm{ST}_S(E) = E$,
- (2) If $E \in M_{\sigma_+}(\mathbf{v})$ becomes semistable on the wall, then $\mathrm{ST}_S(E)$ is σ_- -stable.

To show (1) observe that if E remain stable on the wall, then σ_0 -stability gives $\mathrm{Hom}(E, S) = \mathrm{Hom}(S, E) = 0$. Since $(\mathbf{s}, \mathbf{v}) = 0$, we also get $\mathrm{RHom}(S, E) = 0$. It follows from the definition of spherical twist that then $\mathrm{ST}_S(E) = E$.

To show (2), consider the Jordan-Holder filtration

$$0 \rightarrow S \rightarrow E \rightarrow T \rightarrow 0.$$

Applying $\mathrm{Hom}(S, -)$ to the Jordan-Holder filtration we see that $\mathrm{Hom}(S, E) \cong \mathrm{Hom}(S, S)$ is one dimensional. By σ_+ -stability we get $\mathrm{Hom}(E, S) = 0$, and from $(\mathbf{s}, \mathbf{v}) = 0$ we see $\mathrm{ext}^1(S, E) = 1$. The definition of spherical twist gives a distinguished triangle:

$$S \oplus S[-1] \rightarrow E \rightarrow \mathrm{ST}_S(E) \rightarrow S[1] \oplus S$$

Taking the long exact sequence with respect to the heart \mathcal{A}_{σ_0} we get the two short exact sequences:

$$\begin{aligned} 0 \rightarrow S \rightarrow E \rightarrow R \rightarrow 0 \\ 0 \rightarrow R \rightarrow \mathrm{ST}_S(E) \rightarrow S \rightarrow 0 \end{aligned}$$

The first one shows $R = T$ and it is the Jordan-Holder filtration. The second one then becomes $T \rightarrow \mathrm{ST}_S(E) \rightarrow S$ which shows that $\mathrm{ST}_S(E)$ is σ_- -stable

using Lemma 2.4.7. Starting from $F \in M_{\sigma_-}(\mathbf{v})$, the filtration is reversed, and the analogous argument shows that $\mathrm{ST}_S^{-1}(F)$ is σ_+ -stable. In conclusion, passing to moduli spaces as in Remark 2.3.9, we see that ST_S induces an isomorphism $M_{\sigma_+}(\mathbf{v}) \xrightarrow{\sim} M_{\sigma_-}(\mathbf{v})$.

Hyperbolic case. Let $E \in M_{\sigma_+}(\mathbf{v})$ and assume $\varphi_{\sigma_+}(S) < \varphi_{\sigma_+}(E)$, the other case being proved similarly. Proposition 2.4.10 gives the Jordan-Holder filtration with respect to σ_0 :

$$0 \rightarrow S^{\oplus a} \rightarrow E \rightarrow F \rightarrow 0.$$

Applying $\mathrm{Hom}(S, -)$ to it we get the exact sequence (2.4). In the same way as before we deduce that $\mathrm{RHom}(S, E) = \mathrm{Hom}(S, E) \cong \mathrm{Hom}(S, S)^{\oplus a}$. By choosing a basis for $\mathrm{Hom}(S, E)$ one sees that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(S, E) \otimes S & \xrightarrow{\mathrm{ev}} & E \\ \downarrow \cong & & \parallel \\ S^{\oplus a} & \longrightarrow & E \end{array}$$

In particular, this gives an isomorphism $\mathrm{ST}_S(E) \cong F$.

Now, we have the two distinguished triangles

$$\begin{aligned} \mathrm{Hom}(S, E) \otimes S \rightarrow E \rightarrow \mathrm{ST}_S(E) \rightarrow \mathrm{Hom}(S, E) \otimes S[1] \\ \mathrm{ST}_S(E) \rightarrow \mathrm{ST}_S^2(E) \rightarrow \mathrm{Hom}(S, E) \otimes S \rightarrow \mathrm{ST}_S(E)[1], \end{aligned}$$

where the first one is obtained by definition, and the second one applying the equivalence ST_S to the first. Since $\mathrm{ST}_S(E) \cong F$, we conclude that $\mathrm{ST}_S^2(E)$ is σ_- -stable by Lemma 2.4.7. An analogous argument shows that ST_S^{-2} sends σ_- -stable objects with vector \mathbf{v} to σ_+ -stable objects with vectors \mathbf{v} . Passing to moduli spaces as in Remark 2.3.9 we see that ST_S^2 induces an isomorphism $M_{\sigma_+} \xrightarrow{\sim} M_{\sigma_-}$. \square

Corollary 2.4.11. *Let X be a K3 surface, $\mathbf{v} \in H_{\mathrm{alg}}^*(X, \mathbb{Z})$ primitive with $\mathbf{v}^2 = -2$ or $\mathbf{v}^2 = 0$. Let $\mathcal{W} \subset \mathrm{Stab}^\dagger(X)$ be a wall, $\sigma_0 \in \mathcal{W}$ a generic stability condition on the wall, and σ_\pm generic stability conditions on the adjacent chambers. Then Theorem 2.1.1 holds for $M_{\sigma_+}(\mathbf{v})$ if and only if it holds for $M_{\sigma_-}(\mathbf{v})$.*

Proof. If $\mathbf{v}^2 = -2$ we have to show that if $M_{\sigma_+}(\mathbf{v})$ is a single point, the same is true for $M_{\sigma_-}(\mathbf{v})$. Point (1) of Theorem 2.4.1 gives $M_{\sigma_-}(\mathbf{v}) \neq \emptyset$, so we only have to show uniqueness of stable spherical objects with class \mathbf{v} . Assume that there are two non isomorphic spherical stable objects E, E' with the same vector \mathbf{v} . Up to shift we can assume they are both in the heart of σ_- . By stability, we have $\mathrm{Hom}(E, E') = \mathrm{Hom}^2(E, E') = 0$. Since they are in the heart, we have $0 \leq \mathrm{hom}^1(E, E') = \mathbf{v}^2 = -2$, which is a contradiction.

If $\mathbf{v}^2 = 0$ and primitive, we want to show that if $M_{\sigma_+}(\mathbf{v})$ is a K3 surface, and

$$\theta_{\mathbf{v}} : \mathbf{v}^\perp / \mathbb{Z}\mathbf{v} \rightarrow H^2(M_{\sigma_+}(\mathbf{v}), \mathbb{Z})$$

is a Hodge isometry, the same is true for $M_{\sigma_-}(\mathbf{v})$. This follows from part (2) of Theorem 2.4.1 combined with Proposition 2.3.10. \square

2.5 Wall-crossing for the Hilbert Scheme

In this section we study wall-crossing for the Hilbert scheme of n points on a K3 surface of Picard rank one when the degree is high with respect to the number of points. Of course, this setting is less general than the previous one; nevertheless thanks to the argument in Section 2.6 we will be able to reduce to this case. Throughout this section we assume X is a K3 surface with $\text{Pic}(X) = \mathbb{Z} \cdot H$ with $H^2 = 2d$ and $d = k^2(n-1)$, where $k \in \mathbb{Z}, k > 1$, the Mukai vector is $\mathbf{v} = (1, 0, 1-n)$, and the stability condition is $\sigma_{\alpha, \beta} \in V(X)$. The notation $\sigma_{\alpha, \beta}$ denotes the stability condition $\sigma_{\alpha H, \beta H} \in V(X)$, see Section 2.2.2 for the definition. In particular, the heart is $\text{Coh}^{\alpha H, \beta H}(X)$ and the central charge is

$$Z_{\alpha, \beta}(\mathbf{u}) := (\mathbf{u}, \exp(\beta H + i\alpha H)) = dr(\alpha^2 - \beta^2) + 2dc\beta - s + 2id(c - r\beta)\alpha,$$

where $\mathbf{u} = (r, cH, s)$. One can check that the heart does not depend on α ; in this section we will denote it just by $\text{Coh}^\beta(X)$. It is a stability condition for $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$, provided that $\Im Z(E) \neq 0$ for every spherical torsion-free sheaf E . Hence, the domain $V(X)$ is identified with the upper half plane $\mathbb{R} \times \mathbb{R}_{>0}$ with some isolated points removed. The following is the main result of this section.

Theorem 2.5.1. *Let X be a K3 surface with $\text{Pic}(X) = \mathbb{Z} \cdot H$ with $H^2 = 2d$ and $d = k^2(n-1)$ for $k > 1$ integer. Then $M_{\sigma_{\alpha, \beta}}(\mathbf{v}) = \text{Hilb}^n(X)$ for every $\sigma_{\alpha, \beta} \in V(X)$ with $\beta < 0$, and $\mathbf{v} = (1, 0, 1-n)$.*

For convenience in this section we work with the slope $\nu_{\alpha, \beta}$, instead of the phase $\varphi_{\alpha, \beta}$. It is defined for objects $E \in \text{Coh}^\beta(X)$ as

$$\nu_{\alpha, \beta}(E) := \begin{cases} -\frac{\Re Z_{\alpha, \beta}(\mathbf{u})}{\Im Z_{\alpha, \beta}(\mathbf{u})} = \frac{dr(\beta^2 - \alpha^2) - 2dc\beta + s}{2d(c - r\beta)\alpha} & \text{if } \Im Z_{\alpha, \beta}(\mathbf{u}) \neq 0 \\ +\infty & \text{if } \Im Z_{\alpha, \beta}(\mathbf{u}) = 0 \end{cases},$$

where $\mathbf{u} = v(E) = (r, cH, s)$. It is related to the phase via the formula

$$\varphi_{\alpha, \beta}(E) = \frac{1}{\pi} \cot^{-1}(\nu_{\alpha, \beta}(E)),$$

so it gives the same notion of stability on $\text{Coh}^\beta(X)$. If \mathcal{W} is a wall for \mathbf{v} , and \mathbf{u} is the class of a destabilizing object, we can recover the equation of the wall by $\nu_{\alpha,\beta}(\mathbf{u}) = \nu_{\alpha,\beta}(\mathbf{v})$. Writing $\mathbf{u} = (r, cH, s)$ and expanding this equation we get

$$cd(\alpha^2 + \beta^2) - \beta(r(n-1) + s) + c(n-1) = 0.$$

The following is a well know fact, see [75, Proposition 3.7].

Theorem 2.5.2 (Bertram Nested Wall Theorem). *Let $\mathbf{v} = (r, cH, s) \in H_{\text{alg}}^*(X, \mathbb{Z})$, with $\mathbf{v}^2 > 0$. The walls in $V(X)$ are either semicircles with center in the β -axis or lines parallel to the α -axis. If $r \neq 0$ there is a unique vertical wall at $\beta = \mu(\mathbf{v})$, and there are two sets of nested semicircular walls, one on each side of the vertical wall.*

We are interested in the walls for the vector $\mathbf{v} = (1, 0, 1 - n)$ of the Hilbert scheme. In this case the above Theorem tells us that there is a vertical wall on the line $\beta = 0$; this wall corresponds to the Hilbert-Chow contraction. The moduli space $M_{\alpha,\beta}(\mathbf{v})$ with $\alpha \gg 0$ and $\beta < 0$ is the Hilbert scheme of points $\text{Hilb}^n(X)$, thanks to Theorem 2.2.25. It parameterizes ideal sheaves of subschemes $Y \subset X$ of dimension 0 and length n . On the vertical wall two ideal sheaves become S -equivalent if and only if the corresponding subschemes have the same support.

Recall that to a wall \mathcal{W} we associate the rank two lattice $\mathcal{H}_{\mathcal{W}}$ given by all the classes \mathbf{u} with $Z(\mathbf{u})$ on the same line of $Z(\mathbf{v})$, see Definition 2.4.2. Lemma 2.4.4 implies that $\mathcal{H}_{\mathcal{W}}$ is hyperbolic, since it contains the class \mathbf{v} , which has positive square by assumption.

The idea of the proof of Theorem 2.5.1 is simple. First, we show that the vertical line $\beta = -\frac{1}{k}$ does not meet any semicircular wall. This is because the imaginary part $\Im Z_{t, -\frac{1}{k}}(E)$ takes non-negative integer values (up to a constant) when E varies in $\text{Coh}^\beta(X)$, and $\Im Z_{t, -\frac{1}{k}}(\mathbf{v})$ is the minimal positive value. This is completely analogous to the fact that rank one torsion free sheaves are Gieseker stable with respect to any polarization.

Then we show that, in fact, the line $\beta = -\frac{1}{k}$ should meet every semicircular wall in the left quadrant. This implies that there are no semicircular walls in the left quadrant.

Lemma 2.5.3 ([9, Examples 9.7 and 10.5]). *The stability condition $\sigma_{t, -\frac{1}{k}}$ exists for every $t > 0$, and the moduli space $M_{\sigma_{t, -\frac{1}{k}}}(1, 0, 1 - n)$ is equal to the Hilbert scheme $\text{Hilb}^n(X)$*

Proof. We have

$$\Im Z_{t, -\frac{1}{k}}(r, cH, s) = 2dt \frac{ck + r}{k} \in \frac{2td}{k} \mathbb{Z}, \quad (2.5)$$

for any vector (r, cH, s) . First we show that the stability condition $\sigma_{t, -\frac{1}{k}}$ is defined for every $t > 0$. This means that there is no spherical class $\mathbf{u} = (r, cH, s)$ such that $\Im Z_{t, -\frac{1}{k}}(\mathbf{u}) = 0$. If there were one, it would satisfy

$$\begin{cases} r = -ck \\ 2dc^2 = 2rs - 2 \end{cases}$$

Substituting $d = k^2(n - 1)$ gives a contradiction with $k > 1$.

Now assume that an object E of class $(1, 0, 1 - n)$ becomes strictly semistable for some stability condition $\sigma_{t, -\frac{1}{k}}$. We have a destabilizing short exact sequence in $\text{Coh}^\beta(X)$

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0,$$

with $\nu_{t, -\frac{1}{k}}(F) = \nu_{t, -\frac{1}{k}}(E) = \nu_{t, -\frac{1}{k}}(G) < +\infty$. By definition of Bridgeland stability condition, we have

$$0 \leq \Im Z_{t, -\frac{1}{k}}(F) \leq \Im Z_{t, -\frac{1}{k}}(E) = \frac{2dt}{k},$$

and similarly for G . The equality (2.5) implies that $\Im Z_{t, -\frac{1}{k}}(F) = 0$ or $\Im Z_{t, -\frac{1}{k}}(G) = 0$. In both cases this contradicts the finiteness of the slopes. \square

Lemma 2.5.4. *The vertical line $\beta = -\frac{1}{k}$ meets every semicircular wall in the left quadrant $\beta < 0$.*

Proof. Consider a destabilizing exact sequence on the wall \mathcal{W} :

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0.$$

The equation of the wall is $\nu_{\alpha, \beta}(F) = \nu_{\alpha, \beta}(E)$. Call $\mathbf{u} = v(F)$ and $\mathbf{v} = (1, 0, 1 - n)$. To simplify the computations we change \mathbf{u} in the lattice $\mathcal{H}_{\mathcal{W}}$ with a vector of rank zero, such that the equation of the wall is still given by $\nu_{\alpha, \beta}(\mathbf{v}) = \nu_{\alpha, \beta}(\mathbf{u})$. To do this is sufficient to take \mathbf{u} as the Mukai vector of a semistable object in the heart $\text{Coh}^\beta(X)$ of the same slope as E , with respect to every stability condition on the wall \mathcal{W} . We do it as follows.

- If $r = 0$, we do not change \mathbf{u} .
- If $r > 0$ take $\mathbf{u} := v(E^{\oplus r-1} \oplus G)$.
- If $r < 0$ take $\mathbf{u} := v(E^{\oplus -r} \oplus F)$.

Write $\mathbf{u} = (0, cH, s)$, the equation of the wall and the β coordinate of the center become:

$$\begin{cases} cd(\alpha^2 + \beta^2) - \beta s + c(n - 1) = 0, \\ \beta_0 = \frac{s}{2cd}. \end{cases} \quad (2.6)$$

Since \mathbf{u} is the class of an object in the heart $\text{Coh}^\beta(X)$, it satisfies

$$\Im Z_{\alpha,\beta}(\mathbf{u}) = 2dc\alpha \geq 0.$$

Since $\alpha > 0$ this gives $c \geq 0$. If $c = 0$ we would get the Hilbert-Chow wall, so we have $c > 0$. The center of any semicircular wall is on the negative β -axis, hence $\beta_0 < 0$. The above equation gives $s < 0$.

The classes \mathbf{u} and \mathbf{v} are linearly independent over \mathbb{R} , and the lattice \mathcal{H}_W is hyperbolic, therefore we have

$$\det \begin{pmatrix} \mathbf{v}^2 & (\mathbf{v}, \mathbf{u}) \\ (\mathbf{u}, \mathbf{v}) & \mathbf{u}^2 \end{pmatrix} < 0,$$

where

$$\begin{cases} \mathbf{v}^2 = 2(n-1), \\ \mathbf{u}^2 = 2dc^2, \\ (\mathbf{u}, \mathbf{v}) = -s. \end{cases}$$

This implies

$$4d(n-1)c^2 - s^2 < 0.$$

Substituting $d = k^2(n-1)$, and taking square roots we get:

$$s < -2k(n-1)c \quad \text{or} \quad s > 2k(n-1)c.$$

The second inequality contradicts $s < 0$ and $c > 0$, so we must have

$$s < -2k(n-1)c \tag{2.7}$$

The condition for the wall to meet the vertical line $\beta = -\frac{1}{k}$ is for the equation

$$cd\left(\alpha^2 + \frac{1}{k^2}\right) + \frac{s}{k} + c(n-1) = 0,$$

to have a solution for $\alpha > 0$. Substituting $d = k^2(n-1)$ and rearranging we get:

$$ck^2(n-1)\alpha^2 = -\frac{s}{k} - 2(n-1)c.$$

Since $c > 0$ a solution exists if and only if $\frac{s}{k} + 2(n-1)c < 0$, which is (2.7). \square

Corollary 2.5.5. *Let X be a K3 surface with $\text{Pic}(X) = \mathbb{Z} \cdot H$ with $H^2 = 2d$ and $d = k^2(n-1)$ for $k > 1$ integer. The vertical wall $\beta = 0$ is the only wall for $\mathbf{v} = (1, 0, 1-n)$ in $V(X)$, and the shifted derived dual $\text{R}\mathcal{H}om(-, \mathcal{O}_X)[2]$ induces an isomorphism*

$$M_{\sigma_{\alpha,\beta}}(\mathbf{v}) \xrightarrow{\sim} M_{\sigma_{\alpha,-\beta}}(\mathbf{v}),$$

for $\beta \neq 0$. In particular, Theorem 2.1.1 holds for both of them.

Proof. The functor $\mathrm{R}\mathcal{H}om(-, \mathcal{O}_X)[2]$ induces the desired isomorphism by [8, Proposition 2.11]. Combined with Theorem 2.5.1 this implies that $\beta = 0$ is the only wall in $V(X)$. The moduli space $M_{\sigma_{\alpha,\beta}}(\mathbf{v})$ for $\beta < 0$ is the Hilbert scheme, so Theorem 2.1.1 holds for it (Example 2.3.11) and $\mathrm{R}\mathcal{H}om(-, \mathcal{O}_X)[2]$ is an anti-autoequivalence, so it preserves the Mukai homomorphism by Proposition 2.3.10. \square

2.6 Reduction to the Hilbert scheme

In this section we conclude the proof of Theorem 2.1.1. We fix a K3 surface X , a primitive Mukai vector $\mathbf{v} \in H_{\mathrm{alg}}^*(X, \mathbb{Z})$, with $\mathbf{v}^2 \geq -2$, and a \mathbf{v} -generic stability condition $\sigma \in \mathrm{Stab}^\dagger(X)$. Recall the statement of Theorem 2.1.1

Theorem 2.6.1. *Let X be a K3 surface, $\mathbf{v} \in H_{\mathrm{alg}}^*(X, \mathbb{Z})$ primitive, and $\sigma \in \mathrm{Stab}^\dagger(X)$ a \mathbf{v} -generic stability condition. Then:*

- (1) $M_\sigma(\mathbf{v})$ is non-empty if and only if $\mathbf{v}^2 \geq -2$. Moreover, it is a smooth projective hyperkähler variety of dimension $\mathbf{v}^2 + 2$, deformation-equivalent to the Hilbert scheme of points on a K3 surface.
- (2) If $\mathbf{v}^2 \geq 0$, then the Mukai Homomorphism $\theta_{\mathbf{v}}$ gives a Hodge isometry

$$\theta_{\mathbf{v}}^{-1}: H^2(M_\sigma(\mathbf{v}), \mathbb{Z}) \xrightarrow{\sim} \begin{cases} \mathbf{v}^\perp & \text{if } \mathbf{v}^2 > 0 \\ \mathbf{v}^\perp / \mathbb{Z}\mathbf{v} & \text{if } \mathbf{v}^2 = 0, \end{cases}$$

where the orthogonal is taken in $H^*(X, \mathbb{Z})$.

The proof is in several steps. First, we apply a sequence of autoequivalences to get a Mukai vector of the form $\mathbf{v} = (r, \Delta, s)$ with $r > 0$ and Δ ample. Since Δ is ample, the Hodge locus of \mathbf{v} contains an elliptic K3 surface X' with a section. We deform to X' , where we can find a vector of the form $\mathbf{w} = (0, \alpha f, \beta)$, where f is the class of an elliptic fiber, such that $(\mathbf{w}, \mathbf{v}) = -1$. The moduli space $M = M_H(\mathbf{w})$ is non-empty: a generic point is just a vector bundle supported on a smooth fiber. Moreover, it is a fine moduli space, and the Fourier-Mukai $\Phi_{\mathcal{E}}$ with the universal family as kernel is an equivalence $D^b(X') \xrightarrow{\sim} D^b(M)$. Via this equivalence \mathbf{v} gets mapped to $(1, 0, 1 - n)$, up to tensoring with line bundles. Now assume $n \leq 1$ or equivalently $\mathbf{v}^2 = -2$ or $\mathbf{v}^2 = 0$. The moduli space of Gieseker stable sheaves with vector $(1, 0, 0)$ is a point, and with vector $(1, 0, 1)$ is the K3 surface itself. To conclude the proof of Theorem 2.1.1 in this case we apply the wall-crossing results of Section 2.4.

If $\mathbf{v}^2 > 0$, we take a different K3 surface as a Fourier-Mukai partner: $M = M_\sigma(\mathbf{w})$, where \mathbf{w} is the same vector as before, but the stability condition is the

same one we are studying. We are allowed to do this, because we proved Theorem 2.1.1 for isotropic vectors first. As before, via the Fourier-Mukai transform $\Phi_{\mathcal{E}}$ the vector \mathbf{v} goes to $(1, 0, 1 - n)$, but σ goes to the geometric chamber $U(X)$. Moreover, we show that $\text{Pic}(M)$ is a hyperbolic plane, so we can deform to a K3 surface Y of the type studied in Section 2.5. Being in the geometric chamber is an open condition, so the deformed stability condition remains in the geometric chamber for Y . If we act with $\widetilde{GL}_2(\mathbb{R})$ we end up in the setting of Section 2.5, where the moduli space is just the Hilbert scheme $\text{Hilb}^n(Y)$ up to a shifted derived dual.

In the argument, we apply Proposition 2.3.10 to equivalences of type (1) – (4) of Section 2.2.3, which preserve the distinguished component (Corollary 2.2.26). It is useful to recall their action in cohomology:

- (1) Tensor product with $L \in \text{Pic}(X)$ acts via multiplication with $\exp(c_1(L))$

$$(r, \Delta, s) \left(1, c_1(L), \frac{c_1(L)^2}{2} \right) = \left(r, \Delta + rc_1(L), r \frac{c_1(L)^2}{2} + \Delta \cdot c_1(L) + s \right).$$

- (2) The shift $[1]$ acts as $-\text{id}$.
(3) The spherical twists acts like the reflection around $(1, 0, 1)$

$$\rho_{(1,0,1)}((r, \Delta, s)) = (-s, \Delta, -r).$$

- (4) The Fourier-Mukai $\Phi_{\mathcal{E}}$ acts like the cohomological Fourier-Mukai, with kernel the Mukai vector $v(\mathcal{E})$.

For the deformation arguments we use the notion of a relative stability condition $\underline{\sigma}$ over a base C . It was introduced in [11], and it consists, given a family $\mathcal{X} \rightarrow C$, of a collection $\underline{\sigma} = (\sigma_c)_{c \in C}$ of stability conditions on the fibers \mathcal{X}_c satisfying some technical conditions. There is also a well-behaved notion of relative moduli space. We use the following result, which can be proved by following the arguments given in [11, Corollary 32.1] for cubic fourfolds, but the main existence result is [11, Theorem 24.1].

Theorem 2.6.2. *Let (X, H) be a polarized K3 surface of degree $2d$, \mathbf{v} a primitive vector, and $\sigma \in \text{Stab}^\dagger(X)$ a \mathbf{v} -generic stability condition. Let (X', H') be another polarized K3 surface of the same degree, in the Hodge locus where \mathbf{v} stays algebraic inside the moduli space of polarized K3 surfaces of degree $2d$. Then, there exists a smooth family $\mathcal{X} \rightarrow C$ over a smooth connected quasi-projective curve, and a relative stability condition $\underline{\sigma}$ on $D^b(\mathcal{X})$ such that:*

- (1) *The class \mathbf{v} stays algebraic for all $c \in C$.*
(2) *The stability condition σ_c is in $\text{Stab}^\dagger(\mathcal{X}_c)$ and \mathbf{v} generic for all $c \in C$.*
(3) *$\mathcal{X}_{c_0} = X$, $\mathcal{X}_{c_1} = X'$ and σ_{c_0} is a small deformation of σ such that $M_{X, \sigma}(\mathbf{v}) = M_{X, \sigma_{c_0}}(\mathbf{v})$.*

- (4) The relative moduli space $M_{\underline{\sigma}}(\mathbf{v})$ exists as a smooth and proper algebraic space over C .

The first step in the reduction to the Hilbert scheme is to apply a sequence of autoequivalences to change the Mukai vector \mathbf{v} .

Lemma 2.6.3. *Let X be a K3 surface, $\mathbf{v} = (r, \Delta, s)$ primitive, and let $\sigma \in \text{Stab}^\dagger(X)$ be \mathbf{v} -generic. Then, there exist a primitive $\mathbf{v}' = (r', \Delta', s')$ with $r' > 0$ and Δ' ample, a \mathbf{v}' -generic stability condition $\sigma' \in \text{Stab}^\dagger(X')$, and an isomorphism $M_\sigma(\mathbf{v}) \cong M_{\sigma'}(\mathbf{v}')$.*

Proof. First we reduce to $r > 0$. If $r < 0$ then a shift suffices. If $r = 0$ and $\Delta = 0$, then $\mathbf{v} = (0, 0, \pm 1)$ so after applying either $\text{ST}_{\mathcal{O}_X}[1]$ or $\text{ST}_{\mathcal{O}_X}$ we get $(1, 0, 0)$. If $\Delta \neq 0$, then, up to a shift, we can assume it to be effective. If H is an ample line bundle, tensor product with nH sends \mathbf{v} to $(0, \Delta, s + nH \cdot \Delta)$. By taking $n \gg 0$ we can assume $s > 0$. Applying the shifted spherical twist $\text{ST}_{\mathcal{O}_X}[1]$ we get $r > 0$.

If $r > 0$, to get a Δ ample we can tensor with powers of an ample line bundle. Indeed, Δ goes to $\Delta + rnH$, which is ample if $n \gg 0$. The distinguished component is preserved due to Corollary 2.2.26. \square

The next step is a deformation to an elliptic K3 surface. Consider (X, \mathbf{v}, σ) as in the conclusion of the lemma above, i.e. $\mathbf{v} = (r, \Delta, s)$ with $r > 0$ and Δ ample. We write $\Delta = mH$ with $m \in \mathbb{Z}_{>0}$ and H a primitive polarization on X of degree $H^2 =: 2d$

Lemma 2.6.4. *Let (X, \mathbf{v}, σ) be as in the conclusion on the lemma above. Then, there exists an elliptic K3 surface X' such that \mathbf{v} remains a Hodge class on X' , with $\text{Pic}(X') = \mathbb{Z}\Sigma \oplus \mathbb{Z}f$, where f is the class of the elliptic fiber and Σ the class of a section, and a stability condition σ' on X' such that $M_{X, \sigma}(\mathbf{v})$ is deformation equivalent to $M_{X', \sigma'}(\mathbf{v})$.*

Proof. We only need to check the hypothesis of Theorem 2.6.2. First assume $d > 1$. By the surjectivity of the period map, there exists a K3 surface X' with Picard group as in the statement. Equipping it with the polarization $\Sigma + (d + 1)f$, it defines a point in the moduli space of polarized K3 surfaces of degree $2d$. Since Δ is a multiple of a polarization H it remains algebraic on X' , so we are in the hypothesis of Theorem 2.6.2. The relative moduli space of point (4) gives us the desired deformation.

If $d = 1$, the class $\Sigma + (d + 1)f$ is not ample. In this case, we can apply Theorem 2.6.2 to first deform to a K3 surface X'' with $\rho(X'') > 1$. Indeed, such K3 surfaces are dense in the locus where \mathbf{v} remains a Hodge class. On X'' we can tensor by an ample line bundle and obtain a Mukai vector $\mathbf{v}'' = (r'', \Delta'', s'')$ with $\Delta'' = mH''$ and $(H'')^2 \gg 0$. So we reduced to the case $d > 1$, and the argument above concludes the proof of the Lemma. \square

Remark 2.6.5. Since the previous deformation is given by a relative moduli space, the quasi-universal family deforms, and so does the Mukai homomorphism. In particular, the function

$$\theta_{\mathbf{v}} : \mathbf{v}^\perp \cap H^*(\mathcal{X}_c, \mathbb{Z}) \rightarrow H^2(M_{\mathcal{X}_c, \sigma_c}(\mathbf{v}), \mathbb{Z})$$

is a locally constant on C . Since C is connected, and the Beaville-Bogomolov form is deformation invariant, $\theta_{\mathbf{v}}$ is a Hodge isometry on X' if and only if it is on X .

Now we prove Theorem 2.1.1 for spherical and isotropic classes \mathbf{v} .

Theorem 2.6.6. *Let X be a K3 surface, \mathbf{v} primitive and $\sigma \in \text{Stab}^\dagger(X)$ a \mathbf{v} -generic stability condition. If $\mathbf{v}^2 = -2$, the moduli space $M_\sigma(\mathbf{v})$ is a reduced point. If $\mathbf{v}^2 = 0$, then the moduli space $M_\sigma(\mathbf{v})$ is a projective K3 surface, and the map $\theta_{\mathbf{v}} : \mathbf{v}^\perp / \mathbb{Z}\mathbf{v} \rightarrow H^2(M_\sigma(\mathbf{v}), \mathbb{Z})$ is a Hodge isometry.*

Proof. As a preliminary remark, notice that if $\mathbf{v}^2 = 0$, then $M_\sigma(\mathbf{v})$ is a two-dimensional smooth and proper algebraic space, hence projective, and moreover it is symplectic, see Corollary 2.3.5 and Remark 2.3.6. So, to prove the Theorem it is enough to show that $M_\sigma(\mathbf{v})$ is deformation equivalent to a point if $\mathbf{v}^2 = -2$ or to a K3 surface if $\mathbf{v}^2 = 0$.

From Lemmas 2.6.3 and 2.6.4 we can assume that X is an elliptic K3 surface, with $\text{Pic}(X) = \mathbb{Z}\Sigma \oplus \mathbb{Z}f$, where f is the class of a fiber and s is the class of a section. Moreover, from Lemma 2.6.7 and an application of the shifted spherical twist $\text{ST}_{\mathcal{O}_X}[1]$ we can assume that the Mukai vector $\mathbf{v} = (r, \Delta, s)$ has rank positive rank $r > 0$ and coprime with $\Delta.f$.

Consider a vector $\mathbf{w} = (0, \alpha f, \beta)$; we have $(\mathbf{w}, \mathbf{v}) = \alpha\Delta.f - \beta r$. Since r and $\Delta.f$ are coprime, we can find α and β such that $(\mathbf{v}, \mathbf{w}) = -1$. Since $r > 0$ we can assume also that $\alpha > 0$ and $\beta \neq 0$. Let H be a polarization such that $M := M_H(\mathbf{w})$ is non-empty and parameterizes stable sheaves, as in Example 2.3.12. The moduli space M is fine because $(-\mathbf{v}, \mathbf{w}) = 1$, see [48, Remark 4.6.8].

Consider the Fourier-Mukai transform

$$\Phi_{\mathcal{E}} : D^b(M) \xrightarrow{\sim} D^b(X)$$

given by the universal family; it is an equivalence by Proposition 2.2.20. Since M is a projective symplectic surface, derived equivalent to a K3 surface, it is a K3 surface itself. At the level of cohomology $\Phi_{\mathcal{E}}^H$ is an isometry, and $(\Phi_{\mathcal{E}}^H)^{-1}(\mathbf{w}) = (0, 0, 1)$. Define $\sigma' := \Phi_{\mathcal{E},*}^{-1}(\sigma)$ and $\mathbf{v}' := (\Phi_{\mathcal{E}}^H)^{-1}(\mathbf{v})$. Then

$$r(\mathbf{v}') = -(\mathbf{v}', (0, 0, 1)) = -(\mathbf{v}, \mathbf{w}) = 1$$

Up to twisting for a line bundle we can assume $\mathbf{v}' = (1, 0, 1-n)$, with $n = \frac{\mathbf{v}^2+2}{2} \geq 0$. By Proposition 2.3.10 the moduli space $M_{X,\sigma}(\mathbf{v})$ is isomorphic to the moduli space $M_{M,\sigma'}((1, 0, 1-n))$, and the morphism $\theta_{\mathbf{v}}$ is compatible with this isomorphism.

Consider the wall and chamber decomposition for the vector \mathbf{v}' on $\text{Stab}^\dagger(M)$. Let H' be a polarization on M . From Theorem 2.2.25 there is a chamber where Bridgeland stability is the same as Gieseker H' -stability. If $\mathbf{v}^2 = -2$, then $\mathbf{v}' = (1, 0, 1)$. The moduli space for this vector in the Gieseker chamber is a reduced point corresponding to \mathcal{O}_X . If $\mathbf{v}^2 = 0$, the new vector is $(1, 0, 0)$ and the moduli space in the Gieseker chamber parameterizes ideal sheaves of points, so it is isomorphic to the underlying K3 surface. Moreover, the Mukai homomorphism is just the identity on $H^2(X, \mathbb{Z})$.

Since $\text{Stab}^\dagger(M)$ is connected by definition, we can find a path that connects the Gieseker chamber with the stability condition σ' . This will intersect finitely many walls, because they are locally finite. From Corollary 2.4.11 and Remark 2.6.5 we get the thesis. \square

Lemma 2.6.7. *Let X be an elliptic K3 surface with $\text{Pic}(S) = \mathbb{Z}\Sigma \oplus \mathbb{Z}f$ where f is the class of an elliptic fiber, and Σ is the class of a section. Let $\mathbf{v} = (r, m(\Sigma + df), s)$ primitive, with $r, m, d \in \mathbb{Z}_{>0}$, and assume $\gcd(m, r) \neq 1$. Then, there exist $k \in \mathbb{Z}$ such that, if we write $\mathbf{v} \cdot (k\Sigma) = (r, \Delta_k, s_k)$, we have $\gcd(\Delta_k \cdot f, s_k) = 1$.*

Proof. We have

$$\begin{aligned}\Delta_k &= (m + rk)\Sigma + mdf; \\ s_k &= s + m(d - 2)k - rk^2.\end{aligned}$$

Thus, $\Delta_k \cdot f = m + rk$ and $s_k = s + m(d - 1)k - (m + rk)k$, which give

$$\gcd(\Delta_k \cdot f, s_k) = \gcd(m + rk, s + m(d - 1)k).$$

To conclude the proof we apply the lemma below to the arithmetic progressions $m + rk$ and $s + m(d - 1)k$. We show that the assumptions are satisfied. The primitivity of \mathbf{v} gives

$$\gcd(m, r, s) = 1.$$

If the equality

$$\frac{r}{\gcd(m, r)} = \frac{m(d - 1)}{\gcd(s, m(d - 1))}$$

held, we would have $m \mid \gcd(s, m(d - 1))$, because the left hand side has no common factor with m . In particular, m would divide s and thus the primitivity of \mathbf{v} would force $\gcd(r, m) = 1$, contradicting our assumption. \square

Lemma 2.6.8. *Let $x + yk$ and $u + vk$ be two arithmetic progressions with $y, v > 0$. Assume*

$$\gcd(x, y, u) = 1,$$

and that

$$\frac{y}{\gcd(x, y)} \neq \frac{v}{\gcd(u, v)}.$$

Then, for infinitely many k

$$\gcd(x + yk, u + vk) = 1.$$

Proof. Take k of the form $k = \gcd(x, y)h$ with $h \in \mathbb{Z}$. Write

$$\begin{aligned} x + yk &= \gcd(x, y) \left(\frac{x}{\gcd(x, y)} + yh \right), \\ u + vk &= \gcd(u, v) \left(\frac{u}{\gcd(u, v)} + \frac{v \cdot \gcd(x, y)}{\gcd(u, v)} h \right). \end{aligned}$$

By assumption the arithmetic progressions inside the brackets are different, and they both have coprime generators. Assume without losing generality that the first one is bigger, for $h \gg 0$, than the second one.

From Dirichlet's Theorem on arithmetic progressions we can find infinitely many h for which $\left(\frac{x}{\gcd(x, y)} + yh\right)$ is a prime. If h is large, this prime is bigger than both $\gcd(u, v)$ and $\left(\frac{u}{\gcd(u, v)} + \frac{v \cdot \gcd(x, y)}{\gcd(u, v)} h\right)$. Since $\gcd(x, y)$ and $\gcd(u, v)$ are coprime by assumption, for any such h we have the conclusion. \square

The last step is to prove Theorem 2.1.1 for Mukai vector \mathbf{v} such that $\mathbf{v}^2 > 0$. We first show that the Picard group of the Fourier-Mukai partner M is a hyperbolic plane, and then deform to a K3 surface of Picard rank one.

Lemma 2.6.9. *Let X be an elliptic K3 surface with $\text{Pic}(X) = \mathbb{Z}\Sigma \oplus \mathbb{Z}f$, let $\mathbf{v} = (r, m(\Sigma + (d + 1)f), s)$ primitive, with $r > 0$ and σ generic. There exists another elliptic K3 surface M with $\text{Pic}(M) = \mathbb{Z}\Sigma' \oplus \mathbb{Z}f'$, and an isomorphism $M_{X, \sigma}(\mathbf{v}) \cong M_{M, \sigma'}((1, 0, 1 - n))$ where $n = \frac{\mathbf{v}^2 + 2}{2}$ and $\sigma' \in U(M)$ is generic for $(1, 0, 1 - n)$.*

Proof. We begin as in the proof of Theorem 2.6.6: we apply Lemma 2.6.7 and a spherical twist to reduce to r and $\Delta.f$ coprime, and we consider a vector $\mathbf{w} = (0, \alpha f, \beta)$ such that $(\mathbf{w}, \mathbf{v}) = -1$. Deforming σ if necessary, we can assume it to be \mathbf{w} -generic too. Theorem 2.6.6 applied to the moduli space $M := M_\sigma(\mathbf{w})$, implies that it is non-empty and a K3-surface. It is fine because the wall-crossing isomorphisms are induced by equivalences by Theorem 2.4.1, so they preserve the universal family. The universal family induces a derived equivalence

$$\Phi_{\mathcal{E}} : D^b(M) \xrightarrow{\sim} D^b(X).$$

As in the proof of Theorem 2.6.6, define

$$\sigma' := \Phi_{\mathcal{E},*}^{-1}(\sigma) \text{ and } \mathbf{v}' := (\Phi_{\mathcal{E}}^H)^{-1}(\mathbf{v}).$$

Up to twisting with a line bundle on M we can assume $\mathbf{v}' = (1, 0, 1 - n)$. To conclude the proof, it remains to show that σ' is in $U(M)$ and that $\text{Pic}(M) = \mathbb{Z}\Sigma' \oplus \mathbb{Z}f'$.

From Lemma 2.2.14 and Proposition 2.2.23, we only have to show that the skyscraper sheaves $\{\mathcal{O}_m \mid m \in M\}$ are σ' stable. This is true because $\Phi_{\mathcal{E}}(\mathcal{O}_m)$ are precisely the objects of the moduli space M , which by construction are σ -stable.

For the second statement, consider the two vectors $\mathbf{w}' := (\alpha, \beta\Sigma + (\alpha + \beta)f, \beta)$ and $\mathbf{t} := (\alpha, \beta\Sigma + (\beta - r)f, -\Delta.f)$ on X , where $\beta r - \alpha\Delta.f = 1$. It is a computation to check that they satisfy the following relations.

$$\begin{cases} (\mathbf{w}', \mathbf{w}') = 0, \\ (\mathbf{w}', \mathbf{w}) = 0. \end{cases} \quad \begin{cases} (\mathbf{t}, \mathbf{t}) = -2, \\ (\mathbf{t}, \mathbf{w}) = 0, \\ (\mathbf{t}, \mathbf{w}') = -1. \end{cases}$$

This implies that $(\Phi_{\mathcal{E}}^H)^{-1}(\mathbf{w}') = (0, l, a)$ and $(\Phi_{\mathcal{E}}^H)^{-1}(\mathbf{t}) = (0, t, b)$, with

$$\begin{cases} (l, l) = 0, \\ (t, t) = -2, \\ (l, t) = -1. \end{cases}$$

which means that $\text{Pic}(M)$ contains a hyperbolic plane. Since the Picard rank is a derived invariant for K3 surfaces, the Picard group is a hyperbolic plane. \square

Proof of Theorem 2.1.1. Consider X any K3 surface, $\mathbf{v} = (r, \Delta, s)$ a primitive vector with $\mathbf{v}^2 \geq -2$, and $\sigma \in \text{Stab}^\dagger(X)$ a \mathbf{v} -generic stability condition. The cases $\mathbf{v}^2 = -2$ and $\mathbf{v}^2 = 0$ were proved in Theorem 2.6.6, so we assume $\mathbf{v}^2 > 0$. By Corollary 2.3.5 and Remark 2.3.6 we see that $M_\sigma(\mathbf{v})$ is smooth, symplectic and projective. Since the Hodge numbers are invariant under deformations of projective varieties it is enough to show that $M_\sigma(\mathbf{v})$ is deformation equivalent (via a relative moduli space) to the Hilbert scheme of points on a K3 surface.

Applying Lemma 2.6.3 we can assume $r > 0$ and Δ ample. Under this assumptions, using Lemma 2.6.4 we deform to an elliptic K3 with a section and using Lemma 2.6.7 we can assume r and Δ to be coprime. Finally with Lemma 2.6.9 we reduce to $\mathbf{v} = (1, 0, 1 - n)$ with $n > 1$, a generic $\sigma \in U(X)$, and $\text{Pic}(X) = \mathbb{Z}\Sigma \oplus \mathbb{Z}f$.

Let $d = k^2(n - 1)$, the class $\Sigma + (d + 1)f$ is ample on X of degree $2d$. With a small deformation we reduce to the case of a K3 surface X' of Picard rank one, degree $2d$ and Mukai vector $\mathbf{v} = (1, 0, 1 - n)$. Since stability is an open

property for families of objects, every skyscraper sheaf is still stable with respect to the deformed stability condition. So the deformed stability condition lies in the geometric chamber $U(X')$ by Lemma 2.2.14. By definition of $U(X')$ we can act by the group $\widetilde{GL}_2(\mathbb{R})$ and get a stability condition $\sigma_{\alpha,\beta} \in V(X')$. This brings us in the setting of Corollary 2.5.5, and we conclude the proof. Indeed, at every step of the reduction we get either isomorphisms that preserve the Mukai homomorphism by Proposition 2.3.10, or deformations that also preserve the Mukai homomorphism by Remark 2.6.5. \square

Chapter 3

Cohomology of hyper-Kähler manifolds

3.1 Introduction

This chapter is essentially a review of the papers [71] and [103]. The fundamental result we are interested in is Theorem 3.5.1, which says that the LLV algebra is a derived invariant for HK manifolds. The LLV algebra, introduced in [71] and [107], is a rational Lie algebra

$$\mathfrak{g}(X) \subset \text{End}(H^*(X, \mathbb{Q}))$$

defined for every Kähler manifold X . Heuristically, it is defined by putting together every Hodge-theoretic \mathfrak{sl}_2 -triple associated to a Kähler class. In the case of HK manifolds it is fully understood in terms of the BBF form.

Theorem 3.1.1 (Theorem 3.2.14). *Let X be a hyperkähler manifold.*

(1) *The total Lie algebra $\mathfrak{g}(X)$ lives only in degrees $-2, 0, 2$, so it decomposes as:*

$$\mathfrak{g}(X) = \mathfrak{g}(X)_{-2} \oplus \mathfrak{g}(X)_0 \oplus \mathfrak{g}(X)_2.$$

(2) *There are canonical isomorphisms $\mathfrak{g}(X)_{\pm 2} \cong H^2(X, \mathbb{Q})$.*

(3) *There is a decomposition $\mathfrak{g}(X)_0 = \mathfrak{g}(X)'_0 \oplus \mathbb{Q}h$ with $\mathfrak{g}(X)'_0 \cong \mathfrak{so}(H^2(X, \mathbb{Q}), q)$, where q is the Beauville–Bogolomov–Fujiki quadratic form. Furthermore $\mathfrak{g}(X)'_0$ acts on $H^*(X, \mathbb{Q})$ by derivations.*

The first application that comes to mind is to study the natural action of $\mathfrak{g}(X)$ on the cohomology algebra, and in particular its decomposition in irreducible representations. This is because this decomposition is finer than the other usual decompositions (e.g. the Hodge decomposition, the Lefschetz decomposition), and

so gives a lot of information on the topology of X . This was indeed done in [39], where the decomposition is found for every known examples. Their proof relies on previous work, for example [86] and [29] respectively for OG6 and OG10. See also [97] for more details on this.

The reason we care about the LLV algebra is different, and much more surprising. It was noted by [103], that although $\mathfrak{g}(X)$ depends only on the topology of X , it is also preserved by derived equivalences. The most important consequence of this is the existence of the extended Mukai lattice, see Definition 3.6.1. This is the rational vector space

$$\tilde{H}(X, \mathbb{Q}) := \mathbb{Q}\alpha \oplus H^2(X, \mathbb{Q}) \oplus \mathbb{Q}\beta,$$

equipped with the quadratic form \tilde{q} obtained by extending the BBF form on $H^2(X, \mathbb{Q})$ by declaring that α and β are orthogonal to $H^2(X, \mathbb{Q})$, isotropic and $\tilde{q}(\alpha, \beta) = -1$.

The classes α and β are a priori just abstract classes, but it is not difficult to give them a geometric interpretation. Indeed, the extended Mukai lattice is related to the cohomology $H^*(X; \mathbb{Q})$ by the short exact sequence of Theorem 3.6.4:

$$0 \rightarrow \mathrm{SH}(X) \rightarrow \mathrm{Sym}^n \tilde{H}(X, \mathbb{Q}) \rightarrow \mathrm{Sym}^{n-2} \tilde{H}(X, \mathbb{Q}) \rightarrow 0,$$

where $\mathrm{SH}(X) \subset H^*(X, \mathbb{Q})$ is the Verbitsky component (Definition 3.3.4). This sequence admits an orthogonal splitting

$$T : \mathrm{Sym}^n \tilde{H}(X, \mathbb{Q}) \rightarrow \mathrm{SH}(X).$$

The images of $\alpha^i \beta^{n-i}$ under it generate the monodromy invariant part, as shown in Remark 3.6.13. Furthermore, Taelman [103, Theorems 2.4, 4.8, 4.9] showed that an equivalence $\Phi : \mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ induces Hodge isometries

$$\Phi^{\mathrm{SH}} : \mathrm{SH}(X) \rightarrow \mathrm{SH}(Y) \text{ and } \Phi^{\tilde{H}} : \tilde{H}(X, \mathbb{Q}) \rightarrow \tilde{H}(Y, \mathbb{Q}).$$

These two isometries are compatible via the sequence above, in the sense that the diagram

$$\begin{array}{ccc} \mathrm{SH}(X) & \xrightarrow{\Phi^{\mathrm{SH}}} & \mathrm{SH}(Y) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^n \tilde{H}(X, \mathbb{Q}) & \xrightarrow{\mathrm{Sym}^n \Phi^{\tilde{H}}} & \mathrm{Sym}^n \tilde{H}(Y, \mathbb{Q}) \end{array}$$

commutes up to a sign. This fact is extremely useful to compute the action of an equivalence

$$\Phi : \mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$$

on the cohomology of a HK. Indeed, sometimes (see Proposition 5.2.9) we can compute the isometry $\Phi^{\tilde{H}}$ relatively easily from geometrical consideration, while the isometry Φ^H would take more intense computations.

3.1.1 Structure of the chapter

In Section 3.2 we introduce the LLV algebra following [71]. We start with a general framework and study a close relative of the LLV algebra. Defined in the same way starting from Kähler classes associated to a HK metric. Using this, in Section 3.2.2 we prove Theorem 3.2.14.

In Section 3.3 we introduce and study the Verbitsky component

$$\mathrm{SH}(X) \subset H^*(X, \mathbb{Q}).$$

The main result here is Theorem 3.3.8 which gives an expression for the kernel of the natural map

$$\mathrm{Sym}^* H^2(X, \mathbb{Q}) \rightarrow \mathrm{SH}(X).$$

In the short Section 3.4 we conclude our review of the LLV algebra, and study how the action of the LLV algebra $\mathfrak{g}(X)$ on the cohomology lifts to the algebraic connected groups $\underline{\mathrm{Spin}}(H^2(X, \mathbb{Q}), q)$ and $\underline{\mathrm{SO}}(H^2(X, \mathbb{Q}), q)$.

In Section 3.5, following [103], we prove that the LLV algebra is a derived invariant (see Theorem 3.5.1). We also recall some basics on the Hochschild structure on the derived category of a HK manifold.

Again following [103], in Section 3.6 we introduce the extended Mukai lattice, and study its main properties. We prove Theorem 3.6.4 and Theorem 3.6.8, which we anticipated above.

Lastly, in Section 3.7 we explore the relationship between the Mukai lattice and the second Hochschild cohomology group. Using a morphism

$$\mu : \mathrm{HH}^2(X) \rightarrow \tilde{H}(X, \mathbb{Q}),$$

introduced by Markman [77], we compare the isomorphisms $\Phi^{\tilde{H}}$ and Φ^{HH} .

3.2 The LLV algebra

Let $V = \bigoplus_{k \in \mathbb{Z}} V_k$ be a finite dimensional graded vector space over a field \mathbb{F} of characteristic 0, and denote by h the operator:

$$h|_{V_k} = k \cdot \mathrm{id}.$$

Definition 3.2.1. Let $e: V \rightarrow V$ be a degree 2 endomorphism. We say e has the *Lefschetz property* if

$$e^k: V_{-k} \rightarrow V_k$$

is an isomorphism.

Remark 3.2.2. The degree two operators with the Lefschetz property form a Zariski open subset of $\text{End}_2(V)$.

Theorem 3.2.3 (Jacobson–Morozov, [59, Theorem 3]). *An operator e has the Lefschetz property if and only if there exists a unique degree -2 endomorphism $f: V \rightarrow V$ such that*

$$[e, f] = h.$$

Moreover, if $L \subset \text{End}(V)$ is a semisimple Lie subalgebra and $e, h \in L$, then $f \in L$.

We say that the triple (e, h, f) is an \mathfrak{sl}_2 -triple. The reason is that we can define a representation $\mathfrak{sl}_2(\mathbb{F}) \rightarrow \text{End}(V)$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ on the vector space V as follows

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto h, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto f.$$

In the rest of these notes, we will mostly be interested in the graded rational vector space $V = H^*(X, \mathbb{Q})[N]$, where X is a compact Kähler manifold of dimension N . Here $[m]$ indicates the shift by m , so that $V_0 = H^N(X, \mathbb{Q})$. To any class $a \in H^2(X, \mathbb{Q})$ we can associate the operator in cohomology obtained by taking cup product

$$e_a: H^*(X, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q}), \quad \omega \mapsto a \cdot \omega.$$

The operator h becomes

$$h|_{H^k(X, \mathbb{Q})} := (k - N)\text{id}.$$

From Theorem 3.2.3 we see that if e_a has the Lefschetz property (for example if a is a Kähler class), there is an operator f_a of degree -2 that makes (e_a, h, f_a) an \mathfrak{sl}_2 -triple. Moreover, the map

$$f: H^2(X, \mathbb{Q}) \dashrightarrow \text{End}_{-2}(H^*(X, \mathbb{Q})),$$

that sends a to the operator f_a is defined on a Zariski open subset and rational.

Remark 3.2.4. If $a \in H^{1,1}(X, \mathbb{Q})$ is a Kähler class, it follows from standard Hodge theory that everything can be defined at the level of forms. The dual operator is $f_a = *^{-1}e_a*$, where $*$ is the Hodge star operator. The \mathfrak{sl}_2 -action preserves the harmonic forms, so it induces an action on cohomology.

Definition 3.2.5 ([72, 107]). Let X be a compact Kähler manifold. The *total Lie algebra* $\mathfrak{g}(X)$ of X is the Lie algebra generated by the \mathfrak{sl}_2 -triples

$$(e_a, h, f_a),$$

where $a \in H^2(X, \mathbb{Q})$ is a class with the Lefschetz property.

The following is a general result about this Lie algebra for compact Kähler manifolds. Denote by φ the pairing on $H^*(X, \mathbb{C})$ given by

$$\varphi(\alpha, \beta) = (-1)^q \int_X \alpha \cdot \beta,$$

if α has degree $N + 2q$ or $N + 2q + 1$.

Proposition 3.2.6 ([72, Proposition 1.6]). *The Lie algebra $\mathfrak{g}(X)$ is semisimple and preserves φ infinitesimally. Moreover, the degree-0 part $\mathfrak{g}(X)_0$ is reductive.*

Now let X be a compact hyperkähler manifold of complex dimension $2n$. In this case, the Lie algebra $\mathfrak{g}(X)$ is also called the Looijenga-Lunts-Verbitsky Lie algebra. It is well known that for each hyperkähler metric g on X we get an action of the quaternion algebra \mathbb{H} on the real tangent bundle TX . This means that we have three complex structures I, J, K such that

$$IJ = -JI = K. \tag{3.1}$$

To each of these complex structures we can associate Kähler forms

$$\omega_I := g(I(-), -), \quad \omega_J := g(J(-), -), \quad \omega_K := g(K(-), -),$$

and holomorphic symplectic forms

$$\sigma_I = \omega_J + i\omega_K, \quad \sigma_J = \omega_K + i\omega_I, \quad \sigma_K = \omega_I + i\omega_J.$$

Definition 3.2.7. The *characteristic 3-plane* $F(g)$ of the metric g is

$$F(g) := \langle [\omega_I], [\omega_J], [\omega_K] \rangle = \langle [\omega_I], [\Re\sigma_I], [\Im\sigma_I] \rangle \subset H^2(X, \mathbb{R}).$$

Definition 3.2.8 ([106]). Denote by $\mathfrak{g}_g \subset \text{End}(H^*(X, \mathbb{R}))$ the Lie algebra generated by the \mathfrak{sl}_2 -triples (e_a, h, f_a) where $a \in F(g)$.

Remark 3.2.9. This Lie algebra is generated by the three \mathfrak{sl}_2 -triples associated to the classes $[\omega_I], [\omega_J], [\omega_K]$. Indeed, from the discussion in the following section we will see that the subalgebra generated by these three \mathfrak{sl}_2 -triples is semisimple. From the Jacobson-Morozov Theorem and the linearity of $e: H^2(X, \mathbb{R}) \rightarrow \text{End}(H^*(X, \mathbb{R}))$ we conclude that it contains every \mathfrak{sl}_2 -triple (e_a, h, f_a) with $a \in F(g)$.

3.2.1 The algebra associated to a metric

In this section we study the smaller algebra \mathfrak{g}_g and its action on cohomology. These results are due to Verbitsky [106], see also [52] Proposition 24.2, and [72] Proposition 4.4.

We start with a general algebraic construction. Let \mathbb{H} be the quaternion algebra. As a real vector space it is generated by $1, I, J, K$, where I, J, K satisfy the relations (3.1). We denote by \mathbb{H}_0 the pure quaternions, i.e. the linear combinations of I, J, K .

Let V be a left \mathbb{H} -module, equipped with an inner product

$$\langle -, - \rangle: V \times V \rightarrow \mathbb{R},$$

and assume that I, J, K act on V via isometries. The \mathbb{H} -action gives three complex structures I, J, K on V , satisfying the relations (3.1). Consider the forms

$$\omega_I = \langle I(-), - \rangle,$$

$$\omega_J = \langle J(-), - \rangle,$$

$$\omega_K = \langle K(-), - \rangle$$

and the holomorphic symplectic forms $\sigma_I = \omega_J + i\omega_K, \sigma_J = \omega_K + i\omega_I, \sigma_K = \omega_I + i\omega_J$.

Remark 3.2.10. Note that the operators e_λ for $\lambda = \omega_I, \omega_J, \omega_K$ have the Lefschetz property; the dual operator is given by $f_\lambda = *^{-1}e_\lambda*$, where $*$ is the Hodge star operator on $\Lambda^\bullet V^*$ induced by the inner product.

Definition 3.2.11. Let $\mathfrak{g}(V) \subset \text{End}(\Lambda^\bullet V^*)$ be the Lie algebra generated by the \mathfrak{sl}_2 -triples

$$(e_\lambda, h, f_\lambda)_{\lambda=\omega_I, \omega_J, \omega_K},$$

where h is the shifted degree operator.

In particular, this definition makes sense for the rank one module \mathbb{H} equipped with the standard inner product. This gives a Lie algebra $\mathfrak{g}(\mathbb{H}) \subset \text{End}(\Lambda^\bullet \mathbb{H}^*)$. We denote by $\mathfrak{g}(\mathbb{H})_0$ the degree-0 component of $\mathfrak{g}(\mathbb{H})$ (here the degree is viewed as an endomorphism of the graded vector space). It is a Lie subalgebra, and we denote it by $\mathfrak{g}(\mathbb{H})'_0 := [\mathfrak{g}(\mathbb{H})_0, \mathfrak{g}(\mathbb{H})_0]$ its derived Lie algebra.

Theorem 3.2.12. *With the above notation we have the following.*

- (1) *There is a natural isomorphism $\mathfrak{g}(V) \cong \mathfrak{g}(\mathbb{H})$.*
- (2) *There is an isomorphism $\mathfrak{g}(\mathbb{H}) \cong \mathfrak{so}(4, 1)$.*

(3) The algebra decomposes with respect to the degree as

$$\mathfrak{g}(\mathbb{H}) = \mathfrak{g}(\mathbb{H})_{-2} \oplus \mathfrak{g}(\mathbb{H})_0 \oplus \mathfrak{g}(\mathbb{H})_2.$$

Furthermore, $\mathfrak{g}(\mathbb{H})_{\pm 2} \cong \mathbb{H}_0$ as Lie algebras, and $\mathfrak{g}(\mathbb{H})_0 = \mathfrak{g}(\mathbb{H})'_0 \oplus \mathbb{R}h$ with $\mathfrak{g}(\mathbb{H})'_0 \cong \mathbb{H}_0$; this last isomorphism is compatible with the actions on $\bigwedge^\bullet V^*$.

Proof. (1) Since $\langle -, - \rangle$ is \mathbb{H} -invariant, we can find an orthogonal decomposition

$$V = \mathbb{H} \oplus \cdots \oplus \mathbb{H}.$$

Taking exterior powers we get $\bigwedge^\bullet V^* = \bigwedge^\bullet \mathbb{H}^* \otimes \cdots \otimes \bigwedge^\bullet \mathbb{H}^*$. This gives an injective map $\mathfrak{g}(\mathbb{H}) \rightarrow \text{End}(\bigwedge^\bullet V^*)$, given by the natural tensor product representation. It is a direct check that the image of this morphism is exactly the algebra $\mathfrak{g}(V)$.

(2) Consider the subrepresentation $W \subset \bigwedge^\bullet \mathbb{H}^*$ given by

$$W = \bigwedge^0 \mathbb{H}^* \oplus \langle \omega_I, \omega_J, \omega_K \rangle \oplus \bigwedge^4 \mathbb{H}^*.$$

We equip it with the quadratic form given by setting $\bigwedge^0 \mathbb{H}^* \oplus \bigwedge^4 \mathbb{H}^*$ to be a hyperbolic plane, orthogonal to the 3-plane, and $\{\omega_I, \omega_J, \omega_K\}$ to be an orthonormal basis of the 3-plane. By a direct computation we can see that the action of $\mathfrak{g}(\mathbb{H})$ respects infinitesimally this quadratic form. This gives a map

$$\mathfrak{g}(\mathbb{H}) \rightarrow \mathfrak{so}(W) \cong \mathfrak{so}(4, 1), \quad (3.2)$$

that we next show to be an isomorphism.

Since W has dimension 5, the Lie algebra $\mathfrak{so}(W)$ has dimension 10. Now consider the following 10 elements of $\mathfrak{g}(\mathbb{H})$:

$$h, e_I, e_J, e_K, f_I, f_J, f_K, K_{IJ}, K_{IK}, K_{JK},$$

where $K_{IJ} := [e_I, f_J]$, $K_{IK} = [e_I, f_K]$ and $K_{JK} = [e_J, f_K]$. Verbitsky [106] showed that K_{IJ} acts like the Weil operator associated with the Hodge structure on $\bigwedge^\bullet \mathbb{H}^*$ given by K , and similarly K_{JK} and K_{IK} . This means that it acts on a (p, q) form with respect to K as multiplication by $i(p - q)$. It follows that the ten operators above are linearly independent over W , hence the map is surjective. Moreover they generate $\mathfrak{g}(\mathbb{H})$ as a vector space. In fact, they generate $\mathfrak{g}(\mathbb{H})$ as a Lie algebra, and one has the following relations (see [106]):

$$\begin{aligned} [K_{\lambda, \mu}, K_{\mu, \nu}] &= K_{\lambda, \nu}, & [K_{\lambda, \mu}, h] &= 0, \\ [K_{\lambda, \mu}, e_\mu] &= 2e_\lambda, & [K_{\lambda, \mu}, f_\mu] &= 2f_\lambda, \\ [K_{\lambda, \mu}, e_\nu] &= 0, & [K_{\lambda, \mu}, f_\nu] &= 0, \end{aligned}$$

where $\lambda, \mu, \nu \in \{I, J, K\}$ and $\nu \neq \lambda, \nu \neq \mu$. This implies that they are a basis of $\mathfrak{g}(\mathbb{H})$, hence the map (3.2) is an isomorphism.

Point (3) follows using this explicit basis. Indeed we have

$$\begin{aligned}\mathfrak{g}(\mathbb{H})_{-2} &= \langle f_I, f_J, f_K \rangle, & \mathfrak{g}(\mathbb{H})_2 &= \langle e_I, e_J, e_K \rangle, \text{ and} \\ \mathfrak{g}(\mathbb{H})_0 &= \langle K_{IJ}, K_{JK}, K_{IK} \rangle \oplus \mathbb{R}h.\end{aligned}$$

In particular, we have:

$$\begin{aligned}\mathfrak{g}(\mathbb{H})'_0 &\xrightarrow{\sim} \mathbb{H}_0, \\ K_{IJ} &\mapsto K, \\ K_{JK} &\mapsto I, \\ K_{IK} &\mapsto J.\end{aligned}$$

Since $I, J, K \in \mathbb{H}_0$ act on $\bigwedge^\bullet \mathbb{H}^*$ as Weil operators for the corresponding complex structures on \mathbb{H} , the isomorphism is compatible with the actions. \square

Now we can compute the Lie algebra \mathfrak{g}_g . As above we denote by $(\mathfrak{g}_g)_0$ the degree-0 part, and by $(\mathfrak{g}_g)'_0 := [(\mathfrak{g}_g)_0, (\mathfrak{g}_g)_0]$ its derived Lie algebra.

Proposition 3.2.13. *Let (X, g) be a hyperkähler manifold with a fixed hyperkähler metric.*

- (1) *There is a natural isomorphism of graded Lie algebras $\mathfrak{g}_g \cong \mathfrak{g}(\mathbb{H})$. In particular $\mathfrak{g}_g \cong \mathfrak{so}(4, 1)$.*
- (2) *The semisimple part $(\mathfrak{g}_g)'_0$ acts on $H^*(X, \mathbb{R})$ via derivations.*

Proof. (1). Consider the Lie subalgebra $\hat{\mathfrak{g}}_g \subset \text{End}(\Omega_X^\bullet)$, generated by the \mathfrak{sl}_2 -triples (e_a, h, f_a) with $a \in F(g)$, at the level of forms (in particular $f_a = *^{-1}e_a*$). From Theorem 3.2.12, we see that for every point $x \in X$ there is an inclusion $\mathfrak{g}(\mathbb{H}) \hookrightarrow \text{End}(\Omega_{X,x}^\bullet)$. This gives an inclusion $\mathfrak{g}(\mathbb{H}) \hookrightarrow \prod_{x \in X} \text{End}(\Omega_{X,x}^\bullet)$. It follows from the definitions that the two algebras $\mathfrak{g}(\mathbb{H})$ and $\hat{\mathfrak{g}}_g$ are equal as subalgebras of $\prod_{x \in X} \text{End}(\Omega_{X,x}^\bullet)$.

Since the metric g is fixed, the \mathfrak{sl}_2 -triples (e_a, h, f_a) preserve the harmonic forms $\mathcal{H}^*(X)$, and so does $\hat{\mathfrak{g}}_g$. Since $\mathcal{H}^*(X) \cong H^*(X, \mathbb{R})$ we get a morphism

$$\mathfrak{g}(\mathbb{H}) \cong \hat{\mathfrak{g}}_g \rightarrow \mathfrak{g}_g.$$

This map is surjective, because the image contains the \mathfrak{sl}_2 -triples that generate \mathfrak{g}_g . Moreover, by explicit computations similar to the proof of Theorem 3.2.12, we can see that $\dim \mathfrak{g}_g \geq 10$. Hence the map is an isomorphism.

(2). From the previous proposition we have an isomorphism compatible with the actions on cohomology

$$(\mathfrak{g}_g)'_0 \cong \mathfrak{g}(\mathbb{H})'_0 \cong \mathbb{H}_0.$$

Hence, it suffices to prove the statement for the action of I, J, K . Each of them gives a complex structure, and acts as the Weil operator on the associated Hodge decomposition. So, the action on (p, q) forms is given by multiplication by $i(p - q)$, which is a derivation. \square

3.2.2 The LLV Lie algebra of a HK manifold

The goal of this section is to prove the following result due to Looijenga and Lunts [72, Proposition 4.5] and Verbitsky [107, Theorem 1.6].

Theorem 3.2.14. *Let X be a hyperkähler manifold. With the above notation, we have the following.*

- (1) *The total Lie algebra $\mathfrak{g}(X)$ lives only in degrees $-2, 0, 2$, so it decomposes as:*

$$\mathfrak{g}(X) = \mathfrak{g}(X)_{-2} \oplus \mathfrak{g}(X)_0 \oplus \mathfrak{g}(X)_2.$$

- (2) *There are canonical isomorphisms $\mathfrak{g}(X)_{\pm 2} \cong H^2(X, \mathbb{Q})$.*
 (3) *There is a decomposition $\mathfrak{g}(X)_0 = \mathfrak{g}(X)'_0 \oplus \mathbb{Q}h$ with $\mathfrak{g}(X)'_0 \cong \mathfrak{so}(H^2(X, \mathbb{Q}), q)$, where q is the Beauville–Bogolomov–Fujiki quadratic form. Furthermore $\mathfrak{g}(X)'_0$ acts on $H^*(X, \mathbb{Q})$ by derivations.*

The main geometric input in the proof is the following lemma.

Lemma 3.2.15. *If X is a compact hyperkähler manifold, then $[f_a, f_b] = 0$ for every $a, b \in H^2(X, \mathbb{R})$ for which f is defined.*

The proof relies on the following fact.

Proposition 3.2.16. *The set of characteristic 3-planes is open in the Grassmannian of 3-planes in $H^2(X, \mathbb{R})$.*

In turn, this follows from a celebrated theorem by Yau.

Theorem 3.2.17 (Yau). *Let X be a hyperkähler manifold, and let I be a complex structure on X . If ω is a Kähler class, then there is a unique hyperkähler metric g such that $[\omega_I] = \omega$.*

Proof of Lemma 3.2.15. If we fix a hyperkähler metric g on X , then for every $a, b \in F(g)$ we have $[f_a, f_b] = 0$. This holds already at the level of forms, using the definition $f_a = *^{-1}e_a*$ and the fact that $*$ depends only on the metric. Let $a \in H^2(X, \mathbb{R})$ be a class for which f_a is defined. Since f is a rational endomorphism, the condition $[f_a, f_b] = 0$ is Zariski closed with respect to $b \in H^2(X, \mathbb{R})$. From Proposition 3.2.16 it follows that the set

$$\{b \in H^2(X, \mathbb{R}) \mid a, b \in F(g) \text{ for some metric } g\}$$

is open. Since $[f_a, f_b] = 0$ for every b in this open set, we get $[f_a, f_b] = 0$ for every b where f_b is defined. \square

While the statement of Theorem 3.2.14 is over \mathbb{Q} , we will give the proof over \mathbb{R} following [72].

Proof of Theorem 3.2.14. Consider the subspace

$$V := V_{-2} \oplus V_0 \oplus V_2 \subset \mathfrak{g}(X),$$

where V_2 is the abelian Lie subalgebra generated by e_a with $a \in H^2(X, \mathbb{R})$, V_{-2} is the abelian Lie subalgebra generated by the f_a with $a \in H^2(X, \mathbb{R})$ where f_a is defined, and V_0 is the Lie subalgebra generated by $[e_a, f_b]$. To prove (1) and (2), it is enough to show that V is a Lie subalgebra of $\mathfrak{g}(X)$. Indeed, since $\mathfrak{g}(X)$ is generated by elements contained in V this would imply $V = \mathfrak{g}(X)$. Since V_2 and V_{-2} are abelian, it suffices to show that $[V_0, V_2] \subset V_2$ and $[V_0, V_{-2}] \subset V_{-2}$.

Claim. Define $V'_0 := [V_0, V_0]$. We have $V_0 = V'_0 \oplus \mathbb{R}h$ where V'_0 acts on cohomology via derivations.

Proof of the claim. Proposition 3.2.16 implies that the set $\{(a, b) \in H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \mid a, b \in F(g) \text{ for some metric } g\}$ is open. Arguing as in the proof of Lemma 3.2.15 we see that V_0 is generated by the elements $[e_a, f_b]$ with $a, b \in F(g)$ for some metric g . If we fix a hyperkähler metric g , the elements $[e_a, f_b]$ with $a, b \in F(g)$ generate the Lie algebra $(\mathfrak{g}_g)_0$ and their brackets the Lie subalgebra $(\mathfrak{g}_g)'_0$. Thus, V'_0 is generated by the Lie algebras $(\mathfrak{g}_g)'_0$ and their brackets. Since the Lie algebras $(\mathfrak{g}_g)'_0$ act on cohomology via derivations, the same is true for their brackets, hence V'_0 acts via derivations. Moreover, from point (3) of Theorem 3.2.12 we get the decomposition $V_0 = V'_0 + \mathbb{R}h$. Since $\mathfrak{g}(X)_0$ is reductive (Proposition 3.2.6) and h is in the center, we get $h \notin V'_0 \subset \mathfrak{g}(X)'_0$, so the sum is direct. \square

Now we show that $[V_0, V_2] \subset V_2$. Since the adjoint action of h gives the grading, it is enough to show that $[V'_0, V_2] \subset V_2$. Let $u \in V'_0$ and $e_a \in V_2$. For every $x \in H^2(X, \mathbb{R})$ we have

$$[u, e_a](x) = u(a.x) - a.u(x) = u(a).x = e_{u(a)}(x), \quad (3.3)$$

because u is a derivation.

The inclusion $[V_0, V_{-2}] \subset V_{-2}$ is more difficult. Let $G'_0 \subset GL(H^*(X, \mathbb{R}))$ be the closed Lie subgroup with Lie algebra V'_0 . For every $t \in G'_0$ we have $te_a t^{-1} = e_{t(a)}$ and $tht^{-1} = h$, by integrating the analogous relations at the level of Lie algebras. Since the third element of a \mathfrak{sl}_2 -triple is unique, we get that $tf_a t^{-1} = f_{t(a)}$. This implies that the adjoint action of G'_0 leaves V_{-2} invariant, hence so does the Lie algebra V'_0 .

To summarize, at this point we showed (1) and (2), and also that $\mathfrak{g}(X)'_0$ acts via derivations. It remains to show that $\mathfrak{g}(X)'_0 \cong \mathfrak{so}(H^2(X, \mathbb{R}), q)$.

We begin by defining the map $\mathfrak{g}(X)'_0 \rightarrow \mathfrak{so}(H^2(X, \mathbb{R}), q)$. For this, we consider the restriction of the action of $\mathfrak{g}(X)'_0$ to $H^2(X, \mathbb{R})$, and show that it preserves infinitesimally the Beauville–Bogomolov–Fujiki form q . We can fix a hyperkähler metric g and check this for $(\mathfrak{g}_g)'_0$, because these Lie subalgebras generate $\mathfrak{g}(X)'_0$. From Theorem 3.2.12 it is enough to check it for the Weil operators associated to the three complex structures I, J, K induced from g . Fix one of them, say I ; we have to verify that

$$q(I\alpha, \beta) + q(\alpha, I\beta) = 0,$$

for every $\alpha, \beta \in H^2(X, \mathbb{R})$. This follows from a direct verification using the q -orthogonal decomposition

$$H^2(X, \mathbb{R}) = (H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{R}) \oplus H^{1,1}(X, \mathbb{R}),$$

induced by the Hodge decomposition with respect to the complex structure I .

To conclude the proof it remains to show that this map is bijective; we begin with the surjectivity. Fix a hyperkähler metric g , the image of the Lie algebra $(\mathfrak{g}_g)'_0$ in $\mathfrak{so}(H^2(X, \mathbb{R}), q)$ is generated (as a vector space) by the Weil operators associated to I, J, K . Using this, it is easy to see that $(\mathfrak{g}_g)'_0$ kills the q -orthogonal complement to the characteristic 3-plane $F(g)$, and it maps onto $\mathfrak{so}(F(g), q|_{F(g)})$. One can check that varying the metric g the Lie subalgebras $\mathfrak{so}(F(g), q|_{F(g)})$ generate $\mathfrak{so}(H^2(X, \mathbb{R}))$, hence the surjectivity.

For the injectivity we proceed as follows. Let $SH^2(X, \mathbb{R}) \subset H^*(X, \mathbb{R})$ be the graded subalgebra generated by $H^2(X, \mathbb{R})$; it is a $\mathfrak{g}(X)$ representation for Corollary 3.3.6. By Lemma 3.3.7, the map $\mathfrak{g}(X) \rightarrow \mathfrak{gl}(SH^2(X, \mathbb{R}))$ is injective. Since $\mathfrak{g}(X)'_0$ acts via derivations, the map must be injective already at the level of $H^2(X, \mathbb{R})$. \square

Corollary 3.2.18. *The Hodge structure on $H^*(X, \mathbb{R})$ is determined by the Hodge structure on $H^2(X, \mathbb{R})$ and by the action of $\mathfrak{g}(X)_{2, \mathbb{R}} \cong H^2(X, \mathbb{R})$ on $H^*(X, \mathbb{R})$.*

Proof. Let I, J, K be the three natural complex structures associated to a hyperkähler metric g , and assume I is the given one. As recalled before, the commutator $K_{JK} = [e_J, f_K]$ acts like the Weil operator for I ; hence it recovers the Hodge structure. By definition, it depends only on the classes $[\omega_I], [\omega_K]$ and their action on

$H^*(X, \mathbb{R})$. Since the Hodge structure is given by the class of the symplectic form $[\sigma_I] = [\omega_J] + i[\omega_K]$, the conclusion follows. \square

Recall that if \mathfrak{g} is a Lie algebra, the *universal enveloping algebra* $U\mathfrak{g}$ of \mathfrak{g} is the smallest associative algebra extending the bracket on \mathfrak{g} . It is defined as the quotient of the tensor algebra by the elements of the form:

$$x \otimes y - y \otimes x - [x, y] \quad x, y \in \mathfrak{g}.$$

In particular, if \mathfrak{g} is abelian, then $U\mathfrak{g} = \text{Sym}^* \mathfrak{g}$.

Corollary 3.2.19. *There is a natural decomposition*

$$U\mathfrak{g}_{\text{tot}}(X) = U\mathfrak{g}_{\text{tot}}(X)_2 \cdot U\mathfrak{g}_{\text{tot}}(X)_0 \cdot U\mathfrak{g}_{\text{tot}}(X)_{-2},$$

where \cdot denotes the multiplication in $U\mathfrak{g}_{\text{tot}}(X)$.

Proof. We have to show that every element in $x \in U\mathfrak{g}_{\text{tot}}(X)$ can be written as a sum of elements of the form $x_2 \cdot x_0 \cdot x_{-2}$ with $x_i \in U\mathfrak{g}_{\text{tot}}(X)_i$. It is enough to check this on the images of pure tensors. On these it follows from the fact that the bracket is graded and the decomposition in Theorem 3.2.14. \square

3.3 The Verbitsky component

In this section, we study the relationship between the actions of $\mathfrak{g}(X)$ and $\mathfrak{g}(X)_0$ on $H^*(X, \mathbb{Q})$, where X is a compact hyperkähler manifold of dimension $\dim(X) = 2n$. The main reference is [72], see also [81, Theorem 4.4].

Definition 3.3.1. Let V be a $\mathfrak{g}_{\text{tot}}(X)$ -representation. We define the primitive subspace as:

$$\text{Prim}(V) = \{x \in V \mid (\mathfrak{g}(X)_{-2}).x = 0\}.$$

If $V = H^*(X, \mathbb{Q})$ is the standard representation we denote the primitive subspace as $\text{Prim}(X)$.

Remark 3.3.2. This definition is compatible with the usual notion of primitive element with respect to a Kähler class α in Hodge theory. Indeed, by Lemma 6.24 in [114] we see that an element $x \in H^k(X, \mathbb{R})$ is primitive with respect to α if and only if it is killed by the dual operator f_α .

Remark 3.3.3. The primitive subspace $\text{Prim}(V)$ is a $\mathfrak{g}_{\text{tot}}(X)_0$ -subrepresentation. This follows from the fact that $[\mathfrak{g}(X)_0, \mathfrak{g}(X)_{-2}] \subset \mathfrak{g}(X)_{-2}$.

Definition 3.3.4. The *Verbitsky component* $SH^2(X, \mathbb{Q}) \subseteq H^*(X, \mathbb{Q})$ is the graded subalgebra generated by $H^2(X, \mathbb{Q})$.

Proposition 3.3.5 ([72, Corollary 1.13 and Corollary 2.3]). *The cohomology $H^*(X, \mathbb{Q})$ is generated by $\text{Prim}(X)$ as a $SH^2(X, \mathbb{Q})$ -module. Moreover, if $W \subset \text{Prim}(X)$ is a $\mathfrak{g}(X)_0$ irreducible subrepresentation, then $SH^2(X, \mathbb{Q}).W \subset H^*(X, \mathbb{Q})$ is an irreducible $\mathfrak{g}(X)$ -module.*

Proof. Since $\mathfrak{g}_{\text{tot}}(X)$ is semisimple, we can decompose the cohomology in irreducible $\mathfrak{g}_{\text{tot}}(X)$ -representations:

$$H^*(X, \mathbb{Q}) = V_1 \oplus \cdots \oplus V_k.$$

The primitive part is compatible with this decomposition, so we get the decomposition

$$\text{Prim}(X) = \text{Prim}(V_1) \oplus \cdots \oplus \text{Prim}(V_k),$$

of $\mathfrak{g}(X)_0$ -representations.

We first want to show that $SH^2(X, \mathbb{Q}).\text{Prim}(V_i) = V_i$. We have

$$SH^2(X, \mathbb{Q}).\text{Prim}(V_i) = U\mathfrak{g}(X)_2.\text{Prim}(V_i) = U\mathfrak{g}(X).\text{Prim}(V_i) \subset V_i, \quad (3.4)$$

where the first equality follows from the fact that $\mathfrak{g}(X)_2$ is abelian, and the second from Corollary 3.2.19 and Remark 3.3.3. Thus $SH^2(X, \mathbb{Q}).\text{Prim}(V_i)$ is a $\mathfrak{g}_{\text{tot}}(X)$ subrepresentation of V_i , but V_i is irreducible, so the equality holds. This proves the first part of the proposition.

To prove the second part it is enough to show that each $\text{Prim}(V_i)$ is irreducible as a $\mathfrak{g}(X)_0$ -representation. Assume it is not and write $\text{Prim}(V_i) = W_1 \oplus W_2$. The identities (3.4) show that acting with $SH^2(X, \mathbb{Q})$ gives a decomposition $V_i = SH^2(X, \mathbb{Q}).W_1 \oplus SH^2(X, \mathbb{Q}).W_2$. Again, this contradicts the fact that V_i is an irreducible $\mathfrak{g}(X)$ -representation. \square

Corollary 3.3.6. *The Verbitsky component $SH^2(X, \mathbb{Q}) \subset H^*(X, \mathbb{Q})$ is an irreducible $\mathfrak{g}_{\text{tot}}(X)$ subrepresentation.*

Proof. By definition we have

$$SH^2(X, \mathbb{Q}) = SH^2(X, \mathbb{Q}).H^0(X, \mathbb{Q}), \text{ and } H^0(X, \mathbb{Q}) \subset \text{Prim}(X).$$

So it is enough to observe that $H^0(X, \mathbb{Q})$ is preserved by $\mathfrak{g}(X)_0$, then we conclude by the previous proposition. \square

Lemma 3.3.7. *The restriction map $\mathfrak{g}(X)_{\mathbb{R}} \rightarrow \mathfrak{gl}(SH^2(X, \mathbb{R}))$ is injective.*

Proof. Let $K \subset \mathfrak{g}(X)_{\mathbb{R}}$ be the kernel. It is immediate to see that $K \subset \mathfrak{g}(X)'_0$. The action of K is 0 on $H^2(X, \mathbb{R})$, so by (3.3) we get $[K, \mathfrak{g}(X)_{\mathbb{R}, 2}] = 0$. Taking the Lie group of K and the corresponding adjoint action, we see that $[K, f_a] = 0$ for every $a \in H^2(X, \mathbb{R})$ for which f_a is defined. So K has bracket 0 with $\mathfrak{g}(X)_{\mathbb{R}, 2}$ and $\mathfrak{g}(X)_{\mathbb{R}, -2}$, thus also with $\mathfrak{g}(X)_{\mathbb{R}, 0}$. Since $\mathfrak{g}(X)$ is semisimple this implies $K=0$. \square

3.3.1 Verbitsky's Theorem

In this section we give a proof of a result by Verbitsky on the structure of the irreducible component $SH^2(X)$. This result is particularly useful to study the action of the LLV algebra on the rational cohomology. As a consequence, one can understand $SH^2(X)$ as a highest weight module for $\mathfrak{g}(X)$, see [97]. The argument presented here was given by Bogomolov in [18].

Theorem 3.3.8. *There is a natural isomorphism of algebras and $\mathfrak{g}(X)_0$ -modules:*

$$SH^2(X, \mathbb{C}) \cong \text{Sym}^* H^2(X, \mathbb{C}) / \langle \alpha^{n+1} \mid q(\alpha) = 0 \rangle.$$

The key technical fact is the following lemma from representation theory, of which we omit the proof.

Lemma 3.3.9. *Denote by A the graded \mathbb{C} -algebra $\text{Sym}^* H^2(X, \mathbb{C}) / \langle \alpha^{n+1} \mid q(\alpha) = 0 \rangle$. Then we have:*

- (1) $A_{2n} \cong \mathbb{C}$.
- (2) *The multiplication map $A_k \times A_{2n-k} \rightarrow A_{2n}$ induces a perfect pairing.*

Proof of the theorem. From the local Torelli Theorem we have that $\alpha^{n+1} = 0$ for an open subset of the quadric $\{\alpha \in H^2(X, \mathbb{C}) \mid q(\alpha) = 0\}$. Since the condition $\alpha^{n+1} = 0$ is Zariski closed, we get that it holds for the entire quadric. Consider the multiplication map

$$\text{Sym}^* H^2(X, \mathbb{C}) \rightarrow SH^2(X, \mathbb{C}).$$

The kernel contains $\{\alpha^{n+1} \mid q(\alpha) = 0\}$, hence it factors via the ring A . It is an algebra homomorphism by construction, and a map of $\mathfrak{g}(X)_0$ -representations because $\mathfrak{g}(X)'_0$ acts via derivations.

The induced map $A \rightarrow SH^2(X, \mathbb{C})$ is surjective by construction. If it were not injective, by the above lemma, the kernel would contain A_{2n} . But this is impossible, because in top degree the map $A_{2n} \rightarrow H^{4n}(X, \mathbb{C})$ is non-zero. Indeed if σ is a holomorphic symplectic form, the form $(\sigma + \bar{\sigma})^{2n}$ is non-zero. \square

Corollary 3.3.10. *There are natural isomorphisms defined over \mathbb{Q}*

$$SH^2(X, \mathbb{Q})_{2k} \cong \begin{cases} \text{Sym}^k H^2(X, \mathbb{Q}) & \text{if } k \leq n, \\ \text{Sym}^{2n-k} H^2(X, \mathbb{Q}) & \text{if } n < k \leq 2n. \end{cases}$$

Proof. The properties (1) and (2) in Lemma 3.3.9 hold for $SH^2(X, \mathbb{C})$, as a consequence of Theorem 3.3.8. Up to up to multiplication with a nonzero scalar, they also hold for $SH^2(X, \mathbb{Q})$. The multiplication map $\text{Sym}^k H^2(X, \mathbb{Q}) \rightarrow SH^2(X, \mathbb{Q})_{2k}$ is an isomorphism if $k \leq n$, because it is so over \mathbb{C} . If $k > n$ we have

$$SH^2(X, \mathbb{Q})_{2k} \cong SH^2(X, \mathbb{Q})_{4n-2k}^* \cong \text{Sym}^{2n-k} H^2(X, \mathbb{Q})^* \cong \text{Sym}^{2n-k} H^2(X, \mathbb{Q}),$$

where the last equality is due to the Beauville–Bogomolov–Fujiki form. \square

Example 3.3.11. If X is of $K3^{[2]}$ -type, for dimensional reasons, the Verbitsky component $SH(X)$ is the only irreducible component in the cohomology. For higher values of n the decomposition of $H^*(X, \mathbb{Q})$ in irreducible components is described in [39], for more details on this see [97].

3.4 Spin action

In this section we study how the action of

$$\mathfrak{g}(X)'_0 = \mathfrak{so}(H^2(X, \mathbb{Q}), q)$$

integrates to an action of the simply connected algebraic group $\underline{\mathrm{Spin}}(H^2(X, \mathbb{Q}), q)$. Recall that there is an exact sequence of algebraic groups

$$1 \rightarrow \pm 1 \rightarrow \underline{\mathrm{Spin}}(H^2(X, \mathbb{Q}), q) \rightarrow \underline{\mathrm{SO}}(H^2(X, \mathbb{Q}), q) \rightarrow 1.$$

For more information see [16] and [97].

Proposition 3.4.1 ([81, Theorem 4.4],[109]). *The action of $\mathfrak{so}(H^2(X, \mathbb{Q}), q)$ on $H^*(X, \mathbb{Q})$ integrates to an action of the algebraic group $\underline{\mathrm{Spin}}(H^2(X, \mathbb{Q}), q)$ via ring isomorphisms. On the even cohomology it induces an action of $\underline{\mathrm{SO}}(H^2(X, \mathbb{Q}), q)$.*

Proof. The first part of the statement is clear: we can always lift the action because the algebraic group $\underline{\mathrm{Spin}}(H^2(X, \mathbb{Q}), q)$ is simply connected. The group $\underline{\mathrm{Spin}}(H^2(X, \mathbb{Q}), q)$ acts via ring isomorphisms because the Lie algebra acts via derivations.

To show the second part of the statement we proceed as follows. Fix a hyperkähler metric g and a compatible complex structure I . The Weil operator with respect to I is contained in $(\mathfrak{g}_g)'_0 \cong \mathfrak{so}(H^2(X, \mathbb{Q}))$. The exponential $\exp(\pi I) \in \underline{\mathrm{Spin}}(H^2(X, \mathbb{Q}), q)$ acts on the (p, q) part of $H^k(X, \mathbb{C})$ as multiplication by $e^{i(p-q)\pi}$, which is just multiplication by $(-1)^k$. In particular, on $H^2(X, \mathbb{Q})$ it acts as the identity, so $\exp(\pi I) = -1 \in \underline{\mathrm{Spin}}(H^2(X, \mathbb{Q}), q)$. We have also shown that $-1 \in \underline{\mathrm{Spin}}(H^2(X, \mathbb{Q}), q)$ acts on $H^k(X, \mathbb{Q})$ as $(-1)^k$, which means that the action on even cohomology factors through $\underline{\mathrm{SO}}(H^2(X, \mathbb{Q}), q)$. \square

3.5 Derived invariance of the LLV algebra

A crucial result, upon which the theory of derived categories of hyper-Kähler manifolds lies, is the compatibility between the LLV algebra and the derived category.

Theorem 3.5.1 ([103, Theorem A]). *Let X_1 and X_2 be hyper-Kähler varieties, and let $\Phi : D^b(X_1) \simeq D^b(X_2)$ be a derived equivalence. Then there exists a canonical isomorphism of rational Lie algebras*

$$\Phi^{\mathfrak{g}} : \mathfrak{g}(X_1) \simeq \mathfrak{g}(X_2)$$

and the isometry $\Phi^H : H^*(X_1, \mathbb{Q}) \rightarrow H^*(X_2, \mathbb{Q})$ is equivariant with respect to $\Phi^{\mathfrak{g}}$.

Recall that every derived equivalence $\Phi : D^b(X) \simeq D^b(Y)$ between smooth projective varieties is of Fourier–Mukai type. This means that there exists a unique $\mathcal{E} \in D^b(X \times Y)$ such that

$$\Phi = \Phi_{\mathcal{E}} := p_{Y,*} \circ (\mathcal{E} \otimes -) \circ p_X^*,$$

where $p_X : X \times Y \rightarrow X$ and $p_Y : X \times Y \rightarrow Y$ are the two projections. The cohomological transform $\Phi^H : H^*(X, \mathbb{Q}) \rightarrow H^*(Y, \mathbb{Q})$ is obtained via the correspondence $v(\mathcal{E}) \in H^*(X \times Y)$. It is compatible with the derived equivalence via the Mukai vector, namely the following diagram commutes

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\Phi} & D^b(Y) \\ v \downarrow & & v \downarrow \\ H^*(X, \mathbb{Q}) & \xrightarrow{\Phi^H} & H^*(Y, \mathbb{Q}) \end{array}$$

see [46, Section 5].

The isomorphism Φ^H preserves neither the cup product nor the grading on H^* in general. It does preserve the columns of the Hodge diamond, that is the weight zero Hodge structure given by

$$H^{-k,k} := \bigoplus_{p-q=-k} H^q(X, \Omega_X^p), \quad (3.5)$$

because the Mukai vector $v(\mathcal{E})$ is a Hodge class in $H^*(X \times Y)$. In particular, it preserves the even (and odd) cohomology.

Furthermore, it is an isometry with respect to the generalized Mukai pairing

$$(\mathbf{v}, \mathbf{v}') := \int_X \exp(c_1(X)/2) \cup \mathbf{v}^{\vee} \cup \mathbf{v}', \quad (3.6)$$

where if $\mathbf{v} = \sum_k \mathbf{v}_k$ is the decomposition into graded components, then

$$\mathbf{v}^{\vee} := \sum_k (\sqrt{-1})^k \mathbf{v}_k.$$

The idea behind the proof of Theorem 3.5.1 is to replicate the definition of the LLV algebra Definition 3.2.8 in the context of Hochschild cohomology, which is a derived invariant. It will follow automatically that this new ‘Hochschild’ LLV algebra is a derived invariant, the non-trivial part is to show that it agrees with the usual LLV algebra.

3.5.1 Hochschild cohomology

Let X be a smooth projective variety of dimension n , and let $\Delta : X \rightarrow X \times X$ be the diagonal embedding. The Hochschild cohomology is defined as

$$\mathrm{HH}^k(X) := \mathrm{Ext}^k(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$$

and the Hochschild homology as

$$\mathrm{HH}_k(X) := \mathrm{Ext}^{n-k}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X).$$

The Yoneda product induces maps

$$\mathrm{HH}^n \otimes \mathrm{HH}^m \rightarrow \mathrm{HH}^{n+m} \quad \text{and} \quad \mathrm{HH}^n \otimes \mathrm{HH}_m \rightarrow \mathrm{HH}_{m-n}, \quad (3.7)$$

making HH^\bullet a graded ring, and HH_\bullet a graded module over HH^\bullet . Since the definition can be stated only in terms of the derived category, the following result comes naturally.

Theorem 3.5.2 ([28, Theorem 8.1]). *If $\Phi : \mathrm{D}^b(X_1) \simeq \mathrm{D}^b(X_2)$ is an equivalence, it induces isomorphisms*

$$\Phi^{\mathrm{HH}} : \mathrm{HH}^\bullet(X_1) \rightarrow \mathrm{HH}^\bullet(X_2) \quad \text{and} \quad \Phi_{\mathrm{HH}} : \mathrm{HH}_\bullet(X_1) \rightarrow \mathrm{HH}_\bullet(X_2).$$

Where Φ^{HH} is a graded algebra isomorphism, and Φ_{HH} is equivariant with respect to the action (3.7).

Define

$$\mathrm{HT}^k(X) := \bigoplus_{i+j=k} H^j(X, \bigwedge^i T_X).$$

The Hochschild–Kostant–Rosenberg isomorphism

$$\Delta^*\Delta_*\mathcal{O}_X \simeq \bigoplus_k \Omega_X^k[k]$$

induces isomorphisms

$$I^{\mathrm{HKR}} : \mathrm{HH}^\bullet(X) \simeq \mathrm{HT}^\bullet(X) \quad \text{and} \quad I_{\mathrm{HKR}} : \mathrm{HH}_\bullet(X) \simeq H(X, \mathbb{C}). \quad (3.8)$$

We twist them by the square root of the Todd class as explained in [27]:

$$I^H := (\mathrm{td}_X^{-1/2} \lrcorner -) \circ I^{\mathrm{HKR}} \quad I_H := (\mathrm{td}_X^{-1/2} \wedge -) \circ I_{\mathrm{HKR}}. \quad (3.9)$$

In this way, I^H becomes a graded algebra isomorphism, and the action of $\mathrm{HH}^\bullet(X)$ on $\mathrm{HH}_\bullet(X)$ is mapped under I_H to the action of $\mathrm{HT}^\bullet(X)$ on $H(X, \mathbb{C})$ given by

contraction. Moreover, as shown in [76, Theorem 1.2], via I_H the isomorphism Φ_{HH} becomes the usual Φ^H , that is the following diagram commutes

$$\begin{array}{ccc} \mathrm{HH}_\bullet(X_1) & \xrightarrow{I_H^{X_1}} & H^\bullet(X_1) \\ \downarrow \Phi^{HH\bullet} & & \downarrow \Phi^H \\ \mathrm{HH}_\bullet(X_2) & \xrightarrow{I_H^{X_2}} & H^\bullet(X_2) \end{array}$$

Combining this with Theorem 3.5.2 we get the following.

Proposition 3.5.3. *Let $\Phi : D^b(X_1) \simeq D^b(X_2)$ an equivalence. Then we have isomorphisms*

$$\Phi^{\mathrm{HT}} : \mathrm{HT}^\bullet(X_1) \simeq \mathrm{HT}^\bullet(X_2) \text{ and } \Phi^H : H(X, \mathbb{C}) \rightarrow H(X, \mathbb{C}).$$

Where Φ^{HT} is a graded algebra isomorphism, and Φ^H is equivariant with respect to the action of $\mathrm{HT}(X)$ on $H(X, \mathbb{C})$ given by contraction. That is

$$\Phi^{\mathrm{HT}}(\alpha) \lrcorner \Phi^H(\omega) = \Phi^H(\alpha \lrcorner \omega),$$

for every $\alpha \in \mathrm{HT}^\bullet(X)$ and $\omega \in H(X, \mathbb{C})$.

3.5.2 Hochschild LLV algebra

Consider the operator $h' \in \mathrm{End}(H(X, \mathbb{C}))$ defined by

$$h'|_{H^q(X, \Omega_X^p)} := (q - p) \mathrm{id}.$$

The HKR isomorphism I_H maps the $\mathrm{HH}_k(X)$ the $-k$ -th graded component of the grading induced by h' , which is just the k -th column of the Hodge diamond.

Via the action of $\mathrm{HT}^l(X)$ on $H(X, \mathbb{C})$, we define the operator

$$e'_\mu(-) = \mu \lrcorner - \in \mathrm{End}(H(X, \mathbb{C})) \text{ for every } \mu \in \mathrm{HT}^l(X).$$

It has degree l by the grading induced by h' .

Proposition 3.5.4. *For a holomorphic symplectic variety (X, σ) the operator e'_μ has the Hard Lefschetz property for μ in a dense open set in $\mathrm{HT}^2(X)$.*

To show this fact we briefly explain the structure of the Hochschild cohomology for a symplectic variety. The symplectic form induces an isomorphism $T_X \simeq \Omega_X$, and taking exterior power gives $\bigwedge^i T_X \simeq \Omega_X^i$. Similarly, the holomorphic volume form σ^n induces an isomorphism of sheaves $\bigwedge^i T_X \simeq \Omega^{2n-i}$.

Taking cohomology we get the two isomorphism:

$$\begin{aligned}\gamma : \mathrm{HT}^*(X) &\simeq H^*(X, \mathbb{C}), & \alpha &\mapsto \alpha \lrcorner \sigma^n \\ \sigma : \mathrm{HT}^*(X) &\simeq H^*(X, \mathbb{C}), & \alpha &\mapsto \alpha \lrcorner \sigma.\end{aligned}$$

The isomorphism $\sigma : \mathrm{HT}^\bullet(X) \simeq H(X, \mathbb{C})$ is a graded algebra isomorphism, that is

$$\sigma(\mu \wedge \nu) = \sigma(\mu) \cup \sigma(\nu). \quad (3.10)$$

While γ intertwines the action of HT^\bullet with the wedge product:

$$e'_\mu(\gamma(\alpha)) = \gamma(\mu \wedge \alpha) \quad (3.11)$$

Proof of Proposition 3.5.4. The fact that σ is an isomorphism of algebras implies that the operator $\mu \wedge -$ has the HL property on $\mathrm{HT}^\bullet(X)$. The statement then follows from (3.11). \square

Remark 3.5.5. The composition

$$\eta := \gamma \circ \sigma^{-1} \quad (3.12)$$

is called the *Verbitsky operator*. It is immediate from the definitions that

$$\eta \circ h \circ \eta^{-1} = h'. \quad (3.13)$$

In other words, it exchanges the rows and columns of the Hodge diamond. This allows to use Hochschild (co)homology to gain information on the Hodge structure and viceversa.

Definition 3.5.6. The Hochschild LLV algebra $\mathfrak{g}(X)'$ is the Lie subalgebra of $\mathrm{End}(H(X, \mathbb{C}))$ generated by the \mathfrak{sl}_2 -triples (e'_μ, h', f_μ) for all $\mu \in \mathrm{HT}^2(X)$ with the HL property.

Recall from Section 3.4 that the LLV action integrates to an action

$$\rho : \underline{\mathrm{Spin}}(H^2(X, \mathbb{Q}), q) \rightarrow \mathrm{End}(H(X, \mathbb{C})).$$

The fundamental result we are going to need is the following.

Theorem 3.5.7 ([112, Theorem 9.1 and 9.7(i)]). *The operator $\eta \in \mathrm{End}(H(X, \mathbb{C}))$ is an involution and lives in the image of ρ .*

Theorem 3.5.8 ([103, Proposition 2.8]). *The algebras $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ and $\mathfrak{g}(X)'$ are equal as subalgebras of $\mathrm{End}(H(X, \mathbb{C}))$.*

Proof. It follows from the definition of η and (3.11) that

$$\eta \circ e_{\sigma(\mu)} \circ \eta^{-1} = e'_{\mu}. \quad (3.14)$$

That is

$$\eta(\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}) \eta^{-1} = \mathfrak{g}(X)',$$

but by Theorem 3.5.7 the left hand side is equal to $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. \square

Corollary 3.5.9 ([103, Theorem A]). *Let $\Phi : D^b(X_1) \simeq D^b(X_2)$ be an equivalence between hyper-Kähler manifolds. Then there is an induced isomorphism*

$$\Phi^{\mathfrak{g}} : \mathfrak{g}(X_1) \simeq \mathfrak{g}(X_2)$$

of rational Lie algebras such that

$$\Phi^H : H^*(X_1, \mathbb{C}) \simeq H^*(X_2, \mathbb{C})$$

is equivariant with respect to $\Phi^{\mathfrak{g}}$.

Proof. Via Theorem 3.5.2 and the HKR isomorphism, we get that the isomorphism

$$\Phi^H : H^*(X_1, \mathbb{Q}) \simeq H^*(X_2, \mathbb{Q})$$

is equivariant with respect to the algebra isomorphism $\Phi^{\text{HT}} : \text{HT}^*(X_1, \mathbb{Q}) \simeq \text{HT}^*(X_2, \mathbb{Q})$. Since the algebra $\mathfrak{g}(X)'$ is defined in terms of this action, the result follows from Theorem 3.5.8. \square

Corollary 3.5.10. *The Hodge isometry Φ^H restricts to a Hodge isometry*

$$\Phi^{\text{SH}} : \text{SH}(X_1) \simeq \text{SH}(X_2),$$

where $\text{SH}(X)$ is equipped with the restriction of the Mukai pairing. Moreover, this Hodge isometry is equivariant with respect to the action of the LLV algebra.

Proof. In the proof Corollary 3.3.6 we see that $\text{SH}(X)$ is the unique irreducible subrepresentation of $H^*(X, \mathbb{Q})$ whose Hodge structure contains

$$H^{-2n, 2n} = H^{2n}(X, \mathcal{O}_X).$$

Since Φ^H respects the Hodge structure (3.5) we conclude by Corollary 3.5.9. \square

Remark 3.5.11. On the Verbitsky component the restriction b_{SH} of the generalized Mukai pairing (3.6) becomes

$$(\lambda_1 \cdots \lambda_k, \mu_1 \cdots \mu_{2n-k}) = (-1)^k \int_X \lambda_1 \cup \cdots \cup \lambda_k \cup \mu_1 \cup \cdots \cup \mu_{2n-k}. \quad (3.15)$$

3.6 Extended Mukai lattice

One particularly useful application of Corollary 3.5.9 is the existence of a small dimensional quadratic vector space which is invariant under derived categories. This will make easier the otherwise often impossible task of computing the induced isomorphism Φ^H in cohomology.

Definition 3.6.1. Let X be a hyper-Kähler manifold of dimension $2n$. The *rational extended Mukai lattice* is the rational vector space

$$\tilde{H}(X, \mathbb{Q}) := \mathbb{Q}\alpha \oplus H^2(X, \mathbb{Q}) \oplus \mathbb{Q}\beta.^1$$

It is endowed with the non-degenerate quadratic form \tilde{q} obtained by extending the BBF form q on $H^2(X, \mathbb{Q})$ by declaring that α and β are orthogonal to $H^2(X, \mathbb{Q})$, isotropic and $\tilde{q}(\alpha, \beta) = -1$.

Notice that this looks like the extended Mukai lattice for a K3 surface recalled in Section 2.2.1. In the same flavour one can also equip $\tilde{H}(X, \mathbb{Q})$ with a weight two Hodge structure, compatible with \tilde{q} , given by

$$\tilde{H}(X, \mathbb{C})^{2,0} = H^{2,0}(X), \quad \tilde{H}(X, \mathbb{C})^{1,1} = \mathbb{C}\alpha \oplus H^{1,1}(X) \oplus \mathbb{C}\beta, \quad \tilde{H}(X, \mathbb{C})^{0,2} = H^{0,2}(X),$$

where $\tilde{H}(X, \mathbb{C}) := \tilde{H}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. There is also a grading given by

$$\bar{\alpha} = -2, \quad \bar{\omega} = 0 \text{ for every } \omega \in H^2(X, \mathbb{Q}), \quad \bar{\beta} = 2.$$

The last piece of structure on $\tilde{H}(X, \mathbb{Q})$ we want to describe is the the action of $\mathfrak{g}(X)$. An element $\lambda \in H^2(X, \mathbb{Q})$ induces an operator $e_\lambda \in \mathfrak{so}(\tilde{H}(X, \mathbb{Q}))$ defined as

$$e_\lambda(\alpha) = \lambda, \quad e_\lambda(\omega) = q(\lambda, \omega)\beta, \quad e_\lambda(\beta) = 0.$$

This extends to an action of $\mathfrak{g}(X)$ on $\tilde{H}(X, \mathbb{Q})$, by letting h acts as the grading.

Theorem 3.6.2. *The action induces an isomorphism*

$$\mathfrak{g}(X) \simeq \mathfrak{so}(\tilde{H}(X, \mathbb{Q}), \tilde{q})$$

Proof. Recall that for a rational quadratic space (V, q) there is an isomorphism

$$\bigwedge^2 V \xrightarrow{\simeq} \mathfrak{so}(V, q), \quad x \wedge y \mapsto \frac{1}{2}(q(x, -)y - q(y, -)x)$$

The desired isomorphism follows from this, at least at the level of vector spaces. The computations to show that it is in fact an isomorphism of Lie algebras are carried out in [39, Proposition 2.7]. \square

¹Here α and β are just formal classes.

Corollary 3.6.3. *An element $\omega \in H^2(X, \mathbb{Q})$ has the HL property if and only if $q(\omega) \neq 0$.*

Proof. As before, we denote in the same way the operator e_ω acting on $H^*(X, \mathbb{Q})$ or $\tilde{H}(X, \mathbb{Q})$. By Theorem 3.2.3 the HL property for either of these actions is equivalent to the existence of an operator Λ_ω which completes the pair (e_ω, h) to an \mathfrak{sl}_2 -triple. In particular, e_ω has the HL property on $H^*(X, \mathbb{Q})$ if and only if it does on $\tilde{H}(X, \mathbb{Q})$. For this last space this means just that

$$e_\omega^2 : \mathbb{Q}\alpha \rightarrow \mathbb{Q}\beta, \quad \alpha \rightarrow q(\omega)\beta$$

is an isomorphism. This happens precisely if and only if $q(\omega) \neq 0$. \square

Taelman constructed, for every equivalence $\Phi : D^b(X_1) \simeq D^b(X_2)$ a Hodge isometry $\Phi^{\tilde{H}} : \tilde{H}(X_1, \mathbb{Q}) \simeq \tilde{H}(X_2, \mathbb{Q})$. The construction is different from Φ^H because it is not a ‘cohomological Fourier-Mukai transform’, but rather comes out of a representation-theoretic construction. The following result allows to relate the extended Mukai lattice with the Verbitsky component $\mathrm{SH}(X)$ (see Definition 3.3.4).

Theorem 3.6.4 ([103, Lemma 3.7]). *There is a short exact sequence*

$$0 \rightarrow \mathrm{SH}(X)[2n] \xrightarrow{\Psi} \mathrm{Sym}^n \tilde{H}(X, \mathbb{Q}) \xrightarrow{\Delta} \mathrm{Sym}^{n-2} \tilde{H}(X, \mathbb{Q}) \rightarrow 0.$$

The two maps are given as follows. Denote by $x^{(n)} \in \mathrm{Sym}^n \tilde{H}(X, \mathbb{Q})$ the n -th symmetric power of $x \in \tilde{H}(X, \mathbb{Q})$. Then Ψ is defined as

$$\lambda_1 \dots \lambda_k \mapsto e_{\lambda_1} \dots e_{\lambda_k} (\alpha^{(n)}/n!),$$

where the action of $e_{\lambda_i} \in \mathfrak{g}(X)$ on $\mathrm{Sym}^n \tilde{H}(X, \mathbb{Q})$ is by derivations.

Remark 3.6.5. By [103, Lemma 3.6] Ψ is $\mathfrak{g}(X)$ -equivariant, and in fact is the only $\mathfrak{g}(X)$ -equivariant map mapping 1 to $\alpha^{(n)}/n!$. It is also clear from the definitions that it respects the grading and the Hodge structures.

The map Δ is called the Laplacian, and is defined on generators by

$$x_1 \dots x_n \mapsto \sum_{i < j} \tilde{q}(x_i, x_j) x_1 \dots \hat{x}_i \dots \hat{x}_j \dots x_n.$$

Again, this is $\mathfrak{g}(X)$ equivariant.

Proof of Theorem 3.6.4. Since Ψ is equivariant, and $\mathrm{SH}(X)$ is an irreducible $\mathfrak{g}(X)$ representation, it follows that Ψ is injective. Moreover

$$\Delta(\Psi(1)) = \Delta(\alpha^{(n)}/n!) = 0$$

so the sequence is exact in the middle. The result follows if we know that Δ is surjective with irreducible kernel, which is a well-known result in representation theory. \square

Define a pairing $b_{[n]}$ on $\mathrm{Sym}^n \tilde{H}(X, \mathbb{Q})$ as

$$b_{[n]}(x_1 \cdots x_n, y_1 \cdots y_n) = (-1)^n c_X \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n \tilde{q}(x_i, y_{\sigma(i)}). \quad (3.16)$$

Then, Ψ is an isometry with respect to the pairings b_{SH} and $b_{[n]}$ by [103, Proposition 3.8].

Remark 3.6.6. Recall that, with the normalization we are using, the Fujiki constant c_X is characterized by the property that

$$\int_X \alpha^{2n} = c_X (2n-1)!! q(\alpha)^n$$

for every $\alpha \in H^2(X, \mathbb{Q})$.

Let X be a HK manifold of dimension $2n$. If we take an autoequivalence $\Phi : \mathrm{D}^b(X) \simeq \mathrm{D}^b(X)$, by Corollary 3.5.10 we get a Hodge isometry $\Phi^{\mathrm{SH}} \in O(\mathrm{SH}(X))$. Since it is equivariant with respect to the action of the LLV algebra, it lives in the normalizer group $N(\mathfrak{g}(X))$ of $\mathfrak{g}(X)$ in $GL(\mathrm{SH}(X))$. The following is the main representation-theoretic ingredient that we need.

Theorem 3.6.7 ([103, Proposition 4.1 and 4.4]).

(1) *There is a short exact sequence*

$$1 \rightarrow \{\pm 1\} \rightarrow O(\tilde{H}(X, \mathbb{Q})) \times \mathbb{Q}^* \rightarrow N(\mathfrak{g}(X)) \rightarrow 1$$

(2) *The intersection of the normalizer with the isometries of $\mathrm{SH}(X)$ is:*

$$N(\mathfrak{g}(X)) \cap O(\mathrm{SH}(X)) \simeq \begin{cases} O(\tilde{H}(X, \mathbb{Q})) & \text{if } n \text{ is odd,} \\ O(\tilde{H}(X, \mathbb{Q})) / \{\pm 1\} \times \{\pm 1\} & \text{if } n \text{ is even.} \end{cases}$$

Proof. This essentially follows from Theorem 3.6.4 which exhibits the Verbitsky component as the kernel of the Laplacian. Here we just describe the two maps in (1), for a full proof we refer to [103].

The first map is $\varepsilon \mapsto (\varepsilon, \varepsilon^n)$. The second map is given as follows. If φ is an isometry of $\tilde{H}(X, \mathbb{Q})$ the symmetrization $\mathrm{Sym}^n \varphi$ is an isometry of $\mathrm{Sym}^n \tilde{H}$ commuting with the Laplacian. In particular, it restricts to an isometry of $\mathrm{SH}(X)$ thanks to Theorem 3.6.4. The map then is $(\varphi, \lambda) \mapsto \lambda \mathrm{Sym}^n \varphi|_{\mathrm{SH}(X)}$.

Point (2) follows from (1). Indeed, for $\lambda \mathrm{Sym}^n \varphi|_{\mathrm{SH}(X)}$ to be an isometry we need λ to be ± 1 , and taking the quotient gives the result. \square

So, for every autoequivalence Φ there is an isometry $\Phi^{\tilde{H}} \in O(\tilde{H}(X, \mathbb{Q}))$, which is uniquely determined up to a sign. Define the *sign* of $\Phi^{\tilde{H}}$ as

$$\varepsilon(\Phi^{\tilde{H}}) := \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \det \Phi^{\tilde{H}} & \text{if } n \text{ is even.} \end{cases} \quad (3.17)$$

Then, the proposition above implies the following.

Theorem 3.6.8 ([103, Theorem C]). *Let X be a HK manifold of dimension $2n$, and assume that either n is odd or $b_2(X)$ is odd. Let $\Phi : D^b(X) \simeq D^b(X)$ be an autoequivalence. Then, there exists a Hodge isometry $\Phi^{\tilde{H}}$ making the following diagram commute*

$$\begin{array}{ccc} \mathrm{SH}(X) & \xrightarrow{\varepsilon(\Phi^{\tilde{H}})\Phi^{\mathrm{SH}}} & \mathrm{SH}(X) \\ \downarrow \Psi & & \downarrow \Psi \\ \mathrm{Sym}^n \tilde{H}(X, \mathbb{Q}) & \xrightarrow{\mathrm{Sym}^n \Phi^{\tilde{H}}} & \mathrm{Sym}^n \tilde{H}(X, \mathbb{Q}) \end{array}$$

Proof. The isometry $\Phi^{\tilde{H}}$ is defined above, and the commutativity of the diagram is clear from the definition. Notice that if $b_2(X)$ is odd then:

$$\mathrm{SO}(\tilde{H}(X, \mathbb{Q})) \times \{\pm 1\} \cong O(\tilde{H}(X, \mathbb{Q})) \quad (\varphi, \lambda) \rightarrow \lambda\varphi,$$

with inverse given by $\psi \mapsto ((\det \psi)\psi, \det \psi)$. It only remains to see that it is a Hodge isometry, but this follows from the commutativity and the fact that Ψ is a Hodge isometry. \square

Remark 3.6.9. Everything still applies to the more general case of an equivalence $\Phi : D^b(X_1) \simeq D^b(X_2)$ between two different HK manifolds, provided they are of one the known deformation types. Indeed, X_1 and X_2 are as above, then they are also deformation equivalents, as can be seen just by checking the list. Then one can just choose an isometry $H^2(X_1, \mathbb{Q}) \cong H^2(X_2, \mathbb{Q})$ which allows to identify the quadratic vector spaces $\tilde{H}(X_1, \mathbb{Q})$ and $\tilde{H}(X_2, \mathbb{Q})$. With this identification, the above arguments still work.

Notice also that, for the known deformation types, the assumption of Theorem 3.6.8 are always satisfied.

Example 3.6.10. If $L \in \mathrm{Pic}(X)$ is a line bundle, the tensorization by L is an autoequivalence. Let $\lambda = c_1(L) \in H^2(X, \mathbb{Z})$, then the induced isometry on $\tilde{H}(X, \mathbb{Q})$ is

$$B_\lambda(r\alpha + \mu + s\beta) = r\alpha + \mu + r\lambda + \left(s + q(\lambda, \mu) + r \frac{q(\lambda, \lambda)}{2} \right) \beta. \quad (3.18)$$

Indeed, $B_\lambda \in \mathrm{SO}(\tilde{H}(X, \mathbb{Q}))$ is the exponential of the nilpotent operator $e_\lambda \in \mathfrak{so}(\tilde{H}, \mathbb{Q})$. Hence, it acts on $\mathrm{SH}(X)$ by the exponential of e_λ , which means by multiplication by $\exp(\lambda)$. Since the latter is the action in cohomology induced by tensorization by L , uniqueness of $\Phi^{\tilde{H}}$ gives the result.

3.6.1 Geometric interpretation

So far the classes α and β in $\tilde{H}(X, \mathbb{Q})$ are purely abstract classes, but the exact sequence of Theorem 3.6.4 allows us to give a geometric interpretation for them. Before doing that, we recall the following fact.

Theorem 3.6.11 ([37, 51]). *Let $\omega \in H^{4i}(X, \mathbb{R})$ be a class that remains of type $(2i, 2i)$ on every small deformation of X . Then, there exists a constant $C(\omega) \in \mathbb{R}$ such that*

$$\int_X a^{2n-2i} \omega = C(\omega) q(a)^{n-i},$$

for every $a \in H^2(X, \mathbb{R})$.

This constant $C(\omega)$ is called generalized Fujiki constant. Notice that $C(1)$ differs from c_X of Remark 3.6.6 by the factor $(2n-1)!!$.

Definition 3.6.12 ([17, Section 3]). Let X be a HK manifold of dimension $2n$. For every $1 \leq i \leq n$, denote by $\mathfrak{q}_{2i} \in \mathrm{SH}^{4i}(X, \mathbb{Q})$ the class defined by the property

$$\int_X a^{2n-2i} \mathfrak{q}_{2i} = c_X \frac{(2n-2i)!}{2^{n-i}(n-i)!} q(a)^{n-i} = c_X (2n-2i-1)!! q(a)^{n-i}, \quad (3.19)$$

for every $a \in H^2(X, \mathbb{Q})$. For $i = 0$ we set $\mathfrak{q}_0 := 1$.

Remark 3.6.13. If a non-zero class $\omega \in \mathrm{SH}(X)^{2i}$ is monodromy invariant (i.e. it stays of type (i, i) for every small deformation of X), then $i = 2j$ must be even, and ω is a multiple of \mathfrak{q}_{2j} by Theorem 3.6.11. Indeed, the polynomial on $H^2(X, \mathbb{Q})$

$$P(a) := \int_X a^{2n-i} \cup \omega$$

vanishes on $\{a \in H^2(X, \mathbb{Q}) \mid q(a) = 0\}$. In particular, it has even degree, so i is even.

To relate α and β with these \mathfrak{q}_{2i} , we consider the map

$$T : \mathrm{Sym}^n \tilde{H}(X, \mathbb{Q}) \rightarrow \mathrm{SH}(X) \quad (3.20)$$

defined as the orthogonal splitting of the injection Ψ . It is $\mathfrak{g}(X)$ -equivariant, because it is a projection onto an irreducible subrepresentation. For an element $x \in \tilde{H}(X, \mathbb{Q})$ we denote by $x^{(n)} \in \mathrm{Sym}^n \tilde{H}(X, \mathbb{Q})$ the n -th symmetric power.

Proposition 3.6.14 ([17, Lemma 3.5]). *For $0 \leq i \leq n$ we have*

$$T(\alpha^{(n-i)}\beta^{(i)}) = (n-i)!\mathbf{q}_{2i}.$$

Proof. For $i = 0$ we have $T(\alpha^{(n)}/n!) = T(\Psi(1)) = 1$. For $i \geq 1$, by definition $T(\alpha^{(n-i)}\beta^{(i)}) \in \text{SH}(X)^{4i}$, so we need to prove that

$$\int_X x^{2n-2i} T(\alpha^{(n-i)}\beta^{(i)}) = \frac{(2n-2i)!}{2^{n-i}} q(x)^{n-i},$$

for every $x \in H^2(X, \mathbb{Q})$. We have

$$\int_X x^{2n-2i} T(\alpha^{(n-i)}\beta^{(i)}) = b_{\text{SH}}(x^{2n-2i}, T(\alpha^{(n-i)}\beta^{(i)})) = b_{[n]}(\Psi(x^{2n-2i}), \alpha^{(n-i)}\beta^{(i)}).$$

Using that $\Psi(x^{2n-2i}) = e_x^{2n-2i}(\alpha^{(n)}/n!)$, and doing explicit calculations we conclude. We refer to [17] for details. \square

Following [17, (3.1)] define the constant

$$r_X := \frac{C(c_2(X))2^n n!(2n-1)}{(2n)!24c_X} \in \mathbb{Q} \quad (3.21)$$

The explicit value for the known deformation types is

$$r_X = \begin{cases} \frac{n+3}{4} & \text{for K3}^{[n]} \text{ or OG10 type} \\ \frac{n+1}{4} & \text{for Kum}_n \text{ or OG6 type} \end{cases} \quad (3.22)$$

Corollary 3.6.15 ([17, Lemma 3.3]). *The projection of the square root of the Todd class onto the Verbitsky component is*

$$(\text{td}_X^{1/2})_{\text{SH}} = \sum_i \frac{r_X^i}{i!} \mathbf{q}_{2i} = T\left(\frac{(\alpha + r_X \beta)^{(n)}}{n!}\right).$$

Proof. Again we refer to [17] for the full proof, here we just outline the argument. The idea is a simple but often useful one. We exploit the relation (cf [90])

$$\int_X \text{td}^{1/2} \exp(\omega) = (1 + \lambda(\omega))^n \int_X \text{td}_X^{1/2}, \quad (3.23)$$

where ω is any class in $H^2(X, \mathbb{R})$ and $\lambda(\omega)$ is its characteristic value (cf [90, Definition 17]). Evaluating it in $t\omega$, for $t \in \mathbb{R}$, we get an equation between two polynomials. Comparing the coefficients in the various degrees, and using Proposition 3.6.14 we get the result. \square

3.7 Mukai lattice and Hochschild cohomology

In this section we explore the relationship between the extended Mukai lattice and the second Hochschild cohomology. As a result we will obtain a way of computing the action of an equivalence on HH^2 .

Consider the morphism

$$\mu : \mathrm{HT}^2(X) \rightarrow \tilde{H}(X, \mathbb{C}), \quad \eta \mapsto e'_\eta(\sigma). \quad (3.24)$$

It is the composition

$$\mathrm{HT}^2(X) \xrightarrow{e'} \mathfrak{g}(X) \xrightarrow{\mathrm{ev}_\sigma} \tilde{H}(X, \mathbb{Q}),$$

Lemma 3.7.1 ([77]). *The morphism μ is injective. Moreover, it satisfies*

$$\mu(H^0(\bigwedge^2 T_X)) = \mathbb{C}\alpha, \quad \mu(H^1(T_X)) = H^{1,1}(X), \quad \mu(H^2(\mathcal{O}_X)) = \mathbb{C}\beta.$$

Proof. Consider the analogous morphism

$$H^2(X, \mathbb{C}) \rightarrow \tilde{H}(X, \mathbb{C}) \quad \lambda \mapsto e_\lambda(\alpha).$$

By definition it is injective, and the image is the degree 0 component. This morphism is obtained by conjugating μ with the action of η , as shown in (3.14). It follows that μ is injective. Moreover, by (3.13) the operator η exchanges the eigenspaces of h and h' , which implies the rest of the statement. \square

A general version (without assumptions on the dimension or the b_2) of Theorem 3.6.8 can be stated as follows. The proof follows from Theorem 3.6.7 in the same way.

Lemma 3.7.2. *Let $\Phi : D^b(X_1) \simeq D^b(X_2)$ be an equivalence between HK manifolds. Then, there exists a Hodge similitude $\Phi^{\tilde{H}}$ and a scalar $\lambda \in \mathbb{Q}^*$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathrm{SH}(X) & \xrightarrow{\Phi^{\mathrm{SH}}} & \mathrm{SH}(X) \\ \downarrow \Psi & & \downarrow \Psi \\ \mathrm{Sym}^n \tilde{H}(X, \mathbb{Q}) & \xrightarrow{\lambda \mathrm{Sym}^n \Phi^{\tilde{H}}} & \mathrm{Sym}^n \tilde{H}(X, \mathbb{Q}) \end{array}$$

In particular, $\Phi^{\tilde{H}}$ is equivariant with respect to $\Phi^{\mathfrak{g}}$ that is:

$$\Phi^{\mathfrak{g}}(f) \cdot \Phi^{\tilde{H}}(x) = \Phi^{\tilde{H}}(f \cdot x)$$

for every $f \in \mathfrak{g}(X)$ and $x \in \tilde{H}(X, \mathbb{Q})$.

Here by similitude we mean an element of $O(\tilde{H}(X, \mathbb{Q})) \times \mathbb{Q}^*$. Since $\Phi^{\tilde{H}}$ respect the Hodge structure, there is a constant $c \in \mathbb{C}^*$ such that $\Phi^{\tilde{H}}(\sigma) = c'\sigma$. If moreover $\Phi^{\tilde{H}}$ is an isometry (e.g. if n is odd or $b_2(X)$ is odd), then $|c'| = 1$.

Proposition 3.7.3. *Let $\Phi : D^b(X_1) \simeq D^b(X_2)$ be an equivalence of HK manifolds. Then there is $c \in \mathbb{C}^*$ such that*

$$c \cdot \Phi^{\text{HT}} = \mu^{-1} \circ \Phi^{\tilde{H}} \circ \mu.$$

Proof. First notice that, since $\Phi^{\tilde{H}}$ respects the Hodge structure, it maps the $(1, 1)$ part to itself. The $(1, 1)$ part coincides with the image of μ , hence the statement makes sense. By the construction of $\Phi^{\mathfrak{g}}$ in Corollary 3.5.9, we see that the diagram

$$\begin{array}{ccc} \text{HT}^2(X_1) & \xrightarrow{e'} & \mathfrak{g}(X_1) \\ \simeq \downarrow \Phi^{\text{HT}} & & \simeq \downarrow \Phi^{\mathfrak{g}} \\ \text{HT}^2(X_2) & \xrightarrow{e'} & \mathfrak{g}(X_2) \end{array}$$

is commutative. Which, combined with the lemma above, implies

$$e'_{\Phi^{\text{HT}}(\mu)}(\Phi^{\tilde{H}}(\sigma)) = \Phi^{\tilde{H}}(e'_{\mu}(\sigma)).$$

Using, $\Phi^{\tilde{H}}(\sigma) = c\sigma$ this gives the statement. □

Chapter 4

Atomic and modular sheaves

4.1 Introduction

In the first part of this chapter we review the papers [17, 15, 77], which introduce the notion of an atomic object, and the papers [94, 93] which deal with modular sheaves.

To introduce the notion of atomicity, we take the point of view of Markman [77], and we deal with the obstruction map first. To any object $E \in D^b(X)$ there is associated a map

$$\chi_E : \mathrm{HT}^2(X) \rightarrow \mathrm{Ext}^2(E, E)$$

which measures the obstruction to lift E along a non-commutative deformation. Then we say that E is 1-obstructed if χ_E has rank one. This is a very strong condition on E , and we expect that it should be related to the uniqueness of the symplectic form on a moduli space.

More precisely, assume that \mathcal{M} is a moduli space of stable sheaves on X . In Section 4.8.2 we show that if $\eta \in \mathrm{HT}^*(X)$, the map

$$\tau_\eta : \mathrm{Ext}^1(E, E) \times \mathrm{Ext}^1(E, E) \rightarrow \mathbb{C}, (a, b) \mapsto \mathrm{Tr}(\chi_E(\eta) \circ a \circ b) \quad (4.1)$$

defines a closed two form on the smooth locus of \mathcal{M} . If we want \mathcal{M} to be a smooth projective HK, it should have a unique symplectic form. The expression of τ_η suggests that the sheaves it parametrizes should be 1-obstructed.¹

The issue here is that we don't know if the rank of the obstruction map is constant in families of stable sheaves. To remedy this one can investigate the situation at a cohomological level. In complete analogy to the previous case, there is a cohomological obstruction map

$$\chi_E^H : \mathrm{HT}^2(X) \rightarrow H^*(X, \mathbb{C})$$

¹This turns out not to be true in examples, but we feel it is a useful heuristic.

which measures the obstruction for a class to remain of Hodge type along a non-commutative deformation. We say that E is cohomologically 1-obstructed if χ_E^H has rank one. A priori this is a weaker notion than 1-obstructedness, but it is conjectured that for a stable sheaf the two are equivalent.

The cohomologically 1-obstructed objects are also called atomic (the name comes from [15]) Actually, the original definition of atomic object involves the action of the LLV algebra, see Definition 4.2.16, but it is equivalent to cohomological 1-obstructedness (see Theorem 4.2.17).

The most important feature of an atomic object is its extended Mukai vector

$$\tilde{v}(E) \in \tilde{H}(X, \mathbb{Q}).$$

Besides being invariant, up to constants, via derived equivalences, it is also intimately related to the usual Mukai vector. Indeed, if

$$T : \text{Sym}^n \tilde{H}(X, \mathbb{Q}) \rightarrow \text{SH}(X)$$

denotes the orthogonal projection (3.20), then we have

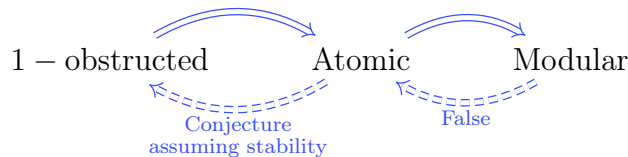
$$T(\tilde{v}(E)^{(n)}) = av(E)$$

where $a \in \mathbb{Q}$ is a non-zero constant, see Proposition 4.3.2.

Using this fact in Section 4.3.2 we give the explicit formula of the Mukai vector of an atomic sheaf on a fourfold. In the same way, in Section 4.3.3, we compute the projection onto the Verbitsky component of the discriminant of an atomic sheaf, which turns out to be

$$\Delta(E)_{\text{SH}} = (\tilde{q}(\tilde{v}(E), \tilde{v}(E)) + 2r_X r(E)^2) \mathfrak{q}_2.$$

This shows also that an atomic torsion-free sheaf is modular. This implication is useful because we are ultimately interested in studying moduli of stable atomic sheaves, and stability behaves for modular sheaves as it does for sheaves on surfaces. The chain of relationships between these notion can be summarized as follows.



The last part of the chapter, Section 4.8 is original work. Here we study the geometry of moduli spaces of atomic sheaves, with the most important result being Theorem 4.8.8.

Theorem 4.1.1 (Theorem 4.8.8). *Let X be a hyper-Kähler manifold of dimension at least four, and $\mathcal{E} \rightarrow S \times X$ a family of slope stable projectively hyperholomorphic vector bundles. Assume moreover that S is integral and normal, and that the function*

$$s \rightarrow \text{ext}^1(\mathcal{E}_s, \mathcal{E}_s)$$

is constant on an open $U \subset S$, with complement $S - U$ of codimension at least two. Then it is constant on all of S .

We see that the important condition for the smoothness of the moduli space is that it parametrizes projectively hyperholomorphic bundles. This is always the case for modular sheaves on a hyper-Kähler manifold of type K3^[2].

We give a brief overview of the argument to give the reader a feeling for the ideas which come in to play. Assume that \mathcal{E} is a vector bundle over $S \times X$, with the property that for every $s \in S$ the sheaf \mathcal{E}_s is slope stable and projectively hyperholomorphic. We anticipated in Theorem 0.3.3 that cup product with $\bar{\sigma} \in H^2(X, \mathcal{O}_X)$ induces an isomorphism

$$\text{Ext}^1(\mathcal{E}_s, \mathcal{E}_s) \rightarrow \text{Ext}^{2n-1}(\mathcal{E}_s, \mathcal{E}_s).$$

It is not hard to see that this isomorphism is induced by Equation (4.1) where $\eta = \bar{\sigma}$. We show in Section 4.8.2 that this form globalizes to a morphism

$$\mathcal{E}xt_{\pi}^1(\mathcal{E}, \mathcal{E}) \rightarrow \mathcal{E}xt^{2n-1}(\mathcal{E}, \mathcal{E})$$

compatible with the base change maps

$$\mathcal{E}xt_{\pi_2}^i(\mathcal{E}, \mathcal{E})|_s \rightarrow \text{Ext}^i(\mathcal{E}_s, \mathcal{E}_s), \text{ for } i = 1, 2n - 1.$$

The idea behind the proof of Theorem 4.8.8 is to use this section to compare the two base change maps, and deduce via the Cohomology and base change Theorem that $\mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})$ is locally free, and the base change

$$\mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})|_s \rightarrow \text{Ext}^1(\mathcal{E}_s, \mathcal{E}_s)$$

is an isomorphism.

4.1.1 Structure of the chapter

In Section 4.2 we begin by reviewing the theory of first order non commutative deformations of a variety X , following [104]. The main result we recall is Proposition 4.2.7, which naturally leads to the definition of the obstruction map. Then we discuss the cohomological obstruction map, we recall the definition of atomic objects, and we show that it coincides with cohomologically 1-obstructed.

In Section 4.3 we investigate the numerical properties of atomic objects. On fourfolds, we compute their Mukai vector in terms of the extended Mukai vector. In general, we compute their discriminant and show that atomic implies modular. Lastly, we compute the Euler pairing $\chi(E, E)$ as a function of the extended Mukai vector $\tilde{v}(E)$.

In Section 4.4 we slightly extend the definition of an atomic sheaf to the twisted case. This will be relevant in Chapter 6. We do not generalize the entire theory, but we limit ourself to explain what is the extended Mukai vector in that case.

In Section 4.5 we recall the first examples of atomic bundles which appeared in the literature. We follow the works of O’Grady [94, 93] and Markman [77].

In Section 4.6 we deal with the case of atomic objects supported on Lagrangians submanifolds. Following [15], we recall the conditions for the structure sheaf of a Lagrangian to be atomic in Theorem 4.6.2. We also present in Theorem 4.6.6 a slight generalization in the case of dimension four. This is based on the computations in Section 4.3 and gives numerical conditions for any object supported on a Lagrangian surface to be atomic. We also review some geometric examples of Lagrangians which satisfy these conditions.

In Section 4.7 we review the theory of slope stability for modular sheaves as developed in [94, 93]. In particular we recall the wall and chamber decomposition, and how to check stability for modular sheaves on a Lagrangian fibration. Of particular importance for later is the notion of a suitable polarization, see Definition 4.7.8.

In Section 4.8 we prove some of the main results of this thesis, namely Theorem 4.8.8. In doing so, we also show that the obstruction map globalizes and it gives rise to a morphism

$$\mathrm{HH}^2(X) \rightarrow H^0(M, \Omega_M^2)$$

which allows to construct holomorphic two forms on the moduli space.

4.2 Obstruction map

4.2.1 Non commutative deformations, after Toda

Let X be a smooth projective variety. A first order deformation of X is a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & \lrcorner & \downarrow \pi \\ \mathrm{Spec}\mathbb{C} & \longrightarrow & \mathrm{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2) \end{array}$$

where π is a flat map. It is well known that first order deformations of X are parametrized by the cohomology group $H^1(X, T_X)$. There is a similar theory for infinitesimal deformations of the abelian category $\text{Coh}(X)$, whose the deformations are controlled by $\text{HH}^2(X)$, see [74, 73]. Here we will not need the full theory, and so we roughly explain how to associate a deformation of $\text{Coh}(X)$ to an element in $\text{HH}^2(X)$, for more details see [104, Section 4]. The construction goes through the (unmodified) HKR isomorphism (3.8)

$$I^{\text{HKR}} : \text{HH}^2(X) \simeq \text{HT}^2(X) = H^2(X, \mathcal{O}_X) \oplus H^1(X, T_X) \oplus H^0(X, \bigwedge^2 T_X).$$

Take an element $v \in \text{HH}^2(X)$ and write $I^{\text{HKR}}(v) = (\alpha, \beta, \gamma) \in \text{HT}^2(X)$. We want a $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ -linear category $\text{Coh}(X, v)$. For this, we first construct a sheaf of $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ algebras $\mathcal{O}_X^{(\beta, \gamma)}$, and then use α to take twisted sheaves in the usual sense. Notice that:

- (1) The element $\beta \in H^1(X, T_X)$ corresponds to a first-order (usual) deformation \mathcal{X} of X , as recalled before.
- (2) The element γ gives rise to a bidifferential operator $\mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X$.

So, as $\mathcal{O}_X^{\beta, \gamma}$ we take the sheaf $\mathcal{O}_{\mathcal{X}}$ with multiplication modified by the bidifferential operator γ . The element $\alpha \in H^2(X, \mathcal{O}_X)$ can be represented by a Čech cocycle $\{\alpha_{ijk}\}_{ijk}$, which can be lifted to a 2-cocycle of $\mathcal{O}_X^{\beta, \gamma}$ by $\tilde{\alpha}_{ijk} := \{1 - \alpha_{ijk}\varepsilon\}_{ijk}$.

Definition 4.2.1 ([104, Definition 4.4]). The category $\text{Coh}(X, v)$ is the category of $\tilde{\alpha}$ -twisted coherent modules over the sheaf of non-commutative algebras $\mathcal{O}_X^{\beta, \gamma}$. We also define the $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ -linear category

$$D^b(X, v) := D^b(\text{Coh}(X, v)).$$

This construction is compatible with the action of derived equivalences, in a way which gives a concrete interpretation for the induced isomorphism $\Phi^{\text{HH}} : \text{HH}^2(X_1) \simeq \text{HH}^2(X_2)$. From the construction of $\mathcal{O}_X^{\beta, \gamma}$ it follows that there is a morphism of algebras

$$i : \mathcal{O}_X^{\beta, \gamma} \rightarrow \mathcal{O}_X.$$

This induces two functors

$$i_* : D^b(X) \rightarrow D^b(X, v) \quad \mathbf{L}i^* : D^b(X, v) \rightarrow D^b(X) \tag{4.2}$$

which realize $D^b(X)$ as the ‘central fiber’ of the first order non commutative deformation $D^b(X, v)$. Lastly, denote by $J : \text{HH}^2(X) \simeq \text{HH}^2(X)$ the automorphism obtained by

$$(\alpha, \beta, \gamma) \mapsto (\alpha, -\beta, \gamma)$$

after applying I^{HKR} .

Theorem 4.2.2 ([104, Theorem 1.2]). *Let $\Phi_{\mathcal{E}} : D^b(X_1) \simeq D^b(X_2)$ be an equivalence between smooth projective varieties with Fourier-Mukai kernel \mathcal{E} , and let $\Phi_{\mathcal{E}}^{HH} : D^b(X_1) \rightarrow D^b(X_2)$ the induced isomorphism. Then, there exists an object $\tilde{\mathcal{E}} \in D_{\text{perf}}(X_1 \times X_2, -J(v_1) \boxplus v_2)$ inducing a Fourier-Mukai equivalence*

$$\Phi_{\tilde{\mathcal{E}}} : D^b(X_1, v_1) \simeq D^b(X_2, v_2)$$

such that $\mathbf{Li}^* \tilde{\mathcal{E}} \cong \mathcal{E}$, if and only if $\Phi^{\text{HH}}(v_1) = v_2$.

For the proof of this result we refer to [104]. The main difficulty can be abstracted by the following general question:

Question: Given an object $\mathcal{E} \in D^b(X)$ and a class $v \in \text{HH}^2(X)$, when can we lift \mathcal{E} to a perfect complex $\tilde{\mathcal{E}} \in D_{\text{perf}}(X, v)$ such that $\mathbf{Li}^* \tilde{\mathcal{E}} \cong \mathcal{E}$?

4.2.2 Obstruction to lifting a deformation

We first recall how to deform a coherent sheaf F along a first order (commutative) deformation of X . Let X be a smooth projective variety. Consider the extension

$$0 \rightarrow \Delta_* \Omega_X^1 \rightarrow \mathcal{O}_{X \times X} / I_{\Delta}^2 \rightarrow \Delta_* \mathcal{O}_X \rightarrow 0 \quad (4.3)$$

coming from the isomorphism $\Delta_* \Omega_X^1 \simeq I_{\Delta} / I_{\Delta}^2$. The universal Atiyah class is the class $\text{At}_X \in \text{Ext}_{X \times X}^1(\mathcal{O}_{\Delta}, \Delta_* \Omega_X^1)$ represented by the sequence above. Taking Fourier-Mukai transforms, we obtain a natural transformation

$$\Phi_{\Delta_*(\mathcal{O}_X)} \rightarrow \Phi_{\Delta_* \Omega_X^1}[1].$$

Since $\Phi_{\Delta_*(\mathcal{O}_X)} = \text{Id}_X$ and $\Phi_{\Delta_* \Omega_X^1} = - \otimes \Omega_X^1$, for every $F \in D^b(X)$ we have a map

$$F \rightarrow F \otimes \Omega_X^1[1] \in \text{Ext}^1(F, F \otimes \Omega_X^1).$$

This is called the Atiyah class $\text{At}_X(F)$ of F .

Remark 4.2.3. Assume F is a locally-free sheaf, and let $J_X^1(F) := \Phi_{\mathcal{O}_{X \times X} / I_{\Delta}^2}(F)$ be the first jet bundle. It is easy to see that a splitting of the sequence

$$0 \rightarrow F \otimes \Omega_X^1 \rightarrow J_X^1(F) \rightarrow F \rightarrow 0$$

exists if and only if there is an algebraic connection $F \rightarrow F \otimes \Omega_X^1$. So we can interpret the Atiyah class $\text{At}(F)$ as the obstruction to the existence of an algebraic connection.

Let F be a coherent sheaf on smooth projective X . Define the obstruction map $\text{ob}_F : H^1(X, T_X) \rightarrow \text{Ext}^2(F, F)$ by the composition

$$\text{ob}_F : \text{Ext}_X^1(\Omega_X^1, \mathcal{O}_X) \xrightarrow{-\otimes \text{id}_F} \text{Ext}^1(\Omega_X^1 \otimes F, F) \xrightarrow{-\circ \text{At}_X(F)} \text{Ext}^2(F, F). \quad (4.4)$$

where \circ denotes the Yoneda pairing. The following classical result justifies the name.

Theorem 4.2.4 ([56, p. IV.3.1.8]). *Let $k \in H^1(X, T_X)$ be a first order deformation of X , and denote $i : X \hookrightarrow \mathcal{X}$ the embedding of X into the total space. There exists a flat coherent sheaf \mathcal{F} on \mathcal{X} such that $\mathcal{F}|_X = F$ if and only if $\text{ob}_F(k) = 0$.*

To extend this to the non commutative deformations in $\text{HT}^2(X)$ we proceed as follows. Combining the Yoneda product with the exterior product on the algebra Ω_X^* one gets a product

$$\cup_E : \text{Ext}^i(E, E \otimes \Omega_X^p) \otimes \text{Ext}^j(E, E \otimes \Omega_X^q) \rightarrow \text{Ext}^{i+j}(E, E \otimes \Omega_X^{p+q}).$$

More explicitly call $\varepsilon : \Omega_X^p \otimes \Omega_X^q \rightarrow \Omega_X^{p+q}$ the natural map. Then for $f \in \text{Ext}^i(E, E \otimes \Omega_X^p)$ and $g \in \text{Ext}^j(E, E \otimes \Omega_X^q)$ the element $f \cup_E g$ is obtained as the following composition

$$E \xrightarrow{f} E \otimes \Omega_X^p[i] \xrightarrow{g \otimes \text{id}_{\Omega_X^p[i]}} E \otimes \Omega_X^q[j] \otimes \Omega_X^p[i] \xrightarrow{\varepsilon^{[i+j]}} E \otimes \Omega_X^{p+q}[i+j].$$

In particular in this way we can define the powers of the Atiyah class, and its exponential

$$\exp(\text{At}_X(E)) = \sum_q \frac{\text{At}_X(E)^q}{q!} \in \bigoplus_q \text{Ext}^q(E, E \otimes \Omega_X^q). \quad (4.5)$$

We can also pair an element in $H^p(X, \bigwedge^q T_X)$ with $\text{At}_X(E)^q$ via

$$\text{Ext}^p(\Omega_X^q, \mathcal{O}_X) \times \text{Ext}^q(E, E \otimes \Omega_X^q) \rightarrow \text{Ext}^{p+q}(E, E), \quad v \mapsto (\text{id}_E \otimes v) \circ \text{At}_X(E)^q.$$

Extending this by additivity to the exponential we define:

Definition 4.2.5. The (non-commutative) obstruction map for an object $E \in \text{D}^b(X)$ is defined by

$$\chi_E : \text{HT}^2(X) \rightarrow \text{Ext}^2(E, E), \quad v \mapsto (\text{id}_E \otimes v) \circ \exp(\text{At}_X(F)).$$

In components it becomes

$$(\alpha, \beta, \gamma) \mapsto \alpha \otimes \text{id}_F + (\beta \otimes \text{id}_F) \circ \text{At}_X(F) + (\gamma \otimes \text{id}_F) \circ \text{At}_X^2(F)/2.$$

Remark 4.2.6. This definition also works for $\mathrm{HT}^k(X)$ for every k :

$$\mathrm{HT}^k(X) \rightarrow \mathrm{Ext}^k(E, E), \quad v \mapsto v \circ \exp(\mathrm{At}_X(F)).$$

In fact, one can put all of these together to get a graded algebra morphism $\mathrm{HT}^*(X) \rightarrow \mathrm{Ext}^*(F, F)$, as we will see later.

As the name suggests this map is the non-commutative version of the map above, in the sense that it gives the obstruction to lift a perfect complex along a non commutative deformation of X .

Proposition 4.2.7 ([104, Proposition 6.1]). *Let $E \in \mathrm{D}^b(X)$ and $v \in \mathrm{HT}^2(X)$. Assume that $\chi_E(v) = 0$, then there exists a perfect complex $\tilde{E} \in \mathrm{D}_{\mathrm{perf}}^b(X, v)$ such that $\mathbf{L}i^*E \simeq \tilde{E}$.*

4.2.3 Relation to Hochschild cohomology

There is a more functorial way to describe the obstruction map via the (unmodified) HKR isomorphism $I^{\mathrm{HKR}} : \mathrm{HH}^*(X) \simeq \mathrm{HT}^*(X)$ introduced in (3.8). Recall that the Fourier–Mukai transform with kernel $\Delta_*\mathcal{O}_X[k] \in \mathrm{D}^b(X \times X)$ is the shift $[k]$ functor. In this way, we can represent the objects in $\mathrm{HH}^k(X) = \mathrm{Ext}^k(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X[k])$ as natural transformations

$$\mathrm{id}_{\mathrm{D}^b(X)} \rightarrow \mathrm{id}_{\mathrm{D}^b(X)}[k].$$

So, for every $E \in \mathrm{D}^b(X)$ we get a morphism of graded algebras

$$\mathrm{ev}_E : \mathrm{HH}^*(X) \rightarrow \mathrm{Ext}^*(E, E), \quad \eta \rightarrow \eta_E : E \rightarrow E[k].$$

That is, we interpret $\eta \in \mathrm{HH}^k(X)$ as a natural transformation and we evaluate it at E . This is related to the obstruction map defined above by the following result.

Proposition 4.2.8 ([43, Theorem A]). *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{HH}^*(X) & \xrightarrow{I^{\mathrm{HKR}}} & \mathrm{HT}^*(X) \\ \mathrm{ev}_E \downarrow & & \swarrow \chi_E \\ \mathrm{Ext}^*(E, E) & & \end{array}$$

In particular, the obstruction map $\chi : \mathrm{HT}^(X) \rightarrow \mathrm{Ext}^*(E, E)$ is a morphism of graded algebras.*

Remark 4.2.9. With this interpretation of the obstruction map it is immediate that it is compatible with derived equivalences.

$$\begin{array}{ccc} \mathrm{HH}^2(X_1) & \xrightarrow{\Phi^{\mathrm{HH}}} & \mathrm{HH}^2(X_2) \\ \downarrow \mathrm{ev}_E & & \downarrow \mathrm{ev}_{\Phi(E)} \\ \mathrm{Ext}^2(E, E) & \xrightarrow{\Phi} & \mathrm{Ext}^2(\Phi(E), \Phi(E)) \end{array}$$

Before giving the proof we explain the relation between the HKR isomorphism and the exponential of the universal Atiyah class:

$$\exp(\mathrm{At}_X) : \Delta_* \mathcal{O}_X \rightarrow \bigoplus_i \Delta_* \Omega_X^i[i].$$

Consider an element $u \in H^p(X, \bigwedge^q T_X)$ as a morphism

$$u : \Omega_X \rightarrow \mathcal{O}_X[p].$$

Then, the element $(I^{\mathrm{HKR}})^{-1}(u) \in \mathrm{HH}^{p+q}(X) = \mathrm{Hom}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X[p+q])$ can be described as the following composition

$$\Delta_* \mathcal{O}_X \xrightarrow{\mathrm{At}_X^q} \Delta_* \Omega_X^q[q] \xrightarrow{\Delta_* u[q]} \Delta_* \mathcal{O}_X[p+q]. \quad (4.6)$$

This follows from [27, Proposition 4.4], and writing down the inverse of the HKR isomorphism as in [27, Proof of Corollary 4.2].

Proof of Proposition 4.2.8. We prove that for $u \in H^p(X, \bigwedge^q T_X)$, we have

$$\mathrm{ev}_E((I^{\mathrm{HKR}})^{-1}(u)) = \chi_E(u).$$

Taking Fourier-Mukai of the composition (4.6) we get natural transformations

$$\mathrm{id}_X \implies - \otimes \Omega_X^q[q] \implies \mathrm{Id}_X[p+q],$$

where the first one is induced by At_X^q and the second one by $\Delta_* u[q]$. Evaluating at E gives

$$E \xrightarrow{\mathrm{At}^q(E)} E \otimes \Omega_X^q[q] \xrightarrow{\mathrm{id}_E \otimes u[q]} E[p+q],$$

which is exactly the formula in Definition 4.2.5. For a more general statement see also [104, Lemma 5.8]. \square

Remark 4.2.10. Consider the case of an object $E \in \mathrm{D}^b(X)$ with non-zero rank. Then if $\bar{\sigma} \in H^2(X, \mathcal{O}_X)$ is the conjugate of the symplectic form, we get that

$$\chi_E(\bar{\sigma}) = \mathrm{id}_E \otimes \bar{\sigma} : E \rightarrow E[2].$$

Under the trace map $\mathrm{Tr} : \mathrm{Ext}^2(E, E) \rightarrow H^2(X, \mathcal{O}_X)$ we see that this class becomes $\mathrm{rk}(E) \cdot \bar{\sigma}$, as showed in [48, Lemma 10.1.3(i)] In particular is non-zero, so that $H^2(X, \mathcal{O}_X) \subset \mathrm{HT}^2(X)$ is not contained in the kernel of the obstruction map.

4.2.4 Cohomological obstruction map

A necessary condition for an object $E \in D^b(X)$ to deform along a (commutative) deformation $\pi : \mathcal{X} \rightarrow \Delta$ of X is that its chern character $\text{ch}(E) \in H^*(X, \mathbb{Q})$ remains of Hodge type on \mathcal{X} . If the base space is geometric (e.g. a smooth curve) this means that the parallel transport of $\text{ch}(E)$ along the fibers of the local system $R^*\pi_*\underline{\mathbb{Q}}$ remains of Hodge type. If we take a first order deformation, one has to take the limit of this condition. That is we ask that under the contraction map

$$H^1(X, T_X) \rightarrow \bigoplus_{p \geq 1} H^{p+1}(X, \Omega_X^{p-1}), \quad v \mapsto v \lrcorner \text{ch}(E)$$

the class k corresponding to the deformation vanishes. Extending to the non commutative deformations we define the cohomological obstruction map.

Definition 4.2.11 ([77, 15]). Let $E \in D^b(X)$ define the cohomological obstruction map

$$\chi_E^H : \text{HT}^2(X) \rightarrow H^*(X, \mathbb{C}), \quad u \mapsto u \lrcorner v(E),$$

where $v(E)$ is the Mukai vector of E .

Remark 4.2.12. As usual we have replaced the Chern character by the Mukai vector, because the latter is more convenient when dealing with derived categories. Indeed, if $\Phi : D^b(X_1) \simeq D^b(X_2)$ is an equivalence, we have a commutative diagram

$$\begin{array}{ccc} \text{HH}^2(X_1) & \xrightarrow{\Phi^{\text{HH}}} & \text{HH}^2(X_2) \\ \downarrow \chi_E \circ I^H & & \downarrow \chi_{\Phi(E)} \circ I^H \\ H^*(X_1, \mathbb{C}) & \xrightarrow{\Phi^H} & H^*(X_2, \mathbb{C}). \end{array}$$

Definition 4.2.13. Let X be a HK manifold and $E \in D^b(X)$. We say that E is (numerically) 1-obstructed if the (cohomological) obstruction map has rank one.

Intuitively we would expect that if an object $E \in D^b(X)$ deforms along $u \in \text{HT}^2(X)$ then so must do its Mukai vector. This is clearly true for commutative deformation, but there is a subtlety which must be accounted for when also considering non-commutative deformation.

Define the Duflo isomorphism

$$D : \text{HT}^2(X) \simeq \text{HT}^2(X), \quad v \mapsto \text{td}_X^{-1/2} \lrcorner v. \quad (4.7)$$

This is the modification required to make the HKR isomorphism compatible with the algebraic structure on Hochschild cohomology.

Proposition 4.2.14. *If $u \in \mathrm{HT}^2(X)$ is such that $\chi_E(u) = 0$, then $D(u) \lrcorner v(E) = 0$. That is*

$$D(\ker(\chi_E)) \subseteq \ker \chi_E^H.$$

In particular, a 1-obstructed object E is numerically 1-obstructed.

Proof. This follows from [43, Lemma 3.2] together with the fact that the modified HKR maps the action of $\mathrm{HH}^*(X)$ on $\mathrm{HH}_*(X)$ to the action of $\mathrm{HT}^*(X)$ by contraction on cohomology. \square

Remark 4.2.15. On a K3 surface S the cohomological obstruction map is a linear map

$$\chi_E^H : \mathrm{HT}^2(S) \rightarrow H^2(S, \mathcal{O}_S).$$

So every object with non-zero Mukai vector is numerically 1-obstructed. Indeed, after tensor product with a line bundle and spherical twists we can always assume that the rank is non-zero, so the cohomological obstruction map does not vanish. This is also immediate using Theorem 4.2.17.

Notice that, even on a K3 surface, it is not true that every object is 1-obstructed. For example, if $L \in \mathrm{Pic}(S)$ is a non-trivial line bundle, then $\mathcal{O}_S \oplus L$ has obstruction map of rank two, as shown in [15, Example 4.4].

On a higher dimensional hyperkähler, the cohomological obstruction map can take different values, and one could try to classify objects depending on this rank. The rank is zero only for very degenerate objects: their Mukai vector would have to be annihilated by the action of the whole LLV algebra, and this imposes very strong numerical conditions, see [15, Appendix A].

4.2.5 Atomic objects

The next step are the objects with cohomological obstruction map of rank one, which are the object of interest in this thesis. They have been independently studied by Markman [77] and Beckmann [15] because their behaviour is very similar to that of objects on a K3 surface.

Definition 4.2.16 ([15, Definition 1.1]). Let X be a HK manifold. An object $E \in \mathrm{D}^b(X)$ is called atomic if there exists a non zero $\tilde{v}(E) \in \tilde{H}(X, \mathbb{Q})$ such that

$$\mathrm{Ann}(v(E)) = \mathrm{Ann}(\tilde{v}(E))$$

as sub-Lie algebras of $\mathfrak{g}(X)$.

The original definition of atomic object is the following equivalent condition.

Theorem 4.2.17 ([15, Theorem 1.2]). *An object $E \in D^b(X)$ is atomic if and only if it is numerically 1-obstructed.*

Before giving the proof, recall that the action of $\mathrm{HT}^2(X)$ on $H^*(X, \mathbb{C})$ factors as the composition

$$\mathrm{HT}^2(X) \hookrightarrow \mathfrak{g}(X) \otimes \mathbb{C} \subset \mathrm{End}(H^*(X, \mathbb{C})),$$

where the first inclusion comes from Theorem 3.5.8. The kernel of the obstruction map is the intersection

$$\mathrm{Ann}_{\mathbb{C}}(v(E)) \cap \mathrm{HT}^2(X) \subset \mathrm{HT}^2(X).$$

In a similar way one can act via $\mathrm{HT}^2(X)$ on the Mukai lattice $\tilde{H}(X, \mathbb{C})$ with operators which have degree two with respect to the grading induced by h' . Hence we get a pairing

$$\mathrm{HT}^2(X) \times \tilde{H}(X, \mathbb{C})^{(1,1)} \rightarrow H^2(X, \mathcal{O}_X) = \mathbb{C}\bar{\sigma}, \quad (4.8)$$

where $\tilde{H}(X, \mathbb{C})_{\mathrm{alg}} = \mathbb{C}\alpha \oplus H^{1,1}(X) \oplus \mathbb{C}\beta$ is the $(1, 1)$ part of the extended Mukai lattice. By [77, p. 6.3] this pairing is non-degenerate.

Lemma 4.2.18. *Let $\mathbf{v} \in H^*(X, \mathbb{Q})$ be a non-zero algebraic class. Assume that $\mathrm{Ann}(x)_{\mathbb{C}} \cap \mathrm{HT}^2(X)$ is a hyperplane in $\mathrm{HT}^2(X)$. Then the Lie algebra $\mathrm{Ann}(x)_{\mathbb{C}}$ is abstractly isomorphic to $\mathfrak{so}(b_2(X) + 1)$.*

Proof. If we conjugate with η , as in Theorem 3.5.8, the decomposition in Theorem 3.2.14 (1), we get a ‘Hochschild’ version of the same decomposition

$$\mathfrak{g}(X)_{\mathbb{C}} = \mathfrak{g}'(X)_{\mathbb{C}, -2} \oplus \mathfrak{g}'(X)_{\mathbb{C}, 0} \oplus \mathfrak{g}'(X)_{\mathbb{C}, 2},$$

where the $\mathfrak{g}'(X)_{\mathbb{C}, i}$ are the weight spaces for the action of h' . Moreover, we have $\mathfrak{g}'(X)_{\mathbb{C}, -2} \cong \mathfrak{g}'(X)_{\mathbb{C}, 2} \cong \mathrm{HT}^2(X)$, and writing

$$\mathfrak{g}'(X)_{\mathbb{C}, 0} = \mathbb{C}h' \oplus \overline{\mathfrak{g}'(X)_{\mathbb{C}, 0}},$$

we get that $\overline{\mathfrak{g}'(X)_{\mathbb{C}, 0}} \simeq \mathfrak{so}(\mathrm{HT}^2(X))$, where $\mathrm{HT}^2(X)$ is equipped with the quadratic form induced by the isomorphism $\mathrm{HT}^2(X) \simeq H^2(X, \mathbb{C})$. Since the class x is algebraic, it has pure weight zero for the action of h' , hence its annihilator is compatible with the decomposition above

$$\mathrm{Ann}(x)_{\mathbb{C}} = \mathrm{Ann}(x)_{\mathbb{C}, -2} \oplus \mathrm{Ann}(x)_{\mathbb{C}, 0} \oplus \mathrm{Ann}(x)_{\mathbb{C}, 2}. \quad (4.9)$$

Moreover, we can write $\mathrm{Ann}(x)_{\mathbb{C}, 0} = \mathbb{C}h' \oplus A_0$ for some A_0 . The subspace

$$W := \mathrm{Ann}(x)_{\mathbb{C}, 2} = \mathrm{Ann}(x)_{\mathbb{C}} \cap \mathrm{HT}^2(X) \subset \mathrm{HT}^2(X)$$

is a hyperplane by assumption. By conjugating (3.3) with η we get that the adjoint action of A_0 on W gives a map

$$A_0 \rightarrow \mathfrak{so}(W),$$

which factors via the restriction $\mathfrak{so}(HT^2(X)) \rightarrow \mathfrak{so}(W)$. Since W is a hyperplane this map is injective, because a skew-symmetric matrix has even rank. Since we also have the inclusion $[W_{-2}, W_2] \subseteq A_0$, we see that $A_0 \simeq \mathfrak{so}(W)$ and the thesis follows as in Theorem 3.6.2. \square

Proof of Theorem Theorem 4.2.17. Assume first that

$$\text{Ann}(v(E)) = \text{Ann}(\tilde{v}(E))$$

for some $\tilde{v}(E) \in \tilde{H}(X, \mathbb{Q})$. Since $v(E)$ is algebraic, we have that $h'(v(E)) = 0$, so $h'(\tilde{v}(E)) = 0$. This means that $\tilde{v}(E) \in \tilde{H}(X, \mathbb{C})^{1,1}$. Since (4.8) is a perfect pairing, we see that the intersection

$$\text{Ann}_{\mathbb{C}}(\tilde{v}(E))_{\mathbb{C}} \cap \text{HT}^2(X)$$

has codimension one. Since this is equal to the kernel of the cohomological obstruction map, we conclude.

Now assume that the kernel of the obstruction map

$$W := \text{Ann}(v(E))_{\mathbb{C}} \cap \text{HT}^2(X).$$

has codimension one. Again, since (4.8) is a perfect pairing, there exists a unique (up to constants) non-zero $\tilde{v} \in \tilde{H}(X, \mathbb{C})^{1,1}$ which pairs trivially with W . We want to show the equality

$$\text{Ann}(v(E))_{\mathbb{C}} = \text{Ann}(\tilde{v})_{\mathbb{C}}.$$

First note that, since both are algebraic classes, the operator h' belongs to both annihilators. Hence, for every $\mu \in W$ with the HL property (which exists by Corollary 3.6.3), we have

$$0 = h'(v(E)) = [e'_{\mu}, \Lambda'_{\mu}](v(E)) = e'_{\mu}(\Lambda'_{\mu}(v(E))). \quad (4.10)$$

Since e'_{μ} has the HL property, we also get that $\Lambda'_{\mu}(v(E)) = 0$. Call \mathfrak{h}_W the sub-Lie algebra of $\mathfrak{g}(X)$ generated by the \mathfrak{sl}_2 triples associated to elements in W . We proved the inclusion

$$\mathfrak{h}_W \subset \text{Ann}(v(E))_{\mathbb{C}}. \quad (4.11)$$

Since \tilde{v} is defined by pairing trivially with W , the same proof works giving

$$\mathfrak{h}_W \subset \text{Ann}(\tilde{v})_{\mathbb{C}}. \quad (4.12)$$

Since the Lie algebra $\mathfrak{g}(X)_{\mathbb{C}}$ is abstractly isomorphic to $\mathfrak{so}(b_2(X) + 2)$, the annihilator of a non-zero element is abstractly isomorphic to $\mathfrak{so}(b_2(X) + 1)$. Moreover, one can prove

$$\mathfrak{h}_W \simeq \mathfrak{so}(b_2(X) + 1). \quad (4.13)$$

with the same computations as Theorem 3.6.2. Hence both the inclusions above are equalities by dimensional reasons. The fact that \tilde{v} is actually defined over \mathbb{Q} is proved in [15, Lemma 3.2] with the argument of [77, Lemma 6.9]. \square

4.3 Extended Mukai vector

The $\tilde{v} \in \tilde{H}(X, \mathbb{Q})$ of Theorem 4.2.17 is called *extended Mukai vector*. Notice that, if such a \tilde{v} exists, it is only defined up to a non-zero constant. In fact, as implicit in [77] the more natural object to work with is the line spanned by \tilde{v} . Nevertheless, in some cases, there is a natural normalization that one can choose, as we will discuss in this section. It is almost immediate from the definition that it is well behaved under derived equivalences.

Proposition 4.3.1 ([77, Theorem 6.14(4)] and [17, Section 4]). *Let $\Phi : D^b(X) \simeq D^b(Y)$ be a derived equivalence between HK manifolds. If $E \in D^b(X)$ is an atomic object, then $\Phi(E)$ is an atomic object and*

$$\langle \Phi^{\tilde{H}}(\tilde{v}(E)) \rangle = \langle \tilde{v}(\Phi(E)) \rangle \subset \tilde{H}(Y, \mathbb{Q}), \quad (4.14)$$

where $\langle x \rangle$ denotes the line in $\tilde{H}(X, \mathbb{Q})$ spanned by x

Proof. If $\Phi : D^b(X) \simeq D^b(Y)$ is a derived equivalence, then Φ^H is Φ^g -equivariant so

$$\Phi^g(\text{Ann}(v(E))) = \text{Ann}(\Phi^H(v(E))) = \text{Ann}(v(\Phi(E))).$$

By the Φ^g equivariance of $\Phi^{\tilde{H}}$ we have

$$\Phi^g(\text{Ann}(\tilde{v}(E))) = \text{Ann}(\Phi^{\tilde{H}}(\tilde{v}(E))).$$

Hence $\text{Ann}(v(\Phi(E))) = \text{Ann}(\Phi^{\tilde{H}}(\tilde{v}(E)))$, which means that $\Phi(E)$ is atomic with extended Mukai vector $\Phi^{\tilde{H}}(\tilde{v}(E))$. \square

The advantage of the extended Mukai vector over the usual one is that it is valued in a much smaller dimensional vector space, but for the moment it is still difficult to compute, since it is constructed in a rather abstract way. So, our next goal is to relate it more explicitly with the usual Mukai vector, generalizing the

computation in Section 3.6.1. Recall that we introduced in (3.20) the orthogonal projection

$$T : \text{Sym}^n \tilde{H}(X, \mathbb{Q}) \rightarrow \text{SH}(X).$$

It is $\mathfrak{g}(X)$ -equivariant, hence also $\text{SO}(\tilde{H}(X, \mathbb{Q}))$ -equivariant.

Proposition 4.3.2 ([15, Proposition 3.3] and [77, Theorem 6.13(2)]). *Let $E \in \text{D}^b(X)$ be an atomic object. Then $T(\tilde{v}(E)^{(n)}) \neq 0$, and the projection onto the Verbitsky component of the Mukai vector $v(E)_{\text{SH}}$ is a rational multiple of $T(\tilde{v}(E)^{(n)})$.*

Proof. The idea is to study the action of the Lie algebra $\text{Ann}(v(E)) = \text{Ann}(\tilde{v}(E))$, and show that in $\text{SH}(X)$ there is a unique (irreducible) trivial subrepresentation. Since both $T(\tilde{v}(E)^{(n)})$ and $v(E)_{\text{SH}}$ are annihilated by $\text{Ann}(v(E))$, the thesis follows.

Notice that the statement is invariant under the action of $\text{SO}(\tilde{H}(X, \mathbb{Q}))$, so we can use it to simplify the computations, in a very similar way to Lemma 2.6.3.

Write $\tilde{v} = r\alpha + \lambda + s\beta$. If $r \neq 0$ we can rescale and assume $r = 1$, and act by $\exp(-\lambda/r)$ to get it to the form

$$\tilde{v} = \alpha + k\beta, \tag{4.15}$$

for some $k \in \mathbb{Q}$. If $r = 0$, we can first act by $\exp(\mu)$ for some $\mu \in H^2(X, \mathbb{Q})$ to obtain $s \neq 0$, and then exchange α and β to obtain $r \neq 0$. A bit of care is needed if $b_2(X)$ is even, since in that case the isometry exchanging α and β has determinant -1 . In this case one can fix this by also acting as -1 on a class in $H^2(X, \mathbb{Q})$ with positive square. Summing up, we can always assume \tilde{v} to have the form (4.15).

Now, take $x \in \text{SH}(X)$, and assume it is killed by $\text{Ann}(\tilde{v})$. Since $e'_\mu \in \text{Ann}(\tilde{v})$ for every $\mu \in H^2(X, \mathbb{Q})$, we get that x is monodromy invariant. Hence, by Remark 3.6.13 we can write

$$x = \sum_{i=1}^n a_i \mathfrak{q}_{2i}, \tag{4.16}$$

for some coefficients $a_i \in \mathbb{Q}$. To determine the coefficients note that if ω has the Hard Lefschetz property, we have

$$\Lambda_\omega(\beta) = \frac{2}{q(\omega)}\omega.$$

This can be seen for example using

$$0 = h\left(\frac{1}{q(\omega)}\omega\right) = [e_\omega, \Lambda_\omega]\left(\frac{1}{q(\omega)}\omega\right),$$

which implies $\Lambda_\omega(\beta) = e_\omega(\Lambda_\omega(\frac{1}{q(\omega)}\omega)) = e_\omega(\frac{2}{q(\omega)}\alpha)$. This implies that $(2ke_\omega - q(\omega)\Lambda_\omega)(\alpha + k\beta) = 0$. By assumption the same must hold for x , and we obtain the relations

$$a_{i+1}q(\omega)\Lambda_\omega(\mathfrak{q}_{2i+2}) = a_i 2k(\mathfrak{q}_{2i} \wedge \omega). \tag{4.17}$$

If we put $k = r_X$, we are in the situations of Corollary 3.6.15; so setting $x = \mathrm{td}_{\mathrm{SH}}^{1/2}$ we obtain

$$\frac{r_X^{i+1}}{(i+1)!} q(\omega) \Lambda_\omega(\mathbf{q}_{2i+2}) = 2 \frac{r_X^{i+1}}{i!} (\mathbf{q}_{2i} \wedge \omega).$$

So that $q(\omega) \Lambda_\omega(\mathbf{q}_{2i+2}) = 2(i+1)(\mathbf{q}_{2i} \wedge \omega)$, and substituting in (4.17) we get $a_{i+1} = \frac{k}{i+1} a_i$. So, up to multiplicative constants we have

$$x = \sum_i \frac{k^i}{i!} \mathbf{q}_{2i},$$

which is precisely $n!T(\tilde{v}^{(n)})$. □

This property for the extended Mukai vector is extremely useful to do computations, and already very interesting on its own. Indeed, in Beckmann's first paper [17] he was just interested in objects having an extended Mukai vector satisfying this property, and only later in [15] he looked atomic objects to also account for the other irreducible subrepresentations.

4.3.1 Normalization

If we want to give an explicit formula for the Mukai vector in terms of the extended Mukai vector, it is useful to choose a representative for the line spanned by $\tilde{v}(E)$. If the rank or the determinant of E do not vanish, we can normalize $\tilde{v}(E)$ as follows. We will see in Section 4.6 what happens if the rank is zero.

Proposition 4.3.3 ([15, Proposition 3.8] and [77, Theorem 6.13(3)]). *Assume $\mathrm{rk}(E) \neq 0$. Then $\tilde{v}(E)$ can be chosen of the form*

$$\mathrm{rk}(E)\alpha + c_1(E) + s(E)\beta,$$

for some $s(E) \in \mathbb{Q}$.

Proof. Write

$$\tilde{v} = r\alpha + \lambda + s\beta,$$

with $r = \mathrm{rk}(E)$. Computing $T(\tilde{v}^{(n)})$ we get

$$T(\tilde{v}^{(n)}) = n!r^n + n!r^{n-1}\lambda + \dots$$

By Proposition 4.3.2 this is a multiple of the projection of the Mukai vector on the Verbitsky component, which is $v(E) = r + c_1(E)$. So we must have $T(\tilde{v}^{(n)}) = n!r^{n-1}v(E)$, and $\lambda = c_1(E)$. □

Sometimes we will call this the normalized extended Mukai vector. As we will see later, this normalization is not always useful when doing computations; for example, it is not preserved under derived equivalences. The question of finding a suitable normalization, compatible with derived equivalences and Bogomolov's inequality (see Corollary 4.3.13), remains open in general.

For certain objects, as those in the orbit of the structure sheaf or of the skyscraper sheaf at a point, it is partially settled in [17]. In the first case, it turns out that the rank is always of the form $\text{rk} = \pm r_0^n$, by [17, Lemma 4.8 (iv)], where n is half the dimension. So, if \tilde{v} denotes the normalized extended Mukai vector, one could choose also

$$\tilde{v}_0 := \frac{\tilde{v}}{r_0^{n-1}}. \quad (4.18)$$

This normalization is preserved under derived equivalences up to sign [17, Equation (4.6)], and satisfies

$$\tilde{q}(\tilde{v}_0) = -2r_X, \quad (4.19)$$

by [17, Lemma 4.8(ii)].

4.3.2 On fourfolds

Using Proposition 4.3.2 we can produce an explicit formula for the Mukai vector of an atomic sheaf on a HK fourfold.

Lemma 4.3.4. *For every $\gamma \in H^2(X, \mathbb{Q})$ we have*

$$T(\alpha^{(n-2)} \cdot \gamma^{(2)}) = (n-2)! (\gamma^2 - q(\gamma, \gamma) \mathfrak{q}_2) \in \text{SH}^4(X).$$

Proof. By definition

$$\Psi(\gamma^2) = e_\gamma \cdot e_\gamma(\alpha^n/n!) = \frac{\alpha^{(n-2)} \cdot \gamma^{(2)}}{(n-2)!} + q(\gamma, \gamma) \frac{\alpha^{(n-1)} \cdot \beta}{(n-1)!}.$$

The map Ψ is a section of T , so $T(\Psi(\gamma^2)) = \gamma^2$. Substituting we get

$$\begin{aligned} T(\alpha^{(n-2)} \cdot \gamma^{(2)}) &= (n-2)! \left(T(\Psi(\gamma^2)) - q(\gamma, \gamma) \frac{T(\alpha^{(n-1)} \cdot \beta)}{(n-1)!} \right) \\ &= (n-2)! (\gamma^2 - q(\gamma, \gamma) \mathfrak{q}_2), \end{aligned}$$

where we used Proposition 3.6.14 in the last equality. \square

Lemma and Definition 4.3.5. *Let X be a hyper-Kähler fourfold, and let $\lambda \in H^2(X, \mathbb{C})$. Denote by $\lambda^\vee \in H^6(X, \mathbb{Q})$ the class such that*

$$\int_X \lambda^\vee \mu = c_X q(\lambda, \mu), \text{ for every } \mu \in H^2(X, \mathbb{Q}). \quad (4.20)$$

Then we have $T(\lambda\beta) = \lambda^\vee$.

Proof. By linearity, we can assume that $q(\lambda, \lambda) \neq 0$. By definition we have

$$\Psi(\lambda^3) = e_\lambda \cdot e_\lambda \cdot e_\lambda \left(\frac{\alpha^{(2)}}{2} \right) = 3q(\lambda, \lambda)\lambda\beta.$$

Using that Ψ is a section of T we obtain $T(\lambda\beta) = \frac{\lambda^3}{3q(\lambda, \lambda)}$, which is easily seen to satisfy the thesis. \square

Corollary 4.3.6. *Let X be a hyper-Kähler fourfold, and $E \in D^b(X)$ an atomic object with non-zero rank. Write*

$$\tilde{v}(E) = r\alpha + \lambda + s\beta.$$

Then we have

$$v(E) = r + \lambda + \frac{\lambda^2 - \tilde{q}(\tilde{v}(E))\mathbf{q}_2}{2r} + \frac{s}{r}\lambda^\vee + \frac{s^2}{2r}\mathbf{q}_4$$

Proof. The symmetric square of $\tilde{v}(E)$ is given by

$$\tilde{v}(E)^{(2)} = r^2\alpha^{(2)} + 2r\alpha\lambda + 2rs\alpha\beta + \lambda^{(2)} + 2s\lambda\beta + s^2\beta^{(2)} \in H^*(X, \mathbb{Q}). \quad (4.21)$$

With the computations above we obtain

$$T(\tilde{v}(E)^{(2)}) = 2r^2 + 2r\lambda + 2rs\mathbf{q}_2 + (\lambda^2 - q(\lambda, \lambda)\mathbf{q}_2) + 2s\lambda^\vee + s^2\mathbf{q}_4.$$

Dividing by $2r$ and rearranging the terms we obtain the formula for the Mukai vector in the statement. \square

4.3.3 Discriminant and modular sheaves

Atomic sheaves are very closely related to modular sheaves. This notion was introduced by O'Grady in [94] a few years before the works of Markman and Beckmann, and it only looks at the commutative deformations. Intuitively, a sheaf on a HK X is modular if it can be deformed to all commutative deformations of X where the first chern class remains of type $(1, 1)$. The reason why this is interesting is that this is a crucial step in the proof (even in the classical one) of Theorem 5.7.5, see Lemma 2.6.4. The idea is to focus the study on sheaves satisfying this modularity property and try to replicate the proof of Theorem 5.7.5 also for higher dimensional HK.

Definition 4.3.7 ([94]). Recall that the discriminant of a torsion-free sheaf F is the class

$$\Delta(F) := -2\mathrm{rk}(F)\mathrm{ch}_2(F) + \mathrm{ch}_1(F)^2 \in H^4(X, \mathbb{Q}).$$

A torsion-free sheaf F is modular if the projection $\Delta(F)_{\mathrm{SH}}$ on the Verbitsky component is monodromy invariant.

Remark 4.3.8. Explicitly this means the following. By Remark 3.6.13 the monodromy invariant part of $\mathrm{SH}(X)^4$ is spanned by \mathfrak{q}_2 . Hence there exists a constant $d(F) \in \mathbb{Q}$ such that

$$\int_X \Delta(F) \cup \omega^{2n-2} = d(F)(2n-3)!!q(\omega)^{n-1},$$

for every $\omega \in H^2(X, \mathbb{Q})$.

Notice that, if F can be deformed along a deformation of X , then the full Chern character remains of Hodge type along the deformation, so of course in particular the c_1 and the discriminant do. It was observed by Verbitsky in [112] that essentially preservation of two classes is the only obstruction to deform a stable vector bundle.²

Theorem 4.3.9 ([112, Theorem 3.19]). *Let $\omega \in H^{1,1}(X, \mathbb{R})$ be a Kähler class, and let E be a (possibly twisted) ω -slope vector bundle. Assume that $\Delta(E)$ remains of type $(2, 2)$ along the twistor line $\mathcal{X}_\omega \rightarrow \mathbb{P}_\omega^1$ spanned by ω . Then there exists a flat deformation \mathcal{E} on \mathcal{X}_ω , such that \mathcal{E}_t is stable for every $t \in \mathbb{P}_\omega^1$.*

A sheaf E satisfying the conclusion is called hyperholomorphic. If instead $\mathrm{End}(E)$ satisfies the conclusion is called projectively hyperholomorphic. For hyperholomorphic bundles we have an even stronger result on deformations.

Theorem 4.3.10 ([110, Proposition 6.3]). *Let X be a HK manifold, and $\omega \in H^{1,1}(X, \mathbb{R})$ a Kähler class. Let E be a ω -projectively hyperholomorphic bundle E , and \mathcal{E} is the deformation along the twistor line $\pi : \mathcal{X}_\omega \rightarrow \mathbb{P}_\omega^1$. Then*

$$R^i \pi_* (\mathrm{End}(\mathcal{E})) \cong \mathcal{O}_{\mathbb{P}_\omega^1}(i) \otimes \mathrm{Ext}_X^i(E, E).$$

In particular, the Ext^ algebra of \mathcal{E}_t is constant along the twistor line.*

By Theorem 3.6.11, the Definition 4.3.7 is a weakening of the condition that the discriminant remains of type $(2, 2)$ on every deformation. Both Markman [77, Theorem 1.2] and Beckmann [15, Proposition 1.5] proved that a torsion-free atomic sheaf is modular. Actually, Markman proved a much stronger result, namely that an atomic sheaf satisfies the conclusion of Theorem 4.3.9 Here we give a different proof of the modularity, by explicitly computing the projection of the discriminant.

Proposition 4.3.11. *Let F be an atomic torsion-free sheaf. Then F is modular, and*

$$\Delta(F)_{\mathrm{SH}} = (\tilde{q}(\tilde{v}(F), \tilde{v}(F)) + 2r_X r(F)^2) \mathfrak{q}_2.$$

²This was later generalized by Markman [78, Corollary 6.11] to stable reflexive sheaves.

Proof. Taking the n -th symmetric power of $\tilde{v}(F)$ we get

$$\begin{aligned}\tilde{v}(F)^{(n)} &= r(F)^n \alpha^{(n)} + nr(F)^{n-1} \alpha^{(n-1)} c_1(F) + \binom{n}{2} r(F)^{n-2} \alpha^{(n-2)} c_1(F)^{(2)} \\ &\quad + nr(F)^{n-1} s(F) \alpha^{(n-1)} \beta + \dots\end{aligned}$$

Using Proposition 3.6.14 and Proposition 3.6.14 we get

$$\begin{aligned}T(\tilde{v}(F)^{(n)}) &= n!r(F)^n + n!r(F)^{n-1} c_1(F) + \binom{n}{2} r(F)^{n-2} (n-2)! (c_1(F))^2 \\ &\quad - c_X q(c_1(F), c_1(F)) \mathfrak{q}_2 + c_X n!r(F)^{n-1} s(F) \mathfrak{q}_2 + \dots\end{aligned}$$

The projection onto $\text{SH}(X)$ of the Mukai vector of F is a rational multiple of this class. Since the rank is non-zero, we deduce that $n!r(F)^{n-1} v(F) = T(\tilde{v}(F)^{(n)})$. Dividing by $n!r(F)^{n-1}$ and comparing the terms of degree four, we get

$$v_2(F)_{\text{SH}} = \frac{1}{2r(F)} (c_1(F)^2 - q(c_1(F), c_1(F)) \mathfrak{q}_2) + s(F) \mathfrak{q}_2. \quad (4.22)$$

On the other hand, by definition $v(F) = \text{ch}(F) \cup (\text{td}_X)^{\frac{1}{2}}$. Lemma 3.3 in [17] gives

$$(\text{td}_X)^{\frac{1}{2}}_{\text{SH}} = 1 + r_X \mathfrak{q}_2 + \dots$$

Taking the product with the projection of the Chern character gives

$$\begin{aligned}\text{ch}_2(F)_{\text{SH}} &= v_2(F)_{\text{SH}} - r_X r(F) \mathfrak{q}_2 = \frac{1}{2r(F)} (c_1(F)^2 - q(c_1(F), c_1(F)) \mathfrak{q}_2) \\ &\quad + (s(F) - r_X r(F)) \mathfrak{q}_2.\end{aligned}$$

Substituting $\text{ch}_2(F)_{\text{SH}}$ in the definition of the discriminant we obtain

$$\begin{aligned}\Delta(F)_{\text{SH}} &= (q(c_1(F), c_1(F)) + 2r_X r(F)^2 - 2r(F)s(F)) \mathfrak{q}_2 \\ &= (\tilde{q}(\tilde{v}(F), \tilde{v}(F)) + 2r_X r(F)^2) \mathfrak{q}_2.\end{aligned}$$

In particular, this implies that F is atomic. □

Remark 4.3.12. The reverse implication is not true. The most natural modular sheaf which is not atomic is, surprisingly, the tangent bundle, see [15, Proposition 8.3].

Corollary 4.3.13. *If F is an atomic torsion-free slope semistable sheaf, then*

$$\tilde{q}(\tilde{v}(F), \tilde{v}(F)) + 2r_X r(F)^2 \geq 0.$$

Proof. If F is slope semistable for a polarization H on X , Bogomolov's inequality gives

$$\int_X \Delta(F) \cup H^{n-2} \geq 0.$$

The thesis follows from the proposition above because

$$\int_X H^{n-2} \mathbf{q}_2 = c_X (2n-3)!! q(H)^{n-1}$$

is non negative for every H ample. \square

Remark 4.3.14. If $X = S$ is a K3-surface, we recover the 'weak' Bogomolov's inequality

$$v(F)^2 \geq -2r(F)^2.$$

It is possible that, similarly to the case of K3 surfaces, a stronger version of the inequality 4.3.13 holds, and equality should be related to F being a \mathbb{P} -object.

A precise formulation of this inequality seems to be related to understanding how to normalize the extended Mukai vector. For example, for the normalization in (4.18) the equality

$$\tilde{q}(\tilde{v}(F), \tilde{v}(F)) = -2r_X$$

holds for objects in the orbit of the structure sheaf (which in particular are \mathbb{P} -objects).

4.3.4 Euler characteristic

Since the euler pairing of two sheaves is cohomological, by Hirzebruch-Riemann-Roch, we can also give a formula for it with similar computations. In this section we do it under the assumption that the Mukai vector is contained in the Verbitsky component. Recall that there is a bilinear product b_{SH} on $\text{SH}(X)$, defined by

$$b_{\text{SH}}(\lambda_1 \cdots \lambda_m, \mu_1 \cdots \mu_{2n-m}) := (-1)^m \int_X \lambda_1 \cup \cdots \cup \lambda_m \cup \mu_1 \cup \cdots \cup \mu_{2n-m}.$$

Lemma 4.3.15. *There exists a constant C such that*

$$b_{\text{SH}}(T(\tilde{v}^{(n)}), T(\tilde{v}^{(n)})) = C \tilde{q}(\tilde{v})^n.$$

Proof. Since both sides of the equality are homogeneous polynomials of degree $2n$, invariant under the action of $\text{SO}(\tilde{H}(X, \mathbb{Q}))$, we can perform the same reductions as in the proof of Proposition 4.3.2 and assume that

$$\tilde{v} = \alpha + s\beta.$$

for some $s \in \mathbb{Q}$. By definition we have $\tilde{q}(\tilde{v}, \tilde{v}) = -2s$. Moreover, we have

$$\tilde{v}^{(n)} = \sum \binom{n}{i} s^i \alpha^{(n-i)} \beta^{(i)}.$$

Applying Proposition 3.6.14 we obtain

$$T(\tilde{v}^{(n)}) = \sum \frac{n!}{i!} s^i \mathfrak{q}_{2i}.$$

By definition of the Mukai pairing b_{SH} we get

$$b_{\text{SH}}(T(\tilde{v}^{(n)}), T(\tilde{v}^{(n)})) = \left(\sum \frac{n!}{i!} \frac{n!}{(n-i)!} \int_X \mathfrak{q}_{2i} \mathfrak{q}_{2n-2i} \right) s^n = C \tilde{q}(\tilde{v}, \tilde{v})^n,$$

for some constant C independent of \tilde{v} . \square

Theorem 4.3.16. *Let $E \in D^b(X)$ be an atomic object with non-zero rank r . Assume that its Mukai vector $v(E)$ is contained in the Verbitsky component. Then*

$$\chi(E, E) = (-1)^n (n+1) r^2 \left(\frac{\tilde{q}(\tilde{v}(E))}{2r_X r^2} \right)^n.$$

Proof. From the Hirzebruch-Riemann-Roch Theorem and the assumption that $v(E)_{\text{SH}} = v(E)$ it follows that

$$\chi(E, E) = b_{\text{SH}}(v(E), v(E)).$$

Since $T(\tilde{v}^{(n)}) = n! r^{n-1} v(E)$, the previous lemma implies

$$b_{\text{SH}}(r^{n-1} v(E), r^{n-1} v(E)) = C \tilde{q}(\tilde{v})^n$$

for some constant C . Dividing both sides by r^n we get

$$b_{\text{SH}} \left(\frac{v(E)}{r}, \frac{v(E)}{r} \right) = C \tilde{q} \left(\frac{\tilde{v}}{r} \right)^n \quad (4.23)$$

To compute the constant C , we substitute $\tilde{v} = \alpha + r_X \beta$, the extended Mukai vector of the structure sheaf. Since $r = 1$ and $\chi(\mathcal{O}_X, \mathcal{O}_X) = n+1$ we get $C = (-1)^n \frac{n+1}{(2r_X)^n}$. Substituting C into (4.23) and rearranging we get the result. \square

Remark 4.3.17. The assumption that the Mukai vector is contained in the Verbitsky component is satisfied for every atomic E in the case of hyper-Kähler varieties of $K3^{[2]}$ -type. In this case, the formula becomes

$$\chi(E, E) = 3 \cdot \left(\frac{\tilde{q}(\tilde{v}(E))}{2r r_X} \right)^2.$$

Note that it gives a non-trivial integral constraint on the difference $\text{ext}^2(E, E) - 2\text{ext}^1(E, E)$. Finding an independent restriction on its possible values, for example in the form of a bound on $\text{ext}^2(E, E)$, could be a path to investigate smoothness of the moduli space of semistable deformations of E .

4.4 Twisted atomic sheaves

In this section we want to extend the theory of atomic sheaves to the case of twisted sheaves. We will need this in Chapter 6. Recall that, if X is an analytic space, given an element

$$\alpha \in H^2(X, \mathcal{O}_X^*)$$

represented by a Čech cocycle $\{\alpha_{ijk}\}$, there is the notion of α -twisted sheaves, see [25]. Roughly speaking, an α -twisted sheaf \mathcal{F} is a collection of sheaves \mathcal{F}_i over an open cover $\{U_i\}$ plus transition functions $\varphi_{ij} : \mathcal{F}_i|_{U_{ij}} \simeq \mathcal{F}_j|_{U_{ij}}$ which satisfy the cycle condition up to α_{ijk} .

In [54] it is shown that a twisted Fourier-Mukai equivalence $\Phi_{\mathcal{E}} : D^b(X) \simeq D^b(Y, \alpha)$ induces an isomorphism in rational cohomology. We briefly recall how it works in our setting, referring to [54, Proposition 1.2] for more details in the general case. Choose a B -field $B \in H^2(Y, \mathbb{Q})$ lifting α , that is a rational cohomology class such that

$$\exp(B^{0,2}) = \alpha \in H^2(Y, \mathcal{O}_Y^*).$$

For an object $E \in D^b(Y, \alpha)$ one defines the twisted Chern character, depending on the B -field,

$$\mathrm{ch}^B(E) \in \exp(B) \left(\bigoplus_p H^{p,p}(Y, \mathbb{Q}) \right), \quad (4.24)$$

and the twisted Mukai vector $v(E) := \mathrm{ch}^B(E) \cdot \mathrm{td}_Y^{1/2}$. The induced isomorphism in cohomology is constructed as usual

$$\Phi_{\mathcal{E}}^H := p_{2,*} \left(v^{p_2^*(B)}(\mathcal{E}) \cdot p_1^*(-) \right) : H^*(X, \mathbb{Q}) \simeq H^*(Y, \mathbb{Q})$$

It depends on the choice of the B -field, but we will often drop this dependence from the notation if it is clear that we have fixed a B -field. On Y one can define a weight zero Hodge structure by

$$H^{-k,k} := \exp(B) \left(\bigoplus_{p-q=-k} H^{p,q}(Y, \mathbb{Q}) \right). \quad (4.25)$$

Then, similarly to the untwisted case we have the following.

Proposition 4.4.1 ([54, Section 4]). *If $\Phi_{\mathcal{E}} : D^b(X) \simeq D^b(Y, \alpha)$ is an equivalence, then $\Phi_{\mathcal{E}}^H$ respects the Mukai pairing and the weight zero Hodge structures.*

We want to generalize Section 4.3 to the twisted case, in the following setting, which will be that of Chapter 6. Let $\pi_X : \mathcal{X} \rightarrow \Delta$ and $\pi_Y : \mathcal{Y} \rightarrow \Delta$ be two families

of hyper-Kähler manifolds (not necessarily projective) and let $\alpha \in H^2(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}^*)$. Let $\mathcal{E} \in \mathrm{D}^b(\mathcal{X} \times_{\Delta} \mathcal{Y}, p_2^*(\alpha))$ be such that

$$\Phi_t := \Phi_{\mathcal{E}_t} : \mathrm{D}^b(X_t) \simeq \mathrm{D}^b(Y_t, \alpha_t)$$

is an equivalence for every $t \in T$. The twist α_t will be torsion, hence lie in the Brauer group $\mathrm{Br}(Y_t) := H^2(Y_t, \mathcal{O}_{Y_t}^*)$ precisely when Y_t is projective. If we assume Δ to be simply connected, then parallel transport gives isomorphisms in cohomology

$$f_{X,t}^H : H^*(X_t, \mathbb{Q}) \simeq H^*(X_0, \mathbb{Q}) \text{ and } f_{Y,t}^H : H^*(Y_t, \mathbb{Q}) \simeq H^*(Y_0, \mathbb{Q}) \quad (4.26)$$

and for the LLV algebra

$$f_{X,t}^{\mathfrak{g}} : \mathfrak{g}(X_t) \simeq \mathfrak{g}(X_0) \text{ and } f_{Y,t}^{\mathfrak{g}} : \mathfrak{g}(Y_t) \simeq \mathfrak{g}(Y_0). \quad (4.27)$$

It is explicitly given in the following way. The rational local system $R^{\bullet}\pi_*\underline{\mathbb{Q}}$ is trivial, because Δ is simply connected. In particular

$$H^*(\mathcal{X}, \mathbb{Q}) \simeq H^0(\Delta, R^{\bullet}\pi_*\underline{\mathbb{Q}}) \quad (4.28)$$

and the restriction map $H^*(\mathcal{X}, \mathbb{Q}) \rightarrow H^*(X_t, \mathbb{Q})$ is an isomorphism for every $t \in \Delta$. Then f_t^H is the composition

$$H^*(X_t, \mathbb{Q}) \simeq H^*(\mathcal{X}, \mathbb{Q}) \simeq H^*(X_0, \mathbb{Q}).$$

The isomorphism $f_t^{\mathfrak{g}}$ is induced from the graded algebra isomorphism f_t^H .

Remark 4.4.2. Notice that, by construction, we have that f_t^H is equivariant with respect to $f_t^{\mathfrak{g}}$.

Since $\alpha_0 = 0$, the equivalence $\Phi_0^{\mathfrak{g}} : \mathrm{D}^b(X_0) \simeq \mathrm{D}^b(Y_0)$ induces an isomorphism of Lie algebras $\Phi_0 : \mathfrak{g}(X) \simeq \mathfrak{g}(Y)$. Define

$$\Phi_t^{\mathfrak{g}} := (f_{Y,t}^{\mathfrak{g}})^{-1} \circ \Phi_0^{\mathfrak{g}} \circ f_{X,t}^{\mathfrak{g}}.$$

To get an easy generalization of the arguments in Section 4.3 we assume that on the fibers where α_t is torsion, hence it is a class in $\mathrm{Br}(Y_t) = H^2(Y_t, \mathcal{O}_{Y_t}^*)_{\mathrm{tor}}$, the induced isometry

$$\Phi_{\mathcal{E}_t}^H : H^*(X, \mathbb{Q}) \rightarrow H^*(Y_t, \mathbb{Q})$$

is equivariant with respect to $\Phi_t^{\mathfrak{g}}$.

4.4.1 Twisted extended Mukai lattice

Let Y be a HK manifold, and $\alpha \in H^2(Y, \mathcal{O}_Y^*)$. Having fixed a B -field $B \in H^2(Y, \mathbb{Q})$, we get an isometry

$$\exp(B) \in \mathrm{SO}(\tilde{H}(Y, \mathbb{Q}))$$

of the rational extended Mukai lattice. We can use it to define a ‘twisted’ weight zero Hodge structure $\tilde{H}(Y, \mathbb{Q}, B)$ by

$$\tilde{H}(Y, \mathbb{C}, B)^{(p,q)} := \exp(B) \left(\tilde{H}(Y, \mathbb{C})^{(p,q)} \right),$$

in analogy with (4.25). This induces a Hodge structure on the symmetric power, by declaring $\mathrm{Sym}^n \tilde{H}(Y, \mathbb{C}, B)^{(p,q)}$ to be the subspace generated by $x_1 \cdots x_n$, with $x_i \in \tilde{H}(Y, \mathbb{C}, B)^{(p_i, q_i)}$ and $\sum_i p_i = p$ and $\sum_i q_i = q$.

Remark 4.4.3. Notice that this Hodge structure $\mathrm{Sym}^n \tilde{H}(Y, \mathbb{C}, B)$ is equal to

$$\mathrm{Sym}^n \tilde{H}(Y, \mathbb{C}, B)^{(p,q)} = \exp(B) \left(\mathrm{Sym}^n \tilde{H}(Y, \mathbb{C})^{(p,q)} \right).$$

Since the morphism Ψ is equivariant with respect to $\mathrm{SO}(\tilde{H}(Y, \mathbb{Q}))$, and it is a morphism of weight zero Hodge structure with the usual ones, we see that it remains a morphism of Hodge structures

$$\Psi : \mathrm{SH}(X, B) \rightarrow \mathrm{Sym}^n \tilde{H}(Y, \mathbb{Q}, B)$$

also with the twisted ones.

Proposition 4.4.4. *Assume that either n or $b_2(X)$ is odd. Then, there exists a Hodge isometry $\Phi_t^{\tilde{H}} : \tilde{H}(X_t, \mathbb{Q}) \simeq \tilde{H}(Y_t, \mathbb{Q}, B_t)$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathrm{SH}(X_t) & \xrightarrow{\varepsilon(\Phi_t^{\tilde{H}})\Phi_t^{\mathrm{SH}}} & \mathrm{SH}(Y_t, B_t) \\ \downarrow \Psi & & \downarrow \Psi \\ \mathrm{Sym}^n \tilde{H}(X_t, \mathbb{Q}) & \xrightarrow{\mathrm{Sym}^n \Phi_t^{\tilde{H}}} & \mathrm{Sym}^n \tilde{H}(Y_t, \mathbb{Q}, B_t). \end{array}$$

Proof. It follows as in the untwisted case from the assumption that $\Phi_{\mathcal{E}_t}^H$ is equivariant with respect to $\Phi_t^{\mathfrak{g}}$ and [103, Propositions 4.1 and 4.4]. \square

Definition 4.4.5. Let Y be a HK manifold, $\alpha \in H^2(Y, \mathcal{O}_Y^*)$ a Brauer class, and B a B -field lifting α . An object $E \in \mathrm{D}^b(Y, \alpha)$ is atomic if there exists a non-zero $\tilde{v}^B(E) \in \tilde{H}(X, \mathbb{Q})$ such that

$$\mathrm{Ann}(v^B(E)) = \mathrm{Ann}(\tilde{v}^B(E))$$

as sub Lie algebras of $\mathfrak{g}(X)$.

Remark 4.4.6. If h' denotes the Weil operator for the usual Hodge structure on H^* , we have

$$h' \circ \exp(-B) \in \text{Ann}(v^B(E))$$

because of (4.24). So, if E is atomic, $h' \circ \exp(-B) \in \text{Ann}(\tilde{v}^B(E))$, which means that $\tilde{v}^B(E) \in \tilde{H}(Y, \mathbb{Q}, B)$ is of Hodge type for the twisted Hodge structure.

Proposition 4.4.7. *In the setting above, for every atomic object $E \in \text{D}^b(X_t)$, the image $\Phi_t(E) \in \text{D}^b(Y_t, \alpha_t)$ is a twisted atomic sheaf.*

Proof. The Hodge isometry $\Phi_t^{\tilde{H}}$ of Proposition 4.4.4 is LLV-equivariant for the same reason as in the untwisted case. Hence the result follows from the definition of twisted atomic object. \square

4.5 First examples

As we have seen, atomic objects have very rigid properties, and they are very hard to find. Not many examples are known, and atomic sheaves with moduli are especially difficult to find. The general strategy to produce more objects is to apply derived equivalences to known ones. Of course, this operation does not change the number of moduli, and it means that we must still start from something, and this usually involves some geometrical considerations. Nevertheless, applying derived equivalences is often interesting, for example to produce vector bundles out of torsion sheaves. In this section we study the very first examples that appeared in the literature, mainly in [94] and [77].

4.5.1 Rigid bundles

One very easy way of getting 1-obstructed (hence atomic) objects, is to ask for the Ext^2 to be one dimensional. This condition is satisfied in particular by the \mathbb{P} -objects. They were introduced in [55] as the hyper-Kähler analogues of spherical objects, see Section 5.3 for more information.

Definition 4.5.1. Let X be a HK manifold of dimension $2n$. An object $E \in \text{D}^b(X)$ is a \mathbb{P}^n -object (or just \mathbb{P} -object) if there is an isomorphism

$$\text{Ext}^*(E, E) \cong H^*(\mathbb{P}^n, \mathbb{C})$$

of graded algebras.

For example, the structure sheaf \mathcal{O}_X is a \mathbb{P} -object.

Lemma 4.5.2. *Let $E \in D^b(X)$, and assume that $\text{Ext}^2(E, E)$ is one dimensional. If the obstruction map χ_E^H is not identically zero, then E is 1-obstructed, hence atomic.*

Proof. We need to show that the obstruction map has rank one, atomicity follows from Theorem 4.2.17. We know by Proposition 4.2.8 that

$$\text{rk } \chi_E \geq \text{rk } \chi_E^H,$$

and this is at least one by assumption. Since $\text{ext}^2(E, E) = 1$, we also have $\text{rk } \chi_E \leq 1$, hence both the ranks are equal to one. \square

Remark 4.5.3. The condition that the cohomological obstruction map does not vanish is very mild. It is satisfied for example if $v(E)_{\text{SH}} \neq 0$, which holds for every sheaf E by [15, Lemma 3.7].

Proposition 4.5.4. *Let X a HK manifold. The structure sheaf $\mathcal{O}_X \in D^b(X)$ is 1-obstructed. In particular it is atomic, and its extended Mukai vector is*

$$\tilde{v} = \alpha + r_X \beta.$$

Proof. The first part follows from the lemma above, we just have to compute the Mukai vector. We computed in Corollary 3.6.15 that

$$v(\mathcal{O}_X)_{\text{SH}} = (\text{td}_X^{1/2})_{\text{SH}} = T \left(\frac{(\alpha + r_X \beta)^{(n)}}{n!} \right).$$

In particular this implies that $\text{Ann}(\tilde{v}) \subseteq \text{Ann}(v(\mathcal{O}_X))$, which must be an equality for dimensional reasons. \square

Remark 4.5.5. For objects in the orbit (under the action of derived equivalences) of the structure sheaf, it is possible to find a normalization of the Mukai vector which is preserved by derived equivalences. See [17, Section 4] for more details.

An immediate consequence is that we can compute the extended Mukai vector of all line bundles. Indeed if $L \in \text{Pic}(X)$ is a line bundle, then we can write $L = L \otimes \mathcal{O}_X$. If $\lambda = c_1(L)$, the action of $- \otimes L$ on $\tilde{H}(X, \mathbb{Q})$ is given by B_λ , see Example 3.6.10. So we have

$$\tilde{v}(L) = B_L(\tilde{v}(\mathcal{O}_X)) = \alpha + \lambda + \left(r_X + \frac{q(\lambda)}{2} \right) \beta. \quad (4.29)$$

As anticipated, a construction of atomic bundles of higher rank, which we now describe, has been carried out in [94, 77]. Let S be a K3 surface, and let F be a spherical vector bundle on S . On the Hilbert scheme $S^{[n]}$, consider the bundle

$$F^{\boxtimes n} := F \boxtimes \cdots \boxtimes F.$$

It is naturally equipped with a Σ_n -linearization λ , given by permuting the factors. So we have an object $(F^{\boxtimes n}, \lambda) \in \mathrm{D}^b_{\Sigma_n}(S^n)$.

Let \mathcal{Z}_n be the isospectral Hilbert scheme. It fits in a diagram

$$\begin{array}{ccc} & \mathcal{Z}_n & \\ q \swarrow & & \searrow p \\ S^{[n]} & & S^n \\ h \searrow & & \swarrow b \\ & S^{(n)} & \end{array}$$

as the reduced fiber product. The map $p : \mathcal{Z}_n \rightarrow S^n$ is the blow-up on the big diagonal, and the map $q : \mathcal{Z}_n \rightarrow S^{[n]}$ is the quotient for the lifted action of Σ_n . As usual, denote by $E_n \subset S^{[n]}$ the divisor of non-reduced subschemes, and by $\delta_n = \frac{E_n}{2} \in \mathrm{Pic}(S^{[n]})$.

Theorem 4.5.6 ([40, Theorem 1]). *The isospectral Hilbert scheme \mathcal{Z}_n is Cohen-Macaulay, and the quotient $q : \mathcal{Z}_n \rightarrow S^{[n]}$ is flat.*

If (E, λ) is an object in $\mathrm{D}^b_{\Sigma_n}(S^n)$, the induced linearization on the object $q_*p^*(E, \lambda)$ is actually an action, because Σ_n acts trivially on $S^{[n]}$. The functor which takes the invariant part of a sheaf with a Σ_n -action is denoted by $(-)^{\Sigma_n}$.

Theorem 4.5.7 ([24, Corollary 1.3]). *The functor*

$$\Phi_{\mathrm{BKR}} : \mathrm{D}^b_{\Sigma_n}(S^n) \rightarrow \mathrm{D}^b(S^{[n]}), \quad (E, \lambda) \mapsto (q_*p^*(E, \lambda))^{\Sigma_n}$$

is an equivalence.

So, starting from F , spherical bundle on the K3 S , we can define (c.f. [94, Definition 5.1]) the object

$$F[n]^+ := \Phi_{\mathrm{BKR}}(F^{\boxtimes n}, \lambda). \quad (4.30)$$

If F is locally free then $q_*p^*F^{\boxtimes n}$ is as well, because q is flat. So the invariant part is again a vector bundle, being a direct summand.

Remark and Definition 4.5.8. Let $(\mathcal{O}_{S^n}, \rho) \in \mathrm{D}^b_{\Sigma_n}(S^n)$ be the structure sheaf equipped with the linearization induced by the sign. Tensorization by $(\mathcal{O}_{S^n}, \rho)$ is an autoequivalence of $\mathrm{D}^b_{\Sigma_n}(S^n)$, so we can also define the object

$$F[n]^- := \Phi_{\mathrm{BKR}}(F^{\boxtimes n}, \lambda \otimes \rho). \quad (4.31)$$

This is also a direct summand of $q_*p^*F^{\boxtimes n}$, hence locally free.

Proposition 4.5.9 ([77, Section 10] and [94, Proposition 5.4]). *The vector bundle $F[n]^\pm$ is a \mathbb{P}^n -object in $D^b(S^{[n]})$, i.e.*

$$\mathrm{Ext}_{S^{[n]}}^*(F[n]^\pm, F[n]^\pm) \cong H^*(\mathbb{P}^n, \mathbb{C}).$$

Proof. By Theorem 4.5.7 we have

$$\mathrm{Ext}_{S^{[n]}}^*(F[n]^\pm, F[n]^\pm) \cong \mathrm{Ext}_{S^n}^*(F^{\boxtimes n}, F^{\boxtimes n})^{\Sigma_n},$$

where the action of Σ_n on

$$\mathrm{Ext}_{S^n}^*(F^{\boxtimes n}, F^{\boxtimes n}) = \mathrm{Ext}_S^*(F, F) \otimes \cdots \otimes \mathrm{Ext}_S^*(F, F)$$

is given by permutation. The invariant part then is $\mathrm{Sym}^n \mathrm{Ext}_S^*(F, F)$ which is isomorphic to $H^*(\mathbb{P}^n, \mathbb{C})$ since F is spherical. \square

By Lemma 4.5.2 the vector bundle $F[n]^\pm$ is 1-obstructed. In particular it is atomic and we would like to compute its extended Mukai vector.

Proposition 4.5.10 ([95, Proposition 3.2]). *Let F be a spherical bundle on S with Mukai vector*

$$v(F) = (r_0, D, s_0) \in \tilde{H}(S, \mathbb{Z}).$$

Then $F^+[n]$ is an atomic locally free sheaf with normalized Mukai vector

$$\tilde{v}(F^+[n]) = r_0^n \alpha + r_0^{n-1} \left(D + \frac{r_0 - 1}{2} \delta_n \right) + r_0^{n-1} \left(s_0 + \frac{(r_0 - 2)(1 - n)}{4} \right) \beta$$

and discriminant

$$\Delta(F^+[n]) = \frac{r_0^{2n-2}(r_0^2 - 1)}{12} c_2(X).$$

Proof. It is clear by the construction that the rank is $\mathrm{rk}(F[n]^\pm) = r_0^n$. The first Chern class can be computed by restricting to Y_* , where $Y = S^{[n]}, \mathcal{Z}_n, S^n$, defined as the locus in Y where at most two points collide. In fact, Y_* has complement of codimension two, so the first Chern class is determined by its restriction. The map

$$q : \mathcal{Z}_{n,*} \rightarrow S_*^{[n]}$$

is $n! : 1$ with a simple $2 : 1$ ramification along the exceptional divisor. Moreover, $\mathcal{Z}_{n,*}$ is non-singular, hence one can use Grothendieck-Riemann-Roch to get the result, see [95, Proposition 3.5] for the full computation. The discriminant is computed with a similar technique, see also [95, Corollary 3.7]. Once the rank and the first Chern class are known, the full Mukai vector is determined by imposing that (4.19) holds. \square

Remark 4.5.11. Notice how in this case there one can factor out r_0^{n-1} , so (4.18) seems to be the more natural normalization.

Restricting to the case of an elliptic K3, O’Grady proves in [95, Section 4] that $F[n]^+$ is slope-stable for some polarizations. We will see more on this in Section 4.7. In [77, Section 10] a different strategy is used to prove stability. Namely, one does the same construction for a non-projective K3 surface, and there one can prove stability directly by showing that $F[n]^+$ has no subsheaves of smaller rank.

In both cases, after proving stability for specific HK manifolds, we can deform to every HK manifold on which the c_1 remains algebraic by Theorem 4.3.9. So we get an existence result for stable atomic sheaves having the numerical invariants of such an $F[n]^+$ for some F . The main result of [94, 95] is that for fixed numerical invariants, this vector bundle is unique. For the explicit form of the invariants, which does not make a reference to $F[n]^+$, we refer to [95, Theorem 1.1].

4.5.2 Semirigid bundles

With a similar construction, Markman obtained stable atomic vector bundles which are non-rigid. In fact, they are deformations of bundles which are image via derived equivalences of the skyscraper sheaf of a point.

Proposition 4.5.12 ([77, Section 3] and [17, Section 4]). *Let X be a HK manifold of dimension $2n$, and let $x \in X$ be a point. Then the skyscraper sheaf $\mathbb{C}(x)$ is 1-obstructed, and*

$$\tilde{v}(\mathbb{C}_x) = \beta.$$

Proof. In this case we can explicitly compute the obstruction map. We have

$$\begin{aligned} \mathrm{Ext}_X^k(\mathbb{C}(x), \mathbb{C}(x) \otimes \Omega_X^1) &\cong H^0(X, \mathcal{E}xt^k(\mathbb{C}(x), \mathbb{C}(x)) \otimes \Omega_X^1) \cong H^0(X, T_{X,x} \otimes \Omega_X^1) \\ &\cong \mathrm{End}\left(\bigwedge^k T_{X,x}\right), \end{aligned}$$

and the k -th power of the Atiyah class $\mathrm{At}_X^k(\mathbb{C}(x))$ is the identity under this isomorphism. The obstruction map

$$\chi_{\mathbb{C}(x)} : H^p(X, \bigwedge^q T_X) \xrightarrow{-\circ \mathrm{At}_X^q(\mathbb{C}(x))} \mathrm{Ext}_X^k(\mathbb{C}(x), \mathbb{C}(x)) \cong \bigwedge^k T_{X,x},$$

vanishes if $q \neq k$, and is the restriction if $q = 0$. In particular for $p + q = 2$ it has rank one. By Proposition 3.6.14 we know that $T(\beta^{(n)})$ is proportional to the class of a point, hence

$$\mathrm{Ann}(\beta) \subseteq \mathrm{Ann}(\mathrm{pt}) = \mathrm{Ann}(v(\mathbb{C}(x))).$$

By atomicity we get the equality. □

The starting point of the semirigid examples is a K3 surface S with a two dimensional moduli space $M := M_S(r_0, D, s_0)$ consisting only of vector bundles. We assume that M is fine, so that there exists a tautological bundle \mathcal{E} on $S \times M$, inducing an equivalence

$$\Phi_{\mathcal{E}} : D^b(M) \rightarrow D^b(S)$$

by Proposition 2.2.20. Following Ploog [100], we consider the equivariant kernel

$$(\mathcal{E}^{\boxtimes n}, \lambda) \in D^b_{\Sigma_n}(M^n \times S^n), \quad (4.32)$$

where Σ_n acts on the product $M^n \times S^n$ via the diagonal action, and λ is the linearization which permutes the factors. Then $\Phi^{\boxtimes n} := \Phi_{(\mathcal{E}^{\boxtimes n}, \lambda)} : D^b_{\Sigma_n}(M^n) \simeq D^b_{\Sigma_n}(S^n)$ is an equivalence. Associated to the same kernel of the equivalence in Theorem 4.5.7 one has also an equivalence in the opposite direction

$$\Phi_{\text{bkr}} := p_* q^* : D^b(M^{[n]}) \simeq D^b_{\Sigma_n}(M^n). \quad (4.33)$$

which is not the inverse.

Proposition 4.5.13 ([77, Section 11]). *Let $\xi \in M^{[n]}$ be a point corresponding to a reduced subscheme. Define*

$$F_{\xi} := (\Phi_{\text{BKR}} \circ \Phi^{\boxtimes n} \circ \Phi_{\text{bkr}})(\mathbb{C}(\xi)).$$

Then F_{ξ} is a locally free atomic sheaf, with normalized extended Mukai vector

$$\tilde{v}(F_{\xi}) = n!r_0^n \alpha + n!r_0^{n-1} \left(D - r_0 \frac{\delta_n}{2} \right) + n!r_0^{n-1} \left(s_0 + \frac{r_0(1-n)}{4} \right)$$

Moreover, if $\xi = \{x_1 + \cdots + x_n\}$, we have

$$F_{\xi} \cong \Phi_{\text{BKR}} \left(\bigoplus_{\sigma \in \Sigma_n} E_{\sigma(1)} \boxtimes \cdots \boxtimes E_{\sigma(n)} \right). \quad (4.34)$$

where E_i is the vector bundle on S parametrized by x_i .

Proof. For a complete proof we refer to [77, Section 11]. The 1-obstructedness is immediate from Proposition 4.5.12 and the definition. Since

$$\Phi_{\text{bkr}}(\mathbb{C}(\{x_1 + \cdots + x_n\})) = \bigoplus_{\sigma \in \Sigma_n} \mathbb{C}(x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

the definition implies (4.34). As in the rigid case, locally freeness follows from the flatness of $q : \mathcal{Z}_n \rightarrow S^{[n]}$. The rank can be computed from (4.34), and the c_1 is computed in [77, Lemma 11.1]. The rest of the extended Mukai vector is computed by imposing that it squares to zero. \square

Again, one can prove stability by considering the case of an analytic K3 surface, as done in [77, Lemma 11.5]. Then deform to get a (possibly twisted) vector bundle with Mukai vector of square zero on every HK of type $\text{K3}^{[n]}$.

4.6 Lagrangian submanifolds

So far we have mostly focused on atomic sheaves of positive rank, so we might wonder what happens if the rank is zero. It turns out that the condition of being atomic imposes strong restrictions on the support of a sheaf.

Proposition 4.6.1 ([15, Proposition 3.11] and [77, Theorem 6.24]). *Let X be a HK manifold of dimension $2n$, and let $F \in \mathbf{D}^b(X)$ be an atomic object of zero rank. Then the Chern classes of F are all isotropic. Moreover, if $\mathrm{ch}_n(F) = 0$, then F is supported in dimension zero.*

Proof. Write the extended Mukai vector as

$$\tilde{v}(F) = \lambda + s\beta.$$

Since $\lambda \in H^{1,1}(X, \mathbb{Q})$ we have $q(\lambda, \sigma) = 0$, so

$$e_\sigma \in \mathrm{Ann}(\tilde{v}(F)) = \mathrm{Ann}(v(F)).$$

This implies that $v_i(E) \cup \sigma = 0$ (or equivalently $\mathrm{ch}_i(F) \cup \sigma = 0$), for all i .

For the last statement, recall that by [Proposition 4.3.2](#) the Mukai vector $v(F)$ is a multiple of $T(\tilde{v}^{(n)})$. In particular $v_n(F) = \mathrm{ch}_n(F)$ is proportional to $T(\lambda^{(n)})$, which does not vanish if $\lambda \neq 0$. If $\lambda = 0$, then

$$T(\tilde{v}^{(n)}) = T(s^n \beta^{(n)}) \in H^{4n}(X, \mathbb{Q})$$

is a multiple of the class of a point. Therefore, if F is atomic and $\mathrm{ch}_n(F) = 0$ we must have $\lambda = 0$, so that F is supported in dimension zero. \square

In particular, if F is a sheaf supported on an irreducible subvariety $Z \subset X$, then Z is either a Lagrangian subvariety or a point. As we have seen in [Proposition 4.5.12](#) every sheaf supported in dimension zero is atomic. For structure sheaves on Lagrangian submanifolds we have the following.

Theorem 4.6.2 ([15, Theorem 1.8]). *Let $i : Z \hookrightarrow X$ a connected Lagrangian submanifold. Then $i_*\mathcal{O}_Z$ is atomic if and only if the restriction map*

$$i^* : H^2(X, \mathbb{Z}) \rightarrow H^2(Z, \mathbb{Z})$$

has rank one, and $\omega_Z \in \mathrm{Im}i^$.*

This goes back, at least at the level of commutative deformations, to a classical result of Voisin.

Theorem 4.6.3 ([116, Corollaire 0.2]). *Let $Z \subset X$ be a Lagrangian submanifold. Let*

$$K := \ker (i^* : H^2(X, \mathbb{Q}) \rightarrow H^2(Z, \mathbb{Q})).$$

Then, the deformation space of the pairs $\text{Def}(X, Z)$ is the space of the deformations of X preserving $K \subset \text{NS}(X)$.

Remark 4.6.4. If the restriction map has rank one, then Z deforms in codimension one, at least as far as the commutative deformations are concerned. Moreover, in this case

$$\ker i^* = \lambda^\perp \text{ for some } \lambda \in H^{1,1}(X, \mathbb{Z}),$$

where the orthogonal is with respect to the BBF form. Indeed, the fact that Z is Lagrangian implies that both $\sigma|_Z$ and $\bar{\sigma}|_Z$ vanish. The restriction preserves the Hodge structure, hence it is non-trivial only on the $H^{1,1}(X)$. So the orthogonal to the kernel is an integral $(1, 1)$ class.

To understand Theorem 4.6.2, look at the cohomological obstruction map

$$\begin{aligned} \text{HT}^2(X) = H^2(X, \mathcal{O}_X) \oplus H^1(X, T_X) \oplus H^0(X, \bigwedge^2 T_X) &\rightarrow H^*(X, \mathbb{Z}), \\ \mu &\mapsto \mu \lrcorner v(\mathcal{O}_Z). \end{aligned}$$

The fact that Z is Lagrangian implies that $\bar{\sigma} \lrcorner v(\mathcal{O}_Z)$ vanishes. The first condition in the statement implies that Z deforms in codimension one in $H^1(X, T_X)$, thanks to Theorem 4.6.3. The second condition controls the non-commutative deformations parametrized by $H^0(X, \bigwedge^2 T_X)$, so that the full cohomological obstruction map has rank one.

Proposition 4.6.5 ([77, Lemma 6.25]). *Let $i : Z \subset X$ an atomic Lagrangian, and write $\lambda := \ker(i^*)^\perp$ and $\omega_Z = t\lambda|_Z$, with $t \in \mathbb{Q}$. Then the extended Mukai vector is*

$$\tilde{v}(\mathcal{O}_Z) = \lambda - t \frac{q(\lambda)}{2} \beta.$$

Proof. The proof of Proposition 4.6.1 already implies that

$$\tilde{v}(\mathcal{O}_Z) = \lambda + s\beta,$$

for some $s \in \mathbb{Q}$. To compute the s , Markman uses the following trick. First observe that, if $(\mathcal{O}_Z)^\vee$ denotes the derived dual, then

$$\tilde{v}((\mathcal{O}_Z)^\vee) = -\lambda + s\beta,$$

as can be seen directly via Proposition 4.3.2 (see also [77, Lemma 6.15]). On the other hand $(\mathcal{O}_Z)^\vee \simeq i_* \omega_Z[-n]$, hence

$$v((\mathcal{O}_Z)^\vee) = (-1)^n \exp(t\lambda)v(\mathcal{O}_Z),$$

which implies $\tilde{v}((\mathcal{O}_Z)^\vee) = -B_{t\lambda} \tilde{v}(\mathcal{O}_Z)$. Comparing the coefficient of β gives the result. \square

4.6.1 Lagrangian surfaces

If X has dimension four and Z is a Lagrangian surface, we can prove a slight generalization of Theorem 4.6.2 using the computation of the Mukai vector in Corollary 4.3.6. For any class $\alpha \in H^2(Z, \mathbb{Q})$ we define the α -twisted Chern character as

$$\mathrm{ch}^\alpha := \mathrm{ch} \cup \exp(\alpha) \in H^*(Z, \mathbb{Q}). \quad (4.35)$$

For any class $\lambda \in H^2(X, \mathbb{Q})$ define the $\lambda|_Z$ -discriminant as

$$\overline{\Delta}_{\lambda|_Z}^{\frac{\omega_Z}{2}}(F) := \left(\mathrm{ch}_1^{\frac{\omega_Z}{2}}(F) \cdot \lambda|_Z \right)^2 - 2(\lambda|_Z \cdot \lambda|_Z) \mathrm{ch}_0^{\frac{\omega_Z}{2}}(F) \cdot \mathrm{ch}_2^{\frac{\omega_Z}{2}}(F), \quad (4.36)$$

where \cdot denotes the Poincaré pairing on Z .

Theorem 4.6.6 (Macrì). *Let X be a HK fourfold, and $i : Z \subset X$ be a Lagrangian submanifold. If $F \in \mathcal{D}^b(Z)$, then i_*F is atomic if and only if there exists a class $\lambda \in H^2(X, \mathbb{Z})$ and a rational number $\gamma \in \mathbb{Q}$ such that*

$$[Z] = \gamma(\lambda^2 - q(\lambda)\mathbf{q}_2) \quad \text{and} \quad \overline{\Delta}_{\lambda|_Z}^{\frac{\omega_Z}{2}}(F) = 0.$$

Proof. Assume first that i_*F is atomic, and write

$$\tilde{v}(i_*F) = \lambda + s\beta,$$

with $\lambda \in H^2(X, \mathbb{Z})$ and $s \in \mathbb{Q}$. By the computations in Corollary 4.3.6 we see that its Mukai vector $v(i_*F)$ is a multiple of

$$T(\tilde{v}^2) = (\lambda^2 - q(\lambda)\mathbf{q}_2) + 2s\lambda^\vee + s^2\mathbf{q}_4.$$

On the other hand, the proof of [15, Lemma 7.4] gives

$$v(i_*F) = i_* \left(\mathrm{ch}(F) \cdot \exp\left(\frac{\omega_Z}{2}\right) \right) = i_* \left(\mathrm{ch}^{\frac{\omega_Z}{2}} \right).$$

Since Z has codimension two in X , we have that

$$v(i_*F)_2 = \mathrm{ch}_2(i_*F) = \mathrm{rk}(F)[Z].$$

It follows that $[Z]$ is proportional to $(\lambda^2 - q(\lambda)\mathbf{q}_2)$, and we define $\gamma \in \mathbb{Q}$ so that the first condition is satisfied. Let $r := \mathrm{rk}(F)$; the Mukai vector must be

$$v(i_*F) = \gamma r (\lambda^2 - q(\lambda)\mathbf{q}_2) + 2\gamma r s \lambda^\vee + \gamma r s^2 \mathbf{q}_4. \quad (4.37)$$

This implies that

$$\int_Z \mathrm{ch}_1^{\frac{\omega_Z}{2}}(F) \cup \lambda|_Z = \int_X i_* \left(\mathrm{ch}_1^{\frac{\omega_Z}{2}}(F) \right) \cup \lambda = 2\gamma r s q(\lambda),$$

and $\text{ch}_2^{\frac{\omega_Z}{2}}(F) = \frac{r\gamma s^2}{c_X} \text{pt}$. By Remark 3.6.6 and Definition 3.6.12 we have

$$\begin{aligned} (\lambda|_Z \cdot \lambda|_Z) &= \int_Z \lambda|_Z \cup \lambda|_Z = \int_X \lambda^2 \cup [Z] = \gamma \int_X \lambda^2 \cdot (\lambda^2 - q(\lambda)q_2) \\ &= \gamma(3c_X q(\lambda)^2 - c_X q(\lambda)^2) = 2\gamma c_X q(\lambda)^2. \end{aligned}$$

Combining all of the above we get

$$\overline{\Delta}_{\lambda|_Z}^{\frac{\omega_Z}{2}}(F) = (2\gamma r s q(\lambda))^2 - 4q(\lambda)^2 \gamma r^2 s^2 = 0.$$

For the reverse direction, note that if $[Z]$ is as in the statement, then it deforms in codimension one. Therefore

$$\ker(i^* : H^2(X, \mathbb{Q}) \rightarrow H^2(Z, \mathbb{Q})) = \lambda^\perp \subset H^2(X, \mathbb{Q}).$$

by Theorem 4.6.3 and Remark 4.6.4. Serre duality and the Universal Coefficients Theorem imply that the pushforward $i_* : H^2(Z, \mathbb{Q}) \rightarrow H^6(X, \mathbb{Q})$ has rank one, and the image is spanned by λ^\vee . In particular, $i_* \text{ch}_1^{\frac{\omega_Z}{2}}(F)$ is a multiple of λ^\vee and the vanishing of the discriminant forces the Mukai vector to be (4.37). \square

4.6.2 Examples

Contrary to the examples discussed in Section 4.5, the examples of atomic Lagrangians have a geometric nature. In this section we will discuss a few of them. We begin with some preliminary observations. It's well known that if $Z \subset X$ is a Lagrangian submanifold, the isomorphism $T_X \cong \Omega_X$ induced by the symplectic form restricts to an isomorphism

$$N_{Z/X} \cong \Omega_Z^1. \quad (4.38)$$

So that the second page of the local-to-global spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{O}_Z, \mathcal{O}_Z)) \implies \text{Ext}^{p+q}(\mathcal{O}_Z, \mathcal{O}_Z)$$

becomes $E_2^{p,q} \cong H^p(X, \Omega_Z^q)$.

Theorem 4.6.7 ([85, Theorem 2.1.7]). *Let X be a HK manifold of dimension $2n$, and $Z \subset X$ a Lagrangian submanifold. The local-to-global spectral sequence degenerates multiplicatively in the second page and gives an isomorphism*

$$\text{Ext}^*(\mathcal{O}_Z, \mathcal{O}_Z) \cong H^*(Z, \mathbb{C}),$$

as graded algebras.

Projective spaces

Let X be a HK of dimension $2n$, and let $P \subset X$ be a subvariety isomorphic to a projective space of dimension n . Since there are no holomorphic two forms on \mathbb{P}^n , it is automatically Lagrangian. Theorem 4.6.7 implies that $\mathcal{O}_P \in D^b(X)$ is a \mathbb{P} -object, and so it is 1-obstructed by Lemma 4.5.2.

Lagrangian tori

Let $T \subset X$ be a Lagrangian complex torus. Then it is projective, because every Lagrangian submanifold is, and it is the fiber of a Lagrangian fibration $\pi : X \rightarrow \mathbb{P}^n$, by the positive answer to Beauville's conjecture in [38]. Moreover, we have the following.

Theorem 4.6.8 ([116, 98]). *Let $X \rightarrow \mathbb{P}^n$ be a Lagrangian fibration. If X_t is a smooth fiber, then the restriction*

$$H^2(X, \mathbb{Z}) \rightarrow H^2(X_t, \mathbb{Z})$$

has rank one.

Corollary 4.6.9. *The structure sheaf \mathcal{O}_T of a Lagrangian torus $T \subset X$ is an atomic object. If $f := c_1(\pi^* \mathcal{O}_{\mathbb{P}^n}(1)) \in H^2(X, \mathbb{Z})$, then the extended Mukai vector is*

$$\tilde{v}(\mathcal{O}_T) = f.$$

Proof. Since the canonical bundle ω_T is trivial, this follows from Theorem 4.6.2 and Proposition 4.6.5. \square

Fano variety of lines

Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold. Let

$$X := F(Y) \subset G(2, 6)$$

be the Fano variety of lines contained in Y . It is a classical result of Beauville and Donagi [14] that $F(Y)$ is a HK manifold of type K3^[2]. Let $h \in \text{NS}(X)$ be the restriction of the Plücker polarization. It is (very) ample with

$$q(h) = 6 \text{ and } \text{div}(h) = 2, \tag{4.39}$$

where the divisibility is defined as the positive generator of the ideal

$$\{q(h, \lambda) \mid \lambda \in H^2(X, \mathbb{Z})\} \subset \mathbb{Z}.$$

If $H \subset \mathbb{P}^5$ is a hyperplane, let $Z \subset X$ be $Z := F(Y \cap H)$ the variety of lines contained in $Y \cap H$. We collect some useful results in the following.

Proposition 4.6.10 ([15, Example 8.1.3] and [77, Example 3.11(2)]). *The submanifold $Z \subset X$ is a Lagrangian surface, with Hodge numbers*

$$h^{1,0}(Z) = 5, \quad h^{2,0}(Z) = 10, \quad h^{1,1}(Z) = 25.$$

The structure sheaf $\mathcal{O}_Z \in D^b(X)$ is an atomic object, and

$$\tilde{v}(\mathcal{O}_Z) = h - 3\beta.$$

Proof. The Hodge numbers are computed in [30, Equations (9.5), (9.12) and (10.12)] The fact that Z is Lagrangian and deforms everywhere X remains the Fano of a cubic is proved in [117]. By Theorem 4.6.3 this means that

$$\ker(H^*(X, \mathbb{Z}) \rightarrow H^2(Z, \mathbb{Z})) = h^\perp.$$

Therefore the first term in \tilde{v} is h . Since Z is the zero locus of a global section of the dual of the tautological bundle on $G(2, 5)$, adjunction formula implies

$$h|_Z \cong \omega_Z. \quad (4.40)$$

Hence the extended Mukai vector follows by Proposition 4.6.5. \square

Double EPW sextics

Let X be a double EPW sextic. It is a HK manifold of $K3^{[2]}$ -type, which is a double cover of a sextic in \mathbb{P}^5 . The natural covering involution is antisymplectic, and we let $Z \subset X$ be the fixed locus. There is a natural polarization $h \in \text{NS}(X)$ with

$$q(h) = 2 \text{ and } \text{div}(h) = 1.$$

Proposition 4.6.11 ([15, Example 8.1.4] and [77, Example 3.11(3)]). *The submanifold $Z \subset X$ is a Lagrangian surface, with Hodge numbers*

$$h^{1,0}(Z) = 0, \quad h^{2,0}(Z) = 45, \quad h^{1,1}(Z) = 100.$$

Its structure sheaf \mathcal{O}_Z is an atomic object and

$$\tilde{v}(\mathcal{O}_Z) = h - 3\beta$$

Proof. It follows from the definition that it is a Lagrangian surface. The Hodge numbers are computed in [34, Section 13.3]. In [34, Proposition 4.22] it is proved that

$$[Z] = 5h^2 - \frac{c_2(X)}{3} \text{ and } \omega_Z = 3h|_Z.$$

Hence, it follows from Theorem 4.6.2 that the extended Mukai vector is as in the statement. \square

4.7 Stability for modular sheaves

The condition of modularity, surprisingly, makes the behaviour of sheaves on HK manifolds very similar to that of sheaves on surfaces when it comes to slope stability. This was first observed by O'Grady in [94]. Recall that the slope of a coherent sheaf F on a polarized variety (X, H) is the rational number

$$\mu_H(F) = \frac{c_1(F) \cdot H^{\dim X - 1}}{\text{rk}(F)}. \quad (4.41)$$

A torsion-free sheaf F is slope semistable if for every subsheaf $E \subset F$ with $\text{rk}(E) < \text{rk}(F)$ we have

$$\mu_H(E) \leq \mu_H(F),$$

and is slope stable if the inequality is strict.

Remark 4.7.1. The condition of slope stability depends on the polarization H , which can be also a real class in $\text{Amp}(X)_{\mathbb{R}}$. Notice that if X has dimension two, the function

$$\text{Amp}(X)_{\mathbb{R}} \rightarrow \mathbb{R}, \quad H \mapsto \mu_H(F)$$

is linear. For this reason the variation of stability only happens along hyperplanes (called walls) in the ample cone, and the family of walls is locally finite. See [48, Chapter 4.C] for a review of this phenomenon.

If $\dim X$ is higher than two, the slope stops being a linear function and the behaviour of the slope is more difficult. On hyper-Kähler manifolds, we can use the BBF form to remedy to this situation. Let $E \subseteq F$ a subsheaf, we define the class

$$\lambda_{E,F} := \text{rk}(F)c_1(E) - \text{rk}(E)c_1(F) \in H^2(X, \mathbb{Z}), \quad (4.42)$$

following [94, Equation (3.2.1)]. Then, Fujiki's formula³ implies that the function $h \mapsto q(\lambda_{E,F}, h)$ can be used as a substitute for the slope to recover linearity.

Lemma 4.7.2 ([94, Lemma 3.7]). *Let (X, h) be a polarized hyper-Kähler manifold, and E, F torsion free sheaves on X . Then*

- (1) $\mu_h(E) > \mu_h(F)$ if and only if $q(\lambda_{E,F}, h) > 0$.
- (2) $\mu_h(E) = \mu_h(F)$ if and only if $q(\lambda_{E,F}, h) = 0$.

This already implies that if F changes its stability, then it must become strictly semistable for some polarization h . Moreover, the polarization h must lie in the hyperplane orthogonal to $\lambda_{E,F}$. Notice that this holds without any assumption on the modularity of E or F .

³It is the polarization formula applied to (3.6.6)

4.7.1 Wall and chamber decomposition

What changes when we look at variation of stability for a modular sheaf is that we can describe which classes give rise to walls. The reason lies in the following observation (c.f. [94, Lemma 3.9]). If

$$0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$$

is a short exact sequence, then

$$\mathrm{rk}(F) \mathrm{rk}(G) \Delta(E) + \mathrm{rk}(F) \mathrm{rk}(E) \Delta(G) = \mathrm{rk}(E) \mathrm{rk}(G) \Delta(F) + \lambda_{E,F}^2. \quad (4.43)$$

Together with Fujiki's formula, this implies the following.

Lemma and Definition 4.7.3 ([94, Definition 3.3 and Proposition 3.10]). *Let (X, h) a polarized HK, and F a torsion-free modular sheaf on X , and let $d(F)$ be as in Remark 4.3.8. Define*

$$a(F) := \frac{\mathrm{rk}(F)^2 d(F)}{4c_X}. \quad (4.44)$$

Assume that F is strictly semistable, and $E \subset F$ is slope destabilizing. Then

$$-a(F) \leq q(\lambda_{E,F}) \leq 0,$$

and $q(\lambda_{E,F}) = 0$ if and only if $\lambda_{E,F} = 0$.

This motivates the following definition.

Definition 4.7.4 ([94, Definitions 3.1 and 3.2]). Let a a positive real number. An a -wall is the intersection

$$\lambda^\perp \cap \mathrm{Amp}(X)_{\mathbb{R}},$$

where $\lambda \in \mathrm{NS}(X)$ with $-a \leq q(\lambda) < 0$. An a -chamber is a connected component of the complement of the union of the a -walls.

Remark 4.7.5. Notice that if $\lambda_{E,F} = 0$, then

$$\frac{c_1(E)}{\mathrm{rk}(E)} = \frac{c_1(F)}{\mathrm{rk}(F)}.$$

In particular, E is destabilizing for every polarization h .

The behaviour of variation of stability for a modular sheaf can be summarized in the following result.

Theorem 4.7.6 ([94, Proposition 3.4]). *Let F be a torsion-free modular sheaf on a HK manifold X .*

- (1) *Assume that h is polarization which lies in a $a(F)$ -chamber. If F is strictly h -semistable, and $E \subset F$ is destabilizing, then $\lambda_{E,F} = 0$.*
- (2) *If h_0 and h_1 belong to the same $a(F)$ -chamber, then F is h_0 -slope-stable if and only if it is h_1 -slope-stable.*

Proof. For (1), if $E \subset F$ is destabilizing a semistable sheaf, by Lemma and Definition 4.7.3 we have

$$-a(F) \leq q(\lambda_{E,F}) \leq 0.$$

By Lemma 4.7.2 we must have $q(\lambda_{E,F}, h) = 0$, but h is in a chamber so is not orthogonal to any λ with $-a(F) \leq q(\lambda) < 0$. It follows that $q(\lambda_{E,F}) = 0$, hence $\lambda_{E,F} = 0$ by Lemma and Definition 4.7.3. For (2), if F is h_0 -slope-stable and h_1 -slope unstable, then there must be an h in the segment joining h_0 and h_1 for which F is strictly semistable. Since the chambers are convex, h also belong in the same $a(F)$ -chamber as h_0, h_1 . So by point (1) we have $\lambda_{E,F} = 0$ for some E . This contradicts the h_0 -stability by Remark 4.7.5. \square

4.7.2 Lagrangian fibrations

Another extremely useful result on the stability of modular sheaves, is the possibility to gain information on it via the restriction to a general fiber of a Lagrangian fibration. In this section we fix a Lagrangian fibration $\pi : X \rightarrow \mathbb{P}^n$, and we denote

$$f := c_1(\pi^*(\mathcal{O}_{\mathbb{P}^n}(1))) \in \text{NS}(X).$$

Recall that f is a nef class, hence it lies in the boundary of the ample cone.

We wish to compare slopes when we restrict to a general fiber. The following Lemma contains the key computation; it can be seen as a fiberwise analogue of Lemma 4.7.2.

Lemma 4.7.7 ([94, Lemma 3.11]). *Let $\pi : X \rightarrow \mathbb{P}^n$ a Lagrangian fibration. Let $E \subseteq F$ be torsion-free sheaves. Let $t \in \mathbb{P}^n$ be a general point. Then, for every ample class h , we have*

- (1) $\mu_h(E_t) < \mu_h(F_t)$ if and only if $q(\lambda_{E,F}, f) < 0$,
 - (2) $\mu_h(E_t) = \mu_h(F_t)$ if and only if $q(\lambda_{E,F}, f) = 0$,
- where $E_t := E|_{\pi^{-1}(t)}$ denotes the restriction to the fiber $\pi^{-1}(t)$.

Proof. Let h the polarization on X , and $h_t := h|_{X_t}$. Fujiki's formula gives

$$\int_X \lambda_{E_t, F_t} \cup h_t^{n-1} = c_X n! \cdot q(h, f)^{n-1} q(\lambda_{E,F}, f). \quad (4.45)$$

Since $q(h, f)$ is strictly positive, we see that the left hand side has the same sign as $q(\lambda_{E,F}, f)$, which is precisely the statement. \square

If F is a modular sheaf, we know by Theorem 4.7.6 that the ample cone is subdivided in chambers. Since f is a nef class, there is one chamber which contains it in its closure. The above lemma suggests that, when investigating stability of F_t , the polarizations in this chamber are privileged.

Definition 4.7.8 ([94, Definition 3.5]). Let a be a positive real number. A class $h \in \text{Amp}(X)$ is called a -suitable if for every $\lambda \in \text{NS}(X)$ as in Definition 4.7.4, either $q(\lambda, h)$ and $q(\lambda, f)$ have the same sign or they are both zero.

The following is one of the main results of [94], and we refer to it for the proof.

Theorem 4.7.9 ([94, Proposition 3.6]). *Let $\pi : X \rightarrow \mathbb{P}^n$ a Lagrangian fibration, and F a modular sheaf on X . Assume that $\text{Sing}(F)$ does not dominate \mathbb{P}^n . Let h be an ample class which is $a(F)$ -suitable. Then the following hold*

- (1) *If the restriction of F to a generic fiber of π is slope stable, then F is slope stable.*
- (2) *If F is slope semistable then the restriction of F is slope semistable.*

The case we will be interested in is one of a vector bundle which is strictly semistable when restricted to the fibers, hence we will need a different result.

Proposition 4.7.10. *Let X be a HK manifold of Picard rank two, $\pi : X \rightarrow \mathbb{P}^n$ a Lagrangian fibration, and F a modular torsion-free sheaf on X . Let h be an ample class which is $a(F)$ -suitable. Assume that $\text{Sing}(F)$ does not dominate \mathbb{P}^n , and that for general t the restriction F_t is semistable. If $E \subset F$ is a destabilizing subsheaf we have*

$$q(\lambda_{E,F}, f) = 0,$$

and $E_t \subset F_t$ is destabilizing for $t \in \mathbb{P}^n$ general.

Proof. Since F_t is semistable we have $\mu(E_t) \leq \mu(F_t)$, which gives

$$q(\lambda_{E,F}, f) \leq 0, \tag{4.46}$$

by Lemma 4.7.7. On the other hand, if $E \subset F$ destabilizes we have

$$q(\lambda_{E,F}, h) \geq 0, \tag{4.47}$$

by Lemma 4.7.2. We can assume h is an $a(F)$ -suitable rational ample class. Any such class can be written as

$$h = f + \varepsilon\lambda,$$

for some ample class λ and $\varepsilon \ll 1$, and viceversa every class of this form is $a(F)$ -suitable provided ε is sufficiently small. Substituting in (4.47) we have

$$q(\lambda_{E,F}, h) = q(\lambda_{E,F}, f) + \varepsilon q(\lambda_{E,F}, \lambda) \geq 0,$$

for every $\varepsilon \ll 1$. Passing to the limit $\varepsilon \rightarrow 0$, we get $q(\lambda_{E,F}, f) \geq 0$. Combined with (4.46), it implies

$$q(\lambda_{E,F}, f) = 0.$$

By Lemma 4.7.7 this implies that $\mu(E_t) = \mu(F_t)$, hence E destabilizes F . \square

Remark 4.7.11. Notice that the only place we need modularity in this statement is for the existence of suitable polarizations.

Corollary 4.7.12. *In the setting of Proposition 4.7.10, assume moreover that $\rho(X) = 2$ and $c_1(F)$ is a multiple of f . If $E \subseteq F$ is destabilizing, then $c_1(E)$ is a multiple of f .*

Proof. We can write

$$c_1(E) = af + b\lambda,$$

where λ restricts to a principal polarization on the general fiber. In particular, $q(\lambda, f) \neq 0$. By Proposition 4.7.10 we get

$$0 = q(\lambda_{E,F}, f) = \text{rk}(F)q(c_1(E), f),$$

because $c_1(F)$ is a multiple of f . This implies that $b = 0$, i.e. $c_1(E)$ is a multiple of f too. \square

4.8 Moduli spaces of stable atomic sheaves

In this section we investigate in as much generality as we can the geometrical properties of the moduli spaces of atomic sheaves. The motivating goal would be to have some sort of analogue of Theorem 2.1.1, which for the moment is far out of reach.

The setting is classical: we let (X, h) be a polarized HK manifold, and $\mathbf{v} \in H^*(X, \mathbb{Q})$. We consider the coarse moduli space

$$M_{X,h}(\mathbf{v}) := \{F \in \text{Coh}(X) \mid F \text{ is } h\text{-Gieseker semi-stable, with } v(F) = \mathbf{v}\} / \sim,$$

where \sim is the S-equivalence relation. As is well-known, this moduli space is always projective because it is constructed via a GIT quotient of a projective scheme.

When dealing with sheaves on high-dimensional HK manifolds, the foundations have a still strong differential geometric nature. In particular, we will make use of results by Verbitsky (c.f. Theorem 4.8.5), which apply to slope stable projectively hyperholomorphic vector bundles.

Recall that a vector bundle E is projectively hyperholomorphic if $\mathcal{E}nd(E)$ is hyperholomorphic, which means that it satisfies the conclusion of Theorem 4.3.9.

This happens, for example, if E is atomic and its Mukai vector is contained in the Verbitsky component.⁴

So, while our results are general, we are tacitly assuming that (at least an irreducible component of) our moduli space parametrizes only slope-stable vector bundles. This will be the case in one of our examples in Section 5.8.

Remark 4.8.1. The fact that the sheaves in $M_{(X,h)}(\mathbf{v})$ are atomic should tell us even more than just the smoothness. In fact we expect a Hodge isometry

$$H^2(M_{X,h}(v), \mathbb{Q}) \simeq \tilde{v}^\perp \subset \tilde{H}(X, \mathbb{Q}),$$

in a similar fashion as Theorem 2.1.1. This is still conjectural, but would be a strong indication that moduli spaces of atomic objects can produce new HK manifolds.

4.8.1 Globalizing the obstruction map

We claim that the 1-obstructedness condition is sometimes related to the existence of a unique symplectic form on the moduli space. We first show how the obstruction map produces such forms, using classical techniques from [48, Chapter 10]. Let $\mathcal{E} \rightarrow X \times S$ an S -flat family of stable sheaves.

Remark 4.8.2. Is not in general true that the moduli space $M_{X,h}(\mathbf{v})$ is fine, but we can always find an étale cover

$$\coprod_i S_i \rightarrow M_{X,h}(\mathbf{v}),$$

such that over each $X \times S_i$ there is a tautological family \mathcal{E}_i . We are mainly interested in the case where the variety S is the normalization of one of these S_i , and \mathcal{E} is the pullback family.

In Section 4.2.2 we introduced the obstruction map

$$\chi_E : \mathrm{HH}^2(X) \rightarrow \mathrm{Ext}^2(E, E)$$

for every object $E \in D^b(X)$. There is a natural relative version of it, which we can construct as follows. If $\pi_2 : X \times S \rightarrow S$ is the second projection, we consider the i -th relative Ext sheaf

$$\mathcal{E}xt_{\pi_2}^i(\mathcal{E}, \mathcal{E}) := \mathcal{H}^i(\pi_{2,*} \circ \mathcal{H}om(\mathcal{E}, \mathcal{E})),$$

⁴If X is of type $K3^{[2]}$ this is always the case.

where both functors are derived. There is a base change map

$$\varphi_i(s) : \mathcal{E}xt_{\pi_2}^i(\mathcal{E}, \mathcal{E})|_s \rightarrow \text{Ext}_X^i(\mathcal{E}_s, \mathcal{E}_s),$$

which satisfies similar properties to the usual cohomology and base change Theorem [42, Theorem 12.11(ii)] as proved in [65].

Proposition 4.8.3. *There is a morphism*

$$\chi : \text{HH}^2(X) \rightarrow H^0(S, \mathcal{E}xt_{\pi_2}^2(\mathcal{E}, \mathcal{E})),$$

such that for every $s \in S$ the composition with the base change map

$$\text{HH}^2(X) \rightarrow H^0(S, \mathcal{E}xt_{\pi_2}^2(\mathcal{E}, \mathcal{E})) \rightarrow \text{Ext}_X^2(\mathcal{E}_s, \mathcal{E}_s)$$

gives the obstruction map $\chi_{\mathcal{E}_s}$.

Proof. Every element $\mu \in \text{HH}^2(X)$ is a morphism

$$\mu : \mathcal{O}_\Delta \rightarrow \mathcal{O}_\Delta \in \text{D}^b(X \times X).$$

Taking Fourier-Mukai with kernel \mathcal{E} we obtain

$$\mathcal{E} \rightarrow \mathcal{E} \in \text{D}^b(X \times S),$$

i.e. an element in $\text{Ext}_{X \times S}^2(\mathcal{E}, \mathcal{E})$. There is a local-to-global spectral sequence

$$E_2^{p,q} = H^p(S, \mathcal{E}xt_{\pi_2}^q(\mathcal{E}, \mathcal{E})) \implies \text{Ext}_{X \times S}^{p+q}(\mathcal{E}, \mathcal{E})$$

which gives a 'localization' morphism

$$\text{Ext}_{X \times S}^2(\mathcal{E}, \mathcal{E}) \rightarrow H^0(S, \mathcal{E}xt_{\pi_2}^2(\mathcal{E}, \mathcal{E})),$$

and we let $\chi : \text{HH}^2(X) \rightarrow H^0(S, \mathcal{E}xt_{\pi_2}^2(\mathcal{E}, \mathcal{E}))$ be the composition. It is a matter of chasing the definitions that for every $s \in S$ the following diagram commutes

$$\begin{array}{ccc} \text{Ext}_{X \times S}^2(\mathcal{E}, \mathcal{E}) & \longrightarrow & H^0(S, \mathcal{E}xt_{\pi_2}^2(\mathcal{E}, \mathcal{E})) \\ \downarrow & & \downarrow \\ \text{Ext}^2(\mathcal{E}_s, \mathcal{E}_s) & \xlongequal{\quad} & \text{Ext}^2(\mathcal{E}_s, \mathcal{E}_s) \end{array}$$

The statement then follows from Definition 4.2.5. □

4.8.2 Symplectic form

How can this be used to construct a symplectic form? For this, we put ourself in the setting of Remark 4.8.2. In particular, S is equipped with a map to the moduli space $M_{X,h}(\mathbf{v})$. Denote by $S_0 \subset S$ the inverse image of the regular locus of $M_{X,h}(\mathbf{v})$, it is smooth because $S \rightarrow M_{X,h}(\mathbf{v})$ is étale on M_0 .

Then, the Kodaira-Spencer map globalizes to a bundle isomorphism

$$T_{S_0} \simeq \mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})|_{S_0}, \tag{4.48}$$

because it is an isomorphism on the fibers.

For every $s \in S_0$, the Yoneda pairing

$$\mathrm{Ext}_X^1(\mathcal{E}_s, \mathcal{E}_s) \times \mathrm{Ext}_X^1(\mathcal{E}_s, \mathcal{E}_s) \rightarrow \mathrm{Ext}_X^2(\mathcal{E}_s, \mathcal{E}_s)$$

is skew-symmetric, because \mathcal{E}_s is a smooth point in the moduli space. For every section $\eta \in H^0(S_0, \mathcal{E}xt_{\pi_2}^2(\mathcal{E}, \mathcal{E})|_{S_0})$ we have a map of vector bundles

$$\begin{aligned} \mathcal{E}xt_{\pi_2}^2(\mathcal{E}, \mathcal{E})|_{S_0} &\rightarrow \bigwedge^2 \mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})^\vee|_{S_0}, \\ c &\mapsto \mathrm{Tr}(\eta^{n-2} \circ c \circ - \circ -). \end{aligned}$$

The Kodaira-Spencer isomorphism (4.48) implies that the target bundle is $\Omega_{S_0}^2$. Taking global section we get a map

$$H^0(S_0, \mathcal{E}xt_{\pi_2}^2(\mathcal{E}, \mathcal{E})|_{S_0}) \rightarrow H^0(S_0, \Omega_{S_0}^2).$$

In particular, looking at the image of η we produce a skew-symmetric form on S_0 , and by abusing the notation still denote by $\tau \in H^0(S_0, \Omega_{S_0}^2)$.

The composition with the (globalized) obstruction map, allows us to associate to every $\mu \in \mathrm{HH}^2(X)$ a skew symmetric form $\tau_\mu \in H^0(S_0, \Omega_{S_0}^2)$. On each point $s \in S_0$ it is given by

$$\tau_\mu(a, b) = \mathrm{Tr}(\chi_{\mathcal{E}_s}(\mu)^{n-1} \circ a \circ b) \tag{4.49}$$

for every $a, b \in \mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_s)$.

Remark 4.8.4. Here we are being slightly sloppy with the notation. The trace of an element in $\mathrm{Ext}^{2n}(\mathcal{E}_s, \mathcal{E}_s)$, is a class in $H^{2n}(X, \mathcal{O}_X)$, which is indeed one dimensional but is not canonically trivialized. To trivialize it we need to choose a symplectic form σ on X . Then, the precise formula for the value of τ_μ at s is

$$\tau_\mu(a, b) = \int_X (\mathrm{Tr}(\chi_{\mathcal{E}_s}(\mu)^{n-1} \circ a \circ b) \cup \sigma_X^n) \in \mathbb{C}. \tag{4.50}$$

So, for every class in the image of the obstruction map we get a skew-symmetric form on S , we want to now when this form is symplectic. First notice that the composition of the map

$$\mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_s) \rightarrow \mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_s)^*, \quad a \mapsto \tau_\mu(a, -)$$

with Serre duality is the Yoneda product:

$$\mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_s) \rightarrow \mathrm{Ext}^{2n-1}(\mathcal{E}_s, \mathcal{E}_s), \quad a \mapsto \chi(\mu)^{n-1} \circ a.$$

Therefore, τ_μ is symplectic if and only if this is an isomorphism. The key result we need is the following.

Theorem 4.8.5 ([108, Theorem 4.2A]). *Let E be a slope stable projectively hyperholomorphic vector bundle on a HK manifold X . For every i the map*

$$\chi_E(\bar{\sigma})^{n-i} \circ - : \mathrm{Ext}^i(E, E) \rightarrow \mathrm{Ext}^{2n-i}(E, E)$$

is an isomorphism.

In particular, if S_0 parametrizes only locally free sheaves, the form $\tau_{\bar{\sigma}} \in H^0(S_0, \Omega_{S_0}^2)$ is non-degenerate. To show symplecticity we must prove that it is closed (with respect to the de-Rham differential), and for this we give an alternative definition of $\tau_{\bar{\sigma}}$.

In [64] the authors associate, to any class $\omega \in H^{n-k-2}(X, \Omega_X^{n-k})$ a closed two-form $\alpha_\omega \in H^0(S_0, \Omega_{S_0}^2)$. On a point $s \in S_0$ its value is

$$\alpha_\omega(a, b) = \int_X (\mathrm{Tr} (\mathrm{At}(\mathcal{E}_s)^k \circ a \circ b) \cup \omega). \quad (4.51)$$

Proposition 4.8.6. *For every $\mu \in \mathrm{HH}^2(X)$ the two-form τ_μ is closed. In particular, if S_0 parametrizes only locally free sheaves, the form $\tau_{\bar{\sigma}}$ is symplectic.*

Proof. Assume by linearity that $\mu \in H^p(X, \wedge^q T_X)$, and let $s \in S_0$. According to (4.50) we have

$$\begin{aligned} \tau_\mu(a, b) &= \int_X (\mathrm{Tr} (\chi_{\mathcal{E}_s}(\mu)^{n-1} \circ a \circ b) \cup \sigma_X^n) \\ &= \int_X (\mathrm{Tr} (\chi_{\mathcal{E}_s}(\mu^{n-1}) \circ a \circ b) \cup \sigma_X^n) \text{ by Proposition 4.2.8,} \\ &= \int_X (\mathrm{Tr} (((\mu^{n-1} \otimes \mathrm{id}_{\mathcal{E}_s}) \circ \mathrm{At}(\mathcal{E}_s)^{q(n-1)}) \circ a \circ b) \cup \sigma_X^n) \text{ by Definition 4.2.5} \\ &= \int_X (\mu^{n-1} \circ \mathrm{Tr} (\mathrm{At}(\mathcal{E}_s)^{q(n-1)} \circ a \circ b) \cup \sigma_X^n), \end{aligned}$$

where the last equality is due to the linearity of the Atiyah class, see [64, Section 1.2]. To conclude notice that there must be some ω which makes this expression equal to α_ω . Indeed, we can consider the functional

$$H^{q(n-1)+2}(X, \Omega_X^{q(n-1)+2}) \rightarrow C, \quad x \mapsto \int_X (\mu^{n-1} \circ x \cup \sigma^n),$$

which must be represented by some $\omega \in H^{n-q(n-1)-2}(X, \Omega_X^{n-q(n-1)})$ by Poincaré duality. \square

Remark 4.8.7. Beckmann conjectures that, for a 1-obstructed object, the image of the obstruction map has the Hard-Lefschetz property on the Ext algebra, see [15, Conjecture A]. Hence, at least conjecturally, a moduli space of 1-obstructed objects should carry a symplectic form on the smooth locus.

4.8.3 Smoothness

In this section we prove smoothness result on the moduli space M . The setting is more special than that of the previous section. Let X be a hyper-Kähler manifold of $\dim X = 2n \geq 4$, and $\mathcal{E} \rightarrow S \times X$ a family of slope stable projectively hyperholomorphic vector bundles.

A general statement we can prove is the following.

Theorem 4.8.8. *In the setting above, assume moreover that S is integral and normal, and that the function*

$$s \rightarrow \text{ext}^1(\mathcal{E}_s, \mathcal{E}_s)$$

is constant on an open $U \subset S$, with complement $S - U$ of codimension at least two. Then it is constant on all of S .

Note that since \mathcal{E} is a vector bundle

$$R\mathcal{H}om(\mathcal{E}, \mathcal{E}) = R\pi_2(\mathcal{E}nd(\mathcal{E})),$$

so that we can apply freely the cohomology and base change Theorems and Grauert's Theorem. A point s in S corresponds to a bundle \mathcal{E}_s : when it does not generate confusion we will simply denote it by E .

Lemma 4.8.9. *The sheaf $\mathcal{E}xt_{\pi_2}^0(\mathcal{E}, \mathcal{E})$ is isomorphic to the structure sheaf, and $\mathcal{E}xt_{\pi_2}^{2n}(\mathcal{E}, \mathcal{E})$ is a line bundle. Moreover, the base change maps*

$$\mathcal{E}xt_{\pi_2}^i(\mathcal{E}, \mathcal{E})|_E \rightarrow \text{Ext}^i(E, E)$$

are isomorphisms for $i = 0, 2n$.

Proof. By stability and Serre duality $\text{ext}^i(E, E) = 1$ for every $E \in S$ and for $i = 0, 2n$. Grauert's Theorem yields that $\mathcal{E}xt_{\pi_2}^i(\mathcal{E}, \mathcal{E})$ are line bundles and that the base change maps are isomorphisms. The $\mathcal{E}xt_{\pi_2}^0(\mathcal{E}, \mathcal{E})$ is trivial because the trace map

$$\text{Tr} : \pi_{2,*}\mathcal{E}nd(\mathcal{E}) \rightarrow \mathcal{O}_M$$

is a fiberwise isomorphism. \square

The next step is to prove that $\mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})$ is a reflexive sheaf. For this we will use Grothendieck-Verdier duality for the projection $\pi_2 : X \times M \rightarrow M$. Notice that since X is smooth and projective, we can apply Grothendieck-Verdier duality, giving

$$R\pi_{2,*}R\mathcal{H}om(\mathcal{E}nd(\mathcal{E}), \mathcal{O}_M[2n]) \simeq R\mathcal{H}om(R\pi_{2,*}(\mathcal{E}nd(\mathcal{E})), \mathcal{O}_M). \quad (4.52)$$

Since $\mathcal{E}nd(\mathcal{E})$ is autodual, this is also isomorphic to $R\pi_{2,*}(\mathcal{E}nd(\mathcal{E})[2n])$.

Lemma 4.8.10. *There is an isomorphism*

$$\mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E}) \simeq \mathcal{E}xt_{\pi_2}^{2n-1}(\mathcal{E}, \mathcal{E})^\vee.$$

In particular, the sheaf $\mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})$ is reflexive.

Proof. There is a spectral sequence

$$E_2^{p,q} = \mathcal{E}xt^p(\mathcal{E}xt_{\pi_2}^{-q}(\mathcal{E}, \mathcal{E}), \mathcal{O}_M) \implies \mathcal{H}^{p+q}(R\mathcal{H}om(R\pi_{2,*}(\mathcal{E}nd(\mathcal{E})), \mathcal{O}_M)).$$

Grothendieck-Verdier duality (4.52) implies that

$$\mathcal{H}^{p+q}(R\mathcal{H}om(R\pi_{2,*}(\mathcal{E}nd(\mathcal{E})), \mathcal{O}_M)) \simeq \mathcal{E}xt_{\pi_2}^{p+q-2n}(\mathcal{E}, \mathcal{E}).$$

Since $-2n \leq q \leq 0$, and $p \geq 0$, for $p + q = -(2n - 1)$ the only non-zero terms are

- $E_2^{0, -(2n-1)} = \mathcal{H}om(\mathcal{E}xt_{\pi_2}^{2n-1}(\mathcal{E}, \mathcal{E}), \mathcal{O}_M) \rightarrow E_2^{1, -2n} = \mathcal{E}xt^2(\mathcal{E}xt_{\pi_2}^{2n}(\mathcal{E}, \mathcal{E}), \mathcal{O}_M)$.
- $E_2^{1, -2n} = \mathcal{E}xt^1(\mathcal{E}xt_{\pi_2}^{2n}(\mathcal{E}, \mathcal{E}), \mathcal{O}_M)$

Now, the terms $E_2^{1, -2n}$ and $E_2^{2, -2n}$ vanish because $\mathcal{E}xt_{\pi_2}^{2n}(\mathcal{E}, \mathcal{E})$ is a line bundle by Lemma 4.8.9. So at this level the spectral sequence degenerates at the second page giving the isomorphism in the statement. \square

Remark 4.8.11. Notice that looking at $p + q = -2n$ gives

$$\mathcal{E}xt_{\pi_2}^0(\mathcal{E}, \mathcal{E}) \simeq (\mathcal{E}xt_{\pi_2}^{2n}(\mathcal{E}, \mathcal{E}))^\vee.$$

Which by Lemma 4.8.9 implies that $\mathcal{E}xt_{\pi_2}^{2n}(\mathcal{E}, \mathcal{E})$ is the trivial line bundle.

Lemma 4.8.12. *Let $\varphi_y : \mathcal{E}xt_{\pi_2}^{2n-1}(\mathcal{E}, \mathcal{E}) \rightarrow (\mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E}))^\vee$ be the map induced by the Yoneda pairing. Then the composition*

$$\mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E}) \xrightarrow{\tau_\eta} \mathcal{E}xt_{\pi_2}^{2n-1}(\mathcal{E}, \mathcal{E}) \xrightarrow{\varphi_y} (\mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E}))^\vee$$

is an isomorphism.

Proof. Notice that since $\mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})$ is reflexive and S is normal, it suffices to prove the result on an open of codimension two. By assumption we know that $\text{ext}^1(\mathcal{E}, \mathcal{E})$ is constant in U which has codimension two. On U , by Grauert's Theorem [42, Corollary 12.9], the sheaf $\mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})$ is locally free, and the base change map

$$\mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})|_E \rightarrow \text{Ext}^1(E, E)$$

is an isomorphism. The same holds for $\mathcal{E}xt_{\pi_2}^{2n-1}(\mathcal{E}, \mathcal{E})$ by (pointwise) Serre duality. Since both maps φ_y and τ_η are constructed starting from the Yoneda pairing, they are compatible with the base change maps.

Therefore, for every $E \in U$ we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})|_E & \xrightarrow{\tau_\eta|_E} & \mathcal{E}xt_{\pi_2}^{2n-1}(\mathcal{E}, \mathcal{E})|_E & \xrightarrow{\varphi_y|_E} & \mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})^\vee|_E \\ \simeq \downarrow & & \simeq \downarrow & & \downarrow \simeq \\ \text{Ext}^1(E, E) & \xrightarrow{\simeq} & \text{Ext}^{2n-1}(E, E) & \xrightarrow{\simeq} & \text{Ext}^1(E, E)^\vee \end{array}$$

Here the first bottom map is an isomorphism by Theorem 4.8.5, and the second one by Serre duality. This gives that the upper composition is an isomorphism. \square

Proof of Theorem 4.8.8. The composition of the map φ_y with the inverse of the isomorphism of Lemma 4.8.12 gives a morphism

$$\mathcal{E}xt_{\pi_2}^{2n-1}(\mathcal{E}, \mathcal{E}) \rightarrow \mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})$$

which is surjective because it is a splitting of τ_η . For every $E \in S$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}xt_{\pi_2}^{2n-1}(\mathcal{E}, \mathcal{E})|_E & \xrightarrow{\varphi_y|_E} & \mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})|_E \\ \simeq \downarrow & & \downarrow \\ \text{Ext}^{2n-1}(E, E) & \xrightarrow{\simeq} & \text{Ext}^1(E, E) \end{array}$$

where the bottom map is the isomorphism in Theorem 4.8.5. The left vertical map is surjective, by the cohomology and base change theorem (c.f. [42, Theorem 12.11(ii)]) because

$$\mathcal{E}xt_{\pi_2}^{2n}(\mathcal{E}, \mathcal{E})|_E \rightarrow \text{Ext}^{2n}(E, E)$$

is an isomorphism. Since the composition is surjective, it follows that the base change map

$$\mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})|_E \rightarrow \text{Ext}^1(E, E) \quad (4.53)$$

is surjective for every $E \in S$.

Another application of [42, Theorem 12.11(ii)] implies that the sheaf $\mathcal{E}xt_{\pi_2}^1(\mathcal{E}, \mathcal{E})$ is locally free because the base change map for $\mathcal{E}xt_{\pi_2}^0(\mathcal{E}, \mathcal{E})$ is an isomorphism. Lastly, by point (i) in *loc. cit.* we get that the base change map (4.53) is an isomorphism. Hence $\text{ext}^1(E, E)$ is constant in S . \square

Chapter 5

Towards a modular construction of OG10

5.1 Introduction

Motivated by the question of realizing OG10 as a moduli space of sheaves on a HK fourfold, in this Chapter we construct a stable atomic bundle whose moduli space is birational to OG10. The main result is the following.

Theorem 5.1.1 (Theorem 5.7.5). *Let X be a hyper-Kähler of $K3^{[2]}$ -type. There exist a stable, atomic vector bundle F_0 with Mukai vector*

$$v(F_0) = 5 \left(1 - \frac{3}{4} \mathbf{q}_2 + \frac{9}{32} \mathbf{pt} \right).$$

The $\mathrm{Ext}^1(F_0, F_0)$ is ten dimensional. The Yoneda pairing is skew-symmetric and induces an isomorphism

$$\bigwedge^2 \mathrm{Ext}^1(F_0, F_0) \xrightarrow{\sim} \mathrm{Ext}^2(F_0, F_0).$$

In particular, its deformation functor is smooth.

We briefly describe the steps involved in the construction.

- (1) If $X \subset \mathbb{P}^5$ is a general cubic fourfold and H is a general hyperplane, then the structure sheaf $\mathcal{O}_{F(X \cap H)}$ is an atomic object in $D^b(F(X))$. We degenerate the cubic to the determinantal cubic and consider the corresponding degeneration of the Fano variety of lines. After a resolution, the central fiber is a moduli space M of torsion sheaves on a general K3 surface of degree two, and the surface $F(X \cap H)$ degenerates to a reducible Lagrangian Z with two components.

- (2) The moduli space M is endowed with a Lagrangian fibration $\pi : M \rightarrow \mathbb{P}^2$. As shown in [3], there is an autoequivalence Φ of M mapping a general point to a line bundle supported on its fiber. The complex $\Phi(\mathcal{O}_Z) \in D^b(M)$ is a locally free sheaf, but it is not stable.
- (3) To make it stable we apply a second autoequivalence: the composition of two (inverses of) \mathbb{P} -twists around line bundles. After twisting by a line bundle, the resulting vector bundle will have $c_1 = 0$. Using atomicity we can easily compute the Mukai vector from this construction.

Along the way we prove a number of interesting results on their own. We highlight especially the following.

Proposition 5.1.2 (Proposition 5.2.6). *Let $M = M_S(0, H, 1 - g)$ be a moduli space of torsion sheaves on a general polarized K3 surface (S, H) of genus g , and let $\pi : M \rightarrow \mathbb{P}^g$ be the Lagrangian fibration. Let $L \subset M$ be a subvariety such that $\pi|_L : L \rightarrow \mathbb{P}^g$ is finite. If V_L is a Cohen-Macaulay sheaf on L , then $\Phi(V_L)$ is a locally free sheaf.*

The proof is based on an analysis done by Arinkin in [7] on the singularities of the Fourier-Mukai kernel of Φ . This is the first technique to produce *locally free* sheaves from structure sheaves of subvarieties in this context. We believe this could be helpful in understanding the relationship between atomic vector bundles and atomic Lagrangians.

The rest of the chapter is devoted to the study of the irreducible component of the moduli space \mathfrak{M} of Gieseker semi-stable sheaves on M containing F_0 . The main result is the following.

Theorem 5.1.3 (Proposition 5.8.2 and Theorem 5.8.6). *The smooth locus \mathfrak{M}_{sm} is equipped with a closed holomorphic 2-form. Moreover, there is a birational map preserving the 2-form*

$$X \dashrightarrow \mathfrak{M},$$

where X is a hyper-Kähler manifold of type OG10.

The birational map is easily described. Recall that M is a moduli space of sheaves on a general polarized K3 surface (S, H) of degree two. The reducible Lagrangian $Z \subset M$ has two components. One is the image P' of a section of the Lagrangian fibration π . The other is a Lagrangian surface L isomorphic to $\text{Sym}^2 C$, where $C \subset S$ is a general curve in $|2H|$.

In the proof of Theorem 5.7.5 it is not necessary to consider the image of the structure sheaf \mathcal{O}_Z , we can allow a slight generalization. Namely, we can consider a line bundle \mathcal{L} on Z of degree zero, which restricts to the structure sheaf on P' and to the symmetric square of a line bundle of degree zero on C .

A degree zero line bundle supported on a general curve in $|2H|$ is a general element in the moduli space $M_S(0, 2H, -4)$. The variety X is the symplectic resolution of this moduli space, and the birational map is given by the steps (1)–(3) applied to \mathcal{L} .

5.1.1 Structure of the chapter

In Section 5.2 we review some basic facts about Beauville-Mukai systems. In particular, we focus on the construction of the Poincaré sheaves following [7, 3] and its properties. We prove Proposition 5.1.2 and compute the action of the equivalence induced by the Poincaré sheaf on the extended Mukai lattice.

Section 5.3 is purely algebraic. It serves to build the computational background to do the semistable reduction in Section 5.7. We compute how the \mathbb{P} -twist around a \mathbb{P} -object \mathcal{E} acts on an object \mathcal{F} under some assumptions on the Ext groups $\text{Ext}^*(\mathcal{E}, \mathcal{F})$.

In Section 5.5 we show we can degenerate the Fano variety of lines of a cubic fourfold to the moduli space $M_S(0, H, -1)$, and that the surface of lines becomes a reducible Lagrangian Z . Nothing here is original work, the degeneration of the fourfold is done in [33] and the one of the surface goes back to [31].

In Section 5.6 we begin the proof of Theorem 5.1.1. Using the results of previous sections, we construct an atomic vector bundle with Mukai vector $v(F_0)$. We compute its Ext algebra, and show that the Yoneda pairing is skew-symmetric and induces the desired isomorphism.

In Section 5.7 we show that the bundle constructed before is not slope stable. The cause of the instability lies in the fact that the Lagrangian $Z \subset M$ is reducible. To get something stable we show that it suffices to apply two \mathbb{P} -twists. This is a wall-crossing phenomenon, very similar to what happens in Section 2.4.

In Section 5.8 we prove Theorem 5.1.3. The key intermediate result is Theorem 5.8.6 where we show that the bundles we are considering are not just atomic but actually 1-obstructed.

5.2 Moduli spaces of torsion sheaves

In this section we collect some results on certain moduli spaces of sheaves on $K3$ surfaces. The setting is the following. Let (S, H) be a polarized $K3$ surface, and assume that $|H|$ parametrizes only irreducible curves, this is the case for example if (S, H) is general. We consider the moduli space $M_d := M_S(0, H, d)$. It is equipped with a Lagrangian fibration

$$\pi_d : M_d \rightarrow |H|$$

realizing it as the relative compactified Picard $\text{Pic}^{d+g-1}(\mathcal{C}/|H|)$ of the the universal curve over the linear system $|H|$. Recall that the compactified Picard of an integral curve C is the moduli space

$$\overline{\text{Pic}}^l(C) = \{\text{torsion-free sheaves on } C \text{ of rank 1 and degree } l\}/\cong.$$

Therefore, a point lying in the regular locus of the map π_d is a line bundle of degree d on a smooth curve in the linear system $|H|$. In particular, if $d = 1 - g$, we get the relative compactified Jacobian $\text{Pic}^0(\mathcal{C}/|H|)$. Notice that in this case, π_{1-g} has a natural section given by

$$|H| \rightarrow M_{1-g}, C \mapsto \mathcal{O}_C.$$

The following is an immediate consequence of Theorem 2.1.1.

Lemma 5.2.1. *Let (S, H) is a general polarized K3 surface of genus g . Then*

$$\text{NS}(M_0) \cong \mathbb{Z}f \oplus \mathbb{Z}\lambda,$$

with

$$q(\lambda, \lambda) = 2g - 2, q(\lambda, f) = 2, q(f) = 0.$$

5.2.1 Relative Poincaré sheaf

If C is an integral curve with planar singularities, Arinkin constructs a Cohen-Macaulay sheaf

$$\mathcal{P}_C \in \text{Coh}(\overline{J}(C) \times \overline{J}(C)),$$

such that the Fourier-Mukai transform

$$\Phi_{\mathcal{P}} : \text{D}^b(\overline{J}(C)) \simeq \text{D}^b(\overline{J}(C))$$

is an equivalence.

In [3] the authors extend the construction of the Poincaré sheaf to the relative compactified Jacobian. We recall the main properties in the following.

Proposition 5.2.2 ([3, 7]). *Let (S, H) be a polarized K3 surface of genus g , such that every curve in $|H|$ is integral, and let $M := M(0, H, 1 - g)$. There exists a coherent sheaf*

$$\mathcal{P} \in \text{Coh}(M \times_{|H|} M),$$

flat with respect to both projection, and such that for every $x \in M$ the restriction $\mathcal{P}_{M \times_{|H|} \{x\}}$ is a Cohen-Macaulay sheaf on $M \times_{|H|} \{x\} = \pi^{-1}(\pi(x))$. The Fourier-Mukai transform with kernel \mathcal{P}

$$\Phi := \Phi_{\mathcal{P}} : \text{D}^b(M) \simeq \text{D}^b(M)$$

is an equivalence.

Remark 5.2.3. The sheaf \mathcal{P} is not uniquely determined by Proposition 5.2.2. For example we can twist it by the pullback of a line bundle on $|H|$, and still get a valid Poincaré sheaf. In our case, we normalize it by imposing that

$$\Phi(\mathcal{O}_{P'}) = \mathcal{O}_M \tag{5.1}$$

where $P' \subset M$ is the natural section of the Lagrangian fibration.

The statement in [3, Proposition 2.1(a)] includes also M_d for every d , with the caveat that in that case \mathcal{P} is a twisted sheaf. We will discuss this later in this section.

Remark 5.2.4. Let $x \in M$ be a point lying over the curve $\pi(x) = C$. The image of a skyscraper sheaf $\mathbb{C}(x)$ is

$$\Phi(\mathbb{C}(x)) \cong \mathcal{P}_{M \times_{|H|} \{x\}}.$$

This is always a Cohen-Macaulay sheaf on the fiber $\bar{J}(C)$, but if x is in the regular locus of the Lagrangian fibration $\Phi(\mathbb{C}(x))$ a line bundle on the fiber.

Since the kernel \mathcal{P} lives in the fiber product, the equivalence Φ is a relative equivalence in the following sense.

Lemma 5.2.5. *Let $M \rightarrow |H|$ be the relative compactified Jacobian, and $\Phi : D^b(M) \simeq D^b(M)$ the autoequivalence induced by the Poincaré sheaf.*

(1) *For every object $E \in D^b(M)$ we have*

$$\Phi_{\mathcal{P}}(E)|_{M_t} = i_{M_t,*} \Phi_{\mathcal{P}_t}(E|_{M_t}), \text{ for a general } t \in (\mathbb{P}^2)^\vee.$$

(2) *If $Z \subset M$ is finite of degree d over the base $|H|$, and $L \in \text{Coh}(Z)$ is a sheaf of generic rank one, then for general $t \in (\mathbb{P}^2)^\vee$ the restriction $\Phi(\mathcal{M}_L)|_t$ is a sum of four non-trivial line bundles of degree 0 on M_t .*

Proof. First note that the equivalence Φ is $(\mathbb{P}^2)^\vee$ -linear, because the kernel \mathcal{P} is defined on the fiber product. In particular there is an isomorphism of functors

$$\Phi(-) \otimes \mathcal{O}_{M_t} = \Phi(- \otimes \mathcal{O}_{M_t}).$$

Projection formula gives the isomorphism of functors $- \otimes \mathcal{O}_{M_t} \cong i_{M_t,*} i_{M_t}^*(-)$. To prove (1) it remains to prove that

$$\Phi(i_{M_t,*}(-)) \cong i_{M_t,*} \Phi_{\mathcal{P}_t}(-),$$

which follows from the base change Theorem as explained in [47, Lemma 11.30]. Point (2) follows from (1) combined with Remark 5.2.4. \square

The most useful property of Φ we are after is its ability to turn supported objects into vector bundles. This can be seen as a globalization of point (2) in the previous Lemma.

Proposition 5.2.6. *Let $M = M_S(0, H, 1 - g)$ be a moduli space of torsion sheaves on a general polarized K3 surface (S, H) of genus g , and let $\pi : M \rightarrow \mathbb{P}^g$ be the Lagrangian fibration. Let $L \subset M$ be a subvariety such that $\pi|_L : L \rightarrow \mathbb{P}^g$ is finite. If V_L is a Cohen-Macaulay sheaf on L , then $\Phi(V_L)$ is a locally free sheaf.*

We first recall the key result from [102] that we need in the proof.

Theorem 5.2.7 ([102, Corollary to Theorem V.4]). *Let X be a smooth variety, and let F and G two Cohen-Macaulay sheaves on X . Assume that $G \otimes F$ is of finite length. Then*

$$\dim \text{Supp } F + \dim \text{Supp } G = \dim X,$$

is equivalent to

$$\mathcal{T}or_k^{\mathcal{O}_X}(F, G) = 0 \quad \forall k < 0.$$

Proof of Proposition 5.2.6. Since the Poincaré sheaf $\mathcal{P} \in \text{Coh}(M \times_{\mathbb{P}^g} M)$ is flat with respect to both projections by [7, Theorem A], $\pi_1^*(\mathcal{M}_L) \otimes \mathcal{P}$ is a sheaf on $M \times_{\mathbb{P}^g} M$. So, $\Phi(V_L)$ is a complex concentrated in non negative degrees. To show that it is a locally free sheaf, it suffices to prove

$$\text{Ext}_M^i(\Phi(V_L), \mathbb{C}(x)) = 0 \quad \text{for } i > 0,$$

for every $x \in M$, where $\mathbb{C}(x)$ denotes the skyscraper sheaf at x . From [7, Proposition 7.1] it follows that $\Phi^{-1}(\mathbb{C}(x)) = \mathcal{P}_{M \times \{x^\vee\}}[g]$, where x^\vee parameterizes the dual sheaf to x . So, we have

$$\text{Ext}_M^i(\Phi(V_L), \mathbb{C}(x)) = \text{Ext}_M^i(V_L, \mathcal{L}_t[g]),$$

where $t := \pi(x)$, and $\mathcal{L}_t := \mathcal{P}_{M \times \{x^\vee\}}$ is a torsion-free rank one sheaf supported on M_t . Since V_L is a Cohen-Macaulay sheaf of dimension g on M , the derived dual $\text{RHom}_M(V_L, \mathcal{O}_M)$ is just $\mathcal{E}xt^g(V_L, \mathcal{O}_M)[-g]$. Hence we have

$$\begin{aligned} \text{Ext}_M^i(V_L, \mathcal{L}_t[g]) &\simeq \mathbb{H}^i(M, \text{RHom}_M(V_L, \mathcal{O}_M) \otimes^L \mathcal{L}_t[g]) \\ &\simeq \mathbb{H}^i(M, \mathcal{E}xt^g(V_L, \mathcal{O}_M) \otimes^L \mathcal{L}_t). \end{aligned}$$

The sheaf

$$\mathcal{H}^i(\mathcal{E}xt^g(V_L, \mathcal{O}_M) \otimes^L \mathcal{L}_t) = \mathcal{T}or_k^{\mathcal{O}_M}(\mathcal{E}xt^g(V_L, \mathcal{O}_M), \mathcal{L}_t)$$

vanishes by Theorem 5.2.7. Indeed, by [7, Theorem A(2)] the sheaf \mathcal{L}_t is Cohen-Macaulay of dimension g on M , and the same holds for $\mathcal{E}xt^g(V_L, \mathcal{O}_M)$. So we have

$$\mathbb{H}^i(M, \mathcal{E}xt^g(V_L, \mathcal{O}_M) \otimes^L \mathcal{L}_t) = H^i(M, \mathcal{E}xt^g(V_L, \mathcal{O}_M) \otimes \mathcal{L}_t) = 0 \quad \text{for } i > 0,$$

because the sheaf $\mathcal{E}xt^g(V_L, \mathcal{O}_M) \otimes \mathcal{L}_t$ is supported on $M_t \cap L$ which is finite. \square

5.2.2 Action on extended Mukai lattice

Let M be the compactified Jacobian M_{1-g} . Our next goal is to compute the isometry induced by Φ on (the algebraic part of) the extended Mukai lattice

$$\Phi^{\tilde{H}} : \tilde{H}(M, \mathbb{Q}) \simeq \tilde{H}(M, \mathbb{Q}).$$

Recall that

$$\tilde{H}(M, \mathbb{Q})^{(1,1)} = \mathbb{Q}\alpha \oplus \mathbb{Q}f \oplus \mathbb{Q}\lambda \oplus \mathbb{Q}\beta,$$

with f and λ as in Lemma 5.2.1. We use the diagram

$$\begin{array}{ccc} \mathrm{SH}(X) & \xrightarrow{\varepsilon(\Phi^{\tilde{H}})\Phi^{\mathrm{SH}}} & \mathrm{SH}(X) \\ \downarrow \Psi & & \downarrow \Psi \\ \mathrm{Sym}^n \tilde{H}(X, \mathbb{Q}) & \xrightarrow{\mathrm{Sym}^n \Phi^{\tilde{H}}} & \mathrm{Sym}^n \tilde{H}(X, \mathbb{Q}) \end{array}$$

of Theorem 3.6.8.

Remark 5.2.8. Notice that if the genus is even, the Theorem applies because the second betti number is $b_2(M) = 23$, in particular is odd. Contrary to the case of odd genus, in this case, the diagram above does not uniquely determine $\Phi^{\tilde{H}}$, but it does so only up to a sign.

Following [17] we introduce the class

$$h := -\frac{\lambda}{2} + \frac{g-1}{4}f + \frac{g+1}{2}\beta. \quad (5.2)$$

The computation for odd genus first appeared in [17, Proposition 10.4].

Proposition 5.2.9. *The equivalence Φ acts on the extended Mukai as follows.*

(1) *If the genus is even*

$$\beta \mapsto f, \quad f \mapsto \beta, \quad \alpha \mapsto h, \quad h \mapsto \alpha.$$

(2) *If the genus is odd*

$$\beta \mapsto f, \quad f \mapsto -\beta, \quad \alpha \mapsto h, \quad h \mapsto -\alpha + (g+1)f.$$

Proof. The main ingredient we need on \mathcal{P} is the autoduality property (c.f. [7, Lemma 6.2 and Equation (7.8)]). Consider the fiberwise dual automorphism

$$\nu : M \simeq M, \quad i_*F \rightarrow i_*(F^\vee),$$

where $i : C \subset S$ is the embedding of a curve in the linear system $|H|$. The autoduality property takes the following form

$$\Phi^{-1}(-) \cong \nu^* \circ \Phi(-) \otimes \mathcal{O}_M((g+1)f)[g]. \tag{5.3}$$

Remark 5.2.4 implies that the skyscraper sheaf of a point gets mapped to a line bundle of degree zero on a fiber. At the level of Mukai vectors this means that

$$\beta \mapsto f,$$

using Proposition 4.5.12 and Corollary 4.6.9. The autoduality property then implies that

$$f \mapsto (-1)^g \beta.$$

Our choice of normalization Equation (5.1) gives that

$$\Phi(\tilde{v}(\mathcal{O}_{P'})) = A \left(\alpha + \frac{g+3}{4} \beta \right),$$

by Proposition 4.5.4, where A is some rational number. The extended Mukai vector of P' can be computed using Proposition 4.6.5,

$$\tilde{v}(\mathcal{O}_{P'}) = \lambda - (g+1)f + (g+1)\beta.$$

Since ν^* commutes with Φ , the autoduality property implies

$$\Phi(\mathcal{O}_M) = \mathcal{O}_{P'} \otimes \mathcal{O}(-(g+1)f)[-g].$$

So

$$\alpha + \frac{g+3}{4} \beta \mapsto B(\lambda - (g+1)f + (g+1)\beta).$$

Imposing $\Phi^{\tilde{H}}$ to be a Hodge isometry allows to determine the coefficients A and B , and rearranging gives the statement. \square

5.3 \mathbb{P} -twist

In this section we collect some purely algebraic computations, which will be very useful later. They are at the core of the semistable reduction in Section 5.7, and they should be seen as an higher dimensional analogue of the computations in Section 2.4.

The setting is the following. We take two coherent sheaves \mathcal{E} and \mathcal{F} on a hyper-Kähler fourfold X . We make the following assumptions

(1) \mathcal{E} is a \mathbb{P} -object, i.e. there is an isomorphism

$$\mathrm{Ext}^*(\mathcal{E}, \mathcal{E}) \cong H^*(\mathbb{P}^2, \mathbb{C})$$

of graded algebras.

(2) There is an isomorphism

$$\mathrm{Ext}^*(\mathcal{E}, \mathcal{F}) \cong \mathbb{C}[-1] \oplus \mathbb{C}[-3]$$

and it is non-trivial as a module over $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E})$.

In particular, there is a unique non-trivial extension

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{E} \rightarrow 0.$$

To a \mathbb{P} -object \mathcal{E} one can associate an autoequivalence $P_{\mathcal{E}}$ of $D^b(X)$ called the \mathbb{P} -twist around \mathcal{E} . Here we briefly recall the definition, for details see [55, Section 2]. If we let $h \in \mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$ be a generator, we can define the map

$$\bar{h}^{\vee} : \mathrm{Ext}^{*-2}(\mathcal{E}, \mathcal{F}) \rightarrow \mathrm{Ext}^*(\mathcal{E}, \mathcal{F})$$

as the precomposition with h . The \mathbb{P} -twist around \mathcal{E} applied to \mathcal{F} can be described as

$$P_{\mathcal{E}}(\mathcal{F}) = C \left(C(\mathrm{Ext}^{*-2}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \xrightarrow{\bar{h}^{\vee} \cdot \mathrm{id} - \mathrm{id} \cdot h} \mathrm{Ext}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E}) \rightarrow \mathcal{F} \right) \quad (5.4)$$

Remark 5.3.1. By the octahedral axiom one can see that $P_{\mathcal{E}}(\mathcal{F})$ can be equivalently described as the cone of the map

$$\mathrm{Ext}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E}[-1] \rightarrow C(\mathrm{Ext}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \rightarrow \mathcal{F}).$$

The goal of this section is to compute the cohomology sheaves of the complex $P_{\mathcal{E}}(\mathcal{F})$. We do it step by step, so we begin with the cone of the evaluation map.

Lemma 5.3.2. *Consider the evaluation map*

$$\mathrm{Ext}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \rightarrow \mathcal{F}.$$

The cohomology sheaves of its cone C are

$$\mathcal{H}^k(C) \cong \begin{cases} \mathcal{G} & \text{for } k = 0, \\ \mathcal{E} & \text{for } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The long exact sequence in cohomology gives the two sequences

$$\begin{aligned} 0 \rightarrow \mathcal{F} \rightarrow \mathcal{H}^0(C) \rightarrow \mathcal{E} \rightarrow 0, \\ 0 \rightarrow \mathcal{H}^2(C) \rightarrow \mathcal{E} \rightarrow 0. \end{aligned}$$

The rest of the long exact sequence shows that there is no cohomology in degrees different from 0 and 2. The first sequence is induced by the evaluation map $\text{Ext}^1(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \rightarrow \mathcal{F}$. Therefore it is not split, and $\mathcal{H}^0(C) \cong \mathcal{G}$. \square

Proposition 5.3.3. *The cohomology sheaves of $P_{\mathcal{E}}(\mathcal{F})$ are given by*

$$\mathcal{H}^k(P_{\mathcal{E}}(\mathcal{F})) \cong \begin{cases} \mathcal{G} & \text{for } k = 0, \\ \mathcal{E} & \text{for } k = 3. \end{cases}$$

In particular, there is a distinguished triangle

$$\mathcal{G} \rightarrow P_{\mathcal{E}}(\mathcal{F}) \rightarrow \mathcal{E}[-3].$$

Proof. Consider the distinguished triangle

$$\text{Ext}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E}[-1] \rightarrow C(\text{Ext}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \rightarrow \mathcal{F}) \rightarrow P_{\mathcal{E}}(\mathcal{F})$$

of Remark 5.3.1. Applying the long exact sequence of cohomology sheaves and Lemma 5.3.2 we get the exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{G} \rightarrow \mathcal{H}^0(P_{\mathcal{E}}(\mathcal{F})) \rightarrow 0, \\ 0 \rightarrow \mathcal{H}^1(P_{\mathcal{E}}(\mathcal{F})) \rightarrow \mathcal{E} \rightarrow \mathcal{E} \rightarrow \mathcal{H}^2(P_{\mathcal{E}}(\mathcal{F})) \rightarrow 0, \\ 0 \rightarrow \mathcal{H}^3(P_{\mathcal{E}}(\mathcal{F})) \rightarrow \mathcal{E} \rightarrow 0. \end{aligned}$$

If we check that the middle map $\mathcal{E} \rightarrow \mathcal{E}$ in the second sequence is the identity we are done. By definition it is induced in \mathcal{H}^2 from the map

$$\text{Ext}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E}[-1] \rightarrow C,$$

which in turn is obtained from the octahedral axiom, composed with the isomorphism in Lemma 5.3.2. Chasing the definitions and the commutativity in the octahedral axiom one sees that the wanted map is induced in \mathcal{H}^2 by the map

$$H[-1] : \text{Ext}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E}[-1] \rightarrow \text{Ext}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E}[1],$$

described explicitly as

$$\begin{array}{ccc} \mathcal{E}[-2] & \xrightarrow{h} & \mathcal{E} \\ \oplus & \searrow \text{id} & \oplus \\ \mathcal{E}[-4] & \xrightarrow{h[-2]} & \mathcal{E}[-2] \end{array}$$

in the proof of [55, Lemma 2.5]. From this description is clear that the induced map in \mathcal{H}^2 is the identity. \square

Corollary 5.3.4. *The object $P_{\mathcal{E}}^{-1}(\mathcal{G})$ sits in a distinguished triangle*

$$\mathcal{E} \rightarrow P_{\mathcal{E}}^{-1}(\mathcal{G}) \rightarrow \mathcal{F}.$$

Proof. From [55, Lemma 2.5] we see that $P_{\mathcal{E}}(\mathcal{E}) \cong \mathcal{E}[-4]$. Applying the equivalence $P_{\mathcal{E}}^{-1}$ to the distinguished triangle

$$\mathcal{G} \rightarrow P_{\mathcal{E}}(\mathcal{F}) \rightarrow \mathcal{E}[-3]$$

of Proposition 5.3.3 we obtain

$$P_{\mathcal{E}}^{-1}(\mathcal{G}) \rightarrow \mathcal{F} \rightarrow \mathcal{E}[1].$$

Rotating this triangle gives the thesis. \square

Corollary 5.3.5. *If \mathcal{E} and \mathcal{F} are as above, we have*

$$\mathrm{Ext}^k(\mathcal{E}, \mathcal{G}) \cong \begin{cases} 0 & \text{if } k \neq 4, \\ \mathbb{C} & \text{if } k = 4. \end{cases}$$

Proof. Setting $\mathcal{G}' := P_{\mathcal{E}}^{-1}(\mathcal{G})$ we get

$$\begin{aligned} \mathrm{Ext}^k(\mathcal{E}, \mathcal{G}) &= \mathrm{Ext}^k(\mathcal{E}, P_{\mathcal{E}}(\mathcal{G}')) = \mathrm{Ext}^k(\mathcal{E}[4], \mathcal{G}') \\ &= \mathrm{Ext}^{k-4}(\mathcal{E}, \mathcal{G}'). \end{aligned}$$

Both objects \mathcal{G} and \mathcal{G}' are sheaves, so the ext groups above vanish for $k \neq 4$. For $k = 4$, the exact sequence

$$\mathrm{Ext}^4(\mathcal{E}, \mathcal{E}) \rightarrow \mathrm{Ext}^4(\mathcal{E}, \mathcal{G}) \rightarrow \mathrm{Ext}^4(\mathcal{E}, \mathcal{F}) = 0,$$

shows that it is at most one dimensional. It is non-zero, because of the map $\mathcal{G} \rightarrow \mathcal{E}$, so it is one dimensional. \square

5.4 Normal crossing Lagrangians

In this section we study the Ext groups of a normal crossing Lagrangian in a HK manifold X of dimension $2n$. By this we mean a subvariety $Z \subset X$ of the form

$$Z = Z_1 \cup Z_2$$

where Z_1 and Z_2 are smooth Lagrangians, and their scheme theoretic intersection $W := Z_1 \cap Z_2$ is smooth of dimension $n - 1$. In particular

$$T_{Z_1}|_W \cap T_{Z_2}|_W = T_W.$$

Lemma 5.4.1 (Markman). *The normal bundle N_{W/Z_1} is dual to N_{W/Z_2} .*

Proof. We begin with an observation. Since $T_W \subset T_{Z_i}|_W$ and each Z_i is Lagrangian, we have

$$\sigma_X(v, w) = 0 \text{ for every } v \in T_{Z_i} \text{ and } w \in T_W,$$

where σ is the symplectic form on X . The sum $T_{Z_1}|_W + T_{Z_2}|_W$ is a subbundle of $T_X|_W$ of rank $n + 1$, so it must be equal to the symplectic orthogonal T_W^\perp to T_W . Consider the following diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & T_W & \xrightarrow{\cong} & T_{Z_1}|_W \cap T_{Z_2}|_W & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & T_W \oplus T_W & \longrightarrow & T_{Z_1}|_W \oplus T_{Z_2}|_W & \longrightarrow & N_{W/Z_1} \oplus N_{W/Z_2} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \cong \\
 0 & \longrightarrow & T_W & \longrightarrow & T_{Z_1}|_W + T_{Z_2}|_W & \longrightarrow & (T_{Z_1}|_W + T_{Z_2}|_W)/T_W \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

The nine lemma implies that the right vertical map is an isomorphism. The previous observation gives that

$$(T_{Z_1}|_W + T_{Z_2}|_W)/T_W \cong (T_W)^\perp/T_W,$$

which is a symplectic rank two bundle, in particular it has trivial determinant. We conclude that

$$N_{W/Z_1} \otimes N_{W/Z_2} \cong \bigwedge^2 (N_{W/Z_1} \oplus N_{W/Z_2}) \cong \mathcal{O}_X.$$

□

Lemma and Definition 5.4.2. *Define the vector bundle¹*

$$\tilde{N} := T_X|_W / (T_{Z_1}|_W + T_{Z_2}|_W).$$

There is an isomorphism of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & N_{W/Z_1} & \longrightarrow & N_{Z_2/X}|_W & \longrightarrow & \tilde{N} \longrightarrow 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 0 & \longrightarrow & N_{W/Z_2}^\vee & \longrightarrow & \Omega_{Z_2}|_W & \longrightarrow & \Omega_W \longrightarrow 0
 \end{array}$$

¹This notation is from [26].

Proof. To get the short exact sequence above, start with

$$0 \rightarrow (T_{Z_1|W} + T_{Z_2|W})/T_{Z_2|W} \rightarrow T_X|W/(T_{Z_2|W}) \rightarrow T_X|W/(T_{Z_1|W} + T_{Z_2|W}) \rightarrow 0,$$

and note that

$$(T_{Z_1|W} + T_{Z_2|W})/T_{Z_2|W} \cong T_{Z_1|W}/T_W.$$

The central vertical map in the diagram is the usual map induced by the restriction of the isomorphism $\sigma_X : T_X \cong \Omega_X$. The composition

$$T_{Z_1} + T_{Z_2} \rightarrow T_X \cong \Omega_X \rightarrow \Omega_W$$

vanishes because $T_{Z_1} + T_{Z_2}$ is symplectic orthogonal to T_W . Therefore the central map factors to give the diagram in the statement. \square

We name the relevant embeddings $W \subset Z_i \subset X$ as in the diagram below

$$\begin{array}{ccc} W & \xrightarrow{i_1} & Z_1 \\ \downarrow i_2 & & \downarrow j_1 \\ Z_2 & \xrightarrow{j_2} & X \end{array}$$

If E_1 and E_2 are locally free sheaves on Z_1 and Z_2 we can compute the Ext groups $\text{Ext}_X^k(j_{1,*}E_1, j_{2,*}E_2)$ using the following spectral sequence.

Theorem 5.4.3 ([26, Theorem A.1]). *With the above notation, there is a convergent spectral sequence*

$$E_2^{p,q} := H^p(W, E_1^\vee|_W \otimes E_2|_W \otimes N_{W/Z_2} \otimes \bigwedge^{q-1} \tilde{N}) \implies \text{Ext}^{p+q}(j_{1,*}E_1, j_{2,*}E_2).$$

Example 5.4.4. Assume that X has dimension 4, and let $E = \mathcal{O}_{Z_1}$ and $F = \mathcal{O}_{Z_2}(-W)$. By Lemma and Definition 5.4.2 the second page of the spectral sequence above takes the form

$$E_2^{p,q} = H^p(N_{W/Z_2}^\vee \otimes N_{W/Z_2} \otimes \Omega_W^{q-1}) = H^p(\Omega_W^{q-1}).$$

It degenerates at the second page, by degree reasons, giving the isomorphism

$$\text{Ext}^k(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}(-W)) \cong H^{k-1}(W, \mathbb{C}).$$

Proposition 5.4.5. *Assume that X has dimension four. Then, there is a long exact sequence*

$$H^k(Z_2, \mathbb{C}) \rightarrow \text{Ext}^k(\mathcal{O}_{Z_2}(-W), \mathcal{O}_Z) \rightarrow H^{k-1}(W, \mathbb{C}) \rightarrow H^{k+1}(Z_2, \mathbb{C}),$$

where the connecting homomorphism is the pushforward in cohomology along the inclusion $W \subset Z_2$.

Proof. Consider the long exact sequence obtained applying $\mathrm{Hom}(\mathcal{O}_{Z_2}(-W), -)$ to

$$0 \rightarrow \mathcal{O}_{Z_2}(-W) \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{Z_1} \rightarrow 0.$$

Since $Z_2 \subset X$ is a Lagrangian surface, by dimensional reasons the local-to-global spectral sequence vanishes and yields

$$\mathrm{Ext}^k(\mathcal{O}_{Z_2}(-W), \mathcal{O}_{Z_2}(-W)) \cong H^k(Z_2, \mathbb{C}). \quad (5.5)$$

Example 5.4.4 implies that

$$\mathrm{Ext}^k(\mathcal{O}_{Z_2}(-W), \mathcal{O}_{Z_1}) \cong H^{k-1}(W, \mathbb{C}). \quad (5.6)$$

Therefore, we only need to show that the connecting homomorphisms

$$\mathrm{Ext}^k(\mathcal{O}_{Z_2}(-W), \mathcal{O}_{Z_1}) \rightarrow \mathrm{Ext}^{k+1}(\mathcal{O}_{Z_2}(-W), \mathcal{O}_{Z_2}(-W))$$

become identified with the pushforwards in cohomology. The Serre dual statement is that the connecting map

$$\mathrm{Ext}^k(\mathcal{O}_{Z_2}(-W), \mathcal{O}_{Z_2}(-W)) \rightarrow \mathrm{Ext}^{k+1}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}(-W))$$

is the restriction $H^k(Z_2, \mathbb{C}) \rightarrow H^k(W, \mathbb{C})$. The isomorphisms (5.5) and (5.6) are induced by the degeneration of the spectral sequences:

$$H^p \mathrm{RHom}(\mathcal{H}^{-q}(j_2^* j_{2,*} \mathcal{O}_{Z_2}(-W)), \mathcal{O}_{Z_2}(-W)) \implies \mathrm{Ext}^{p+q}(\mathcal{O}_{Z_2}(-W), \mathcal{O}_{Z_2}(-W))$$

and

$$H^p \mathrm{RHom}(\mathcal{H}^{-q}(j_2^* j_{1,*} \mathcal{O}_{Z_1}), \mathcal{O}_{Z_2}(-W)) \implies \mathrm{Ext}^{p+q+1}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}(-W)).$$

The connecting homomorphism is induced by pullback along the map

$$\mathcal{O}_{Z_1} \rightarrow \mathcal{O}_{Z_2}(-W)[1].$$

Taking j_2^* and \mathcal{H}^{-q} we get the zero map in cohomology for every q . This implies that the long exact cohomology sequence induced by $j_2^* \mathcal{O}_{Z_1} \rightarrow j_2^* \mathcal{O}_{Z_2}(-W)[1]$ is actually a collection of short exact sequences, represented by maps

$$\mathcal{H}^{-q}(j_2^* j_{1,*} \mathcal{O}_{Z_1}) \rightarrow \mathcal{H}^{-q}(j_2^* j_{2,*} \mathcal{O}_{Z_2}(-W))[1]. \quad (5.7)$$

Pulling back along those maps gives a map on the E_2 page of the spectral sequences, which induces the connecting homomorphism that we wish to understand.

Using [26, Proposition A.6] we obtain $\mathcal{H}^{-q}(j_2^* j_{1,*} \mathcal{O}_{Z_1}) \cong i_{2,*} \bigwedge^q \tilde{N}^\vee$. So the map (5.7) becomes

$$i_{2,*} \bigwedge^q \tilde{N}^\vee \rightarrow \bigwedge^q N_{Z_2/X}^\vee \otimes \mathcal{O}_{Z_2}(-W)[1].$$

Verdier duality gives that $i_2^! = i_2^* \otimes \mathcal{O}_W(W)[-1]$, so the map becomes $\bigwedge^q \tilde{N}^\vee \rightarrow \bigwedge^q N_{Z_2/X}^\vee$, which is identified with the restriction map via Lemma and Definition 5.4.2. \square

5.5 Degenerating the cubic

Let $X_0 \subset \mathbb{P}^5$ be the determinantal cubic, that is the secant variety to the Veronese surface $V \subset \mathbb{P}^5$. It is given in coordinates by

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix} = 0.$$

It is singular along the Veronese surface. If \mathbb{P}^5 is identified with the space of conics on a projective plane, X_0 corresponds to the singular conics and V to the non-reduced ones.

Let $X \subset \mathbb{P}^5$ be a general cubic and let $\mathcal{X} \rightarrow \Delta$ be the pencil spanned by X_0 and X . If $X = \{f = 0\}$, the equation of the pencil is

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix} + tf = 0.$$

Taking the relative Fano variety of lines, we get a family $\mathcal{F} \rightarrow \Delta$ whose general fiber \mathcal{F}_t is the Fano variety of lines $F(\mathcal{X}_t)$ of a general member of the pencil. The central fiber $\mathcal{F}_0 = F(X_0)$ is described in [33, Propositions 3.2.3 and 3.2.4]: it is the union of $F_1 \cong (\mathbb{P}^2)^{[2]}$ and $F_2 \cong \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$, where F_1 is non-reduced with multiplicity four.

Proposition 5.5.1 ([33, Theorem 3.3.7]). *After a base change along a $2 : 1$ map $\Delta' \rightarrow \Delta$ and blowing up \mathcal{F} in F_1 , we get a family $\widehat{\mathcal{F}} \rightarrow \Delta'$ such that*

- (1) *The special fiber has two irreducible components*

$$\widehat{\mathcal{F}}_0 = E \cup \widehat{F}_2.$$

- (2) *The map $\widehat{F}_2 \rightarrow F_2$ is an isomorphism, in particular $\widehat{F}_2 \cong \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$.*
 (3) *The intersection $E \cap \widehat{F}_2 \subset \widehat{F}_2$ is isomorphic to the incidence variety in $\mathbb{P}^2 \times (\mathbb{P}^2)^\vee$.*
 (4) *The blow up $\widehat{\mathcal{F}}$ is smooth along \widehat{F}_2 .*

We describe the family $\widehat{\mathcal{F}}$ in more detail. Since the Veronese surface V has degree two, the intersection $V \cap X$ gives a smooth sextic curve $\Gamma \in \mathbb{P}^2$. Let $p : S \rightarrow \mathbb{P}^2$ be the K3 surface obtained as the double cover of \mathbb{P}^2 ramified over Γ . Let $P \subset S^{[2]}$ be the image of the map

$$\mathbb{P}^2 \rightarrow S^{[2]}, \quad x \mapsto p^{-1}(x),$$

where $p^{-1}(x)$ denotes the schematic fiber. Rephrasing [33, Theorems 3.5.8 and 3.5.11] gives the following result.

Theorem 5.5.2. *There is a smooth family $\overline{\mathcal{F}} \rightarrow \Delta'$ such that the general fiber $\overline{\mathcal{F}}_t = F(\mathcal{X}_t)$ is the Fano variety of lines of the cubic \mathcal{X}_t and the special fiber $\overline{\mathcal{F}}_0$ is isomorphic to $S^{[2]}$. The family $\widehat{\mathcal{F}}$ is the blow-up of $\overline{\mathcal{F}}$ in P . Under this identification \widehat{F}_2 is the exceptional divisor, and $E \cong \text{Bl}_P(S^{[2]})$.*

Consider the moduli space $M := M(0, H, -1)$, where $H := p^*(\mathcal{O}(1))$. As recalled in Section 5.2 this is the relative compactified Jacobian of the universal curve over the linear system $|H|$. It has a Lagrangian fibration $\pi : M \rightarrow |H|$, given by sending a sheaf to its support.

There is a birational map

$$g : S^{[2]} \dashrightarrow M, \quad \xi \mapsto \omega_C \otimes I_\xi, \tag{5.8}$$

where C is the unique curve in $|H|$ containing ξ . This is well defined outside the plane $P \subset S^{[2]}$. The birational map g is the Mukai flop of the plane P , and the dual plane $P' \subset M$ is the image of the section of the Lagrangian fibration.

Remark 5.5.3. Since the cubic X is general, the plane $P \subset S^{[2]}$ does not deform sideways in $\overline{\mathcal{F}} \rightarrow \Delta'$. The argument in the proof of [45, Theorem 3.4] shows that the Mukai flop (5.8) can be deformed to $\overline{\mathcal{F}}$. This implies that $\widehat{\mathcal{F}}$ can also be contracted to a family $\overline{\mathcal{F}}' \rightarrow \Delta'$ with the same general fiber and special fiber $\overline{\mathcal{F}}'_0 \cong M$.

In [31] Collino does the same operations with the Fano variety of lines of a hyperplane section. More precisely, let $H \subset \mathbb{P}^5$ be a general hyperplane. The intersection $V \cap H$ gives a general conic $K \subset \mathbb{P}^2$, and the intersection $X_0 \cap H$ is the secant variety of the image of K via the Veronese embedding. Define $C := p^{-1}(K) \subset S$ as the inverse image of the conic via the double cover, it is a genus five curve.

Let $\mathcal{X}_H \rightarrow \Delta$ be the pencil of the hyperplane sections and let $\mathcal{Z} \subset \mathcal{F}$ be the relative Fano surface of lines. The special fiber is the union of two components $\mathcal{Z}_0 = Z_1 \cup Z_2 \subset F_1 \cup F_2$, where both Z_1 and Z_2 are isomorphic to \mathbb{P}^2 , and Z_1 is non reduced of multiplicity four.

Proposition 5.5.4 ([31, Proposition 2.1]). *After a base change along a $2 : 1$ map $\Delta' \rightarrow \Delta$ and blowing up \mathcal{Z} in Z_1 , we get a smooth family $\widehat{\mathcal{Z}} \rightarrow \Delta'$ with reducible central fiber*

$$\widehat{\mathcal{Z}}_0 = E' \cup \widehat{Z}_2.$$

Moreover the exceptional divisor E' is isomorphic to $\text{Sym}^2 C$ and \widehat{Z}_2 is isomorphic to Z_2 .

We want to understand the image of $\widehat{\mathcal{Z}}$ via the contraction $\widehat{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$ of Remark 5.5.3. First observe that the intersection $Z_1 \cap Z_2$ consists of the lines tangent to K , so it is isomorphic to K^* . Via the embedding $Z_2 \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$ it gets mapped into the incidence variety inside $K \times K^*$. In particular it maps isomorphically to its image under both projections.

Via the contraction $\widehat{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$, the component \widehat{F}_2 in the central fiber $\widehat{\mathcal{F}}_0$ gets mapped to $(\mathbb{P}^2)^\vee$, so it induces a map $\widehat{Z}_2 \rightarrow (\mathbb{P}^2)^\vee$. This map must be an isomorphism. This is because

$$\widehat{Z}_2 \cong Z_2 \cong \mathbb{P}^2,$$

and the contraction maps the intersection $\widehat{Z}_2 \cap E'$ isomorphically to its image K^* . Hence, the special fiber of $\widehat{\mathcal{Z}}$ remains unchanged under the contraction $\widehat{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$. The same argument also works for the contraction $\widehat{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$. Summarizing the argument, and adjusting the notation, we showed the following.

Theorem 5.5.5. *There is a smooth family $\mathcal{F} \rightarrow \Delta$ and a smooth subvariety $\mathcal{Z} \subset \mathcal{F}$ with the following properties.*

- *The general fibers \mathcal{F}_t and \mathcal{Z}_t are respectively the Fano varieties of lines $F(\mathcal{X}_t)$ of the cubic \mathcal{X}_t , and of its hyperplane section $F(\mathcal{X}_t \cap H)$.*
- *The special fiber \mathcal{F}_0 is identified with the moduli space $M = M(0, H, -1)$.*
- *The special fiber \mathcal{Z}_0 is a normal crossing $P' \cup L$, where $L \subset M$ is a Lagrangian surface isomorphic to $\text{Sym}^2 C$. The intersection $L \cap P'$ is isomorphic to K .*

We conclude the section with a more detailed description of the geometry of the central fiber. By Lemma 5.2.1 the Neron-Severi lattice of the moduli space M is

$$\text{NS}(M) = \mathbb{Z}\lambda \oplus \mathbb{Z}f,$$

the Beauville-Bogomolov form with respect to this basis has matrix

$$\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}.$$

From the point of view of the Hilbert scheme, we have

$$\text{NS}(S^{[2]}) = \mathbb{Z}h \oplus \mathbb{Z}\delta,$$

where h is the polarization induced by $H = p^*(\mathcal{O}_{\mathbb{P}^2}(1))$ on S , and δ is half the exceptional divisor of the Hilbert-Chow map. The Mukai flop identifies the divisors

$$\begin{aligned} h &\longleftrightarrow \lambda, \\ h - \delta &\longleftrightarrow f. \end{aligned}$$

Remark 5.5.6. As explained in [33, Section 3.7] the family $\mathcal{F} \rightarrow \Delta$ of Theorem 5.5.5 is a projective family, and comes equipped with an ample line bundle \mathcal{L} . On the general fiber this line bundle is the Plücker polarization, and on the special fiber $\mathcal{F}_0 = M$ is $\mathcal{O}_M(\lambda + f)$. It has square 6 and divisibility 2 on every fiber.

Proposition 5.5.7. *The Lagrangian fibration π is finite of degree 4 when restricted to L .*

Proof. The map $\pi|_L : L \rightarrow (\mathbb{P}^2)^\vee$ is proper, so it suffices to show that it is quasi-finite. The intersection $P' \cap L$ is one dimensional and it maps bijectively onto the dual conic K^* via π . On the complement of P' the Mukai flop is an isomorphism, so it suffices show that $\text{Sym}^2 C - K \rightarrow (\mathbb{P}^2)^\vee$ is quasi-finite.

The fiber of a line $l \in (\mathbb{P}^2)^\vee$ consists of the subschemes $\xi \in S^{[2]}$ mapping to the schematic intersection $l \cap K$. The number of such subschemes is always finite. If l intersects K transversely outside the ramification locus Γ , there are four reduced subschemes mapping to the intersection. \square

5.6 Construction of the sheaf

Let $Z \subset M$ be the central fiber of the family of Lagrangian $\mathcal{Z} \subset \mathcal{F}$ of Theorem 5.5.5. So that Z is the reducible Lagrangian $P' \cup L$, and the intersection $P' \cap L$ is the dual conic K^* .

We make the following construction. Start with \mathcal{L} a line bundle of degree zero on L . Since K^* is rational, the restriction $\mathcal{L}|_{K^*}$ is trivial. Hence \mathcal{L} glues with the structure sheaf of P' and gives a global line bundle $\overline{\mathcal{L}}$ on Z . This means that we have a short exact sequence

$$0 \rightarrow \mathcal{L}(-K^*) \rightarrow \overline{\mathcal{L}} \rightarrow \mathcal{O}_{P'} \rightarrow 0 \tag{5.9}$$

of sheaves on X .

Proposition 5.6.1. *Let $\Phi : D^b(M) \simeq D^b(M)$ be the autoequivalence induced by the Poincaré sheaf. For every $\mathcal{L} \in \text{Pic}^0(L)$, the object*

$$F_{\mathcal{L}} := \Phi(\overline{\mathcal{L}}) \in D^b(X)$$

is an atomic vector bundle of rank five. Its normalized extended Mukai vector is

$$\tilde{v}(F) = 5\alpha + 15f - \frac{15}{4}\beta.$$

Proof. Since \mathcal{L} has degree zero, the Mukai vector of $\overline{\mathcal{L}}$ is the same as $v(\mathcal{O}_Z)$, which is atomic by Proposition 4.6.10. Moreover Theorem 5.5.5 exhibits the Lagrangian

Z as the limit of the surface of lines. Hence, by Proposition 4.6.10 and Remark 5.5.6 we have

$$\tilde{v}(\mathcal{O}_Z) = \lambda + f - 3\beta.$$

The computation of the extended Mukai vector then follows from Proposition 5.2.9.

By our normalization Equation (5.1) of \mathcal{P} , the object $F_{\mathcal{L}}$ fits in a distinguished triangle

$$0 \rightarrow \Phi(\mathcal{L}(-K^*)) \rightarrow F_{\mathcal{L}} \rightarrow \mathcal{O}_M \rightarrow 0. \quad (5.10)$$

Proposition 5.2.6 implies that $\Phi(\mathcal{L}(-K^*))$ is a locally free sheaf. The rank can be computed after restricting to a general fiber, and we see that it is equal to the degree of $\text{Sym}^2 C$ over \mathbb{P}^2 . By Proposition 5.5.7 this degree is four, hence by Equation (5.10) the rank of $F_{\mathcal{L}}$ is five. \square

Remark 5.6.2. We can kill the first Chern class by twisting by $\mathcal{O}(-3f)$, so that the Mukai vector takes a simpler form. The extended Mukai vector of the twisted sheaf $F_0 := F \otimes \mathcal{O}(-3f)$ is

$$\tilde{v}(F_0) = 5 \left(\alpha - \frac{3}{4}\beta \right).$$

So that the Mukai vector is

$$v(F_0) = 5 \left(1 - \frac{3}{4}\mathbf{q}_2 + \frac{9}{32}\mathbf{pt} \right),$$

by Corollary 4.3.6.

5.6.1 Ext groups

Since we are interested in moduli spaces, it is vital to compute the Ext groups of $F_{\mathcal{L}}$. We do this by applying the results of Section 5.4 to the reducible Lagrangian $Z = P' \cup L$.

Lemma 5.6.3. *With the above notation we have isomorphisms*

$$\begin{aligned} \text{Ext}^1(\overline{\mathcal{L}}, \overline{\mathcal{L}}) &\cong H^1(L, \mathbb{C}), \\ \text{Ext}^2(\overline{\mathcal{L}}, \overline{\mathcal{L}}) &\cong \text{Cok}(H^0(K^*, \mathbb{C}) \rightarrow H^2(L, \mathbb{C})). \end{aligned}$$

Proof. The isomorphisms (5.5) and (5.6) remain valid also for a line bundle of degree zero. for the same reasons, thanks to (5.9). The rest of the proof of Proposition 5.4.5 is not affected by the twist, so it remains to show that

$$\text{Ext}^i(\overline{\mathcal{L}}, \overline{\mathcal{L}}) \cong \text{Ext}^i(\mathcal{L}(-K^*), \overline{\mathcal{L}}) \text{ for } i = 1, 2. \quad (5.11)$$

The sheaves $\mathcal{E} = \mathcal{O}_{P^1}$ and $\mathcal{F} = \mathcal{L}(-K^*)$ satisfy the assumptions of Section 5.3 by Theorem 5.4.3. Thus (5.11) is a consequence of the vanishings in Corollary 5.3.5. \square

We can also understand the algebra structure.

Lemma 5.6.4. *There is a commutative diagram*

$$\begin{array}{ccc} H^1(L, \mathbb{C}) \times H^1(L, \mathbb{C}) & \longrightarrow & H^2(L, \mathbb{C}) \\ \downarrow \simeq & & \downarrow \\ \text{Ext}^1(\overline{\mathcal{L}}, \overline{\mathcal{L}}) \times \text{Ext}^1(\overline{\mathcal{L}}, \overline{\mathcal{L}}) & \longrightarrow & \text{Ext}^2(\overline{\mathcal{L}}, \overline{\mathcal{L}}) \end{array}$$

where the vertical maps are the those of the lemma above, and the horizontal maps are respectively the Yoneda pairing and the cup product.

Proof. The sequence (5.9) gives an injective morphism

$$f : \mathcal{L}(-K^*) \rightarrow \overline{\mathcal{L}},$$

which gives rise to

$$\begin{array}{ccc} & \text{Ext}^i(\mathcal{L}(-K^*), \mathcal{L}(-K^*)) & \\ & \downarrow f_* & \\ \text{Ext}^i(\overline{\mathcal{L}}, \overline{\mathcal{L}}) & \xrightarrow[\simeq]{f^*} & \text{Ext}^i(\mathcal{L}(-K^*), \overline{\mathcal{L}}) \end{array}$$

where the horizontal map f^* is an isomorphism for $i = 1, 2$ by Corollary 5.3.5. Unpacking the proof of the lemma above, we see that the vertical maps are the composition

$$H^i(L, \mathbb{C}) \simeq \text{Ext}^i(\mathcal{L}(-K^*), \mathcal{L}(-K^*)) \xrightarrow{f_*} \text{Ext}^i(\mathcal{L}(-K^*), \overline{\mathcal{L}}) \xrightarrow{(f^*)^{-1}} \text{Ext}^i(\overline{\mathcal{L}}, \overline{\mathcal{L}}).$$

The first isomorphism is induced by the degeneration of the local-to-global spectral sequence, which in dimension four is due to degree reasons, and it is the i -th graded piece of an isomorphism of graded algebras, see Theorem 4.6.7. Therefore, to prove the statement we must check that

$$(f^*)^{-1}(f_*(a \circ b)) = (f^*)^{-1}(f_*a) \circ (f^*)^{-1}(f_*b).$$

for every $a, b \in \text{Ext}^1(\mathcal{L}(-K^*), \mathcal{L}(-K^*))$. If we write a, b as maps $\mathcal{L}(-K^*) \rightarrow \mathcal{L}(-K^*)[1]$, there is a commutative diagram

$$\begin{array}{ccccc}
 \mathcal{L}(-K^*) & \xrightarrow{a} & \mathcal{L}(-K^*)[1] & \xrightarrow{b[1]} & \mathcal{L}(-K^*)[2] \\
 \downarrow f & & \downarrow f[1] & & \downarrow f[2] \\
 \overline{\mathcal{L}} & \xrightarrow{(f^*)^{-1}(f_*a)} & \overline{\mathcal{L}}[1] & \xrightarrow{(f^*)^{-1}(f_*b)} & \overline{\mathcal{L}}[2] \\
 & \searrow & & \nearrow & \\
 & & & & (f^*)^{-1}(f_*(a \circ b))
 \end{array}$$

since the Yoneda pairing is precisely the composition in the derived category, we conclude. \square

Corollary 5.6.5. *The Yoneda pairing is skew-symmetric and induces an isomorphism*

$$\bigwedge^2 \text{Ext}^1(\overline{\mathcal{L}}, \overline{\mathcal{L}}) \xrightarrow{\sim} \text{Ext}^2(\overline{\mathcal{L}}, \overline{\mathcal{L}}), \quad a \wedge b \rightarrow a \circ b.$$

Proof. This relies on the fact that the Lagrangian L is in fact the symmetric square of a genus five curve C . In fact we have

$$H^2(L, \mathbb{C}) \cong H^2(C, \mathbb{C}) \oplus \bigwedge^2 H^1(L, \mathbb{C}),$$

where the second summand is embedded via cup product. The fundamental class of $K^* \subset \text{Sym}^2 C$ spans the direct summand $H^2(C, \mathbb{C})$. Thus, taking the cokernel as in Lemma 5.6.3 we get

$$\text{Ext}^2(\overline{\mathcal{L}}, \overline{\mathcal{L}}) \cong \bigwedge^2 H^1(L, \mathbb{C}). \tag{5.12}$$

We conclude by Lemma 5.6.4. \square

5.7 Semistable reduction

In Section 5.6 we showed that the object

$$F_{\mathcal{L}} := \Phi(\overline{\mathcal{L}}) \in D^b(M)$$

is a vector bundle of rank five for any degree zero line bundle $\mathcal{L} \in \text{Pic}^0(L)$. In fact this is true for any CM sheaf on L , but we do not need this now. In this section we investigate its slope stability.

The object $G_{\mathcal{L}} := \Phi(\mathcal{L}(-K^*))$ is itself a vector bundle, as observed in the proof of Proposition 5.6.1, and it sits in the short exact sequence (5.10)

$$0 \rightarrow G_{\mathcal{L}} \rightarrow F_{\mathcal{L}} \rightarrow \mathcal{O}_M \rightarrow 0.$$

The intersection $P' \cap L$ is isomorphic to the dual conic K^* , so Example 5.4.4 implies that the sheaves $\mathcal{E} = \mathcal{O}_{P'}$ and $\mathcal{F} =_{\mathcal{L}}(-K^*)$ are as in the setting of Section 5.3. By Corollary 5.3.4 the inverse \mathbb{P} -twist around $\mathcal{O}_{P'}$ lives in a short exact sequence

$$0 \rightarrow \mathcal{O}_{P'} \rightarrow P_{\mathcal{O}_{P'}}^{-1}(\overline{\mathcal{L}}) \rightarrow \mathcal{L}(-K^*) \rightarrow 0,$$

and we set

$$F'_{\mathcal{L}} := \Phi \left(P_{\mathcal{O}_{P'}}^{-1}(\overline{\mathcal{L}}) \right) \in D^b(M). \quad (5.13)$$

By construction of the \mathbb{P} -twist we have $F'_{\mathcal{L}} \cong P_{\mathcal{O}_M}^{-1}(F)$. Applying the equivalence Φ we get a short exact sequence

$$0 \rightarrow \mathcal{O}_M \rightarrow F'_{\mathcal{L}} \rightarrow G_{\mathcal{L}} \rightarrow 0.$$

In particular, for any line bundle \mathcal{L} , the sheaf $F'_{\mathcal{L}}$ is locally free of rank five.

Lemma 5.7.1. *Let $\mathcal{L} \in \text{Pic}^0(L)$ be a line bundle. The vector bundles $F_{\mathcal{L}}$ and $F'_{\mathcal{L}}$ are unstable for every polarization h on M .*

Proof. By [55, Remark 2.4] any \mathbb{P} -twist acts like the identity in cohomology, so F and F' have the same extended Mukai vector, which we computed in Proposition 5.6.1. In particular

$$\text{rk}(F_{\mathcal{L}}) = \text{rk}(F'_{\mathcal{L}}) = 5 \text{ and } c_1(F_{\mathcal{L}}) = c_1(F'_{\mathcal{L}}) = 15f.$$

Therefore, the slope with respect to any polarization h is

$$\mu(F_{\mathcal{L}}) = \mu(F'_{\mathcal{L}}) = 3q(h, f) > 0.$$

So the sequence (5.10) destabilizes $F_{\mathcal{L}}$.

To destabilize $F'_{\mathcal{L}}$, first recall that the normal bundles of K^* in P' and L are dual to each other

$$\mathcal{O}_L(K^*)|_{K^*} \cong \mathcal{O}_{P'}(2)|_{K^*}^{\vee},$$

as we proved in Lemma 5.4.1. Since the restriction $\mathcal{L}|_{K^*}$ is trivial, because K^* is rational, we

$$\text{Hom}(\mathcal{L}(-K^*), \mathcal{O}_{P'}(2)|_{K^*}) = H^0(K^*, \mathcal{O}_{K^*}) = \mathbb{C}.$$

The unique map

$$\mathcal{L}(-K^*) \twoheadrightarrow \mathcal{O}_{P'}(2)|_{K^*},$$

must be a twisting of the canonical map associated to the embedding $K^* \subset L$, in particular is surjective. Since $\text{Ext}^1(P_{\mathcal{O}_{P'}}^{-1}(\mathcal{O}_Z), \mathcal{O}_{P'}) = 0$, we can lift the composite map

$$P_{\mathcal{O}_{P'}}^{-1}(\mathcal{O}_Z) \rightarrow \mathcal{O}_L(-K^*) \rightarrow \mathcal{O}_{P'}(2)|_{K^*},$$

to a diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{O}_{P'} & \longrightarrow & P_{\mathcal{O}_{P'}}^{-1}(\overline{\mathcal{L}}) & \longrightarrow & \mathcal{L}(-K^*) \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{O}_{P'} & \longrightarrow & \mathcal{O}_{P'}(2) & \longrightarrow & \mathcal{O}_{P'}(2)|_{K^*} \longrightarrow 0
 \end{array} \tag{5.14}$$

where the short exact sequence below is the defining sequence of the inclusion $K^* \subset P'$. When applying Φ , the vertical central map becomes a non-zero morphism $F'_{\mathcal{L}} \rightarrow \mathcal{O}_M(2f)$. The inequality

$$\mu(\mathcal{O}_M(2f)) = 2q(h, f) < 3q(h, f) = \mu(F'_{\mathcal{L}})$$

show that $F'_{\mathcal{L}}$ is destabilized by this map. \square

To obtain a stable bundle $F''_{\mathcal{L}}$, we replicate the construction of $F'_{\mathcal{L}}$ with an additional \mathbb{P} -twist around the line bundle $\mathcal{O}_M(2f)$. Namely, define

$$F''_{\mathcal{L}} := P_{\mathcal{O}_M(2f)}^{-1}(F'_{\mathcal{L}}),$$

and notice that

$$F''_{\mathcal{L}} \simeq \Phi \left(P_{\mathcal{O}_{P'}(2)}^{-1}(P^{-1}(\overline{\mathcal{L}})) \right)$$

by construction. A diagram chase in (5.14) shows that

$$\text{Ker} \left(P_{\mathcal{O}_{P'}}^{-1}(\overline{\mathcal{L}}) \rightarrow \mathcal{O}_{P'}(2) \right) = \mathcal{L}(-2K^*).$$

So, defining $G'_{\mathcal{L}} := \Phi(\mathcal{O}_L(-2K^*))$ we have a short exact sequence

$$0 \rightarrow G'_{\mathcal{L}} \rightarrow F'_{\mathcal{L}} \rightarrow \mathcal{O}_M(2f) \rightarrow 0.$$

From the spectral sequence in Theorem 5.4.3 we see that the pair $\mathcal{E} = \mathcal{O}_M(2f)$ and $\mathcal{F} = \mathcal{O}_L(-2K^*)$ satisfies the assumptions of Section 5.3. Via the equivalence Φ , Corollary 5.3.4 provides a short exact sequence

$$0 \rightarrow \mathcal{O}_M(2f) \rightarrow F''_{\mathcal{L}} \rightarrow G'_{\mathcal{L}} \rightarrow 0, \tag{5.15}$$

from which we deduce that $F''_{\mathcal{L}}$ is a locally free sheaf of rank five. Notice that $\mu(\mathcal{O}_M(2f)) < \mu(F'')$, so this sequence does not destabilize.

Remark 5.7.2. In general, $F_{\mathcal{L}}, F'_{\mathcal{L}}$ and $F''_{\mathcal{L}}$ have all the same Mukai vector. This is easily seen either from the short exact sequences (5.10) and (5.15), or by the fact that any \mathbb{P} -twist acts trivially in cohomology. In particular they are all atomic, because $F_{\mathcal{L}}$ is by Proposition 5.6.1.

5.7.1 Proof of stability

As it turns out, by stopping at $F''_{\mathcal{L}}$ we have performed enough \mathbb{P} -twists to get a slope stable vector bundle. To start, notice that the exact sequence (5.15) this implies that for a general t

$$F''_{\mathcal{L},t} = \mathcal{O}_{M_t} \oplus L_{t,1} \oplus \cdots \oplus L_{t,4},$$

thanks to Lemma 5.2.5. where L_i are line bundles of degree zero on M_t . Hence, the restriction to a generic fiber is semistable, which suggests to try to prove stability with respect to a suitable² polarization h in the sense of Definition 4.7.8.

Lemma 5.7.3. *Assume that there exists an h -destabilizing sequence*

$$0 \rightarrow A \rightarrow F''_{\mathcal{L}} \rightarrow B \rightarrow 0.$$

Then either $\text{rk}(A) = 1$ or $\text{rk}(A) = 4$.

Proof. On a general fiber M_t we can write

$$F''_{\mathcal{L},t} = L_{t,0} \oplus L_{t,1} \oplus \cdots \oplus L_{t,4},$$

where $L_{t,0} = \mathcal{O}_{M_t}$, and $L_{t,i}$ are line bundles of degree zero. The restriction A_t has the same slope as $F''_{\mathcal{L},t}$, hence it is a sub-sum of these line bundles,

$$A_t = L_{t,i_1} \oplus \cdots \oplus L_{t,i_r}.$$

Taking Φ^{-1} , Lemma 5.2.5 gives

$$\Phi^{-1}(A)|_{M_t} = i_{M_t,*} \Phi_{\mathcal{P}_t}^{-1}(A_t) = \mathcal{O}_{M_t, [L_{t,i_1}]} \oplus \cdots \oplus \mathcal{O}_{M_t, [L_{t,i_r}]}.$$

We deduce that over an open $U \subset \mathbb{P}^2$, the support $\text{Supp } \Phi^{-1}(A) \subseteq Z$ and it is finite over the base of degree r . Since $r < 5$ by assumption $\text{Supp } \Phi^{-1}(A)$ is not equal to the whole Z . Hence it must be one of the two components, giving the dichotomy in the statement. \square

Theorem 5.7.4. *The bundle $F''_{\mathcal{L}}$ is slope-stable with respect to any suitable polarization h .*

Proof. Recall that $\text{rk}(F'') = 5$ and $c_1(F'') = 15f$. Let

$$0 \rightarrow A \rightarrow F''_{\mathcal{L}} \rightarrow B \rightarrow 0,$$

be a slope destabilizing short exact sequence. We assume A saturated, so B is torsion-free. By Lemma 5.7.3 either $\text{rk}(A) = 1$ or $\text{rk}(A) = 4$.

²We mean $a(F''_{\mathcal{L}})$ -suitable

Case 1. Assume that $\text{rk}(A) = 1$. Consider the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_M(2f) \cap A & \longrightarrow & A & \longrightarrow & A/(\mathcal{O}_M(2f) \cap A) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_M(2f) & \longrightarrow & F''_{\mathcal{L}} & \longrightarrow & G'_{\mathcal{L}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_M(2f)/(\mathcal{O}_M(2f) \cap A) & \longrightarrow & B & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}$$

The sheaf $A/(\mathcal{O}_M(2f) \cap A)$ has rank 0 and embeds into $G'_{\mathcal{L}}$, which is locally free. Hence it is zero, which gives

$$A \subset \mathcal{O}_M(2f).$$

Using that B is torsion-free, the same argument yields $\mathcal{O}_M(2f) \subset A$. We deduce that $A = \mathcal{O}_M(2f)$, which is not destabilizing.

Case 2. Assume that $\text{rk}(A) = 4$. The quotient B is a torsion-free rank one sheaf, so it injects into its double dual $B^{\vee\vee}$ which is a line bundle on M . Therefore it suffices to show that $\text{Hom}(F'', \mathcal{L}) = 0$ vanishes for every \mathcal{L} line bundle on M with $c_1(\mathcal{L}) = kf$ and $k \leq 3$, thanks to Corollary 4.7.12. By construction we have

$$\text{Hom}(F''_{\mathcal{L}}, \mathcal{O}(2f)) = \text{Hom}(P_{\mathcal{O}_M(2f)}^{-1}(F'_{\mathcal{L}}), \mathcal{O}_M(2f)) = 0.$$

It follows that $\text{Hom}(F'', \mathcal{O}_M(kf)) = 0$ for every $k \leq 2$, because $h^0(M, \mathcal{O}_M(f)) \neq 0$. The only case left is $k = 3$, that is it remains to show that

$$\text{Hom}(F'', \mathcal{O}_M(3f)) = 0.$$

Let $\varphi : F'' \rightarrow \mathcal{O}_M(3f)$ be a morphism, and call A its kernel. Restricting to a general fiber M_t we see that $\text{Hom}(\mathcal{O}_M(2f), A) = 0$, because A_t splits as a sum of four non trivial line bundles of degree 0. This implies that the composition

$$\mathcal{O}_M(2f) \rightarrow F'' \rightarrow \mathcal{O}_M(3f)$$

is not zero, hence it is injective. Applying Φ^{-1} we obtain a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{P'}(2) & \longrightarrow & P_{\mathcal{O}_{P'}(2)}^{-1} \left(P_{\mathcal{O}_{P'}}^{-1}(\overline{\mathcal{L}}) \right) & \longrightarrow & \mathcal{L}(-2K^*) & \longrightarrow & 0 \\
& & \parallel & & \downarrow \Phi^{-1}(\varphi) & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{P'}(2) & \longrightarrow & \mathcal{O}_{P'}(3) & \longrightarrow & \mathcal{O}_l(3) & \longrightarrow & 0
\end{array}$$

where \mathcal{O}_l is the structure sheaf of a line $l \subset P'$, i.e. of the zero locus of a section of $\mathcal{O}_{P'}(1)$. A line l is never contained in L , because the intersection $P' \cap L$ is a smooth conic. It follows that the map $\mathcal{O}_L(-2K^*) \rightarrow \mathcal{O}_l(3)$ is zero, so that the central map factors through $\mathcal{O}_{P'}(2)$. Since $\text{Hom}(F'', \mathcal{O}(2f)) = 0$ is zero, we get that φ is zero. \square

Collecting everything in this section together, we obtain the following result.

Theorem 5.7.5. *Let X be a hyper-Kähler of $K3^{[2]}$ -type. There exist a stable, atomic vector bundle F_0 with Mukai vector*

$$v(F_0) = 5 \left(1 - \frac{3}{4} \mathbf{q}_2 + \frac{9}{32} \mathbf{pt} \right).$$

The $\text{Ext}^1(F_0, F_0)$ is ten dimensional. The Yoneda pairing is skew-symmetric and induces an isomorphism

$$\bigwedge^2 \text{Ext}^1(F_0, F_0) \xrightarrow{\sim} \text{Ext}^2(F_0, F_0).$$

In particular, its deformation functor is smooth.

Proof. The vector bundle $F''_{\mathcal{L}}$ constructed above is stable and atomic. By Remark 5.6.2 we can twist $F''_{\mathcal{L}}$ to obtain a stable atomic vector bundle F_0 with Mukai vector

$$v(F_0) = 5 \left(1 - \frac{3}{4} \mathbf{q}_2 + \frac{9}{32} \mathbf{pt} \right).$$

By Theorem 4.3.9 we can deform F_0 along with every Kähler deformation of M , or equivalently F_0 is hyperholomorphic with respect to any polarization. The Ext algebra remains constant along these deformations by Theorem 4.3.10. In Corollary 5.6.5 we proved that the Yoneda pairing is skew-symmetric. To prove smoothness of the deformation space we argue as follows. The main result in [84] (or [15, Theorem 6.1]) gives formality of the algebra $\text{RHom}(F_0, F_0)$. So the obstruction to lifting a first order deformation is the Yoneda square, which vanishes. \square

5.8 Partial compactification

Here we focus on the semistable deformations of the example constructed in Section 5.5. In this section, we denote by X the moduli space of torsion sheaves $M_S(0, H, -1)$, that before we called M . We recall that we start from a line bundle $\mathcal{L} \in \text{Pic}^0(L)$, and glue it to the structure sheaf of P' to obtain

$$0 \rightarrow \mathcal{L}(-K^*) \rightarrow \overline{\mathcal{L}} \rightarrow \mathcal{O}_{P'} \rightarrow 0.$$

Applying the equivalence Φ induced by the Poincaré sheaf to $\overline{\mathcal{L}}$, we obtain a slope stable locally free sheaf $F_{\mathcal{L}}$. We can tensor by $\mathcal{O}(-3f)$ to kill its first chern class, so that

$$\mathbf{v}_0 := v(F_0) = 5 \left(1 - \frac{3}{4} \mathbf{q}_2 + \frac{9}{32} \mathbf{p}\mathbf{t} \right),$$

as showed in Theorem 5.7.5. Consider the moduli space $M_{X,h}(\mathbf{v}_0)$ parametrizing h -Gieseker semistable sheaves on X with Mukai vector \mathbf{v}_0 , where h is a suitable polarization as in Section 5.7.1.

Remark 5.8.1. Every sheaf in the moduli space $M_{X,h}(\mathbf{v}_0)$ is Gieseker stable, not just semistable. Indeed, since every \mathbb{P} -twist acts as the identity in cohomology, the Euler characteristic is unaffected by the semistable reduction. Example 5.4.4 gives that $\chi(G_{\mathcal{L}}) = -2$, so we have

$$\chi(F_{\mathcal{L}}'') = \chi(F_{\mathcal{L}}) = \chi(G_{\mathcal{L}}) + \chi(\mathcal{O}_M) = 1,$$

which is (tautologically) coprime with the rank. Therefore, every sheaf is Gieseker stable, because the Euler characteristic is the constant term of the Hilbert polynomial.

To study the moduli space $M_{X,h}(\mathbf{v}_0)$ we want to relate it to a known HK manifold. The key observation is the following. The Lagrangian $L \subset X$ is by definition the symmetric square of a curve $C \subset S$ in the linear system $|2H|$. It follows from the Künneth formula, that there is an isomorphism

$$\mathrm{Pic}^0(C) \simeq \mathrm{Pic}^0(\mathrm{Sym}^2 C), \quad G \mapsto G^{(2)} \tag{5.16}$$

where $G^{(2)}$ denotes the symmetrization of the line bundle G . It is defined as the equivariant line bundle

$$G \boxtimes G \in \mathrm{Pic}^0(C \times C),$$

equipped with the linearization which swaps the factors.

Line bundles of degree zero supported on curves in $|2H|$ are generic points of the moduli space $M_S(0, 2H, -4)$. Since the vector $(0, 2H, -4)$ is not primitive, this moduli space is singular, but thanks to a celebrated result by O’Grady [92] its singularities are symplectic. The total space of the symplectic resolution

$$\rho : \widetilde{M}_S(0, 2H, -4) \rightarrow M_S(0, 2H, -4)$$

is a HK variety of type OG10. Similarly to the case considered in Section 5.2, there is a morphism

$$\pi : M_S(0, 2H, -4) \rightarrow |2H|, \quad F \rightarrow \mathrm{Fitt}(F)$$

sending a sheaf F to its Fitting support. The composition

$$\widetilde{M}_S(0, 2H, -4) \xrightarrow{\rho} M_S(0, 2H, -4) \xrightarrow{\pi} |2H|$$

is a Lagrangian fibration.

Proposition 5.8.2. *The autoequivalence*

$$\Psi := (- \otimes \mathcal{O}(-3f)) \circ P_{\mathcal{O}_M(2f)}^{-1} \circ P_{\mathcal{O}_M}^{-1} \circ \Phi$$

induces a morphism

$$\psi : \widetilde{M}_S(0, 2H, -4) \dashrightarrow M_{X,h}(\mathbf{v}_0), \quad G \mapsto F''_{G(2)} \otimes \mathcal{O}(-3f),$$

which is birational onto an irreducible component \mathfrak{M}_0 .

Proof. As remarked previously, a general element of $\widetilde{M}_S(0, 2H, -4)$ is a line bundle of degree zero supported on a curve $C \in |2H|$. It follows from the discussion above that sending $G \in \text{Pic}^0(C)$ to the bundle $F''_{G(2)} \otimes \mathcal{O}(-3f)$ implies that ψ is well-defined. It is invertible where it is defined, because it is the composition of the equivalence Ψ and the isomorphism (5.16). \square

Symplectic forms

Both the source and target of the maps ψ are naturally equipped with symplectic forms on the smooth locus, so it is natural to ask if ψ preserves the symplectic form. It is implicit in Theorem 2.3.2 that the symplectic form on a stable sheaf $G_C \in M_S(0, 2H, -4)$, supported on a curve $C \in |2H|$, is given by Serre duality

$$\text{Ext}^1(G, G) \times \text{Ext}^1(G, G) \rightarrow \mathbb{C}, \quad (a, b) \mapsto \text{Tr}(a \circ b).$$

The degeneration of the local-to-global spectral sequence induces a graded algebra isomorphism

$$\text{Ext}_S^*(G, G) \cong H^*(C, \mathbb{C}),$$

therefore the symplectic form can be identified with Poincaré duality

$$H^1(C, \mathbb{C}) \times H^1(C, \mathbb{C}) \rightarrow \mathbb{C}, \quad (a, b) \mapsto \int_C a \cup b. \quad (5.17)$$

On the other hand, the symplectic form $\tau_{\bar{\sigma}}$ on the locally-free locus of \mathfrak{M}_0 was defined in (4.49). If $F \in \mathfrak{M}_0$ is a stable bundle, it takes the form

$$\text{Ext}^1(F, F) \times \text{Ext}^1(F, F) \rightarrow \mathbb{C}, \quad (a, b) \mapsto \text{Tr}(\chi_{\mathcal{E}_s}(\bar{\sigma}) \circ a \circ b). \quad (5.18)$$

So in order to compare the two, we need first to compute the obstruction map on \mathfrak{M}_0 . For this we exploit the compatibility with derived equivalences, so we can compute

$$\chi_{\overline{\mathcal{L}}_C} : \text{HT}^2(M) \rightarrow \text{Ext}^2(\overline{\mathcal{L}}, \overline{\mathcal{L}}),$$

where \mathcal{L} is a line bundle of degree zero supported on Z . Recall that under the (unmodified) HKR isomorphism (3.9) there is a decomposition

$$\text{HT}^2(M) \cong H^0(M, \bigwedge^2 T_M) \oplus H^1(M, T_M) \oplus H^2(M, \mathcal{O}_M),$$

which gives rise to non commutative deformations as explained in Section 4.2.1. Given a sheaf $G_C \in \overline{M}_S(0, 2H, -4)$ supported on C , we call $\mathcal{L}_C := G_C^{(2)}$ the symmetrization, and by $\overline{\mathcal{L}}_C$ its gluing with $\mathcal{O}_{P'}$.

Lemma 5.8.3. *Under $\chi_{\overline{\mathcal{L}}_C}$, the image of $\text{HT}^2(M)$ and $H^1(M, T_M)$ agree.*

Proof. The obstruction map restricted to $H^2(M, \mathcal{O}_M)$ does not involve the Atiyah class, it is just the tensorization by

$$\overline{\sigma} : \mathcal{O}_M \rightarrow \mathcal{O}_M[2].$$

Since $\overline{\mathcal{L}}_C$ is supported on a Lagrangian submanifold Z , this tensorization vanishes.

To deal with $H^0(M, \bigwedge^2 T_M)$, we use that the equivalence Ψ maps $\overline{\mathcal{L}}_C$ to a stable bundle $F''_{\mathcal{L}_C, 0}$ which deforms to all commutative deformations. This follows from Theorem 4.3.9 using that the first Chern class vanishes. This means that $\chi_{\Psi(\overline{\mathcal{L}}_C)}$ vanishes when restricted to $H^1(M, T_M)$. We also have

$$\chi_{\Psi(\overline{\mathcal{L}}_C)} = \Psi \circ \chi_{\overline{\mathcal{L}}_C} \circ \left(\widetilde{\Psi}^{\text{HT}} \right)^{-1}, \tag{5.19}$$

where $\widetilde{\Psi}^{\text{HT}}$ is obtained from Ψ^{HH} by conjugation with the *modified* HKR. by Proposition 4.2.8 and Remark 4.2.9. To compute Ψ^{HT} we use the morphism

$$\mu : \text{HT}^2(M) \rightarrow \widetilde{H}(X, \mathbb{C}), \quad \eta \mapsto m_\eta(\sigma),$$

of section Section 3.7. Proposition 3.7.3 gives a commutative diagram

$$\begin{array}{ccc} \text{HT}^2(M) & \xrightarrow{\mu} & \widetilde{H}(M, \mathbb{C}) \\ \downarrow \Psi^{\text{HT}} & & \downarrow \Psi^{\widetilde{H}} \\ \text{HT}^2(M) & \xrightarrow{\mu} & \widetilde{H}(M, \mathbb{C}), \end{array}$$

and now Ψ^{HT} is obtained from Ψ^{HH} by conjugation with the *modified* HKR isomorphism (3.9).³ Looking at the definitions gives

$$\Psi^{\text{HT}} = D \circ \tilde{\Psi}^{\text{HT}} \circ D^{-1},$$

where D is the Duflo isomorphism (4.7). It acts on $\text{HT}^2(M)$ as

$$\begin{aligned} \tilde{D}|_{H^2(\mathcal{O}_X)} &= \text{id}, \\ \tilde{D}|_{H^1(T_X)} &= \text{id}, \\ \tilde{D}|_{H^0(\wedge^2 T_X)}(\sigma^\vee) &= 1 \lrcorner \sigma^\vee + \frac{c_2(X)}{6} \lrcorner \sigma^\vee = \sigma^\vee + C \cdot \bar{\sigma}, \end{aligned}$$

for some complex number C . Since the \mathbb{P} -twist acts trivially in cohomology, Proposition 5.2.9 implies that

$$\begin{aligned} (\Psi^{\text{HT}})^{-1}(\sigma^\vee) &\in H^1(M, T_M) \oplus H^2(M, \mathcal{O}_M) \text{ and,} \\ (\Psi^{\text{HT}})^{-1}(f) &\in H^2(M, \mathcal{O}_M). \end{aligned}$$

which implies

$$\chi_{\bar{\mathcal{L}}_C}(\mathbb{C}\sigma^\vee) = \chi_{\Psi(\bar{\mathcal{L}}_C)}(\mathbb{C}\bar{\sigma}) = \chi_{\bar{\mathcal{L}}_C}(\mathbb{C}f),$$

where we are identifying $f \in H^1(M, \Omega_M^1)$ with its image in $H^1(M, T_M)$ via the isomorphism $\Omega_M^1 \cong T_M$. \square

It is only left to compute the restriction of the obstruction map on $H^1(M, T_M)$. Recall that a class $\eta \in HH^2(M)$ represents a natural transformation $\text{id}_M \rightarrow [2]$, and the obstruction map is essentially the evaluation at an object by Proposition 4.2.8. The naturality of η provides a commutative triangle

$$\begin{array}{ccc} \text{HT}^2(M) & & \\ \chi_{\mathcal{L}_C} \downarrow & \searrow \chi_{\bar{\mathcal{L}}_C} & \\ \text{Ext}^2(\mathcal{L}_C, \mathcal{L}_C) & \longrightarrow & \text{Ext}^2(\bar{\mathcal{L}}_C, \bar{\mathcal{L}}_C) \end{array}$$

where the horizontal map is the map inducing the isomorphism of Lemma 5.6.3. For this reason, we focus first on the obstruction map for \mathcal{L}_C . Recall that, by Theorem 4.6.7 there is a graded algebra isomorphism

$$\text{Ext}^*(\mathcal{L}_C, \mathcal{L}_C) \cong H^*(L, \mathbb{C}), \quad (5.20)$$

where the Lagrangian $L \subset M$ is isomorphic to $\text{Sym}^2 C$.

³Because the modification is necessary to have compatibility with the action in cohomology.

Lemma 5.8.4. *The image of $H^1(M, T_M)$ under the obstruction map $\chi_{\mathcal{L}}$ is*

$$\mathbb{C}[\Delta_C] \oplus H^2(C, \mathbb{C}),$$

where $[\Delta_C] \in H^2(L, \mathbb{C})$ is the class of the diagonal.

Proof. By [77, Remark 3.10] there is a commutative diagram

$$\begin{array}{ccc} H^1(M, T_M) & \xrightarrow{\cong} & H^1(M, \Omega_M^1) \\ \downarrow \chi_{\mathcal{L}_C} & & \downarrow \\ \text{Ext}^1(\mathcal{L}_C, \mathcal{L}_C) & \xrightarrow{\cong} & H^1(L, \Omega_L^1) \end{array}$$

where the right vertical map is the restriction map. The restriction can be computed on the other birational model, that is

$$H^2(S^{[2]}, \mathbb{C}) \rightarrow H^2(\text{Sym}^2 C, \mathbb{C}) \cong H^2(C, \mathbb{C}) \oplus \bigwedge^2 H^1(C, \mathbb{C}).$$

The Künneth formula implies that the first summand in

$$H^2(S^{[2]}, \mathbb{C}) \cong H^2(S, \mathbb{C}) \oplus \mathbb{C}\delta.$$

maps to $H^2(C, \mathbb{C})$. To conclude note that, by definition, the class δ maps to a multiple of the class of the diagonal $\Delta_C \subset \text{Sym}^2 C$. \square

Remark 5.8.5. The proof of Lemma 5.6.4 gives also a commutative diagram

$$\begin{array}{ccc} \text{Ext}^2(\mathcal{L}_C, \mathcal{L}_C) \times \text{Ext}^2(\mathcal{L}_C, \mathcal{L}_C) & \longrightarrow & \text{Ext}^4(\mathcal{L}_C, \mathcal{L}_C) \\ \downarrow \Downarrow & & \parallel \\ \text{Ext}^2(\overline{\mathcal{L}}_C, \overline{\mathcal{L}}_C) \times \text{Ext}^2(\overline{\mathcal{L}}_C, \overline{\mathcal{L}}_C) & \longrightarrow & \text{Ext}^2(\overline{\mathcal{L}}_C, \overline{\mathcal{L}}_C) \times \text{Ext}^4(\overline{\mathcal{L}}_C, \overline{\mathcal{L}}_C) \end{array}$$

where the horizontal maps are the Yoneda pairings on L and Z respectively.

Theorem 5.8.6. *The image of the obstruction map for $\overline{\mathcal{L}}_C$ is one dimensional. Under the isomorphism*

$$\text{Ext}^2(\overline{\mathcal{L}}_C, \overline{\mathcal{L}}_C) \cong \bigwedge^2 H^1(C, \mathbb{C})$$

it is spanned by the class representing the Poincaré pairing on C . In particular, ψ preserves the symplectic forms.

Proof. By Lemma 5.8.3, the only piece responsible for the image of the obstruction map is $H^1(M, TM)$. By Lemma 5.8.4 we see that the image of obstruction map for \mathcal{L} is

$$\mathbb{C}[\Delta_C] \oplus H^2(C, \mathbb{C}) \subset \bigwedge^2 H^1(C, \mathbb{C}) \oplus H^2(C, \mathbb{C}).$$

Hence, the projection onto the first factor is spanned by the projection of the class of the diagonal. By Lemma 5.6.4 the Yoneda pairing, under the isomorphism $\text{Ext}^1(\overline{\mathcal{L}_C}, \overline{\mathcal{L}}) \cong H^1(C, \mathbb{C})$ can be identified to

$$(\alpha, \beta) \mapsto \pi_1^*(\alpha) \cup \pi_2^*(\beta) - \pi_1^*(\beta) \cup \pi_2^*(\alpha) \in H^2(\text{Sym}^2 C, \mathbb{C}),$$

where $\alpha, \beta \in H^1(C, \mathbb{C})$ and $\pi_i : C \times C \rightarrow C$ are the two projections. Therefore, by (4.49) and Remark 5.8.5 we can write the symplectic form on $\overline{\mathcal{L}_C}$ as

$$(\alpha, \beta) \mapsto \int_L (\pi_1^*(\alpha) + \pi_2^*(\alpha)) \cup (\pi_1^*(\beta) + \pi_2^*(\beta)) \cup [\Delta_C]$$

To conclude note that we have

$$\int_{\Delta_C} (\pi_1^*(\alpha) + \pi_2^*(\alpha)) \cup (\pi_1^*(\beta) + \pi_2^*(\beta))|_{\Delta_C} = 2 \int_C \alpha \cup \beta.$$

Hence, the image of $[\Delta_C]$ in $\bigwedge^2 H^1(C, \mathbb{C})$ represents the Poincaré pairing on C , and the form $\tau_{\overline{\sigma}}$ is the Poincaré pairing under the isomorphism (5.20)

For a given stable sheaf $G \in M_S(0, 2H, -4)$, the map ψ acts on the Ext groups as the composition of the symmetrization

$$\text{Ext}^i(G, G) \rightarrow \text{Ext}^i(\mathcal{L}_C, \mathcal{L}_C)$$

and the gluing map

$$\text{Ext}^i(\mathcal{L}_C, \mathcal{L}_C) \rightarrow \text{Ext}^i(\overline{\mathcal{L}_C}, \overline{\mathcal{L}_C}).$$

Via the isomorphism (5.20) the symmetrization is the natural symmetrization map

$$H^i(C, \mathbb{C}) \rightarrow H^i(\text{Sym}^2 C, \mathbb{C}).$$

After this isomorphism, the symplectic form on $M_S(0, 2H, -4)$ is the Poincaré pairing on C , thus the above argument implies that ψ is symplectic. \square

Chapter 6

A modular construction of OG10

6.1 Introduction

In this Chapter we degenerate a cubic $Y \subset \mathbb{P}^5$ to a general cubic with the property that its variety of lines $F(Y)$ has a Lagrangian fibration. We show that, in this case, there exists a derived equivalence

$$\Phi : D^b(F(Y)) \simeq D^b(M, \alpha),$$

where M is a four dimensional moduli space of torsion sheaves, and $\alpha \in \text{Br}'(M)$ is a Brauer class. Then, we replicate the construction in Section 5.6 to obtain a slope stable atomic twisted vector bundle on M . More precisely, we consider the Lagrangian surfaces of hyperplane sections

$$F(Y_H) \subset F(Y),$$

and their image under the equivalence Φ . To show that image is locally free (which we do in Proposition 6.5.4), we just have to check the assumptions of Proposition 5.2.6. That is, we need to check that for every hyperplane section $Y_H \subset Y$, the Lagrangian fibration

$$F(Y_H) \rightarrow F(Y)$$

is finite. This is done in Proposition 6.4.14. In contrast to what happens in the previous Chapter, in this case all the surfaces $F(Y_H)$ are integral if Y is general enough. Hence, stability of the image

$$F = \Phi(\mathcal{O}_{F(Y_H)})$$

with respect to a suitable polarization is automatic by a monodromy argument (see Theorem 6.5.6), and we do not need a semistable reduction as in Section 5.7.

The same continues to be true for any line bundle of degree zero supported on $F(Y_H)$. Hence, to study the moduli space of semistable deformations of F , we need to understand the compactified Picard scheme

$$\overline{\text{Pic}}^0(F(Y_H)) \subset \text{Pic}^0(F(Y_H))^\sim$$

which we define in Section 6.4. Our main result is that under our assumptions, every sheaf in $\overline{\text{Pic}}^0(F(Y_H))$ is Cohen-Macaulay.

To see this, we construct a local complete intersection morphism

$$\varphi : F(Y_H) \rightarrow \text{Prym}(C/D)$$

to the compactified Prym of an étale $2 : 1$ cover $f : C \rightarrow D$. Here C and D are integral curves with planar singularities, therefore every sheaf in

$$\overline{\text{Pic}}^0(\text{Prym}(C/D))$$

is CM, as a consequence of the work of Arinkin [7]. Since the morphism φ is lci, it induces a pullback at the level of CM sheaves

$$\varphi^* : \text{Pic}^0(\text{Prym}(C/D))^\sim \rightarrow \text{Pic}^0(F(Y_H))^\sim$$

which turns out to be dominant, as we prove in Lemma 6.4.20. Since the domain is proper, the same is true for the target, which therefore it is equal to the compactified Picard scheme of $F(Y_H)$.

The consequence is that there is an irreducible component of moduli space \mathcal{M}_0 of twisted sheaves on M , which parametrizes only locally free sheaves. Then, the results of Section 4.8 imply that this component is a smooth hyper-Kähler manifold, with the same arguments of Section 5.8. It is easy to see that this moduli space compactifies the intermediate jacobian fibration

$$J_U \rightarrow U,$$

where $U \subset \mathbb{P}^5$ parametrizes smooth hyperplane sections $Y_H \subset Y$. By [66], this admits a HK compactification of type OG10, so \mathcal{M}_0 is of type OG10 itself, giving a positive answer to the question by Markman and O'Grady asked in the introduction.

6.1.1 Structure of the chapter

In Section 6.2 we construct the twisted version of the Poincaré sheaf of Section 5.2.1. We recall the Tate–Shafarevich and Brauer groups, and show that the Poincaré sheaf relates the two.

In Section 6.3 we discuss the geometry of the Fano surfaces $F(Y_H)$ under our assumptions on Y . We recall that (even the singular ones) are integral, and show that they are finite over the base of the Lagrangian fibration $F(Y) \rightarrow \mathbb{P}^2$.

In Section 6.4 we prove Theorem 6.4.1, which says that every sheaf in the compactified Picard scheme is CM. The proof is rather technical, and involves subtle geometrical facts about lines in cubic threefolds.

Section 6.5 is the analogue of Section 5.6. Here we show that applying the equivalence induced by the twisted Poincaré sheaf to the surfaces of lines produces slope stable twisted vector bundles.

Lastly, in Section 6.6 we study the moduli space of these twisted sheaves, and prove Theorem 6.6.3 which is the main result of this thesis.

6.2 Twisted Poincaré sheaf

Let (S, H) be a polarized K3 surface, and assume that every curve in $|H|$ is integral. This happens for example if (S, H) is general. Let $B := |H|$ and $\pi : X := \overline{\text{Pic}}^0(X/B) \rightarrow B$ be the relative compactified Jacobian of the universal curve $\mathcal{C} \rightarrow B$, considered in Section 5.2. Note that π has a section $\sigma : B \rightarrow X$ defined by sending a curve to its structure sheaf.

In Section 5.2.1 we recalled that there exists a relative Poincaré sheaf \mathcal{U} on $X \times_B X$, which induces a derived equivalence

$$\Phi_{\mathcal{U}} : D^b(X) \xrightarrow{\sim} D^b(X).$$

We want to see that if we twist X by an element in the Tate–Shafarevich group, the Poincaré sheaf glues to a twisted sheaf.

6.2.1 Tate–Shafarevich group

The Tate–Shafarevich group of the Lagrangian fibration $X \rightarrow B$ was first introduced in [80, Section 7] as the natural generalization of the Tate–Shafarevich group of an elliptic fibration, and later further studied in [1]. It is defined as

$$\text{III} := H^1(B, \underline{\text{Aut}}_{X/B}^0),$$

where $\underline{\text{Aut}}_{X/B}^0$ is the image of the exponential map

$$\pi_* T_{X/B} \rightarrow \underline{\text{Aut}}_{X/B}.$$

An element $t \in H^1(B, \underline{\text{Aut}}_{X/B}^0)$ can be represented by a Čech cocycle of relative automorphisms

$$t_{ij} \in \text{Aut}^0(\pi^{-1}(U_{ij})/U_{ij}),$$

with respect to an open cover $B = \bigcup_i U_i$. By the cocycle condition, the t_{ij} can be interpreted as transition functions and used to re-glue the manifold X to a new one X^t , equipped with a fibration $\pi^t : X^t \rightarrow B$. In general, the X^t satisfies all the properties for being hyper-Kähler besides being Kähler. In our setting, that is for moduli space of torsion sheaves on a K3 surface, by [80, Proposition 7.7] the manifold X^t is also Kähler, and π^t is a Lagrangian fibration.

As a consequence of [83, Theorem 1.2]

$$R^i \pi_* \mathcal{O}_X \cong \Omega_B^i, \quad (6.1)$$

we get an isomorphism $\pi_* T_{X/B} \cong R^1 \pi_* \mathcal{O}_X$. The group $H^1(B, R^1 \pi_* T_{X/B})$ is denoted by $\widetilde{\text{III}}$ and its image via the exponential map by $\text{III}^0 \subset \text{III}$. There is a short exact sequence

$$0 \rightarrow \text{III}^0 \rightarrow \text{III} \rightarrow H^2(B, \Gamma) \rightarrow 0,$$

where Γ is a sheaf of finitely generated torsion-free abelian groups, see [1, Lemma 3.1].

From now on, we will often abuse the notation denoting by the same letter t both an element in $\widetilde{\text{III}}$ and its image under the exponential map in III . All the twisted fibrations coming from elements in III^0 can be put in a family over $\widetilde{\text{III}}$, as made precise in the following.

Proposition 6.2.1 ([1, Proposition 3.3]). *Let $X \rightarrow B$ be a Lagrangian fibration without multiple fibers in codimension one, for example the one above. There exists a holomorphic family*

$$\mathcal{X} \rightarrow \widetilde{\text{III}}, \quad (6.2)$$

such that the fiber over $t \in \widetilde{\text{III}}$ is X^t . Moreover, there exists a holomorphic fibration

$$\mathcal{X} \rightarrow B \times \widetilde{\text{III}}, \quad (6.3)$$

which restricts to $\pi_t : X^t \rightarrow B$ for every $t \in \widetilde{\text{III}}$.

6.2.2 Brauer group vs Tate–Shafarevich group

Recall that an element

$$\alpha \in \text{Br}'(X) := H^2(X, \mathcal{O}_X^*)$$

can be represented also as a gerbe. In simple terms, this is just a collection $\{\mathcal{L}_{ij}\}$ of line bundles, each over the intersection U_{ij} of some open in a cover of X , satisfying an appropriate cocycle condition, namely

$$\mathcal{L}_{ijk} := \mathcal{L}_{ij} \otimes \mathcal{L}_{jk} \otimes \mathcal{L}_{ki}$$

is trivial. The actual definition is a bit more precise, and we refer to [25, Section 1.1] for details. With this interpretation of a Brauer class, one can also describe an α -twisted sheaf as collection of sheaves $\{\mathcal{F}_i\}$ over the cover $\{U_i\}$ such that

$$\mathcal{F}_i|_{U_{ij}} \cong \mathcal{F}_j|_{U_{ij}} \otimes \mathcal{L}_{ij},$$

for some line bundles \mathcal{L}_{ij} on U_{ij} .

Our next goal is to compare the Brauer group and the Tate–Shafarevich group, and show that, in some sense, the equivalence induced by the Poincaré sheaf exchanges the two.

Lemma 6.2.2. *There is a canonical injection*

$$T : H^1(B, R^1\pi_*\mathcal{O}_X^*) \hookrightarrow H^2(X, \mathcal{O}_X^*).$$

Proof. We analyze the Leray filtration on $H^2(X, \mathcal{O}_X^*)$. The E_2 -page takes the following form:

- $E_2^{2,0} = H^2(B, \mathcal{O}_B^*) = 0$.
- $E_2^{1,1} = H^1(B, R^1\pi_*\mathcal{O}_X^*) \xrightarrow{d_2^{1,1}} E_2^{3,0} = H^3(B, \mathcal{O}_B^*)$.
- $E_2^{0,2} = H^0(B, R^2\pi_*\mathcal{O}_X^*) \xrightarrow{d_2^{0,2}} E_2^{2,1} = H^2(B, R^1\pi_*\mathcal{O}_X^*)$.

So, if we show that $d_2^{1,1}$ vanishes, the degeneration $E_2^{1,1} = E_\infty^{1,1}$ will give the desired injection. For this we proceed in a similar way as in the proof of [80, Lemma 7.3]. The cokernel of the differential $d_2^{1,1}$ is isomorphic to the the image of

$$\pi^* : H^3(B, \mathcal{O}_B^*) \rightarrow H^3(X, \mathcal{O}_X^*).$$

Since π has a section $\sigma : B \rightarrow X$, the composition

$$H^3(B, \mathcal{O}_B^*) \xrightarrow{\pi^*} H^3(X, \mathcal{O}_X^*) \xrightarrow{\sigma^*} H^3(B, \mathcal{O}_B^*)$$

is the identity, so π^* is injective. So $H^3(B, \mathcal{O}_B^*)$ is isomorphic to the cokernel of $d_2^{1,1}$, which means that $d_2^{1,1}$ vanishes. \square

Theorem 6.2.3. *The morphism T is an isomorphism, and fits in the commutative diagram*

$$\begin{array}{ccc} H^1(B, R^1\pi_*\mathcal{O}_X) & \xrightarrow{\cong} & H^2(X, \mathcal{O}_X) \\ \downarrow & & \downarrow \\ H^1(B, R^1\pi_*\mathcal{O}_X^*) & \xrightarrow[T]{\cong} & H^2(X, \mathcal{O}_X^*) \end{array}$$

Proof. The diagram is obtained by the functoriality of the Leray spectral sequence via the exponential map $\mathcal{O}_X \rightarrow \mathcal{O}_X^*$. Using the isomorphism (6.1), we get that the top map is an isomorphism. The right vertical map is surjective, because $H^3(X, \mathbb{Z}) = 0$. Since the diagram is commutative, the bottom map T is also surjective, hence bijective. \square

Remark 6.2.4. Since the fibers of $\pi : X \rightarrow B$ are integral, there is an isomorphism of sheaves of abelian groups

$$R^1\pi_*\mathcal{O}_X^* \simeq \underline{\text{Pic}}_{X/B}.$$

Hence, an element $t \in H^1(B, R^1\pi_*\mathcal{O}_X^*)$ is represented by a collection of relative line bundles $\mathcal{L}_{ij} \in \text{Pic}(\pi^{-1}(U_i))$ for some cover $B = \bigcup_i U_i$, satisfying the cocycle condition. The collection $\{\mathcal{L}_{ij}\}$ represents $T(t)$ as a gerbe. For more on this see the proof of [32, Theorem 3.1].

Theorem 6.2.5. *Let $t \in \widetilde{\text{III}}$, and let X^t be the twisted Lagrangian fibration. Then, there exists a $p_2^*T(t)$ -twisted sheaf \mathcal{U} on $X^t \times X$, supported on $X^t \times_B X$, inducing an equivalence*

$$\text{D}^b(X^t) \simeq \text{D}^b(X, T(t))$$

Proof. Over some cover $B = \bigcup_i U_i$ the class $t \in \text{III}$ is represented by local automorphisms

$$t_{ij} : X_{ij} \xrightarrow{\sim} X_{ji},$$

satisfying the cocycle condition. Denote by P the fiber product $X^t \times_{|H|} X$. Let $X_i := \pi^{-1}(U_i)$ and similarly $P_i := X_i^t \times_{U_i} X_i$. Denote by U_{ij} the double intersections, and similarly X_{ij}, P_{ij} , etc.

Since, by definition $X_i^t = X_i$, by [7] there exists a universal sheaf \mathcal{U}_i over P_i , for details see the proof of [3, Proposition 2.1]. We are interested in understanding how they glue along

$$P_i \supset P_{ij} \xrightarrow[t_{ij} \times \text{id}]{\simeq} P_{ji} \subset P_j.$$

Let X_{ij}° the smooth locus for the fibration π , so that over a curve $C \in U_{ij}$ the fiber is $\text{Pic}^0(C)$. Naturally its complement has codimension two in X_{ij} . Note that when restricted to

$$P_{ij}^\circ := X_{ij}^t \times_{U_{ij}} X_{ij}^\circ,$$

both \mathcal{U}_i and \mathcal{U}_j are line bundles. More precisely, over the open set U_{ij} we have an isomorphism

$$X_{ij}^\circ \cong \text{Pic}^0(X_{ij}^t/U_{ij}), \quad (6.4)$$

and both $\mathcal{U}_i|_{P_{ij}^\circ}$ and $(t_{ij} \times \text{id})^*(\mathcal{U}_j|_{P_{ij}^\circ})$ are universal line bundles. By the definition of the Picard functor, they differ from pullbacks of line bundles on X_{ij}°

$$(t_{ij} \times \text{id})^*(\mathcal{U}_j|_{P_{ij}^\circ}) \cong \mathcal{U}_i|_{P_{ij}^\circ} \otimes p_2^*\mathcal{L}_{ij}. \quad (6.5)$$

Since $X_{ij}^\circ \subset X_{ij}$ has codimension two, the line bundles \mathcal{L}_{ij} extend to X_{ij} . Moreover, the Poincaré sheaves \mathcal{U}_{ij} are Cohen-Macaulay sheaves by Proposition 5.2.2, so the isomorphism (6.5) extends to P_{ij} .

The cocycle condition on the t_{ij} implies that the collection $\{\mathcal{L}_{ij}\}$ is a gerbe on X , representing the obstruction to the existence of a universal sheaf on $X^t \times_B X$. It follows from Remark 6.2.4 that the class of this gerbe is precisely $\alpha = T(t)$. Lastly, the isomorphism (6.5) means that \mathcal{U} glues to a $p_2^*(\alpha)$ -twisted Cohen-Macaulay sheaves on $X^s \times_B X$. It induces an equivalence because it does so on each fiber. \square

Remark 6.2.6. The above arguments also applies to the relative case. More precisely, let $\mathcal{X} \rightarrow \widetilde{\text{III}}$ be the family of Proposition 6.2.1, and define

$$\mathcal{X}' := X \times \widetilde{\text{III}} \rightarrow B,$$

to be the trivial family. We can equip it with a tautological Brauer twist, namely a class

$$\tilde{\alpha} \in \text{Br}'(\mathcal{X}')$$

such that for every $t \in \widetilde{\text{III}}$ it satisfies $\alpha|_t = T(t)$. Then, there exists a twisted sheaf

$$\tilde{\mathcal{U}} \in \text{Coh}(\mathcal{X} \times_{\widetilde{\text{III}}} \mathcal{X}', p_2^*(\tilde{\alpha})) \quad (6.6)$$

such that for every t the restriction $\tilde{\mathcal{U}}|_t$ is the sheaf constructed above.

Remark 6.2.7. Notice that, at least for the first order deformations, we would expect $\tilde{\mathcal{U}}$ to exist by virtue of Theorem 4.2.2. We know that for $t = 0$, there exists a Poincaré sheaf

$$\mathcal{U} \in \text{Coh}(X \times_B X)$$

which induces a derived equivalence. Therefore, to check that it deforms along the family $\mathcal{X} \times_{\widetilde{\text{III}}} \mathcal{X}'$ we just need to check that the action on the second Hochschild cohomology is compatible. The first order deformation associated to \mathcal{X} is the class

$$\sigma^{-1}(f) \in H^1(X, T_X) \subset \text{HH}^2(X),$$

by [80, Theorem 7.11], where $\sigma : T_X \simeq \Omega_X^1$ is the isomorphism induced by the symplectic form. The (non-commutative) first order deformation associated to \mathcal{X} with the tautological twist is

$$\bar{\sigma} \in H^2(X, \mathcal{O}_X) \subset \text{HH}^2(X).$$

Then, Proposition 3.7.3 combined with Proposition 5.2.9 imply that

$$\Phi_{\tilde{\mathcal{U}}}^{\text{HH}}(\sigma^{-1}(f)) = \bar{\sigma},$$

which is what we needed.

6.2.3 Cohomological action

Here we describe the action induced in cohomology by the equivalence

$$\Phi_{\mathcal{U}} : D^b(X^t) \rightarrow D^b(X, T(t)).$$

For this, note that by Remark 6.2.6 we have a relative twisted Poincaré

$$\mathcal{U} \in \text{Coh}(\mathcal{X} \times_{\mathbb{C}} (X \times \mathbb{C}), \alpha)$$

where $\alpha \in H^2(X \times \mathbb{C}, \mathcal{O}_{X \times \mathbb{C}}^*)$ is the universal twist.

Remark 6.2.8. By [1, Theorem 5.19], for a $t \in \mathbb{C} = \text{III}^0(X/\mathbb{P}^n)$ the twist X^t is projective if and only if the image of t in $\text{III}(X/\mathbb{P}^n)$ is torsion. Hence, if X^t is projective, the twist $T(t) \in H^2(X, \mathcal{O}_X^*)$ is torsion, and therefore a Brauer class. This is not the case for every fiber, therefore the global twist will not be a Brauer class, but only on a dense subset of \mathbb{C} . So, at least for projective fibers of the family $\mathcal{X} \rightarrow \mathbb{C}$, it makes sense to talk about the induced action in cohomology.

Recall from Section 4.4 that we call

$$f_t^H : H^*(X_t, \mathbb{Q}) \rightarrow H^*(X_0, \mathbb{Q})$$

the parallel transport operator, and we define

$$\Phi_t^g := \Phi_{\mathcal{U}_0}^g \circ f_t^H : \mathfrak{g}(X_t) \rightarrow \mathfrak{g}(X).$$

The main result of this section is the following.

Proposition 6.2.9. *Let $t \in \text{III}^0(X/\mathbb{P}^n)$ such that $\exp(t) \in \text{III}(X/\mathbb{P}^n)$ is torsion, and consider the equivalence*

$$\Phi_{\mathcal{U}_t} : D^b(X^t) \rightarrow D^b(X, \alpha_t).$$

Let $B \in H^2(X, \mathbb{Q})$ be a B -field lifting α_t . Then the induced Hodge isometry

$$\Phi_{\mathcal{U}_t}^H : H^*(X^t, \mathbb{Q}) \rightarrow H^*(X, B_t, \mathbb{Q})$$

coincides with the isometry $\Phi_{\mathcal{U}_0}^H$. In particular, it is equivariant with respect to Φ_t^g .

Proof. To prove the statement it suffices to see that

$$f_t^H(v(\mathcal{U}_t)) = v(\mathcal{U}_0) \in H^*(X \times X, \mathbb{Q}).$$

The sheaf \mathcal{U} is supported on the fiber product $\mathcal{Z} := \mathcal{X} \times_{\mathbb{C} \times \mathbb{P}^n} (X \times \mathbb{C})$ and has rank one on it. This implies that the restriction of the twist $\alpha|_{\mathcal{Z}} \in \text{Br}(\mathcal{Z})$ vanishes. Because of this, by [25, Theorem 2.2.4] we get a (twisted) pushforward functor

$$i_*^\alpha : \text{Coh}(\mathcal{Z}) \rightarrow \text{Coh}(\mathcal{X} \times_{\mathbb{C}} (X \times \mathbb{C}), \alpha),$$

which is exact because i is a closed embedding. Since \mathcal{P} is Cohen–Macaulay, we have

$$\mathcal{P} = i_*^\alpha i^* \mathcal{P}.$$

We also have the usual untwisted pushforward

$$i_* : \text{Coh}(\mathcal{Z}) \rightarrow \text{Coh}(\mathcal{X} \times_{\mathbb{C}} (X \times \mathbb{C})).$$

This allows us to “untwist” \mathcal{P} by defining

$$\tilde{\mathcal{P}} := i_* i^* \mathcal{P} \in \text{Coh}(\mathcal{X} \times_{\mathbb{C}} (X \times \mathbb{C})). \quad (6.7)$$

This untwisted family is still flat over both factors, because locally it is isomorphic to \mathcal{P} .

Recall the alternative definition of the twisted Mukai vector given in [119]. Let G be an α_t -twisted locally free sheaf on $X_t \times X$, which exists because $\alpha_t \in \text{Br}(X)$, and B a B -field lifting α_t . Then we have

$$v_G^B(-) = \frac{v(G^\vee \otimes -)}{\text{ch}^{-B}(G^\vee)}. \quad (6.8)$$

We have

$$v(G^\vee \otimes \mathcal{P}_t) = v(i_*(i^*(G^\vee \otimes \mathcal{P}_t))) = i_*(v(i^* \mathcal{P}_t) \cdot \text{ch}(i^* G^\vee)),$$

where the first equality follows from the fact that $G^\vee \otimes \mathcal{P}_t$ is a CM untwisted sheaf supported on \mathcal{Z}_t , and the second by the Grothendieck–Riemann–Roch theorem. Substituting in (6.8), and using projection formula we get

$$v(G^\vee \otimes \mathcal{P}_t) = i_* \frac{(v(i^* \mathcal{P}_t) \cdot \text{ch}(i^* G^\vee))}{\text{ch}(i^* G^\vee)} = i_* v(i^* \mathcal{P}_t).$$

Here we have used that

$$i^* \text{ch}_G^{-B}(G^\vee) = \text{ch}(i^* G),$$

because the twist is trivial on \mathcal{Z}_t .

To conclude, observe that, since the untwisted family $\tilde{\mathcal{P}}$ is flat with respect to Δ , we have

$$i_* v(i^* \mathcal{P}_t) = v(\tilde{\mathcal{P}}_t) = v(\tilde{\mathcal{P}}_0) = v(\mathcal{P}_0),$$

because $\alpha_t = 0$ for $t = 0$. □

Having this, the results of Section 4.4 imply that $\Phi_{\mathcal{U}_t}$ maps atomic objects in $\text{D}^b(X^t)$ to twisted atomic objects in $\text{D}^b(X, \alpha_t)$.

6.3 Variety of lines with Lagrangian fibrations

Let \mathcal{K}_6^2 be the moduli space of polarized hyper-Kähler manifolds (X, h) of degree 6 and divisibility 2. A general $(X, h) \in \mathcal{K}_6^2$ can be described as the variety of lines $X = F(Y)$ on a smooth cubic fourfold $Y \subset \mathbb{P}^5$ equipped with the Plücker polarization.

Let $\mathcal{N}(d) \subset \mathcal{K}_6^2$ be the closure of the locus parameterizing polarized hyper-Kähler manifolds with Neron-Severi $\text{NS}(X) = \mathbb{Z}h \oplus \mathbb{Z}f$ and Beauville–Bogomolov matrix

$$\begin{pmatrix} 6 & 2d \\ 2d & 0 \end{pmatrix}.$$

A general $(X, h) \in \mathcal{N}(d)$ has a unique (up to automorphism of \mathbb{P}^2) Lagrangian fibration $\pi : X \rightarrow \mathbb{P}^2$, that satisfies $\pi^* \mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_X(f)$. In this section we collect some results on the geometry of this situation.

Lemma 6.3.1. *Let $\pi : X \rightarrow \mathbb{P}^2$ be the Lagrangian fibration. Then, for every smooth hyperplane section $Y_H \subset Y$, the restriction*

$$\pi|_Z : Z = F(Y_H) \rightarrow \mathbb{P}^2$$

is a finite morphism of degree $5d^2$.

Proof. Let $i : Z \hookrightarrow X$ be the inclusion. Recall that the restriction map

$$i^* : H^2(X, \mathbb{Z}) \rightarrow H^2(Z, \mathbb{Z})$$

has rank one, so every algebraic class in its image is ample. In particular

$$\pi|_Z^*(\mathcal{O}_{\mathbb{P}^2}(1)) = i^* \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$$

is ample on Z , which is equivalent to the finiteness of $\pi|_Z$. To compute the degree it suffices to compute the intersection

$$\int_X [Z] \cup f^2.$$

By [77, Lemma 7.3]

$$[Z] = \frac{5}{8} \left(h^2 - \frac{1}{5} c_2(X) \right) \in H^4(X, \mathbb{Z}).$$

For a HK fourfold, the second Chern class $c_2(X)$ is a multiple of the class \mathfrak{q}_2 , hence

$$\int_X c_2(X) \cup f^2 = 0,$$

because $q(f) = 0$. Moreover, by Fujiki formula we have

$$\int_X h^2 \cup f^2 = 2q(h, f)^2 = 2 \cdot (2d)^2$$

Putting everything together we get

$$\int_X [Z] \cup f^2 = \frac{5}{8} \int_X h^2 \cup f^2 - \frac{1}{8} \int_X c_2(X) \cup f^2 = 5d^2,$$

which is equal to the degree. \square

Remark 6.3.2. The union of the Nother-Lefschetz divisors

$$\bigcup_d \mathcal{N}(d) \subset \mathcal{K}_6^2$$

is dense in the classical topology. This means that all statements which hold for a general $X \in \mathcal{K}_6^2$ remain true for a general element of $\mathcal{N}(d)$ for infinitely many d . The particular result we are interested in is [66, Lemma 2.5], which says that the class of the surface Z indecomposable in $H^4(X, \mathbb{Z})$. This implies that, writing $X = F(Y)$, for every hyperplane section $Y_H \subset Y$, the surface of lines $F(Y_H)$ is integral.

We wish to generalize Lemma 6.3.1 to the singular case. The end goal here is to verify the assumptions of Proposition 5.2.6, but this time, in contrast to Section 5.5, we want to do it for all degenerations.

Consider $Z \subset X$ a singular surface of lines, and let $\nu : \tilde{Z} \rightarrow Z$ be a resolution of singularities. Write $j = i \circ \nu : \tilde{Z} \rightarrow X$ for the composition. Then, by [67, Lemma 2.7] we have

$$\mathrm{rk} \left(j^* : H^2(X, \mathbb{Z}) \rightarrow H^2(\tilde{Z}, \mathbb{Z}) \right) = 1. \quad (6.9)$$

In particular, this implies that

$$j^* : \mathrm{NS}(X) \rightarrow \mathrm{NS}(\tilde{Z})$$

has rank one.

Proposition 6.3.3. *For every hyperplane section $Y_H \subset Y$, the restriction of the Lagrangian fibration*

$$\pi|_Z : Z = F(Y_H) \rightarrow \mathbb{P}^2$$

is a finite morphism of degree $5d^2$.

Proof. As in the smooth case, we wish to show that $\pi|_Z^* \mathcal{O}_{\mathbb{P}^2}(1)$ is ample on Z . By the Nakai–Moishezon criterion, $\pi|_Z^* \mathcal{O}_{\mathbb{P}^2}(1)$ is ample if and only if its class $\mathbf{f} \in \text{NS}(Z)$ satisfies

$$(\mathbf{f}.C) > 0$$

for every curve $C \subset Z$. It follows from (6.9) that

$$j^* \mathcal{O}(f) = j^* \mathcal{O}(h) \in \text{NS}(\tilde{Z}),$$

so we conclude that $\mathcal{O}(f)|_Z = \mathcal{O}(h)|_Z$ if we show that

$$\nu^* : \text{NS}(Z) \rightarrow \text{NS}(\tilde{Z})$$

is injective. Assume that $\nu^*(L) \equiv 0$. Let $C \subset Z$ be an integral curve, and $\tilde{C} \subset \tilde{Z}$ any integral curve in $\nu^{-1}(C)$ mapping surjectively to C . We have

$$\deg(\nu^* L|_{\tilde{C}}) = \deg(\nu^*|_{\tilde{C}}(L|_C)) = \deg(\nu^*|_{\tilde{C}}) \cdot \deg(L|_C).$$

By assumption we have $\deg(\nu^* L|_{\tilde{C}}) = 0$. Since $\deg(\nu^*|_{\tilde{C}}) \neq 0$, this implies that $\deg(L|_C) = 0$, so L is numerically trivial on Z . The degree is computed as in the smooth case. \square

6.4 Compactified Picard group of the surface of lines

Let $Y \subset \mathbb{P}^5$ be a general cubic fourfold. Let $H \subset \mathbb{P}^5$ be any hyperplane, and $Z := F(Y \cap H)$ be the surface parametrizing lines in the intersection $Y \cap H$. Following Altman–Kleiman [4], we denote by

$$\text{Pic}^0(Z)^{\neq} := \{\text{torsion-free sheaves on } Z \text{ of rank 1 and degree 0}\} / \cong.$$

the moduli space of torsion-free sheaves of rank one on Z , and by

$$\overline{\text{Pic}}^0(Z) \subset \text{Pic}^0(Z)^{\neq}$$

the irreducible component of $\text{Pic}^0(Z)^{\neq}$ containing line bundles. The goal of this section is the following result.

Theorem 6.4.1. *For a general cubic $Y \subset \mathbb{P}^5$ and any hyperplane $H \subset \mathbb{P}^5$, the moduli space $\overline{\text{Pic}}^0(Z)$ parametrizes only Cohen–Macaulay sheaves.*

Remark 6.4.2. If \bar{J} denotes the compactified Jacobian of an integral curve C with planar singularities, then every sheaf in $\overline{\text{Pic}}^0(\bar{J})$ is Cohen–Macaulay, as shown in [7, Theorem A and B].

To exploit this fact, we find an integral curve C and a locally complete intersection morphism

$$i : Z \rightarrow \bar{J}(C).$$

Since any lci morphism has finite tor dimension, the pullback is well-defined at the level of CM sheaves. That is, we get an induced morphism

$$i^* : \text{Pic}^0(\bar{J}(C))^- \rightarrow \text{Pic}^0(Z)^-,$$

where, again following [4], we use the notation $\text{Pic}^0(-)^- \subset \text{Pic}^0(-)^=$ to indicate the locus of Cohen-Macaulay sheaves. In particular, by Remark 6.4.2 we can restrict our attention to limits of line bundles, and get a map

$$i^* : \overline{\text{Pic}^0(\bar{J}(C))} \rightarrow \text{Pic}^0(Z)^- \cap \overline{\text{Pic}^0(Z)}, \tag{6.10}$$

Since $\text{Pic}^0(\bar{J}(C))^-$ is complete, if we can prove that i^* is dominant, we conclude that $\overline{\text{Pic}^0(Z)}$ is contained in $\text{Pic}^0(Z)^-$.

6.4.1 Reminders on lines on cubic threefolds

In this section, we let $V \subset \mathbb{P}^4$ denote a (not necessarily smooth) cubic threefold. If $l \subset Y$ is a line, the projection to a complementary plane defines a rational map

$$\pi_l : Y \dashrightarrow \mathbb{P}^2.$$

It is resolved by blowing up the line, and so we obtain a conic bundle

$$Y_l \rightarrow \mathbb{P}^2. \tag{6.11}$$

The discriminant curve, that is the curve parametrizing the singular fibers, is a plane quintic $D_l \subset \mathbb{P}^2$.

If we assume that the fibers of π_l are all reduced, then the curve D_l admits a natural $2 : 1$ étale cover

$$f_l : C_l \rightarrow D_l. \tag{6.12}$$

Denote by $\iota : C_l \rightarrow C_l$ the covering involution; notice that it has no fixed points since the quotient map is étale. The curve C_l can also be realized as the closure of the scheme

$$\{l' \in F(V) \mid l \neq l' \text{ and } l' \cap l \neq \emptyset\} \tag{6.13}$$

of lines meeting l , as shown in [53, Lemma 1.26]. From this point of view, the covering involution is described as

$$l' \mapsto l'', \text{ where } \text{Span}(l, l') \cap V = l \cup l' \cup l''. \tag{6.14}$$

That is, takes a line l' to the residual line of the intersection of the plane spanned by l' and l with V .

Definition 6.4.3 ([66, Definition 2.9]). Let $V \subset \mathbb{P}^4$ be a cubic threefold. A line $l \subset V$ is very good if every fiber of π_l is reduced, and the curve C_l is integral.

Remark 6.4.4. If $p \in V$ a singular point, the locus of lines through p

$$C_p = \{l \in F(V) \mid p \in l\}$$

is a curve, and the singular locus of $F(V)$ is the union of these curves, see for example [6, Lemma 1.5(ii)] It follows that a very good line must be contained in the smooth locus of V , otherwise C_l would have at least two components. This means that a very good line $l \in F(V)$ is a smooth point.

Proposition 6.4.5 ([66, Proposition 2.10]). *If Y is a general cubic fourfold, for every hyperplane section $Y \cap H$ there exists a very good line $l \in F(Y \cap H)$.*

Let $\bar{J}(C_l)$ be the degree zero component compactified Picard of C_l . Following [66, Section 4.1] we can define the compactified Prym variety associated to the double cover $f_l : C_l \rightarrow D_l$ as

$$\overline{\text{Prym}}(C_l, D_l) := \text{Fix}(-i^*)_0 \subset \bar{J}(C_l), \quad (6.15)$$

where $\text{Fix}(-i^*)_0$ denotes the component of the fixed locus which contains the structure sheaf. The Prym variety is related with the Fano variety of lines by the following well-known result.

Proposition 6.4.6. *Let $V \in \mathbb{P}^4$ be a smooth cubic threefold, and $l \in F(V)$ be a general line. Then, there is a closed embedding*

$$F(V) \hookrightarrow \text{Prym}(C_l/D_l).$$

Proof. The generality assumption is needed to make sense of the Prym. So we need that the curve C_l is smooth, and the involution (6.14) has no fixed point. Then by [53, Proposition 3.10] there is an isomorphism

$$\text{Pic}^0(F(V)) \simeq \text{Prym}(C_l/D_l), \quad L \mapsto L|_{C_l} \quad (6.16)$$

induced by the restriction along the inclusion $C_l \subset F(V)$. Moreover, if we fix a line $l_0 \in F(V)$, the Abel-Jacobi map is a closed embedding

$$F(V) \hookrightarrow \text{Pic}^0(F(V)), \quad l \mapsto \mathcal{O}(C_l - C_{l_0}), \quad (6.17)$$

see [53, Corollary 5.3.5]. □

6.4.2 Relative setting

Let $Y \subset \mathbb{P}^5$ a general cubic fourfold, and let $B := |\mathcal{O}_Y(1)| \simeq \mathbb{P}^5$ the space of hyperplane sections. Let

$$\mathcal{V} := \{(x, H) \mid x \in Y \cap H, H \in B\} \subset Y \times B$$

be the universal hyperplane section. The second projection $\mathcal{V} \rightarrow B$ is flat, and the fibers are the hyperplane sections of Y . The first projection $\mathcal{V} \rightarrow Y$ is a \mathbb{P}^4 -bundle, so the space \mathcal{V} is regular of dimension eight.

Analogously, we can consider the relative fano variety of lines

$$\mathcal{F} := \{(l, H) \mid l \in F(Y \cap H), H \in B\} \subset F(Y) \times B.$$

The second projection $\mathcal{F} \rightarrow B$ is flat, and the fiber over H is the surface of lines of $Y \cap H$. The first projection $\mathcal{F} \rightarrow F(Y)$ is a \mathbb{P}^3 -bundle hence the space \mathcal{F} is smooth of dimension seven. Following [66] we denote by $\mathcal{F}_0 \subset \mathcal{F}$ the locus of very good lines. By Proposition 6.4.5 and Remark 6.4.4 the map $\mathcal{F}_0 \rightarrow B$ is smooth and surjective.

We can relativize the construction of the compactified Prym over \mathcal{F}_0 . Consider the universal incidence curve

$$\mathcal{C}^0 := \overline{\{(l, l', H) \mid l \neq l', l \cap l' \neq \emptyset, l, l' \subset Y \cap H\}} \subset \mathcal{F}_0 \times_B \mathcal{F}.$$

It is equipped with a fiberwise involution, and taking the quotient we get a flat family of $2 : 1$ étale cover $\mathcal{C}^0 \rightarrow \mathcal{D}^0$. We can consider the relative compactified Prym

$$\overline{P}_{\mathcal{F}_0} := \overline{\text{Prym}}(\mathcal{C}^0/\mathcal{D}^0) := \text{Fix}(-i^*)_0 \subset \overline{\text{Pic}}^0(\mathcal{C}^0/\mathcal{F}_0),$$

where $\text{Fix}(-i^*)_0$ denotes the component of the fixed locus which contains the zero section.

Theorem 6.4.7 ([66, Proposition 5.1]). *The relative compactified Prym $\overline{P}_{\mathcal{F}_0}$ is smooth, and the morphism $\overline{P}_{\mathcal{F}_0} \rightarrow \mathcal{F}_0$ is flat of relative dimension five.*

6.4.3 Picard group in the singular case

Recall that a morphism $f : X \rightarrow S$ is called locally complete intersection (or lci) if locally it factors as

$$X \hookrightarrow P_S \twoheadrightarrow S,$$

where the first map is Koszul regular¹ and the second map is smooth.

¹This means that is induced by a sequence whose Koszul complex is acyclic.

Remark 6.4.8. The morphisms $\mathcal{V} \rightarrow B$ and $\mathcal{F} \rightarrow B$ are both lci morphisms. Indeed, the embeddings

$$\mathcal{V} \subset Y \times B \text{ and } \mathcal{F} \subset F(Y) \times B$$

are both obtained as zero loci of regular sections of vector bundles. In the first case is just a section of the tautological divisor, and in the second case of the vector bundle $\pi_1^* \mathcal{U}^\vee \otimes \pi_2^* \mathcal{O}_B(1)$.

In order to prove Theorem 6.4.1, our goal is to construct a morphism of schemes over \mathcal{F}_0

$$\varphi : \mathcal{F} \times_B \mathcal{F}_0 \rightarrow \overline{P}_{\mathcal{F}_0}.$$

Since $\mathcal{F} \times_B \mathcal{F}_0 \rightarrow \mathcal{F}_0$ is a lci morphism, and the total space $\overline{P}_{\mathcal{F}_0}$ is regular, the map φ is automatically lci by [Stacks, Lemma 37.60.13]. In this way, taking the fiber over $(l, H) \in \mathcal{F}_0$, we get morphism

$$\varphi_{(l,H)} : F(Y \cap H) \rightarrow \overline{\text{Prym}}(C_l/D_l),$$

which we will show to be lci. Since the morphism is lci, the pullback is well defined at the level of CM sheaves, as in (6.10). Therefore, Theorem 6.4.1 is proved if we can show that this pullback is dominant, which will follow from the smooth case.

Remark 6.4.9. This morphism $\varphi|_{(l,H)}$ will be given by

$$l' \mapsto I_{C'_l \cap C_l} \otimes \mathcal{O}(C_l)|_{C_l}$$

where $I_{C'_l \cap C_l}$ is the ideal sheaf of the intersection $C'_l \subset C_l$ as a subscheme of C_l . If H gives rise to a smooth linear section Y_H , this morphism will recover the inclusion

$$F(Y_H) \hookrightarrow \text{Prym}(C_l/D_l),$$

of Proposition 6.4.6.

The family $\mathcal{C}^0 \rightarrow \mathcal{F}_0$ is a family of curves with integral fibers. In this case, the relative compactified Jacobian has a universal property (i.e. is a fine moduli space), see [4, Theorem 7.9]). So, giving a morphism φ is equivalent to giving a sheaf $\mathcal{G} \in \text{Coh}(\mathcal{F} \times_B \mathcal{C}^0)$, flat over $\mathcal{F} \times_B \mathcal{F}_0$, and such that restricted to each fiber is torsion-free of rank one, and invariant under the action of $-i^*$.

Our first goal is to prove the following.

Proposition 6.4.10. *There exists an integral closed subvariety*

$$\mathcal{C} \subset \mathcal{F} \times_B \mathcal{F}$$

such that:

- (1) The variety \mathcal{C} is CM, and the projection $p_i : \mathcal{C} \rightarrow \mathcal{F}$ is flat, for $i = 1, 2$.
- (2) For a general $(l, H) \in \mathcal{F}$, there is a natural identification of the fiber $p_1^{-1}(l, H) = C_l$ with the curve of lines in $F(Y_H)$ meeting l .

Assuming the existence of such \mathcal{C} , notice that the restriction to \mathcal{F}_0

$$\mathcal{C}^0 = \mathcal{C} \cap (\mathcal{F} \times_B \mathcal{F}_0) \subset \mathcal{F} \times_B \mathcal{F}_0,$$

is a Cartier divisor, because the ambient space is regular. Consider the intersection

$$\Xi := p_{12}^{-1}(\mathcal{C}) \cap p_{23}^{-1}(\mathcal{C}^0) \subseteq \mathcal{F} \times_B \mathcal{F} \times_B \mathcal{F}_0,$$

where

$$p_{12} : \mathcal{F} \times_B \mathcal{F} \times_B \mathcal{F}_0 \rightarrow \mathcal{F} \times_B \mathcal{F} \text{ and } p_{23} : \mathcal{F} \times_B \mathcal{F} \times_B \mathcal{F}_0 \rightarrow \mathcal{F} \times_B \mathcal{F}_0$$

are the projections. Let \mathcal{I} be its ideal sheaf in $\mathcal{F} \times_B \mathcal{C}^0$

$$\mathcal{I} := \mathcal{I}_{\Xi/p_{23}^{-1}(\mathcal{C}^0)} \subset \mathcal{O}_{\mathcal{F} \times_B \mathcal{C}^0}.$$

Our main technical result is the following.

Theorem 6.4.11. *The sheaf $\mathcal{I} \in \text{Coh}(\mathcal{F} \times_B \mathcal{C}^0)$ is a flat family of torsion free sheaves over $\mathcal{F} \times_B \mathcal{F}_0$.*

Family of incidence curves

Now we focus on proving Proposition 6.4.10. Of course, the first guess for the definition of \mathcal{C} is the relative incidence curve

$$\mathcal{C}' := \overline{\{(l, l', H) \in \mathcal{F} \times_B \mathcal{F} \mid l \neq l', l \cap l' \neq \emptyset, l, l' \subseteq H\}}. \quad (6.18)$$

Unfortunately, taking the closure makes proving flatness over \mathcal{F} a difficult task. For this reason we use a different point of view for the construction of the curve C_l , which we now explain. For reference, see [53, Lemma 5.1.26] and the discussion above, and [13, Proposition 1.5].

Let $V \subset \mathbb{P}^4$ be a cubic with only isolated singularities, and let $l \subset V$ be any line. Projecting away from the line gives a rational map

$$V \dashrightarrow \mathbb{P}^2$$

which is resolved by blowing up the line. The resulting regular map

$$\pi : \tilde{V} \rightarrow \mathbb{P}^2$$

is a conic bundle. The discriminant locus $D_l \subset \mathbb{P}^2$ is a plane quintic.

The Hilbert scheme $G(\pi^{-1}(D_l))$ of lines contained in the inverse image $\pi^{-1}(D_l)$ has a flat 2 : 1 map

$$G(\pi^{-1}(D_l)) \rightarrow D_l.$$

It is easy to see (c.f. [53, Lemma 5.1.26]), at least when the line is general, that this is precisely the cover

$$C_l \rightarrow D_l.$$

This construction that generalizes well to the relative situation, as we now show. Let $\mathbb{L}_{\mathcal{V}} \subset \mathcal{V} \times_B \mathcal{F}$ be the universal line. If $\mathcal{H} \subset \mathbb{P}^5 \times B \rightarrow B$ denotes the universal hyperplane section, we can find a family of 2-planes

$$\Pi \subset \mathcal{H} \times_B \mathcal{F}$$

disjoint from the line \mathcal{F} . Blowing up the universal line we get the projection

$$\begin{array}{ccc} \widetilde{\mathcal{V} \times_B \mathcal{F}} & \xrightarrow{\pi} & \Pi \\ & \searrow & \downarrow \\ & & \mathcal{F} \end{array}$$

By the above discussion, the discriminant locus $\mathcal{D} \subset \Pi$ is a hypersurface, which restricts to the plane quintic $D_l \subset \mathbb{P}^2$ for every $(l, H) \in \mathcal{F}$. Define

$$\mathcal{C} := G_{\mathcal{D}}(\pi^{-1}(\mathcal{D})) \rightarrow \mathcal{D}$$

to be the relative Hilbert scheme of lines in the fibers of $\pi : \pi^{-1}(\mathcal{D}) \rightarrow \mathcal{D}$.

Lemma 6.4.12. *The natural map*

$$\mathcal{C} \rightarrow \mathcal{D}$$

is flat, and \mathcal{C} is Cohen-Macaulay.

Proof. Notice that every fiber (even those corresponding to lines of the second type) of the map

$$\pi^{-1}(\mathcal{D}) \rightarrow \mathcal{D}$$

is a singular conic. Since the Hilbert scheme is compatible with base change, the fiber over a point $d \in \mathcal{D}$ is

$$\mathcal{C}|_d = G(\pi^{-1}(d)),$$

the Hilbert scheme of lines in the singular conic $\pi^{-1}(d)$. It consists of two reduced points if the conic is reduced, and a double point if the conic is a double line. So, the map $\mathcal{C} \rightarrow \mathcal{D}$ is finite with fibers of constant length two, hence flat. Moreover, since any length two scheme is CM, the map is also CM by [Stacks, p. 37.22.3]. Therefore, \mathcal{C} is CM by [Stacks, p. 37.22.4]. \square

Lemma 6.4.13. *There is a closed embedding of schemes over \mathcal{F}*

$$\mathcal{C} \hookrightarrow \mathcal{F} \times_B \mathcal{F}.$$

Proof. The product $\mathcal{F} \times_B \mathcal{F}$ is the relative Hilbert scheme of lines of the map

$$\mathcal{V} \times_B \mathcal{F} \rightarrow \mathcal{F}.$$

Therefore, to obtain the desired map it suffices to exhibit a family of lines

$$\mathcal{L} \subset \mathcal{C} \times_{\mathcal{F}} (\mathcal{V} \times_B \mathcal{F}).$$

By construction, \mathcal{C} is a relative Hilbert scheme of lines, so it comes equipped with a family

$$\widetilde{\mathcal{L}} \subset \mathcal{C} \times_{\mathcal{F}} \pi^{-1}(\mathcal{D}) \subset \mathcal{C} \times_{\mathcal{F}} \widetilde{\mathcal{V} \times_B \mathcal{F}}.$$

Moreover, for every $(l, H) \in \mathcal{F}$ the blow-down

$$\text{Bl}_l(Y_H) \rightarrow Y_H$$

does not contract any fiber of the projection π_l . Hence, blowing down gives the desired family \mathcal{L} . This arguments also shows that the map

$$\mathcal{C} \rightarrow \mathcal{F} \times_B \mathcal{F}$$

is a map of Hilbert schemes of lines, which is induced by the closed embedding $\pi^{-1}(\mathcal{D}) \subset \widetilde{\mathcal{V} \times_B \mathcal{F}}$, composed with an isomorphism induced by the blow-down. Hence it is a closed embedding. \square

Proof of Proposition 6.4.10. To prove (1) notice that, since the fibers of the natural map

$$\mathcal{C} \rightarrow \mathcal{F}$$

are finite over the fibers of $\mathcal{D} \rightarrow \mathcal{F}$, they have dimension one. Since \mathcal{C} is CM and \mathcal{F} is regular, this map is flat by miracle flatness. Moreover, it can be identified with the first projection via the embedding of Lemma 6.4.13.

Point (2) is shown in [53, Lemma 5.1.26]. We also remark that for every $(l, H) \in \mathcal{F}$, we have a set-theoretical equality. Indeed, the fiber $p_1^{-1}(l, H)$ contains only lines in Y_H which meet l , and contains l precisely if and only if there is a plane tangent to Y_H in l . This, in turn, happens if and only if $l \in C_l$, see [53, Lemma 5.1.11].

To see that \mathcal{C} is integral, we show that it is equal to the relative incidence curve $\mathcal{C}' \subset \mathcal{F} \times_B \mathcal{F}$ defined in (6.18). Since it is the closure of a scheme which is integral by [66, Lemma 2.14], it is itself integral. Therefore, we have

$$\mathcal{C}' \subset \mathcal{C},$$

by the set-theoretical equality.

The morphism $\mathcal{C} \rightarrow \mathcal{F}$ is projective, and taking the Hilbert polynomials of the fibers with respect to a relatively ample line bundle, we get the inequality

$$P_{\mathcal{C}'|_{(l,H)}}(m) \leq P_{\mathcal{C}|_{(l,H)}}(m).$$

On the other hand, the Hilbert polynomial on the right is constant in the family, because $\mathcal{C} \rightarrow \mathcal{F}$ is flat, and the one on the left can only increase on special fibers by [62, Proposition 28(1)]. Since generically we have equality, it must be that

$$P_{\mathcal{C}'|_{(l,H)}}(m) = P_{\mathcal{C}|_{(l,H)}}(m) \text{ and } \mathcal{C}'|_{(l,H)} = \mathcal{C}|_{(l,H)},$$

proving the statement. □

The proof above says that we can interpret the fiber $\mathcal{C}|_{(l,H)}$ as the correct scheme structure on the curve $C_l \subset F(Y_H)$ to obtain a flat family.

Conclusion of the proof

Now we can conclude the proof of Theorem 6.4.11. To start, we investigate when the intersection $C_l \cap C_{l'}$ is proper, i.e. the two curves have no common components.

Proposition 6.4.14. *The morphism $\Xi \rightarrow \mathcal{F} \times_B \mathcal{F}_0$ is finite away from the diagonal*

$$\Delta_{\mathcal{F}_0} \subset \mathcal{F}_0 \times_B \mathcal{F}_0 \subset \mathcal{F} \times_B \mathcal{F}_0.$$

Proof. We generalize the argument of [53, Remark 1.13]. It is clearly projective, so it suffices to show that it is quasi finite. For this, it suffices to show that set-theoretically, the intersection $C_l \cap C_{l'}$ is finite, where $l \subset F(Y)$ is a very good line and $l' \subset F(Y)$ is any line different from l .

Case 1. Assume that $l \cap l' \neq \emptyset$, so they span a plane. Then the lines in Y meeting both are: the residual line of the intersection $\text{Span}(l, l') \cap Y$, and the lines through the intersection point. Since l is very good, through each point of l there can pass only finitely many lines. Indeed, if there were a curve of lines passing through one point of l , that curve would be an irreducible component of C_l .

Case 2. Assume $l \cap l' = \emptyset$, so they span a three dimensional space. Consider the cubic surface

$$S := \text{Span}(l, l') \cap Y.$$

Every line in Y meeting both l and l' must live in S . If S were reducible, it would have a plane as a component. Since Y is assumed to be general, this does not happen, so S is irreducible.

If there are only finitely many lines in S meeting l (e.g. if S is regular or with normal rational singularities) we are done. Hence, assume otherwise: there is a

curve of lines meeting l in S . Since l is very good, this curve which must be C_l because it is integral.

If S is a cone over an integral plane cubic, every two lines in S must meet, contradicting our assumption that $l \cap l' \neq \emptyset$.

If S is integral, but non normal and not a cone, then is one of two explicit cubics, see [68, Section 2.3]. In both cases S is singular along a line, and the pencil of planes containing l induces a ruling on S . One checks by explicit computations that a line in S is either an isolated point in $F(S)$ or a line in the ruling. Since there is a curve of lines meeting l , this curve must be the ruling, contradicting the assumption that C_l is integral. \square

Over the diagonal, there is a simple way of checking that \mathcal{I} is a flat family of torsion-free sheaves.

Lemma 6.4.15. *Over the open locus $\mathcal{F}_0 \times_B \mathcal{F}_0$, the sheaf \mathcal{I} is a flat family of torsion-free sheaves of rank one.*

Proof. When restricted to $\mathcal{F}_0 \times_B \mathcal{C}^0$, the subscheme $p_{12}^{-1}(\mathcal{C})$ becomes the Cartier divisor $p_{12}^{-1}(\mathcal{C}^0)$. By Proposition 6.4.10 the variety $p_{12}^{-1}(\mathcal{C}^0)$ is integral, so we have a short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{F}_0 \times_B \mathcal{C}^0}(-p_{12}^{-1}(\mathcal{C}^0)) \rightarrow \mathcal{O}_{\mathcal{F}_0 \times_B \mathcal{C}^0} \rightarrow \mathcal{O}_{\Xi|_{\mathcal{F}_0 \times_B \mathcal{F}_0}} \rightarrow 0.$$

Therefore, the sheaf

$$\mathcal{I}|_{\mathcal{F}_0 \times_B \mathcal{C}^0} = \mathcal{O}_{\mathcal{F}_0 \times_B \mathcal{C}^0}(-p_{12}^{-1}(\mathcal{C}^0))$$

is a line bundle on $\mathcal{F}_0 \times_B \mathcal{C}^0$. Since the map $\mathcal{F}_0 \times_B \mathcal{C}^0 \rightarrow \mathcal{F}_0 \times_B \mathcal{F}_0$ is flat, the same is true for \mathcal{I} . The fiber over a point in $\mathcal{F}_0 \times_B \mathcal{F}_0$ is again a line bundle, in particular torsion-free of rank one. Notice although, that the fiber over a point in the diagonal is *not* the ideal sheaf of $\Xi|_{(l,l,H)} \subset C_l$. In fact, the latter scheme is the whole of C_l , so its ideal sheaf agrees with the structure sheaf. \square

Proof of Theorem 6.4.11. By Lemma 6.4.15 we know that the statement is true over $\mathcal{F}_0 \times_B \mathcal{F}_0$, in particular near the diagonal. We now prove that $\Xi \rightarrow \mathcal{F} \times_B \mathcal{F}_0$ is flat away from the diagonal. For this, we consider the embedding

$$\Xi := p_{12}^{-1}(\mathcal{C}) \cap p_{23}^{-1}(\mathcal{C}^0) \subset p_{12}^{-1}(\mathcal{C}).$$

It is a Cartier divisor, because $p_{23}^{-1}(\mathcal{C}^0)$ is. Moreover, away from the diagonal it is fiberwise a proper intersection, by Proposition 6.4.14. Consider the defining short exact sequence

$$0 \rightarrow \mathcal{O}_{p_{12}^{-1}(\mathcal{C})}(-p_{23}^{-1}(\mathcal{C}^0)) \rightarrow \mathcal{O}_{p_{12}^{-1}(\mathcal{C})} \rightarrow \mathcal{O}_{\Xi} \rightarrow 0.$$

Taking the fiber over a point $(l', l, H) \in \mathcal{F} \times_B \mathcal{F}^0$ with $l \neq l'$, we get

$$0 \rightarrow \mathcal{T}or^{\mathcal{O}_{p_{12}^{-1}(\mathcal{C})}}(\mathcal{O}_{\mathcal{C}|_{(l',H)}}, \mathcal{O}_{\Xi|_{(l',l,H)}}) \rightarrow \mathcal{O}_{\mathcal{C}|_{(l',H)}}(-\mathcal{C}^0|_{(l,H)}) \rightarrow \mathcal{O}_{\mathcal{C}|_{(l',H)}} \rightarrow \mathcal{O}_{\Xi|_{(l',l,H)}} \rightarrow 0.$$

Observe that $\mathcal{C}|_{(l',H)}$ has no embedded points, because $\mathcal{C} \rightarrow \mathcal{F}$ is flat, \mathcal{F} is regular, and \mathcal{C} is CM. Hence

$$\mathcal{T}or^{\mathcal{O}_{p_{12}^{-1}(\mathcal{C})}}(\mathcal{O}_{\mathcal{C}|_{(l',H)}}, \mathcal{O}_{\Xi|_{(l',l,H)}}) = 0,$$

because it is supported in dimension zero by Proposition 6.4.14. Therefore, the length of $\Xi|_{(l',l,H)}$ is equal to

$$\chi(\Xi|_{(l',l,H)}, \mathcal{O}_{\Xi|_{(l',l,H)}}) = \chi(\mathcal{C}|_{(l',H)}, \mathcal{O}_{\mathcal{C}|_{(l',H)}}) - \chi(\mathcal{C}|_{(l',H)}, \mathcal{O}_{\mathcal{C}|_{(l',H)}}(-\mathcal{C}^0|_{(l,H)}))$$

which is constant on the complement of the diagonal in $\mathcal{F} \times_B \mathcal{F}^0$. This follows again by the flatness of $\mathcal{C} \rightarrow \mathcal{F}$ and the fact that $\mathcal{O}_{p_{12}^{-1}(\mathcal{C})}(-p_{23}^{-1}(\mathcal{C}^0))$ is a line bundle on $p_{12}^{-1}(\mathcal{C})$, so it is flat as well.

To conclude the proof, we show that the restriction $\mathcal{I}|_{(l',l,H)}$ is torsion-free also when $l \neq l'$, which, combined with Lemma 6.4.15, gives the statement. Now we consider Ξ as a subscheme of $p_{23}^{-1}(\mathcal{C}^0)$. The defining short exact sequence is

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{p_{23}^{-1}(\mathcal{C}^0)} \rightarrow \mathcal{O}_{\Xi} \rightarrow 0.$$

Restricting to the fiber over the point $(l', l, H) \in \mathcal{F} \times \mathcal{F}^0$ gives

$$0 \rightarrow \mathcal{I}|_{((l',l,H))} \rightarrow \mathcal{O}|_{\mathcal{C}^0|_{(l,H)}} \rightarrow \mathcal{O}_{\Xi|_{(l',l,H)}} \rightarrow 0,$$

because Ξ is flat over $\mathcal{F} \times \mathcal{F}^0$. Since

$$\mathcal{C}^0|_{(l,H)} = C_l \subset F(Y_H)$$

is integral by definition of very good line, we conclude that \mathcal{I} is torsion-free. \square

The universal property of the relative compactified Jacobian gives a morphism

$$\mathcal{F} \times_B \mathcal{F}^0 \rightarrow \overline{\text{Pic}}(\mathcal{C}^0/\mathcal{F}^0),$$

which over the line $(l, H) \in \mathcal{F}^0$ acts as

$$l' \in F(Y \cap H) \mapsto \mathcal{I}_{C_l' \cap C_l},$$

where the intersection is seen as a Weil divisor in C_l . We note that the image is not a degree zero divisor, but this is readily fixed by considering the map induced by

$$\mathcal{I} \otimes p_2^* \mathcal{O}_{\mathcal{C}^0}(\mathcal{C}^0) \in \text{Coh}(\mathcal{F}_0 \times \mathcal{C}^0),$$

where $\mathcal{O}_{\mathcal{C}^0}(\mathcal{C}^0)$ is the normal bundle of the embedding $\mathcal{C}^0 \subset \mathcal{F} \times_B \mathcal{F}^0$.

Proposition 6.4.16. *The morphism*

$$\varphi : \mathcal{F} \times_B \mathcal{F}_0 \rightarrow \overline{\text{Pic}}(\mathcal{C}^0/\mathcal{F}^0),$$

induced by the sheaf $\mathcal{I} \otimes p_2^* \mathcal{O}_{\mathcal{C}^0}(\mathcal{C}^0)$, factors through the compactified Prym and gives a morphism.

Proof. Let $B^\circ \subset B$ be the open locus where the hyperplane section Y_H is regular. Over a very good line $l \in F(Y_H)$, the map φ becomes

$$F(Y_H) \rightarrow \text{Prym}(C_l/D_l), \quad l' \mapsto \mathcal{O}_{F(Y \cap H)}(C_l - C'_l)|_{C_l}$$

This is the composition of the map

$$F(Y_H) \rightarrow \text{Pic}^0(F(Y_H)), \quad l' \mapsto \mathcal{O}_{F(Y \cap H)}(C_l - C'_l)$$

described in [53, Lemma 3.1], and the restriction

$$\text{Pic}^0(F(Y_H)) \mapsto \text{Pic}^0(C_l).$$

This last restriction factors via the Prym variety $\text{Prym}(C_l/D_l) \subset \text{Pic}^0(C_l)$ by virtue of [53, Corollary 3.2] as explained in [53, Proposition 3.10]. Therefore, we deduce that over B° the image of φ is contained in the relative Prym, hence taking the closure we get the result. \square

Remark 6.4.17. The map $\varphi : \mathcal{F} \times_B \mathcal{F}_0 \rightarrow \overline{\text{P}}_{\mathcal{F}_0}$ is a local complete intersection morphism because both $\mathcal{F} \times_B \mathcal{F}_0$ and $\overline{\text{P}}_{\mathcal{F}_0}$ are regular. This follows from [Stacks, Lemma 37.60.13].

Now we specialize to a fiber, our first goal is to show that the morphism

$$\varphi_{(l,H)} : F(Y_H) \rightarrow \text{Prym}(C_l/D_l)$$

is a local complete intersection morphism for every hyperplane H and every very good line l .

Lemma 6.4.18. *Consider the following commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow p & \swarrow \\ & & S \end{array}$$

Assume that h is a local complete intersection morphism, S is a smooth variety, and X and Y are flat over S . Then, for every $s \in S$,

$$f_s : X_s \rightarrow Y_s$$

is a local complete intersection morphism.

Proof. Since h is local complete intersection, locally on X is the composition of a smooth morphism with a Koszul regular embedding

$$U \xrightarrow{i} V \xrightarrow{\pi} Y.$$

The diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad} & V \\ & \searrow & \swarrow \\ & S & \end{array}$$

satisfies the same assumptions as our original one, so we can assume that h is a Koszul regular embedding.

For $x \in X$ such that $p(x) = s$ call $R := \mathcal{O}_{Y,x}$. Let (f_1, \dots, f_n) be the equations of X in Y around x , and let (g_1, \dots, g_m) be the equations of Y_s in Y around x . By our assumptions they are both Koszul regular sequences. We want to show that the images $(\overline{f}_1, \dots, \overline{f}_n)$ in $R/(g_1, \dots, g_m)$ form a Koszul regular sequence, which means that $X_s \subset Y_s$ is a regular embedding.

By [Stacks, Lemma 15.30.13] the sequence $(f_1, \dots, f_n, g_1, \dots, g_m)$ is Koszul regular in R . Being Koszul regular is independent of the order, so the sequence $(g_1, \dots, g_m, f_1, \dots, f_n)$ is Koszul regular in R . We conclude by [Stacks, Lemma 15.30.14] that the images $(\overline{f}_1, \dots, \overline{f}_n)$ are Koszul regular in $R/(g_1, \dots, g_m)$. \square

Corollary 6.4.19. *For every hyperplane H and every very good line l , the map*

$$\varphi_{(l,H)} : F(Y_H) \rightarrow \mathrm{Prym}(C_l/D_l)$$

is a local complete intersection morphism.

Therefore, we get an induced map at the level of CM sheaves

$$\mathrm{Pic}^0(\mathrm{Prym}(C_l/D_l))^- \rightarrow \mathrm{Pic}^0(F(Y_H))^-.$$

To prove Theorem 6.4.1 it suffices to show the following.

Lemma 6.4.20. *The pullback*

$$\varphi^* : \mathrm{Pic}^0(\mathrm{Prym}(C_l/D_l)) \rightarrow \mathrm{Pic}^0(F(Y_H))$$

is dominant.

Proof. Consider the composition

$$C_l \xrightarrow{i_{(l,H)}} F(Y_H) \rightarrow \mathrm{Prym}(C_l/D_l).$$

If Y_H is regular, this composition is equal to the Abel-Prym map

$$\text{AP} : C_l \rightarrow \text{Prym}(C_l/D_l),$$

as mentioned in [53, Remark 3.9]. It has the property that the pullback induces an isomorphism

$$\text{AP}^* = \varphi_{(l,H)}^* \circ i_{(l,H)}^* : \text{Pic}^0(\text{Prym}(C_l/D_l)) \simeq \text{Prym}(C_l/D_l). \quad (6.19)$$

Since,

$$\dim \text{Pic}^0(F(Y_H)) = \dim \text{Pic}^0(\text{Prym}(C_l/D_l)) = \dim \text{Prym}(C_l/D_l) = 5$$

the pullback $\varphi_{(l,H)}^*$ must also be surjective, for the composition to be surjective. We can globalize this argument: the composition

$$\mathcal{C}^0 \subset \mathcal{F} \times_B \mathcal{F}_0 \xrightarrow{\varphi} \overline{P}_{\mathcal{F}_0}.$$

is well-defined everywhere and equal to the (singular) Abel-Prym map also for the singular hyperplane sections. This continues to satisfy the isomorphism (6.19), on the open locus of locally free sheaves. So the argument above works and gives that $\varphi_{(l,H)}^*$ is dominant also in the singular case. \square

Now we have all the ingredients to prove the main result of this section.

Proof of Theorem 6.4.1. Let $Y_H \subset Y$ be any hyperplane section, and denote by $Z \subset X$ the corresponding surface of lines. By Corollary 6.4.19 and Lemma 6.4.20 we know that there is a dominant morphism

$$\varphi^* : \text{Pic}^0(\overline{\text{Prym}}(C_l/D_l))^- \rightarrow \text{Pic}^0(Z)^-,$$

where $l \in Z$ is a very good line. The inclusion

$$j : \overline{\text{Prym}}(C_l/D_l) \subset \overline{\text{Pic}}^0(C_l)$$

is also a local complete intersection morphism. This is because the action

$$\overline{\text{Prym}}(C_l/D_l) \times \text{Pic}^0(D_l) \rightarrow \overline{\text{Pic}}^0(C_l), (F, L) \mapsto F \otimes f^*L$$

is étale. The pullback

$$j^* : \text{Pic}^0(\overline{\text{Pic}}^0(C_l)) \rightarrow \text{Pic}^0(\overline{\text{Prym}}(C_l/D_l))$$

is easily seen to be dominant, with a similar argument to Lemma 6.4.20. Hence, putting everything together we have a dominant morphism

$$\text{Pic}^0(\overline{\text{Pic}}^0(C_l))^- \rightarrow \text{Pic}^0(Z)^-.$$

By Remark 6.4.2, we have that

$$\mathrm{Pic}^0\left(\overline{\mathrm{Pic}^0(C_l)}\right)^- = \overline{\mathrm{Pic}^0\left(\mathrm{Pic}^0(C_l)\right)}$$

is proper, hence the same is true for Z

$$\mathrm{Pic}^0(Z)^- = \overline{\mathrm{Pic}^0(Z)},$$

which means that the component $\overline{\mathrm{Pic}^0(Z)}$ parametrizes only CM sheaves. \square

6.5 Locally freenes and stability

The goal of this section is to show that the sheaves obtained via the twisted Poincaré sheaf from the Lagrangian surfaces $Z \subset X$ are locally free and slope stable. To be more precise, let $Y \subset \mathbb{P}^5$, be a cubic, such that its variety of lines $X := F(Y)$ is a general element in the Noether-Lefschetz locus $\mathcal{N}(d) \subset \mathcal{K}_6^2$. To imitate the construction in Section 5.6 we need a derived equivalence induced by some Poincaré sheaf. We start by recalling the following geometrical fact.

Proposition 6.5.1 ([93, Proposition B.4]). *If*

$$d > 5 \cdot 7 = 35 \text{ and } 3 \nmid d, \tag{6.20}$$

then this Lagrangian fibration can be realized as a Tate–Shafarevich twist of the Beauville–Mukai system

$$M := M_S(0, H, -1) \rightarrow \mathbb{P}^2,$$

where (S, H) is a K3 surface of degree two.

The reason for this is [80, Theorem 7.11], which says that via the period map, the base $\widetilde{\mathrm{III}}$ of the family (6.2) gets mapped to the line through M which keeps f of type $(1, 1)$. As showed in [80, Theorem 7.13] this implies that a general such X is birational to a Tate–Shafarevich twist of M , and the numerical conditions (6.20) guarantee that the movable cone of X has no walls.

Remark 6.5.2. Notice that even imposing the condition (6.20) we are left with an infinte collection of Noether-Lefschetz loci. They continue to be dense in the moduli space \mathcal{K}_6^2 , hence the statements in Section 6.4 (which require Y to be general) hold in our situation.

From now on we take a d satisfying (6.20), and consider $X \in \mathcal{N}(d)$ general. by Theorem 6.2.5 there exists a twisted Poincaré sheaf

$$\mathcal{U} \in \text{Coh}(X \times_{\mathbb{P}^2} M, p_2^*(\alpha))$$

for some $\alpha \in \text{Br}(M)$. Let $Y_H \subset Y$ be any hyperplane section (possibly singular), and $Z = F(Y_H)$ its surface of lines. For any $G \in \overline{\text{Pic}}^0(Z)$, we consider the α -twisted object

$$F_G := \Phi_{\mathcal{U}}(G) \in \text{D}^b(M, \alpha). \quad (6.21)$$

Remark 6.5.3. From the proof of Theorem 6.2.5 we see that there is an open cover $B = \bigcup_i U_i$, such that the twist $\alpha \in \text{Br}(M)$ represents the obstruction for patching up the relative Poincaré sheaves on

$$X_{U_i} \times_{U_i} M_{U_i}.$$

Therefore, the twist α vanishes when restricted to $\pi^{-1}(U_i)$. In particular the restriction of any α -twisted sheaf F , to a fiber M_t becomes untwisted. In the particular case of F_G , Lemma 5.2.5 implies that the restriction $F_G|_t$ to a general fiber splits as the sum of line bundles of degree zero.

Proposition 6.5.4. *For every $G \in \overline{\text{Pic}}^0(Z)$, the object*

$$F_G \in \text{D}^b(M, \alpha)$$

is a locally free twisted atomic sheaf of rank $5d^2$.

Proof. The proof is the same as Proposition 5.6.1. To see that it is locally-free, take a cover $\mathbb{P}^2 = \bigcup_i U_i$ such that over each U_i the sheaf \mathcal{U} is untwisted and gives an equivalence

$$\Phi_{\mathcal{U}|_{U_i}} : \text{D}^b(X_{U_i}) \simeq \text{D}^b(M|_{U_i}).$$

Since G is a Cohen-Macaulay sheaf by Theorem 6.4.1, and Z is finite over \mathbb{P}^2 by Proposition 6.3.3, the proof of Proposition 5.2.6 shows that $F|_{M_{U_i}}$ is locally free. Notice that the Poincaré sheaf can be deformed in a one parameter family to an untwisted sheaf, by Remark 6.2.6. Therefore, we are in the setting of Section 4.4 and F_G is a twisted atomic sheaf by Proposition 4.4.7. The rank is equal to the degree of Z over \mathbb{P}^2 , which we computed in Lemma 6.3.1. \square

Recall that on a projective variety X with a fixed Brauer class $\alpha \in \text{Br}(X)$, the definition of slope stability for an α -twisted torsion-free sheaf F is analogous to the classical one. Namely, having chosen a B -field lifting α , we define the twisted slope as

$$\mu_h^B(F) := \frac{c_1^B(F) \cdot h^{\dim X - 1}}{\text{rk}(F)},$$

where h is an ample class on the variety X .

We say that F is h -slope (semi)stable if for every twisted subsheaf

$$E \subset F \in \text{Coh}(X, \alpha), \text{ with } \text{rk}(E) < \text{rk}(F)$$

we have the inequality

$$\mu_h^B(E)(\leq) < \mu_h^B(F).$$

See [119] for more details on the theory of stability for twisted sheaves.

Remark 6.5.5. The computations done in Proposition 4.3.11 apply just as well for the extended Mukai vector $\tilde{v}^B(-)$ of an α -twisted atomic sheaf. This is because the B -field only affects the Hodge structure, and these computations are independent of the Hodge structure. In particular we see that the projection of the discriminant onto the Verbitsky component

$$\Delta(F_G)_{\text{SH}} \in \text{SH}(M),$$

is a multiple of \mathfrak{q}_2 . From this we also deduce that the endomorphism bundle $\mathcal{E}nd(F_G)$ is modular (as an untwisted sheaf), since its discriminant is a multiple of $\Delta(F_G)$.

Theorem 6.5.6. *Consider the α -twisted vector bundle F_G on X , and choose a B -field B lifting α . If h be a polarization which is $a(\mathcal{E}nd(F_G))$ -suitable, then F_G is h -slope stable for every $G \in \overline{\text{Pic}}^0(Z)$.*

Proof. Assume that there exists a destabilizing twisted subsheaf

$$E \hookrightarrow F_G.$$

We can tensor by F_G^\vee to obtain a destabilizing subsheaf

$$E \otimes F_G^\vee \hookrightarrow \mathcal{E}nd(F_G),$$

which now lives in the untwisted category of coherent sheaves. Since h is suitable for $\mathcal{E}nd(F_G)$, we are under the assumptions of Proposition 4.7.10. Indeed, the restriction $\mathcal{E}nd(F_G)|_t$ splits as a sum of line bundles of degree zero, in particular is semistable. Therefore, Corollary 4.7.12 implies that

$$c_1(E \otimes F_G^\vee) \in \mathbb{Z}f \in \text{NS}(M),$$

because $c_1(\mathcal{E}nd(F_G)) = 0$. By Remark 6.5.3, on M_t every α -twisted sheaf becomes untwisted, hence we have

$$0 = c_1(E \otimes F_G^\vee)_t = c_1(E|_t),$$

because $F_G|_t$ splits as the sum of line bundles of degree zero on M_t . From here we conclude as in Lemma 5.7.3. Namely, E_t has to be a subsum of the line bundles which make up $F_G|_t$. Hence $\Phi_{\mathcal{U}}^{-1}(E)$ is an untwisted sheaf whose support is finite over \mathbb{P}^2 and contained in Z . The surface Z is integral by Remark 6.5.2 and Remark 6.3.2. Therefore we must have

$$\text{Supp } \Phi_{\mathcal{U}}^{-1}(E) = Z,$$

which means that $E = F_G$ and F_G is stable. \square

6.6 Compactification

Let $\alpha \in \text{Br}'(M)$ be the twist of F_G constructed above. Let B be a B -field lifting α , and let

$$\mathbf{v}_0^B := v^B(F_G) \in H^*(M, \mathbb{Q}, B)$$

the (twisted) Mukai vector of F_G . Let h be an $a(\text{End}(F_G))$ -suitable polarization, and consider the (coarse) moduli space $M_{X,h}(\mathbf{v}_0^B)$ of h Gieseker semistable, α -twisted sheaves with twisted Mukai vector equal to \mathbf{v}_0^B . The moduli theory for twisted sheaves is very similar to the untwisted case, and it is worked out in [70, 119]. In particular, the moduli space $\mathcal{M}_{M,h}(\mathbf{v}_0^B)$ is a projective scheme by [119, Theorem 2.1]. Let $\mathcal{M}_0 \subset \mathcal{M}_{M,h}(\mathbf{v}_0^B)$ be the irreducible component containing the sheaves F_G constructed above.

Consider the linear system $B = |\mathcal{O}_Y(1)| \simeq \mathbb{P}^5$ of hyperplane sections of the cubic Y . The association

$$H \rightarrow F(Y_H) \subset X$$

embeds B as a component of the Hilbert scheme of deformations of Z in X . Explicitly, if \mathcal{S}^\vee denotes the dual of the tautological bundle on X , then

$$B \simeq \mathbb{P}(H^0(X, \mathcal{S}^\vee))$$

is the space of zero loci of sections of \mathcal{S}^\vee on X . Let $\mathcal{F} \rightarrow B$ be the universal Fano surface of lines (or the universal section of \mathcal{S}^\vee) as in Section 6.4. It is a flat projective morphism with integral Cohen-Macaulay fibers. With the notation of [4, 5], let $\text{Pic}^0(\mathcal{F}/B)^=$ be the moduli space of torsion-free sheaves of rank one and degree zero on the fibers of $\mathcal{F} \rightarrow B$. It is representable by a scheme each of whose component is proper over B by [5, Theorem 3.1]. Also let $\text{Pic}^0(\mathcal{F}/B)^-$ be the locus of such sheaves which are CM. In [4, Theorem 6.6] it is shown that with the natural scheme structure (induced by the moduli problem) is quasi projective over B . A priori, these schemes could have multiple irreducible components. By a slight

abuse, we use the same notation to denote the irreducible component containing the line bundles. With this abuse, in Section 6.4 we have shown that

$$\mathrm{Pic}^0(\mathcal{F}/B)^- = \overline{\mathrm{Pic}^0(\mathcal{F}/B)}^-.$$

We denote this scheme by $\overline{\mathrm{Pic}^0(\mathcal{F}/B)}$; since it is quasi projective and proper, it is projective over B . It is also projective itself, because $B \cong \mathbb{P}^5$ is projective.

Lemma 6.6.1. *The fiber of the natural map*

$$\pi : \overline{\mathrm{Pic}^0(\mathcal{F}/B)} \rightarrow B,$$

over $Z \in B$ is

$$\pi^{-1}(Z) = \overline{\mathrm{Pic}^0(Z)}.$$

In particular, the singular locus in $\overline{\mathrm{Pic}^0(\mathcal{F}/B)}$ has codimension at least two.

Proof. In Section 6.4 we show that there is a surjective morphism of schemes over \mathcal{F}_0

$$\mathrm{Pic}^0(\overline{\mathrm{Pic}^0(\mathcal{C}^0/\mathcal{F}_0)})^- \rightarrow \mathrm{Pic}^0(\mathcal{F}_0 \times_B \mathcal{F}/\mathcal{F})^-.$$

Hence, the fiber $\pi^{-1}(Z)$ is dominated by the fiber of

$$\rho : \mathrm{Pic}^0(\overline{\mathrm{Pic}^0(\mathcal{C}^0/\mathcal{F}_0)})^- \rightarrow \mathcal{F}_0$$

over some very good line $l \subset Z$. Therefore we conclude if we show that every such fiber is integral. On the locus parametrizing line bundles the fibers are connected, and, since it maps surjectively to \mathcal{F}_0 , by Stein factorization every fiber is connected. By [7, Theorem B] the compactified Picard $\overline{\mathrm{Pic}^0}(\overline{\mathrm{Pic}^0}(C_l))$ is a connected component in $\mathrm{Pic}^0(\overline{\mathrm{Pic}^0}(C_l))^-$. The fiber ρ^{-1} clearly contains it, therefore they must be equal

$$\rho^{-1}(l) = \overline{\mathrm{Pic}^0}(\overline{\mathrm{Pic}^0}(C_l)).$$

This implies that every fiber of ρ is integral, and so every fiber of π is. Therefore, the inclusion

$$\overline{\mathrm{Pic}^0}(Z) \subset \pi^{-1}(Z) \tag{6.22}$$

is an equality. The singular locus lives in the complement of the locus of line bundles, which has codimension two by (6.22). \square

The arguments in Section 5.6 give the following.

Proposition 6.6.2. *The equivalence*

$$\Phi_{\mathcal{U}} : \mathrm{D}^b(X) \simeq \mathrm{D}^b(M, \alpha)$$

induces an isomorphism

$$\varphi_{\mathcal{U}} : \overline{\mathrm{Pic}^0(\mathcal{F}/B)} \simeq \mathcal{M}_0.$$

Hence, the component \mathcal{M}_0 parametrizes only locally free twisted sheaves.

Proof. It is well defined at the level of sets by Theorem 6.5.6. Therefore it induces a morphism of stacks, which descends to coarse moduli spaces by the same argument in Remark 2.3.9. It is an isomorphism because $\Phi_{\mathcal{U}}$ is an equivalence. Notice that the key here is that we are taking the natural scheme structure induced by the moduli problem. The fact that every sheaf $F \in \mathcal{M}_0$ is locally free follows from Proposition 6.5.4. Indeed every point in $\overline{\text{Pic}}^0(\mathcal{F}/B)$ is a CM sheaf supported on a $Z \in B$, hence is mapped to a (twisted) vector bundle via $\Phi_{\mathcal{U}}$. \square

Now we can prove the main result of this thesis.

Theorem 6.6.3. *The scheme \mathcal{M}_0 is a smooth projective hyper-Kähler manifold of type OG10, and the map*

$$\pi \circ \varphi_{\mathcal{U}}^{-1} : \mathcal{M}_0 \rightarrow B$$

is a Lagrangian fibration.

Proof. We know it is projective by the discussion above. To show smoothness we apply Theorem 4.8.8, to each open S_i of an étale cover

$$\coprod S_i \rightarrow \mathcal{M}_0,$$

as in Remark 4.8.2. Indeed, we know by Proposition 6.6.2 that \mathcal{M}_0 parametrizes only locally-free (twisted) sheaves. Notice that, although we are dealing with twisted vector bundles, the arguments in Section 4.8.3 depend only on the endomorphism sheaves of local tautological families. These are always untwisted bundles even if \mathcal{M}_0 parametrizes twisted sheaves. It only remains to check that the singular locus has codimension at least two. For this, after applying $\varphi_{\mathcal{U}}^{-1}$ we see that the smooth locus of $\overline{\text{Pic}}^0(\mathcal{F}/B)$ has complement of codimension at least two by Lemma 6.6.1.

The same can be said in regards to the symplectic form constructed in Section 4.8.2. That is, the proof of Theorem 4.8.5 involves only the cohomology of the endomorphisms sheaves. Hence, for $F \in \mathcal{M}_0$, the formula (4.49)

$$\tau_{\mu}(a, b) = \text{Tr} \left(\chi_{\mathcal{E}_s}(\mu)^{n-1} \circ a \circ b \right), \quad a, b \in \text{Ext}^1(F, F)$$

defines a symplectic form on the whole of \mathcal{M}_0 by Proposition 4.8.6.

We have showed that \mathcal{M}_0 is a smooth projective variety equipped with a symplectic form. It is also birational to $\text{Pic}^0(\mathcal{F}/B)$, which admits a HK compactification to a variety of type OG10 by the main result of [66]. It follows that \mathcal{M}_0 is itself a hyper-Kähler manifold, and since is birational to OG10, it is of type OG10. Lastly, the map $\pi \circ \varphi_{\mathcal{U}}^{-1} : \mathcal{M}_0 \rightarrow B$ must be a Lagrangian fibration by [82]. \square

Bibliography

- [1] A. Abasheva and V. Rogov. “Shafarevich–Tate groups of holomorphic Lagrangian fibrations”. In: (2021). eprint: [arXiv:2112.10921](https://arxiv.org/abs/2112.10921).
- [2] D. Abramovich and A. Polishchuk. “Sheaves of t -structures and valuative criteria for stable complexes”. In: *J. Reine Angew. Math.* 590 (2006), pp. 89–130.
- [3] N. Addington, W. Donovan, and C. Meachan. “Moduli spaces of torsion sheaves on K3 surfaces and derived equivalences”. In: *J. Lond. Math. Soc. (2)* 93 (2016), pp. 846–865.
- [4] Allen B. Altman and Steven L. Kleiman. “Compactifying the Picard scheme”. In: *Adv. in Math.* 35.1 (1980), pp. 50–112.
- [5] Allen B. Altman and Steven L. Kleiman. “Compactifying the Picard scheme. II”. In: *Amer. J. Math.* 101.1 (1979), pp. 10–41.
- [6] Allen B. Altman and Steven L. Kleiman. “Foundations of the theory of Fano schemes”. In: *Compositio Math.* 34.1 (1977), pp. 3–47.
- [7] D. Arinkin. “Autoduality of compactified Jacobians for curves with plane singularities”. In: *J. Algebraic Geom.* 22 (2013), pp. 363–388.
- [8] A. Bayer and E. Macrì. “MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations”. In: *Invent. Math.* 198 (2014), pp. 505–590.
- [9] A. Bayer and E. Macrì. “Projectivity and birational geometry of Bridgeland moduli spaces”. In: *J. Amer. Math. Soc.* 27 (2014), pp. 707–752.
- [10] A. Bayer and E. Macrì. “The space of stability conditions on the local projective plane”. In: *Duke Math. J.* 160 (2011), pp. 263–322.
- [11] A. Bayer et al. “Stability conditions in family”. In: (2019). eprint: [arXiv:1902.08184](https://arxiv.org/abs/1902.08184).
- [12] A. Beauville. “Variétés Kähleriennes dont la première classe de Chern est nulle”. In: *J. Differential Geom.* 18 (1983), pp. 755–782.

- [13] Arnaud Beauville. “Variétés de Prym et jacobiniennes intermédiaires”. In: *Ann. Sci. École Norm. Sup. (4)* 10.3 (1977), pp. 309–391.
- [14] Arnaud Beauville and Ron Donagi. “La variété des droites d’une hypersurface cubique de dimension 4”. In: *C. R. Acad. Sci. Paris Sér. I Math.* 301.14 (1985), pp. 703–706.
- [15] T. Beckmann. *Atomic objects on hyper-Kähler manifolds*. eprint [arXiv:2202.01184](https://arxiv.org/abs/2202.01184). 2022.
- [16] T. Beckmann. “Derived categories of hyper-Kähler manifolds via the LLV algebra”. In: *Milan J. Math.* 90.2 (2022), pp. 445–458.
- [17] Thorsten Beckmann. “Derived categories of hyper-Kähler manifolds: extended Mukai vector and integral structure”. In: *Compos. Math.* 159.1 (2023), pp. 109–152.
- [18] Fedor Bogomolov. “On the cohomology ring of a simple hyper-Kähler manifold (on the results of Verbitsky)”. In: *Geom. Funct. Anal.* 6.4 (1996), pp. 612–618.
- [19] Alessio Bottini. *Stable Sheaves on K3 Surfaces via Wall-Crossing*. 2021. eprint: [2103.09661](https://arxiv.org/abs/2103.09661).
- [20] Alessio Bottini. “The Looijenga-Lunts-Verbitsky algebra and Verbitsky’s theorem”. In: *Milan J. Math.* 90.2 (2022), pp. 403–416.
- [21] Alessio Bottini. “Towards a modular construction of OG10”. In: (2022). eprint: [arXiv:2211.09033](https://arxiv.org/abs/2211.09033).
- [22] T. Bridgeland. “Stability conditions on K3 surfaces”. In: *Duke Math. J.* 141 (2008), pp. 241–291.
- [23] T. Bridgeland. “Stability conditions on triangulated categories”. In: *Ann. of Math. (2)* 166 (2007), pp. 317–345.
- [24] Tom Bridgeland, Alastair King, and Miles Reid. “The McKay correspondence as an equivalence of derived categories”. In: *J. Amer. Math. Soc.* 14.3 (2001), pp. 535–554. ISSN: 0894-0347. DOI: [10.1090/S0894-0347-01-00368-X](https://doi.org/10.1090/S0894-0347-01-00368-X). URL: <https://doi.org/10.1090/S0894-0347-01-00368-X>.
- [25] Andrei Horia Caldararu. *Derived categories of twisted sheaves on Calabi-Yau manifolds*. Thesis (Ph.D.)—Cornell University. ProQuest LLC, Ann Arbor, MI, 2000, p. 196.
- [26] A. Căldăraru, S. Katz, and E. Sharpe. “D-branes, B fields, and Ext groups”. In: *Adv. Theor. Math. Phys.* 7 (2003), pp. 381–404.

- [27] Andrei Căldăraru. “The Mukai pairing. II. The Hochschild-Kostant-Rosenberg isomorphism”. In: *Adv. Math.* 194.1 (2005), pp. 34–66.
- [28] Andrei Căldăraru and Simon Willerton. “The Mukai pairing. I. A categorical approach”. In: *New York J. Math.* 16 (2010), pp. 61–98.
- [29] Mark Andrea A. de Cataldo, Antonio Rapagnetta, and Giulia Saccà. “The Hodge numbers of O’Grady 10 via Ngô strings”. In: *J. Math. Pures Appl.* (9) 156 (2021), pp. 125–178.
- [30] C. Herbert Clemens and Phillip A. Griffiths. “The intermediate Jacobian of the cubic threefold”. In: *Ann. of Math.* (2) 95 (1972), pp. 281–356.
- [31] A. Collino. “The fundamental group of the Fano surface”. In: *Algebraic threefolds (Varenna, 1981)*. Vol. 947. Lecture Notes in Math. Berlin-New York: Springer, 1982, pp. 209–220.
- [32] Ron Donagi and Tony Pantev. “Torus fibrations, gerbes, and duality”. In: *Mem. Amer. Math. Soc.* 193.901 (2008). With an appendix by Dmitry Arinkin, pp. vi+90.
- [33] B. van den Dries. “Degenerations of cubic fourfolds and holomorphic symplectic geometry”. PhD thesis. Ridderkerk: Ridderprint, 2012.
- [34] A. Ferretti. “The Chow ring of double EPW sextics”. PhD thesis. Roma: Università di Roma ‘La Sapienza’, 2009.
- [35] L. Fu, C. Li, and X. Zhao. “Stability manifolds of varieties with finite Albanese morphism”. In: (2022). eprint: [arXiv:2103.07728](https://arxiv.org/abs/2103.07728).
- [36] Akira Fujiki. “On primitively symplectic compact Kähler V -manifolds of dimension four”. In: *Classification of algebraic and analytic manifolds (Katata, 1982)*. Vol. 39. Progr. Math. Birkhäuser Boston, Boston, MA, 1983, pp. 71–250.
- [37] Akira Fujiki. “On the de Rham cohomology group of a compact Kähler symplectic manifold”. In: *Algebraic geometry, Sendai, 1985*. Vol. 10. Adv. Stud. Pure Math. North-Holland, Amsterdam, 1987, pp. 105–165.
- [38] Daniel Greb, Christian Lehn, and Sönke Rollenske. “Lagrangian fibrations on hyperkähler manifolds—on a question of Beauville”. In: *Ann. Sci. Éc. Norm. Supér.* (4) 46.3 (2013), 375–403 (2013).
- [39] Mark Green et al. “The LLV decomposition of hyper-Kähler cohomology”. In: (2020). eprint: [arXiv:1906.03432](https://arxiv.org/abs/1906.03432).
- [40] Mark Haiman. “Hilbert schemes, polygraphs and the Macdonald positivity conjecture”. In: *J. Amer. Math. Soc.* 14.4 (2001), pp. 941–1006.

- [41] H. Hartmann. “Cusps of the Kähler moduli space and stability conditions on K3 surfaces”. In: *Math. Ann.* 354.1 (2012), pp. 1–42.
- [42] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg,, 1977, pp. xvi+496. ISBN: 0-387-90244-9.
- [43] Shengyuan Huang. “A note on a question of Markman”. In: *J. Pure Appl. Algebra* 225.9 (2021), Paper No. 106673, 8.
- [44] D. Huybrechts. “Birational symplectic manifolds and their deformations”. In: *J. Diff. Geom.* 45.3 (1997), pp. 488–513.
- [45] D. Huybrechts. “Birational symplectic manifolds and their deformations”. In: *J. Differential Geom.* 45 (1997), pp. 488–513.
- [46] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. Oxford: Oxford University Press, 2006.
- [47] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, 2006.
- [48] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge: Cambridge University Press, 2010.
- [49] D. Huybrechts and P. Stellari. “Equivalences of twisted K3 surfaces”. In: *Math. Ann.* 332.4 (2005), pp. 901–936.
- [50] Daniel Huybrechts. “A global Torelli theorem for hyperkähler manifolds [after M. Verbitsky]”. In: 348. Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027–1042. 2012, Exp. No. 1040, x, 375–403.
- [51] Daniel Huybrechts. “Compact hyper-Kähler manifolds: basic results”. In: *Invent. Math.* 135.1 (1999), pp. 63–113.
- [52] Daniel Huybrechts. “Compact hyperkähler manifolds”. In: *Calabi-Yau manifolds and related geometries (Nordfjordeid, 2001)*. Universitext. Springer, Berlin, 2003, pp. 161–225.
- [53] Daniel Huybrechts. *The geometry of cubic hypersurfaces*. Vol. 206. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2023.
- [54] Daniel Huybrechts and Paolo Stellari. “Equivalences of twisted K3 surfaces”. In: *Math. Ann.* 332.4 (2005), pp. 901–936.
- [55] Daniel Huybrechts and Richard Thomas. “ \mathbb{P} -objects and autoequivalences of derived categories”. In: *Math. Res. Lett.* 13 (2006), pp. 87–98.

- [56] Luc Illusie. *Complexe cotangent et déformations. I*. Springer-Verlag, Berlin-New York, 1971, pp. xv+355.
- [57] M. Inaba. “Smoothness of the moduli space of complexes of coherent sheaves on an abelian or a projective K3 surface”. In: *Adv. Math.* 227.3 (2011), pp. 1399–1412.
- [58] M. Inaba. “Toward a definition of moduli of complexes of coherent sheaves on a projective scheme”. In: *J. Math. Kyoto Univ.* 42.2 (2002), pp. 317–329.
- [59] Nathan Jacobson. “Completely reducible Lie algebras of linear transformations”. In: *Proc. Amer. Math. Soc.* 2 (1951), pp. 105–113.
- [60] D. Kaledin, M. Lehn, and Ch. Sorger. “Singular symplectic moduli spaces”. In: *Invent. Math.* 164.3 (2006), pp. 591–614.
- [61] S. Kobayashi. “Simple vector bundles over symplectic Kähler manifolds”. In: *Proc. Japan Acad. Ser. A Math. Sci.* 62 (1986), pp. 21–24.
- [62] Janos Kollár. “Hulls and Husks”. In: (2009). eprint: [arXiv:0805.0576](https://arxiv.org/abs/0805.0576).
- [63] S. Kuleshov. “Stable bundles on a K3 surface”. In: *Izv. Akad. Nauk SSSR Ser. Mat.* 54.1 (1990), pp. 213–220.
- [64] A. Kuznetsov and D. Markushevich. “Symplectic structures on moduli spaces of sheaves via the Atiyah class”. In: *J. Geom. Phys.* 59 (2009), pp. 843–860.
- [65] Herbert Lange. “Universal families of extensions”. In: *J. Algebra* 83.1 (1983), pp. 101–112.
- [66] Radu Laza, Giulia Saccà, and Claire Voisin. “A hyper-Kähler compactification of the intermediate Jacobian fibration associated with a cubic 4-fold”. In: *Acta Math.* 218.1 (2017), pp. 55–135.
- [67] Christian Lehn. “Deformations of Lagrangian subvarieties of holomorphic symplectic manifolds”. In: *Math. Res. Lett.* 23.2 (2016), pp. 473–497.
- [68] Christian Lehn et al. “Twisted cubics on cubic fourfolds”. In: *J. Reine Angew. Math.* 731 (2017), pp. 87–128.
- [69] M. Lieblich. “Moduli of complexes on a proper morphism”. In: *J. Algebraic Geom.* 15.2 (2006), pp. 175–206.
- [70] Max Lieblich. “Moduli of twisted sheaves”. In: *Duke Math. J.* 138.1 (2007), pp. 23–118.
- [71] E. Looijenga and V. Lunts. “A Lie algebra attached to a projective variety”. In: *Invent. Math.* 129 (1997), pp. 361–412.

- [72] Eduard Looijenga and Valery A. Lunts. “A Lie algebra attached to a projective variety”. In: *Invent. Math.* 129.2 (1997), pp. 361–412.
- [73] Wendy Lowen and Michel Van den Bergh. “Deformation theory of abelian categories”. In: *Trans. Amer. Math. Soc.* 358.12 (2006), pp. 5441–5483.
- [74] Wendy Lowen and Michel Van den Bergh. “Hochschild cohomology of abelian categories and ringed spaces”. In: *Adv. Math.* 198.1 (2005), pp. 172–221.
- [75] Emanuele Macrì and Benjamin Schmidt. “Stability and applications”. In: (2020). eprint: [arXiv:2002.01242](https://arxiv.org/abs/2002.01242).
- [76] Emanuele Macrì and Paolo Stellari. “Infinitesimal derived Torelli theorem for $K3$ surfaces”. In: *Int. Math. Res. Not. IMRN* 17 (2009). With an appendix by Sukhendu Mehrotra, pp. 3190–3220.
- [77] E. Markman. “Stable vector bundles on a hyper-Kähler manifold with a rank 1 obstruction map are modular”. In: *To appear on Kyoto J. Math.* ().
- [78] E. Markman. “The Beauville-Bogomolov class as a characteristic class”. In: *J. Algebraic Geom.* 29 (2020), pp. 199–245.
- [79] Eyal Markman. “A survey of Torelli and monodromy results for holomorphic-symplectic varieties”. In: *Complex and differential geometry*. Vol. 8. Springer Proc. Math. Springer, Heidelberg, 2011, pp. 257–322.
- [80] Eyal Markman. “Lagrangian fibrations of holomorphic-symplectic varieties of $K3^{[n]}$ -type”. In: *Algebraic and complex geometry*. Vol. 71. Springer Proc. Math. Stat. Pp. 241–283.
- [81] Eyal Markman. “On the monodromy of moduli spaces of sheaves on $K3$ surfaces”. In: *J. Algebraic Geom.* 17.1 (2008), pp. 29–99.
- [82] Daisuke Matsushita. “Addendum: “On fibre space structures of a projective irreducible symplectic manifold” [Topology **38** (1999), no. 1, 79–83; MR1644091 (99f:14054)]”. In: *Topology* 40.2 (2001), pp. 431–432.
- [83] Daisuke Matsushita. “Higher direct images of dualizing sheaves of Lagrangian fibrations”. In: *Amer. J. Math.* 127.2 (2005), pp. 243–259.
- [84] F. Meazzini and C. Onorati. “Hyper-holomorphic connections on vector bundles on hyper-Kähler manifolds”. In: *arXiv e-prints* (2022). eprint: [2204.04075](https://arxiv.org/abs/2204.04075).
- [85] B. Mladenov. *Degeneration of spectral sequences and complex Lagrangian submanifolds*. 2019. eprint: [arXiv:1907.04742](https://arxiv.org/abs/1907.04742).

- [86] Giovanni Mongardi, Antonio Rapagnetta, and Giulia Saccà. “The Hodge diamond of O’Grady’s six-dimensional example”. In: *Compos. Math.* 154.5 (2018), pp. 984–1013.
- [87] S. Mukai. “Symplectic structure of the moduli space of sheaves on an abelian or K3 surface”. In: *Invent. Math.* 77 (1984), pp. 101–116.
- [88] S. Mukai. “Symplectic structure of the moduli space of sheaves on an abelian or K3 surface”. In: *Invent. Math.* 77 (1984), pp. 101–116. DOI: [10.1007/BF01388892](https://doi.org/10.1007/BF01388892).
- [89] Shigeru Mukai. “On the Moduli Space of Bundles on K3 Surfaces. I”. In: *Vector Bundles on Algebraic Varieties (Bombay, 1984)*. Tata Inst. Fund. Res. Stud. Math. 11. Bombay: Tata Inst. Fund. Res., 1987, pp. 341–413.
- [90] Marc A. Nieper. “Hirzebruch-Riemann-Roch formulae on irreducible symplectic Kähler manifolds”. In: *J. Algebraic Geom.* 12.4 (2003), pp. 715–739.
- [91] K. O’Grady. “A new six-dimensional irreducible symplectic variety”. In: *J. Algebraic Geom.* 12 (2003), pp. 435–505.
- [92] K. O’Grady. “Desingularized moduli spaces of sheaves on a K3”. In: *J. Reine Angew. Math.* 512 (1999), pp. 49–117.
- [93] Kieran O’Grady. “Modular sheaves on hyperkähler varieties”. In: *Algebr. Geom.* 9 (2022), pp. 1–38.
- [94] Kieran O’Grady. *Moduli of sheaves on K3’s and higher dimensional HK varieties*. 2021. eprint: [arXiv:2109.07425](https://arxiv.org/abs/2109.07425).
- [95] Kieran O’Grady. “Rigid stable vector bundles on hyperkähler varieties of type K3^[n]”. In: (2022). eprint: [arXiv:2211.08970](https://arxiv.org/abs/2211.08970).
- [96] Kieran O’Grady. “The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface”. In: *J. Algebraic Geom.* 6 (1997), pp. 599–644. DOI: [10.1090/S1056-3911-97-00197-4](https://doi.org/10.1090/S1056-3911-97-00197-4).
- [97] Georg Oberdieck and Jieao Song. *Representation theory and cohomology*. 2022.
- [98] Keiji Oguiso. “Picard number of the generic fiber of an abelian fibered hyperkähler manifold”. In: *Math. Ann.* 344.4 (2009), pp. 929–937.
- [99] A. Perego and A. Rapagnetta. “The moduli spaces of sheaves on K3 surfaces are irreducible symplectic varieties”. In: (2018). eprint: [arXiv:1802.01182](https://arxiv.org/abs/1802.01182).
- [100] David Ploog. “Equivariant autoequivalences for finite group actions”. In: *Adv. Math.* 216.1 (2007), pp. 62–74.

- [101] Paul Seidel and Richard Thomas. “Braid group actions on derived categories of coherent sheaves”. In: *Duke Math. J.* 108 (2001), pp. 37–108.
- [102] Jean-Pierre Serre. *Algèbre locale. Multiplicités*. Vol. 11. Lecture Notes in Math. Berlin-New York: Springer, 1965, pp. vii+188.
- [Stacks] The Stacks Project Authors. *Stacks Project*. <https://stacks.math.columbia.edu>. 2021.
- [103] Lenny Taelman. *Derived equivalences of hyperkähler varieties*. 2021. eprint: [arXiv:1906.08081](https://arxiv.org/abs/1906.08081).
- [104] Y. Toda. “Deformations and Fourier-Mukai transforms”. In: *J. Differential Geom.* 81 (2009), pp. 197–224.
- [105] Yukinobu Toda. “Moduli stacks and invariants of semistable objects on K3 surfaces”. In: *Adv. Math.* 217 (2008), pp. 2736–2781. DOI: [10.1016/j.aim.2007.11.013](https://doi.org/10.1016/j.aim.2007.11.013).
- [106] Mikhail Verbitsky. “Action of the Lie algebra of $SO(5)$ on the cohomology of a hyper-Kähler manifold”. In: *Funktsional. Anal. i Prilozhen. translation in Funct. Anal. Appl* 24.3 (1990), pp. 70–71.
- [107] Mikhail Verbitsky. *Cohomology of compact hyperkaehler manifolds*. PhD thesis at Harvard University. 1995.
- [108] Mikhail Verbitsky. “Hyperholomorphic bundles over a hyper-Kähler manifold”. In: *J. Algebraic Geom.* 5.4 (1996), pp. 633–669.
- [109] Mikhail Verbitsky. “Mirror symmetry for hyper-Kähler manifolds”. In: *Mirror symmetry, III (Montreal, PQ, 1995)*. Vol. 10. AMS/IP Stud. Adv. Math. Amer. Math. Soc., Providence, RI, 1999, pp. 115–156.
- [110] Misha Verbitsky. “Coherent sheaves on general K3 surfaces and tori”. In: *Pure Appl. Math. Q.* 4.3, Special Issue: In honor of Fedor Bogomolov. Part 2 (2008), pp. 651–714.
- [111] Misha Verbitsky. “Cohomology of compact hyper-Kähler manifolds and its applications”. In: *Geom. Funct. Anal.* 6 (1996), pp. 601–611.
- [112] Misha Verbitsky. “Hyperholomorphic sheaves and new examples of hyperkähler manifolds”. In: *Hyperkähler manifolds*. Mathematical Physics 12. Somerville, MA: International Press, 1999.
- [113] Misha Verbitsky. “Mapping class group and a global Torelli theorem for hyperkähler manifolds”. In: *Duke Math. J.* 162 (2013), pp. 2929–2986.
- [114] Claire Voisin. *Hodge theory and complex algebraic geometry. I*. English. Vol. 76. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2007, pp. x+322. ISBN: 978-0-521-71801-1.

- [115] Claire Voisin. “Hyper-Kähler compactification of the intermediate Jacobian fibration of a cubic fourfold: the twisted case”. In: *Local and global methods in algebraic geometry*. Vol. 712. Contemp. Math. Amer. Math. Soc., [Providence], RI, 2018, pp. 341–355.
- [116] Claire Voisin. “Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes”. In: *Complex projective geometry (Trieste, 1989/Bergen, 1989)*. Vol. 179. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1992, pp. 294–303.
- [117] Claire Voisin. “Théorème de Torelli pour les cubiques de \mathbf{P}^5 ”. In: *Invent. Math.* 86.3 (1986).
- [118] K. Yoshioka. “Moduli spaces of stable sheaves on abelian surfaces”. In: *Math. Ann.* 321 (2001), pp. 817–884.
- [119] K. Yoshioka. “Moduli spaces of twisted sheaves on a projective variety”. In: *Moduli spaces and arithmetic geometry*. Vol. 45. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2006, pp. 1–30.